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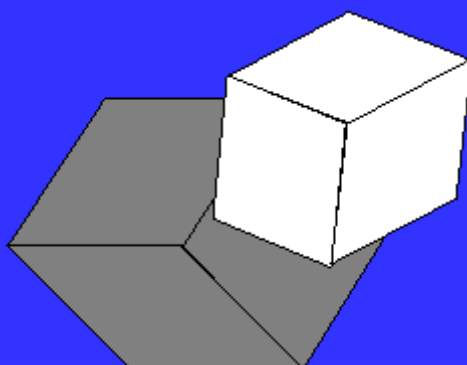
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Table of contents

Moiz Ud Din Khan, Rafaqat Noreen <i>On fuzzy minimal structures</i>	1-15
Samira Naji Kadhim, Muna Jasim Mohammed Ali, Zainab Abed Atiya <i>A note on S-acts and bounded linear operators</i>	16-22
Raja Das, Madhu Sudan Reddy <i>Application of recurrent neural network using Matlab simulink in medicine</i>	23–30
Watheq Bani-Domi <i>Generalized numerical radius inequalities for 2×2 operator matrices</i>	31-38
K.P. Thilagavathy, A. Santha <i>A comparative study on achromatic and B-chromatic number of certain graphs</i>	39-44
Jinbao Li, Guiyun Chen <i>A weaker quantitative characterization of the sporadic simple groups</i>	45-54
Zeinab Khanjanzadeh, Ali Madanshekaf <i>Lawvere-Tierney sheaves, factorization systems, sections and j-essential monomorphisms in a topos</i>	55-72
Baojie Zhang, Hongxing Li, Zitian Li <i>HX-type chaotic (hyperchaotic) system based on fuzzy inference modeling</i>	73-88
H. Shabani, A. R. Ashrafi, E. Haghi, M. Ghorbani <i>The Q-conjugacy character table of dihedral groups</i>	89-96
E. Mohammadzadeh, F. Mohammadzadeh <i>α-nilpotent groups derived from hypergroups with ξ^*-relation</i>	97-106
Hardik P. Patel, G.M. Deheri, R.M. Patel <i>Combined effect of magnetism and roughness on a ferrofluid squeeze film in porous truncated conical plates: effect of variable boundary conditions</i>	107-119
Y. Madhavi Reddy, P. Venkat Raman, E. Keshava Reddy <i>On some properties of rough approximations of subrings via cosets</i>	120-127
N. Jahanbakhsh Basharlou, M.J. Nikmehr, R. Nikandish <i>On generalized zero-divisor graph associated with a commutative ring</i>	128-139
Chang-Wen Peng <i>The fixed point of meromorphic solutions for difference Riccati equation</i>	140-153
Hamed M. Obiedat, Ibraheem Abu-Falahah <i>Structure of (w_1, w_2)-tempered ultradistribution using short-time Fourier transform</i>	154–164
Nedal Tahat, E.S. Ismail, A.K. Alomari <i>Partially blind signature scheme based on chaotic maps and factoring problems</i>	165-177
Xinyang Feng, Jian Tang, Yanfeng Luo <i>A study on pseudoorders in ordered $\ast$-semihypergroups</i>	178-193
A.F. Jameel, N.R. Anakira, M.M. Rashidi, A.K. Alomari, A. Saaban, M.A. Shakhatreh <i>Differential transformation method for solving high order fuzzy initial value problems</i>	194-208

Mohammad Abry, Jafar Zanjani

Some properties of zero gradations on Samanta fuzzy topological spaces 209-219

Mohammad A. Safi

Mathematical analysis of an age-structured quarantine/isolation model 220-242

T. Phaneendra

Banach and Kannan contractions on S -metric space 243-247

T. Phaneendra

Fixed point theorems for one and two self-maps on a G -metric space 248-257

Malik Bataineh, Rashid Abu-Dawwas, Wurood Oteir

Generalizations of prime ternary subsemimodules of ternary semimodules 258-268

G. Basava Kumar, M.N. Srinivas

Impact of harvesting, noise and diffusion on the dynamics of a food chain model with ratio-dependent functional response III 269-289

Jian Tang, Xiaolong Xin, Xiangyun Xie

Characterizations of ordered semihypergroups based on ordered fuzzy points 290-311

M.R. Rakhshani, R.A. Borzooei, G.R. Rezaei

On topological effect algebras 312-325

Tariq Al-Hawary

A certain new familiar class of univalent analytic functions with varying argument of coefficients involving convolution 326-333

A. Namdar, R.A. Borzooei

Special hoop algebras 334-349

Ahmed Ali Atash, Salem Saleh Barahmah

On some generating functions for the two-parameters one-variable Srivastava polynomials 350-358

S.K. Hui, V.N. Mishra, T.Pal, Vandana

Some classes of invariant submanifolds of $(LCS)_n$ -manifolds 359-372

Inderdeep Singh, Sheo Kumar

Haar wavelet collocation method for solving nonlinear Kuramoto–Sivashinsky equation 373-384

Yunhua Zhu

Application of parameter optimization algorithm in digital architecture design 385-392

Doostali Mojdeh, Mohammad Habibi, Leila Badakhshian

Total and connected domination in chemical graphs 393-401

Yili Tan, Xinghuo Wan, Meilin Zhang

The Borda rule comprehensive evaluation method based on spearman rank 402-409

K.A. Challab, M. Darus, F. Ghanim

On subclass of meromorphic univalent functions defined by a linear operator associated with λ -generalized Hurwitz-Lerch zeta function and q -hypergeometric function 410-423

Yi Hu, Linna Wang

The realization of radio frequency identification handset antenna based on internet of things 424-433

K. Ravibabu, Ch. Srinivasa Rao and Ch. Raghavendra Naidu

Coupled fixed and coincidence point theorems for generalized contractions in metric spaces with a partial order 434-450

Mahmood Alizadeh, Mohammad Reza Darafsheh, Saeid Mehrabi

On the k -normal elements and polynomials over finite fields 451-464

Naveed Yaqoob, Irina Cristea, Muhammad Gulistan, Shah Nawaz <i>Left almost polygroups</i>	465-474
H. Shojaei, R. Ameri, S. Hoskova-Mayerova <i>On properties of various morphisms in the categories of general Krasner hypermodules</i>	475-484
Li Zhang, Jianming Zhan <i>A study on bijective soft hemirings</i>	485-497
Jaroslav Seibert, Libor Koudela <i>On the Fibonacci numbers of the molecular graphs of some bent phenylenes</i>	498-507
Arshad Khan, Sucheta Nayak, R.K. Mohanty <i>Off-step discretization for system of nonlinear singular boundary value problems using variable mesh</i>	508-529
A. Javadi, F. Fayazi, A. Gholami <i>Some new results related to subgroup commutativity degrees and p-commutativity degrees of finite groups</i>	530-543
Abdolrahman Razani <i>Chapman-Jouguet travelling wave for a two-steps reaction scheme</i>	544-553
Razieh Mahjoob, Vahid Ghaffari <i>Zariski topology for second subhypermodules</i>	554-568
Mamoon Ahmed and Fida Moh'd <i>The Hecke algebra $H(P_{\mathbb{Q}}, P_{\mathbb{Z}})$ and its relation to the crossed product $H(P_{\mathbb{Q}}^+, P_{\mathbb{Z}}) \times_{\beta} \{1, -1\}$</i>	569-578
Kuldip Raj, S.A. Mohiuddine, M. Ayman Mursaleen <i>Some generalized sequence spaces of invariant means defined by ideal and modulus functions in N-normed spaces</i>	579-595
Chao Wei <i>Parameter estimation for a class of diffusion process from discrete observation</i>	596-607
Nazihah Ahmad, Baravan A. Asaad <i>More properties of an operation on semi-generalized open sets</i>	608-627
M. Ghorbani, M. Songhori, M. Rajabi Parsa <i>Normal edge-transitive Cayley graphs whose order are a product of three primes</i>	628-635
Guochao Zhang, Qingming Gui, Changran Duan, Peng Zhao <i>A Bayesian method to fit an ARMA Model</i>	636-648
Li Zhang, Xueling Ma, Jianming Zhan <i>Rough soft BCK-algebras and their decision making</i>	649-659
K. Bhuvaneswari, T. Kalyani <i>Iso-arrays and conditional communication on P system</i>	660-671
Yongqiang Zhan <i>A high-order accuracy explicit difference scheme with branching stability for solving four-dimensional parabolic equations</i>	672-682
Artion Kashuri, Rozana Liko <i>Hermite-Hadamard type inequalities for generalized (s, m, φ)-preinvex Godunova-Levin functions</i>	683-700
Sarka Hoskova-Mayerova, Antonio Maturo <i>Algebraic hyperstructures and social relations</i>	701-709
Masahiro Igarashi <i>Note on relations among multiple zeta(-star) values</i>	710-756
H. Benbouziane, M. Ech-Cherif El Kettani, A.M. Vadel <i>Linear maps on $M_n(\mathbb{C})$ preserving inner local spectral radius zero</i>	757-763

Jianjun Liu, Guiyun Chen <i>An equivalent definition of a $\mathcal{C}$-group</i>	764-770
M.K. Azam, Fiza Zafar, M.A. Rehman, F. Ahmad, Shahid Qaisar <i>Study of extended Weyl k-fractional integral via Chebyshev inequalities</i>	771-782
S. Al Wadi <i>Modification of accuracy estimation using stock market data</i>	783-792
Wenjuan Chen, Bijan Davvaz <i>Hyper quasi-MV algebras and ideals</i>	793-809
Fabrizio Maturo <i>On an extension of the Dubins conditional probability axiomatic to coherent probability of fuzzy events</i>	810-821
Waqar Khan, Kostaq Hila, Guiyun Chen <i>Sandwich sets and congruences in completely inverse AG^{**}-groupoids</i>	822-838
Pouyan Khamechi, Leila Nouri, Hossein Mohammadzadeh Saany <i>Some generalization of subpullback flat</i>	839-852

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12. Analele Științifice ale Universității “Al. I. Cuza” - Iași	RO
13. Analele Universității din Timișoara - Universitatea din Timișoara	RO
14. Annales Academiae Scientiarum Fennicae Mathematica - Helsinki	SW
15. Annales de la Fondation Louis de Broglie - Paris	F
16. Annales Mathematicae Silesianae – Katowice	PL
17. Annales Scientif. Université Blaise Pascal - Clermont II	F
18. Annales sect. A/Mathematica – Lublin	PL
19. Annali dell’Università di Ferrara, Sez. Matematica	I
20. Annals of Mathematics - Princeton - New Jersey	USA
21. Applied Mathematics and Computer Science -Technical University of Zielona Góra	PL
22. Archivium Mathematicum - Brno	CZ
23. Atti del Seminario di Matematica e Fisica dell’Università di Modena	I
24. Atti dell’Accademia delle Scienze di Ferrara	I
25. Automatika i Telemekhanika - Moscow	RU
26. Boletim de la Sociedade Paranaense de Matematica - San Paulo	BR
27. Bolétin de la Sociedad Matemática Mexicana - Mexico City	MEX
28. Bollettino di Storia delle Scienze Matematiche - Firenze	I
29. Buletinul Academiei de Științe - Seria Matem. - Kishinev, Moldova	CSI
30. Buletinul Științific al Universității din Baia Mare - Baia Mare	RO
31. Buletinul Științific și Tehnic-Univ. Math. et Phys. Series Techn. Univ. - Timișoara	RO
32. Buletinul Universității din Brașov, Seria C - Brașov	RO
33. Bulletin de la Classe de Sciences - Acad. Royale de Belgique	B
34. Bulletin de la Société des Mathématiciens et des Informaticiens de Macedoine	MK
35. Bulletin de la Société des Sciences et des Lettres de Lodz - Lodz	PL
36. Bulletin de la Société Royale des Sciences - Liege	B
37. Bulletin for Applied Mathematics - Technical University Budapest	H
38. Bulletin Mathematics and Physics - Assiut	ET
39. Bulletin Mathématique - Skopje Macedonia	MK
40. Bulletin Mathématique de la S.S.M.R. - Bucharest	RO
41. Bulletin of the Australian Mathematical Society - St. Lucia - Queensland	AUS
42. Bulletin of the Faculty of Science - Assiut University	ET
43. Bulletin of the Faculty of Science - Mito, Ibaraki	J
44. Bulletin of the Greek Mathematical Society - Athens	GR
45. Bulletin of the Iranian Mathematical Society - Tehran	IR
46. Bulletin of the Korean Mathematical Society - Seoul	ROK
47. Bulletin of the Malaysian Mathematical Sciences Society - Pulau Pinang	MAL
48. Bulletin of Society of Mathematicians Banja Luka - Banja Luka	BiH
49. Bulletin of the Transilvania University of Brașov - Brașov	RO
50. Bulletin of the USSR Academy of Sciences - San Pietroburgo	RU
51. Busefal - Université P. Sabatier - Toulouse	F
52. Calculus CNR - Pisa	I
53. Chinese Annals of Mathematics - Fudan University – Shanghai	PRC

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55.	Classification of Commutative FPF Ring - Universidad de Murcia	E
56.	Collectanea Mathematica - Barcelona	E
57.	Collegium Logicum - Institut für Computersprachen Technische Universität Wien	A
58.	Colloquium - Cape Town	SA
59.	Colloquium Mathematicum - Instytut Matematyczny - Warszawa	PL
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65.	Demonstratio Mathematica - Warszawa	PL
66.	Discussiones Mathematicae - Zielona Gora	PL
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69.	Extracta Mathematicae - Badajoz	E
70.	Fasciculi Mathematici - Poznan	PL
71.	Filomat - University of Nis	SRB
72.	Forum Mathematicum - Mathematisches Institut der Universität Erlangen	D
73.	Functiones et Approximatio <i>Commentarii Mathematici</i> - Adam Mickiewicz University	L
74.	Funkcialaj Ekvaciaj - Kobe University	J
75.	Fuzzy Systems & A.I. Reports and Letters - Iași University	RO
76.	General Mathematics - Sibiu	RO
77.	Geometria - Fasciculi Mathematici - Poznan	PL
78.	Glasnik Matematički - Zagreb	CRO
79.	Grazer Mathematische Berichte – Graz	A
80.	Hiroshima Mathematical Journal - Hiroshima	J
81.	Hokkaido Mathematical Journal - Sapporo	J
82.	Houston Journal of Mathematics - Houston - Texas	USA
83.	IJMSI - Iranian Journal of Mathematical Sciences & Informatics, Tarbiat Modares University, Tehran	IR
84.	Illinois Journal of Mathematics - University of Illinois Library - Urbana	USA
85.	Informatica - The Slovene Society Informatika - Ljubljana	SLO
86.	Internal Reports - University of Natal - Durban	SA
87.	International Journal of Computational and Applied Mathematics – University of Qiongzhou, Hainan	PRC
88.	International Journal of Science of Kashan University - University of Kashan	IR
89.	Iranian Journal of Science and Technology - Shiraz University	IR
90.	Irish Mathematical Society Bulletin - Department of Mathematics - Dublin	IRL
91.	IRMAR - Inst. of Math. de Rennes - Rennes	F
92.	Israel Mathematical Conference Proceedings - Bar-Ilan University - Ramat -Gan	IL
93.	Izvestiya: Mathematics - Russian Academy of Sciences and London Mathematical Society	RU
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95.	Journal of Basic Science - University of Mazandaran – Babolsar	IR
96.	Journal of Beijing Normal University (Natural Science) - Beijing	PRC
97.	Journal of Dynamical Systems and Geometric Theory - New Delhi	IND
98.	Journal Egyptian Mathematical Society – Cairo	ET
99.	Journal of Mathematical Analysis and Applications - San Diego California	USA
100.	Journal of Mathematics of Kyoto University - Kyoto	J
101.	Journal of Science - Ferdowsi University of Mashhad	IR
102.	Journal of the Bihar Mathematical Society - Bhagalpur	IND
103.	Journal of the Faculty of Science – Tokyo	J
104.	Journal of the Korean Mathematical Society - Seoul	ROK
105.	Journal of the Ramanujan Mathematical Society - Mysore University	IND
106.	Journal of the RMS - Madras	IND
107.	Kumamoto Journal of Mathematics - Kumamoto	J
108.	Kyungpook Mathematical Journal - Taegu	ROK
109.	L'Enseignement Mathématique - Genève	CH
110.	La Gazette des Sciences Mathématiques du Québec - Université de Montréal	CAN
111.	Le Matematiche - Università di Catania	I
112.	Lecturas Matemáticas, Soc. Colombiana de Matematica - Bogotá	C
113.	Lectures and Proceedings International Centre for Theoretical Physics - Trieste	I

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116. Matematicna Knjiznica - Ljubljana	SLO
117. Mathematica Balcanica – Sofia	BG
118. Mathematica Bohemica - Academy of Sciences of the Czech Republic Praha	CZ
119. Mathematica Macedonica, St. Cyril and Methodius University, Faculty of Natural Sciences and Mathematics - Skopje	MK
120. Mathematica Montisnigri - University of Montenegro - Podgorica	MNE
121. Mathematica Moravica - Cacak	SRB
122. Mathematica Pannonica - Miskolc - Egyetemvaros	H
123. Mathematica Scandinavica - Aarhus - Copenhagen	DK
124. Mathematica Slovaca - Bratislava	CS
125. Mathematicae Notae - Universidad Nacional de Rosario	AR
126. Mathematical Chronicle - Auckland	NZ
127. Mathematical Journal - Academy of Sciences - Uzbekistan	CSI
128. Mathematical Journal of Okayama University - Okayama	J
129. Mathematical Preprint - Dep. of Math., Computer Science, Physics – University of Amsterdam	NL
130. Mathematical Reports - Kyushu University - Fukuoka	J
131. Mathematics Applied in Science and Technology – Sangyo University, Kyoto	J
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133. Mathematics for Applications - Institute of Mathematics of Brnő University of Technology, Brnő	CZ
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135. Mediterranean Journal of Mathematics – Università di Bari	I
136. Memoirs of the Faculty of Science - Kochi University - Kochi	J
137. Memorias de Mathematica da UFRJ - Instituto de Matematica - Rio de Janeiro	BR
138. Memorie lincee - Matematica e applicazioni - Accademia Nazionale dei Lincei	I
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141. Monthly Bulletin of the Mathematical Sciences Library – Abuja	WAN
142. Nagoya Mathematical Journal - Nagoya University, Tokyo	J
143. Neujahrsblatt der Naturforschenden Gesellschaft - Zürich	CH
144. New Zealand Journal of Mathematics - University of Auckland	NZ
145. Nieuw Archief voor Wiskunde - Stichting Mathematicae Centrum – Amsterdam	NL
146. Nihonkai Mathematical Journal - Niigata	J
147. Notas de Algebra y Analisis - Bahia Blanca	AR
148. Notas de Logica Matematica - Bahia Blanca	AR
149. Notas de Matematica Discreta - Bahia Blanca	AR
150. Notas de Matematica - Universidad de los Andes, Merida	YV
151. Notas de Matematicas - Murcia	E
152. Note di Matematica - Lecce	I
153. Novi Sad Journal of Mathematics - University of Novi Sad	SRB
154. Obzornik za Matematiko in Fiziko - Ljubljana	SLO
155. Octogon Mathematical Magazine - Brașov	RO
156. Osaka Journal of Mathematics - Osaka	J
157. Periodica Matematica Hungarica - Budapest	H
158. Periodico di Matematiche - Roma	I
159. Pliska - Sofia	BG
160. Portugaliae Mathematica - Lisboa	P
161. Posebna Izdanja Matematičkog Instituta Beograd	SRB
162. Pre-Publicações de Matematica - Univ. de Lisboa	P
163. Preprint - Department of Mathematics - University of Auckland	NZ
164. Preprint - Institute of Mathematics, University of Lodz	PL
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166. Proceeding of the School of Science of Tokai University - Tokai University	J
167. Proceedings - Institut Teknologi Bandung - Bandung	RI
168. Proceedings of the Academy of Sciences Tasked – Uzbekistan	CSI
169. Proceedings of the Mathematical and Physical Society of Egypt – University of Cairo	ET
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171. Publicaciones - Departamento de Matemática Universidad de Los Andes Merida	YV
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188. Rendiconti del Seminario Matematico - Università di Padova	I
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190. Rendiconti di Matematica e delle sue Applicazioni - Roma	I
191. Rendiconti lincei - Matematica e applicazioni - Accademia Nazionale dei Lincei	I
192. Rendiconti Sem. - Università di Cagliari	I
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194. Reports Math. University of Stockholm - Stockholm	SW
195. Reports - University Amsterdam	NL
196. Reports of Science Academy of Tajikistan - Dushanbe	TAJ
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198. Research Reports - University of Umea - Umea	SW
199. Research Report Collection (RGMIA) Melbourne	AUS
200. Resenhas do Instituto de Matemática e Estatística da Universidade de São Paulo	BR
201. Review of Research, Faculty of Science, Mathematics Series - Institute of Mathematics University of Novi Sad	SRB
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203. Revista Ciencias Matem. - Universidad de la Habana	C
204. Revista Colombiana de Matemáticas - Bogotá	C
205. Revista de Matemáticas Aplicadas - Santiago	CH
206. Revue Roumaine de Mathématiques Pures et Appliquées - București	RO
207. Ricerca Operativa AIRO - Genova	I
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209. Rivista di Matematica - Università di Parma	I
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211. Saitama Mathematical Journal - Saitama University	J
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221. Serdica Mathematical Journal - Bulgarian Academy of Sciences, University of Sofia	BG
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ON FUZZY MINIMAL STRUCTURES

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Abstract. In this paper, we define fuzzy open (closed) M -set, fuzzy M - frontier, fuzzy M -semi frontier, fuzzy rarely M -set, and fuzzy rarely M -continuous functions in fuzzy minimal spaces. We will explore several interesting properties and characterizations of these newly defined notions.

Keywords: fuzzy open (closed) M -set, fuzzy M - frontier, fuzzy M -semi frontier, fuzzy rarely M -set, fuzzy rarely M -continuous function, fuzzy minimal space.

1. Introduction

Fuzziness is one of the most important and useful concepts in the modern scientific studies. The fundamental concepts of Fuzzy sets were originally initiated by Zadeh [17]. Fuzzy sets attained a very important role in the study of Fuzzy topology whose pioneer was Chang [7]. Afterwards, Lowen [10] was a mathematician who worked for fuzzy compactness in fuzzy topological spaces. With this invent a range of aspects of general topology has been investigated and carried out in the Fuzzy sense by quite a few authors.

A fuzzy set is a collection of objects with grades of membership function. Such a set is represented by a membership or characteristic function, which assigns each object a score of membership ranging between 0 and 1. Many notions like intersection, complement, inclusion, convexity, union, relation, etc. are extended to fuzzy sets and various properties of these notions in the context of fuzzy sets are established and further, it has been observed to be very helpful in solving many realistic problems. Du. et al. [8] gave the fuzzified solution to the 9-intersection Egenhofer model for depicting topological relations in Geographic Information system (GIS) query. In [13],[14] El.Naschie proved that the concept of fuzzy topology can also be related to quantum particle physics.

For a set X , we define a fuzzy set in X to be a function $\mu : X \rightarrow [0, 1]$, where $\mu(x)$ represents the degree of membership of x in the fuzzy set μ . It is well

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known that the fuzzy set μ in X is a set of ordered pairs, i.e., $\mu = \{(x, \mu(x)) / x \in X\}$. The family of all fuzzy sets on X is denoted by I^X , consisting of all the mappings from X to $[0, 1]$. Any subset A of a set X can be identified with its characteristic function $\chi_A : X \rightarrow \{0, 1\}$ defined by

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases}$$

and such characteristic functions are fuzzy sets in X . The characteristic functions of subsets of a set X are referred to as the crisp fuzzy sets in X . Agd. El- Monsef and Ramadam introduced the concept of fuzzy supra- topology as follows. A collection of fuzzy sets $F < I^X$ is called a fuzzy supra topology on X if $\bar{0}, \bar{1} \in F$ and F is closed under arbitrary union. In continuation of the study of fuzzy sets, Alimohammady and Roohi [1], [2], [3], [4], [5], introduced and studied the notions of fuzzy minimal structures and fuzzy minimal spaces. It is pertinent to mention that the concept of minimal structures and minimal spaces was introduced by Maki [11],[12] in 1996.

A subfamily m_X of $P(X)$ is called a minimal structure [12] on X , if $\varphi \in m_X$ and $X \in m_X$. Each member of m_X is said to be an m_X -open set, and the complement of an m_X -open set is said to be an m_X -closed set. A minimal structure m_X on a non-empty set X is denoted by (X, m_X) . A family M of fuzzy sets in X is said to be a fuzzy minimal structure [3] on X , if $\bar{0} \in M$ and $\bar{1} \in M$. In this case (X, M) is called a fuzzy minimal space. Each member of M is said to be a fuzzy M -open set, and the complement of a fuzzy M -open set is said to be a fuzzy M -closed set. In [16], for a fuzzy set λ on X , $M-CI(\lambda)$ and $M-Int(\lambda)$ represents the closure and interior respectively with respect to the fuzzy minimal structure and are defined as follows :

$$\begin{aligned} M-CI(\lambda) &= \wedge \{\mu : \lambda \leq \mu; \mu^c \in M\}, \\ M-Int(\lambda) &= \vee \{\mu : \mu \leq \lambda; \mu \in M\}. \end{aligned}$$

where $FMC(X)$ (respectively, $FMO(X)$) represents the collection of all fuzzy M -closed (respectively, fuzzy M -open) sets in X .

2. On fuzzy open- M sets

In this section, we will introduce and study the concept of fuzzy open- M set (fuzzy closed- M set).

Definition 1. Let M be a fuzzy minimal structure on X . A fuzzy set λ on X is said to be an fuzzy open- M (respectively, fuzzy closed - M) set, if $M-Int(\lambda) = \lambda$ (respectively, $M-CI(\lambda) = \lambda$).

Definition 2. A fuzzy M -open (respectively fuzzy M -closed) set is fuzzy open- M (respectively fuzzy closed- M) set, but not conversely which follows from the following example.

Example 3. Let $X = \{a, b, c\}$, $M = \{\bar{0}, \bar{1}, \lambda, \mu, v\}$, where $\lambda = \{(a, 0.2), (b, 0.3), (c, 0)\}$, $\mu = \{(a, 0.2), (b, 0.3), (c, 0.1)\}$, $v = \{(a, 0.7), (b, 0.4), (c, 0.5)\}$.

$M-Int(v) = \vee\{\beta : \beta \leq v \text{ and } \beta \in M\} = \lambda \vee v \vee \mu = v$. This implies that v is fuzzy open- M set as well as fuzzy M -open set.

Again, for $M = \{\bar{0}, \bar{1}, \lambda, \mu, v\}$, where $\lambda = \{(a, 0.3)\}$, $\mu = \{(b, 0.4)\}$, $v = \{(c, 0.5)\}$. Let $\beta = \{(a, 0.3), (b, 0.4), (c, 0.5)\}$. Then $M-Int(\beta) = \lambda \vee \mu \vee v = \beta$. This gives that β is fuzzy open- M set, but it is not fuzzy M -open set, because $\beta \notin M$. Similarly β^c is a fuzzy closed- M set.

Lemma 4. For a fuzzy minimal space (X, M) , a fuzzy set λ on X is a fuzzy open- M set, if and only if $\lambda^c = \bar{1} - \lambda$ is a fuzzy closed- M set.

Theorem 5. In fuzzy minimal space (X, M) , the collection of all fuzzy open- M sets forms a fuzzy supra topology on X .

Proof. The fuzzy sets $\bar{0}, \bar{1}$ are obviously, fuzzy open- M sets. Let $\mu = \vee \{ \Gamma_\alpha : \alpha \in \Delta \}$ be a join of fuzzy open- M sets, we have $M-Int(\Gamma_\alpha) = \Gamma_\alpha$, for each α . Now $\Gamma_\alpha : \alpha \leq \mu$, so $\Gamma_\alpha = M-Int(\Gamma_\alpha) \leq \mu$ or $\vee\{\Gamma_\alpha : \alpha \in \Delta\} = \vee\{M-Int(\Gamma_\alpha : \alpha \in \Delta)\} \leq \mu$, for each α . Hence $\mu = \vee\{\Gamma_\alpha : \alpha \in \Delta\} = \vee\{M-Int(\Gamma_\alpha) : \alpha \in \Delta\} \leq M-Int\{\vee\Gamma_\alpha : \alpha \in \Delta\} \leq M-Int(\mu)$. But we know that $M-Int(\mu) \leq \mu$. This shows that $\mu = M-Int(\mu)$. This proves that arbitrary union of fuzzy open - M sets is an fuzzy open - M set. $\square$

Remark 6. Arbitrary (even finite) intersections (respectively, unions) of fuzzy open- M sets (respectively, fuzzy closed- M) sets need not to be fuzzy open - M set (respectively, fuzzy closed- M set). This is evident from the following example.

Example 7. Let $X = \{a, b, c\}$ and $M = \{\bar{0}, \bar{1}, \lambda, \mu, v\}$, where $\lambda = \{(a, 0.3), (b, 0), (c, 0)\}$, $\mu = \{(a, 0), (b, 0.4), (c, 0.5)\}$, $v = \{(a, 0.2), (b, 0.8), (c, 0)\}$ and let $\alpha = \{(a, 0.3), (b, 0.4), (c, 0.5)\}$, $\beta = \{(a, 0.2), (b, 0.8), (c, 0.5)\}$, $\gamma = \{(a, 0.7), (b, 0.6), (c, 0.5)\}$, $\delta = \{(a, 0.8), (b, 0.2), (c, 0.5)\}$. Then $M-Int(\alpha) = \alpha$, $M-Int(\beta) = \beta$, $M-Int(\alpha \wedge \beta) \neq \alpha \wedge \beta$, $M-Cl(\gamma) = \gamma$, $M-Cl(\delta) = \delta$ and $M-Cl(\gamma \vee \delta) \neq \gamma \vee \delta$.

3. Fuzzy M -frontiers and fuzzy M -semi frontiers

In this section we will give more than a few properties of fuzzy M -frontier of a set and fuzzy M -semi frontiers of a set in fuzzy minimal spaces and these concepts are further supported by examples.

3.1 Fuzzy M -frontiers

Definition 8. Let X be a non-empty set, which has a fuzzy minimal structure M and let Γ be a fuzzy set on X . Then fuzzy M -frontier of Γ in a fuzzy minimal space (X, M) is denoted by $M-Fr(\Gamma)$ and defined by $M-Fr(\Gamma) = M-Cl(\Gamma) \wedge M-Cl(\bar{1} - \Gamma)$.

Theorem 9. Let (X, M) be a fuzzy minimal space and Γ be a fuzzy set, then:

- 1) $M-Fr(\Gamma) = M-Fr(\Gamma^c)$;
- 2) If Γ is a fuzzy M -closed set, then $M-Fr(\Gamma) \leq \Gamma$;
- 3) If Γ is fuzzy M -open set, then $M-Fr(\Gamma) \leq \Gamma$;
- 4) Let $\Gamma \leq \mu$ and $\mu \in FMC(X)$ (respectively, $\mu \in FMO(X)$). Then $M-Fr(\Gamma) \leq \mu$ (respectively, $M-Fr(\Gamma) \leq \mu^c$);
- 5) $(M-Fr(\Gamma))^c = M-Int(\Gamma) \vee M-Int(\Gamma^c)$ for any fuzzy set Γ in X .

Proof. Proof is simple and hence omitted. $\square$

Theorem 10. Let Γ be a fuzzy set in a fuzzy minimal space (X, M) . Then:

- 1) $M-Fr(\Gamma) = M-Cl(\Gamma) - M-Int(\Gamma)$;
- 2) $M-Fr(M-Int(\Gamma)) \leq M-Fr(\Gamma)$;
- 3) $M-Fr(M-Cl(\Gamma)) \leq M-Fr(\Gamma)$;
- 4) $M-Int(\Gamma) \leq \Gamma - M-Fr(\Gamma)$.

Proof. Proof is simple, therefore omitted. $\square$

Theorem 11. Let Γ and μ be fuzzy sets in a fuzzy minimal space (X, M) . Then $M-Fr(\Gamma \vee \mu) \leq M-Fr(\Gamma) \vee M-Fr(\mu)$.

Proof. $M-Fr(\Gamma \vee \mu) = M-Cl(\Gamma \vee \mu) \wedge M-Cl(\Gamma \vee \mu)^c$
 $= (M-Cl(\Gamma) \vee M-Cl(\mu)) \wedge (M-Cl(\Gamma)^c \wedge M-Cl(\mu)^c)$
 $\leq (M-Cl(\Gamma) \vee M-Cl(\mu)) \wedge (M-Cl(\Gamma)^c \wedge M-Cl(\mu)^c)$
 $\leq (M-Cl(\Gamma) \wedge (M-Cl(\Gamma)^c \wedge M-Cl(\mu)^c)) \vee (M-Cl(\mu) \wedge (M-Cl(\Gamma)^c \wedge M-Cl(\mu)^c))$
 $= (M-Fr(\Gamma) \wedge M-Cl(\mu)^c) \vee (M-Fr(\mu) \wedge M-Cl(\Gamma)^c) \leq M-Fr(\Gamma) \vee M-Fr(\mu)$.
 Therefore, $M-Fr(\Gamma \vee \mu) \leq M-Fr(\Gamma) \vee M-Fr(\mu)$. The equality does not hold in general. $\square$

Theorem 12. In a fuzzy minimal space (X, M) , for any two fuzzy sets Γ and μ , we have $M-Fr(\Gamma \wedge \mu) \leq (M-Fr(\Gamma) \wedge M-Cl(\mu)) \vee (M-Fr(\mu) \wedge M-Cl(\Gamma))$.

Proof. $M-Fr(\Gamma \wedge \mu) = M-Cl(\Gamma \wedge \mu) \wedge M-Cl(\Gamma \wedge \mu)^c$
 $\leq (M-Cl(\Gamma) \wedge M-Cl(\mu)) \wedge (M-Cl(\Gamma)^c \vee M-Cl(\mu)^c)$
 $= (M-Cl(\Gamma) \wedge (M-Cl(\mu) \wedge M-Cl(\Gamma)^c)) \vee (M-Cl(\Gamma) \wedge (M-Cl(\mu) \wedge M-Cl(\mu)^c))$
 $= (M-Fr(\Gamma) \wedge M-Cl(\mu)) \vee (M-Fr(\mu) \wedge M-Cl(\Gamma))$. Therefore, $M-Fr(\Gamma \wedge \mu) \leq (M-Fr(\Gamma) \wedge M-Cl(\mu)) \vee (M-Fr(\mu) \wedge M-Cl(\Gamma))$. $\square$

Corollary 13. For any two fuzzy sets Γ and μ in an fuzzy minimal space (X, M) , we have : $M-Fr(\Gamma \wedge \mu) \leq M-Fr(\Gamma) \vee M-Fr(\mu)$.

Theorem 14. For any fuzzy set λ in a fuzzy minimal space (X, M) , we have:

- 1) $M-Fr(M-Fr(\lambda)) \leq M-Fr(\lambda)$;
- 2) $M-Fr(M-Fr(M-Fr(\lambda))) \leq M-Fr(M-Fr(\lambda))$.

Proof. Proof is simple therefore omitted. $\square$

Remark 15. The example below shows that equality in (1) of Theorem 14, does not satisfy always.

Example 16. Let $X = \{e, g, h\}$ and $M = \{\bar{0}, \bar{1}, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6, \mu_7\}$, where $\mu_1 = \{(e, .5), (g, .7), (h, .3)\}$, $\mu_2 = \{(e, .4), (g, .7), (h, .2)\}$, $\mu_3 = \{(e, .5), (g, .7), (h, .1)\}$, $\mu_4 = \{(e, .4), (g, .7), (h, .1)\}$, $\mu_5 = \{(e, .5), (g, .8), (h, .3)\}$, $\mu_6 = \{(e, .6), (g, .9), (h, .3)\}$, $\mu_7 = \{(e, .5), (g, .7), (h, .2)\}$.

Let $\Gamma = \{(e, .2), (g, .5), (h, .9)\}$. Then $M\text{-Fr}(\Gamma) = \bar{1}$, but $M\text{-Fr}(M\text{-Fr}(\Gamma)) = \bar{0}$.

Definition 17 ([15]). A function $f : (X, M_X) \rightarrow (Y, M_Y)$ is fuzzy minimal continuous, if $f^{-1}(B) \in M_X$, for any $B \in M_Y$.

Theorem 18. Let (X, M_X) and (Y, M_Y) be two fuzzy minimal spaces and $f : X \rightarrow Y$ be a fuzzy M - continuous function, then fore very fuzzy set μ in Y , we have : $M_X\text{-Fr}(f^{-1}(\mu)) \leq f^{-1}(M_Y\text{-Fr}(\mu))$.

Proof. Let f be fuzzy M continuous function and μ be a fuzzy set in Y . Then $M_X\text{-Fr}(f^{-1}(\mu)) = M_X\text{-Cl}(f^{-1}(\mu)) \wedge M_X\text{-Cl}(f^{-1}(\mu))^c \leq f^{-1}(M_Y\text{-Cl}(\mu)) \wedge M_X\text{-Cl}(f^{-1}(\mu))^c = f^{-1}(M_Y\text{-Cl}(\mu)) \wedge f^{-1}(M_Y\text{-Cl}(\mu))^c = f^{-1}(M_Y\text{-Cl}(\mu) \wedge M_Y\text{-Cl}(\mu)^c) = f^{-1}(M_Y\text{-Fr}(\mu))$. Therefore, $M_X\text{-Fr}(f^{-1}(\mu)) \leq f^{-1}(M_Y\text{-Fr}(\mu))$. This proves the theorem. $\square$

3.2 Fuzzy M -semi-frontier

In 1960, Levine [9] defined semi-open sets in topological spaces. In this contrast, we define M -semi-open sets in fuzzy minimal space (X, M) . In this section we will study fuzzy M -semi frontier of fuzzy sets in fuzzy minimal space (X, M) . Several interesting relations are explored.

Definition 19. Let Γ be a fuzzy set in X . Then, Γ is called a fuzzy M -semi open set of (X, M) , if there exists an M -open set v in (X, M) , such that $v \leq \Gamma \leq M\text{-Cl } v$. The complement of fuzzy M -semi open set is fuzzy M -semi closed set. where $FSC(X)$ (respectively, $FMSO(X)$) denote the collection of fuzzy M -semi closed (respectively, fuzzy M -semi open) sets in X .

Definition 20. Let Γ be a fuzzy set in fuzzy minimal space (X, M) . Then M -semi closure (briefly $M\text{-sCl}$) and M -semi interior (briefly $M\text{-sInt}$) of Γ are given as:

$$\begin{aligned} M\text{-sCl}(\Gamma) &= \bigwedge \{\beta | \Gamma \leq \beta, \beta \text{ is fuzzy } M\text{-semi closed in } (X, M)\}, \\ M\text{-sInt}(\Gamma) &= \bigvee \{\beta | \beta \leq \Gamma, \beta \text{ is fuzzy } M\text{-semiopen in } (X, M)\}. \end{aligned}$$

Theorem 21. Let Γ and μ be fuzzy sets in fuzzy minimal space (X, M) . Then:

- (1) $M\text{-sInt}(\Gamma \vee \mu) \geq M\text{-sInt}(\Gamma) \vee M\text{-sInt}(\mu)$;
- (2) $M\text{-sInt}(\Gamma \wedge \mu) = M\text{-sInt}(\Gamma) \wedge M\text{-sInt}(\mu)$;
- (3) $M\text{-sCl}(\Gamma \vee \mu) = M\text{-sCl}(\Gamma) \vee M\text{-sCl}(\mu)$;
- (4) $M\text{-sCl}(\Gamma \wedge \mu) \leq M\text{-sCl}(\Gamma) \wedge M\text{-sCl}(\mu)$.

- Proof.** 1. $M-sInt(\Gamma)$ and $M-sInt(\mu)$ are both fuzzy M - semi open sets and $\Gamma \leq \Gamma \vee \mu$, $\mu \leq \Gamma \vee \mu$ imply that $M-sInt(\Gamma) \leq M-sInt(\Gamma \vee \mu)$ and $M-sInt(\mu) \leq M-sInt(\Gamma \vee \mu)$. Combining, $M-sInt(\Gamma) \vee M-sInt(\mu) \leq M-sInt(\Gamma \vee \mu)$ or $M-sInt(\Gamma \vee \mu) \geq M-sInt(\Gamma) \vee M-sInt(\mu)$.
2. $\Gamma \wedge \mu \leq \Gamma$ and $\Gamma \wedge \mu \leq \mu$ imply $M-sInt(\Gamma \wedge \mu) \leq M-sInt(\Gamma)$, $M-sInt(\Gamma \wedge \mu) \leq M-sInt(\mu)$ and therefore, $M-sInt(\Gamma \wedge \mu) \leq M-sInt(\Gamma) \wedge M-sInt(\mu)$. Conversely, $M-sInt(\Gamma) \leq \Gamma$ and $M-sInt(\mu) \leq \mu$ implies $M-sInt(\Gamma) \wedge M-sInt(\mu) \leq \Gamma \wedge \mu$ and $M-sInt(\Gamma) \wedge M-sInt(\mu)$ is fuzzy semi open. But $M-sInt(\Gamma \wedge \mu)$ is the largest fuzzy M -semi open set contained in $\Gamma \wedge \mu$, hence $M-sInt(\Gamma) \wedge M-sInt(\mu) \leq M-sInt(\Gamma \wedge \mu)$. This gives the equality.
3. It follows easily from (2).
4. Since $\Gamma \wedge \mu \leq \Gamma$, $\Gamma \wedge \mu \leq \mu$ $M-sCl(\Gamma \wedge \mu) \leq M-sCl(\Gamma)$, also $M-sCl(\Gamma \wedge \mu) \leq M-sCl(\mu)$. Hence, $M-sCl(\Gamma \wedge \mu) \leq M-sCl(\Gamma) \wedge M-sCl(\mu)$. $\square$

Definition 22. Let Γ be a fuzzy set in fuzzy minimal space (X, M) . Then, the fuzzy M -semi frontier of Γ is defined as $M-sFr(\Gamma) = M-sCl(\Gamma) \wedge M-sCl(\Gamma)^c$. $M-sFr(\Gamma)$ may or may not be a fuzzy M -semi closed set.

Remark 23. We note that $\Gamma \vee M-sFr(\Gamma) \leq M-sCl(\Gamma)$, for an arbitrary fuzzy set Γ in X , the equality in general, does not hold always as it is clear by the following example.

Example 24. Let $X = \{e, g\}$ and fuzzy minimal structure defined on X be $M = \{\bar{0}, \{e.4, g.8\}, \{e.6, g.9\}, \{e.5, g.7\}, \{e.5, g.2\}, \{e.8, g.7\}, \{e.3, g.2\}, \{e.4, g.7\}, \{e.4, g.2\}, \{e.6, g.7\}, \{e.5, g.8\}, \{e.8, g.8\}, \{e.6, g.8\}, \{e.8, g.9\}, 1\}$.

Then M -semi open sets are $\bar{0}, \{e.3, g.2\}, \{e.4, g.2\}, \{e.4, g.7\}, \{e.5, g.2\}, \{e.5, g.7\}, \{e.5, g.8\}, \{e.6, g.7\}, \{e.6, g.8\}, \{e.6, g.9\}, \{e.6, g.1\}, \{e.7, g.9\}, \{e.7, g.1\}, \{e.8, g.8\}, \{e.8, g.9\}, \{e.8, g.1\}, \{e.9, g.9\}, \{e.9, g.1\}, \bar{1}$.

Choose $\Gamma = \{e.4, g.7\}$, then by computation we get: $M-sFr(\Gamma) = M-sCl(\Gamma) \wedge M-sCl(\Gamma)^c = \{a.5, b.3\}$,

$\Gamma \vee M-sFr(\Gamma) = \{a.5, b.7\}$, $M-sCl(\Gamma) = \{a.5, b.8\} \neq \{a.5, b.7\} = \Gamma \vee M-sFr(\Gamma)$.

Theorem 25. Let Γ be a fuzzy set in fuzzy minimal space (X, M) , then the following hold:

- (1) $M-sFr(\Gamma) = M-sFr(\Gamma^c)$;
- (2) If Γ is fuzzy M -semi closed, then $M-sFr(\Gamma) \leq \Gamma$;
- (3) If Γ is fuzzy M - semi open, then $M-sFr(\Gamma) \leq \Gamma^c$;
- (4) Let $\Gamma \leq \mu$ and $\mu \in FMSC(X)$ (respectively, $\mu \in FMSC(X)$). Then, $M-sFr(\Gamma) \leq \mu$ (respectively, $M-sFr(\Gamma) \leq \mu^c$);
- (5) $(M-sFr(\Gamma))^c = M-sInt(\Gamma) \vee M-sInt(\Gamma^c)$;

- (6) $M-sFr(\Gamma) \leq M-Fr(\Gamma)$;
- (7) $M-sCl(M-sFr(\Gamma)) \leq M-Fr(\Gamma)$.

Theorem 26. *Let Γ be a fuzzy set in an fuzzy minimal space X . Then:*

- (1) $M-sFr(\Gamma) = (M-sCl(\Gamma)) - (M-sInt(\Gamma))$;
- (2) $M-sFr(M-sInt(\Gamma)) \leq M-sFr(\Gamma)$;
- (3) $M-sFr(M-sCl(\Gamma)) \leq M-sFr(\Gamma)$;
- (4) $M-sInt(\Gamma) \leq \lambda - (M-sFr(\Gamma))$.

Theorem 27. *Let λ and μ be fuzzy M -sets in a fuzzy minimal structure space (X, M) . Then $M-sFr(\lambda \vee \mu) \leq M-sFr(\lambda) \vee M-sFr(\mu)$.*

The reverse inequality, in general is not true as shown in the following example.

Example 28. In the fuzzy minimal structure M of Example 24, we select fuzzy set $\alpha = \{e.4, g.7\}$ and $\beta = \{e.6, g.2\}$, then computations gives

$$\begin{aligned}
 M-sFr(\alpha) &= M-sCl(\alpha) \wedge M-sCl(\alpha^c) = \{e.5, g.8\} \wedge \{e.6, g.3\} = \{e.5, g.3\} \\
 M-sFr(\beta) &= M-sCl(\beta) \wedge M-sCl(\beta^c) = \{e.6, g.2\} \wedge \{e.5, g.8\} = \{e.5, g.2\} \\
 M-sFr(\alpha \vee \beta) &= M-sCl(\alpha \vee \beta) \wedge M-sCl(\alpha \vee \beta)^c = \{e.6, g.8\} \wedge \{e.4, g.3\} = \\
 &\{e.4, g.3\} \\
 M-sFr\alpha \vee M-sFr\beta &= \{e.5, g.3\} \vee \{e.5, g.2\} = \{e.5, g.3\} \not\leq \{e.4, g.3\} = \\
 &M-sFr(\alpha \vee \beta)
 \end{aligned}$$

Now we show that $M-sFr\gamma \wedge M-sFr\delta \not\leq M-sFr(\gamma \wedge \delta)$

For this choose $\gamma = \{e.4, g.8\}$ and $\delta = \{e.6, g.3\}$. Computations gives

$$\begin{aligned}
 M-sFr(\gamma) &= M-sCl(\gamma) \wedge M-sCl(\gamma^c) = \{e.5, g.8\} \wedge \{e.6, g.3\} = \{e.5, g.3\} \\
 M-sFr(\delta) &= M-sCl(\delta) \wedge M-sCl(\delta^c) = \{e.6, g.3\} \wedge \{e.5, g.8\} = \{e.5, g.3\} \\
 M-sFr(\gamma \wedge \delta) &= M-sCl(\gamma \wedge \delta) \wedge M-sCl(\gamma \wedge \delta)^c = \{e.4, g.3\} \wedge \{e.6, g.8\} = \\
 &\{e.4, g.3\} \\
 M-sFr(\gamma) \wedge M-sFr(\delta) &= \{e.5, g.3\} \wedge \{e.5, g.3\} = \{e.5, g.3\} \\
 M-sFr(\gamma) \wedge M-sFr(\delta) &= \{e.5, g.3\} \not\leq \{e.4, g.3\} = M-sFr(\gamma \wedge \delta)
 \end{aligned}$$

Theorem 29. *For any fuzzy M -sets λ and μ in a fuzzy minimal space (X, M) . $M-sFr(\lambda \wedge \mu) \leq (M-sFr(\lambda) \wedge M-sCl(\mu)) \vee (M-sFr(\mu) \wedge M-sCl(\lambda))$*

Proof. $M-sFr(\lambda \wedge \mu) = M-sCl(\lambda \wedge \mu) \wedge M-sCl(\lambda \wedge \mu)^c$
 $\leq (M-sCl(\lambda) \wedge M-sCl(\mu)) \wedge (M-sCl(\lambda)^c \vee M-sCl(\mu)^c)$
 $= M-sCl(\lambda) \wedge (M-sCl(\mu) \wedge M-sCl(\lambda)^c) \vee (M-sCl(\lambda) \wedge (M-sCl(\mu) \wedge M-sCl(\mu)^c))$
 $= (M-sFr(\lambda) \wedge M-sCl(\mu)) \vee (M-sFr(\mu) \wedge M-sCl(\lambda)).$

Therefore, $M-sFr(\lambda \wedge \mu) \leq (M-sFr(\lambda) \wedge M-sCl(\mu)) \vee (M-sFr(\mu) \wedge M-sCl(\lambda))$ $\square$

The reverse inequality in general is not true as shown by the following example:

Example 30. In the fuzzy minimal structure M of Example 24, we choose fuzzy set $\gamma = \{e.4, g.8\}$ and $\delta = \{e.6, g.3\}$, then computations gives

$$\begin{aligned} (M-sFr(\gamma) \wedge M-sCl(\delta)) &= \{e.5, g.8\} \wedge \{e.6, g.3\} = \{e.5, g.3\} \\ (M-sFr(\delta) \wedge M-sCl(\gamma)) &= \{e.5, g.3\} \wedge \{e.5, g.8\} = \{e.5, g.3\} \\ (M-sFr(\gamma) \wedge M-sCl(\delta)) \vee (M-sFr(\delta) \wedge M-sCl(\gamma)) &= \{e.5, g.3\} \wedge \{e.5, g.3\} \\ &= \{e.5, g.3\} \not\leq \{e.4, g.3\} = M-sFr(\gamma \wedge \delta) \end{aligned}$$

Theorem 31. Let $f : X \rightarrow Y$ be a fuzzy M - continuous function, then $M-sFr(f^{-1}(\mu)) \leq f^{-1}(M-sFr(\mu))$

Proof. Since f is fuzzy M -continuous function and μ be a fuzzy M set in Y . Then

$$\begin{aligned} M_X-sFr(f^{-1}(\mu)) &= M_X-sCl(f^{-1}(\mu)) \wedge M_X-sCl(f^{-1}(\mu)^c) \\ &\leq f^{-1}(M_Y-sCl(\mu)) \wedge M_X-sCl(f^{-1}(\mu)^c) \\ &= f^{-1}(M_Y-sCl(\mu)) \wedge f^{-1}(M_Y-sCl(\mu^c)) \\ &= f^{-1}(M_Y-sCl(\mu) \wedge M_Y-sCl(\mu^c)) \\ &= f^{-1}(M_Y-sFr(\mu)). \end{aligned}$$

Therefore $M_X-sFr(f^{-1}(\mu)) \leq f^{-1}(M_Y-sFr(\mu))$. $\square$

4. Fuzzy rarely- M sets

The aim of this section is to define fuzzy rarely- M sets and fuzzy dense- M sets. Some properties of these sets are studied. It is also shown that the non-zero (respectively, non universal) fuzzy rare- M set (respectively, fuzzy dense- M set) is not fuzzy open- M set (respectively, fuzzy closed- M set).

Definition 32. A fuzzy set λ on X is said to be a fuzzy rare- M set, if $M-Int(\lambda) = \bar{0}$.

Definition 33. Let λ be a fuzzy set on X . It is said to be a fuzzy dense- M set, if $M-Cl(\lambda) = \bar{1}$.

Example 34. Let $X = \{e, g, h\}$, $M = \{\bar{0}, \bar{1}, \Gamma, \mu, \nu\}$, where $\Gamma = \{(e, 0.3), (g, 0.2), (h, 0)\}$, $\mu = \{(e, 0.1), (g, 0), (h, 0.2)\}$, $\nu = \{(e, 0), (g, 0.1), (h, 0.1)\}$. Let $\alpha = \{(e, 0), (g, 0), (h, 0.1)\}$, $\beta = \{(e, 1), (g, 0.9), (h, 1)\}$, $\delta = \{(e, 0.3), (g, 0.2), (h, 0.1)\}$. Then $M-Int(\alpha) = \bar{0}$ gives α is a fuzzy rare M -set. Again $M-Cl(\beta) = \bar{1}$ gives β is fuzzy dense M -set. $M-Int(\delta) \neq \bar{0}$ and $M-Cl(\delta) \neq \bar{1}$. This gives that δ is neither fuzzy rarely- M set and nor fuzzy dense- M set.

Theorem 35. A fuzzy set λ on X is a fuzzy rare- M set, if and only if λ^c is a fuzzy dense - M set.

Proof. Let λ be fuzzy rare- M set. Then $M-Int(\lambda) = \bar{0}$ or $\bar{1} - M-Int(\lambda) = \bar{1} - \bar{0}$. $M-Cl(\bar{1} - \lambda) = \bar{1}$ or $M-Cl(\lambda^c) = \bar{1}$ and conversely, let λ^c be fuzzy rare- M set. Then $M-Cl(\lambda^c) = \bar{1}$ or $\bar{1} - M-Cl(\lambda) = \bar{1} - \bar{1}$. That is $M-Int(\bar{1} - \lambda) = \bar{0}$ or $M-Int(\lambda) = \bar{0}$. $\square$

Theorem 36. *A fuzzy set λ on X is both a fuzzy open $-M$ set and a fuzzy rare $-$ set, if and only if it is the $\bar{0}$ set.*

Proof. Let λ be a fuzzy set which is a fuzzy open $-M$ set and a fuzzy rare- M set, if and only if $M-Int(\lambda) = \lambda = \bar{0}$. $\square$

Corollary 37. *A fuzzy set λ on X is both an fuzzy closed M -set and fuzzy dense M -set, if and only if it is $\bar{1}$ set.*

Remark 38. Since fuzzy M -open sets are fuzzy open- M sets, a fuzzy M -open set other than $\bar{0}$ would be a fuzzy rare- M set gives a contradiction. Hence no fuzzy M -open set other than $\bar{0}$ is a fuzzy rare- M set. Similarly the fuzzy M -closed set, which is also a fuzzy dense $-M$ set is $\bar{1}$.

Remark 39. A fuzzy set λ on X can be both a fuzzy rare- M set and a fuzzy dense $-M$ set. This follows from the following example.

Example 40. Let $X = \{a, b, c\}$ and $M = \{\bar{0}, \bar{1}, \lambda\}$, where $\lambda = \{(a, 0.1), (b, 0.2), (c, 0.6)\}$. Consider the set $\mu = \{(a, 0.1), (b, 0.1), (c, 0.6)\}$. Then $M-Cl(\mu) = \bar{1}$, $M-Int(\mu) = \bar{0}$. So μ is both fuzzy rare- M set and a fuzzy dense $-M$ set.

Theorem 41. *A fuzzy set λ of (X, M) is both a fuzzy rare- M set and a fuzzy dense- M set, if and only if there exists neither a fuzzy M - open set contained in λ nor a fuzzy M -closed set containing λ , except for $\bar{0}$ and $\bar{1}$ respectively.*

Proof. A fuzzy set λ of (X, M) is both a fuzzy rare- M set and a fuzzy dense- M set, then by definition $M-Cl(\lambda) = \bar{1}$ and $M-Int(\lambda) = \bar{0}$. This implies that λ contains neither fuzzy M - open set except $\bar{0}$ nor contained in fuzzy M - closed set except $\bar{1}$ and conversely. $\square$

Remark 42. If a fuzzy set λ of (X, M) is both a fuzzy rare- M set and a fuzzy dense- M set. Then λ is neither an fuzzy M -open set nor a fuzzy M -closed set. Converse needs not be true as follows from the following example.

Example 43. Let $X = \{a, b, c\}$, $M = \{\bar{0}, \bar{1}, \lambda\}$. $\lambda = \{(a, 0.2), (b, 0), (c, 0)\}$. Let $\mu = \{(a, 0), (b, 0.5), (c, 0)\}$. Then μ is neither fuzzy M -open set nor fuzzy M -closed set. But $M-Int(\mu) = \bar{0}$, $M-Cl(\mu) = \{(a, 0.8), (b, 1), (c, 1)\} \neq \bar{1}$, that is μ is not a fuzzy dense- M set.

Theorem 44. *In a fuzzy minimal space (X, M) :*

1. $\bar{1}$ is a fuzzy dense- M set, but it is not a fuzzy rare- M set.
2. $\bar{0}$ is a fuzzy rare- M set, but it is not a fuzzy dense- M set.
3. Arbitrary intersection (respectively union) of fuzzy rare- M (respectively fuzzy dense- M) set is fuzzy rare- M (respectively fuzzy dense- M) set.

Proof. (1) and (2) are obvious . (3) Let $\lambda = \bigwedge \{\Gamma_\alpha : \alpha \in \Delta\}$ be an arbitrary intersection of fuzzy rare M sets, that is $M-Int(\Gamma_\alpha) = \bar{0}$, for each $\alpha \in \Delta$. Then $\bigwedge \{M-Int(\Gamma_\alpha) : \alpha \in \Delta\} = \bar{0}$. We know that $\bar{0} = \bigwedge \{M-Int(\Gamma_\alpha) : \alpha \in \Delta\} \geq M-Int \bigwedge \{\Gamma_\alpha : \alpha \in \Delta\}$. This implies that $\bar{0} \geq M-Int(\Gamma)$ or $M-Int(\Gamma) = \bar{0}$. Similarly, it can be shown that arbitrary union of fuzzy dense $-M$ sets is fuzzy dense $-M$ set. $\square$

Remark 45. Finite unions of fuzzy rare- M sets need not be fuzzy rare- M set. This follows from the following example:

Example 46. Let $X = \{a, b, c\}$, $M = \{\bar{0}, \bar{1}, \lambda, \mu, \nu\}$, where $\lambda = \{(a, .4), (b, .8), (c, .4)\}$, $\mu = \{(a, .1), (b, .4), (c, .1)\}$, $\nu = \{(a, .2), (b, .5), (c, .6)\}$

Let $\alpha = \{(a, .1), (b, .4), (c, 0)\}$ $\beta = \{(a, .1), (b, .3), (c, .5)\}$, $\gamma = \{(a, 0), (b, 0), (c, .2)\}$. Then $M-Int(\alpha) = 0$, $M-Int(\beta) = \bar{0}$. Here $\alpha \vee \beta = \{(a, .1), (b, 0.4), (c, .5)\}$ gives $M-Int(\alpha \vee \beta) \neq \bar{0}$

Theorem 47. A fuzzy set λ of (X, M) is a fuzzy dense $-M$ (respectively fuzzy rare- M) set, if and only if for every fuzzy open- M (respectively fuzzy closed- M) set μ satisfying $\lambda \leq \mu$ (respectively $\mu \leq \lambda$) we have $M-Cl(\lambda) \geq \mu$ (respectively $M-Int(\lambda) \leq \mu$).

Proof. First, assume that λ is a fuzzy dense- M set and take a fuzzy open $-M$ set μ with $\lambda \leq \mu$, then $M-Cl(\lambda) = \bar{1} \geq \mu$. Conversely, let the given condition hold and take $\mu = \bar{1}$. Then μ is an fuzzy open $-M$ set and $\lambda \leq \mu$, so $M-Cl(\lambda) \geq \mu = \bar{1}$, that is, $M-Cl(\lambda) = \bar{1}$. Hence λ is a fuzzy dense- M set. The other part can be proved similarly. $\square$

Remark 48. A fuzzy set λ of (X, M) is a fuzzy rare $-M$ set, if there exist no fuzzy open- M set other $\bar{0}$ contained in λ .

Theorem 49. The union (respectively, intersection) of fuzzy dense- M (respectively fuzzy rare- M) sets and fuzzy closed- M (respectively fuzzy open- M) sets is fuzzy dense- M (respectively fuzzy rare- M) set.

Proof. 1) Let λ be a fuzzy dense- M set and μ is a fuzzy closed $-M$ set. If v is a fuzzy open- M set with $\lambda \vee \mu \leq v$, then $\lambda \leq v$ and so $M-Cl(\lambda) \geq v$ by Theorem 47. Now $M-Cl(\lambda \vee \mu) \geq M-Cl(\lambda) \vee M-Cl(\mu) \geq \mu \vee v \geq v$. That is the union of a fuzzy dense- M set and a fuzzy closed- M set is a fuzzy dense $-M$ set by Theorem 47. Now we prove the result for fuzzy rarely- M set. 2) Let λ be a fuzzy rare- M set and μ is a fuzzy open- M set. If v is a fuzzy closed- M set with $v \leq \lambda \wedge \mu$, then $v \leq \lambda$ and so $M-Int(\lambda) \leq v$ by Theorem 47. Now $M-Int(\lambda \wedge \mu) = M-Int(\lambda) \wedge M-Int(\mu) \leq \mu \wedge v \leq v$. That is the intersection of a fuzzy rare- M set and a fuzzy open- M set is a fuzzy rare $-M$ set by Theorem 47. $\square$

Theorem 50. $M-Cl(\lambda)$ (resp. $M-Int(\lambda)$) is a fuzzy dense- M (respectively fuzzy rare- M) set whenever λ is a fuzzy dense- M (respectively fuzzy rare- M) set.

Proof. Let λ be a fuzzy dense- M set. Hence $M-Cl(\lambda) = \bar{1}$.

This implies that $M-Cl(M-Cl(\lambda)) = M-Cl(1) = \bar{1}$, this proves that $M-Cl(\lambda)$ is be a fuzzy dense- M set. $\square$

Remark 51. Let (X, M) be a fuzzy minimal structure formed from the fuzzy topological space (X, T) . Then, if a subset λ of (X, M) is a fuzzy dense set, it is a fuzzy dense M -set, since $Cl(\lambda) \leq M-Cl(\lambda)$.

Similarly, if a fuzzy set λ of (X, M) is a fuzzy rare set, then it is a fuzzy rare- M set.

Remark 52. A fuzzy set λ of (X, M) is said to be a fuzzy closed rare - M (respectively, fuzzy open dense- M) set if the fuzzy set λ is both a fuzzy closed - M set and a fuzzy rare- M set (respecti vely a fuzzy open- M set and a fuzzy dense- M set).

Example 53. Let $X = \{a, b, c\}$, $M = \{\bar{0}, \bar{1}, \lambda, \mu\}$, where $\lambda = \{(a, 0.7), (b, 0.4), (c, 0)\}$, $\mu = \{(a, 0.5), (b, 1), (c, 0)\}$.

Let $\alpha = \{(a, 0.3), (b, 0), (c, 1)\}$, $M-Int(\alpha) = \bar{0}$ and $M-Cl(\alpha) = \{(a, 0.3), (b, 0), (c, 1)\} = \alpha$. Then α is fuzzy closed rare- M set.

Remark 54. Let λ be a fuzzy dense- M set that is $M-Cl(\lambda) = \bar{1}$. Then there exists an fuzzy open dense - M set μ containing λ .

Example 55. Let $X = \{a, b, c\}$, $M = \{\bar{0}, \bar{1}, \lambda, \mu, \nu\}$, Where $\lambda = \{(a, 0.7), (b, 0.4), (c, 1)\}$, $\mu = \{(a, 0.5), (b, 1), (c, 1)\}$, $\nu = \{(a, 0.6), (b, 1), (c, 1)\}$.

Let $\alpha = \{(a, 0.7), (b, 1), (c, 1)\}$, $\beta = \{(a, 0.6), (b, 0.7), (c, 1)\}$. Then $M-Int(\alpha) = \alpha$ and $M-Cl(\alpha) = \bar{1}$, $M-Cl(\beta) = \bar{1}$ implies $\beta \leq \alpha$.

Theorem 56. A fuzzy set Γ of (X, M) is a fuzzy closed rare- M set, if and only if Γ is a fuzzy closed - M set which does not contain any fuzzy open - M set other than $\bar{0}$.

Proof. Let Γ be a fuzzy closed rare - M set, then by definition $M-Int(\Gamma) = \bar{0}$ and $M-Cl(\Gamma) = \Gamma$. This shows Γ is a fuzzy closed - M set which does not contain any fuzzy open - M set other than $\bar{0}$. $\square$

Theorem 57. A fuzzy set λ of (X, M) is a fuzzy open dense- M set, if and only if λ is a fuzzy closed - M set which does not contain any fuzzy open M -set other than $\bar{1}$.

Proof. Let λ be a fuzzy open dense - M set, then by definition $M-Int(\lambda) = \lambda$ and $M-Cl(\lambda) = \bar{1}$. This shows that Γ is a fuzzy open- M set, which does not contain any fuzzy closed- M set other than $\bar{1}$. $\square$

Theorem 58. If a fuzzy set λ of (X, M) is a fuzzy dense- M set, then $M-Fr(\lambda) = \bar{1} - M-Int(\lambda)$.

Proof. Let λ be a fuzzy dense M set that is $M-Cl(\lambda) = \bar{1}$, then $M-Fr(\lambda) = M-Cl(\lambda) \wedge M-Cl(\bar{1} - \lambda) = \bar{1} \wedge M-Cl(\bar{1} - \lambda) = M-Cl(\bar{1} - \lambda) = \bar{1} - M-Int(\lambda)$ $\square$

Theorem 59. *If fuzzy set λ of (X, M) is a fuzzy rare M set, then $M-Fr(\lambda) = M-Cl(\lambda)$.*

Proof. Let λ be a fuzzy rare M set. Then $M-Int(\lambda) = \bar{0}$. Now $M-Fr(\lambda) = M-Cl(\lambda) \wedge M-Cl(\bar{1} - \lambda) = M-Cl(\lambda) \wedge (\bar{1} - M-Int(\lambda)) = M-Cl(\lambda) \wedge (\bar{1} - \bar{0}) = M-Cl(\lambda)$ $\square$

Theorem 60. *A fuzzy set λ of (X, M) is a both fuzzy dense $-M$ set and a fuzzy rare- M set, if and only if $M-Fr(\lambda) = \bar{1}$.*

Proof. Necessity follows from the Theorem 58. Conversely, let $M-Fr(\lambda) = \bar{1}$, then $M-Cl(\lambda) \wedge M-Cl(\bar{1} - \lambda) = \bar{1}$ i.e. $M-Cl(\lambda) = \bar{1} \rightarrow (i)$ and $M-Cl(\bar{1} - \lambda) = \bar{1} \rightarrow (ii)$. By (i) we get λ is fuzzy dense M set and from (ii) $M-Cl(\bar{1} - \lambda) = \bar{1} - M-Int(\lambda) = \bar{1}$ i.e. $M-Int(\lambda) = \bar{0}$, hence λ is fuzzy rare- M set. This completes the proof. $\square$

Theorem 61. *In fuzzy minimal structure (X, M)*

1. $\bar{0}$ is a fuzzy closed rare- M set;
2. Arbitrary intersections of fuzzy closed rare $-M$ sets are fuzzy closed rare $-M$ set.

Proof. (1) is obvious. (2) Let $\{\lambda_i : i \in I\}$ be a collection of fuzzy closed rare $-M$ sets, i.e. $M-Cl(\lambda_i) = \lambda_i$ and $M-Int(\lambda_i) = \bar{0} \forall i = 1, 2, 3, \dots$. Consider $M-Cl \wedge \{\lambda_i : i \in I\} = \wedge \{M-Cl(\lambda_i) : i \in I\} = \wedge \{\lambda_i : i \in I\}$. This proves that arbitrary intersection of fuzzy closed- M sets is fuzzy closed- M set. Again, since each λ_i is rare- M set, therefore, $M-Int(\lambda_i) = \bar{0}$. Consider $M-Int \wedge \{\lambda_i : i \in I\} = \wedge \{M-Int(\lambda_i) : i \in I\} = \wedge \{\bar{0}\} = \bar{0}$. This completes the proof. $\square$

Remark 62. $\bar{1}$ is not fuzzy closed rare- M set, since $\bar{1}$ is a fuzzy closed $-M$ set but not a fuzzy rare- M set. Also arbitrary unions of fuzzy closed rare- M sets need not be fuzzy closed rare M sets.

5. Fuzzy rarely- M continuous functions

In this section we will define fuzzy rarely- M continuous function and give some characterizations of such functions.

Definition 63. *A function $f : (X, M_X) \rightarrow (Y, M_Y)$ is called fuzzy rarely- M continuous function at x_α , if for a fuzzy open $-M$ set μ containing $f(x_\alpha)$, there exists a fuzzy rare- M set R_μ with $\mu \wedge M-Cl(R_\mu) = \bar{0}$ and a fuzzy open- M set Γ containing x_α , such that $f(\Gamma) \leq \mu \vee R_\mu$.*

Remark 64. Every fuzzy M - continuous function is fuzzy rarely M - continuous. However converse is not true in general.

Theorem 65. Let $g : (X, M_X) \rightarrow (Y, M_Y)$ be a fuzzy M continuous and injection, then g preserves fuzzy rare- M sets.

Proof. Suppose that Γ_u is a fuzzy rare- M set in (X, M_X) , but $g(\Gamma_u)$ is not a fuzzy rare- M set in (Y, M_Y) . Then $M-Int(g(\Gamma_u)) \neq \bar{0}$.

So, there exist a fuzzy open - M set v other than $\bar{0}$ contained in $g(\Gamma_u)$. Since g is injective, $g^{-1}(v) \leq \Gamma_u$. Since g is fuzzy M continuous function, therefore, $g^{-1}(v)$ is a fuzzy open- M set. But this contradicts the fact that Γ_u is fuzzy rare- M set. Hence g preserves fuzzy rare - M sets. $\square$

Theorem 66. The following statements are a like for a function $f : (X, M_X) \rightarrow (Y, M_Y)$.

1. The function f is fuzzy rarely - M continuous at a fuzzy point x_α of (X, M_X) ;
2. For each fuzzy open - M set μ containing $f(x_\alpha)$, there exists a fuzzy open - M set Γ containing x_α , such that $M-Int(f(\Gamma) \wedge (\bar{1} - \mu)) = \bar{0}$;
3. For each fuzzy open - M set μ containing $f(x_\alpha)$, there exists a fuzzy open - M set Γ containing x_α such that $M-Int(f(\Gamma)) \leq M-Cl(\mu)$.

Proof. 1. $\rightarrow$ 2. Let μ be a fuzzy open - M set containing $f(x_\alpha)$ in (Y, M_Y) . Since μ is fuzzy open - M set, therefore, $\mu = M-Int_Y(\mu) < M-Int_Y(M-Cl_Y(\mu))$. Also $M-Int_Y(M-Cl_Y(\mu))$ is a fuzzy open - M set containing $f(x_\alpha)$. Since f is fuzzy rarely M -continuous function, there exists a fuzzy rare- M set R_μ with $M-Int_Y(M-Cl_Y(\mu)) \wedge M-Cl_Y(R_\mu) = \bar{0}$ and a fuzzy open - M set Γ containing x_α , such that $f(\Gamma) \leq M-Int_Y(M-Cl_Y(\mu)) \vee R_\mu$. We have

$$\begin{aligned} M-Int_Y(f(\Gamma) \wedge (\bar{1} - \mu)) &= M-Int_Y(f(\Gamma)) \wedge M-Int_Y(\bar{1} - \mu) \\ &\leq M-Int_Y(M-Cl_Y(\mu) \vee R_\mu) \wedge M-Int_Y(\bar{1} - \mu) \\ &= M-Int_Y(M-Cl_Y(\mu)) \vee (M-Int_Y(R_\mu)) \wedge (\bar{1} - M-Cl_Y(\mu)) \\ &= M-Int_Y(M-Cl_Y(\mu)) \wedge (\bar{1} - M-Cl_Y(\mu)) = \bar{0}. \end{aligned}$$

2. $\rightarrow$ 3. Consider (2) $M-Int(f(\Gamma)) \wedge M-Int(\bar{1} - \mu) = \bar{0}$. This implies $M-Int(f(\Gamma)) \wedge (\bar{1} - M-Cl(\mu)) = \bar{0}$. This implies $M-Int(f(\Gamma)) \leq M-Cl(\mu)$, this proves (3).
3. $\rightarrow$ 1. Let μ be a fuzzy open - M set containing $f(x_\alpha)$. Then by (3), there exists a fuzzy open- M set Γ containing x_α , such that $M-Int(f(\Gamma)) \leq M-Cl(\mu)$. We consider $f(\Gamma) = (f(\Gamma) - M-Int(f(\Gamma))) \vee M-Int(f(\Gamma))$
 $\leq (f(\Gamma) - M-Int(f(\Gamma))) \vee M-Cl(\mu)$

$$\begin{aligned}
&= (f(\Gamma) - M\text{-}Int(f(\Gamma))) \vee \mu \vee (M\text{-}Cl(\mu) - \mu) \\
&= ((f(\Gamma) - M\text{-}Int(f(\Gamma))) \wedge (\bar{1} - \mu)) \vee \mu \vee (M\text{-}Cl(\mu) - \mu).
\end{aligned}$$

put $R_1 = (f(\Gamma) - M\text{-}Int(f(\Gamma))) \wedge (\bar{1} - \mu)$ and $R_2 = (M\text{-}Cl(\mu) - \mu)$. Then R_1 and R_2 are fuzzy rare- M sets. Moreover $R_\mu = R_1 \vee R_2$ is a rare - M set, such that $M\text{-}Cl(R_\mu) \wedge \mu = \bar{0}$ and $f(\Gamma) \leq \mu \vee R_\mu$. This shows that f is rarely- M continuous. $\square$

Theorem 67. Let $f : (X, M_X) \rightarrow (Y, M_Y)$ be a fuzzy rarely- M continuous function and $g : (Y, M_Y) \rightarrow (Z, M_Z)$ a one to one fuzzy M continuous function. Then $g \circ f : (X, M_X) \rightarrow (Z, M_Z)$ is fuzzy rarely- M continuous.

Proof. Let x_α be a fuzzy point of (X, M_X) and $g \circ f(x_\alpha) \in \mu$, where μ is an open M -set in Z . By hypothesis, g is fuzzy M -continuous, therefore there exists a fuzzy open M -set ν in Y containing $f(x_\alpha)$ such that $g(\nu) \leq \mu$. Since f is fuzzy rarely- M continuous therefore there exists a fuzzy rare - M -set R_G in Y with $M\text{-}Cl(R_G) \wedge \nu = \bar{0}$, and a fuzzy open- M set λ containing x_α such that $f(\lambda) \leq \nu \vee R_G$. Since fuzzy rare M -set are preserved under fuzzy M continuous function, therefore $g(R_G)$ is fuzzy rare M -set in Z . Now $g \circ f(\lambda) \leq g(\nu \vee R_G) \leq g(\nu) \vee g(R_G) \leq \mu \vee g(R_G)$. This proves that $g \circ f$ is fuzzy rarely- M continuous function. This completes the proof. $\square$

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A NOTE ON S -ACTS AND BOUNDED LINEAR OPERATORS

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Abstract. In this work, the properties of the certain operator have been studied by looking at the associated S -act of this operator, and conversely. Some operators, for example such operator, one to one, onto operators have been looked. On the other hand, basic mathematical interpretation understanding of S -acts, such as faithful, finitely generated, singular, separated, torsion free and noetherian acts. We have found out the properties may be associated with S -act which has any of these properties. Let V be a inner product space over a field F , T be a bounded operator on V , and let $S = \{e^{x+y} | x, y \text{ are independent variables in } R\}$ be the semigroup. Define $\theta : S \times V \longrightarrow V$ by $\theta(e^{x+y}, v) = e^{T+T^*}(v)$. This function makes V a left S -act, denote by V_{T+T^*} and we call it the associated S -act of $T + T^*$.

Keywords: Bounded linear operator, finite dimensional Banach space, S -act, faithful S -act, Noetherian S -act.

1. Introduction

A non-empty set S with a binary operation $S \times S \rightarrow S$, $(s, s') \mapsto s.s'$, is called a groupoid. The operation of a groupoid is often called multiplication. Instead of $s.s'$ we usually write ss' . The multiplication on a groupoid S is called associative if $a(bc) = (ab)c$ for all $a, b, c \in S$. A groupoid with associative multiplication or for short an associative groupoid is called a semi-group. A semigroup S with 1 is called monoid (see [1]). Let S be a monoid and A a non empty set. If we have a mapping $\mu : A \times S \longrightarrow A$, $(a, s) \mapsto as := \mu(a, s)$ Such that $a(st) = (as)t$, and $a.1 = a$, for $a \in A, s, t \in S$. We call A a right S -act or a right act over S and write A_S , we can define a left S -act and write S_A . In [2], The module of an operator was study, let V be a vector space over a field F . Let T be a linear operator acting on the elements of V on the left. Let $R = F[x]$ be the ring of polynomials in x with coefficients in F . Define $\varphi : R \times V \longrightarrow V$ by $\varphi(p, v) = p.v = p(T)v$. That φ makes V a left R -module denoted V_T , and calls the associated R - module. Let H be a Hilbert space over a field K (K may be

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real or complex), let T be abounded linear operator on H . Define exponential operator e^T as follows: $e^T = \sum_{n=0}^{\infty} T^n/n!$. Note the definition of exponential operator is well defined, this means the sum in the definition of the exponential operator exists [3]. Let V be a Banach space over a field F , T be a bounded operator on V and $S = \{e^x : x \in R\}$ be the semi-group. Define $\mu : S \times V \longrightarrow V$ by $\mu(e^x, v) = e^T(v)$. This function makes V a left S -act, denoted by V_T . We call it the associated S -act of T [4]. In this paper the associated S -act of $T+T^*$ have been study, let $S = \{e^{x+y} | x, y \text{ are independent variables in } R\}$ be the semigroup. Define $\theta : S \times V \rightarrow V$ by $\theta(e^{x+y}, v) = e^{T+T^*}(v)$. This function makes V a left S -act, denote by V_{T+T^*} , the form of every element in V_{T+T^*} is $e^{T+T^*}(v)$, and if two operators T and S are similar then $V_{(T+T^*)}$ is isomorphic to V_{S+S^*} . The relation between a bounded linear operator T and faithful S -act have been study. The relation between finite dimensional Banach space V and Noetherian S -act, discussed have been study by if V is finite dimension then V_{S+S^*} S -act is Noetherian.

2. Main results

Definition 2.1. Let $S = \{e^{x+y} | x, y \text{ are independent variables in } R\}$ be the semigroup. Define $\theta : S \times V \longrightarrow V$ by $\theta(e^{x+y}, v) = e^{T+T^*}(v)$. This function makes V a left S -act, denote by $V_{(T+T^*)}$

We put $p_n(T+T^*) = \sum_{i=0}^n (T+T^*)^i/(i!) = I + (T+T^*) + (T+T^*)^2/2! + (T+T^*)^3/3! + \dots + (T+T^*)^n/n!$.

Proposition 2.2. If $K = \{V_j, j \in \Lambda\}$ is a basis for V , then each element of V_{T+T^*} can be written in the form

$$\lim_{n \rightarrow \infty} \sum_{i=0}^n (T+T^*)^i/(i!) \sum_{j \in \Lambda} a_j v_j = \lim_{n \rightarrow \infty} p_n(T+T^*).V$$

The symbol $\sum_{j \in \Lambda}$ means the sum is taken over a finite subset of Λ .

Proof. We define $\mu : S \times V \longrightarrow V$, by $\mu(e^{x+y}, v) = e^{T+T^*}(v) = \sum_{i=0}^{\infty} (T+T^*)^i/(i!)(v)$ Let $w \in V_{T+T^*}$ then $w = \sum_{i=0}^{\infty} (T+T^*)^i/(i!)(v) = (I + (T+T^*) + (T+T^*)^2/2! + ((T+T^*)^3/3! + \dots))(v)$. Since $K = \{v_j, j \in \Lambda\}$ is a basis for V then $w = I + (T+T^*) + (T+T^*)^2/2! + ((T+T^*)^3/3! + \dots)(\sum_{j \in \Lambda} (a_j v_j)) = I(\sum_{j \in \Lambda} a_j v_j) + (T+T^*)(\sum_{j \in \Lambda} (a_j v_j) + (T+T^*)^2/2!(\sum_{j \in \Lambda} (a_j v_j) + ((T+T^*)^3)/3!(\sum_{j \in \Lambda} (a_j v_j) + \dots$ But the series $\sum_{i=0}^{\infty} T^i/(i!)$ converges in $B(H)$, Where $T \in B(H)$. Therefore $\sum_{i=0}^{\infty} (T+T^*)^i/(i!)$ is converge in $B(H)$ Then we get $w = \lim_{n \rightarrow \infty} \sum_{i=0}^{\infty} T^i/(i!) \sum_{j \in \Lambda} (a_j v_j) = \lim_{n \rightarrow \infty} p_n(T+T^*).V$ $\square$

Examples 2.3. 1. Let $\{v_j | j \in \Lambda\}$ be a basis for a Banach space V . Let O be the zero operator on V , recall $O^o = I$. Let $w \in V_{0+0^*}$, then by proposition 2.2 $w = e^{T+T^*}(v) = e^{0+0^*}(v)$ then $w = I(\sum_{j \in \Lambda} a_j v_j) = \sum_{j \in \Lambda} a_j v_j$, since $e^0 = I$ [2], therefore, $e^{0+0^*} = I$

2. Let $I : V \longrightarrow V$ be the identity operator on V . $\{v_j | j \in \Lambda\}$ be a basis for V , and let $w \in V_{I+I^*}$, then by proposition 2.2 $w = e^{I+I^*}(v) = e^{I+I^*}(v) = e^{I+I^*}(\sum_{j \in \Lambda} a_j v_j)$. Since $I^* = I$ therefore $w = e^{(I+I)}(\sum_{j \in \Lambda} a_j v_j) = (I+I) \sum_{n=0}^{\infty} 1/n! (\sum_{j \in \Lambda} a_j v_j) = 2 \sum_{n=0}^{\infty} 1/n! (\sum_{j \in \Lambda} a_j v_j) = \lim_{n \rightarrow \infty} 2 \sum_{n=0}^n 1/n! (\sum_{j \in \Lambda} a_j v_j)$, put $a_n = \sum_{i=0}^n 1/i!$ then $w = 2 \lim_{n \rightarrow \infty} a_n (\sum_{j \in \Lambda} a_j v_j)$.
3. Let $\{v_j : j \in \Lambda\}$ be a basis for a Banach space V , and T be a nilpotent operator on V (i.e. $T^n = 0$ and $T^{n-1} \neq 0$ for some positive integer n) let $w \in V_{T+T^*}$, then by proposition 2.2 $w = e^{T+T^*}(v) = (I + (T + T^*) + (T + T^*)^2/2! + ((T + T^*)^3)/3! + \dots + ((T + T^*)^n/n!))(v) = [I + (T + T^*) + (T + T^*)^2/2! + ((T + T^*)^3)/3! + \dots + ((T + T^*)^{n-1})/(n-1)!](\sum_{j \in \Lambda} a_j v_j) + (T+T^*)^n/n! (\sum_{j \in \Lambda} a_j v_j) = [I + (T+T^*) + (T+T^*)^2/2! + (T+T^*)^3/3! + \dots + (T+T^*)^{n-1}/(n-1)!](\sum_{j \in \Lambda} a_j v_j) + [(n1)T^{n-1}T^* + (n : 2)T^{n-2}T^{*2} + \dots + (n(n-1))T(T^*)^{n-1}/n!](\sum_{j \in \Lambda} a_j v_j) = \lim_{n \rightarrow \infty} p_{n-1}(T + T^*)(\sum_{j \in \Lambda} a_j v_j) + a_n(\sum_{j \in \Lambda} a_j v_j) = \lim_{n \rightarrow \infty} [p_{n-1}(T + T^*) + a_n].v$, where $a_n = ((n1)T^{n-1}T^* + (n2)T^{n-2}T^{*2} + \dots + (n(n-1))T(T^*)^{n-1}/n!)$.
4. Let $\{v_j : j \in \Lambda\}$ be a basis for a Banach space V , and T be a Self -adjoint operator on H (a bounded linear operator $T : H \longrightarrow H$ on a Hilbert space H is called self -adjoint operator if $T = T^*$ and let $w \in V_{T+T^*}$, then by proposition 2.2 $w = e^{(T+T)}(\sum_{j \in \Lambda} a_j v_j) = e^{2T}(\sum_{j \in \Lambda} a_j v_j) = (e^T)^2(\sum_{j \in \Lambda} a_j v_j) = (I + T + T^2/2! + T^3/3! + \dots)2(\sum_{j \in \Lambda} a_j v_j)$.

Proposition 2.4. *Let T and S be two bounded operators on V . If S and T are similar. Then V_{S+S^*} and V_{T+T^*} are isomorphic.*

Proof. Assume that T and S are similar, and i.e. there exist an invertible operator h on V , such that $hTh^(-1) = S$ [6], then $(hSh^(-1) = T).h$, this gives $h(S+S^*) = (T+T^*)h$. Since $hS = Th$, then $h(S+S^*)n = h(S+S^*)(S+S^*)n - 1 = (T+T^*)h(S+S^*)(S+S^*)n - 2 = (T+T^*)(T+T^*)h(S+S^*)(S+S^*)n - 3 = \dots = (T+T^*)^nh$, then

$$(1) \quad he^{S+S^*} = e^{T+T^*}h.$$

Define $h' : V_{S+S^*} \longrightarrow V_{T+T^*}$ by

$$(2) \quad (e^{S+S^*}(v))h' = e^{T+T^*}(h(v)).$$

To prove h' is isomorphism we must prove:

1. h' is well defined

Let $e^{S+S^*}(v_1) = e^{S+S^*}(v_2)$ then $h(e^{S+S^*}(v_1)) = h(e^{S+S^*}(v_2))$ (Since h is well defined). Then by equation 1 we get $e^{T+T^*}(h(v_1)) = h(e^{S+S^*}(v_1)) = h(e^{S+S^*}(v_2)) = e^{T+T^*}(h(v_2))$, this give

$$(3) \quad e^{(T+T^*)}(h(v_1)) = e^{T+T^*}(h(v_2)).$$

Then, by equations 2, 3 we get $(e^{S+S^*}(v_1))h' = (e^{S+S^*}(v_2))h'$. Thus h' is well defined.

2. h' is one to one.

Let $(e^{S+S^*}(v_1))h' = (e^{S+S^*}(v_2))h'$ Then by equation 2, we get $e^{T+T^*}(h(v_1)) = e^{T+T^*}(h(v_2))$ Then by equation 1 we get $h(e^{S+S^*}(v_1)) = h(e^{S+S^*}(v_2))$ But h is invertible then $h(-1)h(e^{S+S^*}(v_1)) = h(-1)h(e^{S+S^*}(v_2))$. This give $(e^{S+S^*}(v_1)) = (e^{S+S^*}(v_2))$, therefore h' is one to one.

3. h' is onto.

Let $e^T(v) \in V_T$ since $v \in V$ then $h^{-1}(v) \in V$ and $e^{S+S^*}(h^{-1}(v)) \in e^{S+S^*}$, then by equation 2 we get $(e^{S+S^*}(h^{-1}(v)))h' = e^{T+T^*}(h(h^{-1}(v))) = e^{T+T^*}(v)$, then h' is onto. Note that $(e^{S+S^*}(v))h' = e^{T+T^*}(h(v)) = h(e^{S+S^*}(v))$, thus $(e^{S+S^*}(v))h' = h(e^{S+S^*}(v))$ But h it is an operator (linear) on V , thus h is S -homomorphism, this give h' is S -homomorphism. Then h' is an S -isomorphism. $\square$

Recall that the left S -act A_s is faithful if for $s, t \in S$ the equality $sa = ta$ for all $a \in A_s$, implies $s = t$. The relation between faithful S -act and bounded linear operator T have been explained in the following proposition.

Proposition 2.5. *For any bounded linear operator T then V_{T+T^*} is a faithful S -act.*

Proof. We want to show that V_{T+T^*} is a faithful S -act, for any bounded operator T . Let $e^{x_1+y_1}.e^{T+T^*}(v) = e^{x_2+y_2}.e^{T+T^*}(v)$. Since e^T is operator then e^T is linear transformation, this give $e^{x_1+y_1}.e^{T+T^*}(v) = e^{x_2+y_2}.e^{T+T^*}(v) = e^{T+T^*}(e^{x_1+y_1}.v) = e^{T+T^*}(e^{x_2+y_2}.v)$. Since e^T is one to one, therefore e^{T+T^*} is one to one. Hence $e^{x_1+y_1}.v = e^{x_2+y_2}.v$, then $e^{x_1+y_1}.v - e^{x_2+y_2}.v = 0$. Then $(e^{x_1+y_1} - e^{x_2+y_2}).v = 0$, thus $e^{x_1+y_1} = e^{x_2+y_2}$. Therefore V_{T+T^*} is faithful S -act. $\square$

Remark 2.6. If V is a finite dimensional Banach space, then V_{T+T^*} is finitely generated S -act.

In [7], show that a subspace W of V is said to be an invariant subspace of V under T if $Tw \in W$ for all $w \in W$.

The following proposition shows under what condition the vector space V is finite dimension.

Proposition 2.7. *If T is one to one and onto and $V_T + T^*$ is finitely generated, then V is finite dimensional.*

Proof. Assume that V is not finite dimensional. Let $K(T) = \{w \in V | (T + T^*)w = 0\}$. It is clear that K is an invariant subspace of V (since $K \subseteq V$ and $\forall w \in K, (T + T^*)(w) = 0$ but $0 \in K$ then $(T + T^*)(K) \subseteq K$) and by the first isomorphic theorem of S -act, then $(T + T^*)V \cong V/K$ [1], since T is one to one then T^* is onto and T is onto then $(T + T^*)V = V$, therefore $V \cong V/K$. By assuming that V is not finite dimensional then either K is infinite dimensional

or K is finite dimensional. K is an invariant subspace of V then we can consider K_{T+T^*} . If K is finite dimensional then K_{T+T^*} is finite generated, remark 2.6, the subact K_{T+T^*} is generated by the set $\{e^{T+T^*}(w_j) | j \in \Lambda\}$ where $\{w_j | j \in \Lambda\}$ is a basis for K . But $w_j \in K$ given that $Tw_j = 0$, since the restriction of T on K is the zero operator O . Thus $K_{T+T^*} = K_{O+O^*}$, therefore K_{T+T^*} can not be finitely generated (see example 1 from 2.3, this contradiction. Hence K must be infinite dimensional. The subact K_{T+T^*} is generated by the set $\{e^{T+T^*}(w_j) | j \in \Lambda\}$ where $\{w_j | j \in \Lambda\}$ is a basis for K . But $w_j \in K$ given that $Tw_j = 0$, since the restriction of $T + T^*$ on K is the zero operator O . Thus $K_{T+T^*} = K_{O+O^*}$, therefore K_{T+T^*} can not be finitely generated (see example 1 from 2.3, but K_{T+T^*} is a subact of V_{T+T^*} and V_{T+T^*} is finitely generated. This mean infinitely generated contains in finitely generated. This contradiction shows that V is finite dimensional. $\square$

Recall that an S -act A separated if for each $a \neq b$ in A there exists $s \neq e$ such that $sa \neq sb$ [1]. In [8], let M_s be an S -system and H a subset of S , then H is called reductive on M_s if and only if for each $a, b \in M_s, ah = bh$ for all $h \in H$ implies $a = b$, an singular relation ψ_M on M_s by the set $\{(a, b) \in M \times M | ah = bh \text{ for some } h \in H \text{ for some reductive subset } H \text{ of } S\}$.

Proposition 2.8. *For any bounded linear operator T , if V_{T+T^*} is singular S -act then V is generated by one element.*

Proof. Since V_{T+T^*} is singular S -act, then

$$\psi_V = \{(e^{T+T^*}(v_1), e^{T+T^*}(v_2)) \in V_{T+T^*} \times V_{T+T^*} | e^{x+y}e^{T+T^*}(v_1) = e^{x+y}e^{T+T^*}(v_2) \text{ for some } e^x \in H \text{ for some reductive subset } H \text{ of } S\}$$

then $e^{x+y}e^{T+T^*}(v_1) = e^{x+y}e^{T+T^*}(v_2) \dots (2-1)$. This gives $e^{T+T^*}(e^{x+y}v_1) = e^{T+T^*}(e^{x+y}v_2)$ since e^{T+T^*} is one to one, therefore $e^{x+y}v_1 = e^{x+y}v_2$ then $e^{(x+y)}v_1 + (-1)e^{x+y}v_2 = 0 \dots (2-2)$ but H is reductive subset of S , then by (2-1) find $e^{T+T^*}(v_1) = e^{T+T^*}(v_2)$ thus $v_1 = v_2$, we replies $v_1 = v_2$ on (2-2) then $e^{x+y}v_1 + (-1)e^{x+y}v_1 = 0$, therefore V is generated by one element this gives V is a finite dimensional. $\square$

Recall that an S -act is separated if for each $a \neq b$ in A there exists $s \neq e \in S$, where e is the identity element such that $sa \neq sb$ [1]. In the following proposition, we explain the relationship between a bounded linear operator T and separated S -act.

Proposition 2.9. *For any bounded linear operator T then V_{T+T^*} is separated S -act.*

Proof. Let $a \neq b$ in V_{T+T^*} to prove V_{T+T^*} is separated we have to show that there exist s, e in S , $s \neq e$ where e is the identity element such that $sa \neq sb$. Assume $sa = sb$, $e \neq s \in S$, such that $s = e^{x+y}$, e is the identity element, $a, b \in V_{T+T^*}$, this gives $e^{x+y}.e^{T+T^*}(v_1) = e^{x+y}.e^{T+T^*}(v_2)$, $v_1, v_2 \in$

$V, x, y \in R$, since e^{T+T^*} is operator then e^{T+T^*} is linear transformation, this gives $e^{x+y}.e^{T+T^*}(v_1) = e^{x+y}.e^{T+T^*}(v_2)$, thus $e^{T+T^*}(e^{x+y}.v_1) = e^{T+T^*}(e^{x+y}.v_2)$, but e^{T+T^*} is one to one, then $e^{x+y}.v_1 = e^{x+y}.v_2$, hence $(v_1 - v_2)e^{x+y} = 0$, since $e^{x+y} \neq 0$. Then $v_1 = v_2$, this gives either $e^{T+T^*}(v_1) \neq e^{T+T^*}(v_2)$ or $e^{T+T^*}(v_1) = e^{T+T^*}(v_2)$, but if $e^{T+T^*}(v_1) \neq e^{T+T^*}(v_2)$, this give $v_1 \neq v_2$ this contradiction with $v_1 = v_2$, then $e^{T+T^*}(v_1) = e^{T+T^*}(v_2)$, means $a = b$ which a contradiction, then V_{T+T^*} is separated S -act. $\square$

The converse of the proposition 2.9 have been study in the following proposition.

Proposition 2.10. *If V_{T+T^*} is separated S -act then T is one to one.*

Proof. Assume that V_{T+T^*} is separated, we want to prove T is one to one. let $v_1 \neq v_2$, we must prove $T(v_1) \neq T(v_2)$. since $v_1 \neq v_2$ then either $e^{T+T^*}(v_1) \neq e^{T+T^*}(v_2)$ or $e^{T+T^*}(v_1) = e^{T+T^*}(v_2)$. If $e^{T+T^*}(v_1) = e^{T+T^*}(v_2)$, this contradiction with $v_1 \neq v_2$, hence $e^{T+T^*}(v_1) \neq e^{T+T^*}(v_2)$, but V_{T+T^*} is separated S -act then $\exists e \neq s, s = e^{x+y} \in S$ such that $e^{x+y}.e^{T+T^*}(v_1) \neq e^{x+y}.e^{T+T^*}(v_2)$, since e^{T+T^*} is an operator, then e^{T+T^*} is linear transformation, and hence $e^{T+T^*}(e^{x+y}.v_1) \neq e^{T+T^*}(e^{x+y}.v_2)$, this means $(I + (T + T^*) + (T + T^*)^2/2! + (T + T^*)^3/3! + \dots)(e^{x+y}.v_1) \neq ((I + (T + T^*) + (T + T^*)^2/2! + ((T + T^*)^3)/3! + \dots)(e^{x+y}.v_2)$ we get, $e^{x+y}.v_1 + (T + T^*)(e^{x+y}.v_1) + (T + T^*)^2/2!(e^{x+y}.v_1) + ((T + T^*)^3)/3!(e^{x+y}.v_1) + \dots \neq e^{x+y}.v_2 + (T + T^*)(e^{x+y}.v_2) + (T + T^*)^2/2!(e^{x+y}.v_2) + ((T + T^*)^3)/3!(e^{x+y}.v_2) + \dots$ $e^{x+y}.v_1 \neq e^{x+y}.v_2$, this gives $v_1 \neq v_2$ and $(T + T^*)(e^{x+y}.v_1) \neq (T + T^*)(e^{x+y}.v_2)$, but $(T + T^*)$ is an operator, this gives $e^{x+y}.(T + T^*)(v_1) \neq e^{x+y}.(T + T^*)(v_2)$, since $e^{x+y} \neq 0$, then $(T + T^*)(v_1) \neq (T + T^*)(v_2)$, and since $(T + T^*)^2/2!(e^{x+y}.v_1) \neq (T + T^*)^2/2!(e^{x+y}.v_2)$, therefore $(T + T^*)/2!(T + T^*)(e^{x+y}.v_1) \neq (T + T^*)/2!(T + T^*)(e^{x+y}.v_2)$ this give $(T + T^*)(e^{x+y}.v_1) \neq (T + T^*)(e^{x+y}.v_2)$ by using the same way, we get $(T + T^*)(v_1) \neq (T + T^*)(v_2)$. Then we proof $T + T^*$ is one to one, thus T is one to one. $\square$

Recall that An act A_S is torsion free if for any $x, y \in A_S$, and for any right cancellable element $c \in S$, the equality $xc = yc$ this implies $x = y$ (see [1]). In the following proposition the relation between a bounded linear operator T and torsion free S -act have been explain.

Proposition 2.11. *For any bounded linear operator T then V_{T+T^*} is torsion free S -act.*

Proof. Assume $e^{x+y}.e^{T+T^*}(v_1) = e^{x+y}.e^{T+T^*}(v_2), \forall e^{x+y}$ is cancellable element in S , this give $e^{T+T^*}(e^{x+y}.v_1) = e^{T+T^*}(e^{x+y}.v_2)$, since e^{T+T^*} is one to one, therefore $(e^{x+y}.v_1) = (e^{x+y}.v_2)$, then $v_1 = v_2$, thus either $e^{T+T^*}(v_1) = e^{T+T^*}(v_2)$ or $e^{T+T^*}(v_1) \neq e^{T+T^*}(v_2)$, if $e^{T+T^*}(v_1) \neq e^{T+T^*}(v_2)$, we get contradiction with $v_1 = v_2$, then $e^{T+T^*}(v_1) = e^{T+T^*}(v_2)$, this give V_{T+T^*} is torsion free. $\square$

Recall that a monid S is right Noetherian if and only if it satisfies the ascending chain condition for right ideals, this mean for every ascending chain $K_1 \subseteq K_2 \subseteq K_3 \subseteq \cdots \subseteq K_n \subseteq K_{n+1} \subseteq \cdots$, of its right subacts, there exists $n \in \mathbb{N}$ such that $K_n = K_{n+1} = \cdots$.

Theorem 2.12 ([8]). *If S is Noetherian and A is finitely generated S -act then A is Noetherian S -act.*

Proposition 2.13. *Let V be a finite dimensional normed space and T is similar to any operator J from R to R then is Noetherian S -act if and only if S is Noetherian.*

Proof. Since V is finite dimensional then it is finitely generated S -act by remark 2.3, therefore is Noetherian S -act, by theorem 2.12. Let $K_1 \subseteq K_2 \subseteq K_3 \subseteq \cdots \subseteq K_n \subseteq K_{n+1}$ be any ascending sequence ideals of S , then it is a sequence of subacts of S_S denoted by S_J , where J any operator from R to R , since T is similar to J , then by proposition 2.5, V_{T+T^*} is isomorphic S_{J+J^*} , thus S_{J+J^*} is Noetherian S -act, therefore this sequence is finite, then S is Noetherian. $\square$

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APPLICATION OF RECURRENT NEURAL NETWORK USING MATLAB SIMULINK IN MEDICINE

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Abstract. In this paper, a recurrent neural network (RNN) for finding the solution of linear programming problems is presented with better, spontaneous and fast converging. To achieve optimality in accuracy and also in computational effort, an algorithm is also presented. This paper covers the MATLAB Simulink modeling and simulative confirmation of such a recurrent neural network. Modeling and simulative results validate the theoretical analysis and efficiency of the recurrent neural network for finding the solution for linear programming problem. An application RNN in medicine has been presented to show the performance of the recurrent neural network.

Keywords: linear programming problem, Recurrent Neural Network (RRN), MATLAB Simulink.

1. Introduction

Linear programming (LP) techniques are commonly used to find the solution of economic, military, industrial, and social problems. In the last 50 years, researchers have proposed various dynamic tools for solving LP problems. Pyne [1] proposed the dynamic systems approach to solve constrained optimization problems. The recent developments in this field have redefined the application of neural network by dynamic solvers [2, 3, 4].

In recent studies, many researches have been done in relation to the application of the technology to knowledge engineering such as Artificial Neural Network (ANN) to the engineering field. LP is a very traditional discipline of Operations Research. It continues to be the most active branch. LP has been widely applied practically in many sectors such as production, financial, human resources, governing and planning. In the present day scenario, it is possible to construct and solve linear programs with high-speed computers and multi-processors which was not feasible a couple of years ago.

In 1947 Dantzig [5] developed a method for finding the solution to linear programming problems which is the Simplex method. Brown and Koopmans [6] portrayed the first series of interior point method to solve linear programming problems. Karmakar [7] built an algorithm which appears to be more proficient

than the Simplex Method on some intricate real-world problems of scheduling, routing and planning. Conn [7] developed a substitute for solving LP problems by using unconstrained optimization procedure combined with penalty function methods. The first neural approach applied to LP problems was proposed and developed by Tank and Hopfield [2].

The simulation models are designed as a stand alone application using MATLAB Simulink [8]. Matlab Simulation Models have been widely used [9, 10]. The primary aim of this paper is to present a recurrent neural network for finding the solution of linear programming problems. Here I describe circuit implementation of proposed neural network using MATLAB Simulink. The vital advantage of the neural networks is their massive parallel processing ability and rapid convergence properties.

2. Linear programming

Linear Programming is the term used for defining a wide range of optimization problems in which the objective function to be minimized or maximized is linear in the unknown variables and the constraints are a combinations of linear equalities and inequalities. LP is one of the most widely applied techniques of operations research in business, industry and numerous other fields. The objective function may be profit, cost, production capacity or any other measure of effectiveness, which is to be obtained in the best possible or optimal manner. The constraints may be imposed by different resources such as market demand, production process and equipment, storege capacity, raw material availibilty, etc. First, the given problem must be presented in linear programming form. This requires defining the variables of the problem, establising inter-relationship between them and formulating the objective function and constraints. A model, which approximates as closely as possible to the given problem, is then to be developed. If some constraints happen to be non-linear, they are approximated to appropriate linear functions to fit the linear programming format.

3. Problem statement

Consider a standard form of Linear Programming Problems described as

$$\begin{aligned}
 (1) \quad & \text{Minimize } C^T X \\
 (2) \quad & \text{Subject to } AX = B \\
 (3) \quad & X \geq 0,
 \end{aligned}$$

where $X \in R^n$ is a column vector of decision variables, $C \in R^n$ and $B \in R^m$ are column vector of cost coefficient and right hand side parameters, respectively, $A \in R^{m \times n}$ is a constraint coefficient matrix, and the subscript denotes the transpose operator.

4. Solution to the LP Problem

In general, the LP problem can have four possible solution types:

1. Unique Solution: There is only one solution that satisfies all constraints, and the objective function reaches a minimum within the feasible region.
2. Nonunique Solution: There are several feasible solution where the objective function reaches a minimum.
3. An unbounded Solution: The objective function is not bounded in the feasible region and it approaches $-\infty$.
4. No feasible Solution: Constraint provided in (2) and (3) are too restrictive, and the set of feasible solution is empty.

Although theatrically valid, cases 3 and 4 appear rarely in engineering and scientific applications. Furthermore, they can be easily detected, and in further consideration of the LP problem we will assume that it is formulated in such a way that there exists at least one feasible solution

5. The neural network

An artificial Neural Network (ANN) is a dynamic system, consisting of highly interconnected and parallel non-linear processing elements, that is highly efficient in computation. In this paper, a recurrent neural network [7] with equilibrium points representing a solution of the constrained optimization problem has been developed. As introduced in Hopfield [2] these network are composed with feed back connection between nodes. In the standard case, the nodes are fully connected i.e., every node is connected to all other nodes, including itself.

The first step in a neural network implementation for solving the LP problem is to define an energy function that can be optimized in an unconstrained fashion. Therefore, the method of Lagrange multiplier is applied in the network. To accomplish this, the linear constraints and non negativity constraints are appended to the objective function in some convenient way. Commonly the constraints are incorporated as penalty terms that, when ever violated, increase the value of the energy function. One energy function that can be derived using the Lagrange multiplier method are defined as

$$(4) \quad E(X, Y) = C^T X + Y^T (AX - B) - \alpha Y^T Y$$

with $\alpha \geq 0, Y \in R^{m \times 1}$, and $X \geq 0$.

Applying the method steepest descent in discrete time, I compute the gradient of the energy function in (4) with respect to X and obtain

$$\frac{\partial E}{\partial X} = \frac{\partial}{\partial X} \left[C^T X + Y^T (AX - B) - \alpha Y^T Y \right]$$

and

$$(5) \quad \frac{\partial E}{\partial X} = C + A^T Y.$$

In a similar manner we have

$$(6) \quad \frac{\partial E}{\partial Y} = AX - B - \alpha Y$$

Based on (5) and (6), the node equation for the discrete-time network with n -neurons is given by

$$(7) \quad x_i(k+1) = \begin{cases} x_i(k) - \mu [c_i + \sum a_{ji} y_j(k)] & \text{if } x_i(k+1) > 0 \\ 0 & \text{if } x_i(k+1) \leq 0 \end{cases}$$

and

$$y_j(k+1) = y_j(k) + \eta [z_j(k) - \alpha y_j(k)],$$

where $z_j(k) = AX - B$, $\alpha \geq 0$, and $\mu, \eta \geq 0$ are learning rate parameters.

Note that the update equations for the independent-variable vector in (7) guarantee that all the components remain nonnegative. A neural network architecture realization of this process is presented in Figure 1.

6. MATLAB Simulink

The Simulink toolbox is a useful software package to develop simulation models for recurrent neural network applications in the MATLAB Simulink environment. With its graphical user interface and extensive library, it provides researchers with a modern and interactive design tool build simulation models rapidly and easily. Simulink is an input/output device GUI block diagram simulator. It opens with the library browser and library browser is used to build simulation models. The library browser contains continuous system model elements, discontinuous system models elements, and list of math operation elements. In the library browser, Sink elements are used for displaying and Source elements are used for model source functions. Model elements are added by selecting the appropriate elements from the library browser and dragging them into model window to create the required model.

7. Application of RNN in Medicine

First, we give numerical example to demonstrate the solution of linear programming problem through the proposed recurrent neural network.

A patient in a hospital is required to have at least 84 units of drug A and 120 units of drug B each day. Each gram of substance M contains 10 units of drug A and 8 units of drug B, and each gram of substance N contains 2 units of drug A and 4 units of drug B. Now suppose that both M and N contain an

Table 1: Comparison of the simulation result with the exact solution

Variables	Neural Network Result	Exact Solution
x_1	3.85	4
x_2	22.24	22

undesirable drug C, 3 units per gram in M and 1 unit per gram in N. How many grams of substance M and N should be mixed to meet the minimum daily requirements at the same time minimize the intake of drug C? How many units of the undesirable drug C will be this mixture?

To form the mathematical model, we start by identifying the decision variables. Let x_1 be the number of grams of substance and x_2 be the number of substance N used. The objective is to minimize the intake of drug C . In terms of the decision variables, the objective function is $C = 3x_1 + x_2$ which gives the amount of the undesirable drug C in x_1 grams of M and x_2 grams of N . Considering the above constraint inequalities in terms of the decision variables x_1 and x_2 and including the objective function we obtain the following linear programming model.

$$\begin{aligned}
 &\textbf{Minimize } C = 3x_1 + x_2 \\
 &\textbf{Subject to} \\
 &10x_1 + 2x_2 \leq 84 \\
 &8x_1 + 4x_2 \leq 120 \\
 &x_1, x_2 \geq 0.
 \end{aligned}$$

From the problem statement it can be seen that the linear programming problem is not in the standard form. By adding surplus variables x_3 and x_4 , the linear programming problem can be transferred in the standard form as follows:

$$\begin{aligned}
 &\textbf{Minimize } C = 3x_1 + x_2 + 0x_3 + 0x_4 \\
 &\textbf{Subject to} \\
 &10x_1 + 2x_2 - x_3 \leq 84 \\
 &8x_1 + 4x_2 - x_4 \leq 120 \\
 &x_1, x_2, x_3, x_4 \geq 0.
 \end{aligned}$$

To solve this linear programming problem, the MATLAB Simulink model for recurrent neural network in Figure.1 is simulated. The parameters of the network were chosen as $\mu = 0.01$, $\eta = 0.01$. Zero initial conditions were assumed both for X and Y . The network converges in approximately 4,000 iterations.

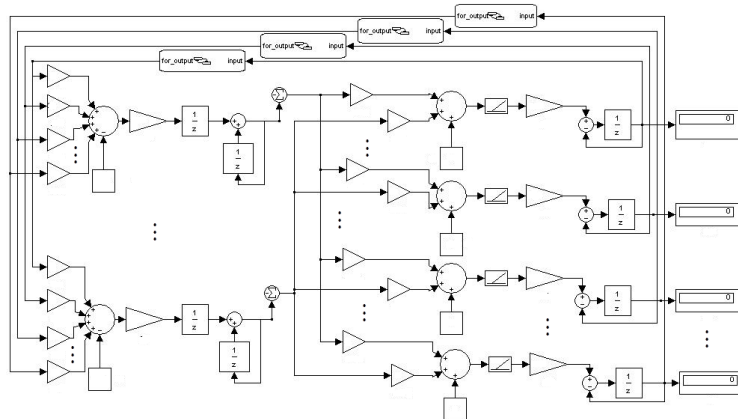


Figure 1: Recurrent Neural Network

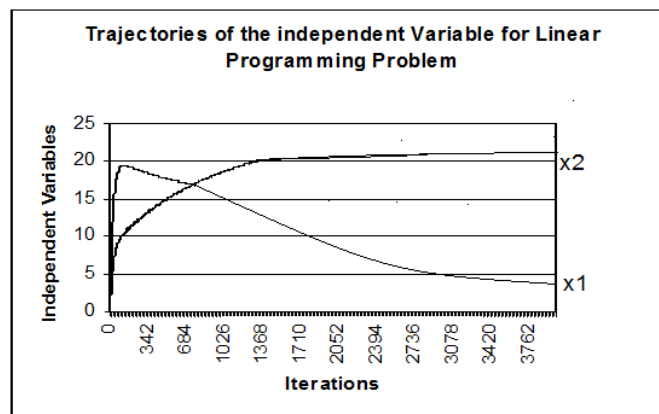


Figure 2: Recurrent Neural Network

8. Conclusion

We investigated in this paper the MATLAB Simulink modeling and simulative verification of such a recurrent neural network. Finding solution of linear programming problems through recurrent neural network approach is an interesting area of research. The energy function of the linear programming is defined in this paper. A circuit has been designed for the purpose. By using click-and-drag mouse operations in MATLAB Simulink environment, we could quickly model and simulate complicated dynamic systems. It has been concluded that recurrent neural network can be used for determining the solution of medicine problem. It converges to the exact solutions of the medicine Problem. Modeling and simulative results substantiate the theoretical analysis and efficacy of the recurrent neural network for solving the linear programming problem.

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GENERALIZED NUMERICAL RADIUS INEQUALITIES FOR 2×2 OPERATOR MATRICES

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Abstract. We prove some new generalized numerical radius inequalities for 2×2 operator matrices, which improve and generalize an earlier numerical radius inequalities.

Keywords: numerical radius, operator norm, operator matrix, off-diagonal part, inequality.

1. Introduction

Let $B(H)$ denote the C^* -algebra of all bounded linear operators on a complex Hilbert space H with inner product $\langle \cdot, \cdot \rangle$. For $A \in B(H)$, let $\omega(A) = \sup \{ |\langle Ax, x \rangle| : x \in H, \|x\| = 1 \}$, $\|A\| = \sup \{ \|Ax\| : x \in H, \|x\| = 1 \}$, where $\|x\|^2 = \langle x, x \rangle$, $r(A) = \sup \{ |\lambda| : \lambda \in \sigma(A) \}$, where $\sigma(A)$ is the spectrum of A , and $|A| = (A^*A)^{\frac{1}{2}}$ denote the numerical radius of A , the usual operator norm of A , the spectral radius of A , and the absolute value of A , respectively.

It is well – known that $\omega(\cdot)$ is a norm on $B(H)$, which is equivalent to the usual operator norm $\|\cdot\|$. In fact, for every $A \in B(H)$, we have

$$(1.1) \quad \frac{1}{2} \|A\| \leq \omega(A) \leq \|A\|.$$

These inequalities are sharp. The first inequality becomes an equality if $A^2 = 0$, and the second inequality becomes an equality if A is normal.

The inequalities in (1.1) have been improved considerably by Kittaneh in [10] and [11]. It has been shown in [10] and [11], respectively, that if $A \in (H)$, then

$$(1.2) \quad \omega(A) \leq \frac{1}{2} (\|A\| + \|A^*\|) \leq \frac{1}{2} (\|A\| + \|A^2\|^{\frac{1}{2}})$$

and

$$(1.3) \quad \frac{1}{4} \|A^*A + AA^*\| \leq \omega^2(A) \leq \frac{1}{2} \|A^*A + AA^*\|.$$

An important property of the numerical radius norm is its weak unitary invariance, that is, for $A \in B(H)$,

$$(1.4) \quad \omega(UAU^*) = \omega(A),$$

for every unitary $U \in B(H)$.

Several numerical radius inequalities improving the inequalities in (1.1) have been recently given in [4], [10], [11], and [12].

Let $H_1, H_2, \dots, H_n$ be complex Hilbert spaces, and consider $H = \bigoplus_{i=1}^n H_i$ with respect to this decomposition, every an $n \times n$ operator matrix representation $A = [A_{ij}]$, with entries $A_{ij} \in B(H_j, H_i)$, the space of all bounded linear operators from H_j to H_i . Operator matrices provide a useful tool for studying Hilbert space operators, which have been extensively studied in the literature (see, e.g., [5]). In [8], Hou and Du established useful estimates for the spectral radius, the numerical radius, and the usual operator norm of an $n \times n$ operator matrix $A = [A_{ij}]$. In particular, they proved that

$$(1.5) \quad r(A) \leq r([\|A_{ij}\|]),$$

$$(1.6) \quad \omega(A) \leq \omega([\|A_{ij}\|]),$$

and

$$(1.7) \quad \|A\| \leq \| [\|A_{ij}\|] \|.$$

Recent numerical radius equalities and inequalities for operator matrices can be found in [1, 2], and [6].

In this paper, we give new generalized numerical radius inequalities for 2×2 operator matrices. In section 2, we establish generalized upper bounds for the numerical radii of the off-diagonal parts of 2×2 operator matrices, i.e., operator matrices of the form $\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$. Also we establish generalized upper bounds for the numerical radii of other 2×2 operator matrices.

2. Main results

The aim of this section is to give generalized upper bounds for the numerical radius of the off-diagonal part of a 2×2 operator matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ defined on $H_1 \oplus H_2$. In order to state our results, we need the following well-known lemmas.

The first lemma is a generalization of the mixed Schwarz inequality which has been proved by Kittaneh [9].

Lemma 1. *Let T be an operator in $B(H)$ and let f and g be nonnegative functions on $[0, \infty)$ which are continuous and satisfying the relation $f(t)g(t) = t$ for all $t \in [0, \infty)$. Then $|\langle Tx, y \rangle| \leq \|f(|T|)x\| \|g(|T^*|)y\|$, for all $x, y \in H$.*

The second lemma contains two parts. Part (a) is well known and can be found in [3, p. 10]. Part (b) is also known (see, e.g., [1]).

Lemma 2. Let $X, Y \in B(H)$. Then

$$(a) \omega \left(\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \right) = \max \{ \omega(X), \omega(Y) \}.$$

$$(b) \omega \left(\begin{bmatrix} X & Y \\ Y & X \end{bmatrix} \right) = \max \{ \omega(X+Y), \omega(X-Y) \}.$$

$$\text{In particular, } \omega \left(\begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix} \right) = \omega(X).$$

The third lemma is very useful in computing the numerical radius for matrices (see [7]).

Lemma 3. If $A = [a_{ij}] \in M_n(\mathbb{C})$, then

$$\omega(A) \leq \omega([|a_{ij}|]) = \frac{1}{2} r([|a_{ij}| + |a_{ji}|]).$$

Our first result is a generalization of the first inequality in (1.2).

Theorem 1. Let $S = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$ be a 2×2 operator matrix in $B(H_1 \oplus H_2)$, and let f and g be nonnegative functions on $[0, \infty)$ which are continuous and satisfying the relation $f(t)g(t) = t$ for all $t \in [0, \infty)$. Then

$$(2.1) \quad \omega(S) \leq \frac{1}{2} \max \{ \|f^2(|C|) + g^2(|B^*|)\|, \|f^2(|B|) + g^2(|C^*|)\| \}.$$

Proof. Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in (H_1 \oplus H_2)$, with $\|x\| = 1$. Then we have

$$\begin{aligned} |\langle Sx, x \rangle| &\leq \langle f^2(|S|)x, x \rangle^{\frac{1}{2}} \langle g^2(|S^*|)x, x \rangle^{\frac{1}{2}} && \text{(by Lemma 1)} \\ &= \left\langle f^2 \left(\begin{bmatrix} |C| & 0 \\ 0 & |B| \end{bmatrix} \right) x, x \right\rangle^{\frac{1}{2}} \left\langle g^2 \left(\begin{bmatrix} |B^*| & 0 \\ 0 & |C^*| \end{bmatrix} \right) x, x \right\rangle^{\frac{1}{2}} \\ &= \left\langle \begin{bmatrix} f^2(|C|) & 0 \\ 0 & f^2(|B|) \end{bmatrix} x, x \right\rangle^{\frac{1}{2}} \left\langle \begin{bmatrix} g^2(|B^*|) & 0 \\ 0 & g^2(|C^*|) \end{bmatrix} x, x \right\rangle^{\frac{1}{2}} \\ &\leq \frac{1}{2} \left(\left\langle \begin{bmatrix} f^2(|C|) & 0 \\ 0 & f^2(|B|) \end{bmatrix} x, x \right\rangle + \left\langle \begin{bmatrix} g^2(|B^*|) & 0 \\ 0 & g^2(|C^*|) \end{bmatrix} x, x \right\rangle \right) \\ &\text{(by the arithmetic - geometric mean inequality)} \\ &= \frac{1}{2} \left\langle \begin{bmatrix} f^2(|C|) + g^2(|B^*|) & 0 \\ 0 & f^2(|B|) + g^2(|C^*|) \end{bmatrix} x, x \right\rangle. \end{aligned}$$

Thus,

$$\begin{aligned} \omega(S) &= \sup \{ |\langle Sx, x \rangle| : x \in (H_1 \oplus H_2), \|x\| = 1 \} \\ &\leq \frac{1}{2} \max \{ \|f^2(|C|) + g^2(|B^*|)\|, \|f^2(|B|) + g^2(|C^*|)\| \}, \end{aligned}$$

as required.

Inequality (2.1) includes several numerical radius inequalities for operator matrices. Samples of inequalities are demonstrated in the following remarks.

Remark 1. For $f(t) = t^\alpha$ and $g(t) = t^{1-\alpha}$, $\alpha \in (0, 1)$, in inequality (2.1), we get the following inequality

$$\omega(S) \leq \frac{1}{2} \max \left\{ \left\| |C|^{2\alpha} + |B^*|^{2(1-\alpha)} \right\|, \left\| |B|^{2\alpha} + |C^*|^{2(1-\alpha)} \right\| \right\}.$$

Remark 2. In Remark 1, if $\alpha = \frac{1}{2}$, then we get

$$\omega(S) \leq \frac{1}{2} \max \{ \| |C| + |B^*| \|, \| |B| + |C^*| \| \}.$$

Remark 3. By letting $H_1 = H_2$ and $B = C$ in Remark 2, and by using Lemma 2(b) it is easy to see that the inequality in Remark 2 generalizes the inequality (1.2), i.e.,

$$\omega(S) = \omega(B) \leq \frac{1}{2} \| |B| + |B^*| \|.$$

In the next theorem, we employ the inequalities in (1.3) to generalize and improve the inequalities in (1.1).

Theorem 2. Let $S = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$ be a 2×2 operator matrix in $B(H_1 \oplus H_2)$.

Then

$$\frac{1}{2} \max \{ \alpha, \beta \} \leq \omega \left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \right) \leq \frac{1}{\sqrt{2}} \max \{ \alpha, \beta \},$$

where $\alpha = \left\| |C|^2 + |B^*|^2 \right\|^{\frac{1}{2}}$ and $\beta = \left\| |B|^2 + |C^*|^2 \right\|^{\frac{1}{2}}$.

In the following results, we establish generalized upper bounds for the numerical radii of a general 2×2 operator matrices.

Theorem 3. Let $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ be a 2×2 operator matrix in $B(H_1 \oplus H_2)$, and let f and g be nonnegative functions on $[0, \infty)$ which are continuous and satisfying the relation $f(t)g(t) = t$, for all $t \in [0, \infty)$. Then

$$\omega(T) \leq \frac{1}{2} \max \{ \|a\|, \|b\| \},$$

where,

$$a = f^2(|A|) + g^2(|A^*|) + f^2(|C|) + g^2(|B^*|)$$

and

$$b = f^2(|D|) + g^2(|D^*|) + f^2(|B|) + g^2(|C^*|).$$

Proof. Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in (H_1 \oplus H_2)$, with $\|x\| = 1$. Then we have

$$\begin{aligned}
\left| \left\langle \begin{bmatrix} A & B \\ C & D \end{bmatrix} x, x \right\rangle \right| &\leq \left\langle f^2 \left(\left\| \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \right\| \right) x, x \right\rangle^{\frac{1}{2}} \left\langle g^2 \left(\left\| \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}^* \right\| \right) x, x \right\rangle^{\frac{1}{2}} \\
&+ \left\langle f^2 \left(\left\| \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \right\| \right) x, x \right\rangle^{\frac{1}{2}} \left\langle g^2 \left(\left\| \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}^* \right\| \right) x, x \right\rangle^{\frac{1}{2}} \text{ (by Lemma 1)} \\
&= \left\langle f^2 \left(\begin{bmatrix} |A| & 0 \\ 0 & |D| \end{bmatrix} \right) x, x \right\rangle^{\frac{1}{2}} \left\langle g^2 \left(\begin{bmatrix} |A^*| & 0 \\ 0 & |D^*| \end{bmatrix} \right) x, x \right\rangle^{\frac{1}{2}} \\
&+ \left\langle f^2 \left(\begin{bmatrix} |C| & 0 \\ 0 & |B| \end{bmatrix} \right) x, x \right\rangle^{\frac{1}{2}} \left\langle g^2 \left(\begin{bmatrix} |B^*| & 0 \\ 0 & |C^*| \end{bmatrix} \right) x, x \right\rangle^{\frac{1}{2}} \\
&= \left\langle \begin{bmatrix} f^2(|A|) & 0 \\ 0 & f^2(|D|) \end{bmatrix} x, x \right\rangle^{\frac{1}{2}} \left\langle \begin{bmatrix} g^2(|A^*|) & 0 \\ 0 & g^2(|D^*|) \end{bmatrix} x, x \right\rangle^{\frac{1}{2}} \\
&+ \left\langle \begin{bmatrix} f^2(|C|) & 0 \\ 0 & f^2(|B|) \end{bmatrix} x, x \right\rangle^{\frac{1}{2}} \left\langle \begin{bmatrix} g^2(|B^*|) & 0 \\ 0 & g^2(|C^*|) \end{bmatrix} x, x \right\rangle^{\frac{1}{2}} \\
&\leq \frac{1}{2} \left(\left\langle \begin{bmatrix} f^2(|A|) & 0 \\ 0 & f^2(|D|) \end{bmatrix} x, x \right\rangle + \left\langle \begin{bmatrix} g^2(|A^*|) & 0 \\ 0 & g^2(|D^*|) \end{bmatrix} x, x \right\rangle \right. \\
&\quad \left. + \left\langle \begin{bmatrix} f^2(|C|) & 0 \\ 0 & f^2(|B|) \end{bmatrix} x, x \right\rangle + \left\langle \begin{bmatrix} g^2(|B^*|) & 0 \\ 0 & g^2(|C^*|) \end{bmatrix} x, x \right\rangle \right) \\
&\text{(by the arithmetic-geometric mean inequality)} \\
&= \frac{1}{2} \left\langle \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} x, x \right\rangle.
\end{aligned}$$

Thus,

$$\omega(T) = \sup \{ |\langle Tx, x \rangle| : x \in (H_1 \oplus H_2), \|x\| = 1 \} \leq \frac{1}{2} \max \{ \|a\|, \|b\| \},$$

as required.

Remark 4. If $f(t) = t^\alpha$ and $g(t) = t^{1-\alpha}$, $\alpha \in (0, 1)$, in Theorem 3, then we get the following inequality

$$\omega \left(\begin{bmatrix} A & B \\ C & D \end{bmatrix} \right) \leq \frac{1}{2} \max \left\{ \left\| |A|^{2\alpha} + |A^*|^{2(1-\alpha)} + |C|^{2\alpha} + |B^*|^{2(1-\alpha)} \right\|, \left\| |D|^{2\alpha} + |D^*|^{2(1-\alpha)} + |B|^{2\alpha} + |C^*|^{2(1-\alpha)} \right\| \right\}.$$

Remark 5. From Remark 4 with $\alpha = \frac{1}{2}$ and Lemma 2(b), If $A = B = C = D$, then we get the following inequality

$$\omega \left(\begin{bmatrix} A & A \\ A & A \end{bmatrix} \right) = 2\omega(A) \leq \frac{1}{2} \|2|A| + 2|A^*|\| = \| |A| + |A^*| \|,$$

and so,

$$\omega(A) \leq \frac{1}{2} \| |A| + |A^*| \|.$$

Remark 6. From Remark 4 with $\alpha = \frac{1}{2}$ and the first inequality in (1.1), If $A = C = D = 0$, then we get the following equality

$$\frac{1}{2} \|B\| = \frac{1}{2} \left\| \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix} \right\| \leq \omega \left(\begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix} \right) \leq \frac{1}{2} \max \{ \| |B^*| \|, \| |B| \| \} = \frac{1}{2} \|B\|.$$

Hence,

$$\omega \left(\begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix} \right) = \omega \left(\begin{bmatrix} 0 & 0 \\ B & 0 \end{bmatrix} \right) = \frac{1}{2} \|B\|.$$

In the following theorem, we present an improvement of the inequality (1.6) when $n = 2$.

Theorem 4. Let $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ be a 2×2 operator matrix in $B(H_1 \oplus H_2)$, and let f and g be nonnegative functions on $[0, \infty)$ which are continuous and satisfying the relation $f(t)g(t) = t$ for all $t \in [0, \infty)$. Then

$$\omega(T) \leq \omega \left(\begin{bmatrix} \omega(A) & \|f(|B|)\| \|g(|B^*|)\| \\ \|f(|C|)\| \|g(|C^*|)\| & \omega(D) \end{bmatrix} \right).$$

Proof. Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in (H_1 \oplus H_2)$, with $\|x\| = 1$. Then we have

$$\begin{aligned} |\langle Tx, x \rangle| &= |\langle Ax_1, x_1 \rangle + \langle Bx_2, x_1 \rangle + \langle Cx_1, x_2 \rangle + \langle Dx_2, x_2 \rangle| \\ &\leq |\langle Ax_1, x_1 \rangle| + |\langle Bx_2, x_1 \rangle| + |\langle Cx_1, x_2 \rangle| + |\langle Dx_2, x_2 \rangle| \\ &\leq \omega(A) \|x_1\|^2 + \|f(|B|)x_2\| \|g(|B^*|)x_1\| \\ &\quad + \|f(|C|)x_1\| \|g(|C^*|)x_2\| + \omega(D) \|x_2\|^2 \\ &\quad \text{(by definition of } \omega(\cdot) \text{ and Lemma 1.1)} \\ &\leq \omega(A) \|x_1\|^2 + \|f(|B|)\| \|g(|B^*|)\| \|x_1\| \|x_2\| \\ &\quad + \|f(|C|)\| \|g(|C^*|)\| \|x_1\| \|x_2\| + \omega(D) \|x_2\|^2 \\ &= \left\langle \begin{bmatrix} \omega(A) & \|f(|B|)\| \|g(|B^*|)\| \\ \|f(|C|)\| \|g(|C^*|)\| & \omega(D) \end{bmatrix} \begin{bmatrix} \|x_1\| \\ \|x_2\| \end{bmatrix}, \begin{bmatrix} \|x_1\| \\ \|x_2\| \end{bmatrix} \right\rangle. \end{aligned}$$

Now, the result follows by taking the supremum over all unit vectors in $(H_1 \oplus H_2)$.

Here, a weaker version of Theorem 4 has been also given in [2] when $n = 2$.

Remark 7. From Theorem 4, if $f(t) = \sqrt{t} = g(t)$, then we get the following inequality

$$\begin{aligned} \omega \left(\begin{bmatrix} A & B \\ C & D \end{bmatrix} \right) &\leq \omega \left(\begin{bmatrix} \omega(A) & \| |B|^{\frac{1}{2}} \| \| |B^*|^{\frac{1}{2}} \| \\ \| |C|^{\frac{1}{2}} \| \| |C^*|^{\frac{1}{2}} \| & \omega(D) \end{bmatrix} \right) \\ &\leq \omega \left(\begin{bmatrix} \omega(A) & \|B\| \\ \|C\| & \omega(D) \end{bmatrix} \right) \\ &\leq \omega \left(\begin{bmatrix} \|A\| & \|B\| \\ \|C\| & \|D\| \end{bmatrix} \right). \end{aligned}$$

Now, from Remark 7 and Lemma 3 we get the inequality

$$\omega \left(\begin{bmatrix} A & B \\ C & D \end{bmatrix} \right) \leq \frac{1}{2} \left(\omega(A) + \omega(D) + \sqrt{(\omega(A) - \omega(D))^2 + (\|B\| + \|C\|)^2} \right),$$

which is a generalized for the second inequality in (1.1) when we take $A = B = C = D$ and use Lemma 2(a). Also this inequality can be employed to give new bounds for the zeros of polynomials (see, e.g., [2, 10], and references therein).

Based on Lemma 2(a), the inequality (2.1) and the property $\omega(X + Y) \leq \omega(X) + \omega(Y)$, we can prove the following corollary.

Corollary 1. Let $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ be a 2×2 operator matrix in $B(H_1 \oplus H_2)$, and let f and g be nonnegative functions on $[0, \infty)$ which are continuous and satisfying the relation $f(t)g(t) = t$ for all $t \in [0, \infty)$. Then

$$\begin{aligned} \omega(T) &\leq \max \{ \omega(A), \omega(D) \} \\ &\quad + \frac{1}{2} \max \{ \|f^2(|C|) + g^2(|B^*|)\|, \|f^2(|B|) + g^2(|C^*|)\| \}. \end{aligned}$$

Remark 8. From Corollary 1 and Lemma 2(b), if $f(t) = \sqrt{t} = g(t)$ and if $A = B = C = D$, then we get the following inequality

$$\omega \left(\begin{bmatrix} A & A \\ A & A \end{bmatrix} \right) = 2\omega(A) \leq \omega(A) + \frac{1}{2} \| |A| + |A^*| \|,$$

and so,

$$\omega(A) \leq \frac{1}{2} \| |A| + |A^*| \|.$$

Remark 9. From Remark 4, Lemma 2(b), $\alpha = \frac{1}{2}$ and If $A = D, B = C$, then we get the following

$$\max \{ \omega(A - B), \omega(A + B) \} \leq \frac{1}{2} \| |A| + |A^*| + |B| + |B^*| \|.$$

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A COMPARATIVE STUDY ON ACHROMATIC AND B -CHROMATIC NUMBER OF CERTAIN GRAPHS

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Abstract. In this paper, we find the achromatic number of central graph of Crown graph and we discussed its structural properties. We compare the achromatic and b -chromatic number of central graph of Sunlet graph, central graph of web graph. Also we study the structural properties of the central graph of Sunlet graph.

Keywords: achromatic number, b -chromatic number, central graph, Crown graph, Sunlet graph.

1. Introduction

Let G be a finite undirectional graph with no loops and multiple edges. The central graph $C(G)$ ([7]) of a graph G is obtained by subdividing each edge of G exactly once and joining all the non adjacent vertices of G . An achromatic colouring ([4]) is a proper vertex colouring such that each pair of colours is adjacent by at least one edge. The largest possible number of colours in an achromatic colouring of G is called the achromatic number and it is denoted by $\psi(G)$. The b -chromatic number $\varphi(G)$ ([3,5]) of a graph G is the largest integer k , such that G admits a proper k -colouring, and every colour class has a representative vertex adjacent at least to one vertex in each other class. This type of colouring is called b -colouring. This concept of b -chromatic number was introduced by Irwing and & Manlove.

The Crown graph S_n for an integer $n > 2$ is the graph with the vertex set $\{x_1, x_2, \dots, x_n, y_1, \dots, y_n\}$ and the edge set $\{(x_i, y_j) : 1 \leq i, j \leq n, i \neq j\}$. The n -Sunlet graph Sl_n is the graph on $2n$ vertices obtained by attaching n pendent edges to the cycle graph C_n . The 3-Sunlet graph is also known as the net graph. The Web graph W_n is a graph consisting of concentric copies of the cycle graph C_n with corresponding vertices connected by spokes and adjoining pendant vertex at each node of the outer cycle.

2. The structural properties of central graph of crown graph

- The number of vertices in the Crown graph S_n is $p = 2n$.

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- The number of vertices in the central graph of Crown graph $C[S_n]$ is $p = n^2 + n$.
- The number of edges in the Crown graph S_n is $q = n(n - 1)$.
- The number of edges in the central graph of Crown graph $C[S_n]$ is $q = 3n^2 - 2n$.
- In the Crown graph all the vertices have the same degree $(n - 1)$. So it is a regular graph.

2.1 Theorem

For any Crown graph $\psi[C(S_n)] = 2n, n \geq 3$

Proof. Let G be the Crown graph with the partition (X, Y) of the vertices where $X = \{x_1, x_2, \dots, x_n\}$ and $Y = \{y_1, y_2, \dots, y_n\}$. In $C(S_n)$, let $v_{i,j}$ be the newly introduced vertex joining x_i and y_j where $i \neq j$ and $i, j = 1, 2, 3, \dots, n$. Consider the two sets of colours $C = \{C_1, C_2, \dots, C_n\}$ and $C' = \{C'_1, C'_2, \dots, C'_n\}$. Assign a proper colouring to G as follows: For $1 \leq i \leq n$, assign C_i to x_i and C'_i to y_i . By the definition of central graph each C_i is adjacent to all the other C_j 's and to C'_i . For a particular i , the $n - 1$ vertices $x_i y_j$ are allocated the $n - 1$ colours $C'_{r,r \neq j}$ in such a way that $x_i y_j$ is given a colour other than C'_j . From the construction it is seen that this is the maximal possible colouring. Hence $\psi[C(S_n)] = 2n, n \geq 3$.

Example.

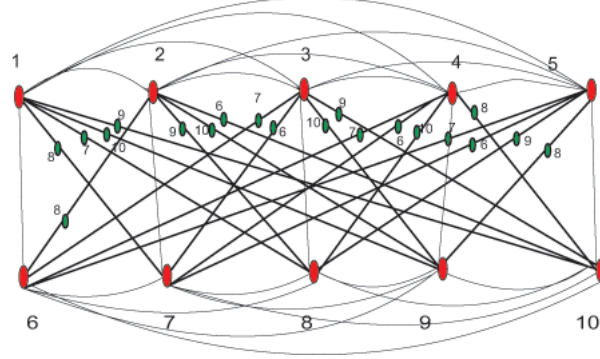


Figure 1: $\psi[C(S_5)] = 10$

3. The structural properties of central graph of sunlet graph

- The number of vertices in the Sunlet graph S_n is $p = 2n$.
- The number of vertices in the central graph of Sunlet graph $C[S_n]$ is $p = 4n$.

- The number of edges in the Sunlet graph S_n is $q = 2n$.
- The number of edges in the central graph of Sunlet graph $C[S_n]$ is $q = 2n^2 + n$.

3.1 Theorem

Theorem: For any Sunlet Graph $\psi[C(Sl_n)] = 2n, n \geq 3$.

Proof. Let C_n be any cycle graph of length n with vertices $v_1, v_2, \dots, v_n$ named in the cyclic order. Name the attached pendant vertices $u_1, u_2, \dots, u_n$ in the same cyclic order. Now in $C(Sl_n)$, let $v_{i,j}$ be the newly introduced vertex on the edge joined v_i and v_j and uv_i represent the newly introduced vertex on the edge joining v_i and u_i . For $1 \leq i \leq n$, the vertex v_i is not adjacent with u_i . For $1 \leq i \leq n-1$, the vertex v_i is not adjacent with v_{i+1} and v_1 is not adjacent with v_n . To make the colouring as achromatic, assign the following colouring procedure: Consider the two set of colours $C = \{C_1, C_2, \dots, C_n\}$ and $C' = \{C'_1, C'_2, \dots, C'_n\}$.

- For $1 \leq i \leq n$, assign C_i to v_i .
- For $1 \leq i \leq n$, assign C'_i to u_i .
- For $1 \leq i \leq n-1$, assign C'_i to $v_{i,i+1}$ and C'_n to $v_{1,n}$.
- For $1 \leq i \leq n-1$, assign C_{i+1} to uv_i and assign the colour C_1 to the vertex uv_n .

By this assignment any pair in the colour set is adjacent by at least one edge. Thus it is complete and it is a maximal one. Hence $\psi[C(Sl_n)] = n + n = 2n, n \geq 3$.

Example.

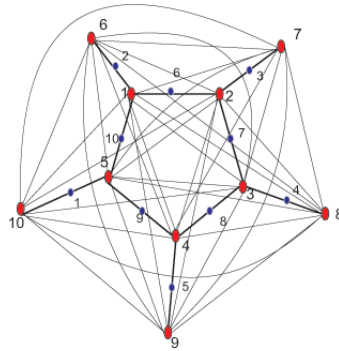


Figure 2: $\psi[C(Sl_5)] = 10$

3.2 Theorem

For any Sunlet Graph $\varphi[C(Sl_n)] = \begin{cases} \frac{3n}{2}, & n = \text{even} \\ n + \lfloor \frac{n}{2} \rfloor, & n = \text{odd} \end{cases}$

Proof. Consider the Sunlet graph which is obtained by attaching n pendent edges to the cycle C_n . Consider the two vertex sets $X = \{x_1, x_2, \dots, x_n\}$ and $Y = \{y_1, y_2, \dots, y_n\}$. First assign the name $x_1, x_2, \dots, x_n$ to the pendant vertices in the cyclic order. Then assign the names $y_1, y_2, \dots, y_n$ to the vertices of the cycle graph in the same cyclic order. Now in $C[S_n]$, let $y_{i,j}$ be the newly introduced vertex on the edge joining y_i and y_j . And let xy_i be the newly introduced vertex on the edge joining x_i and y_i . Here we can observe that

- For $1 \leq i \leq n$, the vertex x_i is not adjacent to y_i .
- For $2 \leq i \leq n-1$, the vertex y_i is not adjacent to y_{i+1} and y_{i-1} .
- y_1 is not adjacent to y_2 and y_n also y_n is not adjacent with y_1 and y_{n-1} .

Case 1. When $n = \text{even}$ Consider the two set of colours

$C = \{C_1, D_1, C_2, D_2, C_{\frac{n}{2}}, D_{\frac{n}{2}}\}$ and $C' = \{C'_1, C'_2, \dots, C'_{\frac{n}{2}}\}$. When $i = \text{odd}$, assign $C_{\frac{i+1}{2}}$ to x_i , when $i = \text{even}$, assign $D_{\frac{i}{2}}$ to x_i . Also assign C'_i to y_{2i-1}, y_{2i} .

- For $1 \leq i \leq \frac{n}{2}$, assign the colour C_i to the vertex y_{2i-1}, y_{2i} .
- For $1 \leq i \leq \frac{n}{2} - 1$, assign the colour D_i to the vertex y_{2i}, y_{2i+1} . And assign the colour $D_{\frac{n}{2}}$ to the vertex $y_{1,n}$.
- For $1 \leq i \leq \frac{n}{2}$, assign the colour $D_{\frac{i+1}{2}}$ to the vertex xy_i , when i is odd.
- For $1 \leq i \leq \frac{n}{2}$, assign the colour $C_{\frac{i}{2}}$ to the vertex xy_i , when i is even.

Case 2. When $n = \text{odd}$ Consider the two set of colours $C = \{C_1, C_2, \dots, C_{\frac{n+1}{2}}, D_1, D_2, \dots, D_{\frac{n-1}{2}}\}$ and $C' = \{C'_1, C'_2, \dots, C'_{\frac{n}{2}}\}$. When $i = \text{odd}$, assign $C_{\frac{i+1}{2}}$ to x_i , when $i = \text{even}$, assign $D_{\frac{i}{2}}$ to x_i . Assign C'_i to y_{2i-1}, y_{2i} and assign $C_{\frac{n+1}{2}}$ to y_n .

- For $1 \leq i \leq \frac{n+1}{2}$, assign the colour C_i to the vertex y_{2i-1}, y_{2i} .
- For $1 \leq i \leq \frac{n-1}{2}$, assign the colour D_i to the vertex y_{2i}, y_{2i+1} . Assign the colour C_1 to the vertex $y_{1,n}$.
- When i is odd, assign the colour $D_{\frac{i+1}{2}}$ to the vertex xy_i .
- When i is even, assign the colour $C_{\frac{i}{2}}$ to the vertex xy_i , where $i \neq n$. Assign the colour C_1 to the vertex xy_n .

If we introduce any new colour to any vertex, that will not be adjacent to all the other colours in the colour set. Hence by this colouring procedure, the above said colouring is b -chromatic and it is the maximal one.

Example.

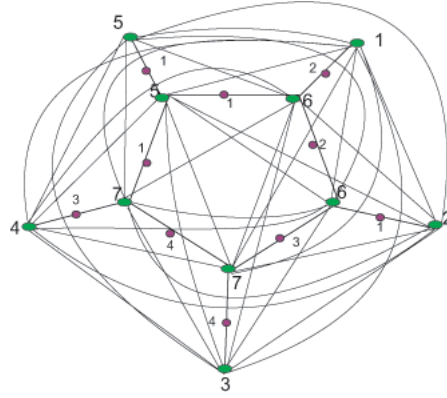


Figure 3: $\varphi[C(Sl_5)] = 7$

4. Observations

- For any Web graph $W_n, \psi[C(W_n)] = 3n, n \geq 3$.
- For any Web graph $W_n, \varphi[C(W_n)] = 2n, n \geq 3$.

5. Conclusion

In this paper, we have found the achromatic number of central graph of Crown graph and have noted that it is equal to the number of vertices in that graph. And we discussed the achromatic and b -chromatic number of central graph of Sunlet graph. Also we find the achromatic and b -chromatic number of central graph of web graph.

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A WEAKER QUANTITATIVE CHARACTERIZATION OF THE SPORADIC SIMPLE GROUPS

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Abstract. It is proved in this paper that all the sporadic simple groups can be characterized by their orders and one special conjugacy class sizes, such as largest conjugacy class sizes, and smallest conjugacy class sizes greater than 1.

Keywords: Sporadic groups, conjugacy class sizes, characterization.

1. Introduction

All groups considered in this paper are finite.

In recent years, it is an interesting topic to characterize finite simple groups by their quantitative properties such as element orders and conjugacy class sizes. For example, in a private communication to W.J. Shi, J.G. Thompson proposed the following conjecture (see [3]).

Thompson's conjecture. *Let G be a group with $Z(G) = 1$ and N is a non-abelian simple group satisfying that $cs(G) = cs(N)$. Then $G \simeq N$.*

In the above conjecture, $cs(G) = \{|G : C_G(g)| : g \in G\}$.

In 1994, G. Y. Chen proved in his Ph. D. dissertation [1] that if G is a group with $Z(G) = 1$, and N a non-abelian simple group with non-connected prime graph such that $cs(G) = cs(N)$, then $G \simeq N$ (also ref. to [2, 3, 4]).

What we can see in Thompson's conjecture is that all the conjugacy class sizes of the sporadic groups are involved. Thus, naturally, one can ask whether this condition can be weakened? For instance, if we just consider some special conjugacy class sizes of a group G , what information about G can we obtain? The purpose of this paper is devoted to this direction.

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In order to state our results, we need the some notation. Let G be a group. Let

$$lcs(G) = \max\{|G : C_G(g)| : g \in G\}$$

and

$$scs(G) = \min\{|G : C_G(g)| : g \in G \setminus Z(G)\}.$$

denote the largest conjugacy class size of G and the smallest conjugacy class size of G greater than 1, respectively. Furthermore, set

$$sscs(G) = \min cs(G) \setminus \{1, scs(G)\}.$$

Our results are as follows.

Theorem 1.1. *Let N be one of the following sporadic groups:*

$$M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, J_1, J_3, J_4,$$

$$Co_1, Co_3, McL, He, Ru, HN, Ly, Th, M.$$

Let G be a group such that $|G| = |N|$ and $scs(G) = scs(N)$. Then $G \simeq N$.

For other sporadic simple groups, we have that

Theorem 1.2. *Let N be one of $J_2, HS, Suz, O'N, Co_2, Fi_{22}, Fi_{23}, Fi'_{24}$. If G is a group with $|G| = |N|$ and $lcs(G) = lcs(N)$, then $G \simeq N$.*

Proposition 1.3. *Let G be a group such that $|G| = |B|$, $scs(G) = scs(B)$ and $sscs(G) = sscs(B)$. Then $G \simeq B$.*

By [1, 2, 3, 4], we know that if G is a group with $Z(G) = 1$ and N is a non-abelian simple group such that $cs(G) = cs(N)$ and the prime graph of N is non-connected, then $|G| = |N|$. Since, the prime graphs of all the sporadic simple groups are not connected, as a corollary, we have

Corollary 1.4. *Let G be a group with $Z(G)$ trivial and N be an arbitrary sporadic group. If $cs(G) = cs(N)$, then $G \simeq N$.*

2. Preliminaries

In this section, we collect some elementary facts which are useful in our proof.

For a group G , define its prime graph $\Gamma(G)$ as follows: the vertices are the primes dividing the order of G , two vertices p and q are joined by an edge if and only if G contains an element of order pq (see [8]). Denote the connected components of the prime graph by $T(G) = \{\pi_i(G) | 1 \leq i \leq t(G)\}$, where $t(G)$ is the number of the prime graph components of G . If the order of G is even, assume that the prime 2 is always contained in $\pi_1(G)$.

For $x \in G$, x^G denotes the conjugacy class in G containing x and $C_G(x)$ denotes the centralizer of x in G . Then $|x^G| = |G : C_G(x)|$.

A simple group whose order has exactly n distinct primes is called a simple K_n -group. For example, it is well known that there are 8 simple K_3 -groups. In addition, for a group G , we call G a 2-Frobenius group if G has a normal series $1 \subseteq H \subseteq K \subseteq G$ such that K and G/H are Frobenius groups with kernels H and K/H respectively.

For other notation and terminologies not mentioned in this paper, the reader is referred to ATLAS [5] if necessary.

Lemma 2.1. *Let G be a group with more than one prime graph component. Then G is one of the following:*

- (i) *a Frobenius or 2-Frobenius group;*
- (ii) *G has a normal series $1 \subseteq H \subseteq K \subseteq G$, where H is a nilpotent π_1 -group, K/H is a non-abelian simple group and G/K is a π_1 -group such that $|G/K|$ divides the order of the outer automorphism group of K/H . Besides, for $i \geq 2$, $\pi_i(G)$ is also a component of $\Gamma(K/H)$.*

Proof. It follows straight forward from Lemmas 1-3 in [8], Lemma 1.5 in [2] and Lemma 7 in [4]. $\square$

Lemma 2.2. *Suppose that G is a Frobenius group of even order and H, K are the Frobenius kernel and the Frobenius complement of G , respectively. Then $t(G) = 2$, $T(G) = \{\pi(H), \pi(K)\}$ and G has one of the following structures:*

- (i) *$2 \in \pi(H)$ and all Sylow subgroups of K are cyclic;*
- (ii) *$2 \in \pi(K)$, H is an abelian group, K is a solvable group, the Sylow subgroups of K of odd order are cyclic groups and the Sylow 2-subgroups of K are cyclic or generalized quaternion groups;*
- (iii) *$2 \in \pi(K)$, H is abelian, and there exists a subgroup K_0 of K such that $|K : K_0| \leq 2$, $K_0 = Z \times SL(2, 5)$, $(|Z|, 2 \times 3 \times 5) = 1$, and the Sylow subgroups of Z are cyclic.*

Proof. This is Lemma 1.6 in [3]. $\square$

Lemma 2.3. *Let G be a 2-Frobenius group of even order. Then $t(G) = 2$ and G has a normal series $1 \subseteq H \subseteq K \subseteq G$ such that $\pi(K/H) = \pi_2$, $\pi(H) \cup \pi(G/K) = \pi_1$, the order of G/K divides the order of the automorphism group of K/H , and both G/K and K/H are cyclic. Especially, $|G/K| < |K/H|$ and G is soluble.*

Proof. This is Lemma 1.7 in [3]. $\square$

The following lemma is well-known (see [7, Theorem 3.3.20]).

Lemma 2.4. *Let $R = R_1 \times \cdots \times R_k$, where R_i is a direct product of n_i isomorphic copies of a non-abelian simple group H_i and H_i and H_j are not isomorphic if $i \neq j$. Then*

$$\text{Aut}(R) \simeq \text{Aut}(R_1) \times \cdots \times \text{Aut}(R_k) \text{ and } \text{Aut}(R_i) \simeq (\text{Aut}(H_i)) \wr S_{n_i}.$$

Moreover,

$$\text{Out}(R) \simeq \text{Out}(R_1) \times \cdots \times \text{Out}(R_k) \text{ and } \text{Out}(R_i) \simeq (\text{Out}(H_i)) \wr S_{n_i}.$$

3. Proof of Theorem 1.1

It has been proved in [6] that the theorem holds for M_{12} and M_{23} . Therefore we need to treat the remaining cases.

From now on, we suppose that G is group satisfying the condition of Theorem 1.1.

Lemma 3.1. *Every minimal normal subgroup of $\overline{G} = G/Z(G)$ is non-soluble. Therefore, if M is the product of all minimal normal subgroups of $\overline{G}$, then $M = S_1 \times S_2 \times \cdots \times S_k$ and $M \leq \overline{G} \leq \text{Aut}(M)$, where every S_i is a non-abelian simple group.*

Proof. By the hypothesis and [5], it is clear that $Z(G)$ is a proper subgroup of G . Let S be any minimal normal subgroup of $\overline{G}$. Suppose that S is soluble. Then S is an elementary abelian group, from which we get the preimage T of S in G is a nilpotent group. If $|S| = r^t$, then the Sylow r -subgroup R of T is normal in G . Moreover, R can not be contained in $Z(G)$. Thus there exists an element y of R which is not contained in $Z(G)$ such that $1 < |y^G| \leq |R|$. However, by [5], we have that $\text{scs}(G) = \text{scs}(N) > |N_p|$, where N_p is any Sylow p -subgroup of M with $p \in \pi(M)$. This contradiction shows that every minimal normal subgroup of $\overline{G}$ is non-soluble, as desired. It follows that the second assertion holds. $\square$

Lemma 3.2. *If $N \simeq M_{11}$, then $G \simeq M_{11}$.*

Proof. By the hypothesis, $|G| = |M| = 2^4 \cdot 3^2 \cdot 5 \cdot 11$ and $\text{scs}(G) = \text{scs}(M) = 3 \cdot 5 \cdot 11$. Then G has an element x such that $|C_G(x)| = 2^4 \cdot 3$ and $|x^G| = 3 \cdot 5 \cdot 11$. Thus, $5, 11 \notin \pi(Z(G))$. Without loss of generality, assume that $11 \in \pi(S_1)$. Then, by [5], we see that S_1 is isomorphic to M_{11} or $L_2(11)$. If $S_1 \simeq L_2(11)$, then $M = L_2(11)$ and $|Z(G)| \geq 6$. Let $y \in G$ with order 11. Then $y \notin Z(G)$ and $|C_G(y)| > |C_G(x)|$, a contradiction. Hence $S_1 \simeq M_{11}$ and so is G . $\square$

Lemma 3.3. *If $N \simeq M_{22}$, then $G \simeq M_{22}$.*

Proof. In this case, $|G| = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$ and $\text{scs}(G) = 3 \cdot 5 \cdot 7 \cdot 11$. Therefore for some $x \in G$, we have $|C_G(x)| = 2^7 \cdot 3$ and $|x^G| = 3 \cdot 5 \cdot 7 \cdot 11$. This implies that $5, 7, 11 \notin \pi(Z(G))$. Assume that $11 \in \pi(S_1)$. Then by [5], we know that S_1 is isomorphic to one of the following groups:

$$L_2(11), M_{11}, M_{22}.$$

If $S_1 \simeq L_2(11)$, then $M = S_1 \times S_2$, where S_2 is a simple K_3 -group with $7 \in \pi(S_2)$. Then we can pick an element $y \in G$ of order 7 such that $1 < |y^G| < |x^G|$, a contradiction. If $S_1 \simeq M_{11}$, then one can conclude that $M = M_{11}$ and so $7 \in \pi(Z(G))$, a contradiction. Hence we have that S_1 must be isomorphic to M_{22} , as wanted. $\square$

Lemma 3.4. *If $N \simeq M_{24}$, then $G \simeq M_{24}$.*

Proof. By the hypothesis, we have that $|G| = 2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$ and $\text{scs}(G) = \text{scs}(M) = 3^2 \cdot 5 \cdot 11 \cdot 23$. Then there is an element $x \in G$ such that $|C_G(x)| = 2^{10} \cdot 3 \cdot 7$ and $|x^G| = \text{scs}(G)$. It is easy to see that $11, 23 \notin \pi(Z(G))$. As before, we assume that $23 \in \pi(S_1)$. Then by [5], we get that S_1 is isomorphic to M_{24} , M_{23} or $L_2(23)$. Similarly, we can rule out $L_2(23)$. If $S_1 \simeq M_{23}$, then $|Z(G)| = 2^3 \cdot 3$ and $G \simeq G' \times Z(G)$, where $G' \simeq M_{23}$. By [5], we know that there is an involution in G' such that $|C_G(y)| > |C_G(x)|$, a contradiction. Hence, S_1 must be isomorphic to M_{24} and so $G \simeq M_{24}$. $\square$

Lemma 3.5. *If $N = J_1$, then $G \simeq J_1$.*

Proof. By the hypothesis, $|G| = 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$ and $\text{scs}(G) = \text{scs}(J_1) = 7 \cdot 11 \cdot 19$. Clearly, $Z(G)$ is a proper subgroup of G and $7, 11, 19 \notin \pi(Z(G))$. By Lemma 2.4, we get that $G/Z(G)$ has a minimal normal subgroup S which is non-abelian simple and $19 \in \pi(S)$. By [5], S must be isomorphic to J_1 and so $G \simeq J_1$. $\square$

Lemma 3.6. *If $N = J_3$, then $G \simeq J_3$.*

Proof. In this case, $|G| = 2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19$ and $\text{scs}(G) = 3^4 \cdot 17 \cdot 19$. Then, for some $x \in G$, we have that $|x^G| = \text{scs}(G)$ and $|C_G(x)| = 2^7 \cdot 3 \cdot 5$. Therefore, $17, 19 \notin \pi(Z(G))$ and by Lemma 2.4, we have that $19 \in \pi(M)$. Without loss of generality, we suppose that $19 \in \pi(S_1)$. Then, $S_1 \simeq L_2(19)$ or J_3 by [5]. If $S_1 \simeq L_2(19)$, then M may be isomorphic to one of the following groups:

$$L_2(19) \times L_2(17), L_2(19) \times L_3(3).$$

In any case, we have that $\pi(Z(G)) \subseteq \{2, 3\}$. Let $y \in G$ with order 19. Then $|C_G(x)| < |C_G(y)| < |G|$ and so $1 < |y^G| < |x^G|$, a contradiction. Therefore, S_1 must be J_3 , which implies that $G \simeq J_3$. $\square$

Lemma 3.7. *If $N = J_4$, then $G \simeq J_4$.*

Proof. By the hypothesis, we have that $|G| = 2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43$ and $\text{scs}(G) = 11^2 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43$.

Let $x \in G$ such that $|x^G| = \text{scs}(G)$. Discussing as above, we see that M is a non-abelian simple group, isomorphic to $L_2(43)$ or J_4 . If $M \simeq L_2(43)$, then $G/Z(G) \simeq L_2(43)$ or $L_2(43).2$. Let $y \in G$ with order 43. Then $y \notin Z(G)$ and $|C_G(y)| > |C_G(x)|$, a contradiction. Hence, $S_1 \simeq J_4$ and consequently $G \simeq J_4$. $\square$

Lemma 3.8. *If $N \simeq McL$, then $G \simeq McL$.*

Proof. In this case, $|G| = 2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$ and $\text{scs}(G) = 3^4 \cdot 5^2 \cdot 11$. Suppose that $x \in G$ such that $|x^G| = \text{scs}(G)$. Then, clearly, $Z(G)$ is a proper subgroup of G and $11 \notin \pi(Z(G))$. Moreover, by Lemma 2.4, $11 \in \pi(M)$. Suppose that $11 \in \pi(S_1)$. Then, by [5], we get that S_1 may be isomorphic to one of the following groups:

$$L_2(11), M_{11}, M_{12}, M_{22}, A_{11}, McL.$$

First, assume $S_1 \simeq L_2(11)$. By Lemma 2.4, we have that 7 does not divide $|Out(M)|$. Let $y \in G$ with order 11. Then, y is not contained in $Z(G)$ and furthermore, we have that in fact $|C_G(y)| > |C_G(x)|$, a contradiction.

Suppose that $S_1 \simeq M_{11}$. Then, by Lemma 2.4, we see that $7 \notin \pi(Out(M))$. If 7 does not divide $|Z(G)|$, then for some y in G of order 7, $1 < |y^G| < |x^G|$, a contradiction. If $7 \in \pi(Z(G))$, then $M \simeq M_{11} \times A_5$ or $M_{11} \times A_6$.

If $M \simeq M_{11} \times A_5$, then, for some $z \in G$ of order 11, we get that $1 < |z^G| < |x^G|$, a contradiction. If $M \simeq M_{11} \times A_6$, then for some involution $g \in G$ such that $gZ(G) \in A_6$, we have that $1 < |g^G| < |x^G|$, also a contradiction. Hence, S_1 can not be isomorphic M_{11} .

Now, assume that $S_1 \simeq M_{12}$. Then it is easy to see that $M = M_{12}$. Therefore $G/Z(G) \simeq M_{12}$ or $M_{12}.2$. If the former case occurs, then, by [5], one can choose an element y in G such that $1 < |y^G| < |x^G|$, a contradiction. If the second case happens, then $|Z(G)| = 3^3 \cdot 5^2 \cdot 7$. By [5], one can pick $z \in G$ of order 2 such that $\bar{z}$ in $\bar{G}$ is a involution and $C_{\bar{G}}(\bar{z}) = 240$. Since $Z(G)$ is a $2'$ -group, we have that

$$C_{\bar{G}}(\bar{z}) = C_G(z)Z(G)/Z(G) = C_G(z)/Z(G),$$

and so $|C_G(z)| > |C_G(x)|$, a contradiction.

If $S_1 \simeq M_{22}$, then $M \simeq M_{22}$. In fact, we have $G/Z(G) \simeq M_{22}$. Then $|Z(G)| = 3^4 \cdot 5^2$. Since there is some involution y in G such that $C_{\bar{G}}(\bar{y}) = 384$, we get that $|C_G(y)| > |C_G(x)|$, a contradiction.

If $S_1 \simeq A_{11}$, then $M = S_1 = A_{11}$ and therefore $G/Z(G) \simeq A_{11}$. It follows that $G'/Z(G') \simeq A_{11}$. Hence, $G' \simeq A_{11}$ or $G' \simeq 2.A_{11}$.

Since 2^8 does not divide $|G|$, we have that G' must be isomorphic to A_{11} , which shows that $G = G' \times Z(G)$. By [5], we can choose an element $y \in G'$ of order 3 such that $C_{G'}(y) = 60840$, which leads to that $|C_G(y)| > |C_G(x)|$, a contradiction.

Thus, S_1 must be isomorphic to McL and so $G \simeq McL$, as desired. $\square$

Lemma 3.9. *If $N \simeq Co_3$, then $G \simeq Co_3$.*

Proof. By the hypothesis, $|G| = |Co_3| = 2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$ and $scs(G) = 3^3 \cdot 5^2 \cdot 11 \cdot 23$. Then, for some $x \in G$, $|x^G| = scs(G)$ and $|C_G(x)| = 2^{10} \cdot 3^4 \cdot 5 \cdot 7$. Obviously, $Z(G)$ is a proper subgroup of G and $11, 23 \notin \pi(Z(G))$. By Lemma 2.4, $11, 23 \in \pi(M)$. Without loss of generality, we assume that $23 \in \pi(S_1)$. Then S_1 may be isomorphic to one of the following groups:

$$Co_3, M_{24}, M_{23}, L_2(23).$$

In order to show $S_1 \simeq Co_3$, we should rule out other 3 cases.

(I) S_1 is not isomorphic to $L_2(23)$.

First, assume that $S_1 \simeq L_2(23)$. By the hypothesis and Lemma 2.4, we have that 5^2 divides the order of M . From now on, we distinguish the following 4 cases.

(1) Both 5 and 7 are in $\pi(Z(G))$.

If this case occurs, then M is isomorphic to one of

$$L_2(23) \times A_5 \times A_5, L_2(23) \times A_5 \times A_6, L_2(23) \times A_6 \times A_6.$$

In all these 3 cases, G has an element y of order 23 such that $1 < |y^G| < |x^G|$, a contradiction.

(2) $7 \in \pi(Z(G))$ but $5 \notin \pi(Z(G))$.

In this case, we have that 5^3 divides the order of M . Then M is isomorphic to one of the following:

$$L_2(23) \times A_5 \times A_5 \times A_5, L_2(23) \times A_5 \times A_5 \times A_6.$$

In any case, we have that $|Z(G)| \geq 3^2 \cdot 7$. Now, as above, one can derive a contradiction.

(3) $5 \in \pi(Z(G))$ but $7 \notin \pi(Z(G))$.

If this case occurs, one can show that M is isomorphic to one of the following groups:

$$L_2(23) \times L_2(7) \times A_5 \times A_5, L_2(23) \times L_2(8) \times A_5 \times A_5.$$

Then, in any case, we obtain that $|Z(G)| \geq 3^2 \cdot 5$. Then, similarly as above, one have a contradiction.

(4) Both 5 and 7 are not in $\pi(Z(G))$.

If this case happens, we see that M may be one of the following:

$$L_2(23) \times A_7 \times A_5 \times A_5, L_2(23) \times U_3(5).$$

Again, we get a contradiction as the foregoing cases.

Hence, we conclude that S_1 can not be isomorphic to $L_2(23)$.

(II) S_1 is not isomorphic to M_{23} .

Suppose that $S_1 \simeq M_{23}$. Then we can conclude that $M \simeq M_{23} \times A_5$. It follows that $G/Z(G)$ is isomorphic to one of the following groups:

$$M_{23} \times A_5, M_{23} \times S_5.$$

If $G/Z(G) \simeq M_{23} \times S_5$, then $|Z(G)|$ is an odd number. Let y be an involution in G such that $\bar{y}$ is also an involution of S_5 . Then one can check that $1 < |y^G| < |x^G|$, a contradiction.

If $G/Z(G) \simeq M_{23} \times A_5$, then one can pick a non-central element z of G such that $\bar{z}$ is contained in A_5 . Then, it is clear that $|y^G| < |x^G|$, a contradiction.

(III) S_1 can not be isomorphic to M_{24} .

Note that $|M_{24}| = 2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$. Hence, if $S_1 \simeq M_{24}$, then $2 \notin \pi(Z(G))$. Then one can choose an involution y of G such that $|C_G(y)| > |C_G(x)|$, a final contradiction.

Finally, we see that S_1 must be isomorphic to Co_3 and therefore $G \simeq Co_3$. Thus, our proof is complete. $\square$

Proof of Theorem 1.1. By Lemmas 3.2-3.9, we have that our assertion holds for $M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, J_1, J_3, J_4, Co_3, McL$. The remaining groups can be dealt with similar methods. Therefore their proofs are omitted.

4. Proof of Theorem 1.2

We divide the proof of Theorem 1.2 into several lemmas. Throughout this section, we assume that G is a group satisfying the condition of Theorem 1.2.

Lemma 4.1. *The prime graph of G is not connected and, in particular, for some $p \in \pi(G)$, $\{p\}$ is a component of the prime graph of G .*

Proof. By [5], we have that for any group N under consideration, $lcs(N) = |x^G|$ with x is of order $p \in \pi(N)$ and $C_G(x) = \langle x \rangle$. Thus, p is an isolated point in $\Gamma(G)$. $\square$

Lemma 4.2. *If $N = J_2$, then $G \simeq J_2$.*

Proof. By [5], we have that $p = 7$. If G is a Frobenius group or a 2-Frobenius group, then G has an element of order 35 in view of Lemmas 2.2 and 2.3, which is a contradiction. Then, by Lemma 2.1, we have that G has a normal series $1 \subseteq H \subseteq K \subseteq G$, where H is a nilpotent π_1 -group, K/H is a non-abelian simple group and G/K is a π_1 -group such that $|G/K|$ divides the order of the outer automorphism group of K/H . Especially, $\{7\}$ is a component of $\Gamma(K/H)$. By [5], K/H may be isomorphic to one of the following groups:

$$L_3(2), L_2(8), A_7, U_3(3), A_8, L_3(4), J_2.$$

If K/H is isomorphic to the former 6 groups, then 5 and 7 are connected in $\Gamma(G)$. Hence, K/H must be isomorphic to J_2 and so $G \simeq J_2$, as desired. $\square$

Lemma 4.3. *If $N = HS$, then $G \simeq HS$.*

Proof. By [5], we know that $p = 7$. By Lemmas 2.2 and 2.3, we see that G is neither a Frobenius group nor a 2-Frobenius group, since, otherwise, 7 and 11 are connected in $\Gamma(G)$. Therefore G has a normal series $1 \subseteq H \subseteq K \subseteq G$, where H is a nilpotent π_1 -group, K/H is a non-abelian simple group and G/K is a π_1 -group such that $|G/K|$ divides the order of the outer automorphism group of K/H . Especially, $\{7\}$ is a component of $\Gamma(K/H)$. By [5], K/H is isomorphic to one of $L_3(2), L_2(8), A_7, A_8, L_3(4), U_3(5), M_{22}, HS$. If K/H is isomorphic to the former 6 groups, then $77 \in \pi(G)$, a contradiction. If $K/H \simeq M_{22}$, then $35 \in \pi(G)$, a contradiction again. Thus, $K/H \simeq HS$ and so $G \simeq HS$, completing the proof. $\square$

Lemma 4.4. *If $N = Suz$, then $G \simeq Suz$.*

Proof. By [5], we have that $p = 11$. By Lemmas 2.2 and 2.3, we obtain that G is neither a Frobenius group nor a 2-Frobenius group. Otherwise, 11 and 13 are connected in $\Gamma(G)$, a contradiction. Now, according to Lemma 2.1, G has a normal series $1 \subseteq H \subseteq K \subseteq G$, where H is a nilpotent π_1 -group, K/H is a non-abelian simple group and G/K is a π_1 -group such that $|G/K|$ divides the order of the outer automorphism group of K/H . Furthermore, $\{11\}$ is a prime graph

component of G . By [5], K/H may be isomorphic to one of the following groups $L_2(11), M_{11}, M_{12}, M_{22}, A_{11}, A_{12}, A_{13}, Suz$. If K/H is isomorphic to the former 6 groups, then 11 is connected to 13 in $\Gamma(G)$. If $K/H \simeq A_{13}$, then $|H_3| = 3^2$, where H_3 is a Sylow 3-subgroup of H . It is easy to see that 11 does not divide the order of $Aut(H_3)$, which implies that $33 \in \pi(G)$, a final contradiction. Thus, K/H must be isomorphic to Suz and so $G \simeq Suz$. $\square$

Lemma 4.5. *If $N = Fi_{22}$, then $G \simeq Fi_{22}$.*

Proof. By [5], $p = 13$. Invoking Lemmas 2.2 and 2.3, one can conclude that G can neither be Frobenius group nor a 2-Frobenius group. Hence, by Lemma 2.1, G has a normal series $1 \subseteq H \subseteq K \subseteq G$, where H is a nilpotent π_1 -group, K/H is a non-abelian simple group and G/K is a π_1 -group such that $|G/K|$ divides the order of the outer automorphism group of K/H . Furthermore, $\{13\}$ is a prime graph component of G . It follows from [5] that K/H may be isomorphic to one of the following groups:

$$L_2(13), L_3(3), L_2(25), L_2(27), Sz(8), U_3(4), L_2(64), G_2(3),$$

$$L_4(3), {}^2F_4(2)', L_3(9), G_2(4), A_{13}, S_6(2), O_7(3), Suz, Fi_{22}.$$

If K/H is isomorphic to one of the former 14 groups, then 11 and 13 are connected in $\Gamma(G)$, a contradiction.

If K/H is isomorphic to A_{13} or Suz , then G has an element of order 26, since 13 does not divide the order of $GL(n, 2)$ with $n \leq 8$.

Thus, K/H must be isomorphic to Fi_{22} and consequently $G \simeq Fi_{22}$. $\square$

Proof of Theorem 1.2. By lemmas 4.2-4.5, our statement holds for J_2 , HS , Suz and Fi_{22} . Analogously, the remaining cases can be verified and so the proof is omitted.

5. Proof of Proposition 1.3

By the hypothesis,

$$\begin{aligned} |G| &= 2^{41} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47, \\ scs(G) &= 2^3 \cdot 3^4 \cdot 5^4 \cdot 23 \cdot 31 \cdot 47, \\ sscs(G) &= 3^7 \cdot 5^3 \cdot 7 \cdot 13 \cdot 17 \cdot 19 \cdot 31 \cdot 47. \end{aligned}$$

Clearly, $Z(G)$ is a proper subgroup of G and $47 \notin \pi(Z(G))$. Let $\overline{G} = G/Z(G)$. Then we claim that every minimal normal subgroup of $\overline{G}$ is non-soluble. If not, assume by way of contradiction that some minimal normal subgroup S of $\overline{G}$ is soluble. Then S is an elementary abelian group. Let $|S| = r^t$ and the pre-image of S in G is T . Then T is a nilpotent group and the Sylow r -subgroup R of T is normal in G . Note that R is not contained in $Z(G)$. By our hypothesis, $r \notin \{3, 5, 7, 11, 13, 17, 19, 23, 31, 47\}$. Now, assume that $r = 2$. Then it is possible that $scs(G) < |R|$. If this occurs, then since $|G_2| < sscs(G)$

with G_2 is a Sylow 2-subgroup of G , we conclude that $scs(G)$ divides $|R| - 1$. However, by direct computation, we see that this is impossible. Hence, S can not be soluble, and so every minimal normal subgroup of $\overline{G}$ is non-soluble, as claimed.

Now let $M = S_1 \times \cdots \times S_k$ be the product of all minimal normal subgroups of $\overline{G}$, where S_i is a non-abelian simple group. It is obvious that $M \leq \overline{G} \leq \text{Aut}(M)$. We suppose that $47 \in \pi(S_1)$. then, by [5], we get that S_1 is isomorphic to $L_2(47)$ or B (Baby Monster). It is easy to rule out the case $S_1 \simeq L_2(47)$. Hence, we obtain that $S_1 \simeq B$ and therefore $G \simeq B$, as desired.

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LAWVERE-TIERNEY SHEAVES, FACTORIZATION SYSTEMS, SECTIONS AND j -ESSENTIAL MONOMORPHISMS IN A TOPOS

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Abstract. Let j be a Lawvere-Tierney topology (a topology, for short) on an arbitrary topos $\mathcal{E}$, B an object of $\mathcal{E}$, and $j_B = j \times 1_B$ the induced topology on the slice topos $\mathcal{E}/B$. In this manuscript, we analyze some properties of the pullback functor $B^* : \mathcal{E} \rightarrow \mathcal{E}/B$ which are dealing with topologies. Then for the left cancellable class $\mathcal{M}$ of all j -dense monomorphisms in a topos $\mathcal{E}$, we achieve some necessary and sufficient conditions for that the pair $(\mathcal{M}, \mathcal{M}^\perp)$ is a factorization system in $\mathcal{E}$, which is related to the factorization systems in slice topoi $\mathcal{E}/B$, where B ranges over the class of objects of $\mathcal{E}$. Among other things, we prove that an arrow $f : X \rightarrow B$ in $\mathcal{E}$ is a j_B -sheaf in $\mathcal{E}/B$ whenever the graph of f , is a section in $\mathcal{E}/B$ as well as the object of sections $S(f)$ of f , is a j -sheaf in $\mathcal{E}$. Furthermore, we introduce a class of monomorphisms in $\mathcal{E}$, which we call each member of the class j -essential. Some equivalent forms and some of their properties are presented. Also, we prove that any presheaf in a presheaf topos has a maximal essential extension. Finally, some similarities and differences of the obtained result are discussed if we put a (productive) weak topology j , studied by some authors, instead of a topology.

Keywords: (weak) Lawvere-Tierney topology, sheaf, factorization system, slice topos, essential monomorphism.

1. Introduction and preliminaries

One of the basic tools to construct new topoi from old ones is the notion of Lawvere-Tierney topology. Recently, applications of Lawvere-Tierney topologies in broad topics such as measure theory [7] and quantum physics [14, 15] are observed. In the spacial case, considerable work has been presented that is dedicated to the study of (weak) Lawvere-Tierney topology on a presheaf topos on a small category and especially on a monoid, see [6, 5]. It is clear that Lawvere-Tierney sheaves in a topos are exactly injective objects (of course,

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with respect to dense monomorphisms, not to merely monomorphisms) which are separated too. Injectivity with respect to a class $\mathcal{M}$ of morphisms in a slice category $\mathcal{C}/B$ (which its objects are $\mathcal{C}$ -arrows with codomain B) has been studied in extensive form; for example we refer the reader to [1, 3]. From this perspective, among other things, in this paper we will establish for Lawvere-Tierney sheaves some categorical characterizations of injective objects in slice topoi. The object of sections $S(f)$ of an arbitrary morphism $f : X \rightarrow B$ is a notion related to injective objects in a slice category, see [3]. This object is very useful in synthetic differential geometry (or SDG, for short) (for details, see [11]). For example, considering D as infinitesimals, for any micro-linear object M we have:

- Let τ be the tangent bundle on M , i.e., $\tau : M^D \rightarrow M$, defined by $\tau(t) = t(0)$. Then $S(\tau)$ is all vector fields on M .
- Consider $\eta : M^{D \times D} \rightarrow M$ which assigns to any micro-square Q of $M^{D \times D}$, the element $Q(0, 0)$. Then, $S(\eta)$ is all distributions of dimension 2 on M .

Throughout this paper, $\mathcal{E}$ is a (elementary) topos, two objects $0, 1$ are the initial and terminal objects and the object Ω together with the arrow $1 \xrightarrow{\text{true}} \Omega$ is the subobject classifier of $\mathcal{E}$. Also, the arrow $\wedge : \Omega \times \Omega \rightarrow \Omega$ is the meet operation on Ω . Now, we express some basic concepts from [12] which will be needed in the sequel.

Definition 1.1. A *Lawvere-Tierney topology* on $\mathcal{E}$ is a map $j : \Omega \rightarrow \Omega$ in $\mathcal{E}$ satisfies the following properties

- (a) $j \circ \text{true} = \text{true}$; (b) $j \circ j = j$; (c) $j \circ \wedge = \wedge \circ (j \times j)$;

$$\begin{array}{ccccc}
 1 & \xrightarrow{\text{true}} & \Omega & & \Omega & \xrightarrow{j} & \Omega & & \Omega \times \Omega & \xrightarrow{\wedge} & \Omega \\
 & \searrow \text{true} & \downarrow j & & \searrow j & \downarrow j & & \downarrow j \times j & & \downarrow j & \\
 & & \Omega & & \Omega & & & \Omega \times \Omega & \xrightarrow{\wedge} & \Omega &
 \end{array}$$

Form now on, we say briefly to a Lawvere-Tierney topology on $\mathcal{E}$, a *topology* on $\mathcal{E}$.

Recall [12] that topologies on $\mathcal{E}$ are in one to one correspondence with universal closure operators. For a topology j on $\mathcal{E}$, considering $\overline{(\cdot)}$ as the universal closure operator corresponding to j , a monomorphism $k : A \rightarrow C$ in $\mathcal{E}$ is called *j -dense* whenever $\overline{A} = C$, as two subobjects of C . Also, we say that k is *j -closed* if we have $\overline{A} = A$, again as subobjects of C .

Definition 1.2. For a topology j on $\mathcal{E}$, an object F of $\mathcal{E}$ is called a *j -sheaf* whenever for any j -dense monomorphism $m : A \rightarrow E$, one can uniquely extend

any arrow $h : A \rightarrow F$ to a map g on all of E ,

$$(1) \quad \begin{array}{ccc} A & \xrightarrow{h} & F \\ m \downarrow & \nearrow g & \\ E & & \end{array}$$

We say that F is *j-separated* if the arrow g exists in (1), it is unique.

We will denote the full subcategories of $\mathcal{E}$ consisting of *j*-sheaves and *j*-separated objects by $\mathbf{Sh}_j(\mathcal{E})$ and $\mathbf{Sep}_j(\mathcal{E})$, respectively.

We now briefly describe the contents of other sections. We start in Section 2, to study basic properties of the pullback functor $B^* : \mathcal{E} \rightarrow \mathcal{E}/B$, for any object B of $\mathcal{E}$, along with the unique map $!_B : B \rightarrow 1$. Afterwards, we would like to achieve, for the left cancellable class $\mathcal{M}$ of all *j*-dense monomorphisms in a topos $\mathcal{E}$, some necessary and sufficient conditions for that $(\mathcal{M}, \mathcal{M}^\perp)$ to be a factorization system in $\mathcal{E}$, which is related to the factorization systems in slice topoi $\mathcal{E}/B$. In section 3, among other things, we prove that an arrow $f : X \rightarrow B$ in $\mathcal{E}$ is a *j_B*-sheaf whenever the graph of f is a section in $\mathcal{E}/B$ as well as the object of sections $S(f)$ of f is a *j*-sheaf in $\mathcal{E}$. In section 4, we introduce a class of monomorphisms in an elementary topos $\mathcal{E}$, which we call them ‘*j*-essential monomorphisms’. Furthermore, we present some equivalent forms to these kind of monomorphisms and get some of their properties. Meanwhile, we prove that any presheaf in a presheaf topos has a maximal essential extension. Also, it is shown that the functor B^* reflects *j*-essential extensions. It is seen that some of these results hold for a (*productive*) *weak topology j*, studied in [10], instead of a topology as well.

2. Pullback functors, dense monomorphisms and factorization systems

The purpose of this section is to present some basic properties of the pullback functor $B^* : \mathcal{E} \rightarrow \mathcal{E}/B$, for any object B of $\mathcal{E}$, along with the unique map $!_B : B \rightarrow 1$. Afterwards, for the left cancellable class $\mathcal{M}$ of all *j*-dense monomorphisms in a topos $\mathcal{E}$ we achieve some necessary and sufficient conditions for that the pair $(\mathcal{M}, \mathcal{M}^\perp)$ to be a factorization system in $\mathcal{E}$, which is related to the factorization systems in slice topoi $\mathcal{E}/B$.

To begin with, the following lemma characterizes sheaves in a topos $\mathcal{E}$.

Lemma 2.1. *Let j be a topology on $\mathcal{E}$. Then an object E of $\mathcal{E}$ is *j*-sheaf iff E is *j*-unique absolute retract; that is, any *j*-dense monomorphism $u : E \rightarrowtail F$, has a unique retraction $v : F \rightarrow E$.*

Proof. *Necessity.* Since E is a *j*-sheaf, for any *j*-dense monomorphism $u : E \rightarrowtail F$, corresponding to the identity map $\text{id}_E : E \rightarrow E$, there exists a unique map

$v : F \rightarrow E$ such that the following diagram commutes.

$$\begin{array}{ccc} E & \xrightarrow{\text{id}_E} & E \\ u \downarrow & \nearrow v & \\ F & & \end{array}$$

Sufficiency. For each j -dense monomorphism $m : U \rightarrow V$ and any map $f : U \rightarrow E$, we construct the following pushout diagram in $\mathcal{E}$.

$$(2) \quad \begin{array}{ccc} U & \xrightarrow{f} & E \\ m \downarrow & & \downarrow n \\ V & \xrightarrow[p.o.]{g} & F \end{array}$$

Since in any topos pushouts transfer j -dense monomorphisms (see [9]), n is j -dense in (2), and hence by assumption, there exists a unique retraction $p : F \rightarrow E$ such that $pn = \text{id}_E$. Now, for the arrow $pg : V \rightarrow E$ we have $pgm = pn f = \text{id}_E f = f$. To prove that $pg : V \rightarrow E$ is unique with this property, let $h : V \rightarrow E$ be an arrow in $\mathcal{E}$ in such a way that $hm = f$. Then, in the pushout diagram (2), according to the maps $h : V \rightarrow E$ and $\text{id}_E : E \rightarrow E$, there exists a unique map $k : F \rightarrow E$ such that $kn = \text{id}_E$ and $kg = h$.

$$\begin{array}{ccc} U & \xrightarrow{f} & E \\ m \downarrow & & \downarrow n \\ V & \xrightarrow{g} & F \end{array} \quad \begin{array}{c} \text{id}_E \\ \nearrow k \\ \searrow h \end{array} \quad \begin{array}{c} \\ \\ E \end{array}$$

Now, k is a retraction of j -dense monomorphism n , so by hypothesis we get $p = k$. Consequently, $pg = kg = h$. $\square$

For an object B of $\mathcal{E}$, we consider the pullback functor $B^* : \mathcal{E} \rightarrow \mathcal{E}/B$ along with the unique map $!_B : B \rightarrow 1$, which assigns to any A of $\mathcal{E}$, the second projection $B^*(A) = \pi_B^A : A \times B \rightarrow B$ and to any $f : A \rightarrow C$, the arrow $f \times \text{id}_B : A \times B \rightarrow C \times B$ in $\mathcal{E}$ such that $\pi_B^C(f \times \text{id}_B) = \pi_B^A$. Recall [12] that the object π_B^Ω together with the arrow

$$\text{true} \times \text{id}_B : \text{id}_B \rightarrow \pi_B^\Omega$$

is the subobject classifier of the slice topos $\mathcal{E}/B$. Also, in a similar vein, we can observe that the meet operation $\wedge_B$ on π_B^Ω is the arrow $\wedge \times 1_B$ in $\mathcal{E}$ such that $\pi_B^\Omega(\wedge \times 1_B) = \pi_B^{\Omega \times \Omega}$,

$$\begin{array}{ccc} \Omega \times \Omega \times B & \xrightarrow{\wedge \times 1_B} & \Omega \times B \\ \pi_B^{\Omega \times \Omega} \searrow & & \downarrow \pi_B^\Omega \\ & & B. \end{array}$$

Now, by Definition 1.1, we easily get the following lemma.

Lemma 2.2. *Let B be any object in a topos $\mathcal{E}$. Then any topology $k : \pi_B^\Omega \rightarrow \pi_B^\Omega$ on $\mathcal{E}/B$ is a pair (l, π_B^Ω) , for some arrow $l : \Omega \times B \rightarrow \Omega$ in $\mathcal{E}$ satisfies the following conditions (as arrows in $\mathcal{E}$)*

- (1) $l \circ (l, \pi_B^\Omega) = l$;
- (2) $l \circ (\text{true} \times 1_B) = \text{true} \circ !_B$;
- (3) $l \circ \wedge_B = \wedge \circ (l \circ (\pi_1, \pi_3), l \circ (\pi_2, \pi_3))$, where π_i is the i -th projection on $\Omega \times \Omega \times B$, for $i = 1, 2, 3$.

By Lemma 2.2, for each topology j on $\mathcal{E}$, considering $l = j \circ \pi_\Omega^B$, it is easily seen that $j \times 1_B = (l, \pi_B^\Omega)$ is a topology on $\mathcal{E}/B$ which we denote it by j_B . In this case j_B is called the *induced topology* on $\mathcal{E}/B$ by j .

One can simply see that if an arrow k is a monomorphism in $\mathcal{E}/B$, then k as an arrow in $\mathcal{E}$, is too. Also, for each monomorphism $k : f \rightarrowtail g$ in $\mathcal{E}/B$, where $f : X \rightarrow B$ and $g : Y \rightarrow B$ in $\mathcal{E}$, we can observe

$$(3) \quad \widetilde{f} \xrightarrow{\widetilde{k}} g = (\overline{X} \xrightarrow{g\bar{k}} B) \xrightarrow{\bar{k}} g,$$

where $\overline{(\cdot)}$ and $\widetilde{(\cdot)}$ are the universal closure operators corresponding to the topologies j and j_B on topoi $\mathcal{E}$ and $\mathcal{E}/B$, respectively, for which the whole and the middle squares of the following diagram are pullbacks in $\mathcal{E}$,

$$(4) \quad \begin{array}{ccccc} \overline{X} & \xrightarrow{\quad} & 1 & & \\ \downarrow \bar{k} & & \downarrow \text{true} & & \\ X & \xrightarrow{\quad} & 1 & & \\ \downarrow k & & \downarrow \text{true} & & \\ Y & \xrightarrow{\quad} & \Omega & \xrightarrow{j} & \Omega \\ & \text{char}(k) & & & \end{array}$$

(for more details, see [12]). One can construct $\widetilde{k}$ in $\mathcal{E}/B$, similar to the above diagram.

Here, we proceed to improve [2, Vol. III, Proposition 9.2.5] as follows:

Lemma 2.3. *Let j be a topology in a topos $\mathcal{E}$. For every object B of $\mathcal{E}$, the pullback functor $B^* : \mathcal{E} \rightarrow \mathcal{E}/B$ preserves and reflects: denseness (closeness) and j -separated objects (j -sheaves).*

Proof. Let j be a topology on $\mathcal{E}$ and B an object of $\mathcal{E}$. Preserving dense (closed) monomorphisms and sheaves (separated objects) in $\mathcal{E}$ by the pullback functor B^* , is standard and may be found in [2, Vol. III, Proposition 9.2.5]. To prove the rest of Lemma, here we just show that B^* reflects dense (closed) monomorphisms. To verify this claim, let $g : A \rightarrow C$ be an arrow in $\mathcal{E}$ for which $B^*(g)$ is a j_B -dense (j_B -closed) monomorphism. We show that g is j -dense (j -closed) monomorphism. As $B^*(g) = g \times \text{id}_B$ being a monomorphism in $\mathcal{E}/B$,

the arrow g is a monomorphism in $\mathcal{E}$ as well. For, let f and h in $\mathcal{E}$ be two arrows such that $gf = gh$, we will have

$$\begin{aligned} gf = gh &\implies (g \times \text{id}_B)(f \times \text{id}_B) = (g \times \text{id}_B)(h \times \text{id}_B) \\ &\implies f \times \text{id}_B = h \times \text{id}_B \quad (g \times \text{id}_B \text{ is a monomorphism}) \\ &\implies f = h. \end{aligned}$$

Considering $\overline{(\cdot)}$ and $\widetilde{(\cdot)}$ as the universal closure operators corresponding to j and j_B , respectively. We get

$$\begin{aligned} \widetilde{B^*(g)} &= \widetilde{g \times \text{id}_B} \\ &= \overline{g \times \text{id}_B} \quad (\text{by (3)}) \\ &= \overline{g} \times \text{id}_B, \end{aligned}$$

where the last equality is true since we have $g \times \text{id}_B = (\pi_C^B)^{-1}(g)$, and because of stability of universal closure operators under pullbacks we get $\overline{(\pi_C^B)^{-1}(g)} = (\pi_C^B)^{-1}(\overline{g})$. The above equalities imply that if $B^*(g)$ is j_B -dense (j_B -closed) in $\mathcal{E}/B$, then g is j -dense (j -closed) in $\mathcal{E}$. $\square$

For any topology j on a topos $\mathcal{E}$, consider $\mathcal{M}$ as the class of all j -dense monomorphisms in $\mathcal{E}$. Also, we denote by $\mathcal{M}^\perp$ the class of all arrows $g : C \rightarrow D$ in $\mathcal{E}$ such that for any $f : A \rightarrow E$ in $\mathcal{M}$ and every commutative square

$$(5) \quad \begin{array}{ccc} A & \xrightarrow{u} & C \\ f \downarrow & \nearrow w & \downarrow g \\ E & \xrightarrow{v} & D \end{array}$$

there exists a unique dashed arrow $w : E \rightarrow C$ such that the resulting triangles are commutative. In this case, we say that g is *right orthogonal* to f . Moreover, we say that the pair $(\mathcal{M}, \mathcal{M}^\perp)$ forms a *factorization system* in $\mathcal{E}$ if any arrow f in $\mathcal{E}$ factors as $f = me$, where $m \in \mathcal{M}$ and $e \in \mathcal{M}^\perp$ (for more information, see [1]).

Lemma 2.4. *Let j be a topology on a topos $\mathcal{E}$. Then for each object B of $\mathcal{E}$, we have $\mathcal{M}_B^\perp \subseteq \mathcal{M}^\perp$, where $\mathcal{M}_B$ is the class of all j_B -dense monomorphisms in the slice topos in $\mathcal{E}/B$.*

Proof. By Lemma 2.3 we get $\mathcal{M}_B \subseteq \mathcal{M}$. To reach the conclusion, let $h : f \rightarrow g$ be an arrow in $\mathcal{M}_B^\perp$, where $f : D \rightarrow B$ and $g : E \rightarrow B$ are arrows in $\mathcal{E}$. Now, consider the commutative square

$$(6) \quad \begin{array}{ccc} A & \xrightarrow{u} & D \\ m \downarrow & & \downarrow h \\ C & \xrightarrow{v} & E \end{array}$$

where $m : A \rightarrow C$ is in $\mathcal{M}$. Since by Lemma 2.3 the arrow $m : fu \rightarrow gv$ in $\mathcal{E}/B$ belongs to $\mathcal{M}_B$ and $h \in \mathcal{M}_B^\perp$, there exists a unique arrow $w : gv \rightarrow f$ in $\mathcal{E}/B$ such that the following diagram commutes

$$(7) \quad \begin{array}{ccc} fu & \xrightarrow{u} & f \\ m \downarrow & \nearrow w & \downarrow h \\ gv & \xrightarrow[v]{} & g \end{array}$$

The arrow $w : C \rightarrow D$ (as an arrow in $\mathcal{E}$) which commutes the resulting triangles, is unique in the diagram (6). To prove this, let $k : C \rightarrow D$ be an arrow in $\mathcal{E}$ such that $km = u$ and $hk = v$. Now, we have $fk = (gh)k = gv$, so $k : gv \rightarrow f$ is an arrow in $\mathcal{E}/B$ making all triangles in (7) commutative. Thus, $k = w$ and we are done. $\square$

Definition 2.5. Let j be a topology on a topos $\mathcal{E}$. We say that $\mathcal{E}$ has enough j -sheaves if for every object A of $\mathcal{E}$ there is a j -dense monomorphism $A \rightarrowtail F$ where F is a j -sheaf.

Following [1] a class $\mathcal{M}$ of morphisms in $\mathcal{E}$ is a *left cancellable class* if $gf \in \mathcal{M}$ implies $f \in \mathcal{M}$. It is well known that for any object B of $\mathcal{E}$, the class $\mathcal{M}_B$ of all j_B -dense monomorphisms in $\mathcal{E}/B$ is left cancellable. In the following we will find some equivalent conditions for $(\mathcal{M}, \mathcal{M}^\perp)$ to be a factorization system in $\mathcal{E}$.

Theorem 2.6. Let j be a topology on a topos $\mathcal{E}$. Then the following are equivalent:

- (i) for any object B of $\mathcal{E}$, $(\mathcal{M}_B, \mathcal{M}_B^\perp)$ is a factorization system in $\mathcal{E}/B$;
- (ii) for any object B of $\mathcal{E}$, $\mathcal{E}/B$ has enough j_B -sheaves;
- (iii) for any object B of $\mathcal{E}$, any object of $\mathcal{E}/B$ is j_B -separated;
- (iv) for any object B of $\mathcal{E}$, any object of $\mathcal{E}/B$ is j_B -sheaf;
- (v) any object of $\mathcal{E}$ is j -sheaf;
- (vi) any object of $\mathcal{E}$ is j -separated;
- (vii) $\mathcal{E}$ has enough j -sheaves;
- (viii) $(\mathcal{M}, \mathcal{M}^\perp)$ is a factorization system in $\mathcal{E}$.

Proof. That any j -sheaf is j -separated in $\mathcal{E}$ yields that (v) $\implies$ (vi) holds.

(vi) $\implies$ (v). That any object of $\mathcal{E}$ is j -separated it follows that $\mathbf{Sep}_j(\mathcal{E})$ is the topos $\mathcal{E}$ and then, every j -separated object is a j -sheaf as in [8, Theorem 2.1].

(iii) $\implies$ (vi). Setting $B = 1$, then any object of $\mathcal{E}$ is j -separated.

(vi) $\implies$ (iii). The claim follows immediately from the fact that for any object B of $\mathcal{E}$,

$$\mathbf{Sep}_{j_B}(\mathcal{E}/B) \cong \mathbf{Sep}_j(\mathcal{E})/B.$$

(see also [9]).

(viii) $\implies$ (vii). By (viii), for any object A of $\mathcal{E}$, the unique arrow $!_A : A \rightarrow 1$ factors as

$$\begin{array}{ccc} A & \xrightarrow{!_A} & 1 \\ & \searrow m & \nearrow !_C \\ & C & \end{array}$$

where $!_C \in \mathcal{M}_1^\perp = \mathcal{M}^\perp$ and $m \in \mathcal{M}_1 = \mathcal{M}$. It is easy to check that for any object B of $\mathcal{E}$, j_B -sheaves in $\mathcal{E}/B$ are exactly the class of all objects of $\mathcal{E}/B$ which belong to $\mathcal{M}_B^\perp$. Since $!_C$ is an object in $\mathcal{E}/1 = \mathcal{E}$ which is in $\mathcal{M}_1^\perp$, so $!_C$ is a j_1 -sheaf, or equivalently, C is a j -sheaf.

(vii) $\implies$ (viii). Consider an arrow $f : A \rightarrow B$ in $\mathcal{E}$, using (vii), there exists a j -dense monomorphism $\iota : A \hookrightarrow F$, where F is a j -sheaf in $\mathcal{E}$. Now, we factor f as the composite arrow $A \xrightarrow{(\iota, f)} F \times B \xrightarrow{\pi_B^F} B$. Since $\pi_F^B(\iota, f) = \iota \in \mathcal{M}$ and $\mathcal{M}$ is a left cancellable class, so $(\iota, f) \in \mathcal{M}$. Also, F being j -sheaf, by Lemma 2.3 we have π_B^F is a j_B -sheaf in $\mathcal{E}/B$. By Lemma 2.4 we have $\pi_B^F \in \mathcal{M}_B^\perp \subseteq \mathcal{M}^\perp$, as required.

(vi) $\implies$ (vii). First of all we know that any j -separated object of $\mathcal{E}$ can be embedded into a j -sheaf (see, e.g. [12, Proposition V.3.4]). Let A be an object of $\mathcal{E}$. Then, by assumption A is j -separated, and there exists an embedding $A \xrightarrow{\iota} F$, where F is a j -sheaf. Now, considering the closure of A in F , since $\overline{A}$ is closed in F , by [12, Lemma V.2.4], it is a j -sheaf. Also since A is j -dense in $\overline{A}$ we get the result.

(vii) $\implies$ (vi). By assumption for any object A of $\mathcal{E}$, there is a j -dense monomorphism $A \hookrightarrow F$ in $\mathcal{E}$, where F is a j -sheaf. Since any subobject of a j -sheaf is j -separated, A is j -separated.

For any object B of $\mathcal{E}$, setting $\mathcal{E}/B$ instead of $\mathcal{E}$ in (v), (vi), (vii) and (viii), we drive the equivalences (i) $\iff$ (ii) $\iff$ (iii) $\iff$ (iv). $\square$

In the following, we will introduce two main classes of dense monomorphisms in a topos $\mathcal{E}$.

Remark 2.7. By diagram (4), one can easily obtain:

(i) Let $j = \text{id}_\Omega$ be the trivial topology on $\mathcal{E}$. Then j -dense monomorphisms are only the identity maps. Therefore, any object of $\mathcal{E}$ is a j -sheaf. Also, j -closed monomorphisms are exactly all monomorphisms.

(ii) Let j be the topology $\text{true}_\Omega!$ on $\mathcal{E}$, that is, the characteristic map of id_Ω . Then, j -dense monomorphisms are exactly all monomorphisms. Furthermore, j -closed monomorphisms are just the identity maps.

Recall [1] that $(\text{Mono}, \text{Mono}^\square)$ is a weak factorization system in any topos $\mathcal{E}$, where Mono is the class of all monomorphisms in $\mathcal{E}$. By Remark 2.7(ii), the class Mono is the class of all j -dense monomorphisms with respect to the topology $j = \text{true}_\Omega!$ on $\mathcal{E}$. Since the class Mono is left cancellable, so we can obtain a special case of Theorem 2.6 as follows. (Notice that by Lemma 2.3 for

the topology $j = \text{true} \circ !_\Omega$ and any object B of $\mathcal{E}$, the class Mono_B will be all monomorphisms in $\mathcal{E}/B$.)

Corollary 2.8. *For the topology $j = \text{true} \circ !_\Omega$ on a topos $\mathcal{E}$, the following are equivalent:*

- (i) *for any object B of $\mathcal{E}$, $(\text{Mono}_B, \text{Mono}_B^\perp)$ is a factorization system in $\mathcal{E}/B$;*
- (ii) *for any object B of $\mathcal{E}$, $\mathcal{E}/B$ has enough j_B -sheaves;*
- (iii) *for any object B of $\mathcal{E}$, any object of $\mathcal{E}/B$ is j_B -sheaf;*
- (iv) *for any object B of $\mathcal{E}$, any object of $\mathcal{E}/B$ is j_B -separated;*
- (v) *any object of $\mathcal{E}$ is j -sheaf;*
- (vi) *any object of $\mathcal{E}$ is j -separated;*
- (vii) *$\mathcal{E}$ has enough j -sheaves;*
- (viii) *$(\text{Mono}, \text{Mono}^\perp)$ is a factorization system in $\mathcal{E}$.*

3. Sheaves and sections of an arrow

In this section, among other things, we investigate a relationship between sheaves and sections of an arrow in a topos $\mathcal{E}$. We start to recall [12] that for any object B of $\mathcal{E}$, the pullback functor $B^* : \mathcal{E} \rightarrow \mathcal{E}/B$ has a right adjoint which we denoted it by $S : \mathcal{E}/B \rightarrow \mathcal{E}$ (according to the terminology of [3]) as for any $f : X \rightarrow B$ we have the following pullback

$$(8) \quad \begin{array}{ccc} S(f) & \longrightarrow & 1 \\ \downarrow & & \downarrow i_B \\ X^B & \xrightarrow{f^B} & B^B \end{array}$$

where i_B is the transpose of $\text{id}_B : 1 \times B \cong B \rightarrow B$ and f^B is the transpose of the composition arrow $X^B \times B \xrightarrow{\text{ev}_X} X \xrightarrow{f} B$ by the exponential adjunction $(-) \times B \dashv (-)^B$; that is, $\text{ev}_B(i_B \times \text{id}_B) = \text{id}_B$ and $\text{ev}_B(f^B \times \text{id}_B) = f \text{ev}_X$, where the natural transformation $\text{ev} : (-)^B \times B \rightarrow (-)$ is the counit of the exponential adjunction. In fact, in the Mitchell-Bénabou language, we can write

$$S(f) = \{h \mid (\forall c \in B) f \circ (h(c)) = c\}.$$

This means that we can call $S(f)$ *the object of sections of f* .

Since any retract of an object in a topos (or in an arbitrary category) is an equalizer, so the topos $\mathbf{Sh}_j(\mathcal{E})$ is closed under retracts. By Lemma 2.3, the pullback functor B^* preserves dense monomorphisms. Furthermore, as $B^* \dashv S$, the functor S preserves sheaves (for details, see [9, Corollary 4.3.12]). (Roughly, for any object $B \in \mathcal{E}$ and any adjoint $F \dashv G : \mathcal{E} \rightarrow \mathcal{E}/B$ one can easily checked that the functor G preserves sheaves whenever F preserves dense monomorphisms.)

In the following theorem we will find a relationship between sheaves in $\mathcal{E}/B$ and the object of sections of an arrow with codomain B .

Theorem 3.1. *Let j be a topology on a topos $\mathcal{E}$ and $f : X \rightarrow B$ be an object of $\mathcal{E}/B$. Then, f is a j_B -sheaf in $\mathcal{E}/B$, whenever the graph of f which stands for the monomorphism $(\text{id}_X, f) : f \rightarrow \pi_B^X$ in $\mathcal{E}/B$, is a section as well as $S(f)$ is a j -sheaf in $\mathcal{E}$.*

Proof. We recall that in [3] it was proved if (id_X, f) is a section in $\mathcal{E}/B$, then f is a retract of $\pi_B^{S(f)}$ in $\mathcal{E}/B$. As $S(f)$ is a j -sheaf, by Lemma 2.3, $\pi_B^{S(f)}$ is a j_B -sheaf in $\mathcal{E}/B$. But $\mathbf{Sh}_{j_B}(\mathcal{E}/B)$ being closed under retracts, therefore f is a j_B -sheaf in $\mathcal{E}/B$. $\square$

Regarding to the converse of Theorem 3.1, that the section functor S preserves sheaves it yields that if $f : X \rightarrow B$ be a j_B -sheaf in $\mathcal{E}/B$, then $S(f)$ is a j -sheaf in $\mathcal{E}$. Also, by Remark 2.7(ii), for $j = \text{true} \circ !_\Omega$, the monomorphism $(\text{id}_X, f) : f \rightarrow \pi_B^X$ is j_B -dense in $\mathcal{E}/B$ and then for a j_B -sheaf $f : X \rightarrow B$, it will be a section in $\mathcal{E}/B$.

In the rest of this section, for a small category $\mathcal{C}$ we restrict our attention to obtain a version of Theorem 3.1 for injective presheaves in trivial slices of the presheaf topos $\widehat{\mathcal{C}} = \mathbf{Sets}^{\mathcal{C}^{op}}$ which is close to the version of this theorem on j -sheaves for the topology $j = \text{true} \circ !_\Omega$ on $\widehat{\mathcal{C}}$. (See Proposition 3.5 below.) Note that the topology $j = \text{true} \circ !_\Omega$ on $\widehat{\mathcal{C}}$ is associated to the *chaotic or indiscrete Grothendieck topology* on $\mathcal{C}$.

Recall [12] that in the presheaf topos $\widehat{\mathcal{C}} = \mathbf{Sets}^{\mathcal{C}^{op}}$, the exponential object G^F is defined in each stage C of $\mathcal{C}$ as $G^F(C) = \text{Hom}_{\widehat{\mathcal{C}}}(Y(C) \times F, G)$, where Y is the Yoneda embedding, that is

$$Y : \mathcal{C} \rightarrow \widehat{\mathcal{C}}; \quad Y(C) = \text{Hom}_{\mathcal{C}}(-, C).$$

Now, for an arrow $\alpha : G \rightarrow F$ in $\widehat{\mathcal{C}}$ consider the arrows $i_F : 1 \rightarrow F^F$ and $\alpha^F : G^F \rightarrow F^F$ in $\widehat{\mathcal{C}}$ as the transposes of $\text{id}_F : 1 \times F \cong F \rightarrow F$ and $\alpha \circ \text{ev}_G : G^F \times F \rightarrow F$, respectively, by the exponential adjunction. We can observe

$$(9) \quad \forall C \in \mathcal{C}, \quad (i_F)_C : 1(C) = \{*\} \longrightarrow F^F(C); \quad (i_F)_C(*) = \pi_F^{Y(C)}.$$

Also, for any two objects C, D of $\mathcal{C}$, any γ in $G^F(C)$ and any (k, y) in $Y(C)(D) \times F(D)$ we have

$$(10) \quad (\alpha_C^F(\gamma))_D(k, y) = \alpha_D(\gamma_D(k, y)).$$

Recall that a presheaf G has a (unique) global section whenever in each stage C of $\mathcal{C}$ there is a (unique) element $\theta_C \in G(C)$ in such a way that for any arrow $k : D \rightarrow C$ in $\mathcal{C}$ we have

$$(11) \quad G(k)(\theta_C) = \theta_D.$$

Here, we give a special case that the exponential object and the object of sections in $\widehat{\mathcal{C}}$ are exactly similar to $\mathbf{Sets}$. First, we state the following which is required.

Lemma 3.2. *Let j be the topology trueo!_Ω on $\widehat{\mathcal{C}}$. Then the following assertions hold:*

(i) *For any j -sheaf G in $\widehat{\mathcal{C}}$, G has a unique global section. More generally, any injective presheaf G of $\widehat{\mathcal{C}}$ has a global section.*

(ii) *For any family $\{G_\lambda\}_{\lambda \in \Lambda}$ in $\widehat{\mathcal{C}}$, the presheaf $G = \prod_{\lambda \in \Lambda} G_\lambda$ is a j -sheaf (injective) in $\widehat{\mathcal{C}}$ iff for all $\lambda \in \Lambda$, G_λ is a j -sheaf (injective) in $\widehat{\mathcal{C}}$.*

Proof. (i) Let G be a j -sheaf in $\widehat{\mathcal{C}}$ and consider the coproduct object $G \sqcup 1$ in $\widehat{\mathcal{C}}$. By Remark 2.7(ii), there exists a unique natural transformation $\eta : G \sqcup 1 \rightarrow G$ in $\widehat{\mathcal{C}}$ such that the following diagram commutes (if G being injective, the arrow η is not necessarily unique)

$$\begin{array}{ccc} G & \xrightarrow{\text{id}_G} & G \\ \downarrow \iota & \nearrow \eta & \\ G \sqcup 1 & & \end{array}$$

in which $\iota : G \rightarrow G \sqcup 1$ is the injection arrow. Now, we will denote $\eta_C(*)$ by an element θ_C in $G(C)$ in each stage C of $\mathcal{C}$. Since $\eta : G \sqcup 1 \rightarrow G$ is natural, so for any arrow $k : D \rightarrow C$ in $\mathcal{C}$ the following square commutes

$$\begin{array}{ccc} (G \sqcup 1)(D) & \xrightarrow{\eta_D} & G(D) \\ (G \sqcup 1)(k) \uparrow & & \uparrow G(k) \\ (G \sqcup 1)(C) & \xrightarrow{\eta_C} & G(C) \end{array}$$

Then, we have

$$\begin{aligned} G(k)(\theta_C) &= G(k)(\eta_C(*)) \\ &= \eta_D((G \sqcup 1)(k)(*)) \\ &= \eta_D(1(k)(*)) = \theta_D. \end{aligned}$$

This is the required result.

(ii) *Necessity.* Let G be a j -sheaf (injective) in $\widehat{\mathcal{C}}$. For any $\lambda, \mu \in \Lambda$, we define $\alpha^{\lambda\mu} : G_\lambda \rightarrow G_\mu$ such that in each stage C of $\mathcal{C}$ and for each $x \in G_\lambda(C)$, we have $\alpha_C^{\lambda\mu}(x) = \theta_C^\mu$, where θ_C^μ is the μ -th component of θ_C corresponding to G in (i). Now, we will show that for any $\lambda, \mu \in \Lambda$, $\alpha^{\lambda\mu}$ is a natural transformation in $\widehat{\mathcal{C}}$, that is, for any arrow $k : D \rightarrow C$ in $\mathcal{C}$ the following diagram is commutative

$$\begin{array}{ccc} G_\lambda(D) & \xrightarrow{\alpha_D^{\lambda\mu}} & G_\mu(D) \\ G_\lambda(k) \uparrow & & \uparrow G_\mu(k) \\ G_\lambda(C) & \xrightarrow{\alpha_C^{\lambda\mu}} & G_\mu(C) \end{array}$$

For, consider an element $x \in G_\lambda(C)$ we get

$$\begin{aligned} G_\mu(k)(\alpha_C^{\lambda\mu}(x)) &= G_\mu(k)(\theta_C^\mu) \\ &= \theta_D^\mu && \text{(by (11))} \\ &= \alpha_D^{\lambda\mu}(G_\lambda(k)(x)). \end{aligned}$$

Now, for any $\lambda \in \Lambda$, consider the family $\{\gamma_\mu : G_\lambda \rightarrow G_\mu\}_{\mu \in \Lambda}$ in $\widehat{\mathcal{C}}$ such that $\gamma_\mu = \alpha^{\lambda\mu}$ and $\gamma_\lambda = \text{id}_{G_\lambda}$, for each $\lambda \neq \mu \in \Lambda$. Since G is the product $\prod_{\lambda \in \Lambda} G_\lambda$, so there is a unique natural transformation $\gamma : G_\lambda \rightarrow G$ such that $p_\mu \gamma = \gamma_\mu$ and $p_\lambda \gamma = \text{id}_{G_\lambda}$, for all $\lambda, \mu \in \Lambda$ and the projections p_λ . Thus, for any $\lambda \in \Lambda$, G_λ is a retract of the j -sheaf (injective) G and then, G_λ is a j -sheaf (injective).

Sufficiency. By the universal property of the product presheaf G , the unique arrow in the definition of a sheaf easily follows. $\square$

We recall [12] that in each stage C of $\mathcal{C}$ the object $\Omega(C)$ of $\widehat{\mathcal{C}}$ is the set of all sieves on C . Also, the arrow $\text{true}_C : 1(C) = \{*\} \rightarrow \Omega(C)$ assigns to $*$, the *maximal sieve* $t(C)$ of $\Omega(C)$, that is the set of all arrows with codomain C of $\mathcal{C}$.

Remark 3.3. Note that the topology $j = \text{true} \circ !_\Omega$ on $\widehat{\mathcal{C}}$ is the unique topology on $\widehat{\mathcal{C}}$ which satisfies Lemma 3.2. To show this, let j be an arbitrary topology on $\widehat{\mathcal{C}}$. For a j -sheaf G of $\widehat{\mathcal{C}}$, consider the injection $\iota : G \rightarrow G \sqcup 1$ in $\widehat{\mathcal{C}}$. In each stage C of $\mathcal{C}$ we have $\text{char}(\iota)_C(*) = \emptyset$. Now, if ι is j -dense monomorphism, then in each stage C of $\mathcal{C}$ we have $j_C(\emptyset) = t(C)$. For any sieve $S \in \Omega(C)$ by Definition 1.1 we get

$$\begin{aligned} t(C) &= j_C(\emptyset) = j_C(\emptyset \cap S) \\ &= j_C(\emptyset) \cap j_C(S) = t(C) \cap j_C(S) = j_C(S). \end{aligned}$$

Thus, j_C is the constant function on $t(C)$, as required.

Let F be the constant presheaf on a set A . One can easily checked that the exponential adjunction $(-) \times F \dashv (-)^F$ is determined by, for any presheaf G in $\widehat{\mathcal{C}}$, the exponential presheaf G^F assigns to any object C of $\mathcal{C}$, the hom-set $\text{Hom}_{\mathbf{Sets}}(A, G(C))$ and to any arrow $f : C \rightarrow D$ of $\mathcal{C}$, the function

$$G^F(f) : \text{Hom}_{\mathbf{Sets}}(A, G(D)) \longrightarrow \text{Hom}_{\mathbf{Sets}}(A, G(C))$$

given by $G^F(f)(g) = G(f) \circ g$. As any function $f : A \rightarrow G(C)$ can be considered as a sequence $(x_a)_{a \in A} \in \prod_A G(C)$, it yields that one has

$$(12) \quad G^F(C) \cong \prod_A G(C) \quad \text{for all objects } C \text{ in } \mathcal{C}.$$

By (12), (9) and (10), it is convenient to see that for each arrow $\alpha : G \rightarrow F$ in $\widehat{\mathcal{C}}$, in which F stands for the constant presheaf on a set A , we get

$$(13) \quad \forall C \in \mathcal{C}, \quad S(\alpha)(C) \cong \prod_{a \in A} \alpha_C^{-1}(a).$$

Now, we will extract a special case of Theorem 3.1 in $\widehat{\mathcal{C}}$. First, let $\alpha : G \rightarrow F$ be an arrow in $\widehat{\mathcal{C}}$ in which F is the constant presheaf on a set A . For each element a of A , consider the subpresheaf H_a of G such that $H_a(C) = \alpha_C^{-1}(a)$, for any object C of $\mathcal{C}$. Since limits in $\widehat{\mathcal{C}}$ are constructed pointwise, so (13) shows that $S(\alpha) \cong \prod_{a \in A} H_a$.

Proposition 3.4. *Let j be the topology $\text{true} \circ !_\Omega$ on $\widehat{\mathcal{C}}$ and $\alpha : G \rightarrow F$ an arrow in $\widehat{\mathcal{C}}$, where F is the constant presheaf on a set A . Then, α is a j_F -sheaf in $\widehat{\mathcal{C}}/F$ iff the monomorphism $(\text{id}_G, \alpha) : \alpha \mapsto \pi_F^G$ is a section in $\widehat{\mathcal{C}}/F$ as well as for any $a \in A$, the subpresheaf H_a of G is a j -sheaf in $\widehat{\mathcal{C}}$.*

Proof. We deduce the result by Theorem 3.1, Lemma 3.2(ii) and (13). $\square$

Since in topoi regular monomorphisms are exactly monomorphisms, so by [3, Theorem 1.2], Lemma 3.2(ii) and (13), we get the following.

Proposition 3.5. *Let $\alpha : G \rightarrow F$ be an arrow in $\widehat{\mathcal{C}}$, where F is the constant presheaf on a set A . Then, α is injective in $\widehat{\mathcal{C}}/F$ iff the monomorphism $(\text{id}_G, \alpha) : \alpha \mapsto \pi_F^G$ is a section in $\widehat{\mathcal{C}}/F$ as well as for any $a \in A$, the subpresheaf H_a of G is injective.*

When $\mathcal{C}$ is a monoid, we obtain the following example.

Example 3.6. Let M be a monoid and $M\text{-Sets}$ the topos of all (right) representations of a fixed monoid M . Since M is a small category with just one object, for two M -sets X, B we have $X^B = \text{Hom}_{M\text{-Sets}}(M \times B, X)$, where $M \times B$ has the componentwise action. Hence, by (9) and (10), for any equivariant map $f : X \rightarrow B$, in the diagram (8) we observe

$$(14) \quad i_B(*) = \pi_B^M : M \times B \rightarrow B,$$

and

$$(15) \quad \forall h \in X^B, \forall (m, b) \in M \times B, (f^B(h))(m, b) = fh(m, b).$$

Note that one writes any equivariant map $h : M \times B \rightarrow X$ in X^B as a sequence $((x_{m,b})_{b \in B})_{m \in M}$, consisting of elements $x_{m,b} = h(m, b)$ of X , for any $(m, b) \in M \times B$. Also, h being equivariant map means that

$$\forall n, m \in M, \forall b \in B, x_{mn, bn} = x_{m, b}n.$$

Hence, we obtain that X^B is equal to

$$(16) \quad \{((x_{m,b})_b)_m \in \prod_{m \in M} \prod_{b \in B} X \mid \forall n, m \in M, \forall b \in B, x_{mn, bn} = x_{m, b}n\}.$$

Now, by (8), (14) and (15) we have

$$\begin{aligned} S(f) &= \{((x_{m,b})_b)_m \in X^B \mid f^B(((x_{m,b})_b)_m) = \pi_B^M = ((b)_b)_m\} \\ &= \{((x_{m,b})_b)_m \in X^B \mid ((f(x_{m,b}))_b)_m = ((b)_b)_m\} \\ &= \{((x_{m,b})_b)_m \in X^B \mid \forall m \in M, \forall b \in B, x_{m,b} \in f^{-1}(b)\}, \end{aligned}$$

Hence, by (16) we interpret a simple form of underlying set of the M -set $S(f)$ in the topos $M\text{-Sets}$ as follows

$$\{((x_{m,b})_b)_m \in \prod_{m \in M} \prod_{b \in B} f^{-1}(b) \mid \forall n, m \in M, \forall b \in B, x_{mn, bn} = x_{m, b}n\}.$$

If B has the trivial action “.”, that is, $\cdot = \pi_1 : B \times M \rightarrow B$ the first projection, then by (12) and (13) we can obtain $X^B \cong \prod_B X$ and $S(f) \cong \prod_{b \in B} f^{-1}(b)$.

Furthermore, recall [12] that for a group G and two G -sets X, B , we have

$$(17) \quad X^B = \{h : B \rightarrow X \mid h \text{ is a function}\} \cong \prod_B X$$

as two sets. According to the action on X^B , under the isomorphism (17), the action on $\prod_B X$ is given by $(x_b)_{b \in B} \cdot g = (x_{bg^{-1}} \cdot g)_{b \in B}$, for any $g \in G$ and $(x_b)_{b \in B} \in \prod_B X$. Also, by (17) for any equivariant map $f : X \rightarrow B$ in $G\text{-Sets}$, in a similar way to (13), we have $S(f) \cong \prod_{b \in B} f^{-1}(b)$.

4. j -essential extensions in a topos

This section is devoted to introduce a class of monomorphisms in an elementary topos, which we call its members ‘ j -essential monomorphisms’. We present some equivalent forms of these monomorphisms and state some of their properties. Meanwhile, we prove that any presheaf in a presheaf topos has a maximal essential extension.

Recall that a monomorphism $\iota : A \rightarrowtail B$ is called *essential* whenever for each arrow $g : B \rightarrow C$ such that $g\iota$ is a monomorphism, then g is a monomorphism also. Now, we define a j -essential monomorphism in a topos $\mathcal{E}$ as follows.

Definition 4.1. For a topology j on $\mathcal{E}$, a monomorphism $\iota : A \rightarrowtail B$ is called *j -essential* whenever it is j -dense as well as essential. In this case, we say that B is a *j -essential extension* of A and we write $A \subseteq_j B$.

We call an arrow $f : A \rightarrow B$ in $\mathcal{E}$ to be *j -dense* whenever the subobject $f(A)$, which is the image of f , is j -dense in B . In this way, any epimorphism in $\mathcal{E}$ becomes j -dense. (For the definition of image of an arrow in a topos, see [12].)

In the following we give some equivalent definitions of j -essential monomorphisms in a topos $\mathcal{E}$.

Lemma 4.2. Let j be a topology on $\mathcal{E}$ and $\iota : A \rightarrowtail B$ a j -dense monomorphism. Then, the following are equivalent:

- (i) for any $g : B \rightarrow C$, g is a monomorphism whenever $g\iota$ is a monomorphism;
- (ii) for any $g : B \rightarrow C$, g is a j -dense monomorphism whenever $g\iota$ is a j -dense monomorphism;
- (iii) for any $g : B \rightarrow C$, g is a monomorphism whenever $g\iota$ is a j -dense monomorphism.

Proof. (i) $\implies$ (ii) and (iii) $\implies$ (ii) are proved by [9, A.4.5.11(iii)].

(ii) $\implies$ (iii) is clear.

(ii) $\implies$ (i). Consider an arrow $g : B \rightarrow C$ for which $g\iota$ is a monomorphism.

We show that g is a monomorphism too. Assume that $B \xrightarrow{k} g(B) \xrightarrow{m} C$ is the

image factorization of the arrow g . Since $g\iota = m(k\iota)$ and $g\iota$ is monomorphism, it follows that the arrow $k\iota$ is a monomorphism. Meanwhile, we get

$$\begin{aligned} g(B) &= k(B) && (\text{as } k \text{ is epic}) \\ &= k(\overline{A}) && (\text{as } \iota \text{ is dense}) \\ &\subseteq \overline{k(A)} \\ &\subseteq g(B). \end{aligned}$$

Therefore, $g(B) = \overline{k(A)} = \overline{k\iota(A)}$. It follows that the compound monomorphism $k\iota : A \rightarrowtail g(B)$ is a dense monomorphism and so is k , by (ii). That k is monomorphism and so isomorphism, yields that g is a monomorphism. $\square$

We point out that the proof of (ii) $\implies$ (i) of Lemma 4.2 shows that any composite $k\iota$, for an epic k and a dense monomorphism ι , is dense.

The following shows that j -essential monomorphisms in $\mathcal{E}$ are closed under composition.

Proposition 4.3. *Let j be a topology on $\mathcal{E}$. Then for two subobjects $A \rightarrowtail A' \rightarrowtail B$ in $\mathcal{E}$, we have $A \subseteq_j B$ iff $A \subseteq_j A'$ and $A' \subseteq_j B$.*

Proof. By [9, A.4.5.11(iii)], one has $\iota'\iota$ is j -dense iff ι' and ι are j -dense.

Necessity. First, by Lemma 4.2(i), we show that $A \subseteq_j A'$. To do so, consider an arrow $f' : A' \rightarrow C$ for which $f'\iota$ is a monomorphism. Now, by [12, Corollary IV. 10. 3], the object C can be embedded into an injective object D as in $C \xrightarrow{\nu} D$ and hence there is an arrow $\tilde{f}' : B \rightarrow D$ such that $\tilde{f}'\iota' = \nu f'$. Since $A \subseteq_j B$ and $\tilde{f}'\iota'\iota = \nu f'\iota$ is a monomorphism, we deduce that $\tilde{f}'$ is a monomorphism. As $\tilde{f}'\iota' = \nu f'$ it follows that f' is a monomorphism.

To prove $A' \subseteq_j B$, choose an arrow $f : B \rightarrow C$ for which $f\iota'$ is a monomorphism. Then, $f\iota'\iota$ is also a monomorphism. Now $A \subseteq_j B$ implies that f is a monomorphism, as required.

Sufficiency. Let $f : B \rightarrow C$ be an arrow in $\mathcal{E}$ such that $f\iota'\iota$ is a monomorphism. Since $A \subseteq_j A'$ and $(f\iota')\iota = f\iota'\iota$ is a monomorphism, it concludes that $f\iota'$ is a monomorphism. Using $A' \subseteq_j B$, we achieve that f is a monomorphism and hence $A \subseteq_j B$. $\square$

In the following, we obtain another property of j -essential monomorphisms in $\mathcal{E}$.

Lemma 4.4. *Let j be a topology on $\mathcal{E}$. If $A \subseteq_j B$ and A is embedded in a j -sheaf F , then B is also embedded in F .*

Proof. Let $\iota : A \rightarrowtail B$ be a j -essential monomorphism and $m : A \rightarrowtail F$ an arbitrary embedding. Since F is a j -sheaf, there exists a unique morphism $f : B \rightarrow F$ making the diagram below commutative;

$$\begin{array}{ccc} A & \xrightarrow{m} & F \\ \iota \downarrow & \nearrow f & \\ B & & \end{array}$$

As $A \subseteq_j B$ being j -essential, f is a monomorphism, as required. $\square$

By Remark 2.7(ii), essential monomorphisms in a topos $\mathcal{E}$ are exactly j -essential monomorphisms in $\mathcal{E}$ with respect to the topology $j = \text{true} \circ !_\Omega$ on $\mathcal{E}$.

Now, we would like to prove that any presheaf in $\widehat{\mathcal{C}}$ has a maximal essential extension.

Theorem 4.5. *Any presheaf in $\widehat{\mathcal{C}}$ has a maximal essential extension.*

Proof. Let F be a presheaf in $\widehat{\mathcal{C}}$ and G an injective presheaf into which F can be embedded. By Lemma 4.4, we can assume that both F and all its essential extensions are subpresheaves of G . Consider Σ as the set of all essential extensions of F which is a poset under subpresheaf inclusion $\subseteq$. Since the arrow id_F is an essential extension of F , it follows that Σ is non-empty. If

$$\dots \subseteq F_i \subseteq \dots,$$

$i \in I$, is a chain in Σ , then the subpresheaf H of G given by $H(C) = \bigcup_{i \in I} F_i(C)$ for any object C in $\mathcal{C}$ is an upper bound of this chain. Now we show that H lies in Σ , i.e., H is an essential extension of F . To achieve this, let $\alpha : H \rightarrow K$ be an arrow in $\widehat{\mathcal{C}}$ such that the restriction arrow $\alpha|_F$ is a monomorphism. We prove that α is a monomorphism. To verify this claim, we show that for any $C \in \widehat{\mathcal{C}}$, the function $\alpha_C : \bigcup_{i \in I} F_i(C) \rightarrow K(C)$ is one to one. Take $a, b \in \bigcup_{i \in I} F_i(C)$, $a \neq b$. Then there is a $j \in I$ such that $a, b \in F_j(C)$. Denote $\alpha|_{F_j}$ by α_j . Since F_j is an essential extension of F and $\alpha_j|_F = \alpha|_F$, it implies that α_j is a monomorphism. Now

$$\alpha_C(a) = (\alpha_j)_C(a) \neq (\alpha_j)_C(b) = \alpha_C(b).$$

Therefore, α is a monomorphism. Thus, $H \in \Sigma$. Now it follows from Zorn's Lemma that there is a maximal element M in Σ . Then, M is a maximal essential extension of F . $\square$

It is straightforward to see that any essential extension of B can be embedded in any injective extension of B .

For a topology j on a topos $\mathcal{E}$, by a j -injective object we mean an injective object with respect to the class of all j -dense monomorphisms in $\mathcal{E}$.

The following shows that the j -injective presheaves (j -sheaves) in $\widehat{\mathcal{C}}$ have no proper j -essential extension.

Proposition 4.6. *Let j be a topology on $\widehat{\mathcal{C}}$ and F a j -injective presheaf (j -sheaf) in $\widehat{\mathcal{C}}$. Then, F has no proper j -essential extension.*

Proof. Suppose that G is a proper j -essential extension of F , so F is a j -dense subpresheaf of G and $F \neq G$. Thus there is an object C of $\mathcal{C}$ such that $G(C) \not\subseteq F(C)$ and then, an $a \in G(C)$ such that $a \notin F(C)$. Since F is j -injective (j -sheaf) implies that there is an arrow $\alpha : G \rightarrow F$ for which $\alpha|_F = \text{id}_F$. That $a \notin F(C)$ and $\alpha_C(a) \in F(C)$ follows that $a \neq \alpha_C(a)$. But $\alpha_C(\alpha_C(a)) = \alpha_C(a)$. Then, α_C and so α is not a monomorphism although $\alpha|_F = \text{id}_F$ is. This shows that G is not a proper j -essential extension of F and it is a contradiction. $\square$

The following shows that the pullback functor B^* reflects j -essential extensions.

Proposition 4.7. *Let j be a topology in a topos $\mathcal{E}$. For every object $B \in \mathcal{E}$, the pullback functor $B^* : \mathcal{E} \rightarrow \mathcal{E}/B$ reflects j -essential monomorphisms.*

Proof. Let $f : A \rightarrow C$ be an arrow in $\mathcal{E}$ such that $B^*(f)$ is a j_B -essential monomorphism. We show that f is a j -essential monomorphism. By Lemma 2.3, f is a j -dense monomorphism in $\mathcal{E}$. Let $g : C \rightarrow D$ be an arrow in $\mathcal{E}$ such that gf is a monomorphism. We show that g is too. Since gf is a monomorphism, the arrow $(g \times \text{id}_B)B^*(f) = (gf) \times \text{id}_B$ is a monomorphism also. As $B^*(f)$ is j_B -essential, $g \times \text{id}_B$ is a monomorphism. Then g is a monomorphism. This is the required result. $\square$

Recall [10] that a *weak topology* on a topos $\mathcal{E}$ is a morphism $j : \Omega \rightarrow \Omega$ such that:

(i) $j \circ \text{true} = \text{true}$;

(ii) $j \circ \wedge \leq \wedge \circ (j \times j)$, in which $\leq$ stands for the internal order on Ω . Meanwhile, a weak topology j on $\mathcal{E}$ is said to be *productive* if $j \circ \wedge = \wedge \circ (j \times j)$.

In what follows, we review the whole paper for a weak topology j on a topos $\mathcal{E}$ instead of a topology.

Remark 4.8. Similar to [9, A.4.5.11(ii)], one can easily check that for a weak topology j on $\mathcal{E}$ pushouts also preserve dense monomorphisms. Hence, we can obtain a version of Lemma 2.1 for a weak topology j on $\mathcal{E}$ as well. One can observe that completely analogous assertions to Lemmas 4.2, 4.4 and 2.3, Proposition 4.6 and Theorem 3.1, hold for a weak topology j on $\mathcal{E}$. But, note that by [10], in the proof of Theorem 2.6, the part (vi) $\implies$ (vii) is true for a productive weak topology j on $\mathcal{E}$. The rest parts of this proof is correct for weak topologies.

Recall [10] that, for a weak topology j on $\mathcal{E}$, it is convenient to see that if the composite subobject mn is dense then so does for m and n . In contrast with topologies [9, A.4.5.11(iii)], the converse is not necessarily true. Hence, the sufficiency part of Proposition 4.3 does not necessarily hold for a weak topology j on a topos $\mathcal{E}$. The necessity part of this proposition is true for a weak topology j as well.

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HX-TYPE CHAOTIC (HYPERCHAOTIC) SYSTEM BASED ON FUZZY INFERENCE MODELING

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Abstract. In this paper, we investigate HX equations of autonomous system. The right-hand side of n -dimensional HX equations theoretically consists of 2^n terms. HX equations seems to be more complicated than the original system. For autonomous chaotic(hyperchaotic) system with polynomial right-hand side, we obtain its HX equations by termwise modeling. We find not all coefficients are variable in HX equations. Even some HX equations are equal to the original chaotic (hyperchaotic) systems. Under some fuzzy partition, HX-type chaotic (hyperchaotic) system is defined if some of its coefficients is exactly variable. By adjusting fuzzy partition, a family of chaotic (hyperchaotic) systems can be defined. They are different but qualitatively similar. Numerical simulations are provided to verify the existence of HX-type chaotic (hyperchaotic) system.

Keywords: HX equations, termwise modeling, HX-type chaotic system.

1. Introduction

HX equation is a kind of special differential equation with variable coefficients. It is constructed on the fuzzy logic system by using interpolation mechanism. This

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kind of modelling method is called modelling method based on fuzzy inference, proposed by Hongxing Li et al.[1]. Many literatures focus on HX equations, such as marginal linearization method for solving HX equations[2], HX equations of time-varying system[3] and some other literatures published[4, 5, 6, 7, 8, 9, 10].

In this paper, we investigate HX equations of continuous autonomous system. The three-dimensional system is considered as an example to get HX equations. The study indicates that the right-hand side of each HX equation is polynomial with eight terms. Similarly, the right-hand side of each HX equation for a four-dimensional system is polynomial with sixteen terms. For n -dimensional system, the right-hand side of its HX equation is polynomial with 2^n terms. While the typical chaotic(hyperchaotic) system is of more concise and more simple, such as the well-known Lorenz system[11], Rössler system[12], etc. We are interested in the characteristics of HX equations of chaotic(hyperchaotic) system, though few literatures[5, 6] considered it. For the system with polynomial right-hand side, we use termwise modeling method. First we divide it into several differential equations with monomial right-hand side. HX equations of the differential equations with monomial right-hand side are obtained. Then we add them together to get HX equations of the original system. The result suggests that not all coefficients of HX equations are varying. Even some HX equations are equal to the original system, such as Lorenz system, Rössler system, Rössler hyperchaotic system[13], etc.

If HX equations of the chaotic (hyperchaotic) system indeed contains variable coefficients and it is chaotic (hyperchaotic) under some fuzzy partition, we call it HX-type chaotic(hyperchaotic) system. For HX-type chaotic (hyperchaotic) system, the change of the fuzzy partition can give rise to the change of HX-type chaotic (hyperchaotic) system. The study on HX-type chaotic (hyperchaotic) system is very interesting for the chaotic switch may improve the security of secret communication.

The rest of the paper is organized as follows. Section 2 investigates HX equations of the continuous autonomous system. Three-dimensional system is taken as an example. In Section 3, HX equations of chaotic(hyperchaotic) system with polynomial right-hand side is considered. Some HX equations of chaotic(hyperchaotic) systems are even equal to the original systems. Section 4 defines HX-type chaotic(hyperchaotic) system. Liu system illustrates the existence of HX-type chaotic system. Finally, conclusions are drawn in Section 5.

2. HX equations of continuous autonomous system

Without loss of generality, we take three dimensional system as example to discuss HX equations of autonomous system. The equations is described as

follows:

$$(1) \quad \begin{cases} \dot{x}_1 = f_1(x_1, x_2, x_3), \\ \dot{x}_2 = f_2(x_1, x_2, x_3), \\ \dot{x}_3 = f_3(x_1, x_2, x_3), \end{cases}$$

where x_1, x_2, x_3 are the states of the system, $f_1, f_2, f_3 : R^3 \rightarrow R^3$ are functions.

Following the fuzzy inference modeling method[1], suppose the universes of $x_1, x_2, x_3, \dot{x}_1, \dot{x}_2, \dot{x}_3$ are respectively $X_1, X_2, X_3, \dot{X}_1, \dot{X}_2, \dot{X}_3$, where $X_1 = [\alpha_1, \beta_1], X_2 = [\alpha_2, \beta_2], X_3 = [\alpha_3, \beta_3], \dot{X}_1 = [\alpha_4, \beta_4], \dot{X}_2 = [\alpha_5, \beta_5], \dot{X}_3 = [\alpha_6, \beta_6]$.

The fuzzy partitions of $X_1, X_2, X_3, \dot{X}_1, \dot{X}_2, \dot{X}_3$ are

$$\begin{aligned} \mathcal{A} &= \{A_i\}_{(1 \leq i \leq p)}, \mathcal{B} = \{B_j\}_{(1 \leq j \leq q)}, \mathcal{C} = \{C_k\}_{(1 \leq k \leq r)}, \\ \mathcal{D} &= \{D_{ijk}\}_{(1 \leq i \leq p, 1 \leq j \leq q, 1 \leq k \leq r)}, \mathcal{E} = \{E_{ijk}\}_{(1 \leq i \leq p, 1 \leq j \leq q, 1 \leq k \leq r)}, \text{ and} \\ \mathcal{F} &= \{F_{ijk}\}_{(1 \leq i \leq p, 1 \leq j \leq q, 1 \leq k \leq r)}, \text{ respectively.} \end{aligned}$$

The peak points of $A_i, B_j, C_k, D_{ijk}, E_{ijk}, F_{ijk}$ are respectively $x_{1i}, x_{2j}, x_{3k}, \dot{x}_{1ijk}, \dot{x}_{2ijk}, \dot{x}_{3ijk}$, which satisfy the conditions $\alpha_1 \leq x_{11} < x_{12} < \dots < x_{1p} \leq \beta_1, \alpha_2 \leq x_{21} < x_{22} < \dots < x_{2q} \leq \beta_2, \alpha_3 \leq x_{31} < x_{32} < \dots < x_{3r} \leq \beta_3$. Here $\dot{x}_{1ijk}, \dot{x}_{2ijk}, \dot{x}_{3ijk}$ don't need satisfy order relation. The fuzzy rule is as follows:

$$(2) \quad \begin{aligned} &R^{ijk} : \text{If } x_1 \text{ is } A_i, x_2 \text{ is } B_j \text{ and } x_3 \text{ is } C_k, \\ &\text{then } \dot{x}_1 \text{ is } D_{ijk}, \dot{x}_2 \text{ is } E_{ijk}, \dot{x}_3 \text{ is } F_{ijk}, \end{aligned}$$

where $i = 1, 2, \dots, p, j = 1, 2, \dots, q, k = 1, 2, \dots, r$.

The fuzzy logic system based on fuzzy rules (2) can be represented as the following piecewise interpolation functions:

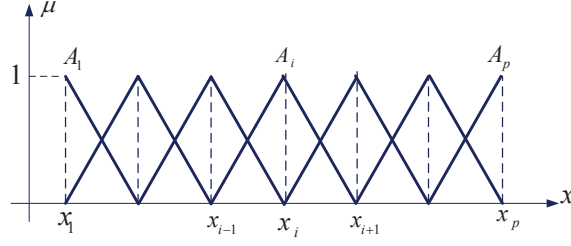
$$(3) \quad \begin{cases} \dot{x}_1 = F_1(x_1, x_2, x_3) \triangleq \sum_{i=1}^p \sum_{j=1}^q \sum_{k=1}^r A_i(x_1) B_j(x_2) C_k(x_3) \dot{x}_{1ijk}, \\ \dot{x}_2 = F_2(x_1, x_2, x_3) \triangleq \sum_{i=1}^p \sum_{j=1}^q \sum_{k=1}^r A_i(x_1) B_j(x_2) C_k(x_3) \dot{x}_{2ijk}, \\ \dot{x}_3 = F_3(x_1, x_2, x_3) \triangleq \sum_{i=1}^p \sum_{j=1}^q \sum_{k=1}^r A_i(x_1) B_j(x_2) C_k(x_3) \dot{x}_{3ijk}. \end{cases}$$

A_i is taken as "triangle wave" membership function:

$$(4) \quad A_i(x_1) = \begin{cases} \frac{x_1 - x_{1(i-1)}}{x_{1i} - x_{1(i-1)}}, & x_{1(i-1)} \leq x_1 \leq x_{1i}, \\ \frac{x_{1(i+1)} - x_1}{x_{1(i+1)} - x_{1i}}, & x_{1i} \leq x_1 \leq x_{1(i+1)}, \\ 0, & \text{otherwise,} \end{cases}$$

where $i = 1, 2, \dots, p$, and let $x_{10} = x_{11}, x_{1(p+1)} = x_{1p}$. The membership function A_i is drawn as Fig.1.

Similarly, the membership functions of B_j and C_k can be given. We can easily draw the conclusion of theorem 1.

Figure 1: Triangle wave membership function of A_i

Theorem 1. *Based on the above assumptions, the fuzzy model (3) of system (1) can be described as the following ordinary differential equations with variable-coefficients*

$$\begin{aligned}
 \dot{x}_1 &= F_1(x_1, x_2, x_3) \\
 &\triangleq a_0(x_1, x_2, x_3) + a_1(x_1, x_2, x_3)x_1 + a_2(x_1, x_2, x_3)x_2 \\
 &\quad + a_3(x_1, x_2, x_3)x_3 + a_{12}(x_1, x_2, x_3)x_1x_2 + a_{13}(x_1, x_2, x_3)x_1x_3 \\
 &\quad + a_{23}(x_1, x_2, x_3)x_2x_3 + a_{123}(x_1, x_2, x_3)x_1x_2x_3, \\
 \dot{x}_2 &= F_2(x_1, x_2, x_3) \\
 &\triangleq b_0(x_1, x_2, x_3) + b_1(x_1, x_2, x_3)x_1 + b_2(x_1, x_2, x_3)x_2 \\
 &\quad + b_3(x_1, x_2, x_3)x_3 + b_{12}(x_1, x_2, x_3)x_1x_2 + b_{13}(x_1, x_2, x_3)x_1x_3 \\
 &\quad + b_{23}(x_1, x_2, x_3)x_2x_3 + b_{123}(x_1, x_2, x_3)x_1x_2x_3, \\
 \dot{x}_3 &= F_3(x_1, x_2, x_3) \\
 &\triangleq c_0(x_1, x_2, x_3) + c_1(x_1, x_2, x_3)x_1 + c_2(x_1, x_2, x_3)x_2 \\
 &\quad + c_3(x_1, x_2, x_3)x_3 + c_{12}(x_1, x_2, x_3)x_1x_2 + c_{13}(x_1, x_2, x_3)x_1x_3 \\
 &\quad + c_{23}(x_1, x_2, x_3)x_2x_3 + c_{123}(x_1, x_2, x_3)x_1x_2x_3.
 \end{aligned}
 \tag{5}$$

Proof. When the state $(x_1, x_2, x_3)^T$ is on (ijk) th piece, i.e. $(x_1, x_2, x_3)^T \in [x_{1i}, x_{1(i+1)}] \times [x_{2j}, x_{2(j+1)}] \times [x_{3k}, x_{3(k+1)}]$, from the piecewise interpolation functions (3) and triangle-wave membership function (4), we get

$$\begin{aligned}
 \dot{x}_1 &= F_1(x_1, x_2, x_3) \triangleq \sum_{i=1}^p \sum_{j=1}^q \sum_{k=1}^r A_i(x_1) B_j(x_2) C_k(x_3) \dot{x}_{1ijk} \\
 &= A_i(x_1) B_j(x_2) C_k(x_3) \dot{x}_{1ijk} + A_i(x_1) B_j(x_2) C_{k+1}(x_3) \dot{x}_{1ijk+1} \\
 &\quad + A_i(x_1) B_{j+1}(x_2) C_k(x_3) \dot{x}_{1ij+1k} + A_i(x_1) B_{j+1}(x_2) C_{k+1}(x_3) \dot{x}_{1ij+1k+1} \\
 &\quad + A_{i+1}(x_1) B_j(x_2) C_k(x_3) \dot{x}_{1i+1jk} + A_{i+1}(x_1) B_j(x_2) C_{k+1}(x_3) \dot{x}_{1i+1jk+1} \\
 &\quad + A_{i+1}(x_1) B_{j+1}(x_2) C_k(x_3) \dot{x}_{1i+1j+1k} + A_{i+1}(x_1) B_{j+1}(x_2) C_{k+1}(x_3) \dot{x}_{1i+1j+1k+1} \\
 &= \frac{x_1 - x_{1(i+1)}}{x_{1i} - x_{1(i+1)}} \frac{x_2 - x_{2(j+1)}}{x_{2j} - x_{2(j+1)}} \frac{x_3 - x_{3(k+1)}}{x_{3k} - x_{3(k+1)}} \dot{x}_{1ijk}
 \end{aligned}$$

$$\begin{aligned}
& + \frac{x_1 - x_{1(i+1)}}{x_{1i} - x_{1(i+1)}} \frac{x_2 - x_{2(j+1)}}{x_{2j} - x_{2(j+1)}} \frac{x_3 - x_{3k}}{x_{3(k+1)} - x_{3k}} \dot{x}_{1ijk+1} \\
& + \frac{x_1 - x_{1(i+1)}}{x_{1i} - x_{1(i+1)}} \frac{x_2 - x_{2j}}{x_{2(j+1)} - x_{2j}} \frac{x_3 - x_{3(k+1)}}{x_{3k} - x_{3(k+1)}} \dot{x}_{1ij+1k} \\
& + \frac{x_1 - x_{1(i+1)}}{x_{1i} - x_{1(i+1)}} \frac{x_2 - x_{2j}}{x_{2(j+1)} - x_{2j}} \frac{x_3 - x_{3k}}{x_{3(k+1)} - x_{3k}} \dot{x}_{1ij+1k+1} \\
& + \frac{x_1 - x_{1i}}{x_{1(i+1)} - x_{1i}} \frac{x_2 - x_{2(j+1)}}{x_{2j} - x_{2(j+1)}} \frac{x_3 - x_{3(k+1)}}{x_{3k} - x_{3(k+1)}} \dot{x}_{1i+1jk} \\
& + \frac{x_1 - x_{1i}}{x_{1(i+1)} - x_{1i}} \frac{x_2 - x_{2(j+1)}}{x_{2j} - x_{2(j+1)}} \frac{x_3 - x_{3k}}{x_{3(k+1)} - x_{3k}} \dot{x}_{1i+1jk+1} \\
& + \frac{x_1 - x_{1i}}{x_{1(i+1)} - x_{1i}} \frac{x_2 - x_{2j}}{x_{2(j+1)} - x_{2j}} \frac{x_3 - x_{3(k+1)}}{x_{3k} - x_{3(k+1)}} \dot{x}_{1i+1j+1k} \\
& + \frac{x_1 - x_{1i}}{x_{1(i+1)} - x_{1i}} \frac{x_2 - x_{2j}}{x_{2(j+1)} - x_{2j}} \frac{x_3 - x_{3k}}{x_{3(k+1)} - x_{3k}} \dot{x}_{1i+1j+1k+1} \\
& = a_0^{(ijk)} + a_1^{(ijk)} x_1 + a_2^{(ijk)} x_2 + a_3^{(ijk)} x_3 + a_{12}^{(ijk)} x_1 x_2 \\
& \quad + a_{13}^{(ijk)} x_1 x_3 + a_{23}^{(ijk)} x_2 x_3 + a_{123}^{(ijk)} x_1 x_2 x_3,
\end{aligned}$$

where

$$\begin{aligned}
a_0^{(ijk)} &= \frac{1}{(x_{1i} - x_{1(i+1)})(x_{2j} - x_{2(j+1)})(x_{3k} - x_{3(k+1)})} \\
&\quad \times (x_{1(i+1)} x_{2(j+1)} (x_{3k} \dot{x}_{1ij(k+1)} - x_{3(k+1)} \dot{x}_{1ijk}) \\
&\quad + x_{1(i+1)} x_{2j} (x_{3(k+1)} \dot{x}_{1i(j+1)k} - x_{3k} \dot{x}_{1i(j+1)(k+1)}) \\
&\quad + x_{1i} x_{2(j+1)} (x_{3(k+1)} \dot{x}_{1(i+1)jk} - x_{3k} \dot{x}_{1(i+1)j(k+1)}) \\
&\quad + x_{1i} x_{2j} (x_{3k} \dot{x}_{1(i+1)(j+1)(k+1)} - x_{3(k+1)} \dot{x}_{1(i+1)(j+1)k})), \\
a_1^{(ijk)} &= \frac{1}{(x_{1i} - x_{1(i+1)})(x_{2j} - x_{2(j+1)})(x_{3k} - x_{3(k+1)})} \\
&\quad \times (x_{2(j+1)} x_{3(k+1)} (\dot{x}_{1ijk} - \dot{x}_{1(i+1)jk}) + x_{2(j+1)} x_{3k} (\dot{x}_{1(i+1)j(k+1)} - \dot{x}_{1ij(k+1)}) \\
&\quad + x_{2j} x_{3(k+1)} (\dot{x}_{1(i+1)(j+1)k} - \dot{x}_{1i(j+1)k}) \\
&\quad + x_{2j} x_{3k} (\dot{x}_{1i(j+1)(k+1)} - \dot{x}_{1(i+1)(j+1)(k+1)})), \\
a_2^{(ijk)} &= \frac{1}{(x_{1i} - x_{1(i+1)})(x_{2j} - x_{2(j+1)})(x_{3k} - x_{3(k+1)})} \\
&\quad \times (x_{1(i+1)} x_{3(k+1)} (\dot{x}_{1ijk} - \dot{x}_{1i(j+1)k}) + x_{1(i+1)} x_{3k} (\dot{x}_{1i(j+1)(k+1)} - \dot{x}_{1ij(k+1)}) \\
&\quad + x_{1i} x_{3(k+1)} (\dot{x}_{1(i+1)(j+1)k} - \dot{x}_{1(i+1)jk}) \\
&\quad + x_{1i} x_{3k} (\dot{x}_{1(i+1)j(k+1)} - \dot{x}_{1(i+1)(j+1)(k+1)})),
\end{aligned}$$

$$\begin{aligned}
a_3^{(ijk)} &= \frac{1}{(x_{1i} - x_{1(i+1)})(x_{2j} - x_{2(j+1)})(x_{3k} - x_{3(k+1)})} \\
&\quad \times (x_{1(i+1)}x_{2(j+1)}(\dot{x}_{1ijk} - \dot{x}_{1ij(k+1)}) + x_{1(i+1)}x_{2j}(\dot{x}_{1i(j+1)(k+1)} - \dot{x}_{1i(j+1)k}) \\
&\quad + x_{1i}x_{2(j+1)}(\dot{x}_{1(i+1)j(k+1)} - \dot{x}_{1(i+1)jk}) \\
&\quad + x_{1i}x_{2j}(\dot{x}_{1(i+1)(j+1)k} - \dot{x}_{1(i+1)(j+1)(k+1)})), \\
a_{12}^{(ijk)} &= \frac{1}{(x_{1i} - x_{1(i+1)})(x_{2j} - x_{2(j+1)})(x_{3k} - x_{3(k+1)})} \\
&\quad \times (x_{3(k+1)}(\dot{x}_{1i(j+1)k} - \dot{x}_{1ijk} + \dot{x}_{1(i+1)jk} - \dot{x}_{1(i+1)(j+1)k}) \\
&\quad + x_{3k}(\dot{x}_{1i(j+1)(k+1)} - \dot{x}_{1ij(k+1)} + \dot{x}_{1(i+1)j(k+1)} - \dot{x}_{1(i+1)(j+1)(k+1)})), \\
a_{13}^{(ijk)} &= \frac{1}{(x_{1i} - x_{1(i+1)})(x_{2j} - x_{2(j+1)})(x_{3k} - x_{3(k+1)})} \\
&\quad \times (x_{2(j+1)}(\dot{x}_{1ij(k+1)} - \dot{x}_{1ijk} + \dot{x}_{1(i+1)jk} - \dot{x}_{1(i+1)j(k+1)}) \\
&\quad + x_{2j}(\dot{x}_{1i(j+1)k} - \dot{x}_{1i(j+1)(k+1)} + \dot{x}_{1(i+1)(j+1)(k+1)} - \dot{x}_{1(i+1)(j+1)k})), \\
a_{23}^{(ijk)} &= \frac{1}{(x_{1i} - x_{1(i+1)})(x_{2j} - x_{2(j+1)})(x_{3k} - x_{3(k+1)})} \\
&\quad \times (x_{1(i+1)}(\dot{x}_{1ij(k+1)} - \dot{x}_{1ijk} + \dot{x}_{1i(j+1)k} - \dot{x}_{1i(j+1)(k+1)}) \\
&\quad + x_{1i}(\dot{x}_{1(i+1)jk} - \dot{x}_{1(i+1)j(k+1)} + \dot{x}_{1(i+1)(j+1)(k+1)} - \dot{x}_{1(i+1)(j+1)k})), \\
a_{123}^{(ijk)} &= \frac{1}{(x_{1i} - x_{1(i+1)})(x_{2j} - x_{2(j+1)})(x_{3k} - x_{3(k+1)})} \\
&\quad \times (\dot{x}_{1ijk} - \dot{x}_{1ij(k+1)} - \dot{x}_{1i(j+1)k} + \dot{x}_{1i(j+1)(k+1)} - \dot{x}_{1(i+1)jk} \\
&\quad + \dot{x}_{1(i+1)j(k+1)} - \dot{x}_{1(i+1)(j+1)(k+1)} + \dot{x}_{1(i+1)(j+1)k}).
\end{aligned}$$

When $(x_1, x_2, x_3)^T \notin [x_{1i}, x_{1(i+1)}] \times [x_{2j}, x_{2(j+1)}] \times [x_{3k}, x_{3(k+1)}]$, suppose $a_0^{(ijk)} = a_1^{(ijk)} = a_2^{(ijk)} = a_3^{(ijk)} = a_{12}^{(ijk)} = a_{13}^{(ijk)} = a_{23}^{(ijk)} = a_{123}^{(ijk)} = 0$, the coefficients can be defined on the whole universe $X_1 \times X_2 \times X_3$, so we have

$$\begin{aligned}
(6) \quad \dot{x}_1 &= a_0^{(ijk)} + a_1^{(ijk)}x_1 + a_2^{(ijk)}x_2 + a_3^{(ijk)}x_3 + a_{12}^{(ijk)}x_1x_2 \\
&\quad + a_{13}^{(ijk)}x_1x_3 + a_{23}^{(ijk)}x_2x_3 + a_{123}^{(ijk)}x_1x_2x_3,
\end{aligned}$$

which is defined on the whole universe.

When i, j, k change, we can get $(p-1) \times (q-1) \times (r-1)$ such equations. Let $a_0(x_1, x_2, x_3) = \sum_{i=1}^{p-1} \sum_{j=1}^{q-1} \sum_{k=1}^{r-1} a_0^{(ijk)}$, $a_1(x_1, x_2, x_3) = \sum_{i=1}^{p-1} \sum_{j=1}^{q-1} \sum_{k=1}^{r-1} a_1^{(ijk)}$, $a_2(x_1, x_2, x_3) = \sum_{i=1}^{p-1} \sum_{j=1}^{q-1} \sum_{k=1}^{r-1} a_2^{(ijk)}$, $a_3(x_1, x_2, x_3) = \sum_{i=1}^{p-1} \sum_{j=1}^{q-1} \sum_{k=1}^{r-1} a_3^{(ijk)}$, $a_{12}(x_1, x_2, x_3) = \sum_{i=1}^{p-1} \sum_{j=1}^{q-1} \sum_{k=1}^{r-1} a_{12}^{(ijk)}$, $a_{13}(x_1, x_2, x_3) = \sum_{i=1}^{p-1} \sum_{j=1}^{q-1} \sum_{k=1}^{r-1} a_{13}^{(ijk)}$, $a_{23}(x_1, x_2, x_3) = \sum_{i=1}^{p-1} \sum_{j=1}^{q-1} \sum_{k=1}^{r-1} a_{23}^{(ijk)}$, $a_{123}(x_1, x_2, x_3) = \sum_{i=1}^{p-1} \sum_{j=1}^{q-1} \sum_{k=1}^{r-1} a_{123}^{(ijk)}$. Then we combine the $(p-1) \times (q-1) \times (r-1)$ equations together and obtain

HX equation of the first equation of system (1), that is

$$(7) \quad \begin{aligned} \dot{x}_1 = & a_0(x_1, x_2, x_3) + a_1(x_1, x_2, x_3)x_1 + a_2(x_1, x_2, x_3)x_2 + \\ & a_3(x_1, x_2, x_3)x_3 + a_{12}(x_1, x_2, x_3)x_1 \\ & + a_{13}(x_1, x_2, x_3) + a_{23}(x_1, x_2, x_3)x_2x_3 + a_{123}(x_1, x_2, x_3)x_1x_2x_3. \end{aligned}$$

Similarly, we can get HX equations of the other two equations. $\square$

From the above analysis, we know that the polynomial right-hand side of the HX equation consists of eight terms. Similarly, the polynomial right-hand side of HX equation about four dimension system contains sixteen terms, The right-hand side of HX equation about n -dimension system is composed by 2^n terms. This seems go against the concise form of the typical autonomous system. For example, the structure of the HX equations of Lorenz system may be more complex than the original system.

Fan and Fang [5, 6] investigated the fuzzy inference modeling of chaotic system, but HX equations of Rössler system given by them is theoretically equal to Rössler system. The system error is caused by round-off error in the numerical calculation.

3. HX equations of chaotic (hyperchaotic) system with polynomial right-hand side

Now we investigate the autonomous chaotic(hyperchaotic) system with polynomial right-hand side by termwise modeling method. Suppose we fully understand the structure of the chaotic (hyperchaotic) system considered. There exist a lot of such systems, such as Lorenz system [11], Rössler system[12], hyperchaotic Lorenz system[19, 23, 22], etc. Specifically, f_1, f_2, f_3 in Eq. (1) are all polynomials. The general form of the equation in system (1) is

$$(8) \quad \dot{w} = \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} \sum_{k_3=0}^{n_3} \lambda_{k_1 k_2 k_3} x_1^{k_1} x_2^{k_2} x_3^{k_3},$$

where $\dot{w}$ denotes $\dot{x}_1, \dot{x}_2$ or $\dot{x}_3$, $\lambda_{k_1 k_2 k_3}$ is the coefficient of $\lambda_{k_1 k_2 k_3} x_1^{k_1} x_2^{k_2} x_3^{k_3}$.

Let $\dot{w}_{k_1 k_2 k_3} = x_1^{k_1} x_2^{k_2} x_3^{k_3}$,

$$(9) \quad \dot{w} = \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} \sum_{k_3=0}^{n_3} \lambda_{k_1 k_2 k_3} \dot{w}_{k_1 k_2 k_3}.$$

The equation (3) is divided into several differential equations with monomial right-hand side. From the linear relation, if we get HX equation $\dot{w}_{k_1 k_2 k_3}^{HX}$ of each monomial differential equation $\dot{w}_{k_1 k_2 k_3}$, and then add them together, we can get HX equation of $\dot{w}$.

$$(10) \quad \dot{w}^{HX} = \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} \sum_{k_3=0}^{n_3} \lambda_{k_1 k_2 k_3} \dot{w}_{k_1 k_2 k_3}^{HX}.$$

Now our aim is to obtain HX equations $\dot{w}_{k_1 k_2 k_3}^{HX}$ of $\dot{w}_{k_1 k_2 k_3}^{HX}$. Following the method of Li[1], on the piece $[x_{1i}, x_{1(i+1)}] \times [x_{2j}, x_{2(j+1)}] \times [x_{3k}, x_{3(k+1)}]$, we have

$$\begin{aligned} \dot{w}_{k_1 k_2 k_3}^{HX(ijk)} &= \frac{x_1 - x_{1i}}{x_{1(i+1)} - x_{1i}} \frac{x_2 - x_{2j}}{x_{2(j+1)} - x_{2j}} \frac{x_3 - x_{3k}}{x_{3(k+1)} - x_{3k}} x_{1(i+1)}^{k_1} x_{2(j+1)}^{k_2} x_{3(k+1)}^{k_3} \\ &+ \frac{x_1 - x_{1i}}{x_{1(i+1)} - x_{1i}} \frac{x_2 - x_{2j}}{x_{2(j+1)} - x_{2j}} \frac{x_3 - x_{3(k+1)}}{x_{3k} - x_{3(k+1)}} x_{1(i+1)}^{k_1} x_{2(j+1)}^{k_2} x_{3k}^{k_3} \\ &+ \frac{x_1 - x_{1i}}{x_{1(i+1)} - x_{1i}} \frac{x_2 - x_{2(j+1)}}{x_{2j} - x_{2(j+1)}} \frac{x_3 - x_{3k}}{x_{3(k+1)} - x_{3k}} x_{1(i+1)}^{k_1} x_{2j}^{k_2} x_{3(k+1)}^{k_3} \\ &+ \frac{x_1 - x_{1i}}{x_{1(i+1)} - x_{1i}} \frac{x_2 - x_{2(j+1)}}{x_{2j} - x_{2(j+1)}} \frac{x_3 - x_{3(k+1)}}{x_{3k} - x_{3(k+1)}} x_{1(i+1)}^{k_1} x_{2j}^{k_2} x_{3k}^{k_3} \\ &+ \frac{x_1 - x_{1(i+1)}}{x_{1i} - x_{1(i+1)}} \frac{x_2 - x_{2j}}{x_{2(j+1)} - x_{2j}} \frac{x_3 - x_{3k}}{x_{3(k+1)} - x_{3k}} x_{1i}^{k_1} x_{2(j+1)}^{k_2} x_{3(k+1)}^{k_3} \\ &+ \frac{x_1 - x_{1(i+1)}}{x_{1i} - x_{1(i+1)}} \frac{x_2 - x_{2j}}{x_{2(j+1)} - x_{2j}} \frac{x_3 - x_{3(k+1)}}{x_{3k} - x_{3(k+1)}} x_{1i}^{k_1} x_{2(j+1)}^{k_2} x_{3k}^{k_3} \\ &+ \frac{x_1 - x_{1(i+1)}}{x_{1i} - x_{1(i+1)}} \frac{x_2 - x_{2(j+1)}}{x_{2j} - x_{2(j+1)}} \frac{x_3 - x_{3k}}{x_{3(k+1)} - x_{3k}} x_{1i}^{k_1} x_{2j}^{k_2} x_{3(k+1)}^{k_3} \\ &+ \frac{x_1 - x_{1(i+1)}}{x_{1i} - x_{1(i+1)}} \frac{x_2 - x_{2(j+1)}}{x_{2j} - x_{2(j+1)}} \frac{x_3 - x_{3(k+1)}}{x_{3k} - x_{3(k+1)}} x_{1i}^{k_1} x_{2j}^{k_2} x_{3k}^{k_3} \\ &= \left(\frac{x_1 - x_{1(i+1)}}{x_{1i} - x_{1(i+1)}} x_{1i}^{k_1} + \frac{x_1 - x_{1i}}{x_{1(i+1)} - x_{1i}} x_{1(i+1)}^{k_1} \right) \\ &\times \left(\frac{x_2 - x_{2(j+1)}}{x_{2j} - x_{2(j+1)}} x_{2j}^{k_2} + \frac{x_2 - x_{2j}}{x_{2(j+1)} - x_{2j}} x_{2(j+1)}^{k_2} \right) \\ &\times \left(\frac{x_3 - x_{3(k+1)}}{x_{3k} - x_{3(k+1)}} x_{3k}^{k_3} + \frac{x_3 - x_{3k}}{x_{3(k+1)} - x_{3k}} x_{3(k+1)}^{k_3} \right), \end{aligned}$$

that is,

$$(11) \quad \begin{aligned} \dot{w}_{k_1 k_2 k_3}^{HX(ijk)} &= \left(\frac{x_{1i}^{k_1} - x_{1(i+1)}^{k_1}}{x_{1i} - x_{1(i+1)}} x_1 + \frac{x_{1i} x_{1(i+1)}^{k_1} - x_{1i}^{k_1} x_{1(i+1)}}{x_{1i} - x_{1(i+1)}} \right) \\ &\times \left(\frac{x_{2j}^{k_2} - x_{2(j+1)}^{k_2}}{x_{2j} - x_{2(j+1)}} x_2 + \frac{x_{2j} x_{2(j+1)}^{k_2} - x_{2j}^{k_2} x_{2(j+1)}}{x_{2j} - x_{2(j+1)}} \right) \\ &\times \left(\frac{x_{3k}^{k_3} - x_{3(k+1)}^{k_3}}{x_{3k} - x_{3(k+1)}} x_3 + \frac{x_{3k} x_{3(k+1)}^{k_3} - x_{3k}^{k_3} x_{3(k+1)}}{x_{3k} - x_{3(k+1)}} \right). \end{aligned}$$

It is easy to see that the content in three parenthesis of equation (11) are all of first degree. We need only to analyze the content of first one. Similar conclusions can be drawn about the other two.

When $k_1 = 0$, that is to say that $\dot{w}_{k_1 k_2 k_3}$ doesn't contain x_1

$$(12) \quad \begin{aligned} & \frac{x_{1i}^{k_1} - x_{1(i+1)}^{k_1}}{x_{1i} - x_{1(i+1)}} x_1 + \frac{x_{1i} x_{1(i+1)}^{k_1} - x_{1i}^{k_1} x_{1(i+1)}}{x_{1i} - x_{1(i+1)}} \\ &= \frac{x_{1i}^0 - x_{1(i+1)}^0}{x_{1i} - x_{1(i+1)}} x_1 + \frac{x_{1i} x_{1(i+1)}^0 - x_{1i}^0 x_{1(i+1)}}{x_{1i} - x_{1(i+1)}} = 0 \times x_1 + 1 = 1. \end{aligned}$$

This suggests $\dot{w}_{k_1 k_2 k_3}^{HX(ijk)}$ don't contain x_1 . In another word, $\dot{w}_{k_1 k_2 k_3}^{HX(ijk)}$ is irrelevant to the fuzzy partition of the universe of x_1 .

When $k_1 = 1$, the degree of x_1 is 1 in $\dot{w}_{k_1 k_2 k_3}$

$$(13) \quad \begin{aligned} & \frac{x_{1i}^{k_1} - x_{1(i+1)}^{k_1}}{x_{1i} - x_{1(i+1)}} x_1 + \frac{x_{1i} x_{1(i+1)}^{k_1} - x_{1i}^{k_1} x_{1(i+1)}}{x_{1i} - x_{1(i+1)}} = \frac{x_{1i} - x_{1(i+1)}}{x_{1i} - x_{1(i+1)}} x_1 + \frac{x_{1i} x_{1(i+1)} - x_{1i} x_{1(i+1)}}{x_{1i} - x_{1(i+1)}} \\ &= 1 \times x_1 + 0 = x_1. \end{aligned}$$

This suggests the degree of x_1 is still 1 in $\dot{w}_{k_1 k_2 k_3}^{HX(ijk)}$. In another word $\dot{w}_{k_1 k_2 k_3}^{HX(ijk)}$ is irrelevant to the fuzzy partition of the universe of x_1 .

When $k_1 \geq 2$

$$(14) \quad \begin{aligned} & \frac{x_{1i}^{k_1} - x_{1(i+1)}^{k_1}}{x_{1i} - x_{1(i+1)}} x_1 + \frac{x_{1i} x_{1(i+1)}^{k_1} - x_{1i}^{k_1} x_{1(i+1)}}{x_{1i} - x_{1(i+1)}} \\ &= \left(\sum_{s=0}^{k_1-1} x_{1i}^s x_{1(i+1)}^{k_1-1-s} \right) x_1 - x_{1i} x_{1(i+1)} \sum_{t=0}^{k_1-2} x_{1i}^t x_{1(i+1)}^{k_1-2-t}. \end{aligned}$$

Obviously, the conclusion is relevant to the fuzzy partition of the universe of x_1 .

Similar conclusions can be obtained for the other two expressions with parenthesis in (11). So we can get

Theorem 2. If $\dot{w}_{k_1 k_2 k_3} = x_1^{k_1} x_2^{k_2} x_3^{k_3}$, HX equation have the following features:
1) If $0 \leq k_i \leq 1$, ($i = 1, 2, 3$), HX equation $\dot{w}_{k_1 k_2 k_3}^{HX_1} = \dot{w}_{k_1 k_2 k_3}$ of $\dot{w}_{k_1 k_2 k_3} = x_1^{k_1} x_2^{k_2} x_3^{k_3}$;

2) If $0 \leq k_i \leq 1$ ($i = 1, 2, 3$) for some k_i , then the coefficients of the HX equation of $\dot{w}_{k_1 k_2 k_3} = x_1^{k_1} x_2^{k_2} x_3^{k_3}$ is irrelated to the fuzzy partition of the i th universe, and HX equation's exponent of the i th variable is still k_i ;

3) If $k_i \geq 2$, ($i = 1, 2, 3$) for some variable, then in the piece $[x_{1i}, x_{1(i+1)}] \times [x_{2j}, x_{2(j+1)}] \times [x_{3k}, x_{3(k+1)}]$, HX equation of $\dot{w}_{k_1 k_2 k_3}$ is related to the fuzzy partition of the i th universe;

4) If $k_i \geq 2$, ($i = 1, 2, 3$) for all variables, then in the piece $[x_{1i}, x_{1(i+1)}] \times [x_{2j}, x_{2(j+1)}] \times [x_{3k}, x_{3(k+1)}]$, HX equation of $\dot{w}_{k_1 k_2 k_3}$ is related to the fuzzy partition of every universe.

Proof. From the equations (12) and (13), we know when $0 \leq k_1 \leq 1$, the coefficient of local HX equation on the (ijk) th piece of $\dot{w}_{k_1 k_2 k_3}$ is still 1, the exponent is still k_1 . In a similar method, when $0 \leq k_2 \leq 1, 0 \leq k_3 \leq 1$, the coefficients of local HX equation on the (ijk) th piece of exponents of $\dot{w}_{k_1 k_2 k_3}$ is still 1, the exponents is still k_2, k_3 . So on the (ijk) th partition, the local HX equation $\dot{w}_{k_1 k_2 k_3}^{HX(ijk)} = x_1^{k_1} x_2^{k_2} x_3^{k_3}$. Meanwhile, if $0 \leq k_i \leq 1$ for k_i , then the coefficients of HX equation is unrelated to the fuzzy partition of the i th universe.

From equation (14), if $k_1 \geq 2$ HX equation is relevant to the fuzzy partition of the universe of x_1 . Similarly, if $k_i \geq 2$ for some k_i , on the piece $[x_{1i}, x_{1(i+1)}] \times [x_{2j}, x_{2(j+1)}] \times [x_{3k}, x_{3(k+1)}]$, HX equation is relevant to the fuzzy partition of the universe x_i . While $k_i \geq 2$ for all variables, on the piece $[x_{1i}, x_{1(i+1)}] \times [x_{2j}, x_{2(j+1)}] \times [x_{3k}, x_{3(k+1)}]$,

$$\begin{aligned} \dot{w}_{k_1 k_2 k_3}^{HX(ijk)} = & \left(\sum_{s_1=0}^{k_1-1} x_{1i}^{s_1} x_{1(i+1)}^{k_1-1-s_1} x_1 - x_{1i} x_{1(i+1)} \sum_{t_1=0}^{k_1-2} x_{1i}^{t_1} x_{1(i+1)}^{k_1-2-t_1} \right) \\ & \times \left(\sum_{s_2=0}^{k_2-1} x_{2j}^{s_2} x_{2(j+1)}^{k_2-1-s_2} x_2 - x_{2j} x_{2(j+1)} \sum_{t_2=0}^{k_2-2} x_{2j}^{t_2} x_{2(j+1)}^{k_2-2-t_2} \right) \\ & \times \left(\sum_{s_3=0}^{k_3-1} x_{3k}^{s_3} x_{3(k+1)}^{k_3-1-s_3} x_3 - x_{3k} x_{3(k+1)} \sum_{t_3=0}^{k_3-2} x_{3k}^{t_3} x_{3(k+1)}^{k_3-2-t_3} \right). \end{aligned}$$

This suggests HX equation relevant to fuzzy partitions of all variables. $\square$

Theorem 2 is not only applicable to three-dimensional differential equation with monomial right-hand side, but also it is can be used to four-dimensional or more high dimensional systems. As discussed above, we can combine HX equations obtained together to get HX equations $\dot{w}^{HX}$ of system . So the following result can be easily obtained.

Theorem 3. *For a chaotic (hyperchaotic) system, if the exponent of some variable is no more than 1, then the exponent of the variable in its HX equations is unchanged; if the exponent of all variables no more than 1, then HX equations is the same as the original one.*

Proof. From 1), 2) of theorem 2, combining with (10), the result easily obtained. $\square$

In conclusion, before solving HX equations of chaotic (hyperchaotic) system with polynomial right-hand side, we analyze the characteristic of the system, carry on fuzzy inference modelling term by term, make linear additivity, and get HX equations of the original system.

4. HX-type chaotic (hyperchaotic) system

From Section 3, we know that HX equations of chaotic (hyperchaotic) system with exponent of variables less than 2 is identical to the original system. Lorenz system and Rössler system et al. are such type systems. For the system which exponent of some variable greater than or equal to 2, some coefficients of HX equations of the system is exactly variable. There are many such systems, such as Liu system [14], hyperchaotic Liu system [15], hyperchaotic Fang system proposed by Fang [16], chaotic system proposed by Li [17], chaotic system proposed by Xie[18], which contain nonlinear terms with exponent 2 of some variable. Chen proposed four-dimension Lorenz-type hyperchaotic system with equilibrium curve[20], which contain nonlinear terms with exponent 3 of some variable.

Definition 1. *If some coefficient of HX equations about chaotic (hyperchaotic) system is variable under some fuzzy partition and the HX equations is chaotic (hyperchaotic), it is called HX-type chaotic (hyperchaotic) system.*

Now we investigate HX equations of Liu system

Example 1. Liu system is described as follows:

$$(15) \quad \begin{cases} \dot{x}_1 = a(x_2 - x_1), \\ \dot{x}_2 = bx_1 - kx_1x_3, \\ \dot{x}_3 = hx_1^2 - cx_3, \end{cases}$$

where $a = 10, b = 40, c = 2.5, k = 1, h = 1$. The Lyapunov exponents are $\lambda_1 = 1.6535, \lambda_2 = 0, \lambda_3 = -14.1446$. The phase diagram of Liu system is shown in Fig.2.

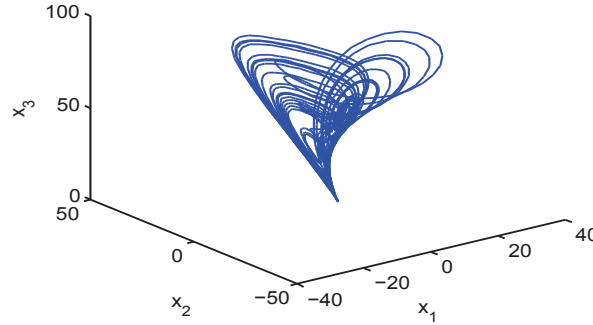


Figure 2: The phase diagram of Liu system

By numerical calculation, we know $x_1 \in [-35.5815, 35.2084]$, $x_2 \in [-60.9790, 62.7936]$, $x_3 \in [6.5037, 108.6809]$. From theorem 2, we know that the first and second equations of system (15) is exact fuzzy modelling. In the third equation, only the quadratic component is not exact modelling, that is to

say, HX equations of Liu system is only related to the fuzzy partition of the universe of x_1 . The universe of x_1 is partitioned by triangular wave membership function. For convenience, suppose the distances of two peak points are same. Avoiding the data overflow the universe, we added a redundancy to the universe. Let $-35.5815-35.5815 \times 0.1 = x_{11} < x_{12} < \dots < x_{1p} = 35.2084+35.2084 \times 0.1$. When $x_1 \in [x_{1i}, x_{1(i+1)}]$, local HX equations is

$$(16) \quad \begin{cases} \dot{x}_1 = a(x_2 - x_1), \\ \dot{x}_2 = bx_1 - kx_1x_3, \\ \dot{x}_3 = h(x_{1i} + x_{1(i+1)})x_1 - cx_3 - hx_{1i}x_{1(i+1)}. \end{cases}$$

Now the system's divergence is $\nabla V = \frac{\partial \dot{x}_1}{\partial x_1} + \frac{\partial \dot{x}_2}{\partial x_2} + \frac{\partial \dot{x}_3}{\partial x_3} = -a + 0 - c = -10 - 2.5 = -12.5 < 0$. This suggests that each local HX equation is dissipative. So HX equations of Liu system is dissipative. If the largest Lyapunov exponent of HX equations greater than zero, HX-type chaotic system is achieved. By using Wolf's method [24], we calculate the Lyapunov exponents of HX equations, as shown in Fig.3, where the peak point number is from $p = 2$ to $p = 100$, i.e. the fuzzy set number is from 2 to 100.

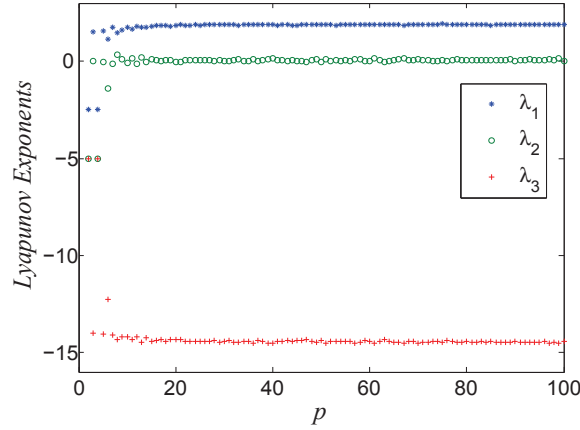


Figure 3: The diagram of Lyapunov exponents as p changing from 2 to 100

From Fig.3, in addition to $p = 2$ and $p = 4$, HX equations of Liu system is chaotic, called HX-type Liu system. When the initial value $\mathbf{x}_0 = (20, 2, 3)^T$, if the peak point number $p = 3$, the phase diagram of HX-type Liu system is shown as Fig. 4, if the peak point number $p = 10$, the phase diagram of HX-type Liu system is shown as Fig. 5, if the peak point number $p = 20$, the phase diagram of HX-type Liu system is shown as Fig.6, if the peak point number $p = 30$, the phase diagram of HX-type Liu system is shown as Fig.7.

From the phase diagram of HX-type Liu system, we know, as the increase of peak points, the phase diagram of HX-type Liu system is more like the phase diagram of Liu system.

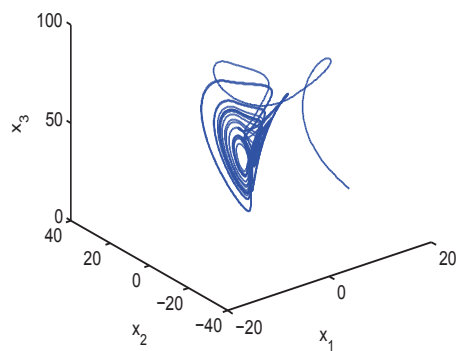


Figure 4: The phase portrait of HX-Type Liu system as $p = 3$

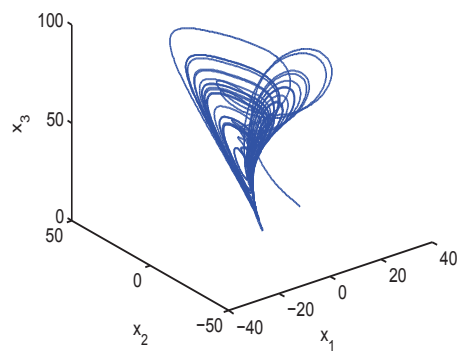


Figure 5: The phase portrait of HX-Type Liu system as $p = 10$

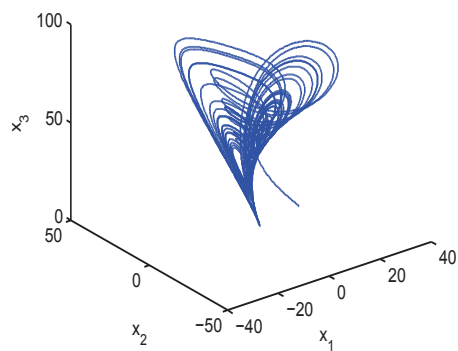


Figure 6: The phase portrait of HX-Type Liu system as $p = 20$

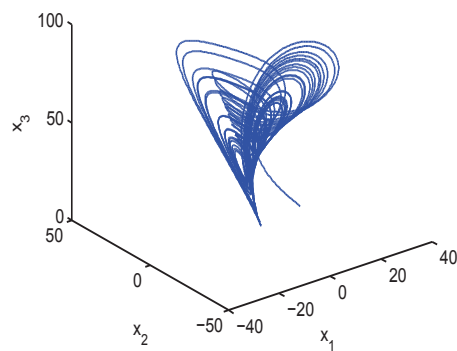


Figure 7: The phase portrait of HX-Type Liu system as $p = 30$

The equilibrium points of the HX equations (16) are obtained by solving the equations

$$(17) \quad \begin{cases} a(x_2 - x_1) = 0, \\ bx_1 - kx_1x_3 = 0, \\ h(x_{1i} + x_{1(i+1)})x_1 - cx_3 - hx_{1i}x_{1(i+1)} = 0. \end{cases}$$

From equations (17), we see that the HX equations (16) has two equilibrium points

$$E_1 \left(0, 0, -\frac{hx_{1i}x_{1(i+1)}}{c} \right), E_2 \left(\frac{bc + khx_{1i}x_{1(i+1)}}{kh(x_{1i} + x_{1(i+1)})}, \frac{bc + khx_{1i}x_{1(i+1)}}{kh(x_{1i} + x_{1(i+1)})}, \frac{b}{k} \right)$$

as $x_1 \in [x_{1i}, x_{1(i+1)}]$.

Remark 1. As we all know, Liu system (15) has three equilibriums:

$$E_1(0, 0, 0), E_2\left(\sqrt{\frac{bc}{kh}}, \sqrt{\frac{bc}{kh}}, \frac{b}{k}\right), E_3\left(-\sqrt{\frac{bc}{kh}}, -\sqrt{\frac{bc}{kh}}, \frac{b}{k}\right),$$

while the HX equations has a family of equilibrium points under the fuzzy partition.

If the fuzzy partition is given, HX-type Liu system is definite. That is to say an appropriate fuzzy partition defined a HX-type Liu system. In the secret communication, changing fuzzy partion can give rise to switch chaotic system, then the security is enhanced.

5. Conclusion

In this paper, we gave HX equations of continuous autonomous system and found that HX equations of high dimensional system is more complicated than the original system. For chaotic (hyperchaotic) system with polynomial right-hand side, we divided it into differential equtions with monomial right-hand side and obtained HX equation of them. By using linear addition, we got HX equations of chaotic (hyperchaotic) system. We discovered that not all coefficients in HX equations are variable. Even HX equations of some chaotic (hyperchatic) system is equal to the original system. Under some fuzzy partition, if HX equations with variable coefficient is chaotic (hyperchaotic), we defined it HX-type chaotic (hyperchaotic) system. The equilibrium points of HX-type chaotic systems were also analyzed. Numerical simulations verified the existence of HX-type chaotic (hyperchaotic) system.

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THE Q -CONJUGACY CHARACTER TABLE OF DIHEDRAL GROUPS

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Abstract. In a seminal paper published in 1998, Shinsaku Fujita introduced the concept of Q -conjugacy character table of a finite group. He applied this notion to solve some problems in combinatorial chemistry. In this paper, the Q -conjugacy character table of dihedral groups is computed in general. As a consequence, Q -conjugacy character table of molecules with point group symmetries $D_3 \cong C_{3v} \cong Dih_6$, $D_4 \cong D_{2d} \cong C_{4v} \cong Dih_8$, $D_5 \cong C_{5v} \cong Dih_{10}$, $D_{3d} \cong D_{3h} \cong D_6 \cong C_{6v} \cong Dih_{12}$, $D_2 \cong C_{2v} \cong Z_2 \times Z_2 \cong Dih_2$, $D_{4d} \cong Dih_{16}$, $D_{5d} \cong D_{5h} \cong Dih_{20}$, $D_{6d} \cong Dih_{24}$ are computed, where Z_2 denotes a cyclic group of order 2 and Dih_n is the dihedral group of even order n .

Keywords: Q -conjugacy character table, dihedral group, conjugacy class.

1. Introduction

A *representation* of a group G is a homomorphism from G into the group of invertible operators of a vector space V . In this case, we can interpret each element of G as an invertible linear transformation $V \rightarrow V$. If $n = \dim V$ and fix a basis for V then we can construct an isomorphism between the set of all “invertible linear transformation $V \rightarrow V$ ” and the set of all “invertible $n \times n$ matrices”. For the existence of this isomorphism, we usually use the term *matrix representation* as representation. The irreducible representations

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are precisely the ones that cannot be broken up into smaller pieces. Suppose α is a representation of the group G . Then the function $\hat{\alpha} : G \rightarrow \mathbb{C}$ given by $\hat{\alpha}(g) = \text{tr}(\alpha(g))$, the sum of all diagonal entries, is called the *character* afforded by α . The character of an irreducible representation is called an *irreducible character*.

It is well-known that the number of conjugacy classes is the same as the number of distinct irreducible representations. Also, the group character afforded by a representation is constant on a conjugacy class. Hence, the values of the irreducible characters of a group G can be written as an array in a square matrix $CT(G)$ known as the *character table* of G . In this matrix, the rows are given the irreducible characters and the columns are given the conjugacy classes of the group G , see [11, 12] for details.

Suppose G is a finite group and $\langle h \rangle$ is denoted the cyclic subgroup of G generated by h . The elements g and h are said to be Q -conjugate to each other if there exist $t \in G$ such that $t^{-1}\langle g \rangle t = \langle h \rangle$. The Q -conjugacy is an equivalence relation and generates equivalence classes which is called the Q -conjugacy classes of G . Suppose that the group G is partitioned into Q -conjugacy classes as $G = K_1 \dot{\cup} K_2 \dot{\cup} \dots \dot{\cup} K_r$. Following Fujita [4], each Q -conjugacy class K_i is called a dominant class."

Suppose Γ is an irreducible Q -representation of a finite group G and γ is the irreducible character afforded by Γ . It is well-known that the Q -character γ is constant on the Q -conjugacy classes of G . So, if $g \in K_i$ then we can write $\gamma(K_i)$ as $\gamma(g)$. The Q -character table of the group G is an $r \times r$ invertible matrix in which the columns and rows are labeled by Q -conjugacy classes and Q -irreducible characters of G , respectively. Shinsaku Fujita used this table in a series of papers to solve some problems in combinatorial chemistry. We encourage the interested readers to consult papers [1–3], [5–10] and references therein.

The dihedral group Dih_{2n} is a group of order $2n$ generated by a, b such that $a^n = 1, b^2 = 1, b^{-1}ab = a^{-1}$. The elements of this group has form $a^\alpha b^\beta$, where $1 \leq \alpha \leq n$ and $\beta = 0, 1$. The character table of Dih_{2n} is shown in Tables 1 and 2, when n is odd or even, respectively.

Table 1: The character table of Dih_{2n} , n is Odd.

D_{2n}	$a^\alpha b^\beta$
χ_1	1
χ_2	$(-1)^\beta$
χ_{j+2}	$(1 - \beta)(\varepsilon^{j\alpha} + \varepsilon^{-j\alpha})$

Table 2: The character table of Dih_{2n} , n is Even.

D_{2n}	$a^\alpha b^\beta$
χ_1	1
χ_2	$(-1)^\beta$
χ_3	$(-1)^\alpha (-1)^\beta$
χ_4	$(-1)^\alpha$
χ_{j+4}	$(1 - \beta)(\varepsilon^{j\alpha} + \varepsilon^{-j\alpha})$

Here, ε denotes the primitive n -th root of unity, i.e. $\varepsilon = \cos(\frac{2\pi}{n}) + i\sin(\frac{2\pi}{n})$. Our other notations are standard and can be taken from the standard books on group theory.

2. Main results

Suppose G is a finite group and T_1, T_2 are representations of G over a field K of characteristic 0. The representations T_1 and T_2 are called *equivalent* if there exists a non-singular matrix S with entries in K such that $S^{-1}T_1(g)S = T_2(g)$, for each $g \in G$. For simplicity, we use the notion K -representation as representation over a field K .

Suppose L and K are fields such that $K \subseteq L$, and U is an L -representation of a group G . We say that U is *realizable* in K if there exists a K -representation T such that U and T are L -equivalent. We now assume that χ is an irreducible character afforded by an irreducible K^* -representation U of G . The field generated by K and all of the values $\chi(g)$ over all elements of G is denoted by $K(\chi)$. The *Schur index* of U with respect to K is defined as $M_K(U) = \min(F : K(\chi))$ in which the minimum is taken over all fields F in which U is realizable. If χ is the character afforded by an irreducible K -representation U of G and K has characteristic 0, then $m_K(\chi)$ denotes the Schur index of χ over K . We use the notation $m(\chi)$, when $K = \mathbb{Q}$ is the field of rational numbers, see [13] for details.

It is merit to mention here that we can construct a group G such that irreducible characters of G are rational values, but the Q -conjugacy character table are from its character table. The reason is the fact that Schur index of irreducible rational values characters can be different from 1 and so calculations given the papers [14, 17] have to be corrected. A counterexample is as follows:

Example 4. Suppose $SmallGroup(n, i)$ denotes the i -th group of order n in the library of GAP, [15]. Then it is easy to see that $SmallGroup(128, 937)$ is a group isomorphic to $Q_8 \times Q_8$. Consider the characters $\chi_{17}, \chi_{18}, \chi_{19}$ and χ_{20} , see Table 1. Then one can see that these irreducible characters are having Schur index 2. Thus the Q -conjugacy character table of this group can be constructed by the character table in which $2\chi_{17}, 2\chi_{18}, 2\chi_{19}$ and $2\chi_{20}$ are substituted by the corresponding irreducible characters in the parent table.

Table 3: The character table of $G = Q_8 \times Q_8$.

G	$1a$	$2a$	$4a$	$4b$	$4c$	$4d$	$2b$	$2c$	$8a$	$8b$	$4e$	$4f$	$4g$	$4h$	$4i$	$8c$	$4j$	$4k$	$4l$	$4m$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	-1	1	1	1	1	1	1	-1	-1	-1	1	1	1	1	-1	1	1	1	1
χ_3	1	1	-1	1	1	1	1	1	-1	1	1	-1	-1	1	1	-1	-1	-1	-1	1
χ_4	1	-1	-1	1	1	1	1	1	1	-1	-1	-1	-1	1	1	1	-1	-1	-1	1
χ_5	1	1	1	-1	1	1	1	1	1	-1	1	-1	1	-1	1	-1	-1	-1	1	-1
χ_6	1	-1	1	-1	1	1	1	1	-1	1	-1	-1	1	-1	1	1	-1	-1	1	-1
χ_7	1	1	-1	-1	1	1	1	1	-1	-1	1	1	-1	-1	1	1	1	1	-1	-1
χ_8	1	1	-1	-1	1	1	1	1	-1	-1	1	1	-1	-1	1	1	1	1	-1	-1
χ_9	1	-1	-1	-1	1	1	1	1	1	1	-1	1	-1	-1	1	-1	1	1	-1	-1
χ_{10}	2	0	0	2	-2	2	2	2	0	0	0	0	-2	-2	0	0	0	0	0	-2
χ_{11}	2	0	0	2	-2	2	2	2	0	0	0	0	0	-2	-2	0	0	0	0	-2
χ_{12}	2	0	-2	0	2	-2	2	2	0	0	0	0	2	0	-2	0	0	0	2	0
χ_{13}	2	0	0	0	-2	-2	2	2	0	0	0	2	0	0	2	0	-2	-2	0	0
χ_{14}	2	0	0	0	-2	-2	2	2	0	0	0	-2	0	0	2	0	2	2	0	0
χ_{15}	4	2	0	0	0	0	-4	4	0	0	-2	0	0	0	0	0	0	0	0	0
χ_{16}	4	-2	0	0	0	0	-4	4	0	0	2	0	0	0	0	0	0	0	0	0
χ_{17}	4	0	0	0	0	0	0	-4	0	0	0	0	2	-2	0	0	2	-2	-2	2
χ_{18}	4	0	0	0	0	0	0	-4	0	0	0	0	-2	-2	0	0	-2	2	2	2
χ_{19}	4	0	0	0	0	0	0	-4	0	0	0	0	2	2	0	0	-2	2	-2	-2
χ_{20}	4	0	0	0	0	0	0	-4	0	0	0	0	-2	2	0	0	2	-2	2	-2

Theorem 1. [16, Example 1]. *The Schur index of any irreducible character of dihedral group Dih_{2n} is equal to 1.*

Let G be a finite group, χ be an irreducible complex character of G and $m_Q(\chi)$ denote the Schur index of χ over Q . If $\Gamma(\chi)$ denotes the Galois group $Q(\chi)$ over Q then it is well-known that $\sum_{\alpha \in \Gamma(\chi)} m(\chi) \chi^\alpha$ is an irreducible Q -conjugacy character of G [11]. Thus, by knowing the character table of a group and its Schur indices one can find the Q -conjugacy character table of the group.

In [6], Fujita computed the Q -conjugacy character table of cyclic groups. For the sake of completeness, we reprove this result by a new method. To do this, we define the Ramanujan sum $c_n(m)$ as $c_n(m) = \sum_{1 \leq k \leq n, (k,n)=1} e^{\frac{2\pi i k m}{n}}$, where m and n are positive integers.

Theorem 2. *The Q -conjugacy character table of cyclic groups Z_n can be computed by $QCT(Z_n) = [a_{ij}]$, where $a_{i,j} = c_{d_i}(\frac{n}{d_j})$.*

In previous result, it is easy to see that

$$a_{i,j} = \frac{\varphi(d_i) \mu(\frac{d_i}{(n/d_j, d_i)})}{\varphi(\frac{d_i}{(d_i, n/d_j)})},$$

where φ is the Euler function and μ is the Möbius function.

Let $\tau(n)$ denote the number of divisors of n . Then we can sort all divisors in a sequence as $d_1 = 1 < d_2 < \dots < d_{\tau(n)}$. In the following theorem we obtain the Q -conjugacy character table of dihedral group Dih_{2n} .

Theorem 3. *The Q -conjugacy character table of Dih_{2n} is recorded in Tables 4 and 5, when n is odd or even, respectively.*

Table 4: The Q -conjugacy character table of Dih_{2n} , n is Odd.

D_{2n}	K_1	K_j	$K_{\tau(n)+1}$
γ_1	1	1	1
γ_i	$\phi(d_i)$	$c_{d_i}(j)$	0
$\gamma_{\tau(n)+1}$	1	1	-1

Table 5: The Q -conjugacy character table of Dih_{2n} , n is Even.

D_{2n}	K_1	K_j	$K_{\tau(n)+1}$	$K_{\tau(n)+2}$
1	1	1	1	1
γ_i	$\phi(d_i)$	$c_{d_i}(j)$	0	0
$\gamma_{\tau(n)}$	1	$(-1)^j$	-1	1
$\gamma_{\tau(n)+1}$	1	$(-1)^j$	1	-1
$\gamma_{\tau(n)+2}$	1	1	-1	-1

Proof. Suppose $[x]$ denotes the conjugacy class with x as a representative. Our main proof consider two cases as follows:

Case 1. n is odd. In this case the dominant classes are $K_1 = \{[id_G]\}$, $K_{\tau(n)+1} = \{[b]\}$ and $K_j = \{[a^r], [a^{n-r}] : (r, n) = \frac{n}{d_j}\}$. Let $\gamma_1 = \chi_1$, $\gamma_{\tau(n)+1} = \chi_2$ and $\psi_i = \chi_{2+i}$. Then $\gamma_i = \sum_{(j,n)=\frac{n}{d_i}} \psi_j$ where,

$$\begin{aligned}
\gamma_i(a^j) &= \sum_{\substack{k=1 \\ (k,n)=\frac{n}{d_i}}}^{k=[n/2]} \psi_k(a^j) \\
&= \sum_{\substack{k=1 \\ (k,n)=\frac{n}{d_i}}}^{k=[n/2]} \varepsilon^{kj} + \varepsilon^{-kj} \\
&= e \sum_{\substack{k=1 \\ (k,n)=\frac{n}{d_i}}}^{k=[n/2]} \varepsilon^{kj} + \sum_{\substack{k=1 \\ (k,n)=\frac{n}{d_i}}}^{k=[n/2]} \varepsilon^{-kj} \\
&= \sum_{\substack{k=1 \\ (k,n)=\frac{n}{d_i}}}^{k=[n/2]} \varepsilon^{kj} + \sum_{\substack{k=[n/2]+1 \\ (k,n)=\frac{n}{d_i}}}^{k=n} \varepsilon^{-kj} \\
&= \sum_{\substack{k=1 \\ (k,n)=\frac{n}{d_i}}}^{k=n} \varepsilon^{kj} \\
&= \sum_{\substack{k=1 \\ (k,d_i)=1}}^{k=d_i} \varepsilon^{kj} \\
&= c_{d_i}(j).
\end{aligned}$$

Case 2. n is even. In this case, the dominant classes are $K_1 = \{[id_G]\}$, $K_j = \{[a^r], [a^{n-r}] : (r, n) = \frac{n}{d_j}\}$, $K_{\tau(n)+1} = \{[ab]\}$ and $K_{\tau(n)+2} = \{[b]\}$. Let $\gamma_1 = \chi_1$, $\gamma_{\tau(n)+2} = \chi_2$, $\gamma_{\tau(n)+1} = \chi_3$, $\gamma_{\tau(n)} = \chi_4$ and $\psi_i = \chi_{2+i}$. Then $\gamma_i = \sum_{(j,n)=\frac{n}{d_i}} \psi_j$. $\square$

We now consider the following isomorphisms between dihedral groups and the point group symmetries of molecules:

- $D_2 \cong C_{2v} \cong Z_2 \times Z_2 \cong Dih_4$,
- $D_3 \cong C_{3v} \cong Dih_6$,
- $D_4 \cong D_{2d} \cong C_{4v} \cong Dih_8$,
- $D_5 \cong C_{5v} \cong Dih_{10}$,
- $D_{3d} \cong D_{3h} \cong D_6 \cong C_{6v} \cong Dih_{12}$,
- $D_{4d} \cong Dih_{16}$,
- $D_{5d} \cong D_{5h} \cong Dih_{20}$,
- $D_{6d} \cong Dih_{24}$.

Using the previous theorem and above isomorphisms, one can calculate the $\mathbb{Q}$ -conjugacy character table of these molecules. In the end of this paper we compute the $\mathbb{Q}$ -conjugacy character table of $C_{5v} \cong Dih_{10}$ and $D_{4d} \cong Dih_{16}$. Our calculations are recorded in Tables 6 and 7.

Table 6: The Q -conjugacy character table of $C_{5v} \cong Dih_{10}$.

D_{10}	1	a	b
γ_1	1	1	1
γ_2	4	-1	0
γ_3	1	1	-1

Table 7: The Q -conjugacy character table of $D_{4d} \cong D_{16}$.

D_{16}	1	a	a^2	a^4	ab	b
γ_1	1	1	1	1	1	1
γ_2	2	0	-2	2	0	0
γ_3	4	0	0	-4	0	0
γ_4	1	-1	1	1	-1	1
γ_5	1	-1	1	1	1	-1
γ_6	1	1	1	1	-1	-1

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α -NILPOTENT GROUPS DERIVED FROM HYPERGROUPS WITH ξ^* -RELATION

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Abstract. This paper deals with hypergroups, as a generalization of classical groups. An important tool in the theory of hyperstructures is the fundamental relation, which brings us into the classical algebra. In this paper for an automorphism α we introduce and study the construction of α -nilpitent fundamental relation in hypergroups. We will characterize α -nilpitent groups via strongly regular relations and several results on the topic are presented.

Keywords: α -nilpitent groups, strongly regular.

1. Introduction and preliminaries

Nilpotent groups in terms of the certain normal series of subgroups are defined in [2]. This approach demonstrates that there is a connection between nilpotent groups and commutators. In [2] Barzegar and Erfanian introduced the α -nilpitent group and α -commutator and its preliminary properties. Also the relative nilpotent groups with respect to a certain automorphism are discussed.

Let G be a group and $\alpha \in \text{Aut}(G)$, for two elements $x, y \in G$, x and y commutes under the automorphism α whenever $yx = xy^\alpha$. Moreover, $x^{-1}y^{-1}xy^\alpha$ is called α -commutator of x, y and denoted by $[x, y]_\alpha$.

It is clear that if α is the identity automorphism, then we have ordinary commutator. One can define a α -commutator of weight n as follows:

$$[x_1, x_2, \dots, x_n]_\alpha = [x_1, [x_2, \dots, x_n]_\alpha]_\alpha.$$

Let X_1 and X_2 be two non-empty subsets of a group G . The α -commutator subgroup of X_1 and X_2 is defined as follows:

$$[X_1, X_2]_\alpha = \langle [x_1, x_2]_\alpha : x_1 \in X_1, x_2 \in X_2 \rangle.$$

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It is obvious that $[X_1, X_2]_\alpha$ is not equal to $[X_2, X_1]_\alpha$ in general. Barzegar and Erfanian introduced α -center of the group G as follows:

$$Z^\alpha(G) = \{y \in G : [x, y]_\alpha = 1 \text{ for all } x \in G\}.$$

Let N be a normal subgroup of G and $N^\alpha = N$. Then $(xN)^{\bar{\alpha}} = x^\alpha N$, where $\bar{\alpha} : G/N \Rightarrow G/N$. It is trivial $[\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n]_{\bar{\alpha}} = [x_1, x_2, \dots, x_n]_\alpha N$.

Definition 1.1. A group G is called α -nilpotent if it has a α -central series, that is, a normal series $\{1\} = G_0 \trianglelefteq G_1 \trianglelefteq \dots \trianglelefteq G_n = G$, such that G_{i+1}/G_i is contained in the $\bar{\alpha}$ -center of G/G_i for all i . The length of a shortest α -central series of G is called relative nilpotency class of G . If G is α -nilpotent of class c_α , then it is nilpotent of class at most c_α .

For the group G and $\alpha \in \text{Aut}(G)$, a *lower central α -series* is the following normal series

$$G = \Gamma_1^\alpha(G) \supseteq \Gamma_2^\alpha(G) \supseteq \dots$$

where, $\Gamma_2^\alpha(G) = \langle [x, y]_\alpha \mid x, y \in G \rangle$ and for $n \geq 1$, $\Gamma_{n+1}^\alpha(G) = [G, \Gamma_n^\alpha(G)]_\alpha$. Moreover, $\Gamma_n^\alpha(G/N) = \Gamma_n^\alpha(G)N/N$, where $N^\alpha = N$.

Also an *upper α -central series* is the following normal series

$$1 \subseteq Z_0^\alpha(G) \trianglelefteq Z_1^\alpha(G) \trianglelefteq \dots,$$

where $Z_1^\alpha(G) = Z^\alpha(G)$ and for $i \geq 1$, $\frac{Z_i^\alpha}{Z_{i-1}^\alpha} = Z^\alpha(\frac{G}{Z_{i-1}^\alpha(G)})$. Clearly, $Z_i^\alpha(G) \trianglelefteq G$ and in general, these two series will not stop, but if so, we will prove that G is α -nilpotent and its converse is valid. Thus we find equivalent definitions for a α -nilpotent group G .

Theorem 1.2 ([2]). Suppose G is a group and $\alpha \in \text{Aut}(G)$. Then

- 1) $Z_n^\alpha(G) = Z_n^{\alpha^{-1}}(G)$;
- 2) $x \in Z_n^\alpha(G)$ if and only if $[g_1, \dots, g_n, x]_\alpha = 1$ for all $g_1, g_2, \dots, g_n \in G$;
- 3) $Z_n^\alpha(G) \subseteq Z_n(G)$.

Theorem 1.3 ([2]). For a group G the following is equivalent.

- 1) G is α -nilpotent;
- 2) There is an integer s such that $Z_s^\alpha(G) = G$.

Hyperstructure theory was first initiated by Marty [11] in 1934 when he defined hypergroups and started to analyze their properties. Since there are extensive application in many branches of mathematics and applied sciences, the theory of algebraic hyperstructures has nowadays become a well-established branch in algebraic theory. Some investigations of the theory hyperstructures are accessible in the book of Corsini [3], Davvaz and Leoreanu-Fotea [5], Corsini and Leoreanu [4] and Vougiouklis [12]. A *hyperstructure (or hypergroupoid)* is a nonempty set H with a *hyperoperation* $\circ$ defined on H , that is, a mapping of $H \times H$ into the family of non-empty subsets of H . If $(x, y) \in H \times H$, its

image under $\circ$ is denoted by $x \circ y$. If A, B are non-empty subsets of H then $A \circ B$ is given by $A \circ B = \bigcup \{x \circ y \mid x \in A, y \in B\}$. $x \circ A$ is used for $\{x\} \circ A$ and $A \circ x$ for $A \circ \{x\}$. Generally, the singleton a is identified with its member a . The structure $(H, \circ)$ is called a *semihypergroup* if $a \circ (b \circ c) = (a \circ b) \circ c$ for all $a, b, c \in H$, and a semihypergroup $(H, \circ)$ is called a *hypergroup* in the sense of Marty if

$$x \circ H = H \circ x = H, \forall x \in H,$$

which is called the *reproduction axiom*. This axiom means that for any $x, y \in H$ there exist $u, v \in H$ such that $y \in x \circ u$ and $y \in v \circ x$.

Definition 1.4. A hypergroup $(P, \circ)$ is said to be a *polygroup* if the following are satisfied:

- (i) P has an identity element, that is there exists an element $e \in P$, such that $x \in e \circ x \cap x \circ e$ for all $x \in P$;
- (ii) every element $x \in P$ has inverse, that is there exists $x^{-1} \in P$, such that $e \in x \circ x^{-1} \cap x^{-1} \circ x$;
- (iii) for all $x, y, z \in P$:

$$x \in y \circ z \Leftrightarrow y \in x \circ z^{-1} \Leftrightarrow z \in y^{-1} \circ x.$$

A function $f : (H, \circ) \rightarrow (H, *)$ is called a *homomorphism* if $f(a \circ b) \subseteq f(a) * f(b)$ for all $a, b \in H$. We say that f is a *good homomorphism* if for all a and b in H , $f(a \circ b) = f(a) * f(b)$. Let $(H, \circ)$ be a hypergroup and $\rho \subseteq H \times H$ be an equivalence relation. For non-empty subsets A and B of H , we define

$$A\bar{\rho}B \iff a\rho b, \forall a \in A, \forall b \in B.$$

The relation ρ is called *strongly regular on the left (on the right)* if $x\bar{\rho}y \implies a \circ x\bar{\rho}a \circ y(x \circ a\bar{\rho}y \circ a, \text{ respectively})$, for all $x, y, a \in H$.

Moreover, ρ is called *strongly regular* if it is strongly regular on the right and on the left.

Theorem 1.5 ([3]). *If $(H, \cdot)$ is a hypergroup and ρ is a strongly regular relation on H , then H/ρ is a group under the operation:*

$$\rho(x) \otimes \rho(y) = \rho(z), \forall z \in x \cdot y.$$

For all $n \geq 1$, we define the relation β_n on a semihypergroup H , as follows: $a\beta_nb \iff \exists(x_1, \dots, x_n) \in H^n : \{a, b\} \subseteq \prod_{i=1}^n x_i$, and $\beta = \cup_{n \geq 1} \beta_n$, where $\beta_1 = \{(x, x); x \in H\}$, is the diagonal relation on H . This relation was introduced by Koskas [9]. Suppose that β^* is the *transitive closure* of β , the relation β^* is a *strongly regular relation* [3]. Also, we have:

Theorem 1.6 ([6]). *If H is hypergroup then $\beta = \beta^*$.*

Freni in [7] introduced a new fundamental relation $\gamma = \cup_{n \geq 1} \gamma_n$, where γ_1 is the diagonal relation and for every integer $n > 1$, γ_n is the relation defined as follows:

$$x\gamma_n y \iff \exists(z_1, \dots, z_n) \in H^n, \exists \tau \in \mathbb{S}_n : x \in \prod_{i=1}^n z_i, y \in \prod_{i=1}^n z_{\tau(i)},$$

where $\mathbb{S}_n$ is the *symmetric group* of order n . Suppose that γ^* is the transitive closure of γ . The relation γ^* is a strongly regular relation [7].

The relation β^* is the least equivalence relation on a hypergroup H , such that the quotient H/β^* is a group, while γ^* is the least equivalence relation on a hypergroup H , such that the quotient H/γ^* is an Abelian group.

As the fundamental relations play an important role in the study of theory of algebraic hyperstructures, it has been studied by many authors (for more details see [8], [3], [13] and [7]). As it is well known that abelian groups are contained in α -nilpotent groups. The smallest equivalence relation γ^* on a hypergroup H such that the quotient H/γ^* , the set of all equivalence classes, is an Abelian group was introduced in [7]. Now in this paper we introduce and analyze a new strongly regular relation ξ^* on a hypergroup H such that the quotient group H/ξ^* is an α -nilpotent group. Also, we study the relationship between α -nilpotent hypergroups and Abelian hypergroups. In particular, we will characterize α -nilpotent groups via strongly regular relations and we obtain several results on the topic.

2. Construct of α -nilpotent groups by a new strongly regular relation ξ_n^*

Now in this paper we introduce and analyze a new strongly regular relation ξ^* on a hypergroup H such that the quotient group H/ξ^* is an α -nilpotent group. Let $\alpha \in \text{Aut}(P)$ for two elements $x, y \in P$, we define α -commutator of x, y as follows:

$$[x, y]_\alpha = \{t | t \in xyx^{-1}y^{-\alpha}\}.$$

Definition 2.1. Let H be a hypergroup and $\alpha \in \text{Aut}(H)$. We define:

- (1) $L_1^\alpha(H) = H$;
- (2) $L_{k+1}^\alpha(H) = \{h \in H | yx \cap hx^\alpha y \neq \emptyset, x \in L_k^\alpha(H), y \in H\}, \forall k \geq 1$.

Theorem 2.2. Let P be a polygroup. Then for all x, y, h in P

$$\{h | yx \cap hx^\alpha y \neq \emptyset\} = \{h | h \in yxy^{-1}x^{-\alpha}\} = [y, x]_\alpha.$$

Proof. Let $h \in \{h | yx \cap hx^\alpha y \neq \emptyset\}$. Then there is $z \in yx \cap hx^\alpha y$, which implies that $h \in z(x^\alpha y)^{-1} \subseteq yxy^{-1}x^{-\alpha}$. Thus $h \in \{h | h \in [y, x]_\alpha\}$. Conversely if $h \in \{h | h \in yxy^{-1}x^{-\alpha}\}$, then $h \in (yx)(x^\alpha y)^{-1}$. Therefore $h \in sr$ for some $s \in (yx)$, $r \in (x^\alpha y)^{-1}$. Consequently $h \in s(x^\alpha y)^{-1}$ implies that $s \in h(x^\alpha y)$. Also $s \in yx$. Hence $yx \cap hx^\alpha y \neq \emptyset$. $\square$

By the above theorem it is obvious that for a polygroup P we have $L_{k+1}^\alpha(P) = \{h \in P \mid h \in [y, x]_\alpha, y \in P, x \in L_k^\alpha\}$

Let $n \in \mathbb{N}$ and $\xi_n = \bigcup_{m \geq 1} \xi_{m,n}$, where $\xi_{1,n}$ is the diagonal relation and for every integer $m \geq 1$, and $\xi_{m,n}$ is the relation defined as follows:

$$x\xi_{m,n}y \iff \exists(z_1, \dots, z_m) \in H^m; \exists \delta \in \mathbb{S}_m : \\ \delta(i) = i \text{ if } z_i \notin L_n^\alpha(H) \text{ such that } x \in \prod_{i=1}^m z_i, y \in \prod_{i=1}^m z_{\delta(i)}.$$

Obviously, for every $n \geq 1$, the relation ξ_n is reflexive and symmetric. Now let ξ_n^* be the transitive closure of ξ_n .

Corollary 2.3. *For every $n \in \mathbb{N}$, we have $\beta^* \subseteq \xi_n^* \subseteq \gamma^*$.*

Theorem 2.4. *For every $n \in \mathbb{N}$, the relation ξ_n^* is a strongly regular relation.*

Proof. Let $n \in \mathbb{N}$. It is obvious that ξ_n^* is an equivalence relation. Now we show that if $x\xi_n y$, then $x.z\xi_n^*y.z$ and $z.x\xi_n^*z.y$. For every $z \in H$, $x\xi_n y$ implies that $m \in \mathbb{N}$ such that $x\xi_{m,n}y$. So there exist $(z_1, z_2, \dots, z_m) \in H^m, \sigma \in \mathbb{S}_m$ with $\sigma(i) = i$ if $z_i \notin L_n^\alpha(H)$, such that $x \in \prod_{i=1}^m z_i, y \in \prod_{i=1}^m z_{\sigma(i)}$. Therefore, for $z \in P$, $x.z \in \prod_{i=1}^m z_i.z, y.z \in \prod_{i=1}^m z_{\sigma(i)}.z$ and $\sigma(i) = i$ if $z_i \notin L_n^\alpha(H)$. Now suppose that $z_{m+1} = z$. We define the permutation $\sigma' \in \mathbb{S}_{m+1}$, with $\sigma'(i) = \sigma(i)$, for all $1 \leq i \leq m$ and $\sigma'(m+1) = m+1$. This implies that $x.z \subseteq \prod_{i=1}^{m+1} z_i.z, y.z \subseteq \prod_{i=1}^{m+1} z_{\sigma'(i)}.z$ such that $\sigma'(i) = i$ if $z_i \notin L_n^\alpha(P)$. Therefore, $x.z\xi_n^*y.z$. Similarly we have $z.x\xi_n^*z.y$. Now, if $x\xi_n^*y$, then there exists $k \in \mathbb{N}$ and $(x = u_0, u_1, \dots, u_k = y) \in P^{k+1}$ such that $x = u_0\xi_n u_1\xi_n u_2 \dots \xi_n u_{k-1}\xi_n u_k = y$. Hence by the above result we have $x.z = u_0.z\xi_n^*u_1.z\xi_n^*u_2.z \dots \xi_n^*u_{k-1}.z\xi_n^*u_k.z = y.z$ and so $x.z\xi_n^*y.z$. Similarly we can prove that $z.x\xi_n^*z.y$ and the proof completes. $\square$

Proposition 2.5. *For every $n \in \mathbb{N}$, we have $\xi_{n+1}^* \subseteq \xi_n^*$.*

Proof. Let $x\xi_{n+1}y$ so $\exists(z_1, \dots, z_m) \in H^m; \exists \delta \in \mathbb{S}_m : \delta(i) = i$ if $z_i \notin L_{n+1}^\alpha(H)$, such that $x \in \prod_{i=1}^m z_i, y \in \prod_{i=1}^m z_{\delta(i)}$. Now let $\delta_1 = \delta$, since $L_{n+1}^\alpha(H) \subseteq L_n^\alpha(H)$ so $x\xi_n y$. $\square$

The next result immediately follows from previous theorem.

Corollary 2.6. *If P is a commutative hypergroup, then $\beta^* = \xi_n^* = \gamma^*$.*

Theorem 2.7. *If H is a hypergroup and φ is a strongly regular relation on H , then $L_{k+1}^{\bar{\alpha}}(H/\varphi) = \{[\bar{s}, \bar{t}]; t \in L_k^\alpha(H), s \in H\}$.*

Proof. The proof is based on induction. Put $G = H/\varphi$ and $\bar{x} = \varphi(x)$, for all $x \in H$ and $\bar{\alpha} \in \text{Aut}(G)$. If $k = 1$, then $L_2^{\bar{\alpha}}(G) = \{[\bar{s}, \bar{t}]_{\bar{\alpha}} \mid t \in L_1^\alpha(H), s \in H\}$. Now, put $\bar{a} = [\bar{s}, \bar{t}]_{\bar{\alpha}}$ where $t \in L_{k+1}^\alpha(H), s \in H$, so there exist $x \in L_k^\alpha(H)$ and

$y \in H$ such that $yx \cap tx^\alpha yt \neq \emptyset$. Then $\bar{t} = [\bar{y}, \bar{x}]_{\bar{\alpha}}$. By induction hypotheses we have $\bar{t} \in L_{k+1}^{\bar{\alpha}}(G)$. Hence $\bar{a} = [\bar{s}, \bar{t}]_{\bar{\alpha}} \in L_{k+2}^{\bar{\alpha}}(G)$. Conversely, let $\bar{a} \in L_{k+2}^{\bar{\alpha}}(G)$. Then $\bar{a} = [\bar{y}, \bar{x}]_{\bar{\alpha}}$, where $\bar{x} \in L_{k+1}^{\bar{\alpha}}(G)$ and $\bar{y} \in G$. So induction hypotheses implies that $\bar{x} = [\bar{v}, \bar{u}]_{\bar{\alpha}}$, where $u \in L_k^\alpha(H)$ and $v \in H$. Since H is a hypergroup there exists $t \in H; vu \cap tuv \neq \emptyset$ such that $\bar{t} = [\bar{v}, \bar{u}]_{\bar{\alpha}} = \bar{x}$ and $t \in L_{k+1}^\alpha(H), y \in H$. Hence, $\bar{a} = [\bar{y}, \bar{x}]_{\bar{\alpha}} = [\bar{y}, \bar{t}]_{\bar{\alpha}} \in \{[\bar{s}, \bar{t}]; t \in L_k^\alpha(H), s \in H\}$. $\square$

Theorem 2.8. H/ξ_n^* is a α -nilpotent group.

Proof. Let $G = H/\xi_n^*$ and $\bar{x} = \xi_n^*(x)$, for all $x \in H$. We show that $L_{n+1-i}^{\bar{\alpha}}(G) \subseteq Z_i^{\bar{\alpha}}(G)$ for all s . Let $i = 0$ then $L_{n+1}^{\bar{\alpha}}(G) \subseteq Z_0^{\bar{\alpha}}(G) = \{\bar{e}\}$. Then $L_{n+1}^{\bar{\alpha}}(G) = \{\bar{e}\}$.

Now let $\bar{a} \in L_{n+1-i-1}^{\bar{\alpha}}(G)$. we show that $a \in Z_{i+1}^{\bar{\alpha}}(G)$. Since $\bar{a} \in L_{n+1-i}^{\bar{\alpha}}(G)$, so $[\bar{s}, \bar{a}]_{\bar{\alpha}} \in L_{n+1-i}^{\bar{\alpha}}(G)$. By hypotheses of induction $L_{n+1-i}^{\bar{\alpha}}(G) \subseteq Z_i^{\bar{\alpha}}(G)$. Thus by theorem 1.2 $[\bar{s}_i, \dots, \bar{s}_1, [\bar{s}, \bar{a}]_{\bar{\alpha}}]_{\bar{\alpha}} = \bar{e}$. Thus $\bar{a} \in Z_{i+1}^{\bar{\alpha}}(G)$. Now let $i = n$ so $L_1^\alpha(G) \subseteq Z_n^{\bar{\alpha}}(G)$. Therefore $G = Z_n^{\bar{\alpha}}(G)$. Hence G is α -nilpotent as desired. $\square$

Example 2.9. Let $H = \{e, a, b, c, d, f, g\}$. Consider the non-commutative hypergroup $(H, \cdot)$, where $\cdot$ is defined on H as follows:

.	e	a	b	c	d	f	g
e	e	a	b	c	d	f, g	f, g
a	a	e	d	f, g	b	c	c
b	b	f, g	e	d	c	a	a
c	c	d	f, g	e	a	b	b
d	d	c	a	b	f, g	e	e
f	f, g	b	c	a	e	d	d
g	f, g	b	c	a	e	d	d

Then $H/\beta^* \cong S_3$ (for more details see [3]). Since S_3 is not α -nilpotent and $H/\xi_n^* \subseteq H/\beta^* \cong (S_3)$. It concluded that $\xi_n^* \neq \beta^*$.

3. On α -nilpotent groups derived from finite polygroups

In this section we try to construct α -nilpotent group from a given hypergroup H and the smallest strongly regular relation on H . Let H be a finite hypergroup. Then we define the relation ξ^* on H by

$$\xi^* = \bigcap_{n \geq 1} \xi_n^*.$$

Theorem 3.1. The relation ξ^* is a strongly regular relation on a finite hypergroup H such that H/ξ^* is a α -nilpotent group.

Proof. Since ξ^* is a strongly regular relation on H we have $\xi^* = \bigcap_{n \geq 1} \xi_n^*$. Now, by Proposition 2.5, there exists $k \in \mathbb{N}$ such that $\xi_{k+1}^* = \xi_k^*$. Hence $\xi^* = \xi_k^*$ for some $k \in \mathbb{N}$. This complete the proof. $\square$

Theorem 3.2. *The relation ξ^* is the smallest strongly regular relation on a finite hypergroup H such that H/ξ^* is a α -nilpotent group.*

Proof. Suppose ρ is a strongly regular relation on H such that $K = H/\rho$ is a α -nilpotent group. If $x\xi y$, then there exists $n \in \mathbb{N}$ such that $x\xi_n y$. Therefore, for some $m \in \mathbb{N}$, $x\xi_{mn} y$ if and only if there exists $(z_1, \dots, z_m) \in H^m$ and $\delta \in \mathbb{S}_m$ such that $\delta(i) = i$ if $z_i \notin L_n^\alpha(H)$ where $x \in \prod_{i=1}^m z_i$, $y \in \prod_{i=1}^m z_{\delta(i)}$. Therefore, $L_{c+1}^\alpha(H/\rho) = \{[\rho(t), \rho(s)] \mid t \in L_c^\alpha(H), s \in H\} = \{\rho(e)\}$. and so for every $z_i \in L_c^\alpha(H)$ and $v \in L_c^\alpha H$, $\rho(z_i)\rho(v) = \rho(v)\rho(z_i)$. This implies that $\rho(x) = \rho(y)$ so $x\rho y$. $\square$

4. Some important properties of ξ^*

Definition 4.1. Let X be a non-empty subset of H . Then we say that X is a ξ -part of H if for every $k \in \mathbb{N}$ and $(z_1, \dots, z_k) \in H^k$ and for every $\sigma \in \mathbb{S}_k$ such that $\sigma(i) = i$ if $z_i \notin \cup_{n \geq 1} L_n^\alpha(H)$, then

$$\prod_{i=1}^k z_i \cap X \neq \emptyset \implies \prod_{i=1}^k z_{\sigma(i)} \subseteq X.$$

Theorem 4.2. *Let X be a non-empty subset of a hypergroup H . Then the following conditions are equivalent:*

- 1) X is a ξ -part of H ,
- 2) $x \in X$, $x\xi y \implies y \in X$,
- 3) $x \in X$, $x\xi^* y \implies y \in X$.

Proof. (1 $\implies$ 2): if $(x, y) \in H^2$ is a pair such that $x \in X$, $x\xi y$, then there exist $(z_1, \dots, z_k) \in H^k$; $x \in \prod_{i=1}^k z_i \cap X$, $y \in \prod_{i=1}^k z_{\sigma(i)}$ and $\sigma(i) = i$ if $z_i \notin \cup_{n \geq 1} L_n^\alpha(H)$. Since X is a ξ -part of H , we have $\prod_{i=1}^k z_{\sigma(i)} \subseteq X$ and so $y \in X$.

(2 $\implies$ 3): Suppose that $(x, y) \in H^2$ is a pair such that $x \in X$ and $x\xi^* y$. Then there is $(z_1, \dots, z_k) \in H^k$ such that $x = z_0 \xi z_1 \xi \dots \xi z_k = y$. Now by using (2) k times we obtain $y \in X$.

(3 $\implies$ 1): Suppose that $x \in \prod_{i=1}^k z_i \cap X$ and $\sigma \in \mathbb{S}_k$ such that $\sigma(i) = i$ if $z_i \notin \cup_{n \geq 1} L_n^\alpha(H)$. Let $y \in \prod_{i=1}^k z_{\sigma(i)}$. Since $x\xi y$, by (3) we have $y \in X$. Consequently, $\prod_{i=1}^k z_{\sigma(i)} \subseteq X$ and so X is a ξ -part. $\square$

Theorem 4.3. *The following conditions are equivalent:*

- (1) for every $a \in H$, $\xi(a)$ is a ξ -part of H ,
- (2) ξ is transitive.

Proof. (1 $\implies$ (2): Suppose that $x\xi^* y$. Then there is $(z_1, \dots, z_k) \in H^k$ such that $x = z_0 \xi z_1 \xi \dots \xi z_k = y$ since $\xi(z_i)$ for all $0 \leq i \leq k$, is a ξ -part, we have $z_i \in \xi(z_{i-1})$, for all $1 \leq i \leq k$. Thus $y \in \xi(x)$, which means that $x\xi y$.

(2 $\implies$ (1): Suppose that $x \in H$, $z \in \xi(x)$ and $z\xi y$. By transitivity of ξ , we have $y \in \xi(x)$. Now according to the last theorem, $\xi(x)$ is a ξ -part of H . $\square$

Definition 4.4. The intersection of all ξ -parts which contain A is called ξ -closure of A in H and it will be denoted by $K(A)$.

In what follows, we determine the set $W(A)$, where A is a non-empty subset of H . We set

- 1) $W_1(A) = A$ and
 - 2) $W_{n+1}(A) = \{x \in H \mid \exists(z_1, \dots, z_k) \in H^k, x \in \prod_{i=1}^k z_{\sigma(i)}, \exists \sigma \in \mathbb{S}_k \text{ such that } \sigma(i) = i, \text{ if } z_i \notin \cup_{t \geq 1} L_t^\alpha(H) \text{ and } \prod_{i=1}^k z_{\sigma(i)} \cap W_n(A) \neq \emptyset\}$.
- We denote $W(A) = \bigcup_{n \geq 1} W_n(A)$.

Theorem 4.5. For any non-empty subset of H , the following statements hold:

- 1) $W(A) = K(A)$,
- 2) $K(A) = \bigcup_{a \in A} K(a)$.

Proof. 1) It is enough to prove:

- (a) $W(A)$ is a ξ -part,
- (b) if $A \subseteq B$ and B is a ξ -part, then $W(A) \subseteq B$.

In order to prove (a), suppose that $\prod_{i=1}^k z_i \cap W(A) \neq \emptyset$ and $\sigma \in \mathbb{S}_k$ such that $\sigma(i) = i$, if $z_i \notin \cup_{n \geq 1} L_n^\alpha(H)$. Therefore, there exist $n \in \mathbb{N}$ such that $\prod_{i=1}^k z_i \cap W_n(A) \neq \emptyset$ where it follows that $\prod_{i=1}^k z_{\sigma(i)} \subseteq W_{n+1}(A) \subseteq W(A)$. Now, we prove (b) by induction on n . We have $W_1(A) = A \subseteq B$. Suppose that $W_n(A) \subseteq B$. We prove that $W_{n+1}(A) \subseteq B$. If $z \in W_{n+1}(A)$, then $z \in \prod_{i=1}^k z_i$ and there exists $\sigma \in \mathbb{S}_k$ such that $\sigma(i) = i$, if $z_i \notin \cup_{t \geq 1} L_t^\alpha(H)$ and also $\prod_{i=1}^k z_{\sigma(i)} \cap W_n(A) \neq \emptyset$. Therefore, $\prod_{i=1}^k z_{\sigma(i)} \cap B \neq \emptyset$ and hence $z \in \prod_{i=1}^k z_i \subseteq B$.

2) It is clear that for all $a \in A$, $K(a) \subseteq K(A)$. By part (1), we have $K(A) = \bigcup_{n \geq 1} W_n(A)$ and $W_1(A) = A = \bigcup_{a \in A} \{a\}$. It is enough to prove that $W_n(A) = \bigcup_{a \in A} W_n(a)$, for all $n \in \mathbb{N}$. We follow by induction on n . Suppose it is true for n . We prove that $W_{n+1}(A) = \bigcup_{a \in A} W_{n+1}(a)$. If $z \in W_{n+1}(A)$, then $z \in \prod_{i=1}^k z_i$ and there exists $\sigma \in \mathbb{S}_k$ such that $\sigma(i) = i$, if $z_i \notin \cup_{t \geq 1} L_t^\alpha(H)$ and also $\prod_{i=1}^k z_{\sigma(i)} \cap W_n(A) \neq \emptyset$. By the hypotheses of induction $\prod_{i=1}^k z_{\sigma(i)} \cap W_n(a') \neq \emptyset$, for some $a' \in A$. Therefore, $z \in W_{n+1}(a')$, and so $W_{n+1}(A) \subseteq \bigcup_{a \in A} W_{n+1}(a)$. Hence $K(A) = \bigcup_{a \in A} K(a)$. $\square$

Theorem 4.6. The following relation is equivalence relation on H .

$$xWy \iff x \in W(y),$$

for every $(x, y) \in H^2$, where $W(y) = W(\{y\})$.

Proof. It is easy to see that W is reflexive and transitive. We prove that W is symmetric. To this end, we check that:

- 1) for all $n \geq 2$ and $x \in H$, $W_n(W_2(x)) = W_{n+1}(x)$,
- 2) $x \in W_n(y)$ if and only if $y \in W_n(x)$.

We prove 1) by induction on n . Suppose that $z \in W_2(W_2(x))$. Then $z \in \prod_{i=1}^k z_i$ and there is $\sigma \in \mathbb{S}_k$ such that $\sigma(i) = i$, if $z_i \notin \cup_{t \geq 1} L_t^\alpha(H)$ and also $\prod_{i=1}^k z_{\sigma(i)} \cap W_2(x) \neq \emptyset$. Thus $z \in W_3(x)$. If $z \in W_{n+1}(W_2(x))$, then $z \in \prod_{i=1}^k z_i$

and there exist $\sigma \in \mathbb{S}_k$ such that $\sigma(i) = i$, if $z_i \notin \cup_{t \geq 1} L_t^\alpha(H)$ and also $\prod_{i=1}^k z_{\sigma(i)} \cap W_n(W_2(x)) \neq \emptyset$. By hypotheses of induction, we have $\prod_{i=1}^k z_{\sigma(i)} \cap W_{n+1}(x) \neq \emptyset$ and so $z \in W_{n+2}(x)$. Now we prove (2) by induction on n , too. It is clear that $x \in W_2(y)$ if and only if $y \in W_2(x)$. Then $x \in \prod_{i=1}^k z_i$ and there exists $\sigma \in \mathbb{S}_k$ such that $\sigma(i) = i$, if $z_i \notin \cup_{t \geq 1} L_t^\alpha(H)$ and also $\prod_{i=1}^k z_{\sigma(i)} \cap W_n(y) \neq \emptyset$. Suppose that $b \in \prod_{i=1}^k z_{\sigma(i)} \cap W_n(y)$. Then we have $y \in W_n(b)$. From $x \in \prod_{i=1}^k z_i \cap W_1(x)$ and $b \in \prod_{i=1}^k z_{\sigma(i)}$ we conclude that $b \in W_2(x)$. Therefore, $y \in W_n(W_2(x)) = W_{n+1}(x)$. $\square$

Definition 4.7. Let H be a hypergroup and $p : H \rightarrow H/\xi$ be the canonical projection. We denote by 1 the identity of the group H/ξ , the set $\rho^{-1}(1)$ is called the ξ -part of H and it is denoted by α_ξ .

Theorem 4.8. α_ξ is the smallest subhypergroup of H , which is also a ξ -part of H .

Proof. First we show that α_ξ is a subhypergroup of H . If $x, y \in \alpha_\xi$ and $z \in x \cdot y$, then $\xi(z) = \xi(x)\xi(y) = 1$, so $z \in \alpha_\xi$. On the other hand, there exists $u \in H$ such that $x \in u \cdot y$ and so $1 = \xi(x) = \xi(u)\xi(y) = \xi(u)$. Therefore, $u \in \alpha_\xi$. This means that $\alpha_\xi y = \alpha_\xi$, for all $y \in H$. Similarly, $y\alpha_\xi = \alpha_\xi$ which follows that α_ξ is a subhypergroup of H . Now we prove that $K(x) = \rho^{-1}(\rho(x)) = \alpha_\xi x = x\alpha_\xi$, for every $x \in H$. Indeed we have $z \in \rho^{-1}(\rho(x))$ if and only if $\rho(z) = \rho(x)$ which means $z\xi x$. Therefore, $z \in V(x) = K(x)$. Hence, $K(x) = \rho^{-1}(\rho(x))$. It is easy to see that $\rho^{-1}(\rho(x)) = \alpha_\xi x = x\alpha_\xi$. From the above we obtain if $x \in \alpha_\xi$, then $K(x) = \alpha_\xi$ which means that α_ξ is a ξ -part of H . Moreover, if L is a subhypergroup of H which is also a ξ -part, then $L = K(L) = \bigcup_{a \in L} K(a) = \bigcup_{a \in L} \alpha_\xi a = \alpha_\xi L$. Thus $\alpha_\xi \subseteq L$. $\square$

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COMBINED EFFECT OF MAGNETISM AND ROUGHNESS ON A FERROFLUID SQUEEZE FILM IN POROUS TRUNCATED CONICAL PLATES: EFFECT OF VARIABLE BOUNDARY CONDITIONS

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Abstract. This article aims to discuss the performance of a ferrofluid squeeze film between transversely rough porous truncated conical plates resorting to special type of boundary conditions depending on the magnetization parameter. Invoking the stochastic averaging model of Christensen and Tonder regarding the roughness characterization, the associated stochastically averaged Reynolds type equation is solved to get the pressure distribution, in turn, which gives the load carrying capacity. The results affirm that suitable boundary condition may help in scaling down the adverse effect of roughness to a large extent appropriately choosing the magnetization parameter. However, in the case of negatively skewed roughness the situation remains relatively better. It is also found that the absence of flow doesn't deter the bearing system from supporting certain amount of load, which is very much unlikely in the case of conventional lubricant based bearing system.

Keywords: squeeze film, truncated conical plates, porosity, magnetization, roughness.

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1. Introduction

Wu [1] examined the effect of porosity on squeeze film behaviour in annular irrotational disks. The load carrying capacity was found to be reduced due to the porosity. Gupta and Vora [2] analyzed the squeeze film behaviour between curved annular plates. The curvature was found to have considerable influence on the performance of the squeeze film. Lin et. al. [3] extended the configuration of [2] to discuss the magneto hydrodynamic squeeze film characteristics between curved annular plates. Patel and Deheri [4] investigated the configuration of [2] by considering the lower plate as well as the upper plate along the surfaces generated by hyperbolic function. Subsequently, Patel and Deheri [5] modified the approach to consider both plates along the surfaces determined by secant functions. The investigations of [4], [5] confirmed that the magnetization had a significantly positive effect on the squeeze film performance in annular plates. Use of magnetic fluid as a lubricant modifying the performance of the bearing system has been very well recognized. Bhat and Deheri [6], [7] analyzed the performance of a magnetic fluid based squeeze film behaviour between curved annular disks and curved circular plates and found that the performance with the magnetic fluid as lubricant was relatively better than with a conventional lubricant.

It is well known that bearing surfaces particularly, after having some run in and wear develop roughness. Various methods have been proposed to study and analyze the effect of surface roughness of the bearing surfaces on the performance of squeeze film bearings. Several investigators have adopted a stochastic approach to mathematically model the randomness of the roughness. A comprehensive general analysis was presented by Christensen and Tonder [8, 9, 10] for surface roughness (both transverse as well as longitudinal) based on a general probability density function. Later on, this approach of Christensen and Tonder [8, 9, 10] laid down the basis of the analysis to discuss the effect of surface roughness on the performance of the bearing systems in a number of investigations. Ting [11] discussed the engagement behaviour of lubricated porous annular disks by considering the effect of surface roughness on the squeeze film. The roughness significantly affected the performance characteristics. Gupta and Deheri [12] studied the effect of transverse surface roughness on the squeeze film performance in a spherical bearing.

Prakash and Vij [13] investigated the load carrying capacity and time height relation for squeeze films between porous plates. Circular, annular, elliptic, rectangular, conical and truncated conical plates were investigated for the squeeze film performance. Deheri et. al. [14] considered the ferrofluid based squeeze film between rough porous truncated conical plates. The negatively skewed roughness provided a better performance for this type of bearing system. Wiercholski and Miszczak [15] presented a method of friction calculation in slide conical micro-bearing occurring in hard disk drives computer disks. Andharia and Deheri [16] analyzed the effect of longitudinal surface roughness on the ferrofluid

based squeeze film between conical plates. The performance of the bearing system was observed to be little better in this case as compared to the case of transverse surface roughness. Vadher et. al. [17] investigated the effect of transverse surface roughness on the performance of hydromagnetic squeeze film between conducting truncated conical plates. This article confirmed that for suitable values of aspect ratio and conductivities, the magnetization parameter offered some measures to counter the adverse effect of porosity and standard deviation associated with roughness.

Deheri et. al. [18] dealt with the behavior of a ferrofluid squeeze film in porous rough conical plates. A suitable combination of magnetization parameter and semi vertical angle of the cone presented a better performance in the case of negatively skewed roughness. Shimpi and Deheri [19] discussed the combined effect of slip velocity and bearing deformation on the behavior of a ferrofluid based squeeze film in rough porous truncated conical plates. For an overall improved performance this article confirmed that the slip velocity was required to be kept at minimum. Same was the case for bearing deformation. Hsu et. al. [20] presented a theoretical study of non-Newtonian effects in conical squeeze film plates that was based on the Rabinowitsch fluid model. The non-Newtonian effect provided better load carrying capacity and lengthened response time.

Here, it has been proposed to analyze the performance characteristics of a transversely rough ferrofluid squeeze film in porous truncated conical plates, taking recourse to a new set of boundary conditions.

2. Analysis

The configuration of the bearing system is presented below

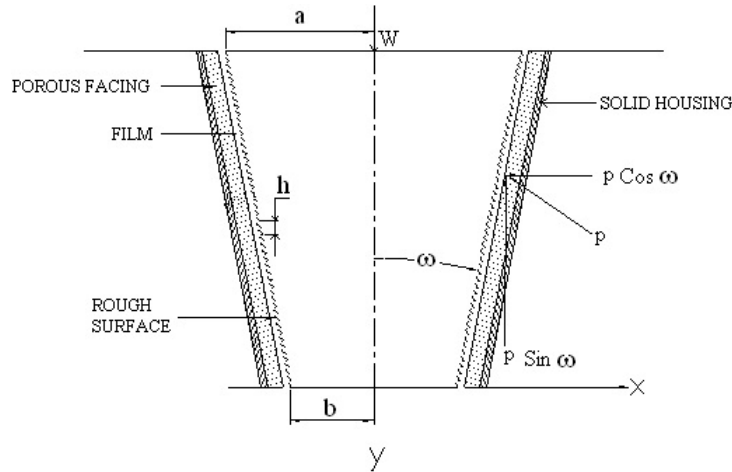


Figure-I. Geometry and configuration of bearing system

The lower plate having porous face is fixed. The upper plate moves towards the lower plate along the normal with a angular velocity $\dot{h} = (\frac{dh}{dt})$. Both the plates are considered to be electrically conducting and an electrically conducting lubricant fills the clearance space. A uniform transverse magnetic field is applied between the plates. The transverse surface roughness of the bearing surface is characterized by a random variable with non zero mean, variance and skewness. Following the discussions of Christensen and Tonder [8, 9, 10], the film thickness $h(x)$ is considered as

$$h(x) = \bar{h}(x) + h_s(x),$$

where $\bar{h}_s(x)$ is the mean film thickness and $h_s(x)$ is the deviation from the mean film thickness characterizing the random roughness of the bearing surfaces. $h_s(x)$ is described by a probability density function $f(h_s)$, defined by

$$f(h_s) = \begin{cases} \frac{35}{32c^7}(C^2 - h_s^2)^3, & \text{if } -C \leq h_s \leq C \\ 0, & \text{elsewhere} \end{cases}$$

C being the maximum deviation from the mean film thickness. The mean α , the standard deviation σ and the parameter ε , which is the measure of symmetry, of random variable h_s , are defined by the relationships

$$\alpha = E(h_s), \quad \sigma^2 = E[(h_s - \alpha)^2]$$

and

$$\varepsilon = E[(h_s - \alpha)^3]$$

where E denotes the expected value defined by

$$E(R) = \int_{-C}^C R f(h_s) dh_s.$$

The details regarding the roughness and characterization can be seen from Christensen and Tonder [8, 9, 10].

A modified form of Darcys law (Prajapati [21]) governs the flow in the porous medium while in the film region the equation of the hydromagnetic lubrication theory holds. Under the traditional assumptions of hydrodynamic lubrication the modified stochastically averaged Reynolds equation for the lubricant film pressure (Prajapati [21]) is found to be

$$(1) \quad \frac{1}{x} \frac{d}{dx} \left\{ x \frac{d}{dx} (p - 0.5\mu_0 \bar{\mu} H^2) \right\} = \frac{12\mu \dot{h} \sin \omega}{A}$$

where

$$A = h^3 \sin^3 \omega + 3\sigma^2 h \sin \omega + 3\alpha^2 h \sin \omega + 3\alpha h^2 \sin^2 \omega + 3\sigma^2 \alpha + \alpha^3 + \varepsilon + 12\phi H_0$$

and

$$H^2 = (a \operatorname{cosec} \omega - x)(x - b \operatorname{cosec} \omega)$$

μ_0 is permeability of the free space, $\bar{\mu}$ is the magnetic permeability, μ is the viscosity of the fluid, ϕ is permeability of porous facing, ω is the semi vertical angle of cone and H_0 is the thickness of porous facing.

The following dimensionless terms are introduced:

$$\sigma^* = \frac{\sigma}{h}, \quad \alpha^* = \frac{\alpha}{h}, \quad \varepsilon^* = \frac{\varepsilon}{h^3}, \quad \psi = \frac{\phi H_0}{h^3}, \quad k = \frac{a}{b}$$

$$\bar{A} = \sin^3 \omega + 3(\sigma^*)^2 \sin \omega + 3(\alpha^*) \sin \omega + 3\alpha^* \sin^2 \omega + 3(\sigma^*)^2 \alpha^* + (\alpha^*)^3 + 12\psi$$

and

$$\mu^* = -\frac{h^3 \mu_0 \bar{\mu}}{\mu \dot{h}}, \quad \bar{p} = -\frac{h^3 p}{\pi \mu \dot{h} (a^2 - b^2) \operatorname{cosec} \omega}, \quad W = -\frac{h^3 w}{\pi \mu \dot{h} (a^2 - b^2)^2 \operatorname{cosec}^2 \omega}.$$

Solution of equation (1) by making use of boundary conditions

$$\frac{d\bar{p}}{d\bar{x}_1} = \frac{\mu^*}{2} \text{ at } \bar{x}_1 = 1$$

and

$$\frac{d\bar{p}}{d\bar{x}_2} = -\frac{\mu^*}{2} \text{ at } \bar{x}_2 = 1$$

where

$$(2) \quad \bar{x}_1 = \frac{x}{a \operatorname{cosec} \omega}, \quad \bar{x}_2 = \frac{x}{b \operatorname{cosec} \omega}, \quad b < x < a$$

determines the pressure distribution as

$$(3) \quad p = 0.5 \mu_0 \bar{\mu} H^2 + \frac{3 \mu \dot{h} \sin \omega}{A} (x^2 - a^2 \operatorname{cosec}^2 \omega) \\ + b \operatorname{cosec} \omega \left\{ \frac{\mu \dot{h} (a^2 - b^2) \mu^*}{2 h^3 b} - 0.5 \mu_0 \bar{\mu} (a - b) \operatorname{cosec} \omega - \frac{6 \mu \dot{h} b}{A} \right\} \ln \left(\frac{x}{\operatorname{cosec} \omega} \right) \\ - \frac{\mu \dot{h} \pi (a^2 - b^2) \mu^* \operatorname{cosec} \omega}{2 h^3}.$$

Then, the load carrying capacity given by

$$(4) \quad w = 2\pi \int_{b \operatorname{cosec} \omega}^{a \operatorname{cosec} \omega} ap(x) dx \\ = 2\pi \left[\left\{ \frac{0.5 \mu_0 \bar{\mu} (a^2 - b^2) (a - b)^2}{12} - \frac{3 \mu \dot{h} \sin \omega (a^2 - b^2)}{4A} \right\} \operatorname{cosec}^4 \omega \right. \\ \left. \left\{ \frac{\mu \dot{h} \pi (a^2 - b^2) \mu^*}{2 h^3 b} - 0.5 \mu_0 \bar{\mu} (a - b) \operatorname{cosec} \omega - \frac{6 \mu \dot{h} b}{A} \right\} \right. \\ \left. \cdot + \left\{ \frac{b^2 - a^2}{4} + \frac{b^2}{a^2} \ln \left(\frac{a}{b} \right) \right\} b \operatorname{cosec}^3 \omega - \frac{\mu \dot{h} \pi (a^2 - b^2)^2 \mu^* \operatorname{cosec}^3 \omega}{4 h^3} \right]$$

is expressed in dimensionless form as

$$(5) \quad W = \frac{\mu^* \operatorname{cosec}^2 \omega}{2} \left\{ \frac{(k-1)}{6\pi(k+1)} + \frac{3}{2} \sin \omega - \frac{\ln k}{k^2-1} \sin \omega \right. \\ \left. + \frac{1}{2\pi(k+1)} - \frac{\ln k}{\pi(k-1)(k+1)^2} \right\} \\ + \frac{3 \operatorname{cosec} \omega}{2\pi A(k^2-1)^2} \{ (k^2-1)(k^2-3) + 4 \ln k \}.$$

Results and discussions:

Equation (5) makes it clear that the increase in load carrying capacity due to the magnetization comes out to be

$$\frac{\mu^* \operatorname{cosec}^2 \omega}{2} \left\{ \frac{(k-1)}{6\pi(k+1)} + \frac{3}{2} \sin \omega - \frac{\ln k}{k^2-1} \sin \omega \right. \\ \left. + \frac{1}{2\pi(k+1)} - \frac{\ln k}{\pi(k-1)(k+1)^2} \right\}$$

in comparison with the conventional lubricant based bearing system.

The variation of load carrying capacity with respect to the magnetization parameter is presented in Figures: 1-5. It is seen that the load carrying capacity increases sharply with increasing magnetization.

The effect of standard deviation on the distribution of load carrying capacity is shown in Figures: 6-8. These figures indicate that the standard deviation has an adverse effect on the bearing performance as the load carrying capacity is considerably reduced.

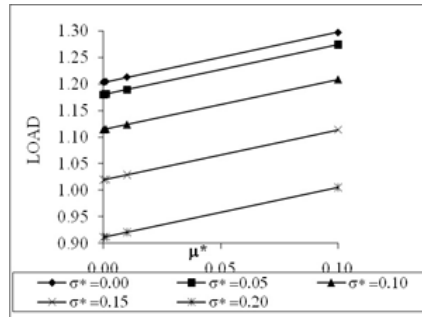


Figure 1: Variation of load carrying capacity with respect to μ^* and σ^*

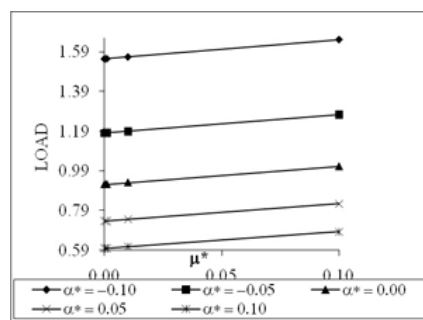


Figure 2: Variation of load carrying capacity with respect to μ^* and α^*

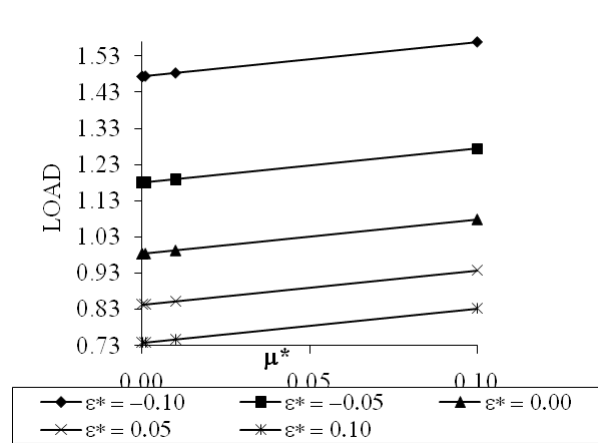


Figure 3: Variation of load carrying capacity with respect to μ^* and ε^*

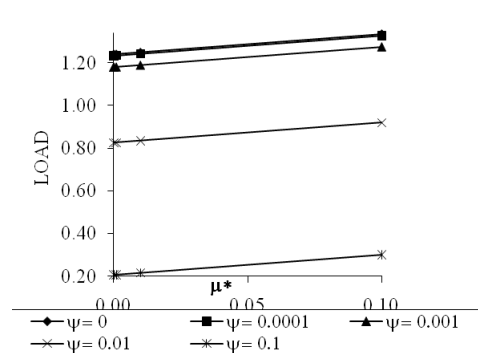
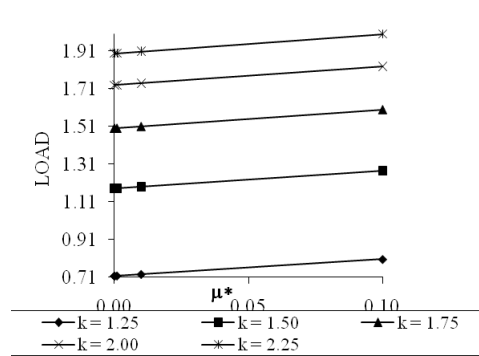
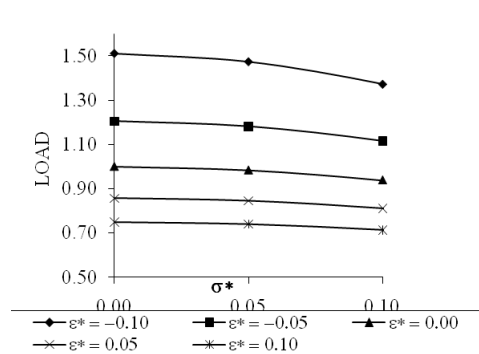
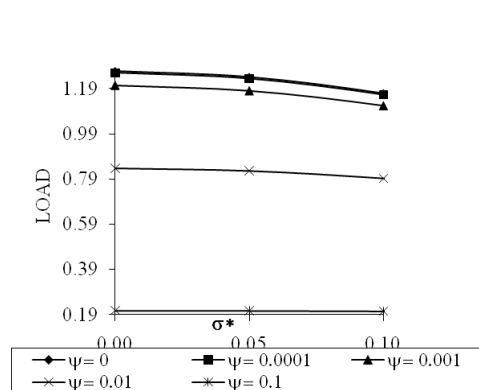


Figure 4: Variation of load carrying capacity with respect to μ^* and ψ

Figure 5: Variation of load carrying capacity with respect to μ^* and k Figure 6: Variation of load carrying capacity with respect to σ^* and ε^* Figure 7: Variation of load carrying capacity with respect to σ^* and ψ

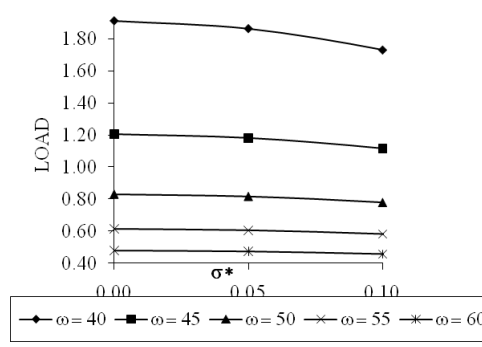


Figure 8: Variation of load carrying capacity with respect to σ^* and ω

The effect of variance depicted in Figures: 9-11 indicates that the variance (+ve) causes reduced load while the load carrying capacity increases sharply owing to variance(-ve).

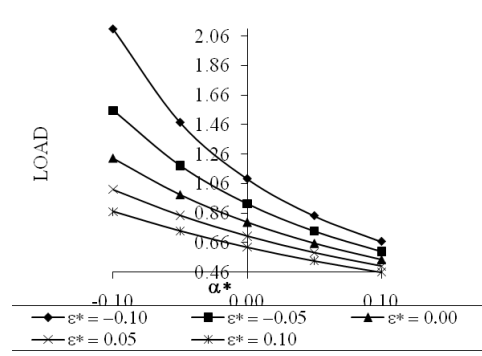


Figure 9: Variation of load carrying capacity with respect to α^* and ϵ^*

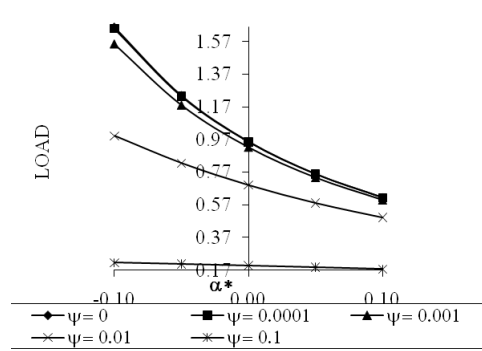


Figure 10: Variation of load carrying capacity with respect to α^* and ψ

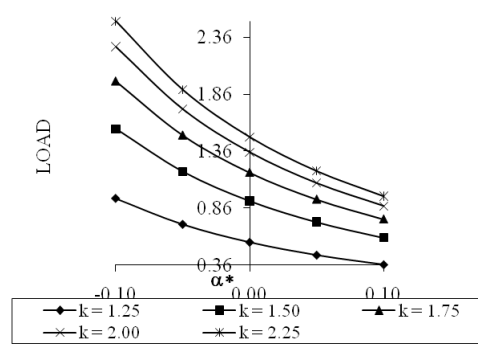


Figure 11: Variation of load carrying capacity with respect to α^* and k

The fact that the skewness goes along the path of variance so far as the trends of load carrying capacity are concerned, is manifest in Figures: 12-13.

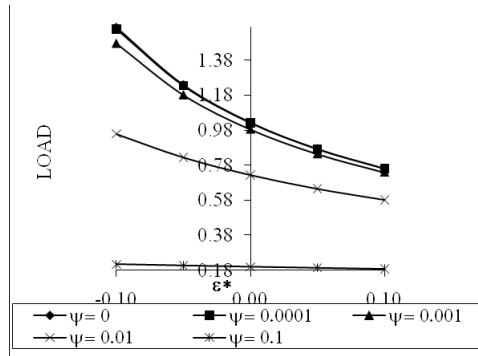


Figure 12: Variation of load carrying capacity with respect to ϵ^* and ψ

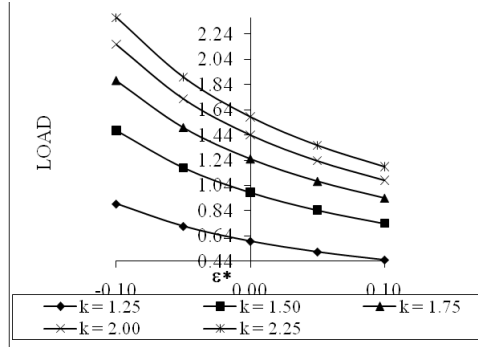


Figure 13: Variation of load carrying capacity with respect to ϵ^* and k

As usual porosity brings down the load carrying capacity. This can be seen from Figure:-14.

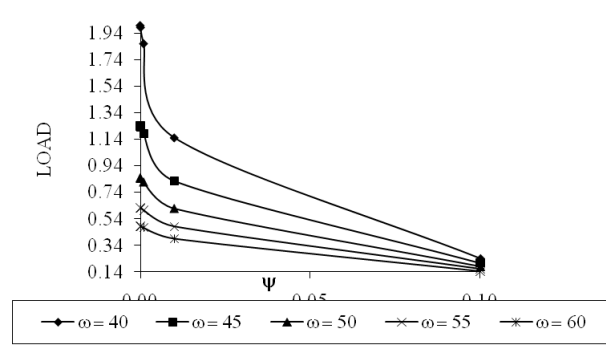


Figure 14: Variation of load carrying capacity with respect to ψ and ω

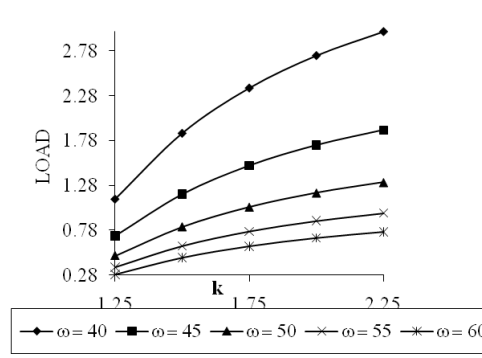


Figure 15: Variation of load carrying capacity with respect to k and ω

The graphical representations suggest that the combined positive effect of negatively skewed roughness and variance (- ve) can be channelized to improve the bearing performance. The magnetic fluid may aid to this positive effect in overcoming the adverse effect of porosity and standard deviation. However, here the semi vertical angle of the cone may play a crucial role. Equally important becomes the role of the aspect ratio (Figure-15), for augmenting the bearing performance.

Conclusions

This investigation reveals that the roughness aspect must be duly addressed while designing this type of bearing system, even if suitable magnetic strength is in place. Also, it can be seen that judiciously choosing the boundary conditions, the performance of the bearing system can be improved by picking up suitable

magnetic strength in spite of the fact that the roughness has an adverse effect in general.

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ON SOME PROPERTIES OF ROUGH APPROXIMATIONS OF SUBRINGS VIA COSETS

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Abstract. In 1982, Zdzislaw Pawlak introduced the theory of Rough sets to deal with the problems involving imperfect knowledge. This present research article studies some interesting properties of Rough approximations of subrings via an equivalence relation involving cosets of an ideal. In this present work, a ring structure is assigned to the universe set and a few results on the Rough approximations of subrings of the universe set are established.

Keywords: ring, subring, ideal, maximal ideal, prime ideal, rough set, upper approximation, lower approximation, information system, equivalence relation.

Introduction. The problem of imperfect knowledge has been tackled for a long time by philosophers, logicians and mathematicians. Recently it became also a crucial issue for computer scientists, particularly in the area of Artificial Intelligence. There are many approaches to the problem of how to understand and manipulate imperfect knowledge. The most successful approaches to tackle this problem are the Fuzzy set theory and the Rough set theory. Theories of Fuzzy sets and Rough sets are powerful mathematical tools for modeling various types of uncertainties. Fuzzy set theory was introduced by L. A. Zadeh in his classical paper [7] of 1965.

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A polish applied mathematician and computer scientist Zdzislaw Pawlak introduced Rough set theory in his classical paper [4] of 1982. Rough set theory is a new mathematical approach to imperfect knowledge. This theory presents still another attempt to deal with uncertainty or vagueness.

The Rough set theory has attracted the attention of many researchers and practitioners who contributed essentially to its development and application. Rough sets have been proposed for a very wide variety of applications.

In particular, the Rough set approach seems to be important for Artificial Intelligence and cognitive sciences, especially for machine learning, knowledge discovery, data mining, pattern recognition and approximate reasoning.

In this present work, we construct Rough sets by considering cosets of an Ideal in a ring. We investigate a few results on lower and upper approximations of subrings.

1. Rings and ideals

In this section, some basic definitions of Rings and Ideals that are necessary for further study of this work are presented.

Definition 1.1. A non-empty set of elements G is said to form a *semi-group* if in G there is defined a binary operation, called the product and denoted by o , such that

- a. $a, b \in G \Rightarrow aob \in G$ (Closure);
- b. $ao(boc) = (aob)oc$, for all a, b, c in G (Associative law).

Definition 1.2. A non-empty set of elements G is said to form a *group* if in G there is defined a binary operation, called the product and denoted by o , such that

- a. $a, b \in G \Rightarrow aob \in G$ (Closure);
- b. $ao(boc) = (aob)oc$, for all a, b, c in G (Associative law);
- c. there exists an element $e \in G$ such that $aoe = eoa = a$, for all $a \in G$. The element e is called the identity element in G . (Existence of identity);
- d. for every $a \in G$ there exists an element $a^{-1} \in G$ such that $aoa^{-1} = a^{-1}oa = e$. The element a^{-1} is called the inverse element of a in G . (Existence of inverse).

We denote a non-empty set G with respect to a binary operation o by the symbol (G, o) .

Definition 1.3. A group G with respect to a binary operation o is said to be an *abelian group* if $aob = boa$, for all $a, b \in G$.

Definition 1.4. A non-empty set U is said to be a *ring* if in there are defined two operations, denoted by $+$ and $\cdot$ respectively, such that

- a. $(U, +)$ is an abelian group;
- b. $(U, \cdot)$ is a semi-group;
- c. $x \cdot (y + z) = x \cdot y + x \cdot z$, for all $x, y, z \in U$;
- d. $(x + y) \cdot z = x \cdot y + x \cdot z$, for all $x, y, z \in U$.

We denote a ring U with respect to the binary operations $+$ and $\cdot$ by the symbol $(U, +, \cdot)$. We denote the identity element in $(U, +)$ by the symbol 0 and we call this element, the zero element of U . If $a \in U$ then the inverse element of a in $(U, +)$ by the symbol $-a$.

Definition 1.5. A non-empty subset S of a ring $(U, +, \cdot)$ is said to be a *subring* of $(U, +, \cdot)$ if and only if $a - b \in S$ for every a and b in S .

Definition 1.6. A non-empty subset I of a ring $(U, +, \cdot)$ is said to be an *ideal* of U if

- a. $x \in I$ and $y \in I \Rightarrow x - y \in I$;
- b. $u \in U$ and $x \in I \Rightarrow u \cdot x \in I$;
- c. $u \in U$ and $x \in I \Rightarrow x \cdot u \in I$.

Definition 1.7. An ideal P of a ring $(U, +, \cdot)$ is said to be a *prime ideal* if $x \cdot y \in P$ then either $x \in P$ or $y \in P$.

Definition 1.8. An ideal M of a ring $(U, +, \cdot)$ is said to be a *maximal ideal* if for any ideal I of U , $M \subseteq I \subseteq U \Rightarrow M = I$ or $I = U$.

2. Rough sets

This section is devoted to the basic concepts of Rough set theory. In what follows ϕ and U stand for the empty set and the universe set respectively.

Definition 2.1. A relation R on a non-empty set S is said to be an *equivalence relation* on S if

- a. xRx , for all $x \in S$ (reflexivity);
- b. $xRy \Rightarrow yRx$ (symmetry);
- c. xRy and $yRz \Rightarrow xRz$ (transitivity).

We denote the equivalence class of an element $x \in S$ with respect to the equivalence relation R by the symbol $R[x]$ and $R[x] = \{y \in S : yRx\}$.

Definition 2.2. Let $X \subseteq U$. Let R be an equivalence relation on U . Then we define the following

- a. The *lower approximation* of X with respect to R is the set of all objects, which can be for certain classified as X using R . That is the set $R_*(X) = \{x : R[x] \subseteq X\}$.
- b. The *upper approximation* of X with respect to R is the set of all objects, which can be possibly classified as X using R . That is the set $R^*(X) = \{x : R[x] \cap X \neq \emptyset\}$.
- c. The *boundary region* of X with respect to R is the set of all objects, which can be classified neither as X nor as $\neg X$ using R . That is the set $B_R(X) = R^*(X) - R_*(X)$.

It is clear that $R_*(X) \subseteq X \subseteq R^*(X)$.

Definition 2.3. A set $X \subseteq U$ is said to be a *Roughset* with respect to an equivalence relation R on U , if the boundary region $B_R(X) = R^*(X) - R_*(X)$ is non-empty.

3. Construction of rough sets

In the Literature of Rough set theory, information systems are considered. An *information system* is a pair (U, A) where A is a set of attributes. Each attribute $a \in A$ is a mapping $a : U \rightarrow V_a$ where V_a is the range set of the attribute $a \in A$. Corresponding to each attribute $a \in A$, an equivalence relation R_a is defined on U such that $xR_a y \Leftrightarrow a(x) = a(y)$. Rough sets are constructed through this relation as usual.

In this section, we slightly deviate from the above traditional setting to construct Rough sets. We consider an equivalence relation on a ring to construct Rough sets and present a few results in this context.

In what follows U stands for a ring and we take our universe set to be the ring $(U, +, \cdot)$.

Definition 3.1. Let I be an ideal of a ring U such that $I \neq \{0\}$ and $I \neq U$.

We define a relation R on U as follows.

For $x, y \in U$, $xRy \Leftrightarrow y - x \in I$.

Proposition 3.1. *The relation R on U is an equivalence relation.*

Remark 3.1. If $x \in U$ then we denote the equivalence class of x under the equivalence relation R by the symbol $R[x]$ and we have $R[x] = \{y \in U : yRx\} = x + I$ thus the equivalence class of $x \in U$ is the left coset $x + I$ of I in U .

Proposition 3.2. *For any subset A of U , the following conditions are equivalent to one another.*

- (a) $R_*(A) = A$;

$$(b) R^*(A) = A;$$

$$(c) R_*(A) = R^*(A).$$

Proof. For any subset A of U , We always have $R_*(A) \subseteq A \subseteq R^*(A)$.

Suppose $R_*(A) = A$ and let $x \in R^*(A)$. Then $R[x] \cap A \neq \phi$

$$\Rightarrow R_*(A) \cap R^*(A) \neq \phi$$

$$\Rightarrow \text{there exists a point } y \text{ in } U \text{ such that } y \in R[x] \cap R_*(A)$$

$$\Rightarrow yRx \text{ and } y \in R_*(A)$$

$$\Rightarrow R[x] = R[y] \text{ and } R[y] \subseteq A$$

$$\Rightarrow R[x] \subseteq A$$

$$\Rightarrow x \in A$$

This shows that $R^*(A) \subseteq A$. Hence $R^*(A) = A$.

This completes the proof of (a) $\Rightarrow$ (b).

Now consider $R^*(A) = A$. Let $x \in A$. Then $x \in R^*(A)$.

$$\Rightarrow R[x] \cap A \neq \phi$$

$$\text{Let } z \in R[x]. \text{ Then } xRz \Rightarrow R[z] = R[x]$$

$$\Rightarrow R[z] \cap A \neq \phi$$

$$\Rightarrow z \in R^*(A) \Rightarrow z \in A.$$

This proves that $R[x] \subseteq A$ and hence $x \in R_*(A)$.

Hence $R_*(A) = A$. This completes the proof of (b) $\Rightarrow$ (a).

Obviously (c) is equivalent to both (a) and (b).

4. Rough approximations of subrings

In this section, we present a few properties of lower and upper rough approximations of subrings of U .

Proposition 4.1. $\{0\}$ is a Rough set.

Proof. Since $R_*(\{0\}) \subseteq \{0\} \subseteq R^*(\{0\})$, either $R_*(\{0\}) = \phi$ or $R_*(\{0\}) = \{0\}$

Assume that $R_*(\{0\}) = \{0\}$.

$$\text{Then } 0 \in R_*(\{0\}) \Rightarrow I \subseteq \{0\} \Rightarrow I = \{0\}.$$

This is a contradiction. Hence $R_*(\{0\}) = \phi$.

$$\text{Let } x \in R^*(\{0\}). \text{ Then } x + I \cap \{0\} \neq \phi.$$

$$\Rightarrow 0 \in x + I.$$

$$\Rightarrow -x \in I.$$

$$\Rightarrow x \in I.$$

This shows that

$$(4.1) \quad R^*(\{0\}) \subseteq I$$

Let $x \in I$. Then $x + I = I$

$$\Rightarrow x + I \cap \{0\} = I \cap \{0\} = \{0\} \neq \phi.$$

$$\Rightarrow x \in R^*(\{0\}).$$

This shows that

$$(4.2) \quad I \subseteq R^*(\{0\})$$

From (4) and (4), $R^*(\{0\}) = I$.

Then $B_R(\{0\}) = I$.

Hence $\{0\}$ is a Rough set.

Remark 4.1. By the above Proposition-4.1, it follows that the lower approximation of a subring of U is not necessarily a subring of U . Even though $\{0\}$ is a subring of U , its lower approximation is empty and hence $R_*(\{0\})$ is not a subring of U .

Proposition 4.2. *I is not a Rough set.*

Proof. Clearly $R_*(I) \subseteq I \subseteq R^*(I)$. Let $x \in I$.

Then $x + I = I \Rightarrow x \in R_*(I)$.

Then $I = R_*(I)$.

By Proposition-3.2, $R_*(I) = I = R^*(I)$.

Hence I is not a Rough set.

Proposition 4.3. *Let K be a subring of U . Then the following are equivalent.*

- (a) $I \subseteq K$;
- (b) $R_*(K) = K = R^*(K)$;
- (c) $R_*(K)$ is a subring of U .

Proof. Let K be a subring of U . Suppose that $I \subset K$

Clearly $R_*(K) \subseteq K \subseteq R^*(K)$.

Let $x \in K$. Since $I \subseteq K$, $x + I \subseteq x + K = K$
 $\Rightarrow x \in R_*(K)$.

This shows that $K \subseteq R_*(K) \Rightarrow K = R_*(K)$.

Hence $R_*(K) = K = R^*(K)$.

This proves (a) $\Rightarrow$ (b).

Suppose that $R_*(K) = K = R^*(K)$.

Then $R_*(K)$ is a subring of U .

This proves (b) $\Rightarrow$ (c).

Suppose that $R_*(K)$ is a subring of U .

$\Rightarrow 0 \in R_*(K)$

$\Rightarrow I = 0 + I \subseteq K$

This proves (c) $\Rightarrow$ (a).

Proposition 4.4. *If K is a subring of U then $R^*(K)$ is a subring of U .*

Proof. Let K be a subring of U .

Let $x \in R^*(K)$.

$\Rightarrow x + I \cap K \neq \emptyset$

$\Rightarrow$ there exists a point z_1 such that $z_1 \in x + I \cap K$
 $\Rightarrow z_1 \in x + I$ and $z_1 \in K$
 $\Rightarrow z_1 - x \in I$ and $z_1 \in K$
 Let $y \in R^*(K)$.
 $\Rightarrow y + I \cap K \neq \phi$
 $\Rightarrow$ there exists a point z_2 such that $z_2 \in y + I \cap K$
 $\Rightarrow z_2 \in y + I$ and $z_2 \in K$
 $\Rightarrow z_2 - y \in I$ and $z_2 \in K$
 Since I is an ideal of U , $z_1 - x \in I$ and $z_2 - y \in I$ we have $z_1 - z_2 - x + y \in I$
 $\Rightarrow z_1 - z_2 \in x - y + I$
 Since K is a subring of U , $z_1 \in K$ and $z_2 \in K$ we have $z_1 - z_2 \in K$
 Hence $z_1 - z_2 \in x - y + I \cap K$.
 This shows that $x - y + I \cap K$ is non-empty.
 $\Rightarrow x - y \in R^*(K)$
 Since $z_2 \in U$ and $z_1 - x \in I$, it follows that $z_1 z_2 - x z_2 \in I$
 Since $x \in U$ and $z_2 - y \in I$, it follows that $x z_2 - x y \in I$
 Then $z_1 z_2 - x z_2 + x z_2 - x y \in I$
 $\Rightarrow z_1 z_2 - x y \in I$
 $\Rightarrow z_1 z_2 \in x y + I$
 Since K is a subring of U , $z_1 \in K$ and $z_2 \in K$ we have $z_1 z_2 \in K$
 $\Rightarrow x y + I \cap K$ is non-empty.
 $\Rightarrow x y \in R^*(K)$
 Hence $R^*(K)$ is a subring of U .

Proposition 4.5. *If K is an ideal of U then $R^*(K)$ is an ideal of U .*

Proof. Since K is an ideal of U , it is a subring of U .

Then $R^*(K)$ is a subring of U .

Let $u \in U$ and $x \in R^*(K)$. Then $(x + I) \cap K \neq \phi$.

$\Rightarrow$ there exists a point $y \in K$ such that $y - x \in I$.

$\Rightarrow uy \in K$ and $uy - ux \in I$

$\Rightarrow uy \in K$ and $uy \in ux + I$

$\Rightarrow uy \in ux + I \cap K$

$\Rightarrow ux + I \cap K \neq \phi$

$\Rightarrow ux \in R^*(K)$

Similarly $xu \in R^*(K)$. Hence $R^*(K)$ is an ideal of U .

Remark 4.2. The proof of the proposition 4.6 follows easily from the definitions and hence can be omitted.

Proposition 4.6. *If K is an maximal ideal of U then $R^*(K)$ is an maximal ideal of U .*

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ON GENERALIZED ZERO-DIVISOR GRAPH ASSOCIATED WITH A COMMUTATIVE RING

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Abstract. Let R be a commutative ring with identity, and let $Z(R)$ be the set of zero-divisors of R . The generalized zero-divisor graph of R is defined as the graph $\Gamma_g(R)$ with the vertex set $Z(R)^* = Z(R) \setminus \{0\}$, and two distinct vertices x and y are adjacent if and only if $\text{ann}_R(x) + \text{ann}_R(y)$ is an essential ideal of R . It follows that each edge (path) of the zero-divisor graph $\Gamma(R)$ is an edge (path) of $\Gamma_g(R)$. It is proved that $\Gamma_g(R)$ is connected with diameter at most three and with girth at most four, if $\Gamma_g(R)$ contains a cycle. Furthermore, all rings with the same generalized zero-divisor and zero-divisor graphs are characterized. Among other results, we show that the generalized zero-divisor graph associated with an Artinian ring is weakly perfect, i.e., its vertex chromatic number equals its clique number.

Keywords: generalized zero-divisor graph, zero-divisor graph, complete graph, chromatic number, clique number.

1. Introduction

Assigning a graph to a ring gives us the ability to translate algebraic properties of rings into graph theory language and vice versa. It leads to arising interesting algebraic and combinatorics problems. Therefore, the study of graphs associated

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with rings has attracted many researches. There are a lot of papers which apply combinatorial methods to obtain algebraic results in ring theory (see for instance [1], [5], [9] and [10]). Moreover, for the most recent study in this field see [6], [12] and [14].

Throughout this paper, all rings are assumed to be commutative with identity and they are not fields. We denote by $\text{Max}(R)$, $\text{Min}(R)$, $\text{Nil}(R)$ and $\text{U}(R)$, the set of all maximal ideals of R , the set of all minimal prime ideals of R , the set of all nilpotent elements of R and the set of all invertible elements of R , respectively. For a subset A of a ring R , we let $A^* = A \setminus \{0\}$. For every ideal I of R , we denote the *annihilator* of I by $\text{ann}_R(I)$. A non-zero ideal I of R is said to be *minimal* if there is no non-trivial ideal of R contained in I . A non-zero ideal I of R is called *essential*, denoted by $I \leq_e R$, if I has a non-zero intersection with every non-zero ideal of R . The *krull dimension* of R , denoted by $\dim(R)$, is the supremum of the lengths of all chains of prime ideals. We say that $\text{depth}(R) = 0$, if every non-unit element of R is a zero-divisor. The ring R is said to be *reduced* if it has no non-zero nilpotent element. A non-zero ideal K of R is called *weakly prime*, if $(0) \neq IJ \subseteq K$ for some ideals I, J of R , then $I \subseteq K$ or $J \subseteq K$ (see [2], for more details). For any undefined notation or terminology in ring theory, we refer the reader to [4, 7].

Let $G = (V, E)$ be a graph, where $V = V(G)$ is the set of vertices and $E = E(G)$ is the set of edges. By $\overline{G}$, $\text{diam}(G)$ and $\text{gr}(G)$, we mean the complement, the diameter and the girth of G , respectively. We write $u - v$, to denote an edge with ends u, v . A graph $H = (V_0, E_0)$ is called a *subgraph of G* if $V_0 \subseteq V$ and $E_0 \subseteq E$. Moreover, H is called an *induced subgraph by V_0* , denoted by $G[V_0]$, if $V_0 \subseteq V$ and $E_0 = \{\{u, v\} \in E \mid u, v \in V_0\}$. Let G_1 and G_2 be two disjoint graphs. The *join* of G_1 and G_2 , denoted by $G_1 \vee G_2$, is a graph with the vertex set $V(G_1 \vee G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 \vee G_2) = E(G_1) \cup E(G_2) \cup \{uv \mid u \in V(G_1), v \in V(G_2)\}$. Also G is called a *null graph* if it has no edge. A complete bipartite graph with part sizes m, n is denoted by $K_{m,n}$. If $m = 1$, then the complete bipartite graph is called *star graph*. Also, a complete graph of n vertices is denoted by K_n . By $\overline{K}_\infty$, we mean a null graph with infinitely many vertices. A *clique* of G is a maximal complete subgraph of G and the number of vertices in the largest clique of G , denoted by $\omega(G)$, is called the *clique number* of G . For a graph G , let $\chi(G)$ denote the *vertex chromatic number* of G , i.e., the minimal number of colors which can be assigned to the vertices of G in such a way that every two adjacent vertices have different colors. Clearly, for every graph G , $\omega(G) \leq \chi(G)$. A graph G is said to be *weakly perfect* if $\omega(G) = \chi(G)$. For any undefined notation or terminology in graph theory, we refer the reader to [13].

The *zero-divisor graph* of a ring R , denoted by $\Gamma(R)$, is a graph with the vertex set $Z(R)^*$ and two distinct vertices x and y are adjacent if and only if $xy = 0$. This is the most important graph associated with a ring and it was first introduced and studied in [3]. Obviously, the properties of zero-divisor graphs are depend on the zero-divisors. Let $x, y \in Z(R)^*$ and $xy \neq 0$ whereas the

behavior of the ideal $\text{ann}_R(x) + \text{ann}_R(y)$ is same to the case that $xy = 0$. These elements motivate us to define a new graph containing the zero-divisor graph. The *generalized zero-divisor graph* of R is defined as the graph $\Gamma_g(R)$ with the vertex set $Z(R)^* = Z(R) \setminus \{0\}$, and two distinct vertices x and y are adjacent if and only if $\text{ann}_R(x) + \text{ann}_R(y)$ is an essential ideal of R . In this paper, we study some connections between the graph-theoretic properties of $\Gamma_g(R)$ and some algebraic properties of rings. Moreover, we investigate the affinity between generalized zero-divisor graph and zero-divisor graph associated with a ring. Especially, we focus on the conditions under which these two graphs are identical. Finally, the coloring of generalized zero-divisor graph is studied.

2. Basic properties of generalized zero-divisor graph

In this section, we study some of fundamental properties of $\Gamma_g(R)$. For instance, it is shown that $\Gamma_g(R)$ is always connected and $\text{diam}(\Gamma_g(R)) \leq 3$. Moreover, we prove that if $\Gamma_g(R)$ contains a cycle, then $\text{girth}(\Gamma_g(R)) \leq 4$. Moreover, all rings with complete or star generalized zero-divisor graph are characterized.

We begin with the following lemma.

Lemma 2.1. *Let R be a ring. Then the following statements hold.*

- (1) *If $x \in R$, then $Rx^n + \text{ann}_R(x)$ is an essential ideal of R , for every positive integer n .*
- (2) *$\text{ann}_R(x)$ is an essential ideal of R , for every $x \in \text{Nil}(R)$.*
- (3) *x is adjacent to all other vertices in $\Gamma_g(R)$, for every $x \in \text{Nil}(R)^*$.*
- (4) *If $x - y$ is an edge of $\Gamma(R)$, then $x - y$ is an edge of $\Gamma_g(R)$.*

Proof. (1) Let $x \in R$ and n be a positive integer. We show that $(Rx^n + \text{ann}_R(x)) \cap Ra \neq (0)$, for every $a \in R$. Suppose that $\text{ann}_R(x) \cap Ra = (0)$, for some $a \in R$. We have to prove that $Rx^n \cap Ra \neq (0)$. Since $\text{ann}_R(x) \cap Ra = (0)$, $ax \neq 0$. If $axx = ax^2 = 0$, then $\text{ann}_R(x) \cap Ra \neq (0)$, a contradiction. This implies that $ax^2 \neq 0$. By continuing this procedure, $ax^n \neq 0$. Hence $ax^n \in Rx^n \cap Ra$ and so $Rx^n \cap Ra \neq (0)$, as desired.

(2) It follows directly from part (1).

(3) Let $x \in \text{Nil}(R)^*$. By part (2), $\text{ann}_R(x)$ is an essential ideal of R and thus $\text{ann}_R(x) + \text{ann}_R(y)$ is essential, for every $y \in Z(R)^*$. Hence x is adjacent to all other vertices.

(4) Let $x - y$ be an edge of $\Gamma(R)$. Then $y \in \text{ann}_R(x)$. By part (1), $Ry + \text{ann}_R(y)$ is an essential ideal. This, together with $Ry \subseteq \text{ann}_R(x)$, imply that $\text{ann}_R(x) + \text{ann}_R(y)$ is essential. Therefore, $x - y$ is an edge of $\Gamma_g(R)$. $\square$

Let R be a ring. By [3, Theorem 2.3], the zero-divisor graph $\Gamma(R)$ is a connected graph and $\text{diam}(\Gamma(R)) \leq 3$. Moreover, if $\Gamma(R)$ contains a cycle, then

$\text{girth}(\Gamma(R)) \leq 4$ (see [11]). By using these facts and Lemma 2.1, we have the following result.

Theorem 2.1. *Let R be a ring. Then $\Gamma_g(R)$ is connected and $\text{diam}(\Gamma_g(R)) \leq 3$. Moreover, if $\Gamma_g(R)$ contains a cycle, then $\text{girth}(\Gamma_g(R)) \leq 4$.*

Proof. By part (4) of Lemma 2.1, $\Gamma(R)$ is a subgraph of $\Gamma_g(R)$ and so $\Gamma_g(R)$ is connected, $\text{diam}(\Gamma_g(R)) \leq 3$ and $\text{girth}(\Gamma_g(R)) \leq 4$, if $\Gamma_g(R)$ contains a cycle. $\square$

To classify all rings with complete or star generalized zero-divisor graph, the following lemma is needed.

Lemma 2.2. *Let R be a ring. Then the following statements hold.*

- (1) *If R is non-reduced and $\Gamma_g(R)$ is complete, then R is an indecomposable ring.*
- (2) *If R is reduced and $x \in Z(R)^*$, then $\text{ann}_R(x) = \text{ann}_R(x^2)$.*

Proof. (1) Suppose that $R \cong R_1 \times R_2$, where R_1 and R_2 are two rings. With no loss of generality, one may assume that $|\text{Nil}(R_1)| \geq 2$. Thus $2 \leq |\text{Nil}(R_1)| \leq |U(R_1)|$. Let $a = (1, 0)$ and $b = (u, 0)$, where $1 \neq u \in U(R_1)$. Then $\text{ann}_R(a) + \text{ann}_R(b) = (0) \times R_2$ and so it is not an essential ideal of R , a contradiction. Thus R is indecomposable.

(2) It is clear by [5, Lemma 3.2]. $\square$

Theorem 2.2. *Let R be a ring. Then the following statements hold.*

- (1) *If R is a non-reduced ring, then the following statements are equivalent:*
 - (i) $\Gamma_g(R)$ is complete.
 - (ii) $\text{ann}_R(x^2)$ is an essential ideal of R , for every $x \in Z(R)^*$.
- (2) *If R is a reduced ring, then the following statements are equivalent:*
 - (i) $\Gamma_g(R)$ is complete.
 - (ii) $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.
 - (iii) $\Gamma_g(R) = K_{1,1}$.

Proof. (1) (i) $\implies$ (ii) Let $x \in Z(R)^*$. If $x \in \text{Nil}(R)$, then by part (2) of Lemma 2.1, $\text{ann}_R(x)$ is an essential ideal of R and thus $\text{ann}_R(x^2)$ is an essential ideal of R . So let $x \in Z(R) \setminus \text{Nil}(R)$. Since $\Gamma_g(R)$ is complete, by Lemma 2.2, R is an indecomposable ring and thus $x \neq x^2$. Hence x is adjacent to x^2 and so $\text{ann}_R(x) + \text{ann}_R(x^2) = \text{ann}_R(x^2)$ is essential.

(ii) $\implies$ (i) Let $x \in Z(R)^*$. We need only to show that $\text{ann}_R(x)$ is an essential ideal of R . Suppose to the contrary, $\text{ann}_R(x) \cap Ra = (0)$, for some $a \in R$. Since $\text{ann}_R(x^2) \cap Ra \neq (0)$, there exists an element $r \in R$ such that

$rax^2 = 0$ and $ra \neq 0$. Since $\text{ann}_R(x) \cap Ra = (0)$, $rax \neq 0$. This means that $\text{ann}_R(x) \cap Ra \neq (0)$, a contradiction. Thus $\text{ann}_R(x)$ is an essential ideal of R .

(2) (i) $\implies$ (ii) Let $x \in Z(R)^*$. As R is a reduced ring, $Rx \cap \text{ann}_R(x) = (0)$ and thus $\text{ann}_R(x)$ is not an essential ideal of R . By part (2) of Lemma 2.2, $\text{ann}_R(x) = \text{ann}_R(x^2)$ and so $\text{ann}_R(x) + \text{ann}_R(x^2) = \text{ann}_R(x^2)$ is not an essential ideal of R . Since $\Gamma_g(R)$ is complete, we deduce that $x = x^2$. Hence $R \cong R_1 \times R_2$, where R_1 and R_2 are two rings. If $a \in R_1 \setminus \{0, 1\}$, then it is easily seen that $(1, 0)$ is not adjacent to $(a, 0)$ (since $\text{ann}_R(a)$ is not essential in R_1). Thus $|R_1| = 2$. Similarly, $|R_2| = 2$. Hence $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

(ii) $\implies$ (iii) and (iii) $\implies$ (i) are clear. $\square$

The following result shows that, if $\Gamma(R) = K_n \vee \overline{K}_m$, for some n, m , then either $\Gamma(R) = \Gamma_g(R)$ or $\Gamma_g(R)$ is a complete graph.

Theorem 2.3. *Let R be a non-reduced ring and $\Gamma(R) = K_n \vee \overline{K}_m$ for some n, m . Then the following statements are equivalent:*

- (1) $\Gamma(R) \neq \Gamma_g(R)$.
- (2) $\Gamma_g(R)$ is a complete graph and $Z(R)^2 \neq (0)$.

Proof. (1) $\implies$ (2) Since $\Gamma(R) = K_n \vee \overline{K}_m$, every vertex of $V(K_n)$ is adjacent to all other vertices and thus by [3, Theorem 2.5], $\text{ann}_R(a) = Z(R)$, for every vertex $a \in V(K_n)$. This implies that $\text{ann}_R(a)$ is an essential ideal of R . Moreover, $\Gamma(R) = K_n \vee \overline{K}_m$ implies that $\text{ann}_R(x) = \{0\} \cup V(K_n)$, for every $x \in Z(R) \setminus \{0\} \cup V(K_n)$. The inequality $\Gamma(R) \neq \Gamma_g(R)$ shows that there exist $x, y \in V(\overline{K}_m)$ such that x is adjacent to y (note that $m \geq 2$). Thus $\text{ann}_R(x) + \text{ann}_R(y)$ is an essential ideal of R . Since $\text{ann}_R(x) = \text{ann}_R(y) = \{0\} \cup V(K_n)$, we conclude that $\{0\} \cup V(K_n)$ is essential. Hence $\text{ann}_R(x)$ is an essential ideal of R , for every $x \in Z(R)^*$ and thus $\Gamma_g(R)$ is a complete graph. Finally, if $Z(R)^2 = (0)$, then $\Gamma(R)$ is complete and since $\Gamma(R)$ is a connected subgraph of $\Gamma_g(R)$, we infer that $\Gamma(R) = \Gamma_g(R) = K_{n+m}$, a contradiction.

(2) $\implies$ (1) Since $Z(R)^2 \neq (0)$, $\Gamma(R)$ is not complete and hence $\Gamma(R) \neq \Gamma_g(R)$. $\square$

The following example explains Theorem 2.3.

Example 2.1. (1) Let $R = \mathbb{Z}_8$. Then $\Gamma(R) = K_{1,2}$, $\Gamma_g(R) = K_3$ and so $n = 1$ and $m = 2$, in Theorem 2.3.

(2) Let $R = \mathbb{Z}_2[X, Y]/(XY, X^2)$, $x = X + (XY + X^2)$ and $y = Y + (XY + X^2)$. Then $Z(R) = (x, y)R$, $\text{Nil}(R) = \{0, x\}$ and $\Gamma(R)$ is an infinite star. It is easily checked that $y \neq y^2$ and y is not adjacent to y^2 in $\Gamma_g(R)$. This means that $\Gamma_g(R)$ is not a complete graph and since $Z(R)^2 \neq (0)$, we deduce that $\Gamma(R) = \Gamma_g(R)$.

The last result of this section is devoted to identifying non-reduced rings with star generalized zero-divisor graph.

Theorem 2.4. *Let R be a non-reduced ring with $|Z(R)^*| \geq 2$. Then the following statements are equivalent:*

- (1) $\Gamma_g(R)$ is a star graph.
- (2) $\text{girth}(\Gamma_g(R)) = \infty$.
- (3) $\text{ann}_R(Z(R))$ is a prime ideal of R and $|\text{ann}_R(Z(R))| = 2$ or $|\text{ann}_R(Z(R))| = |Z(R)| = 3$.
- (4) Either $\Gamma_g(R) = K_{1,1}$ or $\Gamma_g(R) = K_{1,\infty}$.

Proof. (1) $\Rightarrow$ (2) and (4) $\Rightarrow$ (1) are clear.

(2) $\Rightarrow$ (3) Since $\text{girth}(\Gamma_g(R)) = \infty$, by part (3) of Lemma 2.1, $|\text{Nil}(R)| \in \{2, 3\}$. If $|\text{Nil}(R)| = 3$, then part (3) of Lemma 2.1 implies that $|\text{ann}_R(Z(R))| = |Z(R)| = 3$ and so $\text{ann}_R(Z(R))$ is a prime ideal. If $|\text{Nil}(R)| = 2$, then one may let $\text{Nil}(R) = \{0, a\}$ and thus a is adjacent to all other vertices in $\Gamma_g(R)$. Since $\text{girth}(\Gamma_g(R)) = \infty$, $\Gamma_g(R)$ is a star graph. By part (4) of Lemma 2.1, $\Gamma(R)$ is a subgraph of $\Gamma_g(R)$. This implies that $\text{Nil}(R) = \text{ann}_R(Z(R))$ and thus $|\text{ann}_R(Z(R))| = 2$. We show that $\text{Nil}(R)$ is a prime ideal of R . For this, let $xy \in \text{Nil}(R) = \{0, a\}$ such that $x \notin \text{Nil}(R)$ and $y \notin \text{Nil}(R)$. If $xy = 0$, then $x - a - y - x$ is a cycle of length three in $\Gamma_g(R)$, a contradiction. If $xy = a$, then $x^2y^2 = 0$ again a contradiction. Thus $\text{ann}_R(Z(R))$ is a prime ideal of R .

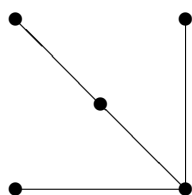
(3) $\Rightarrow$ (4) It is clear that, if $|\text{ann}_R(Z(R))| = |Z(R)| = 3$, then $\Gamma_g(R) = K_{1,1}$. Hence, suppose that $\text{ann}_R(Z(R)) = \{0, a\}$, for some $a \in R$ and $\text{ann}_R(Z(R))$ is a prime ideal. Clearly, $\text{ann}_R(Z(R)) \neq Z(R)$. If $Z(R)$ is a finite set, then since $Z(R)$ is an ideal of R , we deduce that R is an Artinian local ring and this means $Z(R)$ is a prime ideal. Thus we find the chain $\text{ann}_R(Z(R)) \subseteq Z(R)$ of primes, a contradiction (As $\dim(R) = 0$). Hence $Z(R)$ is an infinite set. We claim that $\text{ann}_R(Z(R))$ is not an essential ideal of R . To see this, let $x \in Z(R) \setminus \text{ann}_R(Z(R))$. If $Rx \cap \text{ann}_R(Z(R)) \neq (0)$, then $rx = a$, for some $r \in R$ and thus $rx^2 = 0$, a contradiction. So the claim is proved. Since $\text{ann}_R(Z(R))$ is a prime ideal of R , we deduce that $\text{ann}_R(x) = \text{ann}_R(Z(R))$, for every $x \in Z(R) \setminus \text{ann}_R(Z(R))$. To complete the proof, assume that $x - y$ is an edge of $\Gamma_g(R)$, $x \neq a$ and $y \neq a$. Obviously, $\text{ann}_R(x) + \text{ann}_R(y) = \text{ann}_R(x) = \text{ann}_R(Z(R))$, a contradiction (As $\text{ann}_R(Z(R))$ is not essential) and thus $\Gamma_g(R) = K_{1,\infty}$. $\square$

3. When generalized zero-divisor graphs and zero-divisor graphs are identical?

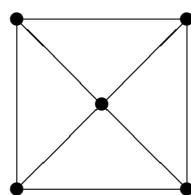
As we observed in the past section, generalized zero-divisor graphs and zero-divisor graphs are close to each other. The following simple example describes two different situations $\Gamma(R) \neq \Gamma_g(R)$ and $\Gamma(R) = \Gamma_g(R)$. Thus it is interesting to study rings whose generalized zero-divisor graphs and zero-divisor graphs are identical. This section is devoted to study of such rings.

Example 3.1.

- (1) Let $R = \mathbb{Z}_2 \times \mathbb{Z}_4$. Then $\Gamma(R) \neq \Gamma_g(R)$.

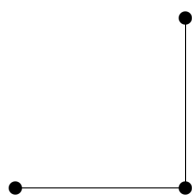


$$\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_4)$$

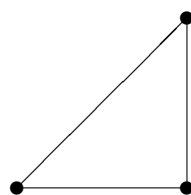


$$\Gamma_g(\mathbb{Z}_2 \times \mathbb{Z}_4)$$

(2) Let $R = \mathbb{Z}_8$. Then $\Gamma(R) \neq \Gamma_g(R)$.

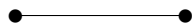


$$\Gamma(\mathbb{Z}_8)$$

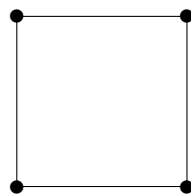


$$\Gamma_g(\mathbb{Z}_8)$$

(3) Let $R = \mathbb{Z}_9$ and $S = \mathbb{Z}_3 \times \mathbb{Z}_3$. Then $\Gamma(R) = \Gamma_g(R)$ and $\Gamma(S) = \Gamma_g(S)$.



$$\Gamma(R) = \Gamma_g(R)$$



$$\Gamma(S) = \Gamma_g(S)$$

To characterize all non-reduced rings with the same generalized zero-divisor graphs and zero-divisor graphs, we need the following lemma.

Lemma 3.1. *Let R be a ring and $x, y \in Z(R)$. Then $\text{ann}_R(x) + \text{ann}_R(y)$ is an essential ideal of R if and only if $\text{ann}_R(x) + \text{ann}_R(y^2)$ is an essential ideal of R .*

Proof. Since $\text{ann}_R(x) + \text{ann}_R(y) \subseteq \text{ann}_R(x) + \text{ann}_R(y^2)$, one side is clear. To prove the other side, suppose that $\text{ann}_R(x) + \text{ann}_R(y^2)$ is an essential ideal of R . Let $(\text{ann}_R(x) + \text{ann}_R(y)) \cap Ra = (0)$, for some $a \in R$. This implies that $rax \neq 0$ and $ray \neq 0$, for every $0 \neq ra \in Ra$. Since $(\text{ann}_R(x) + \text{ann}_R(y^2)) \cap Ra \neq (0)$, there exist $t \in \text{ann}_R(x)$ and $s \in \text{ann}_R(y^2)$ such that $t + s = ra \neq 0$, for some $r \in R$. Hence $ty^2 = ray^2$ and thus $0 \neq ty^2 \in (\text{ann}_R(x) \cap Ra)$, a contradiction. $\square$

Theorem 3.1. *Let R be a non-reduced ring. Then $\Gamma(R) = \Gamma_g(R)$ if and only if all of the following statements hold.*

- (1) $\text{ann}_R(a) = Z(R)$, for every $a \in \text{Nil}(R)^*$.
- (2) If $\text{ann}_R(x)$ is an essential ideal of R , for some $x \in Z(R)^*$, then $x \in \text{Nil}(R)^*$.
- (3) $\text{Nil}(R)$ is a weakly prime ideal of R .

Proof. First suppose that $\Gamma(R) = \Gamma_g(R)$.

If $a \in \text{Nil}(R)^*$, then by part (3) of Lemma 2.1, a is adjacent to all other vertices of $\Gamma(R)$. Since R is non-reduced, [3, Theorem 2.5] follows that $\text{ann}_R(a) = Z(R)$.

Let $\text{ann}_R(x)$ is an essential ideal of R , for some $x \in Z(R)^*$. Hence $\text{ann}_R(x) + \text{ann}_R(y)$ is an essential ideal, for every $y \in Z(R)$, and thus x is adjacent to all other vertices in $\Gamma_g(R)$. Since $\Gamma(R) = \Gamma_g(R)$, x is adjacent to all other vertices in $\Gamma(R)$. By [3, Theorem 2.5], $Z(R) = \text{ann}_R(x)$ and so $x \in \text{Nil}(R)$.

Now, we show that $\text{Nil}(R)$ is a weakly prime ideal of R . Let $0 \neq xy \in \text{Nil}(R)^*$ such that $x \notin \text{Nil}(R)^*$ and $y \notin \text{Nil}(R)^*$. Therefore, $x, y \in Z(R)$ and $x \neq y$. By part (1), $xyx = xy^2 = 0$ and thus $\text{ann}_R(x) + \text{ann}_R(y^2)$ is an essential ideal, as $\Gamma(R) = \Gamma_g(R)$. By Lemma 3.1, $\text{ann}_R(x) + \text{ann}_R(y)$ is essential, a contradiction. Hence $\text{Nil}(R)$ is a weakly prime ideal of R .

Conversely, suppose that all of the statements (1), (2) and (3) are hold. We prove that $\Gamma(R) = \Gamma_g(R)$. Let $x - y$ be an edge of $\Gamma_g(R)$. It is enough to show that $xy = 0$. Assume to the contrary, $xy \neq 0$. Since $\text{ann}_R(x) + \text{ann}_R(y)$ is an essential ideal of R and $\text{ann}_R(y) \subseteq \text{ann}_R(xy)$, we conclude that $\text{ann}_R(x) + \text{ann}_R(xy) = \text{ann}_R(xy)$ is essential. By part (2), $xy \in \text{Nil}(R)^*$ and thus by part (3) follows that $x \in \text{Nil}(R)$ or $y \in \text{Nil}(R)$. This contradicts part (1). Therefore, $xy = 0$, as desired. $\square$

Theorem 3.2. *Let R be a non-reduced ring and $\text{Nil}(R)$ be a prime ideal of R that is not essential. Then the following statements are equivalent:*

- (1) $\Gamma(R) = \Gamma_g(R)$.

- (2) $\Gamma(R) = K_{|\text{Nil}(R)^*|} \vee \overline{K}_\infty$.
- (3) $\Gamma_g(R) = K_{|\text{Nil}(R)^*|} \vee \overline{K}_\infty$ and $\text{ann}_R(x) \in \{\text{Nil}(R), Z(R)\}$, for every $x \in Z(R)^*$.

Proof. (1) $\Rightarrow$ (2) Since $\Gamma(R) = \Gamma_g(R)$, we need only to show that $\Gamma_g(R) = K_{|\text{Nil}(R)^*|} \vee \overline{K}_\infty$. By part (3) of Lemma 2.1, every vertex of $\text{Nil}(R)^*$ is adjacent to all other vertices in $\Gamma_g(R)$. Let $x, y \in Z(R) \setminus \text{Nil}(R)$ and $x \neq y$. Since $\text{Nil}(R)$ is prime that is not essential, $\text{ann}_R(x) + \text{ann}_R(y)$ is not essential and thus x is not adjacent to y . Next we show that $Z(R) \setminus \text{Nil}(R)$ is infinite. If $Z(R) = \text{Nil}(R)$, then $Z(R)$ is an ideal of R which is not an essential ideal of R , a contradiction. Choose $x \in Z(R) \setminus \text{Nil}(R)$. If $x^m = x^n$, for some positive integers $n > m \geq 1$, then $x^m(1 - x^{n-m}) = 0$ and thus $1 - x^{n-m} \in \text{ann}_R(x^m)$. This implies that $Rx^{n-m} + \text{ann}_R(x^m) = R$. This contradicts the fact $\text{ann}_R(x^m) \subseteq \text{Nil}(R)$. Hence $Z(R) \setminus \text{Nil}(R)$ is infinite and so $\Gamma_g(R) = K_{|\text{Nil}(R)^*|} \vee \overline{K}_\infty$.

(2) $\Rightarrow$ (3) Since $\text{Nil}(R)$ is not an essential ideal of R , $Z(R) \neq \text{Nil}(R)$. Let $x \in Z(R) \setminus \text{Nil}(R)$. The equality $\Gamma(R) = K_{|\text{Nil}(R)^*|} \vee \overline{K}_\infty$ implies that $Z(R)$ is an ideal of R and thus $x \neq x^2$. Since $\text{Nil}(R)$ is a prime ideal of R that is not essential, we can easily get x is not adjacent to x^2 . This implies that $\Gamma_g(R)$ is not a complete graph. Thus by Theorem 2.3, $\Gamma(R) = \Gamma_g(R)$. Hence $\Gamma_g(R) = K_{|\text{Nil}(R)^*|} \vee \overline{K}_\infty$. The equality $\Gamma(R) = K_{|\text{Nil}(R)^*|} \vee \overline{K}_\infty$, for every $x \in Z(R)^*$, now shows that $\text{ann}_R(x) \in \{\text{Nil}(R), Z(R)\}$.

(3) $\Rightarrow$ (1) We show that $\text{ann}_R(x) = Z(R)$, for every $x \in \text{Nil}(R)^*$. For if not, one may suppose that $\text{ann}_R(x) = \text{Nil}(R)$, for some $x \in \text{Nil}(R)^*$. Since $\text{Nil}(R)$ is not an essential ideal of R , x is not adjacent to y , for every $y \in Z(R) \setminus \text{Nil}(R)$ (As $\text{Nil}(R)$ is prime), a contradiction. Thus $\text{ann}_R(x) = Z(R)$ and so $\Gamma(R) = K_{|\text{Nil}(R)^*|} \vee \overline{K}_\infty$. Hence $\Gamma(R) = \Gamma_g(R)$. $\square$

All conditions in Theorems 3.1 and 3.2 are necessary. In the following example, we examine condition (1) of Theorem 3.1 and the condition of $\text{Nil}(R)$ not to be essential in Theorem 3.2.

Example 3.2. Let $D = \mathbb{Z}_2[X, Y, W]$, $I = (X^2, Y^2, XY, XW)D$, and $R = D/I$. Consider three elements $x = X + I$, $y = Y + I$, and $w = W + I$ of R . Then $\text{Nil}(R) = (x, y)R$ and $Z(R) = (x, y, w)R$ is an ideal of R . It is not hard to check that $\text{Nil}(R)$ is a prime ideal of R and $\Gamma(R) \neq \Gamma_g(R)$ (As $y - w$ is an edge of $\Gamma_g(R)$ which is not an edge of $\Gamma(R)$). Since $\text{ann}_R(a) \neq Z(R)$, for some $a \in \text{Nil}(R)^*$, we have $\Gamma(R) \neq \Gamma_g(R)$.

Also, $\Gamma_g(R) = K_{|\text{Nil}(R)^*|} \vee \overline{K}_\infty$ and $\text{ann}_R(x) \in \{\text{Nil}(R), Z(R)\}$, for every $x \in Z(R)^*$. Since $\text{Nil}(R)$ is an essential ideal of R , $\Gamma(R) \neq K_{|\text{Nil}(R)^*|} \vee \overline{K}_\infty$.

We close this section with the following result.

Theorem 3.3. *Let R be a reduced ring and $x, y \in Z(R)$. Then the following statements are equivalent:*

- (1) $\Gamma(R) = \Gamma_g(R)$.
- (2) If $xy \neq 0$, then $\text{ann}_R(\text{ann}_R(x) + \text{ann}_R(y)) \neq (0)$.

Proof. (1) $\Rightarrow$ (2) Let $x, y \in Z(R)$ and $xy \neq 0$. If $x = y$, then $\text{ann}_R(x) = \text{ann}_R(y)$ and thus $\text{ann}_R(\text{ann}_R(x) + \text{ann}_R(y)) = \text{ann}_R(\text{ann}_R(x)) \neq (0)$. If $x \neq y$, then the equality $\Gamma(R) = \Gamma_g(R)$ shows that $\text{ann}_R(x) + \text{ann}_R(y)$ is not essential. Hence $\text{ann}_R(\text{ann}_R(x) + \text{ann}_R(y)) \neq (0)$ (note that if $\text{ann}_R(I) = (0)$, then I is an essential ideal of R).

(2) $\Rightarrow$ (1) Let $x - y$ be an edge of $\Gamma_g(R)$. It is enough to show that $xy = 0$. Assume to the contrary, $xy \neq 0$. Let $\text{ann}_R(x) + \text{ann}_R(y) = I$. Then $\text{ann}_R(I) \neq (0)$. Since R is reduced, $I \cap \text{ann}_R(I) = (0)$. This implies that I is not essential, a contradiction. Thus $xy = 0$, as desired. $\square$

4. Coloring of generalized zero-divisor graphs

The main goal of this section is to study the coloring of the generalized zero-divisor graphs associated with Artinian rings. It is shown that the graph $\Gamma_g(R)$ is weakly perfect, for every Artinian ring R . Moreover, the exact value of the $\chi(\Gamma_g(R))$ is given.

It is known that if R is a ring such that $\text{depth}(R) \neq 0$, then R is infinite. Also, Ganesan ([8]) proved that if R is infinite and $Z(R) \neq (0)$, then $Z(R)$ must be infinite (in fact $|R| \leq |Z(R)|^2$ when $2 \leq |Z(R)| < \infty$). In view of these facts, we state the following theorem.

Theorem 4.1. *Let R be a non-reduced indecomposable ring and $\omega(\Gamma_g(R)) < \infty$. Then the following statements are equivalent.*

- (1) $\text{ann}_R(x)$ is an essential ideal of R , for every $x \in Z(R)^*$.
- (2) $\Gamma_g(R)$ is a complete graph.
- (3) $\omega(\Gamma_g(R)) = \chi(\Gamma_g(R)) = |\text{Nil}(R)| - 1$.
- (4) R is an Artinian local ring.

Proof. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (3) Since $\Gamma_g(R)$ is a complete graph, $\omega(\Gamma_g(R)) = \chi(\Gamma_g(R))$. We show that $Z(R) = \text{Nil}(R)$. Let $x \in Z(R) \setminus \text{Nil}(R)$. Since R is an indecomposable ring, $x^m \neq x^n$, for every pair of distinct positive integers n, m . Since $\omega(\Gamma_g(R)) < \infty$, a contradiction and thus $\omega(\Gamma_g(R)) = \chi(\Gamma_g(R)) = |\text{Nil}(R)| - 1$.

(3) $\Rightarrow$ (4). We claim that $Z(R) = \text{Nil}(R)$. Let $x \in Z(R) \setminus \text{Nil}(R)$. By part (3) of Lemma 2.1, every vertex of $\text{Nil}(R)^*$ is adjacent to x in $\Gamma_g(R)$ and thus $\omega(\Gamma_g(R)) \geq |\text{Nil}(R)|$, a contradiction and so the claim is proved. Thus $\Gamma_g(R)$ is a complete graph. Since $\omega(\Gamma_g(R)) < \infty$, we conclude that $|Z(R)| < \infty$, and hence $|R| < \infty$. This means that R is an Artinian ring. As R is indecomposable, we deduce that R is local.

(4) $\Rightarrow$ (1) is clear, by part (2) of Lemma 2.1. $\square$

Theorem 4.2. *Let R be a non-reduced ring and $\dim(R) = 0$. If $|\text{Max}(R)| < \infty$ and $\omega(\Gamma_g(R)) < \infty$, then the following statements hold.*

- (1) $|\text{Max}(R)| \leq \omega(\Gamma_g(R))$.

(2) If R is nonlocal, then

$$\omega(\Gamma_g(R)) = \chi(\Gamma_g(R)) = |J(R)| + |\text{Max}(R)| - 1.$$

Proof. If R is local ($|\text{Max}(R)| = 1$), then there is nothing to prove. Let R be nonlocal. Since R is non-reduced, $|J(R)| > 1$ and hence it is enough to prove (2).

(2) By part (3) of Lemma 2.1, $\Gamma_g(R)[\text{Nil}(R)^*]$ is a complete subgraph of $\Gamma_g(R)$ and thus $|\text{Nil}(R)| < \infty$. This means that $\text{Nil}(R)$ is a nilpotent ideal of R . Since $\dim(R) = 0$, $\text{Min}(R) = \text{Max}(R)$ and so $\text{Nil}(R) = J(R)$. Hence $J(R)$ is a nilpotent ideal of R . Since $\text{Max}(R) < \infty$, $\mathfrak{m}_1^m \cdots \mathfrak{m}_n^m = (0)$, for some positive integers n, m . By the Chinese Remainder Theorem, $R \cong R_1 \times \cdots \times R_n$, where $\dim(R_i) = 0$, for every $1 \leq i \leq n$. This implies that $R_i = \text{Nil}(R_i) \cup U(R_i)$. Put:

$$A := \{(x_1, \dots, x_n) \in V(\Gamma_g(R)) \mid x_i \in \text{Nil}(R_i), \text{ for all } 1 \leq i \leq n\} \text{ and}$$

$$B := \{(x_1, \dots, x_n) \in V(\Gamma_g(R)) \mid x_i \notin \text{Nil}(R_i), \text{ for some } 1 \leq i \leq n\}.$$

It is not hard to check that $V(\Gamma_g(R)) = A \cup B$, $A \cap B = \emptyset$ and so $\{A, B\}$ is a partition of $V(\Gamma_g(R))$. We show that $\Gamma_g(R) = \Gamma_g(R)[A] \vee \Gamma_g(R)[B]$, where $\Gamma_g(R)[A]$ is a complete subgraph of $\Gamma_g(R)$ and $\Gamma_g(R)[B]$ is an n -partite subgraph of $\Gamma_g(R)$ which is not an $(n-1)$ -partite subgraph of $\Gamma_g(R)$. To see this, by Part (3) of Lemma 2.1, $\Gamma_g(R)[A]$ is a complete subgraph of $\Gamma_g(R)$ and every vertex $x \in A$ is adjacent to all other vertices.

Now, let $B_i = \{(x_1, \dots, x_n) \in B \mid x_i \in U(R_i)\}$, for every $1 \leq i \leq n$. It is easy to see that there is no adjacency between two vertices of B_i , for every $1 \leq i \leq n$. This together with this fact that the set $\{(1, 0, 0, \dots, 0), (0, 1, 0, \dots, 0), \dots, (0, 0, \dots, 0, 1)\}$ is a clique of $EG(R)[B]$ imply that $\Gamma_g(R)[B]$ is an n -partite subgraph of $\Gamma_g(R)$ which is not an $(n-1)$ -partite subgraph of $\Gamma_g(R)$. Therefore, $\Gamma_g(R) = \Gamma_g(R)[A] \vee \Gamma_g(R)[B]$ and so $\omega(\Gamma_g(R)) = \chi(\Gamma_g(R)) = \omega(\Gamma_g(R)[A]) + \omega(\Gamma_g(R)[B]) = |\text{Nil}(R)| - 1 + |\text{Max}(R)|$. $\square$

Corollary 4.1. *Let R be a non-reduced ring and $\dim(R) = 0$. Then the following statements hold.*

(1) If $\omega(\Gamma_g(R)) < \infty$, then R is an Artinian ring.

(2) If R is an Artinian ring, then $\omega(\Gamma_g(R)) = \chi(\Gamma_g(R)) = |J(R)| + |\text{Max}(R)| - 1$.

We close this paper with the following example.

Example 4.1. Let $R = \mathbb{Z}_2[X, Y]/(XY, X^2)$, $x = X + (XY + X^2)$ and $y = Y + (XY + X^2)$. Then $Z(R) = (x, y)R$, $\text{Nil}(R) = \{0, x\}$ and $\Gamma_g(R) = K_1 \vee \overline{K}_\infty$. So $\omega(\Gamma_g(R)) = 2 < \infty$. Since $\dim(R) \neq 0$, R is not an Artinian ring.

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THE FIXED POINT OF MEROMORPHIC SOLUTIONS FOR DIFFERENCE RICCATI EQUATION

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Abstract. In this paper, we mainly investigate some properties of the transcendental meromorphic solution $f(z)$ for the difference Riccati equation

$$f(z+1) = \frac{P_1(z)f(z) + P_2(z)}{f(z) + P_3(z)},$$

where $P_i(z)$ ($i = 1, 2, 3$) are polynomials. And we obtain some estimates of exponents of convergence of fixed points and c -points of $f(z)$ and its shift $f(z+n)$.

Keywords: difference Riccati equation, fixed point, admissible solution.

1. Introduction and main results

Early results for difference equations were largely motivated by the work of Kimura [13] on the iteration of analytic functions. Shimomura [19] and Yanagihara [20] proved the following theorems, respectively.

Theorem A ([19]). *For any polynomial $P(y)$, the difference equation*

$$y(z+1) = P(y(z))$$

has a non-trivial entire solution.

Theorem B ([20]). *For any rational function $R(y)$, the difference equation*

$$y(z+1) = R(y(z))$$

has a non-trivial meromorphic solution.

In this paper, we assume that the reader is familiar with the basic notions of Nevanlinna value distribution theory (see, e.g., [11, 14, 21]). In addition, we use the notations $\sigma(f)$ to denote the order of growth of the meromorphic function $f(z)$, $\lambda(f)$ and $\lambda(\frac{1}{f})$ to denote the exponents of convergence of zeros and poles of $f(z)$, respectively. Moreover, we say that a meromorphic function $g(z)$ is small with respect to $f(z)$ if $T(r, g) = S(r, f)$, where $S(r, f) = o(T(r, f))$ outside of a

possible exceptional set of finite logarithmic measure. And we denote by $S(f)$ the family of all meromorphic functions which are small compared to $f(z)$. We say that a meromorphic solution $f(z)$ of a difference equation is admissible if all coefficients of the equation are in $S(f)$. We also use the notation $\tau(f)$ to denote the exponent of convergence of fixed points of $f(z)$ that is defined as

$$\tau(f) = \overline{\lim}_{r \rightarrow \infty} \frac{\log N\left(r, \frac{1}{f-z}\right)}{\log r}.$$

Recently, a number of papers (including [1-6, 8-10, 12,15-18]) focused on complex difference equations and difference analogues of Nevanlinna theory. As the difference analogues of Nevanlinna theory were being investigated, many results on the complex difference equations have been got rapidly. Many papers (including [1, 6, 10, 12]) mainly dealt with the growth of meromorphic solutions of difference equations.

In [10], Halburd and Korhonen used value distribution theory to obtain the following Theorem C.

Theorem C ([10]). *Let $f(z)$ be an admissible finite order meromorphic solution of the equation*

$$(1.1) \quad f(z+1)f(z-1) = \frac{c_2(f(z) - c_+)(f(z) - c_-)}{(f(z) - a_+)(f(z) - a_-)} =: R(z, f(z)),$$

where the coefficients are meromorphic functions, $c_2 \neq 0$ and $\deg_f(R) = 2$. If the order of the poles of $f(z)$ is bounded, then either $f(z)$ satisfies a difference Riccati equation

$$f(z+1) = \frac{p(z)f(z) + q(z)}{f(z) + s(z)},$$

where $p, q, s \in S(f)$, or equation (1.1) can be transformed by a bilinear change in $f(z)$ to one of the equations

$$f(z+1)f(z-1) = \frac{\gamma f^2(z) + \delta \lambda^z f(z) + \gamma \mu \lambda^{2z}}{(f(z) - 1)(f(z) - \gamma)},$$

$$f(z+1)f(z-1) = \frac{f^2(z) + \delta e^{i\pi z/2} \lambda^z f(z) + \mu \lambda^{2z}}{f^2(z) - 1},$$

where $\lambda \in \mathbb{C}$, and $\delta, \mu, \gamma, \underline{\gamma} = \gamma(z-1) \in S(f)$ are arbitrary finite order periodic functions such that δ and γ have period 2 and μ has period 1.

Chen [4] obtained the following Theorem D.

Theorem D ([4]). *Let $P_j(z)$, $j = 1, 2, 3$ be nonzero polynomials, such that*

$$\deg P_3(z) > \max\{\deg P_j(z) : j = 1, 2\}.$$

If $c \in \mathbb{C} \setminus \{0\}$, then every finite order transcendental meromorphic solution $f(z)$ of equation

$$f(z+1) = \frac{P_1(z)f(z)}{P_2(z)f(z) + P_3(z)}$$

satisfies:

- (i) $\lambda(f(z+n) - c) = \sigma(f) \geq 1$, $n = 0, 1, 2, \dots$;
- (ii) if $\deg P_1 \neq \deg P_2$, then $\lambda\left(\frac{\Delta f(z)}{f(z)} - c\right) = \sigma(f)$;
- (iii) if there is a polynomial $h(z)$ satisfying

$$(P_2(z) - P_1(z) + cP_3(z))^2 - 4cP_2(z)P_3(z) = h(z)^2,$$

then $\lambda(\Delta f(z) - c) = \sigma(f)$.

From the above, we see that the difference Riccati equation is an important class of difference equations, it will play an important role in research of difference Painlevé equations. Some papers [2-4] dealt with complex difference Riccati equations.

In this research, we investigate some properties of finite order transcendental meromorphic solution for certain difference Riccati equation, and obtain the following theorems.

Theorem 1.1. Let $P_j(z)$ ($j = 1, 2, 3$) be nonzero polynomials, such that there exists an integer l ($1 \leq l \leq 3$) with

$$(1.2) \quad \deg P_l(z) > \max_{j \neq l} \{\deg P_j(z)\}.$$

If $c \in \mathbb{C}$, then every finite order transcendental meromorphic solution $f(z)$ of the difference equation

$$(1.3) \quad f(z+1) = \frac{P_1(z)f(z) + P_2(z)}{f(z) + P_3(z)},$$

where $P_1(z)f(z) + P_2(z)$ and $f(z) + P_3(z)$ are relatively prime polynomials in f , satisfies $\lambda(f(z+n) - c) = \sigma(f)$, $n = 0, 1, 2, \dots$

Theorem 1.2. Let $P_j(z)$ ($j = 1, 2, 3$) be nonzero polynomials, such that

$$(1.4) \quad \deg P_1(z) > \max\{\deg P_2(z), \deg P_3(z)\} \quad \text{and} \quad \deg P_1(z) \geq 2,$$

or

$$(1.5) \quad \deg P_3(z) > \max\{\deg P_2(z), \deg P_1(z)\} \quad \text{and} \quad \deg P_3(z) \geq 2.$$

Then every finite order transcendental meromorphic solution $f(z)$ of the difference equation (1.3) satisfies $\tau(f(z+n)) = \sigma(f)$, $n = 0, 1, 2, \dots$

2. Proof of Theorem 1.1

We need the following lemmas to prove Theorem 1.1.

Lemma 2.1 ([8,15]). *Let f be a transcendental meromorphic solution of finite order σ of the difference equation*

$$P(z, f) = 0,$$

where $P(z, f)$ is a difference polynomial in $f(z)$ and its shifts. If $P(z, a) \not\equiv 0$ for a slowly moving target function a , i.e. $T(r, a) = S(r, f)$, then

$$m\left(r, \frac{1}{f-a}\right) = S(r, f)$$

outside of a possible exceptional set of finite logarithmic measure.

Remark 2.1. Using the same method as in the proof of Lemma 2.1 (see [8]), we can prove that in Lemma 2.1, if all coefficients $b_\lambda(z)$ of $P(z, f(z))$ satisfy $\sigma(b_\lambda(z)) = \sigma_1 < \sigma(f(z)) = \sigma$, and if $P(z, a) \not\equiv 0$ for a meromorphic function $a(z)$ satisfying $T(r, a) = S(r, f)$, then for a given $\varepsilon (0 < \varepsilon < \sigma - \sigma_1)$

$$m\left(r, \frac{1}{f(z)-a}\right) = S(r, f(z)) + O(r^{\sigma_1+\varepsilon})$$

holds for all r outside of a possible exceptional set of finite logarithmic measure.

Lemma 2.2 ([15]). *Let f be a transcendental meromorphic solution of finite order σ of a difference equation of the form*

$$H(z, f)P(z, f) = Q(z, f),$$

where $H(z, f)$ is a difference product of total degree n in $f(z)$ and its shifts, and where $P(z, f)$, $Q(z, f)$ are difference polynomials such that the total degree $\deg Q(z, f) \leq n$. Then for each $\varepsilon > 0$,

$$m(r, P(z, f)) = O(r^{\sigma-1+\varepsilon}) + S(r, f)$$

possibly outside of an exceptional set of finite logarithmic measure.

Lemma 2.3 (Valiron-Mohon'ko, see [14]). *Let $f(z)$ be a meromorphic function. Then for all irreducible rational function in $f(z)$,*

$$R(z, f(z)) = \frac{a_0(z) + a_1(z)f(z) + \dots + a_m(z)f(z)^m}{b_0(z) + b_1(z)f(z) + \dots + b_n(z)f(z)^n}$$

with meromorphic coefficients $a_i(z) (i = 0, 1, \dots, m)$, $b_j(z) (j = 0, 1, \dots, n)$, the characteristic function of $R(z, f(z))$ satisfies

$$T(r, R(z, f(z))) = dT(r, f) + O(\Psi(r)),$$

where $d = \deg_f R = \max\{m, n\}$ and $\Psi(r) = \max_{i,j}\{T(r, a_i), T(r, b_j)\}$.

In the remark of [10, p.15], it is pointed out that the following Lemma 2.4 holds.

Lemma 2.4 ([2]). *Let f be a nonconstant finite order meromorphic function. Then*

$$N(r+1, f) = N(r, f) + S(r, f), \quad T(r+1, f) = T(r, f) + S(r, f)$$

outside of a possible exceptional set of finite logarithmic measure.

Remark 2.2. In [6], Chiang and Feng proved that let f be a meromorphic function with exponent of convergence of poles $\lambda(\frac{1}{f}) = \lambda < \infty$, $\eta \neq 0$ be fixed, then for each $\varepsilon > 0$,

$$N(r, f(z + \eta)) = N(r, f) + O(r^{\lambda-1+\varepsilon}) + O(\log r).$$

Lemma 2.5 ([6]). *Let $f(z)$ be a meromorphic function with order $\sigma = \sigma(f) < +\infty$, and let η be a fixed nonzero complex number, then for each $\varepsilon > 0$, we have*

$$T(r, f(z + \eta)) = T(r, f) + O(r^{\sigma-1+\varepsilon}) + O(\log r).$$

Lemma 2.6. ([7]). *Let $g : (0, +\infty) \rightarrow R$, $h : (0, +\infty) \rightarrow R$ be non-decreasing functions. If (i) $g(r) \leq h(r)$ outside of an exceptional set of finite linear measure, or (ii) $g(r) \leq h(r)$, $r \notin H \cup (0, 1]$, where $H \subset (1, \infty)$ is a set of finite logarithmic measure, then for any $\alpha > 1$, there exists $r_0 > 0$ such that $g(r) \leq h(\alpha r)$ for all $r > r_0$.*

Lemma 2.7. *Suppose that $P_j(z)$ ($j = 1, 2, 3$) satisfy conditions in Theorem 1.1, $c \in \mathbb{C}$ is a constant, and $f(z)$ is a nonconstant meromorphic function. Then*

$$f_1(z) = (P_1(z) - c)f(z) + P_2(z) - cP_3(z)$$

and

$$f_2(z) = f(z) + P_3(z)$$

have at most finitely many common zeros.

Proof. Suppose that z_0 is a common zero of $f_1(z)$ and $f_2(z)$. Then $f_2(z_0) = f(z_0) + P_3(z_0) = 0$. Thus, $f(z_0) = -P_3(z_0)$. Substituting $f(z_0) = -P_3(z_0)$ into $f_1(z)$, we obtain

$$f_1(z_0) = -(P_1(z_0) - c)P_3(z_0) + P_2(z_0) - cP_3(z_0) = -P_1(z_0)P_3(z_0) + P_2(z_0) = 0.$$

Since $P_1(z)f(z) + P_2(z)$ and $f(z) + P_3(z)$ are relatively prime polynomials in f , we get $-P_1(z)P_3(z) + P_2(z) \not\equiv 0$. And since $-P_1(z)P_3(z) + P_2(z)$ has only finitely many zeros, we see that $f_1(z)$ and $f_2(z)$ have at most finitely many common zeros.

Using the similar method as in the proof of Lemma 2.7, we can prove the following Lemma 2.8.

Lemma 2.8. *Suppose that $P_j(z)$ ($j = 1, 2, 3$) satisfy conditions in Theorem 1.2, and $f(z)$ is a nonconstant meromorphic function. Then*

$$f_1(z) = (P_1(z) - z)f(z) + P_2(z) - zP_3(z)$$

and

$$f_2(z) = f(z) + P_3(z)$$

have at most finitely many common zeros.

Proof of Theorem 1.1. Suppose that $f(z)$ is a finite order transcendental meromorphic solution of equation (1.3). We divide this proof into the following two cases.

Case 1. Suppose that $c = 0$.

Firstly, we suppose that $n = 0$. By (1.3), we get

$$(2.1) \quad H_0(z, f) := f(z+1)(f(z) + P_3(z)) - P_1(z)f(z) - P_2(z) = 0.$$

Thus,

$$H_0(z, 0) = -P_2(z) \neq 0$$

since $P_2(z)$ is a nonzero polynomial. Thus, by Lemma 2.1, we have that

$$(2.2) \quad m\left(r, \frac{1}{f(z)}\right) = S(r, f(z))$$

holds for all r outside of a possible exceptional set with finite logarithmic measure. So, by (2.2), we obtain

$$(2.3) \quad N\left(r, \frac{1}{f(z)}\right) = T(r, f(z)) + S(r, f(z))$$

holds for all r outside of a possible exceptional set with finite logarithmic measure. Hence, by Lemma 2.6 and (2.3), we see that $\lambda(f(z)) = \sigma(f(z))$.

Now suppose that $n = 1$. Set $g(z) = f(z+1)$. Thus, by (1.3), we get

$$(2.4) \quad g(z+1) = \frac{P_1(z+1)g(z) + P_2(z+1)}{g(z) + P_3(z+1)}.$$

For (2.4), applying the conclusion for $n = 0$ above and Lemma 2.5, we obtain

$$\lambda(f(z+1)) = \lambda(g(z)) = \sigma(g(z)) = \sigma(f(z)).$$

Continuing to use the similar method as above, we can obtain

$$\lambda(f(z+n)) = \sigma(f(z)) \quad n = 0, 1, 2, \dots$$

Case 2. Suppose that $c \in \mathbb{C} \setminus \{0\}$. Firstly, we suppose that $n = 0$. By (1.3), we get (2.1). Thus, by (2.1), we get

$$H_0(z, c) = cP_3(z) - cP_1(z) - P_2(z) + c^2.$$

By $c \neq 0$ and (1.2), we see that $H_0(z, c) \not\equiv 0$. Thus, by Lemma 2.1, we have that

$$m\left(r, \frac{1}{f(z) - c}\right) = S(r, f(z))$$

holds for all r outside of a possible exceptional set with finite logarithmic measure. So, we have that

$$(2.5) \quad N\left(r, \frac{1}{f(z) - c}\right) = T(r, f(z)) + S(r, f(z))$$

holds for all r outside of a possible exceptional set with finite logarithmic measure. So, by Lemma 2.6 and (2.5), we obtain $\lambda(f(z) - c) = \sigma(f(z))$.

Now suppose that $n = 1$, By (1.3), we obtain

$$(2.6) \quad f(z+1) - c = \frac{(P_1(z) - c)f(z) + P_2(z) - cP_3(z)}{f(z) + P_3(z)}.$$

If $P_1(z) = c$, then by (2.6) we have that

$$(2.7) \quad f(z+1) - c = \frac{P_2(z) - cP_3(z)}{f(z) + P_3(z)}.$$

By (1.2) and $P_1(z) = c$, we get $\deg P_2(z) \neq \deg P_3(z)$. Thus $P_2(z) - cP_3(z) \not\equiv 0$. From (2.7), we see that if $f(z) = \infty$, then $f(z+1) - c = 0$. Thus, we get $\lambda(f(z+1) - c) = \lambda(\frac{1}{f})$.

By (1.3), we get

$$(2.8) \quad (f(z) + P_3(z))f(z+1) = P_1(z)f(z) + P_2(z).$$

By Lemma 2.2 and (2.8), we have that

$$(2.9) \quad m(r, f(z+1)) = O(r^{\sigma(f)-1+\varepsilon}) + S(r, f)$$

holds for all r outside of a possible exceptional set with finite logarithmic measure. By Lemma 2.3, we get

$$(2.10) \quad T(r, f(z+1)) = T(r, f(z)) + S(r, f).$$

From Lemma 2.4, we obtain

$$(2.11) \quad N(r, f(z+1)) \leq N(r+1, f(z)) = N(r, f(z)) + S(r, f)$$

holds for all r outside of a possible exceptional set with finite logarithmic measure. Thus, by (2.9)-(2.11), we get

$$(2.12) \quad T(r, f(z)) \leq N(r, f(z)) + O(r^{\sigma(f)-1+\varepsilon}) + S(r, f)$$

holds for all r outside of a possible exceptional set with finite logarithmic measure. Hence, by Lemma 2.6 and (2.12), we see that $\lambda\left(\frac{1}{f}\right) = \sigma(f)$. Furthermore, we get $\lambda(f(z+1) - c) = \sigma(f)$.

If $P_1(z) - c \neq 0$ and $cP_3(z) \neq P_2(z)$, then by (2.6) we have that

$$(2.13) \quad f(z+1) - c = \frac{(P_1(z) - c) \left(f(z) - \frac{cP_3(z) - P_2(z)}{P_1(z) - c} \right)}{f(z) + P_3(z)}.$$

By Lemma 2.7, we see that $\left(f(z) - \frac{cP_3(z) - P_2(z)}{P_1(z) - c} \right)$ and $f(z) + P_3(z)$ have at most finitely many common zeros. So by (2.13), we only need to prove that

$$(2.14) \quad \lambda \left(f(z) - \frac{cP_3(z) - P_2(z)}{P_1(z) - c} \right) = \sigma(f(z)).$$

Suppose that $\lambda(f(z) - \frac{cP_3(z) - P_2(z)}{P_1(z) - c}) < \sigma(f(z))$. By $\sigma(f(z) - \frac{cP_3(z) - P_2(z)}{P_1(z) - c}) = \sigma(f(z))$ and Hadamard factorization theorem, $f(z) - \frac{cP_3(z) - P_2(z)}{P_1(z) - c}$ can be rewritten as a form

$$(2.15) \quad f(z) - \frac{cP_3(z) - P_2(z)}{P_1(z) - c} = z^m \frac{b_0(z)}{H_0(z)} e^{h(z)},$$

where $h(z)$ is a polynomial with $\deg h(z) \leq \sigma(f(z))$, $b_0(z)$ and $H_0(z)$ are canonical products ($b_0(z)$ may be a polynomial) formed by nonzero zeros and poles of $f(z) - \frac{cP_3(z) - P_2(z)}{P_1(z) - c}$ respectively, m is an integer, if $m \geq 0$ then $b(z) = z^m b_0(z)$, $H(z) = H_0(z) e^{-h(z)}$; if $m < 0$ then $b(z) = b_0(z)$, $H(z) = z^{-m} H_0(z) e^{-h(z)}$. By property of canonical product, we see that

$$(2.16) \quad \begin{cases} \lambda(b(z)) = \sigma(b(z)) = \lambda \left(f(z) - \frac{cP_3(z) - P_2(z)}{P_1(z) - c} \right) < \sigma(f(z)), \\ \lambda(H(z)) = \sigma(H(z)) = \sigma(f(z)). \end{cases}$$

By (2.15), we obtain

$$(2.17) \quad \begin{cases} f(z) = \frac{cP_3(z) - P_2(z)}{P_1(z) - c} + b(z)y(z), \\ f(z+1) = \frac{cP_3(z+1) - P_2(z+1)}{P_1(z+1) - c} + b(z+1)y(z+1), \end{cases}$$

where $y(z) = \frac{1}{H(z)}$. Thus, by (2.16) and Lemma 2.5, we have that

$$\sigma(y(z)) = \sigma(H(z)) = \sigma(f(z)), \quad \sigma(b(z+1)) = \sigma(b(z)) < \sigma(y(z)).$$

Substituting (2.17) into (1.3), we obtain

$$\begin{aligned}
 (2.18) \quad K_1(z, y) &:= \left\{ \frac{cP_3(z+1) - P_2(z+1)}{P_1(z+1) - c} + b(z+1)y(z+1) \right\} \\
 &\cdot \left\{ \frac{cP_3(z) - P_2(z)}{P_1(z) - c} + b(z)y(z) + P_3(z) \right\} \\
 &- P_1(z) \left\{ \frac{cP_3(z) - P_2(z)}{P_1(z) - c} + b(z)y(z) \right\} - P_2(z) = 0.
 \end{aligned}$$

By (2.18), we see that

$$\begin{aligned}
 K_1(z, 0) &= \frac{cP_3(z+1) - P_2(z+1)}{P_1(z+1) - c} \cdot \left\{ \frac{cP_3(z) - P_2(z)}{P_1(z) - c} + P_3(z) \right\} \\
 &- P_1(z) \frac{cP_3(z) - P_2(z)}{P_1(z) - c} - P_2(z) \\
 &= \frac{P_3(z)P_1(z) - P_2(z)}{P_1(z) - c} \left(\frac{cP_3(z+1) - P_2(z+1)}{P_1(z+1) - c} - c \right).
 \end{aligned}$$

That is

$$K_1(z, 0) = \frac{P_3(z)P_1(z) - P_2(z)}{P_1(z) - c} \cdot \frac{cP_3(z+1) - P_2(z+1) - cP_1(z+1) + c^2}{P_1(z+1) - c}.$$

Since $P_1(z)f(z) + P_2(z)$ and $f(z) + P_3(z)$ are relatively prime polynomials in f , we see that $P_3(z)P_1(z) - P_2(z) \not\equiv 0$. And by (1.2), we get $cP_3(z+1) - P_2(z+1) - cP_1(z+1) + c^2 \not\equiv 0$. Thus, we obtain

$$(2.19) \quad K_1(z, 0) \not\equiv 0.$$

Thus, by (2.16), (2.19), Lemma 2.1 and its Remark 2.1, we obtain for any given $\varepsilon (0 < \varepsilon < \sigma(f(z)) - \sigma(b(z)))$

$$(2.20) \quad N\left(r, \frac{1}{y(z)}\right) = T(r, y(z)) + S(r, y(z)) + O(r^{\sigma(b(z)) + \varepsilon})$$

holds for all r outside of a possible exceptional set with finite logarithmic measure.

On the other hand, by $y(z) = \frac{1}{H(z)}$ and the fact that $H(z)$ is an entire function, we see that

$$N\left(r, \frac{1}{y(z)}\right) = N(r, H(z)) = 0.$$

Thus, (2.20) is a contradiction. Hence, $\lambda\left(f(z) - \frac{cP_3(z) - P_2(z)}{P_1(z) - c}\right) = \sigma(f(z))$.

If $P_1(z) - c \not\equiv 0$ and $cP_3(z) - P_2(z) \equiv 0$, then by (2.6) we have that

$$(2.21) \quad f(z+1) - c = \frac{(P_1(z) - c)f(z)}{f(z) + P_3(z)}.$$

By Lemma 2.7, we see that $(P_1(z) - c)f(z)$ and $f(z) + P_3(z)$ have at most finitely many common zeros. So that we have that $\lambda(f(z+1) - c) = \lambda(f(z))$. By Case 1, we see that $\lambda(f(z)) = \sigma(f(z))$. Thus, we obtain $\lambda(f(z+1) - c) = \sigma(f(z))$.

Now suppose that $n = 2$, By (2.6), we have that

$$(2.22) \quad f(z+2) - c = \frac{(P_1(z+1) - c)f(z+1) + P_2(z+1) - cP_3(z+1)}{f(z+1) + P_3(z+1)}.$$

Set $g(z) = f(z+1)$. Thus, (2.22) is transformed as

$$(2.23) \quad g(z+1) - c = \frac{(P_1(z+1) - c)g(z) + P_2(z+1) - cP_3(z+1)}{g(z) + P_3(z+1)}.$$

Since $\deg P_j(z+1) = \deg P_j(z)$, thus, $P_j(z+1)$ ($j = 1, 2, 3$) satisfy (1.2). For (2.23), applying the conclusion for $n = 1$ above, we obtain $\lambda(f(z+2) - c) = \lambda(g(z+1) - c) = \sigma(g(z)) = \sigma(f(z))$. Continuing to use the same method as above, we can obtain $\lambda(f(z+n) - c) = \sigma(f(z)) \quad n = 0, 1, 2, \dots$

Theorem 1.1 is proved.

3. Proof of Theorem 1.2

Suppose that $f(z)$ is a finite order transcendental meromorphic solution of equation (1.3).

Firstly, we suppose that $n = 0$. Set $f(z) - z = g(z)$. So, $g(z)$ is transcendental, $T(r, g(z)) = T(r, f(z)) + O(\log r)$ and $S(r, g) = S(r, f)$. Substituting $f(z) = z + g(z)$ into (1.3), we get $H_0(z, g) := (g(z+1) + z+1)(g(z) + z + P_3(z)) - P_1(z)(g(z) + z) - P_2(z) = 0$. Thus,

$$(3.1) \quad H_0(z, 0) = (z+1)P_3(z) - zP_1(z) - P_2(z) + z(z+1).$$

By (1.4) or (1.5) and (3.1), we see that $H_0(z, 0) \not\equiv 0$. Thus, by Lemma 2.1 and $H_0(z, 0) \not\equiv 0$, we have that

$$N\left(r, \frac{1}{g(z)}\right) = T(r, g(z)) + S(r, g(z))$$

holds for all r outside of a possible exceptional set with finite logarithmic measure. So, we obtain

$$(3.2) \quad N\left(r, \frac{1}{f(z) - z}\right) = T(r, f(z)) + S(r, f(z))$$

holds for all r outside of a possible exceptional set with finite logarithmic measure. Hence, by Lemma 2.6 and (3.2), we see that $\tau(f(z)) = \sigma(f(z))$.

Now suppose that $n = 1$, By (1.3), we obtain

$$(3.3) \quad f(z+1) - z = \frac{(P_1(z) - z)f(z) + P_2(z) - zP_3(z)}{f(z) + P_3(z)}.$$

If $P_1(z) = z$, then by (3.3) we have that

$$(3.4) \quad f(z+1) - z = \frac{P_2(z) - zP_3(z)}{f(z) + P_3(z)}.$$

By $P_1(z) = z$, we see that $P_1(z)$, $P_2(z)$, $P_3(z)$ satisfy (1.5). Thus $P_2(z) - zP_3(z) \not\equiv 0$. From (3.4), we see that if $f(z) = \infty$, then $f(z+1) - z = 0$. Thus, we get $\tau(f(z+1)) = \lambda\left(\frac{1}{f}\right)$. From the Proof of Theorem 1.1, we see that $\lambda\left(\frac{1}{f}\right) = \sigma(f)$. Furthermore, we get $\tau(f(z+1)) = \sigma(f)$.

If $P_1(z) - z \not\equiv 0$ and $zP_3(z) \not\equiv P_2(z)$, then by (3.3), we have that

$$(3.5) \quad f(z+1) - z = \frac{(P_1(z) - z) \left(f(z) - \frac{zP_3(z) - P_2(z)}{P_1(z) - z} \right)}{f(z) + P_3(z)}.$$

By Lemma 2.8, we see that $\left(f(z) - \frac{zP_3(z) - P_2(z)}{P_1(z) - z} \right)$ and $f(z) + P_3(z)$ have at most finitely many common zeros. So by (3.5), we only need to prove that

$$(3.6) \quad \lambda \left(f(z) - \frac{zP_3(z) - P_2(z)}{P_1(z) - z} \right) = \sigma(f(z)).$$

Suppose that $\lambda(f(z) - \frac{zP_3(z) - P_2(z)}{P_1(z) - z}) < \sigma(f(z))$. By $\sigma(f(z) - \frac{zP_3(z) - P_2(z)}{P_1(z) - z}) = \sigma(f(z))$ and Hadamard factorization theorem, $f(z) - \frac{zP_3(z) - P_2(z)}{P_1(z) - z}$ can be rewritten as a form

$$(3.7) \quad f(z) - \frac{zP_3(z) - P_2(z)}{P_1(z) - z} = z^m \frac{b_0(z)}{H_0(z)} e^{h(z)},$$

where $h(z)$ is a polynomial with $\deg h(z) \leq \sigma(f(z))$, $b_0(z)$ and $H_0(z)$ are canonical products ($b_0(z)$ may be a polynomial) formed by nonzero zeros and poles of $f(z) - \frac{zP_3(z) - P_2(z)}{P_1(z) - z}$ respectively, m is an integer, if $m \geq 0$ then $b(z) = z^m b_0(z)$, $H(z) = H_0(z) e^{-h(z)}$; if $m < 0$ then $b(z) = b_0(z)$, $H(z) = z^{-m} H_0(z) e^{-h(z)}$. By property of canonical product, we see that

$$(3.8) \quad \begin{cases} \lambda(b(z)) = \sigma(b(z)) = \lambda \left(f(z) - \frac{zP_3(z) - P_2(z)}{P_1(z) - z} \right) < \sigma(f(z)), \\ \lambda(H(z)) = \sigma(H(z)) = \sigma(f(z)). \end{cases}$$

By (3.7), we obtain

$$(3.9) \quad \begin{cases} f(z) = \frac{zP_3(z) - P_2(z)}{P_1(z) - z} + b(z)y(z), \\ f(z+1) = \frac{(z+1)P_3(z+1) - P_2(z+1)}{P_1(z+1) - (z+1)} + b(z+1)y(z+1), \end{cases}$$

where $y(z) = \frac{1}{H(z)}$. Thus, by (3.8) and Lemma 2.5, we have that

$$\sigma(y(z)) = \sigma(H(z)) = \sigma(f(z)), \quad \sigma(b(z+1)) = \sigma(b(z)) < \sigma(y(z)).$$

Substituting (3.9) into (1.3), we obtain

$$\begin{aligned}
 (3.10) \quad K_1(z, y) &:= \left\{ \frac{(z+1)P_3(z+1) - P_2(z+1)}{P_1(z+1) - (z+1)} + b(z+1)y(z+1) \right\} \\
 &\cdot \left\{ \frac{zP_3(z) - P_2(z)}{P_1(z) - z} + b(z)y(z) + P_3(z) \right\} \\
 &- P_1(z) \left\{ \frac{zP_3(z) - P_2(z)}{P_1(z) - z} + b(z)y(z) \right\} - P_2(z) = 0.
 \end{aligned}$$

By (3.10), we see that

$$\begin{aligned}
 K_1(z, 0) &= \frac{(z+1)P_3(z+1) - P_2(z+1)}{P_1(z+1) - (z+1)} \cdot \left\{ \frac{zP_3(z) - P_2(z)}{P_1(z) - z} + P_3(z) \right\} \\
 &- P_1(z) \frac{zP_3(z) - P_2(z)}{P_1(z) - z} - P_2(z) \\
 &= \frac{P_3(z)P_1(z) - P_2(z)}{P_1(z) - z} \left(\frac{(z+1)P_3(z+1) - P_2(z+1)}{P_1(z+1) - (z+1)} - z \right).
 \end{aligned}$$

That is

$$\begin{aligned}
 K_1(z, 0) &= \frac{P_3(z)P_1(z) - P_2(z)}{P_1(z) - z} \\
 &\cdot \frac{(z+1)P_3(z+1) - zP_1(z+1) - P_2(z+1) + z(z+1)}{P_1(z+1) - (z+1)}.
 \end{aligned}$$

Since $P_1(z)f(z) + P_2(z)$ and $f(z) + P_3(z)$ are relatively prime polynomials in f , we see that $P_3(z)P_1(z) - P_2(z) \neq 0$. And by (1.4) or (1.5), we get $(z+1)P_3(z+1) - zP_1(z+1) - P_2(z+1) + z(z+1) \neq 0$. Thus, we obtain

$$(3.11) \quad K_1(z, 0) \neq 0.$$

Thus, by (3.8), (3.11), Lemma 2.1 and its Remark 2.1, we obtain for any given $\varepsilon (0 < \varepsilon < \sigma(f(z)) - \sigma(b(z)))$

$$(3.12) \quad N\left(r, \frac{1}{y(z)}\right) = T(r, y(z)) + S(r, y(z)) + O(r^{\sigma(b(z)) + \varepsilon})$$

holds for all r outside of a possible exceptional set with finite logarithmic measure.

On the other hand, by $y(z) = \frac{1}{H(z)}$ and the fact that $H(z)$ is an entire function, we see that

$$N\left(r, \frac{1}{y(z)}\right) = N(r, H(z)) = 0.$$

Thus, (3.12) is a contradiction. Hence, $\lambda(f(z) - \frac{zP_3(z) - P_2(z)}{P_1(z) - z}) = \sigma(f(z))$. Furthermore, $\tau(f(z+1)) = \lambda(f(z+1) - z) = \lambda(f(z) - \frac{zP_3(z) - P_2(z)}{P_1(z) - z}) = \sigma(f(z))$.

If $P_1(z) - z \not\equiv 0$ and $zP_3(z) - P_2(z) \equiv 0$, then by (3.3) we have that $f(z+1) - z = \frac{(P_1(z)-z)f(z)}{f(z)+P_3(z)}$. By Lemma 2.8, we see that $(P_1(z) - z)f(z)$ and $f(z) + P_3(z)$ have at most finitely many common zeros. So that we only need to prove that $\lambda(f(z+1) - z) = \lambda(f(z))$. By Case 1 of Theorem 1.1, we see that $\lambda(f(z)) = \sigma(f(z))$. Thus, we obtain $\lambda(f(z+1) - z) = \sigma(f(z))$, that is $\tau(f(z+1)) = \sigma(f(z))$.

Now suppose that $n = 2$, By (1.3), we have that

$$(3.13) \quad g(z+1) = \frac{P_1(z+1)g(z) + P_2(z+1)}{g(z) + P_3(z+1)},$$

where $g(z) = f(z+1)$. By Lemma 2.5, we have that $\sigma(g(z)) = \sigma(f(z))$. By (1.4) or (1.5), we get $\deg P_1(z+1) > \max\{\deg P_2(z+1), \deg P_3(z+1)\}$ and $\deg P_1(z+1) \geq 2$ or $\deg P_3(z+1) > \max\{\deg P_2(z+1), \deg P_1(z+1)\}$ and $\deg P_3(z+1) \geq 2$. Thus, for (3.13), applying the conclusion for $n = 1$ above, we obtain $\tau(f(z+2)) = \tau(g(z+1)) = \sigma(g(z)) = \sigma(f(z))$. Continuing to use the same method as above, we can obtain $\tau(f(z+n)) = \sigma(f(z))$, $n = 0, 1, 2, \dots$.

Theorem 1.2 is proved.

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STRUCTURE OF (w_1, w_2) -TEMPERED ULTRADISTRIBUTION USING SHORT-TIME FOURIER TRANSFORM

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Abstract. We characterize the space $\mathfrak{S}_{w_1, w_2}$ of test functions of (w_1, w_2) -tempered ultradistribution in terms of their short-time Fourier transform. As a result of this characterization and using Riesz representation theorem, we characterize the space (w_1, w_2) -tempered ultradistribution.

Keywords: short-time Fourier transform, tempered ultradistributions, structure theorem.

1. Introduction

In mathematical analysis, distributions (generalized functions) are objects which generalize functions. They extend the concept of derivative to all integrable functions and beyond, and used to formulate generalized solutions of partial differential equations. They play a crucial role in physics and engineering where many non-continuous problems naturally lead to differential equations whose solutions are distributions, such as the Dirac delta distribution. In late forties when Laurent Schwartz gave his formulation of distribution theory. This formulation leads to extensive applications in mathematical analysis, mathematical physics, and engineering. Recently, the theory of distributions devised by L. Schwartz is used in micro local analysis, signal processing, image processing and wavelets.

The Schwartz space $\mathfrak{S}$, as defined by Laurent Schwartz (see [13]), consists of all $C^\infty(\mathbb{R}^n)$ functions φ such that $\|x^\alpha \partial^\beta \varphi\|_\infty < \infty$ for all $\alpha, \beta \in \mathbb{N}^n$. The topological dual space of $\mathfrak{S}$, is a space of generalized functions, called tempered distributions. Tempered distributions have essential connections with the Fourier transform and partial differential equations. Moreover, they fit in many ways to provide a satisfactory framework of mathematical analysis and mathematical physics.

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In 1963, A. Beurling presented his generalization of tempered distributions. The aim of this generalization was to find an appropriate context for his work on pseudo-analytic extensions (see [2]).

In 1967 (see [3]), G. Björck studied and expanded the theory of Beurling on ultra distributions to extend the work of Hörmander on existence, nonexistence, and regularity of solutions of differential equations with constant coefficient and also consider equations which have no solutions. The Beurling-Björck space $\mathfrak{S}_w$, as defined by G. Björck, consists all $C^\infty(\mathbb{R}^n)$ functions φ such that $\|e^{kw(x)}\partial^\beta\varphi\|_\infty < \infty$ and $\|e^{kw(x)}\partial^\beta\hat{\varphi}\|_\infty < \infty$ for all $\alpha, \beta \in \mathbb{N}^n$, where w is a subadditive weight function satisfying the classical Beurling conditions. The topological dual $\mathfrak{S}'_w$ of $\mathfrak{S}_w$ is a space of generalized functions, called w -tempered ultra distributions. When $w(x) = \log(1 + |x|)$ the Beurling- Björck space $\mathfrak{S}_w$ becomes the Schwartz space $\mathfrak{S}$ (see [1] and [4]). Komatsu (see [10]) proved important structure theorems for these ultradistributions. Some other types of ultradistributions have also been studied by Gel'fand and Shilov (see [9]), called spaces of type S which are well-known in the theory of tempered ultradistributions.

In (see [11]), the authors introduced the space $\mathfrak{S}_{w_1, w_2}$ of all $C^\infty(\mathbb{R}^n)$ functions φ such that $\|e^{kw_1(x)}\partial^\beta\varphi\|_\infty < \infty$ and $\|e^{kw_2(x)}\partial^\beta\hat{\varphi}\|_\infty < \infty$ for all $k \in \mathbb{N}$ and $\beta \in \mathbb{N}^n$, where w_1 and w_2 are two weights satisfying the classical Beurling conditions.

In this paper, we characterize the space $\mathfrak{S}_{w_1, w_2}$ of test functions of (w_1, w_2) -tempered ultradistribution in terms of their short-time Fourier transform. As a result of this characterization and using Riesz representation theorem, we characterize the space (w_1, w_2) -tempered ultradistribution.

The symbols C^∞ , C_0^∞ , L^p , etc., denote the usual spaces of functions defined on $\mathbb{R}^n$, with complex values. We denote $|\cdot|$ the Euclidean norm on $\mathbb{R}^n$, while $\|\cdot\|_p$ indicates the p -norm in the space L^p , where $1 \leq p \leq \infty$. In general, we work on the Euclidean space $\mathbb{R}^n$ unless we indicate other than that as appropriate. Partial derivatives will be denoted ∂^α , where α is a multi-index $(\alpha_1, \dots, \alpha_n)$ in $\mathbb{N}_0^n$. We will use the standard abbreviations $|\alpha| = \alpha_1 + \dots + \alpha_n$, $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$. The Fourier transform of a function f will be denoted $\mathcal{F}(f)$ or $\hat{f}$ and it will be defined as $\int_{\mathbb{R}^n} e^{-2\pi i x \xi} f(x) dx$. With $\mathcal{C}_0$ we denote the Banach space of continuous functions vanishing at infinity with supremum norm.

2. Preliminary definitions and results

In this section, we start with the definition of the space of admissible functions $\mathcal{M}_c$ as they introduced by Björck.

Definition 1 ([3]). *With $\mathcal{M}_c$ we indicate the space of functions $w : \mathbb{R}^n \rightarrow \mathbb{R}$ of the form $w(x) = \Omega(|x|)$, where*

1. $\Omega : [0, \infty) \rightarrow [0, \infty)$ is increasing, continuous and concave,

2. $\Omega(0) = 0$,
3. $\int_{\mathbb{R}} \frac{\Omega(t)}{(1+t^2)} dt < \infty$,
4. $\Omega(t) \geq a + b \ln(1+t)$ for some $a \in \mathbb{R}$ and some $b > 0$.

Standard classes of functions w in $\mathcal{M}_c$ are given by

$$w(x) = |x|^d \text{ for } 0 < d < 1, \text{ and } w(x) = p \ln(1 + |x|) \text{ for } p > 0.$$

Remark 2. Let us observe for future use that if we take $N > \frac{n}{b}$ is an integer, then

$$C_N = \int_{\mathbb{R}^n} e^{-Nw(x)} dx < \infty, \text{ for all } w \in \mathcal{M}_c,$$

where b is the constant in Condition 4 of Definition 1.

Theorem 3 ([11]). *Given $w_1, w_2 \in \mathcal{M}_c$, the space $\mathfrak{S}_{w_1, w_2}$ can be described as a set as well as topologically by*

$$\mathfrak{S}_{w_1, w_2} = \left\{ \begin{array}{l} \varphi : \mathbb{R}^n \rightarrow \mathbb{C} : \varphi \text{ is continuous and for all } \\ k = 0, 1, 2, \dots, p_{k,0}(\varphi) < \infty, \pi_{k,0}(\varphi) < \infty \end{array} \right\},$$

where $p_{k,0}(\varphi) = \|e^{kw_1}\varphi\|_{\infty}$, $\pi_{k,0}(\varphi) = \|e^{kw_2}\widehat{\varphi}\|_{\infty}$. The space $\mathfrak{S}_{w_1, w_2}$, equipped with the family of semi-norms

$$\mathcal{N} = \{p_{k,0}, \pi_{k,0} : k \in \mathbb{N}_0\},$$

is a Fréchet space.

Example 4. From Theorem 3, it is clear that the Gaussian $f(x) = e^{-\pi|x|^2}$ belongs to $\mathfrak{S}_{w_1, w_2}$ for all $w_1, w_2 \in \mathcal{M}_c$.

It is well known that Fourier series are a good tool to represent periodic functions. However, they fail to represent non-periodic functions. To solve this problem, the short-time Fourier transform was introduced by D. Gabor [6]. The short-time Fourier transform works by first cutting off the function by multiplying it by another function called window then the Fourier transform. This technique maps a function of time x into a function of time x and frequency ξ .

Definition 5 ([7], [8]). *The short-time Fourier transform (STFT) of a function or distribution f on $\mathbb{R}^n$ with respect to a non-zero window function g is formally defined as*

$$\nu_g f(x, \xi) = \int_{\mathbb{R}^n} f(t) \overline{g(t-x)} e^{-2\pi i t \cdot \xi} dt = \widehat{(f T_x g)}(\xi) = \langle f, M_{\xi} T_x g \rangle,$$

where $T_x g(t) = g(t-x)$ is the translation operator and $M_{\xi} g(t) = e^{2\pi i t \cdot \xi} g(t)$ is the modulation operator.

The composition of T_x and M_ξ is the time-frequency shift

$$(M_\xi T_x g)(t) = e^{2\pi i x \cdot \xi} g(t - x),$$

and its Fourier transform is given by

$$\widehat{M_\xi T_x g} = e^{2\pi i x \cdot \xi} M_{-x} T_\xi \widehat{g}.$$

The main properties of the short-time Fourier transform is given in the following lemma.

Lemma 6 ([7], [8]). *For $f, g \in \mathfrak{S}_{w_1, w_2}$, the STFT has the following properties.*

1. (Inversion formula)

$$(1) \quad \int \int_{\mathbb{R}^n \times \mathbb{R}^n} \nu_g f(x, \xi) (M_\xi T_x g)(t) dx d\xi = \|g\|_2^2 f.$$

2. (STFT of the Fourier transforms)

$$\nu_{\widehat{g}} \widehat{f}(x, \xi) = e^{-2\pi i x \cdot \xi} \nu_g f(-\xi, x).$$

3. (Fourier transform of the STFT)

$$(2) \quad \widehat{\nu_g f}(x, \xi) = e^{2\pi i x \cdot \xi} f(-\xi) \overline{\widehat{g}(x)}.$$

Remark 7. The space $\nu_g(\mathfrak{S}_{w_1, w_2}) = \{\nu_g f : f \in \mathfrak{S}_{w_1, w_2}\}$ has no functions with compact support.

Now we will introduce two auxiliary results that we will use in the proof of the topological characterization of the space $\mathfrak{S}_{w_1, w_2}$ via the short-time Fourier transform.

Lemma 8 ([8]). *Let f and g be two nonnegative measurable functions. If $N > n$, there exists $C > 0$ such that*

$$\left\| e^{kw} (f * g) \right\|_\infty \leq C \left\| e^{2(N+k)w} f \right\|_\infty \left\| e^{2(N+k)w} g \right\|_\infty,$$

for all $k = 0, 1, 2, \dots$. The constant C does not depend on k .

In the following lemma, we include a proof using the topological characterization of $\mathfrak{S}_{w_1, w_2}$ given in Theorem 3 which imposes no conditions on the derivative. Our proof is an adaptation of the proof of (Proposition 2.6 stated in [8]).

Lemma 9. *Let $g \in \mathfrak{S}_{w_1, w_2}$ be fixed and assume that $F : \mathbb{R}^{2n} \rightarrow \mathbb{C}$ is a measurable function that has a subexponential decay, i.e. such that for each $k = 0, 1, 2, \dots$, there is a constant $C = C_k > 0$ satisfying*

$$|F(x, \xi)| \leq C e^{-k(w_1(x) + w_2(\xi))}.$$

Then the integral

$$f(t) = \int \int_{\mathbb{R}^{2n}} F(x, \xi) (M_\xi T_x g)(t) dx d\xi$$

defines a function in $\mathfrak{S}_{w_1, w_2}$.

Proof. To prove that $f \in \mathfrak{S}_{w_1, w_2}$, we start with

$$\begin{aligned} \left| (e^{kw_1(t)} f)(t) \right| &\leq \int \int_{\mathbb{R}^{2n}} (F(x, \xi) e^{kw_1(t)} (M_\xi T_x g)(t)) dx d\xi \\ &\leq \int \int_{\mathbb{R}^{2n}} |F(x, \xi)| \left| M_\xi T_x (e^{kw_1(t+x)} g)(t) \right| dx d\xi \\ &\leq \int \int_{\mathbb{R}^{2n}} |F(x, \xi)| \left| T_x (e^{kw_1(t+x)} g)(t) \right| dx d\xi \\ &\leq \int \int_{\mathbb{R}^{2n}} e^{kw_1(x)} e^{Nw_2(\xi)} e^{-Nw_2(\xi)} |F(x, \xi)| \left\| e^{kw_1} g \right\|_\infty dx d\xi \\ &\leq \int \int_{\mathbb{R}^{2n}} e^{(k+N)(w_1(x)+w_2(\xi))} e^{-N((w_1(x)+w_2(\xi)))} |F(x, \xi)| \left\| e^{kw_1} g \right\|_\infty dx d\xi \\ &\leq \left\| e^{kw_1} g \right\|_\infty \left\| e^{(N+k)(w_1(x)+w_2(\xi))} F \right\|_\infty \int \int_{\mathbb{R}^{2n}} e^{-N((w_1(x)+w_2(\xi)))} dx d\xi \\ &\leq C \left\| e^{(N+k)(w_1(x)+w_2(\xi))} F \right\|_\infty. \end{aligned}$$

So,

$$(3) \quad \left\| e^{kw_1} f \right\|_\infty \leq C \left\| e^{(N+k)(w_1(x)+w_2(\xi))} F \right\|_\infty.$$

This implies that $\left\| e^{kw_2} f \right\|_\infty < \infty$.

To show that $\left\| e^{kw_2} \hat{f} \right\|_\infty < \infty$, we write

$$\hat{f}(\tau) = \int \int_{\mathbb{R}^{2n}} (F(x, \xi) (M_{-x} T_\xi \hat{g})(\tau)) e^{2\pi i x \cdot \xi} dx d\xi,$$

using

$$(\widehat{M_\xi T_x g})(\tau) = (M_{-x} T_\xi \hat{g})(\tau) e^{2\pi i x \cdot \xi}.$$

Using an argument similar to the one leading to the proof of (3), we have

$$\left| e^{kw_2(\tau)} \hat{f}(\tau) \right| \leq C \left\| e^{(N+k)(w_1(x)+w_2(\xi))} F \right\|_\infty.$$

This completes the proof of Lemma 9. □

Remark 10. Given $w \in \mathcal{M}_c$, for the Gaussian $g(x) = e^{-\pi|x|^2}$ and f with $e^{-kw}f \in L^1$ for some $k \in \mathbb{N}_0$, then $\nu_g f$ is well-defined and continuous. In fact,

$$\begin{aligned}
 |\nu_g f(x, \xi)| &= \left| \int_{\mathbb{R}^n} f(t) \overline{g(t-x)} e^{-2\pi i t \cdot \xi} dt \right| \\
 &\leq \int_{\mathbb{R}^n} |f(t) \overline{g(t-x)} e^{-2\pi i t \cdot \xi}| dt \\
 &= \int_{\mathbb{R}^n} e^{-kw(t)} |f(t)| e^{kw(t)} |\overline{g(t-x)}| dt \\
 &\leq \int_{\mathbb{R}^n} e^{-kw(t)} |f(t)| e^{kw(t-x)} |\overline{g(t-x)}| e^{kw(x)} dt \\
 &= \|e^{-kw}f\|_1 \|e^{kw}g\|_\infty e^{kw(x)}.
 \end{aligned}$$

This shows that $\nu_g f$ is well-defined. Moreover, if we fix $(x_0, \xi_0) \in \mathbb{R}^{2n}$ and let (x_j, ξ_j) be any sequence in $\mathbb{R}^{2n}$ converging to (x_0, ξ_0) as $j \rightarrow \infty$, the function $f(t) \overline{g(t-x_j)} e^{-2\pi i t \cdot \xi_j}$ converges to $f(t) \overline{g(t-x_0)} e^{-2\pi i t \cdot \xi_0}$ pointwise as $j \rightarrow \infty$ and

$$\begin{aligned}
 |f(t) \overline{g(t-x_j)} e^{-2\pi i t \cdot \xi_j}| &\leq |e^{-kw(t)} f(t) e^{kw(t)} \overline{g(t-x_j)} e^{-2\pi i t \cdot \xi_j}| \\
 &\leq |e^{-kw(t)} f(t) e^{kw(t-x_j)} \overline{g(t-x_j)} e^{kw(x_j)}| \\
 &\leq C |e^{-kw(t)} f(t)| \|e^{kw}g\|_\infty \\
 &\leq C |e^{-kw(t)} f(t)|.
 \end{aligned}$$

Since the function $|e^{-kw(t)} f(t)| \in L^1$, we can apply Lebesgue Dominated Convergence Theorem to obtain

$$\nu_g f(x_j, \xi_j) \rightarrow \nu_g f(x_0, \xi_0)$$

as $j \rightarrow \infty$. This shows the continuity of $\nu_g f$.

3. The short-time Fourier transform over $\mathfrak{S}_{w_1, w_2}$

We use the topological characterization as stated in Theorem 3. Our proof imposes no conditions on the derivative.

Theorem 11. *Let $g(x) = e^{-\pi|x|^2}$ be the Gaussian. Then the space $\mathfrak{S}_{w_1, w_2}$ can be described as a set as well as topologically by*

$$(4) \quad \mathfrak{S}_{w_1, w_2} = \{f : \mathbb{R}^n \rightarrow \mathbb{C} : e^{-kw_1} f \in L^1 \text{ for some } k \in \mathbb{N}_0, \forall k \in \mathbb{N}_0, \pi_k(f) < \infty,$$

$$\text{where } \pi_k(f) = \|e^{k(w_1(x) + w_2(\xi))} \nu_g f\|_\infty.$$

Proof. Let us indicate $\mathfrak{B}_{w_1, w_2}$ the space defined in (4). Observe that the condition $e^{-kw_1} f \in L^1$ for some $k \in \mathbb{N}_0$ implies that $\nu_g f$ is continuous by Remark

10, so the formulation of the condition $\|e^{k(w_1(x)+w_2(\xi))}\nu_g f\|_\infty$ makes sense. We define in $\mathfrak{B}_{w_1,w_2}$ a structure of Fréchet space by means of the countable family of semi-norms

$$\mathcal{N} = \{\pi_k : k = 0, 1, 2, \dots\}.$$

We will show that $\mathfrak{B}_{w_1,w_2} = \mathfrak{S}_{w_1,w_2}$. To do so, we first prove that $\mathfrak{B}_{w_1,w_2} \subseteq \mathfrak{S}_{w_1,w_2}$ continuously. Fix $f \in \mathfrak{B}_{w_1,w_2}$, we need to show that $\|e^{kw_2}\widehat{f}\|_\infty$ and $\|e^{kw_1}f\|_\infty$ are finite. Since $f \in \mathfrak{B}_{w_1,w_2}$, then $\pi_k(f) < \infty$ for all $k \in \mathbb{N}_0$ which implies that $\nu_g f$ has a subexponential decay. Then by Lemma 9 and the inversion formula given in Lemma 6, we can write

$$f(t) = \|g\|_2^{-2} \int \int_{\mathbb{R}^{2n}} (\nu_g f(x, \xi)(M_\xi T_x g)(t)) dx d\xi.$$

Using Lemma 9, we have that $\|e^{kw_2}\widehat{f}\|_\infty$ and $\|e^{kw_1}f\|_\infty$ are finite for all $k \in \mathbb{N}_0$. Conversely, let $f \in \mathfrak{S}_{w_1,w_2}$, then we know that f is continuous and for all $k \in \mathbb{N}_0$ $p_{k,0}(f) < \infty$, $\pi_{k,0}(f) < \infty$. It is clear that $e^{-kw_1}f \in L^1$ for some $k \in \mathbb{N}_0$ since $f \in \mathfrak{S}_{w_1,w_2}$. To show that $\pi_k(f) < \infty$ for all $k \in \mathbb{N}_0$, we write

$$\begin{aligned} e^{2kw_1(x)} |\nu_g f(x, \xi)| &= e^{2kw_1(x)} \left| \int_{\mathbb{R}^n} f(t) g(x-t) e^{-2\pi i \xi \cdot t} dt \right| \\ &\leq \|e^{2kw_1}(|f| * |g|)\|_\infty. \end{aligned}$$

Using Lemma 8 we get the following estimate

$$\begin{aligned} e^{2kw_1(x)} |\nu_g f(x, \xi)| &\leq \|e^{2kw_1}(|f| * |g|)\|_\infty \\ &\leq C \|e^{2(N+2k)w_1} f\|_\infty \|e^{2(N+2k)w_1} g\|_\infty \\ &\leq C \|e^{2(N+2k)w_1} f\|_\infty. \end{aligned}$$

Then

$$(5) \quad e^{2kw_1(x)} |\nu_g f(x, \xi)| \leq C \|e^{2(N+2k)w_1} f\|_\infty.$$

Moreover, since we can write $\nu_g f(x, \xi) = e^{-2\pi i \xi \cdot x} \nu_{\widehat{g}} \widehat{f}(\xi, -x)$, we have the following estimate.

$$\begin{aligned} e^{2kw_2(\xi)} |\nu_g f(x, \xi)| &\leq e^{2kw_2(\xi)} |\nu_{\widehat{g}} \widehat{f}(\xi, -x)| \\ &\leq \|e^{2kw_2}(\widehat{f} * \widehat{g})\|_\infty \end{aligned}$$

Once again, using Lemma 8 we obtain

$$\begin{aligned} e^{2kw_2(\xi)} |\nu_g f(x, \xi)| &\leq \|e^{2kw_2}(\widehat{f} * \widehat{g})\|_\infty \\ &\leq C \|e^{2(N+2k)w_2} \widehat{f}\|_\infty \|e^{2(N+2k)w_2} \widehat{g}\|_\infty \\ &\leq C \|e^{2(N+2k)w_2} \widehat{f}\|_\infty. \end{aligned}$$

Then

$$(6) \quad e^{2kw_2(\xi)} |\nu_g f(x, \xi)| \leq C \left\| e^{2(N+2k)w_2} \widehat{f} \right\|_{\infty}.$$

Combining (5) and (6), we have that

$$e^{2k(w_1(x)+w_2(\xi))} |\nu_g f(x, \xi)|^2 \leq C \left(\left\| e^{2(N+2k)w_1} f \right\|_{\infty} \left\| e^{2(N+2k)w_2} \widehat{f} \right\|_{\infty} \right).$$

This implies that

$$(7) \quad \pi_k(f) \leq C \left(\left\| e^{2(N+2k)w_1} f \right\|_{\infty} + \left\| e^{2(N+2k)w_2} \widehat{f} \right\|_{\infty} \right).$$

So, $f \in \mathfrak{B}_{w_1, w_2}$. Hence $\mathfrak{B}_{w_1, w_2} \subseteq \mathfrak{S}_{w_1, w_2}$ and the inclusion is continuous. This completes the proof of Theorem 11. $\square$

Remark 12. Let $g(x) = e^{-\pi|x|^2}$ be the Gaussian. Then for $f \in \mathfrak{S}_{w_1, w_2}(\mathbb{R}^n)$, we have $\nu_g f \in \mathfrak{S}_{w_1, w_2}(\mathbb{R}^{2n})$.

An argument, similar to the one in Theorem 11, we can show the following characterization.

Corollary 13. Let $\alpha, \beta > 1$ and $g(x) = e^{-\pi|x|^2}$ be the Gaussian. Then the Gelfand-Shilov space S_{α}^{β} can be described as a set as well as topologically by $S_{\alpha}^{\beta} = \{f : \mathbb{R}^n \rightarrow \mathbb{C} : e^{-m|x|^{1/\alpha}} f \in L^1 \text{ for some } m \in \mathbb{N}_0 \text{ and } \pi_k(f) < \infty \text{ for all } k \in \mathbb{N}_0, \text{ where } \pi_k(f) = \|e^{k(|x|^{1/\alpha} + |\xi|^{1/\beta})} \nu_g f\|_{\infty}\}.$

4. Representation theorems for functionals in the space $\mathfrak{S}'_{w_1, w_2}$

Theorem 14 ([12]). Given a functional L in the topological dual of the space $\mathcal{C}_0$, there exists a unique regular complex Borel measure μ so that

$$L(\varphi) = \int_{\mathbb{R}^n} \varphi d\mu.$$

Moreover, the norm of the functional L is equal to the total variation $|\mu|$ of the measure μ . Conversely, any such measure μ defines a continuous linear functional on $\mathcal{C}_0$.

Theorem 15. Let $g(x) = e^{-\pi|x|^2}$ be the Gaussian. Then if $L : \mathfrak{S}_{w_1, w_2} \rightarrow \mathbb{C}$, the following statements are equivalent: (i) $L \in \mathfrak{S}'_{w_1, w_2}$ (ii) There exist a regular complex Borel measure μ of finite total variation and $k \in \mathbb{N}_0$ so that

$$L = e^{k(w_1(x)+w_2(\xi))} \nu_g d\mu,$$

in the sense of $\mathfrak{S}'_{w_1, w_2}$.

Proof. (i) $\Rightarrow$ (ii). Given $L \in \mathfrak{S}'_{w_1, w_2}$, there exist k, C so that

$$L(\varphi) \leq C \left\| e^{k(w_1(x)+w_2(\xi))} \nu_g \varphi \right\|_{\infty}$$

for all $\varphi \in \mathfrak{S}_{w_1, w_2}$. Moreover, the map

$$\begin{aligned} \mathfrak{S}_{w_1, w_2}(\mathbb{R}^n) &\rightarrow \mathcal{C}_0(\mathbb{R}^{2n}) \\ \varphi &\rightarrow e^{k(w_1(x)+w_2(\xi))} \nu_g \varphi \end{aligned}$$

is well-defined, linear, continuous and injective. Let $\mathcal{R}$ be the range of this map. We define on $\mathcal{R}$ the map

$$l_1 \left(e^{k(w_1(x)+w_2(\xi))} \nu_g \varphi \right) = L(\varphi),$$

for a unique $\varphi \in \mathfrak{S}_{w_1, w_2}$. The map $l_1 : \mathcal{R} \rightarrow \mathbb{C}$ is linear and continuous. By the Hahn-Banach theorem, there exists a functional L_1 in the topological dual $\mathcal{C}'_0(\mathbb{R}^{2n})$ of $\mathcal{C}_0(\mathbb{R}^{2n})$ such that $\|L_1\| = \|l_1\|$ and the restriction of L_1 to $\mathcal{R}$ is l_1 . Using Theorem 14, there exist a regular complex Borel measure μ of finite total variation so that

$$L_1(f) = \int_{\mathbb{R}^{2n}} f d\mu$$

for all $f \in \mathcal{C}_0(\mathbb{R}^{2n})$. If $f \in \mathcal{R}$, we conclude

$$L(\varphi) = \int_{\mathbb{R}^{2n}} e^{k(w_1(x)+w_2(\xi))} \nu_g \varphi d\mu$$

for all $\varphi \in \mathfrak{S}_{w_1, w_2}$. In the sense of $\mathfrak{S}'_{w_1, w_2}$,

$$L = e^{k(w_1(x)+w_2(\xi))} \nu_g d\mu.$$

(ii) $\Rightarrow$ (i). If μ is a regular complex Borel measure satisfying (ii) and $\varphi \in \mathfrak{S}_{w_1, w_2}$, then

$$L(\varphi) = \int_{\mathbb{R}^{2n}} e^{k(w_1(x)+w_2(\xi))} \nu_g \varphi d\mu.$$

This implies that

$$\begin{aligned} |L(\varphi)| &\leq \left| \int_{\mathbb{R}^{2n}} e^{k(w_1(x)+w_2(\xi))} \nu_g \varphi d\mu \right| \\ &\leq |\mu|(\mathbb{R}^{2n}) \left\| e^{k(w_1(x)+w_2(\xi))} \nu_g \varphi \right\|_{\infty} \\ &\leq C \left(\left\| e^{k(w_1(x)+w_2(\xi))} \nu_g \varphi \right\|_{\infty} \right). \end{aligned}$$

It may be noted that μ , employed to obtain the above inequality, is of finite total variation. This completes the proof of Theorem 15. $\square$

Remark 16. Any $L \in (S_\alpha^\beta)'(\mathbb{R}^n)$, with $\alpha, \beta > 1$, can be written as

$$L = e^{k(e^{|x|^{1/\alpha} + |\xi|^{1/\beta}})} \nu_g d\mu$$

which characterizes the dual space of the Gelfand-Shilov space $(S_\alpha^\beta)'(\mathbb{R}^n)$.

Remark 17. Any $L \in \mathfrak{S}'$ can be written as

$$L = (1 + |x|^k + |\xi|^k) \nu_g d\mu$$

which characterizes tempered distributions.

Corollary 18. *If $L \in \mathfrak{S}'_{w_1, w_2}$ and $\varphi \in \mathfrak{S}_{w_1, w_2}$, then the classical definition of the convolution $L * \varphi$ is defined by*

$$(L * \varphi, \psi)_{\mathfrak{S}'_{w_1, w_2}, \mathfrak{S}_{w_1, w_2}} = (L_x, (\varphi_z, \psi(x + y)))_{\mathfrak{S}'_{w_1, w_2}, \mathfrak{S}_{w_1, w_2}}$$

for all $\psi \in \mathfrak{S}_{w_1, w_2}$. Moreover, the functional $L * \varphi$ coincides with the functional given by the integration against the function

$$f(y) = (L, \varphi(y - \cdot))_{\mathfrak{S}'_{w_1, w_2}, \mathfrak{S}_{w_1, w_2}}.$$

Proof. The Inequality

$$\begin{aligned} |f(y)| &= \left| \int_{\mathbb{R}^{2n}} e^{k(w_1(x) + w_2(\xi))} \nu_g \varphi(y - x, \xi) d\mu \right| \\ &\leq \left| \int_{\mathbb{R}^{2n}} e^{k(w_1(y-x) + w_2(\xi))} e^{kw_1(y)} \nu_g \varphi(y - x, \xi) d\mu \right| \\ &\leq C \left\| e^{k(w_1(x) + w_2(\xi))} \nu_g \varphi \right\|_\infty \end{aligned}$$

implies the continuity of $f(y) = (L, \varphi(y - \cdot))_{\mathfrak{S}'_{w_1, w_2}, \mathfrak{S}_{w_1, w_2}}$. Then

$$\begin{aligned} (L * \varphi, \psi)_{\mathfrak{S}'_{w_1, w_2}, \mathfrak{S}_{w_1, w_2}} &= (L_x, (\varphi_z, \psi(x + y)))_{\mathfrak{S}'_{w_1, w_2}, \mathfrak{S}_{w_1, w_2}} \\ &= \int_{\mathbb{R}^{2n}} e^{k(w_1(x) + w_2(\xi))} \nu_g \left(\int_{\mathbb{R}^n} \varphi(y - x) \psi(y) dy \right) dx d\xi \\ &= \int_{\mathbb{R}^{2n}} e^{k(w_1(x) + w_2(\xi))} \nu_g (\psi * \varphi(x)) dx d\xi \\ &= (e^{k(w_1(x) + w_2(\xi))} \nu_g, \psi * \varphi(x))_{\mathfrak{S}'_{w_1, w_2}, \mathfrak{S}_{w_1, w_2}} \\ &= (e^{k(w_1(x) + w_2(\xi))} \nu_g \varphi(y - x, \xi), \psi(y))_{\mathfrak{S}'_{w_1, w_2}, \mathfrak{S}_{w_1, w_2}} \\ &= ((e^{k(w_1(x) + w_2(\xi))} \nu_g, \varphi(y - x))_{\mathfrak{S}'_{w_1, w_2}, \mathfrak{S}_{w_1, w_2}}, \psi(y))_{\mathfrak{S}'_{w_1, w_2}, \mathfrak{S}_{w_1, w_2}} \\ &= ((e^{k(w_1(x) + w_2(\xi))} \nu_g, (\varphi(y - x), \psi(y)))_{\mathfrak{S}'_{w_1, w_2}, \mathfrak{S}_{w_1, w_2}} \\ &= (L_x, (\varphi(y - x), \psi(y)))_{\mathfrak{S}'_{w_1, w_2}, \mathfrak{S}_{w_1, w_2}}. \end{aligned}$$

□

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PARTIALLY BLIND SIGNATURE SCHEME BASED ON CHAOTIC MAPS AND FACTORING PROBLEMS

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Abstract. Due to the importance of security and efficiency of electronic signatures schemes, there is an increase in interest among scholars to develop such schemes based on mathematical problems to be more secure and efficient. In this paper, we propose a scheme with a low computation cost based on both cryptographic and chaotic system characteristics. The security of the scheme depends upon the intractability of the factorization problem and discrete logarithm of Chebyshev polynomials. The performance comparison demonstrated that the proposed scheme has a lower communication cost than the existing schemes in the literature, such as the one proposed by Tahat et al. To the best of our knowledge, this is the first time a partially blind signature scheme based on chaotic maps and factoring problem has been proposed.

Keywords: chaotic maps, digital signature, factorization, partially blind signature.

1. Introduction

The concept of blind digital signature was introduced by Chaum (1983) [4] and Chaum (1984) [5] to enable spender anonymity in electronic cash system. Such

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signatures require a signer to be able to sign a document without knowing its contents. Moreover, should the signer ever see the document signature pair, he should not be able to determine when or for whom it was signed (although he can verify that the signature is indeed valid). In order to solve the contradiction between the blind signatures' anonymity and controllability, in 1996, Abe and Fujisaki [1] proposed the concept of partially blind signature. Partially blind signatures will divide the signed message into two parts; one of which is public information that is agreed by the signer and the user, for example, the scope of the signed message. The other part is the message which is kept blind as it waits for a signature. Not only does this scheme protect the privacy of the users, it also allows the signer to control parts of the contents of the signature. All developed blind signature schemes in the literature are designed based on a single hard problem such as factoring, discrete logarithm or elliptic curve discrete logarithm problems [4,5,7,8,16,17]. Chun-I et al. (1998) [9] proposed a partially blind scheme based on quadratic residue problem, in which there are no modular exponentiations or inverse computations performed by the signature requesters. Compared to the blind signature schemes proposed in the literature, Chun-I et al. (1998) reduced the number of computations for the signature requesters or users by nearly 98% under a 1024 -bit modulus, but it does not decrease the computation load for the signer. Their scheme is especially suitable for mobile signature requester and smart-card users. Hwang et al. (2002) [10] has shown, however, that it does not meet the untraceability property of a blind signature. Huang et al. (2004) [11] then proposed a new efficient partially blind signature scheme based on discrete logarithm and the Chinese remainder theorem, but unfortunately Zhang and Chen (2005) [22] later showed that their scheme is not secure, as any malicious requester can remove the embedded public common information from the signer's signature and obtain a partially blind signature with a special public information. Recently, Tahat et al. [18] proposed a new partially blind signature scheme based on factoring and discrete logarithm problems.

The first chaotic map-based image encryption algorithm was proposed in 1989 [15]. Recently, there is a growing interest in this area as several approaches have been proposed in the literature [3,6,12,14,19]. The computational costs of chaotic map-based public cryptosystems are very low compared to public cryptosystems based on modular exponential computing or scalar multiplication on elliptic curves. Hence, in this paper, we will propose a new partially blind signature scheme based on chaotic map and factoring problems. The proposed scheme is much more efficient than previous partially blind signature schemes based on two hard problems due to the decreased number of operations.

The remainder of this paper is organized as follows. Section 2 will describe the theory and properties of the extended chaotic maps and two computational problems. In Section 3, the partially blind signature will be proposed. Security requirements will be described in Section 4. In Section 5, we will present the

performance evaluation of the proposed scheme. A numerical example will be given in Section 6. Finally, Section 7 will conclude the paper.

2. Preliminary knowledge

In this section, we briefly introduce the basic concept of Chebyshev chaotic map and its related mathematical properties [13, 21].

2.1 Chepyshev Chaotic Map

Let n be an integer and x be a variable with the interval $[-1, 1]$. The Chebyshev polynomial $T_n(x) : [-1, 1] \rightarrow [-1, 1]$ is defined as

$$(2.1) \quad T_n(x) = \cos(n \cos^{-1}(x)).$$

Chebyshev polynomial map $T_n : R \rightarrow R$ of degree n is defined by the following recurrent relation:

$$(2.2) \quad T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x),$$

where $n \geq 2$, $T_0(x) = 1$, $T_1(x) = x$. some of the other Chebyshev polynomial are $T_2(x) = 2x^2 - 1$, $T_3(x) = 4x^3 - 3x$, $T_4(x) = 8x^4 - 8x^2 + 1$, $T_5(x) = 16x^5 - 20x^3 + 5x$.

The Chebyshev polynomial has the following two interesting properties [3, 13, 21]:

- The semi-group property:

$$(2.3) \quad \begin{aligned} T_r(T_s(x)) &= \cos(r \cos(s \cos^{-1}(x))) = \cos(rs \cos^{-1}(x)) \\ &= T_{sr}(x) = T_s(T_r(x)), \end{aligned}$$

where r and s are positive integers numbers and $x \in [-1, 1]$

- The chaotic property:

The Chebyshev map $T_a(x) = [-1, 1] \rightarrow [-1, 1]$ of degree $a > 1$ is a chaotic map with invariant density $f^*(x) = \frac{1}{\pi\sqrt{1-x^2}}$ for positive Lyapunov exponent $\lambda = \ln(a) > 0$.

In order to improve this property, Zhang [23] proved that the semi-group property holds for Chebyshev polynomials defined on the interval $(-\infty, \infty)$ as follows:

$$(2.4) \quad T_a(x) = 2xT_{a-1}(x) - T_{a-2}(x) \pmod{p},$$

where $a \geq 2$, $x \in (-\infty, \infty)$, and p is a large prime number. Therefore, the property

$$T_r(T_s(x)) = T_{sr}(x) = T_s(T_r(x)) \pmod{p},$$

and the semi group property also holds. The extended Chebyshev polynomials still commute under composition.

Theorem 2.1. Let $f(M) = t^2 - 2Mt + 1$ and α, β be two roots of $f(M)$. If $M = \frac{1}{2}(\alpha + \beta)$, then the number of solutions satisfy

$$T_a(M) = \frac{(M + \sqrt{M^2 - 1})^a + (M - \sqrt{M^2 - 1})^a}{2} \pmod{p}.$$

Theorem 2.2. If a and b are two positive integers and $a > b$, then

$$(2.5) \quad 2T_a(M).T_b(M) = T_{a+b}(M) + T_{a-b}(M).$$

Theorem 2.3. If $a = b + c$ and p is a large prime number, then

$$(2.6) \quad \begin{aligned} & (2T_a(M) T_b(M) T_c(M) + 1) \pmod{p} \\ & = ([T_a(M)]^2 + [T_b(M)]^2 + [T_c(M)]^2) \pmod{p}. \end{aligned}$$

Lemma 2.4. Let g and h be elements of a finite field, i.e. if $g + g^{-1} = h + h^{-1}$ then $g = h$ or $g = h^{-1}$.

Lemma 2.5. For any $g \in GF(p)$ and $y = g^t$ for some integer t , we can find an integer $M \in GF(p)$ and then construct a chaotic maps sequence $\{T_a(M)\}$ such that $\frac{1}{2}(y + y^{-1}) = T_t(M) \in T_a(M)$ in polynomial time.

Theorem 2.6. If an algorithm AL can be used to solve the chaotic maps problem over $GF(p)$, then AL can be used to solve the discrete logarithm problem over $GF(p)$ in polynomial time.

Lemma 2.7. Let p , n and α be defined as above and G be the group generated by α . To find v such that $a = T_{v^2 \pmod{n}}(\alpha) \pmod{p}$, where a is given and $a \in G$, one must solve both chaotic maps problem in G and the factorization of n .

2.2 Computational problems

To prove the security of the proposed scheme, we present some important mathematical properties of Chebyshev chaotic map as follows:

1. If two elements x and y are given, the task of the discrete logarithm problem is to find integers s , such that $T_s(x) = y$.
2. If three elements x , $T_r(x)$, and $T_s(x)$, are given the task of the Diffie-Hellman problem is to compute elements $T_{rs}(x)$

3. The proposed partially blind signature scheme

Throughout the article, we need the following tools to describe our new partially blind signature scheme and to discuss its security analysis and efficiency performances: A large number p and n is a factor of $p - 1$ that is the product of two safe primes $\bar{p}$ and $\bar{q}$ i.e., $n = \bar{p}\bar{q}$. β is an element in $GF(p)$ whose order modulo p is n , and G is the multiplicative group generated by β . A cryptographic hash function $h(\cdot)$ where the output is t -bit length and assume $t = 128$. Note that the two large primes $\bar{p}$ and $\bar{q}$ are kept secret in the system.

3.1 Generating Keys

The signer pick randomly an integer e from $\mathbb{Z}_n^*$ such that $\gcd(e, n) = 1$. Computes an integer d such that $ed \equiv 1 \pmod{\varphi(n)}$. Next select at random an integer x and compute $z = T_x(\beta)$. Here the public and secret keys of the system are given by (n, z) and (d, x) respectively.

3.2 Requesting

Suppose requester A wants to obtain a signature on message, $h(m)$. Firstly, he must notify the signer and then:

1. The signer select an integer $r < n$ such that $\gcd(r, n) = 1$ and compute $\hat{t} = T_r(\beta) \pmod{p}$.
2. Then, the signer check that $\gcd(\hat{t}, n) = 1$. If this not the case, he/she send $\hat{t}$ to the requester A.
3. After receiving $\hat{t}$, requester A checks if $\gcd(\hat{t}, n) = 1$ and prepares the common information c , according to a pre-defined format. Hence, the value c is a common input of both the requester A and the signer.
4. Requester A also randomly selects two blinding factors $u \in \mathbb{Z}_n^*$, $v \in \mathbb{Z}_n^*$ and compute

$$(3.1) \quad t = T_{(u+v)}(\hat{t}) \pmod{p}$$

and checks whether $\gcd(t, n) = 1$. If this is not case, he goes back to select another blinding factor. Otherwise, he computes $\mu \equiv u^{-1}h(m)\hat{t}t^{-1} \pmod{n}$ and sends (μ, c) to the signer.

3.3 Signing and extraction

The signer signs blindly the message $h(m)$ as follows:

The signer signs blindly the message $h(m)$ as follows

1. The signer computes and sends

$$(3.2) \quad \hat{k} \equiv (\mu x c r^{-1} + \hat{t}) \pmod{n}$$

to requester A.

2. Requester A computes and sends

$$k \equiv \hat{k}^{-e} (\hat{k} t \hat{t}^{-1} u + v t) \pmod{n}$$

to the signer.

3. The signer computes and sends

$$(3.3) \quad \hat{R} \equiv (r \ k)^d \pmod{n}$$

to the requester A.

4. The requester A computes

$$(3.4) \quad R \equiv \hat{R} \ \hat{k} \pmod{n}.$$

Then the signature is given by (c, t, R) .

The following theorem shows that if a signature (c, t, R) of a message m is produced by the proposed partially blind signature scheme, then it satisfies

$$(3.5) \quad [T_{R^e(mod \ n)}(\beta)]^2 + [T_{h(m)c \ (mod \ n)}(z)]^2 + [T_t(t)]^2 \\ = (2T_{R^e}(\beta)T_{h(m)c}(z) \ T_t(t) + 1) \pmod{p}.$$

Theorem 3.1. *If (c, t, R) is a signature of the message m produced by a proposed new partially blind signature scheme, then*

$$[T_{R^e(mod \ n)}(\beta)]^2 + [T_{h(m)c \ (mod \ n)}(z)]^2 + [T_t(t)]^2 \\ = (2T_{R^e}(\beta)T_{h(m)c}(z) \ T_t(t) + 1) \pmod{p}.$$

Proof. We have to show that the signature (c, t, R) satisfies:

$$R^e(mod \ n) \equiv (\hat{R} \ \hat{k})^e \equiv (r^d k^d \ \hat{k})^e \equiv r \ k \ \hat{k}^e \\ \equiv r \hat{k}^{-e} (\hat{k} \ t \ \hat{t}^{-1} \ u + v \ t) \ \hat{k}^e \equiv r (\hat{k} \ t \ \hat{t}^{-1} \ u + v \ t) \\ \equiv r ((\mu \ x \ c \ r^{-1} + \hat{t}) t \ \hat{t}^{-1} \ u + vt) \equiv ((\mu \ x \ c + \hat{t}r) t \ \hat{t}^{-1} \ u + vt) \\ \equiv (((u^{-1} h(m) \hat{t} \ t^{-1}) \ x \ c + \hat{t}r) t \ \hat{t}^{-1} \ u + vtr) \equiv (h(m)xc + tur + vtr) \pmod{n},$$

and thus

$$[T_{R^e(mod \ n)}(\beta)]^2 + [T_{h(m)c \ (mod \ n)}(z)]^2 + [T_t(t)]^2 \\ = [T_{((h(m)xc+tur+vtr))}(\beta)]^2 + [T_{h(m)c \ (mod \ n)}T_x(\beta)]^2 + [T_t T_{(u+v)}(T_r(\beta))]^2 \\ = [T_{(h(m)xc+tur+vtr)}(\beta)]^2 + [T_{h(m)xc}(\beta)]^2 + [T_{tur+tvr}(\beta)]^2.$$

Let $a = h(m)xc + tur + vtr$, $b = h(m)xc$, $l = tur + tvr$ and $a = b + l$

$$[T_{(h(m)xc+tur+vtr)}(\beta)]^2 + [T_{h(m)xc}(\beta)]^2 + [T_{tur+tvr}(\beta)]^2 \\ = 2T_{(h(m)xc+tur+vtr)}(\beta)T_{h(m)xc}(\beta)T_{tur+tvr}(\beta) + 1 \\ = (2T_{R^e}(\beta)T_{h(m)c}(z) \ T_t(t) + 1) \pmod{p},$$

which means (c, t, R) is a valid signature of m . Therefore, our proposed protocol provides a partially blind signature scheme.

4. Security analysis

In this section, we discuss some security properties of our partially blind signature scheme. A secure partially blind signature scheme should satisfy the following requirements.

4.1 Partial blindness

The partial blindness of all signatures issued by the signer contains a clear common information c according to the predefined format negotiated and agreed by both the requester and the signer. The requester is unable to change or remove the embedded information c while keeping the verification of signature successful. In the proposed scheme, the requester has to submit the blinded data σ to the signer, and then the signer computes and sends σ to the signer, and then the signer computes and sends $\hat{k} \equiv (\mu x c r^{-1} + \hat{t})(\text{mod } n)$. However, it is difficult to derive the secret key x . Also the signature-requester has to submit the blinded data k to the signer then the signer computes and sends $\hat{R}$ to the requester. The signature-requester cannot change or remove $\hat{R} \equiv r^d k^d (\text{mod } n)$ because it is difficult to derive the secret key d . Hence, in the proposed scheme, the signature-requester cannot change or remove the c and $\hat{R}$ from the corresponding signature (c, t, R) , of message m to forge the unblinded part of the signature.

4.2 Randomization

In the proposed scheme, the signer randomizes the blinded data using the random factor r before signing it in the signing phase. In the requesting phase, the signer selects an integer r and sends such that $\hat{t} = T_r(\beta)(\text{mod } p)$ and submit $\hat{t}$ to the requester. Then the requester A sends μ to the signer and the signer returns $\hat{k} \equiv (\mu x c r^{-1} + \hat{t})(\text{mod } n)$ to the requester A. If the requester A tries to remove r from $\hat{k} \equiv (\mu x c r^{-1} + \hat{t})(\text{mod } n)$ and $\hat{R} \equiv r^d k^d (\text{mod } n)$, then he has to derive x and d which are clearly infeasible difficulty of solving chaotic maps and factoring problem. Hence, in the proposed scheme, the requester A cannot remove the random r from the corresponding signature (c, t, R) of message m .

4.3 Unlinkability

For every instance, the signer can record the transmitted messages (μ_i, k_i) between the signature-requester and the signer during the instance i of the protocol. The pair (μ_i, k_i) is usually referred to as the *view* of the signer to the instance i of the protocol. Thus, we have the following theorem:

Theorem 4.1. *Giving a signature (c, t, R) produced by the proposed scheme, the signer can derive $(\hat{u}_i, \hat{v}_i)$ for every (μ_i, k_i) such that*

$$\mu_i \equiv (\hat{u}_i)^{-1} h(m) \hat{t} t^{-1} (\text{mod } n) \text{ and } k_i \equiv \hat{k}^{-e} (\hat{k} t \hat{t}^{-1} \hat{u}_i + \hat{v}_i t) (\text{mod } n).$$

Proof. $\mu_i \equiv (\hat{u}_i)^{-1} h(m) \hat{t} t^{-1} \pmod{n}$, we have that:

$$\begin{aligned}\mu_i \hat{u}_i &\equiv h(m) \hat{t} t^{-1} \pmod{n} \\ \hat{u}_i &\equiv \mu_i^{-1} h(m) \hat{t} t^{-1} \pmod{n}.\end{aligned}$$

If $k_i \equiv \hat{k}^{-e} (\hat{k} t \hat{t}^{-1} \hat{u}_i + \hat{v}_i t) \pmod{n}$, then we have the following derivations:

$$\begin{aligned}\hat{k}^e &\equiv (\hat{k} t \hat{t}^{-1} \hat{u}_i + \hat{v}_i t) \pmod{n}, \\ \hat{v}_i t &\equiv (k_i \hat{k}^e - \hat{k} t \hat{t}^{-1} \hat{u}_i) \pmod{n}, \\ \hat{v}_i &\equiv (k_i \hat{k}^e - \hat{k} t \hat{t}^{-1} \hat{u}_i) t^{-1} \pmod{n}.\end{aligned}$$

According to the above derivations, the signer can derive $\hat{u}_i, \hat{v}_i$ for every record (μ_i, k_i) . Hence, giving a signature (c, t, R) produced by the proposed scheme, the signer can always derive the two blinding factors (μ_i, k_i) for every transmitted record (μ_i, k_i) . This implies that the signer is unable to find the link between the signer and its corresponding signing process instance. Thus, our scheme satisfies the unlinkability property.

4.4 Unforgeability

The security of our scheme is based on the difficulty of solving the factoring problem and discrete logarithm problem of Chebyshev polynomials. The adversary Adv may try to derive a forged signature using different ways, as shown below.

Attack 1: Adv tries to derive the signature (c, t, R) for a given message m by letting one integer fixed or two and finding the other one.

In this case, Adv randomly fixes either $(c, t), (c, R)$ or (t, R) to find R, t or c respectively to satisfy

$$\begin{aligned}&[T_{R^e \pmod{n}}(\beta)]^2 + [T_{h(m)c \pmod{n}}(z)]^2 + [T_t(t)]^2 \\ &= (2T_{R^e}(\beta)T_{h(m)c}(z) T_t(t) + 1) \pmod{p}\end{aligned}$$

as difficult chaotic maps problems and factorization, simultaneously.

Case 1. Say Adv fixes the value (c, t) and tries to figure out the value R . Adv then needs to solve the following equations that can be reduced from Eq.(3.5)

$$\psi^2 - 2\psi T_{h(m)c}(z) T_t(t) + [T_{h(m)c \pmod{n}}(z)]^2 + [T_t(t)]^2 - 1 = 0 \pmod{p}$$

Therefore, ψ can be recover by the following equation:

$$\begin{aligned}\psi &= \frac{2T_{h(m)c}(z) T_t(t)}{2} \\ (4.1) \quad &\pm \frac{\sqrt{(2T_{h(m)c}(z) T_t(t))^2 - 4([T_{h(m)c \pmod{n}}(z)]^2 + [T_t(t)]^2 - 1)}}{2}.\end{aligned}$$

However, it is infeasible to find y from $\psi \equiv T_{Re(mod\ n)}(\beta) \bmod p$ even if he can get ψ from Eq.(4.1)

From Lemma 1, we can see that this is equivalent to solving the chaotic maps problem in G and factorization of n .

Case 2. Say Adv fixes the value (c, R) and tries to figure out the value t .

Then his task is more difficult than Instance 1 because he must find t from η , where

$$(4.2) \quad \eta^2 - 2\eta T_{h(m)c}(z) T_{Re(mod\ n)}(\beta) + [T_{h(m)c}(z)]^2 + [T_{Re(mod\ n)}(\beta)]^2 - 1 = 0 \pmod{p}.$$

Lemma 1 indicates that this is at least as difficult solving the chaotic maps problems and factorization of n .

Case 3. Say Adv fixes the value (t, R) and tries to figure out the value c .

Then his task is more difficult than Case 1, since he must compute c from

$$\xi^2 - 2\xi T_{Re}(\beta) T_t(t) + [T_{Re(mod\ n)}(\beta)]^2 + [T_t(t)]^2 - 1 = 0 \pmod{p}.$$

Lemma 1 indicates that is at least as hard as solving the chaotic maps problem in G and the factorization n .

Case 4. If Adv randomly chooses one of variable from (c, t, R) , and he wishes to derive the other two variables such that Eq.(2.5), Eq. (2.6), Eq. (3.1) and Eq. (3.5) are upheld, his task is at least as difficult as that of case (1), (2), (3). Furthermore, there is no simpler method than solving the chaotic maps problem in G and the factorization of n .

In the following two attacks, we assume that one the factoring or chaotic maps problems are solvable. The idea is to show that Adv still has to solve the other problem in order to obtain all the secret information.

Attack 2. It is assumed that Ad is able to solve chaotic maps problem. In this case, Adv know x and can generate or calculate the numbers $\hat{k}$ and k . Unfortunately, he does not know d and cannot compute $\hat{R} \equiv r^d k^d \pmod{n}$ and $R \equiv \hat{R} \hat{k} \pmod{n}$, then fails to produce the signature (c, t, R) .

Attack 3. It is assumed that Ad is able to solve factoring problem, which means he knows the prime factorization of n i.e. $\bar{p}$ and $\bar{q}$ and can find the number d . However, he cannot compute $\hat{k}$, since no information on x is available, hence he cannot compute $R \equiv \hat{R} \hat{k}$. Thus fails to produce the signature (c, t, R) .

Attack 4. Adv may also try collecting s valid signature (c_j, t_j, R_j) on message M_j where $j = 1, 2, \dots, s$, and attempts to find the secret keys of the

signature scheme. In this case, Adv has s equations as follows.

$$\begin{aligned} R_1^e &= h(M_1)c_1x + t_1u_1r_1 + t_1v_1r_1 \pmod{n} \\ R_2^e &= h(M_2)c_2x + t_2u_2r_2 + t_2v_2r_2 \pmod{n} \\ &\vdots \\ R_s^e &= h(M_s)c_sx + t_su_sr_s + t_sv_sr_s \pmod{n}. \end{aligned}$$

In the above s equation, there are $(3s + 1)$ variables i.e. r_j, u_j, v_j and x , where $j = 1, 2, \dots, s$, all of which is unknown by Adv. Hence, x remains hard to be obtained as Adv will generate an infinite number of solutions for the above system of equations and cannot figure out which one is correct.

Adv wishes to obtain secret keys (x, d) using all information that is available from the system. In this case, Adv needs to solve $ed \equiv 1 \pmod{\varphi(n)}$ and $z = T_x(\beta) \pmod{p}$ respectively for d and x which are clearly infeasible because the difficulty of solving factoring and chaotic map problems.

5. Performance analysis

Compared to other public key cryptosystems, Chebyshev polynomial computation problem offers smaller key sizes, faster computation as well as memory, energy and bandwidth savings. The computational complexity of ECC is very high, but compared to the ECC encryption algorithm, chaotic maps encryption algorithm avoids scalar multiplication and modular exponentiation computations, effectively improving the efficiency. For the convenience of evaluating the computational cost, we define some notations as follows - T_h : time required to compute hash function, $T_h \approx 0.0005s$; T_C : time required to compute extended chaotic function, $T_C \approx 0.032s$; T_{exp} : time required to compute exponentiation function, $T_{exp} \approx 5.37s$; T_m : time required to compute multiplication function, $T_m \approx 0.00207s$; and T_{inv} : time required to compute inverse function, $T_{inv} \approx 0.0207s$ [2,20].

Table 1 shows the comparison of computational cost between the proposed scheme and the scheme in [18]. We can see that the proposed scheme is more efficient than the scheme in [18]. Our scheme requires only 21.87146 s, while their scheme needs 48.44278 s.

6. Numerical simulation of the scheme

Assume that a signer wishes to sign a hashed message $h(m) = 402$, such that only the intended party can validate the resulting signature

The scheme's set up is done by a signer, with $\bar{p} = 47$, $\bar{q} = 59$, $n = \bar{p}\bar{q} = 2773$, $p = 11093$, $\varphi(n) = (\bar{p} - 1)(\bar{q} - 1) = 2668$, $e = 17$, $d \equiv e^{-1} \equiv 17^{-1} \equiv 157 \pmod{2668}$, $x = 27$, $\beta = 100$, $z = T_{27}(100) \pmod{11093} = 1034$ and the common information $c = 332$.

Table 1: Performance comparisons among our scheme and scheme in [18]

Phases	The proposed scheme	The scheme in [18]		
	Computation cost	Execution time(s)	Computation cost	Execution time(s)
Requesting	$3T_{exp} + 4T_{mul} + 2T_{inv} + T_h$	16.16018	$2T_c + 3T_{mul} + 2T_{inv} + T_h$	0.11202
Singing and Extraction	$2T_{exp} + 9T_{mul} + 2T_{inv}$	10.8003	$2T_{exp} + 10T_{mul} + 3T_{inv}$	10.8228
verification	$4T_{exp} + T_{mul} + T_h$	21.48257	$2T_{exp} + 2T_{mul} + 6T_c + T_h$	10.93664
Total costs	$9T_{exp} + 14T_{mul} + 2T_h + 4T_{inv}$	48.44278	$4T_{exp} + 15T_{mul} + 8T_c + 2T_h + 5T_{inv}$	21.87146

Requesting: A signer select an integer $r = 2551$ such that $\gcd(2551, 2773) = 1$ then computes and send $\hat{t} = T_{2551}(100)(\text{mod } 11093) = 8875$ to the requester. The requester then randomly selects two blinding factors $u = 2331$, $v = 2526$ and computes

$$t = T_{(2331+2526)}(8875) (\text{mod } 11093) \equiv 3292 (\text{mod } 11093),$$

$$\mu \equiv u^{-1}h(m)\hat{t} t^{-1} \equiv (1123)(402)(8875)(1293)(\text{mod } 2773) \equiv 567$$

Then the requester sends (567, 332) to the signer.

Signing and extraction:

The signer computes and sends $\hat{k} = (567 \times 27 \times 332 \times 687 + 8875) (\text{mod } 2773) \equiv 1869$ to the requester, whose next calculates

$$k \equiv 1118(1869 \times 3292 \times 793 \times 2331 + 2526 \times 3292) (\text{mod } 2773) \equiv 34 (\text{mod } 2773)$$

and sends it to the signer. The signer computes and sends

$$\hat{R} \equiv (34 \times 2551)^{157} (\text{mod } 2773) \equiv 2336 (\text{mod } 2773)$$

to the requester.

The requester computes $R \equiv 2336 \times 1869 (\text{mod } 2773) \equiv 1282 (\text{mod } 2773)$. Then the signature is given by $(c, t, R) = (332, 3292, 1282)$. Now the recipient obtain a signature as (332, 3292, 1282) and accept this signature since

$$\begin{aligned} & [T_{Re}(\beta)]^2 + [T_{h(m)c}(z)]^2 + [T_t(t)]^2 (\text{mod } p) \equiv \\ & (10741)^2 + (6483)^2 + (8448)^2 (\text{mod } 11093) \equiv 7228 (\text{mod } 11093) \\ & (2T_{Re}(\beta)T_{h(m)c}(z)T_t(t) + 1) (\text{mod } p) \equiv \\ & (2 \times 10741 \times 6483 \times 8448 + 1) (\text{mod } 11093) \equiv 7228 (\text{mod } 11093) \end{aligned}$$

7. Conclusion

Based on the difficulty to solve chaotic maps and factoring problems, we proposed a partially blind signature scheme in this paper. In the proposed scheme, the security depends on the intractability of both the integer factorization and discrete logarithms of Chebyshev polynomials. The proposed scheme utilizes a smaller number of bits and lower computation cost due to the inherence of Chebyshev polynomials as compared to its partially blind signature scheme counterparts such as the one proposed by Tahat et al. [18]. Providing excellent security, reliability and efficiency, we believe our proposed scheme is more suitable for practical applications.

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A STUDY ON PSEUDOORDERS IN ORDERED *-SEMIHYPERGROUPS

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Abstract. In this paper, we study the pseudoorders on ordered $*$ -semihypergroups in detail. To begin with, we introduce the concept of pseudoorders on an ordered $*$ -semihypergroup, and investigate its related properties. Furthermore, the relationship between strongly regular equivalence relations and pseudoorders on an ordered $*$ -semihypergroup is established, and some homomorphism theorems of ordered $*$ -semihypergroups by pseudoorders are given. Finally, we investigate the direct product of ordered $*$ -semihypergroups, and study the pseudoorders on direct product of ordered $*$ -semihypergroups.

Keywords: ordered $*$ -semihypergroup, pseudoorder, strongly regular equivalence relation, homomorphism, direct product.

1. Introduction

As we know, hyperstructure theory was investigated in 1934, when Marty [17] defined hypergroups, began to analyze their properties and applied them to groups. In the past several decades, a number of different hyperstructures have been widely studied from the theoretical point of view and for their applications to many subjects of pure and applied mathematics by many authors, for example, see [3, 4, 7, 18, 21]. In particular, a semihypergroup is a classic ex-

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ample of algebraic hyperstructure which is a generalization of semigroups and hypergroups. At present, many authors have studied different aspects of semi-hypergroups, for instance, Bonansinga and Corsini [1], Davvaz [5], Fasino and Freni [8], Hasankhani [10], Hila et al. [12], Leoreanu [16], Naz and Shabir [19], Salvo et al. [22], and many others.

Recall that an ordered semigroup is a semigroup together with a partial order that is compatible with the semigroup operation. Many authors, especially Kehayopulu [13], Kehayopulu and Tsingelis [14, 15], Satyanarayana [23] and Xie [26], studied such semigroups with some restrictions. Furthermore, as a generalization of ordered semigroups, Heidari and Davvaz [11] applied the theory of hyperstructures to ordered semigroups and introduced the concept of ordered semihypergroups, also see [2, 6]. It is well known that regular and strongly regular equivalence relations of ordered semihypergroups always play important roles in the study of ordered semihypergroups structure. For more details, the reader is referred to [6, 9]. On the other hand, Nordahl and Scheiblich [20] considered a unary operation $*$ on semigroups and introduced the concept of regularity on $*$ -semigroups. In [25], Wu imposed the $*$ -operation on ordered semigroups under the assumption of order preserving. In the present paper, we shall generalize the concept of ordered $*$ -semigroups to the hyper version and construct a strongly regular equivalence relation of ordered $*$ -semihypergroups by using the notion of pseudoorders on an ordered $*$ -semihypergroup such that the corresponding quotient structure is an ordered $*$ -semigroup.

The rest of this paper is organized as follows. After an introduction, in Section 2 we recall some basic definitions and results of ordered semihypergroups which will be used throughout this paper and introduce the concept of ordered $*$ -semihypergroups. In Section 3, the concept of a pseudoorder on ordered $*$ -semihypergroups is introduced, and the related properties are investigated. In addition, the relationship between ordered regular equivalence relations and pseudoorders on an ordered $*$ -semihypergroup is established, and several homomorphism theorems of ordered $*$ -semihypergroups by pseudoorders are given. In Section 4, we investigate the direct product $S \times T$ of ordered $*$ -semihypergroups S and T , and show that $S \times T$ is also an ordered $*$ -semihypergroup under a suitable hyperoperation and a unary operation. Moreover, the pseudoorders on $S \times T$ are studied.

2. Preliminaries and some notations

For convenience, let us first give some necessary definitions. A mapping $\circ : S \times S \rightarrow \mathcal{P}^*(S)$ is called a hyperoperation on S , where $\mathcal{P}^*(S)$ denotes the family of all nonempty subsets of S . The system $(S, \circ)$ is called a *hypergroupoid*. If A and B are two nonempty subsets of S , then we denote $A \circ B = \bigcup_{a \in A, b \in B} a \circ b$. In particular, for any $x \in S$, we write $x \circ A = \{x\} \circ A$ and $A \circ x = A \circ \{x\}$. Recall that a *semihypergroup* is a hypergroupoid $(S, \circ)$ such that for every $x, y, z \in S$, $x \circ (y \circ z) = (x \circ y) \circ z$ (see [3]).

Let $(S, \circ)$ be a semihypergroup and ρ an equivalence relation on S . If A and B are nonempty subsets of S , then we put

$$A \bar{\rho} B \Leftrightarrow \begin{cases} (\forall a \in A)(\exists b \in B) & a \rho b, \\ (\forall b' \in B)(\exists a' \in A) & a' \rho b', \end{cases}$$

and

$$A \bar{\bar{\rho}} B \Leftrightarrow (\forall a \in A)(\forall b \in B) a \rho b.$$

An equivalence relation ρ on a semihypergroup S is called *regular* [6, 9] if

$$(\forall a, b, x \in S) a \rho b \Rightarrow a \circ x \bar{\rho} b \circ x \text{ and } x \circ a \bar{\rho} x \circ b;$$

ρ is said to be *strongly regular* [6, 9] if

$$(\forall a, b, x \in S) a \rho b \Rightarrow a \circ x \bar{\bar{\rho}} b \circ x \text{ and } x \circ a \bar{\bar{\rho}} x \circ b.$$

Let $(S, \circ)$ be a semihypergroup and ρ an equivalence relation on S . We denote by $a\rho$ the equivalence ρ -class containing a . From [3], we have the following two theorems.

Theorem 2.1. ([3]) *Let $(S, \circ)$ be a semihypergroup and ρ an equivalence relation on S .*

- (1) *If ρ is regular, then S/ρ is a semihypergroup with respect to the following hyperoperation: $x\rho \odot y\rho = \{z\rho \mid z \in x \circ y\}$.*
- (2) *If the above hyperoperation is well defined on S/ρ , then ρ is regular.*

Theorem 2.2. ([3]) *Let $(S, \circ)$ be a semihypergroup and ρ an equivalence relation on S .*

- (1) *If ρ is strongly regular, then S/ρ is a semigroup with respect to the following operation: $x\rho \odot y\rho = z\rho$, for all $z \in x \circ y$.*
- (2) *If the above operation is well defined on S/ρ , then ρ is strongly regular.*

As we know, an ordered semigroup $(S, \cdot, \leq)$ is a semigroup $(S, \cdot)$ with an order relation “ $\leq$ ” such that $a \leq b$ implies $xa \leq xb$ and $ax \leq bx$ for any $x \in S$. Furthermore, an order semigroup S with a unary operation $*$: $S \rightarrow S$ is called an *ordered $*$ -semigroup* if it satisfies $(x^*)^* = x$ and $(xy)^* = y^*x^*$ for any $x, y \in S$ (see [25]).

We now recall the notion of ordered semihypergroups from [11].

Definition 2.3. An algebraic hyperstructure $(S, \circ, \leq)$ is called an *ordered semihypergroup* (also called *po-semihypergroup* in [11]) if $(S, \circ)$ is a semihypergroup and $(S, \leq)$ is a partially ordered set such that: for any $x, y, a \in S$, $x \leq y$ implies $a \circ x \preceq a \circ y$ and $x \circ a \preceq y \circ a$. Here, if $A, B \in \mathcal{P}^*(S)$, then we say that $A \preceq B$ if for every $a \in A$ there exists $b \in B$ such that $a \leq b$. In particular, if $A = \{a\}$, then we write $a \preceq B$ instead of $\{a\} \preceq B$.

In the following, we shall generalize the concept of ordered *-semigroups to the hyper version.

Definition 2.4. An ordered semihypergroup S with a unary operation $*$: $S \rightarrow S$ is called an *ordered *-semihypergroup* if it satisfies:

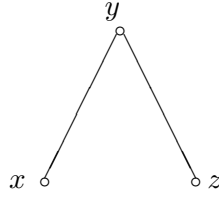
- (1) $(\forall x \in S) (x^*)^* = x$.
- (2) $(\forall x, y \in S) (x \circ y)^* = y^* \circ x^*$, where, for any $A \in \mathcal{P}^*(S)$, the notation A^* is defined by $A^* := \{a^* \in S \mid a \in A\}$.

Such a unary operation $*$ is called an *involution*. If $a \leq b$ implies $a^* \leq b^*$ for any $a, b \in S$, then $*$ is called an *order preserving involution*.

Example 2.5. Let $S = \{x, y, z\}$ be an ordered semihypergroup. The hyperoperation " $\circ$ ", the order " $\leq$ " and the corresponding Hasse diagram are given below. Define the involution $*$ by $x^* = z$, $y^* = y$ and $z^* = x$. It is not difficult to check that $(S, \circ, \leq)$ is an ordered *-semihypergroup with order preserving involution $*$.

$\circ$	x	y	z
x	$\{x\}$	$\{x, z\}$	$\{x, z\}$
y	$\{x, z\}$	$\{y\}$	$\{x, z\}$
z	$\{x, z\}$	$\{x, z\}$	$\{z\}$

$$\leq := \{(x, x), (x, y), (y, y), (z, y), (z, z)\}.$$



Lemma 2.6. Let A, B be nonempty subsets of an ordered *-semihypergroup S . Then the following statements hold :

- (1) $A \subseteq B$ implies $A^* \subseteq B^*$.
- (2) $(A \circ B)^* = B^* \circ A^*$.
- (3) $(A \cup B)^* = A^* \cup B^*$.
- (4) $(A \cap B)^* = A^* \cap B^*$.

Proof. Straightforward. □

Throughout this paper, unless otherwise mentioned, S will denote an ordered *-semihypergroup with order preserving involution $*$. The reader is referred to [4, 24] for notation and terminology not defined in this paper.

3. Pseudoorders on ordered $\ast$ -semihypergroups

In this section we define and characterize the pseudoorders on ordered $\ast$ -semihypergroups, and investigate its related properties. Furthermore, the relationship between strongly regular equivalence relations and pseudoorders on an ordered $\ast$ -semihypergroup is established, and some homomorphism theorems of ordered $\ast$ -semihypergroups by pseudoorders are given.

Definition 3.1. Let $(S; \circ, \ast; \leq)$ be an ordered $\ast$ -semihypergroup. A relation ρ on S is called *pseudoorder* if

- (1) $\leq \subseteq \rho$;
- (2) $a\rho b$ implies $a^\ast \rho b^\ast$;
- (3) $a\rho b$ and $b\rho c$ imply $a\rho c$
- (4) $(\forall c \in S) a\rho b$ implies $a \circ c \bar{\rho} b \circ c$ and $c \circ a \bar{\rho} c \circ b$.

Definition 3.2. An equivalence relation ρ on an ordered $\ast$ -semihypergroup S is called *strongly regular*, if it satisfies

- (1) $(\forall a, b, x \in S) a\rho b \Rightarrow a \circ x \bar{\rho} b \circ x$ and $x \circ a \bar{\rho} x \circ b$;
- (2) $(\forall a, b \in S) a\rho b \Rightarrow a^\ast \rho b^\ast$.

In the following we shall construct a strongly regular relation ρ° on an ordered $\ast$ -semihypergroup S for which S/ρ° is an ordered $\ast$ -semigroup.

Theorem 3.3. Let $(S; \circ, \ast; \leq)$ be an ordered $\ast$ -semihypergroup and ρ be a pseudoorder on S . Then there exists a strongly regular relation ρ° on S such that S/ρ° is an ordered $\ast$ -semigroup.

Proof. Assume that ρ° is the relation on S defined as follows:

$$(\forall a, b \in S) \quad (a, b) \in \rho^\circ \Leftrightarrow (a, b) \in \rho \cap \rho^{-1}.$$

Then, it is easy to see that ρ° is an equivalence relation. Now, we prove that ρ° is a strongly regular relation on S . Let $a\rho^\circ b$ and $c \in S$. Then $a\rho b$ and $b\rho a$. By condition (4) of Definition 3.1, we have

$$a \circ c \bar{\rho} b \circ c, \quad b \circ c \bar{\rho} a \circ c; \quad c \circ a \bar{\rho} c \circ b, \quad c \circ b \bar{\rho} c \circ a.$$

Thus, for every $x \in a \circ c$ and $y \in b \circ c$, by $a \circ c \bar{\rho} b \circ c$, we have $x\rho y$. Also, by $b \circ c \bar{\rho} a \circ c$, we have $y\rho x$. This implies that $x\rho^\circ y$. So, $a \circ c \bar{\rho}^\circ b \circ c$. By using a similar argument, we can deduce that $c \circ a \bar{\rho}^\circ c \circ b$. On the other hand, for any $a, b \in S$, let $a\rho^\circ b$. Then $a\rho b$ and $b\rho a$. Since ρ is a pseudoorder on S , we obtain $a^\ast \rho b^\ast$ and $b^\ast \rho a^\ast$ which imply that $a^\ast \rho^\circ b^\ast$. Hence, ρ° is a strongly regular relation on S . Hence, by Theorem 2.2, S/ρ° with the following operation is a semigroup:

$$x\rho^\circ \odot y\rho^\circ = z\rho^\circ, \text{ for all } z \in x \circ y.$$

Next, we define a relation $\preceq$ on S/ρ° as follows:

$$\preceq := \{(x\rho^\circ, y\rho^\circ) \in S/\rho^\circ \times S/\rho^\circ \mid (x, y) \in \rho\}.$$

We deduce that $(S/\rho^\circ; \odot; \preceq)$ is an ordered semigroup. In fact, for $x\rho^\circ \in S/\rho^\circ$, where $x \in S$, since $(x, x) \in \preceq \subseteq \rho$, it can be obtained that $x\rho^\circ \preceq x\rho^\circ$. If $x\rho^\circ \preceq y\rho^\circ$ and $y\rho^\circ \preceq x\rho^\circ$, then $x\rho^\circ y$ and $y\rho^\circ x$. Hence, $x\rho^\circ y$ means that $x\rho^\circ = y\rho^\circ$. Suppose that $x\rho^\circ \preceq y\rho^\circ$ and $y\rho^\circ \preceq z\rho^\circ$. Then $x\rho^\circ y$ and $y\rho^\circ z$. Hence $x\rho^\circ z$, and we obtain $x\rho^\circ \preceq z\rho^\circ$. Now, let $x\rho^\circ \preceq \rho^\circ$ and $z\rho^\circ \in S/\rho^\circ$. Then $x\rho^\circ y$ and $z \in S$. Since ρ is a pseudoorder on S , we have $x \circ z \bar{\rho} y \circ z$ and $z \circ x \bar{\rho} z \circ y$. Hence, for every $a \in x \circ z$ and $b \in y \circ z$, we have $a\rho^\circ b$. It implies that $a\rho^\circ \preceq b\rho^\circ$. Thus, $x\rho^\circ \odot z\rho^\circ \preceq y\rho^\circ \odot z\rho^\circ$. Similarly, it can be shown that $z\rho^\circ \odot x\rho^\circ \preceq z\rho^\circ \odot y\rho^\circ$.

Finally, we define a unary operation on S/ρ° by:

$$(a\rho^\circ)^* = a^*\rho^\circ, \text{ for all } a \in S.$$

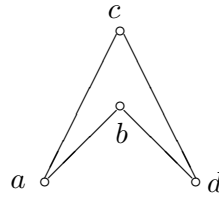
The unary operation $\star$ is well defined. Indeed, let $a\rho^\circ = b\rho^\circ$, i.e., $a\rho^\circ b$. Then $a\rho^\circ b$ and $b\rho^\circ a$. Note that ρ is a pseudoorder on S , we get that $a^*\rho^\circ b^*$ and $b^*\rho^\circ a^*$. So, $a^*\rho^\circ b^*$, that is, $a^*\rho^\circ = b^*\rho^\circ$. Furthermore, it is easy to check that S/ρ° is an ordered $\star$ -semigroup with the unary operation $\star$. Let $\star$ be an order preserving involution on S , and $x\rho^\circ \preceq y\rho^\circ$. Then $x\rho^\circ y$, by the definition of pseudoorders, we have $x^*\rho^\circ y^*$. This means that $x^*\rho^\circ \preceq y^*\rho^\circ$, i.e., $(x\rho^\circ)^* \preceq (y\rho^\circ)^*$. Therefore, we conclude that the operation $\star$ is also an order preserving involution on S/ρ° . $\square$

Example 3.4. Let $S = \{a, b, c, d\}$ be an ordered $\star$ -semihypergroup. The hyperoperation “ $\circ$ ”, the unary operation “ $\star$ ” and the order “ $\leq$ ” are given below.

$\circ$	a	b	c	d
a	$\{a, d\}$	$\{a, d\}$	$\{a, d\}$	$\{a\}$
b	$\{a, d\}$	$\{b\}$	$\{a, d\}$	$\{a, d\}$
c	$\{a, d\}$	$\{a, d\}$	$\{c\}$	$\{a, d\}$
d	$\{a\}$	$\{a, d\}$	$\{a, d\}$	$\{d\}$

$$a^* = a, \quad b^* = c, \quad c^* = b, \quad d^* = d.$$

$$\leq := \{(a, a), (a, b), (a, c), (b, b), (c, c), (d, b), (d, c), (d, d)\}.$$



Let ρ be a pseudoorder on S defined as follows:

$$\rho = \{(a, a), (b, b), (c, c), (d, d), (a, b), (a, c), (a, d), (d, a), (d, b), (d, c)\}.$$

Then, by the definition of ρ° , we have

$$\rho^\circ = \{(a, a), (b, b), (c, c), (d, d), (a, d)\}.$$

Thus, $S/\rho^\circ = \{\alpha, \beta, \gamma\}$, where $\alpha = \{a, d\}, \beta = \{b\}, \gamma = \{c\}$. Immediately, $(S/\rho^\circ; \odot, \star; \preceq)$ is an ordered $\ast$ -semigroup, where the multiplication $\odot$, the unary operation $\star$ and the order $\preceq$ are defined in the following:

$\odot$	α	β	γ
α	α	α	α
β	α	β	α
γ	α	α	γ

$$\alpha^\star = \alpha, \beta^\star = \gamma, \gamma^\star = \beta,$$

and

$$\preceq = \{(\alpha, \alpha), (\alpha, \beta), (\beta, \beta), (\alpha, \gamma), (\gamma, \gamma)\}.$$

Theorem 3.5. *Let $(S; \circ, \ast; \leq)$ be an ordered $\ast$ -semihypergroup and ρ a pseudo-order on S . Let*

$$\mathcal{A} := \{ \theta \mid \theta \text{ is a pseudoorder on } S \text{ such that } \rho \subseteq \theta \}.$$

Let $\mathcal{B}$ be the set of all pseudoorders on S/ρ° . For $\theta \in \mathcal{A}$, define a relation θ' on S/ρ° as follows:

$$\theta' := \{(x\rho^\circ, y\rho^\circ) \in S/\rho^\circ \times S/\rho^\circ \mid (x, y) \in \theta\}.$$

Then the mapping

$$\varphi : \mathcal{A} \rightarrow \mathcal{B}, \theta \mapsto \theta'$$

is a bijection, and $\theta_1 \subseteq \theta_2$ if and only if $\theta'_1 \subseteq \theta'_2$, where $\theta_1, \theta_2 \in \mathcal{A}$.

Proof. Suppose that $\theta \in \mathcal{A}$. We show that θ' is a pseudoorder on S/ρ° . Let $(x\rho^\circ, y\rho^\circ) \in \preceq$. By the proof of Theorem 3.3, we have $(x, y) \in \rho \subseteq \theta$ which implies that $(x\rho^\circ, y\rho^\circ) \in \theta'$. Hence $\preceq \subseteq \theta'$. Also, let $(x\rho^\circ, y\rho^\circ) \in \theta'$ and $z\rho^\circ \in S/\rho^\circ$. Then $(x, y) \in \theta$ and $z \in S$. Thus $x \circ z \theta$ and $z \circ x \theta z \circ y$. So, for all $a \in x \circ z$ and $b \in y \circ z$, we have $a \theta b$. Therefore, $(x\rho^\circ \odot z\rho^\circ) \theta' (y\rho^\circ \odot z\rho^\circ)$. By a similar argument, we deduce that $(z\rho^\circ \odot x\rho^\circ) \theta' (z\rho^\circ \odot y\rho^\circ)$. Moreover, if $(x\rho^\circ, y\rho^\circ) \in \theta'$, then $(x, y) \in \theta$. So, we get $(x^\star, y^\star) \in \theta$, and we have $(x^\star\rho^\circ, y^\star\rho^\circ) \in \theta'$, that is, $((x\rho^\circ)^\star, (y\rho^\circ)^\star) \in \theta'$. Therefore, if $\theta \in \mathcal{A}$, then θ' is a pseudoorder on S/ρ° .

Next, we deduce that the map $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ defined by $\varphi(\theta) = \theta'$ is well defined and one-to-one. In fact, let $\theta_1, \theta_2 \in \mathcal{A}$ and $\theta_1 = \theta_2$. Assume that $(x\rho^\circ, y\rho^\circ) \in \theta'$. Then $(x, y) \in \theta_1 = \theta_2$. This implies that $(x\rho^\circ, y\rho^\circ) \in \theta'_2$. Thus

$\theta'_1 \subseteq \theta'_2$. Similarly, we obtain that $\theta'_2 \subseteq \theta'_1$. On the other hand, let $\theta_1, \theta_2 \in \mathcal{A}$ and $\theta'_1 = \theta'_2$. Suppose that $(x, y) \in \theta_1$. Then $(x\rho^\circ, y\rho^\circ) \in \theta'_1 = \theta'_2$. Thus $(x, y) \in \theta_2$ which implies that $\theta_1 \subseteq \theta_2$. By using a similar argument, it is easy to see that $\theta_2 \subseteq \theta_1$.

Finally, we prove that φ is onto. Take $\delta \in \mathcal{B}$, we define a relation θ on S as follows:

$$\theta = \{(x, y) \in S \times S \mid (x\rho^\circ, y\rho^\circ) \in \delta\}.$$

We claim that θ is a pseudoorder on S and $\rho \subseteq \theta$. To prove our claim, let $(x, y) \in \rho$, by the proof of Theorem 3.3, $(x\rho^\circ, y\rho^\circ) \in \preceq \subseteq \delta$, and so $(x, y) \in \theta$. Thus, $\rho \subseteq \theta$. If $(x, y) \in \leq$, then $(x, y) \in \rho \subseteq \theta$. Hence, $\leq \subseteq \theta$. Suppose that $(x, y) \in \theta$ and $(y, z) \in \theta$. Then $(x\rho^\circ, y\rho^\circ) \in \delta$ and $(y\rho^\circ, z\rho^\circ) \in \delta$. Thus, $(x\rho^\circ, z\rho^\circ) \in \delta$, and we have $(x, z) \in \theta$. Also, let $(x, y) \in \theta$ and $z \in S$. Then $(x\rho^\circ, y\rho^\circ) \in \delta$ and $z\rho^\circ \in S/\rho^\circ$. So, we obtain $(x\rho^\circ \odot z\rho^\circ) \in \delta$ and $(y\rho^\circ \odot z\rho^\circ) \in \delta$. Thus, $(a\rho^\circ, b\rho^\circ) \in \delta$ for all $a \in x \odot z$, $b \in y \odot z$. Thus, $(a, b) \in \theta$, which implies that $x \odot z \theta y \odot z$. Similarly, we can deduce that $z \odot x \theta z \odot y$. Furthermore, let $(x, y) \in \theta$. Then $(x\rho^\circ, y\rho^\circ) \in \delta$. This implies that $((x\rho^\circ)^*, (y\rho^\circ)^*)$, i.e., $(x^*\rho^\circ, y^*\rho^\circ) \in \delta$. Therefore, $(x^*, y^*) \in \theta$. $\square$

Let $(S; \diamond, *, \leq_S)$ and $(T; \circ, \star, \leq_T)$ be two ordered *-semihypergroups, $f : S \rightarrow T$ a mapping from S into T . f is said to be *isotone* if $x \leq_S y$ implies that $f(x) \leq_T f(y)$ for all $x, y \in S$. f is called *reverse isotone* if $f(x) \leq_T f(y)$ implies that $x \leq_S y$. f is called a *strong homomorphism* if (1) f is isotone; (2) for all $x, y \in S$, $f(x) \circ f(y) = f(z)$, where z is an arbitrary element of $x \diamond y$; and (3) $f(x^*) = f^*(x)$ for any $x \in S$. f is said to be *strong isomorphism* if it is strong homomorphism, surjection and reverse isotone. The ordered *-semihypergroups S and T are called *strongly isomorphic*, in symbol $S \cong T$, if there exists a strong isomorphism between them.

Proposition 3.6. *Let $(S; \diamond, *, \leq_S)$ and $(T; \circ, \star, \leq_T)$ be two ordered *-semihypergroups with order preserving involutions $*$ and $\star$, respectively. Let $f : S \rightarrow T$ be a strong homomorphism. The relation ρ on S defined by*

$$\rho := \{(x, y) \in S \times S \mid f(x) \leq_T f(y)\}$$

is a pseudoorder on S .

Proof. Suppose that $(x, y) \in \leq_S$. Since $x \leq_S y$ and f is isotone, we obtain $f(x) \leq_T f(y)$. Thus $(x, y) \in \rho$. If $(x, y) \in \rho$ and $(y, z) \in \rho$. Then $f(x) \leq_T f(y)$, $f(y) \leq_T f(z)$, and thus $f(x) \leq_T f(z)$. This means that $(x, z) \in \rho$. Let $(x, y) \in \rho$, $z \in S$. For any $a \in x \diamond z$ and $b \in y \diamond z$, we have

$$f(a) = f(x) \circ f(z) \leq_T f(y) \circ f(z) = f(b).$$

Therefore, $(a, b) \in \rho$ and we have $x \circ z \stackrel{\bar{\rho}}{=} y \circ z$. Similarly, it can be easily obtained that $z \circ x \stackrel{\bar{\rho}}{=} z \circ y$. Furthermore, let $(x, y) \in \rho$. Then $f(x) \leq_T f(y)$. Since $\star$ is an order preserving involution on T , we have $f^\star(x) \leq_T f^\star(y)$. Note that f is a strong homomorphism, we obtain $f(x^\star) \leq_T f(y^\star)$, that is, $(x^\star, y^\star) \in \rho$. $\square$

Lemma 3.7. *Let $(S; \diamond, *, \leq_S)$ and $(T; \circ, \star, \leq_T)$ be ordered $\ast$ -semihypergroups with order preserving involution $\ast$ and $\star$, respectively. Let $f : S \rightarrow T$ be a strong homomorphism. Then*

$$\ker \varphi := \{(a, b) \in S \times S \mid \varphi(a) = \varphi(b)\}$$

is a strongly regular relation on S and

$$\underline{\varphi}^\circ := \{(a, b) \in S \times S \mid a \underline{\varphi} b \text{ and } b \underline{\varphi} a\} = \ker \varphi,$$

where $\underline{\varphi} := \{(a, b) \in S \times S \mid \varphi(a) \leq_T \varphi(b)\}$ is the pseudoorder on S defined by Proposition 3.6.

Proof. (1). $\ker \varphi$ is a strong regular relation on S . In fact,

Let $a \ker \varphi b$ and $c \in S$. Then $\varphi(a) = \varphi(b)$, which implies that $\varphi(a) \circ \varphi(c) = \varphi(b) \circ \varphi(c)$. Hence, for all $x \in a \circ c$ and all $y \in b \circ c$, we have $\varphi(x) = \varphi(y)$, i.e., $x \ker \varphi y$. Thus $a \circ c \stackrel{\overline{\ker \varphi}}{=} b \circ c$. Similarly, we can show that $c \circ a \stackrel{\overline{\ker \varphi}}{=} c \circ b$.

Suppose that $a \ker \varphi b$ for any $a, b \in S$. Then $\varphi(a) = \varphi(b)$. This means that $\varphi^\star(a) = \varphi^\star(b)$. Since φ is a strong homomorphism, we have $\varphi(a^\star) = \varphi(b^\star)$. That is $a^\star \ker \varphi b^\star$.

(2). $\ker \varphi = \underline{\varphi}^\circ$. Indeed, let $(a, b) \in \underline{\varphi}^\circ$. Then $a \underline{\varphi} b$ and $b \underline{\varphi} a$. That is $\varphi(a) \leq_T \varphi(b)$ and $\varphi(b) \leq_T \varphi(a)$. Hence, $\varphi(a) = \varphi(b)$, i.e., $(a, b) \in \ker \varphi$. On the other hand, let $(a, b) \in \ker \varphi$ for any $a, b \in S$. Then $\varphi(a) = \varphi(b)$. So, $\varphi(a) \leq_T \varphi(b)$, which implies that $a \underline{\varphi} b$ and $\varphi(b) \leq_T \varphi(a)$, and we have $b \underline{\varphi} a$. Thus, $a \underline{\varphi}^\circ b$. $\square$

Let ρ be a pseudoorder on an ordered $\ast$ -semihypergroup $(S; \circ, \ast, \leq)$. Then, by Theorem 3.3, $\rho^\circ = \rho \cap \rho^{-1}$ is a strongly regular equivalence relation on S . We denote by $\rho^\sharp$ the mapping from S onto S/ρ° , i.e., $\rho^\sharp : S \rightarrow S/\rho^\circ \mid x \mapsto x\rho^\circ$, which is a strong homomorphism. In the following, we give out a homomorphism theorem of ordered $\ast$ -semihypergroups by pseudoorders, which is a generalization of [6] and [15].

Theorem 3.8. *Let $(S; \diamond, *, \leq_S)$ and $(T; \circ, \star, \leq_T)$ be ordered $\ast$ -semihypergroups with order preserving involution $\ast$ and $\star$ respectively, $\varphi : S \rightarrow T$ a strong homomorphism. If ρ is a pseudoorder on S such that $\rho \subseteq \underline{\varphi}$, then the mapping $f : S/\rho^\circ \mid a\rho^\circ \mapsto \varphi(a)$ is the unique strong homomorphism of S/ρ° into T such that the diagram*

$$\begin{array}{ccc} S & \xrightarrow{\varphi} & T \\ (\rho^\circ)^\sharp \downarrow & \nearrow f & \\ S/\rho^\circ & & \end{array}$$

commutes. Furthermore, $Im f = Im \varphi$. Conversely, if ρ is a pseudoorder on S for which there exists a strong homomorphism $f : S/\rho^\circ \rightarrow T$ such that the above diagram commutes, then $\rho \subseteq \underline{\varphi}$.

Proof. Let ρ be a pseudoorder on S , $\rho \subseteq \underline{\varphi}$, $f : S/\rho^\circ \rightarrow T$ is defined by $f(a\rho^\circ) = \varphi(a)$. Then f is well defined. In fact,

$$\begin{aligned} a\rho^\circ = b\rho^\circ &\Rightarrow (a, b) \in \rho^\circ \Rightarrow (a, b) \in \rho \text{ and } (b, a) \in \rho \\ &\Rightarrow (a, b) \in \underline{\varphi} \text{ and } (b, a) \in \underline{\varphi} \\ &\Rightarrow \varphi(a) \leq_T \varphi(b) \text{ and } \varphi(b) \leq_T \varphi(a) \\ &\Rightarrow \varphi(a) = \varphi(b). \end{aligned}$$

f is a strong homomorphism and $f \cdot \rho^\# = \varphi$. If $a, b \in S$, then

$$a\rho^\circ \leq_{S/\rho^\circ} b\rho^\circ \Rightarrow (a, b) \in \rho \subseteq \underline{\varphi} \Rightarrow \varphi(a) \leq_T \varphi(b),$$

that is, $f(a\rho^\circ) \leq_T f(b\rho^\circ)$ and f is isotone. Also,

$$f(a\rho^\circ \odot b\rho^\circ) = f(z\rho^\circ) = \varphi(z) = \varphi(a) \circ \varphi(b) = f(a\rho^\circ) \circ f(b\rho^\circ),$$

for all $z \in a \diamond b$. Furthermore, for any $(a\rho^\circ)^* \in S/\rho^\circ$, we have

$$f((a\rho^\circ)^*) = f(a^*\rho^\circ) = \varphi(a^*) = \varphi^*(a) = f^*(a\rho^\circ).$$

Finally, for each $a \in S$, $(f \cdot \rho^\#)a = f(\rho^\#(a)) = f(a\rho^\circ) = \varphi(a)$. That is, $f \cdot \rho^\# = \varphi$. We claim that f is the unique strong homomorphism from S/ρ° to T . To prove our claim, let $g : S/\rho^\circ$ be a strong homomorphism such that $g \cdot \rho^\# = \varphi$. Then $f = g$. In fact, for any $a \in S$, we have

$$f(a\rho^\circ) = \varphi(a) = (g \cdot \rho^\#)(a) = g(\rho^\#(a)) = g(a\rho^\circ).$$

Moreover, $Im f = \{f(a\rho^\circ) \mid a \in S\} = \{\varphi(a) \mid a \in S\} = Im \varphi$.

Conversely, let ρ be a pseudoorder on S , $f : S/\rho^\circ \rightarrow T$ a strong homomorphism, $f \cdot \rho^\# = \varphi$. Then $\rho \subseteq \underline{\varphi}$. Indeed,

$$\begin{aligned} (a, b) \in \rho &\Rightarrow a\rho^\circ \leq_{S/\rho^\circ} b\rho^\circ \Rightarrow f(a\rho^\circ) \leq_T f(b\rho^\circ) \\ &\Rightarrow f(\rho^\#(a)) \leq_T f(\rho^\#(b)) \\ &\Rightarrow (f \cdot \rho^\#)(a) \leq_T (f \cdot \rho^\#)(b) \\ &\Rightarrow \varphi(a) \leq_T \varphi(b) \Rightarrow (a, b) \in \underline{\varphi}. \end{aligned}$$

Hence, the proof is completed. $\square$

Remark 3.9. If S, T are two ordered *-semihypergroups, f a homomorphism and reverse isotone mapping of S into T , then it is easy to check that $S \cong Im f$, since every reverse isotone mapping is injection.

Corollary 3.10. Let $(S; \circ, *, \leq_S)$ and $(T; \diamond, \star, \leq_T)$ be two ordered *-semihypergroups, $\varphi : S \rightarrow T$ a strong homomorphism. Then $S/\ker \varphi \cong Im \varphi$.

Proof. By applying the first part of Theorem 3.8 for $\rho = \underline{\varphi}$. The mapping $f : S/\underline{\varphi}^\circ \rightarrow T \mid a\underline{\varphi}^\circ \rightarrow \varphi(a)$ is a strong homomorphism. In fact, f is reverse isotone. Let $a, b \in S$, $\varphi(a) \leq_T \varphi(b)$. Since $(a, b) \in \underline{\varphi}$, $\underline{\varphi}$ is a pseudoorder on S , we have $(a\underline{\varphi}^\circ, b\underline{\varphi}^\circ) \in \leq_{S/\underline{\varphi}^\circ}$, that is, $a\underline{\varphi}^\circ \leq_{S/\underline{\varphi}^\circ} b\underline{\varphi}^\circ$. Using Remark 3.9, we obtain $S/\underline{\varphi}^\circ \cong \text{Im}f$. Since $\text{Im}f = \text{Im}\varphi$ from the results of Theorem 3.8 and $\underline{\varphi}^\circ = \ker\varphi$, by Lemma 3.7, we conclude $S/\ker\varphi \cong \text{Im}\varphi$. $\square$

Lemma 3.11. Let $(S; \circ, *, \leq)$ be an ordered $*$ -semihypergroup and ρ, σ be pseudoorders on S such that $\rho \subseteq \sigma$. We define a relation σ/ρ on S/ρ° as follows:

$$\sigma/\rho := \{(x\rho^\circ, y\rho^\circ) \in S/\rho^\circ \times S/\rho^\circ \mid (x, y) \in \sigma\}.$$

Then σ/ρ is a pseudoorder on S/ρ° .

Proof. In fact, for any $a, b \in S$, if $a\rho^\circ \leq_{S/\rho^\circ} b\rho^\circ$, then $(a, b) \in \rho \subseteq \sigma$ and $(a\rho^\circ, b\rho^\circ) \in \sigma/\rho$, that is $\leq_{S/\rho^\circ} \subseteq \sigma/\rho$. Let $(a\rho^\circ, b\rho^\circ) \in \sigma/\rho$ and $(b\rho^\circ, c\rho^\circ) \in \sigma/\rho$. Then $(a, b) \in \sigma$ and $(b, c) \in \sigma$, so $(a, c) \in \sigma$, which means that $(a\rho^\circ, c\rho^\circ) \in \sigma/\rho$. Suppose that $(a\rho^\circ, b\rho^\circ) \in \sigma/\rho$, $c\rho^\circ \in S/\rho^\circ$. Then $(a, b) \in \sigma$ and $c \in S$. Thus, $a \circ \bar{c}\bar{\sigma}b \circ c$ and $c \circ a\bar{\sigma}c \circ b$. Thus, for any $x \in a \circ c$, $y \in b \circ c$, we have $(x, y) \in \sigma$. This implies that

$$a\rho^\circ \odot c\rho^\circ = (x\rho^\circ) \sigma/\rho (y\rho^\circ) = b\rho^\circ \odot c\rho^\circ.$$

By using a similar argument, we can deduce that $(c\rho^\circ \odot a\rho^\circ) \sigma/\rho (c\rho^\circ \odot b\rho^\circ)$. Furthermore, for each $a\rho^\circ, b\rho^\circ \in S/\rho^\circ$, $(a\rho^\circ) \sigma/\rho (b\rho^\circ)$, which implies that $(a, b) \in \sigma$. Hence, $(a^*, b^*) \in \sigma$ and $(a^*\rho^\circ) \sigma/\rho (b^*\rho^\circ)$. That is $(a\rho^\circ)^* \sigma/\rho (b\rho^\circ)^*$. $\square$

Moreover, by applying Corollary 3.10 and Lemma 3.11, we have the following theorem.

Theorem 3.12. Let $(S; \circ, *, \leq)$ be an ordered $*$ -semihypergroup, ρ, σ be pseudoorders on S such that $\rho \subseteq \sigma$. Then $(S/\rho^\circ)/(\sigma/\rho)^\circ \cong S/\sigma^\circ$.

Proof. Now, we consider the diagram

$$\begin{array}{ccc} S & \xrightarrow{(\sigma^\circ)^\#} & S/\sigma^\circ \\ (\rho^\circ)^\# \downarrow & \nearrow f & \uparrow \\ S/\rho^\circ & \xrightarrow{(\sigma/\rho)^\#} & (S/\rho^\circ)/(\sigma/\rho)^\circ \end{array}$$

The mapping $f : S/\rho^\circ \rightarrow S/\sigma^\circ$ is defined by $f(a\rho^\circ) = a\sigma^\circ$. Then

(1) f is well defined. In fact,

$$\begin{aligned} a\rho^\circ = b\rho^\circ &\Rightarrow (a, b) \in \rho \subseteq \sigma \quad \text{and} \quad (b, a) \in \rho \subseteq \sigma \\ &\Rightarrow (a, b) \in \sigma^\circ \Rightarrow a\sigma^\circ = b\sigma^\circ. \end{aligned}$$

(2) f is a strong homomorphism. Indeed, let $\odot_\rho, \odot_\sigma$ be the multiplications on S/ρ° and S/σ° , respectively. Then we claim that f is isotone. In fact,

$$a\rho^\circ \leq_{S/\rho^\circ} b\rho^\circ \Rightarrow (a, b) \in \rho \subseteq \sigma \Rightarrow a\sigma^\circ \leq_{S/\sigma^\circ} b\sigma^\circ.$$

Furthermore, we have

$$f(a\rho^\circ \odot_\rho b\rho^\circ) = f(c\rho^\circ) = c\sigma^\circ = a\sigma^\circ \odot_\sigma b\sigma^\circ = f(a\rho^\circ) \odot_\sigma f(b\rho^\circ)$$

and

$$f((a\rho^\circ)^*) = f(a^*\rho^\circ) = a^*\sigma^\circ = (a\sigma^\circ)^* = f^*(a\rho^\circ).$$

Therefore, by Corollary 3.10, $(S/\rho^\circ)/\ker f \cong \text{Im} f$. Define

$$\underline{f} = \{(a\rho^\circ, b\rho^\circ) \in S/\rho^\circ \times S/\rho^\circ \mid f(a\rho^\circ) \leq_{S/\sigma^\circ} f(b\rho^\circ)\}.$$

Then

$$\begin{aligned} (a\rho^\circ, b\rho^\circ) \in \underline{f} &\Leftrightarrow f(a\rho^\circ) \leq_{S/\sigma^\circ} f(b\rho^\circ) \\ &\Leftrightarrow a\sigma^\circ \leq_{S/\sigma^\circ} b\sigma^\circ \\ &\Leftrightarrow (a, b) \in \sigma \\ &\Leftrightarrow (a\rho^\circ, b\rho^\circ) \in \sigma/\rho. \end{aligned}$$

Hence, $\underline{f} = \sigma/\rho$ and $\ker f = \underline{f}^\circ = (\sigma/\rho)^\circ$. Moreover,

$$\text{Im} f = \{f(a\rho^\circ) \mid a \in S\} = \{a\sigma^\circ \mid a \in S\} = S/\sigma^\circ.$$

From the above argument, we conclude that $(S/\rho^\circ)/(\sigma/\rho)^\circ \cong S/\sigma^\circ$. $\square$

4. Direct products of ordered *-semihypergroups

In this section, we investigate the direct product $S \times T$ of ordered *-semihypergroups S and T . Furthermore, the pseudoorders on $S \times T$ are studied.

Let $(S; \circ, *, \leq_S)$ and $(T; \diamond, \dagger, \leq_T)$ be two ordered *-semihypergroups with the order preserving involutions $*$ and $\dagger$, respectively. Define the coordinatewise operation on $S \times T$ as follows:

$$\begin{aligned} (\forall (s_1, t_1), (s_2, t_2) \in S \times T) \quad (s_1, t_1) \odot (s_2, t_2) &= (s_1 \circ s_2) \times (t_1 \diamond t_2) \\ &= \bigcup_{\substack{k \in s_1 \circ s_2 \\ l \in t_1 \diamond t_2}} (k, l), \end{aligned}$$

Obviously, the Cartesian product $S \times T$ of S and T forms a semihypergroup. Define a partial order $\leq$ on $S \times T$ by $(s_1, t_1) \leq (s_2, t_2)$ if and only if $s_1 \leq_S s_2$ and $t_1 \leq_T t_2$. Then $(S \times T; \odot; \leq)$ is an ordered semihypergroup. Furthermore, put a unary operation $\star$ on $S \times T$ as follows:

$$(s, t)^\star = (s^*, t^\dagger).$$

Then, we can deduce that $(S \times T; \odot, \star; \leq)$ is an ordered $\star$ -semihypergroup. In fact, it is easy to see that the unary operation $\star$ is well defined and $[(s, t)^\star]^\star = (s^\star, t^\dagger)^\star = (s, t)$. Moreover,

$$\begin{aligned} [(s_1, t_1) \odot (s_2, t_2)]^\star &= [\bigcup_{\substack{k \in s_1 \circ s_2 \\ l \in t_1 \diamond t_2}} (k, l)]^\star \\ &= \bigcup_{\substack{k \in s_1 \circ s_2 \\ l \in t_1 \diamond t_2}} (k^\star, l^\dagger). \end{aligned}$$

On the other hand,

$$\begin{aligned} (s_2, t_2)^\star \odot (s_1, t_1)^\star &= (s_2^\star, t_2^\dagger) \odot (s_1^\star, t_1^\dagger) \\ &= (s_2^\star \circ s_1^\star) \times (t_2^\dagger \diamond t_1^\dagger) \\ &= \bigcup_{\substack{u \in s_2^\star \circ s_1^\star \\ v \in t_2^\dagger \diamond t_1^\dagger}} (u, v) \\ &= \bigcup_{\substack{u^\star \in s_1 \circ s_2 \\ v^\dagger \in t_1 \diamond t_2}} (u, v) \\ &= \bigcup_{\substack{u \in s_1 \circ s_2 \\ v \in t_1 \diamond t_2}} (u^\star, v^\dagger). \end{aligned}$$

Therefore, $(S \times T; \odot, \star; \leq)$ forms an ordered $\star$ -semihypergroup.

Lemma 4.1. *Let $(S; \circ, \star; \leq_S)$ and $(T; \diamond, \dagger; \leq_T)$ be ordered $\star$ -semihypergroups, ρ, σ be two pseudoorders on S and T , respectively. Define a relation on $S \times T$ by*

$$(s_1, t_1) \delta (s_2, t_2) \Leftrightarrow s_1 \rho s_2 \text{ and } t_1 \sigma t_2.$$

Then δ is a pseudoorder on $S \times T$.

Proof. Suppose that $(s_1, t_1) \leq (s_2, t_2)$. Then $s_1 \leq_S s_2$ and $t_1 \leq_T t_2$. So, $s_1 \rho s_2$ and $t_1 \sigma t_2$. This implies that $(s_1, t_1) \delta (s_2, t_2)$, that is, $\leq \subseteq \delta$. Also, let $(s_1, t_1) \delta (s_2, t_2)$ and $(s_2, t_2) \delta (s_3, t_3)$. Then $s_1 \rho s_2$, $s_2 \rho s_3$, $t_1 \sigma t_2$ and $t_2 \sigma t_3$. Hence $s_1 \rho s_3$ and $t_1 \sigma t_3$. Thus, $(s_1, t_1) \delta (s_3, t_3)$. Furthermore, let $(s_1, t_1) \delta (s_2, t_2)$ and $(k, l) \in S \times T$. For any $u \in s_1 \circ k$, $v \in t_1 \diamond l$, $m \in s_2 \circ k$ and $n \in t_2 \diamond l$. Since $s_1 \rho s_2$, we have $s_1 \circ k \bar{\rho} s_2 \circ k$, that is, $u \rho m$. Also, by $t_1 \sigma t_2$, we deduce that $v \sigma n$. Hence, $(u, v) \delta (m, n)$. This means that $\bigcup_{\substack{u \in s_1 \circ k \\ v \in t_1 \diamond l}} (u, v) \bar{\delta} \bigcup_{\substack{m \in s_2 \circ k \\ n \in t_2 \diamond l}} (m, n)$. We conclude that $[(s_1, t_1) \odot (k, l)] \bar{\delta} [(s_2, t_2) \odot (k, l)]$. By using a similar argument, we deduce that $[(k, l) \odot (s_1, t_1)] \bar{\delta} [(k, l) \odot (s_2, t_2)]$. Finally, assume that $(s_1, t_1) \delta (s_2, t_2)$. Then $s_1 \rho s_2$ and $t_1 \sigma t_2$. Hence,

$s_1^* \rho s_2^*$ and $t_1^\dagger \sigma t_2^\dagger$ since ρ, σ are two pseudoorders on S and T , respectively. Thus, $(s_1^*, t_1^\dagger) \delta (s_2^*, t_2^\dagger)$. That is, $(s_1, t_1)^* \delta (s_2, t_2)^*$. $\square$

Theorem 4.2. *Let $(S; \circ, *, \leq_S)$ and $(T; \diamond, \dagger, \leq_T)$ be two ordered *-semihypergroups, ρ, σ be two pseudoorders on S and T , respectively. Then*

$$(S \times T)/\delta^\circ \cong S/\rho^\circ \times T/\sigma^\circ.$$

Proof. By Theorem 3.3 and Lemma 4.1, it is easy to check that $(S \times T)/\delta^\circ$ and $S/\rho^\circ \times T/\sigma^\circ$ are both ordered *-semigroups with the unary operations $\star$ and $\circledast$ defined as follows:

$$\begin{aligned} [(s, t)\delta^\circ]^\star &= (s, t)^\star \delta^\circ = (s^*, t^\dagger) \delta^\circ, \\ (s\rho^\circ, t\sigma^\circ)^{\circledast} &= (s^*\rho^\circ, t^\dagger\sigma^\circ). \end{aligned}$$

Now, we consider the mapping $\varphi : (S \times T)/\delta^\circ \rightarrow S/\rho^\circ \times T/\sigma^\circ$ defined by $\varphi((s, t)\delta^\circ) = (s\rho^\circ, t\sigma^\circ)$. Then, for any $(s_1, t_1)\delta^\circ, (s_2, t_2)\delta^\circ \in (S \times T)/\delta^\circ$, we have

$$\begin{aligned} (s_1, t_1)\delta^\circ = (s_2, t_2)\delta^\circ &\Leftrightarrow (s_1, t_1) \delta^\circ (s_2, t_2) \\ &\Leftrightarrow (s_1, t_1) \delta (s_2, t_2) \text{ and } (s_2, t_2) \delta (s_1, t_1) \\ &\Leftrightarrow s_1 \rho s_2, s_2 \rho s_1 \text{ and } t_1 \sigma t_2, t_2 \sigma t_1 \\ &\Leftrightarrow s_1 \rho^\circ s_2 \text{ and } t_1 \sigma^\circ t_2 \\ &\Leftrightarrow (s_1\rho^\circ, t_1\sigma^\circ) = (s_2\rho^\circ, t_2\sigma^\circ). \end{aligned}$$

Thus, φ is well defined. Next, we prove that φ is a strong homomorphism. Let $(s_1, t_1)\delta^\circ$ and $(s_2, t_2)\delta^\circ$ be two arbitrary elements of $(S \times T)/\delta^\circ$. Then

$$\begin{aligned} (s_1, t_1)\delta^\circ \leq (s_2, t_2)\delta^\circ &\Leftrightarrow (s_1, t_1) \delta (s_2, t_2) \\ &\Leftrightarrow s_1 \rho s_2 \text{ and } t_1 \sigma t_2 \\ &\Leftrightarrow s_1\rho^\circ \leq_{S/\rho^\circ} s_2\rho^\circ \text{ and } t_1\sigma^\circ \leq_{T/\sigma^\circ} t_2\sigma^\circ \\ &\Leftrightarrow (s_1\rho^\circ, t_1\sigma^\circ) \leq_{(S/\rho^\circ \times T/\sigma^\circ)} (s_2\rho^\circ, t_2\sigma^\circ) \\ &\Leftrightarrow \varphi((s_1, t_1)\delta^\circ) \leq_{(S/\rho^\circ \times T/\sigma^\circ)} \varphi((s_2, t_2)\delta^\circ). \end{aligned}$$

Hence, φ is isotone and reverse isotone. Also,

$$\begin{aligned} \varphi((s_1, t_1)\delta^\circ \bullet (s_2, t_2)\delta^\circ) &= \varphi((s, t)\delta^\circ), \text{ for all } (s, t) \in (s_1, t_1) \odot (s_2, t_2) \\ &= (s\rho^\circ, t\sigma^\circ), \text{ for all } s \in s_1 \circ s_2, t \in t_1 \diamond t_2 \\ &= (s_1\rho^\circ \odot s_2\rho^\circ, t_1\sigma^\circ \otimes t_2\sigma^\circ) \\ &= (s_1\rho^\circ, t_1\sigma^\circ) \times (s_2\rho^\circ, t_2\sigma^\circ) \\ &= \varphi((s_1, t_1)\delta^\circ) \times \varphi((s_2, t_2)\delta^\circ). \end{aligned}$$

Furthermore, $\varphi([(s, t)\delta^\circ]^\star) = \varphi[(s^*, t^\dagger)\delta^\circ] = (s^*\rho^\circ, t^\dagger\sigma^\circ)$. On the other hand, we have

$$\varphi^{\circledast}[(s, t)\delta^\circ] = [\varphi((s, t)\delta^\circ)]^{\circledast} = (s\rho^\circ, t\sigma^\circ)^{\circledast} = (s^*\rho^\circ, t^\dagger\sigma^\circ).$$

Therefore, $\varphi[((s, t)\delta^\circ)^*] = \varphi^{\otimes}[(s, t)\delta^\circ]$ and φ is a strong homomorphism. It is obvious that φ is surjection. Hence, φ is a strong isomorphism and the proof is completed. $\square$

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DIFFERENTIAL TRANSFORMATION METHOD FOR SOLVING HIGH ORDER FUZZY INITIAL VALUE PROBLEMS

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Abstract. In this paper, we develop and analyze the use of the Differential transformation method (DTM) to find the semi analytical solution for high order fuzzy initial

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value problems (FIVPs) involving ordinary differential equations. DTM allows for the solution of FIVPs to be calculated in the form of an infinite series by which the components will be simply computed. Also DTM will be constructed and formulated to obtain a semi-analytical solution of high order FIVPs using the basic properties and definitions of fuzzy set theory. Numerical example involving high order linear FIVPs was solved to illustrate the capability of DTM in this regard. The results obtained by DTM have been compared with the exact solution in the form of figures and tables.

Keywords: fuzzy numbers, fuzzy functions, fuzzy differential equations, differential transformation method.

1. Introduction

Fuzzy set theory is a powerful tool for the modeling of vagueness, and for processing uncertainty or subjective information on mathematical models. The use of fuzzy sets can be an effective tool for a better understanding of the studied phenomena. Many dynamical real life problems may be formulated as a mathematical model. These problems can be formulated either as a system of ordinary or partial differential equations. Fuzzy differential equations are a useful tool to model a dynamical system when information about its behavior. Fuzzy ordinary differential equations may arise in the mathematical modeling of real world problems in which there is some uncertainty or vagueness. Fuzzy Initial value problems (FIVPs) appear when the modeling of these problems was imperfect and its nature is under uncertainty. So the study and solution of FIVPs is extremely necessary in applications, particularly when it involves uncertain parameters or uncertain initial conditions. Fuzzy Initial value problems arise in several areas of mathematics and science including population models [1, 2, 3], mathematical physics [4] and medicine [5, 6]. Approximate-analytical methods such as the Adomian Decomposition Method (ADM), Homotopy Perturbation Method (HPM), Homotopy Analysis Method (HAM) and Variational Iteration Method (VIM) have been used to solve high order FIVPs involving ordinary differential equations. The HPM was utilized to solve high order linear FIVPs [7]. The ADM was employed to solve second order linear FIVP [8] and fourth order linear FIVP in [9]. [10] used VIM to solve linear systems of first order FIVPs. The VIM was implemented also directly to solve second order linear FIVPs [11]. Also HAM was developed and formulated to solve high order linear and nonlinear FIVPs [12].

The differential transformation method (DTM) is one of the Approximate-analytical methods in ordinary differential equations, partial differential equations and integral equations. Since proposed by [13], there are tremendous interests on the applications of the DTM to solve various scientific problems [14, 15, 16]. One of the problems that solvable by this method is the initial value problems (IVPs). Previous studies concluded that the DTM can be easily applied in linear and nonlinear differential equations. This can be observed in [17, 18, 19]. The main aim of this paper is to formulate and employ DTM to solve high order FIVPs directly without reducing into first order system. The

main thrust of this method is that the solution which is expressed as an infinite series converges quick to the exact solutions. To the best of our knowledge, this is the first attempt at solving a high order FIVPs using the DTM. The structure of this paper is organized as follows. We will start in section 2 with some preliminary concepts about fuzzy sets. In section 3 we define the defuzzification procedure of n 'th order FIVP. In section 4, we reviewed the concept of DTM and formulated it to obtain a reliable approximate solution to n th order FIVPs. In section 5, we employ DTM on test examples involving second and third order linear FIVPs and finally, in section 6, we give the conclusion of this study.

2. Preliminaries

Definition [20]. The r -level (or r -cut) set of a fuzzy set $\tilde{A}$, labeled as $\tilde{A}_r$, is the crisp set of all $x \in X$ such that $\mu_{\tilde{A}} \geq r$ i.e.

$$\tilde{A}_r = \{x \in X \mid \mu_{\tilde{A}} > r, r \in [0, 1]\}.$$

Definition [21]. Fuzzy numbers are a subset of the real numbers set, and represent uncertain values. Fuzzy numbers are linked to degrees of membership which state how true it is to say if something belongs or not to a determined set

In this paper the class of all fuzzy subsets of $\mathfrak{R}$ will be denoted by $\tilde{E}$ and satisfy the following properties [21, 22]:

1. $\mu(t)$ is normal, i.e $\exists t_0$ with $\mathfrak{R} \mu(t_0) = 1$.
2. $\mu(t)$ is convex fuzzy set, i.e. $\mu(\lambda t + (1 - \lambda)s) \geq \min\{\mu(t), \mu(s)\}$, $\forall t, s \in \mathfrak{R}, \lambda \in [0, 1]$.
3. μ upper semi-continuous on $\mathfrak{R}$, and $\overline{\{t \in \mathfrak{R} : \mu(t) > 0\}}$ is compact.

$\tilde{E}$ is called the space of fuzzy numbers and $\mathfrak{R}$ is a proper subset of $\tilde{E}$.

Define the r -level set $x \in \mathfrak{R}$, $[\mu]_r = \{x \mid \mu(x) \geq r\}$, $0 \leq r \leq 1$, where $[\mu]_0 = \{x \mid \mu(x) > 0\}$ is compact which is a closed bounded interval and denoted by $[\mu]_r = (\underline{\mu}(t), \bar{\mu}(t))$. In the parametric form, a fuzzy number is represented by an ordered pair of functions $(\underline{\mu}(t), \bar{\mu}(t))$, $r \in [0, 1]$ which satisfies [23]:

1. $\underline{\mu}(t)$ is a bounded left continuous non-decreasing function over $[0, 1]$.
2. $\bar{\mu}(t)$ is a bounded left continuous non-increasing function over $[0, 1]$.
3. $\underline{\mu}(t) \leq \bar{\mu}(t)$, $r \in [0, 1]$.

A crisp number r is simply represented by $\underline{\mu}(r) = \bar{\mu}(r) = r$ for all $r \in [0, 1]$.

Definition [23]. If $\tilde{E}$ be the set of all fuzzy numbers, we say that $\tilde{f}(t)$ is a fuzzy function if $\tilde{f}: \mathfrak{R} \rightarrow \tilde{E}$.

Definition [24]. A mapping $\tilde{f} : T \rightarrow \tilde{E}$ for some interval $T \subseteq \tilde{E}$ is called a fuzzy function process and we denote r -level set by:

$$[\tilde{f}(t)]_r = [\underline{f}(t; r), \overline{f}(t; r)], t \in T, r \in [0, 1].$$

The r -level sets of a fuzzy number are much more effective as representation forms of fuzzy set than the above. Fuzzy sets can be defined by the families of their r -level sets based on the resolution identity theorem.

Definition [25]. Each function $f : X \rightarrow Y$ induces another function $\tilde{f} : F(X) \rightarrow F(Y)$ defined for each fuzzy interval U in X by:

$$\tilde{f}(U)(y) = \begin{cases} \sup_{x \in f^{-1}(y)} U(x), & \text{if } y \in \text{range}(f) \\ 0, & \text{if } y \notin \text{range}(f). \end{cases}$$

This is called the Zadeh extension principle.

Definition [27]. Consider $\tilde{x}, \tilde{y} \in \tilde{E}$. If there exists $\tilde{z} \in \tilde{E}$ such that $\tilde{x} = \tilde{y} + \tilde{z}$, then $\tilde{z}$ is called the H-difference (Hukuhara difference) of x and y and is denoted by $\tilde{z} = \tilde{x} \ominus \tilde{y}$.

Definition [28]. If $\tilde{f} : I \rightarrow \tilde{E}$ and $y_0 \in I$, where $I \in [t_0, T]$. We say that $\tilde{f}$ Hukuhara Differentiable at y_0 , if there exists an element $[\tilde{f}']_r \in \tilde{E}$ such that for all $h > 0$ sufficiently small (near to 0), exists $\tilde{f}(y_0 + h; r) \ominus \tilde{f}(y_0; r)$, $\tilde{f}(y_0; r) \ominus \tilde{f}(y_0 - h; r)$ and the limits are taken in the metric $(\tilde{E}, D)$.

$$\lim_{h \rightarrow 0+} \frac{\tilde{f}(y_0 + h; r) \ominus \tilde{f}(y_0; r)}{h} = \lim_{h \rightarrow 0+} \frac{\tilde{f}(y_0; r) \ominus \tilde{f}(y_0 - h; r)}{h}.$$

The fuzzy set $[\tilde{f}'(y_0)]_r$ is called the Hukuhara derivative of $[\tilde{f}]_r$ at y_0 .

These limits are taken in the space $(\tilde{E}, D)$ if t_0 or T , then we consider the corresponding one-side derivation. Recall that $\tilde{x} \ominus \tilde{y} = \tilde{z} \in \tilde{E}$ are defined on r -level set, where $[\tilde{x}]_r \ominus [\tilde{y}]_r = [\tilde{z}]_r, \forall r \in [0, 1]$ By consideration of definition of the metric D all the r -level set $[\tilde{f}(0)]_r$ are Hukuhara differentiable at y_0 , with Hukuhara derivatives $[\tilde{f}'(y_0)]_r$, when $\tilde{f} : I \rightarrow \tilde{E}$ is Hukuhara differentiable at y_0 with Hukuhara derivative $[\tilde{f}'(y_0)]_r$ it lead to that $\tilde{f}$ is Hukuhara differentiable for all $r \in [0, 1]$ which satisfies the above limits i.e. if f is differentiable at $t_0 \in [t_0 + \alpha, T]$ then all its r -levels $[\tilde{f}'(t)]_r$ are Hukuhara differentiable at t_0 .

Definition [27]. Define the mapping $\tilde{f}' : I \rightarrow \tilde{E}$ and $y_0 \in I$, where $I \in [t_0, T]$. The fuzzy function $\tilde{f}'$ be Hukuhara differentiable at y_0 , if there exists an element $[\tilde{f}^{(n)}]_r \in \tilde{E}$ such that for all $h > 0$ sufficiently small, exists

$$\tilde{f}^{(n-1)}(y_0 + h; r) \ominus \tilde{f}^{(n-1)}(y_0; r), \tilde{f}^{(n-1)}(y_0; r) \ominus \tilde{f}^{(n-1)}(y_0 - h; r)$$

and the limits are taken in the metric $(\tilde{E}, D)$.

$$\lim_{h \rightarrow 0+} \frac{\tilde{f}^{(n-1)}(y_0 + h; r) \ominus \tilde{f}^{(n-1)}(y_0; r)}{h} = \lim_{h \rightarrow 0+} \frac{\tilde{f}^{(n-1)}(y_0; r) \ominus \tilde{f}^{(n-1)}(y_0 - h; r)}{h}$$

exists and equal to $\tilde{f}^{(n)}$ and for $n = 2$ we have second order Hukuhara derivative and so on.

Theorem [27]. Let $\tilde{f} : [t_0 + \alpha, T] \rightarrow \tilde{E}$ be Hukuhara differentiable denoted by

$$[\tilde{f}'(t)]_r = [\underline{f}'(t), \overline{f}'(t)]_r = [\underline{f}'(t; r), \overline{f}'(t; r)].$$

Then the boundary functions $\underline{f}'(t; r), \overline{f}'(t; r)$ are differentiable such that:

$$[\tilde{f}'(t)]_r = \left[(\underline{f}(t; r))', (\overline{f}(t; r))' \right], \forall r \in [0, 1].$$

Theorem [29]. Let $\tilde{f} : [t_0 + \alpha, T] \rightarrow \tilde{E}$ be Hukuhara differentiable denoted by

$$[\tilde{f}'(t)]_r = [\underline{f}'(t), \overline{f}'(t)]_r = [\underline{f}'(t; r), \overline{f}'(t; r)].$$

When the boundary functions $\underline{f}'(t; r), \overline{f}'(t; r)$ are differentiable we can write for n^{th} order fuzzy derivative.

$$[\tilde{f}^{(n)}(t)]_r = \left[(\underline{f}^{(n)}(t; r))', (\overline{f}^{(n)}(t; r))' \right], \forall r \in [0, 1].$$

3. Defuzzification of high order FIVPs

Consider the following general high order FIVP [22]:

$$(1) \quad \tilde{y}^{(n)}(t) = \tilde{f}\left(t, \tilde{y}(t), \tilde{y}'(t), \tilde{y}''(t), \dots, \tilde{y}^{(n-1)}(t)\right) + \tilde{w}(t), \quad t \in [t_0, T]$$

subject to the initial fuzzy conditions

$$(2) \quad \tilde{y}(t_0) = \tilde{y}_0, y'(t_0) = \tilde{y}'_0, \dots, y^{(n-1)}(t_0) = \tilde{y}_0^{(n-1)},$$

where $\tilde{y}$ is a fuzzy function of the crisp variable t with $\tilde{f}$ being a fuzzy function of the crisp variable t , the fuzzy variable $\tilde{y}$ and the fuzzy Hukuhara-derivatives $\tilde{y}'(t), \tilde{y}''(t), \dots, \tilde{y}^{(n-1)}(t)$. Here $y^{(n)}$ is the fuzzy n^{th} order ($n \geq 2$) Hukuhara-derivative and $\tilde{y}(t_0), \tilde{y}'(t_0), \dots, \tilde{y}^{(n-1)}(t_0)$ are convex fuzzy numbers. We denote the fuzzy function y by $\tilde{y} = [\underline{y}, \overline{y}]$ for $t \in [t_0, T]$ and $r \in [0, 1]$. It means that the r -level set of $\tilde{y}(t)$ can be defined as:

$$[\tilde{y}(t)]_r = [\underline{y}(t; r), \overline{y}(t; r)],$$

$$\begin{aligned}
[\tilde{y}'(t)]_r &= [\underline{y}'(t; r), \overline{y}'(t_0; r)], \dots [\tilde{y}^{(n-1)}(t)]_r \\
&= [\underline{y}^{(n-1)}(t; r), \overline{y}^{(n-1)}(t; r)], \\
[\tilde{y}(t_0)]_r &= [\underline{y}(t_0; r), \overline{y}(t_0; r)], [\tilde{y}'(t_0)]_r \\
&= [\underline{y}'(t_0; r), \overline{y}'(t_0; r)], \dots, [\tilde{y}^{(n-1)}(t_0)]_r \\
&= [\underline{y}^{(n-1)}(t_0; r), \overline{y}^{(n-1)}(t_0; r)],
\end{aligned}$$

where is the fuzzy inhomogeneous term such that $[\tilde{w}(t)]_r = [\underline{w}(t; r), \overline{w}(t; r)]$.

Since $\tilde{y}^{(n)}(t) = \tilde{f}(t, \tilde{y}(t), \tilde{y}'(t), \tilde{y}''(t), \dots, \tilde{y}^{(n-1)}(t)) + \tilde{w}(t)$.

Let $\mathcal{Y}(t) = \tilde{y}(t), \tilde{y}'(t), \tilde{y}''(t), \dots, \tilde{y}^{(n-1)}(t)$, such that

$$\begin{aligned}
\tilde{\mathcal{Y}}(t; r) &= [\underline{\mathcal{Y}}(t; r), \overline{\mathcal{Y}}(t; r)] \\
&= [\underline{y}(t; r), \underline{y}'(t; r), \dots, \underline{y}^{(n-1)}(t; r), \overline{y}(t; r), \overline{y}'(t; r), \dots, \overline{y}^{(n-1)}(t; r)].
\end{aligned}$$

Also, we can write

$$[\tilde{f}(t, \tilde{\mathcal{Y}})]_r = [\underline{f}(t, \tilde{\mathcal{Y}}; r), \overline{f}(t, \tilde{\mathcal{Y}}; r)].$$

By using Zadeh extension principles as mentioned in [29, 30], we have $\tilde{f}(t, \tilde{\mathcal{Y}}(t; r)) = [\underline{f}(t, \tilde{\mathcal{Y}}(t; r)), \overline{f}(t, \tilde{\mathcal{Y}}(t; r))]$, such that

$$\begin{aligned}
\underline{f}(t, \tilde{\mathcal{Y}}(t; r)) &= \mathcal{F}(t, \underline{\mathcal{Y}}(t; r), \overline{\mathcal{Y}}(t; r)) = \mathcal{F}(t, \tilde{\mathcal{Y}}(t; r)), \\
\overline{f}(t, \tilde{\mathcal{Y}}(t; r)) &= \mathcal{G}(t, \underline{\mathcal{Y}}(t; r), \overline{\mathcal{Y}}(t; r)) = \mathcal{G}(t, \tilde{\mathcal{Y}}(t; r)).
\end{aligned}$$

Then Eq. (1) can be written as follows:

$$(3) \quad \underline{y}^{(n)}(t; r) = \mathcal{F}(t, \tilde{\mathcal{Y}}(t; r)) + \underline{w}(t; r),$$

$$(4) \quad \overline{y}^{(n)}(t; r) = \mathcal{G}(t, \tilde{\mathcal{Y}}(t; r)) + \overline{w}(t; r),$$

where for all $r \in [0, 1]$, the membership functions $\mathcal{F}(t, \tilde{\mathcal{Y}}(t; r)) + \underline{w}(t; r)$ and $\mathcal{G}(t, \tilde{\mathcal{Y}}(t; r)) + \overline{w}(t; r)$ can be defined as

$$\begin{aligned}
\mathcal{F}(t, \tilde{\mathcal{Y}}(t; r)) + \underline{w}(t; r) &= \min\{\tilde{y}^{(n)}(t, \tilde{\mu}(r)) : \mu | \mu \in [\tilde{\mathcal{Y}}(t; r)]_r\}, \\
\mathcal{G}(t, \tilde{\mathcal{Y}}(t; r)) + \overline{w}(t; r) &= \max\{\tilde{y}^{(n)}(t, \tilde{\mu}(r)) : \mu | \mu \in [\tilde{\mathcal{Y}}(t; r)]_r\}.
\end{aligned}$$

4. Differential transformation method in fuzzy environment

The DTM is developed based on the Taylor series expansion [30]. This method constructs an analytical solution in the form of polynomial.

Definition. If $\tilde{f}:\mathfrak{R} \rightarrow \tilde{E}$, $\exists c \in [t_0, T]$ a Taylor polynomial of degree n is defined as follows:

$$(5) \quad \begin{cases} \underline{P}_n(t; r) = \sum_{k=0}^n \frac{1}{k!} \left(\underline{f}^k(c; r) \right) (t - c)^k, \\ \overline{P}_n(t; r) = \sum_{k=0}^n \frac{1}{k!} \left(\overline{f}^k(c; r) \right) (t - c)^k, \end{cases}$$

where $r \in [0, 1]$ and $t \in [t_0, T]$. Suppose that, the fuzzy function $\tilde{f}(t)$ is a continuously H-differentiable function on the interval $[t_0, T]$.

Definition. If $\tilde{f}:\mathfrak{R} \rightarrow \tilde{E}$, the differential transform of the fuzzy function $\tilde{f}(t)$ for the k th H-derivative is defined as follows:

$$(6) \quad \begin{cases} \mathcal{F}(k; r) = \frac{1}{k!} \left[\frac{d^k \underline{f}(t; r)}{d^k t} \right]_{t=t_0}, \\ \mathcal{G}(k; r) = \frac{1}{k!} \left[\frac{d^k \overline{f}(t; r)}{d^k t} \right]_{t=t_0}, \end{cases}$$

where $\underline{f}(t; r)$ and $\overline{f}(t; r)$ are the original functions with $\mathcal{F}(k; r)$ and $\mathcal{G}(k; r)$ are the lower and upper fuzzy transformed functions for all $r \in [0, 1]$.

Definition. If $\tilde{f}:\mathfrak{R} \rightarrow \tilde{E}$, the inverse differential transform $\mathcal{F}(k; r)$ and $\mathcal{G}(k; r)$ are define as follows:

$$(7) \quad \begin{cases} \underline{f}(t; r) = \sum_{k=0}^n \mathcal{F}(k; r) (t - t_0)^k, \\ \overline{f}(t; r) = \sum_{k=0}^n \mathcal{G}(k; r) (t - t_0)^k. \end{cases}$$

Substitution of Eq. (6) into Eq. (7) yields:

$$(8) \quad \begin{cases} \underline{f}(t; r) = \sum_{k=0}^{\infty} \frac{1}{k!} \left[\frac{d^k \underline{f}(t; r)}{d^k t} \right]_{t=t_0} (t - t_0)^k, \\ \overline{f}(t; r) = \sum_{k=0}^{\infty} \frac{1}{k!} \left[\frac{d^k \overline{f}(t; r)}{d^k t} \right]_{t=t_0} (t - t_0)^k. \end{cases}$$

Note that, this is the Taylor series of the fuzzy functions $\tilde{f}(t; r)$ at $t = t_0$. The basic operations of differential transformation for the crisp problems were given in [30, 15, 16]. Now according to Section 2 and 3, the following basic operations of differential transformation can be deduced from equations Eqs. (6) and (7):

Table 1: Various differential transform operators in fuzzy environment.

<i>Function form</i>		<i> DTM - exact </i>
If $\underline{f}(t; r) = \underline{a}(t; r) \pm \underline{b}(t; r)$	then	$\mathcal{F}(k; r) = \underline{A}(k; r) + \underline{B}(k; r)$
If $\overline{f}(t; r) = \overline{a}(t; r) \pm \overline{b}(t; r)$	then	$\mathcal{G}(k; r) = \overline{A}(k; r) + \overline{B}(k; r)$
If $\underline{f}(t; r) = s\underline{a}(t; r)$	then	$\mathcal{F}(k; r) = s\underline{A}(k; r), s \text{ constant}$
If $\overline{f}(t; r) = s\overline{a}(t; r)$	then	$\mathcal{G}(k; r) = s\overline{A}(k; r), s \text{ constant}$
If $\underline{f}(t; r) = \frac{d^n \underline{a}(t; r)}{d^n t}$	then	$\mathcal{F}(k; r) = \frac{(k+n)!}{k!} \underline{A}(k+n; r)$
If $\overline{f}(t; r) = \frac{d^n \overline{a}(t; r)}{d^n t}$	then	$\mathcal{G}(k; r) = \frac{(k+n)!}{k!} \overline{A}(k+n; r)$
If $\underline{f}(t; r) = \underline{a}(t; r) \underline{b}(t; r)$	then	$\mathcal{F}(k; r) = \sum_{l=0}^k \underline{A}(l; r) \underline{B}(k-l; r)$
If $\overline{f}(t; r) = \overline{a}(t; r) \overline{b}(t; r)$	then	$\mathcal{G}(k; r) = \sum_{l=0}^k \overline{A}(l; r) \overline{B}(k-l; r)$
If $\underline{f}(t; r) = t^n$	then	$\mathcal{F}(k; r) = \underline{\delta}(k-n; r),$ $\underline{\delta}(k-n; r) = \begin{cases} 1, & k = n \\ 0, & k \neq n \end{cases}$
If $\overline{f}(t; r) = t^n$	then	$\mathcal{G}(k; r) = \overline{\delta}(k-n; r),$ $\overline{\delta}(k-n; r) = \begin{cases} 1, & k = n \\ 0, & k \neq n \end{cases}$
If $\underline{f}(t; r) = \exp(\theta t)$	then	$\mathcal{F}(k; r) = \frac{\theta^k}{k!}$
If $\overline{f}(t; r) = \exp(\theta t)$	then	$\mathcal{G}(k; r) = \frac{\theta^k}{k!}$
If $\underline{f}(t; r) = \sin(st + c)$	then	$\mathcal{F}(k; r) = \frac{s^k}{k!} \sin(\pi \frac{k}{2!} + c)$
If $\overline{f}(t; r) = \cos(st + c)$	then	$\mathcal{F}(k; r) = \frac{s^k}{k!} \cos(\pi \frac{k}{2!} + c)$
If $\underline{f}(t; r) = \sin(st + c)$	then	$\mathcal{G}(k; r) = \frac{s^k}{k!} \sin(\pi \frac{k}{2!} + c)$
If $\overline{f}(t; r) = \cos(st + c)$	then	$\mathcal{G}(k; r) = \frac{s^k}{k!} \cos(\pi \frac{k}{2!} + c)$

5. Numerical examples

5.1 Example

Consider the circuit model problem [22] shown in Fig. 2, where $L = 1h$, $R = 2\Omega$, $C = 0.25f$ and $E(t) = 50\cos(t)$. Let $Q(t)$ be the charge on the capacitor at time $t > 0$.

Thus the FIVP of this model is given as follows

$$(9) \quad \begin{aligned} \tilde{Q}''(t; r) + 2\tilde{Q}'(t; r) + 4\tilde{Q}(t; r) &= 50\cos t, \quad t > 0, \\ \tilde{Q}(0; r) &= [4 + r, 6 - r], \tilde{Q}'(0; r) = [r, 2 - r], \quad r \in [0, 1]. \end{aligned}$$

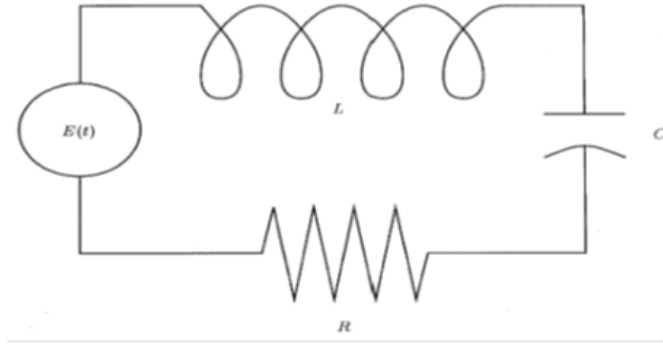


Figure 1: Electric circuit model

The exact analytical solution of Eq. (9) was given by [31]. Taking the differential transform of Eq. (9) by Table 1, we obtain for the lower bound:

$$(10) \quad \mathcal{F}(k+2; r) = \frac{50 \frac{1}{k!} \sin\left(\pi \frac{k}{2!}\right) - 2\{(k+1)\mathcal{F}(k+1; r)\} - 4\mathcal{F}(k; r)}{(k+1)(k+2)}.$$

The differential transform of the initial values of Eq. (10) for all $r \in [0, 1]$ are as follows

$$(11) \quad \mathcal{F}(0; r) = 4 + r, \mathcal{F}(1; r) = r.$$

The differential transform of the upper bound of Eq. (9) is as follows:

$$(12) \quad \mathcal{G}(k+2; r) = \frac{50 \frac{1}{k!} \sin\left(\pi \frac{k}{2!}\right) - 2\{(k+1)\mathcal{G}(k+1; r)\} - 4\mathcal{G}(k; r)}{(k+1)(k+2)}.$$

Now, the differential transform of the initial values of Eq. (10) for all $r \in [0, 1]$ are as follows

$$(13) \quad \mathcal{G}(0; r) = 6 - r, \mathcal{G}(1; r) = 2 - r.$$

Moreover, substituting Eq. (11) into Eq. (10) and Eq. (13) into Eq. (12) and by recursive method we can calculate another value of $\mathcal{F}(k; r)$ and $\mathcal{G}(k; r)$. Finally, by using Eq. 8 the final solution can be obtained. The 15-order DTM approximation solution of Eq. (9) can be obtained by using Mathematica software code as showing in the following table and figures:

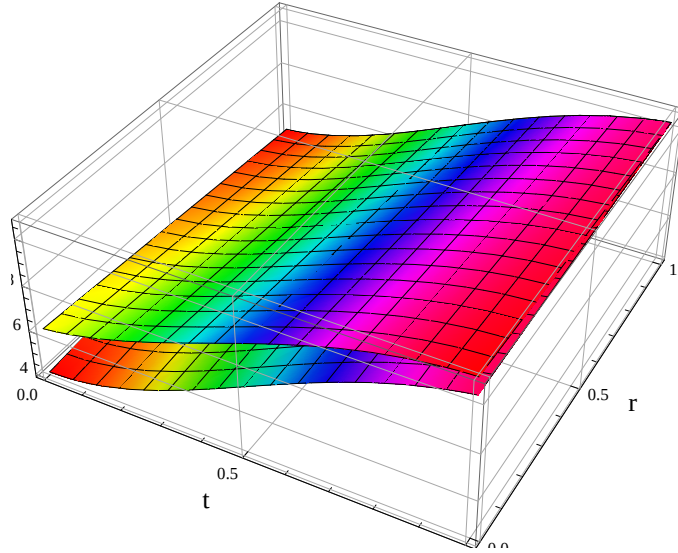


Figure 2: 15-order DTM solution of Eq. (9) for all $r \in [0,1]$ and $t \in [0,1]$.

Table 2: 15-order DTM solution and accuracy of Eq. (9) at $t = 1$.

x	$\underline{Q}(1; r)$	$ DTM - exact $	$\overline{Q}(1; r)$	$ DTM - exact $
0	9.959398092279500	8.172037×10^{-9}	10.679826450125848	6.060271×10^{-9}
0.2	10.031440928064136	7.960863×10^{-9}	10.607783614341212	6.271454×10^{-9}
0.4	10.103483763848772	7.749688×10^{-9}	10.535740772077789	6.482629×10^{-9}
0.6	10.175526599633404	7.538512×10^{-9}	10.463697942771944	6.693806×10^{-9}
0.8	10.247569435418038	7.327333×10^{-9}	10.391655106987310	6.904985×10^{-9}
1	10.319612271202672	7.116156×10^{-9}	10.319612271202672	7.116156×10^{-9}

From Table 1 and Fig. 3 one can note that the 15-order DTM approximate solution of Eq. (9) at $t = 1$ and for all $0 \leq r \leq 1$ satisfy the fuzzy numbers properties in Section 2 by taking the triangular fuzzy number shape.

5.2 Example 2

Consider the following third order fuzzy initial value problem [31]:

$$\begin{aligned}
 \tilde{y}'''(t; r) &= -\tilde{y}''(t; r) - 3\tilde{y}'(t; r) + 5\tilde{y}(t; r), \quad t \in [0, 1] \\
 \tilde{y}(0; r) &= [0.75 - 0.25r, 1.25 - 0.25r], \\
 \tilde{y}'(0; r) &= [1.5 + 0.5r, 2.5 - 0.5r], \quad r \in [0, 1] \\
 \tilde{y}'(0; r) &= [3.75 + 0.25r, 4.25 - 0.25r].
 \end{aligned}
 \tag{14}$$

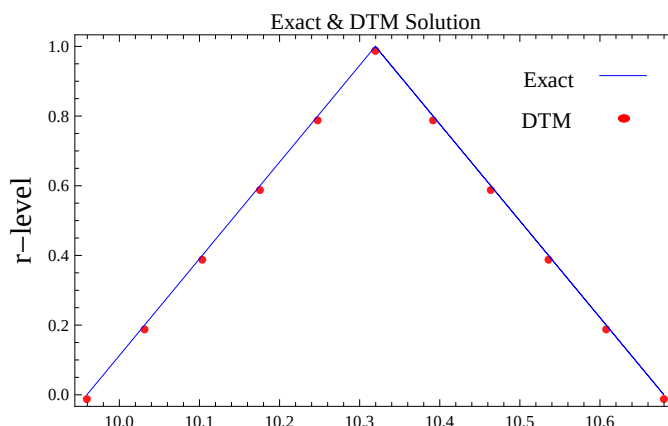


Figure 3: Exact and 15-order DTM solutions of Eq. (9) at $t=1$.

The exact analytical solution of Eq. (14) was given [32]. Taking the differential transform of Eq. (14) by Table 1, we obtain for the lower bound:

$$(15) \quad \begin{aligned} & \mathcal{F}(k+3; r) \\ &= \frac{-\{(k+1)(k+2)\mathcal{F}(k+2; r)\} - 3\{(k+1)\mathcal{F}(k+1; r)\} + 5\mathcal{F}(k; r)}{(k+1)(k+2)(k+3)}. \end{aligned}$$

The differential transform of the initial values of Eq. (10) for all $r \in [0, 1]$ are as follows

$$(16) \quad \mathcal{F}(0; r) = 0.75 + 0.25r, \mathcal{F}(1; r) = 1.5 + 0.5r, \mathcal{F}(2; r) = 0.5(3.75 + 0.25r).$$

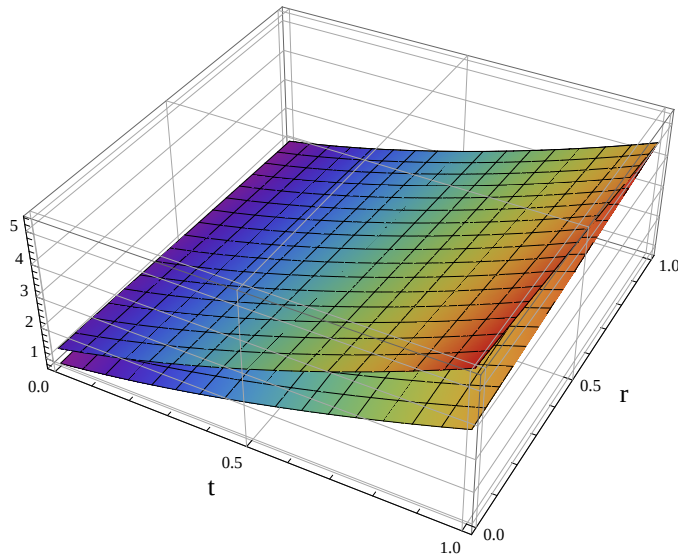
The differential transform of the upper bound of Eq. (14) is as follows:

$$(17) \quad \begin{aligned} & \mathcal{G}(k+3; r) \\ &= \frac{-\{(k+1)(k+2)\mathcal{G}(k+2; r)\} - 3\{(k+1)\mathcal{G}(k+1; r)\} + 5\mathcal{G}(k; r)}{(k+1)(k+2)(k+3)}. \end{aligned}$$

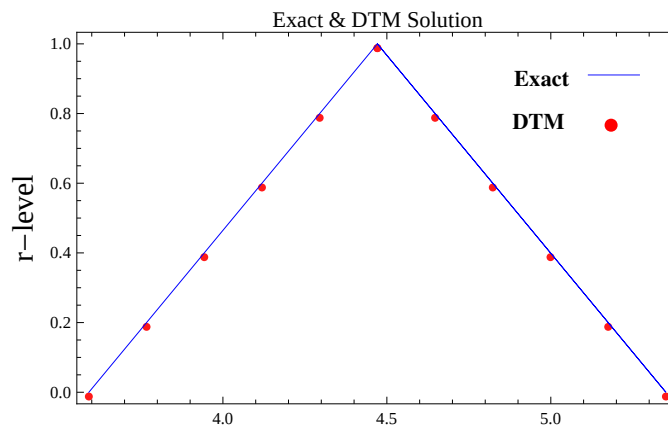
Now, the differential transform of the initial values of Eq. 10 for all $r \in [0, 1]$ are as follows

$$(18) \quad \mathcal{G}(0; r) = 1.25 - 0.25r, \mathcal{G}(1; r) = 2.5 - 0.5r, \mathcal{G}(2; r) = 0.5(4.25 - 0.25r).$$

Moreover, substituting Eq. (16) into Eq. (15) and Eq. (18) into Eq. (17) and by recursive method we can calculate another value of $\mathcal{F}(k; r)$ and $\mathcal{G}(k; r)$. Finally, by using Eq. (8) the final solution can be obtained. The 15-order DTM approximation solution of Eq. (14) can be obtained by using Mathematica software code as showing in the following table and figures:

Figure 4: Exact and 15-order DTM solutions of Eq. (14) at $t=1$.Table 3: Exact and 15-order DTM solutions of Eq. (14) at $t=1$.

x	$y(1;r)$	$ DTM - exact $	$\bar{y}(1;r)$	$ DTM - exact $
0	3.591138112568900	6.257013×10^{-9}	5.351014715840749	4.877771×10^{-9}
0.2	3.767125772896086	6.119089×10^{-9}	5.175027055513564	5.015696×10^{-9}
0.4	3.943113433223270	5.981164×10^{-9}	4.999039395186378	5.153618×10^{-9}
0.6	4.119101093550455	5.843240×10^{-9}	4.823051734859193	5.291543×10^{-9}
0.8	4.295088753877639	5.705315×10^{-9}	4.647064074532009	5.429467×10^{-9}
1	4.471076414204824	5.567392×10^{-9}	4.471076414204824	5.567392×10^{-9}

Figure 5: Exact and 15-order DTM solutions of Eq. (14) at $t = 1$.

As in Example 5.1, from Table 2 and Fig. 5 one can note that the 15-order DTM approximate solution of Eq. (14) at $t = 1$ and for all $0 \leq r \leq 1$ satisfy the fuzzy numbers properties in Section 2 by taking the triangular fuzzy number shape.

6. Conclusions

In this paper, we studied and applied the Differential Transform Method in finding solution of high order fuzzy initial value problems involving linear ordinary differential equations. To the best of our knowledge, this is the first attempt for solving the high order FIVPs with DTM. The method has been formulated to obtain an approximate solution of general high order FIVPs. Two numerical examples including second and third order linear fuzzy initial value problems showed the capability and the efficiency of DTM. Moreover, this technique converges to the exact solution and requires less computational work directly without reduced to first order system. The numerical result that obtained by DTM satisfy the fuzzy numbers properties by taking the convex fuzzy numbers shape.

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SOME PROPERTIES OF ZERO GRADATIONS ON SAMANTA FUZZY TOPOLOGICAL SPACES

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Abstract. Considering the fuzzy topological spaces in the sense of Samanta, the notion of a zero gradation as a fuzzy topological invariant is introduced that might be the first basic step to develop a theory of dimension on the fuzzy topological spaces. Also, some critical properties and applications are established.

Keywords: fuzzy topology, gradation of openness, zero gradation.

1. Introduction

In a recent paper [3], we investigated the zero dimensionality of fuzzy topological spaces in the sense of Lowen and showed how the concept might be sensitive to the choice of definition of fuzzy topology. In this paper and with almost the same purpose, we consider an important modification of definition of a fuzzy topology that has many applications in many branches of science and technology. It has been formulated with respect to the concept of the gradation of openness that originally appeared in a paper by Samanta et al. [12]. Our main goal is to introduce a concept of zero dimensionality so that some required initial properties of a dimension are satisfied while being compatible with the previous relative notions.

The notion of fuzzy topological space was first introduced by Chang [11] over the system of fuzzy sets proposed by Zadeh [22]. In Chang's definition, constant functions between fuzzy topological spaces were not necessarily continuous. Lowen [18] and Hutton [17] have given different ideas for the definition of fuzzy topology. Samanta et al. [12] introduced the concept of fuzzy topology

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by the notion of gradation of openness as a function $\tau : \mathbb{I}^X \rightarrow \mathbb{I}$ satisfying certain axioms. Dimension of fuzzy topological spaces has been studied by several researchers [4], [5], [6], [7], [9], [13]. The subject, however, is still dealing with major obstacles such as lack of a satisfactory concept of a boundary as well as comprehensive definition of a dimension.

In this paper, we introduce the concept of zero gradation which in addition to a natural candida for zero dimensionality of a gradation might be helpful for the characterization of some kind of disconnectedness of fuzzy topological spaces. We prove that the property of being zero gradation is an invariant by strongly gradation preserving maps, and so it is a fuzzy topological invariant. Some examples of zero and nonzero gradations are also presented.

2. Preliminaries

We give below some basic preliminaries required for this paper.

Definition 2.1 ([22]). A *fuzzy set* μ in a nonempty set X is defined as a map $\mu : X \rightarrow \mathbb{I}$, where $\mathbb{I} = [0, 1]$. The family of all fuzzy sets on X is denoted by $\mathbb{I}^X$. A fuzzy set μ is said to be *contained in* a fuzzy set η if $\mu(x) \leq \eta(x)$ for each x in X , denoted by $\mu \leq \eta$. The *union* and *intersection* of a family of fuzzy sets is defined by $\bigvee \mu_\alpha = \sup(\mu_\alpha)$ and $\bigwedge \mu_\alpha = \inf(\mu_\alpha)$, respectively.

Definition 2.2. For every $x \in X$ and every $\alpha \in (0, 1)$, the fuzzy set x_α with membership function

$$x_\alpha(y) = \begin{cases} \alpha, & y = x, \\ 0, & y \neq x, \end{cases}$$

is called a *fuzzy point*. x_α is said to be *contained in* a fuzzy set μ , denoted by $x_\alpha \in \mu$, if $\alpha < \mu(x)$ [21].

We omit the case $\alpha = \mu(x)$ in the above definition for some technical reasons. For instance, every second countable fail to be first countable if we allow the equality. Anyway, any fuzzy set is the union of all points which are contained in it and for every two fuzzy sets μ and ν we have $\mu \leq \nu$ if and only if $x_\alpha \in \mu$ implies $x_\alpha \in \nu$, for every fuzzy point x_α . Any point x_1 is called a *crisp point*. We denote a constant fuzzy set whose unique value is $c \in [0, 1]$ by c_X .

Definition 2.3 ([18]). A *Lowen fuzzy topology* is a family δ of fuzzy sets in X which satisfies the following conditions:

- (i) $\forall c \in \mathbb{I}, c_X \in \delta$,
- (ii) $\forall \mu, \nu \in \delta \Rightarrow \mu \wedge \nu \in \delta$,
- (iii) $\forall (\mu_j)_{j \in J} \subset \delta \Rightarrow \bigvee_{j \in J} \mu_j \in \delta$.

The pair (X, δ) is called a *Lowen fuzzy topological space*. Open fuzzy sets, closed fuzzy sets and fuzzy clopens are defined as usual. In Chang's definition of fuzzy topology, condition (i) should be replaced by (i)' $0, 1 \in \delta$. A base or subbase for a fuzzy space have the same meaning in the classic sense.

It should be noted that the concept of the boundary of a fuzzy set is essential in the definition of inductive dimension. Cuchillo and Tarres [14] proposed a definition of fuzzy boundary of a fuzzy set and we use their definition throughout this paper.

Definition 2.4 ([14]). Let μ be a fuzzy set in a Lowen fuzzy topological space X . The *fuzzy boundary* of μ , denoted by $\text{Fr}(\mu)$, is defined as the infimum of all closed fuzzy sets σ in X with the property $\sigma(x) \geq \bar{\mu}(x)$ for all $x \in X$ for which $\bar{\mu}(x) - \mu^\circ(x) > 0$.

It is ready to see that a fuzzy set μ is clopen if and only if $\text{Fr}(\mu) = 0_X$. The concept of zero gradation is based on the notion of zero dimensionality of Lowen fuzzy topological spaces. Adnadjevic [5], [6] defined two dimension functions $F - \text{ind}$, $F - \text{Ind}$ for the generalized fuzzy spaces. If the definition of the Adnadjevic's dimension function is particularized, in the case of zero dimensionality, the following definition is obtained.

Definition 2.5 ([14]). A Lowen fuzzy topological space (X, δ) is called *zero-dimensional* and it is denoted by $\text{ind}(X) = 0$ if for each fuzzy point x_α in X and every open fuzzy set μ containing x_α , there exists an open fuzzy set σ in X with $\text{Fr}(\sigma) = 0_X$ such that $x_\alpha \in \sigma \leq \mu$.

Example 2.6 ([13]). Let δ be a Lowen fuzzy topology on $X = [0, 1]$ with subbase

$$\{c_X : c \in \mathbb{I}\} \cup \{\mu\},$$

where

$$\mu(x) = \begin{cases} \frac{1}{2}, & 0 \leq x \leq \frac{1}{3}, \\ 0, & \frac{1}{3} < x \leq 1. \end{cases}$$

Clearly any non-constant open fuzzy sets has the form

$$\nu(x) = \begin{cases} a, & 0 \leq x \leq \frac{1}{3}, \\ b, & \frac{1}{3} < x \leq 1, \end{cases} \quad \frac{1}{2} \geq a > b \geq 0.$$

There exists no clopen fuzzy set σ such that $\sigma \leq \mu$ because the constant fuzzy sets are the only clopen fuzzy sets. Thus $\text{ind}(X) \neq 0$.

It is ready to see that for every non empty set X the fuzzy topological space (X, δ) is zero-dimensional, where $\delta = \{c_X : c \in \mathbb{I}\}$.

Remark 2.7 ([20]). Let (X, δ) be a Lowen fuzzy topological space and $Y \subset X$, then the family $\delta_1 = \{\mu|_Y : \mu \in \delta\}$ is a Lowen fuzzy topology for Y and (Y, δ_1) is called a subspace of (X, δ) . If $\text{ind}(X) = 0$, then $\text{ind}(Y) = 0$. Note that restriction of every clopen fuzzy subset of X on subspace Y is a clopen fuzzy set in Y .

3. Zero gradations

In Lowen and Chang's definition of fuzzy topology, fuzziness in the concept of openness of a fuzzy subset has not been considered. The initial request is that the topology be a fuzzy subset of a power set of fuzzy subsets. For this purpose, Samanta et al. gave an axiomatic definition in [12] so called a gradation of openness. Here we use the following modified definition.

Definition 3.1. A fuzzy topology is a mapping $\tau : \mathbb{I}^X \rightarrow \mathbb{I}$ which satisfies the following conditions:

- (i) $\tau(c_X) = 1$, for all $c \in \mathbb{I}$,
- (ii) $\tau(\mu_1 \wedge \mu_2) \geq \tau(\mu_1) \wedge \tau(\mu_2)$,
- (iii) $\tau(\bigvee_i \mu_i) \geq \bigwedge_i \tau(\mu_i)$.

The mapping τ is called a *gradation of openness* on X . The real number $\tau(\mu)$ is the degree of openness of the fuzzy subset $\mu \in \mathbb{I}^X$ and may range from 0 "completely non-open set" to 1 "completely open set". The pair (X, τ) is called a *fuzzy topological space*. In Samanta's definition of fuzzy topology the condition (i) should be replaced by $\tau(\bar{0}) = \tau(\bar{1}) = 1$. Suppose that τ_1 and τ_2 are two fuzzy topologies on a given set X . If $\tau_1 \supset \tau_2$ i.e., $\tau_1(\mu) \geq \tau_2(\mu)$ for every $\mu \in \mathbb{I}^X$, we say that τ_1 is *finer* than τ_2 .

Example 3.2. Let $X = \mathbb{I}$ with the Lowen fuzzy topology generated by the subbasis $\{\eta, c_X\}$, where η is the fuzzy set defined by

$$\eta(x) = \begin{cases} 0, & 0 \leq x \leq \frac{1}{2}, \\ \frac{1}{2}, & \frac{1}{2} < x \leq 1. \end{cases}$$

It is easy to see that the non-constant open fuzzy sets are

$$\eta_{a,b}(x) = \begin{cases} a, & 0 \leq x \leq \frac{1}{2}, \\ b, & \frac{1}{2} < x \leq 1. \end{cases}$$

where $0 \leq a \leq b \leq \frac{1}{2}$. define $\tau : \mathbb{I}^X \rightarrow \mathbb{I}$ by

$$\tau(\mu) = \begin{cases} 1, & \mu = c_X, \\ \frac{1}{2}, & \mu = \eta_{a,b}, \\ 0, & \text{otherwise.} \end{cases}$$

Then τ is a gradation of openness on X .

Definition 3.3. The mapping $\xi : \mathbb{I}^X \rightarrow \mathbb{I}$ is called a *gradation of closedness* on X if it satisfies:

- (i) $\xi(c_X) = 1$, for all $c \in \mathbb{I}$,
- (ii) $\xi(\mu_1 \vee \mu_2) \geq \xi(\mu_1) \wedge \xi(\mu_2)$,
- (iii) $\xi(\bigwedge_i \mu_i) \geq \bigwedge_i \xi(\mu_i)$.

The mapping $\tau : \mathbb{I}^X \rightarrow \mathbb{I}$ is called a *gradation of clopenness* on X if it is a gradation of openness and a gradation of closedness on X . Indeed, the mapping τ is a gradation of clopenness on X if and only if it satisfies:

- (i) $\tau(c_X) = 1$, for all $c \in \mathbb{I}$,
- (ii) $\tau(\mu_i) \geq r \Rightarrow \tau(\bigwedge_i \mu_i) \geq r, \tau(\bigvee_i \mu_i) \geq r$. where $r \in (0, 1]$

For example, the mapping $\tau : \mathbb{I}^X \rightarrow \mathbb{I}$ is defined by

$$\tau(\mu) = \begin{cases} 1, & \mu = c_X, \\ k, & \text{otherwise.} \end{cases} ; k \in [0, 1].$$

is a gradation of clopenness that is called *constant gradation*. Especially for $k = 0$ the gradation of clopenness τ is denoted by τ_0 .

Remark 3.4. If τ is a gradation of openness (closedness) on X , then the mapping $\tau' : \mathbb{I}^X \rightarrow \mathbb{I}$ given by $\tau'(\mu) = \tau(\mu^c)$ will be a gradation of closedness (openness) on X that is called *conjugate gradation* of τ . Therefore, if τ is a gradation of clopenness on X then its conjugate is so.

Proposition 3.5. Let τ be a gradation of openness on X . Then for each $r \in \mathbb{I}$, $\tau_r = \{\mu \in \mathbb{I}^X : \tau(\mu) \geq r\}$ is a Lowen fuzzy topology on X that is called the *r-level fuzzy topology on X with respect to the gradation of openness τ* .

Proof. The first condition for a Lowen fuzzy topology is easy: Since $\tau(c_X) = 1 \geq r$, so $c_X \in \tau_r$. To check the second condition, let μ_1 and μ_2 be two fuzzy sets in τ_r . Hence $\tau(\mu_1) \geq r$ and $\tau(\mu_2) \geq r$. Since τ is a gradation of openness, so $\tau(\mu_1 \wedge \mu_2) \geq \tau(\mu_1) \wedge \tau(\mu_2) \geq r$. Therefore, $\mu_1 \wedge \mu_2 \in \tau_r$. To check the third condition, let $(\mu_i)_i$ be a family of fuzzy sets in τ_r such that $\tau(\mu_i) \geq r$. By (iii) of definition 3.1 $\tau(\bigvee_i \mu_i) \geq \bigwedge_i \tau(\mu_i) \geq r$. Hence, $\bigvee_i \mu_i \in \tau_r$. $\square$

It is ready to see that the family of all r -level fuzzy topologies with respect to τ is a descending family and for each $r \in (0, 1]$, $\tau_r = \bigcap_{s < r} \tau_s$.

Proposition 3.6. Let (X, T) be a Lowen fuzzy topological space. Define for each $r \in (0, 1]$, a mapping $T^r : \mathbb{I}^X \rightarrow \mathbb{I}$ by

$$T^r(\mu) = \begin{cases} 1, & \mu = c_X, \\ r, & \mu \in T, \mu \neq c_X, \\ 0, & \text{otherwise.} \end{cases}$$

Then T^r is a gradation of openness on X such that $(T^r)_r = T$.

Proof. First notice that $T^r(c_X) = 1$. Let $\mu_1, \mu_2 \in \mathbb{I}^X$. Then we have the following possibilities: (i) $\mu_1, \mu_2 \notin T$, (ii) $\mu_1 \in T, \mu_2 \notin T$, (iii) $\mu_1, \mu_2 \in T$. In the case (i), $T^r(\mu_1) = T^r(\mu_2) = 0$, and hence $T^r(\mu_1 \wedge \mu_2) \geq 0 = T^r(\mu_1) \wedge T^r(\mu_2)$. In the case (ii), $T^r(\mu_1) = r$ and $T^r(\mu_2) = 0$. Then $T^r(\mu_1 \wedge \mu_2) \geq 0 = T^r(\mu_1) \wedge T^r(\mu_2)$. In the case (iii), $T^r(\mu_1) = T^r(\mu_2) = r$ and by hypothesis $\mu_1 \wedge \mu_2 \in T$, and hence $T^r(\mu_1 \wedge \mu_2) \geq r = T^r(\mu_1) \wedge T^r(\mu_2)$. Therefore, $T^r(\mu_1 \wedge \mu_2) \geq T^r(\mu_1) \wedge T^r(\mu_2)$. Now let $(\mu_i)_i$ be a family of fuzzy sets in $\mathbb{I}^X$. Thus, $T^r(\bigvee_i \mu_i) = 1$ or r which is greater than or equal to $\bigwedge_i T^r(\mu_i) = r$ or 0 . $\square$

The mapping T^r is called the r -th gradation on X and (X, T^r) is called the r -th graded fuzzy topological space. It should be noted that the family $\{(T^r)_s : 0 < s \leq 1\}$ is a descending family and for each $s \in (0, 1]$, $(T^r)_s = \bigcap_{t < s} (T^r)_t$. Indeed, if $s \geq t$, then obviously, $(T^r)_s \subset (T^r)_t$. Hence, $\{(T^r)_s : 0 < s \leq 1\}$ is a descending family. Clearly $(T^r)_s \subset \bigcap_{t < s} (T^r)_t$, $0 < s \leq 1$. Also, if $\mu \notin (T^r)_s$, then $(T^r)(\mu) < s$ which implies $\exists t \in (0, 1]$ such that $(T^r)(\mu) < t < s$. So, $\mu \notin (T^r)_t$ for some $t < s$. Hence, $\mu \notin \bigcap_{t < s} (T^r)_t$. Consequently $\bigcap_{t < s} (T^r)_t \subset (T^r)_s$.

Definition 3.7. Let τ be a gradation of openness on X and $Y \subset X$. Define a mapping $\tau_Y : \mathbb{I}^Y \rightarrow \mathbb{I}$ by

$$\tau_Y(\mu) = \bigvee \{\tau(\eta) : \eta \in \mathbb{I}^X, \eta|_Y = \mu\}.$$

Then τ_Y is a gradation of openness on Y . We say that τ_Y is a *subgradation* of τ . It is obvious that $\tau_Y(\mu) \geq \tau(\mu_X)$.

Definition 3.8. Let τ_1 and τ_2 be two gradations of openness on X and Y , respectively, and $f : X \rightarrow Y$ be a mapping. Then f is called a *gradation preserving map* (*strongly gradation preserving map*) if for each $\mu \in \mathbb{I}^Y$, $\tau_2(\mu) \leq \tau_1(f^{-1}(\mu))$ ($\tau_2(\mu) = \tau_1(f^{-1}(\mu))$).

For example, let τ_1 and τ_2 be two fuzzy topologies on a given set X such that τ_1 is finer than τ_2 . Thus, the identity map $i : (X, \tau_1) \rightarrow (X, \tau_2)$ is a gradation preserving map.

Definition 3.9. Two gradations of openness σ and τ on X are called *equal* and it is denoted by $\sigma \approx \tau$ if the identity map is a strongly gradation preserving map from (X, τ) to (X, σ) . In this case $\sigma_r = \tau_r$ for all $r \in \mathbb{I}$.

Proposition 3.10. Let τ_1 and τ_2 be two gradations of openness on X and Y , respectively, and $f : X \rightarrow Y$ be a mapping. Then f is a gradation preserving map if and only if $f : (X, (\tau_1)_r) \rightarrow (Y, (\tau_2)_r)$ is continuous with respect to r -level fuzzy topology for all $r \in (0, 1]$.

Proof. Suppose f is a gradation preserving map and $\mu \in (\tau_2)_r$. Then $\tau_1(f^{-1}(\mu)) \geq \tau_2(\mu) \geq r$. Thus $f^{-1}(\mu) \in (\tau_1)_r$. Conversely, suppose $f : (X, (\tau_1)_r) \rightarrow (Y, (\tau_2)_r)$ is continuous with respect to r -level fuzzy topology for all $r \in (0, 1]$. Choose $\mu \in \mathbb{I}^Y$. If $\tau_2(\mu) = 0$, then $\tau_1(f^{-1}(\mu)) \geq \tau_2(\mu)$. If $\tau_2(\mu) = r$, so $\mu \in (\tau_2)_r$. Now by continuity of f it follows that $f^{-1}(\mu) \in (\tau_1)_r$. Hence $\tau_1(f^{-1}(\mu)) \geq r = \tau_2(\mu)$. $\square$

Applications of fuzzy topology and gradation concepts have already been done by S. K. Samanta and others in knowledge engineering, natural language, neural network etc [8], [10]. In these studies, the r -level fuzzy topologies play a major role. In the sequel, we introduce a concept of zero gradation and prove that the property of being zero gradation is invariant under strongly gradation preserving maps. Some examples of zero and nonzero gradations are also presented.

Definition 3.11. Let τ be a gradation of openness on X . Then τ is called a *zero gradation* if the fuzzy topological space (X, τ_r) is zero-dimensional for all $r \in (0, 1]$.

Consider the fuzzy topological space (X, τ_0) . Then the gradation of openness τ_0 is a zero gradation. Note that the r -level fuzzy topology on X with respect to the gradation of openness τ_0 is

$$(\tau_0)_r = \{\mu \in \mathbb{I}^X : \tau_0(\mu) \geq r\} = \{c_X : c \in \mathbb{I}\},$$

for all $r \in (0, 1]$. It is ready to see that the fuzzy topological spaces $(\tau_0)_r$ is zero-dimensional for all $r \in (0, 1]$.

Example 3.12. Let $(\mathbb{I}, \tau)$ be a fuzzy topological space. If the gradation of openness τ be the same as Example 3.2, then τ is not a zero gradation because the fuzzy topological space $(\mathbb{I}, \tau_{\frac{1}{2}})$ is not zero-dimensional. Take the fuzzy point $x_{\frac{1}{3}}$ where $x \in (\frac{1}{2}, 1]$. we have $x_{\frac{1}{3}} \in \eta$ and there exists no clopen fuzzy set $\sigma \in \tau_{\frac{1}{2}}$ such that $x_{\frac{1}{3}} \in \sigma \leq \eta$ since the constant fuzzy sets are the only clopen fuzzy sets. Thus $\text{ind}(\mathbb{I}, \tau_{\frac{1}{2}}) \neq 0$.

It is ready to see that if τ is a zero gradation on X such that $\tau(\mu) = \tau(\mu^c)$ for all $\mu \in \mathbb{I}^X$, then its conjugate τ' is a zero gradation too.

Theorem 3.13. Let τ be a zero gradation on X . If $Y \subset X$ and τ_Y is a subgradation of τ . Then τ_Y is a zero gradation.

Proof. It suffices to prove that the Lowen fuzzy topological space $(Y, (\tau_Y)_r)$ is zero-dimensional for all $r \in (0, 1]$. Choose an arbitrary fuzzy point y_α in Y and let μ be an open fuzzy set containing y_α in the Lowen fuzzy topology $(\tau_Y)_r$. Note that $\tau_Y(\mu) \geq r$. Now we consider the fuzzy point y_α as a fuzzy point in X . It is obvious that there exists an open fuzzy set μ' in τ_r such that $\tau(\mu') \geq r$ and $\mu'|_Y = \mu$. Since the Lowen fuzzy topological space (X, τ_r) is zero-dimensional for all $r \in (0, 1]$, there exists an open fuzzy set η' such that $\text{Fr}(\eta') = 0_X$ and $y_\alpha \in \eta' \leq \mu'$. Now we put $\eta = \eta'|_Y$, then $y_\alpha \in \eta \leq \mu$. $\square$

Remark 3.14. Let $\{T_r\}$ be a non-empty descending family of Lowen fuzzy topologies on X such that $T_r = \bigcap_{s < r} T_s$ for all $r \in (0, 1]$. It is easy to verify that the mapping $\tau : \mathbb{I}^X \rightarrow \mathbb{I}$ defined as

$$\tau(\mu) = \bigvee \{r \in (0, 1] : \mu \in T_r\}$$

is a fuzzy topology on X . If T_r is a zero-dimensional space for all $r \in (0, 1]$, the mapping τ will be a zero gradation since $\tau_r = T_r$ for all $r \in (0, 1]$.

Proposition 3.15. *The infimum of any family of zero gradations on a given set X is a zero gradation.*

Proof. Let $\{\tau_i\}_i$ be an arbitrary family of zero gradations on X . One can readily check that $\bigwedge_i \tau_i$ is a fuzzy topology on X . Because the r -level Lowen fuzzy topology $(\bigwedge_i \tau_i)_r = \{\mu \in \mathbb{I}^X : \bigwedge_i \tau_i(\mu) \geq r\}$ is a subspace of $(\tau_i)_r$, for all $r \in (0, 1]$, we conclude that $\bigwedge_i \tau_i$ is a zero gradation by Remark 2.7. $\square$

Note that the converse of the Proposition 3.15 is not necessarily hold. Choose two fuzzy topologies τ_0 and τ on $\mathbb{I}$, where τ is the same as Example 3.2. Then $\tau_0 \wedge \tau$ is a zero gradation, while by Example 3.12, τ is not zero gradation.

Theorem 3.16. *Let $f : (X, \tau_1) \rightarrow (Y, \tau_2)$ be a bijective strongly gradation preserving map. If τ_1 is a zero gradation, then τ_2 is a zero gradation.*

Proof. We show that Lowen space $(Y, (\tau_2)_r)$ is zero dimensional for all $r \in (0, 1]$. Choose fuzzy subset μ containing fuzzy point y_α with $\tau_2(\mu) \geq r$, for all $r \in (0, 1]$. $f^{-1}(y_\alpha)$ is the fuzzy point $(f^{-1}(y))_\alpha$ in X and $f^{-1}(\mu)$ is a open fuzzy set in X respect to $(\tau_1)_r$. Note that $\tau_1(f^{-1}(\mu)) = \tau_2(\mu) \geq r$. According to zero dimensionality of $(X, (\tau_1)_r)$ there is a clopen fuzzy set λ' in $(\tau_1)_r$ such that $(f^{-1}(y))_\alpha \leq \lambda' \leq f^{-1}(\mu)$. Now we put $\lambda = f(\lambda')$, Thus $y_\alpha \in \lambda \leq \mu$. $\square$

The condition of being strongly for the gradation preserving map f in the Theorem 3.16 is essential. Consider the identity map $i : (X, \tau_1) \rightarrow (X, \tau_2)$, where $X = \mathbb{I}$, τ_1 is the constant gradation (when $k = 1$) and the gradation of openness τ_2 is the same as Example 3.2. $\tau_2(\eta) \neq \tau_1(\eta)$, then the identity map i is not a strongly gradation preserving map. Note that τ_1 is a zero gradation and τ_2 is not a zero gradation.

Proposition 3.17. *Let τ be a zero gradation on X and $\sigma \approx \tau$, then σ is a zero gradation.*

Proof. The proof is obvious since $\{\sigma_r : r \in [0, 1]\} = \{\tau_r : r \in [0, 1]\}$. $\square$

Corollary 3.18. *If σ is a zero gradation, then two gradations of openness τ, σ and $\phi \circ \sigma$ are zero gradation because $\tau, \sigma \approx \sigma$ and $\phi \circ \sigma \approx \sigma$, where τ is the constant gradation on X ($k \neq 0$) and $\phi : \mathbb{I} \rightarrow \mathbb{I}$ is any strictly increasing continuous function with $\phi(1) = 1$.*

Let π_1 and π_2 be two projection maps on $X \times Y$. Then the fuzzy product of two fuzzy space (X, τ) and (Y, σ) is denoted by $(X \times Y, \tau \times \sigma)$ where $\nu = \tau \times \sigma$ is defined as follows

$$\nu(C) = \begin{cases} \tau(A)(\sigma(B)), & C \in K_1 (C \in K_2) \\ \nu(C_1) \wedge \nu(C_2), & C = C_1 \wedge C_2, (C_1, C_2 \in K_1 \cup K_2) \\ \bigvee \nu(C_i), & C = \bigvee C_i, (C_i \in K') \\ 0, & \text{otherwise,} \end{cases}$$

where $K_1 = \{A\pi_1 : \tau(A) > 0\}$, $K_2 = \{B\pi_2 : \sigma(B) > 0\}$ and K' is the set of finite intersections of the elements in $K_1 \cup K_2$. One may easily verify the following.

Theorem 3.19. *The product of two zero gradation is a zero gradation.*

Proposition 3.20. *Let τ and σ be two zero gradations on two disjoint sets X and Y , respectively. Then there is a zero gradation on $X \cup Y$ such that τ and σ are two subgradations of it.*

Proof. Let τ and σ be two gradations of openness on X and Y , respectively. Put $Z = X \cup Y$ and for each fuzzy set μ on Z , Define

$$\delta(\mu) = \min\{\tau(\mu|_X), \sigma(\mu|_Y)\}.$$

It is ready to see that δ is a gradation of openness on Z . To check out that δ is a zero gradation, We must verify the r -levels of δ

$$\begin{aligned} \delta_r &= \{\mu \in \mathbb{I}^Z : \delta(\mu) \geq r\} = \{\mu \in \mathbb{I}^Z : \min\{\tau(\mu|_X), \sigma(\mu|_Y)\} \geq r\} \\ &= \{\mu \in \mathbb{I}^Z : \tau(\mu|_X) \geq r\} \cap \{\mu \in \mathbb{I}^Z : \sigma(\mu|_Y) \geq r\}. \end{aligned}$$

Let z_α be an arbitrary fuzzy point in δ_r and, say, $z \in X$. Let μ be an open fuzzy set containing z_α in the Lowen fuzzy topological space δ_r . So $\tau(\mu|_X) \geq r$. τ is a zero gradation and hence there exists a fuzzy clopen set η in τ_r such that $z_\alpha \in \eta \leq \mu|_X$. Now let η' be a fuzzy set on Z such that $\eta'|_X = \eta$ and $\eta'|_Y = 0$. Then it is ready to see that η' is a clopen fuzzy set in δ_r and $z_\alpha \in \eta' \leq \mu$. Thus, δ is a zero gradation. $\square$

Note that there may be a gradation of openness δ on the space $X \cup Y$, such that δ is not a zero gradation, But the restriction of δ on X and Y is a zero gradation. For example, consider the gradation of openness τ in the Example 3.12 on the $X = [0, \frac{1}{2})$ and $Y = [\frac{1}{2}, 1]$.

A Lowen fuzzy topological space is called almost zero-dimensional if it is T_1 and for every fuzzy point x_α and every open fuzzy set μ contains x_α there exists a fuzzy neighborhood ν of x_α with $x_\alpha \in \nu \leq \mu$ such that ν is an intersection of clopen fuzzy sets [3].

Definition 3.21. A gradation of openness τ on X is called *almost zero gradation* if the fuzzy topological space (X, τ_r) is almost zero-dimensional for all $r \in (0, 1]$.

The following proposition is a direct consequence of [3, Proposition 3.19]

Proposition 3.22. *Let τ be an almost zero gradation on X . If τ_r has a countable base for all $r \in (0, 1]$ then τ is a zero gradation.*

We constructed a fuzzy topological space X in [3, Example 3.21] which mainly use the curious properties of the famous Erdős space as most important example of an almost zero-dimensional space, see [1], [2], [15], [16], [19] for the

more information and generalization of the original concept. As an application if we define $\tau : \mathbb{I}^X \rightarrow \mathbb{I}$ by

$$\tau(\mu) = \begin{cases} 1, & \mu \in \delta, \\ 0, & \mu \in \mathbb{I}^X \setminus \delta \end{cases}$$

then τ is a zero gradation on X .

Remark 3.23. Similar to crisp case there is no relation between being finer and zero gradation. Consider three gradations τ_0, τ_1 and τ , where τ_1 is the constant gradation (when $k = 1$) and the gradation of openness τ is the same as Example 3.2. It is obvious that $\tau_1 \supset \tau \supset \tau_0$. Note that τ_0 and τ_1 are zero gradation and τ is not a zero gradation.

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MATHEMATICAL ANALYSIS OF AN AGE-STRUCTURED QUARANTINE/ISOLATION MODEL

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Abstract. A new age-structured model for disease transmission, subject to the use of quarantine (of asymptomatic cases) and isolation (of individuals with disease symptoms) is presented and rigorously analyzed. The model is, first of all, shown to be properly-posed mathematically by formulating it as an abstract Cauchy problem. For the case where the contact rate is separable (i.e., $\beta(a, b) = \beta_1(a)\beta_2(b)$), rigorous analysis of the model reveals that it has a globally-asymptotically stable disease-free equilibrium whenever its associated reproduction number is less than unity. The model has a unique endemic equilibrium when the threshold quantity exceeds unity. The endemic equilibrium is shown to be locally stable whenever its associated reproduction number exceeds unity. Furthermore, it is shown that adding age-factor to the basic quarantine-isolation model in M. A. Safi and A. B. Gumel (*Discrete Contin. Dyn. Syst. Ser. B.* : 209-231, 2010) does not alter the qualitative dynamics of the autonomous system (with respect to the elimination or persistence of the disease). Numerical simulations show disease elimination whenever its associated reproduction number is less than unity and the disease will persist in this case whenever its associated reproduction number exceeds unity.

Keywords: age-structure, abstract Cauchy problem, C_0 -semigroup, equilibria, stability.

1. Introduction

Quarantine (of individuals suspected of being exposed to the disease) and isolation (of those with clinical symptoms of the disease) have been used in the control of numerous diseases such as leprosy, plague, cholera, typhus, yellow fever, smallpox, diphtheria, tuberculosis, measles, ebola, pandemic influenza, severe acute respiratory syndrome (SARS) and more recently the 2009 swine influenza pandemic [1, 10, 13, 17, 20]. A number of mathematical modeling work have been carried out to assess the impact of quarantine and isolation in combatting the spread of diseases (see, for instance, [1, 4, 10, 13, 17, 18, 23, 24] and some of the references therein). However, many of the models used for assessing the impact of the quarantine and isolation measures tend to be built based on the assumption that the contact rates are age-dependent. However, some studies such as those reported by Elvebac *et al.* [3], suggest that disease trans-

mission models with age-dependent contact rates are more realistic than those that do not consider age-dependent contact rates. Numerous studies, based on the design and analysis of age-structured models, have been presented in the literature (see, for instance, [3, 6, 7, 8, 9, 11, 12, 14, 26, 28, 30] and some of the references therein).

The purpose of this study is to provide a rigorous qualitative analysis of a new age-structured model for disease transmission, subject to the use of quarantine (of asymptomatic cases) and isolation (of individuals with disease symptoms). The model to be designed extends the SEIQRH model given in [21] by considering age-densities of all individuals (susceptible, latent (exposed), infectious, quarantined, isolated (hospitalized) and recovered).

The paper is organized as follows. The model is formulated in Section 2. The locally and globally asymptotic stability of disease-free equilibrium is established in Section 3. The existence and locally asymptotical stability of endemic equilibrium is also studied in Section 4.

2. Model formulation

The total population, at time t and age a (where $0 < a \leq A$, with A being the maximum possible age attained by individuals in the population), is subdivided into six mutually-exclusive compartments of susceptible ($S(a, t)$), latent (those who have been infected but do not show clinical symptoms of the disease yet) ($E(a, t)$), infectious ($I(a, t)$), quarantined ($Q(a, t)$), isolated (hospitalized) ($H(a, t)$) and recovered ($R(a, t)$) individuals, so that

$$N(a, t) = S(a, t) + E(a, t) + I(a, t) + Q(a, t) + H(a, t) + R(a, t).$$

The population density evolves according to the following equation:

$$\frac{\partial N(a, t)}{\partial a} + \frac{\partial N(a, t)}{\partial t} = -\mu(a)N(a, t),$$

where $\mu(a)$ represents the instantaneous age-dependent death rate (the probability that an individual of age a dies before reaching the age $a + \Delta a$ is $\mu(a)\Delta a + O(\Delta a)$).

It is assumed that the net reproductive rate of the population equals to unity, and the total population is at an equilibrium. Thus, the following relations hold [15]:

$$(1) \quad \int_0^A v(a)e^{-\int_0^a \mu(\tau)d\tau} da = 1; \quad N(a, t) = N_\infty(a) = v_0 e^{-\int_0^a \mu(\tau)d\tau},$$

where $v(a)$ is the age-dependent fertility rate, $N_\infty(a)$ is the steady-state population distribution and v_0 is the steady-state number of newborns.

Let $\beta(a, b)$ be the age-dependent contact rate. In other words, the probability of contact between a susceptible of age a and an infectious individual of age b ,

during the time interval $[t, t + \Delta t]$, is given by $\beta(a, b)\Delta t + O(\Delta t)$. Hence, the associated *force of infection* (i.e., the rate at which new cases of infection are generated) at time t , denoted by $\lambda(a, t)$, is given by (this form of the force of infection has been used in [11, 14]):

$$\lambda(a, t) = \int_0^A \beta(a, b)[I(b, t) + \eta H(b, t)]db,$$

where $0 < \eta < 1$ is the modification parameter accounts for the assumed reduction in disease transmission by hospitalized individuals in comparison to non-hospitalized infectious individuals in the I class. Thus, η measures the efficacy of isolation or treatment given to hospitalized individuals (isolation is perfect if $\eta = 0$; leaky if $0 < \eta < 1$ and completely ineffective if $\eta = 1$).

The age-structured dynamics of an infectious disease in the presence of quarantine of exposed individuals and isolation of infectious individuals is given by the following system of partial differential equations :

$$(2) \quad \begin{aligned} \frac{\partial S}{\partial a} + \frac{\partial S}{\partial t} &= -[\mu(a) + \lambda(a, t)]S(a, t), \\ \frac{\partial E}{\partial a} + \frac{\partial E}{\partial t} &= \lambda(a, t)S(a, t) - [\mu(a) + \kappa + \sigma]E(a, t), \\ \frac{\partial I}{\partial a} + \frac{\partial I}{\partial t} &= \kappa E(a, t) - [\mu(a) + \psi + \gamma_1]I(a, t), \\ \frac{\partial Q}{\partial a} + \frac{\partial Q}{\partial t} &= \sigma E(a, t) - [\mu(a) + \alpha]Q(a, t), \\ \frac{\partial H}{\partial a} + \frac{\partial H}{\partial t} &= \psi I(a, t) + \alpha Q(a, t) - [\mu(a) + \gamma_2]H(a, t), \\ \frac{\partial R}{\partial a} + \frac{\partial R}{\partial t} &= \gamma_1 I(a, t) + \gamma_2 H(a, t) - \mu(a)R(a, t), \end{aligned}$$

subject to the boundary and initial conditions:

$$(3) \quad \begin{aligned} S(0, t) &= v_0, \quad E(0, t) = 0, \quad I(0, t) = 0, \quad Q(0, t) = 0, \\ H(0, t) &= 0, \quad R(0, t) = 0, \quad S(a, 0) = S_0(a), \quad E(a, 0) = E_0(a), \\ I(a, 0) &= I_0(a), \quad Q(a, 0) = Q_0(a), \quad H(a, 0) = H_0(a), \quad R(a, 0) = R_0(a). \end{aligned}$$

In (2), $\mu(a)$ is the age-dependent natural death rate, κ is the rate at which latent individuals develop symptoms, σ is the quarantine rate of exposed individuals, α is the hospitalization rate of quarantined individuals, ψ is the hospitalization rate of infectious individuals and γ_i ($i = 1, 2$) is the recovery rate for infectious and hospitalized individuals, respectively. Furthermore, it is assumed that disease-induced mortality is negligible (so that infected individuals do not die of the disease) and that recovery confers permanent immunity against reinfection. The parameters of the model are described in Table 1. The model extends the model in [21] by considering an age-densities of all individuals. In line with [11, 14] the model extends the models in [2, 6, 7] by assuming that the infection

rate (λ) is a function of age. Furthermore, the model (2) is an extension of the SIR models in [6, 11], by including classes for latent (E), quarantined (Q) and Hospitalized (H) individuals.

2.1 Basic properties

To show the existence and uniqueness of the solution for the system (2) with the initial-boundary conditions (3) the natural death rate, $\mu(a)$, is eliminated from the model (2) by the following change of variables [12, 25]:

$$(4) \quad \begin{aligned} x_1(a, t) &= \frac{S(a, t)}{N_\infty(a)}, \quad x_2(a, t) = \frac{E(a, t)}{N_\infty(a)}, \quad x_3(a, t) = \frac{I(a, t)}{N_\infty(a)}, \\ x_4(a, t) &= \frac{Q(a, t)}{N_\infty(a)}, \quad x_5(a, t) = \frac{H(a, t)}{N_\infty(a)}, \quad x_6(a, t) = \frac{R(a, t)}{N_\infty(a)}. \end{aligned}$$

Using (4) in (2) gives the following transformed system:

$$(5) \quad \begin{aligned} \frac{\partial x_1}{\partial a} + \frac{\partial x_1}{\partial t} &= -\lambda_1(a, t)x_1(a, t), \\ \frac{\partial x_2}{\partial a} + \frac{\partial x_2}{\partial t} &= \lambda_1(a, t)x_1(a, t) - (\kappa + \sigma)x_2(a, t), \\ \frac{\partial x_3}{\partial a} + \frac{\partial x_3}{\partial t} &= \kappa x_2(a, t) - (\psi + \gamma_1)x_3(a, t), \\ \frac{\partial x_4}{\partial a} + \frac{\partial x_4}{\partial t} &= \sigma x_2(a, t) - \alpha x_4(a, t), \\ \frac{\partial x_5}{\partial a} + \frac{\partial x_5}{\partial t} &= \psi x_3(a, t) + \alpha x_4(a, t) - \gamma_2 x_5(a, t), \\ \frac{\partial x_6}{\partial a} + \frac{\partial x_6}{\partial t} &= \gamma_1 x_3(a, t) + \gamma_2 x_5(a, t), \end{aligned}$$

and associated boundary and initial conditions:

$$(6) \quad \begin{aligned} x_1(0, t) &= 1, \quad x_2(0, t) = 0, \quad x_3(0, t) = 0, \quad x_4(0, t) = 0, \\ x_5(0, t) &= 0, \quad x_1(a, 0) = f_1(a), \quad x_2(a, 0) = f_2(a), \\ x_3(a, 0) &= f_3(a), \quad x_4(a, 0) = f_4(a), \quad x_5(a, 0) = f_5(a), \quad x_6(a, 0) = f_6(a). \end{aligned}$$

Furthermore, the force of infection, $\lambda(a, t)$, now becomes

$$(7) \quad \lambda_1(a, t) = \int_0^A \beta(a, b)[x_3(b, t) + \eta x_5(b, t)]N_\infty(b)db.$$

Using the relation $x_1(a, t) + x_2(a, t) + x_3(a, t) + x_4(a, t) + x_5(a, t) + x_6(a, t) = 1$ and noting that the variable $x_6(a, t)$ does not feature in the other equations of the model. Furthermore, introducing a new variable $\bar{x}_1(a, t) = x_1(a, t) - 1$ into

the transformed system (5) gives

$$\begin{aligned}
 (8) \quad & \frac{\partial \bar{x}_1}{\partial a} + \frac{\partial \bar{x}_1}{\partial t} = -\lambda_1(a, t)\bar{x}_1(a, t), \\
 & \frac{\partial x_2}{\partial a} + \frac{\partial x_2}{\partial t} = \lambda_1(a, t)\bar{x}_1(a, t) - (\kappa + \sigma)x_2(a, t), \\
 & \frac{\partial x_3}{\partial a} + \frac{\partial x_3}{\partial t} = \kappa x_2(a, t) - (\psi + \gamma_1)x_3(a, t), \\
 & \frac{\partial x_4}{\partial a} + \frac{\partial x_4}{\partial t} = \sigma x_2(a, t) - \alpha x_4(a, t), \\
 & \frac{\partial x_5}{\partial a} + \frac{\partial x_5}{\partial t} = \psi x_3(a, t) + \alpha x_4(a, t) - \gamma_2 x_5(a, t),
 \end{aligned}$$

and associated boundary and initial conditions:

$$\begin{aligned}
 (9) \quad & \bar{x}_1(0, t) = 0, \quad x_2(0, t) = 0, \quad x_3(0, t) = 0, \quad x_4(0, t) = 0, \\
 & x_5(0, t) = 0, \quad x_6(0, t) = 0, \quad \bar{x}_1(a, 0) = f_1(a) - 1, \quad x_2(a, 0) = f_2(a), \\
 & x_3(a, 0) = f_3(a), \quad x_4(a, 0) = f_4(a), \quad x_5(a, 0) = f_5(a).
 \end{aligned}$$

Following [11], the initial-boundary-value problem $\{(8), (9)\}$ is written as an abstract Cauchy problem on the Banach space

$$X = L^1(0, A) \times L^1(0, A) \times L^1(0, A) \times L^1(0, A) \times L^1(0, A)$$

(with the norm $\|\phi\| = \sum_{j=1}^5 \|\phi_j\|_1$, where $\phi(a) = (\phi_1(a), \phi_2(a), \phi_3(a), \phi_4(a), \phi_5(a))^T \in X$ and $\|\phi_j\|_1$ is the norm of $L^1(0, A)$) by defining a linear operator, $\mathfrak{A}$, as follows (by considering the derivatives with respect to age and the negative coefficients of $\phi_i (i = 1, 2, \dots, 5)$ in (8)):

$$(10) \quad (\mathfrak{A}\phi)(a) = \begin{pmatrix} -\frac{d\phi_1(a)}{da} \\ -\frac{d\phi_2(a)}{da} - (\kappa + \sigma)\phi_2 \\ -\frac{d\phi_3(a)}{da} - (\psi + \gamma_1)\phi_3 \\ -\frac{d\phi_4(a)}{da} - \alpha\phi_4 \\ -\frac{d\phi_5(a)}{da} - \gamma_2\phi_5 \end{pmatrix},$$

with $\phi(a) \in D(\mathfrak{A}) = \{\phi \in X : \phi_j \text{ is absolutely continuous on } [0, A], \phi(0) = 0\}$.

Let $F : X \rightarrow X$ be the non-linear operator given by (consisting of the remaining terms, excluding those used in the definition of the operator $\mathfrak{A}$, in (8))

$$(11) \quad (F\phi)(a) = \begin{pmatrix} -(P(\phi_3, \phi_5))(a)(1 + \phi_1) \\ (P(\phi_3, \phi_5))(a)(1 + \phi_1) \\ \kappa\phi_2(a) \\ \sigma\phi_2(a) \\ \psi\phi_3 + \alpha\phi_4(a) \end{pmatrix}, \quad \phi \in X$$

where $P : L^1(0, A) \times L^1(0, A) \rightarrow L^\infty(0, A)$ is a linear bounded operator, given by

$$(P(h, k))(a) = \int_0^A \beta(a, b) N_\infty(b) [h(b) + \eta k(b)] db.$$

Thus, it follows from the above definitions that the initial-boundary-value problem $\{(8), (9)\}$ can be re-written as the following abstract Cauchy problem in X :

$$(12) \quad \frac{du(t)}{dt} = \mathfrak{A}u(t) + F(u(t)), \quad u(0) = u_0 \in X,$$

where $u(t) = (\bar{x}_1(\cdot, t), x_2(\cdot, t), x_3(\cdot, t), x_4(\cdot, t), x_5(\cdot, t))^T$ and $u_0 = (f_1(a)-1, f_2(a), f_3(a), f_4(a), f_5(a))^T$. Furthermore, it follows from (10) and (11) that the operator $\mathfrak{A}$ is the infinitesimal generator of C_0 -semigroup $T(t), t \geq 0$ and F is continuously Frechet differentiable on X [14]. Thus, using Proposition 4.16 of [30], the results below are established

Theorem 1. (i) for each $u_0 \in X$, there exist a maximal interval of existence $[0, b)$ and unique continuous mild solution $t \rightarrow u(t, u_0)$ such that:

$$u(t, u_0) = T(t)u_0 + \int_0^t T(t-s)F(u(s, u_0))ds \quad \text{for all } t \in [0, b),$$

with either $b = \infty$ or $\lim_{t \rightarrow b^-} \|u(t, u_0)\| = \infty$;

(ii) if $u_0 \in D(\mathfrak{A})$, then $u(t, u_0) \in D(\mathfrak{A})$ for $t \in [0, b)$ and the mild solution $u(t, u_0)$ is continuously differentiable and satisfies (12) on the interval $[0, b)$.

Let,

$$\mathcal{D} = \{(\bar{x}_1, x_2, x_3, x_4, x_5) \in X : \bar{x}_1 \geq -1, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0\},$$

and,

$$\begin{aligned} \mathcal{D}_0 = \{(\bar{x}_1, x_2, x_3, x_4, x_5) \in X : -1 \leq \bar{x}_1 \leq 0, 0 \leq x_2 \leq 1, \\ 0 \leq x_3 \leq 1, 0 \leq x_4 \leq 1, 0 \leq x_5 \leq 1\}. \end{aligned}$$

The following result is needed to show that the abstract Cauchy problem (12) has a unique global classical solution.

Lemma 1. The mild solution of the abstract Cauchy problem (12), given by $u(t, u_0)$ with $u_0 \in \mathcal{D}$, enters $\mathcal{D}_0$ in finite time, and the set $\mathcal{D}_0$ is positively-invariant.

Proof. The proof of Lemma 1 is based on the approach given in [11, 14]. First of all, the solution of the system (5), along the characteristic lines, can be represented [9, 29, 30] by fixing arbitrary time (t_1) and age (a_1) and considering the functions $\tilde{x}_1(h) = x_1(a_1 + h, t_1 + h)$ and $\tilde{\lambda}(h) = \lambda_1(a_1 + h, t_1 + h)$. Thus, the first equation of the system (5), along this line, is equivalent to

$$\frac{d\tilde{x}_1}{dh} + \tilde{\lambda}\tilde{x}_1 = 0,$$

which has the following solution

$$(13) \quad \tilde{x}_1(h) = \tilde{x}_1(0) \exp\left(-\int_0^h \tilde{\lambda}(\tau) d\tau\right).$$

Hence,

$$x_1(a_1 + h, t_1 + h) = x_1(a_1, t_1) \exp\left(-\int_0^h \lambda_1(a_1 + \tau, t_1 + \tau) d\tau\right).$$

By setting $(a_1, t_1) = (a - t, 0)$ and $h = t$ for $a > t$ in (13), it follows that

$$x_1(a, t) = f_1(a - t) \exp\left(-\int_0^t \lambda_1(a - t + \tau, \tau) d\tau\right).$$

Similarly, for the case when $a < t$, setting $(a_1, t_1) = (0, t - a)$ and $h = a$ in (13) gives

$$x_1(a, t) = \exp\left(-\int_0^a \lambda_1(\tau, t - a + \tau) d\tau\right).$$

Thus,

$$(14) \quad x_1(a, t) = \begin{cases} f_1(a - t) \exp\left(-\int_0^t \lambda_1(a - t + \tau, \tau) d\tau\right), & a > t \\ \exp\left(-\int_0^a \lambda_1(\tau, t - a + \tau) d\tau\right), & t > a \end{cases}$$

It is clear that $x_1(a, t) \geq 0$ (or, equivalently, $\bar{x} + 1(a, t) \geq -1$) whenever $f_1(a) \geq 0$. Similarly, from the third, fourth and fifth equations of the system (5), the representation for x_3, x_4 and x_5 , can, respectively, be found as follows:

$$(15) \quad x_3(a, t) = \begin{cases} f_3(a - t) \exp(-k_1 t) \\ + \kappa \int_0^t x_2(a - t + \tau, \tau) \exp(-k_1(t - \tau)) d\tau, & a > t \\ \kappa \int_0^a x_2(\tau, t - a + \tau) \exp(-k_1(a - \tau)) d\tau, & t > a, \end{cases}$$

$$(16) \quad x_4(a, t) = \begin{cases} f_4(a - t) \exp(-k_2 t) \\ + \sigma \int_0^t x_2(a - t + \tau, \tau) \exp(-k_2(t - \tau)) d\tau, & a > t \\ \sigma \int_0^a x_2(\tau, t - a + \tau) \exp(-k_2(a - \tau)) d\tau, & t > a, \end{cases}$$

$$(17) \quad x_5(a, t) = \begin{cases} f_5(a - t) \exp(-k_3 t) + \int_0^t [\psi x_3(a - t + \tau, \tau) \\ + \alpha x_4(a - t + \tau, \tau)] \exp(-k_3(t - \tau)) d\tau, & a > t \\ \int_0^t [\psi x_3(a - t + \tau, \tau) \\ + \alpha x_4(a - t + \tau, \tau)] \exp(-k_3(t - \tau)) d\tau, & t > a, \end{cases}$$

where, $k_1 = \psi + \gamma_1$, $k_2 = \alpha$, and $k_3 = \gamma_2$.

Substituting (15), (16) and (17)) into (7) gives

$$(18) \quad \lambda_1(t) = (Gx_2)(a, t),$$

where G is the transformation from $x_2(a, t)$ to $\lambda_1(t)$. The second equation of the system (5) can be written as an abstract Cauchy problem:

$$\frac{dx_2}{dt} = Bx_2(a, t) + (Gx_2(a, t))(x_1), \quad x_2(0) = f_2,$$

where B is an operator given by

$$B = -\frac{d}{da} - (\kappa + \sigma), \quad \text{with } D(B) = D(\mathfrak{A}).$$

Hence, x_2 can be found as follows:

$$(19) \quad x_2(t) = Q(t)f_2(0) + \int_0^t Q(t - \tau)(Gx_2(\tau))(\tau) d\tau,$$

where, $Q(t) = \exp tB$ is the positive C_0 -semigroup generated by the closed operator B [11]. If $f_2 \geq 0, f_3 \geq 0, f_4 \geq 0$ and $f_5 \geq 0$, then G and $Q(t), t \geq 0$ are positive. Thus, (19) shows that x_2 is positive. Since x_2 is positive, it follows that x_3, x_4 and x_5 are positive. Hence, $u(t, u_0) \in \mathcal{D}$ for all $t \geq 0$ whenever $u_0 \in \mathcal{D}_0$. $\square$

It follows from Lemma 1 that the norm of the local solution, $u(t, u_0)$, $u_0 \in D(\mathfrak{A}) \cap \mathcal{D}$, of the abstract Cauchy problem (12) is finite as long as it is defined. Hence, the following result is established:

Theorem 2. *The abstract Cauchy problem (12) has a unique global classical solution on X with respect to initial data $u_0 \in D(\mathfrak{A}) \cap \mathcal{D}$.*

Theorem 2 shows that the initial-boundary-value problem $\{(5), (6)\}$ (or, equivalently, $\{(2), (3)\}$) has a unique positive global solution with respect to the positive initial data.

The analysis of the initial-boundary-value problem $\{(5), (6)\}$ will be explored for a special form of the contact rate $\beta(a, b)$, given by $\beta(a, b) = \beta_1(a)\beta_2(b)$ (this form of the contact rate, known as separable contact rate, has been used in [5, 15, 16]). Using $\beta(a, b) = \beta_1(a)\beta_2(b)$ in (7) gives:

$$(20) \quad \lambda_2(a, t) = \beta_1(a) \int_0^A \beta_2(b) [x_3(b, t) + \eta x_5(b, t)] N_\infty(b) db.$$

3. Stability of the Disease-Free Equilibrium (DFE)

3.1 Local stability of DFE

The DFE of the system (5) is given by

$$\mathcal{E}_0 = (x_1, x_2, x_3, x_4, x_5, x_6) = (1, 0, 0, 0, 0, 0).$$

The local stability of the DFE is studied by considering exponential solutions of the form ([15, 16]):

$$(21) \quad \begin{aligned} x_1(a, t) &= 1 + \bar{x}_1(a)e^{\zeta t}, & x_2(a, t) &= \bar{x}_2(a)e^{\zeta t}, & x_3(a, t) &= \bar{x}_3(a)e^{\zeta t}, \\ x_4(a, t) &= \bar{x}_4(a)e^{\zeta t}, & x_5(a, t) &= \bar{x}_5(a)e^{\zeta t}, & x_6(a, t) &= \bar{x}_6(a)e^{\zeta t}. \end{aligned}$$

Linearizing (5) about the DFE ($\mathcal{E}_0$) gives (noting that $\lambda_1(a, t)$ is now replaced by (20)),

$$(22) \quad \begin{aligned} \zeta \bar{x}_2(a) + \frac{d\bar{x}_2(a)}{da} &= \lambda_2(a) - (\kappa + \sigma)\bar{x}_2(a), \\ \zeta \bar{x}_3(a) + \frac{d\bar{x}_3(a)}{da} &= \kappa \bar{x}_2(a) - (\psi + \gamma_1)\bar{x}_3(a), \\ \zeta \bar{x}_4(a) + \frac{d\bar{x}_4(a)}{da} &= \sigma \bar{x}_2(a) - \alpha \bar{x}_4(a), \\ \zeta \bar{x}_5(a) + \frac{d\bar{x}_5(a)}{da} &= \psi \bar{x}_3(a) + \alpha \bar{x}_4 - \gamma_2 \bar{x}_5(a), \\ \zeta \bar{x}_6(a) + \frac{d\bar{x}_6(a)}{da} &= \gamma_1 \bar{x}_3(a) + \gamma_2 \bar{x}_5(a), \end{aligned}$$

where,

$$\lambda_2(a) = \beta_1(a) \int_0^A \beta_2(b) [\bar{x}_3(b) + \eta \bar{x}_5(b)] N_\infty(b) db = \beta_1(a) V_0,$$

with

$$(23) \quad V_0 = \int_0^A \beta_2(b) [\bar{x}_3(b) + \eta \bar{x}_5(b)] N_\infty(b) db; \quad V_0 \neq 0.$$

Solving system (22) gives,

$$(24) \quad \begin{aligned} \bar{x}_2(a) &= V_0 \int_0^a \beta_1(u) e^{-(\zeta+\kappa+\sigma)(a-u)} du, \\ \bar{x}_3(a) &= V_0 \kappa \int_0^a \int_0^u \beta_1(u_1) e^{-(\zeta+\kappa+\sigma)(u-u_1)} e^{-(\zeta+\psi+\gamma_1)(a-u)} du_1 du \\ &= V_0 Q_1(a, \zeta), \\ \bar{x}_4(a) &= V_0 \sigma \int_0^a \int_0^u \beta_1(u_1) e^{-(\zeta+\kappa+\sigma)(u-u_1)} e^{-(\zeta+\alpha)(a-u)} du_1 du \\ &= V_0 Q_2(a, \zeta) \\ \bar{x}_5(a) &= \int_0^a [\psi \bar{x}_3(u) + \alpha \bar{x}_4(u)] e^{-(\zeta+\gamma_2)(a-u)} du \\ &= V_0 \int_0^a \psi Q_1(u, \zeta) + \alpha Q_2(u, \zeta) e^{-(\zeta+\gamma_2)(a-u)} du, = V_0 Q_3(a, \zeta). \end{aligned}$$

where,

$$\begin{aligned} Q_1(a, \zeta) &= \kappa \int_0^a \int_0^u \beta_1(u_1) e^{-(\zeta+\kappa+\sigma)(u-u_1)} e^{-(\zeta+\psi+\gamma_1)(a-u)} du_1 du, \\ Q_2(a, \zeta) &= \sigma \int_0^a \int_0^u \beta_1(u_1) e^{-(\zeta+\kappa+\sigma)(u-u_1)} e^{-(\zeta+\alpha)(a-u)} du_1 du, \\ Q_3(a, \zeta) &= \int_0^a (\psi Q_1(u, \zeta) + \alpha Q_2(u, \zeta)) e^{-(\zeta+\gamma_2)(a-u)} du. \end{aligned}$$

Substituting the expressions for $\bar{x}_3(a)$ and $\bar{x}_5(a)$ from (24) into (23) gives,

$$(25) \quad V_0 = V_0 \int_0^A \beta_2(b) [Q_1(b, \zeta) + \eta Q_3(b, \zeta)] N_\infty(b) db,$$

so that (by dividing both sides of the equation (25) by V_0)

$$(26) \quad 1 = \int_0^A \beta_2(b) [Q_1(b, \zeta) + \eta Q_3(b, \zeta)] N_\infty(b) db.$$

Let,

$$(27) \quad \mathcal{G}(\zeta) = \int_0^A \beta_2(b) [Q_1(b, \zeta) + \eta Q_3(b, \zeta)] N_\infty(b) db.$$

Define the reproduction number as follows:

$$\mathcal{R}_0 = \int_0^A \beta_2(b) [Q(b, 0) + \eta Q_3(b, 0)] N_\infty(b) db.$$

It is clear that the function $\mathcal{G}$ in (27) satisfies the following properties:

- (i) $\mathcal{G}(0) = \mathcal{R}_0$;
- (ii) $\mathcal{G}$ is a monotone decreasing function in ζ ;
- (iii) $\lim_{\zeta \rightarrow \infty} \mathcal{G}(\zeta) = 0$.

Hence, the result below can be established, based on the properties of the function $\mathcal{G}$ and equation (26):

Theorem 3. *The DFE of the model (5) with (20), given by $\mathcal{E}_0$, is locally-asymptotically stable (LAS) if $\mathcal{R}_0 < 1$, and unstable if $\mathcal{R}_0 > 1$.*

Theorem 3 shows that the spread of the disease can be effectively controlled if the initial sizes of the sub-populations of the model are in the basin of attraction of the DFE ($\mathcal{E}_0$). For effective disease elimination to be independent of such initial sizes, a global asymptotic stability result has to be proved for the DFE. This is explored below.

3.2 Global stability of DFE

Theorem 4. *The DFE of the model (5) with (20) is globally-asymptotically stable (GAS) in $\mathcal{D}_0$ if $\mathcal{R}_0 < 1$.*

Proof. Following [15, 16], let

$$(28) \quad f(a, t) = \lambda_2(a, t)x_1(a, t).$$

Noting that $x_1(a, t) \leq 1$ in $\mathcal{D}_0$, it follows that

$$(29) \quad f(a, t) \leq \lambda_2(a, t).$$

Integrating (5), with (20), along the characteristic lines whenever $a < t$, gives,

$$\begin{aligned}
 (30) \quad x_2(a, t) &= \int_0^a e^{-(\kappa+\sigma)(a-u)} f(u, t) du, \\
 x_3(a, t) &= \kappa \int_0^a e^{-(\psi+\gamma_1)(a-u)} x_2(u, t) du \\
 &= \kappa \int_0^a e^{-(\psi+\gamma_1)(a-u)} \int_0^u e^{-(\kappa+\sigma)(u-u_1)} f(u_1, t) du_1 du, \\
 x_4(a, t) &= \sigma \int_0^a e^{-\alpha(a-u)} x_2(u, t) du \\
 &= \sigma \int_0^a e^{-\alpha(a-u)} \int_0^u e^{-(\kappa+\sigma)(u-u_1)} f(u_1, t) du_1 du, \\
 x_5(a, t) &= \int_0^a e^{-\gamma_2(a-u)} [\psi x_3(u, t) + \alpha x_4(u, t)] du \\
 &= \psi \int_0^a e^{-\gamma_2(a-u)} \kappa \int_0^u e^{-(\psi+\gamma_1)(u-u_1)} \int_0^{u_1} e^{-(\kappa+\sigma)(u_1-u_2)} f(u_2, t) du_2 du_1, \\
 &\quad + \alpha \int_0^a e^{-\gamma_2(a-u)} \kappa \int_0^u e^{-\alpha(u-u_1)} \int_0^{u_1} e^{-(\kappa+\sigma)(u_1-u_2)} f(u_2, t) du_2 du_1,
 \end{aligned}$$

Substituting the expressions for $x_3(a)$ and $x_5(a)$ in (30) into (29) gives,

$$\begin{aligned}
 f(a, t) &\leq \beta_1(a)V(t) = \beta_1(a) \int_0^A N_\infty(b)\beta_2(b) \left[\kappa \int_0^a e^{-(\psi+\gamma_1)(a-u)} \right. \\
 &\quad \cdot \int_0^u e^{-(\kappa+\sigma)(u-u_1)} f(u_1, t) du_1 du \\
 &\quad + \eta\psi \int_0^a e^{-\gamma_2(a-u)} \kappa \int_0^u e^{-(\psi+\gamma_1)(u-u_1)} \\
 (31) \quad &\quad \cdot \int_0^{u_1} e^{-(\kappa+\sigma)(u_1-u_2)} f(u_2, t) du_2 du_1 \\
 &\quad + \eta\alpha \int_0^a e^{-\gamma_2(a-u)} \kappa \int_0^u e^{-\alpha(u-u_1)} \\
 &\quad \cdot \left. \int_0^{u_1} e^{-(\kappa+\sigma)(u_1-u_2)} f(u_2, t) du_2 du_1 \right] db.
 \end{aligned}$$

Let,

$$F(a) = \limsup_{t \rightarrow \infty} f(a, t).$$

Taking the lim sup (when $t \rightarrow \infty$) of both sides of (31), and using Fatou's Lemma [19] gives,

$$(32) \quad F(a) \leq \beta_1(a)L,$$

where,

$$\begin{aligned}
 L &= \int_0^A N_\infty(b)\beta_2(b) \left[\kappa \int_0^a e^{-(\psi+\gamma_1)(a-u)} \right. \\
 &\quad \cdot \int_0^u e^{-(\kappa+\sigma)(u-u_1)} F(u_1) du_1 du \\
 &\quad + \eta\psi \int_0^a e^{-\gamma_2(a-u)} \kappa \int_0^u e^{-(\psi+\gamma_1)(u-u_1)} \\
 (33) \quad &\quad \cdot \int_0^{u_1} e^{-(\kappa+\sigma)(u_1-u_2)} F(u_2) du_2 du_1 \\
 &\quad + \eta\alpha \int_0^a e^{-\gamma_2(a-u)} \kappa \int_0^u e^{-\alpha(u-u_1)} \\
 &\quad \cdot \left. \int_0^{u_1} e^{-(\kappa+\sigma)(u_1-u_2)} F(u_2) du_2 du_1 \right] db.
 \end{aligned}$$

Using inequality (32) in (34) gives

$$\begin{aligned}
 L &\leq \int_0^A N_\infty(b)\beta_2(b) \left[\kappa \int_0^a e^{-(\psi+\gamma_1)(a-u)} \right. \\
 &\quad \cdot \int_0^u e^{-(\kappa+\sigma)(u-u_1)} \beta_1(u_1) L du_1 du \\
 &\quad + \eta\psi \int_0^a e^{-\gamma_2(a-u)} \kappa \int_0^u e^{-(\psi+\gamma_1)(u-u_1)}
 \end{aligned}$$

$$\begin{aligned}
(34) \quad & \cdot \int_0^{u_1} e^{-(\kappa+\sigma)(u_1-u_2)} \beta_1(u_2) L du_2 du_1 \\
& + \eta \alpha \int_0^a e^{-\gamma_2(a-u)} \kappa \int_0^u e^{-\alpha(u-u_1)} \\
& \cdot \int_0^{u_1} e^{-(\kappa+\sigma)(u_1-u_2)} \beta_1(u_2) L du_2 du_1 \Big] db = L \mathcal{R}_0.
\end{aligned}$$

It follows that whenever $\mathcal{R}_0 < 1$, then $L = 0$. Thus,

$$\limsup_{t \rightarrow \infty} f(a, t) = F(a) = 0.$$

Hence (from (28)),

$$(35) \quad \limsup_{t \rightarrow \infty} \lambda_2(a, t) = 0.$$

It follows then (by using (35) in (20)) that

$$\lim_{t \rightarrow \infty} x_3(a, t) = 0, \quad \lim_{t \rightarrow \infty} x_4(a, t) = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} x_5(a, t) = 0.$$

By using comparison theorem [27] it can be shown that

$$\lim_{t \rightarrow \infty} x_2(a, t) = 0, \quad \text{and} \quad \lim_{t \rightarrow \infty} x_6(a, t) = 0.$$

Finally, using the relation $x_1(a, t) = 1 - x_2(a, t) - x_3(a, t) - x_4(a, t) - x_5(a, t) - x_6(a, t)$, gives

$$\lim_{t \rightarrow \infty} x_1(a, t) = 1.$$

Hence, $\lim_{t \rightarrow \infty} (x_1(a, t), x_2(a, t), x_3(a, t), x_4(a, t), x_5(a, t), x_6(a, t)) = (1, 0, 0, 0, 0, 0)$, as required. $\square$

4. Existence and stability of endemic equilibria

In this section, the existence of endemic equilibria (i.e., equilibria where the infected components of the model are non-zero) of the initial-boundary-value problem $\{(5), (6)\}$ will be explored. Let,

$$\mathcal{E} = (S^*(a), E^*(a), I^*(a), Q^*(a), H^*(a), R^*(a))$$

represents any arbitrary equilibrium point of the model $\{(5), (6)\}$ with (20). The associated force of infection (20) can be expressed at steady-state as:

$$(36) \quad \lambda_2^*(a) = \beta_1(a) \int_0^A \beta_2(b) [I^*(b) + \eta H^*(b)] N_\infty(b) db = \beta_1(a) V^*,$$

where,

$$(37) \quad V^* = \int_0^A \beta_2(b) [I^*(b) + \eta H^*(b)] N_\infty(b) db.$$

Solving the equations of the system (5) at steady-state, and noting (36), gives

$$\begin{aligned}
 S^*(a) &= e^{-V^* \int_0^a \beta_1(\tau) d\tau}, \\
 E^*(a) &= V^* e^{-(\kappa+\sigma)a} \int_0^a \beta_1(u_1) e^{-V^* \int_0^{u_1} \beta_1(\tau) d\tau} e^{(\kappa+\sigma)u_1} du_1, \\
 I^*(a) &= V^* \kappa e^{-k_1 a} \int_0^a e^{(k_1-\kappa-\sigma)u_1} \\
 &\quad \cdot \int_0^{u_1} \beta_1(u_2) e^{-V^* \int_0^{u_2} \beta_1(\tau) d\tau} e^{(\kappa+\sigma)u_2} du_2 du_1 = V^* M_1(a, V^*), \\
 Q^*(a) &= V^* \sigma e^{-\alpha a} \int_0^a e^{(\alpha-\kappa-\sigma)u_1} \\
 &\quad \cdot \int_0^{u_1} \beta_1(u_2) e^{-V^* \int_0^{u_2} \beta_1(\tau) d\tau} e^{(\kappa+\sigma)u_2} du_2 du_1 = V^* M_2(a, V^*), \\
 H^*(a) &= e^{-\gamma_2 a} \int_0^a e^{\gamma_2 u_1} [\psi I^*(u_1) + \alpha Q^*(u_1)] du_1 \\
 &= V^* e^{-\gamma_2 a} \int_0^a e^{\gamma_2 u_1} [\psi M_1^*(u_1, V^*) + \alpha M_2(u_1, V^*)] du_1 \\
 &= V^* M_3(a, V^*), \\
 R^*(a) &= \int_0^a \gamma_1 I^*(u_1) + \gamma_2 H^*(u_1) du_1,
 \end{aligned}
 \tag{38}$$

where,

$$\begin{aligned}
 M_1(a, V^*) &= \kappa e^{-k_1 a} \int_0^a e^{(k_1-\kappa-\sigma)u_1} \int_0^{u_1} \beta_1(u_2) e^{-V^* \int_0^{u_2} \beta_1(\tau) d\tau} e^{(\kappa+\sigma)u_2} du_2 du_1, \\
 M_2(a, V^*) &= \sigma e^{-\alpha a} \int_0^a e^{(\alpha-\kappa-\sigma)u_1} \int_0^{u_1} \beta_1(u_2) e^{-V^* \int_0^{u_2} \beta_1(\tau) d\tau} e^{(\kappa+\sigma)u_2} du_2 du_1, \\
 M_3(a, V^*) &= e^{-\gamma_2 a} \int_0^a e^{\gamma_2 u_1} [\psi M_1^*(u_1, V^*) + \alpha M_2(u_1, V^*)] du_1.
 \end{aligned}$$

Substituting the expressions for $I^*(a)$, $Y^*(a)$ and $W^*(a)$ from (38) into (37) gives,

$$V^* = V^* \int_0^A \beta_2(a) [M_1(a, V^*) + \eta M_3(a, \lambda^*)] N_\infty(a) da.
 \tag{39}$$

It follows from equation (39) that the quantity V^* is either 0 (which corresponds to the case where there is no disease in the population) or satisfies the following equation:

$$1 = \int_0^A \beta_2(a) [M_1(a, V^*) + \eta M_3(a, V^*)] N_\infty(a) da.
 \tag{40}$$

Let,

$$\mathcal{H}(V^*) = \int_0^A \beta_2(a) [M_1(a, V^*) + \eta M_3(a, V^*)] N_\infty(a) da.$$

It is clear that the function $\mathcal{H}$ satisfies the following:

- (i) $\mathcal{H}(0) = \mathcal{R}_0$;
- (ii) $\mathcal{H}$ is a monotone decreasing function in V^* ;
- (iii) $\lim_{V^* \rightarrow \infty} \mathcal{H}(V^*) = 0$.

The result below can be established, based on the properties of $\mathcal{H}$ and equation (40):

Theorem 5. *The model (5) with (20) has a unique endemic equilibrium point (EEP), given by (38), whenever $\mathcal{R}_0 > 1$.*

Following [25], the local stability of the unique endemic equilibrium (38) is now explored as follows.

Let $\hat{S}, \hat{E}, \hat{I}, \hat{Q}, \hat{H}$ and $\hat{V}$ be the perturbations of S^*, E^*, I^*, Y^*, W^* and V^* , respectively. Consider, as before, the exponential solutions:

$$(41) \quad \begin{aligned} \hat{S}(a, t) &= \bar{S}(a)e^{\omega t}, \quad \hat{E}(a, t) = \bar{E}(a)e^{\omega t}, \quad \hat{I}(a, t) = \bar{I}(a)e^{\omega t}, \\ \hat{Q}(a, t) &= \bar{Q}(a)e^{\omega t}, \quad \hat{H}(a, t) = \bar{H}(a)e^{\omega t}, \quad \hat{V} = \bar{V}e^{\omega t} \end{aligned}$$

Hence, using (41) in (5) with (36) gives the following system:

$$(42) \quad \begin{aligned} \frac{d\bar{S}(a)}{da} &= -\omega\bar{S}(a) - \beta_1(a)V^*(a)\bar{S}(a) - \beta_1(a)S^*(a)\bar{V}(a) \\ \frac{d\bar{E}(a)}{da} &= -\omega\bar{E}(a) + \beta_1(a)V^*(a)\bar{S}(a) + \beta_1(a)S^*(a)\bar{V}(a) - (\kappa + \sigma)\bar{E}(a) \\ \frac{d\bar{I}(a)}{da} &= -\omega\bar{I}(a) + \kappa\bar{E}(a) - (\psi + \gamma_1)\bar{I}(a) \\ \frac{d\bar{Q}(a)}{da} &= -\omega\bar{Q}(a) + \sigma\bar{E}(a) - \alpha\bar{Q}(a) \\ \frac{d\bar{H}(a)}{da} &= -\omega\bar{H}(a) + \psi\bar{I}(a) + \alpha\bar{Q}(a) - \gamma_2\bar{H}(a) \end{aligned}$$

with initial conditions $\hat{S}(0) = 0, \hat{E}(0) = 0, \hat{I}(0) = 0, \hat{Y}(0) = 0, \hat{W}(0) = 0$. In (42),

$$(43) \quad \bar{V}(a) = \int_0^A \beta_2(b)[\bar{I}(b) + \eta\bar{H}(b)]N_\infty(b)db.$$

It is convenient to introduce the new variables:

$$y_1(a) = \frac{\bar{S}(a)}{\bar{V}(a)}, \quad y_2(a) = \frac{\bar{E}(a)}{\bar{V}(a)}, \quad y_3(a) = \frac{\bar{I}(a)}{\bar{V}(a)}, \quad y_4(a) = \frac{\bar{Q}(a)}{\bar{V}(a)}, \quad y_5(a) = \frac{\bar{H}(a)}{\bar{V}(a)}.$$

Using the above variables into the system (42) gives,

$$\begin{aligned}
 (44) \quad & \frac{dy_1(a)}{da} = -\omega y_1 - \beta_1(a)V^*y_1 - \beta_1(a)S^*, \\
 & \frac{dy_2(a)}{da} = -\omega y_2 + \beta_1(a)V^*y_1 + \beta_1(a)S^* - (\kappa + \sigma)y_2, \\
 & \frac{dy_3(a)}{da} = -\omega y_3 + \kappa y_2 - (\psi + \gamma_1)y_3, \\
 & \frac{dy_4(a)}{da} = -\omega y_4 + \sigma y_2 - \alpha y_4, \\
 & \frac{dy_5(a)}{da} = -\omega y_5 + \psi y_3 + \alpha y_4 - \gamma_2 y_5
 \end{aligned}$$

and,

$$(45) \quad 1 = \int_0^A \beta_2(a)[y_3(a) + \eta y_5(a)]N_\infty(a)da.$$

Solving system (44) gives

$$\begin{aligned}
 (46) \quad & y_1(a) = \int_0^a -\beta_1(u)S^*(u)e^{-\omega(a-u)}e^{\int_u^a \beta_1(u_1)V^*(u_1)du_1}du, \\
 & y_2(a) = \int_0^a [\beta_1(u)S^*(u) + \beta_1(u)V^*(u)y_1(u)]e^{-(\omega+\kappa+\sigma)(a-u)}du, \\
 & y_3(a) = \int_0^a \kappa y_2(u)e^{-(\omega+\psi+\gamma_1)(a-u)}du, \\
 & y_4(a) = \int_0^a \sigma y_2(u)e^{-(\omega+\alpha)(a-u)}du, \\
 & y_5(a) = \int_0^a [\psi y_3(u) + \alpha y_4(u)]e^{-(\omega+\gamma_2)(a-u)}du.
 \end{aligned}$$

Using the equations in (46) into (45) gives

$$(47) \quad 1 = \int_0^A \beta_2(a)[y_3(a) + \eta y_5(a)]N_\infty(a)da.$$

Define,

$$(48) \quad \mathcal{Q}(\omega) = \int_0^A \beta_2(a)[y_3(a) + \eta y_5(a)]N_\infty(a)da.$$

It should be noted that (from system (46) with (47)) the function $\mathcal{Q}(\omega)$ satisfies the following:

- (i) $\mathcal{Q}(\omega)$ is a monotone decreasing function in ω ;
- (ii) $\lim_{\omega \rightarrow \infty} \mathcal{Q}(\omega) = 0$.

Now consider $\mathcal{P}(\omega) = \mathcal{Q}(\omega) - 1$, it is clear that if $\mathcal{Q}(0) < 1$, then $\mathcal{P}(\omega)$ has a negative root. Thus the following result is established.

Theorem 6. *The unique endemic equilibrium of the model (5) with (20) is LAS whenever $\mathcal{R}_0 > 1$ and $\mathcal{Q}(0) < 1$.*

The consequence of Theorem 6 is that the disease will persist (become endemic) in the community whenever $\mathcal{R}_0 > 1$ and $\mathcal{Q}(0) < 1$. In summary, it is shown that the model (5) with (20) has a globally-asymptotically stable DFE in $\mathcal{D}_0$ whenever the associated reproduction number $\mathcal{R}_0$ is less than unity. Furthermore, the model has a unique endemic equilibrium whenever the reproduction number exceeds unity. This equilibrium is shown to be locally-asymptotically stable if another condition holds ($\mathcal{Q}(0) < 1$).

5. Numerical simulation

The age-structured model (2) is now analysed by considering a special case of the model where the natural death rate is constant (i.e., μ is independent of age) and the contact rate ($\beta(a, b)$) is constant (denoted by $\beta(a, b) = \beta$; see also [2, 6, 7]). Consider, now, the model (2) with $\mu(a)$ replaced by the constant μ and $\beta(a, b)$ by a constant β , then it can be shown by following [25] that the model becomes

$$\begin{aligned}
 \frac{d\tilde{S}}{dt} &= \mu N_0 - [\mu + \tilde{\lambda}(t)]\tilde{S}(t), \\
 \frac{d\tilde{E}}{dt} &= \tilde{\lambda}(t)\tilde{S}(t) - (\mu + \kappa + \sigma)\tilde{E}(t), \\
 \frac{d\tilde{I}}{dt} &= \kappa\tilde{E}(t) - (\mu + \psi + \gamma_1)\tilde{I}(t), \\
 \frac{d\tilde{Q}}{dt} &= \sigma\tilde{I}(t) - (\mu + \alpha)\tilde{Q}(t), \\
 \frac{d\tilde{H}}{dt} &= \psi\tilde{I}(t) + \alpha\tilde{Q}(t) - (\mu + \gamma_2)\tilde{H}(t), \\
 \frac{d\tilde{R}}{dt} &= \gamma_1\tilde{I}(t) + \gamma_2\tilde{H}(t) - \mu\tilde{R}(t),
 \end{aligned}
 \tag{49}$$

where,

$$\tilde{\lambda}(t) = \beta(\tilde{I} + \eta\tilde{H}).
 \tag{50}$$

Define the following positively-invariant region for the model (49):

$$\mathcal{D}_c = \left\{ (\tilde{S}, \tilde{E}, \tilde{I}, \tilde{Q}, \tilde{H}, \tilde{R}) \in \mathbb{R}_+^6 : \tilde{S} + \tilde{E} + \tilde{I} + \tilde{Q} + \tilde{H} + \tilde{R} \leq N_0 \right\}.$$

The DFE of the model (49) is given by

$$\mathcal{E}_{0c} = (N_0, 0, 0, 0, 0, 0),
 \tag{51}$$

and the associated reproduction number of the model (49) is given by

$$(52) \quad \mathcal{R}_c = \frac{\beta[\kappa k_3 k_4 + \eta(\psi \kappa k_3 + \alpha \sigma k_2)]}{k_1 k_2 k_3 k_4},$$

where, $k_1 = \mu + \kappa + \sigma$, $k_2 = \mu + \gamma_1 + \psi$, $k_3 = \mu + \alpha$ and $k_4 = \mu + \gamma_2$. It should be mentioned that since the model (49) is a special case of the model (2), it follows that its DFE, $\mathcal{E}_{0c}$, is globally-asymptotically stable whenever $\mathcal{R}_c < 1$ (in line with Theorem 4).

The model (49) is simulated using the parameter values tabulated in Table 2 (until otherwise stated) to gain insight into its quantitative features.

Numerical simulations for the case $R_c < 1$ depicted in Figure 1 show that the combined use of quarantine and isolation can lead to disease elimination (All solutions converged to the DFE) in line with Theorem (4).

Figure 2 depicts the solution profile of the model for the case $R_c > 1$ showing convergence to the EEP in line with Theorem (6), this mean the disease will persist in this case.

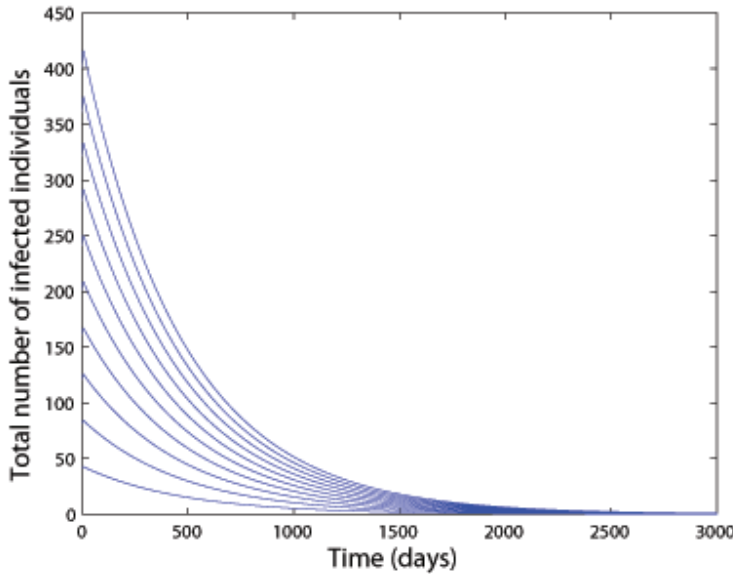


Figure 1: Simulation of the model (49) showing the total number of infected individuals as a function of time for $\mathcal{R}_c < 1$. Parameter values used are as in Table 2 with $\beta = 0.1$ and $\eta = 0.5$ (so that, $\mathcal{R}_c = 0.8065$.)

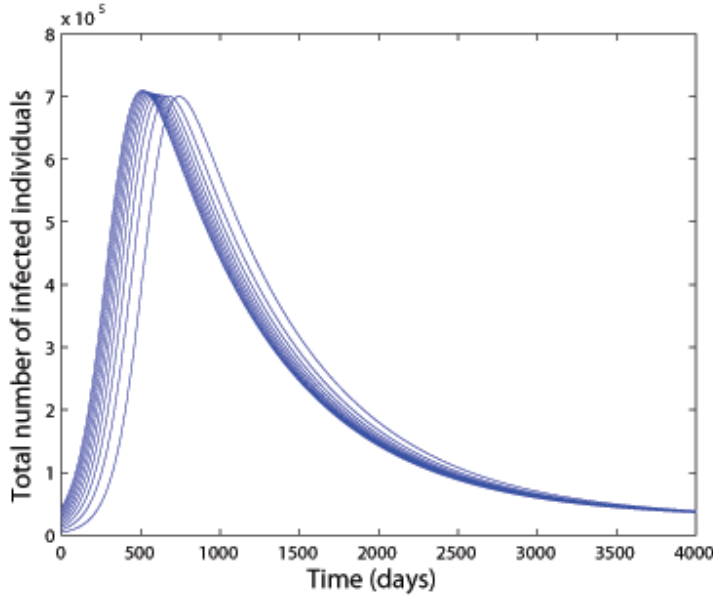


Figure 2: Simulation of the model (49) showing the total number of infected individuals as a function of time for $\mathcal{R}_c > 1$. Parameter values used are as in Table 2 with $\beta = 0.15$ and $\eta = 0.5$ (so that, $\mathcal{R}_c = 1.2097$).

Conclusions

A new age-structured model for disease transmission, subject to the use of quarantine (of asymptomatic cases) and isolation (of individuals with disease symptoms), is presented and rigorously analyzed. The study shows the following:

- (i) The model is shown to be properly-posed mathematically, by formulating it as an abstract Cauchy problem;
- (ii) It is shown, for the case where the effective contact rate is separable (i.e., $\beta(a, b) = \beta_1(a)\beta_2(b)$), that the disease-free equilibrium of the model is locally- and globally-asymptotically stable whenever a certain epidemiological threshold is less than unity.
- (iii) The model has a unique endemic equilibrium when the threshold exceeds unity (this equilibrium is locally-asymptotically stable when another condition holds).
- (iv) Numerical simulations show disease elimination whenever its associated reproduction number is less than unity and the disease will persist in this case whenever its associated reproduction number exceeds unity.

Variable	Description
$S(t)$	Population of susceptible individuals
$E(t)$	Population of exposed individuals
$I(t)$	Population of infectious (symptomatic) individuals
$Q(t)$	Population of quarantined individuals
$H(t)$	Population of hospitalized individuals
$R(t)$	Population of recovered individuals

Parameter	Description
$\mu(a)$	Age-dependent natural death rate
$\beta(a, b)$	Age-dependent Effective contact rate
η	Modification parameter for reduction in infectiousness of hospitalized individuals
κ	Progression rate from exposed to infectious class
σ	Quarantine rate of exposed individuals
α	Hospitalization rate of quarantined individuals
ϕ	Hospitalization rate of infectious individuals
γ_1	Recovery rate of infectious individuals
γ_2	Recovery rate of hospitalized individuals

Table 1: Description of variables and parameters of the model (2).

Parameters	Values (per day)
β	[0.1, 0.2]
μ	0.0000351
γ_1	0.03521
γ_2	0.042553
δ_1	0.04227
δ_2	0.027855
κ	0.156986
α	0.156986
ϕ	0.20619
Π	136
σ	0.1
ψ	0.5
η	(0,1]

Table 2: Estimated values for the parameters of the model (49).

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BANACH AND KANNAN CONTRACTIONS ON S -METRIC SPACE

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Abstract. Unique fixed points are obtained for Banach and Kannan contractions on an S -metric space. Also, the unique fixed points are shown to be S -contractive fixed points.

Keywords: S -metric space, S -Cauchy sequence, fixed point, S -contractive fixed point.

1. Introduction

Let X be a nonempty set. Sedghi et al [10] introduced an S -metric $S : X \times X \times X \rightarrow [0, \infty)$ on X satisfying the following conditions:

(S1) $S(x, y, z) = 0$ if and only if $x, y, z \in X$ with $x = y = z$,

(S2) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$ for all $x, y, z, a \in X$.

The pair (X, S) is called an S -metric space. We obtain from Axiom (S2) that

$$(1.1) \quad S(x, x, y) = S(y, y, x) \text{ for all } x, y \in X.$$

Definition 1.1. A sequence $\langle x_n \rangle_{n=1}^{\infty}$ in a S -metric space (X, S) is said to be S -convergent, if there exists a point x in X such that $S(x_n, x_n, x) \rightarrow 0$ as $n \rightarrow \infty$.

Definition 1.2. A sequence $\langle x_n \rangle_{n=1}^{\infty}$ in a S -metric space (X, S) is said to be S -Cauchy, if $\lim_{n, m \rightarrow \infty} S(x_n, x_n, x_m) = 0$.

Definition 1.3. The space (X, S) is said to be S -complete, if every S -Cauchy sequence in X converges in it.

In his introductory work on S -metric space, Sedghi et al [10] proved the following Banach's contraction mapping theorem:

Theorem 1.1. Let f be a self-map on a complete S -metric space (X, S) such that

$$(1.2) \quad S(fx, fx, fy) \leq \alpha S(x, x, y) \text{ for all } x, y \in X,$$

where $0 < \alpha < 1$. Then f has a unique fixed point.

As an interesting application of the infimum property of the real numbers, unique fixed points of Banach and Kannan contractions are obtained in S -metric space. Then, an S -contractive fixed point is introduced. Also, the unique fixed points for these contractions are shown to be S -contractive fixed points.

2. Main results

The well-known infimum property of real numbers states that, a nonempty and bounded set of real numbers has an infimum in $\mathbb{R}$. In particular,

Lemma 2.1. *If S is a nonempty subset of nonnegative real numbers, then $\alpha = \inf S \geq 0$ and $\lim_{n \rightarrow \infty} p_n = \alpha$ for some sequence $\langle p_n \rangle_{n=1}^\infty$ in S .*

The following proof of Theorem 1.1 differs from that of [10]. In fact, we employ Lemma 2.1, without the concern of iterations:

Step 1 – Existence of the infimum

Define $A = \{S(fx, fx, x) : x \in X\}$. Then by Lemma 2.1, A has an infimum $a \geq 0$, by the infimum property.

Step 2 – Vanishing infimum

If $a > 0$, writing $y = fx$ in (1.2) and using (1.1), we get

$$S(f^2x, f^2x, fx) = S(fx, fx, f^2x) \leq \alpha S(x, x, fx) = \alpha S(fx, fx, x)$$

for all $x \in X$. This implies that $0 < a \leq \alpha a < a$, which is again a contradiction. Hence $a = 0$.

Step 3 – Existence of a sequence

Hence, there exists a sequence $\langle x_n \rangle_{n=1}^\infty$ in X such that

$$(2.1) \quad S(fx_n, fx_n, x_n) \in A \text{ for all } n = 1, 2, 3, \dots \text{ and } \lim_{n \rightarrow \infty} S(fx_n, fx_n, x_n) = 0.$$

Step 4 – $\langle x_n \rangle_{n=1}^\infty$ is S -Cauchy

In fact, by (S2) and (1.1), we have

$$\begin{aligned} S(x_n, x_n, x_m) &\leq S(x_n, x_n, fx_n) + S(x_n, x_n, fx_n) + S(x_m, x_m, fx_n) \\ &= 2S(x_n, x_n, fx_n) + S(x_m, x_m, fx_n) \\ &\leq 2S(x_n, x_n, fx_n) + S(x_m, x_m, fx_m) + S(x_m, x_m, fx_m) \\ &\quad + S(fx_n, fx_n, fx_m) \\ (2.2) \quad &= 2[S(x_n, x_n, fx_n) + S(x_m, x_m, fx_m)] + S(fx_n, fx_n, fx_m). \end{aligned}$$

Now, with $x = x_n$ and $y = x_m$, (1.2) gives,

$$S(fx_n, fx_n, fx_m) \leq \alpha S(x_n, x_n, x_m)$$

Inserting this in (2.2), we get

$$S(x_n, x_n, x_m) \leq 2[S(x_n, x_n, fx_n) + S(x_m, x_m, fx_m)] + \alpha S(x_n, x_n, x_m)$$

or

$$(1 - \alpha)S(x_n, x_n, x_m) \leq 2[S(x_n, x_n, fx_n) + S(x_m, x_m, fx_m)]$$

so that

$$S(x_n, x_n, x_m) \leq \left(\frac{2}{1-\alpha}\right) [S(x_n, x_n, fx_n) + S(x_m, x_m, fx_m)].$$

Applying the limit as $m, n \rightarrow \infty$ in this and using (2.1) we obtain that $\langle x_n \rangle_{n=1}^\infty$ is a S -Cauchy sequence in X .

Step 5 – S -convergence

Since, X is S -complete, we find the point p in X such that

$$(2.3) \quad \lim_{n \rightarrow \infty} x_n = p.$$

Step 6 – S -convergent limit as a fixed point

Again repeatedly using (S2),

$$\begin{aligned} S(fp, fp, p) &\leq S(fp, fp, fx_n) + S(fp, fp, fx_n) + S(p, p, fx_n). \\ &= 2S(fp, fp, fx_n) + S(p, p, fx_n) \\ (2.4) \quad &= 2S(fx_n, fx_n, fp) + S(fx_n, fx_n, p) \end{aligned}$$

Now, from (1.2) with $x = x_n$ and $y = p$, it follows that

$$(2.5) \quad S(fx_n, fx_n, fp) \leq \alpha S(x_n, x_n, p)$$

Substituting (2.5) in (2.4), we get

$$S(fp, fp, p) \leq 2\alpha S(x_n, x_n, p) + S(fx_n, fx_n, p)$$

In the limiting case as $n \rightarrow \infty$, this in view of (2.1) and (2.3) implies $S(fp, fp, p) = 0$ or $fp = p$. Thus p is a fixed point.

Step 7 – Uniqueness of the fixed point

Let q be another fixed point of f . Then, (1.2) with $x = p$ and $y = q$ gives

$$S(p, p, q) = S(fp, fp, fq) \leq \alpha S(p, p, q) \text{ or } (1 - \alpha)S(p, p, q) \leq 0$$

so that $p = q$. That is, p is the unique fixed point of f .

3. S -contractive fixed point

The notion of 2-metric space was introduced by Gähler [1] and a G -metric space was introduced by Mustafa and Sims in [2], as generalizations of a metric space. In these settings, contractive fixed points were introduced in [3] and [4] respectively. For further study on this idea, one can refer to [5, 6, 7, 8, 9].

Now we introduce a contractive fixed point in an S metric space as follows:

Definition 3.1. Let f be a self-map on an S -metric space (X, S) . A fixed point p of f is a contractive fixed point, if for every $x_0 \in X$, the f -orbit $O_f(x_0) = \langle x_0, fx_0, \dots, f^n x_0, \dots \rangle$ converges to p .

We now show that the unique fixed point p is an S -contractive fixed point. In fact, writing $x = f^{n-1}x_0, y = p$ in (1.2), we get

$$(3.1) \quad S(f^n x, f^n x, fp) = S(f^n x, f^n x, p) \leq \alpha S(f^{n-1}x, f^{n-1}x, p) \text{ for } n \geq 1.$$

Proceeding the limit as $n \rightarrow \infty$ in (3.1), we get $S(f^n x, f^n x, p) \rightarrow 0$. Thus $f^n x_0 \rightarrow p$ for each $x_0 \in X$. Thus p is a S -contractive fixed point of f .

Our next result is:

Theorem 3.1. Let f be a self-map on a complete S -metric space (X, S) such that

$$(3.2) \quad S(fx, fx, fy) \leq \alpha[S(fx, fx, x) + S(fy, fy, y)]$$

for all $x, y \in X$, where $0 \leq \alpha < 1/3$. Then f has a unique fixed point p , which is an S -contractive fixed point.

A unique fixed point p for (3.2) is obtained, similar to the previous proof and is omitted here. Now we show that p is an S -contractive fixed point of (3.2), as follows:

Writing $x = f^{n-1}x_0, y = p$ in (3.2), and then using (1.1), we get

$$\begin{aligned} S(f^n x, f^n x, fp) &= S(f^n x, f^n x, p) \\ &\leq \alpha[S(f^n x, f^n x, f^{n-1}x) + S(fp, fp, p)] \\ &= \alpha S(f^n x, f^n x, f^{n-1}x) \\ &\leq \alpha[S(f^n x, f^n x, p) + S(f^n x, f^n x, p) + S(f^{n-1}x, f^{n-1}x, fp)] \\ &= \alpha[2S(f^n x, f^n x, p) + S(f^{n-1}x, f^{n-1}x, fp)] \end{aligned}$$

or

$$(3.3) \quad S(f^n x, f^n x, p) \leq \left(\frac{\alpha}{1-2\alpha} \right) S(f^{n-1}x, f^{n-1}x, p) \rightarrow 0, \text{ as } n \rightarrow \infty,$$

since $\alpha/(1-2\alpha)$ is less than 1. Thus $f^n x_0 \rightarrow p$ for each $x_0 \in X$ so that p is a S -contractive fixed point of f .

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FIXED POINT THEOREMS FOR ONE AND TWO SELF-MAPS ON A G -METRIC SPACE

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Abstract. The proof of a recent result of Vats et al is presented, by employing the infimum property of nonnegative real numbers. Then the unique fixed point is shown to be the G -limit of all the orbits of the form $x, fx, \dots, f^n x, \dots, x \in X$. That is, the unique fixed point is a G -contractive fixed point. Further, a fixed point for a pair of self-maps is obtained as another application of the infimum property.

Keywords: the infimum property, G -metric space, G -Cauchy sequence, fixed point, G -contractive fixed point.

1. Introduction

Let X be a nonempty set and $G : X \times X \times X \rightarrow \mathbb{R}$ such that

- (G1) $G(x, y, z) \geq 0$ for all $x, y, z \in X$ with $G(x, y, z) = 0$ if $x = y = z$,
- (G2) $G(x, x, y) > 0$ for all $x, y \in X$ with $x \neq y$,
- (G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$,
- (G4) $G(x, y, z) = G(x, z, y) = G(y, x, z) = G(z, x, y) = G(y, z, x) = G(z, y, x)$
for all $x, y, z \in X$
- (G5) $G(x, y, z) \leq G(x, w, w) + G(w, y, z)$ for all $x, y, z, w \in X$.

Mustafa and Sims [2] introduced the pair (X, G) as a G -metric space with G -metric G on X . Axioms (G5) is known as the *rectangle inequality* (of G). Note that

$$(1.1) \quad G(x, y, y) \leq 2G(x, x, y) \text{ for all } x, y \in X.$$

Given a G -metric space (X, G) , define

$$(1.2) \quad \rho_G(x, y) = G(x, y, y) + G(x, x, y) \text{ for all } x, y, z \in X.$$

Then it is seen in [2] that ρ_G is a metric on X , and that the family of all G -balls $\{B_G(x, r) : x \in X, r > 0\}$ is the base topology, called the G -metric topology $\tau(G)$ on X , where $B_G(x, r) = \{y \in X : G(x, y, y) < r\}$. Further, it was shown that the G -metric topology coincides with the metric topology induced by the metric ρ_G , which allows us to readily transform many concepts from metric spaces into the setting of G -metric space.

Definition 1.1. A sequence $\langle x_n \rangle_{n=1}^\infty$ in a G -metric space (X, G) is said to be G -convergent with limit $p \in X$, if it converges to p in the G -metric topology $\tau(G)$.

Lemma 1.1 ([2]). *The following statements are equivalent in a G -metric space (X, G) :*

- (a) $\langle x_n \rangle_{n=1}^\infty \subset X$ is G -convergent with limit $p \in X$,
- (b) $\lim_{n \rightarrow \infty} G(x_n, x_n, p) = 0$,
- (c) $\lim_{n \rightarrow \infty} G(x_n, p, p) = 0$.

Definition 1.2. A sequence $\langle x_n \rangle_{n=1}^\infty$ in a G -metric space (X, G) is said to be G -Cauchy, if $G(x_n, x_m, x_m) \rightarrow 0$ as $n, m \rightarrow \infty$.

Definition 1.3. A G -metric space (X, G) is said to be G -complete, if every G -Cauchy sequence in X converges in it.

The infimum property of real numbers states that if $S \subset \mathbb{R}$ be nonempty and bounded, then $\alpha = \inf S$ exists in $\mathbb{R}$. In particular, we have

Lemma 1.2. *If S is a nonempty subset of nonnegative real numbers, then $\alpha = \inf S \geq 0$ and $\lim_{n \rightarrow \infty} p_n = \alpha$ for some sequence $\langle p_n \rangle_{n=1}^\infty$ in S .*

In this paper, two applications of Lemma 1.2 are presented: The first one is to obtain a unique fixed point for a contraction type of Vats et al [7]. This is shown to be a G -contractive fixed point. The second is to obtain a common fixed point for a pair of self-maps.

2. Main result

Vats et al [7] proved the following fixed point theorem:

Theorem 2.1. *Suppose that (X, G) is a complete G -metric space and f be a self-map on X satisfying*

$$(2.1) \quad \begin{aligned} G(fx, fy, fz) \leq k \max \{ & G(x, fx, fx) + G(y, fy, fy) + G(z, fz, fz), \\ & G(x, fy, fy) + G(y, fx, fx) + G(z, fy, fy), \\ & G(x, fz, fz) + G(z, fx, fx) + G(y, fz, fz) \} \\ & \text{for all } x, y, z \in X, \end{aligned}$$

where $0 < k < 1/4$. Then f will have a unique fixed point.

Let $x_0 \in X$ be arbitrary. The proof of Theorem 2.1 in [7] initiates with an orbit $x_0, fx_0, \dots, f^n x_0, \dots$. In fact, $\langle f^n x_0 \rangle_{n=1}^\infty$ is a G -Cauchy in X . Since X is G -complete, $f^n x_0 \rightarrow t$ as $n \rightarrow \infty$ for some $t \in X$. Then, t is shown to be a unique fixed point of f .

Applying Lemma 1.2, we obtain below a fixed point of f satisfying (2.1), which is independent of f -iterations.

Proof. Let $S = \{G(x, fx, fx) : x \in X\}$. In view of Lemma 1.2, S has the infimum, say $a \geq 0$. If $a > 0$, from (2.1) with $y = fx$ and $z = fx$ and the rectangle inequality (G5), we have

$$\begin{aligned} G(fx, f^2x, f^2x) &\leq k \max \{G(x, fx, fx) + G(fx, f^2x, f^2x) + G(fx, f^2x, f^2x), \\ &\quad G(x, fx, fx) + G(fx, fx, fx) + G(fx, f^2x, f^2x), \\ &\quad G(x, f^2x, f^2x) + G(fx, f^2x, f^2x) + G(fx, f^2x, f^2x)\} \\ &\leq k \max \{G(x, fx, fx) + 2G(fx, f^2x, f^2x), \\ &\quad G(x, fx, fx) + G(fx, f^2x, f^2x), \\ &\quad [G(x, fx, fx) + G(fx, f^2x, f^2x)] + 2G(fx, f^2x, f^2x)\} \\ &= k[G(x, fx, fx) + 3G(fx, f^2x, f^2x)], \end{aligned}$$

or

$$(2.2) \quad G(fx, f^2x, f^2x) \leq \frac{k}{1-3k} G(x, fx, fx).$$

Since $k/(1-3k)$ is less than 1, from (2.2), it would follow that

$$G(fx, f^2x, f^2x) < a \text{ where } G(fx, f^2x, f^2x) \in S.$$

In other words, a cannot be a lower bound of S , which is a contradiction.

Therefore, $a = \inf S = 0$, and hence we can choose the points $x_1, x_2, \dots, x_n, \dots$ in X such that

$$(2.3) \quad G(x_n, fx_n, fx_n) \in S \text{ for } n = 1, 2, 3, \dots \text{ and } \lim_{n \rightarrow \infty} G(x_n, fx_n, fx_n) = 0.$$

Now repeated use of (G5) followed by (1.1), we get

$$\begin{aligned} (2.4) \quad G(x_n, x_m, x_m) &\leq G(x_n, fx_n, fx_n) + G(fx_n, x_m, x_m) \\ &\leq G(x_n, fx_n, fx_n) + [G(fx_n, fx_m, fx_m) + G(fx_m, x_m, x_m)] \\ &\leq G(x_n, fx_n, fx_n) \\ &\quad + [G(fx_n, fx_m, fx_m) + 2G(x_m, fx_m, fx_m)]. \end{aligned}$$

Now writing $x = x_n, y = z = x_m$ in (2.1) and using (G5) and (1.1), and then simplifying, we get

$$G(fx_n, fx_m, fx_m) \leq \frac{k}{1-3k} G(x_n, fx_n, fx_n) + \frac{2k}{1-3k} G(x_m, fx_m, fx_m).$$

Inserting this in (2.4), we get

$$\begin{aligned} G(x_n, x_m, x_m) &\leq G(x_n, f x_n, f x_n) + \frac{k}{1-3k} G(x_n, f x_n, f x_n) \\ &\quad + \frac{2k}{1-3k} G(x_m, f x_m, f x_m) + 2G(x_m, f x_m, f x_m). \end{aligned}$$

Employing the limit as $m, n \rightarrow \infty$ in this and using (2.3), it follows that $\langle x_n \rangle_{n=1}^\infty$ is G -Cauchy. Since X is G -complete,

$$(2.5) \quad \lim_{n \rightarrow \infty} x_n = p \text{ for some } p \in X.$$

Again, by repeated application of rectangle inequality (G5), we have

$$\begin{aligned} (2.6) \quad G(p, fp, fp) &\leq G(p, f x_n, f x_n) + G(f x_n, fp, fp) \\ &\leq [G(p, x_n, x_n) + G(x_n, f x_n, f x_n)] + G(f x_n, fp, fp). \end{aligned}$$

Now, (2.1) with $x = x_n, y = z = p$ implies

$$\begin{aligned} G(f x_n, fp, fp) &\leq k \max \{ G(x_n, f x_n, f x_n) + G(p, fp, fp) + G(p, fp, fp), \\ &\quad G(x_n, fp, fp) + G(p, f x_n, f x_n) + G(p, fp, fp), \\ &\quad G(x_n, fp, fp) + G(p, fp, fp) + G(p, f x_n, f x_n) \} \\ &= kM, \end{aligned}$$

where

$$\begin{aligned} M &= \max \{ G(x_n, f x_n, f x_n) + 2G(p, fp, fp), \\ &\quad G(x_n, fp, fp) + G(p, fp, fp) + G(p, f x_n, f x_n) \}. \end{aligned}$$

With this, (2.6) becomes

$$(2.7) \quad G(p, fp, fp) \leq G(p, x_n, x_n) + G(x_n, f x_n, f x_n) + kM.$$

Case (a). Let $M = G(x_n, f x_n, f x_n) + 2G(p, fp, fp)$. Then (2.7) can be written as

$$G(p, fp, fp) \leq G(p, x_n, x_n) + G(x_n, f x_n, f x_n) + k[G(x_n, f x_n, f x_n) + 2G(p, fp, fp)]$$

or

$$(2.8) \quad G(p, fp, fp) \leq \left(\frac{1}{1-2k} \right) G(p, x_n, x_n) + \left(\frac{1+k}{1-2k} \right) G(x_n, f x_n, f x_n).$$

Proceeding the limit as $n \rightarrow \infty$ in (2.8), and then using (2.3), (2.5) and Lemma 1.1, we get $0 \leq G(p, fp, fp) = 0$ or $fp = p$. That is p is a fixed point of f .

Case (b). Let $M = G(x_n, fp, fp) + G(p, fp, fp) + G(p, fx_n, fx_n)$. Then (2.7) can be written as

$$\begin{aligned} G(p, fp, fp) &\leq G(p, x_n, x_n) + G(x_n, fx_n, fx_n) \\ &\quad + k[G(x_n, fp, fp) + G(p, fp, fp) + G(p, fx_n, fx_n)] \\ &\leq G(p, x_n, x_n) + G(x_n, fx_n, fx_n) \\ &\quad + k[G(x_n, p, p) + G(p, fp, fp)] \\ &\quad + G(p, fp, fp) + G(x_n, fx_n, fx_n) + G(p, x_n, x_n) \end{aligned}$$

$$\text{or } G(p, fp, fp) \leq \frac{1+k}{1-2k}[G(p, x_n, x_n) + G(x_n, fx_n, fx_n)] + \frac{k}{1-2k}G(x_n, p, p),$$

Proceeding the limit as $n \rightarrow \infty$ in this, and using (2.3), (2.5) and Lemma 1.1, we obtain that $0 \leq G(p, fp, fp) = 0$ or $fp = p$. That is p is a fixed point of f . The uniqueness of the fixed point follows from (2.1) directly. $\square$

Definition 2.1 ([3]). A fixed point p of f on a G -metric space (X, G) is a G -contractive fixed point, if for each $x \in X$, the orbit $O_f(x) = \{x, fx, \dots, f^n x, \dots\}$ converges to p .

Example 2.1. Let $X = [0, \infty)$ with $G(x, y, z) = 0$ if $x = y = z$, $\max\{x, y, z\}$ otherwise. Define $fx = 0$ if $0 \leq x < 1/2$, qx otherwise, for all $x \in X$, where $0 \leq q < 1$. Then, we see that 0 is the unique fixed point of f and for each $x \in X$, the f -orbit $O_f(x) = \{x, qx, q^2x, \dots, q^n x, \dots\}$ converges to 0. That is, 0 is a G -contractive fixed point of f .

We now show that the unique fixed point p is a G -contractive fixed point. Indeed, taking $y = z = p$ in (2.1) and using (G5), we get

$$\begin{aligned} G(f^n x, p, p) &= G(f^n x, fp, fp) \\ &\leq k \max \{ G(f^{n-1}x, f^n x, f^n x) + G(p, fp, fp) + G(p, fp, fp), \\ &\quad G(f^{n-1}x, fp, fp) + G(p, f^n x, f^n x) + G(p, fp, fp), \\ &\quad G(f^{n-1}x, fp, fp) + G(p, fp, fp) + G(p, f^n x, f^n x) \} \\ &= k \max \{ G(f^{n-1}x, f^n x, f^n x) + 0, \\ &\quad G(f^{n-1}x, p, p) + G(p, f^n x, f^n x) + 0, \\ &\quad G(f^{n-1}x, p, p) + 0 + G(p, f^n x, f^n x) \} \\ &= k \max \{ G(f^{n-1}x, f^n x, f^n x), \\ &\quad G(f^{n-1}x, p, p) + G(p, f^n x, f^n x) + 0 \} \\ &\leq k \max \{ G(p, f^n x, f^n x) \\ &\quad + G(f^{n-1}x, p, p), G(f^{n-1}x, p, p) + 2G(p, p, f^n x) \} \\ &\leq k[2G(p, p, f^n x) + G(f^{n-1}x, p, p)] \end{aligned}$$

or

$$G(f^n x, p, p) \leq c \cdot G(f^{n-1}x, p, p).$$

where $c = k/(1 - 2k)$ is less than 1, by the choice of k . By induction, we have

$$G(f^n x, p, p) \leq c^n G(fx, p, p),$$

which as $n \rightarrow \infty$ gives $f^n x \rightarrow p$ for each $x \in X$, in view of Lemma 1.1. Thus p is a G -contractive fixed point of f . For the G -contractive fixed points under other contraction type conditions, one may refer to [4], [5] and [6].

The following example illustrates Theorem 2.1:

Example 2.2. Let $X = [0, 1]$, with G -metric $G(x, y, z) = \max\{|x - y|, |y - z|, |z - x|\}$. Then (X, d) is a complete G -metric space. Define $f : X \rightarrow X$ by

$$fx = \begin{cases} \frac{x}{12}, & 0 \leq x < \frac{1}{2} \\ \frac{x}{10}, & \frac{1}{2} \leq x \leq 1. \end{cases}$$

We show that f satisfies the inequality (2.1) with $k = 1/5$ in four different cases:

Case (i). Let $x, y, z \in [0, 1/2)$. Then

$$\begin{aligned} G(fx, fy, fz) &= \frac{1}{12} \max\{|x - y|, |y - z|, |z - x|\}, \\ G(x, fx, fx) &= 11x/12, G(y, fy, fy) = 11y/12, G(z, fz, fz) = 11z/12. \end{aligned}$$

Suppose that $x < y < z$. Then

$$G(fx, fy, fz) = \frac{1}{12}(z - x) \leq \frac{1}{5} \frac{5z}{12} \leq \frac{1}{5} \cdot \frac{11z}{12} = \frac{1}{5} G(z, fz, fz),$$

By symmetry, the subcases $y < z < x$ and $z < x < y$ can be handled.

Case (ii). Let $x, y, z \in [1/2, 1]$. Then

$$\begin{aligned} G(fx, fy, fz) &= \frac{1}{10} \max\{|x - y|, |y - z|, |z - x|\}, \\ G(x, fx, fx) &= 9x/10, G(y, fy, fy) = 9y/10, G(z, fz, fz) = 9z/10. \end{aligned}$$

Therefore,

$$G(fx, fy, fz) = \frac{1}{10}(z - x) \leq \frac{1}{5} \frac{5z}{10} \leq \frac{1}{5} \cdot \frac{9z}{10} = \frac{1}{5} G(z, fz, fz).$$

By symmetry, the subcases $y < z < x$ and $z < x < y$ can be handled.

Case (iii). Let $x, y \in [0, 1/2)$, $z \in [1/2, 1]$. Then

$$\begin{aligned} G(fx, fy, fz) &= \max\left\{\frac{|x - y|}{12}, \frac{z}{10} - \frac{y}{12}, \frac{z}{10} - \frac{x}{12}\right\} \\ &\leq \frac{z}{10} = \frac{1}{5} \frac{5z}{10} \leq \frac{1}{5} \cdot \frac{9z}{10} = \frac{1}{5} G(z, fz, fz). \end{aligned}$$

Case (iv). Let $x, y \in [1/2, 1]$, $z \in [0, 1/2)$. Then

$$\begin{aligned} G(fx, fy, fz) &= \max \left\{ \frac{|x-y|}{10}, \frac{y}{10} - \frac{z}{12}, \frac{x}{10} - \frac{z}{12} \right\} \\ &\leq \max \left\{ \frac{y}{10}, \frac{x}{10} \right\} \\ &\leq \frac{1}{5} \left(\frac{9x}{10} + \frac{9y}{10} \right) = \frac{1}{5} [G(x, fx, fx) + G(y, fy, fy)]. \end{aligned}$$

From all the above cases, we observe that

$$G(fx, fy, fz) \leq \frac{1}{5} [G(x, fx, fx) + G(y, fy, fy) + G(z, fz, fz)] \text{ for all } x, y, z \in X,$$

showing that f satisfies the inequality (2.1) with $k = 1/5$. Note that 0 is unique fixed point of f .

3. Common fixed point theorem for two self-maps

Mustafa et al [1] proved the following result:

Theorem 3.1. Suppose that (X, G) is a complete G -metric space and f , a self-map on X satisfying

$$(3.1) \quad \begin{aligned} G(fx, fy, fz) &\leq aG(x, y, z) + bG(x, fx, fx) + cG(y, fy, fy) \\ &+ eG(z, fz, fz) \text{ for all } x, y, z \in X, \end{aligned}$$

where a, b, c and e are nonnegative real numbers with $a + b + c + e < 1$. Then f will have a unique fixed point.

Writing $a = \alpha$, $b = 0$, $c = \beta$ and $e = \gamma$ in Theorem 3.1, we get

Corollary 3.1. Suppose that (X, G) is a complete G -metric space and f , a self-map on X satisfying

$$(3.2) \quad \begin{aligned} G(fx, fy, fz) &\leq \alpha G(x, y, z) + \beta G(y, fy, fy) \\ &+ \gamma G(z, fz, fz), \text{ for all } x, y, z \in X, \end{aligned}$$

where $\alpha, \beta, \gamma \geq 0$ with $\alpha + \beta + \gamma < 1$. Then f will have a unique fixed point.

As another application of Lemma 1.2, we extend Corollary 3.1 to a pair of self-maps to obtain a common fixed point, as follows:

Theorem 3.2. Let (X, G) be a complete G -metric space and f, g be two self maps such that

$$(3.3) \quad \begin{aligned} G(fx, gy, gz) &\leq \alpha G(x, y, z) + \beta G(y, gy, gy) \\ &+ \gamma G(z, gz, gz), \text{ for all } x, y, z \in X, \end{aligned}$$

where $\alpha, \beta, \gamma \geq 0$ with $\alpha + \beta + \gamma < 1$. Then f and g will have a unique common fixed point.

Proof. Let $S(f) = \{G(x, fx, fx) : x \in X\}$ and $S(g) = \{G(y, gy, gy) : y \in X\}$. Suppose that $a \geq 0$ be the infimum of $S(f) \cup S(g)$.

Case (a). Suppose that $a = \inf S(f)$. If $a > 0$, writing $y = z = fx$ in (3.2), we get

$$G(fx, gfx, gfx) \leq \alpha G(x, fx, fx) + \beta G(fx, gfx, gfx) + \gamma G(fx, gfx, gfx)$$

or

$$(3.4) \quad G(fx, gfx, gfx) \leq \frac{\alpha}{1 - \beta - \gamma} \cdot G(x, fx, fx).$$

Since $\alpha/(1 - \beta - \gamma)$ is less than 1, from (3.4), it follows that $G(fx, gfx, gfx) < a$ for some $x \in X$, which contradicts with the choice of a . Therefore $a = 0$.

Case (b). Suppose that $a = \inf S(g)$. Again, if $a > 0$, writing $x = gy$ and $z = y$ in (3.2), we get

$$G(fgy, gy, gy) \leq \alpha G(gy, y, y) + \beta G(y, gy, gy) + \gamma G(y, gy, gy)$$

or

$$(3.5) \quad G(fgy, gy, gy) \leq (\alpha + \beta + \gamma)G(y, gy, gy) < a,$$

which contradicts with the choice of a . Therefore, $a = 0$. Choose the points $x_1, x_2, \dots, x_n, \dots$ in X such that

$$G(x_n, fx_n, fx_n) \in S(f) \text{ for } n = 1, 2, 3, \dots$$

and

$$(3.6) \quad \lim_{n \rightarrow \infty} G(x_n, fx_n, fx_n) = 0.$$

Similarly we can choose the points $x_1, x_2, \dots, x_m, \dots$ in X such that

$$G(x_m, gx_m, gx_m) \in S(g) \text{ for } n = 1, 2, 3, \dots$$

and

$$(3.7) \quad \lim_{n \rightarrow \infty} G(x_m, gx_m, gx_m) = 0.$$

Repeatedly using the rectangle inequality (G5) and (3.2), we see that

$$(3.8) \quad \begin{aligned} G(x_n, x_m, x_m) &\leq G(x_n, fx_n, fx_n) \\ &+ G(fx_n, gx_m, gx_m) + 2G(x_m, gx_m, gx_m). \end{aligned}$$

Now

$$(3.9) \quad \begin{aligned} G(fx_n, gx_m, gx_m) &\leq \alpha G(x_n, x_m, x_m) \\ &+ \beta G(x_m, gx_m, gx_m) + \gamma G(x_m, gx_m, gx_m). \end{aligned}$$

Substituting (3.9) in (3.8), we obtain

$$\begin{aligned} G(x_n, x_m, x_m) &\leq G(x_n, fx_n, fx_n) + \alpha G(x_n, x_m, x_m) + \beta G(x_m, gx_m, gx_m) \\ &+ \gamma G(x_m, gx_m, gx_m) + 2G(x_m, gx_m, gx_m) \end{aligned}$$

or

$$G(x_n, x_m, x_m) \leq \frac{1}{1-\alpha} \cdot G(x_n, fx_n, fx_n) + \frac{2+\beta+\gamma}{1-\alpha} \cdot G(x_m, gx_m, gx_m)$$

As $m, n \rightarrow \infty$ this in view of (3.6) and (3.7), gives $G(x_n, x_m, x_m) \rightarrow 0$, proving that $\langle x_n \rangle_{n=1}^\infty$ is G -Cauchy.

Since X is G -complete, we can find a point $p \in X$ such that

$$(3.10) \quad \lim_{n \rightarrow \infty} x_n = p.$$

Again by repeated application of rectangle inequality (G5), (G4) and (3.2), we have

$$(3.11) \quad G(p, fp, fp) \leq [G(p, x_m, x_m) + G(x_m, gx_m, gx_m)] + 2G(fp, gx_m, gx_m).$$

But, from (3.2), we have

$$(3.12) \quad \begin{aligned} G(fp, gx_m, gx_m) &\leq \alpha G(p, x_m, x_m) \\ &+ \beta G(x_m, gx_m, gx_m) + \gamma G(x_m, gx_m, gx_m). \end{aligned}$$

Substituting (3.12) in (3.11), and then simplifying, it follows that

$$G(p, fp, fp) \leq (1+2\alpha)G(x_m, x_m, p) + (1+\beta+\gamma)G(x_m, gx_m, gx_m).$$

As $m \rightarrow \infty$ this finally yields $d(p, fp, fp) \leq 0$ or $fp = p$.

Now, using (G5) and writing $x = y = z = p$ in (3.2), we have

$$\begin{aligned} G(fp, gp, gp) &\leq \alpha G(p, p, p) + \beta G(p, gp, gp) + \gamma G(p, gp, gp) \\ &\leq (\beta + \gamma)[G(p, fp, fp) + G(fp, gp, gp)] \end{aligned}$$

or

$$(3.13) \quad (1 - \beta - \gamma)G(fp, gp, gp) \leq (\beta + \gamma)G(p, fp, fp).$$

This further implies that $(1 - \beta - \gamma)G(fp, gp, gp) \leq 0$ or $fp = gp$. Since p is a fixed point of f , p will be a common fixed point of f and g .

Uniqueness: Suppose q is another common fixed of f and g . That is $fq = q$ and $gq = q$. Then from (3.2), we have

$$G(p, q, q) = G(fp, gq, gq) \leq \alpha G(p, q, q) + \beta G(q, gq, gq) + \gamma G(q, gq, gq) = 0,$$

which implies that $G(p, q, q) \leq 0$ or $p = q$. That is, p is an unique common fixed point of f and g . $\square$

Remark 3.1. Writing $g = f$, (3.3) reduces to (3.2). Hence Theorem 3.2 is an extension of Corollary 3.1 to two self-maps.

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GENERALIZATIONS OF PRIME TERNARY SUBSEMIMODULES OF TERNARY SEMIMODULES

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Abstract. Let R be a commutative semiring with $1 \neq 0$ and all semimodules are unital. Weakly prime ternary subsemimodules of ternary semimodules have been studied. In this paper we introduce the concept of almost prime ternary subsemimodule of a ternary semimodule over a ternary semiring as a new generalization of prime ternary subsemimodule. We will give some of its properties, characteristics and its relationship among other algebraic structures. Also we carry out this concept under multiplication ternary semimodules.

Keywords: almost prime submodule; subtractive ternary subsemimodule, partitioning ternary subsemimodule, weakly prime subsemimodule, quotient ternary semimodule.

1. Introduction

Through out this paper all semirings are commutative with $1 \neq 0$ and all semimodules are unitary. Weakly prime ideals have been introduced by [1], and almost prime ideals were introduced by [2] and studied by [3]. Later on, these concepts has been studied in modules and semirings by many authors [4, 5, 6, 7]. Further they are extended for semimodules by [8, 9]. The concept of weakly prime ternary subsemimodule of a ternary semimodule over a ternary semiring has introduced by [10]. In this paper we introduce The concept of weakly prime ternary subsemimodule of a ternary semimodule over a ternary semiring and obtain some properties and characteristics of almost prime ternary subsemimodules. For definitions of monoid and semiring see [11, 12] and for ternary semiring see [13, 14]. Throughout, all ternary semirings are commutative with

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$1 \neq 0$. Denote that $Z_0^+(N)$ to be the set of all positive integers where as $Z_0^-(N)$ be the set of negative integers. An ideal I of a ternary semiring R is called a subtractive ideal ($=k$ -ideal) if $a, a+b \in I$ and $b \in R$, then $b \in I$. A proper ideal P of a ternary semiring R is said to be prime if $abc \in P$, then either $a \in P$ or $b \in P$, or $c \in P$. A proper ideal P of a ternary semiring R is said to be weakly prime if $0 \neq abc \in P$, then either $a \in P$ or $b \in P$, or $c \in P$. A proper ideal P of a ternary semiring R is said to be almost prime if $abc \in P - P^2$, then either $a \in P$ or $b \in P$, or $c \in P$. Let R be a ternary semiring. A left ternary R -semimodule is a commutative monoid $(M, +)$ with additive identity 0_M where the function $R \times R \times M \rightarrow M$, defined by $(r_1, r_2, x) \mapsto r_1 r_2 x$ called ternary scalar multiplication, which satisfies the following conditions for all elements r_1, r_2, r_3 and r_4 of R and all elements x and y of M :

- (1) $(r_1 r_2 r_3) r_4 x = r_1 (r_2 r_3 r_4) x = r_1 r_2 (r_3 r_4 x)$;
- (2) $r_1 r_2 (x + y) = r_1 r_2 x + r_1 r_2 y$;
- (3) $r_1 (r_2 + r_3) x = r_1 r_2 x + r_1 r_3 x$;
- (4) $(r_1 + r_2) r_3 x = r_1 r_3 x + r_2 r_3 x$;
- (5) $1_R 1_R x = x$;
- (6) $r_1 r_2 0_M = 0_M = 0_R r_2 x = r_1 0_R x$;

Throughout, by a ternary R -semimodule we mean a left ternary semimodule over a ternary semiring R . By [15], every ternary semiring R is ternary $(Z_0^-, +, \cdot)$ -semimodule. A ternary subsemimodule N of a ternary R -semimodule M is called subtractive ternary subsemimodule ($=$ ternary k -subsemimodule) if $x, x+y \in N$, $y \in M$, then $y \in N$.

Since $0 = 0$ is a subtractive ternary subsemimodule of a ternary R -semimodule, then $(0 : m)$ and $(0 : M)$ are subtractive ideals of R where $m \in M$.

Following [16, Theorem 3.4], for any two subtractive ideals I and J of a ternary semiring R , their union is subtractive ideal of R if and only if their union equals one of the subtractive ideals. Following [10], A proper ternary semimodule N of a ternary R -semimodule M is said to be prime if $r_1 r_2 m \in N$, where, $r_1 r_2 \in R$, $m \in M$, then either $r_1 \in (N : M)$ or $r_2 \in (N : M)$, or $m \in N$. Also, a proper ternary subsemimodule N of a ternary R -semimodule M is said to be weakly prime if $0 \neq r_1 r_2 m \in N$, where $r_1, r_2 \in R$ and $m \in M$, then either $r_1 \in (N : M)$ or $r_2 \in (N : M)$, or $m \in N$.

2. Almost prime ternary subsemimodules

In this section we introduce the concept of almost prime ternary subsemimodule of a ternary semimodule over a ternary semiring and obtain some of its properties and characterizations.

Definition 2.1. A proper ternary subsemimodule N of a ternary R -semimodule M is said to be almost prime ternary if, whenever $r_1, r_2 \in R$ and $m \in M$ such that $r_1 r_2 m \in N - (N : M)^2 N$, then either $r_1 \in (N : M)$ or $r_2 \in (N : M)$ or $m \in N$.

Clearly, any weakly prime ternary subsemimodule of ternary R -semimodule is almost prime ternary subsemimodule. However, the convers is not necessarily true, for example we consider $\mathbb{Z}_0^-$ -semimodule $M = \mathbb{Z}_{-24}$ and the proper ternary cyclic subsemimodule N of M generated by $-\bar{8}$, clearly $(N : M)^2 N = N$ and so N is almost prime ternary subsemimodule.

In contrast $\bar{0} \neq (-4)(-4)(-4) \in N$ but $-\bar{4} \notin N$ and $-4 \notin (N : M)$ which is not weakly.

Recall that, a ternary R -semimodule M is called a cancellation ternary R -semimodule if for all ideals I and J of R , $IRM = JRM$ implies that $I = J$. Following [17], let M be a ternary R -semimodule and N be a ternary subsemimodule of M , then N is called idempotent in M if $(N : M)^2 N = N$. Thus, any proper idempotent ternary subsemimodule of M is almost prime ternary subsemimodule of M . If M is a multiplication ternary R -semimodule and N_1, N_2, N_3 are ternary subsemimodules of M such that $N_1 = IRM$, $N_2 = JRM$, $N_3 = KRM$ for some ideals I, J, K of R , then the ternary multiplication of N_1, N_2 and N_3 is defined in [18], $N_1 N_2 N_3 = (IRM)(JRM)(KRM) = (IJK)RM$. In particular, we have

$$N^3 = NNN = [(N : M)RM][(N : M)RM][(N : M)RM] = (N : M)^3 RM.$$

So, a submodule N is idempotent in M if and only if $N = N^3$.

Recall that, a ternary subsemimodule N of a ternary R -semimodule is called a pure ternary subsemimodule if $I^2 N = N \cap IRM$ for any ideal of R . Following [19], we can prove that if N is a pure ternary subsemimodule in a multiplication R -semimodule M with pure ternary annihilator, then N is idempotent in M almost prime ternary.

Next theorem present the relationship between weakly prime ternary subsemimodule and almost prime ternary subsemimodule.

Theorem 2.2. Let M be a ternary R -semimodule and N be a proper subsemimodule of M . Then N is almost prime in M if and only if $N/(N : M)^2 N$ is weakly prime in $M/(N : M)^2 N$.

Proof. Suppose that N is almost prime ternary subsemimodule in M . Let $r_1, r_2 \in R$ and $m \in M$, such that $\bar{0} \neq r_1 r_2 (m + (N : M)^2 N) \in N/(N : M)^2 N$ in $M/(N : M)^2 N$. Then $r_1 r_2 m \in N - (N : M)^2 N$ and so either $r_1 \in (N : M)$ or $r_2 \in (N : M)$ or $m \in N$ because N is almost prime. Hence, either

$$r_1 \in (N : M) = (N/(N : M)^2 N : M/(N : M)^2 N)$$

or

$$r_2 \in (N : M) = (N/(N : M)^2 N : M/(N : M)^2 N)$$

or $m + (N : M)^2N \in N/(N : M)^2N$ and so $N/(N : M)^2N$ is weakly prime in $M/(N : M)^2N$. Conversely, assume that $N/(N : M)^2N$ is weakly prime ternary in $M/(N : M)^2N$ and let $r_1, r_2 \in R$ and $m \in M$ such that $r_1r_2m \in N - (N : M)^2N$. Then

$$\bar{0} \neq r_1r_2(m + (N : M)^2N) \in N/(N : M)^2N$$

and hence either

$$r_1 \in (N/(N : M)RN : M/(N : M)RN) = (N : M)$$

or

$$r_2 \in (N/(N : M)^2N : M/(N : M)^2N) = (N : M)$$

or $m + (N : M)^2N \in N/(N : M)^2N$ (and so $m \in N$) so N is almost prime ternary. $\square$

Note that, we can generalize Definition 2.1 as follows.

Definition 2.3. A proper ternary subsemimodule N of a ternary R -semimodule M is said to be n -almost prime ternary if, whenever $r_1, r_2 \in R$ and $m \in M$ such that $r_1r_2m \in N - (N : M)^{n-1}N$, then either $r_1 \in (N : M)$ or $r_2 \in (N : M)$ or $m \in N$ ($n \geq 3$).

Theorem 2.4. Let M be a ternary R -semimodule and N be a proper ternary subsemimodule of M . Then for $n \geq 3$ the following statements hold.

- (1) For $r \in R - (N : M)$, $(N : (r)) = N \cup ((N : M)^{n-1}N : (r))$.
- (2) For $r \in R - (N : M)$, $(N : (r)) = N$ or $(N : (r)) = ((N : M)^{n-1}N : (r))$.

Proof. (1) Suppose that N is an almost prime ternary subsemimodule such that $r, r' \notin (N : M)$ for all $r' \in R$. Let $m \in (N : (r))$ so that $rr'm \in N$. If $rr'm \notin (N : M)^{n-1}N$, then N being almost prime implies that $m \in N$. Suppose that $rr'm \in (N : M)^{n-1}N$. Then $m \in ((N : M)^{n-1}N : (r))$ and so $(N : (r)) \subseteq N \cup ((N : M)^{n-1}N : (r))$. The other containment holds for any subsemimodule N .

(2) It is well known that if a ternary subsemimodule is the union of two ternary subsemimodule, then it is equal one of them. $\square$

Theorem 2.5. Let N be a proper ternary subsemimodule of a ternary R -semimodule M . Then for $n \geq 3$ the following are equivalent.

- 1) N is n -almost prime ternary subsemimodule of M .
- 2) For any ideal A and B of R and ternary subsemimodule K of M with $ABK \subseteq N - (N : M)^{n-1}N$, we have $A \subseteq (N : M)$ or $B \subseteq (N : M)$ or $K \subseteq N$.

Proof. (1) $\Rightarrow$ (2) Suppose that N is an n -almost prime ternary subsemimodule of M . Let $ABK \subseteq N - (N : M)^{n-1}N$, where A, B are ideals of R and K is a ternary subsemimodule of M , $B \not\subseteq (N : M)$ and $K \not\subseteq N$. Choose $r_2 \in B$ and $x \in K$ such that $r_2 \notin (N : M)$ and $x \notin N$. Let $r_1 \in A$. Then $r_1r_2x \in ABK \subseteq$

$N - (N : M)^{n-1}N$. Since N is n -almost prime ternary subsemimodule, then either $r_1 \in (N : M)$ or $r_2 \in (N : M)$ or $x \in K$, and so $r_1 \in (N : M)$. Hence $I \subseteq (N : M)$.

(2) $\Rightarrow$ (1) Let $r_1, r_2 \in R$ and $m \in M$ such that $r_1 r_2 m \in N - (N : M)^{n-1}N$. Suppose that $A = (r_1) = RRr_1$, $B = (r_2) = RRr_2$ and $K = (m) = RRm$. Then $ABK \subseteq N - (N : M)^{n-1}N$. So either $A \subseteq (N : M)$ or $B \subseteq (N : M)$ or $K \subseteq N$ and hence $r_1 \in (N : M)$ or $r_2 \in (N : M)$ or $m \in N$. Thus N is an n -almost prime ternary subsemimodule of M . $\square$

Definition 2.6. Let M be a ternary R -semimodule, $S(M)$ be the set of all submodules of M , and $\phi : S(M) \rightarrow S(M) \cup \{\emptyset\}$ be a function. Then a proper subsemimodule N of M is called ϕ -prime subsemimodule if for $r_1, r_2 \in R$ and $m \in M$ with $r_1 r_2 m \in N - \phi(N)$, then either $r_1 \in (N : M)$ or $r_2 \in (N : M)$ or $m \in N$.

For example, define $\phi_n : S(M) \rightarrow S(M) \cup \{\emptyset\}$ with $\phi_n(N) = (N : M)^{n-1}N$, for all $N \in S(M)$; ($n \geq 3$). Hence for $n \geq 3$, ϕ_n -prime submodule of M is an n -almost prime ternary subsemimodule and ϕ_2 -almost prime ternary subsemimodule is an almost prime. Recall that, a ternary subsemimodule N of a ternary R -semimodule M is called subtractive ternary subsemimodule(=ternary k -subsemimodule) if $x, x + y \in N$ and $y \in M$, then $y \in N$.

Lemma 2.7 ([16]). Let I and J be subtractive ideals of a ternary semiring R . Then $I \cup J$ is subtractive ideal of R if and only if $I \cup J = I$ or $I \cup J = J$.

Theorem 2.8. Let R be a ternary semiring and M be a ternary R -semimodule. Let N be a proper ternary subtractive subsemimodule of M . Then the following statements are equivalent:

- (1) If $x \in M - N$, then $(N : x) = (N : M) \cup (\phi(N) : x)$;
- (2) If $x \in M - N$, then $(N : x) = (N : M)$ or $(N : x) = (\phi(N) : x)$.

Proof. (1) Let $x \in M - N$ and $r \in (N : x)$. Then $rr'x \in N$ for all $r' \in R$ and so $r1x \in N$. If $r1x \in \phi(N)$, then for all $a \in R$, we have $rax = arx = 1a(r1x) \in \phi(N)$ and so $r \in (\phi(N) : x)$. Suppose that $r1x \notin \phi(N)$. Then either $r \in (N : M)$ or $1 \in (N : M)$ or $x \in N$, because N is ϕ -prime subtractive ternary subsemimodule. But $1 \in (N : M)$ is impossible and $x \notin N$. Thus $r \in (N : M)$. Hence, $(N : x) = (N : M) \cup (\phi(N) : x)$.

(2) Let $(N : x) = (N : M) \cup (\phi(N) : x)$ for $x \in M - N$. Then either $(N : x) = (N : M)$ or $(N : x) = (\phi(N) : x)$ by Lemma 2.7. Therefore, the inclusion follows. $\square$

Corollary 2.9. Let R be a ternary semiring and M be a ternary R -semimodule. Let N be an almost prime subtractive ternary subsemimodule of M . Then the following statements hold.

- (1) If $x \in M - N$, then $(N : x) = (N : M) \cup ((N : M)^2N : x)$.
- (2) If $x \in M - N$, then $(N : x) = (N : M)$ or $(N : x) = ((N : M)^2N : x)$.

Proof. The proof follows from previous Theorem by taking $\phi(N) = (N : M)^2N$. $\square$

Recall that, a ternary R -semimodule M is called a multiplication ternary R -semimodule if for each ternary subsemimodule N of M , there exists an ideal I of R such that $N = IRM$ (or equivalently, $N = (N : M)RM$).

Also, if N is a ternary subsemimodule of an R -semimodule M , then the radical of N (denoted by $M\text{-rad } N$) is defined as the intersection of all prime ternary subsemimodules of M containing N . It is well known that if M is a multiplication ternary R -semimodule, then $M\text{-rad } N = \sqrt{(N : M)RM}$, where $\sqrt{(N : M)}$ denotes the radical of ideal $(N : M)$ in R .

Theorem 2.10. *Let M be a ternary multiplication R -semimodule. If N is a ternary subsemimodule of M , then, $N \subseteq M\text{-rad}((N : M)^2N)$. Moreover, if N is a prime ternary subsemimodule of M , then, $N = M\text{-rad}((N : M)^2N)$*

Proof. As M is a ternary multiplication R -semimodule, then, $M\text{-rad}((N : M)^2N) = \sqrt{((N : M)^2N : M)RM}$.

As $(N : M)^3 \subseteq ((N : M)^2N : M)$, then $(N : M) \subseteq \sqrt{((N : M)^2N : M)}$ and so $N = (N : M)RM \subseteq \sqrt{((N : M)^2N : M)RM} = M\text{-rad}((N : M)^2N)$. Moreover, let N be a prime ternary in M . If $r \in \sqrt{((N : M)^2N : M)}$, then, $r^n \in ((N : M)^2N : M) \subseteq (N : M)$ for some integer n . As $(N : M)$ is prime in R , then $r \in (N : M)$ and so $\sqrt{((N : M)^2N : M)} \subseteq (N : M)$. Therefore, $\sqrt{((N : M)^2N : M)RM} \subseteq (N : M)RM = N$ and the required holds. $\square$

Recall that, an R -semimodule M is called faithful if $\text{Ann}(M) = 0$ and is called a cancellation ternary R -semimodule if for all ideals I and J of R , $IRM = JRM$ implies that $I = J$.

Lemma 2.11. *let N be a ternary subsemimodule of finitely generated faithful multiplication (and so cancellation) R -semimodule M . Then, we have $(I^2N : M) = I^2(N : M)$ for every ideal I of R .*

Proof. As M is multiplication ternary R -semimodule, then, $I^2(N : M)RM = I^2N = (I^2N : M)RM$. The result follows because M is a cancellation semimodule. $\square$

Theorem 2.12. *Let M be a finitely generated faithful multiplication R -semimodule and N be a proper subsemimodule of M . The following are equivalent*

- (1) N is almost prime ternary in M ;
- (2) $(N : M)$ is almost prime ternary ideal of R ;
- (3) $N = QRM$ for some almost prime ternary ideal Q of R .

Proof. (1) $\Rightarrow$ (2) Suppose that N is an almost prime ternary and let $a, b, c \in R$ such that $abc \in (N : M) - (N : M)^3$. Then $(abc)dM = a(bcd)M = ab(cdM) \subseteq N - (N : M)^2N$ for all $d \in R$. Indeed, if $(abc)dM \subseteq (N : M)^2N$ for all $d \in R$, then by Lemma 2.11, $abc \in ((N : M)^2N : M) = (N : M)^3$, a contradiction.

Now, N is almost prime ternary which implies that $a \in (N : M)$ or $b \in (N : M)$ or $cdM \subseteq N$ for all $d \in R$ (and so $c \in (N : M)$). Hence, $(N : M)$ is almost prime ternary in R .

(2) $\Rightarrow$ (1) In this direction, we need M to be just a multiplication ternary R -subsemimodule. Let $r_1, r_2 \in R$ and $m \in M$, such that $r_1r_2m \in N - (N : M)^2M$. Then $r_1r_2((m) : M) \subseteq ((r_1r_2m) : M) \subseteq (N : M)$. Moreover, $r_1r_2((m) : M) \not\subseteq (N : M)^3$ because otherwise, if $r_1r_2((m) : M) \subseteq (N : M)^3 \subseteq ((N : M)^2 : M)$, then $r_1r_2(m) = r_1r_2((m) : M)RM \subseteq (N : M)^2N$, a contradiction. As $(N : M)$ is almost prime ternary in R , then, either $r_1 \in (N : M)$ or $r_2 \in (N : M)$ or $((m) : M) \subseteq (N : M)$. In third case, we obtain $(m) = ((m) : M)RM \subseteq (N : M)RM = N$ and so N is almost prime ternary in M .

(2) $\Longleftrightarrow$ (3) We choose $Q = (N : M)$. $\square$

Lemma 2.13. *Let R be a semiring, M a faithful multiplication ternary R -semimodule, N a proper ternary subsemimodule of M and I a finitely generated faithful multiplication ternary ideal of R . then the following statements are equivalent.*

- (1) N is weakly prime ternary subsemimodule;
- (2) $(N : M)$ is a weakly ternary prime ideal of R ;
- (3) $N = QRM$ for some weakly prime ternary ideal Q of R .

Proof. (1) $\Rightarrow$ (2) Suppose that N is a weakly prime ternary subsemimodule of M . Let $a, b, c \in R$ such that $0 \neq abc \in (N : M)$. Then $(abc)RM = ab(cRM) \subseteq N$. Since M is faithful, then $abcRM \neq 0$ and since N is a weakly prime ternary, then $a \in (N : M)$ or $b \in (N : M)$ or $cRM \in (N : M)$ (and hence $c \in (N : M)$).

(2) $\Rightarrow$ (1) Let $(N : M)$ be a weakly prime ideal ternary of R . If $0 \neq r_1r_2m \in N$, where $r_1, r_2 \in R, m \in M$, then $r_1r_2(RRm : M) \subseteq (RRr_1r_2m : M) \subseteq (N : M)$. Since M is a multiplication, then $r_1r_2(RRm : M) \neq 0$. As $(N : M)$ is weakly prime ternary, then, either $r_1 \in (N : M)$ or $r_2 \in (N : M)$ or $(RRm : M) \subseteq (N : M)$. In third case, we obtain $RRm = (RRm : M)RM \subseteq (N : M)RM = N$ and so N is weakly prime ternary in M .

(1) $\Rightarrow$ (3). Take $Q = (N : M)$. $\square$

In the following two theorems, we give a new characterization of weakly prime (resp.; almost prime) subsemimodules of finitely generated faithful multiplication ternary R -semimodules.

Theorem 2.14. *Let M be a finitely generated faithful multiplication ternary R -semimodule and P be a proper ternary subsemimodule of M . Then P is weakly prime ternary in M if and only if whenever N, K and L are ternary subsemimodules of M such that $0 \neq NKL \subseteq P$, then, $N \subseteq P$ or $K \subseteq P$ or $L \subseteq P$.*

Proof. Suppose that P is weakly prime ternary. We have $N = (N : M)RM$, $K = (K : M)RM$ and $L = (L : M)RM$, and so $NKL = (N : M)(K : M)(L : M)RM$. Suppose $0 \neq NKL \subseteq P$, but $N \not\subseteq P$, $K \not\subseteq P$ and $L \not\subseteq P$.

Then, $(N : M) \not\subseteq (P : M)$, $(K : M) \not\subseteq (P : M)$ and $(L : M) \not\subseteq (P : M)$. As $(P : M)$ is weakly prime ternary following Lemma 2.13, then, either $(N : M)(K : M)(L : M) \not\subseteq (P : M)$ or $(N : M)(K : M)(L : M) = 0$. In first case, we have $NKL = (N : M)(K : N)(L : N)RM \not\subseteq (P : M)RM = P$, a contradiction. If $(N : M)(K : M)(L : M) = 0$, then, $NKL = 0RM = 0$ and also we get a contradiction. Therefore, either $N \subseteq P$ or $K \subseteq P$ or $L \subseteq P$. Conversely, to prove that P is weakly prime ternary in M , it enough by Lemma 2.13 to prove that $(P : M)$ is weakly prime ternary in R . Let $r_1, r_2, r_3 \in R$, such that $0 \neq r_1 r_2 r_3 \in (P : M)$, but $r_1 \notin (P : M)$, $r_2 \notin (P : M)$ and $r_3 \notin (P : M)$. Let $N = (r_1)RM$, $K = (r_2)RM$ and $L = (r_3)RM$. Then, $0 \neq NKL = (r_1)(r_2)(r_3)RM \subseteq P$. Indeed, if $NKL = (r_1)(r_2)(r_3)RM = 0$, then, $(r_1 r_2 r_3) \subseteq \text{Ann}(M) = 0$, which is a contradiction. By assumption, either $(r_1)RM = N \subseteq P$ or $(r_2)RM = K \subseteq P$ or $(r_3)RM = L \subseteq P$ and so, either $r_1 \in (P : M)$ or $r_2 \in (P : M)$ or $r_3 \in (P : M)$, a contradiction. Therefore, $(P : M)$ is weakly prime ternary in R and so P is weakly prime ternary in M . $\square$

Corollary 2.15. *Let P be a proper ternary subsemimodule of finitely generated faithful multiplication ternary R -semimodule M . Then, P is weakly prime ternary if and only if whenever $m_1, m_2, m_3 \in M$, $0 \neq m_1 m_2 m_3 \in P$ implies $m_1 \in P$ or $m_2 \in P$ or $m_3 \in P$.*

Theorem 2.16. *Let M be a finitely generated faithful multiplication ternary R -semimodule and P be a proper ternary subsemimodules of M . Then, P is almost prime in M if and only if whenever N, K and L are ternary subsemimodules of M such that $NKL \subseteq P - (P : M)^2 P$, then, either $N \subseteq P$ or $K \subseteq P$ or $L \subseteq P$.*

Proof. By Theorem 2.12, P is almost prime in M if and only if $(P : M)$ is almost prime in R . As $(P : M)^3 = ((P : M)^2 P : M)$ by Lemma 2.11, then, the proof is similar to that of Theorem 2.16. $\square$

Recall that, a ternary subsemimodule N of a ternary R -semimodule M is called Q -ternary subsemimodule (= partitioning ternary subsemimodule) if there exists a subset Q of M such that

- 1) $M = \cup \{q + N : q \in Q\}$.
- 2) If $q_1, q_2 \in Q$, then $(q_1 + N) \cap (q_2 + N) \neq \emptyset \Leftrightarrow q_1 = q_2$.

Let N be a Q -ternary subsemimodule of a ternary R -semimodule M . Then $M/N_{(Q)} = \{q + N : q \in Q\}$ forms a ternary R -semimodule under the following addition " $\oplus$ " and ternary scalar multiplication " $\odot$ ", $(q_1 + N) \oplus (q_2 + N) = q_3 + N$ where $q_3 \in Q$ is unique such that $q_1 + q_2 + N \subseteq q_3 + N$, and $r \odot s \odot (q_1 + N) = q_4 + N$ where $q_4 \in Q$ is unique such that $rsq_1 + N \subseteq q_4 + N$. This ternary R -semimodule $M/N_{(Q)}$ is called the quotient ternary semimodule of M by N and denoted by $(M/N_{(Q)}, \oplus, \odot)$ or just $M/N_{(Q)}$.

Lemma 2.17 ([21]). *Let N be a Q -ternary subsemimodule of a ternary R -semimodule M . If A is a subtractive ternary subsemimodule of M such that $N \subseteq A$, then N is a $Q \cap A$ -ternary subsemimodule of A .*

Lemma 2.18 ([18]). *Let N be a Q -ternary subsemimodule of a ternary R -semimodule M . If $r, s \in R$ and $m \in M$, then there exists a unique $q \in Q$ such that $rs m \in r \odot s \odot (q + N)$.*

Theorem 2.19. *Let N be a Q -ternary subsemimodule of a ternary R -semimodule M and P be a subtractive ternary subsemimodule of M with $N \subseteq P$. Then*

- (1) *If P is almost prime ternary subsemimodule of M , then $P/N_{(Q \cap P)}$ is almost prime ternary subsemimodule of $M/N_{(Q)}$.*
- (2) *If $N, P/N_{(Q \cap P)}$ are almost prime ternary subsemimodule of $M, M/N_{(Q)}$ respectively, then P is almost prime ternary subsemimodule of M .*

Proof. Let P be almost prime ternary subsemimodule of M . Let $r, s \in R$ and $q_1 + N \in M/N_{(Q)}$ such that $r \odot s \odot (q_1 + N) \in P/N_{(Q \cap P)} - (P/N_{(Q \cap P)} : M/N_{(Q)})^2 P/N_{(Q \cap P)}$. By Lemma 2.17, N is $Q \cap P$ -ternary subsemimodule of P . Hence, there exists a unique $q_2 \in Q \cap P$ such that $r \odot s \odot (q_1 + N) = q_2 + N$ where $rsq_1 + N \subseteq q_2 + N$. Since $N \subseteq P$, then $rsq_1 \in P$ and since $r \odot s \odot (q_1 + N) \notin (P/N_{(Q \cap P)} : M/N_{(Q)})^2 P/N_{(Q \cap P)} = (P : M)^2 P/N_{(Q \cap P)}$, then $rsq_1 \notin (P : M)^2 N$. As P is almost prime ternary subsemimodule, either $r \in (P : M)$ or $s \in (P : M)$ or $q_1 \in P$. If $q_1 \in P$, then $q_1 \in Q \cap P$ and hence $q_1 + N \in P/N_{(Q \cap P)}$. Without loss of generality suppose that $r \in (P : M)$. For $q_1 + N \in M/N_{(Q)}$ and $s' \in R$, let $r \odot s' \odot (q + N) = q_3 + N$ where q_3 is a unique element of Q such that $rs'q = q_3 + n$ for some $n \in N$. Now $r \in (P : M) \Rightarrow rs'q \in P \Rightarrow q_3 + n \in P \Rightarrow q_3 \in P$, as P is a subtractive ternary subsemimodule of M and $n \in N \subseteq P$. Hence $q_3 \in Q \cap P$. Now $r \odot s' \odot (q + N) = q_3 + N \in P/N_{(Q \cap P)}$ for all $s' \in R$ and $q + N \in M/N_{(Q)}$. Therefore $r \in (P/N_{(Q \cap P)} : M/N_{(Q)})$. Thus $P/N_{(Q \cap P)}$ is almost prime ternary subsemimodule of $M/N_{(Q)}$.

2) Suppose that $N, P/N_{(Q \cap P)}$ are almost prime subsemimodule of $M, M/N_{(Q)}$ respectively. Let $rs m \in P - (P : M)^2 P$ where $r, s \in R, m \in M$. If $rs m \in N - (N : M)^2 N$, then we are through, since N is almost prime ternary subsemimodule of M . So suppose that $rs m \in P - N$. By using Lemma 2.18, there exists a unique $q_1 \in Q$ such that $m \in q_1 + N$ and $rs m \in r \odot s \odot (q_1 + N) = q_2 + N$ where q_2 is a unique element of Q such that $rsq_1 + N \subseteq q_2 + N$. Now $rs m \in P$, $rs m \in q_2 + N$ implies $q_2 \in P$, as P is a subtractive ternary subsemimodule and $N \subseteq P$. Hence $r \odot s \odot (q_1 + N) = q_2 + N \in P/N_{(Q \cap P)} - (P/N_{(Q \cap P)} : M/N_{(Q)})^2 P/N_{(Q \cap P)}$. As $P/N_{(Q \cap P)}$ is almost prime ternary subsemimodule, $r \in (P/N_{(Q \cap P)} : M/N_{(Q)})$ or $s \in (P/N_{(Q \cap P)} : M/N_{(Q)})$ or $q_1 + N \in P/N_{(Q \cap P)}$. If $q_1 + N \in P/N_{(Q \cap P)}$, then $q_1 \in P$. Hence $m \in q_1 + N \subseteq P$. Now without loss of generality assume that $r \in (P/N_{(Q \cap P)} : M/N_{(Q)})$. Let $x \in M$ and $s' \in R$. By using Lemma 2.18, there exists a unique $q_3 \in Q$ such that $x \in q_3 + N$ and $rs'x \in r \odot s \odot (q_3 + N) = q_4 + N$ where q_4 is a unique element of Q such that $rs'q_3 + N \subseteq q_4 + N$. Now $q_4 + N = r \odot s' \odot (q_3 + N) \in P/N_{(Q \cap P)}$ and hence $q_4 \in P$. As $rs'x \in q_4 + N$ and $N \subseteq P$, $rs'x \in P$. So $r \in (P : M)$. Therefore P is almost prime ternary subsemimodule of M . $\square$

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IMPACT OF HARVESTING, NOISE AND DIFFUSION ON THE DYNAMICS OF A FOOD CHAIN MODEL WITH RATIO-DEPENDENT FUNCTIONAL RESPONSE III

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Abstract. The current article is associated with impact of noise and harvesting of a three species eco system which consists of prey–predator–top predator with Holling classification III. The stability of the given system is checked at the interior steady state and bionomial steady states are also evaluated. Model formulation of the optimal harvesting policy is given and its solution is derived at interior steady state by using Pontryagins Maximum principle. We also examined the population intensities of variations at the positive steady state due to environmental attribute, we have also highlighted the diffusive steadiness of the structure along with some numerical simulations.

Keywords: bionomic harvesting,optimal harvesting,prey,predator, stability, stochasticity white noise, diffusion.

1. Introduction

The prey-predator structure is a very affective model which has received extensive attention [1-5]. But, all these works not completely supported the effect of harvesting of species along with functional response. However in the real world almost all species have the age structure of prey, predator and top predator. Liu and Chen [6] surveyed the progression stage structured population dynamics. There are several debates on ratio dependent predation have drawn the attention of technologist is on Holling classification. Since this classification is necessary and essential for prey-predator interaction. It is also important for the dynamics of food web models, Bio pest controlling [7]. The choosing of the form of functional response gives a very interesting dynamics and surprising effects on statistical prediction.

Freedmen [8] discussed about the food chain model consisting of these species, food chain systems are very complex and interesting and are dependent in the en-

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vironment. Insignificant modifications in the characteristics of such type of food web system give the drastic consequences. Many food chain models have been thoroughly explored, many other authors discovered and examining models have three or four traffic levels [8]-[9] and they presented that the food chain models are rich in dynamics. In many field situation like tea plant-pest-top predator food chain have becomes extremely important and predator may determine the fitness of the plant and destroying pest[10]. In view of social necessities the utilization of organic assets and harvesting populace are usually practiced in yield life controlling, fishery and forestry etc. There is a wide range of interest use of bionomic modelling to gain insight in the technological utilization of well known fields like forestry, fisheries [11-13]. In [13] Xiao et.al, presented the following model

$$x'(t) = rx \left(1 - \frac{x}{k}\right) - \frac{cxy}{my + x}, \quad y'(t) = y \left(-D - \frac{fx}{my + x}\right).$$

Our article is associated with the prey which is of profitable important. The predator and top predator are constantly participated in harvesting with an agency. This process do not disturb the prey populace directly. This article mainly concentrates on analysis of the changing aspects of tri-tropic food web constitutes of victim, hunter and chief hunter, and which is having ratio dependent three type functional response, along with harvesting efforts. Jostet al., [14] discussed and studied the underlying forces of a two species model with Holling classification. The main aim of the current article is to study the dynamics properties of the model system defined above, the effects of noise and diffusion on the above said structure are also analysed. Inspired by [15] and [16] we consider the tri tropic prey-predator-top predator system incorporating harvesting.

The organization of this article consists of several subdivisions. Subdivision 2 defines the elementary scientific model. Section 3 consists of boundedness of the structure and investigation of fixed points, whereas steadiness is analysed in section 4. Section 5 includes bionomic equilibrium analysis. Section 6 includes optimal harvesting approach. The consequences of noise of the given food web model termed as stochastic model is described in section 7. We also have given the spatiotemporal analysis in section 8. In section 9, numerical simulations are given. Finally concluding remarks are in section 10.

2. Mathematical model

We consider a computational mathematical model with harvesting, includes 3 species say a prey N_1 , a predator N_2 and a top predator N_3 . In this construction, we assume that top predator preys on predator only and predator, preys on prey species only. Also we assume that the nutrient recycling is not considered. In view of both biological and methodical point of view, this supposition is a remarkable and a real-world one. It is also assumed that a ratio dependent

type-III Holling classification for both $N_1 \leftrightarrow N_2$ and $N_2 \leftrightarrow N_3$ interfaces. From the above technological discussion the framed system is as follows

$$(2.1) \quad N_1'(T) = r_1 N_1 \left(1 - \frac{N_1}{k_1} \right) - \frac{s_1 N_1^2 N_2}{(b_1 N_2^2 + N_1^2) c_1} - q_1 E_1 N_1,$$

$$(2.2) \quad N_2'(T) = \frac{s_1 N_1^2 N_2}{(b_1 N_2^2 + N_1^2)} - m_1 N_2 - \frac{s_2 N_2^2 N_3}{(b_2 N_3^2 + N_2^2) c_2} - q_2 E_2 N_2,$$

$$(2.3) \quad N_3'(T) = \frac{s_2 N_2^2 N_3}{(b_2 N_3^2 + N_2^2)} - m_2 N_3 - q_3 E_3 N_3,$$

where

$$N_1(0) > 0, N_2(0) > 0, N_3(0) > 0.$$

Here $N_i, i = 1, 2, 3$ represents bio mass densities of prey, predator and super predator species, $q_i, i = 1, 2, 3$ represents catchability coefficients of species respectively. $E_i, i = 1, 2, 3$ represents efforts of applied to harvest to prey, predator and super predators respectively, r_1 represents intrinsic growth rate of prey species, $s_i, i = 1, 2$ represents maximal predator growth of predator and top predator, $m_i, i = 1, 2$ represents natural death rate of predator and top predator, $b_i, i = 1, 2$ represents half saturation constants of predator and top predator, $c_i, i = 1, 2$ represents yield constants of prey and predator. The term $(N_1^2 N_2)/(b_1 N_2^2 + N_1^2)$ represents ratio dependent type III functional response for prey and predator and $(N_2^2 N_3)/(b_2 N_3^2 + N_2^2)$ represents ratio dependent type III functional response for predator and super predator. Here, throughout our analysis we are assuming that $r_1 - q_1 E_1 > 0, p_2 - q_4 - q_3 E_3 > 0, p_1 - q_5 - q_2 E_2 > 0$. We make an observable supposition that all the attributes are positive. Since the densities of the population cannot be negative, the state space of system (2.1)-(2.3) is given by R_+^3 . To make a methodical analysis easy, we reduce (2.1)-(2.3) into a non-dimensionalized one by using

$$t = r_1 T, n_1 = \frac{N_1}{k_1}, n_2 = \frac{N_2 \sqrt{b_1}}{k_1}, n_3 = \frac{N_3 \sqrt{b_1 b_2}}{k_1}.$$

Then the system (2.1)-(2.3) takes the form

$$(2.4) \quad n_1'(t) = n_1(1 - n_1) - \frac{l_1 n_1^2 n_2}{n_1^2 + n_2^2} - \frac{q_1 E_1 n_1}{r_1},$$

$$(2.5) \quad n_2'(t) = \frac{p_1 n_1^2 n_2}{n_1^2 + n_2^2} - q_5 n_2 - \frac{l_2 n_2^2 n_3}{n_2^2 + n_3^2} - \frac{q_2 E_2 n_2}{r_1},$$

$$(2.6) \quad n_3'(t) = \frac{p_2 n_2^2 n_3}{n_2^2 + n_3^2} - q_4 n_3 - \frac{q_3 E_3 n_3}{r_1},$$

where

$$n_1(0) > 0, n_2(0) > 0, n_3(0) > 0,$$

$$l_1 = \frac{s_1}{c_1 r_1 \sqrt{b_1}}, l_2 = \frac{s_2}{c_2 r_1 \sqrt{b_2}}, p_1 = \frac{s_1}{r_1}, q_5 = \frac{m_1}{r_1}, q_4 = \frac{m_2}{r_1}, p_2 = \frac{s_2}{r_1}.$$

3. Analysis of steady states

3.1 Steady states

It is obvious that the interior equilibrium point $E^*(n_1^*, n_2^*, n_3^*)$ of system (2.1)-(2.3) exist in the interior point of the first octant if there is a positive solution of the algebraic equations

$$\begin{aligned} n_1(1 - n_1) - \frac{l_1 n_1^2 n_2}{n_1^2 + n_2^2} - \frac{q_1 E_1 n_1}{r_1} &= 0, \\ \frac{p_1 n_1^2 n_2}{n_1^2 + n_2^2} - q_5 n_2 - \frac{l_2 n_2^2 n_3}{n_3^2 + n_2^2} - \frac{q_2 E_2 n_2}{r_1} &= 0, \\ \frac{p_2 n_2^2 n_3}{n_3^2 + n_2^2} - q_4 n_3 - \frac{q_3 E_3 n_3}{r_1} &= 0. \end{aligned}$$

Then, by solving these equations, we get,

$$\begin{aligned} n_1^* &= \left(\frac{p_1 p_2 r_1^2 (r_1 - q_1 E_1) - l_1 r_1 \sqrt{L(p_1 p_2 r_1^2 - L)}}{p_1 p_2 r_1^3} \right), \\ n_2^* &= \left(\sqrt{\frac{p_1 p_2 r_1^2 - L}{L}} \right) n_1^*, \\ n_3^* &= \left(\sqrt{\frac{(p_2 + q_4) r_1 - q_3 E_3}{q_4 r_1 + q_3 E_3}} \right) n_2^*, \end{aligned}$$

where

$$L = p_2 q_2 - l_2 r_1 \sqrt{(p_2 - q_2 - q_3 E_3)(q_4 + q_3 E_3)} - p_2 q_2 E_2.$$

For positiveness of these values, we must have,

$$p_1 p_2 r_1^2 > L, (p_2 + q_4) r_1 > q_3 E_3.$$

4. Stability analysis

4.1 Local stability analysis

Now to investigate the confined steadiness of inner steady state $E^*(n_1^*, n_2^*, n_3^*)$.

Next we have to construct a matrix $M(E^*)$

$$M(E^*) = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}.$$

Where

$$b_{11} = \left(1 - 2n_1 - \frac{2l_1 n_2^3 n_1}{(n_1^{*2} + n_2^{*2})^2} - \frac{q_1 E_1}{r_1} \right), b_{12} = \left(-\frac{2l_1 n_1^{*4}}{(n_1^{*2} + n_2^{*2})^2} \right),$$

$$\begin{aligned}
b_{13} &= 0, b_{21} = \left(\frac{2p_1 n_2^{*3} n_1^*}{(n_1^{*2} + n_2^{*2})^2} \right), \\
b_{22} &= \left(\frac{2p_1 n_1^{*3}}{(n_1^{*2} + n_2^{*2})^2} \right) - q_5 - l_2 \left(\frac{2n_3^{*3} n_2^*}{(n_2^{*2} + n_3^{*2})^2} \right) - \frac{q_2 E_2}{r_1}, \\
b_{23} &= - \left(\frac{2n_2^{*4} l_2}{(n_2^{*2} + n_3^{*2})^2} \right), b_{31} = 0, b_{32} = \left(\frac{2p_2 n_2 n_3^3}{(n_3^{*2} + n_2^{*2})^2} \right), \\
b_{33} &= \left(\frac{p_2 n_2^{*2} (n_2^{*2} - n_3^{*2})}{(n_3^{*2} + n_2^{*2})^2} \right) - q_4 - \frac{q_3 E_3}{r_1}.
\end{aligned}$$

The analogous specific equation is

$$(4.1.1) \quad \lambda^3 + A_1 \lambda^2 + A_2 \lambda + A_3 = 0.$$

Where

$$A_1 = -(b_{11} + b_{22} + b_{33})$$

$$A_1 = - \left(\left(1 - 2n_1 - \frac{2l_1 n_2^{*3} n_1}{(n_1^{*2} + n_2^{*2})^2} - \frac{q_1 E_1}{r_1} \right) + \left(\begin{aligned} &\left(\frac{2p_1 n_1^{*3}}{(n_1^{*2} + n_2^{*2})^2} \right) \\ &- q_5 - \left(\frac{2l_2 n_3^{*3} n_2^*}{(n_2^{*2} + n_3^{*2})^2} \right) - \frac{q_2 E_2}{r_1} \end{aligned} \right) + \left(\begin{aligned} &\left(\frac{p_2 n_2^{*2} (n_2^{*2} - n_3^{*2})}{(n_3^{*2} + n_2^{*2})^2} \right) - q_4 - \frac{q_3 E_3}{r_1} \end{aligned} \right) \right)$$

$$A_2 = (b_{11}b_{22} + b_{11}b_{33} + b_{22}b_{33} - b_{23}b_{32} - b_{21}b_{12})$$

$$\begin{aligned}
A_2 &= \left(1 - 2n_1 - \frac{2l_1 n_2^{*3} n_1}{(n_1^{*2} + n_2^{*2})^2} - \frac{q_1 E_1}{r_1} \right) \cdot \left(\begin{aligned} &\left(\frac{2p_1 n_1^{*3}}{(n_1^{*2} + n_2^{*2})^2} \right) - q_5 - l_2 \left(\frac{2n_3^{*3} n_2^*}{(n_2^{*2} + n_3^{*2})^2} \right) \\ &- \frac{q_2 E_2}{r_1} + \left(\frac{p_2 n_2^{*2} (n_2^{*2} - n_3^{*2})}{(n_3^{*2} + n_2^{*2})^2} \right) - q_4 - \frac{q_3 E_3}{r_1} \end{aligned} \right) \\
&+ \left(\left(\frac{2p_1 n_1^{*3}}{(n_1^{*2} + n_2^{*2})^2} \right) - q_5 - l_2 \left(\frac{2n_3^{*3} n_2^*}{(n_2^{*2} + n_3^{*2})^2} \right) - \frac{q_2 E_2}{r_1} \right) \cdot \left(\begin{aligned} &\left(\frac{p_2 n_2^{*2} (n_2^{*2} - n_3^{*2})}{(n_3^{*2} + n_2^{*2})^2} \right) - q_4 - \frac{q_3 E_3}{r_1} \end{aligned} \right) \\
&- \left(- \left(\frac{2n_2^{*4} l_2}{(n_2^{*2} + n_3^{*2})^2} \right) \left(\frac{2p_2 n_2 n_3^3}{(n_3^{*2} + n_2^{*2})^2} \right) + \left(- \frac{2l_1 n_1^{*4}}{(n_1^{*2} + n_2^{*2})^2} \right) \left(\frac{2p_1 n_2^{*3} n_1^*}{(n_1^{*2} + n_2^{*2})^2} \right) \right)
\end{aligned}$$

$$A_3 = [b_{11}b_{23}b_{33} - b_{11}b_{23}b_{32} - b_{12}b_{21}b_{33}]$$

$$\begin{aligned} A_3 = & \left(1 - 2n_1 - \frac{2l_1n_2^{*3}n_1}{(n_1^{*2} + n_2^{*2})^2} - \frac{q_1E_1}{r_1} \right) \cdot \\ & \cdot \left(\left(\frac{2p_1n_1^{*3}}{(n_1^{*2} + n_2^{*2})^2} \right) - q_5 - l_2 \left(\frac{2n_3^{*3}n_2^*}{(n_2^{*2} + n_3^{*2})^2} \right) - \frac{q_2E_2}{r_1} \right) \cdot \\ & \cdot \left(\frac{p_2n_2^{*2}(n_2^{*2} - n_3^{*2})}{(n_3^{*2} + n_2^{*2})^2} \right) - q_4 - \frac{q_3E_3}{r_1} \\ & - \left(\left(1 - 2n_1 - \frac{2l_1n_2^{*3}n_1}{(n_1^{*2} + n_2^{*2})^2} - \frac{q_1E_1}{r_1} \right) \left(-\frac{2n_2^{*4}l_2}{(n_2^{*2} + n_3^{*2})^2} \right) \left(\frac{2p_2n_2n_3^3}{(n_3^{*2} + n_2^{*2})^2} \right) \right) \\ & - \left(\left(-\frac{2l_1n_1^{*4}}{(n_1^{*2} + n_2^{*2})^2} \right) \left(\frac{2p_1n_2^{*3}n_1^*}{(n_1^{*2} + n_2^{*2})^2} \right) \left(\left(\frac{p_2n_2^{*2}(n_2^{*2} - n_3^{*2})}{(n_3^{*2} + n_2^{*2})^2} \right) - q_4 - \frac{q_3E_3}{r_1} \right) \right) \end{aligned}$$

clearly, the proposed system is locally asymptotically stable by Routh-Hurwitz criteria.

4.2 Global steadiness

Theorem. The positive steady state (n_1^*, n_2^*, n_3^*) of the above proposed system is globally asymptotically stable if

$$\begin{aligned} V(t) = & \left(n_1 - n_1^* - n_3^* \ln \left(\frac{n_1}{n_1^*} \right) \right) + l_1 \left(n_2 - n_2^* - n_2^* \ln \left(\frac{n_2}{n_2^*} \right) \right) \\ & + l_2 \left(n_3 - n_3^* - n_3^* \ln \left(\frac{n_3}{n_3^*} \right) \right), \end{aligned}$$

$l_1 > 0$, $l_2 > 0$ and $P < \min(Q, R)$, where

$$\begin{aligned} P &= \alpha(l_1c_2m_3n_2^* + l_2p_2m_4n_3^*) + \beta(c_1m_1n_1^* + l_1p_1m_2n_2^*), \\ Q &= 2l_1(\beta p_1m_2n_1^* + \alpha c_2m_3n_3^*), \\ R &= 2\alpha l_2p_2m_4y^* + \beta(2\alpha + c_1m_1y^*). \end{aligned}$$

Proof. To check the global steadiness of inner equilibrium point, construct a Lyapunov function $V(t)$ as

$$\begin{aligned} V(t) &= \left(\frac{n_1 - n_1^*}{n_1} \right) n_1(t) + l_1 \left(\frac{n_2 - n_2^*}{n_2} \right) n_2(t) + l_2 \left(\frac{n_3 - n_3^*}{n_3} \right) n_3(t) \\ V'(t) &= (n_1 - n_2^*) \left[1 - n_1 - \frac{c_1n_1n_2}{n_1^2 + n_2^2} - \frac{q_1E_1}{r} \right] \\ &+ l_1(n_2 - n_2^*) \left[\frac{p_1n_1^2}{n_1^2 + n_2^2} - q_5 - \frac{c_2n_2n_3}{n_2^2 + n_3^2} - \frac{q_2E_2}{r} \right] \end{aligned}$$

$$+ l_2 (n_3 - n_3^*) \left[\frac{p_2 n_2^2}{n_2^2 + n_3^2} - q_4 - \frac{q_3 E_3}{r} \right].$$

At the equilibrium point (n_1^*, n_2^*, n_3^*) , we have

$$\begin{aligned} V'(t) = & (n_1 - n_2^*) \left[1 - n_1 - \frac{c_1 n_1 n_2}{n_1^2 + n_2^2} - 1 + n_1^* + \frac{c_1 n_1^* n_2^*}{n_1^{*2} + n_2^{*2}} \right] \\ & + l_1 (n_2 - n_2^*) \left[\frac{p_1 n_1^2}{n_1^2 + n_2^2} - \frac{c_2 n_2 n_3}{n_2^2 + n_3^2} - \frac{p_1 n_1^{*2}}{n_1^{*2} + n_2^{*2}} + \frac{c_2 n_2^* n_3^*}{n_2^{*2} + n_3^{*2}} \right] \\ & + l_2 (n_3 - n_3^*) \left[\frac{p_2 n_2^2}{n_2^2 + n_3^2} - \frac{p_2 n_2^{*2}}{n_2^{*2} + n_3^{*2}} \right]. \end{aligned}$$

Therefore, we have

$$\begin{aligned} V'(t) = & - (n_1 - n_2^*)^2 - \frac{c_1 n_1^*}{\alpha} (n_2 n_2^* - n_1 n_1^*) (n_1 - n_1^*) (n_2 - n_2^*) \\ & - \frac{c_1 n_2^*}{\alpha} (n_2 n_2^* - n_1 n_1^*) (n_1 - n_1^*)^2 \\ & + \frac{l_1 p_1 n_2^*}{\alpha} (n_1 n_2^* + n_1^* n_2) (n_1 - n_2^*) (n_2 - n_2^*) - \frac{l_1 p_1 n_1^*}{\alpha} (n_1 n_2^* + n_1^* n_2) (n_2 - n_2^*)^2 \\ & + \frac{l_1 c_2 n_2^*}{\beta} (n_3 n_3^* - n_2 n_2^*) (n_2 - n_2^*) (n_3 - n_3^*) - \frac{l_1 c_2 n_3^*}{\beta} (n_3 n_3^* - n_2 n_2^*) (n_2 - n_2^*)^2 \\ & + \frac{l_2 p_2 n_3^*}{\beta} (n_2 n_3^* + n_2^* n_3) (n_2 - n_2^*) (n_3 - n_3^*) \\ & - \frac{l_2 p_2 n_2^*}{\beta} (n_2 n_3^* + n_2^* n_3) (n_3 - n_3^*)^2, \end{aligned}$$

where

$$\begin{aligned} \alpha = & (n_1^2 + n_2^2) (n_1^{*2} + n_2^{*2}), \beta = (n_2^2 + n_3^2) (n_2^{*2} + n_3^{*2}), \\ V'(t) \leq & (n_1 - n_1^*)^2 \left[-1 + \frac{c_1 m_1 n_1^*}{2\alpha} - \frac{c_1 m_1 n_2^*}{\alpha} + \frac{l_1 p_1 m_2 n_2^*}{2\alpha} \right] \\ & + (n_2 - n_2^*)^2 \left[\frac{c_1 m_1 n_1^*}{2\alpha} + \frac{l_1 p_1 m_2 n_2^*}{2\alpha} - \frac{l_1 p_1 m_2 n_1^*}{\alpha} + \frac{l_1 c_2 m_3 n_2^*}{2\beta} \right. \\ & \left. - \frac{l_1 c_2 m_3 n_3^*}{\beta} + \frac{l_2 p_2 m_4 n_3^*}{2\beta} \right] \\ & + (n_3 - n_3^*)^2 \left[\frac{l_1 c_2 m_3 n_1^*}{2\beta} + \frac{l_2 p_2 m_4 n_3^*}{2\beta} - \frac{l_2 p_2 m_4 n_2^*}{\beta} \right], \end{aligned}$$

where $m_1 = n_2 n_2^* - n_1 n_1^* > 0$, $m_2 = n_1 n_2^* + n_1^* n_2$, $m_3 = n_3 n_3^* - n_2 n_2^* > 0$, $m_4 = n_2 n_3^* + n_2^* n_3$. The derivative of Lyapunov function becomes negative, if

$$\alpha(l_1 c_2 m_3 n_1^* + l_2 p_2 m_4 n_2^*) + \beta(c_1 m_1 n_1^* + l_1 p_1 m_2 n_2^*) < 2\alpha l_2 p_2 m_4 n_2^* + 2\beta(\alpha + c_1 m_1 n_2^*).$$

Hence proved.

5. Bionomic steadiness

The aim of this section is to capture the changing aspects of several live tools and data using inexpensive models. Let c_3 be the harvesting cost per unit effort for prey species. Let c_4 be the harvesting cost per unit effort for predator species. Let c_5 be the harvesting cost per unit effort for super species. Let p_3 be the price for unit biomass for prey species. Let p_4 be the price for unit biomass for predator species. Let p_5 be the price for unit biomass for super predator species. Thus net revenue is $NR = NR_1 + NR_2 + NR_3$. Where $NR_1 = (p_3q_1n_1 - c_3)E_1$. $NR_2 = (p_4q_2n_2 - c_4)E_2$, $NR_3 = (p_5q_3n_3 - c_5)E_3$. The Bionomic steady state is $((n_1)_\infty, (n_2)_\infty, (n_3)_\infty, (E_1)_\infty, (E_2)_\infty, (E_3)_\infty)$ and

$$(5.1) \quad n_1(1 - n_1) - \frac{l_1n_1^2n_2}{n_1^2 + n_2^2} - \frac{q_1E_1n_1}{r_1} = 0,$$

$$(5.2) \quad \frac{p_1n_1^2n_2}{n_1^2 + n_2^2} - q_5n_2 - \frac{l_2n_2^2n_3}{n_3^2 + n_2^2} - \frac{q_2E_2n_2}{r_1} = 0,$$

$$(5.3) \quad \frac{p_2n_2^2n_3}{n_3^2 + n_2^2} - q_4n_3 - \frac{q_3E_3n_3}{r_1} = 0,$$

$$(5.4) \quad \begin{aligned} NR &= NR_1 + NR_2 + NR_3 = (p_3q_1n_1 - c_3)E_1 + (p_4q_2n_2 - c_4)E_2 \\ &+ (p_5q_3n_3 - c_5)E_3 = 0. \end{aligned}$$

To obtain the bionomic steady state, it is needed to verify the cases

Case (i). If $c_3 > p_3q_1n_1, c_4 > p_4q_2n_2, c_5 > p_5q_3n_3$, which is not nearer to our aim.

Case (ii). if $c_3 < p_3q_1n_1, c_4 < p_4q_2n_2, c_5 < p_5q_3n_3$, which is very nearer to our aim and recommended for the above working model.

$$(5.5) \quad (n_1)_\infty = \left(\frac{c_3}{p_3q_1} \right), \quad (n_2)_\infty = \left(\frac{c_4}{p_4q_2} \right), \quad (n_3)_\infty = \left(\frac{c_5}{p_5q_3} \right).$$

From (5.1), (5.2) and (5.3) we get

$$(5.6) \quad (E_1)_\infty = \frac{r_1}{q_1} \left(\frac{p_3q_1 - c_3}{p_3q_3} - \frac{l_1c_3^2c_4p_3p_1p_4q_2}{c_3^2p_4^2q_2^2 + c_4^2p_3^2q_1^2} \right),$$

$$(5.7) \quad (E_2)_\infty = \frac{r_1}{q_2} \left(\frac{p_1c_3^2p_4^2q_2^2}{c_3^2p_4^2q_2^2 + c_4^2p_3^2q_1^2} - q_5 - l_2 \left(\frac{c_4c_5p_4p_5q_2q_3}{c_4^2p_5^2q_3^2 + c_5^2p_4^2q_2^2} \right) \right),$$

$$(5.8) \quad (E_3)_\infty = \frac{r_1}{q_3} \left(\frac{p_2c_4p_4q_2p_5^2q_3^2}{c_4^2p_5^2q_3^2 + c_5^2p_4^2q_2^2} - q_4 \right).$$

For, $(E_1)_\infty, (E_2)_\infty, (E_3)_\infty$ are to be positive if

$$(5.9) \quad A_P > B_P, C_P > D_P, E_P > F_P,$$

where

$$A_p = \frac{p_3 q_1 - c_3}{p_3 q_3}, B_p = \frac{l_1 c_3^2 c_4 p_3 p_1 p_4 q_2}{c_3^2 p_4^2 q_2^2 + c_4^2 p_3^2 q_1^2}, C_p = \frac{p_1 c_3^2 p_4^2 q_2^2}{c_3^2 p_4^2 q_2^2 + c_4^2 p_3^2 q_1^2} - q_5,$$

$$D_p = l_2 \left(\frac{c_4 c_5 p_4 p_5 q_2 q_3}{c_4^2 p_5^2 q_3^2 + c_5^2 p_4^2 q_2^2} \right), E_P = \frac{p_2 c_4 p_4 q_2 p_5^2 q_3^2}{c_4^2 p_5^2 q_3^2 + c_5^2 p_4^2 q_2^2}, F_p = q_4.$$

Thus, bionomic steady state

$$((n_1)_\infty, (n_2)_\infty, (n_3)_\infty, (E_1)_\infty, (E_2)_\infty, (E_3)_\infty)$$

occurs if (5.9) satisfied.

6. Optimal harvesting strategy

This section deals with feasible harvesting procedure of (2.4)-(2.6). we use the pontryagin's principle to attain an optimal path. For this we construct present value function

$$(6.1) \quad J = \int_0^\infty P(n_1, n_2, n_3, E_1, E_2, E_3, t) e^{-\delta t}.$$

Where

$$(6.2) \quad \begin{aligned} P(n_1, n_2, n_3, E_1, E_2, E_3, t) = & (p_1 q_1 n_1 E_1 - c_3 E_1) + (p_2 q_2 n_2 E_2 - c_4 E_2) \\ & + (p_3 q_3 n_3 E_3 - c_5 E_3) \end{aligned}$$

And δ is the instantaneous annual rate of the discount. To maximize J with respect to (2.4)-(2.6), consider the Hamiltonian function

$$(6.3) \quad \begin{aligned} H = & e^{-\delta t} (p_1 q_1 n_1 - c_3) E_1 + e^{-\delta t} (p_2 q_2 n_2 - c_4) E_2 + e^{-\delta t} (p_3 q_3 n_3 - c_5) E_3 \\ & + \lambda_1 \left[n_1 (1 - n_1) - \frac{l_1 n_1^2 n_2}{(n_2^2 + n_1^2)} - \frac{q_1 E_1 n_1}{r_1} \right] \\ & + \lambda_2 \left(\frac{p_1 n_1^2 n_2}{(n_2^2 + n_1^2)} - q_5 n_2 - \frac{l_2 n_2^2 n_3}{(n_3^2 + n_2^2)} - \frac{q_2 E_2 n_2}{r_1} \right) \\ & + \lambda_3 \left(\frac{p_2 n_2^2 n_3}{(n_3^2 + n_2^2)} - q_4 n_3 - \frac{q_3 E_3 n_3}{r_1} \right), \end{aligned}$$

where $\lambda_1, \lambda_2, \lambda_3$ are adjoint variables and E_1, E_2, E_3 are the control variables satisfying the constraints $0 \leq E_1 \leq (E_1)_{\max}, 0 \leq E_2 \leq (E_2)_{\max}, 0 \leq E_3 \leq (E_3)_{\max}$, and $\psi_1(t) = e^{-\delta t} [p_1 q_1 n_1 - c_3] - \lambda_1 q_1 n_1, \psi_2(t) = e^{-\delta t} [p_2 q_2 n_2 - c_4] - \lambda_2 q_2 n_2, \psi_3(t) = e^{-\delta t} [p_3 q_3 n_3 - c_5] - \lambda_3 q_3 n_3$.

In order to find a feasible steady state $((n_1)_\delta, (n_2)_\delta, (n_3)_\delta, (E_1)_\delta, (E_2)_\delta, (E_3)_\delta)$ if H is linear in E_1 and E_2 ideal control can be extreme, thus we have $E_i = (E_i)_{\max}, \psi_i(t) > 0, E_i = 0, \psi_i(t) < 0$, for $i = 1, 2, 3$.

If $\psi_1(t) = \psi_2(t) = \psi_3(t) = 0$, then $\lambda_1 e^{\delta t} = p_1 - \frac{c_3}{q_1 n_1}$, $\lambda_2 e^{\delta t} = p_2 - \frac{c_4}{q_2 n_2}$, $\lambda_3 e^{\delta t} = p_3 - \frac{c_5}{q_3 n_3}$

$$(6.4) \quad \frac{\partial H}{\partial E_i} = 0,$$

where $i = 1, 2, 3$ With the help of ideal controls, and (6.4) are necessary conditions for the maximization H . By pontryagin's rule, the equations are

$$(6.5) \quad \begin{aligned} \frac{\partial \lambda_1}{\partial t} = & -\frac{\partial H}{\partial n_1} - \left[e^{-\delta t} p_1 q_1 + \lambda_1 \left[1 - 2n_1 - \frac{l_1 n_2 2n_1 n_2^2}{(n_1^2 + n_2^2)^2} - \frac{q_1 E_1}{r_1} \right] \right. \\ & \left. + \frac{\lambda_2 p_1 n_2 2n_1 n_2^2}{(n_1^2 + n_2^2)^2} \right], \end{aligned}$$

$$(6.6) \quad \begin{aligned} \frac{\partial \lambda_2}{\partial t} = & -\frac{\partial H}{\partial n_2} = - \left[e^{-\delta t} p_2 q_2 + \lambda_1 \left[\frac{l_1 n_1^2 (n_2^2 - n_1^2)}{(n_1^2 + n_2^2)^2} \right] \right. \\ & \left. + \lambda_2 \left[p_1 n_1^2 \left[\frac{n_2^2 - n_1^2}{(n_1^2 + n_2^2)^2} \right] - q_5 \right] \right. \\ & \left. + 2\lambda_2 \left[\left[\frac{l_2 n_3^3 n_2}{(n_3^2 + n_2^2)^2} \right] - \frac{q_2 E_2}{r_1} \right] + \lambda_3 \left[\frac{2p_2 n_3^3 n_2}{(n_3^2 + n_2^2)^2} \right] \right], \end{aligned}$$

$$(6.7) \quad \begin{aligned} \frac{\partial \lambda_3}{\partial t} = & -\frac{\partial H}{\partial n_3} = - \left[e^{-\delta t} p_3 q_3 - \lambda_2 \left[\frac{l_2 n_2^2 (n_3^2 - n_2^2)}{(n_3^2 + n_2^2)^2} \right] \right. \\ & \left. + \lambda_3 \left[\frac{p_2 n_2^2 (n_3^2 - n_2^2)}{(n_3^2 + n_2^2)^2} - q_4 - \frac{q_3 E_3}{r_1} \right] \right]. \end{aligned}$$

At $E^*(n_1^*, n_2^*, n_3^*)$ and from (6.4), (6.5), (6.6) and (6.7) we have

$$\frac{d\lambda_1}{dt} + \lambda_1 M_1 = -e^{-\delta t} M_2, \quad \frac{d\lambda_2}{dt} + \lambda_2 M_3 = -e^{-\delta t} M_4, \quad \frac{d\lambda_3}{dt} + \lambda_3 M_5 = -e^{-\delta t} M_6.$$

And solutions are

$$(6.8) \quad \lambda_1 = \frac{M_2}{(-M_1 + \delta)} e^{-\delta t}, \quad \lambda_2 = \frac{M_4}{(\delta - M_3)} e^{-\delta t}, \quad \lambda_3 = \frac{M_6}{(\delta - M_5)} e^{-\delta t},$$

where

$$\begin{aligned} M_1 = & \left[1 - 2n_1 - l_1 n_2 \frac{2n_1 n_2^2}{(n_1^2 + n_2^2)^2} - \frac{q_1 E_1}{r_1} \right], \\ M_2 = & \left[p_1 q_1 + \left(p_2 - \frac{c_4}{q_2 n_2} \right) \left[p_1 n_2 \left[\frac{2n_1 n_2^2}{(n_1^2 + n_2^2)^2} \right] \right] \right], \end{aligned}$$

$$\begin{aligned}
M_3 &= p_1 n_1^2 \left[\frac{n_2^2 - n_1^2}{(n_1^2 + n_2^2)^2} \right] - q_5 + 2 \left[\frac{l_2 n_3^3 n_2}{(n_3^2 + n_2^2)^2} \right] - \frac{q_2 E_2}{r_1}, \\
M_4 &= -p_2 q_2 - \left(p_1 - \frac{c_3}{q_2 n_2} \right) \left[\frac{l_1 n_1^2 (n_2^2 - n_1^2)}{(n_1^2 + n_2^2)^2} \right] + \left(p_3 - \frac{c_5}{q_3 n_3} \right) \left[\frac{2 p_2 n_3^3 n_2}{(n_3^2 + n_2^2)^2} \right], \\
M_5 &= \frac{p_2 n_2^2 (n_3^2 - n_2^2)}{(n_3^2 + n_2^2)^2} - q_4 - \frac{q_3 E_3}{r_1}, \\
M_6 &= (p_3 q_3 - (p_2 - \frac{c_4}{q_2 n_2}) \left[\frac{l_2 n_2^2 (n_3^2 - n_2^2)}{(n_3^2 + n_2^2)^2} \right].
\end{aligned}$$

It is obviously that $\lambda_1(t), \lambda_2(t), \lambda_3(t)$ are bounded as $t \rightarrow \infty$. From (6.4) & (6.8), we obtain a singular paths

$$p_1 - \frac{c_3}{q_1 N_1^*} = \frac{M_2}{\delta - M_1}, p_2 - \frac{c_4}{q_2 N_2^*} = \frac{M_4}{\delta - M_3}, p_3 - \frac{c_5}{q_3 N_3^*} = \frac{M_6}{\delta - M_5}.$$

These singular paths can be written as

$$F_1(N_1^*) = \left(p_1 - \frac{c_3}{q_1 N_1^*} \right) - \frac{M_2}{\delta - M_1},$$

$$F_2(N_2^*) = \left(p_2 - \frac{c_4}{q_2 N_2^*} \right) - \frac{M_4}{\delta - M_3}, F_3(N_3^*) = \left(p_3 - \frac{c_5}{q_3 N_3^*} \right) - \frac{M_5}{\delta - M_5}.$$

There exists a unique roots $n_i^* = (n_i)_\delta, i = 1, 2, 3$ of $F_i(n_i^*) = 0$ in the interval $0 < (n_i)_\infty < k_i$ if the inequalities $F_i(0) < 0, F_i(k_i) > 0, F_i^1(n_i^*) > 0$ for $n_i^* > 0$. Therefore, the feasible steady state population are is $n_i^* = (n_i)_\delta, i = 1, 2, 3$.

Then the feasible efforts are

$$\begin{aligned}
E_1^* &= (E_1)_\delta = \frac{r_1}{q_1} \left((1 - n_1) - \frac{l_1 n_1 n_2}{n_1^2 + n_2^2} \right), \\
E_2^* &= (E_2)_\delta = \frac{r_1}{q_2} \left(\frac{p_1 n_1^2}{n_1^2 + n_2^2} - q_5 - \frac{l_2 n_3 n_2}{n_3^2 + n_2^2} \right), \\
E_3^* &= (E_3)_\delta = \frac{r_1}{q_3} \left(\frac{p_2 n_2^2}{n_3^2 + n_2^2} - q_4 \right).
\end{aligned}$$

For $(E_i)_\delta > 0, i = 1, 2, 3$, are to be affirmative, if

$$(1 - n_1) > \frac{l_1 n_1 n_2}{n_1^2 + n_2^2}, \frac{p_1 n_1^2}{n_1^2 + n_2^2} > q_5 + \frac{l_2 n_3 n_2}{n_3^2 + n_2^2}, \frac{p_2 n_2^2}{n_3^2 + n_2^2} > q_4$$

exists. Hence $(n_i)_\delta, (E_i)_\delta, i = 1, 2, 3$ are resolute, and from (6.3), (6.4) and (6.8), we establish that $\lambda_i(t) e^{\delta t}, (i = 1, 2, 3)$ do not vary with time in feasible steady state. Therefore they keep on bounded as $t \rightarrow \infty$.

7. The Stochastic model

Now, this section is meant for the extension of the deterministic model (2.4)-(2.6), which is obtained by adding noisy term. There are several ways in which environmental noise may be incorporated in the model system (2.4)-(2.6). External noise may arise from random fluctuations of finite number of parameters around some known mean values of the populace densities around some fixed values. The populace intensities of oscillations are calculated near the inner steady states due to environmental attributes by applying the method of [17] and [18]. Since the aquatic ecosystem which always has unsystematic fluctuations of the environment, it is difficult to define the usual phenomenon as a deterministic ideal. The stochastic investigation benefits us to get an extra intuition about the continuous changing aspects of any ecological unit. The deterministic model (2.4)-(2.6) with the effect of random noise of the environmental results in a stochastic system ((7.1)-(7.3)) given in the following discussion.

$$(7.1) \quad n_1'(t) = n_1(1 - n_1) - \frac{l_1 n_1^2 n_2}{n_1^2 + n_2^2} - \frac{q_1 E_1 n_1}{r_1} + \eta_1 \varphi_1(t),$$

$$(7.2) \quad n_2'(t) = \frac{p_1 n_1^2 n_2}{n_1^2 + n_2^2} - q_5 n_2 - \frac{l_2 n_2^2 n_3}{n_1^2 + n_2^2} - \frac{q_2 E_2}{r_1} n_2 + \eta_2 \varphi_2(t),$$

$$(7.3) \quad n_3'(t) = \frac{p_2 n_1^2 n_3}{n_3^2 + n_2^2} - q_4 n_3 - \frac{q_3 E_3}{r_1} n_3 + \eta_3 \varphi_3(t),$$

where η_1, η_2, η_3 are the real constants and $\varphi_1(t) = [\varphi_1(t), \varphi_2(t), \varphi_3(t)]$ is a three dimensional Gaussian white noise process satisfying $E(\varphi_1(t)) = 0, i = 1, 2, 3$, $E[\varphi_i(t)\varphi_j(t)] = \delta_{ij}\delta(t-t^1)$, $i = j = 1, 2, 3$ where δ_{ij} is the Kronecker delta function, δ is the Dirac -delta function. In this analysis, we focus on the dynamics of the model (7.1)-(7.3) and we compute the population variances around E^* .

Let

$$n_1(t) = u_1(t) + S^*, n_2(t) = u_2(t) + P^*, n_3(t) = u_3(t) + T^*,$$

then

$$\frac{dn_1}{dt} = \frac{du_1}{dt}, \frac{dn_2}{dt} = \frac{du_2}{dt}, \frac{dn_3}{dt} = \frac{du_3}{dt}.$$

Utilizing the above and taking only the linear part, we have

$$\frac{du_1}{dt} = -u_1 s^* - u_2 (S^*)^2 + \eta_1 \varphi_1(t),$$

$$\frac{du_2}{dt} = u_2 (P^*)^2 + \eta_2 \varphi_2(t), \frac{du_3}{dt} = \eta_3 \varphi_3(t).$$

Now applying Fourier transform, we get

$$i\omega \tilde{u}_1(\omega) = -S^* \tilde{u}_1(\omega) - (S^*)^2 \tilde{u}_2(\omega) + \eta_1 \tilde{\varphi}_1(t),$$

$$i\omega \tilde{u}_2(\omega) = -(P^*)^2 \tilde{u}_2(\omega) + \eta_2 \tilde{\varphi}_2(t), i\omega \tilde{u}_3(\omega) = \eta_3 \tilde{\varphi}_3(t).$$

The matrix form can be written as

$$(7.4) \quad \tilde{\varphi}(\omega) = A(\omega)\tilde{u}(\omega)$$

where

$$A(\omega) = \begin{bmatrix} (i\omega + S^*) & (S^*)^2 & 0 \\ 0 & i\omega + (P^*)^2 & 0 \\ 0 & 0 & i\omega \end{bmatrix}, \tilde{u}(\omega) = \begin{bmatrix} \tilde{u}_1(\omega) \\ \tilde{u}_2(\omega) \\ \tilde{u}_3(\omega) \end{bmatrix}, \tilde{\varphi}(\omega) = \begin{bmatrix} \eta_1 \tilde{\varphi}_1(\omega) \\ \eta_2 \tilde{\varphi}_2(\omega) \\ \eta_3 \tilde{\varphi}_3(\omega) \end{bmatrix}$$

(7.4) can also be expressed as $\tilde{u}(\omega) = [A(\omega)]^{-1}\tilde{\varphi}(\omega) = B(\omega)\tilde{\varphi}(\omega)$, and $B(\omega) = [A(\omega)]^{-1}$. The solution of (7.4) is

$$\tilde{u}(\omega) = \sum_{j=1}^3 B_{ij}(\omega)\eta_j \tilde{\varphi}_j(\omega), i = 1, 2, 3.$$

The strengths of oscillations of u_i , $i = 1, 2, 3$ are

$$\sigma_{u_i}^2 = \frac{1}{2\pi} \sum_{j=1}^3 \int_{-\infty}^{\infty} \eta_j |B_{ij}(\omega)|^2 d\omega, i = 1, 2, 3,$$

where

$$B_{ij} = \frac{G_{ij}(\omega)}{\det A(\omega)}, i = 1, 2, 3.$$

Using (7.4), The populace variances of (7.1)-(7.3) are as follows,

$$\sigma_{u_i}^2 = \frac{\eta_1}{2\pi} \int_{-\infty}^{\infty} \frac{|G_{ij}(\omega)|^2}{|\det A(\omega)|^2} d\omega + \frac{\eta_2}{2\pi} \int_{-\infty}^{\infty} \frac{|G_{ij}(\omega)|^2}{|\det A(\omega)|^2} d\omega + \frac{\eta_3}{2\pi} \int_{-\infty}^{\infty} \frac{|G_{ij}(\omega)|^2}{|\det A(\omega)|^2} d\omega,$$

where

$$G_{mn} = X_{mn} + iY_{mn}, m, n = 1, 2, 3, X_{11} = -\omega^2, Y_{11} = \omega P^*, X_{12} = 0, Y_{12} = \omega S^*$$

$$X_{13} = 0, Y_{13} = 0, X_{21} = 0, Y_{21} = 0, X_{22} = \omega^2, Y_{22} = \omega S^*, X_{23} = 0, Y_{23} = 0,$$

$$X_{31} = 0, Y_{31} = 0, X_{32} = 0, Y_{32} = 0, X_{33} = -\omega^2 + S^* P^{*2}, Y_{33} = \omega(S^* + P^{*2}).$$

Hence the population variances of prey, predator and top predator are as

$$\sigma_{u_1}^2 = \frac{\eta_1}{2\pi} \int_{-\infty}^{\infty} \frac{|G_{11}(\omega)|^2}{|\det A(\omega)|^2} d\omega + \frac{\eta_2}{2\pi} \int_{-\infty}^{\infty} \frac{|G_{12}(\omega)|^2}{|\det A(\omega)|^2} d\omega,$$

$$\sigma_{u_2}^2 = \frac{1}{2\pi} \left\{ \eta_2 \int_{-\infty}^{\infty} \frac{|G_{22}(\omega)|^2}{|\det A(\omega)|^2} d\omega \right\}, \sigma_{u_3}^2 = \frac{1}{2\pi} \left\{ \eta_3 \int_{-\infty}^{\infty} \frac{|G_{33}(\omega)|^2}{|\det A(\omega)|^2} d\omega \right\}.$$

Now, we come across, the following cases.

Case (i). If $\eta_1 = \eta_2 = 0$, then

$$\sigma_{u_1}^2 = 0, \sigma_{u_2}^2 = 0, \sigma_{u_3}^2 = \frac{\eta_3}{2\pi} \int_{-\infty}^{\infty} \frac{((- \omega^2 + S^* P^{*2})^2 + (\omega(S^* + P^*)^2))}{R^2(\omega) + I^2(w)} d\omega$$

Case(ii). If $\eta_2 = \eta_3 = 0$, then

$$\sigma_{u_1}^2 = \frac{\eta_1}{2\pi} \int_{-\infty}^{\infty} \frac{(\omega^4 + \omega^2 P^{*2})}{R^2(\omega) + I^2(w)} d\omega, \sigma_{u_2}^2 = 0, \sigma_{u_3}^2 = 0$$

Case (iii). If $\eta_1 = \eta_3 = 0$, then

$$\sigma_{u_1}^2 = \frac{\eta_1}{2\pi} \int_{-\infty}^{\infty} \frac{(\omega^2 S^{*2})}{R^2(\omega) + I^2(w)} d\omega, \sigma_{u_2}^2 = \frac{\eta_2}{2\pi} \int_{-\infty}^{\infty} \frac{(\omega^4 + \omega^2 S^{*2})}{R^2(\omega) + I^2(w)} d\omega, \sigma_{u_3}^2 = 0.$$

Further, for stochastic system, populace variances play an affective role in stability analysis of the system. In the presence of environmental variable, the parameters of the system oscillate in populace densities around the inner state. Hence we conclude that inclusion of stochastic perturbation creates a significant change in the entire changing aspects and in the proposed model system due to change of responsive parameters can create more effective large environmental oscillations.

8. Diffusion analysis

The current article deals with a class of extended tri trophic prey predator systems in environmental science, modelled by diffusion equations. Although the dispersal system is a relatively simple model for the raid of prey species by predators in a spatial domain, the solutions exhibit an extensive spectrum of ecologically pertinent behaviour. Spatiotemporal dynamics includes chaos, target patterns [19,20,21]. The study of such spatiotemporal dynamics is an intensive area of research and there are still many unanswered questions concerning these solution types [21,22,23]. By constructing a structure consists of prey, predator and top predator system with constant harvesting rates. The populace of the system are prey, predator and top predator. The populations are subject to dispersal. The spread of the population is observed by the pattern. These are two kinds of spread (i) The propagation of continuous travelling population fronts of high species density. (ii) The formation & movement of paths of high density separated by areas with density close to zero. The actual dynamics of the species spread is a result of the inter play between diffusion and deterministic factors. We shall study the effect of diffusion of ecological population on the model system. Let us consider the diffusive equation system

as

$$(8.1) \quad n_1'(t) = n_1(1 - n_1) - \frac{l_1 n_1^2 n_2}{n_1^2 + n_2^2} - \frac{q_1 E_1 n_1}{r_1} + D_1 \frac{\partial^2 n_1}{\partial u^2},$$

$$(8.2) \quad n_2'(t) = \frac{p_1 n_1^2 n_2}{n_1^2 + n_2^2} - q_5 n_2 - \frac{l_2 n_2^2 n_3}{n_1^2 + n_2^2} - \frac{q_2 E_2}{r_1} n_2 + D_2 \frac{\partial^2 n_2}{\partial u^2},$$

$$(8.3) \quad n_3'(t) = \frac{p_2 n_1^2 n_3}{n_3^2 + n_2^2} - q_4 n_3 - \frac{q_3 E_3}{r_1} n_3 + D_3 \frac{\partial^2 n_3}{\partial u^2}.$$

In this D_1, D_2, D_3 represents the constant diffusion coefficients of the prey, predator and super predator. The model system (8.1)-(8.3) are inhomogeneous as the reaction diffusion system. For such introduction of the diffusion term of the populations, it has become a spatiotemporal dynamical system. We consider the following conditions of the population $n_1(u, t)$, $n_2(u, t)$ and $n_3(u, t)$ in $0 \leq u \leq L, L > 0$ as follows

$$\frac{\partial n_1(0, t)}{\partial t} = \frac{\partial n_1(L, t)}{\partial t} = \frac{\partial n_2(0, t)}{\partial t} = 0, \quad \frac{\partial n_2(L, t)}{\partial t} = \frac{\partial n_3(0, t)}{\partial t} = \frac{\partial n_3(L, t)}{\partial t} = 0.$$

The zero isoclines of model equations (8.1)-(8.3) also give the steady state which are same as we have obtained for homogeneous system. Now we linearize the system (8.1)-(8.3) by putting $n_1 = n_1^* + N_{11}, n_2 = n_2^* + N_{22}, n_3 = n_3^* + N_{33}$, in view of inner steady state and we get

$$(8.4) \quad N_{11}'(t) = -l_1 r_1 N_{11} N_{22} n_1^* + D_1 \frac{\partial^2 N_{11}}{\partial u^2},$$

$$(8.5) \quad N_{22}'(t) = D_2 \frac{\partial^2 N_{22}}{\partial u^2},$$

$$(8.6) \quad N_{33}'(t) = D_3 \frac{\partial^2 N_{33}}{\partial u^2}.$$

The solution of (8.4)-(8.6) can be expressed as

$$n_1(u, t) = \alpha_1 e^{\lambda t} e^{iku}, \\ n_2(u, t) = \alpha_2 e^{\lambda t} e^{iku}, n_3(u, t) = \alpha_3 e^{\lambda t} e^{iku}.$$

Then the model becomes

$$(8.7) \quad N_{11}'(t) = -l_1 r_1 N_{11} N_{22} n_1^* + D_1 (-k^2 N_{11}),$$

$$(8.8) \quad N_{22}'(t) = D_2 (-k^2 N_{22}),$$

$$(8.9) \quad N_{33}'(t) = D_3 (-k^2 N_{33}).$$

The characteristic equation of (8.7)-(8.9) is

$$(8.10) \quad \lambda^3 + A\lambda^2 + B\lambda + C = 0$$

where $A = k^2(D_1 + D_2 + D_3) + l_1 r_1 n_1^* N_{22}$, $C = l_1 r_1 n_1^* N_{22} D_1 D_3 k^4 + D_1 D_2 D_3 k^6$, $B = l_1 r_1 N_{22} n_1^* k^2(D_3 + D_2) + k^4(D_1 D_2 + D_2 D_3 + D_1 D_3)$.

By applying Routh-Hurwitz criterion, to satisfy and make it possible if and only if $A > 0$, $C > 0$, $D = C - AB < 0$ (which is definitely possible).

Theorem. The system in the absence of spatiotemporal attributes at the inner steady state (n_1^*, n_2^*, n_3^*) attains steadiness, then the corresponding uniform steady state of the model (8.1)-(8.3) in the presence of spatiotemporal attributes also attains steadiness.

Proof. Consider a function $V_1(t)$ as $V_1(t) = \int_0^R V(n_1, n_2, n_3) du$

$$\begin{aligned} V(n_1, n_2, n_3) &= \left[(n_1 - n_1^*) - n_1^* \ln\left(\frac{n_1}{n_1^*}\right) \right] + l_1 \left[(n_2 - n_2^*) - n_2^* \ln\left(\frac{n_2}{n_2^*}\right) \right] \\ &\quad + l_2 \left[(n_3 - n_3^*) - n_3^* \ln\left(\frac{n_3}{n_3^*}\right) \right], \\ V_1^1(t) &= \int_0^R \left(\frac{\partial v}{\partial n_1} \cdot \frac{\partial n_1}{\partial t} + \frac{\partial v}{\partial n_2} \cdot \frac{\partial n_2}{\partial t} + \frac{\partial v}{\partial n_3} \cdot \frac{\partial n_3}{\partial t} \right) du = I_1 + I_2 \end{aligned}$$

where $I_1 = \int_0^R \frac{dv}{dt} du$ and $I_2 = \int_0^R \left(D_1 \frac{\partial v}{\partial n_1} \frac{\partial^2 n_1}{\partial u^2} + D_2 \frac{\partial v}{\partial n_2} \frac{\partial^2 n_2}{\partial u^2} + D_3 \frac{\partial v}{\partial n_3} \frac{\partial^2 n_3}{\partial u^2} \right) du$

$$I_2 = -D_1 \int_0^R \frac{\partial^2 v}{\partial n_1} \left(\frac{\partial n_1}{\partial u} \right)^2 du - D_2 \int_0^R \frac{\partial^2 v}{\partial n_2} \left(\frac{\partial n_2}{\partial u} \right)^2 du - D_3 \int_0^R \frac{\partial^2 v}{\partial n_3} \left(\frac{\partial n_3}{\partial u} \right)^2 du$$

$$I_2 = -D_1 \int_0^R \frac{n_1^*}{n_1} \left(\frac{\partial n_1}{\partial u} \right)^2 du - D_2 \int_0^R \frac{n_2^*}{n_2} \left(\frac{\partial n_2}{\partial u} \right)^2 du - D_3 \int_0^R \frac{n_3^*}{n_3} \left(\frac{\partial n_3}{\partial u} \right)^2 du$$

It is observed that, if $I_1 < 0$ then $\frac{dV_1}{dt} < 0$. Hence the theorem holds.

9. Numerical simulations

In this section we have validated the analytical findings through numerical data using Technical tool MATLAB

Example 1. For the parameters $l_1 = 8$, $l_2 = 5.3$, $p_1 = 3.1$, $p_2 = 2$, $q_5 = 0.3$, $q_4 = 0.6$, $E_1 = 10$, $E_2 = 15$, $E_3 = 5$, $r_1 = 10$, $q_1 = 0.15$, $q_2 = 0.01$, $q_3 = 0.01$

Example 2. For $l_1 = 8$, $l_2 = 11$, $p_1 = 10$, $p_2 = 2$, $q_5 = 1.5$, $q_4 = 1.5$, $E_1 = 2$, $E_2 = 3$, $E_3 = 4$, $r_1 = 4$, $q_1 = 0.15$, $q_2 = 0.01$, $q_3 = 0.01$

Example 3. For $l_1 = 2$, $l_2 = 11$, $p_1 = 10$, $p_2 = 2$, $q_5 = 1.5$, $q_4 = 1.5$, $E_1 = 2$, $E_2 = 3$, $E_3 = 4$, $r_1 = 4$, $q_1 = 0.15$, $q_2 = 0.01$, $q_3 = 0.01$

Example 4. For $l_1 = 2$, $l_2 = 5.3$, $p_1 = 3.1$, $p_2 = 2$, $q_5 = 0.3$, $q_4 = 0.6$, $E_1 = 10$, $E_2 = 15$, $E_3 = 5$, $r_1 = 10$, $q_1 = 0.15$, $q_2 = 0.01$, $q_3 = 0.01$, $D_1 = 20$, $D_2 = 30$, $D_3 = 40$

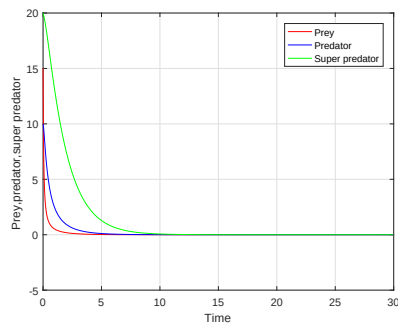
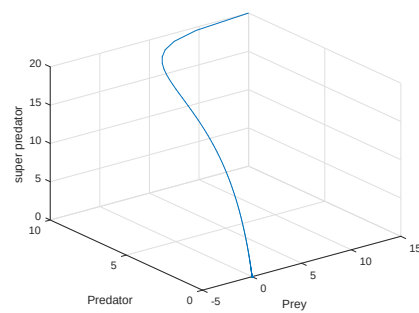
**Figure 1.a****Figure 1.b**

Figure 1: a: shows the differences in populace against time for the set of values of example 1.

Figure 1.b: shows graphically the differences in the population among N_1 , N_2 & N_3 predator values for the values of example 1

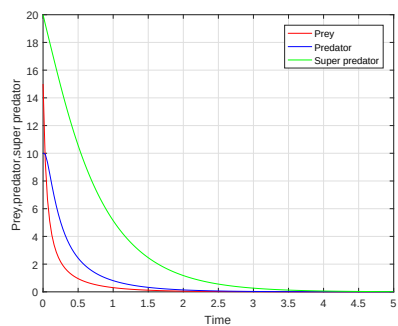
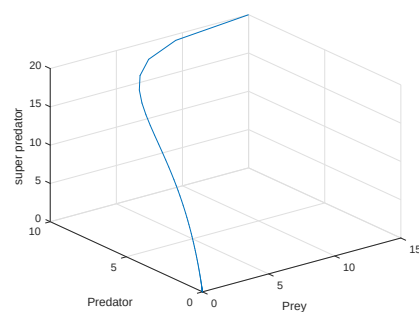
**Figure 2.a****Figure 2.b**

Figure 2: a: shows the differences in populace against time for the set of values of example 2.

Figure 2.b: shows graphically the differences in the population among N_1 , N_2 & N_3 predator values for the values of example 2

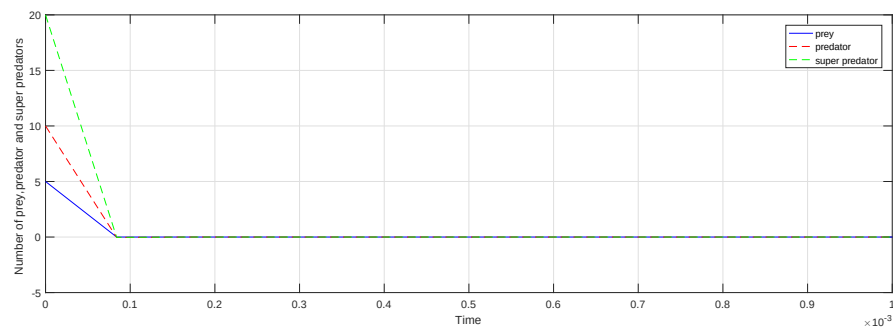


Figure 3

Figure 3: shows populace shades against time for the set of values of example 3.

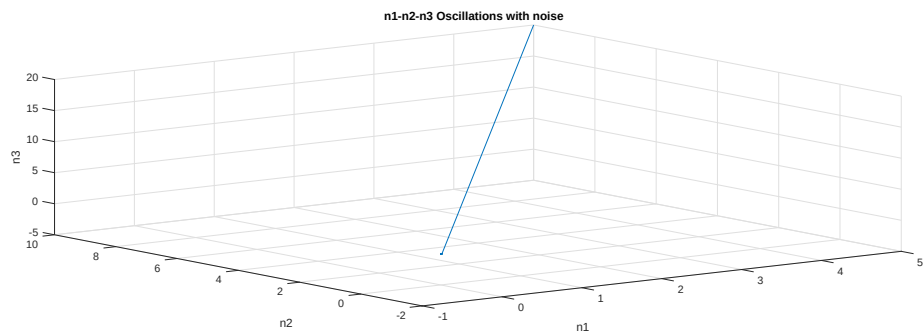


Figure 4

Figure 4: shows graphically how the populace shades differ among N_1 , N_2 , & N_3 for the set of values of example 3.

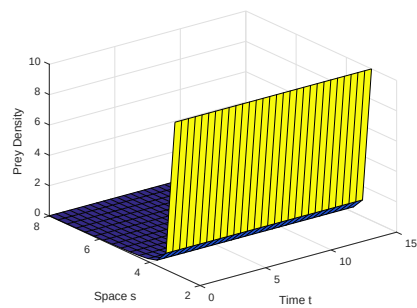


Figure 5.a

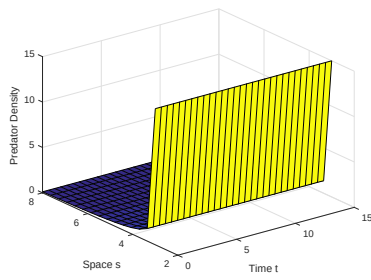
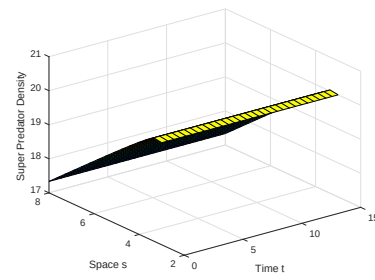
**Figure 5.b****Figure 5.c**

Figure 5: a , 5.b , 5.c represents the stable oscillations of $n_1(u, t)$, $n_2(u, t)$, and $n_3(u, t)$, respectively against time and space for the attributes of example 4.

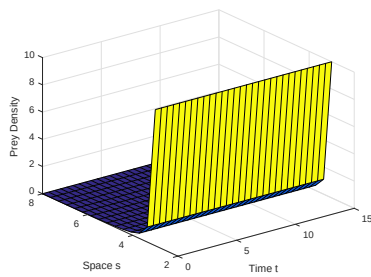
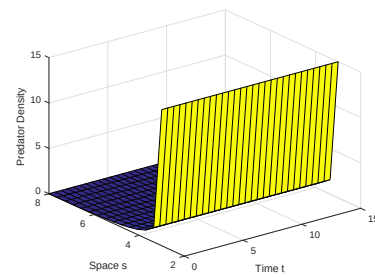
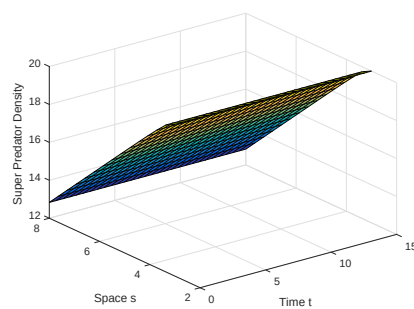
**Figure 6.a****Figure 6.b**

Figure 6: a , 6.b , 6.c represents the stable oscillations of $n_1(u, t)$, $n_2(u, t)$, and $n_3(u, t)$, respectively against time and space for the attributes of example 4.

**Figure 6.c**

10. Concluding remarks

This article mainly concentrates and aims at the most interesting changing aspects of a prey, predator and top predator food chain model with functional response III. We obtain the possible equilibrium points and analysed. Bio-economic and feasible harvesting strategies have been computed using maximum principle. It is shown that the dynamics of deterministic system in the figures (1.a), (1.b), (2.a) and (2.b). It is also studied about the stochasticity of the given system and observed the effect of environmental flux around the positive steady state in the presence of an environmental attribute. The population variances computed and stability concept is also analysed. Graphically Figures (3),(4) shows the impact of noise with suitable set of values. We also verified the steadiness of the spatiotemporal model of the system (8.1)-(8.3) graphically. Figures (5.a), (5.b), (5.c), (6.a), (6.b), (6.c) shows the spatiotemporal steadiness.

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CHARACTERIZATIONS OF ORDERED SEMIHYPERGROUPS BASED ON ORDERED FUZZY POINTS

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Abstract. In this paper, we introduce the concepts of quasi-prime and quasi-semiprime fuzzy left hyperideals of ordered semihypergroups, and investigate their related properties. Furthermore, we give some characterizations of strongly semisimple ordered semihypergroups in terms of ordered fuzzy points and fuzzy left hyperideals. Especially, we prove that an ordered semihypergroup S is strongly semisimple if and only if every fuzzy left hyperideal of S can be expressed as the intersection of all quasi-prime fuzzy left hyperideals of S containing it.

Keywords: ordered semihypergroup, ordered fuzzy point, quasi-prime fuzzy left hyperideal, quasi-semiprime fuzzy left hyperideal, strongly semisimple ordered semihypergroup.

1. Introduction

The important concept of a fuzzy set put forth by L.A. Zadeh in 1965 [33] has opened up keen insights and applications in a wide range of scientific fields. Since its inception, the theory of fuzzy sets has developed in many directions and found applications in a wide variety of fields. The study of fuzzy sets and its application to various mathematical contexts has given rise to what is now

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commonly called “fuzzy mathematics”. Fuzzy algebra is an important branch of fuzzy mathematics. The study of fuzzy algebraic structures was started with the introduction of the concept of fuzzy subgroups of a group in the pioneering paper of A. Rosenfeld [24]. The fuzzy algebraic structures play an important role in Mathematics with wide applications in computer sciences, coding theory, theoretical physics, information sciences and topological spaces [11, 21]. Since then, fuzzy sets have been applied to diverse branches of algebra. In [16], N. Kehayopulu and M. Tsingelis applied the concept of fuzzy sets to the theory of ordered semigroups. Then they defined “fuzzy” analogue of several notations, which appeared to be useful in the theory of ordered semigroups. The theory of fuzzy sets on ordered semigroups has been recently developed. For more details, the reader is referred to [17, 18, 25, 28, 31].

In 1934, F. Marty introduced the theory of hyperstructures [20]. He analyzed different properties of hypergroups and applied them to the theory of groups. Thus one can say that hypergroups are suitable generalization of classical groups. Later on, many researchers have worked on algebraic hyperstructures and generalized various classical algebraic structures, for example [9, 15]. One of the main reason which attracts researches towards hyperstructures is its unique property that in hyperstructures composition of two elements is a set, while in classical algebraic structures the composition of two elements is an element. Thus hyperstructures are natural extension of classical algebraic structures. After the pioneering work of F. Marty, algebraic hyperstructures have been intensively studied, both from the theoretical point of view and especially for their applications in other fields such as Euclidean and non-Euclidean geometries, graphs and hypergraphs, fuzzy sets, automata, cryptography, artificial intelligence, codes, probabilities, lattices and so on (see [3]). Several papers and books have been written on algebraic hyperstructures theory, for example, see [5, 8, 9, 14, 29].

We noticed that the relationships between the fuzzy sets and algebraic hyperstructures have been already considered by P. Corsini, B. Davvaz, V. Leoreanu, W.A. Dudek, J. Zhan, K. Hila and others, for instance, the reader can refer to [1, 4, 6, 10, 13, 19, 32, 34, 35]. Recently, D. Heidari and B. Davvaz [12] applied the theory of hyperstructures to ordered semigroups and introduced the concept of ordered semihypergroups, which is a generalization of the concept of ordered semigroups. Also see [7, 22, 23, 26]. It is now natural to investigate similar type of the existing fuzzy subsystems of ordered semihypergroups. As a further study of ordered semihypergroups theory, we attempt in the present paper to study fuzzy left hyperideals of ordered semihypergroups in detail.

The rest of this paper is organized as follows. In Section 2, we recall some basic definitions and results of ordered semihypergroups which will be used throughout this paper. In Section 3, we introduce the concepts of quasi-prime and quasi-semiprime fuzzy left hyperideals in ordered semihypergroups, and give some characterizations of them. We also introduce the concept of fuzzy m -systems of an ordered semihypergroup S , and prove that a fuzzy left hyperideal f of S is quasi-prime if and only if $1 - f$ is a fuzzy m -system of S . In Section 4,

some characterizations of strongly semisimple ordered semihypergroups based on ordered fuzzy points and fuzzy left hyperideals are given. In particular, it is proven that an ordered semihypergroup S is strongly semisimple if and only if every fuzzy left hyperideal of S can be expressed as the intersection of all quasi-prime fuzzy hyperideals of S containing it.

2. Preliminaries and some notations

In this section, we present some definitions and results which will be used throughout this paper.

Recall that a *hypergroupoid* $(S, \circ)$ is a nonempty set S together with a hyperoperation, that is a map $\circ : S \times S \rightarrow P^*(S)$, where $P^*(S)$ denotes the set of all nonempty subsets of S (see [2]). The image of the pair (x, y) is denoted by $x \circ y$. If $x \in S$ and A, B are nonempty subsets of S , then $A \circ B$ is defined by $A \circ B = \bigcup_{a \in A, b \in B} a \circ b$. The notations $A \circ x$ and $x \circ A$ are used for $A \circ \{x\}$ and $\{x\} \circ A$, respectively.

We say that a hypergroupoid $(S, \circ)$ is a *semihypergroup* if the hyperoperation “ $\circ$ ” is associative, that is, $(x \circ y) \circ z = x \circ (y \circ z)$ for all $x, y, z \in S$ (see [3]).

As we know, an ordered semigroup $(S, \cdot, \leq)$ is a semigroup $(S, \cdot)$ with an order relation “ $\leq$ ” such that $a \leq b$ implies $xa \leq xb$ and $ax \leq bx$ for any $x \in S$. In the following, we shall extend the concept of ordered semigroups to the hyper version, and introduce the concept of ordered semihypergroups from [12].

Definition 2.1. An algebraic hyperstructure $(S, \circ, \leq)$ is called an *ordered semihypergroup* (also called *po-semihypergroup* in [12]) if $(S, \circ)$ is a semihypergroup and $(S, \leq)$ is a partially ordered set such that: for any $x, y, a \in S$, $x \leq y$ implies $a \circ x \preceq a \circ y$ and $x \circ a \preceq y \circ a$. Here, if $A, B \in P^*(S)$, then we say that $A \preceq B$ if for every $a \in A$ there exists $b \in B$ such that $a \leq b$. In particular, if $A = \{a\}$, then we write $a \preceq B$ instead of $\{a\} \preceq B$.

Clearly, every ordered semigroup can be regarded as an ordered semihypergroup (see [26]).

Throughout this paper, unless otherwise mentioned, S will denote an ordered semihypergroup.

Let S be an ordered semihypergroup. For $\emptyset \neq H \subseteq S$, we define

$$(H) := \{t \in S \mid t \leq h \text{ for some } h \in H\}.$$

For $H = \{a\}$, we write (a) instead of $(\{a\})$.

By a *subsemihypergroup* of an ordered semihypergroup S we mean a nonempty subset A of S such that $A \circ A \subseteq A$. A nonempty subset A of an ordered semihypergroup S is called a *left* (resp. *right*) *hyperideal* of S if (1) $S \circ A \subseteq A$ (resp. $A \circ S \subseteq A$) and (2) If $a \in A$ and $S \ni b \leq a$, then $b \in A$. If A is both a left and a right hyperideal of S , then it is called a (*two-sided*) *hyperideal* of S (see [12]). We denote by $L(a)$ the left hyperideal of S generated by a ($a \in S$). One can easily prove that $L(a) = (a \cup S \circ a)$. Let L be a left hyperideal of an

ordered semihypergroup S . L is called *quasi-prime* if for any two left hyperideals L_1, L_2 of S such that $L_1 \circ L_2 \subseteq L$, we have $L_1 \subseteq L$ or $L_2 \subseteq L$.

Lemma 2.2 ([27]). *Let S be an ordered semihypergroup. Then the following statements hold:*

- (1) $A \subseteq (A], \quad \forall A \subseteq S$.
- (2) If $A \subseteq B \subseteq S$, then $(A] \subseteq (B]$.
- (3) $(A] \circ (B] \subseteq (A \circ B]$ and $((A] \circ (B]) = (A \circ B], \quad \forall A, B \subseteq S$.
- (4) $((A]) = (A], \quad \forall A \subseteq S$.
- (5) For every left hyperideal T of S , we have $(T] = T$.
- (6) If A, B are left hyperideals of S , then $(A \circ B]$ and $A \cap B$ are left hyperideals of S .
- (7) For every $a \in S$, $(S \circ a]$ is a left hyperideal of S .
- (8) For any two nonempty subsets A, B of S such that $A \preceq B$, we have $C \circ A \preceq C \circ B$ and $A \circ C \preceq B \circ C$ for any nonempty subset C of S .

Definition 2.3. Let M be a nonempty subset of an ordered semihypergroup S . M is called a *m-system* if for any $a, b \in M$, there exists $x \in S$ such that $(a \circ x \circ b) \cap M \neq \emptyset$.

We next state some fuzzy logic concepts.

Let S be an ordered semihypergroup. By a *fuzzy subset* of S , we mean a function from S into the real closed interval $[0, 1]$, that is, $f : S \rightarrow [0, 1]$. For an ordered semihypergroup S , the fuzzy subset 1 of S is defined as follows:

$$1 : S \rightarrow [0, 1], \quad x \mapsto 1(x) := 1, \quad \forall x \in S.$$

Let f and g be two fuzzy subsets of S . Then the inclusion relation $f \subseteq g$ is defined by $f(x) \leq g(x)$ for all $x \in S$, and $1 - f, f \cap g, f \cup g$ are defined by

$$\begin{aligned} (1 - f)(x) &= 1 - f(x), \\ (f \cap g)(x) &= f(x) \wedge g(x), \\ (f \cup g)(x) &= f(x) \vee g(x), \end{aligned}$$

for all $x \in S$, respectively. We denote by $F(S)$ the set of all fuzzy subsets of S . One can easily show that $(F(S), \subseteq, \cap, \cup)$ forms a complete lattice with the maximum element 1 and the minimum element 0, which is a mapping from S into $[0, 1]$ defined by

$$0 : S \rightarrow [0, 1], \quad x \mapsto 0(x) := 0, \quad \forall x \in S.$$

Let $(S, \circ, \leq)$ be an ordered semihypergroup. For $x \in S$, we define $H_x := \{(y, z) \in S \times S \mid x \preceq y \circ z\}$. For any $f, g \in F(S)$, the product $f * g$ of f and g is defined by

$$(\forall x \in S) (f * g)(x) = \begin{cases} \bigvee_{(y,z) \in H_x} [f(y) \wedge g(z)], & \text{if } H_x \neq \emptyset, \\ 0, & \text{if } H_x = \emptyset. \end{cases}$$

As we know, the multiplication “ $*$ ” on $F(S)$ is associative and $(F(S), *, \subseteq)$ forms an ordered semigroup (see [27]).

Let S be an ordered semihypergroup. A fuzzy subset f of S is called a *fuzzy left* (resp. *right*) *hyperideal* of S if

- (1) $x \leq y$ implies $f(x) \geq f(y)$, for all $x, y \in S$, and
- (2) $\bigwedge_{z \in x \circ y} f(z) \geq f(y)$ (resp. $\bigwedge_{z \in x \circ y} f(z) \geq f(x)$) for all $x, y \in S$. Equivalently, $1 * f \subseteq f$ (resp. $f * 1 \subseteq f$).

A *fuzzy hyperideal* of S is a fuzzy subset of S which is both a fuzzy left and a fuzzy right hyperideal of S (see [23, 27]).

Lemma 2.4. *Let $\{f_i \mid i \in I\}$ be a family of fuzzy left hyperideals of an ordered semihypergroup S . Then $f := \bigcup_{i \in I} f_i$ is a fuzzy left hyperideal of S , where $(\bigcup_{i \in I} f_i)(x) = \bigvee_{i \in I} (f_i(x))$.*

Proof. The proof is straightforward verification, and hence we omit the details.

Definition 2.5. Let S be an ordered semihypergroup and $f \in F(S)$. The set

$$f_t := \{x \in S \mid f(x) \geq t\}, \text{ where } t \in (0, 1]$$

is called a *level subset* of f .

Lemma 2.6 ([27]). *Let S be an ordered semihypergroup and $f \in F(S)$. Then f is a fuzzy left hyperideal of S if and only if the level subset f_t ($t \in (0, 1]$) of f is a left hyperideal of S for $f_t \neq \emptyset$.*

Let A be a nonempty subset of an ordered semihypergroup S . We define a fuzzy subset λf_A ($\lambda \in (0, 1]$) of S as follows:

$$(\forall x \in S) \lambda f_A(x) = \begin{cases} \lambda, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

Clearly, λf_A is a generalization of the characteristic mapping f_A of A .

Lemma 2.7 ([27]). *Let A, B be any nonempty subsets of an ordered semihypergroup S . Then the following statements are true:*

- (1) $A \subseteq B$ if and only if $\lambda f_A \subseteq \lambda f_B$.
- (2) $\lambda f_A * \lambda f_B = \lambda f_{(A \circ B)}$. In particular, $f_A * f_B = f_{(A \circ B)}$.
- (3) A is a left hyperideal of S if and only if λf_A is a fuzzy left hyperideal of S .

Let S be an ordered semihypergroup, $a \in S$ and $\lambda \in [0, 1]$. An *ordered fuzzy point* a_λ of S is defined by the rule that

$$(\forall x \in S) a_\lambda(x) = \begin{cases} \lambda, & \text{if } x \in (a], \\ 0, & \text{if } x \notin (a]. \end{cases}$$

It is evident that every ordered fuzzy point of S is a fuzzy subset of S . For any fuzzy subset f of S , we also denote $a_\lambda \subseteq f$ by $a_\lambda \in f$ in the sequel (see [27]).

Definition 2.8 ([27]). Let f be a fuzzy subset of an ordered semihypergroup S . We define $(f]$ by the rule that

$$(f](x) = \bigvee_{y \geq x} f(y),$$

for all $x \in S$. A fuzzy subset f of S is called *strongly convex* if $f = (f]$.

Lemma 2.9 ([27]). If f is a strongly convex fuzzy subset of an ordered semihypergroup S , then $f = \bigcup_{a_\lambda \in f} a_\lambda$.

Lemma 2.10 ([27]). Let a_λ, b_μ ($\lambda > 0, \mu > 0$) be ordered fuzzy points of an ordered semigroup S , and $f, g, h \in F(S)$. Then the following statements are true:

$$(1) (\forall x \in S) (1 * a_\lambda)(x) = \begin{cases} \lambda, & \text{if } x \in (S \circ a], \\ 0, & \text{if } x \notin (S \circ a], \end{cases} \text{ and } 1 * a_\lambda \text{ is a fuzzy left}$$

hyperideal of S .

(2) $(a_\lambda * b_\mu) * c_\delta = a_\lambda * (b_\mu * c_\delta) = \bigcup_{d \in (a \circ b \circ c]} d_{\lambda \wedge \mu \wedge \delta}$ for any ordered fuzzy point a_λ, b_μ and c_δ of S .

$$(3) L(a_\lambda) = a_\lambda \cup 1 * a_\lambda.$$

$$(4) (L(a_\lambda))^2 \subseteq 1 * a_\lambda.$$

$$(5) \text{ If } S \text{ is commutative, then } f * 1 = 1 * f.$$

$$(6) (g \cup h) * f = (g * f) \cup (h * f).$$

The reader is referred to [3, 30] for notation and terminology not defined in this paper.

3. Quasi-prime and quasi-semiprime fuzzy left hyperideals of ordered semihypergroups

In what follows, we denote by $\mathbb{Z}^+$ the set of positive integers. In the current section we define and study the quasi-prime and quasi-semiprime fuzzy left hyperideals of ordered semihypergroups, and give some characterizations of them.

Definition 3.1. Let S be an ordered semihypergroup. A fuzzy left hyperideal f of S is called *quasi-prime* if for any two fuzzy left hyperideals g and h of S , $g * h \subseteq f$ implies $g \subseteq f$ or $h \subseteq f$.

Theorem 3.2. Let L be a nonempty subset of an ordered semihypergroup S . Then a left hyperideal L is quasi-prime if and only if the characteristic function f_L of L is a quasi-prime fuzzy left hyperideal of S .

Proof. Let L be a quasi-prime left hyperideal of S . Then, by Lemma 2.7(3), f_L is a fuzzy left hyperideal of S . For any two fuzzy left hyperideals g and h of S ,

if $g * h \subseteq f_L$, then $g \subseteq f_L$ or $h \subseteq f_L$. In fact, if $g \not\subseteq f_L$ and $h \not\subseteq f_L$, then there exist $x, y \in S$ such that $g(x) > f_L(x), h(y) > f_L(y)$. Thus we have

$$g(x) > 0, h(y) > 0, f_L(x) = f_L(y) = 0.$$

It implies that $x, y \notin L$. We now show that there exists $s \in S$ such that $(x \circ s \circ y] \not\subseteq L$. Indeed, if $(x \circ S \circ y] \subseteq L$, then $(S \circ x] \circ (S \circ y] \subseteq L$. By Lemma 2.2(7), $(S \circ x]$ and $(S \circ y]$ are left hyperideals of S . Since L is a quasi-prime left hyperideal of S , it can be obtained that $(S \circ x] \subseteq L$ or $(S \circ y] \subseteq L$. Let $(S \circ x] \subseteq L$. Then

$$(L(x))^2 = (x \cup S \circ x] \circ (x \cup S \circ x] \subseteq (S \circ x] \subseteq L.$$

Thus $x \in L(x) \subseteq L$, which is impossible. From $(S \circ y] \subseteq L$, similarly, we get a contradiction. Now if $a \in (x \circ s \circ y]$ such that $a \notin L$, then $f_L(a) = 0$, and there exists $z \in s \circ y$ such that $a \preceq x \circ z$. Thus

$$\begin{aligned} (g * h)(a) &= \bigvee_{(p,q) \in H_a} [g(p) \wedge h(q)] \geq g(x) \wedge h(z) \\ &\geq g(x) \wedge \left(\bigwedge_{z \in s \circ y} h(z) \right) \geq g(x) \wedge h(y) > 0, \end{aligned}$$

which contradicts the fact that $g * h \subseteq f_L$. Therefore, f_L is a quasi-prime fuzzy left hyperideal of S .

Conversely, suppose that f_L is a quasi-prime fuzzy left hyperideal of S . Let L_1, L_2 are left hyperideals of S such that $L_1 \circ L_2 \subseteq L$. Then, by Lemma 2.2, $(L_1 \circ L_2] \subseteq (L] = L$. Thus, by Lemma 2.7, we have

$$f_{L_1} * f_{L_2} = f_{(L_1 \circ L_2]} \subseteq f_L.$$

By hypothesis and Lemma 2.7(3), since f_L is quasi-prime, it can be shown that $f_{L_1} \subseteq f_L$ or $f_{L_2} \subseteq f_L$, which implies that $L_1 \subseteq L$ or $L_2 \subseteq L$. This completes the proof.

Lemma 3.3. *Let S be an ordered semihypergroup. If f is a nonconstant quasi-prime fuzzy left hyperideal of S , then $|Im(f)| = 2$.*

Proof. Since f is a nonconstant quasi-prime fuzzy left hyperideal of S , we have $|Im(f)| \geq 2$. Suppose $|Im(f)| \geq 3$. Then there exist $x, y, z \in S$ such that $f(x), f(y)$ and $f(z)$ are different from each other. Without loss of generality, it can be assumed that

$$f(x) < f(y) < f(z).$$

Thus there exist $r, t \in (0, 1)$ such that

$$f(x) < r < f(y) < t < f(z).$$

Then, for any $u \in S$, we have

$$(L(x_r) * L(y_t))(u) = \begin{cases} r \wedge t = r, & u \in (L(x) \circ L(y)), \\ 0, & \text{otherwise.} \end{cases}$$

If $u \in (L(x) \circ L(y))$, then there exist $a \in L(x), b \in L(y)$ such that $u \in (a \circ b)$, and there exists $c \in a \circ b$ such that $u \leq c$. Since f is a fuzzy left hyperideal of S , we have

$$f(u) \geq \bigwedge_{\substack{c \in a \circ b \\ u \leq c}} f(c) \geq \bigwedge_{c \in a \circ b} f(c) \geq f(b).$$

Since $b \in L(y) = (y \cup S \circ y) = (y] \cup (S \circ y]$, we have $b \in (y]$ or $b \in (S \circ y]$. Similar to the previous proof, it can be obtained that $f(b) \geq f(y)$. Hence $f(u) \geq f(y) > r$. It follows that $L(x_r) * L(y_t) \subseteq f$. Thus $L(x_r) \subseteq f$ or $L(y_t) \subseteq f$ because f is a quasi-prime fuzzy left hyperideal of S . Let $L(x_r) \subseteq f$. Then we have $f(x) \geq L(x_r)(x) = r$, which is impossible. From $L(y_t) \subseteq f$, similarly, we get a contradiction. This completes the proof.

Theorem 3.4. *Let S be an ordered semihypergroup. If f is a nonconstant quasi-prime fuzzy left hyperideal of S , then there exists $x_0 \in S$ such that $f(x_0) = 1$.*

Proof. By Lemma 3.3, $|Im(f)| = 2$. If $f(x) \neq 1$ for all $x \in S$, then $Im(f) = \{s, t\}$, $s < t < 1$. Hence there exist $x, y \in S$ and $m \in (0, 1]$ such that

$$f(x) = s < t = f(y) < m \leq 1.$$

Let $t_1, t_2 \in (0, 1)$ such that $s < t_1 < t < t_2 < m$. Then by the similar way of the proof of Lemma 3.3, we have $L(x_{t_1}) * L(y_{t_2}) \subseteq f$. Since f is a quasi-prime fuzzy left hyperideal of S , we have $L(x_{t_1}) \subseteq f$ or $L(y_{t_2}) \subseteq f$. It implies that $f(x) \geq t_1$ or $f(y) \geq t_2$, which is impossible. Thus there exists $x_0 \in S$ such that $f(x_0) = 1$.

Theorem 3.5. *Let S be an ordered semihypergroup. If f is a quasi-prime fuzzy left hyperideal of S , then the level subset f_t ($t \in (0, 1]$) of f is a quasi-prime left hyperideal of S for $f_t \neq \emptyset$.*

Proof. Suppose that f is a quasi-prime fuzzy left hyperideal of S . By Lemma 2.6, for any $t \in (0, 1]$, f_t is a left hyperideal of S for $f_t \neq \emptyset$. To prove that f_t is quasi-prime, let L_1 and L_2 be left hyperideals of S such that $L_1 \circ L_2 \subseteq f_t$, and let $g = tf_{L_1}$ and $h = tf_{L_2}$. Then, by Lemma 2.7(3), g and h are fuzzy left hyperideals of S . Furthermore, we have $g * h \subseteq f$, that is, $(g * h)(x) \leq f(x)$ for all $x \in S$. Indeed, if $(g * h)(x) = 0$, then it is obvious. If $(g * h)(x) \neq 0$, then $H_x \neq \emptyset$, and there exist $y, z \in S$ such that $x \preceq y \circ z$, $0 < g(y) \wedge h(z) \leq t$. Thus $y \in L_1$ and $z \in L_2$, and so $x \in (L_1 \circ L_2] \subseteq (f_t] = f_t$, and $f(x) \geq t$. Consequently,

$$(g * h)(x) = \bigvee_{(y,z) \in H_x} [g(y) \wedge h(z)] \leq t \leq f(x).$$

Hence it can be obtained that $g * h \subseteq f$. Since f is a quasi-prime fuzzy left hyperideal of S , it can be followed that $g \subseteq f$ or $h \subseteq f$. Say $g \subseteq f$, then for any $x \in L_1$, $g(x) = t \leq f(x)$, and $x \in f_t$. Thus $L_1 \subseteq f_t$. Similarly, say $h \subseteq f$, we have $L_2 \subseteq f_t$. Therefore, f_t is a quasi-prime left hyperideal of S for $f_t \neq \emptyset$.

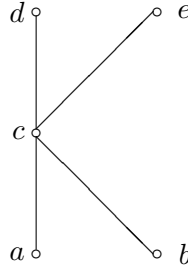
Example 3.6. We consider a set $S := \{a, b, c, d, e\}$ with the following hyperoperation “ $\circ$ ” and the order “ $\leq$ ”:

$\circ$	a	b	c	d	e
a	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$
b	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$
c	$\{a, b\}$	$\{a, b\}$	$\{c\}$	$\{c\}$	$\{e\}$
d	$\{a, b\}$	$\{a, b\}$	$\{c\}$	$\{d\}$	$\{e\}$
e	$\{a, b\}$	$\{a, b\}$	$\{c\}$	$\{c\}$	$\{e\}$

$\leq := \{(a, a), (a, c), (a, d), (a, e), (b, b), (b, c), (b, d), (b, e), (c, c), (c, d), (c, e), (d, d), (e, e)\}$.

We give the covering relation “ $\prec$ ” and the figure of S as follows:

$$\prec = \{(a, c), (b, c), (c, d), (c, e)\}.$$



Then $(S, \circ, \leq)$ is an ordered semihypergroup. With a small amount of effort one can verify that the sets $\{a, b\}$, $\{a, b, c, d\}$, $\{a, b, c, e\}$ and S are all quasi-prime left hyperideals of S . Now let f be a fuzzy subset of S such that $f(a) = f(b) = 0.8$, $f(c) = f(d) = 0.7$, $f(e) = 0.6$. Then

$$f_t = \begin{cases} S, & \text{if } t \in (0, 0.6], \\ \{a, b, c, d\}, & \text{if } t \in (0.6, 0.7], \\ \{a, b\}, & \text{if } t \in (0.7, 0.8], \\ \emptyset, & \text{if } t \in (0.8, 1]. \end{cases}$$

Thus all nonempty level subsets f_t ($t \in (0, 1]$) of f are quasi-prime left hyperideals of S and by Theorem 3.5, f is a quasi-prime fuzzy left hyperideal of S .

By Theorems 3.4 and 3.5, we immediately obtain the following corollary:

Corollary 3.7. *Let S be an ordered semihypergroup. If f is a nonconstant quasi-prime fuzzy left hyperideal of S , then f_1 is a quasi-prime left hyperideal of S .*

Remark 3.8. The inverse of Theorem 3.5 is not true. For example, let L be a quasi-prime left hyperideal of S , $L \neq S$, and

$$f(x) = \begin{cases} \lambda, & \text{if } x \in L, \\ 0, & \text{if } x \notin L, \end{cases}$$

for any $x \in S$, where $0 < \lambda < 1$. Then f is a fuzzy left hyperideal of S . For any $t \in (0, 1]$, if $f_t \neq \emptyset$, then $f_t = L$, which is a quasi-prime left hyperideal of S . But f is not quasi-prime since $f_1 = \emptyset$.

Now, quasi-prime fuzzy left hyperideals of ordered semihypergroups can be characterized.

Theorem 3.9. *Let f be a nonconstant fuzzy subset of an ordered semihypergroup S . Then f is a quasi-prime fuzzy left hyperideal of S if and only if f satisfies the following conditions:*

- (1) $|Im(f)| = 2$.
- (2) $f_1 \neq \emptyset$, and f_1 is a quasi-prime left hyperideal of S .

Proof. Suppose that f is a nonconstant quasi-prime fuzzy left hyperideal of S . Then, by Lemma 3.3, Theorem 3.4 and Corollary 3.7, the conditions (1) and (2) hold.

Conversely, assume that the conditions (1) and (2) hold. Since $|Im(f)| = 2$, by hypothesis we have $Im(f) = \{t, 1\}$ ($t < 1$). Thus

(A) f is a fuzzy left hyperideal of S . To prove this assertion, let $x, y \in S$. We consider the following two cases:

Case 1. If $y \in f_1$, then $f(y) = 1$, and by (2), we have $x \circ y \subseteq S \circ f_1 \subseteq f_1$, which implies that $f(z) = 1$ for any $z \in x \circ y$. Hence $\bigwedge_{z \in x \circ y} f(z) = 1 = f(y)$.

Case 2. If $y \notin f_1$, then $f(y) = t$. Consequently, by hypothesis, $\bigwedge_{z \in x \circ y} f(z) \geq t = f(y)$.

Thus, in both cases, $\bigwedge_{z \in x \circ y} f(z) \geq f(y)$ for all $x, y \in S$. Furthermore, let $x, y \in S$ such that $x \leq y$. Then $f(x) \geq f(y)$. In fact, if $y \notin f_1$, then $f(y) = t \leq f(x)$. If $y \in f_1$, then, since f_1 is a left hyperideal of S , we have $x \in f_1$. Thus $f(x) = 1 = f(y)$.

(B) f is quasi-prime. In fact, let g and h be fuzzy left hyperideals of S such that $g * h \subseteq f$. We claim that $g \subseteq f$ or $h \subseteq f$. If $g \not\subseteq f$ and $h \not\subseteq f$, then there exist $x, y \in S$ such that $g(x) > f(x)$ and $h(y) > f(y)$. Hence $x, y \notin f_1$, which implies $(x \circ S \circ y) \not\subseteq f_1$. Otherwise, by Lemma 2.2, we have

$$(S \circ x] \circ (S \circ y) \subseteq (S \circ (x \circ S \circ y)) \subseteq (S \circ f_1) \subseteq (f_1] \subseteq f_1.$$

Since f_1 is a quasi-prime left hyperideal of S , by Lemma 2.2(7) we have $(S \circ x] \subseteq f_1$ or $(S \circ y) \subseteq f_1$. Say $(S \circ x] \subseteq f_1$, we can deduce that $(L(x))^2 \subseteq (S \circ x] \subseteq f_1$. It follows that $x \in L(x) \subseteq f_1$ because f_1 is quasi-prime. Impossible. Say $(S \circ y) \subseteq f_1$, similarly, we get a contradiction. Thus $(x \circ S \circ y) \not\subseteq f_1$, and there exists $a \in (x \circ S \circ y)$ such that $a \notin f_1$. Then $f(a) = t$ and there exists $s \in S$ such

that $a \preceq x \circ s \circ y$. Thus there exists $b \in s \circ y$ such that $a \preceq x \circ b$. Since $x, y \notin f_1$, by hypothesis we have $f(x) = f(y) = t$. Consequently,

$$\begin{aligned} (g * h)(a) &= \bigvee_{(u,v) \in H_a} [g(u) \wedge h(v)] \geq g(x) \wedge h(b) \\ &\geq g(x) \wedge \left(\bigwedge_{b \in s \circ y} h(b) \right) \geq g(x) \wedge h(y) \\ &\quad (\text{Since } h \text{ is a fuzzy left hyperideal of } S) \\ &> f(x) \wedge f(y) = t = f(a), \end{aligned}$$

which contradicts the fact that $g * h \subseteq f$. Therefore, f is a quasi-prime fuzzy left hyperideal of S .

Definition 3.10. Let S be an ordered semihypergroup. A fuzzy left hyperideal f of S is called *proper* if $f \neq 1$.

Theorem 3.11. Let S be an ordered semihypergroup. If f is a nonconstant quasi-prime fuzzy left hyperideal of S , then there exists a proper quasi-prime fuzzy left hyperideal g of S such that $f \subset g$.

Proof. Let f be a nonconstant quasi-prime fuzzy left hyperideal of S . By Theorem 3.9, there exists $x_0 \in S$ such that $f(x_0) = 1$, and $Im(f) = \{t, 1\}$, where $t < 1$. Let g be a fuzzy subset of S defined by

$$(\forall x \in S) \quad g(x) = \frac{1}{2}f(x) + \frac{1}{2}.$$

Then, it is easy to show that g is a fuzzy left hyperideal of S , and $|Im(g)| = 2$.

On the other hand, since $g_1 = f_1$, by Theorem 3.9, g_1 is a quasi-prime left hyperideal of S and g is a quasi-prime fuzzy left hyperideal of S . Let $y \in S$ such that $f(y) = t$. Then

$$f(y) < \frac{1}{2}(f(y) + 1) = g(y) < 1,$$

which implies that g is a proper quasi-prime fuzzy left hyperideal of S and $f \subset g$. The proof is completed.

We now characterize the quasi-prime fuzzy left hyperideals by ordered fuzzy points.

Theorem 3.12. Let S be an ordered semihypergroup. Then a fuzzy left hyperideal f of S is quasi-prime if and only if for any two ordered fuzzy points x_r, y_t of S ($r > 0, t > 0$), $x_r * 1 * y_t \subseteq f$ implies that $x_r \in f$ or $y_t \in f$.

Proof. Let x_r and y_t are ordered fuzzy points of S such that $x_r * 1 * y_t \subseteq f$. Then

$$(1 * x_r) * (1 * y_t) = 1 * (x_r * 1 * y_t) \subseteq 1 * f \subseteq f.$$

By Theorem 2.10(1), $1 * x_r$ and $1 * y_t$ are fuzzy left hyperideals of S . Since f is quasi-prime, we have $1 * x_r \subseteq f$ or $1 * y_t \subseteq f$. Say $1 * x_r \subseteq f$, then, by Theorem 2.10(4), $(L(x_r))^2 \subseteq 1 * x_r \subseteq f$. Thus $x_r \in L(x_r) \subseteq f$. Similarly, say $1 * y_t \subseteq f$, we have $y_t \in L(y_t) \subseteq f$.

Conversely, let g, h be fuzzy left hyperideals of S such that $g * h \subseteq f$. If $g \not\subseteq f, h \not\subseteq f$, then there exist $x, y \in S$ such that $g(x) > f(x), h(y) > f(y)$. Let $r = g(x), t = h(y)$. Then $r > 0, t > 0, x_r \in g, y_t \in h$, since h is a fuzzy left hyperideal of S , we have

$$x_r * 1 * y_t \subseteq g * 1 * h \subseteq g * h \subseteq f.$$

By hypothesis, $x_r \in f$ or $y_t \in f$. If $x_r \in f$, then $f(x) \geq r = g(x)$, which is impossible. Similarly, if $y_t \in f$, then we get a contradiction. Therefore, f is a quasi-prime fuzzy left hyperideal of S .

In order to characterize the quasi-prime fuzzy left hyperideals of ordered semihypergroups, we need the following concept.

Definition 3.13. Let S be an ordered semihypergroup. A fuzzy subset f of S is called *fuzzy m -system* if for any $s, t \in [0, 1)$ and $a, b \in S$, $f(a) > s, f(b) > t$ imply that there exists $x \in S$ such that $f(y) > s \vee t$ for some $y \in (a \circ x \circ b]$.

Theorem 3.14. Let M be a nonempty subset of an ordered semihypergroup S . Then M is a m -system of S if and only if the characteristic function f_M of M is a fuzzy m -system of S .

Proof. For any $s, t \in [0, 1)$ and $a, b \in S$, if $f_M(a) > s, f_M(b) > t$, then $a, b \in M$. Since M is a m -system of S , there exists $x \in S$ such that $(a \circ x \circ b] \cap M \neq \emptyset$. Let $y \in (a \circ x \circ b] \cap M$. Then $f_M(y) = 1$. Hence $f_M(y) > s \vee t$ for some $y \in (a \circ x \circ b]$. It thus follows that f_M is a fuzzy m -system of S .

Conversely, suppose that f_M is a fuzzy m -system of S . Let $a, b \in M$. Then $f_M(a) = f_M(b) = 1$. Thus for any $s, t \in [0, 1)$, we have

$$f_M(a) > s, f_M(b) > t,$$

which imply that there exists an element $x \in S$ such that $f_M(y) > s \vee t$ for some $y \in (a \circ x \circ b]$ and that $f_M(y) = 1$, that is, $y \in M$. It can be followed that $(a \circ x \circ b] \cap M \neq \emptyset$. Hence M is a m -system of S .

Theorem 3.15. Let f be a proper fuzzy left hyperideal of an ordered semihypergroup S . Then f is quasi-prime if and only if $1 - f$ is a fuzzy m -system of S .

Proof. Suppose that f is a quasi-prime fuzzy left hyperideal of S . For any $s, t \in [0, 1)$, $a, b \in S$, if $(1 - f)(a) > s, (1 - f)(b) > t$, then $f(a) < 1 - s, f(b) < 1 - t$. It implies that $a_{1-s} \notin f$ and $b_{1-t} \notin f$. Since f is a quasi-prime fuzzy left hyperideal

of S , by Theorem 3.12, Lemmas 2.9 and 2.10(2), there exists an ordered fuzzy point x_r of S such that

$$a_{1-s} * x_r * b_{1-t} = \bigcup_{y \in (a \circ x \circ b]} y_{(1-s) \wedge (1-t) \wedge r} \notin f.$$

Thus, there exists $y \in (a \circ x \circ b]$ such that

$$f(y) < (1-s) \wedge (1-t) \wedge r \leq (1-s) \wedge (1-t) = 1 - (s \vee t).$$

which implies that $(1-f)(y) > s \vee t$. We have thus shown that $1-f$ is a fuzzy m -system of S .

Conversely, assume that $1-f$ is a fuzzy m -system of S . Let a_s, b_t of S ($t > 0, s > 0$) such that $a_s * 1 * b_t \subseteq f$. If $a_s \notin f$ and $b_t \notin f$, then there exist $a_1 \in (a], b_1 \in (b]$ such that $f(a_1) < s, f(b_1) < t$. Thus we have

$$(1-f)(a_1) > 1-s, (1-f)(b_1) > 1-t.$$

By hypothesis, there exists an element $x \in S$ such that

$$(1-f)(y) > (1-s) \vee (1-t) = 1-s \wedge t$$

for some $y \in (a_1 \circ x \circ b_1]$, that is, $f(y) < s \wedge t$. Since S be an ordered semihypergroup, it can be obtained that $y \in (a \circ x \circ b]$. It thus follows, by Lemma 2.10(2), that $a_s * x_{s \wedge t} * b_t = \bigcup_{y \in (a \circ x \circ b]} y_{s \wedge t} \notin f$, which is a contradiction. Consequently, f is a quasi-prime fuzzy left hyperideal of S .

In the following we shall define and study the quasi-semiprime fuzzy left hyperideals of ordered semihypergroups.

Definition 3.16. Let S be an ordered semihypergroup. A fuzzy left hyperideal f of S is called *quasi-semiprime* if for any fuzzy left hyperideal g of S , $g * g \subseteq f$ implies $g \subseteq f$.

Lemma 3.17. If f and g are fuzzy left hyperideals of an ordered semihypergroup S , then $f * g$ is also a fuzzy left hyperideal of S .

Proof. Let f, g be two fuzzy left hyperideals of S . Then we have

$$1 * (f * g) = (1 * f) * g \subseteq f * g.$$

Furthermore, if $x \leq y$, then $(f * g)(x) \geq (f * g)(y)$. Indeed, if $H_y = \emptyset$, then $(f * g)(y) = 0$. Since $f * g$ is a fuzzy subset of S , we have $(f * g)(x) \geq 0 = (f * g)(y)$. If $H_y \neq \emptyset$, then, since $x \leq y$, we have $H_y \subseteq H_x$. Thus we have

$$(f * g)(y) = \bigvee_{(u,v) \in H_y} [f(u) \wedge g(v)] \leq \bigvee_{(u,v) \in H_x} [f(u) \wedge g(v)] = (f * g)(x).$$

Therefore, $f * g$ is a fuzzy left hyperideal of S .

Theorem 3.18. *Let S be an ordered semihypergroup and f a fuzzy left hyperideal of S . Then f is quasi-semiprime if and only if for any fuzzy left hyperideal g of S , $g^n \subseteq f$, $n \in \mathbb{Z}^+$ implies that $g \subseteq f$.*

Proof. $\Leftarrow$. This is obvious.

$\Rightarrow$. Let f be a quasi-semiprime fuzzy left hyperideal of S . Here we prove the result by induction. Clearly the result holds for $n = 2$. Let $k \geq 2$ be any positive integer and let the result holds for every positive integer n , $1 \leq n \leq k$. We claim that $g^{k+1} \subseteq f$ implies $g \subseteq f$. We consider the following two cases:

Case 1. If k is odd, let $k = 2m + 1$. Then $g^{k+1} = g^{2(m+1)} = (g^{m+1})^2$.

Case 2. If k is even, let $k = 2m$. Then, by Lemma 3.17, we have

$$g^{k+1} = g^{2m+1} \supseteq 1 * g^{2m+1} \supseteq g * g^{2m+1} = g^{2m+2} = (g^{m+1})^2.$$

Thus, in both cases, if $g^{k+1} \subseteq f$, then $g^{m+1} \subseteq f$. Since $m + 1 \leq k$, the induction hypothesis insures that $g \subseteq f$. The proof is completed.

Remark 3.19. By Theorem 3.18, we have characterized quasi-semiprime fuzzy left hyperideals of an ordered semihypergroup S . The characterization, however, make no reference to the grade of membership of an element of S . The purpose of following theorem is to characterize quasi-semiprime fuzzy left hyperideal in terms of its effect on the elements of S . We shall see that the following theorem is simpler to use.

Theorem 3.20. *Let S be an ordered semihypergroup and f a fuzzy left hyperideal of S . Then f is quasi-semiprime if and only if $f(a) = \bigwedge_{b \in (a \circ S \circ a]} f(b)$ for all $a \in S$.*

Proof. Assume that $f(a) = \bigwedge_{b \in (a \circ S \circ a]} f(b)$ for any $a \in S$. Let g be any fuzzy left hyperideal of S such that $g * g \subseteq f$. If $g \not\subseteq f$, then there exists $a \in S$ such that $g(a) > f(a)$. Since $f(a) = \bigwedge_{b \in (a \circ S \circ a]} f(b)$, there exists $t \in S$ such that $b \preceq a \circ t \circ a$ and $f(a) = f(b)$. Then there exists $c \in a \circ t \circ a$ such that $b \leq c$, and there exists $x \in t \circ a$ such that $c \in a \circ x$. Since f is a fuzzy left hyperideal of S , we have

$$f(c) \leq f(b) = f(a) < g(a).$$

Furthermore, according to $g * g \subseteq f$, we have

$$\begin{aligned} g(a) &> f(c) \geq (g * g)(c) = \bigvee_{(u,v) \in H_c} [g(u) \wedge g(v)] \\ &\geq g(a) \wedge g(x) \geq g(a) \wedge \left(\bigwedge_{x \in t \circ a} g(x) \right) \\ &\geq g(a) \wedge g(a) = g(a), \end{aligned}$$

which is a contradiction.

Conversely, suppose that f is a quasi-semiprime fuzzy left hyperideal of S . If $f(a) \neq \bigwedge_{b \in (a \circ S \circ a)} f(b)$ for some $a \in S$, then $f(a) < \bigwedge_{b \in (a \circ S \circ a)} f(b)$. In fact, for any $b \in (a \circ S \circ a)$, there exists $t \in S$ such that $b \preceq a \circ t \circ a$. Then there exists $c \in a \circ t \circ a$ such that $b \leq c$, and there exists $x \in a \circ t$ such that $c \in x \circ a$. By the fact that f is a fuzzy left hyperideal of S , we have

$$f(b) \geq f(c) \geq \bigwedge_{c \in x \circ a} f(c) \geq f(a).$$

Let $\bigwedge_{b \in (a \circ S \circ a)} f(b) = m$. Define a fuzzy subset g of S as follows:

$$(\forall x \in S) \quad g(x) = \begin{cases} m, & \text{if } x \in (S \circ a), \\ 0, & \text{if } x \notin (S \circ a). \end{cases}$$

Then, by Lemma 2.10(1), g is a fuzzy left hyperideal of S . Furthermore, we can show that $g * g \subseteq f$. It is enough to prove that $(g * g)(x) \leq f(x)$ for all $x \in S$. Indeed, if $(g * g)(x) = 0$, then it is obvious that $(g * g)(x) \leq f(x)$. Let $(g * g)(x) = m$. Then we have

$$\bigvee_{(y,z) \in H_x} [g(y) \wedge g(z)] = m,$$

which means there exist $u, v \in (S \circ a)$ such that $x \preceq u \circ v$. Put $u \preceq s \circ a$, $v \preceq t \circ a$ for some $s, t \in S$. Then, by Lemma 2.2(8), we have

$$x \preceq u \circ v \preceq (s \circ a) \circ (t \circ a) = s \circ (a \circ t \circ a),$$

and there exists $y \in a \circ t \circ a$ such that $x \preceq s \circ y$. Then there exists $z \in s \circ y$ such that $x \leq z$. Since f is a fuzzy left hyperideal of S , we have

$$\begin{aligned} f(x) &\geq \bigwedge_{\substack{x \leq z \\ z \in s \circ y}} f(z) \geq \bigwedge_{z \in s \circ y} f(z) \geq f(y) \geq \bigwedge_{y \in a \circ t \circ a} f(y) \\ &\geq \bigwedge_{y \in (a \circ t \circ a)} f(y) \geq \bigwedge_{y \in (a \circ S \circ a)} f(y) = m = (g * g)(x). \end{aligned}$$

It implies that $g * g \subseteq f$. By hypothesis, $g \subseteq f$. Again define a fuzzy subset h of S as follows:

$$(\forall x \in S) \quad h(x) = \begin{cases} m, & \text{if } x \in L(a), \\ 0, & \text{if } x \notin L(a). \end{cases}$$

Clearly, $h = m f_{L(a)}$. Then, by Lemma 2.7(3), h is a fuzzy left hyperideal of S . Moreover, $h * h \subseteq f$. Indeed, since $(h * h)(x) = \bigvee_{x \preceq x_1 \circ x_2} [h(x_1) \wedge h(x_2)] = m$ only if there exist $u, v \in L(a)$ such that $x \preceq u \circ v$. We can easily verify that

$$x \preceq u \circ v \subseteq S \circ a,$$

which implies that $x \in (S \circ a]$. Thus $(h * h)(x) = m$ implies $g(x) = m$. Consequently, $h * h \subseteq g \subseteq f$. Since f is quasi-semiprime, by Proposition 3.7 we have $h \subseteq f$. Thus $m = h(a) \leq f(a)$, which contradicts the fact that $f(a) < m$. This completes the proof.

Definition 3.21. Let S be an ordered semihypergroup. A fuzzy subset f of S is called *fuzzy n -system* if for any $s \in [0, 1)$ and $a \in S$, $f(a) > s$ implies that there exists $x \in S$ such that $f(y) > s$ for some $y \in (a \circ x \circ a]$.

We now give characterizations of quasi-semiprime fuzzy left hyperideals of an ordered semihypergroup.

Theorem 3.22. Let S be an ordered semihypergroup and f a proper fuzzy left hyperideal of S . Then the following statements are equivalent:

- (1) f is quasi-semiprime.
- (2) For every ordered fuzzy point x_r of S ($r > 0$), $x_r * 1 * x_r \subseteq f$ implies that $x_r \in f$.
- (3) $1 - f$ is a fuzzy n -system of S .

Proof. The proof is similar to that of Theorems 3.12 and 3.15 with a slight modification, we omit it.

4. Characterizations of strongly semisimple ordered semihypergroups

In this section, we investigate mainly the properties of strongly semisimple ordered semihypergroups. In particular, we discuss the characterizations of strongly semisimple ordered semihypergroups by fuzzy left hyperideals generated by ordered fuzzy points.

Definition 4.1. An ordered semihypergroup S is called *strongly semisimple* if $(L^2] = L$ holds for every left hyperideal L of S .

Lemma 4.2. Let S be an ordered semihypergroup. Then the following statements are equivalent:

- (1) S is strongly semisimple.
- (2) $a \in (S \circ a \circ S \circ a]$ for all $a \in S$.

Proof. (1) $\Rightarrow$ (2). Let $a \in S$. Then, by Lemma 2.2, we have

$$\begin{aligned}
 (L(a) \circ L(a)) &= ((a \cup S \circ a] \circ (a \cup S \circ a]) \\
 &= ((a \cup S \circ a) \circ (a \cup S \circ a)) \\
 &= (a \circ a \cup a \circ S \circ a \cup S \circ a \circ a \cup S \circ a \circ S \circ a) \\
 &\subseteq (S \circ a].
 \end{aligned}$$

Thus, by (1), we have

$$\begin{aligned} a &\in L(a) = (L(a) \circ L(a)) = (((L(a) \circ L(a)) \circ (L(a) \circ L(a))) \\ &\subseteq ((S \circ a] \circ (S \circ a]) = ((S \circ a) \circ (S \circ a)) = (S \circ a \circ S \circ a). \end{aligned}$$

(2) $\Rightarrow$ (1). Let L be a left hyperideal of S . Then $(L^2] = (L \circ L) \subseteq (S \circ L) \subseteq (L] = L$. On the other hand, let $x \in L$. Then, by (2) and Lemma 2.2, we have

$$\begin{aligned} x &\in (S \circ x \circ S \circ x] \subseteq (S \circ L \circ S \circ L] \\ &= ((S \circ L) \circ (S \circ L)) \subseteq (L \circ L) = (L^2], \end{aligned}$$

which means that $L \subseteq (L^2]$. Therefore, S is strongly semisimple.

Now, we give some characterizations of a strongly semisimple ordered semi-hypergroup by ordered fuzzy points and fuzzy left hyperideals.

Theorem 4.3. *Let S be an ordered semihypergroup. Then the following statements are equivalent:*

- (1) S is strongly semisimple.
- (2) $f \cap g \subseteq f * g$ for all fuzzy left hyperideals f and g of S .
- (3) $f * f = f$ for every fuzzy left hyperideal f of S .
- (4) $(L(a_r))^2 = L(a_r)$ for every ordered fuzzy point a_r of S .
- (5) $a_r \in S \circ a_r \circ S \circ a_r$ for every ordered fuzzy point a_r of S .
- (6) Every fuzzy left hyperideal of S is quasi-semiprime.
- (7) Every fuzzy left hyperideal of S is the intersection of all quasi-prime fuzzy left hyperideals of S containing it.

Proof. (1) $\Rightarrow$ (2). Let S be a strongly semisimple ordered semihypergroup and $a \in S$. Then, by Lemma 5.2, we have $a \in (S \circ a \circ S \circ a]$, and there exist $x, y \in S$ such that $a \preceq x \circ a \circ y \circ a$. Then there exist $b \in x \circ a, c \in y \circ a$ such that $a \preceq b \circ c$. For any two fuzzy left hyperideals f and g of S , we have

$$\begin{aligned} (f * g)(a) &= \bigvee_{(u,v) \in H_a} [f(u) \wedge g(v)] \geq f(b) \wedge g(c) \\ &\geq \left(\bigwedge_{b \in x \circ a} f(b) \right) \wedge \left(\bigwedge_{c \in y \circ a} g(c) \right) \\ &\geq f(a) \wedge g(a) = (f \cap g)(a), \end{aligned}$$

which implies that $f \cap g \subseteq f * g$.

(2) $\Rightarrow$ (3). Let f be any fuzzy left hyperideal of S . Then, by (2), we have $f * f \supseteq f \cap f = f$. On the other hand, since f is a fuzzy left hyperideal of S , it can be obtained that $f * f \subseteq 1 * f \subseteq f$. Therefore, $f * f = f$.

(3) $\Rightarrow$ (1). Let L be any left hyperideal of S . Then, by Lemma 2.7(3), the characteristic function f_L of L is a fuzzy left hyperideal of S . Thus, by (2), we have $f_L * f_L = f_L$. By Lemma 2.7(2), we have $f_{(L^2]} = f_L$, and thus $(L^2] = L$. Hence S is a strongly semisimple ordered semihypergroup.

(3) $\Rightarrow$ (4). Clearly.

(4) $\Rightarrow$ (5). Let a_r be any ordered fuzzy point of S . By (4), $(L(a_r))^2 = L(a_r)$. Then we have

$$a_r \in L(a_r) = (L(a_r))^4 = (L(a_r))^3 * L(a_r).$$

By Lemma 2.10(4), $(L(a_r))^2 \subseteq 1 * a_r$. Then

$$(L(a_r))^3 = (L(a_r))^2 * L(a_r) \subseteq 1 * a_r * 1.$$

Thus we have

$$\begin{aligned} (L(a_r))^4 &= (L(a_r))^3 * L(a_r) \\ &\subseteq (1 * a_r * 1) * (a_r \cup 1 * a_r) \\ &= 1 * a_r * 1 * a_r \cup 1 * a_r * 1 * 1 * a_r \quad (\text{By Lemma 2.10(6)}) \\ &\subseteq 1 * a_r * 1 * a_r. \end{aligned}$$

Therefore, $a_r \in 1 * a_r * 1 * a_r$.

(5) $\Rightarrow$ (6). Suppose that f is a fuzzy left hyperideal of S . Let g be a fuzzy left hyperideal of S such that $g * g \subseteq f$. Then, for any $a_r \in g$, by (5), we have

$$a_r \in 1 * a_r * 1 * a_r \subseteq 1 * g * 1 * g \subseteq g * g \subseteq f.$$

By Lemma 2.9, $g = \bigcup_{a_r \in g} a_r$, and thus $g \subseteq f$. Consequently, f is quasi-semiprime.

(6) $\Rightarrow$ (3). Let f be any fuzzy left hyperideal of S . Then, by Lemma 3.17, $f * f$ is also a fuzzy left hyperideal of S . Since $f * f \subseteq f * f$, by (6) we have $f \subseteq f * f$. Clearly, $f * f \subseteq f$. It thus follows that $f * f = f$.

(2) $\Rightarrow$ (7). Let f be a fuzzy left hyperideal of S , and let

$$\mathcal{N} = \{g_\alpha \mid g_\alpha \text{ is a quasi-prime fuzzy left hyperideal of } S \text{ such that } f \subseteq g_\alpha\}.$$

We claim that $f = \bigcap_{g_\alpha \in \mathcal{N}} g_\alpha$. Indeed, it is obvious that $f \subseteq \bigcap_{g_\alpha \in \mathcal{N}} g_\alpha$. Conversely, for any $a_r \in \bigcap_{g_\alpha \in \mathcal{N}} g_\alpha$, if $a_r \notin f$, then $r > 0, f(a) < r$. Let

$$\mathcal{B} = \{h_\beta \mid h_\beta \text{ is a fuzzy left hyperideal of } S \text{ such that } f \subseteq h_\beta, f(a) = h_\beta(a)\}.$$

Clearly, $\mathcal{B} \neq \emptyset$ because $f \in \mathcal{B}$. Thus $(\mathcal{B}, \subseteq)$ is an ordered set. Let $\mathcal{C}$ be a chain in $\mathcal{B}$. Then, by Lemma 2.4, the set $\bigcup_{h_\beta \in \mathcal{C}} h_\beta$ is a fuzzy left hyperideal of S and $f \subseteq \bigcup_{h_\beta \in \mathcal{C}} h_\beta$. Since for any $h_\beta \in \mathcal{C}$, $f(a) = h_\beta(a)$, we have

$$(\bigcup_{h_\beta \in \mathcal{C}} h_\beta)(a) = f(a).$$

Thus the fuzzy left hyperideal $\bigcup_{h_\beta \in \mathcal{C}} h_\beta$ is an upper bound of $\mathcal{C}$ in $\mathcal{B}$. By Zorn's Lemma, $\mathcal{B}$ has a maximal element. Denote it by $h_{\max}$. Then $a_r \notin h_{\max}$. We now

show that $h_{\max}$ is a quasi-prime fuzzy left hyperideal of S . Let f_1 and f_2 be two fuzzy left hyperideals of S with $f_1 * f_2 \subseteq h_{\max}$. Then, by (2), we have

$$f_1 \cap f_2 \subseteq f_1 * f_2 \subseteq h_{\max}.$$

It thus follow that $h_{\max} = h_{\max} \cup (f_1 \cap f_2) = (h_{\max} \cup f_1) \cap (h_{\max} \cup f_2)$. We claim that $h_{\max} = h_{\max} \cup f_1$ or $h_{\max} = h_{\max} \cup f_2$, that is, $f_1 \subseteq h_{\max}$ or $f_2 \subseteq h_{\max}$. In fact, by $h_{\max} = (h_{\max} \cup f_1) \cap (h_{\max} \cup f_2)$, we have

$$f(a) = h_{\max}(a) = (h_{\max} \cup f_1)(a) \wedge (h_{\max} \cup f_2)(a).$$

This implies $(h_{\max} \cup f_1)(a) = f(a)$ or $(h_{\max} \cup f_2)(a) = f(a)$. Since $h_{\max}$ is maximal with respect to the property that $f \subseteq h_{\max}$ and $h_{\max}(a) = f(a)$, we have $h_{\max} = h_{\max} \cup f_1$ or $h_{\max} = h_{\max} \cup f_2$. Hence $h_{\max}$ is a quasi-prime fuzzy left hyperideal of S . Thus, by hypothesis, $a_r \in h_{\max}$. This is a contradiction. Therefore, $f = \bigcap_{g \in \mathcal{N}} g$.

(7) $\Rightarrow$ (3). Let f be any fuzzy left hyperideal of S . Then, by Lemma 3.17, $f * f$ is also a fuzzy left hyperideal of S . By (7), we have

$$f * f = \bigcap_{g \in \mathcal{M}} g,$$

where $\mathcal{M}$ is the set of all quasi-prime fuzzy left hyperideals of S containing $f * f$. Furthermore, we prove that $f * f = f$. In fact, for any $g \in \mathcal{M}$, clearly, $f * f \subseteq g$. Since g is quasi-prime, it can be obtained that $f \subseteq g$. Then we have $f \subseteq \bigcap_{g \in \mathcal{M}} g = f * f$. On the other hand, since f is a fuzzy left hyperideal of S , we have $f * f \subseteq 1 * f \subseteq f$. Thus $f * f = f$.

Theorem 4.4. *Let S be a commutative ordered semihypergroup. Then the fuzzy left hyperideals of S are quasi-prime if and only if they form a chain and S is strongly simisimple.*

Proof. Suppose that the fuzzy left hyperideals of S are quasi-prime. Let g and h be fuzzy left hyperideals of S . By Lemma 3.17, $g * h$ is a fuzzy left hyperideal of S . Then, by hypothesis, $g * h$ is quasi-prime. From $g * h \subseteq g * h$, by Lemma 2.10(5), we have $g \subseteq g * h \subseteq 1 * h \subseteq h$ or $h \subseteq g * h \subseteq g * 1 = 1 * g \subseteq g$. Thus the fuzzy left hyperideals of S form a chain. Moreover, for any fuzzy left hyperideal f of S , obviously, $f * f \subseteq f$. Since $f * f \subseteq f * f$, by hypothesis we have $f \subseteq f * f$. It thus follows that $f * f = f$. By Theorem 4.3, S is strongly simisimple.

Conversely, assume that f is a fuzzy left hyperideal of S . Let g, h be any fuzzy left hyperideals of S such that $g * h \subseteq f$. By hypothesis, we have $g \subseteq h$ or $h \subseteq g$. Say $g \subseteq h$, then, by Theorem 4.3, $g = g * g \subseteq g * h \subseteq f$. Similarly, say $h \subseteq g$, we have $h \subseteq f$. Therefore, f is quasi-prime.

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ON TOPOLOGICAL EFFECT ALGEBRAS

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Abstract. In this paper the notions of topological and paratopological effect algebras are defined and their properties are investigated. Then by considering the notion of uniformity space and using of Riesz ideals in effect algebras, a topology and a uniformity space on any effect algebra is obtained. Finally, by definition of an equivalence relation on the set of all Cauchy nets of an effect algebra, and getting a quotient structure by that set, a completion for above uniformity space is constructed.

Keywords: effect algebra, topology, paratopology, uniformity, completion.

1. Introduction

Topology and algebra, the two fundamental domains of mathematics, play complementary roles. Topology studies continuity and convergence and provides a general framework to study the concept of a limit. Algebra considers all kinds of operations and provides a basis for algorithms and calculations. Many of the most important objects of mathematics represent a blend of algebraic and topological structures. For example, in applications, in higher level domains of mathematics, such as functional analysis, dynamical systems, representation theory, topology and algebra come in contact naturally. Several mathematicians have endowed a number of algebraic structures associated with logical systems

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with a topology and have found some of their properties. For example, R. A. Borzooei et al. [1, 2, 14, 15] who defined semitopological and topological BL -algebras and MV -algebras and M. Haveshti et al. [11] introduced the topology induced by uniformity on BL -algebras. Effect algebras have been introduced by D. J. Foulis and M. K. Bennett in 1994 [6] for modelling unsharp measurement in a quantum mechanical system. They are a generalization of many structures which arises in quantum physics [3] and in mathematical economics. In the last few years, the theory of effect algebras has enjoyed a rapid development. As an important tool of studying, the topological structures of effect algebras not only can help us to describe the convergence properties, but also can help us to characterize some algebra properties of effect algebras. In this paper we present the main properties of effect algebras, and introduce topologies that cause the operation $\oplus$ and $\prime$ to be continuous, and study their properties. We define (para) topological effect algebra and some examples are presented. We apply the notion of ideals, especially Riesz ideals in an effect algebra and we produce a binary relation which is an congruence relation. Then we show that an effect algebra with the topology that is defined by Riesz ideals, form a uniform space. Finally, we introduce a completion on an effect algebra.

2. Preliminary

Recall that a set A with a family $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ of its subsets is called a *topological space*, denoted by $(A, \mathcal{U})$, if $A, \emptyset \in \mathcal{U}$, the intersection of any finite numbers of members of $\mathcal{U}$ is in $\mathcal{U}$ and the arbitrary union of members of $\mathcal{U}$ is in $\mathcal{U}$. The members of $\mathcal{U}$ are called *open sets* of A and the complement of $U \in \mathcal{U}$, that is $A \setminus U$, is said to be a *closed set*. If B is a subset of A , the smallest closed set containing B is called the *closure* of B and denoted by $\overline{B}$ (or $cl_u B$). A subset P of A is said to be a *neighborhood* of $x \in A$, if there exists an open set U such that $x \in U \subseteq P$. A subfamily $\{U_\alpha\}$ of $\mathcal{U}$ is said to be a *base* of $\mathcal{U}$ if for each $x \in U \in \mathcal{U}$ there exists an $\alpha \in I$ such that $x \in U_\alpha \subseteq U$, or equivalently, each U in $\mathcal{U}$ is the union of members of $\{U_\alpha\}$. A subfamily $\{U_\beta\}$ of $\mathcal{U}$ is a *subbase* for $\mathcal{U}$ if the family of finite intersections of members of $\{U_\beta\}$ forms a base of $\mathcal{U}$. Let $\mathcal{U}_x$ denote the totality of all neighborhoods of x in A . Then a subfamily $\mathcal{V}_x$ of $\mathcal{U}_x$ is said to form a *fundamental system* of neighborhoods of x , if for each U_x in $\mathcal{U}_x$, there exists V_x in $\mathcal{V}_x$ such that $V_x \subseteq U_x$ (See [3]). Let $(A, *)$ be an algebra of type 2 and $\mathcal{U}$ be a topology on A . Then $\mathcal{A} = (A, *, \mathcal{U})$ is called a *left (right) topological algebra*, if for all $a \in A$ the map $*$: $A \rightarrow A$ is defined by $x \rightarrow a * x$ ($x \rightarrow x * a$) is continuous, or equivalently, for any x in A and any open set U of $a * x$ ($x * a$), there exists an open set V of x such that $a * V \subseteq U$ ($V * a \subseteq U$), *topological algebra*, if the operation $*$ is continuous, or equivalently, if for any x, y in A and any open set (neighborhood) W of $x * y$ there exist two open sets (neighborhoods) U and V of x and y , respectively, such that $U * V \subseteq W$ (See [2]). An *effect algebra* is algebraic structure $(E, \oplus, 0, 1)$,

where $0, 1$ are distinct elements of E and $\oplus$ is a partial binary operation on E that satisfies the following conditions:

(E1) (Commutative law) If $a \oplus b$ is defined, then $b \oplus a$ is defined and $a \oplus b = b \oplus a$.

(E2) (Associative law) If $a \oplus b$ and $(a \oplus b) \oplus c$ are defined, then $b \oplus c$ and $a \oplus (b \oplus c)$ are defined and $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.

(E3) (Orthosupplementation law) For each $a \in E$ there exist a unique $b \in E$ such that $a \oplus b$ is defined and $a \oplus b = 1$.

(E4) (Zero-Unit law) If $a \oplus 1$ is defined, then $a = 0$.

For each $a \in E$, we denote the unique b in condition (E3) by a' and call it the orthosupplement of a . The sense is that if a presents a proposition, then a' corresponds to the negation. We define $a \leq b$, if there exists $c \in E$ such that $a \oplus c = b$. Such an element c is unique, and therefore we can introduce a dual operation $\ominus$ in E by $a \ominus b = c$ if and only if $a = c \oplus b$. Partial order $\leq$ on E which is defined as above, is a totally order (See [6]). Let A and B be subsets of a effect algebra E . Then $A \oplus B$ denotes the set $\{a \oplus b : a \in A, b \in B\}$, and A' denotes the set $\{a' : a \in A\}$. If $a \oplus b$ is defined, then we say that a and b are *orthogonal* and write $a \perp b$.

Example 2.1 ([4]). (i) Let $E = [0, 1] \subseteq R$, where $a \oplus b$ is defined if and only if $a + b \leq 1$, in which case we define, $a \oplus b = a + b$. Then $(E, \oplus, 0, 1)$ is an effect algebra.

(ii) Let $C_n = \{0, a, 2a, \dots, na = 1\}$, where $ia \oplus ja$ on C_n is defined if and only if $i + j \leq n$ for any $i, j = 0, 1, 2, \dots, n$, and in this case $ia \oplus ja = (i + j)a$. Then n -chain $(C_n, \oplus, 0, 1)$ is an effect algebra.

Example 2.2. Let $E = \{0, a, b, 1\}$, where $0 < a, b < 1$. Consider the following table:

$\oplus$	0	a	b	1
0	0	a	b	1
a	a	a	1	-
b	b	1	b	-
1	1	-	-	-

Then, $(E, \oplus, 0, 1)$ is an effect algebra.

Proposition 2.3 ([6]). The following properties hold for any effect algebra:

- (i) $a'' = a$,
- (ii) $1' = 0$ and $0' = 1$,
- (iii) $0 \leq a \leq 1$,
- (iv) $a \oplus 0 = a$,
- (v) $a \oplus b = 0 \Rightarrow a = b = 0$,
- (vi) $a \leq a \oplus b$,
- (vii) $a \leq b \Rightarrow b' \leq a'$,
- (viii) $b \ominus a = (a \oplus b')'$,
- (ix) $a \oplus b' = (b \ominus a)'$,

- (x) $a = a \oplus 0$,
- (xi) $a \oplus a = 0$,
- (xii) $a' = 1 \oplus a$ and $a = 1 \oplus a'$.

Let $(E, \oplus, 0, 1)$ be an effect algebra. A nonempty subset I of E is said to be an ideal of E , if for all $a, b \in E$, $a \in I$ and $b \leq a$ implies $b \in I$ and $a \oplus b \in I$ and $b \in I$ implies $a \in I$.

Equivalently, if $a \oplus b$ is defined, then $a \oplus b \in I \Leftrightarrow a, b \in I$. (See [13])

A binary relation $\sim$ on effect algebra E is called a *congruence relation* if:

- (C1) $\sim$ is an equivalence relation,
- (C2) $a \sim a_1, b \sim b_1, a \perp b$ and $a_1 \perp b_1$ then $a \oplus b \sim a_1 \oplus b_1$.
- (C3) if $a \sim b$ and $b \perp c$, then there exists $d \in E$ such that $d \sim c$ and $a \perp d$.

Note: Condition (C3) is equivalent to the conditions:

- (C4) If $a \sim b$ and $a \oplus a_1 \sim b \oplus b_1$, then $a_1 \sim b_1$.

(C5) If $a \sim b \oplus c$, then there are a_1, a_2 such that $a = a_1 \oplus a_2$ and $a_1 \sim b, a_2 \sim c$.
and condition (C4) is equivalent to condition

- (C6) if $a \sim b$, then $a' \sim b'$. (See [8])

If $\sim$ is a congruence relation on effect algebra E , then quotient $E/\sim$ is an effect algebra ([5]). If I is an ideal on effect algebra E , we define a binary relation $\sim_I$ on E by $a \sim_I b$ if and only if there are $i, j \in I$, such that $i \leq a, j \leq b$ and $a \oplus i = b \oplus j$ or equivalently, there exists $k \in E$, such that $k \leq a, b, a \oplus k$ and $b \oplus k \in I$. Let $(E, \oplus, 0, 1)$ be an effect algebra and I be an ideal of E . We say that I is a *Riesz ideal* of E if for any $i \in I$ and $a, b \in E$ if $a \perp b$ and $i \leq a \oplus b$, then there exist $a_1, b_1 \in I$ such that $a_1 \leq a, b_1 \leq b$ and $i \leq a_1 \oplus b_1$.

Theorem 2.4 ([7]). *Let I be an ideal in an effect algebra E . Then $\sim_I$ is a congruence relation on E if and only if I is a Riesz ideal of E .*

Note: From now on, in this paper we let $(E, \oplus, 0, 1)$ or E is an effect algebra.

3. On topological effect algebras

In this section we construct a topology and paratopology on effect algebras.

From now on, we consider the operation ι as a single operation from E into E by $\iota(a) = a'$.

Definition 3.1. Let $\mathcal{T}$ be a topology on E . Then we say that $(E, \mathcal{T})$ is a:

- (i) *paratopological effect algebra* if the operation $\oplus$ is continuous, or equivalently, for any $x, y \in E$ and any open neighborhood W of $x \oplus y$, there exist two open neighborhoods U and V of x and y , respectively, such that $U \oplus V \subseteq W$,
- (ii) *topological effect algebra* if the operation $\oplus$ and ι are continuous.

Example 3.2. (i) Let $E = [0, 1]$ be effect algebra as Example 2.1(i) with the interval topology $\mathcal{T}$ of R . Then E is a topological effect algebra. For this, let $W = (a \oplus b - \varepsilon, a \oplus b + \varepsilon)$ be an open neighborhood of $a \oplus b$. It is clearly

$U = (a - \varepsilon/2, a + \varepsilon/2)$ and $V = (b - \varepsilon/2, b + \varepsilon/2)$ are two open neighborhoods of x and y , respectively. Since $U \oplus V = (a \oplus b - \varepsilon, a \oplus b + \varepsilon) \subseteq W$, hence $\oplus$ is continuous. Now we prove that ι is continuous, too. Let $W = (a' - \varepsilon, a' + \varepsilon)$ be an neighborhood of a' and $\delta \leq \varepsilon$. Since $(a - \delta, a + \delta)$ is a neighborhood of a and

$$\begin{aligned} \iota(a - \delta, a + \delta) &= ((a + \delta)', (a - \delta)') = (1 - (a + \delta), 1 - (a - \delta)) \\ &= ((1 - a) - \delta, (1 - a) + \delta) \\ &= (a' - \delta, a' + \delta) \\ &\subseteq W. \end{aligned}$$

Hence E is a topological effect algebra.

(ii) Consider effect algebra C_4 as Example 2.1(ii). Then we have $C_4 = \{0, 1/4, 2/4, 3/4, 1\}$. It is clearly $\mathcal{T} = \{\phi, \{0\}, \{0, 1/4\}, C_4\}$ is a topology on C_4 . Now $(C_4, \mathcal{T})$ is a paratopological effect algebra. For this, it is enough to show that the mapping $\oplus$ is continuous. Let W be an open neighborhood of $a \oplus b$. If $a \oplus b \geq 1/2$, then C_4 is the unique open neighborhood of $a \oplus b$. Hence we can consider C_4 as the open neighborhood of a and b such that $C_4 \oplus C_4 \subseteq C_4$. If $a = b = 0$, then $a \oplus b = 0$, and so $W = \{0, 1/4\}$ or $\{0\}$ or C_4 . Consider $U = V = \{0\}$. Thus, $U \oplus V = \{0\} \subseteq W$. If $a = 0$ and $b = 1/4$, then $a \oplus b = 1/4$ and so $W = \{0, 1/4\}$ or C_4 , which are the neighborhoods of $1/4$. Now, if $U = \{0\}$ and $V = \{0, 1/4\}$, then $U \oplus V = \{0, 1/4\} \subseteq W$. Therefore, the mapping $\oplus$ is continuous, and so C_4 is a paratopological effect algebra. But C_4 is not a topological effect algebra, because $(3/4)' \in \{0, 1/4\}$, C_4 is the only open neighborhood of $3/4$ and $C_4' = C_4 \not\subseteq W = \{0, 1/4\}$. Therefore, the mapping ι is not continuous.

Theorem 3.3. *Let $\mathcal{T}$ be a topology on E . Then $(E, \mathcal{T})$ is a topological effect algebra if and only if the function $F : E \times E \rightarrow E$ defined by $F(a, b) = a \oplus b'$ is continuous.*

Proof. Suppose $(E, \mathcal{T})$ is a topological effect algebra. Since the mapping $i(a) = a$ and $\iota(a) = a'$ on E are continuous, then the mapping $f(a, b) = (i(a), \iota(b)) = (a, b')$ is continuous on $E \times E$, too. Hence $F = \oplus \circ f$ is continuous. Conversely, let F be continuous. Then for any $b \in E$, $K = F(0, b)$ is continuous. We define $h : E \rightarrow E \times E$ by $h(b) = (0, b)$. Clearly, h is continuous. Hence, $\iota = K \circ h$ is continuous. Now we show that $\oplus$ is continuous. Let $f : E \times E \rightarrow E \times E$ be defined by $f(a, b) = (a, b')$. Since the identity map and ι are continuous, f is continuous, too. Now, since $\oplus = F \circ f$, so $(F \circ f)(a, b) = F(a, b') = a \oplus (b')' = a \oplus b$. Hence $\oplus$ is continuous. $\square$

Notation. For $a \in E$, we define the maps $T_a, L_a, R_a : E \rightarrow E$ as follows:

$$T_a(x) = a \oplus x, \quad L_a(x) = a \ominus x, \quad R_a(x) = x \ominus a.$$

Proposition 3.4. *Let $(E, \mathcal{T})$ be a paratopological effect algebra. Then $(E, \mathcal{T})$ is a topological effect algebra if and only if the mapping ι is an open map.*

Proof. ($\Leftarrow$) Let $(E, \mathcal{T})$ be a paratopological effect algebra and mapping ι be an open map in $(E, \mathcal{T})$. We show that for any $a \in E$, ι is continuous, for any $a \in U \in \mathcal{T}$, there exists $V \in \mathcal{T}$ such that $a' \in V$ and $V' \subseteq U$. Put $V = U'$, so $a' \in V$. Since ι is an open map, hence $V \in \mathcal{T}$. On the other hand by Proposition 2.3(i), we have $V' = (U')' = U \subseteq U$.

($\Rightarrow$) The proof is clear. $\square$

Theorem 3.5. *Let $(E, \mathcal{T})$ be a topological effect algebra. Then the operation $\ominus$ is continuous.*

Proof. Let $(E, \mathcal{T})$ be a topological effect algebra. Then the mapping $f(a, b) = (b, a')$ on $E \times E$ is continuous. On the other hand, since the maps $\oplus$ and ι are continuous, thus $\iota \circ \oplus \circ f$ is continuous. Now by Proposition 2.3(viii), we have;

$$(\iota \circ \oplus \circ f)(a, b) = \iota(\oplus(f(a, b))) = \iota(\oplus(b, a')) = \iota(b \oplus a') = (b \oplus a')' = a \ominus b.$$

Hence $\ominus$ is continuous. $\square$

Theorem 3.6. *Let $\mathcal{T}$ be a topology on E and $\ominus$ and ι are continuous on it. Then $(E, \mathcal{T})$ is a topological effect algebra.*

Proof. Let $f : E \times E \rightarrow E \times E$ is defined by $f(a, b) = (a', b)$, for any $a, b \in E$. Since ι , $\ominus$ and f are continuous on E , thus $\iota \circ \ominus \circ f$ is continuous, too. Hence

$$(\iota \circ \ominus \circ f)(a, b) = \iota(\ominus(a', b)) = \iota(a' \ominus b) = (a' \ominus b)' = b \oplus (a')' = b \oplus a.$$

and so $\oplus$ is continuous on E . $\square$

Lemma 3.7. *Let $(E, \mathcal{T})$ be a topological effect algebra. Then:*

- (i) *for any $a, b \in E$, there is a continuous map f on E such that $f(a) = b$,*
- (ii) *the mapping $\iota(a) = a'$ on E is a homeomorphism,*
- (iii) *for any $a \in E$ if L_a or R_a be an open map, then T_a is an open map, too.*

Proof. (i) Let $a, b \in E$. As E is a topological effect algebra, then T_b and T_a are continuous. Then $f = T_b \circ T_a$ is continuous, too. Now by Proposition 2.3(xi, iv),

$$f(a) = T_b \circ L_a(a) = T_b(a \ominus a) = T_b(0) = 0 \oplus b = b.$$

(ii) By Proposition 2.3(i), $(x')' = x$, for each $x \in E$. Then ι is an invertible map and $(\iota)^{-1} = \iota$. Since ι is continuous, thus ι is a homeomorphism.

(iii) Let $a \in E$ and R_a be an open map. Suppose that U is an open set of $\mathcal{T}$. By (ii), U' is an open set, too. Since R_a is an open map, then $R_a(U') = U' \ominus a$ is open. Again by (ii), $(U' \ominus a)'$ is open. On the other hand by Proposition 2.3(ix), $(U' \ominus a)' = a \oplus (U')' = a \oplus U$. Hence $a \oplus U$ or $T_a(U)$ is open and this means that T_a is an open map. The proof of the other case is similar. $\square$

Proposition 3.8. Let $\mathcal{N}$ be a fundamental system of open neighborhoods of 0 in a topological effect algebra $(E, \mathcal{T})$. If $1 \in \cap \mathcal{N}$, then there exists a fundamental system $\mathcal{N}_0$ of open neighborhoods of 0 such that $U = U'$, for each $U \in \mathcal{N}_0$.

Proof. Let $\mathcal{N}_0 = \{V \cap V' : V \in \mathcal{N}\}$. Since $1 \in V$, we get $0 \in V'$ and so $0 \in V \cap V'$. By Lemma 3.7(ii), the map ι is a homeomorphism, so for each $V \in \mathcal{N}$, $V \cap V'$ is an open neighborhood of 0. Let O be an open neighborhood of 0. Because $0 \oplus 0 = 0 \in O$, there is $W \in \mathcal{N}$ such that $W = 0 \oplus W \subseteq O$. On the other hand, since $0' = 1 \in W$, there is $V \in \mathcal{N}$ such that $V' \subseteq W$. Hence $V \cap V' \subseteq V' \subseteq W \subseteq O$. This shows that $\mathcal{N}_0$ is a fundamental system of open neighborhoods of 0. Clearly, for each $V \in \mathcal{N}$,

$$(V \cap V')' = V' \cap (V')' = V' \cap V.$$

Hence for any $U \in \mathcal{N}_0$, we have $U = U'$. □

Definition 3.9. Let $\mathcal{N}$ be a family of subsets in E . We call $\mathcal{N}$ is a *system of 0* if $0 \in \cap \mathcal{N}$ and

- (i) for every $x \in U \in \mathcal{N}$, there is $V \in \mathcal{N}$ such that $x \oplus V \subseteq U$,
- (ii) for every $U \in \mathcal{N}$, there is $V \in \mathcal{N}$ such that $V \oplus V \subseteq U$,
- (iii) for every $U, V \in \mathcal{N}$, there is $W \in \mathcal{N}$ such that $W \subseteq U \cap V$.

Proposition 3.10. Let $\mathcal{N}$ be a fundamental system of open neighborhoods of 0 in a paratopological effect algebra $(E, \mathcal{T})$. Then $\mathcal{N}$ is a system of 0 in E .

Proof. Clearly, $0 \in \cap \mathcal{N}$. Let $a \in U \in \mathcal{N}$. Because $\oplus$ is continuous and $a \oplus 0 = a \in U$, then there exists an open neighborhood W of 0 such that $a \oplus W \subseteq U$. Since $\mathcal{N}$ is a fundamental system of open neighborhoods of 0, there is $V \in \mathcal{N}$ such that $0 \in V \subseteq W$. Hence, $a \oplus V \subseteq a \oplus W \subseteq U$. Now, let $U \in \mathcal{N}$. Because $0 \oplus 0 = 0 \in U$ and the mapping $\oplus$ is continuous, there are open neighborhoods W_0 and W_1 of 0 such that $W_0 \oplus W_1 \subseteq U$. We consider $W = W_0 \cap W_1$. As $\mathcal{N}$ is a fundamental system of open neighborhoods of 0, there is $V \in \mathcal{N}$ such that $0 \in V \subseteq W$. Hence $V \oplus V \subseteq W_0 \oplus W_1 \subseteq U$. Finally, let $U, V \in \mathcal{N}$. Since $\mathcal{N}$ is a fundamental system of neighborhoods of 0, hence it is closed under finite intersection, that is $W \subseteq U \cap V$. Therefore, $\mathcal{N}$ is a system of 0 in E . □

Theorem 3.11. Let $\mathcal{N}$ be a system of 0 in effect algebra E . Then there exists a topology $\mathcal{T}$ on E such that $(E, \mathcal{T})$ is a paratopological effect algebra.

Proof. Let $\mathcal{T} = \{W \subseteq E : \forall x \in W, \exists U \in \mathcal{N} \text{ s.t. } x \oplus U \subseteq W\}$. First we show that, $x \oplus U \in \mathcal{T}$, for each $x \in E$ and $U \in \mathcal{N}$. Let $y \in x \oplus U$. Then $y = x \oplus a$, for some $a \in U$. Since $\mathcal{N}$ is a system, there is $V \in \mathcal{N}$ such that $a \oplus V \subseteq U$. Hence $y \oplus V = x \oplus a \oplus V \subseteq x \oplus U$ which implies that $x \oplus U \in \mathcal{T}$. Let $\{W_i : i \in I\}$ be a subfamily of $\mathcal{T}$. We show that $\cup W_i \in \mathcal{T}$. Let $x \in \cup W_i$. Then there exists $i \in I$ such that $x \in W_i$. Hence, there exists $U \in \mathcal{N}$ such that $x \oplus U \subseteq W_i$. Thus $x \oplus U \subseteq \cup W_i$.

Now let $W_1, W_2 \in \mathcal{T}$ and $W = W_1 \cap W_2$. We prove that $W \in \mathcal{T}$. For $x \in W$, there exist $U_1, U_2 \in \mathcal{N}$ such that $x \oplus U_1 \subseteq W_1$ and $x \oplus U_2 \subseteq W_2$. Since $\mathcal{N}$ is a system of 0, there is $U \in \mathcal{N}$ such that $U \subseteq U_1 \cap U_2$. Because

$$x \oplus U \subseteq x \oplus (U_1 \cap U_2) \subseteq (x \oplus U_1) \cap (x \oplus U_2) \subseteq W_1 \cap W_2 = W,$$

then $\mathcal{T}$ is a topology on E . Now, it is enough to prove that $\oplus$ is continuous. Let $z = x \oplus y$ and W be an open neighborhood of z . Then there is $U \in \mathcal{N}$ such that $z \oplus U \subseteq W$. By definition, there exists $V \in \mathcal{N}$ such that $V \oplus V \subseteq U$. On the other hands $x \oplus V$ and $y \oplus V$ are two open neighborhoods of x and y , respectively, such that

$$(x \oplus V) \oplus (y \oplus V) = x \oplus y \oplus V \oplus V \subseteq x \oplus y \oplus U = z \oplus U \subseteq W.$$

Therefore, $(E, \mathcal{T})$ is a paratopological effect algebra. $\square$

Example 3.12. Let $E = [0, 1]$ be effect algebra as Example 2.1(i) and $\mathcal{N} = \{[0, 1/n) : n \in \mathbb{N}\}$. It is clearly that $\mathcal{N}$ is a system of 0 in E . By Theorem 3.11, $\mathcal{T} = \{W \subseteq E : \forall x \in W, \exists U \in \mathcal{N} \text{ s.t. } x \oplus U \subseteq W\}$ is a topology on E . We show that the operation $\oplus$ is continuous. Let $x \oplus y \in W \in \mathcal{T}$. Then there exists $n \in \mathbb{N}$ such that $x \oplus y \oplus [0, 1/n) \subseteq W$. Consider $U = [x, x \oplus 1/k)$ and $V = [y, y \oplus 1/m)$ such that $k, m > 2$ and $1/k + 1/m \leq 1/n$. Thus, U and V are two open neighborhoods of x and y , respectively such that $U \oplus V \subseteq W$. Therefore $(E, \mathcal{T})$ is a paratopological effect algebra.

4. Uniformity on effect algebras

In this section we study the concept of uniformity on effect algebras and define a topology on effect algebras by Riesz ideals.

Notations. Let X be a nonempty set and U, V are nonempty subsets of $X \times X$. Then we define:

$$\begin{aligned} U \circ V &= \{(a, b) : (c, b) \in U, (a, c) \in V, \text{ for some } c \in X\}, \\ U^{-1} &= \{(a, b) : (b, a) \in U\}, \\ \Delta &= \{(a, a) : a \in X\}. \end{aligned}$$

Definition 4.1 ([10]). By a *uniformity* on X we shall mean a nonempty collection $\mathcal{U}$ of subset of $X \times X$ which satisfies the following conditions:

- (U₁) $\Delta \subseteq U$ for any $U \in \mathcal{U}$,
- (U₂) if $U \in \mathcal{U}$, then $U^{-1} \in \mathcal{U}$,
- (U₃) if $U \in \mathcal{U}$, then there exists $V \in \mathcal{U}$ such that $V \circ V \subseteq U$,
- (U₄) if $U, V \in \mathcal{U}$, then $U \cap V \in \mathcal{U}$,
- (U₅) if $U \in \mathcal{U}$, and $U \subseteq V \subseteq X \times X$, then $V \in \mathcal{U}$.

If $\mathcal{U}$ is a uniformity on X , then the pair $(X, \mathcal{U})$ is called a *uniform structure* or *uniform space*.

Notations. Let Λ be a family of Riesz ideals in E , such that it is closed under intersection. Then for any $I \in \Lambda$, we define

$$U_I = \{(a, b) \in E \times E : a \sim_I b\} \quad , \quad \mathcal{U}_\Lambda = \{U \subseteq E \times E : \exists I \in \Lambda \text{ s.t. } U_I \subseteq U\}$$

Theorem 4.2. $(E, \mathcal{U}_\Lambda)$ is a uniform space.

Proof. (U_1) : Since I is a Riesz ideal of E , $a \sim_I a$, for any $a \in E$. Hence $\Delta \subseteq U_I \subseteq U$, for all $U \in \mathcal{U}_\Lambda$.

(U_2) : For any $U \in \mathcal{U}_\Lambda$, we have

$$\begin{aligned} (a, b) \in U^{-1} &\Leftrightarrow (b, a) \in U \Leftrightarrow (b, a) \in U_I \subseteq U, \exists I \in \Lambda \Leftrightarrow b \sim_I a, \exists I \in \Lambda \\ &\Leftrightarrow a \sim_I b, \exists I \in \Lambda \Leftrightarrow (a, b) \in U_I, \exists I \in \Lambda \\ &\Leftrightarrow (a, b) \in U \end{aligned}$$

(U_3) : For any $U \in \mathcal{U}_\Lambda$, the transitivity of $\sim_I$ implies that $U \circ U \subseteq U$.

(U_4) : For any $U, V \in \mathcal{U}_\Lambda$, there exist $I, J \in \Lambda$ such that $U_I \subseteq U$ and $U_J \subseteq V$. We claim that $U_I \cap U_J = U_{I \cap J}$. Let $(a, b) \in U_I \cap U_J$. Then $a \sim_I b$ and $a \sim_J b$. Hence, for any $i \in I$ and $a, b \in E$, $i \leq a \oplus b$ implies that there exist $a_1, b_1 \in I$ such that $a_1 \leq a$ and $b_1 \leq b$ with $i \leq a_1 \oplus b_1$. Furthermore, for any $j \in J$ and $a, b \in E$, $j \leq a \oplus b$ implies that there are $a_2, b_2 \in J$ such that $a_2 \leq a$ and $b_2 \leq b$ with $j \leq a_2 \oplus b_2$. If $K = I \cap J$, then $a \sim_K b$ and this means that $(a, b) \in U_{I \cap J}$. Conversely, let $(a, b) \in U_{I \cap J}$. Then $a \sim_{I \cap J} b$. Hence, for any $k \in I \cap J$ and $a, b \in E$, $k \leq a \oplus b$ implies that there are $a_1, b_1 \in I \cap J$ such that $a_1 \leq a$ and $b_1 \leq b$ with $k \leq a_1 \oplus b_1$. Since $k \in I$ we have $a \sim_I b$, which means that $(a, b) \in U_I$. Now, since $k \in J$, similarly $(a, b) \in U_J$ and so $(a, b) \in U_I \cap U_J$. Hence $U_I \cap U_J = U_{I \cap J}$. Since $I, J \in \Lambda$, then $I \cap J \in \Lambda$ and $U_I \cap U_J = U_{I \cap J} \subseteq U \cap V$. Therefore $U \cap V \in \mathcal{U}_\Lambda$.

(U_5) : Let $U \in \mathcal{U}_\Lambda$ and $U \subseteq V \subseteq E \times E$. Then there exists $U_I \subseteq U \subseteq V$, which means that $V \in \mathcal{U}_\Lambda$. $\square$

Theorem 4.3. Let for any $a \in E$, $U[a] = \{b \in E : (a, b) \in U\}$. Then

$$\mathcal{T}_\Lambda = \{G \subseteq E : \forall a \in G, \exists U \in \mathcal{U}_\Lambda \text{ s.t. } U[a] \subseteq G\}$$

is a topology on E .

Proof. Clearly, $\emptyset, E \in \mathcal{T}_\Lambda$. Let Σ be a non-empty directed set, $\{G_\alpha\}_{\alpha \in \Sigma}$ be a subfamily of $\mathcal{T}_\Lambda$, and $a \in \bigcup_{\alpha \in \Sigma} G_\alpha$. Then, there exists $\alpha_0 \in \Sigma$ such that $a \in G_{\alpha_0}$. Since $G_{\alpha_0} \in \mathcal{T}_\Lambda$, there exist $U \in \mathcal{U}_\Lambda$ and $a \in U[a] \subseteq G_{\alpha_0} \subseteq \bigcup_{\alpha \in \Sigma} G_\alpha$. Hence $\bigcup_{\alpha \in \Sigma} G_\alpha \in \mathcal{T}_\Lambda$. Let $G_1, G_2 \in \mathcal{T}_\Lambda$, and $a \in G_1 \cap G_2$. Then there are $U_1, U_2 \in \mathcal{U}_\Lambda$ such that $a \in U_1[a]$ and $a \in U_2[a]$. By (U_4) , $U_1 \cap U_2 \in \mathcal{U}_\Lambda$ and $a \in (U_1 \cap U_2)[a] \subseteq G_1 \cap G_2$. Therefore $G_1 \cap G_2 \in \mathcal{T}_\Lambda$. Hence $\mathcal{T}$ is a topology on E . $\square$

Lemma 4.4. Let $a, b \in E$ and $I \in \Lambda$. Then $U_I[a \oplus b] = U_I[a] \oplus U_I[b]$.

Proof. Let $t \in U_I[a \oplus b]$. Then, $(t, a \oplus b) \in U_I$ and $t \sim_I a \oplus b$. Since I is a Riesz ideal of E , there are $t_1, t_2 \in E$ such that $t = t_1 \oplus t_2$, $t_1 \oplus t_2 \sim_I a \oplus b$ and $t_1 \sim_I a, t_2 \sim_I b$. Thus $t_1 \in U_I[a], t_2 \in U_I[b]$ and $t_1 \oplus t_2 \in U_I[a] \oplus U_I[b]$. Hence $t \in U_I[a] \oplus U_I[b]$ and so $U_I[a \oplus b] \subseteq U_I[a] \oplus U_I[b]$. By the similar way, we have $U_I[a] \oplus U_I[b] \subseteq U_I[a \oplus b]$. Therefore $U_I[a \oplus b] = U_I[a] \oplus U_I[b]$. $\square$

Theorem 4.5. $(E, \mathcal{T}_\Lambda)$ is a topological effect algebra.

Proof. We prove that operations $\oplus$ and ι are continuous on E . Let G be an open set of E and $a \oplus b \in G$. Then there exists $U \in \mathcal{U}_\Lambda$ such that $U[a \oplus b] \subseteq G$, and there exists Riesz ideal I of E such that $U_I[a \oplus b] \subseteq U[a \oplus b]$. On the other hand, By Lemma 4.4, we have $U_I[a \oplus b] = U_I[a] \oplus U_I[b]$. Thus $\oplus(U_I[a], U_I[b]) = U_I[a] \oplus U_I[b] = U_I[a \oplus b] \subseteq U[a \oplus b] \subseteq G$ and this means that $\oplus$ is continuous. Now we show the mapping $\iota : E \rightarrow E$ by $\iota(a) = a'$ is continuous. Let G be an open set of E and a' in E . Then there exists $U \in \mathcal{U}_\Lambda$ such that $U[a'] \subseteq G$, and there is a Riesz ideal I of E such that $U_I[a'] \subseteq U[a'] \subseteq G$. Since $I = I'$, $\iota(U_I[a]) = U_{I'}[a'] = U_I[a'] \subseteq U[a'] \subseteq G$. Hence ι is continuous. $\square$

Proposition 4.6. For any $i \in \Lambda$ and $x \in E$, $U_I[x]$ is a clopen subset of E .

Proof. Since the pair $(E, \mathcal{U}_\Lambda)$ is a uniform space, $U_I[x]$ is an open subset of E . We prove $U_I[x]$ is closed in E . Let $y \in (U_I[x])^c$. Hence $y \not\sim_I x$ and so $U_I[x] \cap U_I[y] = \emptyset$. This means that $U_I[y] \subseteq (U_I[x])^c$, and so $U_I[x]$ is closed. Therefore, $U_I[x]$ is a clopen subset of E . $\square$

Recall that a uniform space $(X, \mathcal{U})$ is called totally bounded if for each $U \in \mathcal{U}$, there exist $x_1, x_2, \dots, x_n \in X$ such that $X = \bigcup_{i=1}^n U[x_i]$,

Theorem 4.7. The following statements are equivalent:

- (i) topological space $(E, \mathcal{T}_\Lambda)$ is compact.
- (ii) uniformity space $(E, \mathcal{U}_\Lambda)$ is totally bounded.

Proof. (i) $\Rightarrow$ (ii) This is clear by [10].

(ii) $\Rightarrow$ (i) Let $\{U_\alpha\}_{\alpha \in \Sigma}$ be a cover open sets of E and $J = \bigcap_{I \in \Lambda} I$. Since $(E, \mathcal{U}_\Lambda)$ is totally bounded and Λ is closed under intersection, we have $J \in \Lambda$ and there exists $x_1, x_2, \dots, x_n \in E$ such that $E = \bigcup_{i=1}^n U_J[x_i]$. Now for any $1 \leq i \leq n$, there exists $\alpha_i \in \Sigma$ such that $x_i \in U_{\alpha_i}$. Hence $U_J[x_i] \subseteq U_{\alpha_i}$ and so

$$E = \bigcup_{i=1}^n U_J[x_i] \subseteq \bigcup_{i=1}^n U_{\alpha_i}.$$

Therefore, E is compact. $\square$

Corollary 4.8. In uniform space $(E, \mathcal{U}_\Lambda)$, E is compact if and only if E is totally bounded.

5. Completion of an effect algebra

The study of completion in uniformity spaces are very important. Hence in this section we are going to get some conditions that completion for uniformity space be an effect algebra.

Note. In this section, we suppose Λ is closed under finite intersection.

Definition 5.1 ([9]). Let $(X, \mathcal{U})$ be a uniformity space. Then uniformity space $(X^*, \mathcal{U}^*)$ is called a *completion* of $(X, \mathcal{U})$ if:

- (i) $(X^*, \mathcal{U}^*)$ is complete.
- (ii) for any uniformity space $(Y', \mathcal{U}')$ and map $i : (X, \mathcal{U}) \rightarrow (Y, \mathcal{U}')$ there exist an extension $i^* : (X^*, \mathcal{U}^*) \rightarrow (Y, \mathcal{U}')$ such that: $i^*|_X = i$.

Recall that, for a nonempty set Σ , we say that $(\Sigma, \leq)$ is a *directed set*, if the binary relation $\leq$ on Σ has the following properties: (i): for every $\alpha \in \Sigma$, $\alpha \leq \alpha$, (ii): if $\alpha \leq \beta$ and $\beta \leq \gamma$, then $\alpha \leq \gamma$, (iii): for any $\alpha, \beta \in \Sigma$ there exists a $\gamma \in \Sigma$ such that $\alpha \leq \gamma$ and $\beta \leq \gamma$. A *net* in a topological space X is a function from a non-empty directed set Σ to the space X . Nets will be denoted by $\{x_\alpha\}_{\alpha \in \Sigma}$, where x_α is the point of X assigned to the element $\alpha \in \Sigma$. The net of $\{x_\alpha\}_{\alpha \in \Sigma}$ is called a *Cauchy net* if and only if for each $U \in \mathcal{U}$, there exists $\alpha_0 \in \Sigma$ such that for all $\alpha, \beta \geq \alpha_0$, then $(x_\alpha, x_\beta) \in U$. (See [9])

Lemma 5.2. Let Σ be a nonempty directed set and $\tilde{E} = \{\{x_\alpha\}_{\alpha \in \Sigma} : \{x_\alpha\}_{\alpha \in \Sigma} \text{ is a Cauchy net in } E\}$. Suppose that the binary relation $\sim$ on $\tilde{E}$ is defined by $\{x_\alpha\}_{\alpha \in \Sigma} \sim \{y_\beta\}_{\beta \in \Sigma}$ if and only if for all $U \in \mathcal{U}_\Lambda$, there exist $\alpha_0, \beta_0 \in \Sigma$ such that for all $\alpha \geq \alpha_0$ and $\beta \geq \beta_0$, $(x_\alpha, y_\beta) \in U$. Then $\sim$ is an equivalence relation on $\tilde{E}$.

Proof. Since $(E, \mathcal{U}_\Lambda)$ is a uniform space, for any $U \in \mathcal{U}_\Lambda$ and $\alpha \in \Sigma$, $\Delta = (x_\alpha, x_\alpha) \in U$. Hence, $\{x_\alpha\}_{\alpha \in \Sigma} \sim \{x_\alpha\}_{\alpha \in \Sigma}$. Let $\{x_\alpha\}_{\alpha \in \Sigma} \sim \{y_\beta\}_{\beta \in \Sigma}$. Then for any $U \in \mathcal{U}_\Lambda$, there are $\alpha_0, \beta_0 \in \Sigma$ such that for all $\alpha \geq \alpha_0$ and $\beta \geq \beta_0$, $(x_\alpha, y_\beta) \in U$. Since $U \in \mathcal{U}_\Lambda$, we get $U^{-1} \in \mathcal{U}_\Lambda$ and so $(y_\beta, x_\alpha) \in U^{-1}$. Thus $\{y_\beta\}_{\beta \in \Sigma} \sim \{x_\alpha\}_{\alpha \in \Sigma}$.

Now, let $\{x_\alpha\}_{\alpha \in \Sigma} \sim \{y_\beta\}_{\beta \in \Sigma}$ and $\{y_\beta\}_{\beta \in \Sigma} \sim \{z_\gamma\}_{\gamma \in \Sigma}$. Hence for all $U \in \mathcal{U}_\Lambda$, there exist $\alpha_0, \beta_0 \in \Sigma$ such that for all $\alpha \geq \alpha_0$ and $\beta \geq \beta_0$, $(x_\alpha, y_\beta) \in U$ and there exist $\beta_1, \gamma_0 \in \Sigma$ such that for all $\beta \geq \beta_1$ and $\gamma \geq \gamma_0$, $(y_\beta, z_\gamma) \in U$. By Definition 4.1(U_3), for all $\alpha \geq \alpha_0$ and $\beta \geq \beta_0, \beta_1$ and $\gamma \geq \gamma_0$, $(x_\alpha, z_\gamma) \in U$. Thus $\{x_\alpha\}_{\alpha \in \Sigma} \sim \{z_\gamma\}_{\gamma \in \Sigma}$. Therefore $\sim$ is an equivalence relation. $\square$

Notations. Let $E^* = \tilde{E} / \sim$, for any $U \in \mathcal{U}_\Lambda$,

$$U^* = \{([\{x_\alpha\}], [\{y_\beta\}]) : \exists \alpha_0, \beta_0 \in \Sigma; \forall \alpha \geq \alpha_0, \beta \geq \beta_0, (x_\alpha, y_\beta) \in U\}$$

and $\mathcal{U}_\Lambda^* = \{U^* : U \in \mathcal{U}_\Lambda\}$.

Theorem 5.3. $(E^*, \mathcal{U}_\Lambda^*)$ is a uniform space.

Proof. (U_1) : Let $[\{x_\alpha\}_{\alpha \in \Sigma}] \in E^*$. Since $\{x_\alpha\}_{\alpha \in \Sigma} \sim \{x_\alpha\}_{\alpha \in \Sigma}$ for any $U^* \in \mathcal{U}_\Lambda^*$, there exists $\alpha \geq \alpha_0$ such that $(x_\alpha, x_\alpha) \in U$ and so $\Delta \in \mathcal{U}_\Lambda^*$.

(U_2) : Let $U^* \in \mathcal{U}_\Lambda^*$. Then there exist $\{x_\alpha\}_{\alpha \in \Sigma}, \{y_\beta\}_{\beta \in \Sigma} \in \tilde{E}$ such that there are $\alpha \geq \alpha_0$ and $\beta \geq \beta_0$ which $(x_\alpha, y_\beta) \in U$. Since $(E, \mathcal{U}_\Lambda)$ is a uniform space, hence $U = U^{-1}$ and so $(x_\alpha, y_\beta) \in U^{-1}$. Then $(x_\alpha, y_\beta) \in (U^{-1})^* = (U^*)^{-1}$. Therefore, $(U^*)^{-1} \in \mathcal{U}_\Lambda^*$.

(U_3) : For any $U^* \in \mathcal{U}_\Lambda^*$, the transitivity of $\sim$ implies that $U^* \circ U^* \subseteq U^*$.

(U_4) : Let $U^*, V^* \in \mathcal{U}_\Lambda^*$. Then there exist $\{x_\alpha\}_{\alpha \in \Sigma}, \{y_\beta\}_{\beta \in \Sigma} \in \tilde{E}$ such that there are $\alpha_0, \beta_0 \in \Sigma$ which for all $\alpha \geq \alpha_0$ and $\beta \geq \beta_0$, $(x_\alpha, y_\beta) \in U$. Similarly there are $\alpha_1, \beta_1 \in \Sigma$ such that for all $\alpha \geq \alpha_1$ and $\beta \geq \beta_1$, $(x_\alpha, y_\beta) \in V$. Thus for all $\alpha \geq \alpha_0, \alpha_1$ and $\beta \geq \beta_0, \beta_1$, we get $(x_\alpha, y_\beta) \in U \cap V$. Hence $U^* \cap V^* \in \mathcal{U}_\Lambda^*$.

(U_5) : Let $U^* \in \mathcal{U}_\Lambda^*$ and $U^* \subseteq V^* \subseteq E^* \times E^*$. Since $(x_\alpha, y_\beta) \in U$ and $U \subseteq V$, $(x_\alpha, y_\beta) \in V$. Hence $V^* \in \mathcal{U}_\Lambda^*$. $\square$

Lemma 5.4. For any $I \in \Lambda$, $U_I^*([\{x_\alpha \oplus y_\beta\}]) = U_I^*([\{x_\beta\}]) \oplus U_I^*([\{y_\beta\}])$.

Proof. Let $I \in \Lambda$. Then $\{z_\alpha \oplus t_\beta\} \in U_I^*([\{x_\alpha \oplus y_\beta\}]) \Leftrightarrow z_\alpha \oplus t_\beta \sim_I x_\alpha \oplus y_\beta \Leftrightarrow z_\alpha \sim_I x_\alpha, t_\beta \sim_I y_\beta \Leftrightarrow [\{z_\alpha\}] \in U_I^*([\{x_\alpha\}])$ and $[\{t_\beta\}] \in U_I^*([\{y_\beta\}]) \Leftrightarrow \{z_\alpha \oplus t_\beta\} \in U_I^*([\{x_\alpha\}]) \oplus U_I^*([\{y_\beta\}])$ $\square$

Notation. We denote:

$$\mathcal{T}_{\mathcal{U}_\Lambda^*} = \{G \subseteq E^* : \forall [\{x_\alpha\}] \in G, \exists U^* \in \mathcal{U}_\Lambda^* \text{ s.t. } U^*([\{x_\alpha\}]) \subseteq G\}.$$

Theorem 5.5. $(E^*, \mathcal{T}_{\mathcal{U}_\Lambda^*})$ is a topological effect algebra.

Proof. Let $\oplus : E^* \times E^* \rightarrow E^*$ has been defined by $([\{x_\alpha\}], [\{y_\beta\}]) \mapsto [\{x_\alpha \oplus y_\beta\}]$. We show that the operation $\oplus$ is continuous. Let G be an open set of E^* such that $[\{x_\alpha \oplus y_\beta\}] \in G$. Then there is $U^* \in \mathcal{U}_\Lambda^*$ such that $U^*([\{x_\alpha \oplus y_\beta\}]) \subseteq G$. Hence there exists a Riesz ideal $I \in \Lambda$ such that $U_I^*([\{x_\alpha \oplus y_\beta\}]) \subseteq U^*([\{x_\alpha \oplus y_\beta\}]) \subseteq G$. By Lemma 5.4, we have $U_I^*([\{x_\alpha\}]) \oplus U_I^*([\{y_\beta\}]) \subseteq G$, where $U_I^*([\{x_\alpha\}])$ and $U_I^*([\{y_\beta\}])$ are two open neighbourhoods of $[\{x_\alpha\}]$ and $[\{y_\beta\}]$, respectively, and this means that the operation $\oplus$ is continuous. Now we prove the mapping $\iota : E^* \rightarrow E^*$ which is defined by $[\{x_\alpha\}] \mapsto [\{x'_\alpha\}]$ is continuous. Let G be an open set of E^* and $[\{x'_\alpha\}] \in G$. Hence there is $U^* \in \mathcal{U}_\Lambda^*$ such that $U^*([\{x'_\alpha\}]) \subseteq G$. Then, there exists $I \in \Lambda$ such that $U_I^*([\{x'_\alpha\}]) \subseteq U^*([\{x'_\alpha\}]) \subseteq G$. On the other hand, $\iota(U_I^*([\{x_\alpha\}])) = U_I^*([\{x'_\alpha\}])$ and $U^*([\{x_\alpha\}])$ is an open neighborhood of $[\{x_\alpha\}]$. So ι is continuous. Thus $(E^*, \mathcal{T}_{\mathcal{U}_\Lambda^*})$ is a topological effect algebra. $\square$

Theorem 5.6. $(E^*, \mathcal{U}_\Lambda^*)$ is a completion of $(E, \mathcal{U}_\Lambda)$.

Proof. Let $i : E \rightarrow E^*$ is defined by $x \mapsto [\{x_\alpha\}_{\alpha \in \Sigma}]$, where $x_\alpha = x$, for any $\alpha \in \Sigma$. Clearly if $i(x) = i(y)$ then $x = y$. Now we show that $i(E)$ is dense in E^* . Let $U^*([\{x_\alpha\}]) \in \mathcal{T}_{\mathcal{U}_\Lambda^*}$. Hence there is $\alpha_0 \in \Sigma$ such that for all $\alpha \geq \alpha_0$, $x_\alpha \in U[x]$ and so $x_\alpha \rightarrow x$. Hence $\{x_\alpha\}_{\alpha \in \Sigma} \sim \{t_\alpha\}_{\alpha \in \Sigma}$, where $t_\alpha = x$ for any

$\alpha \in \Sigma$ and this means $\{t_\alpha\} \in U^*([\{x_\alpha\}]) \cap i(E)$. Thus $U^*([\{x_\alpha\}]) \cap i(E) \neq \phi$. At present we prove that $(E^*, \mathcal{U}_\Lambda^*)$ is complete. Let $\{[(x_\alpha^\gamma)]\}_{\gamma \in \Sigma}$ be a Cauchy net in E^* . We show this net is convergent. As this net is Cauchy, so for every $U \in \mathcal{U}_\Lambda$ there is $\alpha_0 \in \Sigma$ such that for all $\alpha \geq \alpha_0$, $x_\alpha^\gamma \in U$. Now we define the net of $\{z_t\}_{t \in \Sigma}$ with $z_t = x_{\alpha_0}^\gamma$. We show that the net $\{[(x_\alpha^\gamma)]_{\alpha \in \Sigma}\}_{\gamma \in \Sigma}$ is convergent to $[\{z_t\}_{t \in \Sigma}]$. For this purpose, let U^* be a neighborhood of $[\{z_t\}_{t \in \Sigma}]$. Hence for all $\alpha \geq \alpha_0$, $[\{x_\alpha^\gamma\}] \in U^*([\{z_t\}])$ and the proof is completed. $\square$

6. Conclusion

In this paper we have studied topological structures on effect algebras and we introduced the notions of topological and paratopological effect algebras. Next researches can study separation axioms on topological effect algebras, quotient effect algebras and many of the other concepts of topology.

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A CERTAIN NEW FAMILIAR CLASS OF UNIVALENT ANALYTIC FUNCTIONS WITH VARYING ARGUMENT OF COEFFICIENTS INVOLVING CONVOLUTION

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Abstract. In this paper, we used the generalization of the modified-Hadamard products to obtain some interesting characterization theorems for certain general subclass of uniformly functions with positive coefficients.

Keywords: analytic, univalent, uniformly starlike, uniformly convex, Hadamard product (or convolution).

1. Introduction and preliminaries

Let $\mathcal{A}$ denote the class of functions of the form:

$$(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic and univalent in the open unit disc $\mathbb{U} = \{z : |z| < 1\}$. This class of functions has been extensively exploited in some recent articles to study subclasses of functions satisfy certain conditions [16, 17, 18, 19, 20]. For functions $f, g \in \mathcal{A}$ of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, their Hadamard product or convolution $f(z) * g(z)$ is defined by

$$(1.2) \quad f(z) * g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in \mathbb{U}.$$

Definition 1.1 ([14]). Let $\kappa - ST(\alpha, \beta)$ denote the subclass of $\mathcal{A}$ consisting of functions $f(z)$ of the form (1.1) and satisfy the following inequality

$$(1.3) \quad \operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} - \alpha > \kappa \left| \frac{zf'(z)}{f(z)} - \beta \right|$$

$$(0 \leq \alpha < \beta \leq 1; \kappa(1 - \beta) < 1 - \alpha; z \in \mathbb{U}).$$

Also let $\kappa - UCV(\alpha, \beta)$ denote the subclass of $\mathcal{A}$ consisting of functions $f(z)$ of the form (1.1) and satisfy the following inequality

$$(1.4) \quad \operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} - \alpha > \kappa \left| 1 + \frac{zf''(z)}{f'(z)} - \beta \right|$$

$$(0 \leq \alpha < \beta \leq 1; \kappa(1 - \beta) < 1 - \alpha; z \in \mathbb{U}).$$

The class $\kappa - ST(\alpha, \beta)$ denote the class of κ -uniformly starlike functions of order α and type β and the class $\kappa - UCV(\alpha, \beta)$ denote the class of κ -uniformly convex functions of order α and type β .

Specializing the parameters α, β and κ , we obtain many subclasses studied by various authors (see [1-4] and [7-13]).

We now introduce the familiar subclass $\kappa - \mathcal{SC}(\Phi, \Psi; \alpha, \beta)$ of the functions in class $\mathcal{A}$ as follows.

Definition 1.2. Given

$$\Phi(z) = z + \sum_{n=2}^{\infty} \lambda_n z^n \quad \text{and} \quad \Psi(z) = z + \sum_{n=2}^{\infty} \mu_n z^n$$

be analytic in $\mathbb{U}$, such that $\lambda_n \geq 0, \mu_n \geq 0$ and $\lambda_n \geq \mu_n$ for $n \geq 2$, we say that $f(z) \in \mathcal{A}$ is in the class $\kappa - \mathcal{SC}(\Phi, \Psi; \alpha, \beta)$ if $f(z) * \Psi(z) \neq 0$ and

$$(1.5) \quad \operatorname{Re} \left\{ \frac{f(z) * \Phi(z)}{f(z) * \Psi(z)} \right\} - \alpha > \kappa \left| \frac{f(z) * \Phi(z)}{f(z) * \Psi(z)} - \beta \right|,$$

$$(0 \leq \alpha < \beta \leq 1; \kappa(1 - \beta) < 1 - \alpha; z \in \mathbb{U}).$$

It is easy to check that various subclasses of $\mathcal{A}$ referred to above can be represented as $\kappa - \mathcal{SC}(\Phi, \Psi; \alpha, \beta)$ for suitable choices of Φ, Ψ . For example

(i) $\kappa - \mathcal{SC} \left(\frac{z}{(1-z)^2}, \frac{z}{1-z}; \alpha, \beta \right) = \kappa - ST(\alpha, \beta)$ and $\kappa - \mathcal{SC} \left(\frac{z+z^2}{(1-z)^3}, \frac{z}{(1-z)^2}; \alpha, \beta \right) = \kappa - UCV(\alpha, \beta)$ (see Sim et al. [14]);

(ii) $\kappa - \mathcal{SC} \left(\frac{z}{(1-z)^2}, \frac{z}{1-z}; \alpha, 1 \right) = SD(\kappa, \alpha)$ and $\kappa - \mathcal{SC} \left(\frac{z+z^2}{(1-z)^3}, \frac{z}{(1-z)^2}; \alpha, 1 \right) = KD(\kappa, \alpha)$ (see Shams et al. [12]);

(iii) $\kappa - \mathcal{SC} \left(\frac{z}{(1-z)^2}, \frac{z}{1-z}; 0, 1 \right) = \kappa - ST$ and $\kappa - \mathcal{SC} \left(\frac{z+z^2}{(1-z)^3}, \frac{z}{(1-z)^2}; 0, 1 \right) = \kappa - UCV$ (see Kanas and Wisniowska [9, 10]);

(iv) $1 - \mathcal{SC} \left(\frac{z}{(1-z)^2}, \frac{z}{1-z}; \alpha, 1 \right) = S_p(\alpha)$ and $1 - \mathcal{SC} \left(\frac{z+z^2}{(1-z)^3}, \frac{z}{(1-z)^2}; \alpha, 1 \right) = UCV(\alpha)$ (see Ronning [4]);

(v) $1 - \mathcal{SC} \left(\frac{z}{(1-z)^2}, \frac{z}{1-z}; 0, 1 \right) = S_p$ and $1 - \mathcal{SC} \left(\frac{z+z^2}{(1-z)^3}, \frac{z}{(1-z)^2}; 0, 1 \right) = UCV$ (see Goodman [1, 2], Ma and Minda [13] and Ronning [3, 4]).

Another subclasses are the subclasses

$$1 - \mathcal{SC} \left(\frac{z}{(1-z)^2}, \frac{z}{1-z}; \alpha, \beta \right) = US(\alpha, \beta) \equiv \operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} - \alpha > \left| \frac{zf'(z)}{f(z)} - \beta \right|$$

and

$$\begin{aligned} & 1 - \mathcal{SC} \left(\frac{z + z^2}{(1 - z)^3}, \frac{z}{(1 - z)^2}; \alpha, \beta \right) \\ &= UC(\alpha, \beta) \equiv \operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} - \alpha > \left| 1 + \frac{zf''(z)}{f'(z)} - \beta \right|. \end{aligned}$$

For $p_i \geq 1$ and $\sum_{i=1}^{\infty} \frac{1}{p_i} = 1$, the Hölder inequality is defined by (see [15]):

$$(1.6) \quad \sum_{i=2}^{\infty} \left(\prod_{j=1}^m a_{i,j} \right) \leq \prod_{j=1}^m \left(\sum_{i=2}^{\infty} a_{i,j}^{p_i} \right)^{\frac{1}{p_i}}.$$

Let $f_j \in \mathcal{A}$ ($j = 1, 2$) be given by

$$(1.7) \quad f_j(z) = z + \sum_{n=2}^{\infty} a_{n,j} z^n \quad (j = 1, 2).$$

Then the modified Hadamard product or (convolution) $f_1 * f_2$ is defined by

$$(1.8) \quad (f_1 * f_2)(z) = z + \sum_{n=2}^{\infty} a_{n,1} a_{n,2} z^n.$$

For any real numbers p and q , the modified generalized Hadamard product $(f_1 \Delta f_2)(p, q; z)$ defined by (see Choi et al. [6]):

$$(1.9) \quad (f_1 \Delta f_2)(p, q; z) = z + \sum_{n=2}^{\infty} (a_{n,1})^p (a_{n,2})^q z^n.$$

In the special case, if we take $p = q = 1$, then

$$(1.10) \quad (f_1 \Delta f_2)(1, 1; z) = (f_1 * f_2)(z) \quad (z \in \mathbb{U}).$$

In order to prove our results, we shall need the following lemma.

Lemma 1.3 ([5]). *Let the function $f(z)$ be given by (1.1). If*

$$(1.11) \quad \sum_{n=2}^{\infty} [(1 + \kappa)\lambda_n - (\alpha + \kappa\beta)\mu_n] |a_n| \leq 1 - \alpha - \kappa(1 - \beta),$$

where $\lambda_n \geq 0, \mu_n \geq 0$ and $\lambda_n \geq \mu_n$, then $f(z) \in \kappa - \mathcal{SC}(\Phi, \Psi; \alpha, \beta)$.

In the present paper, we will obtain several results for the generalized Hadamard product of functions in the class $\kappa - \mathcal{SC}(\Phi, \Psi; \alpha, \beta)$.

2. Main results

Unless otherwise mentioned, we assume in the reminder of this paper that;
 $0 \leq \alpha < \beta \leq 1; \kappa(1 - \beta) < 1 - \alpha; z \in \mathbb{U}$.

Theorem 2.1. *If the function f_j ($j = 1, 2$) defined by (1.7) belongs to the subclass $\kappa - \mathcal{SC}(\Phi, \Psi; \alpha_j, \beta)$ ($j = 1, 2$), then*

$$(2.1) \quad (f_1 \Delta f_2) \left(\frac{1}{p}, \frac{1}{q}; z \right) \in \kappa - \mathcal{SC}(\Phi, \Psi; \sigma, \beta),$$

where $p, q > 1$ and σ is given by

$$\sigma = \min_{n \geq 2} \left\{ \frac{(1 - \kappa(1 - \beta)) \left(\frac{(1 + \kappa)\lambda_n - (\alpha_1 + \kappa\beta)\mu_n}{1 - \alpha_1 - \kappa(1 - \beta)} \right)^{\frac{1}{p}} \left(\frac{(1 + \kappa)\lambda_n - (\alpha_2 + \kappa\beta)\mu_n}{1 - \alpha_2 - \kappa(1 - \beta)} \right)^{\frac{1}{q}} + \kappa\beta\mu_n - (1 + \kappa)\lambda_n}{\left(\frac{(1 + \kappa)\lambda_n - (\alpha_1 + \kappa\beta)\mu_n}{1 - \alpha_1 - \kappa(1 - \beta)} \right)^{\frac{1}{p}} \left(\frac{(1 + \kappa)\lambda_n - (\alpha_2 + \kappa\beta)\mu_n}{1 - \alpha_2 - \kappa(1 - \beta)} \right)^{\frac{1}{q}} - \mu_n} \right\}.$$

Proof. Let $f_j \in \kappa - \mathcal{SC}(\Phi, \Psi; \alpha_j, \beta)$. Then by using Lemma 1.3, we have

$$(2.2) \quad \sum_{n=2}^{\infty} \frac{(1 + \kappa)\lambda_n - (\alpha_j + \kappa\beta)\mu_n}{1 - \alpha_j - \kappa(1 - \beta)} |a_{n,j}| \leq 1 \quad (j = 1, 2).$$

Moreover,

$$(2.3) \quad \left\{ \sum_{n=2}^{\infty} \frac{(1 + \kappa)\lambda_n - (\alpha_1 + \kappa\beta)\mu_n}{1 - \alpha_1 - \kappa(1 - \beta)} |a_{n,1}| \right\}^{\frac{1}{p}} \leq 1,$$

and

$$(2.4) \quad \left\{ \sum_{n=2}^{\infty} \frac{(1 + \kappa)\lambda_n - (\alpha_2 + \kappa\beta)\mu_n}{1 - \alpha_2 - \kappa(1 - \beta)} |a_{n,2}| \right\}^{\frac{1}{q}} \leq 1.$$

Applying the Hölder inequality (1.6) to (2.3) and (2.4), we obtain

$$(2.5) \quad \sum_{n=2}^{\infty} \left[\frac{(1 + \kappa)\lambda_n - (\alpha_1 + \kappa\beta)\mu_n}{1 - \alpha_1 - \kappa(1 - \beta)} \right]^{\frac{1}{p}} \cdot \left[\frac{(1 + \kappa)\lambda_n - (\alpha_2 + \kappa\beta)\mu_n}{1 - \alpha_2 - \kappa(1 - \beta)} \right]^{\frac{1}{q}} |a_{n,1}|^{\frac{1}{p}} |a_{n,2}|^{\frac{1}{q}} \leq 1.$$

Since

$$(2.6) \quad (f_1 \Delta f_2) \left(\frac{1}{p}, \frac{1}{q}; z \right) = z + \sum_{n=2}^{\infty} (a_{n,1})^{\frac{1}{p}} (a_{n,2})^{\frac{1}{q}} z^n,$$

we see that

$$(2.7) \quad \sum_{n=2}^{\infty} \left[\frac{(1+\kappa)\lambda_n - (\sigma + \kappa\beta)\mu_n}{1 - \sigma - \kappa(1-\beta)} \right] |a_{n,1}|^{\frac{1}{p}} |a_{n,2}|^{\frac{1}{q}} \leq 1,$$

with

$$\sigma \leq \min_{n \geq 2} \left[\frac{(1 - \kappa(1 - \beta)) \left(\frac{(1+\kappa)\lambda_n - (\alpha_1 + \kappa\beta)\mu_n}{1 - \alpha_1 - \kappa(1-\beta)} \right)^{\frac{1}{p}} \left(\frac{(1+\kappa)\lambda_n - (\alpha_2 + \kappa\beta)\mu_n}{1 - \alpha_2 - \kappa(1-\beta)} \right)^{\frac{1}{q}} + \kappa\beta\mu_n - (1 + \kappa)\lambda_n}{\left(\frac{(1+\kappa)\lambda_n - (\alpha_1 + \kappa\beta)\mu_n}{1 - \alpha_1 - \kappa(1-\beta)} \right)^{\frac{1}{p}} \left(\frac{(1+\kappa)\lambda_n - (\alpha_2 + \kappa\beta)\mu_n}{1 - \alpha_2 - \kappa(1-\beta)} \right)^{\frac{1}{q}} - \mu_n} \right].$$

Thus, by using Lemma 1.3, the proof of Theorem 2.1 is completed. $\square$

Putting $\alpha_j = \alpha$ ($j = 1, 2$) in Theorem 2.1, we obtain the following corollary.

Corollary 2.2. *If the functions f_j ($j = 1, 2$) defined by (1.7) are in the subclass $\kappa - \mathcal{SC}(\Phi, \Psi; \alpha, \beta)$. Then*

$$(2.8) \quad (f_1 \Delta f_2) \left(\frac{1}{p}, \frac{1}{q}; z \right) \in \kappa - \mathcal{SC}(\Phi, \Psi; \alpha, \beta) \quad (p, q > 1).$$

Theorem 2.3. *If the function f_j ($j = 1, 2, \dots, m$) defined by (1.7) belongs to the subclass $\kappa - \mathcal{SC}(\Phi, \Psi; \alpha_j, \beta)$ ($j = 1, 2, \dots, m$), and $\mathcal{G}_m(z)$ defined by*

$$(2.9) \quad \mathcal{G}_m(z) = z + \sum_{n=2}^{\infty} \left(\sum_{j=1}^m (a_{n,j})^p \right) z^n,$$

then

$$(2.10) \quad \mathcal{G}_m(z) \in \kappa - \mathcal{SC}(\Phi, \Psi; \sigma_m, \beta),$$

where

$$\sigma_m = \min_{n \geq 2} \left\{ 1 - \kappa(1 - \beta) - \frac{(\kappa(1 - 2\beta) - 1)m\mu_n + (1 - \kappa)m\lambda_n}{\left[\frac{(1+\kappa)\lambda_n - (\alpha + \kappa\beta)\mu_n}{1 - \alpha - \kappa(1-\beta)} \right]^p - m\mu_n} \right\}, \alpha = \min_{1 \leq j \leq m} \{\alpha_j\}.$$

Proof. Since $f_j \in \kappa - \mathcal{SC}(\Phi, \Psi; \alpha_j, \beta)$ by using Lemma 1.3, we have

$$(2.11) \quad \sum_{n=2}^{\infty} \frac{(1 + \kappa)\lambda_n - (\alpha_j + \kappa\beta)\mu_n}{1 - \alpha_j - \kappa(1 - \beta)} |a_{n,j}| \leq 1 \quad (j = 1, 2, \dots, m; n \geq 2).$$

and

$$(2.12) \quad \begin{aligned} & \sum_{n=2}^{\infty} \left\{ \frac{(1+\kappa)\lambda_n - (\alpha_j + \kappa\beta)\mu_n}{1 - \alpha_j - \kappa(1-\beta)} \right\}^p |a_{n,j}|^p \\ & \leq \left\{ \sum_{n=2}^{\infty} \frac{(1+\kappa)\lambda_n - (\alpha_j + \kappa\beta)\mu_n}{1 - \alpha_j - \kappa(1-\beta)} |a_{n,j}| \right\}^p \leq 1, \end{aligned}$$

it follows from (2.12), that

$$(2.13) \quad \sum_{n=2}^{\infty} \left(\frac{1}{m} \sum_{j=1}^m \left\{ \frac{(1+\kappa)\lambda_n - (\alpha_j + \kappa\beta)\mu_n}{1 - \alpha_j - \kappa(1-\beta)} \right\}^p |a_{n,j}|^p \right) \leq 1.$$

Putting

$$(2.14) \quad \alpha = \min_{1 \leq j \leq m} \{\alpha_j\},$$

and by virtue of Lemma 1.3, we find that

$$(2.15) \quad \begin{aligned} & \sum_{n=2}^{\infty} \frac{(1+\kappa)\lambda_n - (\sigma_m + \kappa\beta)\mu_n}{1 - \sigma_m - \kappa(1-\beta)} \sum_{j=1}^m |a_{n,j}|^p \\ & \leq \sum_{n=2}^{\infty} \left\{ \frac{1}{m} \left[\frac{(1+\kappa)\lambda_n - (\alpha + \kappa\beta)\mu_n}{1 - \alpha - \kappa(1-\beta)} \right]^p \sum_{j=1}^m |a_{n,j}|^p \right\} \\ & \leq \sum_{n=2}^{\infty} \left\{ \frac{1}{m} \sum_{j=1}^m \left[\frac{(1+\kappa)\lambda_n - (\alpha_j + \kappa\beta)\mu_n}{1 - \alpha_j - \kappa(1-\beta)} \right]^p |a_{n,j}|^p \right\} \leq 1, \end{aligned}$$

if

$$(2.16) \quad \sigma_m \leq \min_{n \geq 2} \left\{ 1 - \kappa(1-\beta) - \frac{(\kappa(1-2\beta) - 1)m\mu_n + (1-\kappa)m\lambda_n}{\left[\frac{(1+\kappa)\lambda_n - (\alpha + \kappa\beta)\mu_n}{1 - \alpha - \kappa(1-\beta)} \right]^p - m\mu_n} \right\}.$$

Thus the proof of Theorem 2.3 is completed. $\square$

Taking $p = 2$ and $\alpha_j = \alpha (j = 1, 2, \dots, m)$ in Theorem 2.3, we obtain the following corollary:

Corollary 2.4. *Let the functions $f_j(z)$ ($j = 1, 2, \dots, m$) defined by (1.7) be in the class $\kappa - SC(\Phi, \Psi; \alpha, \beta)$ and let the function $\mathcal{G}_m(z)$ be defined by*

$$(2.17) \quad \mathcal{G}_m(z) = z + \sum_{n=2}^{\infty} \left(\sum_{j=1}^m (a_{n,j})^2 \right) z^n, \quad z \in \mathbb{U}.$$

Then $\mathcal{G}_m(z) \in \kappa - \mathcal{SC}(\Phi, \Psi; \zeta_m, \beta)(z \in \mathbb{U})$, where

$$(2.18) \quad \zeta_m = \min_{n \geq 2} \left\{ 1 - \kappa(1 - \beta) - \frac{(\kappa(1 - 2\beta) - 1)m\mu_n + (1 - \kappa)m\lambda_n}{\left[\frac{(1+\kappa)\lambda_n - (\alpha+\kappa\beta)\mu_n}{1-\alpha-\kappa(1-\beta)}\right]^2 - m\mu_n} \right\}.$$

Taking $m = 2$ in Corollary 2.4, we obtain

Corollary 2.5. *Let the functions $f_j(z)$ ($j = 1, 2$) defined by (1.7) be in the class $\kappa - \mathcal{SC}(\Phi, \Psi; \alpha, \beta)$ and let the function $\mathcal{G}_2(z)$ defined by*

$$(2.19) \quad \mathcal{G}_2(z) = z + \sum_{n=2}^{\infty} (a_{n,1}^2 + a_{n,2}^2)z^n, \quad z \in \mathbb{U}.$$

Then $\mathcal{G}_2(z) \in \kappa - \mathcal{SC}(\Phi, \Psi; \zeta_2, \beta)(z \in \mathbb{U})$, where

$$(2.20) \quad \zeta_2 = \min_{n \geq 2} \left\{ 1 - \kappa(1 - \beta) - \frac{(\kappa(1 - 2\beta) - 1)\mu_n + (1 - \kappa)\lambda_n}{\left[\frac{(1+\kappa)\lambda_n - (\alpha+\kappa\beta)\mu_n}{\sqrt{2}(1-\alpha-\kappa(1-\beta))}\right]^2 - \mu_n} \right\}.$$

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SPECIAL HOOP ALGEBRAS

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Abstract. Hoops are naturally ordered commutative residuated integral monoids, introduced by B. Bosbach in [5, 6]. In this paper, we introduce the concepts of special hoop algebra and special filter in hoop algebras and study some properties of them. We establish relation between special hoops with other structures such as simple hoops, local hoops, locally finite hoops, perfect hoops, semi-De Morgan algebras and Boolean algebras. Then, by define the notion of special filter in bounded hoops, we study the relationship between special filters and implicative (positive implicative, maximal, obstinate) filters on bounded and special hoops. Finally, we investigated the properties of a quotient structure, when it is produced by a special filter.

Keywords: Hoop algebra, special hoop, simple hoops, local hoops, locally finite hoops, perfect hoops, semi-De Morgan algebras, Boolean algebra, special filter, implicative (positive implicative, maximal, obstinate) filter.

1. Introduction

Bosbach [5, 6] undertook the investigation of a class of residuated structures that were related to considerably more general than the Brouwerian semilattices and the algebras associated with Łukasiewicz's calculus mentioned above. The requirement he added was that the partial order be *natural*; in the commutative case (to which we will restrict ourselves in this paper) this means that $a \leq b$ if and only if there is an element c such that $a = b \odot c$. Brouwerian semilattices as well as the models of many-valued logic satisfy this requirement, but the models of linear logic do not in general. He showed that the resulting class of structures can be viewed as an equational class, and that the class is congruence distributive and congruence permutable. In a manuscript by J. R. Büchi and

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T. M. Owens [7], devoted to a study of Bosbachs algebras, written in the mid-seventies, the commutative members of this equational class were given the name hoops. The manuscript is a rich source of ideas, but of a preliminary nature and was never published. Some of the results obtained there can be found in two joint papers with Blok [2]; in particular the description of subdirectly irreducible hoops ([2], Theorem 2.9) will play a crucial role in this paper. In the last years, hoops theory was enriched with deep structure theorems (see [1, 8]). Many of these results have a strong impact with fuzzy logic. The algebraic structures corresponding to Hájek's propositional basic logic, BL -algebras, are particular cases of hoops. Kondo in [9], considered fundamental properties of some types of filters (implicative, positive implicative and fantastic filters) of hoops. R. A. Borzooei and M. Aaly Kologani investigate the relation between these filters in [3]. Also they introduce in [4], the concepts of local and perfect semihoops and state and prove some related results. Specially, they defined the concepts of locally finite semihoop and found a relation between local and perfect semihoops. N. Mohtashamnia and A. Borumand Saeid in 2012 introduced the notion of special type of BL -algebras [10].

The aim of this paper is the introduction of a new structure from hoop as a special and compare it with other structures hoop. After that we define special filters and consider relationships between special filters and some other filters. We show that in some examples any special filter is not (obstinate, fantastic, implicative, maximal, prime and perfect) filter and conversely is not true, too. Also, we study the relationship between special filter and congruence relations on hoop and obtain equivalent relation with special filter.

2. Preliminaries

In this section, we recollect some definitions and results which will be used not cite them every time they are used.

Definition 2.1 ([1]). A *hoop algebra* or *hoop* is an algebra $(A, \odot, \rightarrow, 1)$ of type $(2, 2, 0)$ such that for all $x, y, z \in A$:

- (HP1) $(A, \odot, 1)$ is a commutative monoid,
- (HP2) $x \rightarrow x = 1$,
- (HP3) $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z)$,
- (HP4) $x \odot (x \rightarrow y) = y \odot (y \rightarrow x)$.

On hoop A , we define $x \leq y$ if and only if $x \rightarrow y = 1$. It is easy to see that $\leq$ is a partial order relation on A . A hoop A is *bounded* if there is an element $0 \in A$ such that $0 \leq x$, for all $x \in A$. We let $x^0 = 1$, $x^n = x^{n-1} \odot x$, for any $n \in \mathbb{N}$. Let A be a bounded hoop. We define a negation "′" on A by, $x' = x \rightarrow 0$, for all $x \in A$. If $x'' = x$, for all $x \in A$, then the bounded hoop A is said to have the *double negation property*, or (*DNP*), for short. The *order* of $1 \neq x \in A$, in symbols $ord(x)$ is the smallest $n \in \mathbb{N}$ such that $x^n = 0$. If no such n exists, then $ord(x) = \infty$. A hoop is called *locally finite* if for any $x \in A$, $x \neq 1$, has

finite order. An element $x \in A$ is called *dense* if and only if $x' = 0$ and the set of all dense elements of A shown that by $D_e(A) = \{x \in A \mid x' = 0\}$. An element $a \in A$ is called *atom* if it is minimal among elements in bounded hoop $A \setminus \{0\}$. (See [8])

Proposition 2.2 ([5, 6]). *In any hoop $(A, \odot, \rightarrow, 1)$ the following properties hold, for all $x, y, z \in A$:*

- (i) $(A, \leq)$ is a meet-semilattice with $x \wedge y = x \odot (x \rightarrow y)$,
- (ii) $x \odot y \leq x, y$ and $x^n \leq x$, for any $n \in \mathbb{N}$,
- (iii) $1 \rightarrow x = x$, $x \rightarrow x = 1$, $x \rightarrow 1 = 1$,
- (iv) $x \leq y \rightarrow x$,
- (v) $x \odot (y \rightarrow z) \leq y \rightarrow x \odot z$.

Proposition 2.3 ([5, 6]). *Let A be a bounded hoop. Then the following properties hold, for all $x, y, z \in A$:*

- (i) $x \odot x' = 0$, $0' = 1$, $1' = 0$, $x''' = x'$, $0 \rightarrow x = 1$ and $0 \odot x = 0$,
- (ii) $x' \leq x \rightarrow y$,
- (iii) if $x \leq y$, then $y' \leq x'$,
- (iv) $x \rightarrow y' = y \rightarrow x' = (x \odot y)'$,
- (v) $x \leq x''$.

Definition 2.4 ([8]). Let A be a bounded hoop and for any $x, y \in A$, we define $x \vee y = ((x \rightarrow y) \rightarrow y) \wedge ((y \rightarrow x) \rightarrow x)$. If $\vee$ is the join operation on A , then A is called a $\vee$ -hoop.

Definition 2.5 ([8]). Let A be a hoop. A non-empty subset F of A is called a *filter* of A if,

- (F1) $x \in F$ and $x \leq y$, $y \in A$, then $y \in F$,
- (F2) $x \odot y \in F$, for any $x, y \in F$.

Clearly, $1 \in F$, for all filter of A . A filter F of A is called *proper filter* if $F \neq A$. It can be easily to see that, if A is a bounded hoop, then a filter is proper if and only if it is not containing 0.

Definition 2.6 ([3]). A proper filter F of a $\vee$ -hoop A is called *prime filter* of A if $x \vee y \in F$ implies $x \in F$ or $y \in F$, for any $x, y \in A$. A *maximal filter* is a proper filter M of hoop A such that it is not included in any other proper filter.

Proposition 2.7 ([4]). *A proper filter M of bounded hoop A is a maximal filter of A if and only if $x \notin M$, then there exists $n \in \mathbb{N}$ such that $(x^n)' \in M$.*

Definition 2.8 ([4, 9, 11]). Let F be a subset of A such that $1 \in F$. Then for any $x, y, z \in A$:

- (i) F is called a *positive implicative filter* of A , if $x \rightarrow (y \rightarrow z) \in F$ and $x \rightarrow y \in F$, then $x \rightarrow z \in F$.
- (ii) F is called an *implicative filter* of A , if $x \rightarrow ((y \rightarrow z) \rightarrow y) \in F$ and $x \in F$, then $y \in F$.

(iii) F is called a *fantastic filter* of A , if $z \rightarrow (y \rightarrow x) \in F$ and $z \in F$, then $((x \rightarrow y) \rightarrow y) \rightarrow x \in F$.

(iv) F is called a *perfect filter* of A , if F is a filter such that, for any $x \in A$, $(x^n)' \in F$, for some $n \in \mathbb{N}$ if and only if $((x')^m)' \notin F$, for any $m \in \mathbb{N}$.

(v) F is called an *obstinate filter* of A , if F is a proper filter such that, for any $x, y \notin F$, $x \rightarrow y \in F$ and $y \rightarrow x \in F$.

(vi) F is called a *primary filter* of A , if for all $x, y \in A$, $(x \odot y)' \in F$ implies $(x^n)' \in F$ or $(y^n)' \in F$, for some $n \in \mathbb{N}$.

Definition 2.9 ([8]). Let A and B be two bounded hoops. A map $f : A \rightarrow B$ is called a *hoop homomorphism* if and only if for all $x, y \in A$, $f(0) = 0$, $f(1) = 1$, $f(x \odot y) = f(x) \odot f(y)$ and $f(x \rightarrow y) = f(x) \rightarrow f(y)$.

Proposition 2.10 ([8]). Let A be a $\vee$ -hoop. Then $(A, \vee, \wedge)$ is a distributive lattice.

Proposition 2.11 ([3, 11]). (i) If F is an implicative filter of A , then $x'' \rightarrow x \in F$, for any $x \in A$.

(ii) F is an implicative filter of A if and only if $(x' \rightarrow x) \rightarrow x \in F$, for any $x \in A$.

(iii) Any obstinate filter of A is a maximal filter of A .

Definition 2.12 ([1, 4]). (i) A *simple hoop* is a hoop which has just two trivial filters.

(ii) A *cancellative hoop* is a hoop, where the monoid $(A, \odot, 1)$ is cancellative.

(iii) A *basic hoop* is a hoop and for any $x, y, z \in A$, $(x \rightarrow y) \rightarrow z \leq ((y \rightarrow x) \rightarrow z) \rightarrow z$.

(iv) A *finitely subdirectly irreducible hoop* is a hoop which any pair of non-trivial principal filters has a non-trivial intersection.

(v) A *local hoop* is a hoop, where $\text{ord}(x) < \infty$ or $\text{ord}(x') < \infty$, for all $x \in A$.

(vi) A *perfect hoop* is a hoop, where for any $x \in A$, if $\text{ord}(x) < \infty$, then $\text{ord}(x') = \infty$, and if $\text{ord}(x) = \infty$, then $\text{ord}(x') < \infty$.

Definition 2.13 ([12]). An algebra $(L, \vee, \wedge, ', 0, 1)$ of type $(2, 2, 1, 0, 0)$ is called a *semi-De Morgan algebra* if $(L, \vee, \wedge, 0, 1)$ is a distributive lattice, $0' = 1$, $1' = 0$, and for any $x, y \in L$, $(x \vee y)' = x' \wedge y'$, $(x \wedge y)'' = x'' \wedge y''$ and $x''' = x'$.

3. Special hoop

In this section, we introduce the notion of special hoop and investigate some properties of them. Also, we compare relation between special hoop and other structures of hoops.

Definition 3.1. A bounded hoop A is called *special hoop* if for any $x, y \in A \setminus \{0\}$,

$$(x \rightarrow y)' = (y \rightarrow x)'$$

By the following example we show the relationship between special hoop and other structures of hoops.

Example 3.2. (i) Let $(A = \{0, a, b, c, 1\}, \leq)$, be a poset with $0 < c < a, b < 1$, but a, b are incomparable. Define the operations $\odot$ and $\rightarrow$ on A as follows:

$\rightarrow$	0	c	a	b	1
0	1	1	1	1	1
c	0	1	1	1	1
a	0	b	1	b	1
b	0	a	a	1	1
1	0	c	a	b	1

$\odot$	0	c	a	b	1
0	0	0	0	0	0
c	0	c	c	c	c
a	0	c	a	c	a
b	0	c	c	b	b
1	0	c	a	b	1

Then $(A, \odot, \rightarrow, 1, 0)$ is a special hoop. But it is not Wajsberg hoop, because $1 = (a \rightarrow 0) \rightarrow 0 \neq (0 \rightarrow a) \rightarrow a = a$. Also, it is not a finitely subdirectly irreducible hoop, because $[a] \cap [b] = \{1\}$.

(ii) Let $(A = \{0, a, b, 1\}, \leq)$ be a chain, that is $0 < a < b < 1$. Define the operations $\odot$ and $\rightarrow$ on A as follows:

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

Then $(A, \odot, \rightarrow, 1, 0)$ is a bounded hoop which is a finitely subdirectly irreducible hoop. But it is not a special hoop, because $0 = (a \rightarrow b)' \neq (b \rightarrow a)' = a$.

(iii) Let $A = \{0, 1\}$ be a two-element chain with the following operations.

$\rightarrow$	0	1
0	1	1
1	0	1

$\odot$	0	1
0	0	0
1	0	1

Then A is a special hoop. But A is not a cancellative hoop, because $0 \odot 1 = 0 \odot 0$, and $1 \neq 0$.

(iv) Let $G = (G, +, -, \vee, \wedge, 0)$ be an arbitrary l -group and $N(G)$ be the negative cone of G , that is $N(G) = \{a \in G \mid a \leq 0\}$. Define the operations $\odot$ and $\rightarrow$ on $N(G)$ as follows:

$$a \odot b = a + b \quad \text{and} \quad a \rightarrow b = (b - a) \wedge 0$$

Then $(N(G), \odot, \rightarrow, 0)$ is a basic hoop and cancellative hoop (See [8]). But it is not a special hoop, because it is not bounded.

(v) Let $A = [0, 1]$ and operations $\odot$ and $\rightarrow$ on A are defined by $x \odot y = \min\{x, y\}$ and

$$x \rightarrow y = \begin{cases} 1, & \text{if } x \leq y \\ y, & \text{if otherwise} \end{cases}$$

Then $(A, \odot, \rightarrow, 1, 0)$ is a special hoop.

Proposition 3.3. *For a bounded hoop A , the following conditions are equivalent, for any $x, y \in A$ and $x \neq 0$:*

- (i) A is a special hoop,
- (ii) $x' = 0$,
- (iii) $x'' = 1$,
- (iv) $x' \rightarrow y = 1$,
- (v) $y \odot x' = 0$,
- (vi) $y \rightarrow (x' \rightarrow y') = 1$,
- (vii) $(x^n)' = 0$, for any $n \in \mathbb{N}$.

Proof. (i) $\Rightarrow$ (ii) Let A be a special hoop. Then we have $(x \rightarrow y)' = (y \rightarrow x)'$, for any $x, y \in A \setminus \{0\}$. Consider $y = 1$, hence we have $0 = 1' = (x \rightarrow 1)' = (1 \rightarrow x)' = x'$.

(ii) $\Rightarrow$ (i) Let $x, y \in A \setminus \{0\}$. By Proposition 2.2(iv), $y \leq x \rightarrow y$ and $x \leq y \rightarrow x$ and so by Proposition 2.3(iii), $(x \rightarrow y)' \leq y' = 0$ and $(y \rightarrow x)' \leq x' = 0$. Hence $(x \rightarrow y)' = 0 = (y \rightarrow x)'$ and this means that A is a special hoop.

(ii) $\Leftrightarrow$ (iii) By Proposition 2.3(i).

(ii) $\Rightarrow$ (iv), (v), (vi) The proof are clear by Proposition 2.3.

(iv) $\Rightarrow$ (ii) Let $y = 0$. Then $x'' = x' \rightarrow 0 = x' \rightarrow y = 1$ and so by Proposition 2.3(i), $x' = x''' = 0$.

(v) $\Rightarrow$ (ii) Let $y = 1$. Then $x' = 1 \odot x' = y \odot x' = 0$.

(vi) $\Rightarrow$ (ii) Let $y = 1$. Then by Propositions 2.2(iii) and 2.3(i), $1 = y \rightarrow (x' \rightarrow y') = 1 \rightarrow (x' \rightarrow 0) = 1 \rightarrow x'' = x''$. So by Proposition 2.3(i), $x' = 0$.

(ii) $\Rightarrow$ (vii) Let $0 \neq x \in A$. If $x^2 = 0$, then

$$1 = 0 \rightarrow 0 = x^2 \rightarrow 0 = (x \odot x) \rightarrow 0 = x \rightarrow (x \rightarrow 0) = x \rightarrow x' = x \rightarrow 0 = x' = 0$$

which is impossible. Moreover, if $x^3 = 0$, then

$$1 = 0 \rightarrow 0 = x^3 \rightarrow 0 = (x^2 \odot x) \rightarrow 0 = x^2 \rightarrow (x \rightarrow 0) = x^2 \rightarrow x' = x^2 \rightarrow 0 = 0$$

which is impossible. Hence by the same way, for any $n \in \mathbb{N}$, $x^n \neq 0$. Therefore, by (ii), $(x^n)' = 0$.

(vii) $\Rightarrow$ (ii) The proof is clear. $\square$

Proposition 3.4. *In any special hoop A , the following properties hold:*

- (i) $x \rightarrow x' = x'$, for all $x \in A$,
- (ii) $x \odot y \neq 0$, for any $x, y \in A \setminus \{0\}$,
- (iii) $D_e(A) = \{x \in A \mid x' = 0\} = A \setminus \{0\}$,

- (iv) $D_e(F) = F$, for any filter F of A ,
- (v) $x \rightarrow y'' = y \rightarrow x''$, for all $x, y \in A \setminus \{0\}$,
- (vi) $x'' \rightarrow x = x$, for all $x \in A \setminus \{0\}$,
- (vii) $(x' \rightarrow y) \rightarrow x = x$, for all $x, y \in A \setminus \{0\}$.

Proof. (i) By Proposition 3.3(ii), it is clear.

(ii) Let $x, y \in A \setminus \{0\}$, and $x \odot y = 0$, by the contrary. Since by Proposition 3.3(ii), $x' = 0$ and $y' = 0$, so by Proposition 2.3(i),(iv), $1 = 0' = (x \odot y)' = x \rightarrow y' = x' = 0$, which is a contradiction. Hence $x \odot y \neq 0$.

(iii) By Proposition 3.3(ii), the proof is clear.

(iv) Let F be a filter of A . Then by (iii), $D_e(F) = D_e(A) \cap F = A \setminus \{0\} \cap F = F$.

(v) Let $x, y \in A \setminus \{0\}$. Then by Propositions 3.3(iii) and 2.2(iii),

$$x \rightarrow y'' = x \rightarrow 1 = 1 = y \rightarrow 1 = y \rightarrow x''$$

(vi) By Propositions 3.3(iii) and 2.2(iii), $x'' \rightarrow x = 1 \rightarrow x = x$, for all $x \in A \setminus \{0\}$.

(vii) By Propositions 3.3(ii) and 2.2(iii), $(x' \rightarrow y) \rightarrow x = (0 \rightarrow y) \rightarrow x = 1 \rightarrow x = x$, for any $x, y \in A \setminus \{0\}$. $\square$

In the following example we show that the converse of some properties of above proposition is not correct, in general.

Example 3.5. (i) Let $(A = \{0, a, b, c, 1\}, \leq)$ be a poset with $0 < a, b < c < 1$ but a, b are incomparable. Define the operations $\odot$ and $\rightarrow$ on A as follows:

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	0	a	a
b	0	0	b	b	b
c	0	a	b	c	c
1	0	a	b	c	1

Then $(A, \odot, \rightarrow, 1, 0)$ is a bounded hoop. It is clear that $x \rightarrow x' = x'$, for all $x \in A$. But it is not a special hoop, since $a = (a \rightarrow b)' \neq (b \rightarrow a)' = b$.

(ii) Consider Example 3.2(i), we can see that the unique maximal filter of A is $\{1, b\} = D_e(A)$, but it is not a special hoop.

(iii) In Example 3.2(i), it is clear that for all filter of A , $D_e(F) = F$, but A is not a special hoop.

Theorem 3.6. Let A be a special hoop. Then:

- (i) A has only one atom,
- (ii) $\text{ord}(x) = \infty$, for any $x \in A \setminus \{0\}$.

Proof. (i) Let x, y be two atoms of A . Since by Proposition 2.2(ii), $x \odot y \leq x, y$ and x, y are atoms, we get $x \odot y = 0$. Hence by Propositions 2.2(iii), 2.3(iv) and 3.3(ii), $1 = 0 \rightarrow 0 = (x \odot y) \rightarrow 0 = x \rightarrow y' = x' = 0$, which is a contradiction.

(ii) Let $x \in A$ and there exists $n \in N$ such that $\text{ord}(x) = n$, by the contrary. Then $x^n = 0$ but $x, x^{n-1} \neq 0$, and so by Proposition 3.4(ii), $x^n = x \odot x^{n-1} \neq 0$, which is a contradiction. Therefore, for any $x \in A \setminus \{0\}$, $\text{ord}(x) = \infty$. $\square$

Example 3.7. (i) Let A be as in Example 3.2(ii). Then A has only one atom, but it is not special hoop.

(ii) Let A be as in Example 3.5(i). Then $\text{ord}(x) = \infty$, for any $x \in A \setminus \{0\}$, but it is not a special hoop.

Definition 3.8. A hoop is called *meet zero divisor hoop* or *mzd-hoop*, if $x \wedge x' = 0$, for any $x \in A$.

Example 3.9. (i) Let A be as in Example 3.2(i). Then A is a *mzd-hoop*.

(ii) Let A be as in Example 3.2(ii). Then A is not a *mzd-hoop*. Because $a \wedge a' = a \neq 0$.

Proposition 3.10. (i) If A is a special hoop, then A is a *mzd-hoop*,

(ii) If A is a linear *mzd-hoop*, then A is a special hoop.

Proof. (i) Let A be a special hoop. Then by Proposition 3.3(ii), if $x \neq 0$, then $x' = 0$. So $x \wedge x' = 0$. Therefore, A is a *mzd-hoop*.

(ii) Let A be a linear *mzd-hoop*. Then for any $x \in A$, $x \leq x'$ or $x' \leq x$. Since A is *mzd-hoop*, $x \wedge x' = 0$ and so $x = 0$ or $x' = 0$. Hence if $x \neq 0$, then $x' = 0$ and so by Proposition 3.3(ii), A is a special hoop. $\square$

In the following example we show that the converse of Proposition 3.10(ii), is not correct, in general.

Example 3.11. Let A be as Example 3.5(i). Then A is *mzd-hoop*, but it is not a special hoop.

By the following example we study the relationship between a special hoop and a simple hoop.

Example 3.12. (i) Let $A = \{0, a, 1\}$ be a chain. Define the operations $\odot$ and $\rightarrow$ on A as follows:

$\rightarrow$	0	a	1
0	1	1	1
a	a	1	1
1	0	a	1

$\odot$	0	a	1
0	0	0	0
a	0	0	a
1	0	a	1

Then $(A, \odot, \rightarrow, 1, 0)$ is a bounded simple hoop. But it is not a special hoop, because $a' = a \neq 0$.

(ii) Let A be as in Example 3.2(i). Then A is a special hoop, but it is not a simple hoop.

Theorem 3.13. *If A is a special $\vee$ -hoop, then A is a semi-De Morgan algebra.*

Proof. Since A is a $\vee$ -hoop, by Proposition 2.10, $(A, \vee, \wedge, 0, 1)$ is a bounded distributive lattice and $(x \vee y)' = x' \wedge y'$. It is sufficient to show that $(x \wedge y)'' = x'' \wedge y''$. If $x = 0$ or $y = 0$, then it is true. If $x, y \neq 0$, then by Theorem 3.6(ii), x and y are not atoms, together. Hence $x \wedge y \neq 0$, and so by Proposition 3.3(iii), $(x \wedge y)'' = 1 = x'' \wedge y''$. $\square$

In the following example we show that the converse of Theorem 3.13, is not correct, in general.

Example 3.14. Let A be as in Example 3.2(ii). Then A is a semi-De Morgan algebra, but it is not a special hoop.

Proposition 3.15. *Special hoop A is a locally finite hoop if and only if $A = \{0, 1\}$.*

Proof. Let A be a special hoop and there is $x \in A \setminus \{0, 1\}$ such that $x^n = 0$ for any $n \in \mathbb{N}$. Then by Proposition 2.3(i), $(x^n)' = 1$. On the other hands, by Proposition 3.3(vii), $(x^n)' = 0$, which is a contradiction. The converse is clear. $\square$

Example 3.16. Hoop A in Example 3.12(i), is a locally finite hoop, but it is not a special hoop. Also, hoop A in Example 3.2(ii), is a special hoop, but it is not a locally finite hoop, because $c^2 = c \neq 0$.

Proposition 3.17. *Every special hoop is a local and perfect hoop.*

Proof. Let A be a special hoop. Then by Proposition 3.3(ii), we have $\text{ord}(x') < \infty$, for any $0 \neq x \in A$. Therefore, A is a local hoop. Moreover, if A is a special hoop, then by Proposition 3.3(ii), and Theorem 3.6(iii), we have $\text{ord}(x') < \infty$ and $\text{ord}(x) = \infty$. Therefore, A is a perfect hoop. $\square$

We show that by the following example, every local or perfect hoop is not a special hoop, in general.

Example 3.18. (i) Let $A = \{0, a, b, 1\}$ be a chain that is $0 < a < b < 1$. Define the operations $\odot$ and $\rightarrow$ on A as follows:

$\rightarrow$	0	a	b	1	$\odot$	0	a	b	1
0	1	1	1	1	0	0	0	0	0
a	b	1	1	1	a	0	0	0	a
b	a	b	1	1	b	0	0	a	b
1	0	a	b	1	1	0	a	b	1

Then $(A, \odot, \rightarrow, 1, 0)$ is a bounded local hoop. But it is not a special hoop, because $0 = (a \rightarrow b)' \neq (b \rightarrow a)' = a$.

(ii) Let $A = \{0, a, b, 1\}$ be a chain. Define the operations $\odot$ and $\rightarrow$ on A as follows:

$\rightarrow$	0	a	b	1	$\odot$	0	a	b	1
0	1	1	1	1	0	0	0	0	0
a	b	1	1	1	a	0	0	0	a
b	a	a	1	1	b	0	0	b	b
1	0	a	b	1	1	0	a	b	1

Then $(A, \odot, \rightarrow, 1, 0)$ is a bounded perfect hoop. But it is not a special hoop, because $0 = (a \rightarrow b)' \neq (b \rightarrow a)' = b$.

Proposition 3.19. *Let A be a special hoop. Then A is a Boolean algebra if and only if $A = \{0, 1\}$.*

Proof. Let A be a Boolean algebra and $0 \neq x \in A$. Then by Proposition 3.3(ii), $x' = 0$. Hence $x = x \vee 0 = x \vee x' = 1$. Therefore, $A = \{0, 1\}$. The proof of converse is clear. $\square$

4. Filters in special hoops

In this section, we study some properties of filters in a special hoop.

Proposition 4.1. *Let A be a special hoop. Then the following properties hold:*

- (i) *A has only one maximal filter,*
- (ii) *any filter of A is a perfect filter,*
- (iii) *A has only one obstinate filter,*
- (iv) *any filter of A is a primary filter,*
- (v) *any implicative filter of A is a maximal filter.*

Proof. (i) Let F and G be two maximal filters of A such that $x \in G \setminus F$. Then by Proposition 2.7, there exists $n \in \mathbb{N}$ such that $(x^n)' \in F$. Since A is a special hoop, by Proposition 3.3(vii), $0 = (x^n)' \in F$, which is a contradiction. This maximal filter is $F = A \setminus \{0\}$.

(ii) Let $x = 0$. Then for any $n \in \mathbb{N}$, $(x^n)' = 1 \in F$, and $((x')^m)' = 0 \notin F$. If $0 \neq x \in A$, then by Proposition 3.3(vii), $0 = (x^n)' \notin F$ and $((x')^m)' = 1 \in F$, for any $n, m \in \mathbb{N}$.

(iii) Let F be an obstinate filter of A . Then for any $x \in A$, by Proposition 3.3(ii), $x' = 0 \notin F$, hence $x \in F$ and so $F = A \setminus \{0\}$.

(iv) Let F be a proper filter of A . If $x = 0$ or $y = 0$ and $(x \odot y)' \in F$, then $(x^n)' \in F$ or $(y^n)' \in F$, for $n \in \mathbb{N}$. If $x, y \neq 0$, then $(x \odot y)' = x \rightarrow y' = x \rightarrow 0 = 0 \notin F$. Therefore, F is a primary filter of A .

(v) Let F be an implicative filter of A . Then by Proposition 2.11(i), $x'' \rightarrow x \in F$, for any $x \in A$. So by Propositions 2.2(iii) and 3.3(iii), $1 \rightarrow x = x \in F$. Hence, $F = A \setminus \{0\}$ and so F is a maximal filter of A . $\square$

Example 4.2. Let $A = \{0, a, b, c, 1\}$, with $0 < a < b, c < 1$, but c, b are incomparable. Define the operations $\odot$ and $\rightarrow$ on A as follows:

$\rightarrow$	0	a	b	c	1	$\odot$	0	a	b	c	1
0	1	1	1	1	1	0	0	0	0	0	0
a	0	1	1	1	1	a	0	a	a	a	a
b	0	c	1	c	1	b	0	a	b	a	b
c	0	b	b	1	1	c	0	a	a	c	c
1	0	a	b	c	1	1	0	a	b	c	1

Then A is a special hoop and $F = \{1, b\} \neq A \setminus \{0\}$ is a positive implicative filter and fantastic filter of A , but it is not a maximal filter of A .

Corollary 4.3. *Let F be a filter of special hoop A . Then F is an implicative filter of A if and only if F is an obstinate filter of A if and only if F is a maximal filter of A if and only if $F = A \setminus \{0\}$.*

Theorem 4.4. *Let A be a special hoop and F be a maximal ((positive)implicative, obstinate, fantastic) filter of A . Then A/F :*

- (i) *is a mzd-hoop,*
- (ii) *is a Boolean algebra,*
- (iii) *is a special hoop,*
- (iv) *is a local hoop,*
- (v) *is a perfect hoop.*

Proof. (i) Let $x/F \in A/F$. Then $x = 0$ or $x \neq 0$. If $x = 0$, then $x/F = 0/F$. If $x \neq 0$, then by Proposition 3.3(ii), $x'/F = 0/F$. Hence $x/F \wedge x'/F = 0/F$.

(ii) By Corollary 4.3, the proof is clear.

(iii) Let $0/F \neq x/F \in A/F$. Then by Proposition 3.3(ii), $x' = 0$, and so $x'/F = 0/F$.

(iv) By (iii), A/F is a special hoop. Then by Proposition 3.17, A/F is a local hoop.

(v) By (iii) and Proposition 3.17, the proof is clear. $\square$

Proposition 4.5. *Let A/P be a special hoop. Then P is a primary filter of A .*

Proof. Assume that A/P is a special hoop and $(x \odot y)' = y \rightarrow x' \in P$, for some $x, y \in A$. Then $y/P \rightarrow x'/P = (y \rightarrow x')/P = 1/P$, and so $y/P \leq x'/P$. Assume that $(x^n)' \notin P$, for all $n \in \mathbb{N}$. Then $(x^n)'/P \neq 1/P$. Hence x^n/P and $x'/P \neq 0/P$. Since A/P is a special hoop, $x'/P = 0/P$. Also, $y^m/P \leq (x')^m/P = 0/P$, for some $m \in \mathbb{N}$. Hence $(y^m)'/P = 1/P$, i.e. $(y^m)' \in P$. Therefore, P is a primary filter of A . $\square$

In the following example we show that the converse of above Proposition is not correct, in general.

Example 4.6. Let A be as in Example 3.2(ii). Then $F = \{1, b\}$ is a primary filter of A , but A/F is not a special hoop. Because $a/F = a'/F \neq 0/F$.

In the following Theorem we show that if in Theorem 4.4, A is not a special hoop, then this Theorem is not correct, in general.

Theorem 4.7. *Let A be a bounded hoop and F be a subset of A . If F is an implicative (maximal, obstinate) filter of A . Then A/F is not a special hoop.*

Proof. Let F be an implicative filter of A . Then by Proposition 2.11(i), $x'' \rightarrow x \in F$. So $x''/F \rightarrow x/F = 1/F$ and by Proposition 2.3(v), $x/F \rightarrow x''/F = 1/F$. Hence $x/F = x''/F$ and so $D_e(A/F) = \{1/F\}$. So $x' \neq 0$, for all $1 \neq x \in A/F$. Then by Proposition 3.3(ii), A is not a special hoop.

If F is a maximal filter of A and $x/F \neq 0/F$, $x \notin F$, then by Proposition 2.7, $(x^n)' \in F$. So $(x^n)'/F = 1/F$. If A/F is a special hoop, then Proposition 3.3(vii), $(x^n)'/F = 0/F$, which is a contradiction.

If F is an obstinate filter of A and $x/F \neq 0/F$, $x \notin F$, then $x' \in F$. So $x'/F = 1/F$, which is a contradiction by Proposition 3.3(ii). Then A/F is not a special hoop. $\square$

Proposition 4.8. *Let A be a hoop with DNP. Then A/F is a special hoop if and only if F is a maximal filter of A .*

Proof. Let A/F be a special hoop. Then by Proposition 3.3(ii), for any $0/F \neq x/F \in A/F$, $x'/F = 0/F$. By Proposition 2.3(i), $x''/F = 1/F$ and so $x'' \in F$, for all $0 \neq x \in A$. By assumption, $x \in F$, for all $0 \neq x \in A$. Hence $F = A \setminus \{0\}$. The converse is similar, too. $\square$

5. Special filter

In this section, we introduce the concept of special filter in a bounded hoop and investigate some properties of them. Also, we study relation between special filter and some other filters in hoops.

Note. In this section, we consider A , as a bounded hoop.

Definition 5.1. A proper filter F of A is called *special filter* of A if and only if $(x \rightarrow y)' = (y \rightarrow x)'$, for all $x, y \in F$.

Example 5.2. (i) $\{1\}$ is a special filter in A .

(ii) Let $A = \{0, a, b, c, 1\}$ be as in Example 4.2. Then $F = \{1, c\}$ is a special filter of A .

Proposition 5.3. *F is a proper special filter of A if and only if $D_e(F) = \{x \in F \mid x' = 0\} = F$.*

Proof. It is clear that $D_e(F) \subseteq F$. If $x \in F$, then $(x \rightarrow 1)' = (1 \rightarrow x)'$ and so $x' = 0$. Hence $x \in D_e(F)$.

Conversely, if $F = D_e(F)$, then $x' = y' = 0$, for all $x, y \in F$. In the other hand, by Propositions 2.2(iv) and 2.3(iii), $x \leq y \rightarrow x$, then $(y \rightarrow x)' \leq x' = 0$. Hence $(x \rightarrow y)' = (y \rightarrow x)' = 0$, for all $x, y \in F$. Therefore, F is a special filter of A . $\square$

Proposition 5.4. *For all proper filter F of A , $D_e(F) = F$ if and only if A is a special hoop.*

Proof. If $D_e(F) = F$, for all filter F of A , then by Proposition 5.3, A is a special hoop.

Conversely, if A is a special hoop, then $x' = 0$, for all $0 \neq x \in A$. Therefore, $D_e(F) = F$, for all filter F of A . $\square$

Corollary 5.5. *Any filter of special hoop A is a special filter of A .*

Proposition 5.6. *Let F and G be two special filters of A . Then:*

- (i) $[F \cup G]$ is special filter of A ,
- (ii) $F \cap H$ is special filter of A , for any $H \in \mathcal{F}(A)$.

Proof. (i) Let $x \in [F \cup G]$. Then for $f \in F$ and $g \in G$, $x \geq f \odot g$. Since $f \in F$, $g \in G$, and F and G are special filters, then $f' = g' = 0$. Also, by Proposition 2.3(iii), (iv), $x' \leq (f \odot g)' = f \rightarrow g' = f' = 0$. By Proposition 5.3, $[F \cup G]$ is a special filter of A .

(ii) By Proposition 5.3, $D_e(F) = F$. If $x \in F \cap H$, then $x \in F$ and $x' = 0$. Hence $D_e(F \cap H) = F \cap H$. $\square$

In the following example we show that the converse of Proposition 5.6, is not correct, in general.

Example 5.7. Let A be a hoop in Example 3.5(i). Then for $F = \{1, c, b\}$ and $G = \{1, c, a\}$, we have $F \cap G = \{1, c\}$ is a special filter of A . But F and G are not a special filters of A . Because $a' = b \neq 0$ and $b' = a \neq 0$.

Proposition 5.8. (i) *Let F be a special filter and G be an obstinate filter of A . Then $F \subseteq G$.*

- (ii) *If F is a special filter and G is an implicative filter of A , then $F \subseteq G$.*

Proof. (i) Let $F \not\subseteq G$. Then there exists $x \in F \setminus G$. Hence by Proposition 5.3, since G is an obstinate filter, $0 = x' \in G$, which is a contradiction.

(ii) Let $F \not\subseteq G$. Then there exists $x \in F \setminus G$. By Propositions 2.11(i), 2.2(iii) and 3.3(iii), $x'' \rightarrow x = 1 \rightarrow x = x \in G$, which is a contradiction. $\square$

Proposition 5.9. (i) *Let F be a special filter of A and $x \in A$ be a dense. Then $F(x)$ is a special filter of A .*

- (ii) $[x]$ is a special filter of A , if and only if x is a dense of A .
- (iii) Let x be a dense and $x \leq y$, for $y \in A$. Then $[y]$ is a special filter of A .

Proof. (i) Let $y \in F(x)$. Then $f \odot x^n \leq y$, for $f \in F$ and $n \in \mathbb{N}$. Since $f \in F$ and F is a special filter, then $f' = 0$. By Proposition 2.3(iii) and (iv), $y' \leq (f \odot x^n)' = x^n \rightarrow f' = (x^n)' = x^{n-1} \rightarrow x' = \dots = x' = 0$. Hence, by Proposition 5.3, $F(x)$ is a special filter of A .

(ii) If x is a dense of A , then for any $a \in [x]$, $x^n \leq a$. By Propositions 2.3(iii), and 5.3, $a' \leq (x^n)' = 0$, and so $[x]$ is a special filter of A . The converse is clear.

(iii) If x is a dense, then by (ii), $[x]$ is a special filter of A . On the other hands, $[y] \subseteq [x]$. Let $z \in [y]$. Then $z \in [x]$ and $z' = 0$. Therefore, by Proposition 5.3, $[y]$ is a special filter of A . $\square$

Corollary 5.10. *If A is a special hoop, then $[x]$ is a special filter of A , for any $x \in A$.*

Proof. By Theorem 3.3(ii), x is a dense of A . Hence by Proposition 5.9(ii), the proof is clear. $\square$

We determine the relationship between the special filter and the other types of filters in hoop.

Example 5.11. Let $A = \{0, a, b, 1\}$ be as in Example 3.2(i). Then $F = \{b, 1\}$ is a special filter of A .

(i) F is not an obstinate filter of A , because $a, 0 \notin F$ and $a \rightarrow 0 \notin F$.

(ii) F is not a positive implicative filter of A , because $a \rightarrow (a \rightarrow 0) = a \rightarrow a = 1 \in F$ and $a \rightarrow a = 1 \in F$. But $a \rightarrow 0 = a \notin F$.

(iii) F is not an implicative filter A , because $1 \rightarrow ((a \rightarrow 0) \rightarrow a) = 1 \in F$ and $1 \in F$. But $a \notin F$.

(iv) F is not a perfect filter of A , because $(a^2)' = 1 \in F$ and $((a')^2)' = 1 \in F$.

Example 5.12. Let $A = \{0, a, b, c, 1\}$ be as in Example 3.5(i). Then $G = \{1, c\}$ is a special filter of A .

(i) G is not a maximal filter, because $G \subseteq \{1, c, a\}$.

(ii) G is not a primary filter, because $(a \odot b)' = 1 \in G$, but $(a^n)' = a' = b \notin G$ and $(b^n)' = b' = a \notin G$.

Example 5.13. Let $(A = \{0, a, b, c, d, 1\}, \leq)$ be a poset. Define the operations $\odot$ and $\rightarrow$ on A as follows:

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	1	d	1	1
b	c	d	1	c	d	1
c	b	b	b	1	1	1
d	a	b	b	d	1	1
1	0	a	b	c	d	1

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	a	0	0	a
b	0	a	b	0	a	b
c	0	0	0	c	c	c
d	0	0	a	c	c	d
1	0	a	b	c	d	1

Then $(A, \odot, \rightarrow, 1, 0)$ is a bounded hoop. It is clear that $F = \{1, c, d\}$ is (obstinate, prime, (positive) implicative, maximal, perfect, primary and fantastic) filter of A . But it is not a special filter, because $0 = (c \rightarrow d)' \neq (d \rightarrow c)' = a$

Proposition 5.14. *Let F is a special filter of A , and $x' \rightarrow x = x$, for any $x \in A \setminus F$. Then F is an implicative filter of A .*

Proof. Let F be a special filter of A . If $x \in F$, then by Propositions 2.2(iii) and 5.3, $(x' \rightarrow x) \rightarrow x = 1 \rightarrow x = x \in F$. If $x \notin F$, then by assumption, $(x' \rightarrow x) \rightarrow x = x \rightarrow x = 1 \in F$. By Proposition 2.11(ii), F is an implicative filter of A . $\square$

Proposition 5.15. *If A/F is a special hoop, then for any $0 \neq x, y \in A$, $x' \rightarrow (x'' \odot y) \in F$.*

Proof. Let $0/F \neq x/F \in A/F$. Then by Proposition 3.3(iii), $x''/F = 1/F$. So $x'' \in F$. Now, by Proposition 2.3(ii), $x'' \leq x' \rightarrow y$. By (F1), $x' \rightarrow y \in F$. Now, by Proposition 2.2(v) and (F2), $x'' \odot (x' \rightarrow y) \leq x' \rightarrow (x'' \odot y)$. Therefore, $x' \rightarrow (x'' \odot y) \in F$. $\square$

Proposition 5.16. *If F is a special filter and obstinate filter of A , then A/F is a locally finite hoop.*

Proof. By Proposition 2.11(iii), F is a maximal filter of A . Let $x/F \neq 1/F$ be an arbitrary element of A/F . Since $x \notin F$, by Proposition 2.7, there exists $n \in \mathbb{N}$ such that $(x^n)' \in F$. Then $(x^n)'/F = 1/F$ and $(x^n)''/F = 0/F$. Hence $x^n/F = 0/F$. Therefore, A/F is a locally finite hoop. $\square$

By the following example we show that F is a special filter of A , but A/F is not a special hoop.

Example 5.17. Let A be as in Example 3.2(i). Then it is clear that $F = \{1, b\}$, is a special filter of A , but A/F is not a special hoop. Because $a/F \neq 0/F$ and $a'/F = a/F \neq 0/F$.

6. Conclusion and future research

Hoops are a particular class of algebraic structures which were introduced in an unpublished manuscript by Büchi and Owens in the mid-1970s. In fact, hoops are partially ordered commutative residuated integral monoids satisfying a further divisibility condition.

In this note, we introduced the notion special hoop and we show that every special hoop is a local and perfect hoop. But the converse is not true. Then we studied special filter and relationships between special filter and some other filters.

Some important issues for future work are:

- (i) define fuzzy special filter on hoop and special hoop,
- (ii) investigate congruence relations of special filter in hoop and some of the application of them.

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ON SOME GENERATING FUNCTIONS FOR THE TWO-PARAMETERS ONE-VARIABLE SRIVASTAVA POLYNOMIALS

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Abstract. In the present paper we prove a general theorems on generating functions involving the two-parameter one-variable Srivastava polynomials, Hermite and Laguerre polynomials of two variables. It is also shown how these theorems can be used to derive several bilateral generating functions (known or new) involving Hermite and Laguerre polynomials of two variables and other classical polynomials of one variable which are contained by the two-parameter one-variable Srivastava polynomials.

Keywords: Generating functions, Srivastava polynomials, Hermite polynomials, Laguerre polynomials.

1. Introduction

In 1972, Srivastava [8] introduced the following family of polynomials:

$$(1.1) \quad S_n^N(x) = \sum_{k=0}^{\lfloor \frac{n}{N} \rfloor} \frac{(-n)_{Nk}}{k!} A_{n,k} x^k \quad (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; N \in \mathbb{N}),$$

where $\mathbb{N}$ is the set of positive integers, $\{A_{n,k}\}_{n,k=0}^{\infty}$ is a bounded double sequence of real or complex numbers, $[a]$ denotes the greatest integer in $a \in \mathbb{R}$ and $(\lambda)_n$ denotes the well-known Pochhammers symbol.

In [4], Gonzalez *at al.* extended the Srivastava polynomials $S_n^N(x)$ as follows:

$$(1.2) \quad S_{n,m}^N(x) = \sum_{k=0}^{\lfloor \frac{n}{N} \rfloor} \frac{(-n)_{Nk}}{k!} A_{n+m,k} x^k \quad (m, n \in \mathbb{N}_0; N \in \mathbb{N}).$$

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In 2013, Kaanoglu and Ozarslan [5] introduced the following family of two-parameter one-variable Srivastava polynomials:

$$(1.3) \quad S_n^{p,q}(x) = \sum_{k=0}^n \frac{(-n)_k}{k!} A_{p+q+n,q+k} x^k \quad (p, q, n, k \in \mathbb{N}_0),$$

where $\{A_{n,k}\}$ is a bounded double sequence of real or complex numbers. Also the following remarks are given in [5]:

Remark 1.1. Choosing $A_{m,n} = (-\alpha - m)_n$, $(m, n \in \mathbb{N}_0)$ in (1.3), we get

$$(1.4) \quad S_n^{p,q}\left(\frac{-1}{x}\right) = (-1)^q (\alpha + p + n + 1)_q \frac{n!}{(-x)^n} L_n^{(\alpha+p)}(x),$$

where $L_n^{(\alpha)}(x)$ are the classical Laguerre polynomials [9]

$$(1.5) \quad L_n^{(\alpha)}(x) = \frac{(-x)^n}{n!} {}_2F_0 \left[-n, -\alpha - n; -; \frac{-1}{x} \right].$$

Remark 1.2. Choosing $A_{m,n} = \frac{(\alpha+\beta+1)_{2m}(-\beta-m)_n}{(\alpha+\beta+1)_m(-\alpha-\beta-2m)_m}$, $(m, n \in \mathbb{N}_0)$ in (1.3), we get

$$(1.6) \quad S_n^{p,q}\left(\frac{2}{1+x}\right) = \frac{(\alpha+\beta+1)_{2p+2q+2n}(-\beta-p-q-n)_q(1+\alpha+\beta+2p+q)_n}{(\alpha+\beta+1)_{p+q+n}(-\alpha-\beta-2p-2q-2n)_q(1+\alpha+\beta+2p+q)_{2n}} \\ \times n! \left(\frac{2}{1+x}\right)^n P_n^{(\alpha+p+q, \beta+p)}(x),$$

where $P_n^{(\alpha,\beta)}(x)$ are the classical Jacobi polynomials [7]

$$(1.7) \quad P_n^{(\alpha,\beta)}(x) = \binom{\alpha+\beta+2n}{n} \times \\ \times \left(\frac{1+x}{2}\right)^n {}_2F_1 \left[-n, -\beta - n; -\alpha - \beta - 2n; \frac{2}{1+x} \right].$$

Further, we add the following remark:

Remark 1.3. Choosing $A_{m,n} = \frac{(\alpha)_{2m}}{(\alpha)_{m+n}}$, $(m, n \in \mathbb{N}_0)$ in (1.3), we get

$$(1.8) \quad S_n^{p,q}(x) = n! (\alpha + p + 2q + 2n)_p R_n(\alpha + p + 2q; x),$$

where $R_n(\alpha, x)$ are the Shivelys pseudo Laguerre polynomials [7]

$$(1.9) \quad R_n(\alpha, x) = \frac{(\alpha)_{2n}}{n! (\alpha)_n} {}_1F_1[-n, \alpha + n; x].$$

The Hermite polynomials of two variables are defined by [6]

$$(1.10) \quad H_n(x, y) = \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^r n! H_{n-2r}(x) x^{2r} y^{n-2r}}{r!(n-2r)!},$$

where $H_n(x)$ are the well-known Hermite polynomials [7].

The Laguerre polynomials of two variables are defined by (see [2],[3])

$$(1.11) \quad L_n(x, y) = n! \sum_{k=0}^n \frac{(-1)^k x^k y^{n-k}}{(k!)^2 (n-k)!}.$$

Also, we note that the Hermite and Laguerre polynomials of two variables (1.10) and (1.11) are satisfy the following generating functions respectively :

$$(1.12) \quad \sum_{n=0}^{\infty} \frac{H_n(x, y) t^n}{n!} = \exp[2xyt - (x^2 + y^2)t^2],$$

$$(1.13) \quad \sum_{n=0}^{\infty} \frac{(c)_n H_n(x, y) t^n}{n!} = [1 - 2xyt]^{-c} \\ \times F \begin{matrix} 2 : 0; 0 \\ 0 : 0; 0 \end{matrix} \left[\begin{matrix} \frac{c}{2}, \frac{c}{2} + \frac{1}{2} & : & - & ; & - & ; & \frac{-4x^2 t^2}{(1-2xyt)^2}, \frac{-4y^2 t^2}{(1-2xyt)^2} \end{matrix} \right],$$

where $F \begin{matrix} A : B; D \\ E : G; H \end{matrix} [x, y]$ is the Kamp de Friet function [9]

$$(1.14) \quad \sum_{n=0}^{\infty} \frac{(a)_n L_n(x, y) t^n}{n!} = (1 - yt)^{-a} {}_1F_1 \left[a; 1; \frac{-xt}{(1-yt)} \right] \quad (|yt| < 1),$$

$$(1.15) \quad \sum_{n=0}^{\infty} \frac{L_n(x, y) t^n}{n!} = \exp(yt) C_0(xt),$$

where $C_n(x)$ denotes the n^{th} order Tricomi function [9]

$$(1.16) \quad C_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!(n+k)!}.$$

2. Main results

In this section ,we have proved the following theorems :

Theorem 2.1. *The following family of bilateral generating functions involving the two-parameter one-variable Srivastava polynomials and Hermite polynomials of two variables holds true:*

$$(2.1) \quad \sum_{p,q,n=0}^{\infty} H_n(x, y) S_n^{p,q}(z) \frac{u^p}{p!} \frac{v^q}{q!} \frac{t^n}{n!}$$

$$= \sum_{p,q=0}^{\infty} H_{p+q}(x, y) A_{p+q,q} \frac{(u+t)^p}{p!} \frac{(v-zt)^q}{q!}.$$

Theorem 2.2. *The following family of bilateral generating functions involving the two-parameter one-variable Srivastava polynomials and Laguerre polynomials of two variables holds true:*

$$(2.2) \quad \begin{aligned} & \sum_{p,q,n=0}^{\infty} L_n(x, y) S_n^{p,q}(z) \frac{u^p}{p!} \frac{v^q}{q!} \frac{t^n}{n!} \\ &= \sum_{p,q=0}^{\infty} L_{p+q}(x, y) A_{p+q,q} \frac{(u+t)^p}{p!} \frac{(v-zt)^q}{q!}. \end{aligned}$$

Proof of 2.1. Denoting the left hand side of (2.1) by S , expressing $S_n^{p,q}(z)$ as in (1.3) and using the result [9]

$$(2.3) \quad (-n)_k = \frac{(-1)^k n!}{(n-k)!}, \quad 0 \leq k \leq n,$$

we obtain

$$(2.4) \quad S = \sum_{p,q,n=0}^{\infty} H_{p+q+n}(x, y) \sum_{k=0}^n \frac{(-z)_k}{k!} A_{p+q+n,q+k} \frac{u^p}{p!} \frac{v^q}{q!} \frac{t^n}{(n-k)!}.$$

Using the following result [9]:

$$(2.5) \quad \sum_{n=0}^{\infty} \sum_{k=0}^n A(k, n) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k, n+k),$$

we get

$$(2.6) \quad S = \sum_{p,q,n,k=0}^{\infty} H_{p+q+n+k}(x, y) A_{p+q+n+k,q+k} \frac{u^p}{p!} \frac{v^q}{q!} \frac{t^n}{n!} \frac{(-zt)^k}{k!}.$$

Now, using the following results [9]:

$$(2.7) \quad \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k, n) = \sum_{n=0}^{\infty} \sum_{k=0}^n A(k, n-k),$$

$$(2.8) \quad \sum_{n=0}^{\infty} (\lambda)_n \frac{x^n}{n!} = (1-x)^{-\lambda},$$

we get

$$(2.9) \quad S = \sum_{p,q,k=0}^{\infty} H_{p+q+k}(x, y) A_{p+q+k,q+k} \frac{(u+t)^p}{p!} \frac{v^q}{q!} \frac{(-zt)^k}{k!}.$$

Finally using the results (2.7) and (2.8), after a little simplification, we arrive at the right-hand side of (2.1). This completes the proof of Theorem 2.1. The Theorem 2.2. can be established similarly .

Remark 2.1. On taking $u = -t$ in Theorems 2.1. and 2.2., we obtain the following family of bilateral generating functions:

Corollary 2.1.

$$(2.10) \quad \sum_{p,q,n=0}^{\infty} H_{p+q+n}(x, y) S_n^{p,q}(z) \frac{(-t)^p}{p!} \frac{v^q}{q!} \frac{t^n}{n!} = \sum_{q=0}^{\infty} H_q(x, y) A_{q,q} \frac{(v - zt)^q}{q!}.$$

Corollary 2.2.

$$(2.11) \quad \sum_{p,q,n=0}^{\infty} L_{p+q+n}(x, y) S_n^{p,q}(z) \frac{(-t)^p}{p!} \frac{v^q}{q!} \frac{t^n}{n!} = \sum_{q=0}^{\infty} L_q(x, y) A_{q,q} \frac{(v - zt)^q}{q!}.$$

Remark 2.2. On taking $v = 0$ in Theorems 2.1. and 2.2. and using the relation $S_n^{p,0}(z) = S_{n,p}^1(z)$, we obtain the following family of bilateral generating functions :

Corollary 2.3.

$$(2.12) \quad \sum_{p,n=0}^{\infty} H_{p+n}(x, y) S_{n,p}^1(z) \frac{u^p}{p!} \frac{t^n}{n!} = \sum_{p,q=0}^{\infty} H_{p+q}(x, y) A_{p+q,q} \frac{(u + t)^p}{p!} \frac{(-zt)^q}{q!}.$$

Corollary 2.4.

$$(2.13) \quad \sum_{p,n=0}^{\infty} L_{p+n}(x, y) S_{n,p}^1(z) \frac{u^p}{p!} \frac{t^n}{n!} = \sum_{p,q=0}^{\infty} L_{p+q}(x, y) A_{p+q,q} \frac{(u + t)^p}{p!} \frac{(-zt)^q}{q!},$$

where $S_{n,m}^N(z)$ is the extended Srivastava polynomials (1.2).

3. Applications

I. In (2.10) and (2.11) Choosing $A_{m,n} = (-\alpha - m)_n$ and using (1.4), we get

$$(3.1) \quad \begin{aligned} & \sum_{p,q,n=0}^{\infty} (1 + \alpha + p + n)_q H_{p+q+n}(x, y) L_n^{(\alpha+p)}(z) \frac{t^p}{p!} \frac{v^q}{q!} \left(\frac{t}{z}\right)^n \\ &= \sum_{q=0}^{\infty} (\alpha + 1)_q H_q(x, y) \frac{(v + z/t)^q}{q!} \end{aligned}$$

and

$$(3.2) \quad \begin{aligned} & \sum_{p,q,n=0}^{\infty} (1 + \alpha + p + n)_q L_{p+q+n}(x, y) L_n^{(\alpha+p)}(z) \frac{t^p}{p!} \frac{v^q}{q!} \left(\frac{t}{z}\right)^n \\ &= \sum_{q=0}^{\infty} (\alpha + 1)_q L_q(x, y) \frac{(v + z/t)^q}{q!}. \end{aligned}$$

Now, by using (1.13) and (1.14) in (3.1) and (3.2) respectively, we get

$$(3.3) \quad \sum_{p,q,n=0}^{\infty} (1+\alpha+p+n)_q H_{p+q+n}(x,y) L_n^{(\alpha+p)}(z) \frac{t^p}{p!} \frac{v^q}{q!} \left(\frac{t}{z}\right)^n = [1-2xyw]^{-\alpha-1} \\ \times F \begin{matrix} 2:0;0 \\ 0:0;0 \end{matrix} \left[\begin{matrix} \frac{\alpha}{2}, \frac{\alpha}{2} + \frac{1}{2} & : & - & ; & - & ; & \frac{-4x^2w^2}{(1-2xyw)^2}, \frac{-4y^2w^2}{(1-2xyw)^2} \end{matrix} \right],$$

where $w = v + \frac{t}{z}$ and

$$(3.4) \quad \sum_{p,q,n=0}^{\infty} (1+\alpha+p+n)_q L_{p+q+n}(x,y) L_n^{(\alpha+p)}(z) \frac{t^p}{p!} \frac{v^q}{q!} \left(\frac{t}{z}\right)^n \\ = \left(\frac{z}{z-yt-yvz}\right)^{1+\alpha} {}_1F_1 \left[\alpha+1; 1; \frac{-x(vz+t)}{(z-yt-yvz)} \right].$$

Further, if we take $v = 0$ in (3.3) and (3.4) respectively we obtain

$$(3.5) \quad \sum_{p,n=0}^{\infty} H_{p+n}(x,y) L_n^{(\alpha+p)}(z) \frac{t^p}{p!} \left(\frac{t}{z}\right)^n = [1-2xyt/z]^{-\alpha-1} \\ \times F \begin{matrix} 2:0;0 \\ 0:0;0 \end{matrix} \left[\begin{matrix} \frac{\alpha}{2}, \frac{\alpha}{2} + \frac{1}{2} & : & - & ; & - & ; & \frac{-4x^2t^2}{(z-2xyt)^2}, \frac{-4y^2t^2}{(z-2xyt)^2} \end{matrix} \right]$$

and

$$(3.6) \quad \sum_{p,n=0}^{\infty} L_{p+n}(x,y) L_n^{(\alpha+p)}(z) \frac{t^p}{p!} \left(\frac{t}{z}\right)^n = \left(\frac{z}{z-yt}\right)^{1+\alpha} {}_1F_1 \left[\alpha+1; 1; \frac{-xt}{z-yt} \right].$$

II. In (2.10) and (2.11) Choosing $A_{m,n} = \frac{(\alpha+\beta+1)_{2m}(-\beta-m)_n}{(\alpha+\beta+1)_m(-\alpha-\beta-2m)_m}$ and using (1.6), we get

$$(3.7) \quad \sum_{p,q,n=0}^{\infty} \frac{(1+\alpha+\beta+p+q+n)_{p+q+n}(-\beta-p-q-n)_q}{(1+\alpha+\beta+2p+q+n)_n(-\alpha-\beta-2p-2q-2n)_q} H_{p+q+n}(x,y) \\ \times P_n^{(\alpha+p+q,\beta+p)}(z) \frac{(-t)^p}{p!} \frac{v^q}{q!} \left(\frac{2t}{1+z}\right)^n = \sum_{q=0}^{\infty} (\beta+1)_q H_q(x,y) \frac{(v-2t/(1+z))^q}{q!}$$

and

$$(3.8) \quad \sum_{p,q,n=0}^{\infty} \frac{(1+\alpha+\beta+p+q+n)_{p+q+n}(-\beta-p-q-n)_q}{(1+\alpha+\beta+2p+q+n)_n(-\alpha-\beta-2p-2q-2n)_q} L_{p+q+n}(x,y) \\ \times P_n^{(\alpha+p+q,\beta+p)}(z) \frac{(-t)^p}{p!} \frac{v^q}{q!} \left(\frac{2t}{1+z}\right)^n = \sum_{q=0}^{\infty} (\beta+1)_q L_q(x,y) \frac{(v-2t/(1+z))^q}{q!}.$$

Now, using (1.13) and (1.14) in (3.7) and (3.8) respectively, we get

$$(3.9) \quad \sum_{p,q,n=0}^{\infty} \frac{(1+\alpha+\beta+p+q+n)_{p+q+n}(-\beta-p-q-n)_q}{(1+\alpha+\beta+2p+q+n)_n(-\alpha-\beta-2p-2q-2n)_q} H_{p+q+n}(x, y) \\ \times P_n^{(\alpha+p+q, \beta+p)}(z) \frac{(-t)^p}{p!} \frac{v^q}{q!} \left(\frac{2t}{1+z} \right)^n = [1-2xyw]^{-\beta-1} \\ {}_F \begin{matrix} 2:0;0 \\ 0:0;0 \end{matrix} \left[\begin{matrix} \frac{\beta+1}{2}, \frac{\beta+2}{2} & : & - & ; & - & ; & \frac{-4x^2w^2}{(1-2xyw)^2}, \frac{-4y^2w^2}{(1-2xyw)^2} \end{matrix} \right]$$

and

$$(3.10) \quad \sum_{p,q,n=0}^{\infty} \frac{(1+\alpha+\beta+p+q+n)_{p+q+n}(-\beta-p-q-n)_q}{(1+\alpha+\beta+2p+q+n)_n(-\alpha-\beta-2p-2q-2n)_q} L_{p+q+n}(x, y) \\ \times P_n^{(\alpha+p+q, \beta+p)}(z) \frac{(-t)^p}{p!} \frac{v^q}{q!} \left(\frac{2t}{1+z} \right)^n = (1-yw)^{-1-\beta} {}_1F_1 \left[\beta+1; 1; \frac{-xw}{1-yw} \right]$$

where $w = v - \frac{2t}{1+z}$.

Further, if we take $v = 0$ in (3.9) and (3.10) respectively we obtain

$$(3.11) \quad \sum_{p,n=0}^{\infty} (1+\alpha+\beta+p+n)_p H_{p+n}(x, y) P_n^{(\alpha+p, \beta+p)}(z) \frac{(-t)^p}{p!} \left(\frac{2t}{1+z} \right)^n \\ = \left(\frac{1+z+4xyt}{1+z} \right)^{-\beta-1} \\ \times {}_F \begin{matrix} 2:0;0 \\ 0:0;0 \end{matrix} \left[\begin{matrix} \frac{\beta+1}{2}, \frac{\beta+2}{2} & : & - & ; & - & ; & \frac{-(4xt)^2}{(1+z+4xyt)^2}, \frac{-(4yt)^2}{(1+z+4xyt)^2} \end{matrix} \right]$$

and

$$(3.12) \quad \sum_{p,n=0}^{\infty} (1+\alpha+\beta+p+n)_p L_{p+n}(x, y) P_n^{(\alpha+p, \beta+p)}(z) \frac{(-t)^p}{p!} \left(\frac{2t}{1+z} \right)^n \\ = \left(\frac{1+z}{1+z+2yt} \right)^{1+\beta} {}_1F_1 \left[\beta+1; 1; \frac{2xt}{1+z+2yt} \right].$$

III. In (2.10) and (2.11) choosing $A_{m,n} = \frac{(\alpha)_{2m}}{(\alpha)_{m+n}}$ and using (1.8), we get

$$(3.13) \quad \sum_{p,q,n=0}^{\infty} (\alpha+p+2q+2n)_p H_{p+q+n}(x, y) R_n(\alpha+p+2q; x) \frac{(-t)^p}{p!} \frac{v^q}{q!} t^n \\ = \sum_{q=0}^{\infty} H_q(x, y) \frac{(v-zt)^q}{q!},$$

$$\begin{aligned}
 (3.14) \quad & \sum_{p,q,n=0}^{\infty} (\alpha + p + 2q + 2n)_p L_{p+q+n}(x, y) R_n(\alpha + p + 2q; x) \frac{(-t)^p}{p!} \frac{v^q}{q!} t^n \\
 &= \sum_{q=0}^{\infty} L_q(x, y) \frac{(v - zt)^q}{q!}.
 \end{aligned}$$

Now, by using (1.12) and (1.15) in (3.13) and (3.14) respectively, we get

$$\begin{aligned}
 (3.15) \quad & \sum_{p,q,n=0}^{\infty} (\alpha + p + 2q + 2n)_p H_{p+q+n}(x, y) R_n(\alpha + p + 2q; x) \frac{(-t)^p}{p!} \frac{v^q}{q!} t^n \\
 &= \exp[2xyt(v - zt) - (x^2 + y^2)(v - zt)^2],
 \end{aligned}$$

and

$$\begin{aligned}
 (3.16) \quad & \sum_{p,q,n=0}^{\infty} (\alpha + p + 2q + 2n)_p L_{p+q+n}(x, y) R_n(\alpha + p + 2q; x) \frac{(-t)^p}{p!} \frac{v^q}{q!} t^n \\
 &= \exp(y(v - zt)) C_0(x(v - zt)).
 \end{aligned}$$

Further, if we take $v = 0$ in (3.15) and (3.16) respectively we obtain

$$\begin{aligned}
 (3.17) \quad & \sum_{p,n=0}^{\infty} (\alpha + p + 2n)_p H_{p+n}(x, y) R_n(\alpha + p; x) \frac{(-t)^p}{p!} t^n \\
 &= \exp[t^2(-2xyz - z^2(x^2 + y^2))]
 \end{aligned}$$

and

$$\begin{aligned}
 (3.18) \quad & \sum_{p,q,n=0}^{\infty} (\alpha + p + 2n)_p L_{p+n}(x, y) R_n(\alpha + p; x) \frac{(-t)^p}{p!} t^n \\
 &= \exp(-yzt) C_0(-xzt).
 \end{aligned}$$

Remark 3.1. The results (3.6), (3.12) and (3.18) are a known results of Al-Gonah [1].

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SOME CLASSES OF INVARIANT SUBMANIFOLDS OF $(LCS)_n$ -MANIFOLDS

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Abstract. The object of the present paper is to study the pseudoparallel, Ricci generalized pseudoparallel and η -parallel invariant submanifolds of $(LCS)_n$ -manifolds and we obtained some equivalent conditions of invariant submanifolds of $(LCS)_n$ -manifolds under which the submanifolds are totally geodesic. Among others we found the necessary

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and sufficient condition of the second fundamental form to be η -parallel in an invariant submanifold of a $(LCS)_n$ -manifold. Finally an example of invariant submanifold of $(LCS)_5$ -manifold is constructed.

Keywords: $(LCS)_n$ -manifold, invariant submanifold, pseudo parallel submanifold, Ricci generalized pseudo parallel submanifold, η -parallel submanifold, totally geodesic.

1. Introduction

In 2003 Shaikh [29] introduced the notion of Lorentzian concircular structure manifolds (briefly, $(LCS)_n$ -manifolds), with an example, which generalizes the notion of LP-Sasakian manifolds introduced by Matsumoto [18] and also by Mihai and Rosca [20]. Then Shaikh and Baishya ([32], [33]) investigated the applications of $(LCS)_n$ -manifolds to the general theory of relativity and cosmology. It is to be noted that the most interesting fact is $(LCS)_n$ -manifold remains invariant under a D-homothetic transformation, which does not hold for an LP-Sasakian manifold [31]. The $(LCS)_n$ -manifolds have been also studied by Atceken [3], Narain and Yadav [21], Prakasha [26], Hui et. al ([4], [9],[13], [14]), Shaikh [30], Shaikh, Basu and Eyasmin ([35], [36]), Shaikh and Binh [34], Shaikh and Hui [37], Sreenivasa, Venkatesha and Bagewadi [39], Yadav, Dwivedi and Suthar [41] and others.

In modern analysis the geometry of submanifolds have become a subject of growing interest for its significant application in applied mathematics and theoretical physics. For instance, the notion of invariant submanifold is used to discuss properties of non-linear autonomous system [12]. Also the notion of geodesics play an important role in the theory of relativity [19]. For totally geodesic submanifolds, the geodesics of the ambient manifolds remain geodesics in the submanifolds. Hence, totally geodesic submanifolds are also very much important in physical sciences. The study of geometry of invariant submanifolds was initiated by Bejancu and Papaghuic [7]. In general the geometry of an invariant submanifold inherits almost all properties of the ambient manifold. The invariant submanifolds have been studied by many geometers to different extent such as [1], [2], [5], [6], [11], [16], [17], [22], [23], [25], [27], [28], [40], [43] and many others. Recently Shaikh et. al [38] studied invariant submanifolds of $(LCS)_n$ -manifolds.

In the present paper we study some classes of invariant submanifolds of $(LCS)_n$ -manifolds. The paper is organized as follows. Section 2 is concerned with preliminaries of $(LCS)_n$ -manifolds. Section 3 deals with the study of some classes of invariant submanifolds of $(LCS)_n$ -manifolds. We obtain the necessary and sufficient conditions for some classes of invariant submanifolds of $(LCS)_n$ -manifolds to be totally geodesic. It is shown that pseudoparallelism and Ricci generalized pseudoparallelism of an invariant submanifold of a $(LCS)_n$ -manifold are equivalent with a certain condition. We also find the necessary and sufficient condition of the second fundamental form h to be η -parallel in an invariant sub-

manifold of a $(LCS)_n$ -manifold. Finally, we construct an example of invariant submanifold of $(LCS)_5$ -manifold.

2. Preliminaries

An n -dimensional Lorentzian manifold M is a smooth connected paracompact Hausdorff manifold with a Lorentzian metric g , that is, M admits a smooth symmetric tensor field g of type $(0,2)$ such that for each point $p \in M$, the tensor $g_p : T_p M \times T_p M \rightarrow \mathbb{R}$ is a non-degenerate inner product of signature $(-, +, \dots, +)$, where $T_p M$ denotes the tangent vector space of M at p and $\mathbb{R}$ is the real number space. A non-zero vector $v \in T_p M$ is said to be timelike (resp., non-spacelike, null, spacelike) if it satisfies $g_p(v, v) < 0$ (resp., ≤ 0 , $= 0$, > 0) [24].

Definition 2.1 ([42]). In a Lorentzian manifold (M, g) a vector field P defined by

$$g(X, P) = A(X),$$

for any $X \in \Gamma(TM)$, is said to be a concircular vector field if

$$(\bar{\nabla}_X A)(Y) = \alpha\{g(X, Y) + \omega(X)A(Y)\}$$

where α is a non-zero scalar and ω is a closed 1-form and $\bar{\nabla}$ denotes the operator of covariant differentiation of M with respect to the Lorentzian metric g .

Let M be an n -dimensional Lorentzian manifold admitting a unit timelike concircular vector field ξ , called the characteristic vector field of the manifold. Then we have

$$(2.1) \quad g(\xi, \xi) = -1.$$

Since ξ is a unit concircular vector field, it follows that there exists a non-zero 1-form η such that for

$$(2.2) \quad g(X, \xi) = \eta(X),$$

the equation of the following form holds

$$(2.3) \quad (\bar{\nabla}_X \eta)(Y) = \alpha\{g(X, Y) + \eta(X)\eta(Y)\}, \quad \alpha \neq 0$$

for all vector fields X, Y and α is a non-zero scalar function satisfies

$$(2.4) \quad \bar{\nabla}_X \alpha = (X\alpha) = d\alpha(X) = \rho\eta(X),$$

ρ being a certain scalar function given by $\rho = -(\xi\alpha)$.

Let us take

$$(2.5) \quad \phi X = \frac{1}{\alpha} \bar{\nabla}_X \xi,$$

then from (2.3) and (2.5) we have

$$(2.6) \quad \phi X = X + \eta(X)\xi,$$

from which it follows that ϕ is a symmetric (1,1) tensor and called the structure tensor of the manifold. Thus the Lorentzian manifold M together with the unit timelike concircular vector field ξ , its associated 1-form η and an (1,1) tensor field ϕ is said to be a Lorentzian concircular structure manifold (briefly, $(LCS)_n$ -manifold), [29]. Especially, if we take $\alpha = 1$, then we can obtain the LP-Sasakian structure of Matsumoto [18]. In a $(LCS)_n$ -manifold ($n > 2$), the following relations hold ([29], [30]):

$$(2.7) \quad \eta(\xi) = -1, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y),$$

$$(2.8) \quad \phi^2 X = X + \eta(X)\xi,$$

$$(2.9) \quad \bar{S}(X, \xi) = (n-1)(\alpha^2 - \rho)\eta(X),$$

$$(2.10) \quad \bar{R}(X, Y)\xi = (\alpha^2 - \rho)[\eta(Y)X - \eta(X)Y],$$

$$(2.11) \quad \bar{R}(\xi, Y)Z = (\alpha^2 - \rho)[g(Y, Z)\xi - \eta(Z)Y],$$

$$(2.12) \quad \bar{R}(\xi, X)\xi = (\alpha^2 - \rho)[\eta(X)\xi + X],$$

$$(2.13) \quad (\bar{\nabla}_X \phi)(Y) = \alpha\{g(X, Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X\},$$

$$(2.14) \quad (X\rho) = d\rho(X) = \beta\eta(X),$$

$$(2.15) \quad \bar{R}(X, Y)Z = \phi\bar{R}(X, Y)Z + (\alpha^2 - \rho)\{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\}\xi$$

for all $X, Y, Z \in \Gamma(TM)$ and $\beta = -(\xi\rho)$ is a scalar function, where $\bar{R}$ is the curvature tensor and $\bar{S}$ is the Ricci tensor of the manifold.

For a $(0, l)$ tensor field T , $l \geq 1$, and a symmetric $(0, 2)$ tensor field B , we have

$$(2.16) \quad \begin{aligned} & Q(B, T)(X_1, \dots, X_l; X, Y) \\ &= -T((X \wedge_B Y)X_1, X_2, \dots, X_l) - \dots - T(X_1, \dots, X_{l-1}, (X \wedge_B Y)X_l), \end{aligned}$$

where

$$(2.17) \quad (X \wedge_B Y)Z = B(Y, Z)X - B(X, Z)Y.$$

Putting $T = h$ and $B = g$ or $B = S$, we obtain $Q(g, h)$ and $Q(S, h)$ respectively.

Let N be a submanifold of a $(LCS)_n$ -manifold M with induced metric g and let ∇ and $\bar{\nabla}$ be the Levi-Civita connection of N and M respectively. Also let ∇ and $\nabla^\perp$ be the induced connection on the tangent bundle TN and the normal bundle $T^\perp N$ of N respectively. Then the Gauss and Weingarten formulae are given by

$$(2.18) \quad \bar{\nabla}_X Y = \nabla_X Y + h(X, Y)$$

and

$$(2.19) \quad \bar{\nabla}_X V = -A_V X + \nabla_X^\perp V$$

for all $X, Y \in \Gamma(TN)$ and $V \in \Gamma(T^\perp N)$, where h and A_V are second fundamental form and the shape operator (corresponding to the normal vector field V) respectively for the immersion of N into M . The second fundamental form h and the shape operator A_V are related by [44]

$$(2.20) \quad g(h(X, Y), V) = g(A_V X, Y)$$

for any $X, Y \in \Gamma(TN)$ and $V \in \Gamma(T^\perp N)$. We note that $h(X, Y)$ is bilinear and since $\nabla_{fX} Y = f \nabla_X Y$ for any smooth function f on a manifold, we have

$$(2.21) \quad h(fX, Y) = fh(X, Y).$$

For the second fundamental form h , the covariant derivative of h is defined by

$$(2.22) \quad (\bar{\nabla}_X h)(Y, Z) = \nabla_X^\perp(h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$$

for any vector fields X, Y, Z tangent to N . Then $\bar{\nabla}h$ is a normal bundle valued tensor of type $(0,3)$ and is called the third fundamental form of N , $\bar{\nabla}$ is called the Vander-Waerden-Bortolotti connection of M , i.e. $\bar{\nabla}$ is the connection in $TN \oplus T^\perp N$ built with ∇ and $\nabla^\perp$. If $\bar{\nabla}h = 0$, then N is said to have parallel second fundamental form [44].

An immersion is said to be pseudo parallel if

$$(2.23) \quad \bar{R}(X, Y) \cdot h = (\bar{\nabla}_X \bar{\nabla}_Y - \bar{\nabla}_Y \bar{\nabla}_X - \bar{\nabla}_{[X, Y]})h = L_1 Q(g, h)$$

for all vector fields X, Y tangent to N [10]. If in particular, $L_1 = 0$ then the manifold is said to be semiparallel. Again the submanifold N of a $(LCS)_n$ -manifold M is said to be Ricci generalized pseudoparallel [10] if its second fundamental form h satisfies

$$(2.24) \quad \bar{R}(X, Y) \cdot h = L_2 Q(S, h), \text{ where } L_2 \text{ is constant.}$$

Also the second fundamental form h of submanifold N of a $(LCS)_n$ -manifold M is said to be η -parallel [44] if

$$(2.25) \quad (\nabla_X h)(\phi Y, \phi Z) = 0$$

for all vector fields X, Y and Z tangent to N . A submanifold N of a $(LCS)_n$ -manifold M is said to be totally umbilical if

$$(2.26) \quad h(X, Y) = g(X, Y)H$$

for any vector fields $X, Y \in \Gamma(TN)$, where H is the mean curvature of N . Moreover if $h(X, Y) = 0$ for all $X, Y \in \Gamma(TN)$, then N is said to be totally geodesic and if $H = 0$ then N is minimal in M .

Definition 2.2 ([7]). A submanifold N of a $(LCS)_n$ -manifold M is said to be invariant if the structure vector field ξ is tangent to N at every point of N and ϕX is tangent to N for any vector field X tangent to N at every point of N , that is $\phi(TN) \subset TN$ at every point of N .

From the Gauss and Weingarten formulae we obtain

$$(2.27) \quad \bar{R}(X, Y)Z = R(X, Y)Z + A_{h(X, Z)}Y - A_{h(Y, Z)}X,$$

where $\bar{R}(X, Y)Z$ denotes the tangential part of the curvature tensor of the submanifold.

Now we have

$$(2.28) \quad \begin{aligned} (\bar{R}(X, Y) \cdot h)(Z, U) &= R^\perp(X, Y)h(Z, U) \\ &\quad - h(R(X, Y)Z, U) - h(Z, R(X, Y)U) \end{aligned}$$

for all vector fields X, Y, Z and U , where

$$R^\perp(X, Y) = [\nabla_X^\perp, \nabla_Y^\perp] - \nabla_{[X, Y]}^\perp.$$

In an invariant submanifold N of a $(LCS)_n$ -manifold M , the following relations hold [38]:

$$(2.29) \quad \nabla_X \xi = \alpha \phi X,$$

$$(2.30) \quad h(X, \xi) = 0,$$

$$(2.31) \quad R(X, Y)\xi = (\alpha^2 - \rho)[\eta(Y)X - \eta(X)Y],$$

$$(2.32) \quad S(X, \xi) = (n - 1)(\alpha^2 - \rho)\eta(X), \quad \text{i.e., } Q\xi = (n - 1)(\alpha^2 - \rho)\xi,$$

$$(2.33) \quad (\nabla_X \phi)(Y) = \alpha\{g(X, Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X\},$$

$$(2.34) \quad h(X, \phi Y) = \phi h(X, Y) = h(\phi X, Y) = h(X, Y).$$

3. Totally geodesic submanifolds of $(LCS)_n$ -manifolds

In this section we prove the following:

Theorem 3.1. *Let N be an invariant submanifold of a $(LCS)_n$ -manifold M with $L_1 \neq -(\alpha^2 - \rho)$, then N is totally geodesic if and only if N is pseudoparallel.*

Proof. Let N be an invariant submanifold of a $(LCS)_n$ -manifold M with $L_1 \neq -(\alpha^2 - \rho)$.

Since N is pseudoparallel, we have

$$\bar{R}(X, Y) \cdot h = L_1 Q(g, h),$$

i.e.,

$$\begin{aligned} (3.1) \quad & R^\perp(X, Y)h(Z, U) - h(R(X, Y)Z, U) - h(Z, R(X, Y)U) \\ &= L_1[g(Y, Z)h(X, U) - g(X, Z)h(Y, U) + g(Y, U)h(X, Z) - g(X, U)h(Y, Z)] \end{aligned}$$

for all vector fields X, Y, Z, U on N .

Putting $X = U = \xi$ in (3.1) and using (2.30), we obtain

$$(3.2) \quad h(Z, R(\xi, Y)\xi) = -L_1 h(Y, Z).$$

Feeding (2.30) and (2.31) in (3.2) we get

$$(\alpha^2 - \rho + L_1)h(Y, Z) = 0,$$

which implies that $h(Y, Z) = 0$ for all Y, Z on N , i.e., N is totally geodesic, since $L_1 \neq -(\alpha^2 - \rho)$. The converse part is trivial.

Hence the theorem is proved.

Corollary 3.1. Let N be an invariant submanifold of a $(LCS)_n$ -manifold M with $\alpha^2 - \rho \neq 0$. Then N is totally geodesic if and only if N is semiparallel.

Theorem 3.2. *Let N be an invariant submanifold of a $(LCS)_n$ -manifold M with $\alpha^2 - \rho \neq 0$ and $L_2 \neq \frac{1}{n-1}$, then N is totally geodesic if and only if N is Ricci generalized pseudoparallel.*

Proof. Let N be a Ricci generalized pseudoparallel invariant submanifold of a $(LCS)_n$ -manifold M with $\alpha^2 - \rho \neq 0$ and $L_2 \neq \frac{1}{n-1}$. Then

$$R(X, Y) \cdot h = L_2 Q(S, h),$$

i.e.,

$$\begin{aligned} (3.3) \quad & R^\perp(X, Y)h(Z, U) - h(R(X, Y)Z, U) - h(Z, R(X, Y)U) \\ &= -L_2[S(Y, Z)h(X, U) - S(X, Z)h(Y, U) \\ &\quad + S(Y, U)h(X, Z) - S(X, U)h(Y, Z)] \end{aligned}$$

for all vector fields X, Y, Z, U on N .

Putting $X = U = \xi$ in (3.3) and using (2.30) we get

$$(3.4) \quad h(Z, R(\rho, \xi)\xi) = -L_2 S(\xi, \xi)h(Y, Z).$$

Using (2.31) and (2.32) in (3.4) we get

$$(3.5) \quad [1 - (n-1)L_2](\alpha^2 - \rho)h(Y, Z) = 0,$$

which implies that $h(Y, Z) = 0$, since $\alpha^2 - \rho \neq 0$ and $L_2 \neq \frac{1}{n-1}$.

Thus N is totally geodesic. The converse part is trivial.

By virtue of Theorem 3.1 and Theorem 3.2, we can state that

Theorem 3.3. *Let N be an invariant submanifold of a $(LCS)_n$ -manifold M , then the following statements are equivalent:*

- (i) N is totally geodesic.
- (ii) N is pseudoparallel with $L_1 \neq -(\alpha^2 - \rho)$.
- (iii) N is Ricci generalized pseudoparallel with $(\alpha^2 - \rho) \neq 0$ and $L_2 \neq \frac{1}{n-1}$.

Suppose that N be an invariant submanifold of a $(LCS)_n$ -manifold M whose second fundamental tensor h is pseudo parallel [15], i.e. h satisfies

$$(3.6) \quad (\nabla_X h)(Y, Z) = 2A(X)h(Y, Z) + A(Y)h(X, Z) + A(Z)h(X, Y).$$

Setting $Z = \xi$ in (3.6) and using (2.30) we get,

$$(3.7) \quad (\nabla_X h)(Y, \xi) = A(\xi)h(X, Y).$$

Now, $(\nabla_X h)(Y, \xi) = \nabla_X h(Y, \xi) - h(\nabla_X Y, \xi) - h(Y, \nabla_X \xi) = -h(Y, \alpha\phi X)$, i.e.,

$$(3.8) \quad (\nabla_X h)(Y, \xi) = -\alpha h(Y, \phi X).$$

Using (3.8) in (3.7) we get,

$$(3.9) \quad -\alpha h(Y, \phi X) = A(\xi)h(X, Y).$$

In view of (2.6) and (2.30), (3.9) yields,

$$(3.10) \quad [\alpha + A(\xi)]h(X, Y) = 0,$$

which implies that $h(X, Y) = 0$, provided $\alpha \neq -A(\xi)$.

This leads to the following theorem:

Theorem 3.4. *Let N be an invariant submanifold of $(LCS)_n$ -manifold M with pseudo parallel second fundamental tensor, then N is totally geodesic if $\alpha \neq -A(\xi)$.*

We now consider that N be an invariant submanifold of a $(LCS)_n$ -manifold M such that h is η -parallel. Since h is η -parallel we have

$$(3.11) \quad \nabla_X h(\phi Y, \phi Z) = h(\nabla_X \phi Y, \phi Z) + h(\phi Y, \nabla_X \phi Z).$$

From (2.34) we get

$$(3.12) \quad h(\phi Y, \phi Z) = h(Y, Z).$$

Using (3.12) in (3.11) we get

$$\begin{aligned} (3.13) \quad \nabla_X h(Y, Z) &= h(\{(\nabla_X \phi)Y + \phi(\nabla_X Y)\}, \phi Z) \\ &\quad + h(\phi Y, \{(\nabla_X \phi)Z + \phi(\nabla_X Z)\}) \\ &= h((\nabla_X \phi)Y, \phi Z) + h(\nabla_X Y, Z) \\ &\quad + h(\phi Y, (\nabla_X \phi)Z) + h(Y, \nabla_X Z). \end{aligned}$$

Feeding (2.30) and (2.33) in (3.13), we obtain

$$\begin{aligned} (3.14) \quad (\nabla_X h)(Y, Z) &= h(\alpha\{g(X, Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X\}, \phi Z) \\ &\quad + h(\phi Y, \alpha\{g(X, Z)\xi + 2\eta(X)\eta(Z)\xi + \eta(Z)X\}) \\ &= \alpha[\eta(Y)h(X, \phi Z) + \eta(Z)h(\phi Y, X)] \\ &= \alpha[\eta(Y)h(X, Z) + \eta(Z)h(Y, X)]. \end{aligned}$$

Conversely let N be an invariant submanifold of a $(LCS)_n$ -manifold M such that the relation (3.14) holds. Then from (3.14) it follows that $(\nabla_X h)(\phi Y, \phi Z) = 0$, i.e., the second fundamental form h is η -parallel. Thus we can obtain the following:

Theorem 3.5. *let N be an invariant submanifold of a $(LCS)_n$ -manifold M , then the second fundamental form h is η -parallel if and only if (3.14) holds.*

We now prove the following:

Theorem 3.6. *let N be an invariant submanifold of a $(LCS)_n$ -manifold M , then*

$$(3.15) \quad \phi(A_V X) = A_V X = A_{\phi V} X$$

for all $X \in \Gamma(TN)$ and $V \in \Gamma(T^\perp N)$.

Proof. By virtue of (2.6) we have

$$(3.16) \quad g(\phi(A_V X), Y) = g(A_V X, \phi Y).$$

In view of (2.20) and (2.34), (3.16) yields,

$$g(\phi(A_V X), Y) = g(h(X, \phi Y), V) = g(h(X, Y), V) = g(A_V X, Y).$$

Thus we have

$$(3.17) \quad \phi(A_V X) = A_V X.$$

Again using (2.20) and (2.34), we obtain

$$\begin{aligned} g(A_{\phi V} X, Y) &= g(h(X, Y), \phi V) = g(\phi(h(X, Y)), V) \\ &= g(h(X, Y), V) = g(A_V X, Y). \end{aligned}$$

Hence we get

$$(3.18) \quad A_{\phi V} X = A_V X.$$

From (3.17) and (3.18) we get the desired result.

We now provide an example of invariant submanifold of $(LCS)_n$ -manifold.

Example 3.1. Let us consider the 5-dimensional manifold $\overline{M} = \{(x, y, z, u, v) \in \mathbb{R}^5 : (x, y, z, u, v) \neq (0, 0, 0, 0, 0)\}$, where (x, y, z, u, v) are the standard coordinates in $\mathbb{R}^5$. The vector fields

$$e_1 = e^z \frac{\partial}{\partial x}, \quad e_2 = e^{z-ax} \frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z}, \quad e_4 = e^z \frac{\partial}{\partial u}, \quad e_5 = e^{z-u} \frac{\partial}{\partial v}$$

are linearly independent at each point of M where 'a' is a scalar.

For $\alpha \neq 0$, let $\bar{g}$ be the metric defined by

$$\begin{aligned} \bar{g}(e_i, e_j) &= \frac{1}{\alpha}, \quad \text{for } i = j \neq 3, \\ &= 0, \quad \text{for } i \neq j, \\ &= -\frac{1}{\alpha}, \quad \text{for } i = j = 3. \end{aligned}$$

Here i and j runs over 1 to 5.

Let η be the 1-form defined by $\eta(X) = \bar{g}(X, e_3)$ for any vector field X tangent to $\overline{M}$.

Let ϕ be the (1,1) tensor field defined by

$$\phi e_1 = -e_1, \quad \phi e_2 = -e_2, \quad \phi e_3 = 0, \quad \phi e_4 = -e_4, \quad \phi e_5 = -e_5.$$

Then using the linearity property of ϕ and $\bar{g}$ we have $\eta(e_3) = -\frac{1}{\alpha}$. Thus for $\xi = \alpha e_3$ we get $\eta(\xi) = -1$ and $\phi^2 X = X + \eta(X)\xi$.

Let $\bar{\nabla}$ be the Levi-Civita connection on $\overline{M}$ with respect to the metric $\bar{g}$. Then we have

$$\begin{aligned} [e_1, e_2] &= -ae^z e_2, \quad [e_1, e_3] = -e_1, \quad [e_1, e_4] = 0, \quad [e_1, e_5] = 0, \quad [e_2, e_3] = -e_2, \\ [e_2, e_4] &= 0, \quad [e_2, e_5] = 0, \quad [e_3, e_4] = e_4, \quad [e_4, e_5] = -e^z e_5. \end{aligned}$$

Now, using Koszul's formula for $\bar{g}$, it can be calculated that

$$\begin{aligned}\bar{\nabla}_{e_1}e_1 &= e_3, & \bar{\nabla}_{e_1}e_2 &= 0, & \bar{\nabla}_{e_1}e_3 &= -e_1, & \bar{\nabla}_{e_1}e_4 &= 0, & \bar{\nabla}_{e_1}e_5 &= 0, \\ \bar{\nabla}_{e_2}e_1 &= ae^ze_2, & \bar{\nabla}_{e_2}e_2 &= -ae^ze_2 + e_3, & \bar{\nabla}_{e_2}e_3 &= -e_2, & \bar{\nabla}_{e_2}e_4 &= 0, & \bar{\nabla}_{e_2}e_5 &= 0, \\ \bar{\nabla}_{e_3}e_1 &= 0, & \bar{\nabla}_{e_3}e_2 &= 0, & \bar{\nabla}_{e_3}e_3 &= 0, & \bar{\nabla}_{e_3}e_4 &= 0, & \bar{\nabla}_{e_3}e_5 &= 0, \\ \bar{\nabla}_{e_4}e_1 &= 0, & \bar{\nabla}_{e_4}e_2 &= 0, & \bar{\nabla}_{e_4}e_3 &= -e_4, & \bar{\nabla}_{e_4}e_4 &= 0, & \bar{\nabla}_{e_4}e_5 &= 0, \\ \bar{\nabla}_{e_5}e_1 &= 0, & \bar{\nabla}_{e_5}e_2 &= 0, & \bar{\nabla}_{e_5}e_3 &= -e_5, & \bar{\nabla}_{e_5}e_4 &= e^ze_5, & \bar{\nabla}_{e_5}e_5 &= e_3 - e^ze_5.\end{aligned}$$

From the above calculations we see that the (ϕ, ξ, η, g) structure satisfies $\eta(\xi) = -1$ and $\bar{\nabla}_X\xi = \alpha\phi X$. Hence $\bar{M}$ is an $(LCS)_5$ -manifold. Let f be an isometric immersion from M to $\bar{M}$ defined by $f(x, y, z) = (x, y, z, 0, 0)$.

Let $M = \{(x, y, z) \in \mathbb{R}^3 : (x, y, z) \neq (0, 0, 0)\}$, where (x, y, z) are the standard coordinates in $\mathbb{R}^3$. The vector fields $e_1 = e^z \frac{\partial}{\partial x}$, $e_2 = e^{z-ax} \frac{\partial}{\partial y}$, $e_3 = \frac{\partial}{\partial z}$ are linearly independent at each point of M where 'a' is a scalar.

For $\alpha \neq 0$, let g be the metric defined by

$$g(e_1, e_3) = g(e_2, e_3) = g(e_1, e_2) = 0, \quad g(e_1, e_1) = g(e_2, e_2) = \frac{1}{\alpha}, \quad g(e_3, e_3) = -\frac{1}{\alpha}.$$

Let η be the 1-form defined by $\eta(X) = g(X, e_3)$ for any vector field X tangent to M .

Let ϕ be the (1,1) tensor field defined by $\phi e_1 = -e_1$, $\phi e_2 = -e_2$, $\phi e_3 = 0$. Thus for $\xi = \alpha e_3$ we get $\eta(\xi) = -1$, $\phi^2 X = X + \eta(X)e_3$ for any vector field X tangent to M .

Let ∇ be the Levi-Civita connection on M with respect to the metric g . Then we have $[e_1, e_2] = -ae^ze_2$, $[e_1, e_3] = -e_1$, $[e_2, e_3] = -e_2$.

Now, using Koszul's formula for g , it can be calculated that

$$\begin{aligned}\nabla_{e_1}e_1 &= e_3, & \nabla_{e_1}e_2 &= 0, & \nabla_{e_1}e_3 &= -e_1, & \nabla_{e_2}e_1 &= ae^ze_2, & \nabla_{e_2}e_2 &= -ae^ze_2 + e_3, \\ \nabla_{e_2}e_3 &= -e_2, & \nabla_{e_3}e_1 &= 0, & \nabla_{e_3}e_2 &= 0, & \nabla_{e_3}e_3 &= 0.\end{aligned}$$

Therefore (ϕ, ξ, η, g) structure satisfies $\eta(\xi) = -1$, $\nabla_X\xi = \alpha\phi X$. Hence M is a $(LCS)_3$ -manifold. It is obvious that M is a submanifold of $\bar{M}$. Also $\phi X \in TM$ for $X \in TM$. Hence M is invariant.

Let $U = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 \in TM$ and $V = \mu_1 e_1 + \mu_2 e_2 + \mu_3 e_3 \in TM$, where λ_i, μ_i are scalars such that $i = 1, 2, 3$. Then

$$h(U, V) = \sum_{i=1}^3 \lambda_i \mu_i h(e_i, e_j) = \sum_{i=1}^3 \lambda_i \mu_i h(\bar{\nabla}_{e_i}e_j - \nabla_{e_i}e_j) = 0.$$

Hence the submanifold is totally geodesic.

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HAAR WAVELET COLLOCATION METHOD FOR SOLVING NONLINEAR KURAMOTO–SIVASHINSKY EQUATION

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Abstract. A collocation method based on Haar wavelet is presented for solving numerical solution of fourth order nonlinear Kuramoto-Sivashinsky equation. Efficiency and accuracy of the present method has been established by comparing the numerical results with exact solutions.

Keywords: Haar wavelet, Kuramoto-Sivashinsky equation, Numerical observations.

1. Introduction

Wavelet methods are more appealing, unsophisticated and reliable to obtain the numerical solutions of partial differential equations. A variety of methods have been developed for solving nonlinear partial differential equations. Haar wavelet is simple, computationally fast and give more accurate numerical results. It is discontinuous and therefore not differentiable. So, it is impossible to find the numerical results of partial differential equation using Haar wavelet method directly. Due to integrability of this function, it is utilised as a powerful mathematical tool for solving nonlinear equations. In [3], Haar wavelet method has been used for numerical solution of generalized Burger-Huxley equation. Haar wavelet method for solving lumped and distributed-parameter systems has been presented in [4]. Haar wavelet method has been presented for solving Fisher's equation, Fitzhugh–Nagumo equation and evolution equation in [6, 7, 12]. Haar and Legendre wavelets collocation methods has been used for finding numerical solution of Schrodinger and wave equation in [10]. Numerical solutions of higher degree partial differential equations using Haar wavelet have been presented in [1, 13, 15, 17, 18]. Consider the general Kuramoto-Sivashinsky equation [14]:

$$(1) \quad u_t + uu_x + \mu u_{xx} + \nu u_{xxx} = 0,$$

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with initial and boundary conditions

$$(2) \quad u(x, 0) = f(x), u_x(0, t) = 0, u_{xx}(0, t) = 0, u(0, t) = g_1(t), u(1, t) = g_2(t),$$

where $f(x)$, $g_1(t)$ and $g_2(t)$ are known functions.

Kuramoto-Sivashinsky (KS) equation which is a canonical nonlinear evolution equation that arises in a variety of physical contexts. This equation was originally derived in the context of plasma instabilities, flame front propagation, and phase turbulence in reaction-diffusion system [16]. It occurs in context of long waves on the interface between two viscous fluids [9], unstable drift waves in plasmas, reaction-diffusion systems [11], and flame front instability.

The main aim of this research is to find an accurate and efficient numerical method for solving Kuramoto-Sivashinsky equation. In Section 2, theory of Haar wavelets has been presented. In Section 3, we describe function approximation. Description of Haar wavelet method for solving such equations has been given in Section 4. Error analysis of Haar wavelet method has been presented in Section 6. In Section 7, numerical observations have been solved using the present methods and compared with exact solutions.

2. Haar wavelet

Haar wavelet is discontinuous function and is defined as:

$$(3) \quad \mathcal{H}_i(x) = \begin{cases} 1, & \alpha \leq x < \beta, \\ -1, & \beta \leq x < \gamma, \\ 0, & \text{elsewhere,} \end{cases}$$

where $\alpha = \frac{k}{m}$, $\beta = \frac{k+0.5}{m}$, $\gamma = \frac{k+1}{m}$, $m = 2^j$, $j = 0, 1, 2, \dots, J$. J denotes the level of resolution. The integer $k = 0, 1, 2, \dots, m-1$ is the translation parameter. The index i is calculated as: $i = m + k + 1$. The minimal value of $i = 2$. The maximal value of i is 2^{j+1} . The collocation points are calculated as:

$$(4) \quad x_l = \frac{l - 0.5}{2M}, \quad l = 1, 2, 3, \dots, 2M.$$

The operational matrices $\mathcal{P}$, which are $2M \times 2M$, are calculated as below:

$$(5) \quad \mathcal{P}_{1,i}(x) = \int_0^x \mathcal{H}_i(x) dx,$$

and

$$(6) \quad \mathcal{P}_{n+1,i}(x) = \int_0^x \mathcal{P}_{n,i}(x) dx.$$

In general, operational matrices can be obtained directly from the following relation (see, for example [8])

$$(7) \quad \mathcal{P}_{n,i}(x) = \begin{cases} 0, & x < \alpha, \\ \frac{1}{n!}(x - \alpha)^n, & x \in [\alpha, \beta], \\ \frac{1}{n!}\{(x - \alpha)^n - 2(x - \beta)^n\}, & x \in [\beta, \gamma], \\ \frac{1}{n!}\{(x - \alpha)^n - 2(x - \beta)^n + (x - \gamma)^n\}, & x > \gamma. \end{cases}$$

3. Function approximation

The function $y(t) \in L^2(0, 1)$ can be approximated as:

$$(8) \quad y(t) = \sum_{i=0}^{\infty} \mathcal{C}_i \mathcal{H}_i(t),$$

where the coefficient $\mathcal{C}_i$ are determined as:

$$(9) \quad \mathcal{C}_i = 2^j \int_0^1 y(t) \mathcal{H}_i(t),$$

where $i = 2^j + k$, $j \geq 0$, $0 \leq k < 2^j$. The series expansion of $y(t)$ contains infinite terms. If $y(t)$ is piecewise constant by itself, or may be approximated as piecewise constant during each subinterval, then $y(t)$ will be terminated at finite terms, that is:

$$(10) \quad y(t) = \sum_{i=0}^{m-1} \mathcal{C}_i \mathcal{H}_i(t) = \mathcal{C}_m^T \mathcal{H}_m(x),$$

where $\mathcal{C}_m^T = [\mathcal{C}_0, \mathcal{C}_1, \dots, \mathcal{C}_{m-1}]$ and $\mathcal{H}_m = [\mathcal{H}_0, \mathcal{H}_1, \dots, \mathcal{H}_{m-1}]^T$, where T is transpose.

4. Description of method for solving fourth order Kuramoto-Sivashinsky equation

Consider the approximation

$$(11) \quad \dot{u}'''(x, t) = \sum_{i=0}^{2M} \mathcal{C}_i \mathcal{H}_i(x).$$

Here $(\cdot)$ represents the differentiation with respect to t and $(')$ represents the differentiation with respect to x . Integrating (11) one time with respect to t , from t_s to t , we obtain

$$(12) \quad u'''(x, t) = u'''(x, t_s) + (t - t_s) \sum_{i=0}^{2M} \mathcal{C}_i \mathcal{H}_i(x).$$

Now, integrating (12) four times with respect to x , from 0 to x , we obtain

$$(13) \quad u'''(x, t) = u'''(x, t_s) + u'''(0, t) - u'''(0, t_s) + (t - t_s) \sum_{i=0}^{2M} \mathcal{C}_i \mathcal{P}_{1,i}(x),$$

$$(14) \quad \begin{aligned} u''(x, t) &= u''(x, t_s) + u''(0, t) - u''(0, t_s) + x \left[u'''(0, t) - u'''(0, t_s) \right] \\ &\quad + (t - t_s) \sum_{i=0}^{2M} \mathcal{C}_i \mathcal{P}_{2,i}(x), \end{aligned}$$

$$(15) \quad \begin{aligned} u'(x, t) &= u'(x, t_s) + u'(0, t) - u'(0, t_s) + x \left[u''(0, t) - u''(0, t_s) \right] \\ &\quad + \left(\frac{x^2}{2} \right) \left[u'''(0, t) - u'''(0, t_s) \right] + (t - t_s) \sum_{i=0}^{2M} \mathcal{C}_i \mathcal{P}_{3,i}(x), \end{aligned}$$

and

$$(16) \quad \begin{aligned} u(x, t) &= u(x, t_s) + u(0, t) - u(0, t_s) + x \left[u'(0, t) - u'(0, t_s) \right] \\ &\quad + \left(\frac{x^2}{2} \right) \left[u''(0, t) - u''(0, t_s) \right] \\ &\quad + \left(\frac{x^3}{6} \right) \left[u'''(0, t) - u'''(0, t_s) \right] + (t - t_s) \sum_{i=0}^{2M} \mathcal{C}_i \mathcal{P}_{4,i}(x), \end{aligned}$$

Substituting $x = 1$ in (14), we obtain

$$(17) \quad \begin{aligned} \left[u'''(0, t) - u'''(0, t_s) \right] &= u''(1, t) - u''(1, t_s) - u''(0, t) \\ &\quad + u''(0, t_s) - (t - t_s) \sum_{i=0}^{2M} \mathcal{C}_i \mathcal{P}_{2,i}(1). \end{aligned}$$

Using (17), from (14)-(16), we obtain

$$(18) \quad \begin{aligned} u''(x, t) &= u''(x, t_s) + u''(0, t) - u''(0, t_s) \\ &\quad + x \left[u''(1, t) - u''(1, t_s) - u''(0, t) + u''(0, t_s) \right] \\ &\quad + (t - t_s) \sum_{i=0}^{2M} \mathcal{C}_i \left[\mathcal{P}_{2,i}(x) - x \mathcal{P}_{2,i}(1) \right], \end{aligned}$$

$$(19) \quad \begin{aligned} u'(x, t) &= u'(x, t_s) + u'(0, t) - u'(0, t_s) + x \left[u''(0, t) - u''(0, t_s) \right] \\ &\quad + \left(\frac{x^2}{2} \right) \left[u''(1, t) - u''(1, t_s) - u''(0, t) + u''(0, t_s) \right] \\ &\quad + (t - t_s) \sum_{i=0}^{2M} \mathcal{C}_i \left[\mathcal{P}_{3,i}(x) - \frac{x^2}{2} \mathcal{P}_{2,i}(1) \right], \end{aligned}$$

and

$$\begin{aligned}
 u(x, t) &= u(x, t_s) + u(0, t) - u(0, t_s) + x[u'(0, t) - u'(0, t_s)] \\
 (20) \quad &+ \left(\frac{x^2}{2}\right)[u''(0, t) - u''(0, t_s)] + \left(\frac{x^3}{6}\right)[u''(1, t) - u''(1, t_s) - u''(0, t) + u''(0, t_s)] \\
 &+ (t - t_s) \sum_{i=0}^{2M} \mathcal{C}_i \left[\mathcal{P}_{4,i}(x) - \frac{x^3}{6} \mathcal{P}_{2,i}(1) \right],
 \end{aligned}$$

Now, differentiating (20) with respect to t , we obtain

$$\begin{aligned}
 \dot{u}(x, t) &= \dot{u}(0, t) + x\dot{u}'(0, t) + \left(\frac{x^2}{2}\right)\dot{u}''(0, t) + \left(\frac{x^3}{6}\right)[\dot{u}''(1, t) - \dot{u}''(0, t)] \\
 (21) \quad &+ \sum_{i=0}^{2M} \mathcal{C}_i \left[\mathcal{P}_{4,i}(x) - \frac{x^3}{6} \mathcal{P}_{2,i}(1) \right].
 \end{aligned}$$

From finite difference scheme, we obtain

$$(22) \quad \dot{u}''(0, t) = \left[\frac{u''(0, t) - u''(0, t_s)}{t - t_s} \right],$$

$$(23) \quad \dot{u}'(0, t) = \left[\frac{u'(0, t) - u'(0, t_s)}{t - t_s} \right],$$

and

$$(24) \quad \dot{u}(0, t) = \left[\frac{u(0, t) - u(0, t_s)}{t - t_s} \right].$$

$$\begin{aligned}
 \dot{u}(x, t) &= \left[\frac{u(0, t) - u(0, t_s)}{t - t_s} \right] + x \left[\frac{u'(0, t) - u'(0, t_s)}{t - t_s} \right] \\
 &+ \left(\frac{x^2}{2}\right) \left[\frac{u''(0, t) - u''(0, t_s)}{t - t_s} \right] \\
 (25) \quad &+ \left(\frac{x^3}{6}\right) \left[\left(\frac{u''(1, t) - u''(1, t_s)}{t - t_s} \right) - \left(\frac{u''(0, t) - u''(0, t_s)}{t - t_s} \right) \right] \\
 &+ \sum_{i=0}^{2M} \mathcal{C}_i \left[\mathcal{P}_{4,i}(x) - \frac{x^3}{6} \mathcal{P}_{2,i}(1) \right].
 \end{aligned}$$

Discretising (12), (18)-(21) by substituting $x \rightarrow x_l$ and $t \rightarrow t_{s+1}$, we obtain

$$(26) \quad u''''(x_l, t_{s+1}) = u''''(x_l, t_s) + (t_{s+1} - t_s) \sum_{i=0}^{2M} \mathcal{C}_i \mathcal{H}_i(x_l),$$

$$\begin{aligned}
 u''(x_l, t_{s+1}) &= u''(x_l, t_s) + u''(0, t_{s+1}) - u''(0, t_s) \\
 (27) \quad &+ x_l [u''(1, t_{s+1}) - u''(1, t_s) - u''(0, t_{s+1}) + u''(0, t_s)] \\
 &+ (t_{s+1} - t_s) \sum_{i=0}^{2M} \mathcal{C}_i \left[\mathcal{P}_{2,i}(x_l) - x_l \mathcal{P}_{2,i}(1) \right],
 \end{aligned}$$

$$\begin{aligned}
(28) \quad u'(x_l, t_{s+1}) &= u'(x_l, t_s) + u'(0, t_{s+1}) - u'(0, t_s) + x_l \left[u''(0, t_{s+1}) - u''(0, t_s) \right] \\
&+ \left(\frac{x_l^2}{2} \right) \left[u''(1, t_{s+1}) - u''(1, t_s) - u''(0, t_{s+1}) + u''(0, t_s) \right] \\
&+ (t_{s+1} - t_s) \sum_{i=0}^{2M} \mathcal{C}_i \left[\mathcal{P}_{3,i}(x_l) - \frac{x_l^2}{2} \mathcal{P}_{2,i}(1) \right],
\end{aligned}$$

$$\begin{aligned}
(29) \quad u(x_l, t_{s+1}) &= u(x_l, t_s) + u(0, t_{s+1}) - u(0, t_s) + x_l \left[u'(0, t_{s+1}) - u'(0, t_s) \right] \\
&+ \left(\frac{x_l^2}{2} \right) \left[u''(0, t_{s+1}) - u''(0, t_s) \right] \\
&+ \left(\frac{x_l^3}{6} \right) \left[u''(1, t_{s+1}) - u''(1, t_s) - u''(0, t_{s+1}) + u''(0, t_s) \right] \\
&+ (t_{s+1} - t_s) \sum_{i=0}^{2M} \mathcal{C}_i \left[\mathcal{P}_{4,i}(x_l) - \frac{x_l^3}{6} \mathcal{P}_{2,i}(1) \right],
\end{aligned}$$

and

$$\begin{aligned}
(30) \quad \dot{u}(x_l, t_{s+1}) &= \left[\frac{u(0, t_{s+1}) - u(0, t_s)}{t_{s+1} - t_s} \right] \\
&+ x_l \left[\frac{u'(0, t_{s+1}) - u'(0, t_s)}{t_{s+1} - t_s} \right] + \left(\frac{x_l^2}{2} \right) \left[\frac{u''(0, t_{s+1}) - u''(0, t_s)}{t_{s+1} - t_s} \right] \\
&+ \left(\frac{x_l^3}{6} \right) \left[\left(\frac{u''(1, t_{s+1}) - u''(1, t_s)}{t_{s+1} - t_s} \right) - \left(\frac{u''(0, t_{s+1}) - u''(0, t_s)}{t_{s+1} - t_s} \right) \right] \\
&+ \sum_{i=0}^{2M} \mathcal{C}_i \left[\mathcal{P}_{4,i}(x_l) - \frac{x_l^3}{6} \mathcal{P}_{2,i}(1) \right].
\end{aligned}$$

The nonlinear term in the partial differential equation (1) is linearized, using the following time discretized form

$$(31) \quad u_t(x_l, t_{s+1}) + u(x_l, t_s) u_x(x_l, t_s) + \mu \cdot u_{xx}(x_l, t_{s+1}) + \nu \cdot u_{xxx}(x_l, t_{s+1}) = 0.$$

Substituting the values from (25) – (30) in (31), we obtain

$$\begin{aligned}
&\sum_{i=0}^{2M} \left[\mathcal{P}_{4,i}(x_l) - \left(\frac{x_l^3}{6} \right) \mathcal{P}_{2,i}(1) + \nu \cdot (t_{s+1} - t_s) \mathcal{H}_i(x_l) \right. \\
&\quad \left. + \mu \cdot (t_{s+1} - t_s) \left[\mathcal{P}_{2,i}(x_l) - x_l \mathcal{P}_{2,i}(1) \right] \right] \\
&= -\nu \cdot u'''(x_l, t_s) - u(x_l, t_s) u'(x_l, t_s) \\
&\quad - \mu \cdot u''(x_l, t_s) - \mu \cdot u''(0, t_{s+1}) + \mu \cdot u''(0, t_s) \\
&\quad - \mu \cdot x_l \left[u''(1, t_{s+1}) - u''(1, t_s) - u''(0, t_{s+1}) + u''(0, t_s) \right]
\end{aligned}$$

$$\begin{aligned}
(32) \quad & - \left[\frac{u(0, t_{s+1}) - u(0, t_s)}{t_{s+1} - t_s} \right] \\
& - x_l \left[\frac{u'(0, t_{s+1}) - u'(0, t_s)}{t_{s+1} - t_s} \right] - \left(\frac{x_l^2}{2} \right) \left[\frac{u''(0, t_{s+1}) - u''(0, t_s)}{t_{s+1} - t_s} \right] \\
& - \left(\frac{x_l^3}{6} \right) \left[\left(\frac{u''(1, t_{s+1}) - u''(1, t_s)}{t_{s+1} - t_s} \right) - \left(\frac{u''(0, t_{s+1}) - u''(0, t_s)}{t_{s+1} - t_s} \right) \right].
\end{aligned}$$

After applying initial and boundary conditions in (32), we obtain the system of equations. The wavelet coefficients are obtained from this system of linear equations. The numerical solution of (1) is obtained by substituting the values of wavelet coefficients into (29).

5. Error analysis of Haar wavelet method

Let $u(x, t)$ be a differentiable function and assume that $u(x, t)$ have bounded first derivative on $[0, 1]$, that is, there exist $K > 0$, such that

$$(33) \quad |u'(x, t)| \leq K, \quad x \in [0, 1].$$

Consider the Haar wavelet approximation as below

$$(34) \quad u_{2M}(x, t) = \sum_{i=1}^{2M} C_i \mathcal{H}_i(x).$$

L_2 -error norm for Haar wavelet approximation [2] is given by

$$(35) \quad \| u(x, t) - u_{2M}(x, t) \|^2 \leq \frac{K^2}{3} \frac{1}{(2M)^2}.$$

After simplification, from (35), we obtain

$$(36) \quad \| u(x, t) - u_{2M}(x, t) \| \leq \frac{1}{(M)}.$$

As J is the maximal level of resolution and $M = 2^J$. From (36), we obtain

$$(37) \quad \| u(x, t) - u_{2M}(x, t) \| \leq \frac{1}{(2^J)}.$$

From (37), we conclude that error is inversely proportional to the level of resolution. It ensures the convergence of Haar wavelet approximation at higher level of resolution J .

6. Numerical observations

Here, we present some numerical observations to establish the efficiency and accuracy of the present collocation method based on Haar wavelet.

Example 1. Consider nonlinear Kuramoto-Sivashinsky equation with $\mu = 1$ and $\nu = 1$. The exact solution of the problem is given in [14] and is

$$(38) \quad u(x, t) = \rho + \frac{15}{19} \sqrt{\frac{11}{19}} [-9 \tanh(\sigma(x - \rho t - x_0)) + 11 \tanh^3(\sigma(x - \rho t - x_0))].$$

The initial and boundary conditions are obtained from exact solution. Table 1 shows the comparison of absolute errors at different values of t, ρ, x_0 with $\sigma = \frac{1}{2} \sqrt{\frac{11}{19}}$. Figure 1 shows the comparison absolute errors of Example 1 at $J = 3$.

xL/32	Absolute errors for $t = 0.1, \rho = 5, \sigma = -25$	Absolute errors for $t = 1, \rho = 5, \sigma = -35$	Absolute errors for $t = 1, \rho = 10, \sigma = -35$
1	7.8636E-010	6.8077E-010	7.1905E-008
3	4.0585E-009	6.2014E-010	6.5375E-008
5	7.0443E-009	8.2882E-010	6.0913E-008
7	9.6528E-009	1.4740E-009	5.9622E-008
9	1.1790E-008	2.7120E-009	6.2531E-008
11	1.3362E-008	4.6840E-009	7.0576E-008
13	1.4282E-008	7.5119E-009	8.4563E-008
15	1.4475E-008	1.1294E-008	1.0514E-007
17	1.3878E-008	1.6100E-008	1.3279E-007
19	1.2452E-008	2.1971E-008	1.6775E-007
21	1.0180E-008	2.8908E-008	2.1006E-007
23	7.0749E-009	3.6877E-008	2.5946E-007
25	3.1816E-009	4.5800E-008	3.1544E-007
27	1.4170E-009	5.5553E-008	3.7717E-007
29	6.5976E-009	6.5964E-008	4.4347E-007
31	1.2192E-008	7.6810E-008	5.1285E-007

Table 1: Comparison of absolute errors of Example 1 for $J = 3$ and different x_0, ρ, σ .

Example 2. Consider nonlinear Kuramoto-Sivashinsky equation with $\mu = -1$ and $\nu = 1$. The exact solution of the problem is given in [14] and is

$$(39) \quad u(x, t) = \rho + \frac{15}{19\sqrt{19}} [-3 \tanh(\sigma(x - \rho t - x_0)) + \tanh^3(\sigma(x - \rho t - x_0))].$$

The initial and boundary conditions are obtained from exact solution. Table 2 shows the comparison of absolute errors at different values of t, ρ, x_0 with $\sigma = \frac{1}{2} \sqrt{\frac{1}{19}}$. Figure 2 shows the comparison absolute errors of Example 2 at $J = 3$.

$xL/32$	Absolute errors for $t = 0.1, \rho = 5, \sigma = -25$	Absolute errors for $t = 1, \rho = 5, \sigma = -35$	Absolute errors for $t = 1, \rho = 10, \sigma = -35$
1	3.9296E-008	3.1345E-007	6.1496E-006
3	1.9555E-007	3.1226E-006	1.3651E-005
5	3.3756E-007	1.3809E-005	4.3476E-005
7	4.5837E-007	3.7423E-005	1.0993E-004
9	5.5089E-007	7.8717E-005	2.2649E-004
11	6.0821E-007	1.4202E-004	4.0540E-004
13	6.2385E-007	2.3112E-004	6.5740E-004
15	5.9213E-007	3.4914E-004	9.9132E-004
17	5.0845E-007	4.9840E-004	1.4138E-003
19	3.6959E-007	6.8030E-004	1.9286E-003
21	1.7403E-007	8.9517E-004	2.5370E-003
23	7.7708E-008	1.1422E-003	3.2363E-003
25	3.8278E-007	1.4191E-003	4.0205E-003
27	7.3566E-007	1.7223E-003	4.8791E-003
29	1.1278E-006	2.0466E-003	5.7973E-003
31	1.5474E-006	2.3848E-003	6.7551E-003

Table 2: Comparison of absolute errors of Example 2 for $J = 3$ and different x_0, ρ, σ .

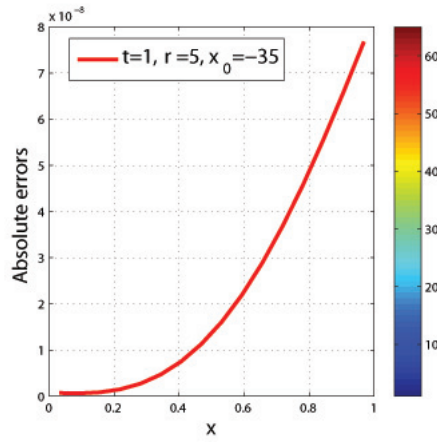


Figure 1: Absolute errors of Example 1 at $J = 3$.

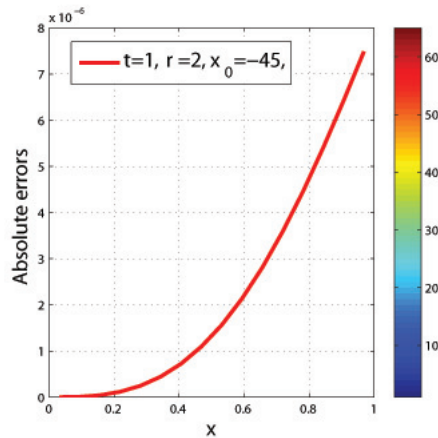


Figure 2: Absolute errors of Example 2 at $J = 3$.

7. Conclusion

From above, it is concluded that Haar wavelet method is a powerful mathematical tool for solving Kuramoto-Sivashinsky equation. The numerical solutions are much closer to the exact solutions. Also, it is concluded that Haar wavelet method is simpler, efficient and take low computational time for solving such equations.

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APPLICATION OF PARAMETER OPTIMIZATION ALGORITHM IN DIGITAL ARCHITECTURE DESIGN

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Abstract. In recent years, digital technology has been used in all aspects of peoples work and life more and more frequently and has exerted a certain impact on the design of buildings. More and more architectural design teams add mathematical logic and digital technology to the early stages of their design to better control the design work and implement the construction. Digital building design mainly includes parametric design and algorithm generation design, the former of which is mainly studied in this paper. As a modeling design algorithm which puts an emphasis on logic and reason, parametric design emphasizes the scientific nature of architectural design. In order to achieve the optimal design of the combination of building physics, institutional performance and parameterization, this paper applies the Dijkstra algorithm and the most energy efficient scheme generation (MEESG) energy consumption prediction method to the optimization design of wiring and performance and achieves good results, suggesting that the parameter optimization algorithm has a good impetus to the design of digital building.

Keywords: Digitization, parameterization, optimization algorithm, architectural design.

Introduction

Nowadays, digital technology has played an increasingly important role in architectural design. Under the influence of digital technology, the contemporary architectural design and construction has undergone a significant change [1]. Through computer-aided design (CAD) software, architects can do some simple design work such as construction drawing, etc. However, shortages, for example, sunshine distance can not be timely calculated, exist [2]. Zhou C et al. [3] applied the digital technology to the design and planning of the landscape architecture industry. Oxman R [4] introduced the architectural design concept and digital design model and introduced the concept of digital architecture into a series of architectural design programs and studied them. Hence, the emergence of parametric design makes up for the above shortcomings. Parametricism based on the theory of complexity science is the new architectural design paradigm acknowledged by the majority of architects [5]. Through enhancing the advan-

tages and avoiding the disadvantages of modern, postmodern and minimalist design, parametricism not only makes architectural performance more scientific, but also breaks the limitation of traditional architectural style and design [6, 7]. Besides, parametricism is applied to the creation of ancient architectural styles through book calligraphy design and can be applied to designs of different architectural styles [8]. This paper introduces three parameter optimization algorithms: Dijkstra algorithm, simulated annealing algorithm and most energy efficient scheme generation (MEESG) energy consumption prediction method and applies the Dijkstra algorithm into the optimization design of the wiring parameters and the MEESG energy consumption prediction method into the performance optimization design and achieves good effect, suggesting that the parameter optimization algorithm can produce a good effect on the design of digital buildings, which is worth popularizing.

1. Digital building

Digital building refers to the buildings designed by applying computer technology, the process of which includes digital design and digital construction [9]. Digital design is embodied in the design concept and the concept modeling. The architects carried out conceptual design and idea modeling of the simple schematic diagram through imagination earlier, which was inefficient and time consuming with great limitations. When the digital design intelligence tool appears, architects can design a more scientific solution through three-dimensional model, spatial modeling and color design [10] combining with the real environment. Digital technology has a good application prospect for the design and special analysis of buildings. Through the qualitative and quantitative analysis of the physical, environmental and other design, determining the indicators and clarifying the range values which are the most consistent with the requirement, the architectural design can be more scientific and rational. In the special analysis, the parameters can be adjusted to obtain a variety of programs to improve the efficiency of architectural design and the geometric algorithm and intelligent algorithm are used to design a complex and efficient architectural form. In addition, by encoding the information into the form of space, the parameters of the program are adjusted to prepare for a variety of changes [11]. In general, the parametric design transforms the 2 dimensional drawing parameters into digital information and plays a large role throughout the life cycle of the building.

2. Parameter optimization algorithm

2.1 Dijkstra algorithm

Taking the starting point as the center, Dijkstra algorithm extends outward layer by layer until reaching the end point. In short, the Dijkstra algorithm uses the vertices currently found as the initial point to find the path closest to the destination. Taking as the vertex set and the center to extend the collection,

the computation formula is as follows: write W as a set of nodes in the graph, make $W = \{1\}$ and the nodes not on W is denoted by

$$(1) \quad \text{dist}[x] = \text{Distance}(1, x),$$

$$(2) \quad \text{dist}[x] = \infty$$

equation (1) represents that node x and node 1 are linked directly, equation (2) represents that node x and node 1 are not linked directly. When applying the computer to seek a solution, a value that is longer than any path can be used to replace ∞ , such as $\infty = 100000$. Then, make y the node not on W and the value of $\text{dist}[y]$ the minimum, add y to W ; for all the nodes x that are not on W , a smaller value of $[\text{dist}[x], \text{dist}[y] + \text{Distance}(y, x)]$ is used to replace the original $\text{dist}[x]$ until all the nodes are added to W .

2.2 Simulated annealing algorithm

The simulated annealing algorithm is a heuristic stochastic optimization algorithm, which firstly determines the initial temperature and then simulates the temperature decrease of solid annealing, i.e., the process of internal ion movement from continuous strengthening, slow weakening to the equilibrium state, with the energy tending to be minimal. Due to the impact of high temperature, the molecular movement is constantly strengthened at first, with the equilibrium point becoming farther and farther and the search range of the algorithm gradually increasing. With the decrease of annealing temperature, the search area is decreasing, and the probability of sudden jump sampling is repeated to get the optimal solution. The specific algorithm is as follows: take an initial temperature T_1 and make $T = T_1$; take an initial solution A_1 and determine the number of iterations during each T , i.e., the Metropolis chain length M . According to temperature T and $n = 1, 2, \dots, M$ a new solution A_2 is obtained through solving stochastic perturbations and increments $df = f(A_2) - f(A_1)$ of A_2 is calculated, with $f(A_1)$ as the evaluation function of A_1 . When $df < 0$, A_2 is selected as the current new solution and $A_1 = A_2$; otherwise, the acceptance probability of is a uniformly distributed random number rand during $(0, 1)$ interval of $\exp(-df/T)$; when $\exp(-df/T) > \text{rand}$, A_2 is selected as the current new solution and $A_1 = A_2$; in other cases, A_1 is selected as the current solution.

2.3 MEESG energy consumption prediction method

A large number of literature shows that the early decision of the architectural design of the building has great impact on lighting, ventilation, sun exposure of the building. Besides, the size of the building's energy consumption is related to the lighting of the building. Hence, a relationship expression needs to be established. Taking architectural lighting and natural light rate as an example, the relationship between the two can be expressed as follows: $Q = DA \times S \times \rho$, where Q refers to the lighting energy consumption, DA refers to the lighting

rate, S refers to the lighting outer region area and ρ refers to power density. The lighting total energy consumption of both the outer and inner lighting region is $Q_{all} = \sum (1 - DA_i) \times \rho \times S_{out} + \rho \times S_{in}$, where DA_i refers to the lighting rate at different building orientations.

3. Wiring optimization based on Dijkstra algorithm

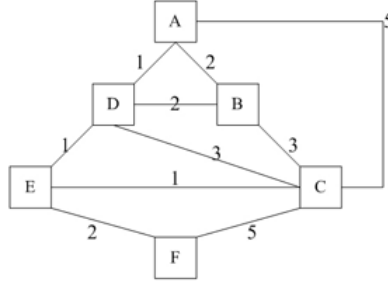


Figure 1. Line laying network map

As shown in figure 1, A is the wiring room and B, C, D, E and F are the wiring intakes. The weight (arc length) is the distance between the intakes. Through the Dijkstra algorithm, the shortest distance between the wiring points of each room and the wiring room is determined.

	W	$dist[2]$	$dist[3]$	$dist[4]$	$dist[5]$	$dist[6]$
initialization	$\{1\}$	2	5	1	∞	∞
1	$\{1,4\}$	2	4	(1)	2	∞
2	$\{1,4,5\}$	2	3	1	(2)	4
3	$\{1,2,4,5\}$	(2)	3	1	2	4
4	$\{1,2,3,4,5\}$	2	(3)	1	2	4
5	$\{1,2,3,4,5,6\}$	2	3	1	2	(4)

Table 1. Values obtained through Dijkstra algorithm

As shown in the table, the minimum value of $dist[]$ is calculated for five times. The numbers in the brackets are the solutions of each calculation, as follows: $prev[2] = 1$, $prev[3] = 5$, $prev[4] = 1$, and $prev[6] = 5$. Hence, the shortest path from A to F is $A-D-E-F$.

4. Simulated annealing algorithm

In the study, the central air conditioning system of the college is taken as an example. The size of the central air-conditioning system room is $1m \times 0.5m \times 0.5m$, which can work for four classrooms. The refrigeration capacity of the compressor is 3060W and the heat transfer capacity and condensation temperature of the condenser are 4000W and $40^{\circ}C$; The heat transfer capacity and the evaporation temperature of the evaporator are 3060W and $^{\circ}C$; The heat transfer capacity and the rated water quantity of the cooling tower are 7550W and 0.8kg/s; the power and the rated flow of the pump are 400W and 0.8kg/s. The initial annealing temperature is $T_1 = 100^{\circ}C$. Suppose A_1 and A_2 are one of the solutions and the attenuation factor is 0.95.

During the annealing process, when the inner balance is 5, the annealing temperature is $60^{\circ}C$ and the terminating algebra is 100. The results obtained by simulated annealing algorithm are shown in the following figure:

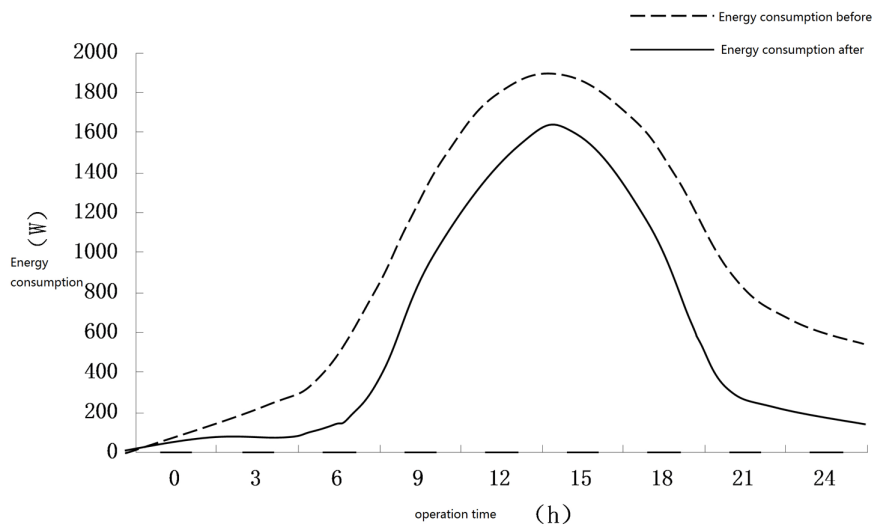


Figure 2. Comparison of energy consumption

From figure 2, we can see that the energy consumption showed a decline with the application of the algorithm, which reduced by 330W during the operation time of 12-15h and 24 hours after operation, suggesting that the energy consumption was greatly reduced.

5. Optimization design of parametric performance

A large number of literature shows that the early decision of an architectural design largely influences the lighting and ventilation of a building. Here, Neijiang City in Sichuan Province was taken as an example. Neijiang City has a sub-

tropical humid monsoon climate, where the weather is mild and light and heat are adequate, with a total solar radiation of $89.3\text{km}/\text{cm}^2$. Combined with the meteorological information of the city, the following parameters are obtained: the simulated window ratio is 0.3, 0.4, 0.5 and the outer depth is 4, 6 and 8 m. The physical parameters of the school building are as follows: the construction area is 5000m^2 , the floor number is 5, the height of each floor is 3 m, the length of the south direction is 45m and the permeability is $0.4\text{ac}/\text{h}$.

With the MEESG energy consumption prediction method, the consumption when the equipment heat is $20\text{W} / \text{m}^2$ is obtained, as shown in table 2.

Window wall ratio	Outer region depth 4				Outer region depth 6				Outer region depth 8			
	heating	air conditioning	illumination	amount	heating	air conditioning	illumination	amount	heating	air conditioning	illumination	amount
0.3	19.49	21.47	16.38	57.34	19.46	21.46	15.04	55.96	21.18	21.78	16.29	59.25
0.4	19.97	22.64	55.89	98.5	19.50	22.49	11.02	86.01	22.01	22.86	12.14	57.01
0.5	20.47	23.89	57.96	102.32	19.94	23.79	8.45	52.18	22.56	24.03	8.31	54.88

Table 2. Energy consumption values under certain equipment heat ($\text{kWh}/\text{m}^2\cdot\text{a}$)

As the depth of the outer region increases, the impact on heating and air conditioning is small while that on illumination is relatively bigger. The calculation results of the deviation values show that the relative deviation of air conditioning energy consumption is negative in all cases and the relative deviation of the three increases with the increase of outer depth and window ratio.

6. Discussion

By combining building design with digital technology, not only the rising material needs of human society can be met, but also the contemporary architectural philosophy and culture can be inherited. The use of digital technology in architecture has changed the performance of traditional architecture so that peoples building space awareness is becoming more diversified. In addition to improving the efficiency of existing architectural design, parametric technology can achieve the design ideas which were unable to be achieved and control the complex forms which were not possible earlier. It can cost-effectively implement complex geometries, efficiently create multiple options, encode other information efficiently and translate them into space forms, and can respond to changes in the project in a manner that can be efficiently adjusted by parameters. Niroumand et al. [12] studied the sustainable development of buildings in Malaysia, Iran and the United Kingdom; Cho J H et al. [13] used the OLAP evaluation system to define the characteristics of the architectural design as parameters of quantitative changes and established a parametric model to predict the cost of the building.

7. Conclusion

Based on the parameter optimization algorithm, this paper studied the wiring, energy consumption and construction performance in building design and found that the application of parameter optimization algorithm to architectural design can greatly help the design of the building and make the architectural design more scientific.

Acknowledgements

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TOTAL AND CONNECTED DOMINATION IN CHEMICAL GRAPHS

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Abstract. For a given graph G a subset D of the vertex-set $V(G)$ of G is called a total dominating set if every vertex $v \in V(G)$ is adjacent to at least one vertex of D . The total domination number $\gamma_t(G)$ is the cardinality of the smallest total dominating set. Also D is called a connected dominating set if every vertex $v \in V(G) - D$ is adjacent to at least one vertex in D and the induced subgraph $\langle D \rangle$ is connected. The connected domination number $\gamma_c(G)$ is the minimum cardinality taken over all connected dominating sets of G . In this paper, we determine the domination number, the total domination number and the connected domination number for some chemical graphs.

Keywords: Total domination number, connected domination number, bondage number, hexagonal chains, pyrene.

1. Introduction

Domination in graph theory is a natural model form any location problems in operations research. Domination has many other applications in dominating queens problem, school bus routing problem, computer communication network problems, social network theory [5]. Also chemical structures are conveniently represented by graphs, where atoms correspond to vertices and chemical bonds correspond to edges. This representation inherits many useful information about chemical properties of molecules. Also it has been shown in QSAR and QSPR studies that many physical and chemical properties of molecules are well correlated with graph theoretical invariants that are termed topological indices or molecular descriptors.

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Let G be a graph with vertex and edge sets $V(G)$ and $E(G)$, respectively. By the neighborhood of a vertex v of G we mean the set $N_G(v) = N(v) = \{u \in V(G) : uv \in E(G)\}$. The degree of a vertex v , denoted by $d_G(v)$, is the cardinality of its neighborhood. A subset $D \subseteq V(G)$ is a dominating set of G if every vertex of $V(G) - D$ has a neighbor in D . The domination number of G , denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G . For a comprehensive survey of domination in graphs, see [4, 5, 11, 12]. A subset $D \subseteq V(G)$ is a total dominating set, abbreviated TDS, of G if every vertex of G has a neighbor in D . The total domination number of G , denoted by $\gamma_t(G)$ and introduced by Cockayne, Dawes, and Hedetniemi [1].

A set $D \subseteq V(G)$ is called a connected dominating set if every vertex $v \in V(G) - D$ is adjacent to at least one vertex in D and the induced subgraph $\langle D \rangle$ is connected. The connected domination number $\gamma_c(G)$ is the minimum cardinality taken over all connected domination sets of G , see [6]. The cardinality of smallest dominating set D such that $\langle D \rangle$ has no edges is the independent domination number. The independent domination number of G , denoted by $\gamma_i(G)$.

The bondage number $b(G)$, to be the minimum number of edges whose removal increases the domination number.

In [9, 13] authors computed some domination number for Linear and double hexagonal chain. Here we continue this progress by computing the domination number, total domination number and connected domination number for some hexagonal chains. We shall need the following theorems [5].

Theorem 1.1. *Let D be a dominating set with the property that if every vertex $v_i \in V(G)$ is dominated by exactly one vertex of D , then D is minimum dominating set.*

Total domination number is easily calculated for cycles and paths, [7].

Theorem 1.2. *The total domination number of a cycle C_n or a path P_n on $n \geq 3$ vertices is given by:*

$$\gamma_t(C_n) = \gamma_t(P_n) = \begin{cases} \frac{n}{2}, & \text{if } n \equiv 0 \pmod{4} \\ \frac{n+2}{2}, & \text{if } n \equiv 2 \pmod{4} \\ \frac{n+1}{2}, & \text{otherwise.} \end{cases}$$

Hexagonal chains are important for theoretical chemistry because they are natural graph representations of benzenoid hydrocarbons, a great deal of investigation in mathematical chemistry has been developed to hexagonal chains [2, 3].

A hexagonal system is a connected plane graph without cut-vertices in which all inner faces are hexagons (and all hexagons are faces), such that two hexagons are either disjoint or have exactly one common edge, and no three hexagons share a common edge. Hexagonal systems are geometric objects obtained by arranging mutually congruent regular hexagons in the plane.

We call hexagonal system catacondensed if it does not possess internal vertices, otherwise we call it pericondensed. A hexagonal chain is a catacondensed hexagonal system in which every hexagon is adjacent to at most two hexagons.

Hexagons sharing a common edge are said to be adjacent or neighboring. Two hexagons of a hexagonal system may have either two common vertices (if they are adjacent) or none (if they are not adjacent). A vertex of a hexagonal system belongs to at most three hexagons.

A vertex shared by three hexagons is called an internal vertex the respective hexagonal system. A hexagonal chain with n hexagons, $n \geq 2$, possesses two terminal hexagons and $n - 2$ hexagons that have two neighbors.

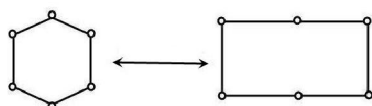


Figure 1: Isomorphism Graphs.

We have same domination number for two isomorphism graphs. Therefore, each hexagon will be represented with its isomorphic graph are shown in Figure 1.

In what follows for the sake of brevity a hexagonal chain with n hexagons will be referred to as dimension n . G_n is the hexagonal chain with n hexagons represented by Figure 2. All hexagonal chains with n hexagons have $4n + 2$ vertices and $5n + 1$ edges. Spiral chain is a kind of hexagonal chain that be shown with S_n (Figure 5).

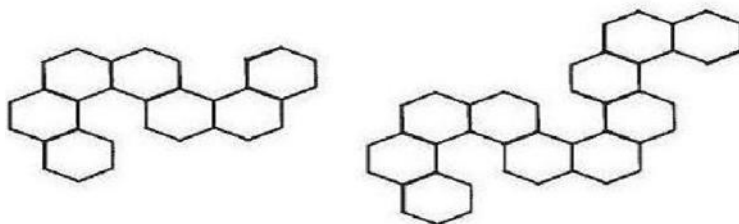
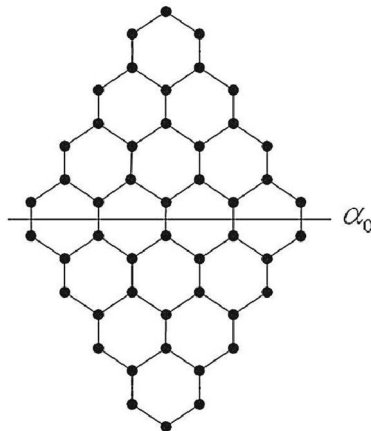


Figure 2: Hexagonal chain with 7 and 10 hexagons.

Pyrene is a typical polycyclic aromatic hydrocarbon (PAH). It has interesting photo physical properties, such as long excited state lifetime, high quantum yield of fluorescence.

Pyrene exhibits sensitive behavior, with the relative intensity of emission bands that depends on the solvent polarity. Pyrene is used to make dyes, plastics and pesticides. In order to study domination number in pyrene, it is necessary to introduce its topological properties. Let α_0 be the center line perpendicular

Figure 3: A Pyrene $PY(4)$

to the vertical edge direction of hexagon of $PY(n)$ as shown in Figure 3. The number of vertices and edges of $PY(n)$ are $2n^2 + 4n$ and $3n^2 + 4n - 1$ respectively.

2. Main results

In this section, we compute domination number, total domination and connected domination for G_n .

Theorem 2.1. *Let G_n be a Hexagonal chain with dimension n .*

Then we have:

$$\gamma(G_n) = n + \left\lfloor \frac{n}{6} \right\rfloor + 1,$$

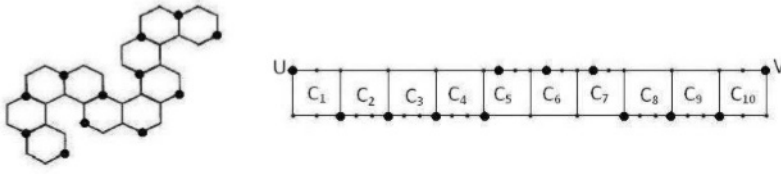
$$\gamma_i(G_n) = n + \left\lfloor \frac{n}{6} \right\rfloor + 1.$$

Proof. From Figure 4, we can see that G_n has exactly n cycles, $C_1, C_2, C_3, \dots, C_n$. Let D be any minimum dominating set of G_n . For computing $\gamma(G_n)$, it is enough to calculate $|D|$. To do this, we must choose $u, v \in D$ in first row, by Figure 4, and consider two cases for other vertices $w \in D$.

Case 1. If k is odd then for $3k - 2 \leq i \leq 3k + 1$, we have vertices $w \in D$ that be common vertex between C_i and C_{i+1} in second row.

Case 2. If k is even then we choose one vertex from cycles C_{3k-1}, C_{3k} and C_{3k+1} .

Now, we consider above cases and compute measure of dominating set D :

Figure 4: Dominating set of G_{10} .

1. If $n = 6k$, then we need to choose $\frac{2n}{3}$ common vertices (Case 1) and $\lfloor \frac{n}{6} \rfloor + \frac{n}{3} - 1$ internal vertices (Case 2) and so we have:

$$|D| = 2 + \frac{2n}{3} + (\lfloor \frac{n}{6} \rfloor + \frac{n}{3} - 1) = n + \lfloor \frac{n}{6} \rfloor + 1.$$

2. If $n = 6k + 1, k \geq 1$, then number of vertices that dominated G_n :

$$|D| = 2 + \frac{2(n-1)}{3} + (\lfloor \frac{n}{6} \rfloor + \frac{n-1}{3}) = n + \lfloor \frac{n}{6} \rfloor + 1.$$

3. If $n = 6k + 2, k \geq 1$, then we can compute domination number as follow:

$$|D| = 2 + (\frac{2(n-2)}{3} + 1) + (\lfloor \frac{n}{6} \rfloor + \frac{n-2}{3}) = n + \lfloor \frac{n}{6} \rfloor + 1.$$

4. If $n = 6k + 3, k \geq 1$, then we have :

$$|D| = 2 + (\frac{2(n-3)}{3} + 2) + (\lfloor \frac{n}{6} \rfloor + \frac{n-3}{3}) = n + \lfloor \frac{n}{6} \rfloor + 1.$$

5. If $n = 6k + 4, k \geq 1$, then number of vertices that dominated G_n :

$$|D| = 2 + (\frac{2(n-4)}{3} + 3) + (\lfloor \frac{n}{6} \rfloor + \frac{n-4}{3}) = n + \lfloor \frac{n}{6} \rfloor + 1.$$

6. If $n = 6k + 5, k \geq 1$, then we have:

$$|D| = 2 + (\frac{2(n-5)}{3} + 4) + (\lfloor \frac{n}{6} \rfloor + \frac{n-5}{3}) = n + \lfloor \frac{n}{6} \rfloor + 1.$$

Therefore domination number G_n for any n is given by:

$$\gamma(G_n) = n + \lfloor \frac{n}{6} \rfloor + 1.$$

We consider $\langle D \rangle$, that D is dominating set of G_n , it is easy to see the subgraph generated by D is an empty graph. So, by considering the fact that $\gamma(G_n) \leq \gamma_i(G_n)$, we have

$$\gamma_i(G_n) = n + \lfloor \frac{n}{6} \rfloor + 1.$$

The proof is completed. $\square$

Lemma 2.2. *Let G_n be a Hexagonal chain with dimension n . Then we have:*

$$\gamma_t(G_n) = 2n + 2,$$

$$\gamma_c(G_n) = 2n + 2.$$

Proof. Let G be the graph Figure 4, it is easy to see that G is isomorphism to the hexagonal chain, so $\gamma(G) = \gamma(G_n)$. It is sufficient to show that $\gamma_t(G_n) = \gamma_c(G_n) = 2n + 2$.

We can separate G_n to $n - 1$ paths with length four and note that Lemma 1.2, we have $\gamma_t(P_4) = 2$.

Therefore we can compute $\gamma_t(G_n)$. $\gamma_t(G_n) = 4 + 2(n - 1) = 2n + 2$.

Obviously, for connected domination of G_n we have $\gamma_c(G_n) \geq \gamma_t(G_n)$. So this set

$$D = \{(1, 2), (1, 3), \dots, (1, n + 2), (2, 2), (2, 3), (2, 6), \dots, (2, 3(n - 1)), (2, 2n + 2)\}$$

is a connected domination. In one side $|D| = \gamma_t(G_n)$ and so $\gamma_c(G_n) = 2n + 2$. $\square$

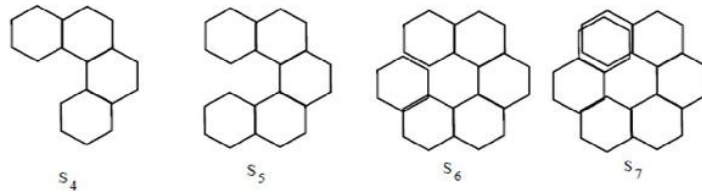


Figure 5: Spiral chain with 4,5,6, and 7 hexagons.

Theorem 2.3. *If G is a hexagonal chain with dimension n , then $b(G_n) = 1$.*

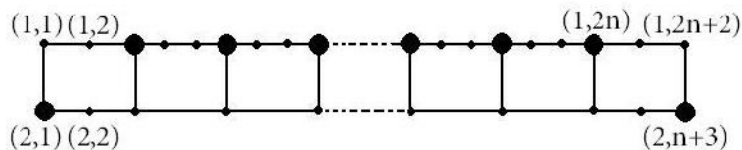
Proof. Since all the saturated vertices form a minimum dominating set of G by Theorem 2.1. Removal of an edge which is adjacent to saturated vertices, increases the domination number to $n + \lfloor \frac{n}{6} \rfloor + 2$. $\square$

Lemma 2.4. *Let S_n be a spiral chain hexagonal with n hexagons shown in Figure 5, then $\gamma(S_n) = n + 1$.*

Proof. We consider spiral chain hexagonal, that have $4n + 2$ vertices and isomorphism graph with S_n is shown in Figure 6, also we determine point of dominating set in Figure 6, the dominating set of S_n is:

$$D = \{(2, 1), (2, n + 3), (1, 3), (1, 6), \dots, (1, 2n)\}.$$

So we conclude $\gamma(S_n) = n + 1$. $\square$

Figure 6: Dominating set of S_n

Lemma 2.5. *Let S_n be a spiral chain with dimension n , then we have:*

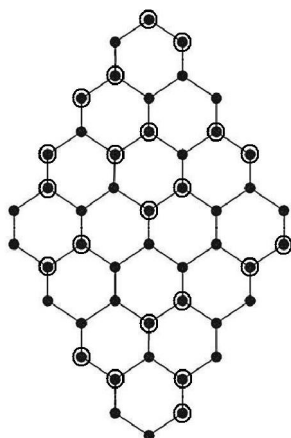
$$\gamma_t(S_n) = 2n + 2, \quad \gamma_c(S_n) = 2n + 2.$$

Proof. We can use the same sketch proof of Lemma 2.3 and determine γ_t and γ_c for spiral chain hexagonal S_n . $\square$

In [10] authors calculated domination number of $PY(n)$ as follow:

$$\gamma(PY(n)) = \begin{cases} \left(\frac{2n^2+4n}{4}\right) + 1, & \text{if } n \text{ is even} \\ \lceil \frac{2n^2+4n}{4} \rceil, & \text{if } n \text{ is odd.} \end{cases}$$

Here we compute the total domination and the connected domination number of pyrene.

Figure 7: Total dominating set of $PY(4)$

Theorem 2.6. *Let G be a pyrene of dimension n . Then we have:*

$$\gamma_t(G) = n(n + 1) + 2, \quad \gamma_c(G) = n(n + 3) + 2.$$

Proof. It is easy to see that $\gamma(G) \leq \gamma_t(G) \leq 2\gamma(G)$, therefore we have: $\frac{1}{2}n^2 + n \leq \gamma_t(G) \leq n^2 + 2n$ also in Figure 7 is shown vertices of total dominating set, so that $\gamma_t(G) = n^2 + n + 2$. Since any nontrivial connected dominating set is also a total dominating set, $\gamma(G) \leq \gamma_t(G) \leq \gamma_c(G)$ for any connected graph G with $\Delta(G) < n - 1$. So $\gamma_c(G) \geq n^2 + n + 2$, by the same sketch from Lemma 2.2, we can compute $\gamma_c(G)$, so we have $\gamma_c(G) = n^2 + 3n + 2$. $\square$

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THE BORDA RULE COMPREHENSIVE EVALUATION METHOD BASED ON SPEARMAN RANK

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Abstract. In the perspectives of systematization and axiomatization, the improved Borda rule was used to assign the evaluation results to avoid getting the same value when evaluation objects were different. Then the ideal evaluation result which was determined by criteria was summed up. Finally the Spearman rank correlation coefficient of the ideal evaluation result and all commonly used comprehensive evaluation methods was calculated through comparison. According to the size of the correlation coefficient, each method was sorted based on rationality. Finally the rationality of these methods was simulated by numerical values.

Keywords: Borda rule, Criterion, Spearman coefficient of rank correlation.

1. Introduction

Comprehensive evaluation refers to selecting multiple factors or indicators according to different evaluation purposes and using different evaluation forms and then transforming them to information or single-factor indicator that can reflect the general characteristics of evaluated objects.

Today comprehensive evaluation has been widely used, and it appears in different forms. Therefore there are a wide range of comprehensive evaluation methods. We can always get a result when we evaluate a system using different methods. But it is not sure if the result is consistent with the objective practice or reasonable. As we all know, there is nearly no systematical literature concerning these issues. Rationality of comprehensive evaluation methods is the core of decision-making. In daily management, a problem that how to sort or classify a large number of evaluated objects and choose the best one is often encountered. Such a kind of problem reflects the issues that relate to decision-making. For this reason, it is necessary to study methods used in decision-making and the selection process to provide a scientific reference to decision-makers. The rationality of comprehensive evaluation methods is the key to successfully achieve decision-

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making. Since the comprehensive evaluation has been proposed, many scholars have studied it in the aspects of the application of comprehensive evaluation, the classification and the comparison of comprehensive evaluation methods, the problems of the comprehensive evaluation, the research trends, etc. But there were few studies concerning the rationality of comprehensive evaluation methods. Therefore, studying the rationality of comprehensive evaluation methods has important practical significance.

2. Borda criteria and Spearman rank correlation

2.1 Borda criteria

Definition 2.1. Borda count method is a simple sort voting method. Each option obtains its point by ballot order, and the option with the highest point wins.

Voters arrange candidates in order according to the preferences. The candidate ranking the first gets a score, and next is the second candidate. Finally, candidate with the highest score wins.

2.2 Equal treatment of Borda number (improved)

Level-P connection matrix idea in Graph theory was used to deal with the situation when the Borda count score of two objects was the same. A level-1 connection matrix was supposed as follows:

	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>	<i>e</i>	<i>f</i>	Row sum
<i>a</i>	0	4	4	4	4	4	20
<i>b</i>	1	0	5	5	5	5	21
<i>c</i>	1	0	0	5	5	5	16
<i>d</i>	1	0	0	0	5	5	11
<i>e</i>	1	0	0	0	0	5	6
<i>f</i>	1	0	0	0	0	0	1

Matrix (1)

(Considering the number of levels, matrix (2) was obtained)

Suppose level P connectivity matrix as (P_{ij}) , then $(P + 1)$ level connection matrix $(p + 1)$ was:

$$(1) \quad (P + 1)_{ij} = \left(\sum_{k=1}^n P_{ik} + \sum_{k=1}^n P_{jk} \right) X_{ij}.$$

	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>	<i>e</i>	<i>f</i>	Row sum
<i>a</i>	0	4	8	12	16	20	60
<i>b</i>	5	0	5	10	15	20	21
<i>c</i>	4	0	0	5	10	15	16
<i>d</i>	3	0	0	0	5	10	11
<i>e</i>	2	0	0	0	0	5	6
<i>f</i>	1	0	0	0	0	0	1

Matrix (2)

Where i and j refer to the subscripts of the subject being evaluated and X_{ij} is the value in level-1 connection matrix which was at point (i, j) . When the row sum of matrix (1) and (2) was the same, the element value in the next level of matrix which was at point (i, j) could be calculated according to formula (1); as a result a new matrix and a new row sum were obtained. This process continued until the row sum could be distinguished. It was assumed that there were five participating objects, i.e., a, b, c, d, e (therefore $i, j = a, b, c, d, e$). The evaluation results are shown in matrix (3) (i.e., level-1 connection matrix): The row sum

	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>	<i>e</i>	Row sum
<i>a</i>	0	1	1	0	1	3
<i>b</i>	0	0	0	1	0	1
<i>c</i>	0	1	0	0	1	2
<i>d</i>	1	0	1	0	0	2
<i>e</i>	0	1	0	1	0	2

suggested that c, d and e had the same score; therefore level-2 connection matrix should be calculated, and the step were as follows. Firstly the numerical value on (a, b) was obtained by multiplying the sum of row a and $b(3 + 1 = 4)$ by the numerical value on (a, b) in the level-1 matrix, i.e., $4 * 1 = 4$. According to equation (1), the level-2 connection matrix and the corresponding row sum were as follows: As we can see, the values of players c and e still remained the same. To distinguish the order, it is necessary to calculate the level-3 connection matrix. According to equation (1), the level-3 connection matrix and the corresponding row sum were as follows: According to the row sum, the order was a, d, e, c, b ,

	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>	<i>e</i>	Row sum
<i>a</i>	0	4	5	0	5	14
<i>b</i>	0	0	0	3	0	3
<i>c</i>	0	3	0	0	4	7
<i>d</i>	5	0	4	0	0	9
<i>e</i>	0	3	0	7	0	7

	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>	<i>e</i>	Row sum
<i>a</i>	0	17	21	0	21	59
<i>b</i>	0	0	0	12	0	12
<i>c</i>	0	10	0	0	14	24
<i>d</i>	23	0	16	0	0	39
<i>e</i>	0	10	0	16	0	26

from excellent to poor.

2.3 Spearman rank correlation

Definition 2.3. Spearman rank correlation is mainly used to solve the problem which concerns nominal data and sequential data. It is applied to two-column variable and has rank variable properties and materials with rank variable property and linear relationship.

Spearman rank correlation formula was:

$$\rho_i = 1 - \frac{6 \sum_{i=1}^n d_i^2}{n^3 - n}.$$

Where n refers to the number of ranks and d refers to the rank difference of two-column paired variates.

3. The evaluation criteria of the rationality of the comprehensive evaluation methods

To further discuss and analyze the comprehensive evaluation methods, each evaluation method was regarded as a voter participating in Borda count. Borda

count was performed according to the evaluation results of each method; then a comprehensive result was obtained. Then the result obtained by Borda count method was regarded as a reasonable and accurate result. The correlation of the obtained comprehensive result and the evaluation results of the other methods was calculated using Spearman rank correlation theories. Then the methods were ranked based on the correlation. It was concluded that the comprehensive evaluation method with a larger correlation was more reasonable than that with a smaller correlation.

Specific steps were as follows.

Step one: Suppose that there were n evaluation objects in the evaluation system, which were denoted as $D_1, D_2, \dots, D_n$, there were m comprehensive evaluation methods (equivalent to m voters), which were denoted as $P_1, P_2, \dots, P_m$. The comprehensive evaluation results are as follows.

Evaluation methods/Evaluation objects	D_1	D_2		D_n
P_1	a_{11}	a_{12}		a_{1n}
P_2	a_{21}	a_{22}		a_{2n}
$\vdots$	$\vdots$	$\vdots$		$\vdots$
P_m	a_{m1}	a_{m2}		a_{mn}

$a_{i1}, a_{i2}, \dots, a_{in}$ was a sequence of $1, 2, \dots, n$.

Step two: The sum of evaluation results on $D_i (i = 1, 2, \dots, n)$ using different comprehensive evaluation methods ($P_i (i = 1, 2, \dots, m)$) were calculated.

- The sum of evaluation results on D_1 using different comprehensive evaluation methods was $\sum_{i=1}^m a_{i1}$;
- The sum of evaluation results on D_2 using different comprehensive evaluation methods was $\sum_{i=1}^m a_{i2}$;
- The sum of evaluation results on D_n using different comprehensive evaluation methods was $\sum_{i=1}^m a_{in}$.

Step three: $\sum_{i=1}^m a_{i1}, \sum_{i=1}^m a_{i2}, \dots, \sum_{i=1}^m a_{in}$ were compared.

1. If $\sum_{i=1}^m a_{i1}, \sum_{i=1}^m a_{i2}, \dots, \sum_{i=1}^m a_{in}$ were completely different (absence of two equal values), then there was a sequence of $1, 2, \dots, n$. It was determined as the ideal evaluation result by the criteria, which was denoted as P_L .

2. If there were two equal values among $\sum_{i=1}^m a_{i1}, \sum_{i=1}^m a_{i2}, \dots, \sum_{i=1}^m a_{in}$, the evaluation results in Table 1 were assigned by Borda count method, until they were completely different (absence of two equal values). Then there was a sequence of $1, 2, \dots, n$. It was determined as the ideal evaluation result by the criteria, which was denoted as P_L .

Step Four: Spearman rank correlation coefficients of the comprehensive evaluation results of $P_1, P_2, \dots, P_m$ and the ideal evaluation result P_L were calculated. The size of the correlation coefficient was regarded as the rationality of each comprehensive evaluation method.

4. Numerical simulation on rationality evaluation criteria for comprehensive evaluation

There were ten objects to be evaluated in the evaluation system, which were denoted as $D_1, D_2, \dots, D_{10}$. Six comprehensive evaluation methods were used, which were denoted as $P_1, P_2, \dots, P_6$ (the weighted average method P_1 , the fuzzy comprehensive evaluation method P_2 , analytic hierarchy process P_3 , principal component analysis; P_4 , comprehensive index method P_5 and the efficacy coefficient method P_6). The rationality of the evaluation criterion was simulated, and the following numerical values were randomly generated. The comprehensive evaluation results are as follows. Due to the equal comprehensive evaluation

Methods	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8	D_9	D_{10}
P_1	1	2	3	4	5	6	7	8	9	10
P_2	5	2	7	1	4	10	3	6	9	8
P_4	8	6	7	3	4	9	2	1	10	5
P_5	9	5	10	4	7	2	1	8	6	3
P_6	8	7	6	3	2	1	5	4	10	9

results, $D_1 > D_4 > D_5 > D_2 > D_8 > D_1 > D_6 > D_3 > D_{10} > D_9$ was obtained after assignment with Borda count method. According to the criteria, the result was P_L .

It was found that $\rho_2 > \rho_4 > \rho_6 > \rho_3 > \rho_5 > \rho_1$ based on the size of the correlation coefficient. Therefore, the fuzzy comprehensive evaluation method was more reasonable than the others and the rationality of weighted average method was the worst.

In the perspectives of systematization and axiomatization, the results of several commonly used comprehensive evaluation methods were analyzed, and the improved Borda rule was used to assign the results to avoid the same values. Then the ideal evaluation result which was determined by the criteria

was obtained. Through comparing it with the commonly used comprehensive evaluation methods, the Spearman rank correlation coefficients of them were obtained. According to the size of the correlation coefficient, the rationality of the various methods was sorted. In this study, the quantified characteristics of comprehensive evaluation methods and process selected in decision-making were investigated to provide adaptive strategies for decision-makers.

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ON SUBCLASS OF MEROMORPHIC UNIVALENT FUNCTIONS DEFINED BY A LINEAR OPERATOR ASSOCIATED WITH λ -GENERALIZED HURWITZ-LERCH ZETA FUNCTION AND q -HYPERGEOMETRIC FUNCTION

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Abstract. In this article, a linear operator associated with the λ -generalized Hurwitz-Lerch zeta function and q -hypergeometric function by using the Hadamard product (or convolution) is defined by the authors, a different interesting properties of certain subclass of meromorphic univalent functions related to a linear operator in the punctured unit disk are introduced and investigated. The authors also consider some closely related (known or new) corollaries and consequences of the main results presented in this paper.

Keywords: analytic functions, meromorphic functions, univalent functions, Hadamard product or (convolution), λ -generalized Hurwitz-Lerch zeta function, q -hypergeometric function, Srivastava-Attiya operator.

1. Introduction, definitions and preliminaries

Normally, we are considering the class of meromorphic function f of the form

$$(1.1) \quad f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n,$$

in the punctured open unit disc $\mathbb{U}^* = \{z : z \in \mathbb{C}, \text{ and } 0 < |z| < 1\} = \mathbb{U} \setminus \{0\}$ and denoted by Σ .

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The set of complex numbers is, as usual $\mathbb{C}$. By using $\Sigma S^*(\beta)$ and $\kappa(\beta)$ ($\beta \geq 0$), we denote the subclasses of Σ that encompass all of the meromorphic functions, which are, starlike of the order β and convex of order β in $\mathbb{U}^*$, respectively (see also the recent studies [33], [34]).

In the case of the functions $f_j(z)$ ($j = 1, 2$) which have been defined by

$$f_j(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_{n,j} z^n, \quad (j = 1, 2),$$

the Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ can be denoted by using

$$(f_1 * f_2)(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_{n,1} a_{n,2} z^n.$$

For complex parameters $\beta_1, \dots, \beta_m$ ($\beta_j \neq 0, -1, \dots; j = 1, 2, \dots, m$) and $\alpha_1, \dots, \alpha_l$ the q -hypergeometric function ${}_l\Psi_m(z)$ can be defined as

$$\begin{aligned} {}_l\Psi_m(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; q, z) &:= \sum_{n=0}^{\infty} \frac{(\alpha_1, q)_n \dots (\alpha_l, q)_n}{(q, q)_n (\beta_1, q)_n \dots (\beta_m, q)_n} \\ (1.2) \quad &\times \left[(-1)^n q^{\binom{n}{2}} \right]^{1+m-l} z^n, \end{aligned}$$

with $\binom{n}{2} = n(n-1)/2$ where $q \neq 0$ when $l > m+1$ ($l, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; z \in \mathbb{U}$).

Also the q -shifted factorial can be defined for $\alpha, q \in \mathbb{C}$ as a product of n factors by using

$$(1.3) \quad (\alpha; q)_n = \begin{cases} (1-\alpha)(1-\alpha q) \dots (1-\alpha q^{n-1}), & (n \in \mathbb{N}) \\ 1, & (n = 0), \end{cases}$$

and in terms of basic analogue of the gamma function

$$(1.4) \quad (q^\alpha; q)_n = \frac{\Gamma_q(\alpha+n)(1-q)^n}{\Gamma_q(\alpha)}, \quad n > 0.$$

Interesting to note is that, $\lim_{q \rightarrow -1} ((q^\alpha; q)_n / (1-q)^n) = (\alpha)_n = \alpha(\alpha+1) \dots (\alpha+n-1)$ is the common Pochhammer symbol, and

$$(1.5) \quad {}_l\Psi_m(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \dots (\alpha_l)_n}{(\beta_1)_n \dots (\beta_m)_n} \frac{z^n}{n!}.$$

Now for $z \in \mathbb{U}$, $0 < |q| < 1$, and $l = m+1$, the form of

$$(1.6) \quad {}_l\Psi_m(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; q, z) = \sum_{n=0}^{\infty} \frac{(\alpha_1, q)_n \dots (\alpha_l, q)_n}{(q, q)_n (\beta_1, q)_n \dots (\beta_m, q)_n} z^n,$$

is taken by the basic hypergeometric function which is defined in (1.2), and converges absolutely in the open unit disk $\mathbb{U}$ see [1].

Huda and Darus [1] and K.A Challab et al. [[2], [5]] introduced and studied a q -analogue of the Liu-Srivastava operator in correspondence with the function ${}_l\Psi_m(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; q, z)$ for the meromorphic functions of $f \in \Sigma$, which consist of the functions in the form of (1.1), and as presented below:

$$\begin{aligned}
 (1.7) \quad & {}_l\Upsilon_m(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; q, z) * f(z) \\
 &= \frac{1}{z} {}_l\Psi_m(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; q, z) * f(z) \\
 &= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{\prod_{i=1}^l (\alpha_i, q)_{n+1}}{(q, q)_{n+1} \prod_{i=1}^m (\beta_i, q)_{n+1}} a_n z^n,
 \end{aligned}$$

where $\prod_{k=1}^s (\alpha_k, q)_{n+1} = (\alpha_1, q)_{n+1} (\alpha_2, q)_{n+1} \dots (\alpha_s, q)_{n+1}$, where $z \in \mathbb{U}^* := \{z \in \mathbb{C} : 0 < |z| < 1\}$.

Recently, Ghanim ([8]; see also [9]) introduced the function $G_{s,a}$ which defined by

$$\begin{aligned}
 (1.8) \quad & G_{s,a} := (a+1)^s \left[\Phi(z, s, a) - a^s + \frac{1}{z(a+1)^s} \right], \\
 & G_{s,a} = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{a+1}{a+n} \right)^s z^n, \quad (z \in \mathbb{U}^*).
 \end{aligned}$$

Also, the function $\Phi(z, s, a)$ be the well-known Hurwitz-Lerch zeta function as was defined by (see, e.g. [[28], p. 121 et seq.]; see also [[24], [23], p. 194 et seq.])

$$\Phi(z, s, a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s} \quad (a \in \mathbb{C} \setminus \mathbb{Z}_0^-; s \in \mathbb{C} \text{ when } |z| < 1; \Re(s) > 1 \text{ when } |z| = 1).$$

We recollect here that, Srivastava introduced and systematically investigated the following new group of λ -generalised Hurwitz-Lerch zeta functions (see for example, [3], [4], [17], [18], [19], [21], [22], [26], [27], [29], and [32]):

$$\Phi_{\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_r}^{(\rho_1, \dots, \rho_p, \sigma_1, \dots, \sigma_r)}(z, s, a; b, \lambda)$$

$$(1\oplus) \frac{1}{\lambda \Gamma(s)} \cdot \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (\lambda_j)_{n\rho_j}}{(a+n)^s \cdot \prod_{j=1}^r (\mu_j)_{n\sigma_j}} H_{0,2}^{2,0} \left[(a+n)b^{\frac{1}{\lambda}} \mid (s, 1), \left(0, \frac{1}{\lambda}\right) \right] \frac{z^n}{n!}$$

$$(\min\{\Re(a), \Re(s)\} > 0; \Re(b) > 0; \lambda > 0),$$

where

$$\left(\lambda_j \in \mathbb{C} (j = 1, \dots, p) \text{ and } \mu_j \in \mathbb{C} \setminus \mathbb{Z}_0^- (j = 1, \dots, r); \rho_j > 0 (j = 1, \dots, p); \right. \\ \left. \sigma_j > 0 (j = 1, \dots, r); 1 + \sum_{j=1}^r \sigma_j - \sum_{j=1}^p \rho_j \geq 0 \right)$$

it was found that the equality in the convergence condition remained true for the suitably bounded values of $|z|$, which

$$|z| < \nabla := \left(\prod_{j=1}^p \rho_j^{-\rho_j} \right) \cdot \left(\prod_{j=1}^r \sigma_j^{\sigma_j} \right)$$

had given.

Definition 1.1. *The H -function which was involved on the right-hand side of (1.9) was the well-known Fox's H -function [[12], Definition 1.1] (see also [16], [20]) that*

$$H_{p,r}^{m,n}(z) = H_{p,r}^{m,n} \left[z \mid \begin{matrix} (a_1, A_1), \dots, (a_p, A_p) \\ (b_1, B_1), \dots, (b_r, B_r) \end{matrix} \right] \\ = \frac{1}{2\pi i} \int_{\ell} \Xi(s) z^{-s} ds \quad (z \in \mathbb{C} \setminus \{0\}; |\arg(z)| < \pi),$$

had defined, where

$$\Xi(s) = \frac{\prod_{j=1}^m \Gamma(b_j + B_j s) \cdot \prod_{j=1}^n \Gamma(1 - a_j - A_j s)}{\prod_{j=n+1}^p \Gamma(a_j + A_j s) \cdot \prod_{j=m+1}^r \Gamma(1 - b_j - B_j s)},$$

a hollow product is also depicted as 1, m, n, p and r are integers such that

$$1 \leq m \leq r \text{ and } 0 \leq n \leq p, \\ A_j > 0 \quad (j = 1, \dots, p) \text{ and } B_j > 0 \quad (j = 1, \dots, r), \\ a_j \in \mathbb{C} \quad (j = 1, \dots, p) \text{ and } b_j \in \mathbb{C} \quad (j = 1, \dots, r)$$

and ℓ is a suitable Mellin-Barnes type contour separating the poles of the gamma functions

$$\{\Gamma(b_j + B_j s)\}_{j=1}^m$$

from the poles of the gamma functions

$$\{\Gamma(1 - a_j + A_j s)\}_{j=1}^n.$$

It is worth mentioning that, if we use the fact that [[32], p. 1496, Remark 7]

$$\lim_{b \rightarrow 0} H_{0,2}^{2,0} \left[(a+n)b^{\frac{1}{\lambda}} \mid (s, 1), \left(0, \frac{1}{\lambda} \right) \right] = \lambda \Gamma(s) \quad (\lambda > 0),$$

equation (1.8) is reduced to the form of the following:

$$(1.10) \quad \Phi_{\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_r}^{(\rho_1, \dots, \rho_p, \sigma_1, \dots, \sigma_r)}(z, s, a; b, \lambda) = \Phi_{\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_r}^{(\rho_1, \dots, \rho_p, \sigma_1, \dots, \sigma_r)}(z, s, a) \\ \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (\lambda_j)_{n\rho_j}}{(a+n)^s \cdot \prod_{j=1}^r (\mu_j)_{n\sigma_j}} \frac{z^n}{n!}.$$

Definition 1.2. The function $\Phi_{\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_r}^{(\rho_1, \dots, \rho_p, \sigma_1, \dots, \sigma_r)}(z, s, a)$ which was involved in (1.10) was a multiparameter extension and generalisation of the Hurwitz-Lerch zeta function $\Phi(z, s, a)$ that Srivastava et al. introduced [17, p.503, Equation (6.2)] and was defined by

$$\Phi_{\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_r}^{(\rho_1, \dots, \rho_p, \sigma_1, \dots, \sigma_r)}(z, s, a) =: \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (\lambda_j)_{n\rho_j}}{(a+n)^s \cdot \prod_{j=1}^r (\mu_j)_{n\sigma_j}} \frac{z^n}{n!} \\ \left(p, q \in \mathbb{N}_0; \lambda_j \in \mathbb{C} (j = 1, \dots, p); a, \mu_j \in \mathbb{C} \setminus \mathbb{Z}_0^- (j = 1, \dots, r); \rho_j, \sigma_k \in \mathbb{R}^+ \right. \\ \left. (j = 1, \dots, p; k = 1, \dots, r); \Delta > -1 \text{ when } s, z \in \mathbb{C}; \Delta = -1 \text{ and } s \in \mathbb{C} \right. \\ \left. \text{when } |z| < \nabla^*; \Delta = -1 \text{ and } \Re(\Xi) > \frac{1}{2} \text{ when } |z| = \nabla^* \right)$$

with

$$\nabla^* := \left(\prod_{j=1}^p \rho_j^{-\rho_j} \right) \cdot \left(\prod_{j=1}^r \sigma_j^{\sigma_j} \right), \\ \Delta := \sum_{j=1}^r \sigma_j - \sum_{j=1}^p \rho_j \text{ and } \Xi := s + \sum_{j=1}^r \mu_j - \sum_{j=1}^p \lambda_j + \frac{p-r}{2}.$$

Srivastava et al. [30, 31] presented and developed a new linear operator by applying new family of meromorphic λ -generalised Hurwitz-Lerch zeta functions. The convolution operator which was studied by Dziok-Srivastava [6], [7] is a generalisation of two other operators: they are the Ruscheweyh [15] operator and the Hohlov [10] operator. As a matter of fact, the Dziok-Srivastava convolution operator is in and of itself, a special case of the Srivastava-Wright operator (see, for more details [11], [25]).

We have considered the new linear operator $K^{\alpha_l} f(z)$ in this study such that,

$$K^{\alpha_l} f(z) \equiv K_{(\lambda_p), (\mu_r), b}^{s, a, \lambda, \alpha_l, \beta_m} f(z) : \Sigma \rightarrow \Sigma,$$

this has been defined by

$$(1.11) \quad K^{\alpha_l} f(z) = G_{(\lambda_p), (\mu_r), b}^{s, a, \lambda}(z) *_l \Upsilon_m(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; q, z),$$

where the Hadamard product (or convolution) of the analytical functions has been denoted by $*$, and

$$(1.12) \quad G_{(\lambda_p),(\mu_r),b}^{s,a,\lambda}(z) := (a+1)^s \cdot \left[\Phi_{\lambda_1,\dots,\lambda_p;\mu_1,\dots,\mu_q}^{(1,\dots,1,1,\dots,1)}(z, s, a; b, \lambda) - \frac{a^{-s}}{\lambda\Gamma(s)} \Lambda(a, b, s, \lambda) + \frac{(a+1)^{-s}}{z} \right]$$

$$= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{\prod_{j=1}^p (\lambda_j)_n}{\prod_{j=1}^r (\mu_j)_n} \left(\frac{a+1}{a+n} \right)^s \frac{\Lambda(a+n, b, s, \lambda)}{\lambda\Gamma(s)} \frac{z^n}{n!}$$

gave the function $G_{(\lambda_p),(\mu_r),b}^{s,a,\lambda}(z)$, with

$$\Lambda(a, b, s, \lambda) := H_{0,2}^{2,0} \left[ab^{\frac{1}{\lambda}} \mid (s, 1), \left(0, \frac{1}{\lambda} \right) \right].$$

Now

$$K^{\alpha_l} f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{\prod_{i=1}^l (\alpha_i, q)_{n+1}}{(q, q)_{n+1} \prod_{i=1}^m (\beta_i, q)_{n+1}} \frac{\prod_{j=1}^p (\lambda_j)_n}{\prod_{j=1}^r (\mu_j)_n} \left(\frac{a+1}{a+n} \right)^s$$

$$\cdot \frac{\Lambda(a+n, b, s, \lambda)}{\lambda\Gamma(s)} \frac{z^n}{n!}$$

$$K^{\alpha_l} f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \Omega_{\lambda_p, \mu_r, q, b}^{\alpha_l, \beta_m, a, s} a_n \frac{z^n}{n!}$$

$$(1.13) \quad (z \in \mathbb{U}^*; \alpha, \lambda_j \in \mathbb{C} \quad (j = 1, \dots, p); \beta, \mu_j \in \mathbb{C} \setminus \mathbb{Z}_0^- \quad (j = 1, \dots, r); p \leq r+1)$$

is obtained if (1.11) and (1.12) are combined, with

$$\min\{\Re(a), \Re(s)\} > 0; \quad \lambda > 0 \quad \text{if} \quad \Re(b) > 0$$

and

$$s \in \mathbb{C} \quad a \in \mathbb{C} \setminus \mathbb{Z}_0^- \quad \text{if} \quad b = 0$$

where

$$\Omega_{\lambda_p, \mu_r, q, b}^{\alpha_l, \beta_m, a, s} = \frac{\prod_{i=1}^l (\alpha_i, q)_{n+1}}{(q, q)_{n+1} \prod_{i=1}^m (\beta_i, q)_{n+1}} \frac{\prod_{j=1}^p (\lambda_j)_n}{\prod_{j=1}^r (\mu_j)_n} \left(\frac{a+1}{a+n} \right)^s \frac{\Lambda(a+n, b, s, \lambda)}{\lambda\Gamma(s)}.$$

Let the class of all functions $f(z) \in \Sigma$ such that

$$\Re \left((1-\rho) \left(\frac{K^{\alpha_l} f(z)}{K^{\alpha_l} g(z)} \right)^{\mu} + \rho \left(\frac{K^{\alpha_l+1} f(z)}{K^{\alpha_l+1} g(z)} \right) \cdot \left(\frac{K^{\alpha_l} f(z)}{K^{\alpha_l} g(z)} \right)^{\mu-1} \right) > \gamma,$$

$$(z \in \mathbb{U}^*; 0 \leq \gamma < 1),$$

be denoted by $\Sigma_{(\lambda_p),(\mu_r),b}^{s,a,\lambda,\alpha_l,\beta_m}(\gamma, \delta, \mu, \rho)$, where the following inequality of:

$$(1.14) \quad \Re \left(\frac{K^{\alpha_l} g(z)}{K^{\alpha_l+1} g(z)} \right) > \delta \quad (0 \leq \delta < 1; z \in \mathbb{U}^*)$$

is satisfied by $g(z) \in \Sigma$.

At this point and as follows, γ and μ are real numbers such that $0 \leq \gamma < 1$ and $\mu > 0$ and $\rho \in \mathbb{C}$ with $\Re(\rho) > 0$.

Different properties of certain subclass of the meromorphically analytical function class Σ in the punctured unit disk $\mathbb{U}^*$ have been investigated. One of these function class was introduced, first, and then the properties of the linear operator

$$K^{\alpha_l+1} \equiv K_{(\lambda_p),(\mu_r),b}^{s,a,\lambda,\alpha_l+1,\beta_m} f(z)$$

were investigated.

2. Main results

The following Lemmas were needed so that the main results could be proved:

Lemma 2.1. (see[13]). *Let Ω be a set in the complex plane $\mathbb{C}$ and let the function $\Psi : \mathbb{C}^2 \rightarrow \mathbb{C}$ satisfy the following condition:*

$$\Psi(ir_2, a_1) \notin \Omega \quad \text{for all real } r_2, a_1 \leq \frac{1}{2}(1 + r_2^2).$$

If $q(z)$ is analytic in $\mathbb{U}^*$ with $q(0) = 1$ and $\Psi(q(z), zq'(z)) \in \Omega$ ($z \in \mathbb{U}^*$), then $\Re\{q(z)\} > 0$.

Our first main result is now stated and proved as Theorem 2.1, which is presented below:

Theorem 2.1. *Let $f(z) \in \Sigma_{(\lambda_p),(\mu_r),b}^{s,a,\lambda,\alpha_l,\beta_m}(\gamma, \delta, \mu, \rho)$, $\alpha \in \mathbb{R} \setminus \{0\}$ and $\rho \geq 0$. Then*

$$(2.1) \quad \Re \left(\frac{K^{\alpha_l} f(z)}{K^{\alpha_l} g(z)} \right) > \frac{2\alpha\gamma\mu + \delta\rho}{2\alpha\mu + \delta\rho} \quad (0 \leq \gamma < 1; \mu > 0; z \in \mathbb{U}^*),$$

where the condition (1.14) is satisfied by the function $g(z) \in \Sigma$.

Proof. Let

$$\xi = \frac{2\alpha\gamma\mu + \delta\rho}{2\alpha\mu + \delta\rho}$$

Now, let us suppose that

$$(2.2) \quad q(z) = \frac{1}{1-\xi} \left[\left(\frac{K^{\alpha_l} f(z)}{K^{\alpha_l} g(z)} \right)^\mu - \xi \right]$$

defines the function $q(z)$.

In that case, the function $q(z)$ is analytical in $\mathbb{U}^*$ and $q(0) = 1$. If

$$(2.3) \quad h(z) = \frac{K^{\alpha_l} g(z)}{K^{\alpha_l+1} g(z)},$$

is put in, then, according to the hypothesis of Theorem 2.1, we get $\Re\{h(z)\} > \delta$.

Then, since $f(z) \in \Sigma_{(\lambda_p), (\mu_r), b}^{s, a, \lambda, \alpha_l, \beta_m}(\gamma, \delta, \mu, \rho)$

$$\begin{aligned} & (1 - \rho) \left(\frac{K^{\alpha_l} f(z)}{K^{\alpha_l} g(z)} \right)^\mu + \rho \left(\frac{K^{\alpha_l+1} f(z)}{K^{\alpha_l+1} g(z)} \right) \cdot \left(\frac{K^{\alpha_l} f(z)}{K^{\alpha_l} g(z)} \right)^{\mu-1} \\ &= [\xi + (1 - \xi)q(z)] + \frac{\rho(1 - \xi)}{\alpha\mu} h(z) z q'(z), \end{aligned}$$

is obtained by differentiating (2.2) with respect to z .

Let us use

$$\Psi(r, s) = \xi + (1 - \xi)r + \left(\frac{\rho(1 - \xi)}{\alpha\mu} \right) h(z)s$$

to define the function $\Psi(r, s)$. Then, if we use (2.3) and the fact that

$$\begin{aligned} & f(z) \in \Sigma_{(\lambda_p), (\mu_r), b}^{s, a, \lambda, \alpha_l, \beta_m}(\gamma, \delta, \mu, \rho), \\ & \{\Psi(q(z), zq'(z)); z \in \mathbb{U}^*\} \subset \Omega = \{w \in \mathbb{C} : \Re(w) > \gamma\} \end{aligned}$$

is obtained.

At this point, for all real numbers $r_2, a_1 \leq \frac{1}{2}(1 + r_2^2)$, we get

$$\begin{aligned} \Re\{\Psi(ir_2, a_1)\} &= \xi + \left(\frac{\rho(1 - \xi)}{\alpha\mu} \right) \Re\left(\frac{K^{\alpha_l} g(z)}{K^{\alpha_l+1} g(z)} \right) \\ &\leq \xi - \frac{\rho\delta(1 - \xi)(1 + r_2^2)}{2\alpha\mu} \leq \xi - \frac{\rho\delta(1 - \xi)}{\alpha\mu} =: \gamma. \end{aligned}$$

Hence, for every $z \in \mathbb{U}^*$, we have $\Psi(ir_2, a_1) \notin \Omega$. Therefore, by using Lemma 2.1, we get that $\Re\{q(z)\} > 0$, which is,

$$\Re\left(\frac{K^{\alpha_l} f(z)}{K^{\alpha_l} g(z)} \right)^\mu > \xi \quad (z \in \mathbb{U}^*).$$

The proof of Theorem 2.1 is obviously completed at this point. $\square$

Corollary 2.1. *If the functions of $f(z)$ and $g(z)$ are in the class Σ and also suppose that condition (1.14) is satisfied by the function of $g(z)$, if the $\alpha \in \mathbb{R} \setminus \{0\}$, $\rho \geq 0$ and*

$$(2.4) \quad \Re\left((1 - \rho) \frac{K^{\alpha_l} f(z)}{K^{\alpha_l} g(z)} + \rho \frac{K^{\alpha_l+1} f(z)}{K^{\alpha_l+1} g(z)} \right) > \gamma \quad (0 \leq \gamma < 1; z \in \mathbb{U}^*).$$

Then,

$$\Re\left(\frac{K^{\alpha_l+1} f(z)}{K^{\alpha_l+1} g(z)} \right) > \eta := \frac{\gamma(2\alpha + \delta) + \delta(\rho - 1)}{2\alpha + \rho\delta}.$$

Proof. We have

$$\lambda \frac{K^{\alpha_l+1}f(z)}{K^{\alpha_l+1}g(z)} = \left((1-\rho) \frac{K^{\alpha_l}f(z)}{K^{\alpha_l}g(z)} + \rho \frac{K^{\alpha_l+1}f(z)}{K^{\alpha_l+1}g(z)} \right) + (\rho-1) \frac{K^{\alpha_l}f(z)}{K^{\alpha_l}g(z)}.$$

We can deduce the following desired inequality

$$\Re \left(\frac{K^{\alpha_l+1}f(z)}{K^{\alpha_l+1}g(z)} \right) > \eta := \frac{\gamma(2\alpha + \delta) + \delta(\rho - 1)}{2\alpha + \rho\delta},$$

if we use of (2.4) and (2.1) (for $\mu = 1$), and since $\rho > 1$. $\square$

Corollary 2.2. Let $\alpha \in \mathbb{R} \setminus \{0\}$ and $\Re\{\rho\} > 0$. If the following condition:

$$\Re\{(1-\rho)(zK^{\alpha_l}f(z))^\mu + \rho(zK^{\alpha_l+1}f(z))(zK^{\alpha_l}f(z))^{\mu-1}\} > \gamma$$

($0 \leq \gamma < 1; \mu > 0; z \in \mathbb{U}^*$), is satisfied by $f(z) \in \Sigma$, then

$$(2.5) \quad \Re\{(zK^{\alpha_l}f(z))^\mu\} > \frac{2\alpha\mu\gamma + \Re\{\rho\}}{2\mu\alpha + \Re\{\rho\}}.$$

In addition, if $\rho \geq 1$ and $\alpha \in \mathbb{R} \setminus \{0\}$, and if $f(z) \in \Sigma$ satisfies the following condition $\Re((1-\rho)zK^{\alpha_l}f(z) + \rho(zK^{\alpha_l+1}f(z))) > \gamma$, then

$$(2.6) \quad \Re(zK^{\alpha_l+1}f(z)) > \frac{(2\alpha + 1)\gamma + \rho - 1}{2\alpha + \rho} \quad (0 \leq \gamma < 1; z \in \mathbb{U}^*).$$

Proof. The results of (2.5) and (2.6) are achieved by putting $g(z) = \frac{1}{z}$ in Theorem 2.1 and Corollary 2.1, respectively. $\square$

Remark 1. (i) By putting $\rho = 1$ and for $\alpha_i, \beta_j > 0$ ($i = 1, 2, \dots, l$) and ($j = 1, 2, \dots, m$) in Corollary 2.2, we get $\Re(zK^{\alpha_l+1}f(z).(zK^{\alpha_l}f(z))^{\mu-1}) > \gamma$, this implies that

$$\Re\{(zK^{\alpha_l}f(z))^\mu\} > \frac{2\alpha\gamma + \Re\{\rho\}}{2\alpha + \Re\{\rho\}} \quad (z \in \mathbb{U}^*).$$

(ii) For $\rho \in \mathbb{C} \setminus \{0\}$ with $\Re\{\rho\} > 0$, $\mu = 1$ and $\alpha_i, \beta_j > 0$ ($i = 1, 2, \dots, l$) and ($j = 1, 2, \dots, m$) in Corollary 2.2, we get

$$\Re\{(1-\rho)zK^{\alpha_l}f(z) + \rho(zK^{\alpha_l+1}f(z))\} > \gamma \quad (0 \leq \gamma < 1; z \in \mathbb{U}^*),$$

this implies that

$$\Re(zK^{\alpha_l}f(z)) > \frac{2\alpha\gamma + \Re\{\rho\}}{2\alpha + \Re\{\rho\}} \quad (0 \leq \gamma < 1; z \in \mathbb{U}^*).$$

(iii) For $\rho = 1$, $s = 0$, $\alpha_i = \beta_j = 1$ ($i = 1, 2, \dots, l$) and ($j = 1, 2, \dots, m$), $p-1 = r = 0$ and $\lambda_1 = 1$, if we proceed to the limit as $b \rightarrow 0$ in Corollary 2.2, we have

$$\Re \left(\frac{zf'(z)}{f(z)} (zf(z))^\mu \right) > \gamma \quad (0 \leq \gamma < 1; \mu > 0; z \in \mathbb{U}^*),$$

which implies that

$$\Re([zf(z)]^\mu) > \frac{2\gamma\mu + 1}{2\mu + 1} \quad (0 \leq \gamma < 1; \mu > 0; z \in \mathbb{U}^*).$$

(iv) For $\rho \in \mathbb{C} \setminus \{0\}$ with $\Re\{\rho\} > 0$, $\alpha_i = \beta_j = 1$ ($i = 1, 2, \dots, l$) and ($j = 1, 2, \dots, m$), $\mu = 1$, $s = 0$, $p - 1 = r = 0$, and $\lambda_1 = 1$, if we take the limit as $b \rightarrow 0$ in Corollary 2.2, we get

$$\Re([(1 - \rho)zf(z) + \rho(z^2f'(z))]) > \gamma \quad (0 \leq \gamma < 1; \mu > 0; z \in \mathbb{U}^*),$$

which implies that

$$\Re\{zf(z)\} > \frac{2\gamma + \Re\{\rho\}}{2 + \Re\{\rho\}} \quad (0 \leq \gamma < 1; \mu > 0; z \in \mathbb{U}^*).$$

(v) Replacing $f(z)$ by $-zf'(z)$ in Remark 1 (ii) above, we have

$$-\Re\{(1 - \rho)z^2f(z) + \rho(z^3f''(z))\} > \gamma,$$

which implies that

$$-\Re\{z^2f'(z)\} > \frac{2\gamma + \Re\{\rho\}}{2 + \Re\{\rho\}} \quad (z \in \mathbb{U}^*).$$

(vi) For $\rho \in \mathbb{R}$ with $\rho \geq 1$, $\mu = 1$, $s = 0$, $\alpha_i = \beta_j = 1$ ($i = 1, 2, \dots, l$) and ($j = 1, 2, \dots, m$), $p - 1 = r = 0$, and $\lambda_1 = 1$, if we take the limit as $b \rightarrow 0$ in Corollary 2.2, we obtain $\Re\{(1 - \rho)zf(z) + \rho(z^2f'(z))\} > \gamma$, which implies that

$$\Re\{zf(z)\} > \frac{3\gamma + \rho - 1}{2 + \rho}.$$

The following theorem gives a further extension of the previous result.

Theorem 2.2. *Let the functions $f(z)$ and $g(z)$ be in the class Σ . Let us also suppose that, if*

$$(2.7) \quad \Re\left(\frac{K^{\alpha_l+1}f(z)}{K^{\alpha_l+1}g(z)} - \frac{K^{\alpha_l}f(z)}{K^{\alpha_l}g(z)}\right) > \frac{(1 - \gamma)\delta}{2\alpha},$$

$$(0 \leq \gamma < 1; \alpha \in \mathbb{R} \setminus \{0\}; 0 \leq \delta < 1; z \in \mathbb{U}^*),$$

then the condition (1.14) is satisfied by the function $g(z)$, hence

$$(2.8) \quad \Re\left(\frac{K^{\alpha_l}f(z)}{K^{\alpha_l+1}g(z)}\right) > \gamma, \quad (0 \leq \gamma < 1; \alpha \in \mathbb{R} \setminus \{0\}; 0 \leq \delta < 1; z \in \mathbb{U}^*)$$

and

$$(2.9) \quad \Re\left\{\frac{K^{\alpha_l+1}f(z)}{K^{\alpha_l+1}g(z)}\right\} > \frac{(2\alpha + 1 + \delta)\gamma - \delta}{2\alpha},$$

$$(0 \leq \gamma < 1; \alpha \in \mathbb{R} \setminus \{0\}; 0 \leq \delta < 1; z \in \mathbb{U}^*).$$

Proof. Let

$$q(z) = \frac{1}{1-\gamma} \left(\frac{K^{\alpha_l} f(z)}{K^{\alpha_l} g(z)} - \gamma \right).$$

Then, the function $q(z)$ is analytic in $\mathbb{U}^*$ with $q(0) = 1$. At this point, by setting

$$\Phi(z) = \frac{K^{\alpha_l} g(z)}{K^{\alpha_l+1} g(z)}$$

then, by using the hypothesis, we observe that $\Re\{\Phi(z)\} > \delta, (z \in \mathbb{U}^*)$. Now,

$$\frac{(1-\gamma)zq'(z)\Re\{\Phi\}}{\alpha} = \frac{K^{\alpha_l+1}f(z)}{K^{\alpha_l+1}g(z)} - \frac{K^{\alpha_l}f(z)}{K^{\alpha_l}g(z)} = \Psi(q(z), zq'(z)),$$

is shown by a simple computation, where

$$\Psi(r, a) = \frac{(1-\gamma)\Phi(z)a}{\alpha} \quad (\alpha \in \mathbb{R} \setminus \{0\}).$$

Therefore, we get

$$\Psi(q(z), zq'(z))_{(z \in \mathbb{U}^*)} \subset \Omega := \left\{ w : w \in \mathbb{C} \text{ and } \Re(w) > -\frac{\delta(1-\gamma)}{2\alpha} \right\}$$

if we use hypothesis (2.7). Now, for all real $r_2, a_1 \leq -(1+r_2^2)/2$, or $r_2, a_1 \leq (1+r_2^2)/2$, we obtain

$$\Re\{\Psi(ir_2, a_1)\} = \frac{a_1(1-\gamma)\Re\{\Phi\}}{\alpha} \leq -\frac{\delta(1-\gamma)(1-r_2^2)}{2\alpha} \leq -\frac{\delta(1-\gamma)}{2\alpha}.$$

This shows that $\Re\{\Psi(ir_2, a_1)\} \notin \Omega, (z \in \mathbb{U}^*)$. Then, we get $\Re\{q(z)\} > 0, (z \in \mathbb{U}^*)$, with Lemma 2.1. The assertion of (2.8) is thus proved.

The assertion of (2.9) is proved if (2.8) and (2.9) are used in the following identity,

$$\frac{K^{\alpha_l+1}f(z)}{K^{\alpha_l+1}g(z)} = \left(\frac{K^{\alpha_l+1}f(z)}{K^{\alpha_l+1}g(z)} - \frac{K^{\alpha_l}f(z)}{K^{\alpha_l}g(z)} \right) + \frac{K^{\alpha_l}f(z)}{K^{\alpha_l}g(z)}.$$

The proof of Theorem 2.2 is now obviously completed. $\square$

Remark 2. Upon putting $\alpha_i = \beta_j = 1 \quad (i = 1, 2, \dots, l) \text{ and } (j = 1, 2, \dots, m),$
 $s = 0, g(z) = \frac{1}{z}, p-1 = r = 0$, and $\lambda_1 = 1$, if we take the limit as $b \rightarrow 0$ in Theorem 2.2, we obtain

$$\Re\{zf(z) + z^2f'(z)\} > -\frac{\delta(1-\gamma)}{2} \quad (0 \leq \gamma < 1; 0 \leq \delta < 1; z \in \mathbb{U}^*),$$

which implies that $\Re\{zf(z)\} > \gamma \quad (0 \leq \gamma < 1; z \in \mathbb{U}^*)$ and

$$\Re\{2zf(z) + z^2f'(z)\} > \frac{\gamma(2+\delta) - \delta}{2} \quad (0 \leq \gamma < 1; 0 \leq \delta < 1; z \in \mathbb{U}^*).$$

3. Concluding remarks and observations

A remarkably general group of linear operators related to the λ generalised Hurwitz-Lerch zeta functions has been successfully applied in our present investigation. By using this general linear operator, we have a variety of properties of some new subclass of meromorphically univalent functions in the punctured unit disk $\mathbb{U}^*$ which were introduced and investigated. Several closely related (new or known) consequences and corollaries of the main results (Theorem 2.1 and 2.2) presented in this paper have also been considered.

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THE REALIZATION OF RADIO FREQUENCY IDENTIFICATION HANDSET ANTENNA BASED ON INTERNET OF THINGS

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Abstract. Radio frequency identification (RFID) as an automatic identification technology is one of the core technologies of Internet of Things. RFID handset is known for its light and portable characteristics; therefore, reducing the volume of RFID antenna which is the forefront and important component of RFID system without influencing working distance is an important direction of its design. According to the rules of information exchange and communication based on Internet of Things, this study investigated the effects of antenna parameters on system performance using a passive RFID system. It was found that, when the read-write distance was fixed, the higher the frequency was, the larger the one-way space loss would be; for antennae with the same structure size, higher frequency could bring better gain effects; the polarization property of antennae was the same as the polarization property of electromagnetic waves; the matching of the polarization of receiving antenna and incoming wave could induce polarization loss. A kind of RFID near-field antenna which was suitable for near-distance reading was designed through analyzing the design principles of a kind of microstrip-fed monopole near-field antenna. Then the antenna was manufactured based on simulation using High Frequency Structure Simulator (HFSS). The actual test suggested that the front end of the antenna could read the tags which were 6 cm away and 3 to 4 tags at once, which was applicable to actual application environment.

Keywords: RFID, Internet of Things, tag, antenna.

1. Introduction

The Internet of Things is an extended network based on the Internet, with its user extending to between any two articles for information exchange and communication, which can be used to realize the identification, positioning, tracking, monitoring and management of target objects by taking advantage of relevant technologies [1]. RFID is one of the core technologies [2], with antenna

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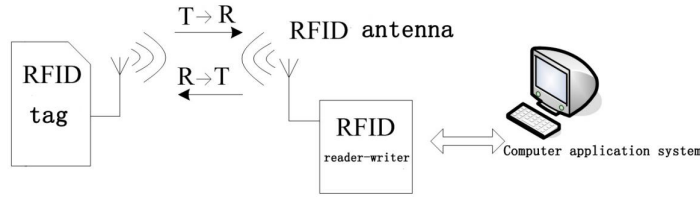


Figure 1: The main framework of RFID system

as a key component in information exchange, which is a main factor influencing the performance of RFID system.

RFID handset which is light, portable and cheap has been extensively applied in life. To adapt to the characteristic of convenience, small volume of antenna and large working distance are required for the antenna of handset. Therefore, how to design antenna with a small size without influencing precision is one of the hot spots for development. Currently, ultra high frequency band RFID antenna miniaturization develops fastest. Foreign experts have made many studies on RFID antenna miniaturization. To reduce the size of spiral curve antenna, Raunonen et al. [3] from Finland successfully applied electromagnetic band gap structure into reader-writer. N. Michishita [4] proposed a kind of evenly spaced meander-line dipole antenna and the design minimized the size of the antenna to $16\text{mm} \times 13\text{mm}$. Based on relevant technical data in China and abroad, this study investigated the design principles and miniaturization of handset antenna and successfully designed and manufactured a kind of handheld antenna which was suitable for near-distance reading. The antenna was found being able to satisfy application requirements after test.

2. RFID system

RFID, a kind of non-contact automatic identification technology, realizes the communication between host computer and information carrier using radio-frequency signals through space coupling (alternating magnetic field or electromagnetic field). A typical RFID system is composed of application system software, reader, middleware and electronic tag [5], as shown in figure 1.

The working process of RFID is as follows. Firstly, an antenna installed on a reader gives off radio-frequency signals within a specified frequency range. The electronic tag receives the radio-frequency signals released by the reader after entering into the magnetic field, obtains energy from induced current generated by space coupling, and sends product information stored in the chip. Or, the reader captures electromagnetic waves radiated by an active electronic tag, reads the information, decodes them, and sends them to the central application system for processing.

3. Effects of RFID antenna parameters on system performance

According to different functions and roles of antenna in RFID system, it can be typed into tag antenna and reader antenna [6]. Tags in passive RFID system keep working relying on electromagnetic waves launched by reader-writer. Thus, we will explain the effects of main antenna parameters on the performance of passive RFID system [7].

(1) Gain

Gain refers to the ratio of density of signal power generated by real antenna and ideal radiating element at the same point of a space under the condition of equal input power [8]. As to the effect of antenna gain on read-write distance, only when transmitting power and antenna gain increase by at least 12 dB can the read-write distance double. When transmitting power remains unchanged, the maximum identification distance of RFID system can be affected by antenna gain and working frequency band. Suppose the transmitting power of the reader-writer as $P_{transmitted}^{reader}$ and R as the power density of tag beyond read-write distance. Then the power density of the tag at R distance was:

$$(1) \quad S_1 = \frac{G_{reader} P_{transmitted}^{reader}}{4\pi R^2}.$$

(1) Power received by tag $p_{received}^{tag} = S_1 A_{tag}$ and $A_{tag} = \frac{G_{tag} \lambda^2}{4\pi}$. Therefore, we have:

$$(2) \quad P_{received}^{tag} = \left(\frac{l}{4\pi R} \right)^2 G_{reader} G_{tag} P_{transmitted}^{reader}$$

The echo power density of tags at the position of the reader-writer was:

$$(3) \quad S_2 = \frac{G_{tag} P_{received}^{tag}}{4\pi R^2}.$$

Similarly, we have:

$$(4) \quad P_{back}^{reader} = \left(\frac{l}{4\pi R} \right)^4 G_{reader}^2 G_{tag}^2 P_{transmitted}^{reader}.$$

Where G_{reader} stands for the gain of reader-writer antenna, A_{tag} stands for the effective area of tag antenna, G_{tag} stands for the gain of tag antenna, and A_{reader} stands for the effective area of reader-writer antenna.

If the sensitivity of the reader-writer was $P_{sensitivity}^{reader}$, then the read-writer distance was:

$$(5) \quad R = \frac{l}{4\pi} \sqrt[4]{\frac{P_{transmitted}^{reader} G_{reader}^2 G_{tag}^2}{P_{sensitivity}^{reader}}}.$$

Generally, the working frequency of backscattered RFID system is 915 MHz, 2.45 GHz and 5.8 GHz; the corresponding wavelength is 0.328 m, 0.122 m and

0.051 m. It could be known from formula (5) that, the maximum reader-writer distance R was in a proportion relation with working wavelength λ , i.e., when the distance was equal, the higher the frequency was, the larger the one-way space loss was. Space loss was:

$$(6) \quad SL = \left(\frac{4\pi R}{l} \right)^2.$$

Generally, the size of antenna is in a direct proportion to its working wavelength, i.e., the lower the frequency, the longer the wavelength, the larger the size of antenna, and the larger the size of tag. Therefore, for antenna with the same structure size, high frequency can bring higher gain. The size of antenna is a major limiting factor in the miniaturization of tag and reader-writer. Thus in the selection of working frequency of RFID system, factors such as antenna gain, space loss and structure size should be considered as well.

(2) Polarization

The polarization property of antenna refers to the variation rules of electric field vector at the maximum radiation direction or the maximum receiving direction along with time variation [9]. At some point in a space, the figure drawn by the end of electric field vector includes linear polarization, circular polarization, etc. If the polarization of transmitting antenna and receiving antenna is different or incoming wave polarization does not match with antenna polarization, then signals received will be smaller, which is called polarization loss. Suppose the electronic field of incoming wave as $\vec{E}_l = \hat{\rho}_w E_l$ and the electronic field polarization of receiving antenna as $\vec{E}_a = \hat{\rho}_w$, among which, $\hat{\rho}_w$ stands for the unitary vector of incoming wave, $\hat{\rho}_a$ stands for the polarization direction of antenna, and $\hat{\rho}_0$ stands for the polarization direction orthogonal to the polarization direction of antenna. Introducing polarization factor $PLF = |\hat{\rho}_w g \hat{\rho}_a|^2 = |\cos \varphi_p|$, we have $PLF(dB) = 10 \log_{10} PLF$. Then the power received by antenna was $P_r = P_{\max} g PLF$ and $P_r(dB) = P_{\max}(dB) + PLF(dB)$ ($P_{\max}$: the maximum power received in polarization matching). If the electromagnetic wave was circular polarization, polarization unitary vector could be expressed as:

$$\hat{\rho}_w = \frac{\sqrt{2}}{2}(\hat{\rho}_a \pm j\hat{\rho}_0),$$

then $PLF = 1/2$ and $PLF(dB) = -3dB$.

In conclusion, the weakening of receiving power of antenna induced by polarization matching could affect system performance. Thus the selection of antenna with specified polarization property is also an important step in the establishment of RFID system and antenna design.

4. Internet of things based RFID handset antenna

Internet of things realizes communication and information exchange by extending user side to objects and integrating computer interconnect network. Internet

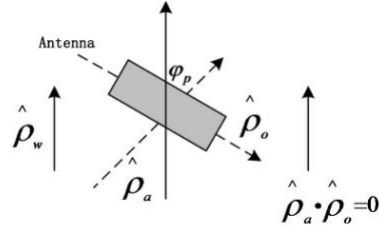


Figure 2: Antenna polarization and the polarization direction of incoming wave

of things is composed of perception layer, network layer and application layer. Perception layer is composed of various sensors, which is the source of information. RFID is the most important technology in sensing layer. In complex and various application environments, RFID system performance determines the quality of Internet of things to some extent [10].

In recent years, RFID handset has rapidly developed for its favorable characteristics such as portable shape and stable read. Reader-antenna, subsequent signal processing circuit and power supply unit are all inserted into a small handset. Therefore, antenna miniaturization is a key research subject currently. Microstrip antenna, yagi antenna and doublet antenna are commonly used in RFID system.

RFID system can be divided into remote system, short-range system and super short-range system. Super short-range distance systems such as access identification system and bus IC card charging system identify targets which are 0.1 cm to 10 cm away. Taking super short-range system as an example, this study explored the near-field antenna of RFID handset used in short-distance reading occasions.

4.1 Design principles of near-field antenna

Monopole antenna is a kind of common radiating antenna unit. Figure 3 shows the structure of ordinary monopole antenna which is composed of an above-ground conductor with a length of $\lambda/4$ and the infinite floor [11]. The effect of ground is simulated by the mirror image of antenna. In this way, monopole antenna can be equivalent to symmetrical dipole with $l/2$ long arm in free space. Certainly, such equivalence is only effective for half space above ground because there is no radiation field under floor.

Bending technology is adopted because the near-field antenna of handset put forward high requirement on the size of antenna [12]. The simplest and most classical example of bending technology is inverted-L antenna (Figure 3b) or folded monopole antenna which is obtained by bending the top of monopole antenna for 90 degrees. To realize impedance matching more conveniently, inverted-F antenna emerges (Figure 3c). Planar inverted f antenna which is

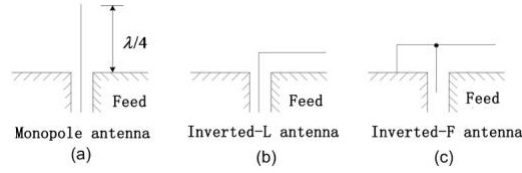


Figure 3: The basic structure of monopole antenna

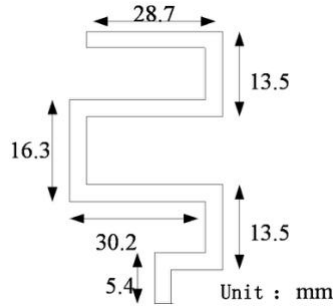


Figure 4: The basic structure of microstrip bending monopole antenna

famous later is obtained by transforming inverted-F antenna into a plane structure. Thus microstrip bending monopole was used to meet the requirement on antenna miniaturization.

To make the resonant frequency of antenna be ultra high, there are requirements on the overall length and bending mode of antenna radiating unit. Figure 4 demonstrated the basic structure of microstrip bending monopole antenna.

The radiation of traditional monopole shaped like a ring is strong on two sides and weak in the middle. As the near-field antenna designed in this study requires that radiation gain should not be too high, the middle part with weak radiation was taken as the transmitting terminal and the radiation of the right side and reverse side was limited using certain methods. In this way, the two sides would not misread other tag data on the premise that the front end could read tag.

4.2 Antenna processing and testing data

A series of variables were set as parameters for scanning according to the design methods of microstrip monopole antenna [13] and using HFSS software; then a structure figure with simulation data was obtained, as shown in figure 3. Based on that, a piece of RF4 dielectric-slab was used in the design of microstrip monopole antenna model with UHF. In the process of testing, due to the strong radiation performance of microstrip monopole, it was easy to read the next

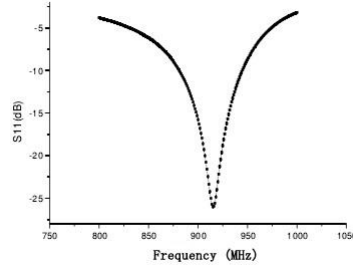


Figure 5: Return loss testing data of the near-field antenna of handset

target tag during the joint debugging of system, which did not conform to actual application environment. Thus, an earth terminal was designed for the antenna using plated-through hole. Moreover, 50Ω of impedance matching was added at end to directly guide a part of energy to the metal grounding structure to consume it compulsively, prevent it to radiate to spatial field, and avoid excessively large field intensity. Though this method could effectively reduce the reading distance of antenna, asymmetric antenna radiation was found in the process of testing. As the large impact of the metal grounding structure at the edge of the long side of the dielectric-slab on the radiation performance of antenna leads to the weakened radiation to antenna grounding structure, the radiation to antenna was asymmetric. Therefore, the end of the antenna with relatively strong radiation was attached with a piece of FR4 dielectric material to prevent the spatial field radiation of antenna artificially and make radiation relatively asymmetric.

The near-field antenna of handset ($60mm \times 40mm \times 4mm$) was processed based on the model and simulation data; coaxial SMA feed was adopted. After the matched resistance and feed interface were welded and the two layers of dielectric-slabs were fixed, a vector network analyzer was connected [14]. Return loss testing data obtained are shown in figure 5.

The above testing data of the near-field antenna of handset suggested that, the antenna satisfied the application requirement of UHF antenna. However, actual measurement has not been performed. Next, the actual reading effect of the antenna was tested by establishing the actual application environment of the near-field antenna. Firstly, to ensure the actual application effect of the product, the antenna should be used along with corresponding readers and tags. The picture of the handset is shown in figure 6.

A tag was pasted on the back of each target object, numbered as 01, 02, 03, 04 and 05. All the objects were placed vertically. The near-field antenna of the handset was connected with the reader. The antenna was ensured to put vertically. The end of the antenna with relatively weak radiation was used in near-field environment to avoid the tags to be misread by the end with relatively



Figure 6: The picture of the handset

Testing distance	Tag number	01	02	03	04	05
5cm		v	v	v	v	v
6cm		v	v	v	v	v
7cm		v	v	x	v	v
8cm		v	x	x	v	v

Figure 7: Testing results of joint debugging of antenna of the handset

strong radiation. The height of the antenna was equal to the height where the tags were pasted. The input power was set as 30 dBm, i.e., 1w. Whether near-field antenna can read tags which was 10 cm away was observed.

Joint debugging testing results of the system are shown in table 4. In the table, $\checkmark$ indicated that the tag could be read in that distance and $\times$ indicated it could not be read. The analysis of the testing data suggested that, the front end of the antenna could ensure it to read tag which was 6 cm away and read 3 $\sim$ 4 tags once, which conformed to practical application environment.

However, the field intensity of the two ends of the antenna was higher than that in the middle, which was consistent with the radiation property of the near-field antenna. Thus, the height of the antenna should be ensured. Tags will be misread if the right side or reverse side is close to tags, because the field intensity of plane radiation of monopole antenna is large. But in practical package, antenna is installed inside metal shell, which can restrain the strong radiation of the two ends of antenna.

5. Conclusions

RFID is one of the core technologies of Internet of Things, in which, antenna is a key component responsible for information exchange. This study demonstrated the key points and difficulties of antenna design through investigating the effects of antenna parameters on system performance. It was concluded that, when the read-write distance was fixed, the higher the frequency was, the larger the one-way space loss was; for antenna with the same structure size, higher frequency could bring higher gain; the polarization property of antenna was the polarization property of electromagnetic wave; the matching of the po-

larization of receiving antenna and incoming waves could generate polarization loss. Therefore, the gain, structure size and polarization matching of antenna should be considered together in the antenna design of RFID system.

In this study, a kind of RFID near-field antenna which is applicable to short-distance read was designed. The antenna was made based on the simulation model which was obtained using HFSS software and the size of the antenna was reduced using bending technology. Actual measurement suggested that, the antenna satisfied the application requirement of UHF antenna.

As the radiation near-field of antenna is difficult to be accurately described and complete design guidance theories are still lack of, the design scheme in this study which was determined through data testing is deficient inevitably and thus the adaptability of antenna in practical application environment remains to be discussed.

6. Acknowledgement

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COUPLED FIXED AND COINCIDENCE POINT THEOREMS FOR GENERALIZED CONTRACTIONS IN METRIC SPACES WITH A PARTIAL ORDER

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Abstract. In this paper, we establish results on the existence and uniqueness of coupled common fixed point theorems and coupled coincidence fixed point theorems for such non-linear contraction mappings having a mixed monotone property in partially ordered complete metric spaces with out using continuity. Our results generalize and extend the results of V. Lakshmikantham and L. Ćirić [13], Sintunavarat and Poom Kumam [16].

Keywords: coupled fixed point, coupled coincidence point, mixed monotone property, partially ordered set.

1. Introduction

Recently V. Lakshmikantham and L. Ćirić [13] generalized the concept of coupled fixed point theorems for non-linear contractions in partially ordered metric spaces. Subsequently Sintunavarat and Poom Kumam [16] studied unique coupled fixed point theorem in partially ordered metric spaces. The aim of this paper is to extend the results of T.G. Bhaskar and V. Lakshmikantham [5] and V. Lakshmikantham and L. Ćirić [13] and Sintunavarat and Poom Kumam [16]

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for a mixed monotone non-linear contractive mapping and to generalize the notion of a mixed monotone mapping

We proved some coupled coincidence and coupled common fixed point theorems for a pair of mappings. Our results extend the recent fixed point theorems due to V. Lakshmikantham and L. Ćirić [13], fixed point theorems due to Sintunavarat and Poom Kumam [16] and include several recent developments.

Suppose $(X, \leq)$ is a partially ordered set. Let $F : X \rightarrow X$ be such that for $x, y \in X$, $x \leq y \Rightarrow F(x) \leq F(y)$. Then the mapping F is said to be non-decreasing, similarly a non-increasing mapping is defined, Bhaskar and Lakshmikantham [5] introduced the following notions of a coupled fixed point theorems.

Before going to prove the main result, we need some basic definitions and results from the literature.

2. Preliminaries

Definition 2.1 (Bhaskar and Lakshmikantham [5]). Let $(X, \leq)$ be a partially ordered set and $F : X \times X \rightarrow X$. The mapping F is said to have the mixed monotone property, if F is monotone non-decreasing in its first argument and monotone non-increasing in its second argument, that is, for any $x, y \in X$,

$$x_1, x_2 \in X, x_1 \leq x_2 \Rightarrow F(x_1, y) \leq F(x_2, y)$$

and

$$y_1, y_2 \in X, y_1 \leq y_2 \Rightarrow F(x, y_1) \geq F(x, y_2).$$

Definition 2.2 (Bhaskar and Lakshmikantham [5]). Let X be a nonempty set. An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $F : X \times X \rightarrow X$, if $F(x, y) = x$ and $F(y, x) = y$

Bhaskar and Lakshmikantham [5] proved the following two coupled fixed point theorems.

Theorem 2.3 (Bhaskar and Lakshmikantham [5], Theorem 2.1). *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X , such that (X, d) is a complete metric space. Let $F : X \times X \rightarrow X$ be a continuous mapping having the mixed monotone property on X . Assume there exists a $k \in [0, 1)$ with*

$$(2.3.1) \quad d(F(x, y), F(u, v)) \leq \frac{k}{2}(d(x, u) + d(y, v))$$

for each $x \geq u$ and $y \leq v$.

If there exist $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$, then there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$.

Theorem 2.4 (Bhaskar and Lakshmikantham [5], Theorem 2.2). *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X , such that (X, d) is a complete metric space. Assume that X has the following property:*

- i) if a non decreasing sequence $\{x_n\} \rightarrow x$, then $x_n \leq x$, for all $n \in N$.
- ii) if a non increasing sequence $\{y_n\} \rightarrow y$, then $y \leq y_n$, for all $n \in N$.

Let $F : X \times X \rightarrow X$ be a continuous mapping having the mixed monotone property on X . Assume that there exists a $k \in [0, 1)$ with

$$d(F(x, y), F(u, v)) \leq \frac{k}{2}(d(x, u) + d(y, v)) \text{ for all } x, y, u, v \in X,$$

for which $x \geq u$ and $y \leq v$.

If there exists $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$, then there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$.

Definition 2.5. (V. Lakshmikantham and L. Ćirić [13]). Let $(X, \leq)$ be a partially ordered set and $F : X \times X \rightarrow X$ and $g : X \rightarrow X$. The mapping F is said to have the mixed g -monotone property, if F is monotone g -non decreasing in its first argument and is monotone g -non increasing in its second argument, that is, for any $x, y \in X$

$$x_1, x_2 \in X, g(x_1) \leq g(x_2) \Rightarrow F(x_1, y) \leq F(x_2, y)$$

and

$$y_1, y_2 \in X, g(y_1) \leq g(y_2) \Rightarrow F(x, y_1) \geq F(x, y_2).$$

Definition 2.6 (Lakshmikantham and Ćirić [12]). Let X be a nonempty set. An element $(x, y) \in X \times X$ is called a coupled coincidence point of a mapping $F : X \times X \rightarrow X$ and $g : X \rightarrow X$, if $x = g(x) = F(x, y)$ and $y = g(y) = F(y, x)$.

Definition 2.7 (Lakshmikantham and Ćirić [12]). Let X be a nonempty set and $F : X \times X \rightarrow X$, $g : X \rightarrow X$. We say F and g are commutative, if $g(F(x, y)) = F(g(x), g(y))$, for all $x, y \in X$.

Theorem 2.8 (Lakshmikantham and Ćirić [12], Theorem 2.1). Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X , such that (X, d) is a complete metric space. Assume there is a function $\varphi : [0, \infty) \rightarrow [0, \infty)$ with $\varphi(t) < t$ and $\lim_{r \rightarrow t^+} \varphi(r) < t$ for each $t > 0$, and also suppose that $F : X \times X \rightarrow X$ and $g : X \rightarrow X$ are such that F has the mixed g -monotone property and

$$(2.8.1) \quad d(F(x, y), F(u, v)) \leq \varphi \left(\frac{d(g(x), g(u)) + d(g(y), g(v))}{2} \right),$$

for all $x, y, u, v \in X$ for which $g(x) \leq g(u)$ and $g(y) \geq g(v)$.

Suppose $F(X \times X) \subseteq g(X)$, g is continuous and commutes with F and also suppose either

- (a) F is continuous (or)
- (b) X has the following property.
 - i) if a non decreasing sequence $\{x_n\} \rightarrow x$, then $x_n \leq x$, for all $n \in N$.
 - ii) if a non increasing sequence $\{y_n\} \rightarrow y$, then $y \leq y_n$, for all $n \in N$.

If there exist $x_0, y_0 \in X$ such that $g(x_0) \leq F(x_0, y_0)$ and $g(y_0) \geq F(y_0, x_0)$, then there exist $x, y \in X$ such that $g(x) = F(x, y)$ and $g(y) = F(y, x)$, that is, F and g have a coupled coincidence point.

Theorem 2.9 (Lakshmikantham and Ćirić [12], Theorem 2.2). *In addition to the hypothesis of Theorem 2.8, suppose that F and g are commutative and for every $(x, y), (z, t) \in X \times X$, there exists a $(u, v) \in X \times X$ such that $(F(u, v), F(v, u))$ is comparable to $(F(x, y), F(y, x))$ and $(F(z, t), F(t, z))$. Then F and g have a unique coupled common fixed point, that is, there exists a unique $(x, y) \in X \times X$ such that*

$$x = g(x) = F(x, y) \text{ and } y = g(y) = F(y, x).$$

In 2013, Sintunavarat and Kumam [16] gave an extension of the result of Bhaskar and Lakshmikantham [5], Lakshmikantham and Ćirić [13]. They used this concept to establish the existence of coupled coincidence point and coupled common fixed point theorem.

Theorem 2.10 (Sintunavarat and Kumam [16], Theorem 2.1). *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X , such that (X, d) is a complete metric space. Assume there is a function $\varphi : [0, \infty) \rightarrow [0, \infty)$ with $\varphi(t) < t$ and $\lim_{r \rightarrow t^+} \varphi(r) < t$, for each $t > 0$ and also suppose that $F : X \times X \rightarrow X$ is such that F has the mixed monotone property and*

$$(2.10.1) \quad d(F(x, y), F(u, v)) \leq \varphi \left(\frac{d(x, u) + d(y, v)}{2} \right),$$

for all $x, y, u, v \in X$ for which $x \leq u$ and $y \geq v$.

Suppose either

(a) F is continuous (or);

(b) X has the following property.

i) if a non decreasing sequence $\{x_n\} \rightarrow x$, then $x_n \leq x$ for all $n \in \mathbb{N}$;

ii) if a non increasing sequence $\{y_n\} \rightarrow y$, then $y \leq y_n$ for all $n \in \mathbb{N}$.

If there exist $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$, then there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$, that is, F has a coupled fixed point.

Theorem 2.11 (Sintunavarat and Kumam [16], Theorem 2.2). *In addition to the hypothesis of Theorem 2.10, suppose that for every $(x, y), (z, t) \in X \times X$, there exists a $(u, v) \in X \times X$, which is comparable to (x, y) and (z, t) . Then F has a unique coupled fixed point.*

Corollary 2.12 (Sintunavarat and Kumam [16], Corollary 2.4). *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X , such that (X, d) is a complete metric space. Suppose that $F : X \times X \rightarrow X$ and $g : X \rightarrow X$ are such that F has the mixed g -monotone property and assume there is a $k \in [0, 1)$ such that*

$$(2.12.1) \quad d(F(x, y), F(u, v)) \leq \frac{k}{2}(d(g(x), g(u)) + d(g(y), g(v))),$$

for all $x, y, u, v \in X$ for which $g(x) \leq g(u)$ and $g(y) \geq g(v)$.

Suppose $F(X \times X) \subseteq g(X)$, g is continuous and suppose either

(a) F is continuous (or)

(b) X has the following property.

i) if a non decreasing sequence $\{x_n\} \rightarrow x$, then $x_n \leq x$ for all $n \in N$.

ii) if a non increasing sequence $\{y_n\} \rightarrow y$, then $y \leq y_n$ for all $n \in N$.

If there exist $x_0, y_0 \in X$ such that $g(x_0) \leq F(x_0, y_0)$ and $g(y_0) \geq F(y_0, x_0)$, then there exist $x, y \in X$ such that $g(x) = F(x, y)$ and $g(y) = F(y, x)$, that is, F and g have a coupled coincidence fixed point.

3. Main result

In this section, we improve the results in section 2, by replacing the conditions (i) $\varphi(t) < t$ and (ii). $\lim_{r \rightarrow t+} \varphi(r) < t$ by the single condition: $\varphi(t+0) < t$, and the average in the argument of φ by maximum.

We introduce the class Φ of functions as follows:

$$\Phi = \{\varphi/\varphi : [0, \infty) \rightarrow [0, \infty), \varphi \text{ is increasing and } \varphi(t+0) < t \forall t > 0\}.$$

We observe that $\varphi \in \Phi \Rightarrow \varphi(t) < t$ and $\lim_{r \rightarrow t+} \varphi(r) < t \forall t > 0$.

Now we prove our main result.

Theorem 3.1. Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X , such that (X, d) is a complete metric space. Assume there is an increasing function $\varphi : [0, \infty) \rightarrow [0, \infty)$ with $\varphi(t+0) < t$ for each $t > 0$, and also suppose that $F : X \times X \rightarrow X$ is such that F has the mixed monotone property and

$$(3.1.1) \quad d(F(x, y), F(u, v)) \leq \varphi\{\max(d(x, u), d(y, v))\},$$

for all $x, y, u, v \in X$ for which $x \leq u$ and $y \geq v$. Suppose either

a) F is continuous (or);

b) X has the following property.

i) if a non decreasing sequence $\{x_n\} \rightarrow x$, then $x_n \leq x$, for all $n \in N$.

ii) if a non increasing sequence $\{y_n\} \rightarrow y$, then $y \leq y_n$, for all $n \in N$.

If there exist $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$, then there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$, that is, F has a coupled fixed point.

Proof. Suppose

$$(3.1.2) \quad u_0 \leq F(u_0, v_0) \text{ and } v_0 \geq F(v_0, u_0).$$

Define the sequences $\{u_n\}$ and $\{v_n\}$ by $u_1 = F(u_0, v_0)$ and $v_1 = F(v_0, u_0)$.

In general $u_{n+1} = F(u_n, v_n)$ and $v_{n+1} = F(v_n, u_n)$, for $n = 0, 1, 2, \dots$

From (3.1.2), $u_0 \leq F(u_0, v_0) = u_1$. Therefore $u_0 \leq u_1$ and $v_0 \geq F(v_0, u_0) = v_1$. Therefore $v_0 \geq v_1$. Now $u_2 = F(u_1, v_1) \geq F(u_0, v_1) \geq F(u_0, v_0) = u_1$.

Therefore $u_2 \geq u_1$. And $v_2 = F(v_1, u_1) \leq F(v_0, u_1) \leq F(v_0, u_0) = v_1$. Therefore $v_2 \leq v_1$. Similarly $u_3 \geq u_2$ and $v_3 \leq v_2$. In general $u_{n+1} \geq u_n$ and $v_{n+1} \leq v_n$. Therefore

$$u_0 \leq u_1 \leq u_2 \leq \dots \leq u_n \leq u_{n+1}$$

and

$$v_0 \geq v_1 \geq v_2 \dots \geq v_n \geq v_{n+1}.$$

Therefore $\{u_n\}$ is $\uparrow$ and $\{v_n\}$ is $\downarrow$. First we show that $\{u_n\}$ and $\{v_n\}$ are Cauchy sequences. If possible assume either $\{u_n\}$ or $\{v_n\}$ fails to be Cauchy. Then either

$$\lim_{m,n \rightarrow \infty} d(u_m, u_n) \neq 0$$

or

$$\lim_{m,n \rightarrow \infty} d(v_m, v_n) \neq 0.$$

Therefore,

$$\max\left\{\lim_{m,n \rightarrow \infty} d(u_m, u_n), \lim_{m,n \rightarrow \infty} d(v_m, v_n)\right\} \neq 0,$$

i.e

$$\lim_{m,n \rightarrow \infty} \max\left\{\lim_{m,n \rightarrow \infty} d(u_m, u_n), \lim_{m,n \rightarrow \infty} d(v_m, v_n)\right\} \neq 0,$$

i.e, there exist $\varepsilon > 0$, for which we can find sub sequences $\{m_k\}$ and $\{n_k\}$ of positive integers with $n_k > m_k > k$ such that

$$(3.1.3) \quad \max\{d(u_{m_k}, u_{n_k}), d(v_{m_k}, v_{n_k})\} \geq \varepsilon.$$

Further, we choose $\{n_k\}$ to be the smallest positive integer such that $n_k > m_k$ satisfying (3.1.1).

Hence, we have $\max\{d(u_{m_k}, u_{n_k}), d(v_{m_k}, v_{n_k})\} \geq \varepsilon$ and

$$(3.1.4) \quad \max\{d(u_{m_k}, u_{n_k-1}), d(v_{m_k}, v_{n_k-1})\} < \varepsilon.$$

Now, we prove that

- I. $\lim_{k \rightarrow \infty} \max\{d(u_{n_k}, u_{m_k}), d(v_{n_k}, v_{m_k})\} = \varepsilon$;
- II. $\lim_{k \rightarrow \infty} \max\{d(u_{n_k-1}, u_{m_k-1}), d(v_{n_k-1}, v_{m_k-1})\} = \varepsilon$;
- III. $\lim_{k \rightarrow \infty} \max\{d(u_{m_k}, u_{n_k-1}), d(v_{m_k}, v_{n_k-1})\} = \varepsilon$.

First we prove I:

From the triangular inequality we have

$$(3.1.5) \quad d(u_{n_k}, u_{m_k}) \leq d(u_{n_k}, u_{n_k-1}) + d(u_{n_k-1}, u_{m_k}) < d(u_{n_k}, u_{n_k-1}) + \varepsilon$$

and

$$(3.1.6) \quad d(v_{n_k}, v_{m_k}) \leq d(v_{n_k}, v_{n_k-1}) + d(v_{n_k-1}, v_{m_k}) < d(v_{n_k}, v_{n_k-1}) + \varepsilon.$$

From (3.1.3), (3.1.5) and (3.1.6)

$$(3.1.7) \quad \varepsilon \leq \max\{d(u_{n_k}, u_{m_k}), d(v_{n_k}, v_{m_k})\} < \max\{d(u_{n_k}, u_{n_k-1}), d(v_{n_k}, v_{n_k-1})\}.$$

On letting $k \rightarrow \infty$,

$$\varepsilon \leq \lim_{k \rightarrow \infty} \max\{d(u_{n_k}, u_{m_k}), d(v_{n_k}, v_{m_k})\} \leq \varepsilon.$$

Therefore

$$\lim_{k \rightarrow \infty} \max\{d(u_{n_k}, u_{m_k}), d(v_{n_k}, v_{m_k})\} = \varepsilon$$

Therefore (I) holds.

Now we prove II:

$$\begin{aligned} d(u_{n_k-1}, u_{m_k-1}) &\leq d(u_{m_k-1}, u_{m_k}) + d(u_{m_k}, u_{n_k-1}) \\ &< d(u_{m_k-1}, u_{m_k}) + \varepsilon \quad (\text{by (3.1.4)}) \end{aligned}$$

and

$$\begin{aligned} d(v_{n_k-1}, v_{m_k-1}) &\leq d(v_{m_k-1}, v_{m_k}) + d(v_{m_k}, v_{n_k-1}) \\ &< d(v_{m_k-1}, v_{m_k}) + \varepsilon \quad (\text{by (3.1.4)}). \end{aligned}$$

Therefore

$$\begin{aligned} &\max\{d(u_{n_k-1}, u_{m_k-1}), d(v_{n_k-1}, v_{m_k-1})\} \\ &\leq \max\{d(u_{m_k-1}, u_{m_k}), d(v_{m_k-1}, v_{m_k})\} + \varepsilon. \end{aligned}$$

Therefore

$$(3.1.8) \quad \limsup \max\{d(u_{n_k-1}, u_{m_k-1}), d(v_{n_k-1}, v_{m_k-1})\} \leq \varepsilon.$$

Now

$$d(u_{n_k}, u_{m_k}) \leq d(u_{n_k}, u_{n_k-1}) + d(u_{n_k-1}, u_{m_k-1}) + d(u_{m_k-1}, u_{m_k})$$

and

$$d(v_{n_k}, v_{m_k}) \leq d(v_{n_k}, v_{n_k-1}) + d(v_{n_k-1}, v_{m_k-1}) + d(v_{m_k-1}, v_{m_k}).$$

Therefore

$$\begin{aligned} \{d(u_{n_k}, u_{m_k}), d(v_{n_k}, v_{m_k})\} &\leq \max\{d(u_{n_k}, u_{n_k-1}), d(v_{n_k-1}, v_{n_k})\} \\ &\quad + \max\{d(v_{n_k-1}, v_{m_k-1}), d(v_{m_k-1}, u_{m_k-1})\} \\ &\quad + \max\{d(u_{m_k-1}, u_{m_k}), d(v_{m_k-1}, v_{m_k})\}. \end{aligned}$$

Letting $k \rightarrow \infty$, we get

$$\begin{aligned} 0 &\leq \liminf \max\{d(u_{n_k-1}, u_{m_k-1}), d(v_{n_k-1}, v_{m_k-1})\} \\ &\leq \limsup \max\{d(u_{n_k-1}, u_{m_k-1}), d(v_{n_k-1}, v_{m_k-1})\} \leq \varepsilon. \end{aligned}$$

Therefore

$$\lim \max\{d(u_{n_k-1}, u_{m_k-1}), d(v_{n_k-1}, v_{m_k-1})\} = \varepsilon.$$

Therefore (II) holds. Now

$$\begin{aligned} d(u_{n_k}, u_{m_k}) &\leq d(u_{n_k}, u_{n_k-1}) + d(u_{n_k-1}, u_{m_k}) \\ &\leq d(u_{n_k}, u_{n_k-1}) + d(u_{n_k-1}, u_{m_k-1}) + d(u_{m_k-1}, u_{m_k}) \end{aligned}$$

and

$$\begin{aligned} d(v_{n_k}, v_{m_k}) &\leq d(v_{n_k}, v_{n_k-1}) + d(v_{n_k-1}, v_{m_k}) \\ &\leq d(v_{n_k}, v_{n_k-1}) + d(v_{n_k-1}, v_{m_k-1}) + d(v_{m_k-1}, v_{m_k}). \end{aligned}$$

Therefore

$$\begin{aligned} \max\{d(u_{n_k}, u_{m_k}), d(v_{n_k}, v_{m_k})\} &\leq \max\{d(u_{n_k}, u_{n_k-1}), d(v_{n_k}, v_{n_k-1})\} \\ &\quad + \max\{d(u_{n_k-1}, u_{m_k}), d(v_{n_k-1}, v_{m_k})\} \\ &\leq \max\{d(u_{n_k}, u_{n_k-1}), d(v_{n_k}, v_{n_k-1})\} \\ &\quad + \max\{d(u_{n_k-1}, u_{m_k-1}), d(v_{n_k-1}, v_{m_k-1})\} \\ &\quad + \max\{d(u_{m_k-1}, u_{m_k}), d(v_{m_k-1}, v_{m_k})\}. \end{aligned}$$

On letting $k \rightarrow \infty$, from (I), (II), we get (III), since

$$\varepsilon \leq 0 + \lim \max\{d(u_{n_k-1}, u_{m_k}), d(v_{n_k-1}, v_{m_k})\} \leq \varepsilon.$$

Now we have

$$\begin{aligned} \varepsilon &\leq \liminf \max\{d(u_{n_k-1}, u_{m_k-1}), d(v_{n_k-1}, v_{m_k-1})\} \\ &\leq \limsup \max\{d(u_{n_k-1}, u_{m_k-1}), d(v_{n_k-1}, v_{m_k-1})\} = \varepsilon. \end{aligned}$$

Since $u_{n_k-1} \geq u_{m_k-1}$ and $v_{n_k-1} \leq v_{m_k-1}$. From (3.1.1), we get

$$\begin{aligned} (3.1.9) \quad d(u_{n_k}, u_{m_k}) &= d(F(u_{n_k-1}, v_{n_k-1}), F(u_{m_k-1}, v_{m_k-1})) \\ &\leq \varphi\{\max\{d(u_{n_k-1}, u_{m_k-1}), d(v_{n_k-1}, v_{m_k-1})\}\} \end{aligned}$$

similarly

$$\begin{aligned} (3.1.10) \quad d(v_{n_k}, v_{m_k}) &= d(F(v_{n_k-1}, u_{m_k-1}), F(v_{m_k-1}, u_{n_k-1})) \\ &\leq \varphi\{\max\{d(v_{n_k-1}, v_{m_k-1}), d(u_{m_k-1}, u_{n_k-1})\}\}. \end{aligned}$$

From (3.1.9) and (3.1.10), we have

$$(3.1.11) \quad \varepsilon \leq s_k \leq \varphi(p_k) < p_k.$$

Where $s_k = \max\{d(u_{n_k}, u_{m_k}), d(v_{n_k}, v_{m_k})\}$. Therefore $s_k \rightarrow \varepsilon$. and $p_k = \max\{d(u_{n_k-1}, u_{m_k-1}), d(v_{n_k-1}, v_{m_k-1})\}$. On letting $k \rightarrow \infty$, from (3.1.11), we have $\varepsilon \leq \lim \varphi(p_k) \leq \varepsilon$. Therefore $\varepsilon = \lim \varphi(p_k) < \varepsilon$, by (I), a Contradiction. Hence $\{u_n\}$ and $\{v_n\}$ are Cauchy sequences. Suppose $u_n \rightarrow u$ and $v_n \rightarrow v$. Suppose (a) holds. Then F is continuous, hence $u_{n+1} = F(u_n, v_n) \rightarrow F(u, v)$. So $u = F(u, v)$, since $u_{n+1} \rightarrow u$. Similarly $v_{n+1} = F(v_n, u_n) \rightarrow F(v, u)$. So

$v = F(v, u)$, since $v_{n+1} \rightarrow v$. Therefore (u, v) is a coupled fixed point of F . Now suppose (b) holds. Then $u_n \leq u$ and $v_n \geq v$ for all n .

$$\begin{aligned} d(u_{n+1}, F(u, v)) &= d(F(u_n, v_n), F(u, v)) \\ &\leq \varphi\{\max(d(u_n, u), d(v_n, v))\} \\ &\leq \max\{d(u_n, u), d(v_n, v)\} \rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

i.e, $d(u_{n+1}, F(u, v)) \rightarrow 0$ as $n \rightarrow \infty$. i.e, $u_{n+1} \rightarrow F(u, v)$. So $u = F(u, v)$. Similarly $v = F(v, u)$. Therefore (u, v) is a coupled fixed point of F .

Lemma 3.2. *Under the hypothesis of Theorem 3.1, suppose (x, y) is a coupled fixed point of F and (u, v) is comparable to (x, y) . Write $u_0 = u$ and $v_0 = v$ and construct the sequences $\{u_n\}$ and $\{v_n\}$ by $u_{n+1} = F(u_n, v_n)$ and $v_{n+1} = F(v_n, u_n)$ for $n = 0, 1, 2, 3, \dots$. Then $\{u_n\}$ and $\{v_n\}$ are Cauchy sequences and $\{u_n\} \rightarrow x$ and $\{v_n\} \rightarrow y$.*

Proof. Case (i): Suppose $(u, v) \geq (x, y)$, so that

$$(3.2.1) \quad u \geq x \text{ and } v \leq y.$$

Write

$$(3.2.2) \quad u_0 = u \text{ and } v_0 = v,$$

and construct the sequences $\{u_n\}$ and $\{v_n\}$ by

$$(3.2.3) \quad u_{n+1} = F(u_n, v_n) \text{ and } v_{n+1} = F(v_n, u_n).$$

From (3.2.1) and (3.2.2) $u_0 \geq x$ and $v_0 \leq y$ for every n . Now we have to show that $u_n \geq x$ and $v_n \leq y$ for every n . From (3.2.3), $u_1 = F(u_0, v_0) \geq F(x, v_0) \geq F(x, y) = x$. Therefore $u_1 \geq x$. And $v_1 = F(v_0, u_0) \leq F(y, u_0) \leq F(y, x) = y$. Therefore $v_1 \leq y$. Similarly $u_2 \geq x$ and $v_2 \leq y$. Hence by induction $u_n \geq x$ and $v_n \leq y$ for every n .

As in theorem 3.1, we can show that $\{u_n\}$ and $\{v_n\}$ are Cauchy sequences.

Suppose $\{u_n\} \rightarrow a$ and $\{v_n\} \rightarrow b$. Now

$$\begin{aligned} d(u_{n+1}, x) &= d(F(u_n, v_n), F(x, y)) \\ (3.2.4) \quad &\leq \varphi\{\max(d(u_n, x), d(v_n, y))\} \\ &< \max\{d(u_n, x), d(v_n, y)\} \text{ (since } u_n \geq x \text{ and } v_n \leq y) \end{aligned}$$

and

$$\begin{aligned} d(v_{n+1}, y) &= d(F(v_n, u_n), F(y, x)) \\ (3.2.5) \quad &\leq \varphi\{\max(d(v_n, y), d(u_n, x))\} \\ &< \max\{d(v_n, y), d(u_n, x)\} \text{ (since } u_n \geq x \text{ and } v_n \leq y). \end{aligned}$$

Write $A_n = \max\{d(u_n, x), d(v_n, y)\}$. Then by (3.2.4) and (3.2.5)

$$(3.2.6) \quad A_{n+1} \leq \varphi(A_n) < A_n.$$

Write $\lim_{n \rightarrow \infty} A_n = \alpha$. Then by (3.2.6), $\lim \varphi(A_n) = \alpha$. But by hypothesis $\lim \varphi(A_n) = \varphi(\alpha + 0) < \alpha$, if $\alpha > 0$, which is a contradiction. Therefore $\alpha = 0$. Therefore $\lim_{n \rightarrow \infty} A_n = 0$. Therefore

$$(3.2.7) \quad \max\{d(u_{n+1}, x), d(v_{n+1}, y)\} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Therefore $d(u_{n+1}, x) = 0$ and $d(v_{n+1}, y) = 0$ i.e., $d(a, x) = 0$ since $u_{n+1} \rightarrow a$. Therefore $a = x$ and $d(b, y) = 0$ since $v_{n+1} \rightarrow b$. Therefore $b = y$. Therefore $\{u_n\} \rightarrow x$ and $\{v_n\} \rightarrow y$.

Case (ii): Suppose $(u, v) \leq (x, y)$.

Then $u \leq x$ and $v \geq y$. We assume that $(u_0, v_0) = (u, v) \leq (x, y)$. Now $u_1 = F(u_0, v_0) \leq F(x, v_0) \leq F(x, y) = x$. Therefore $u_1 \leq x$ and $v_1 = F(v_0, u_0) \geq F(y, u_0) \geq F(y, x) = y$. Therefore $v_1 \geq y$. Similarly $u_2 = F(u_1, v_1) \leq F(x, v_1) \leq F(x, y) = x$. Therefore $u_2 \leq x$ and $v_2 = F(v_1, u_1) \geq F(y, u_1) \geq F(y, x) = y$. Therefore $v_2 \geq y$. Thus by induction follows that $u_n \leq x$ and $v_n \geq y$ for every n . As in Theorem (3.1), we can show that $\{u_n\}$ and $\{v_n\}$ are Cauchy sequences. Suppose $\{u_n\} \rightarrow a$ and $\{v_n\} \rightarrow b$. Now

$$\begin{aligned} d(u_{n+1}, x) &= d(F(u_n, v_n), F(x, y)) \\ &\leq \varphi\{\max(d(u_n, x), d(v_n, y))\} \\ &< \max\{d(u_n, x), d(v_n, y)\} \text{ (since } u_n \geq x \text{ and } v_n \leq y) \end{aligned}$$

and

$$\begin{aligned} d(v_{n+1}, y) &= d(F(v_n, u_n), F(y, x)) \\ &\leq \varphi\{\max(d(v_n, y), d(u_n, x))\} \\ &< \max\{d(v_n, y), d(u_n, x)\} \text{ (since } u_n \geq x \text{ and } v_n \leq y). \end{aligned}$$

Therefore $\max\{d(u_{n+1}, x), d(v_{n+1}, y)\} \rightarrow 0$ as $n \rightarrow \infty$ (as in (3.2.7)) i.e., $d(u_{n+1}, x) = 0$ and $d(v_{n+1}, y) = 0$ i.e., $d(a, x) = 0$ since $u_{n+1} \rightarrow a$. Therefore $a = x$, and $d(b, y) = 0$ since $v_{n+1} \rightarrow b$. Therefore $b = y$, $\{u_n\} \rightarrow x$ and $\{v_n\} \rightarrow y$. $\square$

Theorem 3.3. *In addition to the hypothesis of Theorem 3.1, suppose that for every $(x, y), (z, t) \in X \times X$, there exists a $(u, v) \in X \times X$, which is comparable to (x, y) and (z, t) . Then F has a unique coupled fixed point.*

Proof. Suppose (x, y) and (z, t) are coupled fixed points of F .

Suppose (u, v) is comparable with (x, y) and (z, t) . Let $\{u_n\}$ and $\{v_n\}$ be as in Lemma 3.2, since (u, v) is comparable with (x, y) , $\{u_n\} \rightarrow x$ and $\{v_n\} \rightarrow y$. Similarly, since (u, v) is comparable with (z, t) , again by Lemma 3.2, $\{u_n\} \rightarrow z$ and $\{v_n\} \rightarrow t$. Therefore $x = z$ and $y = t$. Therefore F has a unique coupled fixed point. $\square$

Theorem 3.4. *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X , such that (X, d) is a complete metric space. Assume there is an increasing function $\varphi : [0, \infty) \rightarrow [0, \infty)$ with $\varphi(t+0) < t$ for each $t > 0$, and also suppose that $F : X \times X \rightarrow X$ and $g : X \rightarrow X$ are such that F has the mixed g -monotone property and*

$$(3.4.1) \quad d(F(x, y), F(u, v)) \leq \varphi\{\max(d(g(x), g(u)), d(g(y), g(v)))\},$$

for all $x, y, u, v \in X$ for which $g(x) \leq g(u)$ and $g(y) \geq g(v)$.

Suppose that $F(X \times X) \subseteq g(X)$, g is continuous and commutes with F and also Suppose either

a) F is continuous (or);

b) X has the following property.

i) if a non decreasing sequence $\{x_n\} \rightarrow x$, then $x_n \leq x$ for all $n \in N$.

ii) if a non increasing sequence $\{y_n\} \rightarrow y$, then $y \leq y_n$ for all $n \in N$.

If there exist $x_0, y_0 \in X$ such that $g(x_0) \leq F(x_0, y_0)$ and $g(y_0) \geq F(y_0, x_0)$, then there exist $x, y \in X$ such that $g(x) = F(x, y)$ and $g(y) = F(y, x)$, that is, F and g have a coupled coincidence point.

Proof. Suppose that $x_0, y_0 \in X$ such that

$$(3.4.2) \quad g(x_0) \leq F(x_0, y_0) \text{ and } g(y_0) \geq F(y_0, x_0).$$

Since

$$(3.4.3) \quad F(X \times X) \subseteq g(X)$$

we can choose $x_1, y_1 \in X$ such that $g(x_1) = F(x_0, y_0)$ and $g(y_1) = F(y_0, x_0)$. Again from (3.4.3), we can choose $x_2, y_2 \in X$ such that $g(x_2) = F(x_1, y_1)$ and $g(y_2) = F(y_1, x_1)$. Continuing this process, inductively we construct the sequences $\{g(x_n)\}$ and $\{g(y_n)\}$ in X such that

$$(3.4.4) \quad g(x_{n+1}) = F(x_n, y_n) \text{ and } g(y_{n+1}) = F(y_n, x_n) \text{ for } n = 0, 1, 2, \dots$$

We show that

$$(3.4.5) \quad g(x_n) \leq g(x_{n+1}) \text{ for } n = 0, 1, 2, \dots$$

and

$$(3.4.6) \quad g(y_n) \geq g(y_{n+1}) \text{ for } n = 0, 1, 2, \dots$$

From (3.4.2), $g(x_0) \leq F(x_0, y_0)$ and $g(y_0) \geq F(y_0, x_0)$. Therefore from (3.4.3), $g(x_1) = F(x_0, y_0)$ and $g(y_1) = F(y_0, x_0)$. Therefore $g(x_0) \leq g(x_1)$ and $g(y_0) \geq g(y_1)$. Thus (3.4.5) and (3.4.6) hold for $n = 0$. Suppose that (3.4.5) and (3.4.6) hold for some $n \geq 0$. Then, since $g(x_n) \leq g(x_{n+1})$ and $g(y_n) \geq g(y_{n+1})$, and as F has the mixed g -monotone property, from (3.4.3),

$$(3.4.7) \quad g(x_{n+1}) = F(x_n, y_n) \leq F(x_{n+1}, y_n), F(y_{n+1}, x_n) \leq F(y_n, x_n) = g(y_{n+1}).$$

Again from (3.4.3),

$$(3.4.8) \quad \begin{aligned} g(x_{n+2}) &= F(x_{n+1}, y_{n+1}) \geq F(x_{n+1}, y_n), \\ F(y_{n+1}, x_n) &\geq F(y_{n+1}, x_{n+1}) = g(y_{n+2}). \end{aligned}$$

From (3.4.7) and (3.4.8), we get $g(x_{n+1}) \leq g(x_{n+2})$ and $g(y_{n+1}) \geq g(y_{n+2})$. Thus by mathematical induction we conclude that (3.4.5) and (3.4.6) hold for all $n \geq 0$. Therefore

$$(3.4.9) \quad g(x_0) \leq g(x_1) \leq g(x_2) \leq \dots g(x_n) \leq g(x_{n+1}) \leq \dots$$

and

$$(3.4.10) \quad g(y_0) \geq g(y_1) \geq g(y_2) \geq \dots g(y_n) \geq g(y_{n+1}) \geq \dots$$

Therefore $\{g(x_n)\}$ is $\uparrow$ and $\{g(y_n)\}$ is $\downarrow$.

Denote $\delta_n = \max\{d((g(x_n), g(x_{n+1}))), d((g(y_n), g(y_{n+1})))\}$. We show that

$$(3.4.11) \quad \delta_n \leq \varphi(\delta_{n-1}).$$

Since $g(x_{n-1}) \leq g(x_n)$ and $g(y_{n-1}) \geq g(y_n)$ from (3.4.3) and (3.4.1), we have

$$(3.4.12) \quad \begin{aligned} d((g(x_n), g(x_{n+1}))) &= d((F(x_{n-1}, y_{n-1}), (F(x_n, y_n))) \\ &\leq \varphi\{\max((d(g(x_{n-1}), g(x_n))), (d(g(y_{n-1}), g(y_n))))\} \\ &= \varphi(\delta_{n-1}). \end{aligned}$$

Similarly

$$(3.4.13) \quad \begin{aligned} d((g(y_n), g(y_{n+1}))) &= d((F(y_{n-1}, x_{n-1}), (F(y_n, x_n))) \\ &\leq \varphi\{\max((d(g(y_{n-1}), g(y_n))), (d(g(x_{n-1}), g(x_n))))\} \\ &= \varphi(\delta_{n-1}). \end{aligned}$$

From (3.4.12) and (3.4.13) we obtain (3.4.11). From (3.4.11), since $\varphi(t) < t$ for $t > 0$, it follows that sequence $\{\delta_n\}$ is monotone decreasing. Therefore there is some $\delta \geq 0$ such that $\lim_{n \rightarrow \infty} \delta_n = \delta$. We show that $\delta = 0$. Suppose to the contrary that $\delta > 0$. Then taking the limit as $n \rightarrow \infty$ on both sides of (3.4.11), we have $\delta = \lim_{n \rightarrow \infty} \delta_n \leq \lim_{n \rightarrow \infty} \varphi(\delta_{n-1}) = \varphi(\delta + 0) < \delta$, a contradiction. Thus $\delta = 0$,

$$(3.4.14) \quad \lim_{n \rightarrow \infty} \{\max((d(g(x_n), g(x_{n+1}))), d(g(y_n), g(y_{n+1})))\} = 0.$$

Now we prove that $\{g(x_n)\}$ and $\{g(y_n)\}$ are Cauchy sequences.

Suppose to the contrary that at least one of $\{g(x_n)\}$ or $\{g(y_n)\}$ is not a Cauchy sequence.

Then there exist an $\epsilon > 0$, and two sub sequences of integers $\{l_k\}$ and $\{m_k\}$, $m_k > l_k \geq k$ with

$$(3.4.15) \quad r_k = \max(d(g(x_{l_k}), g(x_{m_k})), d(g(y_{l_k}), g(y_{m_k}))) > \epsilon \text{ for } k = 1, 2, 3, \dots$$

We may also assume

$$(3.4.16) \quad \max(d(g(x_{l_k}), g(x_{m_k})), d(g(y_{l_k}), g(y_{m_k}))) \leq \epsilon$$

by choosing m_k to be the smallest number exceeding l_k for which (3.4.15) holds. From (3.4.15), (3.4.16) and by the triangular inequality

$$\epsilon < \max\{(d(g(x_{l_k}), g(x_{m_k-1})) + d(g(x_{m_k-1}), g(x_{m_k}))), \\ (d(g(y_{l_k}), g(y_{m_k-1})) + d(g(y_{m_k-1}), g(y_{m_k})))\}.$$

Taking the limit as $k \rightarrow \infty$ we get by (3.4.14)

$$(3.4.17) \quad \lim_{k \rightarrow \infty} r_k = \epsilon^+$$

Since from (3.4.3) and (3.4.1), $g(x_{l_k}) \leq g(x_{m_k})$ and $g(y_{l_k}) \geq g(y_{m_k})$ we have

$$(3.4.18) \quad \begin{aligned} (d(g(x_{l_{k+1}}), g(x_{m_{k+1}}))) &= d(F(x_{l_k}, y_{l_k}), F(x_{m_k}, y_{m_k})) \\ &\leq \varphi\{\max((d(g(x_{l_k}), g(x_{m_k}))), d(g(y_{l_k}), g(y_{m_k})))\} \\ &= \varphi(r_k). \end{aligned}$$

Similarly

$$(3.4.19) \quad \begin{aligned} (d(g(y_{l_{k+1}}), g(y_{m_{k+1}}))) &= d(F(y_{l_k}, x_{l_k}), F(y_{m_k}, x_{m_k})) \\ &\leq \varphi\{\max((d(g(y_{l_k}), g(y_{m_k}))), d(g(x_{l_k}), g(x_{m_k})))\} \\ &= \varphi(r_k). \end{aligned}$$

From (3.4.18) and (3.4.19), $\epsilon < r_{k+1} \leq \varphi(r_k)$. Taking $k \rightarrow \infty$, using (3.4.14) and (3.4.17) we get

$$\epsilon \leq \lim_{k \rightarrow \infty} (\varphi(r_k)) = \varphi(\epsilon + 0) < \epsilon, \text{ a contradiction.}$$

Thus our supposition is wrong. Therefore $\{g(x_n)\}$ and $\{g(y_n)\}$ are Cauchy sequences. Since X is complete, there exist $x, y \in X$ such that

$$(3.4.20) \quad \lim_{n \rightarrow \infty} g(x_n) = x \text{ and } \lim_{n \rightarrow \infty} g(y_n) = y.$$

From (3.4.20) and continuity of g ,

$$(3.4.21) \quad \lim_{n \rightarrow \infty} g(g(x_n)) = g(x) \text{ and } \lim_{n \rightarrow \infty} g(g(y_n)) = g(y).$$

Since F and g commute, from (3.4.21)

$$(3.4.22) \quad g(g(x_{n+1})) = g(F(x_n, y_n)) = F(g(x_n), g(y_n))$$

and

$$(3.4.23) \quad g(g(y_{n+1})) = g(F(y_n, x_n)) = F(g(y_n), g(x_n)).$$

We show that $g(x) = F(x, y)$ and $g(y) = F(y, x)$.

Suppose (a) holds. Taking the limit as $n \rightarrow \infty$ in (3.4.22) and (3.4.23) by (3.4.20), (3.4.21) and continuity of F , we get

$$\begin{aligned} g(x) &= \lim_{n \rightarrow \infty} g(g(x_{n+1})) = \lim_{n \rightarrow \infty} F(g(x_n), g(y_n)) \\ &= F(\lim_{n \rightarrow \infty} g(x_n), \lim_{n \rightarrow \infty} g(y_n)) \\ &= F(x, y) \end{aligned}$$

and

$$\begin{aligned} g(y) &= \lim_{n \rightarrow \infty} g(g(y_{n+1})) = \lim_{n \rightarrow \infty} F(g(y_n), g(x_n)) \\ &= F(\lim_{n \rightarrow \infty} g(y_n), \lim_{n \rightarrow \infty} g(x_n)) \\ &= F(y, x). \end{aligned}$$

Therefore $g(x) = F(x, y)$ and $g(y) = F(y, x)$.

Suppose (b) holds. Since $\{g(x_n)\}$ is non decreasing and $g(x_n) \rightarrow x$ and $\{g(y_n)\}$ is non increasing and $g(y_n) \rightarrow y$ from hypothesis we have $g(x_n) \leq x$ and $g(y_n) \geq y$, for all n . Then by the triangular inequality (3.4.22), (3.4.23) and (3.4.1), we get

$$\begin{aligned} d(g(x), F(x, y)) &\leq d(g(x), g(g(x_{n+1}))) + d(g(g(x_{n+1})), F(x, y)) \\ &= \{d(g(x), g(g(x_{n+1}))) + d(F(g(x_n), g(y_n)), F(x, y))\} \\ &\leq \{d(g(x), g(g(x_{n+1}))) + \\ &\quad \varphi(\max(d(g(g(x_n), g(x))), d(g(g(y_n), g(y))))\} \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Therefore $d(g(x), F(x, y)) \leq 0$. Therefore $g(x) = F(x, y)$. Similarly $g(y) = F(y, x)$. Therefore F and g have a coupled coincidence point. $\square$

Lemma 3.5. Suppose (x, y) is coupled coincidence point of F and g and $(g(u), g(v))$ is comparable with $(g(x), g(y))$. Write $u_0 = u$ and $v_0 = v$ and construct the sequences $\{g(u_n)\}$ and $\{g(v_n)\}$ by $g(u_{n+1}) = F(u_n, v_n)$ and $g(v_{n+1}) = F(v_n, u_n)$ for $n = 0, 1, 2, \dots$. Then $g(u_n) \rightarrow g(x)$ and $g(v_n) \rightarrow g(y)$.

Proof. Case (i): $(g(u), g(v)) \leq (g(x), g(y))$. Then $g(u) \leq g(x)$ and $g(v) \geq g(y)$. Write $u_0 = u$ and $v_0 = v$. Then $(g(u_0), g(v_0)) = (g(u), g(v)) \leq (g(x), g(y))$. Therefore $g(u_0) \leq g(x)$ and $g(v_0) \geq g(y)$. Choose u_1, v_1 such that $g(u_1) = F(u_0, v_0)$ and $g(v_1) = F(v_0, u_0)$. Therefore $g(u_1) = F(u_0, v_0) \leq F(x, v_0) \leq F(x, y) = g(x)$. Therefore $g(u_1) \leq g(x)$ and $g(v_1) = F(v_0, u_0) \geq F(y, u_0) \leq F(y, x) = g(y)$. Therefore $g(v_1) \geq g(y)$. In general $(g(u_n), g(v_n)) \leq (g(x), g(y))$. Now

$$\begin{aligned} d(g(u_n), g(x)) &= d(F(u_{n-1}, v_{n-1}), F(x, y)) \\ &\leq \varphi\{\max(d(g(u_{n-1}), g(x)), d(g(v_{n-1}), g(y)))\} \end{aligned}$$

and

$$\begin{aligned} d(g(v_n), g(y)) &= d(F(v_{n-1}, u_{n-1}), F(y, x)) \\ &\leq \varphi\{\max(d(g(v_{n-1}), g(y)), d(g(u_{n-1}), g(x)))\}. \end{aligned}$$

Let $r_k = \max\{d(g(u_n), g(x)), d(g(v_n), g(y))\} \leq \varphi(r_{k-1}) < r_{k-1}$. Therefore $\{r_k\}$ is a decreasing sequence. Suppose $r_k \downarrow \alpha$. Hence $r_k \leq \varphi(r_{k-1}) < r_{k-1}$. Letting $k \rightarrow \infty$, we get $\lim_{k \rightarrow \infty} \varphi(r_k) = \alpha$. But by hypothesis $\lim \varphi(r_k) = \varphi(\alpha + 0) < \alpha$, if $\alpha > 0$. Therefore $\alpha = 0$. Therefore $\lim_{k \rightarrow \infty} r_k = 0$. Therefore $\max\{d(g(u_n), g(x)), d(g(v_n), g(y))\} \rightarrow 0$ as $n \rightarrow \infty$. Therefore $d(g(u_n), g(x)) \rightarrow 0$ as $n \rightarrow \infty$ and $d(g(v_n), g(y)) \rightarrow 0$ as $n \rightarrow \infty$. Therefore $g(u_n) \rightarrow g(x)$ and $g(v_n) \rightarrow g(y)$.

Case(ii): Suppose $(g(u), g(v)) \geq (g(x), g(y))$. In this case the proof is similar to case (i). $\square$

Theorem 3.6. *In addition to the hypothesis of Theorem 3.4, suppose $(x, y), (z, t) \in X \times X \Rightarrow$ there exist $u, v \in X \times X$ such that $(g(u), g(v))$ is comparable with $(g(x), g(y))$ and $(g(z), g(t))$. Then F and g have unique coupled coincidence point, in the sense that $g(x) = g(z)$ and $g(y) = g(t)$.*

Proof. Suppose (x, y) and (z, t) are coupled coincidence points of F and g .

Suppose $(g(u), g(v))$ is comparable with $(g(x), g(y))$ and $(g(z), g(t))$.

Let $\{u_n\}$ and $\{v_n\}$ be in Lemma 3.5, $g(u_n) \rightarrow g(x)$ and $g(v_n) \rightarrow g(y)$.

Similarly, since $(g(u), g(v))$ is comparable with $(g(z), g(t))$, again by Lemma 3.5, $g(u_n) \rightarrow g(z)$ and $g(v_n) \rightarrow g(t)$. Therefore $g(x) = g(z)$ and $g(y) = g(t)$. Now $F(x, y) = g(x) = g(z) = F(z, t)$ and $F(y, x) = g(y) = g(t) = F(t, z)$. $\square$

Note: In view of the comment made at the beginning of this section, Theorem 2.3, 2.4, 2.8, 2.10 and 2.11 follow as corollaries to Theorems 3.1, 3.3, 3.4 and 3.6 of this section.

Corollary 3.7. *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X , such that (X, d) is a complete metric space. Suppose that $F : X \times X \rightarrow X$ and $g : X \rightarrow X$ are such that F has the mixed g -monotone property and assume there is a $k \in [0, 1)$ such that*

$$d(F(x, y), F(u, v)) \leq k\{\max(d(g(x), g(u)), d(g(y), g(v)))\},$$

for all $x, y, u, v \in X$ for which $g(x) \leq g(u)$ and $g(y) \geq g(v)$.

Suppose that $F(X \times X) \subseteq g(X)$, g is continuous and also Suppose either

(a) F is Continuous (or);

(b) X has the following property.

i) if a non decreasing sequence $\{x_n\} \rightarrow x$, then $x_n \leq x$ for all $n \in \mathbb{N}$.

ii) if a non increasing sequence $\{y_n\} \rightarrow y$, then $y \leq y_n$ for all $n \in \mathbb{N}$.

If there exist $x_0, y_0 \in X$ such that $g(x_0) \leq F(x_0, y_0)$ and $g(y_0) \geq F(y_0, x_0)$, then there exist $x, y \in X$ such that $g(x) = F(x, y)$ and $g(y) = F(y, x)$, that is, F and g have a coincidence point.

Proof. Taking $\varphi(t) = kt$ where $k \in [0, 1)$ in Theorem 3.4, we obtain Corollary 3.7. $\square$

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ON THE k -NORMAL ELEMENTS AND POLYNOMIALS OVER FINITE FIELDS

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Abstract. An element $\alpha \in \mathbb{F}_{q^n}$ is normal over $\mathbb{F}_q$ if the set $\{\alpha, \alpha^q, \dots, \alpha^{q^{n-1}}\}$ is a basis of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$. The k -normal elements over finite fields are defined and characterized by Huczynska, Mullen, Panario and Thomson (2013). For $0 \leq k \leq n-1$, the element $\alpha \in \mathbb{F}_{q^n}$ is said to be a k -normal element if $\gcd(x^n - 1, \sum_{i=0}^{n-1} \alpha^{q^i} x^{n-1-i})$ has degree k . It is well known that a 0-normal element is a normal element. So, the k -normal elements are a generalization of normal elements. By analogy with the case of normal polynomials, a monic irreducible polynomial of degree n is called a k -normal polynomial if its roots are k -normal elements of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$. In this paper, a new characterization and construction of k -normal elements and polynomials over finite fields are given.

Keywords: finite field, normal basis, k -normal element, k -normal polynomial.

1. Introduction

Let $\mathbb{F}_q$ be the Galois field of order $q = p^m$, where p is a prime and m is a natural number, and $\mathbb{F}_q^*$ be its multiplicative group. A *normal basis* of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$ is a basis of the form $N = \{\alpha, \alpha^q, \dots, \alpha^{q^{n-1}}\}$, i.e. a basis that consists of the algebraic conjugates of a fixed element $\alpha \in \mathbb{F}_{q^n}^*$. Such an element $\alpha \in \mathbb{F}_{q^n}$ is

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said to generate a normal basis for $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$, and for convenience called a *normal element*.

A monic irreducible polynomial $F(x) \in \mathbb{F}_q[x]$ is called *normal polynomial* or *N-polynomial* if its roots are linearly independent over $\mathbb{F}_q$. Since the elements in a normal basis are exactly the roots of some N -polynomials, there is a canonical one-to-one correspondence between N -polynomials and normal bases. Normal bases have many applications, including coding theory, cryptography and computer algebra systems. For further details, see [9].

Recently, the k -normal elements over finite fields are defined and characterized by Huczynska et al [8]. For $0 \leq k \leq n-1$, the element $\alpha \in \mathbb{F}_{q^n}$ is called a k -normal element if $\deg(\gcd(x^n - 1, \sum_{i=0}^{n-1} \alpha^{q^i} x^{n-1-i})) = k$.

By analogy with the case of normal polynomials, a monic irreducible polynomial $P(x) \in \mathbb{F}_q[x]$ of degree n is called a k -normal polynomial (or N_k -polynomial) over $\mathbb{F}_q$ if its roots are k -normal elements of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$. Here, $P(x)$ has n distinct conjugate roots, of which $(n-k)$ are linearly independent. Recall that an element $\alpha \in \mathbb{F}_{q^n}$ is called a proper element of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$ if $\alpha \notin \mathbb{F}_{q^v}$ for any proper divisor v of n . So, the element $\alpha \in \mathbb{F}_{q^n}$ is a proper k -normal element of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$ if α is a k -normal and proper element of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$.

Using the above mention, a normal polynomial (or element) is a 0-normal polynomial (or element). Since the proper k -normal elements of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$ are the roots of a k -normal polynomial of degree n over $\mathbb{F}_q$, hence the k -normal polynomials of degree n over $\mathbb{F}_q$ is just another way of describing the proper k -normal elements of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$. Some results on the constructions of special sequences of k -normal polynomials over $\mathbb{F}_q$, in the cases $k = 0$ and 1 can be found in [2, 4, 5, 10, 11] and [6], respectively.

In this paper, in Sec. 2 some definitions, notes and results which are useful for our study have been stated. Section 3 is devoted to characterization and construction of k -normal elements. Finally, in Sec. 4 a recursive method for constructing k -normal polynomials of higher degree from a given k -normal polynomial is given.

2. Preliminary notes

We use the definitions, notations and results given by Huczynska [8], Gao [7] and Kyuregyan [10, 11], where similar problems are considered. We need the following results for our further study.

The trace of α in $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$, is given by $Tr_{\mathbb{F}_{q^n}|\mathbb{F}_q}(\alpha) = \sum_{i=0}^{n-1} \alpha^{q^i}$. For convenience, $Tr_{\mathbb{F}_{q^n}|\mathbb{F}_q}$ is denoted by $Tr_{q^n|q}$.

Let $\mathbb{F}$ be a field and $f(x) = \sum_{i=0}^n f_i x^i$ and $g(x) = \sum_{j=0}^m g_j x^j$ with all $f_i, g_j \in \mathbb{F}$. The Sylvester matrix $S_{f,g}$ is the $(m+n) \times (m+n)$ matrix given by:

$$(1) \quad S_{f,g} = \begin{pmatrix} f_n & f_{n-1} & \dots & f_1 & f_0 & \dots & \dots \\ 0 & f_n & \dots & \dots & \dots & f_0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & f_n & \dots & \dots & \dots & f_0 \\ g_m & g_{m-1} & \dots & g_1 & g_0 & \dots & \dots \\ 0 & g_m & g_{m-1} & \dots & \dots & g_0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & g_m & \dots & \dots & \dots & g_0 \end{pmatrix}$$

Proposition 2.1 ([8]). *Let $\mathbb{F}$ be a field. For two non-zero polynomials $f, g \in \mathbb{F}[x]$, we have*

$$\text{rank}(S_{f,g}) = \deg(f) + \deg(g) - \deg(\gcd(f, g)).$$

Proposition 2.2 ([8]). *Let $\alpha \in \mathbb{F}_{q^n}$. Then the following properties are equivalent:*

- i) α is k -normal over $\mathbb{F}_q$;
- ii) α gives rise to a basis $\{\alpha, \alpha^q, \dots, \alpha^{q^{n-k-1}}\}$ of a q -modules of degree $n - k$ over $\mathbb{F}_q$;
- iii) $\text{rank}(A_\alpha) = n - k$, where

$$A_\alpha = \begin{pmatrix} \alpha & \alpha^q & \dots & \alpha^{q^{n-1}} \\ \alpha^q & \alpha^{q^2} & \dots & \alpha \\ \vdots & \vdots & \vdots & \vdots \\ \alpha^{q^{n-1}} & \alpha & \dots & \alpha^{q^{n-2}} \end{pmatrix}.$$

Proposition 2.3 ([6]). *Let p divide n , then $n = n_1 p^e$, for some $e \geq 1$ and $a, b \in \mathbb{F}_q^*$. Theefore the element α is a proper k -normal element of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$ if and only if $a + b\alpha$ is a proper k -normal element of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$.*

Let p denote the characteristic of $\mathbb{F}_q$ and let $n = n_1 p^e = n_1 t$, with $\gcd(p, n_1) = 1$ and suppose that $x^n - 1$ has the following factorization in $\mathbb{F}_q[x]$:

$$(2) \quad x^n - 1 = (\varphi_1(x)\varphi_2(x) \cdots \varphi_r(x))^t,$$

where $\varphi_i(x) \in \mathbb{F}_q[x]$ are the distinct irreducible factors of $x^n - 1$. For each s , $0 \leq s < n$, let there is a $u_s > 0$ such that $R_{s,1}(x), R_{s,2}(x), \dots, R_{s,u_s}(x)$ are all of the s degree polynomials dividing $x^n - 1$. So, from (2) we can write $R_{s,i}(x) = \prod_{j=1}^r \varphi_j^{t_{ij}}(x)$, for each $1 \leq i \leq u_s$, $0 \leq t_{ij} \leq t$. Let

$$(3) \quad \phi_{s,i}(x) = \frac{x^n - 1}{R_{s,i}(x)},$$

for $1 \leq i \leq u_s$. Then, there is a useful characterization of the k -normal polynomials of degree n over $\mathbb{F}_q$ as follows:

Proposition 2.4 ([6]). *Let $F(x)$ be an irreducible polynomial of degree n over $\mathbb{F}_q$ and α be a root of it. Let $x^n - 1$ factor as (2) and let $\phi_{s,i}(x)$ be as in (3). Then $F(x)$ is a N_k -polynomial over $\mathbb{F}_q$ if and only if, there is j , $1 \leq j \leq u_k$, such that*

$$L_{\phi_{k,j}}(\alpha) = 0,$$

and also

$$L_{\phi_{s,i}}(\alpha) \neq 0,$$

for each s , $k < s < n$, and $1 \leq i \leq u_s$, where u_s is the number of all s degree polynomials dividing $x^n - 1$ and $L_{\phi_{s,i}}(x)$ is the linearized polynomial defined by

$$L_{\phi_{s,i}}(x) = \sum_{v=0}^{n-s} t_{iv} x^{q^v} \text{ if } \phi_{s,i}(x) = \sum_{v=0}^{n-s} t_{iv} x^v.$$

The following propositions and lemma are useful for constructing N_k -polynomials over $\mathbb{F}_q$.

Proposition 2.5 ([3]). *Let $x^p - \delta_2 x + \delta_0$ and $x^p - \delta_2 x + \delta_1$ be relatively prime polynomials in $\mathbb{F}_q[x]$ and $P(x) = \sum_{i=0}^n c_i x^i$ be an irreducible polynomial of degree $n \geq 2$ over $\mathbb{F}_q$, and let $\delta_0, \delta_1 \in \mathbb{F}_q$, $\delta_2 \in \mathbb{F}_q^*$, $(\delta_0, \delta_1) \neq (0, 0)$. Then*

$$F(x) = (x^p - \delta_2 x + \delta_1)^n P\left(\frac{x^p - \delta_2 x + \delta_0}{x^p - \delta_2 x + \delta_1}\right)$$

is an irreducible polynomial of degree np over $\mathbb{F}_q$ if and only if $\delta_2^{\frac{q-1}{p-1}} = 1$ and

$$\text{Tr}_{q|p}\left(\frac{1}{A^p}\left((\delta_1 - \delta_0)\frac{P'(1)}{P(1)} - n\delta_1\right)\right) \neq 0,$$

where $A^{p-1} = \delta_2$, for some $A \in \mathbb{F}_q^*$.

Proposition 2.6 ([1]). *Let $x^p - x + \delta_0$ and $x^p - x + \delta_1$ be relatively prime polynomials in $\mathbb{F}_q[x]$ and let $P(x)$ be an irreducible polynomial of degree $n \geq 2$ over $\mathbb{F}_q$, and $0 \neq \delta_1, \delta_0 \in \mathbb{F}_p$, such that $\delta_0 \neq \delta_1$. Define*

$$F_0(x) = P(x)$$

$$F_k(x) = (x^p - x + \delta_1)^{t_{k-1}} F_{k-1}\left(\frac{x^p - x + \delta_0}{x^p - x + \delta_1}\right), \quad k \geq 1$$

where $t_k = np^k$ denotes the degree of $F_k(x)$. Suppose that

$$\text{Tr}_{q|p}\left(\frac{(\delta_1 - \delta_0)F'_0(1) - n\delta_1 F_0(1)}{F_0(1)}\right) \cdot \text{Tr}_{q|p}\left(\frac{(\delta_1 - \delta_0)F'_0\left(\frac{\delta_0}{\delta_1}\right) + n\delta_1 F_0\left(\frac{\delta_0}{\delta_1}\right)}{F_0\left(\frac{\delta_0}{\delta_1}\right)}\right) \neq 0.$$

Then $(F_k(x))_{k \geq 0}$ is a sequence of irreducible polynomials over $\mathbb{F}_q$ of degree $t_k = np^k$, for every $k \geq 0$.

Lemma 2.7. *Let γ be a proper element of $\mathbb{F}_{q^n}$ and $\theta \in \mathbb{F}_p^*$, where $q = p^m$, ($m \in \mathbb{N}$). Then we have*

$$(4) \quad \sum_{j=0}^{p-1} \frac{1}{\gamma + j\theta} = -\frac{1}{\gamma^p - \gamma}.$$

Proof. By observing that

$$\sum_{j=0}^{p-1} \frac{1}{\gamma + j\theta} = \frac{1}{\gamma^p - \gamma} \left(\sum_{j=0}^{p-1} \frac{\gamma^p - \gamma}{\gamma + j\theta} \right),$$

it is enough to show that

$$\sum_{j=0}^{p-1} \frac{\gamma^p - \gamma}{\gamma + j\theta} = -1.$$

We note that

$$(5) \quad \begin{aligned} \sum_{j=0}^{p-1} \frac{\gamma^p - \gamma}{\gamma + j\theta} &= \sum_{j=0}^{p-1} \left((\gamma + j\theta)^{p-1} - 1 \right) \\ &= \sum_{j=0}^{p-1} (\gamma + j\theta)^{p-1} \\ &= \sum_{j=1}^{p-1} \theta^j \binom{p-1}{j} \left(\sum_{i=1}^{p-1} i^j \right), \end{aligned}$$

where

$$\binom{p-1}{j} = \frac{(p-1)!}{(p-1-j)!j!}, \quad j \in \mathbb{F}_p^*.$$

On the other side, we know that

$$(6) \quad \sum_{i=1}^{p-1} i^j = \begin{cases} 0 \pmod{p}, & \text{if } p-1 \nmid j \\ -1 \pmod{p}, & \text{if } p-1 \mid j \end{cases}$$

and also $\theta^{p-1} = 1$. Thus by (5) and (6), the proof is completed. $\square$

3. Characterization and construction of k -normal elements

In this section, we extend some existence results on the characterization and construction of normal elements into k -normal elements over finite fields. In the case $k = 0$, the following theorems had been obtained in [7] and [13].

Theorem 3.1. *Suppose that α is a proper element of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$. Let $\alpha_i = \alpha^{q^i}$ and $t_i = \text{Tr}_{q^n|q}(\alpha_0 \alpha_i)$, $0 \leq i \leq n-1$. Then α is a k -normal element of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$ if and only if $\deg(\gcd(g(x), x^n - 1)) = k$, where $g(x) = \sum_{i=0}^{n-1} t_i x^i$.*

Proof. Let

$$A_\alpha = \begin{pmatrix} \alpha & \alpha^q & \dots & \alpha^{q^{n-1}} \\ \alpha^q & \alpha^{q^2} & \dots & \alpha \\ \vdots & \vdots & \ddots & \vdots \\ \alpha^{q^{n-1}} & \alpha & \dots & \alpha^{q^{n-2}} \end{pmatrix}.$$

So, by setting

$$\begin{aligned} \Delta = A_\alpha A_\alpha^T &= \begin{pmatrix} \text{Tr}_{q^n|q}(\alpha_0 \alpha_0) & \text{Tr}_{q^n|q}(\alpha_0 \alpha_1) & \dots & \text{Tr}_{q^n|q}(\alpha_0 \alpha_{n-1}) \\ \text{Tr}_{q^n|q}(\alpha_1 \alpha_0) & \text{Tr}_{q^n|q}(\alpha_1 \alpha_1) & \dots & \text{Tr}_{q^n|q}(\alpha_1 \alpha_{n-1}) \\ \vdots & \vdots & \ddots & \vdots \\ \text{Tr}_{q^n|q}(\alpha_{n-1} \alpha_0) & \text{Tr}_{q^n|q}(\alpha_{n-1} \alpha_1) & \dots & \text{Tr}_{q^n|q}(\alpha_{n-1} \alpha_{n-1}) \end{pmatrix} \\ &= \begin{pmatrix} t_0 & t_1 & \dots & t_{n-1} \\ t_{n-1} & t_0 & \dots & t_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ t_1 & t_2 & \dots & t_0 \end{pmatrix}, \end{aligned}$$

we get

$$\text{rank}(A_\alpha A_\alpha^T) = \text{rank}(A_\alpha) = \text{rank}(\Delta).$$

Now, it is enough to show that $\deg(\gcd(\sum_{i=0}^{n-1} t_i x^i, x^n - 1)) = k$ if and only if the matrix Δ has rank $n - k$. The Sylvester matrix $S_{f,g}$ (see Equation 1) with $f(x) = x^n - 1$ can be converted, by a sequence of column operations, into the block matrix

$$\begin{pmatrix} I_{n-1} & 0_{n-1} \\ 0_{n-1} & \Delta \end{pmatrix}.$$

From this block decomposition, it follows that

$$\text{rank}(S_{f,g}) = \text{rank}(\Delta) + \text{rank}(I_{n-1}) = \text{rank}(\Delta) + (n - 1).$$

By Proposition 2.1,

$$\text{rank}(S_{f,g}) = n + (n - 1) - \deg(\gcd(f(x), g(x))).$$

Combining these two expressions yields

$$\deg(\gcd(f(x), g(x))) = n - \text{rank}(\Delta).$$

The proof is complete. $\square$

Theorem 3.2. *Let α be a k -normal element of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$. Then the element $\gamma = \sum_{i=0}^{n-1} a_i \alpha^{q^i}$, where $a_i \in \mathbb{F}_q$, is a k -normal element of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$ if and only if the polynomial $\gamma(x) = \sum_{i=1}^{n-1} a_i x^i$ is relatively prime to $x^n - 1$.*

Proof. Since α is a k -normal element of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$, so by Proposition 2.2, $\text{rank}(A_\alpha) = n - k$, where

$$A_\alpha = \begin{pmatrix} \alpha & \alpha^q & \dots & \alpha^{q^{n-1}} \\ \alpha^q & \alpha^{q^2} & \dots & \alpha \\ \vdots & \vdots & \ddots & \vdots \\ \alpha^{q^{n-1}} & \alpha & \dots & \alpha^{q^{n-2}} \end{pmatrix}.$$

Let

$$A_\gamma = \begin{pmatrix} \gamma & \gamma^q & \dots & \gamma^{q^{n-1}} \\ \gamma^q & \gamma^{q^2} & \dots & \gamma \\ \vdots & \vdots & \ddots & \vdots \\ \gamma^{q^{n-1}} & \gamma & \dots & \gamma^{q^{n-2}} \end{pmatrix}.$$

By Proposition 2.2, it is enough to show that $\text{rank}(A_\gamma) = n - k$. We note that $A_\gamma = A \cdot A_\alpha$, where

$$A = \begin{pmatrix} a_0 & a_1 & \dots & a_{n-1} \\ a_1 & a_2 & \dots & a_0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_0 & \dots & a_{n-2} \end{pmatrix}.$$

Since $\gamma(x) = \sum_{i=0}^{n-1} a_i x^i$ is relatively prime to $x^n - 1$, thus A is non-singular and so

$$\text{rank}(A_\gamma) = \text{rank}(A \cdot A_\alpha) = \text{rank}(A_\alpha) = n - k.$$

The proof is complete. $\square$

Theorem 3.3. *Let t and v are two positive integers with $1 < t < v < 2t$ and α is a k -normal element of $\mathbb{F}_{q^{vt}}$ over $\mathbb{F}_q$ for $v - t \leq k \leq t - 1$. If $\gamma = \text{Tr}_{q^{vt}|q^t}(\alpha)$ is a proper element of $\mathbb{F}_{q^t}$ over $\mathbb{F}_q$, then γ is a proper k -normal element of $\mathbb{F}_{q^t}$ over $\mathbb{F}_q$.*

Proof. Since α is a k -normal element of $\mathbb{F}_{q^{vt}}$ over $\mathbb{F}_q$, so by Proposition 2.2 the elements $\alpha, \alpha^q, \alpha^{q^2}, \dots, \alpha^{q^{vt-k-1}}$ form a basis for a q -module of degree $vt - k$ over $\mathbb{F}_q$. By hypothesis and considering $\gamma = \text{Tr}_{q^{vt}|q^t}(\alpha)$, the elements $\gamma, \gamma^q, \dots, \gamma^{q^{v-k-1}}$ are non-overlapping sums of the $vt - k$ conjugates of α , which are assumed to be linearly independent over $\mathbb{F}_q$. So the $v - k$ conjugates of γ are linearly independent over $\mathbb{F}_q$. On the other side, for each $0 \leq s \leq k - 1$,

$$\begin{aligned} \gamma^{q^{v-k+s}} &= \sum_{i=1}^v \alpha^{q^{vt-k+(v+s-it)}} \\ &= \sum_{i=1}^{vt-k} c_i \alpha^{q^{vt-k-i}}, \quad c_i \in \mathbb{F}_q \\ &= \sum_{j=1}^{v-k} d_j \gamma^{q^{v-k-j}}, \quad d_j \in \mathbb{F}_{q^t}. \end{aligned}$$

So γ gives rise to a basis $M = \{\gamma, \gamma^q, \dots, \gamma^{q^{v-k-1}}\}$ of a q -modules of degree $v - k$ over $\mathbb{F}_q$. By Proposition 2.2, the proof is complete. $\square$

Theorem 3.4. *Let t and v are two positive integers with $\gcd(v, t) = 1$ and α is a k -normal element of $\mathbb{F}_{q^v}$ over $\mathbb{F}_q$, for $0 \leq k \leq v - 1$. Then α is also a k -normal element of $\mathbb{F}_{q^{vt}}$ over $\mathbb{F}_{q^t}$.*

Proof. Since α is a k -normal element of $\mathbb{F}_{q^v}$ over $\mathbb{F}_q$, so by Proposition 2.2, $\text{rank}(A_\alpha) = v - k$, where

$$A_\alpha = \begin{pmatrix} \alpha & \alpha^q & \dots & \alpha^{q^{v-1}} \\ \alpha^q & \alpha^{q^2} & \dots & \alpha \\ \vdots & \vdots & \ddots & \vdots \\ \alpha^{q^{v-1}} & \alpha & \dots & \alpha^{q^{v-2}} \end{pmatrix}.$$

The element α is also a k -normal element of $\mathbb{F}_{q^{vt}}$ over $\mathbb{F}_{q^t}$ if $\text{rank}(A'_\alpha) = v - k$, where

$$A'_\alpha = \begin{pmatrix} \alpha & \alpha^{q^t} & \dots & \alpha^{q^{(v-1)t}} \\ \alpha^{q^t} & \alpha^{q^{2t}} & \dots & \alpha \\ \vdots & \vdots & \ddots & \vdots \\ \alpha^{q^{(v-1)t}} & \alpha & \dots & \alpha^{q^{(v-2)t}} \end{pmatrix}.$$

Since $\gcd(v, t) = 1$, when j runs through $0, 1, 2, \dots, v - 1$ modulo v , tj also runs through $0, 1, 2, \dots, v - 1$ modulo v . Note that since $\alpha \in \mathbb{F}_{q^v}$, we have $\alpha^{q^v} = \alpha$ and thus $\alpha^{q^{jt}} = \alpha^{q^{k_j}}$ whenever $jt \equiv k_j \pmod{v}$ and k_j runs through $0, 1, 2, \dots, v - 1$. So $\text{rank}(A'_\alpha) = \text{rank}(A_\alpha) = v - k$ and the proof is complete. $\square$

4. Recursive construction N_k -polynomials

In this section we establish theorems which will show how propositions 2.4, 2.5 and 2.6 can be applied to produce infinite sequences of N_k -polynomials over $\mathbb{F}_q$. Recall that, the polynomial $P^*(x) = x^n P\left(\frac{1}{x}\right)$ is called the reciprocal polynomial of $P(x)$, where n is the degree of $P(x)$. In the case $k = 0$, some similar results of the following theorems have been obtained in [2], [4] and ([5], Theorems 3.3.1 and 3.4.1). We use of an analogous technique to that used in the above results, where similar problems are considered.

Theorem 4.1. *Let $P(x) = \sum_{i=0}^n c_i x^i$ be an N_k -polynomial of degree n over $\mathbb{F}_q$, for each $n = rp^e$, where $e \in \mathbb{N}$ and r equals 1 or is a prime different from p and q a primitive element modulo r . Suppose that $\delta \in \mathbb{F}_q^*$ and*

$$(7) \quad F(x) = (x^p - x + \delta)^n P^* \left(\frac{x^p - x}{x^p - x + \delta} \right).$$

Then $F^*(x)$ is an N_k -polynomial of degree np over $\mathbb{F}_q$ if $k < p^e$ and

$$\text{Tr}_{q|p} \left(\delta \frac{P^{*'}(1)}{P^*(1)} \right) \neq 0.$$

Proof. Since $P^*(x)$ is an irreducible polynomial over $\mathbb{F}_q$, so Proposition 2.5 and theorem's hypothesis imply that $F(x)$ is irreducible over $\mathbb{F}_q$.

Let $\alpha \in \mathbb{F}_{q^n}$ be a root of $P(x)$. Since $P(x)$ is an N_k -polynomial of degree n over $\mathbb{F}_q$ by theorem's hypothesis, then $\alpha \in \mathbb{F}_{q^n}$ is a proper k -normal element over $\mathbb{F}_q$.

Since q is a primitive modulo r , so in the case $r > 1$ the polynomial $x^{r-1} + \cdots + x + 1$ is irreducible over $\mathbb{F}_q$. Thus $x^n - 1$ has the following factorization in $\mathbb{F}_q[x]$:

$$(8) \quad x^n - 1 = (\varphi_1(x) \cdot \varphi_2(x))^t,$$

where $\varphi_1(x) = x - 1$, $\varphi_2(x) = x^{r-1} + \cdots + x + 1$ and $t = p^e$.

Letting that for each $0 \leq s < n$ and $1 \leq i \leq u_s$, $R_{s,i}(x)$ is the s degree polynomial dividing $x^n - 1$, where u_s is the number of all s degree polynomials dividing $x^n - 1$. So, from (8), we can write $R_{s,i}(x) = (x - 1)^{s_{1,i}} \cdot (x^{r-1} + \cdots + x + 1)^{s_{2,i}}$, where $s = s_{1,i} + s_{2,i} \cdot (r - 1)$ for each $0 \leq s_{1,i}, s_{2,i} \leq t$, except when $s_{1,i} = s_{2,i} = t$. So, we have

$$(9) \quad \phi_{s,i}(x) = \frac{x^n - 1}{R_{s,i}(x)} = \frac{x^n - 1}{(x - 1)^{s_{1,i}} \cdot (x^{r-1} + \cdots + x + 1)^{s_{2,i}}} = \sum_{v=0}^{n-s} t_{s,i,v} x^v.$$

Since $P(x)$ is an N_k -polynomial of degree n over $\mathbb{F}_q$, so by Proposition 2.4, there is a j , $1 \leq j \leq u_k$, such that

$$L_{\phi_{k,j}}(\alpha) = 0,$$

and also

$$L_{\phi_{s,i}}(\alpha) \neq 0,$$

for each $k < s < n$ and $1 \leq i \leq u_s$. Further, we proceed by proving that $F^*(x)$ is a k -normal polynomial. Let α_1 be a root of $F(x)$. Then $\beta_1 = \frac{1}{\alpha_1}$ is a root of its reciprocal polynomial $F^*(x)$. Note that by (8), the polynomial $x^{np} - 1$ has the following factorization in $\mathbb{F}_q[x]$:

$$(10) \quad x^{np} - 1 = (\varphi_1(x) \cdot \varphi_2(x))^{pt},$$

where $\varphi_1(x) = x - 1$, $\varphi_2(x) = x^{r-1} + \cdots + x + 1$ and $t = p^e$.

Letting that for each $0 \leq s' < np$ and $1 \leq i' \leq u'_{s'}$, $R'_{s',i'}(x)$ is the s' degree polynomial dividing $x^{np} - 1$, where $u'_{s'}$ is the number of all s' degree polynomials dividing $x^{np} - 1$. So, from (10) we can write $R'_{s',i'}(x) = (x - 1)^{s'_{1,i'}} \cdot (x^{r-1} + \cdots + x + 1)^{s'_{2,i'}}$, where $s' = s'_{1,i'} + s'_{2,i'} \cdot (r - 1)$ for each $0 \leq s'_{1,i'}, s'_{2,i'} \leq pt$, except when $s'_{1,i'} = s'_{2,i'} = pt$. Therefore by considering

$$(11) \quad H'_{s',i'}(x) = \frac{x^{np} - 1}{R'_{s',i'}(x)},$$

and Proposition 2.4, $F^*(x)$ is an N_k -polynomial of degree np over $\mathbb{F}_q$ if and only if there is a j' , $1 \leq j' \leq u'_k$, such that

$$L_{H'_{k,j'}}(\beta_1) = 0,$$

and also

$$L_{H'_{s',i'}}(\beta_1) \neq 0,$$

for each $k < s' < np$ and $1 \leq i' \leq u'_{s'}$. Consider

$$(12) \quad H_{s,i}(x) = \frac{x^{np} - 1}{R_{s,i}(x)} = \frac{x^n - 1}{R_{s,i}(x)} \left(\sum_{j=0}^{p-1} x^{jn} \right),$$

for each $0 \leq s < n$ and $1 \leq i \leq u_s$. By (9) we obtain

$$H_{s,i}(x) = \phi_{s,i}(x) \left(\sum_{j=0}^{p-1} x^{jn} \right) = \sum_{v=0}^{n-s} t_{s,i,v} \left(\sum_{j=0}^{p-1} x^{jn+v} \right).$$

It follows that

$$L_{H_{s,i}}(\beta_1) = \sum_{v=0}^{n-s} t_{s,i,v} \left(\sum_{j=0}^{p-1} (\beta_1)^{p^{jmn}} \right)^{p^{mv}},$$

or

$$(13) \quad L_{H_{s,i}}(\beta_1) = L_{H_{s,i}} \left(\frac{1}{\alpha_1} \right) = \sum_{v=0}^{n-s} t_{s,i,v} \left(\sum_{j=0}^{p-1} \left(\frac{1}{\alpha_1} \right)^{p^{jmn}} \right)^{p^{mv}}.$$

From (7), if α_1 is a zero of $F(x)$, then $\frac{\alpha_1^p - \alpha_1 + \delta}{\alpha_1^p - \alpha_1}$ is a zero of $P(x)$, and therefore it may assume that

$$\alpha = \frac{\alpha_1^p - \alpha_1 + \delta}{\alpha_1^p - \alpha_1},$$

or

$$(14) \quad \frac{\alpha - 1}{\delta} = (\alpha_1^p - \alpha_1)^{-1}.$$

Now, by (14) and observing that $P(x)$ is an irreducible polynomial of degree n over $\mathbb{F}_q$, we obtain

$$(15) \quad \frac{\alpha - 1}{\delta} = \left(\frac{\alpha - 1}{\delta} \right)^{p^{mn}} = (\alpha_1^{p^{mn+1}} - \alpha_1^{p^{mn}})^{-1}.$$

It follows from (14) and (15) that

$$(16) \quad (\alpha_1^{p^{mn+1}} - \alpha_1^{p^{mn}})^{-1} = (\alpha_1^p - \alpha_1)^{-1}.$$

Also observing that $F(x)$ is an irreducible polynomial of degree np over $\mathbb{F}_q$, we have $(\alpha_1^p - \alpha_1) \neq 0$ and $(\alpha_1^{p^{mn+1}} - \alpha_1^{p^{mn}}) \neq 0$. Hence by (16)

$$(17) \quad (\alpha_1^{p^{mn}} - \alpha_1)^p = (\alpha_1^{p^{mn}} - \alpha_1).$$

It follows from (17) that $\alpha_1^{p^{mn}} - \alpha_1 = \theta \in \mathbb{F}_p^*$. Hence $\alpha_1^{p^{mn}} = \alpha_1 + \theta$ and

$$\alpha_1^{p^{2mn}} = (\alpha_1 + \theta)^{p^{mn}} = \alpha_1^{p^{mn}} + \theta^{p^{mn}} = \alpha_1 + \theta + \theta = \alpha_1 + 2\theta.$$

It is easy to show that $\alpha_1^{p^{jmn}} = \alpha_1 + j\theta$, for $1 \leq j \leq p-1$, or

$$(18) \quad \left(\frac{1}{\alpha_1}\right)^{p^{jmn}} = \frac{1}{\alpha_1 + j\theta}, \quad \text{for } 1 \leq j \leq p-1.$$

From (13) and (18), we immediately obtain

$$(19) \quad L_{H_{s,i}}(\beta_1) = \sum_{v=0}^{n-s} t_{s,i,v} \left(\sum_{j=0}^{p-1} \frac{1}{\alpha_1 + j\theta} \right)^{p^{mv}}.$$

Thus, by (14), (19) and Lemma 2.7 we have

$$(20) \quad \begin{aligned} L_{H_{s,i}}(\beta_1) &= \sum_{v=0}^{n-s} t_{s,i,v} \left(-\frac{1}{\alpha_1^p - \alpha_1} \right)^{p^{mv}} \\ &= \frac{1}{\delta} \sum_{v=0}^{n-s} t_{s,i,v} (1 - \alpha)^{p^{mv}} = L_{\phi_{s,i}} \left(\frac{1 - \alpha}{\delta} \right). \end{aligned}$$

Since α is a zero of $P(x)$, then α will be a k -normal element in $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$. Thus according to Proposition 2.3, the element $\frac{1-\alpha}{\delta}$ will also be a k -normal element. since $\frac{1-\alpha}{\delta}$ is a root of $P(-\delta x + 1)$, so by (20) and Proposition 2.4, there is a j , $1 \leq j \leq u_k$, such that $L_{H_{k,j}}(\beta_1) = 0$, and also $L_{H_{s,i}}(\beta_1) \neq 0$, for each s , $k < s < n$ and $1 \leq i \leq u_s$. So, there is a j' , $1 \leq j' \leq u'_k$, such that, $L_{H'_{k,j'}}(\beta_1) = L_{H_{k,j}}(\beta_1) = 0$. On the other side, by (11) and (12), for each s' , $k < s' < np$ and $1 \leq i' \leq u'_{s'}$, there is s , $k < s < n$ and $1 \leq i \leq u_s$ such that $H'_{s',i'}(x)$ divide $H_{s,i}(x)$. It follows that $L_{H'_{s',i'}}(\beta_1) \neq 0$, for each s' , $k < s' < np$ and $1 \leq i' \leq u'_{s'}$. The proof is completed. $\square$

In the following theorem, a computationally simple and explicit recurrent method for constructing higher degree N_k -polynomials over $\mathbb{F}_q$ starting from an N_k -polynomial is described.

Theorem 4.2. *Let $P(x)$ be an N_k -polynomial of degree n over $\mathbb{F}_q$, for each $n = rp^e$, where $e \in \mathbb{N}$ and r equals 1 or is a prime different from p and q a primitive element modulo r . Define*

$$F_0(x) = P^*(x)$$

$$(21) \quad F_u(x) = (x^p - x + \delta)^{np^{u-1}} F_{u-1} \left(\frac{x^p - x}{x^p - x + \delta} \right),$$

where $\delta \in \mathbb{F}_p^*$. Then $(F_u^*(x))_{u \geq 0}$ is a sequence of N_k -polynomials of degree np^u over $\mathbb{F}_q$ if $k < p^e$ and

$$\text{Tr}_{q|p} \left(\frac{P^{*'}(0)}{P^*(0)} \right) \cdot \text{Tr}_{q|p} \left(\frac{P^{*'}(1)}{P^*(1)} \right) \neq 0,$$

where $P^{*'}(0)$ and $P^{*'}(1)$ are the formal derivative of $P^*(x)$ at the points $x = 0$ and $x = 1$, respectively.

Proof. By Proposition 2.6 and hypotheses of theorem for each $u \geq 1$, $F_u(x)$ is an irreducible polynomial over $\mathbb{F}_q$. Consequently, $(F_u^*(x))_{u \geq 0}$ is a sequence of irreducible polynomials over $\mathbb{F}_q$. The proof of k -normality of the irreducible polynomials $F_u^*(x)$, for each $u \geq 1$ is implemented by mathematical induction on u . In the case $u = 1$, by Theorem 4.1 $F_1^*(x)$ is a k -normal polynomial.

For $u = 2$ we show that $F_2^*(x)$ is also a k -normal polynomial. To this end we need to show that the hypothesis of Theorem 4.1 are satisfied. By Theorem 4.1, $F_2^*(x)$ is a k -normal polynomial if

$$\text{Tr}_{q|p} \left(\frac{F_1'(1)}{F_1(1)} \right) \neq 0,$$

since $\delta \in \mathbb{F}_p^*$. We apply (21) to compute

$$(22) \quad F_u(0) = F_u(1) = \delta^{un} P^*(0), \quad u = 1, 2, \dots$$

We calculate the formal derivative of $F_1'(x)$ at the points $x = 0$ and $x = 1$. According to (21) the first derivative of $F_1(x)$ is

$$\begin{aligned} F_1'(x) &= -n(x^p - x + \delta)^{n-1} F_0' \left(\frac{x^p - x}{x^p - x + \delta} \right) \\ &\quad + (x^p - x + \delta)^n \cdot \left(\frac{(px^{p-1} - 1)(x^p - x + \delta) - (px^{p-1} - 1)(x^p - x)}{(x^p - x + \delta)^2} \right) \\ &\quad \cdot F_0' \left(\frac{x^p - x}{x^p - x + \delta} \right) \\ &= -\delta(x^p - x + \delta)^{n-2} \cdot P^{*'} \left(\frac{x^p - x}{x^p - x + \delta} \right), \end{aligned}$$

and at the points $x = 0$ and $x = 1$

$$(23) \quad F_1'(0) = -F_1'(1) = -\delta^{n-1} P^{*'}(0)$$

which is not equal to zero by the condition $\text{Tr}_{q|p} \left(\frac{P^{*'}(0)}{P^*(0)} \right) \neq 0$ in the hypothesis of theorem, since $\delta \in \mathbb{F}_p^*$. From (23) and (22)

$$(24) \quad \text{Tr}_{q|p} \left(\frac{F_1'(1)}{F_1(1)} \right) = \text{Tr}_{q|p} \left(\frac{-\delta^{n-1} P^{*'}(0)}{\delta^n P^*(0)} \right) = -\frac{1}{\delta} \text{Tr}_{q|p} \left(\frac{P^{*'}(0)}{P^*(0)} \right),$$

which is not equal to zero by hypothesis of theorem. Hence the polynomial $F_2^*(x)$ is a k -normal polynomial. If induction holds for $u-1$, then it must hold also for u , that is by assuming that $F_{u-1}^*(x)$ is a k -normal polynomial, we show that $F_u^*(x)$ is also a k -normal polynomial.

Let $u \geq 3$. By Theorem 4.1, $F_u^*(x)$ is a k -normal polynomial if

$$\text{Tr}_{q|p} \left(\frac{F'_{u-1}(1)}{F_{u-1}(1)} \right) \neq 0,$$

since $\delta \in \mathbb{F}_p^*$. We calculate the formal derivative of $F'_{u-1}(x)$ at the points 1 and 0. By (21) the first derivative of $F_{u-1}(x)$ is

$$\begin{aligned} F'_{u-1}(x) &= (x^p - x + \delta)^{np^{u-2}} \left(\frac{(px^{p-1} - 1)(x^p - x + \delta) - (px^{p-1} - 1)(x^p - x)}{(x^p - x + \delta)^2} \right) \\ &\quad \cdot F'_{u-2} \left(\frac{x^p - x}{x^p - x + \delta} \right) \\ &= -\delta(x^p - x + \delta)^{np^{u-2}-2} F'_{u-2} \left(\frac{x^p - x}{x^p - x + \delta} \right), \end{aligned}$$

and at the point $x = 0$ and $x = 1$

$$F'_{u-1}(0) = F'_{u-1}(1) = -\delta^{np^{u-2}-1} F'_{u-2}(0) = -\delta^{n-1} F'_{u-2}(0).$$

So we have

$$F'_{u-1}(0) = F'_{u-1}(1) = (-1)^{u-2} \delta^{(n-1)(u-2)} F'_1(0),$$

which is not equal to zero by (23) and the condition $\text{Tr}_{q|p} \left(\frac{P'^*(0)}{P^*(0)} \right) \neq 0$ in the hypothesis of theorem, since $\delta \in \mathbb{F}_p^*$. Also

$$\text{Tr}_{q|p} \left(\frac{F'_{u-1}(1)}{F_{u-1}(1)} \right) = (-1)^{u-2} \frac{1}{\delta^{(u-2)}} \text{Tr}_{q|p} \left(\frac{F'_1(1)}{F_1(1)} \right),$$

which is not equal to zero by (24) and hypothesis of theorem. The theorem is proved. $\square$

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LEFT ALMOST POLYGROUPS

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Abstract. In this introductory note we define the concept of left almost polygroups and provide several examples. We also discuss the quotient structure and isomorphism theorems for left almost polygroups.

Keywords: left almost polygroups, direct product, hyperorder, homomorphism.

1. Introduction

The work on left almost structures starts with the notion of left almost semi-groups (abbreviated as LA-semigroups) defined by Kazim and Naseeruddin [8] in 1972. This structure, known also as Abel Grassmann-groupoid (abbreviated as AG-groupoid), is a groupoid where the left invertive law holds. Later on, Mushtaq and Kamran [10] introduced a new concept of a non-associative group, called the left almost group (LA-group).

The idea of defining left almost hyperstructures belongs to Hila and Dine [7], who proposed the study of left almost semihypergroups, further explored by Yaqoob et al. [11] and Amjad et al. [1]. Recently, in 2015, Gulistan et al. [6]

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extended it in the form of H_v -LA-semigroups, where the weak left invertive law holds.

The theory of polygroups has been initiated by Comer [2, 3, 4], who applied these quasicanonical hypergroups in color schemes theory, graph theory or cylindric algebras. A comprehensive overview of the most relevant results on polygroups is summarized in Davvaz' book [5].

The aim of this introductory note is to define the concept of left almost polygroups (abbreviated LA-polygroups). In particular, we study some basic properties of them, related with their subhyperstructures, and propose new constructions of such polygroups. Results concerning the homomorphism problems related to the left almost polygroups are covered in Section 4. The paper's conclusions are gathered in the last section.

2. Preliminaries

In this section, we first recall some basic definitions and results concerning LA-semigroups, and then we focus on those related to polygroups.

A groupoid $(S, \cdot)$ is called a *left almost semigroup* (abbreviate LA-semigroup) [8], if it satisfies the *left invertive law*: $(a \cdot b) \cdot c = (c \cdot b) \cdot a$, for all $a, b, c \in S$.

It is well known [8] that, in an LA-semigroup $(S, \cdot)$ the *medial law* $(a \cdot b) \cdot (c \cdot d) = (a \cdot c) \cdot (b \cdot d)$, for all $a, b, c, d \in S$ holds, too. If the LA-semigroup S contains a left identity, the *paramedial law* $(a \cdot b) \cdot (c \cdot d) = (d \cdot c) \cdot (b \cdot a)$, for all $a, b, c, d \in S$, is valid. Moreover, in such an LA-semigroup, by using the medial law, we get $a \cdot (b \cdot c) = b \cdot (a \cdot c)$, for all $a, b, c, d \in S$.

Extending the left invertive property to hyperstructures, one obtained the following notion.

Definition 1 ([7, 11]). *A hypergroupoid $(S, \circ)$, which is left invertive, that is $(x \circ y) \circ z = (z \circ y) \circ x$, for all $x, y, z \in S$, is called an LA-semihypergroup.*

Example 1 ([11]). Let $S = \mathbb{Z}$. If we define on S the following hyperproduct $x \circ y = y - x + 3\mathbb{Z}$, where $x, y \in \mathbb{Z}$, then $(S, \circ)$ becomes an LA-semihypergroup.

Definition 2 ([2]). *A hypergroup $(H, \circ)$ is called a polygroup if*

- (i) *there exists $e \in H$ such that $e \circ x = x = x \circ e$, for all $x \in H$,*
- (ii) *for all $x \in H$ there exists an unique element, say $x^{-1} \in H$, such that $e \in x \circ x^{-1} \cap x^{-1} \circ x$,*
- (iii) *for all $x, y, z \in H$, $z \in x \circ y \Rightarrow x \in z \circ y^{-1} \Rightarrow y \in x^{-1} \circ z$.*

3. Left almost polygroups

In this section, we introduce and study some properties of the concept of left almost polygroup (abbreviate LA-polygroup) and provide some examples on

how to construct new such hyperstructures. We also discuss some useful results related to LA-polygroups.

Definition 3. A multivalued system $\langle P, \circ, e, {}^{-1} \rangle$, where $e \in P$, ${}^{-1} : P \longrightarrow P$ and $\circ : P \times P \longrightarrow \mathcal{P}^*(P)$ is called an LA-polygroup, if, for all $x, y, z \in P$, the following axioms hold:

- (i) left invertive law: $(x \circ y) \circ z = (z \circ y) \circ x$,
- (ii) reproducibility axiom: $x \circ P = P \circ x = P$,
- (iii) there exists a left identity $e \in P$ such that $e \circ x = x$,
- (iv) $e \in x \circ x^{-1} \cap x^{-1} \circ x$ (we call x^{-1} the inverse of x),
- (v) $x \in y \circ z \implies y \in x \circ z^{-1}$.

In the above definition $\mathcal{P}^*(P)$ is the set of all non-empty subsets of P and e is just the left identity. The following elementary facts about LA-polygroups follow easily from the above axioms $e^{-1} = e$ and $(x^{-1})^{-1} = x$.

Example 2. Let $P = \{e, x, y\}$ and the binary hyperoperation " $\circ$ " be defined as in the following table:

$\circ$	e	x	y
e	e	x	y
x	y	$\{e, x, y\}$	$\{x, y\}$
y	x	$\{x, y\}$	$\{e, x, y\}$

Here all the elements of P satisfy the left invertive law and e is the left identity. ${}^{-1}$ is a unitary operation on P taken as

	e	x	y
${}^{-1}$	e	x	y

Now, for the fifth condition in Definition 3, see the following calculations:

$$\begin{array}{llll}
 e & = & e \circ e & \implies & e & = & e \circ e^{-1} & = & e \circ e & = & e \\
 x & = & e \circ x & \implies & e & \in & x \circ x^{-1} & = & x \circ x & = & \{e, x, y\} \\
 y & = & e \circ y & \implies & e & \in & y \circ y^{-1} & = & y \circ y & = & \{e, x, y\} \\
 y & = & x \circ e & \implies & x & = & y \circ e^{-1} & = & y \circ e & = & x \\
 e & \in & x \circ x & \implies & x & = & e \circ x^{-1} & = & e \circ x & = & x \\
 x & \in & x \circ x & \implies & x & \in & x \circ x^{-1} & = & x \circ x & = & \{e, x, y\} \\
 y & \in & x \circ x & \implies & x & \in & y \circ x^{-1} & = & y \circ x & = & \{x, y\} \\
 x & \in & x \circ y & \implies & x & \in & x \circ y^{-1} & = & x \circ y & = & \{x, y\} \\
 y & \in & x \circ y & \implies & x & \in & y \circ y^{-1} & = & y \circ y & = & \{e, x, y\} \\
 x & = & y \circ e & \implies & y & = & x \circ e^{-1} & = & x \circ e & = & y \\
 x & \in & y \circ x & \implies & y & \in & x \circ x^{-1} & = & x \circ x & = & \{e, x, y\} \\
 y & \in & y \circ x & \implies & y & \in & y \circ x^{-1} & = & y \circ x & = & \{x, y\} \\
 e & \in & y \circ y & \implies & y & = & e \circ y^{-1} & = & e \circ y & = & y \\
 x & \in & y \circ y & \implies & y & \in & x \circ y^{-1} & = & x \circ y & = & \{x, y\} \\
 y & \in & y \circ y & \implies & y & \in & y \circ y^{-1} & = & y \circ y & = & \{e, x, y\}
 \end{array}$$

Hence $\langle P, \circ, e, {}^{-1} \rangle$ is an LA-polygroup.

Example 3. Consider a finite set P with at least 3 elements. Define a hyperoperation " $*$ " on P as follows:

$$x_i * x_j = \begin{cases} x_j, & \text{for } i = 1, \\ x_k, & \text{for } j = 1 \text{ and } k \equiv 2 - i \pmod{|P|}, \\ P, & \text{for } i = j, i \neq 1, j \neq 1, \\ P \setminus \{x_1\}, & \text{for } i \neq j, i \neq 1, j \neq 1. \end{cases}$$

Then P under the hyperoperation " $*$ " forms an LA-polygroup with the left identity x_1 and ${}^{-1}$ is a unitary operation on P taken as

$$\begin{array}{c|ccccc} & x_1 & x_2 & x_3 & \cdot & \cdot & \cdot & x_{|P|} \\ \hline {}^{-1} & x_1 & x_2 & x_3 & \cdot & \cdot & \cdot & x_{|P|} \end{array}$$

Besides " $*$ " is non-associative because $\{x_2\} = (x_2 * x_1) * x_1 \neq x_2 * (x_1 * x_1) = \{x_{|P|}\}$. Consider $P = \{x_1, x_2, x_3, x_4, x_5\}$. Then under the binary hyperoperation $*$ defined above, P is an LA-polygroup. See the following Cayley's table:

$*$	x_1	x_2	x_3	x_4	x_5
x_1	x_1	x_2	x_3	x_4	x_5
x_2	x_5	P	$\{x_2, x_3, x_4, x_5\}$	$\{x_2, x_3, x_4, x_5\}$	$\{x_2, x_3, x_4, x_5\}$
x_3	x_4	$\{x_2, x_3, x_4, x_5\}$	P	$\{x_2, x_3, x_4, x_5\}$	$\{x_2, x_3, x_4, x_5\}$
x_4	x_3	$\{x_2, x_3, x_4, x_5\}$	$\{x_2, x_3, x_4, x_5\}$	P	$\{x_2, x_3, x_4, x_5\}$
x_5	x_2	$\{x_2, x_3, x_4, x_5\}$	$\{x_2, x_3, x_4, x_5\}$	$\{x_2, x_3, x_4, x_5\}$	P

One can see that " $*$ " satisfies the left invertive law, x_1 is the left identity, ${}^{-1}$ is a unitary operation on P taken as

$$\begin{array}{c|ccccc} & x_1 & x_2 & x_3 & x_4 & x_5 \\ \hline {}^{-1} & x_1 & x_2 & x_3 & x_4 & x_5 \end{array}$$

but " $*$ " is non-associative because $\{x_2\} = (x_2 * x_1) * x_1 \neq x_2 * (x_1 * x_1) = \{x_5\}$.

Theorem 1. Let $\langle P, \circ, e, {}^{-1} \rangle$ be an LA-polygroup satisfying the condition $e = a \circ a^{-1}$, for any $a \in P$. Then P is a polygroup if and only if $a \circ (b \circ c) = (c \circ b) \circ a$ holds, for all $a, b, c \in P$.

Proof. Let $\langle P, \circ, e, {}^{-1} \rangle$ be a polygroup. Then, by associativity, we have $(a \circ b) \circ c = a \circ (b \circ c)$ for all $a, b, c \in P$, and since P is an LA-polygroup, it follows that $(a \circ b) \circ c = (c \circ b) \circ a$, therefore $a \circ (b \circ c) = (c \circ b) \circ a$, for all $a, b, c \in P$. Conversely, suppose $a \circ (b \circ c) = (c \circ b) \circ a$ holds, for all $a, b, c \in P$.

(i) Since $\langle P, \circ, e, {}^{-1} \rangle$ is an LA-polygroup, we have $a \circ (b \circ c) = (c \circ b) \circ a = (a \circ b) \circ c$.

(ii) Based on the following identities $a \circ e = a \circ (e \circ e) = (e \circ e) \circ a = e \circ a = a$, it results that e is also the right identity.

(iii) Let $a, b, c \in P$ such that $a \in b \circ c \Rightarrow b \in a \circ c^{-1}$. Now,

$$\begin{aligned}
 c &= e \circ c = (b^{-1} \circ b) \circ c \\
 &= (c \circ b) \circ b^{-1} \quad (\text{left invertive law}) \\
 &= b^{-1} \circ (b \circ c) \quad (\text{by } a \circ (b \circ c) = (c \circ b) \circ a) \\
 &\subseteq b^{-1} \circ ((a \circ c^{-1}) \circ c) \\
 &= b^{-1} \circ ((c \circ c^{-1}) \circ a) \quad (\text{left invertive law}) \\
 &= b^{-1} \circ (e \circ a) \\
 &= b^{-1} \circ a.
 \end{aligned}$$

We can conclude that $\langle P, \circ, e, {}^{-1} \rangle$ is a polygroup. $\square$

Definition 4. A non-empty subset K of an LA-polygroup $\langle P, \circ, e, {}^{-1} \rangle$ is called LA-subpolygroup of P if, under the hyperoperation in P , K itself forms an LA-polygroup.

Example 4. Let $P = \{e, x, y, z\}$ and the hyperoperation on P be defined in the following table.

$\circ$	e	x	y	z
e	e	x	y	z
x	y	$\{x, y\}$	$\{e, x\}$	z
y	x	$\{e, y\}$	$\{x, y\}$	z
z	z	z	z	$\{e, x, y\}$

Here all the elements of P satisfy the left invertive law and e is the left identity. ${}^{-1}$ is a unitary operation on P taken as

	e	x	y	z
${}^{-1}$	e	y	x	z

Besides P is not a polygroup since $\{x, y\} = (x \circ e) \circ y \neq x \circ (e \circ y) = \{e, x\}$. So $\langle P, \circ, e, {}^{-1} \rangle$ is an LA-polygroup and $K = \{e, x, y\}$ is an LA-subpolygroup of P .

Lemma 1. A non-empty subset K of the LA-polygroup $\langle P, \circ, e, {}^{-1} \rangle$ is an LA-subpolygroup of it, if and only if the following relations are satisfied.

(i) For all $a, b \in K \Rightarrow a \circ b \subseteq K$.

(ii) For all $a \in K \Rightarrow a^{-1} \in K$.

Proof. Let K be an LA-subpolygroup of P . Then relations (i) and (ii) are obvious.

Conversely, suppose that relations (i) and (ii) are true. Since K is a non-empty subset of P , it follows that the left invertive law holds in K . Now from (ii) we have the implication $a \in K \Rightarrow a^{-1} \in K$, and from (i) it results that $e \in a \circ a^{-1} \subseteq K$, for $a, a^{-1} \in K$. Moreover, let $c \in a \circ b$. It implies that $a \in c \circ b^{-1}$ by (ii). Hence, K is an LA-subpolygroup of P . $\square$

Lemma 2. *In an LA-polygroup $\langle P, \circ, e, {}^{-1} \rangle$ the following laws holds:*

- (i) *Medial law: $(a \circ b) \circ (c \circ d) = (a \circ c) \circ (b \circ d)$,*
- (ii) *$a \circ (b \circ c) = b \circ (a \circ c)$,*
- (ii) *Paramedial law: $(a \circ b) \circ (c \circ d) = (d \circ c) \circ (b \circ a)$, for all $a, b, c, d \in P$.*

Proof. Straightforward. $\square$

Lemma 3. *If K is an LA-subpolygroup of the LA-polygroup $\langle P, \circ, e, {}^{-1} \rangle$, then, for every $a, b \in P$, we have:*

- (i) *$K = K \circ K$.*
- (ii) *$e \circ K = K \circ e = K$.*
- (ii) *$a \circ K = (K \circ a) \circ e$,*
- (iv) *$(a \circ b) \circ K = K \circ (b \circ a)$.*

Proof. Straightforward. $\square$

It is important to note that there is no concept of polygroup theoretic normality in LA-polygroups, meaning that we can factor an LA-polygroup by any of its LA-subpolygroups. We know that if $\langle P, \circ, e, {}^{-1} \rangle$ is a polygroup and K is its subpolygroup, then $(K \circ a) \circ (K \circ b) \neq K \circ (a \circ b)$, unless K is normal in P . But for LA-polygroups we have no such condition because of the medial property, that is, if $K \circ a, K \circ b$ belong to P/K , then

$$(K \circ a) \circ (K \circ b) = (K \circ K) \circ (a \circ b) \quad (\text{by the medial law}) = K \circ (a \circ b),$$

without having an extra condition on P .

Remark 1. An LA-polygroup can be partitioned only into right cosets (or left cosets) and we do not require the two side decomposition.

Theorem 2. *If $\langle P, \circ, e, {}^{-1} \rangle$ is an LA-polygroup and K is an LA-subpolygroup of P , then $P/K = \{K \circ a \mid a \in P\}$ is an LA-polygroup, too.*

Proof. Let us define the hyperoperation in P/K as $K \circ a \boxtimes K \circ b = K \circ (a \circ b)$, which is obviously closed.

- (i) Let $K \circ a, K \circ b$ and $K \circ c \in P/K$. Then one obtains

$$\begin{aligned} (K \circ a \boxtimes K \circ b) \boxtimes K \circ c &= K \circ (a \circ b) \boxtimes K \circ c \\ &= K \circ ((a \circ b) \circ c) = K \circ ((c \circ b) \circ a) \\ &= K \circ (c \circ b) \boxtimes K \circ a \\ &= (K \circ c \boxtimes K \circ b) \boxtimes K \circ a. \end{aligned}$$

- (ii) There exists $K \circ e \in P/K$ such that $K \circ e \boxtimes K \circ b = K \circ (e \circ b) = K \circ b$, so $K \circ e = K$ is the left identity in P/K .

- (iii) Let $K \circ a, K \circ b$ and $K \circ c \in P/K$ such that $K \circ a \subseteq K \circ b \boxtimes K \circ c = K \circ (b \circ c)$ i.e. $K \circ a \subseteq K \circ (b \circ c)$, so $a \in b \circ c$. This implies that $b \in a \circ c^{-1}$ and therefore $K \circ b \subseteq K \circ (a \circ c^{-1})$. It results that $K \circ b \subseteq K \circ a \boxtimes K \circ c^{-1}$. Hence $P/K = \{K \circ a : a \in P\}$ is an LA-polygroup. We conclude this section with some constructions of LA-polygroups. $\square$

Definition 5. Let $\langle P_1, \circ_1, e_1, {}^{-1} \rangle$ and $\langle P_2, \circ_2, e_2, {}^{-1} \rangle$ be two LA-polygroups. Then on $P_1 \times P_2$ we can define the hyperproduct as follows:

$$(a_1, b_1) \circ (a_2, b_2) = \{(c, d) \mid c \in a_1 \circ_1 a_2, d \in b_1 \circ_2 b_2\},$$

for all $(a_1, b_1), (a_2, b_2) \in P_1 \times P_2$.

Proposition 1. The direct product of two LA-polygroups is an LA-polygroup, too.

Proof. Straightforward. $\square$

Corollary 1. If K_1, K_2 are LA-subpolygroups of the LA-polygroups P_1, P_2 respectively, then $K_1 \times K_2$ is an LA-subpolygroup of $P_1 \times P_2$ and $(P_1 \times P_2)/(K_1 \times K_2) \cong P_1/K_1 \times P_2/K_2$.

Definition 6. Let $\langle P, \circ, e, {}^{-1} \rangle$ be an LA-polygroup, and let $a, b \in P$. We write $a \blacktriangleright b$ if $a \circ c \subseteq b \circ c$, for all $c \in P$, and we call $\blacktriangleright$ a hyperorder on P .

If $a \blacktriangleright b$ and $b \blacktriangleright a$, then we say a is hyperequal to b , and we write $a \sim b$. It is clear that the relation " $\sim$ " is an equivalence relation on P .

Proposition 2. Let $\langle P, \circ, e, {}^{-1} \rangle$ be an LA-polygroup. We define the class $[a] = \{b \in P \mid a \sim b\}$ represented by a , and let $C(P) = \{[a] \mid a \in P\}$ denote the set of all classes of the elements in P . If we define the hyperoperation on $C(P)$ as $[a] \bullet [b] = \{[n] \mid n \in a \circ b\}$, then $\langle C(P), \bullet, e, {}^{-1} \rangle$ is an LA-polygroup.

Proof. (i) Let $[a], [b], [c] \in C(P)$. One finds that

$$\begin{aligned} ([a] \bullet [b]) \bullet [c] &= (\{[n] \mid n \in a \circ b\}) \bullet [c] = \{[m] \mid m \in n \circ c\} \\ &= \{[m] \mid m \in (a \circ b) \circ c\} = \{[m] \mid m \in (c \circ b) \circ a\} \\ &= \{[m] \mid m \in n \circ a\} = (\{[n] \mid n \in c \circ b\}) \bullet [a] \\ &= ([c] \bullet [b]) \bullet [a]. \end{aligned}$$

(ii) Let $[e], [a] \in C(P)$. Then $([e] \bullet [a]) = (\{[n] \mid n \in e \circ a\}) = (\{[n] \mid n = a\}) = [a]$. (iii) Let $[a], [b], [c] \in C(P)$ and consider $[a] \in [b] \bullet [c] = \{[x] \mid x \in b \circ c\}$. Therefore, there exist $y \in [b]$ and $z \in [c]$ such that $x \in y \circ z$, so $y \in x \circ z^{-1}$. This implies that $[y] \in [x] \bullet [z^{-1}]$, thus $[b] \in [a] \bullet [c^{-1}]$. Hence $\langle C(P), \bullet, e, {}^{-1} \rangle$ is an LA-polygroup. $\square$

4. Homomorphisms of LA-polygroups

This section is devoted to the study of some homomorphism problems related to LA-polygroups.

Definition 7. Let $\langle P_1, \circ_1, e_1, {}^{-1} \rangle$ and $\langle P_2, \circ_2, e_2, {}^{-1} \rangle$ be two LA-polygroups. Let f be a mapping from P_1 into P_2 such that $f(e_1) = e_2$. Then, f is called

- (i) an inclusion homomorphism if $f(x \circ_1 y) \subseteq f(x) \circ_2 f(y)$.
- (ii) a good homomorphism if $f(x \circ_1 y) = f(x) \circ_2 f(y)$.

Example 5. (i) Let $P_1 = \{e, x, y\}$ and $P_2 = \{g, a, b\}$ be two LA-polygroups with the hyperoperations defined in the following tables:

$\circ_1$	e	x	y	$\circ_2$	g	a	b
e	e	x	y	g	g	a	b
x	y	$\{x, y\}$	$\{e, x\}$	a	b	$\{a, b\}$	P_2
y	x	$\{e, y\}$	$\{x, y\}$	b	a	P_2	$\{a, b\}$

and let $f : P_1 \rightarrow P_2$ be defined by $f(e) = g$, $f(x) = a$, $f(y) = b$. Then, clearly, f is an inclusion homomorphism.

(ii) Let $P_1 = \{e, x, y, z\}$ and $P_2 = \{g, a, b, c\}$ be two LA-polygroups with the hyperoperations defined in the following tables:

$\circ_1$	e	x	y	z	$\circ_2$	g	a	b	c
e	e	x	y	z	g	g	a	b	c
x	y	$\{x, y, z\}$	$\{e, x, z\}$	$\{x, y\}$	a	c	$\{a, b, c\}$	$\{a, c\}$	$\{g, a, b\}$
y	x	$\{e, y, z\}$	$\{x, y, z\}$	$\{x, y\}$	b	b	$\{a, c\}$	$\{g, b\}$	$\{a, c\}$
z	z	$\{x, y\}$	$\{x, y\}$	$\{e, z\}$	c	a	$\{g, b, c\}$	$\{a, c\}$	$\{a, b, c\}$

and let $f : P_1 \rightarrow P_2$ be defined by $f(e) = g$, $f(x) = a$, $f(y) = c$, $f(z) = b$. Then, clearly f is a good homomorphism.

Lemma 4. Let f be a good homomorphism from $\langle P_1, \circ_1, e_1, {}^{-1} \rangle$ into $\langle P_2, \circ_2, e_2, {}^{-1} \rangle$. Let K_1 and K_2 be LA-subpolygroups of P_1 and P_2 , respectively. Then the following statements are valid.

- (i) The image $f(K_1)$ of K_1 under f is an LA-subpolygroup of P_2 .
- (ii) The inverse image $f^{-1}(K_2)$ of K_2 under f is an LA-subpolygroup of P_1 .

Proof. Straightforward. □

Lemma 5. Let f be a good homomorphism from $\langle P_1, \circ_1, e_1, {}^{-1} \rangle$ into $\langle P_2, \circ_2, e_2, {}^{-1} \rangle$. Then:

- (i) $f(e_1) = e_2$,
- (ii) $f(a^{-1}) \subseteq f(a)^{-1}$.

Proof. Straightforward. □

Lemma 6. Let f be a good homomorphism from $\langle P_1, \circ_1, e_1, {}^{-1} \rangle$ into $\langle P_2, \circ_2, e_2, {}^{-1} \rangle$. Then f is injective if and only if $\ker f = \{e_1\}$.

Proof. Let f be injective and assume that $x \in \ker f$. By Lemma 5, we have $f(e_1) = e_2$. Therefore $f(x) = e_2 = f(e_1) \Rightarrow x = e_1$ and hence $\ker f = \{e_1\}$.

Conversely, let $\ker f = \{e_1\}$ and assume that $f(x) = f(y)$ for $x, y \in P_1$. Now considering $f(x) = f(y)$, we have $f(x) \circ_2 f(x^{-1}) = f(y) \circ_2 f(x^{-1})$. It follows that $f(e_1) \in f(x \circ_1 x^{-1}) = f(y \circ_1 x^{-1})$. So there exists $t \in y \circ_1 x^{-1}$ such that $e_2 = f(e_1) = f(t)$. Thus $e_1 = t \in y \circ_1 x^{-1}$, whence $x = y$. □

Theorem 3. Let f be a good homomorphism from $\langle P_1, \circ_1, e_1, {}^{-1} \rangle$ into $\langle P_2, \circ_2, e_2, {}^{-1} \rangle$ with the kernel κ , such that κ is an LA-subpolygroup of $\langle P_1, \circ_1, e_1, {}^{-1} \rangle$. Then $P_1/\kappa \cong P_2$.

Proof. Let f be a good homomorphism, i.e. $f(e_1) = e_2$ and $f(a \circ_1 b) = f(a) \circ_2 f(b)$, for all $a, b \in P_1$. Define a mapping $\lambda : P_1/\kappa \rightarrow P_2$ by $\lambda(\kappa x) = f(x)$, for all $x \in P_1$. We first show that λ is well-defined. For $x, y \in P_1$, $\kappa x = \kappa y \implies x \circ_1 y^{-1} \subseteq \kappa$. Let $t \in x \circ_1 y^{-1}$. Consequently, $f(t) = e_2$ and $f(t) \subseteq f(x) \circ_2 f(y^{-1}) = f(x) \circ_2 f(y)^{-1}$. Thus $f(x) = f(y)$. Clearly λ is onto. Now we have to show that λ is one to one. Suppose $f(x) = f(y)$. Then $e_2 \in f(x \circ_1 y^{-1})$ and so there exists $t \in x \circ_1 y^{-1}$ with $t \in \ker f$. Therefore $x \circ_1 y^{-1} \subseteq \kappa$, which implies that $\kappa x = \kappa y$, and so λ is one to one. Now it remains only to prove that λ is a good homomorphism. Let $\kappa x, \kappa y \in P_1/\kappa$. It results that $\lambda(\kappa x \circ_1 \kappa y) = \lambda(\kappa(x \circ_1 y)) = f(x \circ_1 y) = f(x) \circ_2 f(y) = \lambda(\kappa x) \circ_2 \lambda(\kappa y)$ and also $\lambda(\kappa e_1) = f(e_1) = e_2$. Hence $P_1/\kappa \cong P_2$. $\square$

Theorem 4. If K and N are LA-subpolygroups of the LA-polygroup $\langle P, \circ, e, {}^{-1} \rangle$, then $K/(N \cap K) \cong KN/N$.

Proof. Let us define $f : K \rightarrow KN/N$ by $f(k) = Nk$, for all $k \in K$. It is easy to show that f is a good homomorphism. Since each element of KN/N has the form of knN , where $k \in K$ and $n \in N$, it follows that $nN = N$. Therefore each element of KN/N is of the form KN , which is the image of K under f . Hence f is onto. Therefore, by Theorem 3, $K/\ker f \cong KN/N$. Now we need to show that $\ker f = N \cap K$.

$$\begin{aligned} \ker f &= \{k \in K : f(k) = \text{identity of } KN/N\} \\ &= \{k \in K : kN = N\} \\ &= \{k \in K : k \in N\} = N \cap K. \end{aligned}$$

Hence $K/(N \cap K) \cong KN/N$. $\square$

Theorem 5. If K and N are LA-subpolygroups of the LA-polygroup $\langle P, \circ, e, {}^{-1} \rangle$ such that $N \subseteq K$, then $(P/N)/(K/N) \cong P/K$.

Proof. Let us define a mapping $f : P/N \rightarrow P/K$ by $f(Na) = Ka$, for any $a \in P$. It is easy to show that f is a good homomorphism. Since, for each $Na \in P/N$ there exist $Ka \in P/K$ such that $f(Na) = Ka$, it results that the mapping f is onto. So, by Theorem 3, $(P/N)/\ker f \cong P/K$. Now we show that $\ker f = K/N$. This follows from the following identities:

$$\begin{aligned} \ker f &= \{Na \in P/N : a \in P \text{ and } f(Na) = \text{identity of } P/K\} \\ &= \{Na \in P/N : a \in P \text{ and } Ka = K\} \\ &= \{Na \in P/N : a \in K\} = K/N. \end{aligned}$$

Hence $(P/N)/(K/N) \cong P/K$. $\square$

5. Conclusions

Substituting in a polygroup the associativity with the left invertive law, one obtains the notion of left almost polygroup, by short LA-polygroup. Most of the properties of polygroups are valid also for LA-polygroups, but the normality concept is different here. This means that an LA-polygroup may be factorize by any of its LA-subpolygroup K , without asking K to be a normal LA-subpolygroup, as it happens in polygroups framework. This property is assured by the medial law, that holds in an LA-polygroup. Consequently, the three isomorphism theorems are simplified for LA-polygroups, with respect to polygroups.

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ON PROPERTIES OF VARIOUS MORPHISMS IN THE CATEGORIES OF GENERAL KRASNER HYPERMODULES

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Abstract. In a recent paper entitled *Pre-semihyperadditive Categories*, we introduced some categories in which for objects A and B , the class of all morphisms from A to B denoted by $Mor(A, B)$, admits an algebraic hyperstructures such as semihypergroup or hypergroup. Then after defining and fixing a general Krasner hyperring R , we introduced and studied the categories of general Krasner R -hypermultiples, ${}_R\mathcal{G}.mod$, ${}_{R_s}\mathcal{G}.mod$, ${}_{R_w}\mathcal{G}.mod$ and etc. In this paper we present some properties of multi-valued homomorphisms as morphisms of these categories and study various concepts related to these morphisms in connection with the fundamental relation of their domain or codomain.

Keywords: pre-semihyperadditive category, R – mv -homomorphism, injectivity, equality, monicness, fundamental relation, fundamental module.

1. Introduction

Hypergroup as a generalization of group in Algebra was introduced by Marty during the 8th Congress of the Scandinavian Mathematicians [15] in 1934. This concept has resulted in a new branch of mathematics science named Hyperstructures Theory. This theory as a new field of modern Algebra has been developed in the view points of theory and applications by many researchers, see e.g. [2, 8, 9, 10, 11, 12]. Among others, Ameri [1], Corsini [3], Corsini and Leoreanu [4], Cristea et al. [5,6,7], Novák [16,17], Massouros [13,14], Vougiouklis

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[20], studied hypercompositional structures defined in terms of binary relations. In [19], the authors described the interaction between hyperstructure theory and category theory. As defined in [19], in a pre-semihyperadditive category $\mathcal{C}$ for two objects A and B the class of morphisms from A into B denoted by $Mor_{\mathcal{C}}(A, B)$ is at least a semihypergroup.

As two examples of such categories, letting R be a general Krasner hyperring, we introduced the category ${}_R\mathcal{G}.mod$ (resp., ${}_R\mathbf{G}.mod$) consisting of general Krasner hypermodules as objects and $R - mv$ -homomorphism (resp., R -homomorphism) as morphisms. Every morphism of ${}_R\mathbf{G}.mod$ is a function that satisfies some conditions and can be considered as a special morphism of ${}_R\mathcal{G}.mod$. Behavior of a morphism of ${}_R\mathcal{G}.mod$ differs from that of one (usual) function and we naturally do not expect to be the same. For example, an $R - mv$ -homomorphism has various types of injectivity and monicness. In this regards we introduce some kinds of these concepts and obtain some results explaining the related properties of a morphism of ${}_R\mathcal{G}.mod$.

2. Preliminary

Let H be a nonempty set and $P^*(H)$ denotes the set of all nonempty subsets of H . A hyperoperation $\cdot$ on H is a map $\cdot : H \times H \longrightarrow P^*(H)$ sending (a, b) to $a \cdot b \subseteq H$. In this case $(H, \cdot)$ is called a *hypergroupoid*. The hyperoperation $\cdot$ is extended to subsets of H in a natural way. Indeed, $A \cdot B$ or AB is given by

$$(2.1) \quad A \cdot B = \bigcup_{(a,b) \in A \times B} a \cdot b.$$

A hypergroupoid $(H, \cdot)$ is said to be a semihypergroup if $\cdot$ is associative. A hypergroupoid $(H, \cdot)$ is a hypergroup if it is a semihypergroup satisfying (*reproductivity* property) $Hx = xH = H$, for every $x \in H$. A semihypergroup or hypergroup H is called commutative if $xy = yx$ for every $x, y \in H$. An element y of semihypergroup $(H, +)$ is called identity if for all $x \in H$, $y \in x + y \cap y + x$.

Let x be an element of semihypergroup $(H, +)$ (resp., $(H, \cdot)$) such that $x + y = y$ (resp., $x \cdot y = y$). Then x is called a left scalar identity (resp., unit). Similarly, a right scalar identity (resp., unit) is defined with the affection on the right.

An element x of semihypergroup $(H, +)$ (resp., $(H, \cdot)$) is called a scalar identity (resp., unit) if it is a left and right scalar identity (resp., unit). We denote the scalar identity (resp., unit) of H by 0_H (resp., 1_H). Every scalar identity or scalar unit in a semihypergroup H is unique.

Definition 2.1. A nonempty set H together with the hyperoperation “+” is called a *canonical hypergroup* if the following axioms hold:

1. $(H, +)$ is a commutative semihypergroup,
2. there is a scalar identity 0_H ,

3. for every $x \in H$, there is a unique element denoted by $-x$ such that $0_H \in x + (-x)$ which for simplicity we write $0_H \in x - x$,
4. $x \in y + z$ implies $y \in x - z$ (and thus $z \in -y + x$).

Definition 2.2 ([12]). A non-empty set R together with the hyperoperation $+$ and the operation $\cdot$ is called a *Krasner hyperring* if the following axioms hold:

1. $(R, +)$ is a canonical hypergroup;
2. $(R, \cdot)$ is a semigroup including 0_R as a bilaterally absorbing element, that is $0_R \cdot x = x \cdot 0_R = 0_R$ for all $x \in R$;
3. $(y + z) \cdot x = (y \cdot x) + (z \cdot x)$ and $x \cdot (y + z) = x \cdot y + x \cdot z$ for all $x, y, z \in R$.

Definition 2.3 ([18]). Let $(R, +, \cdot)$ be a hyperring. A canonical hypergroup $(A, +)$ together with left external multiplication $* : R \times A \longrightarrow A$ is called a *left Krasner hypermodule* over R if for all $r_1, r_2 \in R$ and for all $a_1, a_2 \in A$ the following axioms hold:

1. $r_1 * (a_1 + a_2) := \bigcup_{b \in a_1 + a_2} r_1 * b = r_1 * a_1 + r_1 * a_2$;
2. $(r_1 + r_2) * a_1 := \bigcup_{r \in r_1 + r_2} r * a_1 = r_1 * a_1 + r_2 * a_1$;
3. $(r_1 \cdot r_2) * a_1 = r_1 * (r_2 * a_1)$;
4. $0_R * a_1 = 0_A$.

Definition 2.4 ([19]). A nonempty set R together with two hyperoperations $+$ and $\cdot$ is called a *general Krasner hyperring* if the following axioms hold:

1. $(R, +)$ is a canonical hypergroup (with scalar identity 0_R),
2. $(R, \cdot)$ is a semihypergroup including 0_R as a bilaterally absorbing element, i.e., $0_R \cdot a = a \cdot 0_R = 0_R$ for all $a \in A$,
3. $(y + z) \cdot x \subseteq (y \cdot x) + (z \cdot x)$ and $x \cdot (y + z) \subseteq x \cdot y + x \cdot z$ for all $x, y, z \in R$.

We say a general Krasner hyperring $(R, +, \cdot)$ has the scalar unit 1_R if $1_R \cdot r = r \cdot 1_R = r$ for all $r \in R$.

Definition 2.5 ([19]). Let R be a general Krasner hyperring. A nonempty set A is called a *left general Krasner hypermodule over R* , for short a left general Krasner R -hypermodule, if $(A, +)$ is a canonical hypergroup together with the map $* : R \times A \longrightarrow P^*(A)$ satisfying the following axioms for all $r_1, r_2 \in R$ and $a_1, a_2 \in A$:

1. $r_1 * (a_1 + a_2) := \bigcup_{b \in a_1 + a_2} r_1 * b \subseteq r_1 * a_1 + r_1 * a_2$;
2. $(r_1 + r_2) * a_1 := \bigcup_{r \in r_1 + r_2} r * a_1 \subseteq r_1 * a_1 + r_2 * a_1$;

3. $(r_1 \cdot r_2) * a_1 \subseteq r_1 * (r_2 * a_1)$ in which $(r_1 \cdot r_2) * a_1 := \bigcup_{r \in r_1 \cdot r_2} r * a_1$ and $r_1 * (r_2 * a_1) := \bigcup_{a \in r_2 * a_1} r * a$;
4. $0_R * a_1 = 0_A$.

A left general Krasner R -hypermodule A is called unitary if R has the scalar unit 1_R with $1_R * a = a$ for all $a \in A$. Every general Krasner hyperring R with 1_R is a unitary left general Krasner R -hypermodule.

In an obvious way, one can consider the external multiplication map $*$: $A \times R \rightarrow P^*(A)$ to define the right general Krasner R -hypermodule. From now on, R denotes a general Krasner hyperring. Also, for convenience, by a hyperring R and an R -hypermodule we mean a general Krasner hyperring and a left general Krasner R -hypermodule, respectively.

In order to have a category whose objects are the class of all R -hypermodules, we need morphisms. For this, we start with the following concept.

Definition 2.6. For two R -hypermodules A and B , let f be a function from A into $P^*(B)$, that is a multi-valued function from A to B , that satisfies the conditions

1. $f(x + y) \subseteq f(x) + f(y)$,
2. $f(r * x) \subseteq r * f(x)$,

for all $r \in R$ and all $x, y \in A$. In this case, f is said to be a *multi-valued R -homomorphism*, for short we write *R -mv-homomorphism* from A to B . Sometimes an R -mv-homomorphism is called an *inclusion R -mv-homomorphism*.

Note that $+$ in Definition 2.6 is given by (2.1). If f satisfies the conditions

1. $f(x + y) = f(x) + f(y)$,
2. $f(r * x) = r * f(x)$,

for all $r \in R$ and all $x, y \in A$, then it is called a *strong R -mv-homomorphism*. Also, if f satisfies the conditions

1. $f(x + y) \cap [f(x) + f(y)] \neq \emptyset$,
2. $f(r * x) \cap r * f(x) \neq \emptyset$,

for all $r \in R$ and all $x, y \in A$, then f is said to be a *weak R -mv-homomorphism*. The class of all R -mv-homomorphisms, strong R -mv-homomorphisms and weak R -mv-homomorphisms from A to B as morphisms from A to B is denoted by $Hom_R(A, B)$, $Hom_R^s(A, B)$ and $Hom_R^w(A, B)$, respectively.

Let $f \in Hom_R(A, B)$ and $g \in Hom_R(B, C)$. Define the composition $g \circ f$ as:

$$(2.2) \quad (g \circ f)(a) = \bigcup_{b \in f(a)} g(b), \quad \forall a \in A.$$

Following [19], ${}_R\mathcal{G}.\mathbf{mod}$, ${}_{R_s}\mathcal{G}.\mathbf{mod}$ and ${}_{R_w}\mathcal{G}.\mathbf{mod}$ denote the categories formed by the class of all R -hypermultiples together with the class of all R - mv -homomorphisms, strong R - mv -homomorphisms and weak R - mv -homomorphisms, respectively, with the composition of morphisms as (2.2).

One can consider a function f from A into B satisfying two conditions in Definition 2.6 as a morphism. We call such morphism an (inclusion) R -homomorphism. Similarly, we can define strong R -homomorphisms and weak R -homomorphisms from A to B . We use $hom_R(A, B)$, $hom_R^s(A, B)$ and $hom_R^w(A, B)$ for the class of (inclusion) R -homomorphisms, strong R -homomorphisms and weak R -homomorphisms from A to B , respectively. Also, we denote the corresponding categories by ${}_R\mathbf{G}.\mathbf{mod}$, ${}_{R_s}\mathbf{G}.\mathbf{mod}$ and ${}_{R_w}\mathbf{G}.\mathbf{mod}$, respectively. Any singleton set is identified with its element. Thus we may write $f(a) = b$ instead of $f(a) = \{b\}$. Therefore, any single-valued $f \in Hom_R(A, B)$ is an element of $hom_R(A, B)$, and conversely, any element of $hom_R(A, B)$ is a single-valued element of $Hom_R(A, B)$.

Remark 2.7. It is necessary to emphasize that according to [18], the external multiplication $*$ in a Krasner hypermodule over a Krasner hyperring R is single-valued and for every morphism between two Krasner hypermodules such as $f : A \rightarrow P^*(B)$ and $f : A \rightarrow B$, we have $f(r * x) = r * f(x)$ for every $x \in A$ and $r \in R$, while in the categories of general Krasner hypermodules over a general Krasner hyperring R , $*$ is a multi-valued map and for an R - mv -homomorphism f , we have $f(r * x) \subseteq r * f(x)$ for every $x \in A$ and $r \in R$.

In the sequel, we write ra instead of $r * a$ if there is no confusion. For more details about hyperrings and hypermodules, see e.g. [1, 11, 13, 14]. For some concepts related to category theory the reader can refer to [3].

3. Some properties of morphisms of ${}_R\mathcal{G}.\mathbf{mod}$

From now on, fixing a general Krasner hyperring $(R, +, \cdot)$ and following [20], let Γ^* be the smallest equivalence relation such that $(R/\Gamma^*, \oplus, \otimes)$ is a ring with

$$\begin{aligned} \forall x, y \in R : \quad \Gamma^*(x) \oplus \Gamma^*(y) &= \Gamma^*(z) \quad \forall z \in \Gamma^*(x) + \Gamma^*(y), \\ \forall x, y \in R : \quad \Gamma^*(x) \otimes \Gamma^*(y) &= \Gamma^*(z) \quad \forall z \in \Gamma^*(x) \cdot \Gamma^*(y). \end{aligned}$$

Also, let A be a general Krasner R -hypermodule and ϵ_A^* be the smallest equivalence relation such that firstly, $(A/\epsilon_A^*, \oplus)$ is a (commutative) group with

$$\forall x, y \in R : \quad \Gamma^*(x) \oplus \Gamma^*(y) = \Gamma^*(z) \quad \forall z \in \Gamma^*(x) + \Gamma^*(y),$$

and secondly, A/ϵ_A^* is an R/Γ^* -module with the external multiplication

$$*': R/\Gamma^* \times A/\epsilon_A^* \rightarrow A/\epsilon_A^*, \quad \Gamma^*(r) *' \epsilon_A^*(a) = \epsilon_A^*(z) \quad \forall z \in \Gamma^*(r) * \epsilon_A^*(a)$$

for all $(r, a) \in R \times A$. Here, Γ^* , ϵ_A^* , R/Γ^* and A/ϵ_A^* are called *fundamental relation of R* , *fundamental relation of A* , *fundamental ring of R* and *fundamental module of A* , respectively.

In this section, we study some properties of an arbitrary R – mv -homomorphism $f \in \text{Hom}_R(A, B)$ in connection with the equivalence relations ϵ_A^* and ϵ_B^* .

Definition 3.1. A nonempty subset $B \subseteq A$ is called an R -subhypermodule of the R -hypermodule A if B is an R -hypermodule itself.

Let $f \in \text{Hom}_R(A, B)$. Set

$$\begin{aligned} \text{Ker}(f) &:= \{x \in A \mid f(x) = 0_B\}, \\ K_f &:= \{x \in A \mid 0_B \in f(x)\}, \\ \text{Im}(f) &:= \{y \in B \mid \exists x \in A \quad y \in f(x)\}. \end{aligned}$$

Remark 3.2. Clearly K_f and $\text{Im}(f)$ are not necessarily R -subhypermodules of A and B , respectively. It is easy to see that the set $\text{Ker}(f)$ is always an R -subhypermodule of A .

Example 3.3. Clearly every Krasner hyperring R is a Krasner R -hypermodule. Also, every Krasner hypermodule is a general Krasner hypermodule. Consider the set of nonnegative real numbers denoted by $\mathbb{R}^{+0}$. It is easy to verify that $\mathbb{R}^{+0}$ under the hyperoperation $+$ defined by $x + y = \max\{x, y\}$ if $x \neq y$, and $[0, x]$ if $x = y$, is a canonical hypergroup. Considering the usual multiplication $*$: $\mathbb{R}^{+0} \times \mathbb{R}^{+0} \rightarrow \mathbb{R}^{+0}$, we can check that $(\mathbb{R}^{+0}, +, *)$ is a Krasner hyperring, and indeed $A := \mathbb{R}^{+0}$ is a Krasner A -hypermodule. Consequently, A is a unitary general Krasner A -hypermodule. Now define $f : A \rightarrow P^*(A)$ with $f(x) = [0, x]$ for every $x \in A$. Then $f \in \text{Hom}_A^s(A, A)$ with $\text{Ker}(f) = \{0\}$ and $K_f = \text{Im}(f) = A$.

In the sequel, we introduce few kinds of equality. First note that for an R -hypermodule A and $X \subseteq A$, we define $\overline{\epsilon_A^*(X)} = \{\epsilon_A^*(x) \mid x \in X\}$ as a subset of $\frac{A}{\epsilon_A^*}$. Note that $\overline{\epsilon_A^*(X)}$ is different from the subset $\bigcup_{x \in X} \epsilon_A^*(x) \subseteq A$.

Definition 3.4.

- (i) We say $a_1, a_2 \in A$ are ϵ_A^* -equal if $\epsilon_A^*(a_1) = \epsilon_A^*(a_2)$ and *product-equal*, for short *pro-equal*, if $\overline{\epsilon_A^*(ra_1)} \cap \overline{\epsilon_A^*(ra_2)} \neq \emptyset$, $\forall r \in R$.
- (ii) We say $a_1, a_2 \in A$ are *R-equal* if $\overline{\epsilon_A^*(Ra_1)} \cap \overline{\epsilon_A^*(Ra_2)} \neq \emptyset$.

It is necessary to emphasize ra_1 and ra_2 are two subsets of A in Definition 3.4.

Proposition 3.5. Let R be a hyperring with 1_R and A be a unitary R -hypermodule.

1. If $a_1, a_2 \in A$ are *pro-equal*, then they are ϵ_A^* -equal.
2. If $a_1, a_2 \in A$ are *pro-equal*, then they are *R-equal*.

Proof. 1. Let for all $r \in R$, $\overline{\epsilon_A^*(ra_1)} \cap \overline{\epsilon_A^*(ra_2)} \neq \emptyset$. If 1_R is the scalar unit of R , then $1_R * a_1 = \{a_1\}$ and $1_R * a_2 = \{a_2\}$. Thus

$$\overline{\epsilon_A^*(\{a_1\})} \cap \overline{\epsilon_A^*(\{a_2\})} \neq \emptyset \implies \{\epsilon_A^*(a_1)\} \cap \{\epsilon_A^*(a_2)\} \neq \emptyset.$$

Hence $\epsilon_A^*(a_1) = \epsilon_A^*(a_2)$.

2. Let $a_1, a_2 \in A$ be pro-equal. Then for all $r \in R$, $\overline{\epsilon_A^*(ra_1)} \cap \overline{\epsilon_A^*(ra_2)} \neq \emptyset$. So $\forall r \in R, \exists x \in ra_1, \exists y \in ra_2 : \epsilon_A^*(x) = \epsilon_A^*(y)$. Clearly, $x \in Ra_1$ and $y \in Ra_2$ imply $\epsilon_A^*(x) \in \epsilon_A^*(Ra_1)$ and $\epsilon_A^*(y) \in \epsilon_A^*(Ra_2)$. Thus the result is followed. $\square$

Definition 3.6. Let $g, h \in \text{Hom}_R(A, B)$.

- (i) (equality of morphisms) We say g and h are *equal* if $g(a) = h(a), \forall a \in A$. In this case we write $g = h$.
- (ii) (*weak equality* of morphisms) We say g and h are *weakly equal* if $g(a) \cap h(a) \neq \emptyset, \forall a \in A$. In this case we write $g \approx h$.
- (iii) (ϵ_B^* -equality of morphisms) We say g and h are ϵ_B^* -equal if $\overline{\epsilon_B^*(g(a))} = \overline{\epsilon_B^*(h(a))}, \forall a \in A$. In this case we write $g \doteq h$.
- (iv) (*weak ϵ_B^* -equality* of morphisms) We say g and h are *weakly ϵ_B^* -equal* if $\overline{\epsilon_B^*(g(a))} \cap \overline{\epsilon_B^*(h(a))} \neq \emptyset, \forall a \in A$. In this case we write $g \ddot{=} h$.

Clearly $g = h \implies g \approx h \implies g \ddot{=} h$, and $g \doteq h \implies g \ddot{=} h$.

In the case that both f and g are single-valued, the above types of equality is as the following:

- 1. $g = h$ if and only if $g \approx h$.
- 2. $g \doteq h$ if and only if $g \ddot{=} h$.

Definition 3.7. A morphism $f \in \text{Hom}_R(A, B)$ is said

- (i) *injective* if for all $a_1, a_2 \in A$, $f(a_1) = f(a_2)$ implies $a_1 = a_2$.
- (ii) *strongly injective* if for all $a_1, a_2 \in A$, $f(a_1) \cap f(a_2) \neq \emptyset$ implies $a_1 = a_2$.

For a single-valued $f \in \text{Hom}_R(A, B)$, injectivity is equivalent to strong injectivity.

Example 3.8. The morphism f mentioned in Example 3.3 is injective.

Definition 3.9. A morphism $f \in \text{Hom}_R(A, B)$ is said

- (i) ϵ_A^* -injective if for all $a_1, a_2 \in A$, $f(a_1) = f(a_2) \implies \epsilon_A^*(a_1) = \epsilon_A^*(a_2)$.
- (ii) *strongly ϵ_A^* -injective* if for all $a_1, a_2 \in A$, $f(a_1) \cap f(a_2) \neq \emptyset \implies \epsilon_A^*(a_1) = \epsilon_A^*(a_2)$.

Every (strongly) injective $f \in \text{Hom}_R(A, B)$ is (strongly) ϵ_A^* -injective. It is clear that if f is strongly ϵ_A^* -injective, then it is ϵ_A^* -injective. Also a single-valued $f \in \text{Hom}_R(A, B)$ is strongly ϵ_A^* -injective if and only if it is ϵ_A^* -injective.

Definition 3.10. $f \in \text{Hom}_R(A, B)$ is said to be

- (i) $\epsilon_{B,A}^*$ -injective if for all $a_1, a_2 \in A$,

$$\overline{\epsilon_B^*(f(a_1))} = \overline{\epsilon_B^*(f(a_2))} \implies \epsilon_A^*(a_1) = \epsilon_A^*(a_2).$$

(ii) *strongly $\epsilon_{B,A}^*$ -injective* if for all $a_1, a_2 \in A$,

$$\overline{\epsilon_B^*(f(a_1))} \cap \overline{\epsilon_B^*(f(a_2))} \neq \emptyset \implies \epsilon_A^*(a_1) = \epsilon_A^*(a_2).$$

Obviously, each strongly $\epsilon_{B,A}^*$ -injective is $\epsilon_{B,A}^*$ -injective. A single-valued $f \in \text{Hom}_R(A, B)$ is $\epsilon_{B,A}^*$ -injective if and only if it is strongly $\epsilon_{B,A}^*$ -injective.

In every category $\mathcal{C}$, a morphism $f \in \text{Mor}_{\mathcal{C}}(B, C)$ is said to be *mono* if for every $g, h \in \text{Mor}_{\mathcal{C}}(A, B)$, the following implication holds $f \circ g = f \circ h \implies g = h$.

Definition 3.11. Let $f \in \text{Hom}_R(B, C)$. Then f is said to be

- (i) *monic* or *R -mv-monomorphism* if f is a mono of ${}_R\mathcal{G}\mathbf{mod}$ (in the sense of category theory).
- (ii) *$\epsilon_{C,B}^*$ -monic* if for all $g, h \in \text{Hom}_R(A, B)$, $f \circ g \doteq f \circ h \implies g \doteq h$.
- (iii) *partially $\epsilon_{C,B}^*$ -monic* if for all $g, h \in \text{Hom}_R(A, B)$, $f \circ g \ddot{=} f \circ h \implies g \ddot{=} h$.

Proposition 3.12. Every strongly $\epsilon_{C,B}^*$ -injective R -mv-homomorphism $f \in \text{Hom}_R(B, C)$ is partially $\epsilon_{C,B}^*$ -monic.

Proof. Let $f \in \text{Hom}_R(B, C)$ be strongly $\epsilon_{C,B}^*$ -injective and $g, h \in \text{Hom}_R(A, B)$. Assume $f \circ g \ddot{=} f \circ h$. Then

$$\begin{aligned} \forall a \in A : \quad & \overline{\epsilon_C^*[(f \circ g)(a)]} \cap \overline{\epsilon_C^*[(f \circ h)(a)]} \neq \emptyset \\ \implies \forall a \in A : \quad & \overline{\epsilon_C^*[f(g(a))]} \cap \overline{\epsilon_C^*[f(h(a))]} \neq \emptyset. \end{aligned}$$

Hence $\forall a \in A : \exists x \in g(a), \exists y \in h(a) : \overline{\epsilon_C^*(f(x))} \cap \overline{\epsilon_C^*(f(y))} \neq \emptyset$. Now by assumption, $\epsilon_B^*(x) = \epsilon_B^*(y)$. On the other hand, $x \in g(a)$ and $y \in h(a)$ implies $\overline{\epsilon_B^*(g(a))} \ni \epsilon_B^*(x) = \epsilon_B^*(y) \in \overline{\epsilon_B^*(h(a))}$. So $\overline{\epsilon_B^*(g(a))} \cap \overline{\epsilon_B^*(h(a))} \neq \emptyset$. Thus we have $g \ddot{=} h$. $\square$

Proposition 3.13. Let R be a hyperring with 1_R and $f \in \text{Hom}_R^s(A, B)$ be partially $\epsilon_{B,A}^*$ -monic. If for $a_1, a_2 \in A$, $f(a_1) \cap f(a_2) \neq \emptyset$, then a_1 and a_2 are pro-equal.

Proof. Assume $f(a_1) \cap f(a_2) \neq \emptyset$. Then for an arbitrary $r \in R$, $f(ra_1) \cap f(ra_2) = rf(a_1) \cap rf(a_2) \neq \emptyset$. Now consider $j_1, j_2 \in \text{Hom}_R(R, A)$, with $j_1(r) = ra_1$ and $j_2(r) = ra_2$ for all $r \in R$. Thus

$$\begin{aligned} f(j_1(r)) \cap f(j_2(r)) \neq \emptyset & \implies \overline{\epsilon_C^*(f(j_1(r)))} \cap \overline{\epsilon_C^*(f(j_2(r)))} \neq \emptyset \\ & \implies \overline{\epsilon_C^*(f \circ j_1)(r)} \cap \overline{\epsilon_C^*(f \circ j_2)(r)} \neq \emptyset \end{aligned}$$

for all $r \in R$. So $f \circ j_1 \ddot{=} f \circ j_2$. According to the assumption, we obtain $j_1 \ddot{=} j_2$, i.e., $\overline{\epsilon_B^*(j_1(r))} \cap \overline{\epsilon_B^*(j_2(r))} \neq \emptyset$ for all $r \in R$. Therefore $\overline{\epsilon_A^*(ra_1)} \cap \overline{\epsilon_A^*(ra_2)} \neq \emptyset$ for all $r \in R$ and the proof is complete. $\square$

Proposition 3.14. Let R be a hyperring with 1_R and A be a unitary R -hypermodule. If $f \in \text{Hom}_R^s(A, B)$ is partially $\epsilon_{B,A}^*$ -monic, then f is strongly ϵ_A^* -injective.

Proof. Suppose $f(a_1) \cap f(a_2) \neq \emptyset$. Now Proposition 3.13 implies $\overline{\epsilon_A^*(ra_1)} = \overline{\epsilon_A^*(ra_2)}$. Now it immediately follows from Case 1 of Proposition 3.5, $\epsilon_A^*(a_1) = \epsilon_A^*(a_2)$. $\square$

Clearly, every (strongly) $\epsilon_{B,A}^*$ -injective R - mv -homomorphism $f \in \text{Hom}_R(A, B)$ is (strongly) ϵ_A^* -injective.

Proposition 3.15. *Let R be a hyperring. Suppose R has 1_R and B is a unitary R -hypermodule. If $f \in \text{Hom}_R^s(A, B)$ is monic in ${}_R\mathcal{G}\text{-mod}$, then f is injective.*

Proof. Let $f(a) = f(a')$ and consider $\bar{a}, \bar{a}' \in \text{Hom}_R(R, A)$ with $\bar{a}(r) = \{ra\}$ and $\bar{a}'(r) = \{ra'\}$. Then $(f \circ \bar{a})(r) = f(ra) = rf(a) = rf(a') = f(ra') = (f \circ \bar{a}')(r)$. Hence $\bar{a}(r) = \bar{a}'(r)$. Since $\bar{a}(1_R) = \bar{a}'(1_R)$ and A is unitary, we have $a = a'$. $\square$

Proposition 3.16. *If $f \in \text{Hom}_R(B, C)$ is strongly injective, then f is monic in ${}_R\mathcal{G}\text{-mod}$.*

Proof. Suppose $f \circ g = f \circ h$ for $g, h \in \text{Hom}_R(A, B)$. Let $a \in A$ and $b \in g(a)$. Clearly, $f(b) \subseteq f(g(a)) = f(h(a))$. This means $f(b) \cap f(b') \neq \emptyset$ for some $b' \in h(a)$. Thus $b = b'$. So $g(a) \subseteq h(a)$. Similarly, $g(a) = h(a)$. Hence $g = h$. $\square$

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A STUDY ON BIJECTIVE SOFT HEMIRINGS

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Abstract. The concept of bijective soft hemirings is firstly proposed and some of related properties are investigated. Especially the basic operations of soft bijective hemirings are discussed and some good examples are also given. We also define a bijective soft ideal and discuss the primary operations of bijective soft h -ideals (k -ideals, strong k -ideals) and idealistic soft hemirings.

Keywords: bijective soft hemirings, bijective soft h -ideals, hemirings.

1. Introduction

The traditional classical models usually fail to conquer the complexities of inconclusive data in many important areas. In 1999, the initial conception of soft sets was firstly proposed by Molodtsov [28]. Molodtsov had undergone great increase and applications in many fields. Maji et al. [26] gave an application of soft set theory in decision making problems through using rough sets, and constructed a theoretical investigation on soft sets in a particular method. Many authors researched on applications of soft sets. Ma et al. [22] gave a survey of decision making methods based on certain hybrid soft set models. Two significant notions of a novel uncertain soft model and a novel soft rough set were given in [36, 39]. Alcantud [4, 5] proposed a novel algorithm for fuzzy soft sets and investigated some formal relationships among soft sets and their extensions. Sun et al. [29, 30] investigated fuzzy rough sets and its many important applications in our life. In 2017, Alcantud et al. [6, 7] researched the problem of collective identity in a fuzzy environment, and put forward the rational fuzzy and sequential fuzzy choice. And then, the study on the soft set theory has been comprehensively learned by a lot of authors. Çağman et al. [3, 13] put forward a rational definition of products of soft sets and uni-int decision method to tackle the uncertain problems. In 2010, a sub soft set of the cartesian product of the soft sets and many associated notions were introduced by Babitha et al. [12]. Xiao et al. [32] come up with the concept of exclusive disjunctive soft sets

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and investigated some of its operations. Gong et al. [16] defined the notion of bijective soft sets, investigated some operations of it, and obtained its several properties. The concept of soft groups and many notions of group theory can be expanded in an elementary way to develop the theory of soft groups were introduced by Aktaş and Çağman [13]. Jun and Park [20] investigated idealistic soft BCK/BCI-algebras and soft ideals. Acar et al. [1] put forward the initial notion of soft rings. Aygünöğlu et al. [11] proposed the notions of fuzzy soft groups and its structural features and properties were researched. Atagun and Sezgin [10] proposed the concepts of soft ideal of a ring, soft sub-ring, and sub fields of a field and researched them. In 2015, Aktaş [2] put forward the definition of bijective soft groups and demonstrated its basic properties with a theoretical theory research, by using Aktaş's definition of soft group and Gong's definition of bijective soft sets. In 2017, Han et al. [17] aimed to find better algorithms for solving parameter reduction problems of soft sets, and gave their potential applications. In these papers, Aktaş discussed some significant properties of soft groups. Meanwhile, this theory is very important in some different investigate fields such as approximate reasoning, information sciences with intelligent systems, decision making and decision support systems, we can see some examples in [14, 15, 27, 32, 41]. Based on soft sets, many algebraic structures have been introduced such as soft ordered semigroups [19], soft rings [1], soft int-groups [13], soft BCI-algebras [21].

As we know that the hemiring is a special and important algebraic structure, some authors investigated the speciality of hemiring. Torre [31] introduced a definition of h -ideals of hemirings and showed its properties. He established some ring theorems for the hemirings, by using h -ideals. Especially, Jun investigated some properties of hemirings in [18]. Ma and Zhan [23, 40] studied some characteristics of h -hemiregular hemirings. Furthermore, Yin [34, 33] constructed some properties of h -semisimple and h -intra-hemiregular hemirings. Seeing from [9, 23, 24], Allen pointed some generalized fuzzy h -ideals of hemirings.

This article consists of five parts. In section 2, we review some basic knowledge of soft sets, bijective soft groups, hemirings, ideals (h -ideals, k -ideals, strong k -ideals) and so on. In section 3, the concept of bijective soft hemirings is given, its basic properties are investigated and some fundamental applications of bijective soft hemirings are mentioned. In section 4, we give a definition of bijective soft ideals (h -ideals, k -ideals, strong k -ideals) and some primary operations of them.

2. Basic terminologies

Some basic terminologies about soft sets, ideals and hemirings are given. A semiring is an algebraic system $(S, +, \cdot)$ consisting of a non-empty set S together with two binary operations on S called addition and multiplication (denoted in the usual manner) such that $(S, +)$ and $(S, \cdot)$ are semigroups and the following distributive laws:

For the sake of simplicity, we shall write ab for $a \cdot b$ ($a, b \in S$),
 $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$ are satisfied for all $a, b, c \in S$.

By *zero* of a semiring $(S, +, \cdot)$, we mean an element $0 \in S$ such that $0 \cdot x = x \cdot 0 = 0$ and $0 + x = x + 0 = x$ for all $x \in S$. A semiring with *zero* and a commutative semigroup $(S, +)$ is called a hemiring.

A subhemiring of a hemiring S is a subset A of S closed under addition and multiplication. A subset A of S is called a left (right) ideal of S if A is closed under addition and $SA \subseteq A$ ($AS \subseteq A$).

A subhemiring (left ideal, right ideal, ideal) of S is called an h -subhemiring (left h -ideal, right h -ideal, h -ideal) of S , respectively, if for any $x, z \in S$, $a, b \in A$, $x + a + z = b + z$ it follows $x \in A$.

The h -closure $\bar{A}$ of a subset A of S is defined as: $\bar{A} = \{x \in S \mid x + a + z = b + z \text{ for some } a, b \in A, z \in S\}$.

From now on, S is a hemiring, U is an initial universe, E is a set of parameters, $P(U)$ is the power set of U and $A, B, C \subseteq E$.

Definition 2.1 ([28]). A pair (F, A) is called a soft set over U , where $A \subseteq E$, F is a mapping given by $F: A \rightarrow P(U)$. In other words, a soft set over U is a parameterized family of subsets of the universe U . For $\varepsilon \in A$, $F(\varepsilon)$ may be considered as the set of ε -approximate.

Definition 2.2 ([31]). An ideal I of S is called a k -ideal of S , if $x \in S$, $a, b \in I$, $x + a = b$ implies $x \in I$.

Definition 2.3 ([35]). An ideal I of S is called a strong h -ideal, if $x, y, z \in S$, $a, b \in I$ and $x + a + z = y + b + z$ implies $x \in y + I$.

Definition 2.4 ([35]). Let (F, A) be a non-null soft set over S . Then:

- (1) (F, A) is called a soft hemiring over S if $F(x)$ is a subhemiring of S for all $x \in \text{supp}(F, A)$, where $\text{supp}(F, A) = \{a \in A \mid F(a) \neq \emptyset\}$,
- (2) (F, A) is called an idealistic soft semiring over S if $F(x)$ is an ideal of S for all $x \in \text{supp}(F, A)$, where $\text{supp}(F, A) = \{a \in A \mid F(a) \neq \emptyset\}$.

The bi-idealistic (k -idealistic, h -idealistic and strong h -idealistic) soft semirings are defined similarly.

Definition 2.5 ([8, 26]). Let (F, A) and (G, B) be two soft sets over a common universe U .

(1) The restricted-intersection of (F, A) and (G, B) is defined to the soft set (H, C) , where $C = A \cap B$, and $H: c \rightarrow P(U)$ is a mapping given by $H(c) = F(c) \cap G(c)$ for all $c \in C$. This is denoted by $(F, A) \tilde{\cap} (G, B) = (H, C)$,

(2) (F, A) AND (G, B) denoted by $(F, A) \tilde{\wedge} (G, B) = (H, A \times B)$ where $H(x, y) = F(x) \cap G(y)$ for all $(x, y) \in A \times B$,

(3) The extended union of (F, A) and (G, B) , denoted by $(F, A) \tilde{\cup} (G, B)$, is defined as the soft set (H, C) , where $C = A \cup B$ and $\forall x \in C$,

$$H(x) = \begin{cases} F(x), & \text{if } x \in A - B, \\ G(x), & \text{if } x \in B - A, \\ F(x) \cup G(x), & \text{if } x \in A \cap B. \end{cases}$$

Definition 2.6 ([2]). Let (F, A) be a soft group over G , where F is a mapping $F : A \rightarrow P(G)$ and A is a nonempty parameter set. We say that (F, A) is a bijective soft group, if (F, A) such that:

- (1) $\bigcup_{a \in A} F(a) = G$,
- (2) For any two parameters $a_i, a_j \in A, a_i \neq a_j, F(a_i) \cap F(a_j) = \{e\}$.

3. Bijective soft hemirings

In this section, we give a new concept of bijective soft hemirings and investigate some properties of it.

Definition 3.1. Let (F, A) be a soft hemiring over H , where H is a hemiring. Then (F, A) is a bijective soft hemiring, if (F, A) satisfies:

- (1) $\bigcup_{e \in A} F(e) = H$,
- (2) For any two parameters $e_i, e_j \in A, e_i \neq e_j, F(e_i) \cap F(e_j) = \{0\}$.

Example 3.2. Let $S = \{0, 1, 2, 3\}$ with the following Cayley tables:

Table 2					Table for a hemiring S				
+	0	1	2	3	·	0	1	2	3
0	0	1	2	3	0	0	0	0	0
1	1	1	2	3	1	0	1	1	1
2	2	2	2	3	2	0	1	1	1
3	3	3	3	2	3	0	1	1	1

We can obtain that the S is a finite hemiring. Let $A = \{0, 1\}$. Certainly, A is also a hemiring. Now, we define a mapping $G(x) = \{y | y\rho x \Leftrightarrow y = 2xa, a \in A\}$, $\forall x \in A$. From the operations, we can get that $G(0) = \{0\}$, $G(1) = \{0, 1\}$. It is easy to check that (G, A) is a soft hemiring of A . Since $G(0) \cap G(1) = \{0\}$ and $G(0) \cup G(1) = \{0, 1\} = A$. By the Definition 3.1, we can get that (G, A) is a bijective soft hemirings of A .

Definition 3.3. (1) (F, A) is said to be an identity bijective soft hemiring over H , if $F(x) = \{0\}$ for all $x \in A$, where $\{0\}$ is the identity element of hemiring.

(2) (F, A) is said to be an absolute bijective soft hemiring over H , if $F(x) = H$ for all $x \in A$.

Theorem 3.4. Let (F, A) and (G, B) be two bijective soft hemirings over S with $A \cap B \neq \emptyset$. Then their intersection $(F, A) \tilde{\cap} (G, B)$ is still a bijective soft hemiring over S .

Proof. By Definition 2.5 (1), $(F, A) \tilde{\cap} (G, B) = (H, C)$, where $H(c) = F(c) \cap G(c)$ for all $c \in C$ where $C = A \cap B \neq \emptyset$. Since (F, A) and (G, B) are two bijective soft hemirings over common S . Then, we can obtain that every $H(c)$ is a hemiring of S , for all $c \in C$, where $C = A \cap B$. That is $(F, A) \tilde{\cap} (G, B)$ is a soft hemiring of S . By Definition 3.1, we have that $\bigcup_{x \in A} F(x) = S$, and $F(x_i) \cap F(x_j) = \{0\}$, for any two parameters $x_i, x_j \in A$, $x_i \neq x_j$; $\bigcup_{y \in B} G(y) = S$, $G(y_i) \cap G(y_j) = \{0\}$ for any two parameters $y_i, y_j \in B$, $y_i \neq y_j$. Then $\bigcup_{x \in C=A \cap B} H(x) = \bigcup_{x \in C} (F(x) \cap G(x)) = (\bigcup_{x \in C} F(x)) \cap (\bigcup_{x \in C} G(x)) = S \cap S = S$. Let $\forall a \in C, b \in C, a \neq b$ where $C = A \cap B$, since $H(a) \cap H(b) = (F(a) \cap G(a)) \cap (F(b) \cap G(b)) = (F(a) \cap F(b)) \cap (G(a) \cap G(b)) = \{0\} \cap \{0\} = \{0\}$. Thus, $(F, A) \tilde{\cap} (G, B)$ is a bijective soft hemiring over S . $\square$

Theorem 3.5. If (F, A) and (G, B) are two bijective soft hemirings over S , then $(F, A) \text{ AND } (G, B)$ is also a bijective soft hemiring.

Proof. Since (F, A) and (G, B) are two bijective soft hemirings, then $F(x)$ and $G(y)$ are two subhemirings of S . Thus, we can get that $H(x, y) = F(x) \cap G(y)$ is also a subhemiring of S , for all $(x, y) \in A \times B$. Then, $(F, A) \tilde{\wedge} (G, B)$ is a soft hemiring of S . Because the $\bigcup_{(x,y) \in A \times B} H(x, y) = \bigcup_{(x,y) \in A \times B} F(x) \cap G(y) = (\bigcup_{x \in A} F(x)) \cap (\bigcup_{y \in B} G(y)) = S \cap S = S$. Suppose that $(a_i, a_j) \in A \times B$ where $a_i = (a_1, b_1)$, $a_j = (a_2, b_2)$ and $a_i \neq a_j$. That is $a_1 \neq a_2$ or $b_1 \neq b_2$, $a_1 \neq a_2$ and $b_1 \neq b_2$. When $a_1 \neq a_2$ or $b_1 \neq b_2$, we have

$$\begin{aligned} H(a_i) \cap H(a_j) &= (F(a_1) \cap G(b_1)) \cap (F(a_2) \cap H(b_2)) \\ &= (F(a_1) \cap F(a_2)) \cap (H(b_1) \cap H(b_2)) \\ &= \{0\} \cap (H(b_1) \cap H(b_2)) \\ &= \{0\}. \end{aligned}$$

or

$$\begin{aligned} H(a_i) \cap H(a_j) &= (F(a_1) \cap G(b_1)) \cap ((F(a_2) \cap H(b_2))) \\ &= (F(a_1) \cap F(a_2)) \cap (H(b_1) \cap H(b_2)) \\ &= (F(a_1) \cap F(a_2)) \cap \{0\} \\ &= \{0\}. \end{aligned}$$

When $a_1 \neq a_2$ and $b_1 \neq b_2$, we obtain that

$$\begin{aligned} H(a_i) \cap H(a_j) &= (F(a_1) \cap G(b_1)) \cap (F(a_2) \cap H(b_2)) \\ &= (F(a_1) \cap F(a_2)) \cap (H(b_1) \cap H(b_2)) \\ &= \{0\} \cap \{0\} \\ &= \{0\}. \end{aligned}$$

That is $H(a_i) \cap H(b_j) = \{0\}$. Thus, we prove that $(F, A) \tilde{\wedge} (G, B)$ is a bijective soft hemiring over S . $\square$

Theorem 3.6. Let (F, A) be a bijective soft hemiring over S and (H, B) be an identity soft hemiring, then $(F, A) \tilde{\cup} (H, B)$ is a bijective soft hemiring.

Proof. Since (F, A) is a bijective soft hemiring over S and (H, B) is an identity soft hemiring, by Definition 2.5 (3), for $\forall x \in C$, where $C = A \cup B$. We have:

$$H(x) = \begin{cases} F(x), & \text{if } x \in A - B, \\ 0, & \text{if } x \in B - A, \\ F(x) \cup G(x), & \text{if } x \in A \cap B. \end{cases}$$

If $A \cap B = \emptyset$, then $H(x) = F(x)$ or $\{0\}$. If $H(x) = F(x)$ for $x \in A - B$. Since (F, A) is a bijective soft hemiring over S , so is (H, C) . If $H(x) = \{0\}$ for $x \in B - A$, thus (H, C) is a trivial bijective soft hemiring over S . If $A \cap B \neq \emptyset$, then $H(x) = F(x) \cup \{0\} = F(x)$. It is easy to check that (H, C) is a bijective soft hemiring. Above all, thus $(F, A) \tilde{\cup} (G, B)$ is a bijective soft hemiring. $\square$

Definition 3.7. Let (F, A) and (H, K) be two bijective soft hemirings over S . Then (H, K) is a bijective soft subhemiring of (F, A) , written $(H, K) \tilde{<} (F, A)$, if:

- (1) $K \subseteq A$,
- (2) $H(x) \subseteq F(x)$ for all $x \in K$.

Example 3.8. Consider that $S = \{0, a, b, c\}$ is set with a multiplication operation an addition $(+, \cdot)$ as follow:

Table 2					Table for a hemiring S				
$+$	0	a	b	c	$\cdot$	0	a	b	c
0	0	a	b	c	0	0	0	0	0
a	a	0	b	c	a	0	a	0	a
b	b	c	0	a	b	0	0	b	b
c	c	b	a	0	c	0	a	b	c

Let $A = S$, $K = \{a, b, c\}$, $K \subset A$. It is obvious that S is a hemiring. We give a mapping, where $F(x) = \{y \in S \mid xpy \Leftrightarrow y = x^n \text{ for some } n \in N\}$ for all $x \in A$. Here $x^0 = 0$. Then $F(0) = \{0\}$, $F(a) = \{0, a\}$, $F(b) = \{0, b\}$, $F(c) = \{0, c\}$, which are subhemirings of S . Since $F(e_i) \cap F(e_j) = \{0\}$, for $\forall e_i$, $e_i \neq e_j$, $e_j \in A$, and $F(0) \cup F(a) \cup F(b) \cup F(c) = S$. Hence, it is easy to check that (F, A) is a bijective soft hemiring. Since $K \subset A$, $F(a) \cup F(b) \cup F(c) = S$ and $F(e_i) \cap F(e_j) = \{0\}$, for $\forall e_i$, $e_i \neq e_j$, $e_j \in K$. So, we obtain that (F, K) is a bijective soft hemiring of S . It is easy to check that $F(x) \subseteq F(x)$ for all $x \in K$. Then (F, K) is a bijective soft subhemiring of (F, A) . That is $(F, K) \tilde{<} (F, A)$.

Definition 3.9. Let (F, A) be a bijective soft hemiring over S and (H, B) be a soft subhemiring of (F, A) . Then (H, B) is called a bijective soft hemirng of (F, A) , if it satisfies :

- (1) $\bigcup_{a \in B} H(a)$ is a subhemiring of S ,
- (2) For any two parameters $a_i, a_j \in B, a_i \neq a_j, H(a_i) \cap H(a_j) = \{0\}$.

If $(H, B) = \{S\}$ and $(H, B) = \{0\}$, then (H, B) is a bijective soft subhemiring of (F, A) . In this case, (H, B) is called trivial bijective soft hemiring.

Example 3.10. Let (H, B) be a soft set over Z_6 , where $A=B=S=\{\bar{0}, \bar{1}\}$ and $H : B \rightarrow P(Z_6)$ is defined by $H(x) = \{y \in Z_6 | x\rho y \iff xy = \bar{0}\}$ for $x \in B$. And suppose that (F, A) is a absolute bijective soft hemiring of S , that is $F(x) = S$ for all $x \in B$. Then $H(\bar{0}) = Z_6$ is a subhemiring of $F(\bar{0})$, $H(\bar{1}) = \{\bar{0}\}$ is a subhemiring of $F(\bar{1})$, although $\bar{0} \neq \bar{1}$, $H(\bar{1}) \cap H(\bar{0}) = \{0\}$, $H(\bar{1}) \cup H(\bar{0}) = \{0, 1\} = S$. Hence, by the Definition 3.7, we have that (H, B) is a bijective soft subhemiring of (F, A) .

Proposition 3.11. Let (F, A) be a bijective soft hemiring over S and $\{(H_i, (K_i) | i \in I\}$

- (1) If $\bigcap_{i \in I} K_i \neq \emptyset$, then $\bigcap_{i \in I} (H_i, K_i)$ is a bijective soft subhemiring of (F, A) ,
- (2) $\bigwedge_{i \in I} (H_i, K_i)$ is a bijective soft subhemiring of $\bigwedge_{i \in I} (F, A)$,
- (3) If $K_i \cap K_j = \emptyset$ for all $i, j \in I$ with $i \neq j$, then $\bigcup_{i \in I} (H_i, K_i)$ is a bijective soft subhemiring of (F, A) ,

Proof. It is straightforward. □

Definition 3.12. Let (F, A) and (H, B) be two bijective soft hemirings over H and K , respectively, and define $f : H \rightarrow K$ and $g : A \rightarrow B$ as two functions. Then we say (f, g) is a bijective soft homomorphism of soft hemiring to (H, B) , denoted by $(F, A) \sim (H, B)$, if the following conditions are satisfied:

- (1) f is a homomorphism from G to K ,
- (2) g is a mapping from A to B ,
- (3) $f(F(x)) = H(g(x))$ for all $x \in A$.

Proposition 3.13. Let (F, A) be a bijective soft hemiring over S , (H, B) be a soft hemiring over K and (f, g) be a soft hemiring homomorphism from (F, A) to (H, B) . If f is a one to one function, then (H, B) is a bijective soft hemiring over K .

Proof. Assume that (f, g) is a soft homomorphism, f is a one to one function and $g(x), g(y) \in B$ such that $g(x) \neq g(y)$. Then we have $H(g(x)) \cap H(g(y)) = f(F(x)) \cap f(F(y)) = f(F(x) \cap F(y)) = f(\{0\}) = \{0\}$. Since (F, A) is a bijective soft hemiring. We have $\bigcup_{g(x) \in B} H(g(x)) = \bigcup_{x \in A} f(F(x)) = f(\bigcup_{x \in A} F(x)) = f(S) = K$. This is completed the proof. □

Proposition 3.14. Let (F, B) and (H, B) two be bijective soft hemirings of two hemirings S and K , respectively, over S . If (H, B) is a bijective soft subhemiring of (F, A) and f is a homomorphism over S to another hemiring K , then $(f(F), B)$ is a bijective soft subhemiring of $(f(F), A)$.

Proof. Since f is a homomorphism from S to K , $f(F(x))$ and $f(H(y))$ are bijective soft hemirings of K for any $x \in A$, and $y \in B$. Then $(f(F), A)$ and $(f(H), B)$ are both bijective soft hemirings over K . If (H, B) is a bijective soft subhemiring of (F, A) , then $H(y)$ is a bijective soft subhemiring of $F(y)$ and $f(H(y))$ is a subhemiring of $f(F(y))$ for any $y \in B$. $f(H(y_1)) \cap f(H(y_2)) = f(H(y_1) \cap H(y_2)) = f(0) = \{0\}$ and $\bigcup_{x \in B} f(H(x)) = f(\bigcup_{x \in B} H(x)) = f(S) = K$. By Definition 3.6, we obtain immediately that $(f(H), B)$ is a bijective soft subhemiring of $(f(F), A)$. $\square$

4. Bijective soft ideals

In this section, we put forward a new notion of bijective soft ideals and research some important properties of it. And some basic operations of these ideals are also studied.

Definition 4.1. Let (F, A) be a bijective soft hemiring over H . A non-null soft set (I, B) over S is called a bijective soft ideal (h -ideal, k -ideal, k -ideal, strong k -ideal) of (F, A) , if it satisfies the following conditions:

- (1) $B \subseteq A$,
- (2) $I(x)$ is an ideal (h -ideal, k -ideal, k -ideal, strong k -ideal) of $F(x)$ and $\bigcup_{x \in B} I(B) = H$, $I(x_i) \cap I(x_j) = \{0\}$, $x_i \neq x_j$, $x_i, x_j \in B$, for all $x \in \text{Supp}(I, B)$.

Example 4.2. Let $H = \{0, a, b\}$ be a set with a multiplication operation and an addition $(+, \cdot)$ as follow:

Table 3				Table for a hemiring H			
$+$	0	a	b	$\cdot$	0	a	b
0	0	a	b	0	0	0	0
a	a	0	b	a	0	0	0
b	b	b	b	b	0	0	0

Let $B = \{0, a, b\}$, $A = \{a, b\}$. From the table, we can see that A and B are two hemirings. Two functions are given: $I(x) = F(x) = \{y \in S \mid x\rho y \Leftrightarrow y = x^n \text{ for some } n \in N\}$ for all $x \in A$. Here $x^0 = 0$. Then we can get $F(0) = \{0\}$, $I(a) = F(a) = \{0, a\}$, $I(b) = F(b) = \{0, b\}$. It is proved that $IF \subseteq I$, and $FI \subseteq I$. Besides that $\bigcup_{x_i \in A} I(x_i) = H$, and $I(x_i) \cap I(x_j) = \{0\}$, $x_i \neq x_j$, $x_i, x_j \in A$. By the Definition 4.1, we obtain that (I, B) is a bijective ideal of (F, A) .

Theorem 4.3. Let (F, A) and (G, B) be two bijective soft h -ideals (strong h -ideals, k -ideals) over a hemiring H . Then $(F, A) \tilde{\cap} (G, B)$ is a bijective soft h -ideal (k -ideal, strong k -ideal) over H if it is non-null.

Proof. By Definition 2.5 (1), we can write $(F, A)\tilde{\cap}(G, B) = (\gamma, C)$, where $C = A \cap B$ and $\gamma(c) = F(c) \cap G(c)$ for all $c \in C$. Suppose that (γ, C) is a non-null soft set over H . If $c \in C$, then $\gamma(c) = F(c) \cap G(c) \neq \emptyset$. Since (F, A) and (G, B) are two bijective soft h -ideals, thus the nonempty sets $F(c)$ and $G(c)$ are h -ideals of H . It follows that $\gamma(x)$ is a h -ideal of H , for all $c \in I$.

In the following, we will prove that $\bigcup_{x \in I} \gamma(x) = H$, and $I(x_i) \cap I(x_j) = \{0\}$ for every $x_i \neq x_j$, $x_i, x_j \in I$.

(1) $\bigcup_{x \in I} \gamma(x) = \bigcup_{x \in A \times B} \gamma(x) = \bigcup_{a \in A} \bigcup_{b \in B} (F(a) \cap G(b)) = (\bigcup_{a \in A} F(a)) \cap (\bigcup_{b \in B} G(b)) = H \cap H = H$.

(2) Suppose that $(a_i, a_j) \in A \times B$, where $a_i = (a_1, b_1)$ and $a_j = (a_2, b_2)$ and $a_i \neq a_j$. Then $\gamma(a_i) \cap \gamma(a_j) = (F(a_1) \cap G(b_1)) \cap (F(a_2) \cap G(b_2)) = (F(a_1) \cap F(a_2)) \cap (G(b_1) \cap G(b_2)) = \{0\} \cap \{0\} = \{0\}$.

Above all, thus $(F, A)\tilde{\cap}(G, B)$ is a bijective soft h -ideal over H . The k -ideals and strong k -ideals have the similar properties, and the proofs are also similar. $\square$

Theorem 4.4. Let (F, A) and (G, B) be two bijective soft h -ideals (k -ideals, strong k -ideals) over H . If A and B are disjoint, then $(F, A)\tilde{\wedge}(G, B)$ is also a bijective soft h -ideal over H .

Proof. The proof is similar to Theorem 4.3. $\square$

Theorem 4.5. Let (I_1, A_1) and (I_2, A_2) be two bijective soft h -ideals of two bijective soft hemirings, (F, A) and (G, B) over H respectively. Then $(I_1, A_1)\tilde{\cap}(I_2, A_2)$ is a bijective soft h -ideal of $(F, A)\tilde{\cap}(G, B)$.

Proof. The proof is easily omitted. $\square$

Theorem 4.6. Let $(F_i, A_i)_{i \in I}$ be a non-empty family of bijective soft h -ideals of a bijective soft hemiring (F, A) over H . Then:

- (1) $\bigcap_{i \in I} (F_i, A_i)$ is a bijective soft h -ideal of (F, A) , if it is non-null.
- (2) $\bigwedge_{i \in I} (F_i, A_i)$ is a bijective soft h -ideal of (F, A) , if it is non-null.
- (3) $\bigcup_{i \in I} (F_i, A_i)$ is a bijective soft h -ideal of (F, A) , if $\{A_i, i \in I\}$ are pair wise disjoint, whenever it is non-null.

Proof. It is straightforward. $\square$

Proposition 4.7. Let (F, A) be a bijective h -idealistic (k -idealistic) soft hemiring over H . If $B \subset A$, then (F, B) is an h -idealistic (k -idealistic) soft hemiring over H , if $B = A - \{x_i\}$, where $F(x_i) = \{0\}$ and it is non-null.

Proof. Since (F, A) is an h -idealistic bijective soft hemiring over H , then $F(x)$ is an h -ideal of H , for all $x \in \text{supp}(F, A)$. Since $B = A - \{x_i\}$, $F(x)$ is also an h -ideal of H for all $x \in \text{supp}(F, B)$. Because (F, A) is a bijective soft hemiring over H and $F(x_i) = \{0\}$, we can get $\bigcup_{x \in A} F(x) = \bigcup_{x \in B} F(x) \cup F(x_i) = \bigcup_{x \in B} F(x) =$

H , $F(x_j) \cap F(x_k) = \{0\}$, $x_j \neq x_k$ for all $x_j, x_k \in A$. It is easy to check that $\bigcup_{x \in B} F(x) = H$ and $F(x'_i) \cap F(x'_j) = \{0\}$ for $x'_i \neq x'_j$, $x'_i, x'_j \in B$. Thus (F, B) is an h -idealistic soft hemiring over H . $\square$

Example 4.8. Let $A = \{0, a, b, c\}$, $B = \{a, b, c\}$, we define a mapping F , where $F(0) = \{0\}$, $F(a) = \{0, a\}$, $F(b) = \{0, b\}$, $F(c) = \{0, c\}$. Then we can easily to check that (F, A) is a bijective h -idealistic soft hemiring.

Theorem 4.9. Let (I_1, A_1) and (I_2, A_2) be bijective soft h -ideal of a bijective soft hemiring (F, A) over H . Then :

- (1) $(I_1, A_1) \tilde{\cap} (I_2, A_2)$ is a bijective soft h -ideal of (F, A) if it is non-null.
- (2) $A_1 \cap A_2 = \emptyset$, then $(I_1, A_1) \tilde{\cup} (I_2, A_2)$ is a bijective soft h -ideal of (F, A) .

Proof. (1) Let $(I_1, A_1) \tilde{\cap} (I_2, A_2) = (I, B)$, where $B = A_1 \cap A_2 \subset A$. We can prove $I_1(x)$ is a bijective soft h -ideal of $F(x)$ for all $x \in A_1$, so is $I_2(x)$. Therefore $I(x) = I_1(x) \cap I_2(x) \neq \emptyset$. Since (I, B) is non-null. We can obtain $I(x)$ is a bijective h -ideal of $F(x)$, for all $x \in B$. That is (I, B) is a bijective soft h -ideal of (F, A) .

(2) Let $(I_1, A_1) \tilde{\cup} (I_2, A_2) = (I, C)$, where $C = A_1 \cup A_2$ and $\forall x \in C \subset A$. By Definition 2.5 (3), $I(x) = I_1(x)$ is a non-empty h -ideal of $F(x)$, $\forall x \in \text{supp}(I, C)$. Also $I(x) = I_1(x)$ is a non-empty h -ideal of $F(x)$, $\forall x \in \text{supp}(I, C)$. Therefore $I(x)$ is a non-empty h -ideal of $F(x)$, $\forall x \in \text{supp}(I, C)$. Hence (I, C) is a bijective soft h -ideal of (F, A) . $\square$

5. Conclusion

In [16], Gong et al. firstly defined the concept of bijective soft sets, then Aktas [2] put forth a concept of bijective soft groups based on the bijective soft sets. Now, based on the bijective soft groups, the concept of bijective soft hemirings is proposed. But, it is different from the bijective soft groups. For example, some conditions of the two notions are different. For any two parameters $e_i, e_j \in A$, $e_i \neq e_j$, $F(e_i) \cap F(e_j) = \{e\}$ in the bijective soft groups, but $F(e_i) \cap F(e_j) = \{0\}$ in the bijective soft hemirings. Also, the definition of dependent and independent element of bijective soft groups cannot suit the bijective soft hemirings. They are two different structures, so the operations and characterizations are also different. The groups are closed under the multiplication but the hemirings are both closed under the addition and multiplication. We hope that our works can invoke some research topics for algebraic systems and provide more applications in many important fields.

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ON THE FIBONACCI NUMBERS OF THE MOLECULAR GRAPHS OF SOME BENT PHENYLENES

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Abstract. The Fibonacci number $f(G)$ of a graph $G = (V, E)$ is defined as the number of all subsets U of V such that no two vertices in U are adjacent. Phenylenes represent a class of condensed polycyclic conjugated compounds which have the molecular graph possessing both six-membered and four-membered circuits. In this paper we are concerned with special types of bent phenylenes expanding our previous results on the linear phenylenes. The explicit formulas for the Fibonacci numbers of the bent phenylenes are found as functions of the number n of hexagons in both mentioned branches of phenylene.

Keywords: Molecular graph, Fibonacci number, bent phenylene.

1. Introduction

A molecular graph in chemical graph theory is a representation of the structural formula of a chemical compound in terms of graph theory. Vertices of it correspond to the atoms of the compound and edges correspond to the chemical bonds. Many chemical structures and compounds are usually modeled by a molecular graph to analyze underlying theoretical properties.

Phenylenes are an important class of conjugated hydrocarbons. Characteristic structural features of the phenylenes are alternating fused benzene and cyclobutadiene rings (circuits) which can be arranged in linear, angular or branched geometries. It means that the six-membered circuits (hexagons) are adjacent only to four-membered circuits, and every four-membered circuit is adjacent to a pair of nonadjacent hexagons. If each six-membered circuit in the molecular graph of a phenylene is adjacent only to two four-membered circuits, we say that it is a $[N]$ phenylene chain, where N signifies the number of benzene units. The molecular graphs of several phenylenes are presented in Fig. 1. In

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particular, there are the linear [3]phenylene (a), the angular [3]phenylene (b) and the triangular [4]phenylene (c) as the case of a branched phenylene.

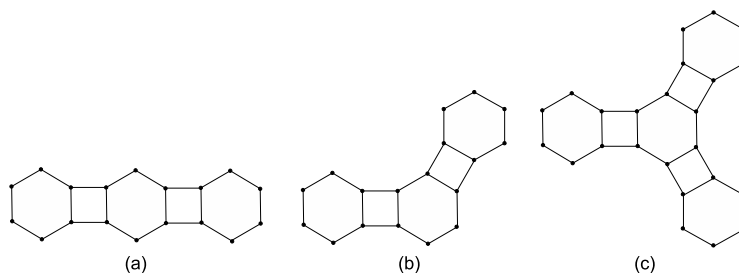


Figure 1: Examples of the molecular graphs of phenylenes

A topological index of a graph can be viewed as a numerical quantity which is invariant under isomorphism of graphs. Many topological indices are closely correlated with some physico-chemical characteristics of the respective compounds. Hexagonal systems are of the great importance for theoretical chemistry because they are the natural graph representations of the benzenoid hydrocarbons [2]. The structure of these graphs is apparently the simplest among all hexagonal systems [3]. Therefore the first results on topological indices were achieved for hexagonal chains. One of the most famous and interested topological indices is the Fibonacci number of a molecular graph. For the general graph-theoretic terminology we refer the reader to any of standard monographs, e.g. [10].

In the number theory the Fibonacci numbers F_n are defined by the second order recurrence $F_{n+2} = F_{n+1} + F_n$ with $F_0 = 0$, $F_1 = 1$. Similarly, the Lucas numbers L_n satisfy the same recurrence with the initial terms $L_0 = 2$, $L_1 = 1$. The total number of subsets of $\{1, 2, \dots, n\}$ such that no two elements are adjacent is the Fibonacci number F_{n+2} . In view of this fact Prodinger and Tichy introduced in 1982 the Fibonacci number of a graph [6].

Definition 1. Let $G = (V, E)$ be a simple graph. The Fibonacci number $f(G)$ of G is defined as the number of all subsets U of V such that no two vertices in U are adjacent.

The subset U of k mutually independent vertices is called the k -independent set of G . We denote $i(G, k)$ the number of the k -independent sets of G and $i(G, 0) = 1$ by definition for any graph G . Then, the Fibonacci number of G is given by the relation $f(G) = \sum_k i(G, k)$, where the summation is taken over all nonnegative integers k .

The chemists Merrifield and Simmons [4] elaborated a theory aimed at describing molecular structure by means of finite set topology. As their graph-topological considerations containing independent sets of vertices attracted wide attention there is used the name the Merrifield-Simmons index in chemistry instead of the Fibonacci number of a graph. However, we will use primarily the

name the Fibonacci number of a graph in this paper. In recent years, a lot of works have been published on the extremal problem for the Fibonacci number of graphs. Wagner and Gutman gave in [9] a survey which collects and classifies these results, and also provides some useful auxiliary tools and techniques that are used in the study of this type of problems.

Directly from Definition 1 it is easy to find the Fibonacci numbers for paths and circuits (rings).

Theorem 1. *Let P_n be a path with n vertices and C_n a circuit with n vertices. Then $f(P_n) = F_{n+2}$ and $f(C_n) = L_n$.*

The Fibonacci numbers for various classes of graph have been found. For example, Yeh [11] computed algorithmically the Fibonacci numbers of the lattice product graphs, Ren, He and Yang [7] found the Fibonacci number of zig-zag tree-type hexagonal systems and Alameddine [1] found upper and below bounds for the Fibonacci numbers of maximal outerplanar graphs on a given number of vertices.

2. Preliminary results

In this section, we remind some important and useful results for the following calculations.

Theorem 2 ([6]). *If G_1, G_2 are disjoint graphs then $f(G_1 \sqcup G_2) = f(G_1)f(G_2)$.*

Theorem 3 ([5]). *Let G be a graph with at least two vertices and v be its arbitrary vertex. Then for the Fibonacci number of G the formula $f(G) = f(G - v) + f(G - (v))$ holds, where $G - v$ is the subgraph of G obtained from G by deletion of the vertex v and $G - (v)$ is the subgraph of G obtained by deletion of the vertex v and all the vertices adjacent to v .*

Theorem 4 ([5]). *If vertices u, v are adjacent in a graph G then*

- a) $f(G) = f(G - \{u, v\}) + f(G - (u)) + f(G - (v))$, where $G - \{u, v\}$ is the subgraph of G obtained by deletion of the vertices u and v of G ,
- b) $f(G) = f(G - uv) + f(G - (u, v))$, where $G - uv$ is the subgraph of G obtained by deletion of the edge uv of G and $G - (u, v)$ is the subgraph of G obtained by deletion of the vertices u, v and all the vertices adjacent to them.

In [8] we expressed the Fibonacci number of the linear phenylene as a function of the number of its hexagons. We mention the principle of our considerations as it will be used in the next section.

The following formulas were derived from Theorem 3 by suitable choices of the vertex v in the particular cases.

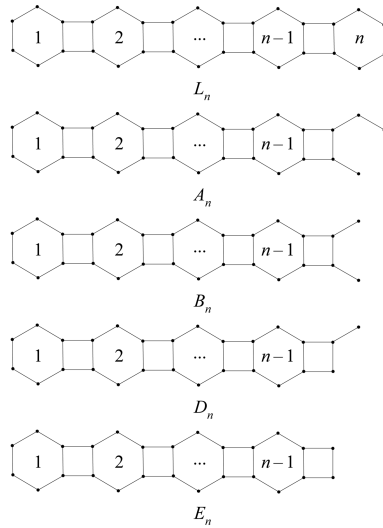


Figure 2: The molecular graph of the linear phenylene and its subgraphs

Lemma 1 ([8]). *Let L_n be the linear phenylene with n hexagons and let A_n , B_n , D_n , E_n be graphs as in Fig. 2. Then the following relations hold for any positive integer $n > 1$.*

- (1) $f(L_n) = f(A_n) + f(D_n),$
- (2) $f(A_n) = f(B_n) + f(D_n),$
- (3) $f(B_n) = 4f(L_{n-1}) + 4f(A_{n-1}),$
- (4) $f(D_n) = f(L_{n-1}) + f(A_{n-1}) + f(E_n),$
- (5) $f(E_n) = f(L_{n-1}) + 2f(A_{n-1}).$

We denote $f(L_n) = l_n$, $f(A_n) = a_n$, $f(B_n) = b_n$, $f(D_n) = d_n$ and $f(E_n) = e_n$ for short.

Theorem 5. *The values of the Fibonacci numbers for the graphs A_n are $a_n = (1/(\gamma - \delta)) [(199 - 13\delta)\gamma^{n-1} - (199 - 13\gamma)\delta^{n-1}]$ for any positive integer n , where $\gamma = (15 + \sqrt{241})/2$, $\delta = (15 - \sqrt{241})/2$.*

The proof is based on Lemma 1. After elimination of the remaining variables identities (1)-(5) lead to the homogeneous linear difference equation of the second order with constant coefficients $a_{n+2} - 15a_{n+1} - 4a_n = 0$. The general solution of this equation has the form $a_n = K_1\gamma^n + K_2\delta^n$, where K_1, K_2 are arbitrary real numbers. It is easy to calculate that $a_1 = 13$ and $a_2 = 199$, and therefore $K_1 = \frac{199-13\delta}{\gamma(\gamma-\delta)}$, $K_2 = \frac{199-13\gamma}{\delta(\gamma-\delta)}$, which gives the expression for a_n .

Using the expression for a_n and the relation $l_n = \frac{1}{6}a_{n+1} - \frac{7}{6}a_n$ we have the following result.

Theorem 6. *The Fibonacci number of the linear phenylene with n hexagons can be expressed in the form*

$$(6) \quad f(L_n) = l_n = \frac{1}{\gamma - \delta} [(18\gamma + 4)\gamma^{n-1} - (18\delta + 4)\delta^{n-1}]$$

for any positive integer n .

Remark. The closed expression for b_n , d_n , e_n are obtained by the similar way. For any positive integer we have

$$\begin{aligned} b_n &= \frac{1}{\gamma - \delta} [(124\gamma + 32)\gamma^{n-2} - (124\delta + 32)\delta^{n-2}], \\ d_n &= \frac{5}{\gamma - \delta} (\gamma^n - \delta^n), \\ e_n &= \frac{1}{\gamma - \delta} [(44\gamma + 12)\gamma^{n-2} - (44\delta + 12)\delta^{n-2}]. \end{aligned}$$

3. Main results

Now we consider the molecular graph $L_{n,n}$ of the bent phenylene which consists of two linear phenylenes L_n of the same length of $n \geq 2$. The phenylenes have the common hexagon (Fig.3). It is possible to use the results from the previous section.

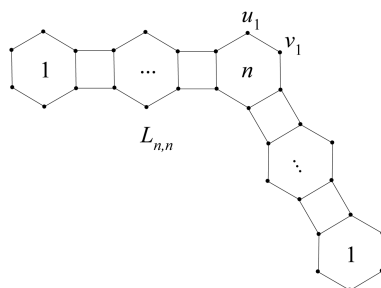


Figure 3: The molecular graph of the bent phenylene of the first type studied in section 3

First, we prove the following Lemma.

Lemma 2. *The terms of the sequence $\{f(L_{n,n})\}$ satisfy the relation*

$$(7) \quad f(L_{n,n}) = d_n^2 - 4a_{n-1}^2 - l_{n-1}^2 - 2l_{n-1}a_{n-1}$$

for any positive integer $n \geq 2$.

Proof. The relation can be derived by repeatedly using Theorem 2 and the both statements of Theorem 4. First, we choose the edge u_1v_1 in $L_{n,n}$ (see Fig. 3) and use statement (b) of Theorem 4. Then we choose the edge u_2v_2 in $L_{n,n} - u_1v_1 = L^{(1)}$ and use the statement (b) of Theorem 4. Finally, we use the vertices u_3, v_3 in $L_{n,n} - (u_1, v_1) = L^{(2)}$ and the statement (a) of Theorem 4 (see Fig.4).

So we have successively

$$\begin{aligned}
 f(L_{n,n}) &= f(L_{n,n} - u_1v_1) - f(L_{n,n} - (u_1, v_1)) \\
 &= f(L^{(1)}) - f(L^{(2)}) \\
 &= f(L^{(1)} - u_2v_2) - f(L^{(1)} - (u_2, v_2)) - (f(L^{(2)} - \{u_3, v_3\}) \\
 &\quad + f(L^{(2)} - (u_3)) + f(L^{(2)} - (v_3))) \\
 &= d_n d_n - 2a_{n-1}2a_{n-1} - (l_{n-1}l_{n-1} + l_{n-1}a_{n-1} + a_{n-1}l_{n-1})
 \end{aligned}$$

which gives the desired expression. $\square$

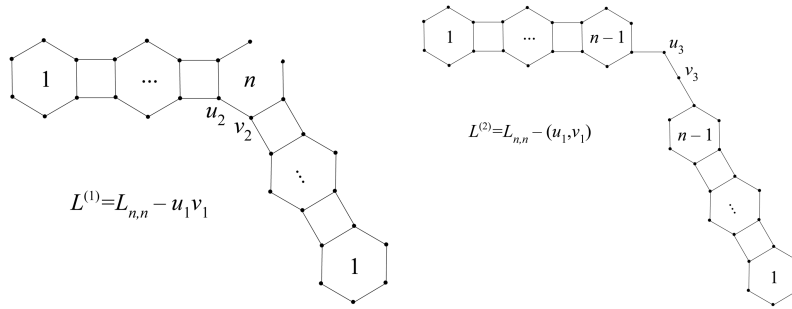


Figure 4: Subgraphs of the graph $L_{n,n}$

Lemma 3. For the roots $\gamma = (15 + \sqrt{241})/2$, $\delta = (15 - \sqrt{241})/2$ of the equation $x^2 - 15x - 4 = 0$ the following relations hold: $\gamma\delta = -4$, $\gamma^2 = 15\gamma + 4$, $\gamma^4 = 3495\gamma + 916$, $\delta^2 = 15\delta + 4$, $\delta^4 = 3495\delta + 916$.

Proof. These identities are trivial consequences of roots properties of a quadratic equation. For instance, $\gamma^4 = (15\gamma + 4)^2 = 225(15\gamma + 4) + 120\gamma + 16 = 3495\gamma + 916$. $\square$

Theorem 7. The Fibonacci number of the graph $L_{n,n}$ for any positive integer n can be written in the form

$$\begin{aligned}
 f(L_{n,n}) &= \frac{1}{(\gamma - \delta)^2} [(64547\gamma + 16916)\gamma^{2n-4} \\
 (8) \quad &\quad + (64547\delta + 16916)\delta^{2n-4} - 200(-4)^{n-2}].
 \end{aligned}$$

Proof. We will use Lemma 2 and the explicit formulas for l_n , a_n and d_n . Then we can write successively for any positive integer $n \geq 2$

$$\begin{aligned}
 f(L_{n,n}) &= \frac{1}{(\gamma - \delta)^2} (25(\gamma^n - \delta^n)^2 \\
 &\quad - 4[(199 - 13\delta)\gamma^{n-2} - (199 - 13\gamma)\delta^{n-2}]^2 - [(18\gamma + 4)\gamma^{n-2} - (18\delta + 4)\delta^{n-2}]^2
 \end{aligned}$$

$$\begin{aligned}
& -2[(199-13\delta)\gamma^{n-2} - (199-13\gamma)\delta^{n-2}][(18\gamma+4)\gamma^{n-2} - (18\delta+4)\delta^{n-2}] \\
& = \frac{1}{(\gamma-\delta)^2}([25\gamma^4 - 4(199-13\delta)^2 - (18\gamma+4)^2 - 2(199-13\delta)(18\gamma+4)]\gamma^{2n-4} \\
& \quad + [25\delta^4 - 4(199-13\gamma)^2 - (18\delta+4)^2 - 2(199-13\gamma)(18\delta+4)]\delta^{2n-4} \\
& \quad + [-800 + 8(199-13\delta)(199-13\gamma) + 2(18\gamma+4)(18\delta+4) \\
& \quad + 2(199-13\delta)(18\delta+4) + 2(199-13\gamma)(18\gamma+4)](-4)^{n-2} \quad \text{as } \gamma\delta = -4.
\end{aligned}$$

This expression can be simplified using the identities of Lemma 3. Further details of this part of the proof are left to readers.

Then

$$\begin{aligned}
f(L_{n,n}) &= \frac{1}{(\gamma-\delta)^2}[(75207\gamma + 10660\delta - 142984)\gamma^{2n-4} \\
& \quad + (75207\delta + 10660\gamma - 142984)\delta^{2n-4} \\
& \quad + (307480 - 20512\gamma - 20512\delta)(-4)^{n-2}] \\
&= \frac{1}{(\gamma-\delta)^2}[(64547\gamma + 10660(\gamma+\delta) - 142984)\gamma^{2n-4} \\
& \quad + (64547\delta + 10660(\gamma+\delta) - 142984)\delta^{2n-4} \\
& \quad + (307480 - 20512(\gamma+\delta))(-4)^{n-2}].
\end{aligned}$$

As $\gamma + \delta = 15$ and $\gamma - \delta = \sqrt{241}$, the formula (8) is obtained immediately for $n \geq 2$.

Moreover,

$$\begin{aligned}
f(L_{1,1}) &= \frac{1}{(\gamma-\delta)^2}[(64547\gamma + 16916)\gamma^{-2} + (64547\delta + 16916)\delta^{-2} - 200(-4)^{-1}] \\
&= \frac{1}{(\gamma-\delta)^2} \left[(64547\gamma + 16916)\frac{\delta^2}{16} + (64547\delta + 16916)\frac{\gamma^2}{16} - 200\left(-\frac{1}{4}\right) \right] \\
&= \frac{1}{(\gamma-\delta)^2} \left[\left(\frac{64547}{16}\gamma + \frac{4229}{4} \right) (15\delta+4) + \left(\frac{64547}{16}\delta + \frac{4229}{4} \right) (15\gamma+4) + 50 \right] \\
&= \frac{1}{241} \left[\frac{968205}{8}\gamma\delta + \frac{63991}{2}(\gamma+\delta) + 8508 \right] = 18 = f(C_6).
\end{aligned}$$

As $L_{1,1} = C_6$, the statement holds also for $n = 1$. □

Example. The previous function expression of $f(L_{n,n})$ can be used to find $f(L_{2,2})$ and $f(L_{3,3})$. In this case, $L_{2,2}$ and $L_{3,3}$ represent the molecular graphs of a bent [3]phenylene and of a bent [5]phenylene, respectively.

Thus,

$$\begin{aligned}
f(L_{2,2}) &= \frac{1}{(\gamma-\delta)^2}[(64547\gamma + 16916)\gamma^0 + (64547\delta + 16916)\delta^0 - 200(-4)^0] \\
&= \frac{1}{241}[64547(\gamma+\delta) + 33832 - 200] = 4157.
\end{aligned}$$

Similarly,

$$\begin{aligned}
 f(L_{3,3}) &= \frac{1}{(\gamma - \delta)^2} [(64547\gamma + 16916)\gamma^2 + (64547\delta + 16916)\delta^2 - 200(-4)] \\
 &= \frac{1}{(\gamma - \delta)^2} [(64547\gamma + 16916)(15\gamma + 4) + (64547\delta + 16916)(15\delta + 4) + 800] \\
 &= \frac{1}{(\gamma - \delta)^2} [968205(15\gamma + 4) + 511928\gamma + 67664 + 968205(15\delta + 4) + 511928\delta \\
 &\quad + 67664 + 800] = \frac{1}{241} [15035003(\gamma + \delta) + 7880968 + 800] = 968493.
 \end{aligned}$$

Further we consider the molecular graph $Z_{n,n}$ of the bent phenylene which consists of two linear phenylenes L_n of the same length of $n \geq 1$. In this case, the linear phenylenes are linked using a square (Fig. 5).

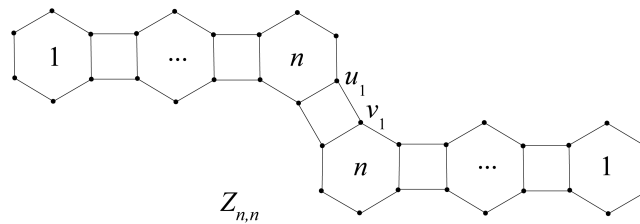


Figure 5: The molecular graph of the bent phenylene of the second type studied in section 3

First, we prove the following Lemma.

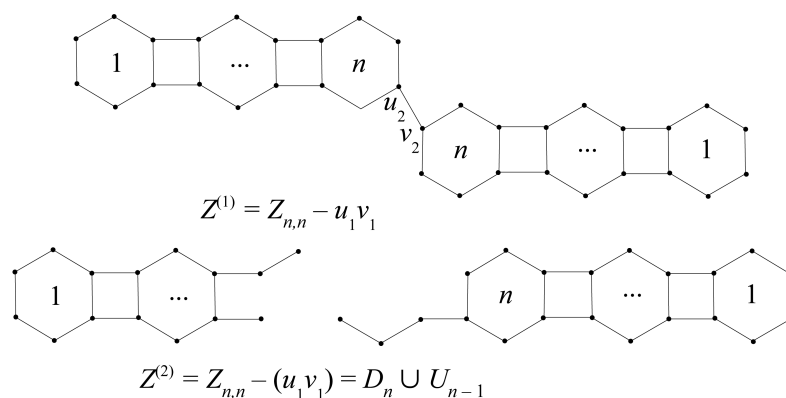
Lemma 4. *The terms of the sequence $\{f(Z_{n,n})\}$ satisfy the relation*

$$(9) \quad f(Z_{n,n}) = l_n^2 - 2d_n(3l_{n-1} + 2a_{n-1})$$

for any positive integer $n \geq 2$.

Proof. The relation can be derived by repeatedly using Theorem 2 and the both statements of Theorem 4. First we choose the edge u_1v_1 in $Z_{n,n}$ (see Fig. 5) and use the statement (b) of Theorem 4. Then we choose the edge u_2v_2 in $Z_{n,n} - u_1v_1 = Z^{(1)}$ and use the statement (b) of Theorem 4. Hence $f(Z_{n,n}) = f(Z_{n,n} - u_1v_1) - f(Z_{n,n} - (u_1, v_1)) = f(Z^{(1)}) - f(Z^{(2)})$.

It can be easily seen (Fig. 6) that $f(Z^{(2)}) = f(D_n)f(U_{n-1})$. If we choose the vertices u_3, v_3 in U_{n-1} and use the statement (a) of Theorem 4 (see Fig. 6), we obtain (with the help of Theorem 2) $f(U_{n-1}) = 2l_{n-1} + l_{n-1} + 2a_{n-1}$. Then $f(Z_{n,n}) = l_n l_n - d_n f(U_{n-1}) - d_n f(U_{n-1}) = l_n^2 - 2d_n(3l_{n-1} + 2a_{n-1})$, which was to be shown. $\square$

Figure 6: Subgraphs of the graph $Z_{n,n}$

Theorem 8. *The Fibonacci number of the graph $Z_{n,n}$ has the closed function expression*

$$(10) \quad f(Z_{n,n}) = \frac{1}{(\gamma - \delta)^2} [(4204\gamma + 1112)\gamma^{2n-2} + (4204\delta + 1112)\delta^{2n-2} - 3000(-4)^{n-2}]$$

for any positive integer n .

Proof. Using Lemma 2 and the explicit formulas for l_n , a_n and d_n , we get for $n \geq 2$

$$\begin{aligned} f(Z_{n,n}) &= \frac{1}{(\gamma - \delta)^2} \{ [(18\gamma + 4)\gamma^{n-1} - (18\delta + 4)\delta^{n-1}]^2 \\ &\quad - 10(\gamma^n - \delta^n)(3[(18\gamma + 4)\gamma^{n-2} - (18\delta + 4)\delta^{n-2}] \\ &\quad + 2[(199 - 13\delta)\gamma^{n-2} - (199 - 13\gamma)\delta^{n-2}]) \} \\ &= \frac{1}{(\gamma - \delta)^2} \{ [(18\gamma + 4)^2 - 800\gamma - 200]\gamma^{2n-2} + [(18\delta + 4)^2 - 800\delta - 200]\delta^{2n-2} \\ &\quad + [-2(18\gamma + 4)(18\delta + 4) + 10\delta^2(80\gamma + 20) + 10\gamma^2(80\delta + 20)](-4)^{n-2} \} \\ &= \frac{1}{(\gamma - \delta)^2} [(4204\gamma + 1112)\gamma^{2n-2} + (4204\delta + 1112)\delta^{2n-2} \\ &\quad + (24000\gamma\delta + 6200\gamma + 6200\delta)(-4)^{n-2}]. \end{aligned}$$

Since $\gamma + \delta = 15$ and $\gamma\delta = -4$, we arrive at the expression (10) for $f(Z_{n,n})$, if $n \geq 2$. Moreover, $f(Z_{1,1}) = \frac{1}{(\gamma - \delta)^2} [(4204\gamma + 1112)\gamma^0 + (4204\delta + 1112)\delta^0 - 3000(-4)^{-1}] = \frac{1}{(\gamma - \delta)^2} [4204(\gamma + \delta) + 2224 - 3000(-\frac{1}{4})] = \frac{1}{241} [63060 + 2224 + 750] = 274 = f(L_2)$ as $Z_{1,1} = L_2$. It completes the proof for all $n \geq 1$. $\square$

4. Conclusion

The total number of independent subsets of graph vertices finds its application mainly in organic chemistry. In particular, there exists the link between the

Merrifield-Simmons index and a boiling point of organic compounds. It is the reason why the name “Merrifield-Simmons index” is preferred in the literature to originally pure mathematical “Fibonacci number”. In case of molecular graphs, there exists a lot of works devoted to the calculation of the Merrifield-Simmons index for various classes of graphs. The method of calculation used in this paper and the obtained results can be generalized for other classes of molecular graphs.

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OFF-STEP DISCRETIZATION FOR SYSTEM OF NONLINEAR SINGULAR BOUNDARY VALUE PROBLEMS USING VARIABLE MESH

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Abstract. In this paper, we propose two generalized variable mesh schemes based on off-step points to solve the system of nonlinear singular two point boundary value problems. Theoretical analysis proves that the proposed methods have second, third and fourth order convergence. Both the methods are applicable to singular boundary value problems. Numerical results are also provided to show the accuracy and efficiency of the proposed methods.

Keywords: off-step, variable mesh, singular, nonlinear, system.

1. Introduction

In this paper, we present some effective numerical techniques using variable mesh based on off-step points to solve system of M nonlinear singular boundary value problems(BVPs) of the following type:

$$(1) \quad u_{xx}^{(i)} = F^{(i)}(x, u^{(1)}, \dots, u^{(i)}, \dots, u^{(M)}, u_x^{(1)}, \dots, u_x^{(i)}, \dots, u_x^{(M)}), a \leq x \leq b,$$

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subject to boundary conditions

$$(2) \quad u^{(i)}(a) = A_i, u^{(i)}(b) = B_i, \text{ where } u_x^{(i)} = \frac{du^{(i)}}{dx}, u_{xx}^{(i)} = \frac{d^2u^{(i)}}{dx^2}, i = 1(1)M$$

We assume that for $-\infty < a \leq x \leq b < \infty$ and $-\infty < u^{(i)}, u_x^{(i)} < \infty$, we have

- (i) $F^{(i)}(x, u^{(1)}, u^{(2)}, \dots, u^{(i)}, \dots, u^{(M)}, u_x^{(1)}, u_x^{(2)}, \dots, u_x^{(i)}, \dots, u_x^{(M)})$ is continuous;
- (ii) $\frac{\partial F^{(i)}}{\partial u^{(j)}}$ and $\frac{\partial F^{(i)}}{\partial u_x^{(j)}}$ exist and are continuous;
- (iii) $\frac{\partial F^{(i)}}{\partial u^{(j)}} > 0$ and $|\frac{\partial F^{(i)}}{\partial u_x^{(j)}}| \leq C$, for some positive constant C and $i, j = 1(1)M$.

These conditions, proved by Keller [10] assure us the existence of a unique solution of the system of boundary value problem (1)-(2).

In the present paper, we have derived generalized schemes of second and third order using variable mesh based on off-step points for solving system of two point boundary value problems (1)-(2). Such systems decomposes several higher order problems and then solves them efficiently. The higher order boundary value problems models various phenomena in the field of astrophysics, hydrodynamics, fluid dynamics, astronomy, beam and wave theory, engineering and applied physics (see [2], [3], [6], [12], [19]). The existence and uniqueness of such higher order boundary value problems are discussed by Aftabizadeh [1], Regan [4], Agarwal [20]. Several authors have also solved such boundary value problems using numerical techniques. To name a few, Akram et. al. [8] used kernel space method to solve eighth order boundary value problems and also used non-polynomial spline technique [9] to solve sixth order boundary value problems. Talwar et. al. [13] developed finite difference method to solve fourth order BVPs on uniform and variable mesh respectively. Noor et. al. [14] and Pandey [17] used homotopy perturbation method and finite difference method respectively to solve sixth order boundary value problems. So, previous work motivated us to solve higher order boundary value problems using system of boundary value problems.

Presently, the higher order boundary value problems are decomposed into system of second order boundary value problems (1). The boundary conditions are also modified accordingly and when incorporated into the scheme containing system of discretized second order boundary value problems, we get a block tri-diagonal Jacobian. In case of linear boundary value problem, we have used block Gauss elimination method to solve the Jacobian and in case of nonlinear problem we have used block Newton's method.

The sections of this paper are organized as follows. In section 2, we give details of derivation of the scheme using second order linear boundary value problem and in section 3, we provide generalization of the scheme. In section 4, we present the application of the proposed schemes on a fourth order singular

boundary value problem. In section 5, we discuss convergence analysis of variable as well as uniform mesh schemes and in section 6, we provide numerical illustrations to demonstrate the accuracy of the proposed schemes. Finally in section 7, we provide concluding remarks.

2. Derivation of the schemes

We consider a second order nonlinear boundary value problem of the following type:

$$(3) \quad u_{xx} = F(x, u, u_x), \text{ such that } u(a) = A, u(b) = B$$

We discretise the solution region $[a, b]$ such that $a = x_0 < x_1 < x_2 \dots x_{N-1} < x_N = b$. Let $h_j = x_j - x_{j-1}$, $j = 1(1)N$ be the mesh size and the mesh ratio be $\sigma_j = \frac{h_{j+1}}{h_j} > 0$, $j = 1(1)N-1$. When $\sigma_j = 1$ the mesh reduces to a uniform mesh i.e., $h_{j+1} = h_j = h$. Now, without loss of generality, we choose $\sigma_j = \sigma$ a constant $\forall j$. Also, assume u_j and U_j be the approximate and exact solution of (3) at the grid points x_j , $j = 1, 2, \dots, N$. Now, we start with following approximations at the grid points,

$$(4) \quad S_j = \sigma(\sigma + 1),$$

$$(5) \quad \bar{u}_{j+\frac{1}{2}} = \frac{u_{j+1} + u_j}{2},$$

$$(6) \quad \bar{u}_{j-\frac{1}{2}} = \frac{u_{j-1} + u_j}{2},$$

$$(7) \quad \bar{u}_{x_{j+\frac{1}{2}}} = \frac{u_{j+1} - u_j}{h_j \sigma},$$

$$(8) \quad \bar{u}_{x_{j-\frac{1}{2}}} = \frac{u_j - u_{j-1}}{h_j},$$

$$(9) \quad \bar{u}_{x_j} = \frac{u_{j+1} + (\sigma^2 - 1)u_j - \sigma^2 u_{j-1}}{h_j S_j}.$$

Using (3), we define the following:

$$(10) \quad \bar{F}_{j+\frac{1}{2}} = f(x_{j+\frac{1}{2}}, \bar{u}_{j+\frac{1}{2}}, \bar{u}_{x_{j+\frac{1}{2}}}),$$

$$(11) \quad \bar{F}_{j-\frac{1}{2}} = f(x_{j-\frac{1}{2}}, \bar{u}_{j-\frac{1}{2}}, \bar{u}_{x_{j-\frac{1}{2}}}),$$

$$(12) \quad \bar{F}_j = f(x_j, u_j, \bar{u}_{x_j}).$$

Now, to get a higher order approximation for u_j and u_{x_j} , we define the following approximations:

$$(13) \quad \hat{u}_j = u_j + \delta h_j^2 (\bar{F}_{j+\frac{1}{2}} + \bar{F}_{j-\frac{1}{2}}),$$

$$(14) \quad \hat{u}_{x_j} = \bar{u}_{x_j} + \gamma h_j (\bar{F}_{j+\frac{1}{2}} - \bar{F}_{j-\frac{1}{2}}),$$

where δ, γ are parameters to be determined. Therefore, the modified $\bar{F}_j$ is as follows:

$$(15) \quad \hat{F}_j = f(x_j, \hat{u}_j, \hat{u}_{x_j}).$$

It is easy to see that

$$(16) \quad \hat{u}_j = u_j + \delta h_j^2 (2u_{xx_j}) + O(h_j^3), \sigma \neq 1$$

$$(17) \quad \hat{u}_{x_j} = u_{x_j} + \frac{h_j^2}{6} u_{xxx_j} (\sigma + 3\gamma(1 + \sigma)) + O(h_j^3), \sigma \neq 1$$

$$(18) \quad \bar{F}_{j+\frac{1}{2}} = F_{j+\frac{1}{2}} + \frac{(h_j \sigma)^2}{8} u_{xx_j} G + \frac{(h_j \sigma)^2}{24} u_{xxx_j} H + O(h_j^3),$$

$$(19) \quad \bar{F}_{j-\frac{1}{2}} = F_{j-\frac{1}{2}} + \frac{h_j^2}{8} u_{xx_j} G + \frac{h_j^2}{24} u_{xxx_j} H + O(h_j^3),$$

$$(20) \quad \hat{F}_j = F_j + 2h_j^2 \delta u_{xx_j} G + \frac{h_j^2}{6} (\sigma + 3\gamma(1 + \sigma)) u_{xxx_j} H + O(h_j^3),$$

where

$$G = \frac{\partial f}{\partial u_j}, H = \frac{\partial f}{\partial u_{x_j}}.$$

Now, by Taylor's expansion we derive the following off-step discretization schemes for $\sigma \neq 1, j = 1(1)N - 1$:

$$(21) \quad u_{j+1} - (1 + \sigma)u_j + \sigma u_{j-1} = \frac{h_j^2}{6} (A_j F_{j+\frac{1}{2}} + B_j F_{j-\frac{1}{2}}),$$

$$(22) \quad u_{j+1} - (1 + \sigma)u_j + \sigma u_{j-1} = \frac{h_j^2}{6} (P_j F_{j+\frac{1}{2}} + R_j F_{j-\frac{1}{2}} + Q_j F_j),$$

where A_j, B_j, P_j, Q_j and R_j are given as follows:

$$(23) \quad A_j = \sigma(2\sigma + 1), B_j = \sigma(2 + \sigma),$$

$$(24) \quad P_j = 2\sigma^2, R_j = 2\sigma, Q_j = \sigma(\sigma + 1).$$

Further, we discretize the boundary value problem(3) by using the first scheme (21) as :

$$(25) \quad u_{j+1} - (1 + \sigma)u_j + \sigma u_{j-1} = \frac{h_j^2}{6} (A_j \bar{F}_{j+\frac{1}{2}} + B_j \bar{F}_{j-\frac{1}{2}}) + T_j^2.$$

Here, we can easily show that $T_j^2 = O(h_j^4)$ by using the approximations (18)-(19). Similarly, using the second scheme(22) along with the approximation (18)-(20) we discretize (3) at each grid points x_j :

$$(26) \quad \begin{aligned} \sigma u_{j-1} - (1 + \sigma)u_j + u_{j+1} &= \frac{h_j^2}{6} (P_j \bar{F}_{j+\frac{1}{2}} + Q_j \hat{F}_j + R_j \bar{F}_{j-\frac{1}{2}}) \\ &- h_j^4 \left[\left(\frac{P_j \sigma^2}{24} + \frac{Q_j}{6} (\sigma + 3\gamma(\sigma + 1)) + \frac{R_j}{24} \right) u_{xxx_j} H \right. \\ &+ \left. \left(\frac{P_j \sigma^2}{8} + 2\delta Q_j + \frac{R_j}{8} \right) u_{xx_j} G \right] + T_j^3. \end{aligned}$$

To make the local truncation error T_j^3 of proposed scheme as $O(h_j^5)$, the coefficients of h_j^4 is equated to zero. Hence, we get $\delta = -\frac{(\sigma^2+1-\sigma)}{8}$, $\gamma = -\frac{(\sigma+1+\sigma^2)}{6(1+\sigma)}$. The same truncation error for uniform mesh becomes $O(h^6)$. Note that the coefficients A_j, B_j, P_j, Q_j and R_j are positive for $\sigma > 0$, which is a necessary condition for convergence of the methods[16].

Hence, both the proposed three point discretization schemes for $j = 1(1)N-1$ are as follows:

$$(27) \quad u_{j+1} - (1 + \sigma)u_j + \sigma u_{j-1} = \frac{h_j^2}{6}(A_j \bar{F}_{j+\frac{1}{2}} + B_j \bar{F}_{j-\frac{1}{2}})$$

and

$$(28) \quad u_{j+1} - (1 + \sigma)u_j + \sigma u_{j-1} = \frac{h_j^2}{6}(P_j \bar{F}_{j+\frac{1}{2}} + Q_j \hat{F}_j + R_j \bar{F}_{j-\frac{1}{2}}).$$

3. Generalization of the schemes

We generalize our method for the solution of the system of M nonlinear boundary value problems (1). At the grid point $x_j, j = 1(1)N-1$ and for $i = 1(1)M$, we use the following approximations and schemes:

$$(29) \quad S_j = \sigma(\sigma + 1),$$

$$(30) \quad \bar{u}_{j+\frac{1}{2}}^{(i)} = \frac{u_{j+1}^{(i)} + u_j^{(i)}}{2},$$

$$(31) \quad \bar{u}_{j-\frac{1}{2}}^{(i)} = \frac{u_{j-1}^{(i)} + u_j^{(i)}}{2},$$

$$(32) \quad \bar{u}_{x_{j+\frac{1}{2}}}^{(i)} = \frac{u_{j+1}^{(i)} - u_j^{(i)}}{h_j \sigma},$$

$$(33) \quad \bar{u}_{x_{j-\frac{1}{2}}}^{(i)} = \frac{u_j^{(i)} - u_{j-1}^{(i)}}{h_j},$$

$$(34) \quad \bar{u}_{x_j}^{(i)} = \frac{u_{j+1}^{(i)} + (\sigma^2 - 1)u_j^{(i)} - \sigma^2 u_{j-1}^{(i)}}{h_j S_j},$$

$$(35) \quad \begin{aligned} \bar{F}_{j+\frac{1}{2}}^{(i)} &= f^{(i)}(x_{j+\frac{1}{2}}, \bar{u}_{j+\frac{1}{2}}^{(1)}, \bar{u}_{j+\frac{1}{2}}^{(2)}, \dots, \bar{u}_{j+\frac{1}{2}}^{(i)}, \dots, \bar{u}_{j+\frac{1}{2}}^{(M)}), \\ &\quad \bar{u}_{x_{j+\frac{1}{2}}}^{(1)}, \bar{u}_{x_{j+\frac{1}{2}}}^{(2)}, \dots, \bar{u}_{x_{j+\frac{1}{2}}}^{(i)}, \dots, \bar{u}_{x_{j+\frac{1}{2}}}^{(M)}), \end{aligned}$$

$$(36) \quad \begin{aligned} \bar{F}_{j-\frac{1}{2}}^{(i)} &= f^{(i)}(x_{j-\frac{1}{2}}, \bar{u}_{j-\frac{1}{2}}^{(1)}, \bar{u}_{j-\frac{1}{2}}^{(2)}, \dots, \bar{u}_{j-\frac{1}{2}}^{(i)}, \dots, \bar{u}_{j-\frac{1}{2}}^{(M)}), \\ &\quad \bar{u}_{x_{j-\frac{1}{2}}}^{(1)}, \bar{u}_{x_{j-\frac{1}{2}}}^{(2)}, \dots, \bar{u}_{x_{j-\frac{1}{2}}}^{(i)}, \dots, \bar{u}_{x_{j-\frac{1}{2}}}^{(M)}), \end{aligned}$$

$$(37) \quad \bar{F}_j^{(i)} = f^{(i)}(x_j, u_j^{(1)}, u_j^{(2)}, \dots, u_j^{(i)}, \dots, u_j^{(M)}, \bar{u}_{x_j}^{(1)}, \bar{u}_{x_j}^{(2)}, \dots, \bar{u}_{x_j}^{(i)}, \dots, \bar{u}_{x_j}^{(M)}),$$

$$(38) \quad \hat{u}_j^{(i)} = u_j^{(i)} + \delta_i h_j^2 (\bar{F}_{j+\frac{1}{2}}^{(i)} + \bar{F}_{j-\frac{1}{2}}^{(i)}),$$

$$(39) \quad \hat{u}_{x_j}^{(i)} = \bar{u}_{x_j}^{(i)} + \gamma_i h_j (\bar{F}_{j+\frac{1}{2}}^{(i)} - \bar{F}_{j-\frac{1}{2}}^{(i)}),$$

$$(40) \quad \hat{F}_j^{(i)} = f^{(i)}(x_j, \hat{u}_j^{(1)}, \hat{u}_j^{(2)}, \dots, \hat{u}_j^{(i)}, \dots, \hat{u}_j^{(M)}, \hat{u}_{x_j}^{(1)}, \hat{u}_{x_j}^{(2)}, \dots, \hat{u}_{x_j}^{(i)}, \dots, \hat{u}_{x_j}^{(M)}),$$

$$(41) \quad u_{j+1}^{(i)} - (1 + \sigma)u_j^{(i)} + \sigma u_{j-1}^{(i)} = \frac{h_j^2}{6} (A_j \bar{F}_{j+\frac{1}{2}}^{(i)} + B_j \bar{F}_{j-\frac{1}{2}}^{(i)}),$$

$$(42) \quad u_{j+1}^{(i)} - (1 + \sigma)u_j^{(i)} + \sigma u_{j-1}^{(i)} = \frac{h_j^2}{6} (P_j \bar{F}_{j+\frac{1}{2}}^{(i)} + Q_j \hat{F}_j^{(i)} + R_j \bar{F}_{j-\frac{1}{2}}^{(i)}).$$

where

$$(43) \quad A_j = \sigma(2\sigma + 1), B_j = \sigma(2 + \sigma)$$

$$(44) \quad P_j = 2\sigma^2, R_j = 2\sigma, Q_j = \sigma(\sigma + 1).$$

4. Application to fourth order singular boundary value problem

We consider a fourth order singular boundary value problem of the following type:

$$(45) \quad u_{xxxx}(x) = a(x)u_{xxx}(x) + b(x)u_{xx}(x) + d(x), 0 \leq x \leq 1$$

subject to boundary conditions:

$$(46) \quad u(0) = \alpha_1, u_{xx}(0) = \alpha_2, u(1) = \beta_1, u_{xx}(1) = \beta_2,$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2$ are real constants and $a(x)$ is singular at $x = 0$. Using (1), we write the problem (45) – (46) as follows:

$$(47) \quad u_{xx}(x) = v(x),$$

$$(48) \quad v_{xx}(x) = a(v)v_x(x) + b(x)v(x) + d(x),$$

subject to

$$(49) \quad u(0) = \alpha_1, v(0) = \alpha_2, u(1) = \beta_1, v(1) = \beta_2.$$

Applying the difference scheme (41) to the coupled second order boundary value problem (47) – (48) we obtain the following difference scheme:

$$(50) \quad \sigma_j u_{j-1} - (1 + \sigma_j)u_j + u_{j+1} = \frac{h_j^2}{6} (A_j \bar{v}_{j+\frac{1}{2}} + B_j \bar{v}_{j-\frac{1}{2}}),$$

$$(51) \quad \begin{aligned} \sigma_j v_{j-1} - (1 + \sigma_j)v_j + v_{j+1} = & \frac{h_j^2}{6} [A_j (a_{j+\frac{1}{2}} \bar{v}_{x_{j+\frac{1}{2}}} + b_{j+\frac{1}{2}} \bar{v}_{j+\frac{1}{2}} + d_{j+\frac{1}{2}}) \\ & + B_j (a_{j-\frac{1}{2}} \bar{v}_{x_{j-\frac{1}{2}}} + b_{j-\frac{1}{2}} \bar{v}_{j-\frac{1}{2}} + d_{j-\frac{1}{2}})]. \end{aligned}$$

Then, we define the following relations for $a_{j\pm\frac{1}{2}}$ for the coupled finite difference scheme (47) – (48)

$$(52) \quad a_{j-\frac{1}{2}}^* = a_j - \frac{h_j}{2}a_{x_j} + O(h_j^2),$$

$$(53) \quad a_{j+\frac{1}{2}}^* = a_j + \frac{\sigma h_j}{2}a_{x_j} + O(h_j^2),$$

$$(54) \quad a_{j-\frac{1}{2}}^{**} = a_j - \frac{h_j}{2}a_{x_j} + \frac{(h_j)^2}{8}a_{xx_j} + O(h_j^3),$$

$$(55) \quad a_{j+\frac{1}{2}}^{**} = a_j + \frac{\sigma h_j}{2}a_{x_j} + \frac{(\sigma h_j)^2}{8}a_{xx_j} + O(h_j^3).$$

Similar relations for $b_{j\pm\frac{1}{2}}, d_{j\pm\frac{1}{2}}$ can be defined. Using the relations (52) – (53) in (50) – (51) we get,

$$(56) \quad \sigma u_{j-1} - (1 + \sigma)u_j + u_{j+1} = \frac{h_j^2}{6}(A_j \bar{v}_{j+\frac{1}{2}} + B_j \bar{v}_{j-\frac{1}{2}}),$$

$$(57) \quad \begin{aligned} \sigma v_{j-1} - (1 + \sigma)v_j + v_{j+1} &= \frac{h_j^2}{6} [A_j(a_{j+\frac{1}{2}}^* \bar{v}_{x_{j+\frac{1}{2}}} + b_{j+\frac{1}{2}}^* \bar{v}_{j+\frac{1}{2}} + d_{j+\frac{1}{2}}^*) \\ &+ B_j(a_{j-\frac{1}{2}}^* \bar{v}_{x_{j-\frac{1}{2}}} + b_{j-\frac{1}{2}}^* \bar{v}_{j-\frac{1}{2}} + d_{j-\frac{1}{2}}^*)]. \end{aligned}$$

Finally, substituting (29)-(36) in (56)-(57) we obtain the vector difference equation of boundary value problem (45)-(46) as follows:

$$(58) \quad \begin{aligned} &\begin{bmatrix} sub_j^{11} & sub_j^{12} \\ sub_j^{21} & sub_j^{22} \end{bmatrix} \begin{bmatrix} u_{j-1} \\ v_{j-1} \end{bmatrix} + \begin{bmatrix} diag_j^{11} & diag_j^{12} \\ diag_j^{21} & diag_j^{22} \end{bmatrix} \begin{bmatrix} u_j \\ v_j \end{bmatrix} \\ &+ \begin{bmatrix} sup_j^{11} & sup_j^{12} \\ sup_j^{21} & sup_j^{22} \end{bmatrix} \begin{bmatrix} u_{j+1} \\ v_{j+1} \end{bmatrix} = \begin{bmatrix} \phi_j^1 \\ \phi_j^2 \end{bmatrix}, \end{aligned}$$

where

$$(59) \quad \begin{cases} sub_j^{11} = -\sigma, & sub_j^{12} = \frac{h_j^2 \sigma (\sigma + 2)}{12}, \\ sub_j^{21} = 0, & sub_j^{22} = -\sigma + \frac{h_j \sigma (\sigma + 2)}{12} (-2a_j + h_j(a_{x_j} + b_j) - \frac{h_j^2 b_{x_j}}{2}). \end{cases}$$

$$(60) \quad \begin{cases} diag_j^{11} = (1 + \sigma), & diag_j^{12} = \frac{3h_j^2 \sigma (\sigma + 1)}{12}, \\ diag_j^{21} = 0, & diag_j^{22} = (1 + \sigma) + \frac{h_j (2\sigma + 1)}{12} (-2a_j - h_j \sigma (a_{x_j} - b_j) \\ & + \frac{(h_j \sigma)^2 b_{x_j}}{2}) \end{cases}$$

$$(61) \quad \begin{cases} sup_j^{11} = -1, & sup_j^{12} = \frac{h_j^2 \sigma (2\sigma + 1)}{12}, \\ sup_j^{21} = 0, & sup_j^{22} = -1 + \frac{h_j (2\sigma + 1)}{12} (2a_j + h_j \sigma (a_{x_j} + b_j) + \frac{(h_j \sigma)^2 b_{x_j}}{2}) \end{cases}$$

$$(62) \quad \begin{cases} \phi_j^1 = 0, & \phi_j^2 = -\frac{h_j^2}{6} [d_j (3\sigma^2 + \sigma) + d_{x_j} h_j (\sigma^3 + \sigma)] \end{cases}.$$

Similarly, using the difference scheme (42) for the boundary value problem (45) – (46), we obtain the following difference scheme :

$$\begin{aligned}
 (63) \quad \sigma u_{j-1} - (1 + \sigma)u_j + u_{j+1} &= \frac{h_j^2}{6}(P_j \bar{v}_{j+\frac{1}{2}} + Q_j \hat{v}_j + R_j \bar{v}_{j-\frac{1}{2}}), \\
 \sigma v_{j-1} - (1 + \sigma)v_j + v_{j+1} &= \frac{h_j^2}{6}(G_j[a_{j+\frac{1}{2}}^{**} \bar{v}_{x_{j+\frac{1}{2}}} + b_{j+\frac{1}{2}}^{**} \bar{v}_{j+\frac{1}{2}} + d_{j+\frac{1}{2}}^{**}] \\
 &\quad + Q_j[a_j \hat{v}_{x_j} + b_j \hat{v}_j + d_j] \\
 (64) \quad &\quad + H_j[a_{j-\frac{1}{2}}^{**} \bar{v}_{x_{j-\frac{1}{2}}} + b_{j-\frac{1}{2}}^{**} \bar{v}_{j-\frac{1}{2}} + d_{j-\frac{1}{2}}^{**}]),
 \end{aligned}$$

where $G_j = P_j + Q_j(a_j \gamma_2 h_j + b_j \delta_2 h_j^2)$, $H_j = R_j - Q_j(a_j \gamma_2 h_j + b_j \delta_2 h_j^2)$, $j = 1(1)N - 1$. This scheme can be simplified upto $O(h_j^4)$ terms by using (29)-(40).

5. Convergence analysis

The convergence analysis of scalar singular boundary value problem has been given by R.K. Mohanty [18]. We consider the vector form of convergence analysis of scheme (41) for the coupled second order boundary value problem (47) – (49). Now, once the boundary conditions are incorporated in the the vector difference equation (58), it can be written in matrix form as:

$$(65) \quad D\hat{u} + \hat{\phi} + T(\hat{h}_j) = \begin{bmatrix} sub_j & diag_j & sup_j \end{bmatrix} \begin{bmatrix} \hat{u}_{j-1} \\ \hat{u}_j \\ \hat{u}_{j+1} \end{bmatrix} + \hat{\phi}_j + T(\hat{h}_j) = \hat{0},$$

where D is a block tridiagonal matrix of order $N - 1$; $sub_j, sup_j, diag_j$ are block matrices of order 2×2 in D,

$$\begin{aligned}
 \hat{u} &= [\hat{u}_1, \hat{u}_2, \dots, \hat{u}_j, \dots, \hat{u}_{N-1}]^T, \text{ where } \hat{u}_j = [u_j, v_j]^T \\
 \hat{\phi} &= [\hat{\phi}_1 + sub_1[\alpha_1, \alpha_2]^T, \hat{\phi}_2, \dots, \hat{\phi}_j, \dots, \hat{\phi}_{N-1} + sup_{N-1}[\beta_1, \beta_2]^T]^T, \\
 &\quad \text{where } \hat{\phi}_j = [\phi_j^1, \phi_j^2]^T \\
 T(\hat{h}_j) &= [T_1^2, T_2^2, \dots, T_{N-1}^2]^T \\
 \hat{0} &= [[0, 0]^T, [0, 0]^T, \dots, [0, 0]^T]^T.
 \end{aligned}$$

Let

$$(66) \quad U = [[U_1, V_1]^T, [U_2, V_2]^T, [U_j, V_j]^T, \dots, [U_{N-1}, V_{N-1}]^T]^T \cong \hat{u} \text{ satisfy } DU + \hat{\phi} = 0, \text{ where D is defined in (65) .}$$

Let $\hat{e}_j = [U_j - u_j, V_j - v_j]^T \equiv [e_{j_u}, e_{j_v}]^T$ be the discretization error in absence of round off error, then $U - \hat{u} = E = [\hat{e}_1, \hat{e}_2, \dots, \hat{e}_{N-1}]^T$.

Subtracting (65) from (66), we obtain the error equation as follows

$$(67) \quad DE = T.$$

Let $|a_j| \leq K_1, |a_{x_j}| \leq K_2, |b_j| \leq K_3$ and $|b_{x_j}| \leq K_4$ for some positive constants K_1, K_2, K_3, K_4 , then using (59) and (61) we get,

$$(68) \quad \|sup_j\|_\infty \leq \max_{1 \leq j \leq N-2} \left\{ 1 + \frac{h_j^2 \sigma (1 + 2\sigma)}{(1 + \frac{12}{2\sigma})} [2h_j K_1 + h_j^2 \sigma (K_2 + K_3) + \frac{h_j^3 \sigma^2}{2} K_4] + O(h_j^4) \right.$$

$$(69) \quad \|sub_j\|_\infty \leq \max_{2 \leq j \leq N-1} \left\{ \sigma + \frac{h_j^2 \sigma (2 + \sigma)}{12} [2h_j K_1 + h_j^2 (K_2 + K_3) + \frac{h_j^3}{2} K_4] + O(h_j^4) \right. \quad \cdot$$

Thus, for sufficiently small h_j , we get $\|sub_j\|_\infty \leq \sigma$ and $\|sup_j\|_\infty \leq 1$. Hence, D is irreducible. Now, let sum_{rowj} be the sum of elements of row_j of D

$$(70) \quad sum_{rowj} = \begin{cases} \sigma + \frac{h_j^2}{12} \sigma (4 + 5\sigma), & j = 1 \\ \sigma + \frac{h_j \sigma (2 + \sigma) a_j}{6} + \frac{h_j^2 \sigma}{12} [- (2 + \sigma) a_{x_j} + (4 + 5\sigma) b_j] \\ + O(h_j^3), & j = 2 \end{cases}$$

$$(71) \quad sum_{rowj} = \begin{cases} \frac{h_j^2}{2} S_j, & j = 3, 5..N-4 \\ \frac{h_j^2}{2} b_j S_j, & j = 4, 6..N-3 \end{cases}$$

$$(72) \quad sum_{rowj} = \begin{cases} 1 + \frac{h_j^2}{12} \sigma (5 + 4\sigma), & j = N-2 \\ 1 - \frac{h_j (1 + 2\sigma) a_j}{6} + \frac{h_j^2}{12} [- \sigma (1 + 2\sigma) a_{x_j} + \sigma (5 + 4\sigma) b_j] \\ + O(h_j^3), & j = N-1 \end{cases}$$

Let

$$(73) \quad 0 < K_{min} \leq \min(K_1, K_2, K_3, K_4) \leq K_{max}.$$

Using (70)-(72) and for sufficiently small h_j , we can easily prove that D is Monotone. Therefore, D^{-1} exist and $D^{-1} \geq 0$. Hence by (67) we have,

$$(74) \quad \|E\| = \|D^{-1}\| \|T\|.$$

Now, for sufficiently small h_j , by (70) – (73) we can say that:

$$(75) \quad sum_{rowj} > \begin{cases} \frac{h_j^2}{12}\sigma(4+5\sigma), & j=1 \\ \frac{h_j^2}{12}\sigma(4+5\sigma)K_{min}, & j=2 \end{cases}$$

$$(76) \quad sum_{rowj} > \begin{cases} \frac{h_j^2}{2}S_j, & j=3, 5\dots N-4 \\ \frac{h_j^2}{2}S_jK_{min}, & j=4, 6\dots N-3 \end{cases}$$

$$(77) \quad sum_{rowj} > \begin{cases} \frac{h_j^2}{12}\sigma(5+4\sigma), & j=N-2 \\ \frac{h_j^2}{12}\sigma(5+4\sigma)K_{min}, & j=N-1 \end{cases}.$$

Since $\sigma > 0$ we can say that:

$$(78) \quad sum_{rowj} > \max[\frac{h_j^2}{12}\sigma(4+5\sigma), \frac{h_j^2}{12}\sigma(4+5\sigma)K_{min}]$$

$$= \frac{h_j^2}{12}\sigma(4+5\sigma)K_{min}, \text{ for } j=1, 2$$

$$(79) \quad sum_{rowj} > \max[\frac{h_j^2}{2}S_j, \frac{h_j^2}{2}K_{min}S_j] = \frac{h_j^2}{2}S_jK_{min}, \text{ for } j=3, 4\dots, N-3$$

$$(80) \quad sum_{rowj} > \max[\frac{h_j^2}{12}\sigma(5+4\sigma), \frac{h_j^2}{12}\sigma(5+4\sigma)K_{min}]$$

$$= \frac{h_j^2}{12}\sigma(5+4\sigma)K_{min}, \text{ for } j=N-2, N-1.$$

Let $D_{i,j}^{-1}$ be the $(i, j)^{th}$ element of D^{-1} , then by theory of matrices [21] for $i = 1(1)N-1$

$$(81) \quad D_{i,j}^{-1} \leq \frac{1}{sum_{rowj}}.$$

By using (78)-(80), we have

$$(82) \quad \frac{1}{sum_{rowj}} \leq \begin{cases} \frac{12}{h_j^2\sigma(4+5\sigma)K_{min}}, & j=1, 2 \\ \frac{2}{h_j^2S_jK_{min}}, & j=3, 4, 5\dots, N-3 \\ \frac{12}{h_j^2\sigma(5+4\sigma)K_{min}}, & j=N-2, N-1 \end{cases}.$$

Now let us define

$$(83) \quad \| D_{i,j}^{-1} \| = \max_{1 \leq i \leq N-1} \sum_{j=1}^{N-1} | D_{i,j}^{-1} | \text{ and } \| T_j \| = \max_{1 \leq j \leq N-1} \sum_{j=1}^{N-1} | T_j^2 |.$$

Therefore, using (67) and (81) – (83) we get,

$$(84) \quad \| E \| \leq \frac{12}{h_j^2 K_{min} \sigma} \left(\frac{1}{4 + 5\sigma} + \frac{1}{5 + 4\sigma} + \frac{1}{6(1 + \sigma)} \right) O(h_j^4) = O(h_j^2).$$

Hence, the second order convergence of scheme (41) for boundary value problems of the type (45) – (46) follows. Therefore, without loss of generality we can say that finite difference scheme (41) has second order convergence for boundary value problems (1)-(2). Similarly, we can prove the third order convergence of the difference scheme (42).

Theorem 1. The scheme (41) for the numerical solution of system of nonlinear singular boundary value problem (1) – (2) with sufficiently small $h_j, 0 < \sigma \neq 1$ has second order convergence under appropriate conditions.

5.1 Convergence Analysis of Fourth Order Method

Let us consider a fourth order singular boundary value problem of the following type:

$$(85) \quad u_{xxxx}(x) = F(x, u(x), u_x(x), u_{xx}(x), u_{xxx}(x)), 0 \leq x \leq 1$$

subject to boundary conditions:

$$(86) \quad u(0) = \alpha_1, u_{xx}(0) = \alpha_2, u(1) = \beta_1, u_{xx}(1) = \beta_2$$

where $F = a(x)u(x) + g(x)$ and $a(x)$ is singular at $x = 0$. Using (1), we write the problem (85) – (86) as follows:

$$(87) \quad u_{xx}(x) = v(x),$$

$$(88) \quad v_{xx}(x) = a(x)u(x) + g(x),$$

subject to

$$(89) \quad u(0) = \alpha_1, v(0) = \alpha_2, u(1) = \beta_1, v(1) = \beta_2.$$

Next, we convert the approximations (30) – (40) and variable mesh scheme (42) by putting $\sigma = 1$ into uniform mesh:

$$(90) \quad \bar{u}_{j+\frac{1}{2}}^{(i)} = \frac{u_{j+1}^{(i)} + u_j^{(i)}}{2}$$

$$(91) \quad \bar{u}_{j-\frac{1}{2}}^{(i)} = \frac{u_{j-1}^{(i)} + u_j^{(i)}}{2}$$

$$(92) \quad \bar{u}_{x_{j+\frac{1}{2}}}^{(i)} = \frac{u_{j+1}^{(i)} - u_j^{(i)}}{h}$$

$$(93) \quad \bar{u}_{x_{j-\frac{1}{2}}}^{(i)} = \frac{u_j^{(i)} - u_{j-1}^{(i)}}{h}$$

$$(94) \quad \bar{u}_{x_j}^{(i)} = \frac{u_{j+1}^{(i)} - u_{j-1}^{(i)}}{2h}$$

$$(95) \quad \begin{aligned} \bar{F}_{j+\frac{1}{2}}^{(i)} &= f^{(i)}(x_{j+\frac{1}{2}}, \bar{u}_{j+\frac{1}{2}}^{(1)}, \bar{u}_{j+\frac{1}{2}}^{(2)}, \dots, \bar{u}_{j+\frac{1}{2}}^{(i)}, \dots, \bar{u}_{j+\frac{1}{2}}^{(M)}, \\ &\quad \bar{u}_{x_{j+\frac{1}{2}}}^{(1)}, \bar{u}_{x_{j+\frac{1}{2}}}^{(2)}, \dots, \bar{u}_{x_{j+\frac{1}{2}}}^{(i)}, \dots, \bar{u}_{x_{j+\frac{1}{2}}}^{(M)}) \end{aligned}$$

$$(96) \quad \begin{aligned} \bar{F}_{j-\frac{1}{2}}^{(i)} &= f^{(i)}(x_{j-\frac{1}{2}}, \bar{u}_{j-\frac{1}{2}}^{(1)}, \bar{u}_{j-\frac{1}{2}}^{(2)}, \dots, \bar{u}_{j-\frac{1}{2}}^{(i)}, \dots, \bar{u}_{j-\frac{1}{2}}^{(M)}, \\ &\quad \bar{u}_{x_{j-\frac{1}{2}}}^{(1)}, \bar{u}_{x_{j-\frac{1}{2}}}^{(2)}, \dots, \bar{u}_{x_{j-\frac{1}{2}}}^{(i)}, \dots, \bar{u}_{x_{j-\frac{1}{2}}}^{(M)}) \end{aligned}$$

$$(97) \quad \begin{aligned} \bar{F}_j^{(i)} &= f^{(i)}(x_j, u_j^{(1)}, u_j^{(2)}, \dots, u_j^{(i)}, \dots, u_j^{(M)}, \\ &\quad \bar{u}_{x_j}^{(1)}, \bar{u}_{x_j}^{(2)}, \dots, \bar{u}_{x_j}^{(i)}, \dots, \bar{u}_{x_j}^{(M)}) \end{aligned}$$

$$(98) \quad \hat{u}_j^{(i)} = u_j^{(i)} - \frac{h^2}{8}(\bar{F}_{j+\frac{1}{2}}^{(i)} + \bar{F}_{j-\frac{1}{2}}^{(i)})$$

$$(99) \quad \hat{u}_{x_j}^{(i)} = \bar{u}_{x_j}^{(i)} - \frac{h}{4}(\bar{F}_{j+\frac{1}{2}}^{(i)} - \bar{F}_{j-\frac{1}{2}}^{(i)})$$

$$(100) \quad \begin{aligned} \hat{F}_j^{(i)} &= f^{(i)}(x_j, \hat{u}_j^{(1)}, \hat{u}_j^{(2)}, \dots, \hat{u}_j^{(i)}, \dots, \hat{u}_j^{(M)}, \\ &\quad \hat{u}_{x_j}^{(1)}, \hat{u}_{x_j}^{(2)}, \dots, \hat{u}_{x_j}^{(i)}, \dots, \hat{u}_{x_j}^{(M)}) \end{aligned}$$

$$(101) \quad u_{j+1}^{(i)} - 2u_j^{(i)} + u_{j-1}^{(i)} = \frac{h^2}{3}(\bar{F}_{j+\frac{1}{2}}^{(i)} + \hat{F}_j^{(i)} + \bar{F}_{j-\frac{1}{2}}^{(i)}).$$

Now, we use the uniform mesh scheme (101) to solve the coupled second order boundary value problem (87)-(89) and obtain the following difference scheme:

$$(102) \quad u_{j-1} - 2u_j + u_{j+1} = \frac{h^2}{3}(\bar{v}_{j+\frac{1}{2}} + \hat{v}_j + \bar{v}_{j-\frac{1}{2}})$$

$$(103) \quad \begin{aligned} v_{j-1} - 2v_j + v_{j+1} &= \frac{h^2}{3}[(a_{j+\frac{1}{2}}\bar{u}_{j+\frac{1}{2}} + g_{j+\frac{1}{2}}) \\ &+ (a_j\hat{u}_j + g_j) + (a_{j-\frac{1}{2}}\bar{u}_{j-\frac{1}{2}} + g_{j-\frac{1}{2}})]. \end{aligned}$$

Also, we define the following relations for $a_{j\pm\frac{1}{2}}$ in the coupled finite difference scheme (102)-(103)

$$(104) \quad a_{j-\frac{1}{2}}^* = a_j - \frac{h}{2}a_{x_j} + \frac{h^2}{8}a_{xx_j} + O(h^3),$$

$$(105) \quad a_{j+\frac{1}{2}}^* = a_j + \frac{h}{2}a_{x_j} + \frac{h^2}{8}a_{xx_j} + O(h^3).$$

Similar relations for $g_{j\pm\frac{1}{2}}$ can also be defined. Using the relations (104) – (105) in (102) – (103) we get,

$$(106) \quad \begin{aligned} u_{j-1} - 2u_j + u_{j+1} &= \frac{h^2}{3}(\bar{v}_{j+\frac{1}{2}} + \hat{v}_j + \bar{v}_{j-\frac{1}{2}}) \\ v_{j-1} - 2v_j + v_{j+1} &= \frac{h^2}{3}[(a_{j+\frac{1}{2}}^* \bar{u}_{j+\frac{1}{2}} + g_{j+\frac{1}{2}}) \\ (107) \quad &+ (a_j \hat{u}_j + g_j) + (a_{j-\frac{1}{2}}^* \bar{u}_{j-\frac{1}{2}} + g_{j-\frac{1}{2}})]. \end{aligned}$$

Finally, we simplify the difference scheme (106) – (107) upto order $O(h^5)$ terms and obtain the analogue of vector difference equation (58). The components of the vector difference equation are as follows

$$(108) \quad \begin{cases} sub_j^{11} = -1 - h^4 \frac{a_j}{48} + h^5 \frac{a_{x_j}}{96}, & sub_j^{12} = \frac{h^2}{6}, \\ sub_j^{21} = h^2 \frac{a_j}{6} - h^3 \frac{a_{x_j}}{12} + h^4 \frac{a_{xx_j}}{48}, & sub_j^{22} = -1 - h^4 \frac{a_j}{48}. \end{cases}$$

$$(109) \quad \begin{cases} diag_j^{11} = 2 - h^4 \frac{a_j}{24}, & diag_j^{12} = \frac{2}{3}h^2, \\ diag_j^{21} = h^2 \frac{2a_j}{3} + \frac{h^4 a_{xx_j}}{24}, & diag_j^{22} = 2 - h^4 \frac{a_j}{24} \end{cases}$$

$$(110) \quad \begin{cases} sup_j^{11} = -1 - h^4 \frac{a_j}{48} - h^5 \frac{a_{x_j}}{96}, & sup_j^{12} = \frac{h^2}{6}, \\ sup_j^{21} = h^2 \frac{a_j}{6} - h^3 \frac{a_{x_j}}{12} + h^4 \frac{a_{xx_j}}{48}, & sup_j^{22} = -1 - h^4 \frac{a_j}{48}. \end{cases}$$

$$(111) \quad \begin{cases} \phi_j^1 = -2h^4 g_j, & \phi_j^2 = -h^2(g_j + \frac{h^2}{12}g_{xx_j}). \end{cases}$$

Once we incorporate the boundary conditions in the vector difference equation (58) we get the matrix form (65) and as done in section 5 we similarly obtain the error equation (67). Further, let $|a_j| \leq K_1, |a_{x_j}| \leq K_2, |a_{xx_j}| \leq K_3$ for some positive constant K_1, K_2, K_3 . Now, using (108) and (110) we get,

$$(112) \quad \|sup_j\|_\infty \leq \max_{1 \leq j \leq N-2} \begin{cases} 1 + \frac{h^2}{6} + h^4 \frac{K_1}{48} + h^5 \frac{K_2}{96}, \\ 1 + h^2 \frac{K_1}{6} + h^3 \frac{K_2}{12} + h^4 \frac{K_1 + K_3}{48} \end{cases}$$

$$(113) \quad \|sub_j\|_\infty \leq \max_{2 \leq j \leq N-1} \begin{cases} 1 + \frac{h^2}{6} + h^4 \frac{K_1}{48} + h^5 \frac{K_2}{96}, \\ 1 + h^2 \frac{K_1}{6} + h^3 \frac{K_2}{12} + h^4 \frac{K_1 + K_3}{48} \end{cases}.$$

Thus for sufficiently small h , we get $\|sub_j\|_\infty \leq \sigma$ and $\|sup_j\|_\infty \leq 1$. Hence, D is irreducible.

Let sum_{rowj} be the sum of elements of row_j of D

$$(114) \quad sum_{rowj} = \begin{cases} 1 + h^2 \frac{5}{6} - h^4 \frac{a_j}{16} - h^5 \frac{a_{xj}}{96}, & j = 1 \\ 1 + h^2 \frac{5a_j}{6} + h^3 \frac{a_{xj}}{12} + h^4 \left(\frac{-a_j}{16} + \frac{3a_{xxj}}{48} \right), & j = 2 \end{cases}$$

$$(115) \quad sum_{rowj} = \begin{cases} h^2 - h^4 \frac{a_j}{12}, & j = 3, 5..N-4 \\ h^2 a_j + h^4 \left(\frac{a_{xxj}}{12} - \frac{a_j}{12} \right), & j = 4, 6..N-3 \end{cases}$$

$$(116) \quad sum_{rowj} = \begin{cases} 1 + h^2 \frac{5}{6} - h^4 \frac{a_j}{16} + h^5 \frac{a_{xj}}{96}, & j = N-2 \\ 1 + h^2 \frac{5a_j}{6} - h^3 \frac{a_{xj}}{12} - h^4 \left(\frac{-a_j}{16} + \frac{3a_{xxj}}{48} \right), & j = N-1 \end{cases}.$$

Let

$$(117) \quad 0 < K_{min} \leq \min(K_1, K_2, K_3) \leq K_{max}.$$

Using (114) – (117) and for sufficiently small h , we can easily prove that D is Monotone. Therefore, D^{-1} exist and $D^{-1} \geq 0$. Hence by (67) we have,

$$(118) \quad \|E\| = \|D^{-1}\| \|T\|,$$

where $T = O(h^6)$ as discussed in section 2. Now for sufficiently small h , by (114) – (116) we can say that:

$$(119) \quad sum_{rowj} > \begin{cases} h^2 \frac{5}{6}, & j = 1 \\ h^2 \frac{5}{6} K_{min}, & j = 2 \end{cases}$$

$$(120) \quad sum_{rowj} \geq \begin{cases} h^2, & j = 3, 5..N-4 \\ h^2 K_{min}, & j = 4, 6..N-3 \end{cases}$$

$$(121) \quad sum_{rowj} > \begin{cases} h^2 \frac{5}{6}, & j = N-2 \\ h^2 \frac{5}{6} K_{min}, & j = N-1 \end{cases}.$$

Since $\sigma > 0$ we can say that:

$$(122) \quad sum_{rowj} > \max[h^2 \frac{5}{6}, h^2 \frac{5}{6} K_{min}] = h^2 \frac{5}{6} K_{min}, \text{ for } j = 1, 2$$

$$(123) \quad sum_{rowj} \geq \max[h^2, h^2 K_{min}] = h^2 K_{min}, \text{ for } j = 3, 4.., N-3$$

$$sum_{rowj} > \max[h^2 \frac{5}{6}, h^2 \frac{5}{6} K_{min}]$$

$$(124) \quad = h^2 \frac{5}{6} K_{min}, \text{ for } j = N-2, N-1.$$

Let $D_{i,j}^{-1}$ be the $(i,j)^{th}$ element of D^{-1} , then as discussed in section 5,

$$(125) \quad D_{i,j}^{-1} \leq \frac{1}{sum_{row j}}.$$

By using (122)-(124), we have

$$(126) \quad \frac{1}{sum_{row j}} \leq \begin{cases} \frac{6}{5h^2 K_{min}}, & j = 1, 2 \\ \frac{1}{h^2 K_{min}}, & j = 3, 4, 5, \dots, N-3 \\ \frac{6}{5h^2 K_{min}}, & j = N-2, N-1 \end{cases}.$$

Now let us define,

$$(127) \quad \| D_{i,j}^{-1} \| = \max_{1 \leq i \leq N-1} \sum_{j=1}^{N-1} | D_{i,j}^{-1} |$$

Therefore, using (67), (125) - (127) we get,

$$(128) \quad \| E \| \leq \frac{6}{h^2 K_{min}} \left(\frac{2}{5} + \frac{1}{6} \right) O(h^6) = \frac{17}{5K_{min}} O(h^4).$$

Hence, the fourth order convergence of uniform mesh scheme for system of boundary value problems of the type (87) – (89) follows. Therefore, without loss of generality we can say that fourth order convergence follows for system of boundary value problems of the type (1) – (2).

6. Numerical illustration

To illustrate the proposed methods, we solve following six problems. The root mean square (RMS) errors in case of variable mesh and maximum absolute (MA) error for uniform mesh are tabulated in the Tables 1-6. For simplicity, we have chosen $\sigma_j = \sigma$ and hence $h_1 = \frac{(\sigma-1)}{(\sigma^{N-1}-1)}, \sigma \neq 1$. Therefore, the rest of the h_j 's can be obtained as $h_{j+1} = \sigma h_j, j = 1(1)N-1$. In case of the presence of boundary layer near the left or right end of the domain, take $h_1 = \begin{cases} \frac{\sigma-1}{\sigma^{N-1}-1}, \sigma > 1 \\ \frac{1-\sigma}{1-\sigma^N}, \sigma < 1 \end{cases}$. This ensures the mesh points in the boundary layer region near the left or right end of the interval. The linear system of difference equations have been solved by Block Gauss Elimination method and the nonlinear system of difference equations by Block Newton's method in which we have considered $u_0 = 0$ as the initial approximation. The computational order of convergence (COC*) is also given for fourth order uniform mesh method. All calculations have been done in Matlab 07. Also, in the following questions $u^{(i)}(x)$ means i^{th} order derivative of $u(x)$.

Example 5.1. We consider fourth order nonlinear boundary value problem [5]: $u^{(4)}(x) = 6 \exp(-4 u(x)) - \frac{12}{(1+x)^4}, 0 < x < 1, u(0) = 0, u^{(2)}(0) = -1, u(1) = .6931, u^{(2)}(1) = -.25$. The exact solution is given by $u(x) = \log(1+x)$. The RMS errors for $\sigma = 0.9$ and MA error for $\sigma = 1$ are tabulated in Table 1.

Table 1: Example 5.1

N	RMS error		MA error	
	$O(h_j^2)method$	$O(h_j^3)method$	Twizell[5]	$O(h^4)method$
8	5.2564e-05	4.8610e-06	.37e-05	7.2499e-07
16	6.5587e-05	1.2961e-06	.29e-06	4.6937e-08
32	6.0144e-05	6.7628e-07	.19e-07	2.9600e-09

Example 5.2. Consider a sixth order nonlinear boundary value problem ([7],[11]): $u^{(6)}(x) = \exp(-x)u(x), 0 < x < 1, u(0) = u^{(2)}(0) = u^{(4)}(0) = 1, u(1) = u^{(2)}(1) = u^{(4)}(1) = e$. The exact solution is given by $u(x) = \exp(x)$. The RMS errors for $\sigma = 0.9$ and MA error for $\sigma = 1$ are tabulated in Table 2.

Table 2: Example 5.2

N	RMS error		MA Error		
	$O(h_j^2) method$	$O(h_j^3)method$	Haq et. al.[7]	Inayat et.al.[11]	$O(h^4)method$
.1	2.5623e-04	7.5612e-08	-1.2e-04	1.1106e-07	6.7273e-09
.2	4.0612e-04	1.1260e-07	-2.3e-04	2.1138e-07	1.2324e-08
.3	4.7519e-04	1.2429e-07	-3.2e-04	2.9128e-07	1.6686e-08
.4	4.8429e-04	1.2000e-07	-3.8e-04	3.4229e-07	7.3657e-08
.5	4.5014e-04	1.0613e-07	-4.0e-04	3.6143e-07	2.1209e-08
.6	3.8597e-04	8.6985e-08	-3.9e-04	3.4461e-07	2.1072e-08
.7	3.0201e-04	6.5347e-08	-3.3e-04	2.9390e-07	1.9102e-08
.8	2.0606e-04	4.2992e-08	-2.4e-04	2.1404e-07	1.5094e-08
.9	1.0397e-04	2.1001e-08	-1.2e-04	1.1271e-07	8.8147e-09

Example 5.3. Consider fourth order linear boundary value problem of the form [15]: $u^{(4)}(x) - u(x) = -8x \cos(x) - 12 \sin(x), 0 \leq x \leq 1, u(0) = u(1) = 0, u^{(2)}(0) = 0, u^{(2)}(1) = 2 \sin(1) + 4 \cos(1)$. The exact solution is given by $u(x) = (x^2 - 1) \sin(x)$. The RMS errors for a fixed value $\sigma = 0.9$ and MA error for $\sigma = 1$ are tabulated in Table 3.

Table 3: Example 5.3

N	RMS error		MA error	
	$O(h_j^2)method$	$O(h_j^3)method$	$O(h^4)method$	Ramadan[15]
8	6.4952e-04	8.1257e-06	7.8386e-07	3.010 e-05
16	5.9397e-04	2.1152e-06	4.9504e-08	1.8318 e-06
32	5.1433e-04	1.0688e-06	3.0919e-09	1.1179e-07

Example 5.4. We consider the sixth order linear boundary value problem $([9],[22]): (\frac{d^6}{dx^6} + 1)u(x) = 6(2x \cos(x) + 5 \sin(x)), 0 \leq x \leq 1, u(0) = u(1) = 0, u^{(2)}(0) = 0, u^{(2)}(1) = 2 \sin(1) + 4 \cos(1), u^{(4)}(0) = 0, u^{(4)}(1) = -12 \sin(1) - 8 \cos(1)$. The exact solution is given by $u(x) = (x^2 - 1) \sin(x)$. The RMS errors for a fixed value $\sigma = 0.9$ and MA error for $\sigma = 1$ are tabulated in Table 4.

Table 4: *Example 5.4*

RMS error			MA error		
N	$O(h_j^2)method$	$O(h_j^3)method$	$O(h^4)method$	Akram et.al.[9]	Siddiqi et.al.[22]
8	6.4952e-04	4.2946e-06	6.5904e-07	1.5379 e-06	8.1514e-05
16	5.9397e-04	9.9183e-07	4.1834e-08	1.9790 e-07	2.1052 e-05
32	5.1433e-04	4.7874e-07	2.6137e-09	4.0596 e-08	5.3084 e-06

Example 5.5. Consider a fourth order nonlinear singular boundary value problem of the form: $(\frac{d^4}{dx^4} + \frac{4}{x} \frac{d^3}{dx^3})u = u^2 + \sin(x) - 4 \frac{\cos(x)}{x}, 0 < x \leq 1$. The exact solution is given by $u(x) = \sin(x)$. The boundary conditions are obtained from the exact solution by test procedure. The RMS errors for a fixed value $\sigma = 0.9$ and MA error for $\sigma = 1$ are tabulated in Table 5.

Table 5: *Example 5.5*

RMS error			MA error	
N	$O(h_j^2)method$	$O(h_j^3)method$	$O(h^4)method$	COC*
8	2.1632e-04	5.1244e-06	1.8261e-06	-
16	1.3961e-04	1.3673e-06	1.2672e-07	3.8490
32	1.0763e-04	7.0981e-07	8.3028e-09	3.9320
64	1.0184e-04	6.2357e-07	5.3206e-10	3.9639

Example 5.6. We consider a sixth order nonlinear singular boundary value problem of the form: $(\frac{d^6}{dx^6} + \frac{6}{x} \frac{d^5}{dx^5} + 2)u = e^u + 6e^x(\frac{1+x}{x}), 0 < x \leq 1$. The exact solution is given by $u(x) = \exp(x)$. The boundary conditions are obtained from the exact solution by test procedure. The RMS errors for a fixed value $\sigma = 0.9$ and MA error for $\sigma = 1$ are tabulated in Table 6.

Table 6: *Example 5.6*

RMS error			MA error	
N	$O(h_j^2)method$	$O(h_j^3)method$	$O(h^4)method$	COC*
8	6.1864e-04	6.4701e-07	5.7095e-07	-
16	3.1623e-04	1.6324e-07	4.5354e-08	3.654
32	2.2084e-04	8.3600e-08	3.4251e-09	3.727
64	2.0512e-04	7.3184e-08	2.5097e-10	3.770

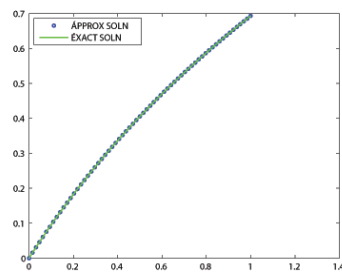


Figure 1: Graph of the exact solution $u(x) = \log(1+x)$ versus the approximate solution in fourth order uniform mesh method for $N = 64$ and $\sigma = 1$ for Example 5.1

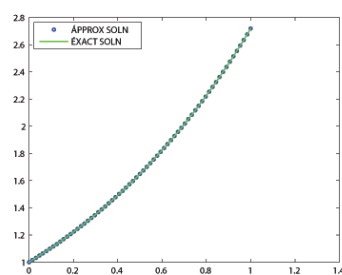


Figure 2: Graph of the exact solution $u(x) = \exp(x)$ versus the approximate solution in fourth order uniform mesh method for $N = 64$ and $\sigma = 1$ for Example 5.2

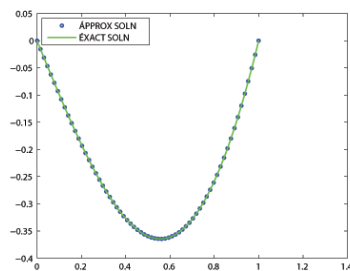


Figure 3: Graph of the exact solution $u(x) = (x^2 - 1)\sin(x)$ versus the approximate solution in fourth order uniform mesh method for $N = 64$ and $\sigma = 1$ for Example 5.3

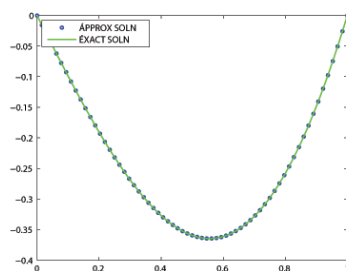


Figure 4: Graph of the exact solution $u(x) = (x^2 - 1) \sin(x)$ versus the approximate solution in fourth order uniform mesh method for $N = 64$ and $\sigma = 1$ for Example 5.4

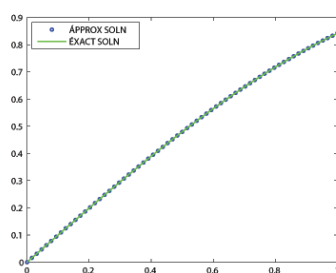


Figure 5: Graph of the exact solution $u(x) = \sin(x)$ versus the approximate solution in fourth order uniform mesh method for $N = 64$ and $\sigma = 1$ for Example 5.5

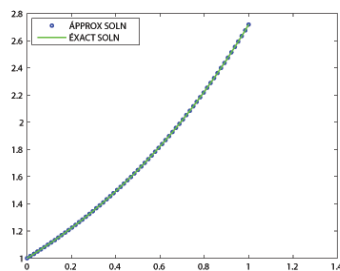


Figure 6: Graph of the exact solution $u(x) = \exp(x)$ versus the approximate solution in fourth order uniform mesh method for $N = 64$ and $\sigma = 1$ for Example 5.6

7. Conclusion

We derived second and third order variable mesh schemes using off-step points for solving linear and nonlinear even order boundary value problems. Although in this paper, only fourth order and sixth order nonlinear and linear boundary value problems are considered, but the method is general enough to be implemented in case of higher even order linear and nonlinear boundary value problems. Table 1 – 4 shows presence of refinement in results when compared with other nonlinear and linear boundary value problems solved by using extrapolation, collocation method using Haar wavelets, iterative method and non polynomial splines. Computationally our methods seems to be more viable due to usage of only three grid points at a time which leads to solving of a tri-diagonal matrix. Also, in the end we have solved fourth and sixth order nonlinear singular boundary value problems. Due to the usage of off-step points presence of singularity is overcome. As per the literature available, such class of boundary value problems has not been solved so far. Therefore, due to unavailability of any prior results we were unable to present a comparative study. Hence we have compared our own results in Table 5 and 6. We have also provided the computational order of convergence (COC^*) for the uniform mesh method. Our methods are applicable to problems in cartesian as well as polar coordinates and even higher order singularly perturbed boundary value problems can be solved easily due to usage of variable mesh.

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SOME NEW RESULTS RELATED TO SUBGROUP COMMUTATIVITY DEGREES AND p -COMMUTATIVITY DEGREES OF FINITE GROUPS

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Abstract. In 1970 Gallagher introduced the concept of commutativity degree of finite groups. Then some authors extend this concept to some variety of groups. In this paper, we define commutativity degree and p -commutativity degree with respect to Burnside variety of groups and study their subgroups and some properties of finite groups in this variety.

Keywords: commutativity degree, Burnside variety, p -commutativity degree.

1. Introduction

Through this paper we assume that all groups are finite. In [1] Gallagher introduced the concept of commutativity degree of finite groups. Lescot [4] defined commutativity degree on abelian variety and Moghaddam [5] defined commutativity degree with respect to the nilpotent variety. In this paper, we define the commutativity degree with respect to Burnside variety for any prime number p and study some results of Lescot in 1995 and Moghaddam in 2005 (see [4,5]). Also we define subgroup commutativity degree with respect to Burnside variety of groups and investigate some properties of Tarnauceanu in [7]. Finally, we define p -commutativity degree and also subgroup p -commutativity degree of finite groups.

Let F be a free group freely generated by a countable set $X = \{x_1, x_2, \dots\}$ and V a non-empty subset of F . Let $\mathcal{V}$ be a variety of groups defined by the set of laws V . There exists two important subgroups associated to a given group G with respect to a variety, as follows

$$V(G) = \langle v(g_1, g_2, \dots, g_r) \mid g_i \in G, 1 \leq i \leq r, v \in V \rangle$$

$$V^*(G) = \{g \in G \mid v(g_1, g_2, \dots, g_i g, \dots, g_r) = v(g_1, g_2, \dots, g_r) \mid g_i \in G, 1 \leq i \leq r, v \in V\}$$

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which are called verbal subgroup and marginal subgroup of G with respect to the variety $\mathcal{V}$. Now, if we take $v(x) = x^p$, then this word corresponds to the Burnside variety $\mathcal{B}$ that means all groups of exponent p . The marginal subgroup corresponding to $\mathcal{B}$ is the set of all central element of exponent p , that is, $V^*(G) = \{x \in Z(G) | x^p = 1\}$ and the verbal subgroup is $V(G) = \{g^p | g \in G\}$.

Definition 1.1. Let $\mathcal{V}$ be a variety of groups defined by the set of laws V , and let G and H be two groups. Then the pair (ϕ_1, ϕ_2) is said to be $\mathcal{V}$ -isologism between G and H , and we say that G and H are $\mathcal{V}$ -isologic, if $\phi_1 : \frac{G}{V^*(G)} \rightarrow \frac{H}{V^*(H)}$ and $\phi_2 : V(G) \rightarrow V(H)$ are isomorphisms such that for all $v(x_1, x_2, \dots, x_r) \in V$ and for all $(g_1, g_2, \dots, g_r) \in G$, we have $\phi_2 v(g_1, g_2, \dots, g_r) = v(h_1, \dots, h_r)$ whenever $h_i \in \phi_1(g_i V^*(G))$, $i=1, \dots, r$.

It is easy to see that G and H are n -isoclinic, when $\mathcal{V}$ is the variety of all nilpotent groups of class at most n .

2. Commutativity degrees of finite groups in Burnside variety

In this section we define the commutativity degree of finite groups with respect to Burnside variety. Let G be a group and $\mathcal{B}$ be Burnside variety, then commutativity degree of group G with respect to Burnside variety is called $\mathcal{B}$ -degree and is as follows

$$d_{\mathcal{B}}(G) = \frac{1}{|G|} |\{x \in G | x^p = 1\}|$$

obviously $0 < d_{\mathcal{B}}(G) \leq 1$.

Clearly if $G \in \mathcal{B}$, then $d_{\mathcal{B}}(G) = 1$.

Theorem 2.1. If G and H are two isologic groups, then $d_{\mathcal{B}}(G) = d_{\mathcal{B}}(H)$.

Proof. Let G and H be two isologic groups. Then there is a pair (α, β) with $\alpha : \frac{G}{V^*(G)} \rightarrow \frac{H}{V^*(H)}$ and $\beta : V(G) \rightarrow V(H)$ and we have $\beta(g^p) = (\alpha(g V^*(G)))^p$, such that the following diagram is commutative

$$\begin{array}{ccc} \frac{G}{V^*(G)} & \xrightarrow{\quad} & \frac{H}{V^*(H)} \\ a_G \downarrow & & a_H \downarrow \\ V(G) & \xrightarrow{\quad} & V(H) \end{array} .$$

Where, $a_G(g V^*(G)) = g^p$ and $a_H(h V^*(H)) = h^p$,

$$\begin{aligned} \frac{|G|}{|V^*(G)|} d_{\mathcal{B}}(G) &= \frac{1}{|V^*(G)|} |\{x \in G | x^p = 1\}| \\ &= \frac{1}{|V^*(G)|} |\{x \in G | a_G(x V^*(G)) = 1\}| \\ &= |\{x V^*(G) \in \frac{G}{V^*(G)} | \beta a_G(x V^*(G)) = 1\}| \\ &= |\{x V^*(G) \in \frac{G}{V^*(G)} | a_H \alpha(x V^*(G)) = 1\}| \end{aligned}$$

$$\begin{aligned}
&= |\{hV^*(H) \in \frac{H}{V^*(H)} | a_H(hV^*(H)) = 1\}| \\
&= |\{hV^*(H) \in \frac{H}{V^*(H)} | h^p = 1\}| \\
&= \frac{1}{|V^*(H)|} |\{h \in H | h^p = 1\}| = \frac{|H|}{|V^*(H)|} d_{\mathcal{B}}(H)
\end{aligned}$$

so $d_{\mathcal{B}}(G) = d_{\mathcal{B}}(H)$.

In order to simplify our notation, we define the Burnside sign function by

$$\begin{aligned}
\phi_{\mathcal{B}} : G &\rightarrow \{0, 1\} \\
\phi_{\mathcal{B}} &= \begin{cases} 1, & x^p = 1_G \\ 0, & x^p \neq 1_G. \end{cases}
\end{aligned}$$

So, $d_{\mathcal{B}}(G) = \frac{1}{|G|} \sum_{x \in G} \phi_{\mathcal{B}}(x)$.

Theorem 2.2. Let G be a group and N be a normal subgroup of G , then

$$d_{\mathcal{B}}(G) \leq d_{\mathcal{B}}(N) d_{\mathcal{B}}\left(\frac{G}{N}\right).$$

Proof. If $x \notin N$ and $x^p \in N$ then $\phi_{\mathcal{B}}(x) = 0$ but $\phi_{\mathcal{B}}(xN) = 1$,

$$\begin{aligned}
|G| d_{\mathcal{B}}(G) &= |\{x \in G | x^p = 1\}| \\
&= \sum_{x \in G} \phi_{\mathcal{B}}(x) \\
&= \sum_{s \in \frac{G}{N}} \sum_{x \in s} \phi_{\mathcal{B}}(x) \leq \sum_{y=xN \in \frac{G}{N}} \phi_{\mathcal{B}}(y) \sum_{x \in N} \phi_{\mathcal{B}}(x) \\
&= \left|\frac{G}{N}\right| \phi_{\mathcal{B}}\left(\frac{G}{N}\right) |N| \phi_{\mathcal{B}}(N)
\end{aligned}$$

which complete the proof.

Corollary 2.3. Let G be a group and N is a normal subgroup of G , then

$$d_{\mathcal{B}}(G) \geq d_{\mathcal{B}}(N)$$

and

$$d_{\mathcal{B}}(G) \leq d_{\mathcal{B}}\left(\frac{G}{N}\right).$$

In the following theorem we show that $\mathcal{B}$ -degree of a direct product of groups is the same as the product of those of direct product.

Theorem 2.4. If G and H are two groups, then

$$d_{\mathcal{B}}(H \times G) = d_{\mathcal{B}}(H) \times d_{\mathcal{B}}(G).$$

Proof. By definition we have

$$\begin{aligned}
 d_{\mathcal{B}}(H \times G) &= \frac{1}{|H \times G|} |\{(h, g) \in H \times G | (h, g)^p = 1\}| \\
 &= \frac{1}{|H||G|} |\{(h, g) \in H \times G | (h^p, g^p) = (1_H, 1_G)\}| \\
 &= \frac{|\{h \in H | h^p = 1\}|}{|H|} \frac{|\{g \in G | g^p = 1\}|}{|G|} \\
 &= d_{\mathcal{B}}(H) \times d_{\mathcal{B}}(G),
 \end{aligned}$$

which complete the proof.

Corollary 2.5. If q is a prime number and $(p, q) = 1$ and G is a q -group, then

$$|G|d_{\mathcal{B}}(G) = 1.$$

Now we introduce subgroup commutativity degree with respect to Burnside variety. The lattice formed by all subgroups of a group G will be denoted by $L(G)$ and will be called the subgroup lattice of the group G (see also [8]). Recall that $L(G)$ is a complete bounded lattices with respect to set inclusion, having initial element the trivial subgroup $\{1\}$ and final element G , and its binary operations $\wedge, \vee$ are defined by $H \wedge K = H \cap K$, $H \vee K = \langle H \cup K \rangle$, for all $H, K \in L(G)$. Now, we define the subgroup Burnside commutativity degree by

$$sd_{\mathcal{B}}(G) = \frac{1}{|L(G)|} |\{H \in L(G) | H^p = 1\}|$$

and the Burnside sign function will be define as following:

$$\begin{aligned}
 \phi_{\mathcal{B}} : L(G) &\rightarrow \{0, 1\}, \\
 \phi_{\mathcal{B}}(H) &= \begin{cases} 1, & H^p = 1 \\ 0, & H^p \neq 1 \end{cases}
 \end{aligned}$$

so

$$sd_{\mathcal{B}}(G) = \frac{1}{|L(G)|} \sum_{H \in L(G)} \phi_{\mathcal{B}}(H).$$

Corollary 2.6. If G is a q -group where $q \mid p$, then $sd_{\mathcal{B}}(G) = 1$.

Theorem 2.7. Let N be a normal subgroup of a group G , then

$$sd_{\mathcal{B}}(G) \geq \frac{1}{|L(G)|} (sd_{\mathcal{B}}(\frac{G}{N})|L(\frac{G}{N})| + sd_{\mathcal{B}}(N)|L(N)| - 1).$$

Proof. Let N be a normal subgroup of G and $A_1 = \{H \in L(G) | N \subseteq H\}$ and $A_2 = \{H \in L(G) | H \subset N\}$. but $A_1 \cup A_2 \subseteq L(G)$ so

$$\begin{aligned}
 (1) \quad sd_{\mathcal{B}}(G) &\geq \frac{1}{|L(G)|} \sum_{H \in A_1 \cup A_2} \phi_{\mathcal{B}}(H) \\
 &= \frac{1}{|L(G)|} \left(\sum_{H \in A_1} \phi_{\mathcal{B}}(H) + \sum_{H \in A_2} \phi_{\mathcal{B}}(H) \right).
 \end{aligned}$$

We can calculate the right side of (1) as follows

$$(2) \quad \sum_{H \in A_1} \phi_{\mathcal{B}}(H) = sd_{\mathcal{B}} \frac{G}{N} |L(\frac{G}{N})|,$$

$$(3) \quad \sum_{H \in A_2} \phi_{\mathcal{B}}(H) = \sum_{H \in A_2 \cup N} \phi_{\mathcal{B}}(H) - \phi_{\mathcal{B}}(N) \geq sd_{\mathcal{B}}(N) |L(N)| - 1$$

by insertion (2) and (3) in (1), the result hold.

Corollary 2.8. (i) If N and $\frac{G}{N}$ are q - groups where $q \mid p$, then

$$sd_{\mathcal{B}}(G) \geq \frac{1}{|L(G)|} (|L(\frac{G}{N})| + |L(N)| - 1),$$

(ii) If N is a normal subgroup of prime index in G , then

$$sd_{\mathcal{B}}(G) \geq \frac{1}{|L(G)|} (2|sd_{\mathcal{B}}(\frac{G}{N}) + sd_{\mathcal{B}}(N)|L(N)| - 1);$$

(iii) If N is a normal subgroup of index q in G where $q \mid p$, then

$$sd_{\mathcal{B}}(G) \geq \frac{1}{|L(G)|} (sd_{\mathcal{B}}(N)|L(N)| + 1);$$

(iv) If $(G_i)_{i=1}^k$ is a family of finite groups with coprime orders, then

$$sd_{\mathcal{B}}(\times_{i=1}^k G_i) = \pi_{i=1}^k sd_{\mathcal{B}}(G_i);$$

(v) If G is a finite nilpotent group and $(p_i)_{i=1, \dots, k}$ are the sylow subgroups of G , then

$$sd_{\mathcal{B}}(G) = \pi_{i=1}^k sd_{\mathcal{B}}(G_i).$$

3. p -commutativity degrees of finite groups

In this section we define p -commutativity degree and subgroup p -commutativity degree of finite groups. Let G be a finite group then the p -commutativity degree, $d_p(G)$, is defined as follows

$$\begin{aligned} d_p(G) &= \frac{1}{|G|^2} |\{(x, y) \in G^2 | [x^p, y^p] = 1\}| \\ &= \frac{1}{|G|^2} |\{(x, y) \in G^2 | x^p y^p = y^p x^p\}| \end{aligned}$$

obviously if $G \in \mathcal{B}$ then $d_p(G) = 1$, otherwise $0 < d_p(G) \leq 1$.

In the following Lemma we state some properties of $d_p(G)$.

Lemma 3.1. Let G and H be two isoclinic groups then $d_p(G) = d_p(H)$.

Proof. Since G and H are isoclinic, the following diagram is commutative

$$\begin{array}{ccc} \frac{G}{Z(G)} \times \frac{G}{Z(G)} & \longrightarrow & \frac{H}{Z(H)} \times \frac{H}{Z(H)} \\ \downarrow a_G & & \downarrow a_H \\ G' & \longrightarrow & H' \end{array}$$

$$\begin{aligned} \left| \frac{G}{Z(G)} \right|^2 d_p(G) &= \frac{1}{|Z(G)|^2} \{ (x, y) \in G^2 \mid [x^p, y^p] = 1 \} \\ &= \{ (xZ(G), yZ(G)) \in (\frac{G}{Z(G)})^2 \mid \phi_2 a_G(x^p Z(G), y^p Z(G)) = 1 \} \\ &= \{ (\gamma, \eta) \in (\frac{G}{Z(G)})^2 \mid \phi_2(a_G(\gamma^p, \eta^p)) = 1 \} \\ &= \{ (\gamma, \eta) \in (\frac{G}{Z(G)})^2 \mid a_H(\phi_1^2((\gamma^p, \eta^p))) = 1 \} \\ &= \{ (\alpha, \beta) \in (\frac{H}{Z(H)})^2 \mid a_H(\alpha^p Z(H), \beta^p Z(H)) = 1 \} \\ &= \frac{1}{|Z(H)|^2} \{ (\alpha, \beta) \in H^2 \mid [\alpha^p, \beta^p] = 1 \} \\ &= \left| \frac{H}{Z(H)} \right|^2 d_p(H) \end{aligned}$$

so $d_p(G) = d_p(H)$.

Definition 3.2. We define the function f_p by

$$f_p : G \times G \rightarrow \{0, 1\}$$

$$f_p(x, y) = \begin{cases} 1, & [x^p, y^p] = 1 \\ 0, & [x^p, y^p] \neq 1 \end{cases}$$

$$\text{so } d_p(G) = \frac{1}{|G|^2} \sum_{(x,y) \in G \times G} f_p(x, y).$$

In 1970 Gallagher [1] proved that if G is a finite group with a normal subgroup N , then $d(G) \leq d(N)d(\frac{G}{N})$. Now we generalize this result for the class of all group G in which the centralizer p -element is normal in G . Such group is called CN-group.

Lemma 3.3. Let G be a CN-group or centralizer p -element with a normal subgroup N , then

$$d_p(G) = \frac{1}{|G|} \sum_{x \in G} d_B(\frac{G}{C_G(x^p)}).$$

Proof. Let $y^{-p} = y^p C_G(x^p)$ for any $x, y \in G$,

$$\begin{aligned} |G|^2 d_p(G) &= |\{(x, y) \in G^2 \mid [x^p, y^p] = 1\}| = \sum_{x \in G} |\{y \in G \mid y^p \in C_G(x^p)\}| \\ &= \sum_{x \in G} |\{\bar{y} \in \frac{G}{C_G(x^p)} \mid \bar{y}^p = \bar{1}\}| |C_G(x^p)| \\ &= \sum_{x \in G} |C_G(x^p)| \left| \frac{G}{C_G(x^p)} \right| d_{\mathcal{B}}\left(\frac{G}{C_G(x^p)}\right) \\ &= \sum_{x \in G} |G| d_{\mathcal{B}}\left(\frac{G}{C_G(x^p)}\right), \end{aligned}$$

which complete the proof.

Theorem 3.4. Let G be a CN-group then for any normal subgroup N

$$d_p(G) \leq d_p(N) d_p\left(\frac{G}{N}\right).$$

Proof. It is clear that if two elements x^p and y^p are commutative, then $x^p N$ and $y^p N$ are commutative. Now for any $x \in G$

$$(4) \quad \frac{C_G(x^p)}{N} \subseteq C_{\frac{G}{N}}(x^p N)$$

on the other hand we know

$$(5) \quad d_{\mathcal{B}}\left(\frac{G}{N}\right) \geq d_{\mathcal{B}}(G)$$

by Lemma 1.4,

$$(6) \quad d_p(G) = \frac{1}{|G|} \sum_{x \in G} d_{\mathcal{B}}\left(\frac{G}{C_G(x^p)}\right)$$

it is obvious $C_N(x^p) = C_G(x^p) \cap N$ for any $x \in G$, so

$$(7) \quad \frac{N}{C_N(x^p)} \cong \frac{C_G(x^p)N}{C_G(x^p)} \triangleleft \frac{G}{C_G(x^p)}$$

and so,

$$(8) \quad \frac{\frac{G}{C_G(x^p)}}{\frac{N}{C_N(x^p)}} \cong \frac{\frac{G}{N}}{\frac{C_G(x^p)N}{N}}.$$

Now by (1),

$$(9) \quad \frac{\frac{\left(\frac{G}{N}\right)}{\left(\frac{C_G(x^p)N}{N}\right)}}{\frac{C_{\frac{G}{N}}(x^p N)}{\left(\frac{C_G(x^p)N}{N}\right)}} \cong \frac{\frac{G}{N}}{C_{\frac{G}{N}}(x^p N)}$$

by (3) and (4),

$$|G|d_p(G) = \sum_{x \in G} d_{\mathcal{B}}\left(\frac{G}{C_G(x^p)}\right) \leq \sum_{x \in G} d_{\mathcal{B}}\left(\frac{N}{C_N(x^p)}\right) d_{\mathcal{B}}\left(\frac{\left(\frac{G}{C_G(x^p)}\right)}{\left(\frac{N}{C_N(x^p)}\right)}\right)$$

by (5),

$$= \sum_{x \in G} d_{\mathcal{B}}\left(\frac{N}{C_N(x^p)}\right) d_{\mathcal{B}}\left(\frac{\left(\frac{G}{N}\right)}{\left(\frac{C_G(x^p)N}{N}\right)}\right)$$

by (2) and (6),

$$\leq \sum_{x \in G} d_{\mathcal{B}}\left(\frac{N}{C_N(x^p)}\right) d_{\mathcal{B}}\left(\frac{\frac{G}{N}}{C_{\frac{G}{N}}(x^p N)}\right)$$

by (3),

$$\begin{aligned} |G|d_p(G) &\leq \sum_{x \in N} d_{\mathcal{B}}\left(\frac{N}{C_N(x^p)}\right) d_{\mathcal{B}}\left(\frac{\frac{G}{N}}{C_{\frac{G}{N}}(x^p N)}\right) \\ &\quad \sum_{y=xN \in \frac{G}{N}} d_{\mathcal{B}}\left(\frac{N}{C_N(x^p)}\right) d_{\mathcal{B}}\left(\frac{\frac{G}{N}}{C_{\frac{G}{N}}(x^p N)}\right) \\ &\leq \sum_{x \in N} d_{\mathcal{B}}\left(\frac{N}{C_N(x^p)}\right) \sum_{y \in \frac{G}{N}} d_{\mathcal{B}}\left(\frac{\frac{G}{N}}{C_{\frac{G}{N}}(x^p N)}\right) \\ &= |N|d_p(N) \left|\frac{G}{N}\right| d_p\left(\frac{G}{N}\right) \end{aligned}$$

and we know $|G| = |N| \left|\frac{G}{N}\right|$ so the proof is complete.

In the following theorem, we show that p-degree of a direct product of groups is the same as the product of those of direct product.

Theorem 3.5. If G and H are two groups, then

$$d_p(H \times G) = d_p(H) \times d_p(G).$$

Proof. By theorem 1.5, we have

$$\begin{aligned} d_p(H \times G) &= \frac{1}{|H \times G|} \sum_{(x,y) \in G \times H} d_{\mathcal{B}}\left(\frac{G \times H}{C_{G \times H}(x^p, y^p)}\right) \\ &= \frac{1}{|H||G|} \sum_{(x,y) \in G \times H} d_{\mathcal{B}}\left(\frac{G}{C_G(x^p)} \times \frac{H}{C_H(y^p)}\right) \\ &= \frac{1}{|H||G|} \sum_{(x,y) \in G \times H} d_{\mathcal{B}}\left(\frac{G}{C_G(x^p)}\right) \times d_{\mathcal{B}}\left(\frac{H}{C_H(y^p)}\right) \\ &= \frac{1}{|G|} \sum_{x \in G} d_{\mathcal{B}}\left(\frac{G}{C_G(x^p)}\right) \frac{1}{|H|} \sum_{y \in H} d_{\mathcal{B}}\left(\frac{H}{C_H(y^p)}\right) \\ &= d_p(H) \times d_p(G) \end{aligned}$$

which complete the proof.

Lemma 3.6. If r is number of the conjugacy classes of $V(G)$, then

$$d_p(G) = \frac{r|V(G)|}{|G|^2}$$

Proof. Let $c_1, c_2, \dots, c_r$ be conjugacy classes of $V(G)$ for $i \in \{1, \dots, r\}$ and $x_i^p \in c_i$. For any $y_i^p \in c_i$ there is $g^p \in V(G)$ such that $g^{-p}x_i^p g^p = y_i^p$ so

$$C_{V(G)}(y_i^p) = C_{V(G)}(x_i^p)$$

now we have

$$\begin{aligned} |G|^2 d_p(G) &= |\{(x, y) \in G^2 | x^p y^p = y^p x^p\}| \geq |\{(x, y) \in V(G) \times V(G) | x^p y^p = y^p x^p\}| \\ &= \sum_{x^p \in V(G)} |C_{V(G)}(x^p)| = \sum_{i=1}^r \sum_{x^p \in c_i} |C_{V(G)}(x^p)| = \sum_{i=1}^r |c_i| [V(G) : c_i] \\ \sum_{i=1}^r |V(G)| &= r|V(G)| \end{aligned}$$

and the proof is complete.

Corollary 3.7. If $K(V(G))$ is number of the conjugacy classes of $V(G)$, then

$$K(V(G)) \leq \frac{|G|^2}{|V(G)|}.$$

Corollary 3.8. If G is free of initial member of order p , then the above equality is pointed.

Corollary 3.9. If G is q -group where $(p, q) = 1$, then

$$d_p(N) d_p\left(\frac{G}{N}\right) = \frac{K(V(N)) K(V(\frac{G}{N}))}{|G|}.$$

Proof. G is a q -group that $(p, q) = 1$ so, $V(N) = (N^p)$, $V(\frac{G}{N}) = (\frac{G}{N})^p$

$$\begin{aligned} d_p(N) d_p\left(\frac{G}{N}\right) &= \frac{|N^p|}{|N|^2} K(N^p) \frac{|(\frac{G}{N})^p|}{|\frac{G}{N}|^2} K(V(\frac{G}{N})) \\ &= \frac{|V(N)| |V(\frac{G}{N})|}{|G|^2} K(V(N)) K(V(\frac{G}{N})) \end{aligned}$$

since $|G| = |V(G)| = |V(N)| |V(\frac{G}{N})|$ so the result hold.

Definition 3.10. For two subgroups H and K of finite group G , $HK \in L(G)$ if and only if $KH = HK$ it means H and K are commutative. So $H^p K^p \in L(G)$ if and only if $H^p K^p = K^p H^p$ for any prime number of p . We define the subgroup p -commutativity degree $sd_p(G)$ of a finite group G , by

$$sd_p(G) = \frac{|\{(H, K) \in L(G) \times L(G) | H^p K^p = K^p H^p\}|}{|L(G)^2|}.$$

It is clear that, $0 < sd_p(G) \leq 1$. Now for any subgroup H of G we define

$$sd_p(G) = \frac{1}{|L(G)^2|} \sum_{H \in L(G)} |C_p(H)|,$$

where

$$C_p(H) = \{k \in L(G) | H^p K^p = K^p H^p\}.$$

In order to simplify our notation, we define the following function

$$f_p : L(G)^2 \rightarrow \{0, 1\}$$

$$f(H, K) = \begin{cases} 1, & H^p K^p = K^p H^p \\ 0, & H^p K^p \neq K^p H^p \end{cases}$$

obviously $|C_p(H)| = \sum_{K \in L(G)} f(H, K)$, for any $H \in L(G)$, so

$$sd_p(G) = \frac{1}{|L(G)^2|} \sum_{H, K \in L(G)} f(H, K).$$

Theorem 3.11. If $(G_i)_{i=1}^k$ be a family of finite groups having coprime orders, then

$$sd_p(\times_{i=1}^k G_i) = \pi_{i=1}^k sd_p(G_i).$$

Corollary 3.12. If G is a finite nilpotent group and $(p_i)_{i=1, \dots, k}$ are the sylow subgroups of G , then

$$sd_{\mathcal{B}}(G) = \pi_{i=1}^k sd_{\mathcal{B}}(G_i).$$

Theorem 3.13. Let N be a normal subgroup of a group G , then

$$sd_p(G) \geq \frac{1}{|L(G)|^2} [(|L(N)| + |L(\frac{G}{N})| - 1)^2 + (sd_p(N) - 1)|L(N)|^2 \\ + (sd_p(\frac{G}{N}) - 1)|L(\frac{G}{N})|^2].$$

Proof. Let N be a normal subgroup of G and $A_1 = \{H \in L(G) | N \subseteq H\}$ and $A_2 = \{H \in L(G) | H \subset N\}$ but $A_1 \cup A_2 \subseteq L(G)$ so

$$\begin{aligned}
 (10) \quad sd_p(G) &\geq \frac{1}{|L(G)|^2} \sum_{H, K \in A_1 \cup A_2} f(H, K) \\
 &= \frac{1}{|L(G)|^2} \left(\sum_{H, K \in A_1} f(H, K) \right. \\
 &\quad \left. + \sum_{H, K \in A_2} f(H, K) + 2 \sum_{H \in A_1} \sum_{K \in A_2} f(H, K) \right).
 \end{aligned}$$

We can calculate the right side of (1) as follows

$$\begin{aligned}
 \sum_{H, K \in A_1} f(H, K) &= sd_p\left(\frac{G}{N}\right) |L\left(\frac{G}{N}\right)|^2 \\
 \sum_{H, K \in A_2} f(H, K) &= \sum_{H, K \in A_2 \cup N} f(H, K) - 2 \sum_{H, K \in A_2 \cup N} f(H, N) + 1 \\
 &= sd_p(N) |L(N)|^2 - 2|L(N)| + 1
 \end{aligned}$$

and

$$2 \sum_{x \in A_1} \sum_{x \in A_2} f(H, K) = 2|A_1||A_2| = 2|L\left(\frac{G}{N}\right)|(|L(N)| - 1)$$

so by (1) the result is hold.

Corollary 3.14. If N is a normal subgroup of G such that N and $\frac{G}{N}$ are p -abelian groups, then

$$sd_p(G) \geq \left(\frac{|L(N)| + |L\left(\frac{G}{N}\right)| - 1}{|L(G)|} \right)^2.$$

Now we can expand the definition of $ssd(G)$ to $ssd_p(G)$. Let, $G^p = V(G) = \{g^p | g \in G\}$, then

$$\begin{aligned}
 d_p(G) &= \frac{|\{(x, y) \in G^2 | [x^p, y^p] = 1\}|}{|G|^2} \\
 &= \frac{1}{|G|^2} \sum_{x \in G} |C_{V(G)}(x^p)|,
 \end{aligned}$$

where

$$C_{V(G)}(x^p) = \{y^p \in V(G), |[x^p, y^p] = 1\}.$$

If we take a subgroup H in G , denoted by $d_p(H, G)$ as follows

$$\begin{aligned}
 d_p(H, G) &= \frac{|\{(x, y) \in H \times G | [x^p, y^p] = 1\}|}{|H||G|} \\
 &= \frac{1}{|G||H|} \sum_{h \in H} |C_{V(G)}(h^p)|
 \end{aligned}$$

where $d_p(G, G) = d_p(G)$, so we define the relative subgroup p -commutativity degree of a subgroup H of G

$$sd_p(H, G) = \frac{1}{|L(G)||L(H)|} |\{(H_1, G_1) \in L(H_1) \times L(G_1) | H_1^p G_1^p = G_1^p H_1^p\}|$$

$sd_p(G) = sd_p(G, G)$ and also we can define $sd_p(H, K)$ for any group of G . But we know that by given subgroups H and K of G , the product $H^p K^p$ is not always a subgroup of G .

Now H and K p -permute if $H^p K^p = K^p H^p$ and must be p -permutable in G , if H^p permutes with every power of p of subgroup of G . We will define strong subgroup p -commutativity degree of G by

$$ssd_p(G) = \frac{|\{(H, K) \in L(G) \times L(G) | [H^p, K^p] = 1\}|}{|L(G)|^2}$$

it is easy to see that

$$ssd_p(G) = \frac{1}{|L(G)|^2} \sum_{H \in L(G)} |comm_{p-G}(H)|.$$

Where

$$comm_{p-G}(H) = \{k \in L(G) | [H^p, K^p] = 1\}.$$

It is obvious that $sd_p(G) = 1$ if and only if any subgroups of G be p -permutable. for a finite group G , $ssd_p(G) \leq sd(G)$ because we have

$$ssd_p(G) = \frac{1}{|L(G)|^2} \sum_{H \in L(G)} |comm_{p-G}(H)| \leq \frac{1}{|L(G)|^2} \sum_{H \in L(G)} |C_p(H)| = sd_p(G)$$

and it means

$$|\{(H, K) \in L(G) \times L(G) | H^p K^p = K^p H^p\}| \geq |\{(H, K) \in L(G) \times L(G) | [H^p, K^p] = 1\}|.$$

Theorem 3.15. If H and K are two subgroups of G , then

$$ssd_p(G) < \frac{|G|^2}{|L(G)|^2} \sum_{H, K \in L(G)} d_p(H, K).$$

Proof.

$$\begin{aligned} ssd_p(G) &= \frac{1}{|L(G)|^2} |\{(H, K) \in L(G) \times L(G) | [H^p, K^p] = 1\}| \\ &\leq \frac{1}{|L(G)|^2} \sum_{H, K \in L(G)} |H||K|d_p(H, K) \\ &< \frac{1}{|L(G)|^2} \sum_{H, K \in L(G)} |G|^2 d_p(H, K) \end{aligned}$$

and the result is obtained.

In general, $L(G_1 \times G_2) \neq (G_1) \times L(G_2)$, and if G_1 and G_2 having coprime orders then the equality is obtained and if $(G_i)_{i=1}^k$ be a family of finite groups having coprime orders, then

$$ssd_p(\times_{i=1}^k G_i) = \pi_{i=1}^k ssd_p(G_i).$$

Theorem 3.16. If H and K are two groups of G , then

$$sd_p(G) \geq \frac{1}{|L(G)|^2} \sum_{H \in L(G)} \left| \bigcap_{h \in H} C_{V(K)}(h^p) \right|.$$

Proof.

$$\bigcap_{h \in H} C_{V(K)}(h^p) = C_{V(K)}(V(H)) \subseteq C_p(H)$$

so

$$\sum_{H \in L(G)} \left| \bigcap_{h \in H} C_{V(K)}(h^p) \right| \leq \sum_{H \in L(G)} |C_p(H)|$$

and the result hold.

Theorem 3.17. If H and K are two groups of G , then

$$\sum_{H, K \in L(G)} d_p(H, K) |H| |K| \geq \sum_{H, K \in L(G)} \left| \bigcap_{h \in H} C_{V(K)}(h^p) \right|.$$

Proof. The proof is obvious with the following inequality

$$\sum_{h \in H} |C_{V(K)}(h^p)| \geq \left| \bigcap_{h \in H} C_{V(K)}(h^p) \right|.$$

Corollary 3.18. For any subgroups L and M of H that H is a subgroup of G ,

$$\frac{1}{|L(G)|^2} \sum_{L \in L(H)} \left| \bigcap_{l \in L} C_{V(M)}(l^p) \right| \leq sd_p(H) \leq sd_p(G).$$

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CHAPMAN-JOUGUET TRAVELLING WAVE FOR A TWO-STEPS REACTION SCHEME

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Abstract. In this paper, a version of the Majda's model for a two-steps reaction is studied. Then by studying the stable and unstable manifolds of the resulted system of ordinary differential equations, the existence of a heteroclinic orbit is proved.

Keywords: Detonation wave, Majda's model, Combustion, Shock waves, Travelling waves.

1. Introduction

Majda [7] wrote the Navier-Stokes equations of reacting flow for a one-step reaction as a model which is expected to retain important aspects of the strongly coupled interactions between the nonlinear motion of a gas mixture and chemical reactions involving the different species of gas making up the mixture. This model in the abstract form is as follows.

$$(1) \quad \begin{cases} (U + qZ)_t + f(U)_x = BU_{xx}, \\ Z_t = -k\phi(U)Z \end{cases}$$

where the unknown function $U = U(x, t)$ is real valued(should be thought of as a stand-in for density, velocity, and temperature), $0 \leq Z = Z(x, t) \leq 1$ measures the fraction by mass of reactant (fuel) in a simple one-step reaction scheme, the flux f is a nonlinear convex function, ϕ is the ignition function and finally k, q , and B are positive constants measuring reaction rate, heat release, and viscosity, respectively.

Following [3], we restrict attention to the Majda's model for a two-steps reaction scheme $R \rightarrow I, I \rightarrow P$ which is modeled by

$$(2) \quad \begin{cases} \beta T_{xx} = \frac{d}{d\tau}T + \{f(T)\}_x - q_0 Z_{0x} - q_1 Z_{1x}, \\ Z_{0x} = \kappa_1 \Phi_1(T)Z_0, \\ Z_{1x} = \kappa_2 \Phi_2(T)(Z_0 - Z_1), \end{cases}$$

where $\kappa_j > 0$, $j = 1, 2$ are constant, T temperature, Z_0 mass fraction of unburned gas (note that the completely unburned gas is $Z_0 = 1$ and a totally

burned gas is $Z_0 = 0$), Z_1 mass fraction of intermediate gas (note that the completely intermediate gas is $Z_1 = 1$ and a totally intermediate gas is $Z_1 = 0$), τ a variable proportional to the arc length along the characteristics, the coordinate x is not the space coordinate but is determined as a scaled space-time coordinate representing distance to the reaction zone (the x -differentiation occurs because Z_0 in (2) is convected at the much slower fluid velocity rather than the much faster reacting shock speed (see [1] or [9] for details)), q_j for $j = 1, 2$ denote heats of reactions with either (i) $q_0, q_1 > 0$ (exothermic/exothermic), or (ii) $q_0 > 0, q_1 < 0$ (exothermic/endothermic), and finally $\beta > 0$ is a lumped viscosity-thermal-conductivity coefficient. The function $\Phi_1(T)$, which is called the “reaction rate function” is defined by:

$$(3) \quad \Phi_1(T) = \begin{cases} 0, & \text{for } T < T_i, \\ \Phi'(T), & \text{for } T \geq T_i, \end{cases}$$

where $\Phi'(T)$ is a smooth positive function and T_i is the “ignition temperature” of the reaction. A typical example for $\Phi'(T)$ is the Arrhenius law, *i.e.* $\Phi'(T) = T^\gamma e^{-\frac{A}{T}}$ for some positive constants γ and A . Notice that $\Phi_1(T)$ is discontinuous at the point T_i . Moreover, $\Phi_2(T)$ is ignition function of the usual form, so $\Phi_2(T) > 0$ and is continuous. Also $f(T)$ is a convex and strongly nonlinear function satisfying (see [7])

$$(4) \quad \begin{aligned} \frac{\partial f}{\partial T} &= a(T) > 0, \quad \frac{\partial^2 f}{\partial T^2} > \delta > 0, \\ \lim_{T \rightarrow +\infty} f(T) &= +\infty, \end{aligned}$$

and for example you can choose $f(T) = \frac{1}{2}aT^2$, ($a > 0$) (see [9, page 1100]).

We study the existence of Chapman-Jouguet detonation wave for the system (2) when this model involving two reactions which are exothermic, *i.e.* $q_0, q_1 > 0$. In order to prove the existence of travelling wave solution for the system (2), the system (2) is reduced to a system of differential equations and the rest points of the resulted system are studied, in Section 2. Finally, in Section 3 the existence of CJ detonation wave for (2) is proved.

2. Travelling wave solution

First, the definition of the travelling wave solution of the system (2) is recalled.

Definition 2.1. A travelling wave solution between two states $(T_l, Z_{0l}, Z_{1l})^T$ and $(T_r, Z_{0r}, Z_{1r})^T$ of the system (2) is a solution $(T(x, \tau), Z_0(x, \tau), Z_1(x, \tau))^T$ of the system (2), if there is a constant $s \in \mathbb{R}$, which is called the speed of combustion shock wave, satisfying “the jump and entropy conditions”, moreover this solution depends only on the variable $\xi = x - s\tau$ [13].

This means that a travelling wave solution of (2) has the following form:

$$(T(x - s\tau), Z_0(x - s\tau), Z_1(x - s\tau))^T.$$

Thus the system (2) reduces to the following system of equations (for travelling wave solutions):

$$(5) \quad \begin{cases} \beta T_{\xi\xi} = -sT_{\xi} + \{f(T)\}_{\xi} - q_0 Z_{0\xi} - q_1 Z_{1\xi}, \\ Z_{0\xi} = \kappa_1 \Phi_1(T) Z_0, \\ Z_{1\xi} = \kappa_2 \Phi_2(T) (Z_0 - Z_1). \end{cases}$$

Integrate the first equation in (5), to get $\beta T_{\xi} = f(T) - sT - q_0 Z_0 - q_1 Z_1 + C$, where C is the constant of integration. Thus the system (5) reduces to the following system:

$$(6) \quad \begin{cases} \beta T_{\xi} = f(T) - sT - q_0 Z_0 - q_1 Z_1 + C := g(T, Z_0, Z_1), \\ Z_{0\xi} = \kappa_1 \Phi_1(T) Z_0, \\ Z_{1\xi} = \kappa_2 \Phi_2(T) (Z_0 - Z_1). \end{cases}$$

In order to find the rest points of the system (6), we have

$$(7) \quad \begin{cases} f(T) - sT - q_0 Z_0 - q_1 Z_1 + C = 0, \\ \kappa_1 \Phi_1(T) Z_0 = 0, \\ \kappa_2 \Phi_2(T) (Z_0 - Z_1) = 0. \end{cases}$$

Since $\Phi_1(T) = 0$ for $T < T_i$, where T_i is the ignition temperature and this set is contained in the region $0 < Z_0 \leq 1$. Thus from the second equation of (7), at a rest point, we must have $Z_0 = 0$ or $T < T_i$. Thus we have two cases as follows:

Case 1: $Z_0 = 0$. Since $\Phi(T) > 0$, from the last equation of (7), we have $Z_1 = 0$. Also the first equation of (7), at a rest point, we have

$$(8) \quad g_{00}(T) := g(T, 0, 0) = f(T) - sT + C = 0.$$

Case 2: $\Phi_1(T) = 0$. This means $T < T_i$. Thus the second equation of (7) gives $Z_0 = m_0$, for $0 < m_0 \leq 1$. Also the last equation of (7), implies $Z_1 = m_0$, since $\phi_2(T) > 0$. Finally, the first equation of (7) gives a set of rest points. This set is

$$(9) \quad g_{m_0 m_0}(T) := g(T, m_0, m_0) = f(T) - sT - q_0 m_0 - q_1 m_0 + C = 0.$$

Now we have the following lemmas.

Lemma 2.2. $g_{m_0 m_0}(T)$ has an absolute minimum, where $0 < m_0 \leq 1$, i.e. $\frac{dg_{m_0 m_0}(T)}{dT} = 0$ for precisely one value of T .

Proof. By computing the first and second derivative $g_{m_0 m_0}(T)$ with respect to T , we have

$$\frac{dg_{m_0 m_0}}{dT} = f'(T) - s = a(T) - s, \quad \frac{d^2 g_{m_0 m_0}}{dT^2} = f''(T) > \delta > 0.$$

Notice that $a(T) - s = 0$ for precisely one value of T and $\frac{d^2 g_{m_0 m_0}}{dT^2} > \delta > 0$. Thus $g_{m_0 m_0}(T)$ has an absolute minimum. $\square$

Lemma 2.3. *There is a number $C_{00} \in \mathbb{R}$ such that for $C > C_{00}$, the system (8) admits no solutions. For $C = C_{00}$ it admits one solution, and for $C < C_{00}$ it admits two solutions.*

Proof. It is trivial to see. $\square$

Remark 2.4. With respect to Lemma 2.3 for $C = C_{00}$ we have one rest point $(T_{CJ}, 0, 0)$ which represent the Chapman-Jouguet state.

Now consider $C = C_{00}$, for $m_0 = 0$, Lemma 2.2 along with (4) show that the equation (8) has exactly one root T_{CJ} with $s = f'(T_{CJ})$.

Consider now all the rest points of the system (6) corresponding to unburned gas states. These points are described by $g(T_{m_0}, m_0, m_0) = 0$, $T_{m_0} \leq T_i$, where $0 < m_0 \leq 1$ and T_i is the ignition temperature which will be obtained later.

By considering the above results the rest points of the system (6) are

$$(10) \quad u_{CJ} = (T_{CJ}, 0, 0), \quad u_{m_0 0} = (T_{m_0 0}, m_0, m_0), 0 < m_0 \leq 1, T_{m_0 0} \leq T_i,$$

where $T_i < T_{CJ}$.

In the present work it is assumed that the rest point u_{CJ} exists.

Corollary 2.5. *If the rest point u_{CJ} exists, then the rest point $u_{m_0 0}$ exists for some $0 < m_0 \leq 1$.*

Now, we recall the definition of Chapman-Jouguet detonation wave.

Definition 2.6. A combustion shock wave between u_{CJ} and $u_{m_0 0}$ is called a Chapman-Jouguet detonation wave.

From mathematical point of view, the existence of Chapman-Jouguet detonation wave corresponds to the existence of some complete orbit of the system (6) which is running from the rest point u_{CJ} to $u_{m_0 0}$ for some $0 < m_0 \leq 1$. Such an orbit is called a travelling wave solution for the system (2).

3. Chapman-Jouguet detonation

We consider the linearized system of (6) at the rest point u_{CJ} , which can be written as $\dot{u} = M_{CJ}(u - u_{CJ})$, where

$$M_{CJ} = \begin{bmatrix} \frac{1}{\beta}(f'(T_{CJ}) - s) & -\frac{q_0}{\beta} & -\frac{q_1}{\beta} \\ 0 & \kappa_1 \Phi'(T_{CJ}) & 0 \\ 0 & \kappa_2 \Phi_2(T_{CJ}) & -\kappa_2 \Phi_2(T_{CJ}) \end{bmatrix},$$

where the entries of the matrix must be considered at the rest point u_{CJ} . The next lemma says about the sign of the eigenvalues and its proof is easy.

Lemma 3.1. *Let $\lambda_k, k = 1, 2, 3$, be the eigenvalues of the matrix M_{CJ} at the rest point $u_{CJ} = (T_{CJ}, 0, 0)$. Then at the rest point u_{CJ} , $\lambda_1 = 0, \lambda_2 > 0, \lambda_3 < 0$.*

About the eigenvectors at this rest point we have the following theorem.

Theorem 3.2. *Let $(y_1, y_2, y_3)^T$ be an eigenvector corresponding to the positive eigenvalue λ_2 . At u_{CJ} , either $y_1 < 0$ and $y_2 > 0, y_3 > 0$, or the reverses inequalities hold.*

Proof. Notice that

$$\begin{bmatrix} 0 & -\frac{q_0}{\beta} & -\frac{q_1}{\beta} \\ 0 & \kappa_1 \Phi'(T_{CJ}) & 0 \\ 0 & \kappa_2 \Phi_2(T_{CJ}) & -\kappa_2 \Phi_2(T_{CJ}) \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \lambda_2 y_1 \\ \lambda_2 y_2 \\ \lambda_2 y_3 \end{bmatrix}.$$

Thus

$$(11) \quad \begin{cases} \lambda_2 y_1 = -\frac{q_0}{\beta} y_2 - \frac{q_1}{\beta} y_3, \\ \kappa_1 \Phi'(T_{CJ}) y_2 = \lambda_2 y_2, \\ \kappa_2 \Phi_2(T_{CJ}) y_2 - \kappa_2 \Phi_2(T_{CJ}) y_3 = \lambda_2 y_3. \end{cases}$$

The last equation of (11) implies $\kappa_2 \Phi_2(T_{CJ}) y_2 = (\kappa_2 \Phi_2(T_{CJ}) + \lambda_2) y_3$. Since $\lambda_2 > 0$, this means that $\text{sgn} y_2 = \text{sgn} y_3$. Also, the first equation of (11) implies $\text{sgn} y_1 = -\text{sgn} y_2 = -\text{sgn} y_3$. $\square$

Now, we define:

$$D = \{u \in \mathbb{R}^3 : 0 < Z_0 < 1, T < T_{CJ}, g(T, Z_0, Z_1) < 0, 0 < Z_0 - Z_1, Z_1 > 0\}.$$

Notice that the rest point u_{CJ} is located on ∂D . Moreover, by Lemma 3.1, the unstable manifold at u_{CJ} , is one dimensional.

Lemma 3.3. *Let D be as above, the unstable manifold at u_{CJ} intersects D on a curve.*

Proof. The linearized system of (6) at the rest point u_{CJ} , has the following form:

$$(12) \quad \begin{cases} \beta \dot{T} = -q_0 Z_0 - q_1 Z_1 = g_{1l}(T, Z_0, Z_1), \\ \dot{Z}_0 = \kappa_1 \Phi'(T_{CJ}) Z_0 = g_{2l}(Z_0, Z_1), \\ \dot{Z}_1 = \kappa_2 \Phi_2(T_{CJ}) Z_0 - \kappa_2 \Phi_2(T_{CJ}) Z_1 = g_{3l}(Z_0, Z_1), \end{cases}$$

Let $(y_1, y_2, y_3)^T$ be an eigenvector corresponding to the positive eigenvalue $\lambda_2 = \kappa_1 \Phi'(T_{CJ})$, at the rest point u_{CJ} . Now consider the solution $u(\xi) = (T(\xi), Z_0(\xi), Z_1(\xi))^T = (y_1, y_2, y_3)^T e^{\lambda_2 \xi} + u_{CJ}$, of the linear system (12). Then $u(\xi) \in D_{CJ}$, where $D_{CJ} = \{u \in \mathbb{R}^3 : 0 < Z_0 < 1, 0 < Z_0 - Z_1, Z_1 > 0\}$. To see this notice that $(g_{1l}(T, Z_0, Z_1), g_{2l}(Z_0, Z_1), g_{3l}(Z_0, Z_1))^T = M(u - u_{CJ}) = MY e^{\lambda_2 \xi} = \lambda_2 Y e^{\lambda_2 \xi} = (\lambda_2 y_1, \lambda_2 y_2, \lambda_2 y_3)^T e^{\lambda_2 \xi}$.

By Theorem 3.2, we may assume that $y_1 < 0$ and $y_2 > 0, y_3 > 0$. Since $\lambda_2 > 0$, it follows from the last equality that $(g_{1l}(T, Z_0, Z_1), g_{2l}(Z_0, Z_1), g_{3l}(Z_0, Z_1)) \in$

D_{CJ} . This means that the unstable manifold of (12), at the rest point u_{CJ} , which is the line $M_{CJ} = \{u \in \mathbb{R}^3 : u - u_{00} = (y_1, y_2, y_3)^T s, s \in \mathbb{R}\}$, lies in D_{CJ} for $s > 0$, and lies in D for $s > 0$ and small. Thus the unstable manifold of the system (12), at the rest point u_{CJ} intersect D on a curve. $\square$

Now consider the following system of ordinary differential equations:

$$(13) \quad \begin{cases} \beta T_\xi = f(T) - sT - q_0 Z_0 - q_1 Z_1 + C = g(T, Z_0, Z_1), \\ Z_{0\xi} = \kappa_1 \Phi'(T) Z_0 := g_4(T, Z_0, Z_1), \\ Z_{1\xi} = \kappa_2 \Phi_2(T)(Z_0 - Z_1) := g_5(T, Z_0, Z_1). \end{cases}$$

where $\Phi'(T)$ is defined by (3). Notice that the above system leads us to the proof of the existence of travelling waves for Chapman-Jouguet detonation wave. Before doing this, we recall the following theorem from [8].

Theorem 3.4. *Suppose the function f in*

$$(14) \quad \frac{dx}{dt} = f(x), \quad x = (x_1, x_2, \dots, x_n)^T,$$

is locally Lipschitz in a neighborhood of the closure of a bounded open set D which is homeomorphic to the cylinder $\{x \in \mathbb{R}^n : \sum_{i=1}^{n-1} x_i^2 < 1, 0 < x_n < 1\}$, and (14) is gradient-like with respect to a function h in D . Moreover, suppose the following conditions hold:

C_1 : *The set $\{x \in \bar{D} : h(x) = c\}$ corresponds to the set $\{x \in \mathbb{R}^n : \sum_{i=1}^{n-1} x_i^2 \leq 1, x_n = c\}$ for $0 \leq c \leq 1$, under the homeomorphism.*

C_2 : *The system (14) has finitely many rest points which are located in the set $\{x \in \partial D : h(x) = 0\}$. Moreover this system has no other rest point in $\bar{D}$.*

C_3 : *The flow comes in D on $\{x \in \partial D : 0 < h(x) < 1\}$.*

C_4 : *Let $\tilde{x}$ be one of the rest points of the system (14), and the unstable manifold of this system at $\tilde{x}$ intersects D in a nonempty set.*

Then there is a point $p \in \{x \in \partial D : h(x) = 1\}$ such that $\lim_{t \rightarrow -\infty} p.t = \tilde{x}$. Moreover, if the intersection of D and the unstable manifold at $\tilde{x}$ is one dimensional, this point is unique. If this dimension is more than one, then there are infinitely many of such points.

Lemma 3.5. *Let D be as above. Then there is a unique orbit of the system (13) which lies in D , its α -limit set is u_{CJ} , and this orbit intersects the set $\Delta = \{u \in \bar{D} : g(T, Z_0, Z_1) < 0, T < T_{CJ}, Z_0 = 1, Z_1 = 1\}$. Along this orbit $T(\xi)$ is decreasing and $Z_0(\xi)$ and $Z_1(\xi)$ are increasing.*

Proof. First of all note that the system (13) is gradient like with respect to $h(u) = Z_0$ in D , and is locally Lipschitz in a neighborhood of $\bar{D}$. Now we shall show that the system (13) together with D (as D), u_{CJ} , (as the rest point $\tilde{x}$) and the real valued function $h(u) = Z_0$ (as h) satisfy all of the conditions of Theorem 3.4.

Conditions C_1 , C_2 and C_4 trivially hold. We shall show that Condition C_3 of Theorem 3.4 is fulfilled. To see this, let $u_0 \in \{u \in \partial D : 0 < Z_0 < 1\}$. Then $g(u_0) = 0$ or $T = T_{CJ}$ or $Z_0 = 0$ or $Z_0 = 1$ or $Z_1 = 0$ and $Z_0 - Z_1 = 0$.

Now, suppose $g(u_0) = 0$. If we differentiate $g(T, Z_0, Z_1)$ along the orbit of (13) we obtain $\frac{dg(u)}{d\xi} = (\frac{\partial f(T)}{\partial T} - s - (q_0)_T Z_0 - (q_1)_T Z_1) \dot{T} - q_0 \dot{Z}_0 - q_1 \dot{Z}_1$. Hence

$$\begin{aligned} \frac{dg(u)}{d\xi} \Big|_{g(u_0)=0} &= ((\frac{\partial f(T)}{\partial T} - s - (q_0)_T Z_0 - (q_1)_T Z_1) \frac{g(u)}{\beta} - q_0 \dot{Z}_0 - q_1 \dot{Z}_1) \Big|_{g(u_0)=0} \\ &= -q_0 \dot{Z}_0 - q_1 \dot{Z}_1 < 0. \end{aligned}$$

Thus the flow comes in $\bar{D}$ on $g(u_0) = 0$.

Let $T = T_{01}$ and differentiate T along the orbit to obtain:

$$\frac{dT}{d\xi} \Big|_{T=T_{CJ}} = \frac{1}{\beta} (f(T) - sT - q_0 Z_0 - q_1 Z_1 + C) \Big|_{T=T_{CJ}} = \frac{1}{\beta} g(T_{CJ}, Z_0, Z_1) < 0.$$

Thus the flow comes in $\bar{D}$ on $T = T_{CJ}$.

Let $Z_0 - Z_1 = 0$ and differentiate $Z_0 - Z_1$ along the orbit to obtain:

$$\begin{aligned} \frac{Z_0 - Z_1}{d\xi} \Big|_{Z_0 - Z_1 = 0} &= \dot{Z}_0 - \dot{Z}_1 \Big|_{Z_0 - Z_1 = 0} = [\kappa_1 \Phi'(T) Z_0 - \kappa_2 \Phi_2(T) (Z_0 - Z_1)] \Big|_{Z_0 - Z_1 = 0} \\ &= \kappa_1 \Phi'(T) Z_0 > 0 \end{aligned}$$

Thus the flow comes in $\bar{D}$ on $Z_0 - Z_1 = 0$.

Let $Z_1 = 0$ and differentiate Z_1 along the orbit to obtain:

$$\frac{Z_1}{d\xi} \Big|_{Z_1=0} = \dot{Z}_1 \Big|_{Z_1=0} = \kappa_2 \Phi_2(T) (Z_0 - Z_1) \Big|_{Z_1=0} > 0.$$

Thus the flow comes in $\bar{D}$ on $Z_1 = 0$.

Let $Z_0 = 0$ and differentiate Z_0 along the orbit to obtain:

$$\frac{Z_0}{d\xi} \Big|_{Z_0=0} = \dot{Z}_0 \Big|_{Z_0=0} = \kappa_1 \Phi'(T) Z_0 \Big|_{Z_0=0} > 0.$$

Thus the flow comes in $\bar{D}$ on $Z_0 = 0$.

Let $Z_0 = 1$ and differentiate Z_0 along the orbit to obtain:

$$\frac{Z_0}{d\xi} \Big|_{Z_0=1} = \dot{Z}_0 \Big|_{Z_0=1} = \kappa_1 \Phi'(T) Z_0 \Big|_{Z_0=1} > 0.$$

Thus the flow goes out $\bar{D}$ on $Z_0 = 1$.

Hence Condition C_3 of Theorem 3.4 holds too. Thus by Theorem 3.4, there must be an orbit of the system (13) lying in D , initiating at a point on the surface $Z_0 = 1$ and running to the point u_{CJ} as $\xi \rightarrow -\infty$. Finally from the system (13) and the set D it follows that along this orbit $Z_0(\xi)$ and $Z_1(\xi)$ are increasing and $T(\xi)$ is decreasing. $\square$

Let $\tilde{u}(\xi), \xi \in (-\infty, \xi_0]$ be the orbit which is given by the above lemma. Then $\tilde{u}(\xi_0) \in \{u \in \bar{D} : Z_0 = 1\}$ and $\lim_{\xi \rightarrow -\infty} \tilde{u}(\xi) = u_{CJ}$. About the orbit $\tilde{u}(\xi)$ we have the following lemma.

Lemma 3.6. *Let $\tilde{u}(\xi)$ be as above. Then there is $0 < \tilde{Z}_0 \leq 1$, such that for $\kappa_1\beta$ and $\kappa_2\beta$ small enough, the orbit $\tilde{u}(\xi)$ meets the line $T = T_i$, at $\tilde{u}(\tilde{\xi}) = (T_i, \tilde{Z}_0, \tilde{Z}_1)$, for some $\tilde{\xi} \in (-\infty, \xi_0)$.*

Proof. Let u_{CJ} be as (10). Choose the line $P : T - T_{CJ} = 0$, and $u_{CJ} \in \{u | T < T_{CJ}\}$. Let $(T_i, Z_{0i}, Z_{1i})^T$ be the unique solution of the equation $g(u) = 0$, $T = T_i$, $Z_{0i} - Z_{1i} = 0$, this shows $Z_{0i} = Z_{1i}$ and from $g(u) = 0$, we have $Z_{0i} = Z_{1i} := Z_i = \frac{1}{q_0 + q_1}(f(T_i) - sT_i + C)$. Also from $T_m < T_i < T_{CJ}$, it follows that $0 < Z_i < 1$ and $\{u \in \bar{D} : g(u) = 0, Z_i < Z_0 < 1, Z_i < Z_1 < 1\} \subset \{u \in \bar{D} : T \leq T_i\}$. Now consider the line $P' : T - T_i = 0$, since $u_{m0} \in \{u \in \bar{D} : (T - T_i) < 0\}$, we can choose $Z_i < Z'_0 < 1$, $Z_i < Z'_1 < 1$ such that

$$\{u \in \bar{D} : g(u) = 0, Z'_0 < Z_0 < 1, Z'_1 < Z_1 < 1\} \subset \{u \in \bar{D} : T - T_i < 0\}.$$

Let $D_0 = \{u \in D : T_i < T < T_{CJ}, Z'_0 < Z_0 < 1\} \cup \{u \in D : T_i < T < T_{CJ}, Z'_1 < Z_1 < 1\}$ and $\delta = \max_{u \in \bar{D}_0} g(u)$. Then $\delta < 0$.

Now suppose the orbit $\tilde{u}(\xi), \xi \in (-\infty, \xi_0]$, does not meet the set $\{u \in D : T = T_i, 0 < Z_0 \leq 1, 0 < Z_1 \leq 1\}$. Let $\xi_1 < \xi_0$ and $\xi_2 < \xi_3 < \xi_0$ be the solutions of the equations $\tilde{Z}_0(\xi) = Z'_0$, $\tilde{Z}_1(\xi) = Z'_1$ and $\tilde{Z}_1(\xi) = 1$, respectively, where $\tilde{Z}_0(\xi)$ and $\tilde{Z}_1(\xi)$ are the second and third components of $\tilde{u}(\xi)$, respectively. Since $\frac{dT}{d\xi} = \frac{1}{\beta}g(u) < 0$, T is decreasing along the orbit $\tilde{u}(\xi)$, it follows that $\tilde{u}(\xi)$ remains in D_0 for $\min\{\xi_3, \xi_1\} < \xi < \xi_0$. Now along the orbit $\tilde{u}(\xi)$ in D_0 we have:

$$\begin{aligned} -\frac{dT}{dZ_0} &= \frac{1}{\frac{dZ_0}{d\xi}} \left(-\frac{dT}{d\xi}\right) = \frac{1}{\kappa_1 Z_0 \Phi'(T)} \left(\frac{-1}{\beta}g(u)\right) \geq \frac{\sigma_1(-\delta)}{\kappa_1 \beta} > 0, \\ -\frac{dT}{dZ_1} &= \frac{1}{\frac{dZ_1}{d\xi}} \left(-\frac{dT}{d\xi}\right) = \frac{1}{\kappa_2 (Z_0 - Z_1) \Phi_2(T)} \left(\frac{-1}{\beta}g(u)\right) \geq \frac{\sigma_2(-\delta)}{\kappa_2 \beta} > 0, \end{aligned}$$

where $\frac{1}{\sigma_1} = \max_{u \in \bar{D}} \Phi'(T) Z_0$ and $\frac{1}{\sigma_2} = \max_{u \in \bar{D}} \Phi_2(T) (Z_0 - Z_1)$, respectively.

Let $T_0 = T(\xi_0)$, then $T_i < T_0$, if $\tilde{u}(\xi)$ does not meet T_i . Therefore

$$\begin{aligned} T_{CJ} - T_i &\geq T_{CJ} - T_0 = -\int_{-\infty}^{\xi_0} \frac{1}{\beta} g(u) d\xi = \int_{-\infty}^{\xi_0} \frac{1}{\beta} (-g(u)) d\xi \\ &\geq \int_{\xi_1}^{\xi_0} \frac{1}{\beta} (-g(u)) d\xi = \int_{Z'_0}^1 \frac{1}{\kappa_1 Z_0 \Phi'(T)} \left(\frac{-1}{\beta} g(u)\right) dZ_0 \geq \frac{\sigma_1(-\delta)}{\kappa_1 \beta} (1 - Z'_0), \end{aligned}$$

which is impossible for $\kappa_1\beta$ small enough. Let $T_0 = T(\xi_0)$, then $T_i < T_0$, if $\tilde{u}(\xi)$ does not meet T_i . Therefore

$$\begin{aligned} T_{CJ} - T_i &\geq T_{CJ} - T_0 = - \int_{-\infty}^{\xi_0} \frac{1}{\beta} g(u) d\xi = \int_{-\infty}^{\xi_0} \frac{1}{\beta} (-g(u)) d\xi \\ &\geq \int_{\xi_2}^{\xi_3} \frac{1}{\beta} (-g(u)) d\xi = \int_{Z'_1}^1 \frac{1}{\kappa_2(Z_0 - Z_1)\Phi_2(T)} \left(\frac{-1}{\beta} g(u) \right) dZ_1 \\ &\geq \frac{\sigma_2(-\delta)}{\kappa_2\beta} (1 - Z'_1), \end{aligned}$$

which is impossible for $\kappa_2\beta$ small enough. Thus there is a $\tilde{\xi} \in (-\infty, \xi_0)$ such that the orbit $\tilde{u}(\xi)$ meets the line $T = T_i$ at the point $\tilde{u}_i = (\tilde{T}_i, \tilde{Z}_{0i}, \tilde{Z}_{1i})^T$, where $\tilde{T}_i = T_i$, $\tilde{Z}_{0i} = \tilde{Z}_0(\tilde{\xi})$ and $\tilde{Z}_{1i} = \tilde{Z}_1(\tilde{\xi})$. $\square$

From now on we assume that $\kappa_1\beta$ and $\kappa_2\beta$ are small enough, or the orbit $\tilde{u}(\xi)$ meets the line $T = T_i$ at the point $\tilde{u}_i = (\tilde{T}_i, \tilde{Z}_{0i}, \tilde{Z}_{1i})^T$, where $\tilde{T}_i = T_i$, $\tilde{Z}_{0i} = \tilde{Z}_0(\tilde{\xi})$ and $\tilde{Z}_{1i} = \tilde{Z}_1(\tilde{\xi})$. We call the point $\tilde{u}_i$ the ignition point. According to Lemma 3.5, this point for Chapman-Jouget detonation is unique.

Theorem 3.7. *Suppose that the system (6) admits the rest points u_{CJ} and u_{m0} , for some $0 < m \leq 1$. If $\kappa_1\beta$ and $\kappa_2\beta$ are small enough, then there is a unique orbit of the system (6) which is running from u_{CJ} to u_{m0} , for some $0 < m \leq 1$.*

Proof. In the region $T < T_i$, the second equation of (6) becomes $\dot{Z}_0 = 0$. Thus, in this region, along the orbit of this system $Z_0(\xi)$ is constant. Let $Z_0(\xi) = \tilde{Z}_{0i}$, where $\tilde{Z}_{0i}$ is the second component of $\tilde{u}_i$, (the ignition point). On the surface $Z_0 = \tilde{Z}_{0i}$, the system (6) reduces to the following two dimensional system of equations, in the region $T \leq T_i$:

$$(15) \quad \begin{cases} \beta \dot{T} = f(T) - sT - q_0 \tilde{Z}_{0i} - q_1 Z_1 + C, \\ \dot{Z}_1 = \kappa_2 \Phi_2(T) (\tilde{Z}_{0i} - Z_1). \end{cases}$$

By solving the second equation of (15), we have $\frac{dZ_1}{\tilde{Z}_{0i} - Z_1} = \kappa_2 \Phi_2(T) d\xi$ and so $Z_1 = \tilde{Z}_{0i} - e^{-\int_{\xi}^{+\infty} \kappa_2 \Phi_2(T(\eta)) d\eta}$. Therefore the first equation of (15) is

$$(16) \quad \beta \dot{T} = f(T) - sT - q_0 \tilde{Z}_{0i} - q_1 (\tilde{Z}_{0i} - e^{-\int_{\xi}^{+\infty} \kappa_2 \Phi_2(T(\eta)) d\eta}) + C := F_1(T).$$

Now consider the region $D' = \{T \in \mathbb{R} : F_1(T) < 0, T < T_i\}$. Notice that $T = T_i \in \partial D'$. Also this is trivial to see that any orbit of (16) initiating at a point on $\partial D' \cap \{T : T = T_i\}$ approaches to the unique rest point of the system (16) which is located in the region $T < T_i$, as ξ tends to $+\infty$. We denote this rest point by $\bar{T}_i$.

Now, consider again the ignition point $\tilde{u}_i = (\tilde{T}_i, \tilde{Z}_{0i}, \tilde{Z}_{1i})$. By the above argument, there is a unique orbit of the system (6), say

$$\tilde{u}(\xi) = \begin{cases} (\tilde{T}(\xi), \tilde{Z}_0(\xi), \tilde{Z}_1(\xi)), & \tilde{\xi} < \xi < +\infty \\ (\tilde{T}_i, \tilde{Z}_{0i}, \tilde{Z}_{1i}), & \xi = \tilde{\xi} \end{cases}$$

with $\tilde{Z}_0(\xi) = \tilde{Z}_{0i}$ for $\tilde{\xi} \leq \xi$ and $\lim_{\xi \rightarrow +\infty} \tilde{u}(\xi) = (\bar{T}_i, \tilde{Z}_{0i}, \bar{Z}_{1i})$. Along this orbit $T(\xi)$ is decreasing, $Z_1(\xi)$ is increasing and $Z_0(\xi)$ is constant, respectively. This orbit lies in D , the domain which is used in the proof of Lemma 3.5.

Now define

$$u(\xi) = \begin{cases} \tilde{u}(\xi), & \xi < \tilde{\xi}, \\ \tilde{u}(\xi), & \xi \geq \tilde{\xi}. \end{cases}$$

Then $u(\xi)$ is a complete orbit of the system (6) lying in D and is running from u_{CJ} to u_{m0} for some $0 < m \leq 1$. $\square$

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ZARISKI TOPOLOGY FOR SECOND SUBHYPERMODULES

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Abstract. The purpose of this paper is introduce the concept of second subhypermodules over commutative hyperrings and topologize the collection of all second subhypermodules and also investigate the properties of this topological space.

Keywords: R -hypermodule, subhypermodule, second subhypermodule, top hypermodule.

1. Introduction

Hyperstructure theory was first initiated by Marty [8] in 1934 at the 8th congress of scandinavian mathematicians. He defined hypergroups as a generalization of groups and proved its unility in solving some problems of groups, algebraic functions and rational functions. Survey of the theory of hyperstructures can be found in the book of Corsini [4], Vougiouklis [12], Corsini and Leoreanu [5]. The Krasner hyperring [7] is a well known type of hyperring, with the property that the addition is a hyperoperation and the multiplication is a binary operation. This concept has been studied in depth by many authors, for example see [6]. The concept of hypermodule over a Krasner hyperring has been introduced and investigated by Massouros [9].

As a dual notion of prime submodules, Yassemi [13], introduced the notion of second submodules of a given nonzero module over a commutative ring.

An R -endomorphism r^* on M is called homothety if $r^*(x) = rx$, for all $x \in M$. The nonzero submodule K of M is said to be second if for each $r \in R$ the homothety $r^* : K \rightarrow K$ is either surjective or zero. This implies that $\text{Ann}(K) = (0 : M) = p$ is a prime ideal of R , and K is said to be p -second. Also, we say that M is a second module if M is a second submodule of itself.

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The notion of second submodule can be generalized to second subhypermodule. In fact, in this paper we introduce the concept of second subhypermodules over a commutative hyperring and we topologize the spectrum of second subhypermodules and investigate the properties of the induced topology.

2. Preliminaries

The following definitions and preliminaries are required in the sequel of our work and hence presented in brief.

Let H be a nonempty set and $P^*(H)$ be the set of all nonempty subsets of H . A hyperoperation on H is a map $\circ : H \times H \longrightarrow P^*(H)$ and the couple $(H, \circ)$ is called a hypergroupoid.

For any two nonempty subsets A and B of H , we define

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b.$$

If $x \in H$, by $x \circ A$ and $A \circ x$, we mean $\{x\} \circ A$ and $A \circ \{x\}$, respectively.

A hypergroupoid $(H, \circ)$ is called semihypergroup if $a \circ (b \circ c) = (a \circ b) \circ c$, for all $a, b, c \in H$.

A hypergroupoid $(H, \circ)$ is called quasihypergroup if $a \circ H = H = H \circ a$, for all $a \in H$. If $(H, \circ)$ is semihypergroup and quasihypergroup, then $(H, \circ)$ is called hypergroup.

A subhypergroup $(K, \circ)$ of $(H, \circ)$ is a nonempty subset K of H such that $k \circ K = K = K \circ k$, for all $k \in K$.

A hypergroup $(H, \circ)$ is called canonical [4], if:

- (i) $(H, \circ)$ is commutative, which means $x \circ y = y \circ x$, for all $x, y \in H$;
- (ii) there exists $e \in H$, such that $\{x\} = (x \circ e) \cap (e \circ x)$ for all $x \in H$;
- (iii) for all $x \in H$ there exists a unique $x^{-1} \in H$, such that $e \in x \circ x^{-1}$;
- (iv) $x \in y \circ z$ implies that $y \in x \circ z^{-1}$.

In (ii), an element e is called identity (scalar identity) of $(H, \circ)$.

There are several kinds of hyperrings and hypermodules that can be defined on a nonempty set. In what follows, we consider some of the most general types of hyperrings and hypermodules.

An algebraic system $(R, +, \cdot)$ is called Krasner hyperring, if

- (i) $(R, +)$ is a canonical hypergroup;
- (ii) $(R, \cdot)$ is a semigroup having zero as a bilaterally absorbing element;
- (iii) $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(y + z) \cdot x = y \cdot x + z \cdot x$, for all $x, y, z \in R$.

The hyperring and R -hypermodule was studied in [10]. In this paper, a little change is done, which consists in using " $=$ " instead of " $\subseteq$ ", as in [10].

Definition 2.1 ([10]). *The triple $(R, +, \cdot)$ is a hyperring if:*

- (i) $(R, +)$ is a canonical hypergroup with scalar identity 0_R (we denote the opposite of $r \in R$, that is r^{-1} , by $-r$);
- (ii) $(R, \cdot)$ is a semihypergroup;
- (iii) $r \cdot (s + t) = r \cdot s + r \cdot t$ and $(s + t) \cdot r = s \cdot r + t \cdot r$, for all $r, s, t \in R$.

Note that " $\cdot$ " is a hyperoperation on R . The hyperring $(R, +, \cdot)$ satisfies the conditions:

for all $r, s \in R$, we have

$$r \cdot 0_R = \{0_R\} = 0_R \cdot r \text{ and } r \cdot (-s) = (-r) \cdot s = -(r \cdot s) = \{-t \mid t \in r \cdot s\}.$$

We abbreviate a hyperring $(R, +, \cdot)$ by R .

The hyperring R is said to be commutative, if R is commutative with respect to hyperoperation " $\cdot$ ". In the sequel, by a hyperring we mean a commutative hyperring. We say that a hyperring R is with identity element 1, if for all $r \in R$, $r \in r \cdot 1$.

Definition 2.2 ([10]). *Let R be a hyperring with identity element 1. An R -hypermodule is a structure $(M, +, \circ)$ such that $(M, +)$ is a canonical hypergroup with scalar identity 0_M and " $\circ$ " is a multivalued scalar operation, i.e.,*

$$\circ : R \times M \longrightarrow P^*(M)$$

such that for all $a, b \in R$ and $x, y \in M$,

- (i) $a \circ (x + y) = a \circ x + a \circ y$;
- (ii) $(a + b) \circ x = a \circ x + b \circ x$;
- (iii) $(a \cdot b) \circ x = a \circ (b \circ x)$;
- (iv) $x \in 1 \circ x$.

The hypermodule $(M, +, \circ)$ also satisfies the conditions: for all $a \in R$ and $x \in M$, we have

$$\begin{aligned} a \circ 0_M &= \{0_M\} = 0_R \circ x \text{ and} \\ a \circ (-x) &= (-a) \circ x = -(a \circ x) = \{-y \mid y \in a \circ x\} \end{aligned}$$

(we denote the opposite of $x \in M$, that is x^{-1} , by $-x$).

We abbreviate an R -hypermodule $(M, +, \circ)$ by M . This definition is a generalization of the concept of hypermodule over a Krasner hyperring (see [4]).

A nonempty subset N of an R -hypermodule M is subhypermodule, if N is an R -hypermodule under the same hyperoperations on M .

Proposition 2.3. *Let N be a nonempty subset of an R -hypermodule M . Then N is a subhypermodule of M if and only if for all $r \in R$ and $x, y \in N$, we have $r \circ x \subseteq N$ and $x - y \subseteq N$.*

Proof. Let for all $r \in R$ and $x, y \in M$, $r \circ x \subseteq N$ and $x - y \subseteq N$.

Since N is nonempty, then there exists $a \in N$. So $0 \in a - a \subseteq N$. If $y \in N$, then $-y = 0 - y \in N$. Also, if $x, y \in N$, then $x - (-y) = x + y \subseteq N$. All other conditions for N to be a hypermodule follows as hereditary properties from M . Thus N is an R -hypermodule and so it is a subhypermodule of M . The converse is obvious. $\square$

For a family of subhypermodules $\{N_i\}_{i \in I}$ of M , we define:

$$\sum_{i \in I} N_i = \cup \left\{ \sum_{i \in I} a_i \mid a_i \in N_i, \text{ for } i \in I \text{ and } \exists n \in \mathbb{N}; a_i = 0, \forall i \geq n \right\}.$$

By Proposition 2.3, we can see that $\sum_{i \in I} N_i$ is a subhypermodule of M .

Let R be a hyperring and M an R -hypermodule. For any nonempty subsets A, B of R and nonempty subset X of M , we set

$$\begin{aligned} A \circ X &= \bigcup_{i \in I} \{a_i \circ x_i \mid a_i \in A \text{ and } x_i \in X\}; \\ [A \circ X] &= \cup \left\{ \sum_{i=1}^n a_i \circ x_i \mid a_i \in A \text{ and } x_i \in X, \text{ for all } 1 \leq i \leq n \right\}; \\ [A \cdot B] &= \cup \left\{ \sum_{i=1}^n a_i \cdot b_i \mid a_i \in A \text{ and } b_i \in B, \text{ for all } 1 \leq i \leq n \right\}. \end{aligned}$$

A nonempty subset I of a commutative hyperring R is called hyperideal, if $a - b \subseteq I$ and $r \cdot a \subseteq I$, for all $r \in R$ and $a, b \in I$. If I and J are hyperideals of R , then $I + J$ and $[I \cdot J]$ are hyperideals of R .

A proper hyperideal P of R is called prime if $a \cdot b \subseteq P$ implies that either $a \in P$ or $b \in P$, for $a, b \in R$ (see [1]).

Let A be a nonempty subset of a hyperring R . Define $\langle A \rangle$ to be the smallest hyperideal of R containing A .

Lemma 2.4 ([10]). *Let A be a nonempty subset of a hyperring R with identity. Then $\langle A \rangle = [R \cdot A]$.*

Let I be a hyperideal of a hyperring R . Put $\frac{R}{I} = \{r + I \mid r \in R\}$. It is easy to see that $r + I = s + I$ if and only if $r - s \subseteq I$ for $r, s \in R$. So $\frac{R}{I}$ with the hyperoperations defined as follows, is a hyperring.

$$\begin{aligned} (r + I) \oplus (s + I) &= \{t + I \mid t \in r + s\}; \\ (r + I) \odot (s + I) &= \{u + I \mid u \in r \cdot s\}, \end{aligned}$$

for all $r, s \in R$.

Denote the opposite of $r + I \in \frac{R}{I}$, by $(-r) + I$.

Proposition 2.5 ([10]). *Let M be an R -hypermodule, $r \in R$, I a hyperideal of R , N and K subhypermodules of M and X a nonempty subset of M . Then $r \circ N$, $[I \circ N]$ and $[R \circ X]$ are subhypermodules of M .*

Let X be a nonempty subset of an R -hypermodule M . Define $\langle X \rangle$ to be the smallest subhypermodule of M containing X . It is easy to see that, $\langle X \rangle = [R \circ X]$.

Let M be an R -hypermodule and X, Y be subsets of M . In general, $\langle X + Y \rangle \subseteq \langle X \rangle + \langle Y \rangle$. Now if $\{0_M\} \subseteq X \cap Y$, then $\langle X + Y \rangle \supseteq \langle X \rangle + \langle Y \rangle$. Because $[R \circ X] = [R \circ (X + 0_M)] \subseteq [R \circ (X + Y)]$. Similarly, $[R \circ Y] \subseteq [R \circ (X + Y)]$.

Corollary 2.6. *Let M be an R -hypermodule and X, Y be subsets of M such that $\{0_M\} \subseteq X \cap Y$. Then $\langle X + Y \rangle = \langle X \rangle + \langle Y \rangle$.*

Now it is natural to speak about homomorphism of hypermodules.

Definition 2.7. *Suppose that M and M' are R -hypermodules. A map $f : M \rightarrow M'$ is called homomorphism if for all $x, y \in M$ and $r \in R$, we have*

$$f(x + y) \subseteq f(x) + f(y) \quad \text{and} \quad f(r \circ x) = r \circ f(x).$$

Also, f is called strong homomorphism if for all $x, y \in M$ and $r \in R$,

$$f(x + y) = f(x) + f(y) \quad \text{and} \quad f(r \circ x) = r \circ f(x).$$

Suppose that M and M' are R -hypermodules, f is a homomorphism from M into M' and N is a subhypermodule of M . Then for all $r \in R$,

$$f(r \circ N) = r \circ f(N).$$

Moreover, if f is a strong homomorphism, then $f(N)$ is a subhypermodule of M' .

Recall that two subhypermodules N, K of an R -hypermodule M are independent, if $K \cap N = \{0_M\}$. If K and N are independent, then $N + K$ is denoted by $N \oplus K$. Also, a subhypermodule N of M is called a direct summand of M , if $M = N \oplus L$, for some subhypermodule L of M (see [11]).

Note that if A, B, C are subhypermodules of M , then it need not be that

$$A \cap (B + C) = A \cap B + A \cap C.$$

For example, let $M = \{(x, y) | x, y \in \mathbb{Z}\}$ with trivial hyperoperations on the hyperring $\mathbb{Z}$. Also, let

$$A = \{(x, x) | x \in \mathbb{Z}\}, B = \{(x, 0) | x \in \mathbb{Z}\}, C = \{(0, x) | x \in \mathbb{Z}\}.$$

Then A, B, C are subhypermodules of M and we have $A \cap (B + C) = A$ and $A \cap B + A \cap C = \{(0, 0)\}$.

Lemma 2.8 ([11]). *Suppose that M is an R -hypermodule and A, B, C are subhypermodules of M such that $B \subseteq A$. Then $A \cap (B + C) = A \cap B + A \cap C$.*

Definition 2.9. *A subhypermodule N of an R -hypermodule M is said to be relatively divisible if $r \circ N = (r \circ M) \cap N$, for all $r \in R$.*

Lemma 2.10. *Let M be an R -hypermodule. Then every direct summand N of M is relatively divisible.*

Proof. There exists a subhypermodule K of M such that $M = N \oplus K$ (Since N is a direct summand of M). So

$$r \circ M = r \circ (N \oplus K) = (r \circ N) \oplus (r \circ K).$$

Since $r \circ N \subseteq N$, by Lemma 2.8, we have

$$\begin{aligned} N \cap (r \circ M) &= N \cap ((r \circ N) \oplus (r \circ K)) \\ &= (r \circ N) \oplus (N \cap (r \circ K)) = (r \circ N) \oplus 0_M = r \circ N. \end{aligned}$$

□

Definition 2.11. *An R -hypermodule M is called divisible if $r \circ M = M$, for every $r \in R$.*

Example 2.12. Let M be a divisible R -module, N a submodule of M and I an ideal of R . Assume that $\bar{a} = a + I$ is the additive class for $a \in R$, and $\frac{R}{I}$ is the set of all these classes. For all $\bar{a}, \bar{b} \in \frac{R}{I}$, we define

$$\bar{a} \oplus \bar{b} = \{\bar{c} | c \in \bar{a} + \bar{b}\} \text{ and } \bar{a} \odot \bar{b} = \{\overline{ab}\}.$$

Then $(\frac{R}{I}, \oplus, \odot)$ is a hyperring.

Define the equivalence relation β on M by

$$x \beta y \text{ if and only if } x + N = y + N.$$

For all $\bar{x}, \bar{y} \in \frac{M}{N}, \bar{a} \in \frac{R}{I}$, we define

$$\bar{x} \oplus \bar{y} = \{\bar{z} | z \in \bar{x} + \bar{y}\} \text{ and } \bar{a} \circ \bar{x} = \{\overline{ax}\}.$$

Then $(\frac{M}{N}, \oplus, \circ)$ is an $\frac{R}{I}$ -hypermodule. Also,

$$\bar{r} \circ \frac{M}{N} = r \left(\frac{M}{N} \right) = \frac{M}{N},$$

for all $r \in R$ (since M is divisible).

Moreover, if M is an R -hypermodule and $\{N_i\}_{i \in \Lambda}$ is a family of divisible subhypermodules of M , then $\sum_{i \in \Lambda} N_i$ is a divisible subhypermodule of M . Because

$$r \circ \left(\sum_{i \in \Lambda} N_i \right) = \sum_{i \in \Lambda} (r \circ N_i) = \sum_{i \in \Lambda} N_i$$

for all $r \in R$.

3. Second subhypermodules

In this section, we introduce the notion of second subhypermodules and investigate some properties of them.

Definition 3.1. Let M be an R -hypermodule. A nonzero subhypermodule N of M is called second if for all $r \in R$, either $r \circ N = N$ or $r \circ N = \{0_M\}$. Also, we say that M is a second R -hypermodule if M is a second subhypermodule of itself.

By $\text{Spec}^s(M)$, the spectrum of second subhypermodules of M , we mean the set of all second subhypermodules of M .

Example 3.2. Every divisible R -hypermodule is second.

Example 3.3. Every relatively divisible subhypermodule of a second R -hypermodule is second.

It is not necessary that every subhypermodule of a second R -hypermodule is second. For example, Suppose that $(M, +, \cdot)$ is a divisible hypermodule over a Krasner hyerring R . Let I be a hyperideal of R and N, K subhypermodules of M such that for all $r \in R$, $\{0_M\} \subset r \cdot K \subset K$. We define

$$\circ : R \times M \longrightarrow P^*(M) \text{ by } r \circ x = r \cdot x + N$$

for all $r \in R$ and $x \in M$. Then $(M, +, \circ)$ is a second R -hypermodule and K is not a second subhypermodule of M .

Lemma 3.4. Let M and M' be R -hypermodules and f a strong homomorphism of M into M' . If $N \in \text{Spec}^s(M)$, then $f(N) \in \text{Spec}^s(M')$.

Proof. Since f is a strong homomorphism of M into M' and N is second, then $r \circ f(N) = f(r \circ N) = f(0_M) = \{0_{M'}\}$ or $r \circ f(N) = f(r \circ N) = f(N)$, for all $r \in R$. $\square$

For an R -hypermodule M , we set

$$\text{Ann}(M) = \{r \in R \mid r \circ M = \{0_M\}\}.$$

Since $0_R \in \text{Ann}(M)$, then $\text{Ann}(M) \neq \emptyset$.

Lemma 3.5. Let M be an R -hypermodule. Then $\text{Ann}(M)$ is a hyperideal of R .

Proof. Let $a, b \in \text{Ann}(M)$ and $r \in R$. Then

$$\{0_M\} = 0_M + 0_M = a \circ M + b \circ M = (a + b) \circ M.$$

So $a + b \subseteq \text{Ann}(M)$. Also,

$$\{0_M\} = r \circ 0_M = r \circ (a \circ M) = (r \cdot a) \circ M.$$

Hence $r \cdot a \subseteq \text{Ann}(M)$. $\square$

Proposition 3.6. *Let N be a second subhypermodule of an R -hypermodule M . Then $p = \text{Ann}(N)$ is a prime hyperideal of R .*

Proof. Let $a, b \in R$ and $a \cdot b \subseteq \text{Ann}(N)$ and $b \notin \text{Ann}(N)$. Then $b \circ N = N$ and $(a \cdot b) \circ N = \{0_M\}$. So $\{0_M\} = a \circ (b \circ N) = a \circ N$. Hence $a \in \text{Ann}(N)$. $\square$

In the meantime, we say that N is a p -second subhypermodule of an R -hypermodule M if N is a second subhypermodule of M with $p = \text{Ann}(N)$.

Lemma 3.7. *Let N be a subhypermodule of an R -hypermodule M and $W(N) = \{r \in R \mid r \circ N \neq N\}$. Then the following statements are equivalent:*

- (i) N is a p -second subhypermodule of M ;
- (ii) $\text{Ann}(N) = W(N) = p$.

Proof. The proof is straightforward. $\square$

Definition 3.8. *Let M be an R -hypermodule. A nonzero element $x \in M$ is called torsion-free if for $t \in R$, $0_M \in t \circ x$ implies that $t = 0_R$. Also, M is called torsion-free R -hypermodule, if all nonzero elements of M are torsion-free.*

Let M be a torsion-free R -hypermodule and I a hyperideal of R such that $I \subseteq \text{Ann}(M)$ and $\odot$ be a multivalued scalar operation define by

$$(r + I) \odot x := r \circ x,$$

for all $r \in R$. Then $(M, +, \odot)$ is an $\frac{R}{I}$ -hypermodule.

Lemma 3.9. *Let N be a subhypermodule of a torsion-free R -hypermodule M and $\text{Ann}(N) = p$. Then the following statements are equivalent:*

- (i) N is a p -second subhypermodule of M ;
- (ii) N is a divisible $\frac{R}{p}$ -hypermodule;
- (iii) $r \circ N = N$ for all $r \in R - p$;
- (iv) $W(N) = p$.

Proof. The proof is straightforward. $\square$

Definition 3.10. *Let M be an R -hypermodule and N a subhypermodule of M . N is called a minimal subhypermodule, if it is a minimal element in the set of all subhypermodules of M . On the other word, it is not strictly contains any other nonzero subhypermodules of M .*

Theorem 3.11. *Every minimal subhypermodule of a hypermodule is second.*

Proof. Let M be an R -hypermodule and N a minimal subhypermodule of M . Consider $r \in R$. Since $r \circ N \subseteq N$, then either $r \circ N = N$ or $r \circ N = \{0_M\}$ (by minimality of N). $\square$

Let M be an R -hypermodule. We set

$$\sqrt{\text{Ann}(M)} = \{r \in R \mid r^n \circ M = \{0_M\}, \text{ for some } n \in \mathbb{N}\}.$$

Lemma 3.12. *Let M be an R -hypermodule. Then $\sqrt{\text{Ann}(M)}$ is a hyperideal of R .*

Proof. Let $a, b \in \sqrt{\text{Ann}(M)}$ and $r \in R$. Then $-b \in \sqrt{\text{Ann}(M)}$ and there exist $m, n \in \mathbb{N}$ such that $a^m \circ M = \{0_M\} = (-b)^n \circ M$. But

$$(a - b)^{m+n} = \sum_{r=0}^{m+n} \binom{m+n}{r} (a^{m+n-r} \cdot (-b)^r).$$

So $(a - b)^{m+n} \circ M = \{0_M\}$ and hence $a - b \in \sqrt{\text{Ann}(M)}$. Also,

$$(r \cdot a)^m \circ M = (r^m \cdot a^m) \circ M = r^m \circ (a^m \circ M) = r^m \circ \{0_M\} = \{0_M\}.$$

Thus $r \cdot a \in \sqrt{\text{Ann}(M)}$. □

Definition 3.13. *An R -hypermodule M is called secondary if for all $r \in R$, either $r \circ M = M$ or there exists $n \in \mathbb{N}$ such that $r^n \circ M = \{0_M\}$.*

Obviously, every second R -hypermodule is secondary.

Lemma 3.14. *Let M be a secondary R -hypermodule. Then $\sqrt{\text{Ann}(M)} = p$ is a prime hyperideal of R .*

Proof. Let $a \cdot b \in \sqrt{\text{Ann}(M)}$ and $b \notin \sqrt{\text{Ann}(M)}$. Then there exists $n \in \mathbb{N}$ such that $\{0_M\} = (a \cdot b)^n \circ M$. So $\{0_M\} = a^n \circ (b^n \circ M)$. Since M is secondary and $b \notin \sqrt{\text{Ann}(M)}$, then $a^n \circ M = \{0_M\}$. Hence $a \in \sqrt{\text{Ann}(M)}$. □

We say that N is a p -secondary subhypermodule of an R -hypermodule M if N is a secondary subhypermodule of M with $\sqrt{\text{Ann}(M)} = p$.

Proposition 3.15. *Let M be an R -hypermodule and N a subhypermodule of M . Then the following statements hold:*

- (i) *If N is secondary, then N is second if and only if $\text{Ann}(N)$ is a prime hyperideal of R .*
- (ii) *If N is p -secondary contained in a p -second subhypermodule, then N is a p -second subhypermodule of M .*

Proof. (i) This is obvious.

(ii) Let K be a p -second subhypermodule of M and $N \subseteq K$. Then

$$p = \text{Ann}(K) \subseteq \text{Ann}(N) \subseteq \sqrt{\text{Ann}(N)} = p.$$

So $\text{Ann}(N) = p$ and by (i), N is a p -second subhypermodule of M . □

Lemma 3.16. *Let M be an R -hypermodule, N a secondary subhypermodule of M and $r \in R$. Then $r \circ N$ is a secondary subhypermodule of M .*

Proof. Let $r \circ N \subsetneq N$. Since N is secondary, then there exists $n \in \mathbb{N}$ such that $\{0_M\} = r^n \circ N$. So $r^{n-1} \circ (r \circ N) = \{0_M\}$. Thus $r \circ N$ is a secondary subhypermodule of M . $\square$

We call a subhypermodule N of an R -hypermodule M a minimal secondary subhypermodule of M if N is a secondary subhypermodule which is not strictly contains any other secondary subhypermodule of M .

Corollary 3.17. *Let M be an R -hypermodule and N be a minimal secondary subhypermodule of M . Then N is second and $W(N) = \text{Ann}(N)$.*

Proof. Let $r \in W(N)$. Then $r \circ N$ is a secondary subhypermodule of M (by Lemma 3.16). Since N is minimal secondary and $r \circ N \subseteq N$, then $r \circ N = \{0_M\}$. Hence N is a second subhypermodule of M and $r \in \text{Ann}(N)$ and $W(N) = \text{Ann}(N)$. $\square$

4. Top hypermodules

In this section, we topologize $\text{Spec}^s(M)$ and investigate the properties of the induced topology. First we recall some definitions.

An R -module M is called uniserial, if for any submodules K, L of M , either $K \subseteq L$ or $L \subseteq K$. Also, A submodule N of M is called strongly hollow, if for any submodules K, L of M , $N \subseteq K + L$ implies that $N \subseteq K$ or $N \subseteq L$ (see [2]). So we can define uniserial R -hypermodule and strongly hollow subhypermodule in the same way.

Example 4.1. Let M be a uniserial R -module and N be a submodule of M . We define $\circ : R \times M \rightarrow P^*(M)$ by $r \circ x = rx + N$, for all $r \in R$ and $x \in M$. Then $(M, +, \circ)$ is a uniserial R -hypermodule.

Example 4.2. Let M be an R -module and N a submodule of M . We define $\circ : R \times M \rightarrow P^*(M)$ by $r \circ x = rx + N$, for all $r \in R$ and $x \in M$. Then $(M, +, \circ)$ is an R -hypermodule and N is a strongly hollow subhypermodule of M .

By $SH(M)$, we mean the set of all strongly hollow subhypermodules of M . For every subhypermodule N of an R -hypermodule M , we set

$$\begin{aligned} V^s(N) &= \{K \in \text{Spec}^s(M) \mid K \subseteq N\} \\ \chi^s(N) &= \{K \in \text{Spec}^s(M) \mid K \not\subseteq N\}. \end{aligned}$$

Also, for an R -hypermodule M we set

$$\begin{aligned} \zeta^s(M) &= \{V^s(N) \mid N \text{ is a subhypermodule of } M\} \\ \tau^s(M) &= \{\chi^s(N) \mid N \text{ is a subhypermodule of } M\} \\ Z^s(M) &= (\text{Spec}^s(M), \tau^s(M)). \end{aligned}$$

Theorem 4.3. *Let M be an R -hypermodule and $\{N_i\}_{i \in I}$ be a family of subhypermodules of M . Then the following statements hold:*

- (i) $V^s(\{0_M\}) = \emptyset$, $V^s(M) = \text{Spec}^s(M)$;
- (ii) $\bigcap_{i \in I} V^s(N_i) = V^s(\bigcap_{i \in I} N_i)$;
- (iii) $V^s(N) \cup V^s(K) \subseteq V^s(N + K)$ for subhypermodules N, K .

Proof. Immediate. □

In general, $\zeta^s(M)$ is not closed under finite unions. If $\zeta^s(M)$ is closed under finite unions, then we call M is top hypermodule.

For example, if $\text{Spec}^s(M) = \emptyset$, then M is a top hypermodule. Also, if M is a uniserial R -hypermodule, then M is a top hypermodule.

Lemma 4.4. *Let M be an R -hypermodule. If $\text{Spec}^s(M) \subseteq SH(M)$, then M is a top hypermodule.*

Proof. It is obvious. □

Lemma 4.5. *Let M be an R -hypermodule. If M is a top hypermodule, then the closure of any subset $A \subseteq \text{Spec}^s(M)$ is*

$$\overline{A} = V^s(\sum \{K | K \in A\}).$$

Proof. Let $A \subseteq \text{Spec}^s(M)$. Since $A \subseteq V^s(\sum \{K | K \in A\})$, then

$$\overline{A} \subseteq V^s(\sum \{K | K \in A\}).$$

Conversely, Let $N \in V^s(\sum \{K | K \in A\}) \setminus A$ and $\chi^s(L)$ be a neighborhood of N . Then $N \not\subseteq L$. So there exists $T \in A$ such that $T \not\subseteq L$. That is $T \in \chi^s(L) \cap (A \setminus \{N\})$ and N is a cluster point of A . Thus

$$V^s(\sum \{K | K \in A\}) \subseteq \overline{A}.$$

□

Theorem 4.6. *Let M be an R -hypermodule. If M is a top hypermodule, then*

$$B = \{\chi^s(N) | N = \langle X \rangle, X \text{ is a finite subset of } M\}$$

is a basis of open sets for $Z^s(M)$.

Proof. Any $K \in \text{Spec}^s(M)$ is contained in some $\chi^s(N)$ for some subhypermodule N of M such that $N = \langle X \rangle$ and X is a finite subset of M .

Now let $\{\chi^s(N_1), \chi^s(N_2)\} \subseteq B$ and L a subhypermodule of M such that $L \in \chi^s(N_1) \cap \chi^s(N_2)$. Since M is a top hypermodule, then

$$\chi^s(N_1) \cap \chi^s(N_2) = \chi^s(N_1 + N_2).$$

By Corollary 2.9, $N_1 + N_2 = \langle X + Y \rangle$ and so $\chi^s(N_1 + N_2) \in B$. □

A nonempty topological space X is called irreducible if X is not the union of two proper closed subsets (see [3]).

Lemma 4.7 ([2]). *Let X be a nonempty topological space and $A \subseteq X$. Then the following are equivalent:*

- (1) A is irreducible;
- (2) for any closed subsets A_1, A_2 of X , $A \subseteq A_1 \cup A_2$ implies that $A \subseteq A_1$ or $A \subseteq A_2$;
- (3) for any open subsets U_1, U_2 of X , $U_1 \cap A \neq \emptyset \neq U_2 \cap A$ implies that $(U_1 \cap U_2) \cap A \neq \emptyset$.

Proposition 4.8. *Let M be an R -hypermodule and $A \subseteq \text{Spec}^s(M)$. If M is a top hypermodule and A is irreducible, then $\sum_{N \in A} N$ is a second subhypermodule of M .*

Proof. Let $r \in R$ and suppose that

$$r \circ \left(\sum_{N \in A} N \right) \neq \sum_{N \in A} N$$

and

$$r \circ \left(\sum_{N \in A} N \right) \neq \{0_M\}.$$

We set

$$A_1 = \{K \in A \mid r \circ K = K\} \text{ and } \\ A_2 = \{K \in A \mid r \circ K = \{0_M\}\}.$$

So $A \subseteq V^s(\sum_{N \in A_1} N) \cup V^s(\sum_{N \in A_2} N)$.

But $A \not\subseteq V^s(\sum_{N \in A_1} N)$ and $A \not\subseteq V^s(\sum_{N \in A_2} N)$ which is a contradiction. $\square$

Proposition 4.9. *Let M be an R -hypermodule and*

$$A \subseteq \text{Spec}^s(M) \subseteq SH(M).$$

If $\sum_{N \in A} N$ is a nonzero strongly hollow subhypermodule of M , then A is irreducible.

Proof. Let $A \subseteq V^s(N_1) \cup V^s(N_2)$. Then $\sum_{N \in A} N \subseteq N_1 + N_2$.

Since $(\sum_{N \in A} N) \in SH(M)$, then $\sum_{N \in A} N \subseteq N_1$ or $\sum_{N \in A} N \subseteq N_2$. Hence

$$A \subseteq V^s(N_1) \text{ or } A \subseteq V^s(N_2).$$

$\square$

Definition 4.10. A nonzero R -hypermodule M is simple, if the only subhypermodules of M are $\{0_M\}$ and M .

Example 4.11. Let $R = \{0, 1\}$ with

$$\begin{aligned} 0 + 0 &= \{0\} \\ 0 + 1 &= 1 + 0 = \{0, 1\} \\ 1 + 1 &= \{1\} \\ 0 \cdot 0 &= \{0\} \\ 0 \cdot 1 &= 1 \cdot 0 = \{0\} \\ 1 \cdot 1 &= \{0, 1\}. \end{aligned}$$

Then R is a simple R -hypermodule.

By $S(M)$, we mean the set of all simple subhypermodules of an R -hypermodule M .

We say that an R -hypermodule M has the min-property, if for any simple subhypermodule L of M , $L \not\subseteq \sum_{K \in S(M) \setminus \{L\}} K$.

For example, every R -hypermodule M with at most one simple subhypermodule has the min-property.

Definition 4.12. Let M be an R -hypermodule. Then

- (i) M is called semisimple, if for every subhypermodule K of M , there exists a subhypermodule P of M such that $M = K \oplus P$,
- (ii) M is called atomic, if any nonzero subhypermodule N of M contains a simple subhypermodule,
- (iii) M is called uniform, if for any nonzero subhypermodules N_1, N_2 of M , we have $N_1 \cap N_2 \neq \{0_M\}$.

For example, any simple R -hypermodule is semisimple and any uniserial R -hypermodule is uniform.

Definition 4.13 ([2]). We say that a nonempty topological space X

- (i) is countably compact if every open cover of X has a countable finite subcover;
- (ii) is ultraconnected if the intersection of any two nonempty closed subsets of X is nonempty.

Proposition 4.14. Let M be an R -hypermodule and atomic top hypermodule. M is uniform if and only if M is ultraconnected.

Proof. Assume that M is uniform and $V^s(N_1), V^s(N_2)$ are nonempty closed subsets of $\text{Spec}^s(M)$. Then $N_1 \neq \{0_M\}$ and $N_2 \neq \{0_M\}$. Since M is uniform, then $N_1 \cap N_2 \neq \{0_M\}$. But $V^s(N_1) \cap V^s(N_2) = V^s(N_1 \cap N_2)$. Since M is an atomic R -hypermodule, then there exists a simple subhypermodule L of M such that $L \subseteq N_1 \cap N_2$. So $V^s(N_1 \cap N_2) \neq \emptyset$.

Conversely, let L_1, L_2 be nonzero subhypermodules of M . So $V^s(L_1) \neq \emptyset$ and $V^s(L_2) \neq \emptyset$. Since M is ultraconnected and $V^s(L_1) \cap V^s(L_2) = V^s(L_1 \cap L_2)$, then $L_1 \cap L_2 \neq \{0_M\}$ and hence M is uniform. $\square$

Theorem 4.15. *Let M be an R -hypermodule and atomic top hypermodule. If $S(M)$ is countable, then $\text{Spec}^s(M)$ is countably compact.*

Proof. Assume that $S(M) = \{N_{\lambda_k}\}_{k \geq 1}$ is countable. Let $\{\chi^s(L_\alpha)\}_{\alpha \in I}$ be an open cover of $\text{Spec}^s(M)$. Since $S(M) \subseteq \text{Spec}^s(M)$, then for some $\alpha_k \in I$, $N_{\lambda_k} \not\subseteq L_{\alpha_k}$ for each $k \geq 1$.

Let $\bigcap_{k \geq 1} L_{\alpha_k} \neq \{0_M\}$. Since M is an atomic R -hypermodule, then there exists a simple subhypermodule N of M such that $N \subseteq \bigcap_{k \geq 1} L_{\alpha_k}$ which is contradiction.

Hence $\bigcap_{k \geq 1} L_{\alpha_k} = \{0_M\}$ and so

$$\text{Spec}^s(M) = \chi^s(\{0_M\}) = \chi^s\left(\bigcap_{k \geq 1} L_{\alpha_k}\right) = \bigcup_{k \geq 1} \chi^s(L_{\alpha_k}).$$

$\square$

Proposition 4.16. *Let M be an R -top hypermodule and*

$$\text{Spec}^s(M) = S(M).$$

- (i) *If M has the min-property, then $\text{Spec}^s(M)$ is discrete.*
- (ii) *M has a unique simple subhypermodule if and only if M has the min-property and $\text{Spec}^s(M)$ is connected.*

Proof. (i) Let M has the min-property. Then for every $K \in S(M)$,

$$\chi^s\left(\sum_{N \in S(M) \setminus \{K\}} N\right) = \{K\}.$$

Since every singleton set is open, then $\text{Spec}^s(M)$ is discrete.

(ii) Let M has a unique simple subhypermodule. Clearly, M has the min-property and $\text{Spec}^s(M)$ is connected.

Conversely, by (i) $\text{Spec}^s(M)$ is discrete. Since $\text{Spec}^s(M)$ is connected, then $\text{Spec}^s(M)$ has only one point. $\square$

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THE HECKE ALGEBRA $H(P_{\mathbb{Q}}, P_{\mathbb{Z}})$ AND ITS RELATION TO THE CROSSED PRODUCT $H(P_{\mathbb{Q}}^+, P_{\mathbb{Z}}) \times_{\beta} \{1, -1\}$

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Abstract. The algebra $H(P_{\mathbb{Q}}^+, P_{\mathbb{Z}})$ arose in number theory has been studied by Bost and Connes in [2]. In [1] a related Hecke algebra $H(P_{\mathbb{Q}}, P_{\mathbb{Z}})$ is considered wherein it is shown to be a universal $*$ -algebra generated by the elements $\{\mu_n : n \in \mathbb{N}^*\}$, $\{e(r) : r \in \mathbb{Q}/\mathbb{Z}\}$ and an element $u = \left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right]$. The goal of this paper is to study the relationship between the Hecke algebra of Bost and Connes and the Hecke algebra $H(P_{\mathbb{Q}}, P_{\mathbb{Z}})$. By showing the existence of a $*$ -automorphism α of $H(P_{\mathbb{Q}}^+, P_{\mathbb{Z}})$, we construct a covariant representation (ι, U) of $H(P_{\mathbb{Q}}^+, P_{\mathbb{Z}}) \times_{\beta} \{1, -1\}$ on $H(P_{\mathbb{Q}}, P_{\mathbb{Z}})$. This leads to our main result that $H(P_{\mathbb{Q}}, P_{\mathbb{Z}})$ is realized as the crossed product $H(P_{\mathbb{Q}}^+, P_{\mathbb{Z}}) \times_{\beta} \{1, -1\}$. **Keywords:** Hecke algebras, the Hecke algebra $H(P_{\mathbb{Q}}, P_{\mathbb{Z}})$, $*$ -automorphism, $*$ -isomorphism, covariant representation.

1. Introduction

A Hecke pair (G, S) consists of a discrete group G and a subgroup S of G such that every double coset consists of finitely many left cosets.

The Hecke algebra $H(P_{\mathbb{Q}}^+, P_{\mathbb{Z}})$ first arose in Bost and Connes' study [2] and they have proved that it is a universal $*$ -algebra generated by elements $\{\mu_n : n \in \mathbb{N}^*\}$ and $\{e(r) : r \in \mathbb{Q}/\mathbb{Z}\}$ subject to six relations.

In [1] we introduced a new Hecke pair $(P_{\mathbb{Q}}, P_{\mathbb{Z}})$, where

$$P_{\mathbb{Q}} = \left\{ \begin{pmatrix} 1 & a \\ 0 & r \end{pmatrix} : a, r \in \mathbb{Q}, r \neq 0 \right\}.$$

Then we showed that this closely related Hecke algebra $H(P_{\mathbb{Q}}, P_{\mathbb{Z}})$ is a universal $*$ -algebra generated by elements $\{\mu_n : n \in \mathbb{N}^*\}$, $\{e(r) : r \in \mathbb{Q}/\mathbb{Z}\}$ and an element

$$u = \left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right].$$

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We begin with a preliminaries section in which we define Hecke algebras, set up our notation and give information about the Hecke algebra $H(P_{\mathbb{Q}}, P_{\mathbb{Z}})$. Then we review dynamical systems and their covariant representations and recall the basic properties. In section 3, we show the existence of a $*$ -automorphism α of $H(P_{\mathbb{Q}}^+, P_{\mathbb{Z}})$. In section 4, we show the existence of a $*$ -homomorphism ϕ from $H(P_{\mathbb{Q}}, P_{\mathbb{Z}})$ into $H(P_{\mathbb{Q}}^+, P_{\mathbb{Z}}) \times_{\beta} \{1, -1\}$. We then construct a covariant representation (ι, U) of $H(P_{\mathbb{Q}}^+, P_{\mathbb{Z}}) \times_{\beta} \{1, -1\}$ on $H(P_{\mathbb{Q}}, P_{\mathbb{Z}})$. This enables us to show our main theorem that $H(P_{\mathbb{Q}}, P_{\mathbb{Z}})$ is realized as the crossed product $H(P_{\mathbb{Q}}^+, P_{\mathbb{Z}}) \times_{\beta} \{1, -1\}$.

2. Preliminaries

In this section we give the background required for this paper, give the necessary information about the Hecke algebra $H(P_{\mathbb{Q}}, P_{\mathbb{Z}})$, and setup our notations.

Definition 2.1. *Let G be a discrete group and S a subgroup of G . The pair (G, S) is called a Hecke pair if each double coset StS can be written as a finite union of left cosets.*

The following Proposition can be found in [1, Proposition 2.2].

Proposition 2.2. *Let (G, S) be a Hecke pair. Then the set*

$$H(G, S) = \{f : S \backslash G / S \longrightarrow \mathbb{C} : f \text{ has finite support}\}$$

is a $*$ -algebra with

$$(2.1) \quad (f * g)(StS) = \sum_{rS \in G/S} f(SrS)g(Sr^{-1}tS)$$

and

$$f^*(StS) = \overline{f(St^{-1}S)}.$$

This $*$ -algebra is called a Hecke algebra.

Bost and Connes defined

$$P_{\mathbb{Q}}^+ = \left\{ \begin{pmatrix} 1 & a \\ 0 & r \end{pmatrix} : a, r \in \mathbb{Q}, r > 0 \right\},$$

and

$$P_{\mathbb{Z}} = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}.$$

Then they showed that $H(P_{\mathbb{Q}}^+, P_{\mathbb{Z}})$ is the universal $*$ -algebra generated by the elements $\{\mu_n : n \in \mathbb{N}^*\}$ and $\{e(r) : r \in \mathbb{Q}/\mathbb{Z}\}$ subject to the following four relations according to the improvement of their theorem given by Laca and Raeburn in [4]:

- (a) $\mu_n^* \mu_n = 1$ for all $n \in \mathbb{N}^*$.
- (b) $\mu_{mn} = \mu_m \mu_n$ for all m, n .
- (c) $e(r)^* = e(-r)$, $e(r_1 + r_2) = e(r_1)e(r_2)$ for all $r_1, r_2 \in \mathbb{Q}/\mathbb{Z}$.
- (d) $\mu_n e(r) \mu_n^* = (1/n) \sum_{j=1}^n e(r/n + j/n)$ for all n and all r .

Where,

$$\mu_n = n^{-1/2} \begin{bmatrix} 1 & 0 \\ 0 & n \end{bmatrix}, \text{ and } e(r) = \begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix}.$$

In [1] we defined the group

$$P_{\mathbb{Q}} = \left\{ \begin{pmatrix} 1 & a \\ 0 & r \end{pmatrix} : a, r \in \mathbb{Q}, r \neq 0 \right\},$$

and the subgroup

$$P_{\mathbb{Z}} = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}.$$

Then we proved the following theorem.

Theorem 2.3 (Theorem 7.4 of [1]). *$H(P_{\mathbb{Q}}, P_{\mathbb{Z}})$ is the universal $*$ -algebra generated by elements $\{\mu_n : n \in \mathbb{N}^*\}, \{e(r) : r \in \mathbb{Q}/\mathbb{Z}\}$, and $u = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ subject to the relations*

- (a) $\mu_n^* \mu_n = 1$ for all $n \in \mathbb{N}^*$.
- (b) $\mu_{mn} = \mu_m \mu_n$ for all m, n .
- (c) $e(r)^* = e(-r)$, $e(r_1 + r_2) = e(r_1)e(r_2)$ for all $r_1, r_2 \in \mathbb{Q}/\mathbb{Z}$.
- (d) $\mu_n e(r) \mu_n^* = (1/n) \sum_{j=1}^n e(r/n + j/n)$ for all n and all r .
- (e) $u^* = u$, $u^2 = 1$.
- (f) $u \mu_n = \mu_n u$ for all $n \in \mathbb{N}^*$.
- (g) $e(r)u = ue(-r)$ for all $r \in \mathbb{Q}/\mathbb{Z}$.

Remark 2.4. An action of a group G on a $*$ -algebra A is a homomorphism $\beta : G \rightarrow \text{Aut}(A)$, where $\text{Aut}(A)$ is the group of $*$ -automorphisms of A . The pair (A, G) is referred to as a dynamical system. We usually write $\beta_s(a)$ for $\beta(s)(a)$.

Background 2.5. Let A be a unital $*$ -algebra, G a group and β an action of the group G on A . A covariant representation of the dynamical system (A, G, β) on a unital $*$ -algebra B is a pair (π, U) consisting of a unital $*$ -algebra homomorphism $\pi : A \rightarrow B$ and a unitary homomorphism $U : G \rightarrow u(B)$, such that

$$\pi(\beta_t(a)) = U_t \pi(a) U_t^* \text{ for all } a \in A, t \in G.$$

Let (A, G, β) be a dynamical system; we shall assume that A has an identity 1_A . Define the crossed product $A \rtimes_{\beta} G$ to be $k(G, A)$, which is the vector space of finitely supported functions $f : G \rightarrow A$, with operations given by

$(\lambda f + \gamma h)(s) = \lambda f(s) + \gamma h(s)$. By [4, Lemma 42] $k(G, A)$ is a $*$ -algebra with multiplication and involution given by

$$f *_{\beta} h(t) := \sum_{s \in G} f(s) \beta_s(h(s^{-1}t))$$

and

$$f^*(s) := \beta_s(f(s^{-1})).$$

Also if (π, U) is a covariant representation of (A, G, β) on a unital $*$ -algebra B , then there is a unital $*$ -representation $\pi \times U$ of $k(G, A)$ on B such that

$$(2.2) \quad \pi \times U(f) = \sum_{s \in G} \pi(f(s)) U_s \text{ for } f \in k(G, A).$$

To go from representations of $k(G, A)$ to covariant representations of the system, we define $i_G : G \rightarrow k(G, A)$ by $i_G(s) := \delta_s 1_A$, i.e.

$$i_G(s)(t) = \begin{cases} 1_A, & \text{if } s = t \\ 0, & \text{otherwise,} \end{cases}$$

and $i_A : A \rightarrow k(G, A)$ by $i_A(a) := \delta_e a$. By [4, proposition 44] i_G is a homomorphism of G into the group $u(k(G, A))$ of unitary elements in the $*$ -algebra $k(G, A)$, and i_A is a $*$ -homomorphism of A into $k(A, G)$.

3. The automorphism of $H(P_{\mathbb{Q}}^+, P_{\mathbb{Z}})$

Lemma 3.1. *There is a $*$ -automorphism α of $H(P_{\mathbb{Q}}^+, P_{\mathbb{Z}})$ such that*

- (i) $\alpha(\mu_n) = \mu_n$ for all $n \in \mathbb{N}^*$.
- (ii) $\alpha(e(r)) = e(-r)$ for all $r \in \mathbb{Q}/\mathbb{Z}$.
- (iii) $\alpha^2 = id$.

Proof. Define $\{\tilde{\mu}_n = \mu_n : n \in \mathbb{N}^*\}$ and $\{\tilde{e}(r) = e(-r) : r \in \mathbb{Q}/\mathbb{Z}\}$, then these elements are in $H(P_{\mathbb{Q}}^+, P_{\mathbb{Z}})$. If these elements satisfy the relations (a)-(d) of Theorem 5.1 in [1], then the proof follows.

Parts (a) and (b) are trivially true because $\tilde{\mu}_n = \mu_n$.

For (c), let us start with

$$\tilde{e}(r)^* = e(-r)^* = e(r) = \tilde{e}(-r)$$

and for the second part of (c) we have that

$$\tilde{e}(r_1 + r_2) = e(-(r_1 + r_2)) = e(-r_1)e(-r_2) = \tilde{e}(r_1)\tilde{e}(r_2).$$

For (d),

$$\begin{aligned}
 \tilde{\mu}_n \tilde{e}(r) \tilde{\mu}_n^* &= \mu_n e(-r) \mu_n^* \\
 &= \frac{1}{n} \sum_{j=1}^n e(-r/n + j/n) \\
 &= \frac{1}{n} \sum_{j=1}^n e(-(r/n - j/n)) \\
 &= \frac{1}{n} \sum_{j=1}^n \tilde{e}(r/n - j/n) \\
 &= \frac{1}{n} \sum_{j=1}^n \tilde{e}(r/n + j/n) \text{ since we are summing over } 1 \leq j \leq n.
 \end{aligned}$$

Thus by [1, Theorem 5.1], there exists a $*$ -homomorphism $\alpha : H(P_{\mathbb{Q}}^+, P_{\mathbb{Z}}) \longrightarrow H(P_{\mathbb{Q}}^+, P_{\mathbb{Z}})$ satisfying the relations (i) and (ii).

To show part (iii), notice that α is a $*$ -homomorphism and $H(P_{\mathbb{Q}}^+, P_{\mathbb{Z}})$ is generated by elements μ_n and $e(r)$. So to show that $\alpha^2 = id$ it is enough to check the equality for μ_n and $e(r)$.

$$\alpha^2(\mu_n) = \alpha(\alpha(\mu_n)) = \alpha(\mu_n) = \mu_n$$

and for $e(r)$

$$\alpha^2(e(r)) = \alpha(\alpha(e(r))) = \alpha(e(-r)) = e(r).$$

□

Proposition 3.2. *Let G be a group, A a unital $*$ -algebra and $\beta : G \longrightarrow \text{Aut}(A)$ be an action of the group G on A . Then $k(G, A)$ is the universal $*$ -algebra generated by elements $\{i_A(a) : a \in A\}$ and $\{i_G(s) : s \in G\}$ such that*

- (a) i_A is a unital $*$ -homomorphism.
- (b) i_G is a homomorphism from G into the group $u(k(G, A))$.
- (c) $i_A(\beta_s(a)) = i_G(s)i_A(a)i_G(s)^*$ for all $a \in A$ and $s \in G$.

Proof. That $k(G, A)$ is a $*$ -algebra generated by the elements $i_A(a)$ and $i_G(s)$ such that the relations (a)-(c) are satisfied follows directly from [4, Lemma 42 and Lemma 43]. So we just need to check that i_A is unital and that $k(G, A)$ is a universal $*$ -algebra.

That i_A is unital is pretty clear since

$$i_A(1_A) = \delta_e 1_A = 1_{k(G, A)}.$$

To show that $k(G, A)$ is a universal $*$ -algebra, suppose that $\{\hat{i}_A(a) : a \in A\}$ and $\{\hat{i}_G(s) : s \in G\}$ are elements in a $*$ -algebra B which also satisfies (a)-(c). We need to find a $*$ -homomorphism $\phi : k(G, A) \longrightarrow B$ such that $\phi(i_A(a)) = \hat{i}_A(a)$ and $\phi(i_G(s)) = \hat{i}_G(s)$. □

Claim. *The map $\phi : k(G, A) \longrightarrow B$ defined by $\phi(f) = \sum_{s \in G} \hat{i}_A(f(s))\hat{i}_G(s)$ is a $*$ -homomorphism.*

Proof. Notice that ϕ is a linear combination of the linear maps $f \mapsto \hat{i}_A(f(s))$, hence ϕ is linear. Let $a \in A$. Then

$$\begin{aligned} \phi(i_A(a)) &= \sum_{s \in G} \hat{i}_A(i_A(a)(s))\hat{i}_G(s) \\ &= \sum_{s \in G} \hat{i}_A(\delta_e a(s))\hat{i}_G(s) \\ &= \hat{i}_A(a)\hat{i}_G(e) \\ &= \hat{i}_A(a)1_B \\ &= \hat{i}_A(a). \end{aligned}$$

Let $t \in G$. Then

$$\begin{aligned} \phi(i_G(t)) &= \sum_{s \in G} \hat{i}_A(i_G(t)(s))\hat{i}_G(s) \\ &= \sum_{s \in G} \hat{i}_A(\delta_t 1_A(s))\hat{i}_G(s) \\ &= \hat{i}_A(1_A)\hat{i}_G(t) \\ &= 1_B\hat{i}_G(t) \\ &= \hat{i}_G(t). \end{aligned}$$

Next, we compute

$$\begin{aligned} \phi(f^*) &= \sum_{s \in G} \hat{i}_A(f^*(s))\hat{i}_G(s) \\ &= \sum_{s \in G} \hat{i}_A(\beta_s(f(s^{-1})^*))\hat{i}_G(s) \\ &= \sum_{s \in G} \hat{i}_A(\beta_s(f(s^{-1}))^*)\hat{i}_G(s) \\ &= \sum_{s \in G} \hat{i}_A(\beta_s(f(s^{-1})))^*\hat{i}_G(s) \end{aligned}$$

From relation (c) we conclude that

$$\hat{i}_G(s)^*\hat{i}_A(\beta_s(a)) = \hat{i}_A(a)\hat{i}_G(s)^*,$$

and this is equivalent to

$$\hat{i}_A(\beta_s(a))^*\hat{i}_G(s) = \hat{i}_G(s)\hat{i}_A(a)^*.$$

By noting that

$$\hat{i}_G(s)^* = \hat{i}_G(s^{-1}),$$

we have

$$\begin{aligned}
 \phi(f^*) &= \sum_{s \in G} \hat{i}_G(s) \hat{i}_A(f(s^{-1}))^* \\
 &= \sum_{s \in G} \hat{i}_G(s^{-1})^* \hat{i}_A(f(s^{-1}))^* \\
 &= \sum_{p \in G} \hat{i}_G(p)^* \hat{i}_A(f(p))^* \\
 &= \phi(f)^*.
 \end{aligned}$$

Finally, let $f, h \in k(G, A)$. Then

$$\begin{aligned}
 \phi(f *_{\beta} h) &= \sum_{s \in G} \hat{i}_A(f *_{\beta} h(s)) \hat{i}_G(s) \\
 &= \sum_{s \in G} \hat{i}_A\left(\sum_{t \in G} f(t) \beta_t(h(t^{-1}s))\right) \hat{i}_G(s) \\
 &= \sum_{s \in G} \left(\sum_{t \in G} \hat{i}_A(f(t)) \hat{i}_A(\beta_t(h(t^{-1}s)))\right) \\
 &= \sum_{s \in G} \left(\sum_{t \in G} \hat{i}_A(f(t)) \hat{i}_G(t) \hat{i}_A(h(t^{-1}s)) \hat{i}_G(t)^* \hat{i}_G(s)\right) \\
 &= \sum_{s \in G} \left(\sum_{t \in G} \hat{i}_A(f(t)) \hat{i}_G(t) \hat{i}_A(h(t^{-1}s)) \hat{i}_G(t^{-1}s)\right).
 \end{aligned}$$

By writing $d = t^{-1}s$ and noting that $\sum_{s \in G} = \sum_{d \in G}$ (this is true since all sums are finite) we have

$$\phi(f *_{\beta} h) = \sum_{t \in G} \hat{i}_A(f(t)) \hat{i}_G(t) \sum_{d \in G} \hat{i}_A(h(d)) \hat{i}_G(d) = \phi(f) \phi(h).$$

Thus ϕ is multiplicative. □

4. The relation between $H(P_{\mathbb{Q}}^+, P_{\mathbb{Z}})$ and $H(P_{\mathbb{Q}}, P_{\mathbb{Z}})$

Proposition 4.1. *Consider the group $G = \{1, -1\}$ and the algebra $A = H(P_{\mathbb{Q}}^+, P_{\mathbb{Z}})$. Then there is a $*$ -homomorphism $\phi : H(P_{\mathbb{Q}}, P_{\mathbb{Z}}) \rightarrow H(P_{\mathbb{Q}}^+, P_{\mathbb{Z}}) \times_{\beta} \{1, -1\}$ such that*

- (i) $\phi(e(r)) = i_A(e(r))$ for all $r \in \mathbb{Q}/\mathbb{Z}$.
- (ii) $\phi(\mu_n) = i_A(\mu_n)$ for all $n \in \mathbb{N}^*$.
- (iii) $\phi(u) = i_G(-1)$.

Proof. Define a map $\beta : G \rightarrow \text{Aut}(A)$ by

$$\beta_1 = id \text{ and } \beta_{-1} = \alpha \text{ (the } * \text{-homomorphism of Lemma 3.1).}$$

To show that β is an action we only need to check that $\beta_{(1)(-1)} = \beta_1 \beta_{-1}$ and $\beta_{-1}^2 = \beta_1$. Now $\beta_{(1)(-1)} = \beta_{-1} = \alpha = id\alpha = \beta_1 \beta_{-1}$, and by Lemma 3.1 we

have $\beta_{-1}^2 = \alpha^2 = id = \beta_1$. Thus β is an action of the group G on the group of automorphisms of A . Suppose that $\{\hat{\mu}_n = i_A(\mu_n) : n \in \mathbb{N}^*\}$, $\{\hat{e}(r) = i_A(e(r)) : r \in \mathbb{Q}/\mathbb{Z}\}$ and $\hat{u} = i_G(-1)$. If these elements satisfy the relations (a)-(g) of [1, Theorem 7.4] we are done.

For (a)

$$\hat{\mu}_n^* \hat{\mu}_n = i_A(\mu_n)^* i_A(\mu_n) = i_A(\mu_n^* \mu_n) = 1_{k(G,A)}.$$

For (b)

$$\hat{\mu}_{mn} = i_A(\mu_{mn}) = i_A(\mu_m \mu_n) = i_A(\mu_m) i_A(\mu_n) = \hat{\mu}_m \hat{\mu}_n.$$

For (c), let us start with

$$\hat{e}(r)^* = i_A(e(r))^* = i_A(e(r)^*) = i_A(e(-r)) = \hat{e}(-r).$$

and for the second part

$$\hat{e}(r_1 + r_2) = i_A(e(r_1 + r_2)) = i_A(e(r_1)e(r_2)) = i_A(e(r_1))i_A(e(r_2)) = \hat{e}(r_1)\hat{e}(r_2).$$

For (d)

$$\begin{aligned} \hat{\mu}_n \hat{e}(r) \hat{\mu}_n^* &= i_A(\mu_n) i_A(e(r)) i_A(\mu_n)^* \\ &= i_A(\mu_n e(r) \mu_n^*) \\ &= i_A\left(\frac{1}{n} \sum_{j=1}^n e(r/n + j/n)\right) \\ &= \frac{1}{n} \sum_{j=1}^n i_A(e(r/n + j/n)) \\ &= \frac{1}{n} \sum_{j=1}^n \hat{e}(r/n + j/n). \end{aligned}$$

For (e), let us start with $\hat{u}^* = i_G(-1)^* = i_G(-1) = \hat{u}$ and for the second part of (e) $\hat{u}^2 = i_G(-1)i_G(-1) = i_G(1) = 1_{k(G,A)}$.

For (f), by relation (c) of Proposition 3.2

$$\begin{aligned} \hat{u} \hat{\mu}_n &= i_G(-1) i_A(\mu_n) \\ &= i_A(\beta_{-1}(\mu_n)) i_G(-1) \\ &= i_A(\alpha(\mu_n)) i_G(-1) \\ &= i_A(\mu_n) i_G(-1) \\ &= \hat{\mu}_n \hat{u}. \end{aligned}$$

For (g), by relation (c) of Proposition 3.2

$$\begin{aligned} \hat{u} \hat{e}(-r) &= i_G(-1) i_A(e(-r)) \\ &= i_A(\beta_{-1}(e(-r))) i_G(-1) \\ &= i_A(\alpha(e(-r))) i_G(-1) \\ &= i_A(e(r)) i_G(-1) \\ &= \hat{e}(r) \hat{u}. \end{aligned}$$

Thus, [1, Theorem 7.4] says that there exists a $*$ -homomorphism $\phi: H(P_{\mathbb{Q}}, P_{\mathbb{Z}}) \rightarrow A \times_{\beta} G$. $\square$

Lemma 4.2. *The pair (ι, U) is a covariant representation of $H(P_{\mathbb{Q}}^+, P_{\mathbb{Z}}) \times_{\beta} \{1, -1\}$ on $H(P_{\mathbb{Q}}, P_{\mathbb{Z}})$ where*

- (i) $\iota: H(P_{\mathbb{Q}}^+, P_{\mathbb{Z}}) \rightarrow H(P_{\mathbb{Q}}, P_{\mathbb{Z}})$ is the unital $*$ -homomorphism of [1, Lemma 2.8].
- (ii) $U: \{1, -1\} \rightarrow u(H(P_{\mathbb{Q}}, P_{\mathbb{Z}}))$ is defined by $U_1 = \mu_1 = 1_{H(P_{\mathbb{Q}}, P_{\mathbb{Z}})}$ and $U_{-1} = u$.
- (iii) β is the group action defined in Proposition 4.1.

Proof. The map U is a homomorphism by relation (e) of [1, Theorem 7.4]. We still need to show that $\iota(\beta_1(a)) = U_1 \iota(a) U_1^*$ and $\iota(\beta_{-1}(a)) = U_{-1} \iota(a) U_{-1}^*$ for all $a \in H(P_{\mathbb{Q}}^+, P_{\mathbb{Z}})$.

On one hand, the first relation is true since both sides are $\iota(a)$, and on the other, it is enough to check the second relation when $a = \mu_n$ and $a = e(r)$.

If $a = \mu_n$, then we have relation (f) in [1, Theorem 7.4] and if $a = e(r)$ we have relation (g) in [1, Theorem 7.4]. Both β_{-1} and $a \mapsto uau^*$ are $*$ -homomorphisms, so this implies the second relation is true for all $a \in H(P_{\mathbb{Q}}^+, P_{\mathbb{Z}})$. $\square$

Now we give our main theorem which allows us to realize the Bost and Connes Hecke Algebra $H(P_{\mathbb{Q}}, P_{\mathbb{Z}})$ as the crossed product $H(P_{\mathbb{Q}}^+, P_{\mathbb{Z}}) \times_{\beta} \{1, -1\}$.

Theorem 4.3. *The map ϕ in Proposition 4.1 is a $*$ -isomorphism of $H(P_{\mathbb{Q}}, P_{\mathbb{Z}})$ onto $H(P_{\mathbb{Q}}^+, P_{\mathbb{Z}}) \times_{\beta} \{1, -1\}$ with inverse $\iota \times U$ where*

- (i) β is the group action defined in Proposition 4.1.
- (ii) The pair (ι, U) is the covariant representation in Lemma 4.2.

Proof. Let $A = H(P_{\mathbb{Q}}^+, P_{\mathbb{Z}})$ and $G = \{1, -1\}$. Lemma 4.2 and [4, Lemma 42] yield the map $\iota \times U: H(P_{\mathbb{Q}}^+, P_{\mathbb{Z}}) \times_{\beta} \{1, -1\} \rightarrow H(P_{\mathbb{Q}}, P_{\mathbb{Z}})$ defined by

$$\iota \times U(f) = \sum_{s \in \{1, -1\}} \iota(f(s)) U_s$$

is a $*$ -homomorphism. So if we show that $(\iota \times U) \circ \phi = id$ and $\phi \circ (\iota \times U) = id$ the proof of this theorem will follow. Since the elements $\mu_n, e(r)$ and u generate the $*$ -algebra $H(P_{\mathbb{Q}}, P_{\mathbb{Z}})$ and $(\iota \times U) \circ \phi$ is a $*$ -homomorphism. To show that $(\iota \times U) \circ \phi = id$, it is enough to check that it is true for $\mu_n, e(r)$ and u .

For μ_n we have

$$\begin{aligned} (\iota \times U) \circ \phi(\mu_n) &= (\iota \times U)(\phi(\mu_n)) \\ &= (\iota \times U)(i_A(\mu_n)) \\ &= \iota(\mu_n) \\ &= \mu_n. \end{aligned}$$

The same argument works for $e(r)$. For u we have

$$\begin{aligned}(\iota \times U) \circ \phi(u) &= (\iota \times U)(\phi(u)) \\ &= (\iota \times U)(i_G(-1)) \\ &= U_{-1} \\ &= u.\end{aligned}$$

For the second part of the proof, notice that the elements $i_A(\mu_n)$, $i_A(e(r))$ and $i_G(-1)$ generate the $*$ -algebra $A \times_\beta G$, and $\phi \circ (\iota \times U)$ is a $*$ -homomorphism. So to show that $\phi \circ (\iota \times U) = id$, it is enough to check the equality for $i_A(\mu_n)$, $i_A(e(r))$ and $i_G(-1)$.

For $i_A(\mu_n)$

$$\begin{aligned}\phi \circ (\iota \times U)(i_A(\mu_n)) &= \phi(\iota \times U(i_A(\mu_n))) \\ &= \phi(\iota(\mu_n)) \text{ [4, proposition 44]} \\ &= \phi(\mu_n) \\ &= i_A(\mu_n).\end{aligned}$$

The same argument works for $i_A(e(r))$. For $i_G(-1)$ we have

$$\begin{aligned}\phi \circ (\iota \times U)(i_G(-1)) &= \phi(\iota \times U(i_G(-1))) \\ &= \phi(U_{-1}) \text{ [4, proposition 44]} \\ &= \phi(u) \\ &= i_G(-1).\end{aligned}$$

Consequently, the two maps are inverses of each other, and hence the two $*$ -algebras are isomorphic. $\square$

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SOME GENERALIZED SEQUENCE SPACES OF INVARIANT MEANS DEFINED BY IDEAL AND MODULUS FUNCTIONS IN N -NORMED SPACES

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Abstract. In the present paper we introduce and study some generalized difference sequence spaces of invariant means defined by ideal and a sequence of modulus functions over n -normed space. We study some topological properties and prove some inclusion results between these spaces. Further, we also study some results on statistical convergence.

Keywords: ideal, Difference sequence space, modulus function, Lacunary sequence, Invariant mean, Statistical convergence.

1. Introduction and preliminaries

Let σ be the mapping of the set of positive integers into itself. A continuous linear functional φ on l_∞ is said to be an invariant mean or σ -mean (c.f. [5], [34], [35]) if and only if

1. $\varphi(x) \geq 0$ when the sequence $x = (x_k)$ has $x_k \geq 0$, for all k ,

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2. $\varphi(e) = 1$, where $e = (1, 1, 1, \dots)$ and

3. $\varphi(x_{\sigma(k)}) = \varphi(x)$, for all $x \in l_\infty$.

If $x = (x_n)$, write $Tx = Tx_n = (x_{\sigma(n)})$. It can be shown in [41] that

$$V_\sigma = \left\{ x \in l_\infty : \lim_k t_{k_n}(x) = l, \text{ uniformly in } n, l = \sigma - \lim x \right\},$$

where

$$t_{k_n}(x) = \frac{x_n + x_{\sigma^1 n} + \dots + x_{\sigma^{k_n} n}}{k + 1}.$$

In the case σ is the translation mapping $n \rightarrow n + 1$, σ -mean is often called a Banach limit and V_σ , the set of bounded sequences all of whose invariant means are equal, is the set of almost convergent sequences [24] (also see [27]). By using the concept of invariant means Mursaleen et al. [36] introduced the following sequence spaces:

$$\begin{aligned} w_\sigma &= \left\{ x : \lim_{n \rightarrow \infty} \frac{1}{n+1} \sum_{k=0}^n t_{k_m}(x_k - l) \rightarrow 0, \text{ uniformly in } m \right\}, \\ [w]_\sigma &= \left\{ x : \lim_{n \rightarrow \infty} \frac{1}{n+1} \sum_{k=0}^n |t_{k_m}(x_k - l)| \rightarrow 0, \text{ uniformly in } m \right\}, \\ [w_\sigma] &= \left\{ x : \lim_{n \rightarrow \infty} \frac{1}{n+1} \sum_{k=0}^n t_{k_m}(|x_k - l|) \rightarrow 0, \text{ uniformly in } m \right\}, \end{aligned}$$

and investigate some of its properties.

The notion of statistical convergence has been introduced by Fast [11] in 1951 and later developed by Fridy [12], Šalát [40], Mohiuddine and Belen [31] and many others. Furthermore, Kostyrko et al. [21] presented a very interesting generalization of statistical convergence called as I -convergence. The detailed history and development in this regard can be found by Connor [6], Maddox [25] and many others.

By a lacunary sequence $\theta = (k_r)$ where $k_0 = 0$, we shall mean an increasing sequence of non-negative integers with $k_r - k_{r-1} \rightarrow \infty$ as $r \rightarrow \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$. We write $h_r = k_r - k_{r-1}$. The ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r . The space of lacunary strongly convergent sequence was defined by Freedman et al. [15] as follows:

$$N_\theta = \left\{ x = (x_k) : \lim_{r \rightarrow \infty} \frac{1}{h_r} \sum_{k \in I_r} |x_k - L| = 0 \text{ for some } L \right\}.$$

Fridy and Orhan [13] generalized the concept of statistical convergence by using lacunary sequence which is called lacunary statistical convergence. Further, lacunary sequences have been studied by Fridy and Orhan [14]. Quite recently, Karakaya [22] combined the approach of lacunary sequence with invariant means and introduced the notion of strong σ -lacunary statistically convergence as follows:

Definition 1.1. [22] Let $\theta = (k_r)$ be a lacunary sequence. A sequence $x = (x_k)$ is said to be lacunary strong σ lacunary statistically convergent if for every $\varepsilon > 0$,

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} \{k \in I_r : |t_{k_m}(x - L)| \geq \varepsilon\} = 0 \quad \text{uniformly in } m$$

where St_{θ_σ} denotes the set of all lacunary strong σ -lacunary statistically convergent sequences.

Let $\mathbb{N}$ be a non empty set. Then a family of sets $I \subseteq 2^{\mathbb{N}}$ (Power set of $\mathbb{N}$) is said to be an ideal if I is additive i.e $A, B \in I \Rightarrow A \cup B \in I$ and $A \in I, B \subseteq A \Rightarrow B \in I$. A non empty family of sets $\mathcal{L}(I) \subseteq 2^{\mathbb{N}}$ is said to be filter on $\mathbb{N}$ if and only if $\Phi \notin \mathcal{L}(I)$ for $A, B \in \mathcal{L}(I)$ we have $A \cap B \in \mathcal{L}(I)$ and for each $A \in \mathcal{L}(I)$ and $A \subseteq B$ implies $B \in \mathcal{L}(I)$. An ideal $I \subseteq 2^{\mathbb{N}}$ is called non trivial if $I \neq 2^{\mathbb{N}}$. A non trivial ideal $I \subseteq 2^{\mathbb{N}}$ is called admissible if $\{\{x\} : x \in \mathbb{N}\} \subseteq I$. A non-trivial ideal is maximal if there cannot exist any non trivial ideal $J \neq I$ containing I as a subset. For each ideal I , there exist a filter $\mathcal{L}(I)$ corresponding to I i.e $\mathcal{L}(I) = \{K \subseteq \mathbb{N} : K^c \in I\}$, where $K^c = \mathbb{N} \setminus K$. Recently, Das et al. [7] unified the idea of lacunary statistical convergence with ideal convergence and presented the following interesting generalization of statistical convergence.

Definition 1.2. [7] Let $\theta = (k_r)$ be a lacunary sequence. A sequence $x = (x_k)$ is said to be I -lacunary statistical convergent or $S_\theta(I)$ -convergent to L , if for every $\varepsilon > 0$ and $\delta > 0$,

$$\left\{r \in \mathbb{N} : \frac{1}{h_r} |\{k \in I_r : |x_k - L| \geq \varepsilon\}| \geq \delta\right\} \in I.$$

In this case $x_k \rightarrow L(S_\theta(I))$ or $S_\theta(I) - \lim_{k \rightarrow \infty} x_k = L$. The set of all I -lacunary statistically convergent sequences will be denoted by $S_\theta(I)$.

Definition 1.3. [7] Let $\theta = (k_r)$ be a lacunary sequence. A sequence $x = (x_k)$ is said to be $N_\theta(I)$ -convergent to L , if for every $\varepsilon > 0$ we have

$$\left\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} |x_k - L| \geq \varepsilon\right\} \in I.$$

In this case $x_k \rightarrow L(N_\theta(I))$.

The concept of 2-normed spaces was initially developed by Gähler [16] in the mid of 1960's, while that of n -normed spaces one can see in Misiak [26]. Since then, many others have studied this concept and obtained various results, see Gunawan ([17],[18]) and Gunawan and Mashadi [19]. Let $n \in \mathbb{N}$ and X be a linear space over the field $\mathbb{R}$ of reals of dimension d , where $d \geq n \geq 2$. A real valued function $\|\cdot, \dots, \cdot\|$ on X^n satisfying the following four conditions:

1. $\|x_1, x_2, \dots, x_n\| = 0$ if and only if $x_1, x_2, \dots, x_n$ are linearly dependent in X ;

2. $\|x_1, x_2, \dots, x_n\|$ is invariant under permutation;
3. $\|\alpha x_1, x_2, \dots, x_n\| = |\alpha| \|x_1, x_2, \dots, x_n\|$ for any $\alpha \in \mathbb{R}$, and
4. $\|x + x', x_2, \dots, x_n\| \leq \|x, x_2, \dots, x_n\| + \|x', x_2, \dots, x_n\|$

is called a n -norm on X , and the pair $(X, \|\cdot, \dots, \cdot\|)$ is called a n -normed space over the field $\mathbb{R}$.

For example, we may take $X = \mathbb{R}^n$ being equipped with the Euclidean n -norm $\|x_1, x_2, \dots, x_n\|_E =$ the volume of the n -dimensional parallelopiped spanned by the vectors $x_1, x_2, \dots, x_n$ which may be given explicitly by the formula

$$\|x_1, x_2, \dots, x_n\|_E = |\det(x_{ij})|,$$

where $x_i = (x_{i1}, x_{i2}, \dots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \dots, n$. Let $(X, \|\cdot, \dots, \cdot\|)$ be a n -normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \dots, a_n\}$ be linearly independent set in X . Then the following function $\|\cdot, \dots, \cdot\|_\infty$ on X^{n-1} defined by

$$\|x_1, x_2, \dots, x_{n-1}\|_\infty = \max\{\|x_1, x_2, \dots, x_{n-1}, a_i\| : i = 1, 2, \dots, n\}$$

defines an $(n-1)$ -norm on X with respect to $\{a_1, a_2, \dots, a_n\}$.

A sequence (x_k) in a n -normed space $(X, \|\cdot, \dots, \cdot\|)$ is said to converge to some $L \in X$ if

$$\lim_{k \rightarrow \infty} \|x_k - L, z_1, \dots, z_{n-1}\| = 0 \text{ for every } z_1, \dots, z_{n-1} \in X.$$

A sequence (x_k) in a n -normed space $(X, \|\cdot, \dots, \cdot\|)$ is said to be Cauchy if

$$\lim_{k, p \rightarrow \infty} \|x_k - x_p, z_1, \dots, z_{n-1}\| = 0 \text{ for every } z_1, \dots, z_{n-1} \in X.$$

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the n -norm. Any complete n -normed space is said to be n -Banach space.

Definition 1.4. Let $I \subseteq 2^{\mathbb{N}}$. A sequence $x = (x_k)$ in a n -normed space $(X, \|\cdot, \dots, \cdot\|)$ is said to be I -convergent to a number L if for every $\epsilon > 0$, the set $A(\epsilon) = \{k \in \mathbb{N} : \|x_k - L, z_1, \dots, z_{n-1}\| \geq \epsilon\} \in I$. In this case we write $I - \lim_{k \rightarrow \infty} \|x_k, z_1, \dots, z_{n-1}\| = \|L, z_1, \dots, z_{n-1}\|$.

Definition 1.5. A sequence $x = (x_k)$ in a n -normed space $(X, \|\cdot, \dots, \cdot\|)$ is said to be statistical convergent to some $L \in X$ if for each $\epsilon > 0$, the set $A(\epsilon) = \{k \in \mathbb{N} : \|x_k - L, z_1, \dots, z_{n-1}\| \geq \epsilon\}$ having its natural density zero.

The notion of difference sequence spaces was introduced by Kızmaz [20] who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et. and Çolak [9] by introducing the spaces $l_\infty(\Delta^n)$,

$c(\Delta^n)$ and $c_0(\Delta^n)$. Let w be the space of all complex or real sequences $x = (x_k)$ and let m, v be non-negative integers, then for $Z = l_\infty, c, c_0$ we have sequence spaces $Z(\Delta_v^m) = \{x = (x_k) \in w : (\Delta_v^m x_k) \in Z\}$, where $\Delta_v^m x = (\Delta_v^m x_k) = (\Delta_v^{m-1} x_k - \Delta_v^{m-1} x_{k+1})$ and $\Delta_v^0 x_k = x_k$, for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta_v^m x_k = \sum_{s=0}^m (-1)^s \binom{m}{s} x_{k+vs}.$$

Taking $v = 1$, we get the spaces which were introduced and studied by Et. and Çolak [9]. Taking $m = v = 1$, we get the spaces which were studied by Kızmaz [20]. For more details about sequence spaces (see [23, 32, 33, 36, 37, 39]) and reference therein.

A modulus function is a function $f : [0, \infty) \rightarrow [0, \infty)$ such that

1. $f(x) = 0$ if and only if $x = 0$,
2. $f(x + y) \leq f(x) + f(y)$, for all $x, y \geq 0$,
3. f is increasing,
4. f is continuous from the right at 0.

It follows that f must be continuous everywhere on $[0, \infty)$. The modulus function may be bounded or unbounded. For example, if we take $f(x) = \frac{x}{x+1}$, then $f(x)$ is bounded. If $f(x) = x^p, 0 < p < 1$ then the modulus function $f(x)$ is unbounded. Subsequently, modulus function has been discussed in ([2], [38]) and references therein.

Lemma 1.6. *Let $\mathcal{F} = (f_k)$ be a sequence of modulus functions and $0 < \delta < 1$. Then for each $x > \delta$ we have $f_k(x) \leq \frac{2f_k(1)x}{\delta}$.*

Let $\mathcal{F} = (f_k)$ be a sequence of modulus functions, $(X, \|\cdot, \dots, \cdot\|)$ be a n -normed space, $p = (p_k)$ be a bounded sequence of strictly positive real numbers and $u = (u_k)$ be any sequence of positive real numbers. By $S(n - X)$ we denote the space of all sequences defined over $(X, \|\cdot, \dots, \cdot\|)$. In this paper we define the following sequence spaces:

$$\begin{aligned} & w_\sigma^0 \left[\mathcal{F}, I, u, p, \|\cdot, \dots, \cdot\| \right]_\theta (\Delta_v^m) \\ &= \left\{ x \in S(n - X) : \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} u_k \left[f_k \left(\left\| \frac{t_{k_n}(\Delta_v^m x_k)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \geq \varepsilon \right\} \in I, \right. \end{aligned}$$

for some $\rho > 0$ $\Big\}$,

$$\begin{aligned} & w_\sigma \left[\mathcal{F}, I, u, p, \|\cdot, \dots, \cdot\| \right]_\theta (\Delta_v^m) \\ &= \left\{ x \in S(n - X) : \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} u_k \left[f_k \left(\left\| \frac{t_{k_n}(\Delta_v^m x_k - l)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \geq \varepsilon \right\} \in I, \right. \end{aligned}$$

for some l and $\rho > 0$, $\Big\}$

and

$$\begin{aligned} & w_{\sigma}^{\infty} \left[\mathcal{F}, I, u, p, \|\cdot, \dots, \cdot\| \right]_{\theta} (\Delta_v^m) \\ &= \left\{ x \in S(n-X) : \exists K > 0, \right. \\ & \left. \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} u_k \left[f_k \left(\left\| \frac{t_{k_n}(\Delta_v^m x_k)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \geq K \right\} \in I, \text{ for some } \rho > 0 \right\}, \end{aligned}$$

uniformly in n .

If we take $\mathcal{F}(x) = x$, we get the spaces

$$\begin{aligned} & w_{\sigma}^0 \left[I, u, p, \|\cdot, \dots, \cdot\| \right]_{\theta} (\Delta_v^m) \\ &= \left\{ x \in S(n-X) : \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} u_k \left[\left(\left\| \frac{t_{k_n}(\Delta_v^m x_k)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \geq \varepsilon \right\} \in I, \right. \end{aligned}$$

for some $\rho > 0$ \}

$$\begin{aligned} & w_{\sigma} \left[I, u, p, \|\cdot, \dots, \cdot\| \right]_{\theta} (\Delta_v^m) \\ &= \left\{ x \in S(n-X) : \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} u_k \left[\left(\left\| \frac{t_{k_n}(\Delta_v^m x_k - l)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \geq \varepsilon \right\} \in I, \right. \end{aligned}$$

for some l and $\rho > 0$, \}

and

$$\begin{aligned} & w_{\sigma}^{\infty} \left[I, u, p, \|\cdot, \dots, \cdot\| \right]_{\theta} (\Delta_v^m) \\ &= \left\{ x \in S(n-X) : \exists K > 0, \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} u_k \cdot \right. \right. \\ & \quad \cdot \left. \left[\left(\left\| \frac{t_{k_n}(\Delta_v^m x_k)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \geq K \right\} \in I, \text{ for some } \rho > 0 \right\}. \end{aligned}$$

If we take $p = (p_k) = 1$, we get the spaces

$$\begin{aligned} & w_{\sigma}^0 \left[\mathcal{F}, I, u, \|\cdot, \dots, \cdot\| \right]_{\theta} (\Delta_v^m) \\ &= \left\{ x \in S(n-X) : \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} u_k \cdot \right. \right. \\ & \quad \cdot \left. \left[f_k \left(\left\| \frac{t_{k_n}(\Delta_v^m x_k)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right] \geq \varepsilon \right\} \in I, \text{ for some } \rho > 0 \right\}, \\ & w_{\sigma} \left[\mathcal{F}, I, u, \|\cdot, \dots, \cdot\| \right]_{\theta} (\Delta_v^m) \\ &= \left\{ x \in S(n-X) : \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} u_k \cdot \right. \right. \\ & \quad \cdot \left. \left[f_k \left(\left\| \frac{t_{k_n}(\Delta_v^m x_k - l)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right] \geq \varepsilon \right\} \in I, \text{ for some } l \text{ and } \rho > 0, \right\} \end{aligned}$$

and

$$\begin{aligned} & w_{\sigma}^{\infty} \left[\mathcal{F}, I, u, \|\cdot, \dots, \cdot\| \right]_{\theta} (\Delta_v^m) \\ &= \left\{ x \in S(n-X) : \exists K > 0, \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} u_k \cdot \right. \right. \\ & \quad \cdot \left. \left[f_k \left(\left\| \frac{t_{k_n}(\Delta_v^m x_k)}{\rho} \right\|, z_1, \dots, z_{n-1} \right) \right] \geq K \right\} \in I, \text{ for some } \rho > 0 \right\}. \end{aligned}$$

The following inequality will be used throughout the paper. If $0 \leq p_k \leq \sup p_k = H$, $K = \max(1, 2^{H-1})$ then

$$(1.1) \quad |a_k + b_k|^{p_k} \leq D\{|a_k|^{p_k} + |b_k|^{p_k}\}$$

for all k and $a_k, b_k \in \mathbb{R}$. Also $|a|^{p_k} \leq \max(1, |a|^H)$, for all $a \in \mathbb{R}$.

The main aim of the present paper is to study some topological properties and prove some inclusion relations between above defined sequence spaces.

2. Main results

Theorem 2.1. *Let $\mathcal{F}=(f_k)$ be a sequence of modulus functions, $p=(p_k)$ be a bounded sequence of strictly positive real numbers and $u=(u_k)$ be any sequence of positive real numbers. Then the classes of sequences $w_{\sigma}^0[\mathcal{F}, I, u, p, \|\cdot, \dots, \cdot\|]_{\theta}(\Delta_v^m)$, $w_{\sigma}[\mathcal{F}, I, u, p, \|\cdot, \dots, \cdot\|]_{\theta}(\Delta_v^m)$ and $w_{\sigma}^{\infty}[\mathcal{F}, I, u, p, \|\cdot, \dots, \cdot\|]_{\theta}(\Delta_v^m)$ are linear spaces over the real field $\mathbb{R}$.*

Proof. We shall prove the assertion for $w_{\sigma}^0[\mathcal{F}, I, u, p, \|\cdot, \dots, \cdot\|]_{\theta}(\Delta_v^m)$ only and the others can be proved similarly.

Let $x = (x_k)$, $y = (y_k) \in w_{\sigma}^0[\mathcal{F}, I, u, p, \|\cdot, \dots, \cdot\|]_{\theta}(\Delta_v^m)$ and $\alpha, \beta \in \mathbb{R}$. Then there exist positive real numbers ρ_1 and ρ_2 such that for every $\varepsilon > 0$ and $z_1, \dots, z_{n-1} \in X$, we have

$$A_{\theta}\left(\frac{\varepsilon}{2}\right) = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} u_k \left[f_k \left(\left\| \frac{t_{k_n}(\Delta_v^m x_k)}{\rho_1} \right\|, z_1, \dots, z_{n-1} \right) \right]^{p_k} \geq \frac{\varepsilon}{2} \right\}$$

and

$$B_{\theta}\left(\frac{\varepsilon}{2}\right) = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} u_k \left[f_k \left(\left\| \frac{t_{k_n}(\Delta_v^m y_k)}{\rho_2} \right\|, z_1, \dots, z_{n-1} \right) \right]^{p_k} \geq \frac{\varepsilon}{2} \right\}$$

belongs to I . Let $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since $\|\cdot, \dots, \cdot\|$ is a n -norm on X and f_k 's are modulus functions so by using inequality (1.1), we have

$$\begin{aligned} & \frac{1}{h_r} \sum_{k \in I_r} u_k \left[f_k \left(\left\| \frac{t_{k_n}(\alpha \Delta_v^m x_k + \beta \Delta_v^m y_k)}{\rho_3}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \\ & \leq \frac{1}{h_r} \sum_{k \in I_r} u_k \left[f_k \left(|\alpha| \left\| \frac{t_{k_n}(\Delta_v^m x_k)}{\rho_3}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \\ & \quad + \frac{1}{h_r} \sum_{k \in I_r} u_k \left[f_k \left(|\beta| \left\| \frac{t_{k_n}(\Delta_v^m y_k)}{\rho_3}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \\ & \leq D \cdot \frac{1}{h_r} \sum_{k \in I_r} \frac{1}{2^{p_k}} u_k \left[f_k \left(\left\| \frac{t_{k_n}(\Delta_v^m x_k)}{\rho_1}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \\ & \quad + D \cdot \frac{1}{h_r} \sum_{k \in I_r} \frac{1}{2^{p_k}} u_k \left[f_k \left(\left\| \frac{t_{k_n}(\Delta_v^m y_k)}{\rho_2}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k}. \end{aligned}$$

For given $\varepsilon > 0$ and for all $z_1, \dots, z_{n-1} \in X$, we have the following containment

$$\begin{aligned} & \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} u_k \left[f_k \left(\left\| \frac{t_{k_n}(\alpha \Delta_v^m x_k + \beta \Delta_v^m y_k)}{\rho_3}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \geq \varepsilon \right\} \\ & \subseteq \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} u_k \left[f_k \left(\left\| \frac{t_{k_n}(\Delta_v^m x_k)}{\rho_1}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \geq \frac{\varepsilon}{2D} \right\} \\ & \cup \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} u_k \left[f_k \left(\left\| \frac{t_{k_n}(\Delta_v^m y_k)}{\rho_2}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \geq \frac{\varepsilon}{2D} \right\}. \end{aligned}$$

By using the property of an ideal the set on the left hand side in the above expression belongs to I . Thus, $\alpha x + \beta y \in w_\sigma^0[\mathcal{F}, I, u, p, \|\cdot, \dots, \cdot\|]_\theta(\Delta_v^m)$. This completes the proof. $\square$

Theorem 2.2. Let $p = (p_k)$ be a bounded sequence of strictly positive real numbers and $u = (u_k)$ be a sequence of positive real numbers. Then for $m \geq 1$, we have:

- (i) $w_\sigma^0[\mathcal{F}, I, u, p, \|\cdot, \dots, \cdot\|]_\theta(\Delta_v^{m-1}) \subset w_\sigma^0[\mathcal{F}, u, p, \|\cdot, \dots, \cdot\|]_\theta(\Delta_v^m)$ is strict.
- (ii) $w_\sigma[\mathcal{F}, I, u, p, \|\cdot, \dots, \cdot\|]_\theta(\Delta_v^{m-1}) \subset w_\sigma[\mathcal{F}, u, p, \|\cdot, \dots, \cdot\|]_\theta(\Delta_v^m)$ is strict.
- (iii) $w_\sigma^\infty[\mathcal{F}, I, u, p, \|\cdot, \dots, \cdot\|]_\theta(\Delta_v^{m-1}) \subset w_\sigma^\infty[\mathcal{F}, u, p, \|\cdot, \dots, \cdot\|]_\theta(\Delta_v^m)$ is strict.

Proof. We shall prove the result for $w_\sigma^0[\mathcal{F}, I, u, p, \|\cdot, \dots, \cdot\|]_\theta(\Delta_v^{m-1})$ only. The others can be proved similarly. Suppose $x \in w_\sigma^0[\mathcal{F}, I, u, p, \|\cdot, \dots, \cdot\|]_\theta(\Delta_v^{m-1})$, by definition for every $\varepsilon > 0$ and $z_1, \dots, z_{n-1} \in X$, we have

$$(2.1) \quad \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} u_k \left[f_k \left(\left\| \frac{t_{k_n}(\Delta_v^{m-1} x_k)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \geq \varepsilon \right\} \in I,$$

uniformly in n . Since $\mathcal{F} = (f_k)$ is a sequence of modulus functions, we have the following inequality:

$$\begin{aligned} & \frac{1}{h_r} \sum_{k \in I_r} u_k \left[f_k \left(\left\| \frac{t_{k_n}(\Delta_v^m x_k)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \\ & \leq \frac{1}{h_r} \sum_{k \in I_r} \left[u_k \left[f_k \left(\left\| \frac{t_{k_n}(\Delta_v^{m-1} x_k)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right] \right. \\ & \quad \left. + u_k \left[f_k \left(\left\| \frac{t_{k_n}(\Delta_v^{m-1} x_{k+1})}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right] \right]^{p_k} \\ & \leq D \frac{1}{h_r} \sum_{k \in I_r} u_k \left[f_k \left(\left\| \frac{t_{k_n}(\Delta_v^{m-1} x_k)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \\ & \quad + D \frac{1}{h_r} \sum_{k \in I_r} u_k \left[f_k \left(\left\| \frac{t_{k_n}(\Delta_v^{m-1} x_{k+1})}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \end{aligned}$$

uniformly in n .

Now for given $\varepsilon > 0$, we have

$$\begin{aligned} & \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} u_k \left[f_k \left(\left\| \frac{t_{k_n}(\Delta_v^m x_k)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \geq \varepsilon \right\} \\ & \subseteq \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} u_k \left[f_k \left(\left\| \frac{t_{k_n}(\Delta_v^{m-1} x_k)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \geq \frac{\varepsilon}{2D} \right\} \\ & \cup \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} u_k \left[f_k \left(\left\| \frac{t_{k_n}(\Delta_v^{m-1} x_{k+1})}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \geq \frac{\varepsilon}{2D} \right\}, \end{aligned}$$

uniformly in n .

Both the sets on the right hand side in the above containment belong to I by (2.1). It follows that $x \in w_\sigma^0[\mathcal{F}, I, u, p, \|\cdot, \dots, \cdot\|]_\theta(\Delta_v^m)$. Clearly the inclusion is strict because if we take $x = (x_k) = k^{m-1}$, $f_k(x) = x$, $p_k = 1$, $u_k = 1$, $t_{0_n}(x) = (x_n)$ and $\theta = (2^r)$ for all $k \in \mathbb{N}$, then $x_k \in w_\sigma^0[\mathcal{F}, u, p, \|\cdot, \dots, \cdot\|]_\theta(\Delta_v^m)$ but $x_k \notin w_\sigma^0[\mathcal{F}, I, u, p, \|\cdot, \dots, \cdot\|]_\theta(\Delta_v^{m-1})$. $\square$

Theorem 2.3. Let $\mathcal{F}' = (f'_k)$ and $\mathcal{F}'' = (f''_k)$ be two sequences of modulus functions. If $\limsup_{t \rightarrow \infty} \frac{f'_k(t)}{f''_k(t)} = P > 0$, then $w_\sigma^0[\mathcal{F}', u, p, \|\cdot, \dots, \cdot\|]_\theta(\Delta_v^m) \subset w_\sigma^0[\mathcal{F}'', I, u, p, \|\cdot, \dots, \cdot\|]_\theta(\Delta_v^m)$.

Proof. Let $\limsup_{t \rightarrow \infty} \frac{f'_k(t)}{f''_k(t)} = P$, then there exists a positive number $K > 0$ such that $f'_k(t) \geq K f''_k(t)$, for all $t \geq 0$. Therefore, for each $z_1, \dots, z_{n-1} \in X$,

we have

$$\begin{aligned} & \frac{1}{h_r} \sum_{k \in I_r} u_k \left[f'_k \left(\left\| \frac{t_{k_n}(\Delta_v^m x_k)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \\ & \geq (K)^H \frac{1}{h_r} \sum_{k \in I_r} u_k \left[f''_k \left(\left\| \frac{t_{k_n}(\Delta_v^m x_k)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k}, \end{aligned}$$

uniformly in n . Then for every $\varepsilon > 0$ and $z_1, \dots, z_{n-1} \in X$, we have following relationship

$$\begin{aligned} & \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} u_k \left[f''_k \left(\left\| \frac{t_{k_n}(\Delta_v^m x_k)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \geq \varepsilon \right\} \\ & \subseteq \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} u_k \left[f'_k \left(\left\| \frac{t_{k_n}(\Delta_v^m x_k)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \geq \varepsilon (K)^H \right\}, \end{aligned}$$

uniformly in n . Therefore, the above containment gives the result. $\square$

Theorem 2.4. Suppose $\mathcal{F} = (f_k)$, $\mathcal{F}' = (f'_k)$ and $\mathcal{F}'' = (f''_k)$ are sequences of modulus functions, then

- (i) $w_\sigma[\mathcal{F}', I, u, p, \|\cdot, \dots, \cdot\|]_\theta(\Delta_v^m) \subset w_\sigma[\mathcal{F} \circ \mathcal{F}', I, u, p, \|\cdot, \dots, \cdot\|]_\theta(\Delta_v^m)$.
- (ii) $w_\sigma[\mathcal{F}', I, u, p, \|\cdot, \dots, \cdot\|]_\theta(\Delta_v^m) \cap w_\sigma[\mathcal{F}'', I, u, p, \|\cdot, \dots, \cdot\|]_\theta(\Delta_v^m) \subset w_\sigma[\mathcal{F}' + \mathcal{F}'', I, u, p, \|\cdot, \dots, \cdot\|]_\theta(\Delta_v^m)$.

Proof. (i) Let $x = (x_k) \in w_\sigma[\mathcal{F}', I, u, p, \|\cdot, \dots, \cdot\|]_\theta(\Delta_v^m)$, then for every $\varepsilon > 0$ choose $\delta \in (0, 1)$ such that $f_k(t) < \varepsilon$, for all $0 < t < \delta$, we have

$$(2.2) \quad A_\delta = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} u_k \left[f'_k \left(\left\| \frac{t_{k_n}(\Delta_v^m x_k - l)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \geq \delta \right\} \in I,$$

uniformly in n . On the other hand, we have

$$\begin{aligned} & \frac{1}{h_r} \sum_{k \in I_r} u_k \left[(f_k \circ f'_k) \left(\left\| \frac{t_{k_n}(\Delta_v^m x_k - l)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \\ & = \frac{1}{h_r} \sum_{k \in I_r \& u_k \left[f'_k \left(\left\| \frac{t_{k_n}(\Delta_v^m x_k - l)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} < \delta} \cdot u_k \left[(f_k \circ f'_k) \left(\left\| \frac{t_{k_n}(\Delta_v^m x_k - l)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \\ & + \frac{1}{h_r} \sum_{k \in I_r \& u_k \left[f'_k \left(\left\| \frac{t_{k_n}(\Delta_v^m x_k - l)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \geq \delta} \cdot u_k \left[(f_k \circ f'_k) \left(\left\| \frac{t_{k_n}(\Delta_v^m x_k - l)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \\ & \leq (\varepsilon)^H + \max(1, (2 \cdot \frac{f_k(1)}{\delta})^H) \frac{1}{h_r} \sum_{k \in I_r} u_k \left[f'_k \left(\left\| \frac{t_{k_n}(\Delta_v^m x_k - l)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \end{aligned}$$

uniformly in n by lemma (1.6). Then for any $\eta > 0$,

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} u_k \left[(f_k \circ f'_k) \left(\left\| \frac{t_{k_n}(\Delta_v^m x_k - l)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \geq \eta \right\} \\ \subseteq \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} u_k \left[f'_k \left(\left\| \frac{t_{k_n}(\Delta_v^m x_k - l)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \geq \frac{\eta - \varepsilon}{K} \right\},$$

where $K = \max(1, (2 \cdot \frac{f_k(1)}{\delta})H)$.

By using (2.2), we obtain $x \in w_\sigma[\mathcal{F} \circ \mathcal{F}', I, u, p, \|\cdot, \dots, \cdot\|]_\theta(\Delta_v^m)$.

(ii) This part of the theorem proved by using the following inequality

$$\frac{1}{h_r} \sum_{k \in I_r} u_k \left[(f'_k + f''_k) \left(\left\| \frac{t_{k_n}(\Delta_v^m x_k - l)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \\ \leq \frac{D}{h_r} \sum_{k \in I_r} u_k \left[f'_k \left(\left\| \frac{t_{k_n}(\Delta_v^m x_k - l)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \\ + \frac{D}{h_r} \sum_{k \in I_r} u_k \left[f''_k \left(\left\| \frac{t_{k_n}(\Delta_v^m x_k - l)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k},$$

where $\sup_k p_k = H$ and $D = \max(1, 2^{H-1})$. □

Theorem 2.5. Let $\mathcal{F} = (f_k)$ be a sequence of modulus functions and $p = (p_k)$ be a bounded sequence of strictly positive real numbers, then

$$w_\sigma[I, u, p, \|\cdot, \dots, \cdot\|]_\theta(\Delta_v^m) \subseteq w_\sigma[\mathcal{F}, I, u, p, \|\cdot, \dots, \cdot\|]_\theta(\Delta_v^m).$$

Proof. This can be proved by using the techniques similar to those used in Theorem 2.4 (i). □

Theorem 2.6. Let $\mathcal{F} = (f_k)$ be a sequence of modulus functions and $p = (p_k)$ be a bounded sequence of strictly positive real numbers, if $\limsup_{t \rightarrow \infty} \frac{f_k(t)}{t} = Q > 0$, then

$$w_\sigma[\mathcal{F}, I, u, p, \|\cdot, \dots, \cdot\|]_\theta(\Delta_v^m) \subseteq w_\sigma[u, p, \|\cdot, \dots, \cdot\|]_\theta(\Delta_v^m).$$

Proof. Suppose $x = (x_k) \in w_\sigma[\mathcal{F}, I, u, p, \|\cdot, \dots, \cdot\|]_\theta(\Delta_v^m)$ and $\limsup_{t \rightarrow \infty} \frac{f_k(t)}{t} = Q > 0$, then there exists a constant $R > 0$ such that $f_k(t) \geq Rt$, for all $t \geq 0$. Thus we have

$$\frac{1}{h_r} \sum_{k \in I_r} u_k \left[f_k \left(\left\| \frac{t_{k_n}(\Delta_v^m x_k - l)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \\ \geq (R)^H \frac{1}{h_r} \sum_{k \in I_r} u_k \left[\left(\left\| \frac{t_{k_n}(\Delta_v^m x_k - l)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k},$$

uniformly in n and for each $z_1, \dots, z_{n-1} \in X$. Which gives the result. □

Theorem 2.7. *If $0 < p_k \leq q_k$ and $(\frac{q_k}{p_k})$ be bounded, then*

$$w_\sigma[\mathcal{F}, I, u, q, \|\cdot, \dots, \cdot\|]_\theta(\Delta_v^m) \subset w_\sigma[\mathcal{F}, I, u, p, \|\cdot, \dots, \cdot\|]_\theta(\Delta_v^m)$$

Proof. The proof is easy so omitted. $\square$

3. Statistical convergence

The concept of convergence of a sequence of real numbers had been extended to statistical convergence by Fast [11] and later studied by many authors. We refer to the recent work in [1, 3, 4, 8, 28, 29, 30] for some applications of statistical summability to approximation theorems. Here, we define the notion of $S_{\theta_\sigma}^{\Delta_v^m}[I, u]$ -convergence with the help of an ideal and invariant means. We also made an effort to establish a strong connection between the spaces $S_{\theta_\sigma}^{\Delta_v^m}[I, u, \|\cdot, \dots, \cdot\|]$ and $w_\sigma[\mathcal{F}, I, u, p, \|\cdot, \dots, \cdot\|]_\theta(\Delta_v^m)$.

Definition 3.1. Let $I \subseteq P(\mathbb{N})$ be a non-trivial ideal. A sequence $x = (x_k) \in X$ is said to be $S_{\theta_\sigma}^{\Delta_v^m}[I, u]$ -convergent to a number l provided that for every $\varepsilon > 0$, $\delta > 0$ and $z_1, \dots, z_{n-1} \in X$, the set

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \left| \left\{ k \in I_r : u_k \left\| \frac{t_{k_n}(\Delta_v^m x_k - l)}{\rho}, z_1, \dots, z_{n-1} \right\| \geq \varepsilon \right\} \right| \geq \delta \right\} \in I.$$

uniformly in n . In this case, we write $S_{\theta_\sigma}^{\Delta_v^m}[I, u] - \lim_{k \rightarrow \infty} x_k = l$.

Let $S_{\theta_\sigma}^{\Delta_v^m}[I, u, \|\cdot, \dots, \cdot\|]$, denotes the set of all $S_{\theta_\sigma}^{\Delta_v^m}[I, u]$ -convergent sequences in X .

Theorem 3.2. *Let $\mathcal{F} = (f_k)$ be a sequence of modulus functions and $0 < \inf_k p_k = h \leq p_k \leq \sup_k p_k = H < \infty$. Then $w_\sigma[\mathcal{F}, I, u, p, \|\cdot, \dots, \cdot\|]_\theta(\Delta_v^m) \subset S_{\theta_\sigma}^{\Delta_v^m}[I, u, \|\cdot, \dots, \cdot\|]$.*

Proof. Suppose $x = (x_k) \in w_\sigma[\mathcal{F}, I, u, p, \|\cdot, \dots, \cdot\|]_\theta(\Delta_v^m)$ and $\varepsilon > 0$ be given. Then for each $z_1, \dots, z_{n-1} \in X$, we obtain

$$\begin{aligned} & \frac{1}{h_r} \sum_{k \in I_r} u_k \left[f_k \left(\left\| \frac{t_{k_n}(\Delta_v^m x_k - l)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \\ &= \frac{1}{h_r} \sum_{k \in I_r \& f_k \left(\left\| \frac{t_{k_n}(\Delta_v^m x_k - l)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \geq \varepsilon} u_k \left[f_k \left(\left\| \frac{t_{k_n}(\Delta_v^m x_k - l)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \\ &+ \frac{1}{h_r} \sum_{k \in I_r \& f_k \left(\left\| \frac{t_{k_n}(\Delta_v^m x_k - l)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) < \varepsilon} u_k \left[f_k \left(\left\| \frac{t_{k_n}(\Delta_v^m x_k - l)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \\ &\geq \frac{1}{h_r} \sum_{k \in I_r \& f_k \left(\left\| \frac{t_{k_n}(\Delta_v^m x_k - l)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \geq \varepsilon} u_k \left[f_k \left(\left\| \frac{t_{k_n}(\Delta_v^m x_k - l)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \end{aligned}$$

$$\begin{aligned}
&\geq \frac{1}{h_r} \sum_{k \in I_r} [f_k(\varepsilon)]^{p_k} \geq \sum_{k \in I_r} \min([f_k(\varepsilon)]^h, [f_k(\varepsilon)]^H) \\
&\geq K \frac{1}{h_r} \left| \left\{ k \in I_r : \left\| \frac{t_{k_n}(\Delta_v^m x_k - l)}{\rho}, z_1, \dots, z_{n-1} \right\| \geq \varepsilon \right\} \right|,
\end{aligned}$$

where $K = \min([f_k(\varepsilon)]^h, [f_k(\varepsilon)]^H)$. Then for every $\delta > 0$ and $z_1, \dots, z_{n-1} \in X$, we have

$$\begin{aligned}
&\left\{ r \in \mathbb{N} : \frac{1}{h_r} \left| \left\{ k \in I_r : u_k \left\| \frac{t_{k_n}(\Delta_v^m x_k - l)}{\rho}, z_1, \dots, z_{n-1} \right\| \geq \varepsilon \right\} \right| \geq \delta \right\} \\
&\subseteq \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} u_k \left[f_k \left(\left\| \frac{t_{k_n}(\Delta_v^m x_k - l)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \geq K\delta \right\}.
\end{aligned}$$

Since $x_k \in w_\sigma[\mathcal{F}, I, u, p, \|\cdot, \dots, \cdot\|]_\theta(\Delta_v^m)$ so that $x \in S_{\theta_\sigma}^{\Delta_v^m}[I, u, \|\cdot, \dots, \cdot\|]$. $\square$

Theorem 3.3. Let $\mathcal{F} = (f_k)$ be a sequence of modulus functions and $p = (p_k)$ be a bounded sequence of strictly positive real numbers. If $0 < \inf_k p_k = h \leq p_k \leq \sup_k p_k = H < \infty$ then $S_{\theta_\sigma}^{\Delta_v^m}[I, u, \|\cdot, \dots, \cdot\|] \subset w_\sigma[\mathcal{F}, I, u, p, \|\cdot, \dots, \cdot\|]_\theta(\Delta_v^m)$.

Proof. Using the same technique of ([10], Theorem 3.5), it is easy to prove. $\square$

Theorem 3.4. Let $\mathcal{F} = (f_k)$ be a bounded sequence of modulus functions and $p = (p_k)$ be a bounded sequence of strictly positive real numbers. If $0 < \inf_k p_k = h \leq p_k \leq \sup_k p_k = H < \infty$ then

$$S_{\theta_\sigma}^{\Delta_v^m}[I, u, \|\cdot, \dots, \cdot\|] = w_\sigma[\mathcal{F}, I, u, p, \|\cdot, \dots, \cdot\|]_\theta(\Delta_v^m)$$

if and only if $\mathcal{F} = (f_k)$ is bounded.

Proof. Direct part can be obtained by combining Theorems 3.2. and 3.3. Conversely suppose $\mathcal{F} = (f_k)$ be unbounded defined by $f_k(k) = k$, for all $k \in \mathbb{N}$. Let $\theta = (2^r)$ be a lacunary sequence. We take a fixed set $B \in I$, where I be an admissible ideal and define $x = (x_k)$ as follows

$$x_k = \begin{cases} k^{m+1}, & \text{if } r \notin B, 2^{r-1} + 1 \leq k \leq 2^{r-1} + [\sqrt{h_r}], \\ k^{m+1}, & \text{if } r \in B, 2^{r-1} < k \leq 2^{r-1} + h_r, \\ 0, & \text{otherwise.} \end{cases}$$

where $I_r = (2^{r-1}, 2^r]$ and $h_r = 2^r - 2^{r-1}$. For given $\varepsilon > 0$ and for each $z_1, \dots, z_{n-1} \in X$ we have,

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} \left| \left\{ k \in I_r : u_k \left\| \frac{t_{k_n}(\Delta_v^m x_k - 0)}{\rho}, z_1, \dots, z_{n-1} \right\| \geq \varepsilon \right\} \right| < \frac{[\sqrt{h_r}]}{h_r} \rightarrow 0,$$

for all $r \notin B$, uniformly in n . Hence for $\delta > 0$, there exists a positive integer r_0 such that $\frac{1}{h_r} |\{k \in I_r : u_k \| \frac{t_{kn}(\Delta_v^m x_k - 0)}{\rho}, z_1, \dots, z_{n-1} \| \geq \varepsilon\}| < \delta$ for $r \notin B$ and $r \geq r_0$. Now we have

$$\begin{aligned} & \{r \in \mathbb{N} : \frac{1}{h_r} |\{k \in I_r : u_k \| \frac{t_{kn}(\Delta_v^m x_k - 0)}{\rho}, z_1, \dots, z_{n-1} \| \geq \varepsilon\}| \geq \delta\} \\ & \subset \{B \cup (1, 2, \dots, r_0 - 1)\}. \end{aligned}$$

Since I be an admissible ideal.

It follows that $S_{\theta_\sigma}^{\Delta_v^m}[I, u] - \lim_{k \rightarrow \infty} \| \frac{x_k}{\rho}, z_1, \dots, z_{n-1} \| \rightarrow 0$ for each $z_1, \dots, z_{n-1} \in X$. On the other hand, if we take $p = (p_k) = 1$, for all $k \in \mathbb{N}$ then $x_k \notin w_\sigma[\mathcal{F}, I, u, p, \|\cdot, \dots, \cdot\|]_\theta(\Delta_v^m)$. This contradicts the fact $S_{\theta_\sigma}^{\Delta_v^m}[I, u, \|\cdot, \dots, \cdot\|] = w_\sigma[\mathcal{F}, I, u, p, \|\cdot, \dots, \cdot\|]_\theta(\Delta_v^m)$, so our supposition is wrong. $\square$

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PARAMETER ESTIMATION FOR A CLASS OF DIFFUSION PROCESS FROM DISCRETE OBSERVATION

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Abstract. This paper is concerned with the parameter estimation problem for a class of diffusion process with drift coefficient $\alpha X_t^{2\gamma-1}$ and diffusion coefficient σX_t^γ from discrete observation. Euler-Maruyama scheme and iterative method are used to get the joint conditional probability density function. The maximum likelihood approach is applied for obtaining the parameter estimators and the explicit expressions of the error of estimation are given. The strong consistency of the estimators and asymptotic normality of the error of estimation are proved by using the law of large numbers for martingales, the strong law of large numbers and central-limit theorem. Hypothesis testing is made to verify the effectiveness of the estimation method used in this paper.

Keywords: Diffusion process, discrete observation, parameter estimation, strong consistency, asymptotic normality.

1. Introduction

Each field is more or less influenced by the random factors and diffusion process is an important tool to study the random phenomenon. Moreover, diffusion processes defined by stochastic differential equation are widely used for model building in astronomy, engineering, medical science and physical ([3, 13]). A recent application is in the area of financial economics ([5, 21]). The Black-Scholes option pricing model described by a geometric Brownian motion ([6]) and the Vasicek and Cox-Ingersoll-Ross models developed based on two specific mean-reverting diffusion processes ([22, 8, 9]) are widely used models in economic cases. However, in engineering practice, due to the interference of random factors, part or all of parameters in diffusion process are always unknown. Parameters are needed to be estimated for the purpose of obtaining proper structures. Therefore, statistical inference for diffusion processes is of great importance from the theoretical as well as from an application point of view in model building.

In the past few decades, some methods have been used to estimate the parameters for diffusion process from continuous-time observation. For example, Kutoyants[16] used Bayes method to study the parameter estimation problem for diffusion process defined by nonlinear stochastic differential equation.

Yoshida[24] applied M-estimation to discuss the consistency of the parameter estimators. Khasminskii[14] considered the effectiveness of the estimators by using likelihood ratio function. Barczy[2] analyzed the consistency of the estimator by applying maximum likelihood estimation. Wei[23] used maximum likelihood approach to study the strong consistency of the estimator and asymptotic normality of the error of estimation. However, in fact, it is impossible to observe a process continuously in time. Therefore, parametric inference based on sampled data is important in dealing with practical problems. In earlier literatures, some methods have been applied to research the parameter estimation problem for continuous-time diffusion process from discrete observation. Prakasa Rao[20] used least squares estimation to study the consistency of the estimator. Florens-Zmirou[10] considered the weak convergence of the minimum contrast estimator. Bibby[4] constructed the martingale estimating function with zero mean to estimate the parameter for ergodic diffusion process. Jacod[12] discussed the convergence in probability of the estimator by using the construct function. Kuang[15] studied the Berry-Esseen boundedness of the estimator. Other methods such as generalized method of moments ([11]), Bayesian estimation([18, 19]) and approximation of the transition function([1, 7, 17]) have been used to estimate the parameters for diffusion processes as well.

In this paper, the parameter estimation problem for a class of diffusion process defined by a nonlinear stochastic differential equation is studied from discrete observation. Although parameter estimation for diffusion process has been investigated by many authors from discrete observation, the asymptotic normality of the estimator for the parameter in diffusion item and the hypothesis testing have not been discussed in earlier literatures. In our work, Euler-Maruyama scheme is used to discrete the process and the joint conditional probability density function is given. The explicit expression of the parameter estimators and the error of estimation are obtained. The strong consistency of the estimators and asymptotic normality of the error of estimation are proved by using the law of large numbers for martingales, the strong law of large numbers and central-limit theorem. Hypothesis testing is made to verify the effectiveness of the estimation method.

This paper is organized as follows. In Section 2, the joint conditional probability density function and the explicit expression of the parameter estimators are provided. In Section 3, the strong consistency of the estimators and asymptotic normality of the error of estimation are proved. In Section 4, hypothesis testing is made to verify the effectiveness of the estimators. Conclusion is given in Section 5.

2. Problem formulation and preliminaries

In this paper, we study the parameter estimation problem for a class of diffusion process described by the following nonlinear stochastic differential equation:

$$(1) \quad \begin{cases} dX_t = \alpha X_t^{2\gamma-1} dt + \sigma X_t^\gamma dB_t \\ X_0 = x_0. \end{cases}$$

where B_t is a standard Wiener process, α and σ are two unknown parameters, γ is a constant and $\gamma \in (1, \frac{3}{2}]$.

When $\gamma = 1$, (1) is a popular economic model called Black-Scholes Model. As it is a linear model, we do not consider it here. Due to the complexity of the transitional density function, it is difficult to obtain the commonly used expression for the unknown parameters. Therefore, numerical method should be used to obtain the approximate likelihood function.

From now on we shall work under the assumptions below.

Assumption 1. $\alpha < 0$, $\sigma > 0$. x_0 is positive and independent with B_t .

Assumption 2. $\sup_t \mathbb{E}|X_t| < \infty$, $\sup_t \mathbb{E} \frac{1}{|X_t|} < \infty$.

Now the specific steps for obtaining the approximate likelihood function and the estimators is given below.

Let $Y_t = X_t^{1-\gamma}$, then equation (1) is changed to an equivalent equation, which is:

$$(2) \quad dY_t = (1-\gamma)(\alpha - \frac{1}{2}\gamma\sigma^2)\frac{1}{Y_t}dt + (1-\gamma)\sigma dB_t.$$

It is assumed that the process is observed at times $\{t_0, t_1, \dots, t_n\}$ where $t_i = i\Delta, i = 1, 2, \dots, n$ and $\Delta > 0$. Discretizing equation (1), it follows that

$$(3) \quad Y_{t_i} - Y_{t_{i-1}} = (1-\gamma)(\alpha - \frac{1}{2}\gamma\sigma^2)\frac{1}{Y_{t_{i-1}}}\Delta + (1-\gamma)\sigma\sqrt{\Delta}\varepsilon_{t_i},$$

where $t_i = i\Delta$, ε_{t_i} is a i.i.d. $N(0,1)$ sequence and for every i , ε_{t_i} is independent with $\{Y_{t_j}, j < i\}$.

Let $\mathcal{F}_{i-1} = \sigma(Y_{t_j}, j \leq i-1)$. For the given $\mathcal{F}_{i-1}$, the conditional probability density function of Y_{t_i} is:

$$(4) \quad f(Y_{t_i}|\mathcal{F}_{i-1}) = \frac{1}{\sqrt{2\pi\Delta}(1-\gamma)\sigma} \cdot \exp\left\{-\frac{(Y_{t_i} - Y_{t_{i-1}} - (1-\gamma)(\alpha - \frac{1}{2}\gamma\sigma^2)\frac{1}{Y_{t_{i-1}}}\Delta)^2}{2(1-\gamma)^2\sigma^2\Delta}\right\}.$$

Thus, for the given $\mathcal{F}_0$, the joint conditional probability density function of $\{Y_{t_1}, Y_{t_2}, \dots, Y_{t_n}\}$ is:

$$(5) \quad f(Y_{t_1}, Y_{t_2}, \dots, Y_{t_n}|\mathcal{F}_0) = \left(\frac{1}{\sqrt{2\pi\Delta}(1-\gamma)\sigma}\right)^n \cdot \prod_{i=1}^n \exp\left\{-\frac{(Y_{t_i} - Y_{t_{i-1}} - (1-\gamma)(\alpha - \frac{1}{2}\gamma\sigma^2)\frac{1}{Y_{t_{i-1}}}\Delta)^2}{2(1-\gamma)^2\sigma^2\Delta}\right\}.$$

Therefore, the likelihood function is given as follows:

$$(6) \quad L_n(\alpha, \sigma^2) = -\frac{n}{2} \ln \sigma^2 - \frac{1}{2(1-\gamma)^2 \sigma^2 \Delta} \cdot \sum_{i=1}^n (Y_{t_i} - Y_{t_{i-1}} - (1-\gamma)(\alpha - \frac{1}{2}\gamma\sigma^2) \frac{1}{Y_{t_{i-1}}} \Delta)^2.$$

Solving the equation set:

$$(7) \quad \begin{cases} \frac{\partial L_n(\alpha, \sigma^2)}{\partial \alpha} = 0 \\ \frac{\partial L_n(\alpha, \sigma^2)}{\partial \sigma^2} = 0, \end{cases}$$

we obtain the estimators:

$$(8) \quad \begin{cases} \widehat{\sigma^2} = \frac{\sum_{i=1}^n (Y_{t_i} - Y_{t_{i-1}})^2 \sum_{i=1}^n \frac{1}{Y_{t_{i-1}}^2} - (\sum_{i=1}^n \frac{Y_{t_i} - Y_{t_{i-1}}}{Y_{t_{i-1}}})^2}{n\Delta(1-\gamma)^2 \sum_{i=1}^n \frac{1}{Y_{t_{i-1}}^2}} \\ \widehat{\alpha} = \frac{\gamma}{2} \widehat{\sigma^2} + \frac{\sum_{i=1}^n \frac{Y_{t_i} - Y_{t_{i-1}}}{Y_{t_{i-1}}}}{\Delta(1-\gamma) \sum_{i=1}^n \frac{1}{Y_{t_{i-1}}^2}}. \end{cases}$$

3. Main results and proofs

In the following theorem, the strong consistency of two parameter estimators are proved by using the law of large numbers for martingales and the strong law of large numbers.

Theorem 1. *Under the Assumptions (1) and (2), $\widehat{\sigma^2}$ and $\widehat{\alpha}$ are strongly consistent.*

Proof. From (3), one has

$$(9) \quad \begin{aligned} & \sum_{i=1}^n (Y_{t_i} - Y_{t_{i-1}})^2 \sum_{i=1}^n \frac{1}{Y_{t_{i-1}}^2} \\ &= (1-\gamma)^2 (\alpha - \frac{1}{2}\gamma\sigma^2)^2 \Delta^2 (\sum_{i=1}^n \frac{1}{Y_{t_{i-1}}^2})^2 + (1-\gamma)^2 \sigma^2 \Delta \sum_{i=1}^n \frac{1}{Y_{t_{i-1}}^2} \sum_{i=1}^n \varepsilon_{t_i}^2 \\ &+ 2(1-\gamma)^2 \sigma (\alpha - \frac{1}{2}\gamma\sigma^2) \Delta^{\frac{3}{2}} \sum_{i=1}^n \frac{\varepsilon_{t_i}}{Y_{t_{i-1}}} \sum_{i=1}^n \frac{1}{Y_{t_{i-1}}^2}, \end{aligned}$$

and

$$\begin{aligned}
 & \left(\sum_{i=1}^n \frac{Y_{t_i} - Y_{t_{i-1}}}{Y_{t_{i-1}}} \right)^2 \\
 (10) \quad &= (1-\gamma)^2 \left(\alpha - \frac{1}{2} \gamma \sigma^2 \right)^2 \Delta^2 \left(\sum_{i=1}^n \frac{1}{Y_{t_{i-1}}^2} \right)^2 + (1-\gamma)^2 \sigma^2 \Delta \left(\sum_{i=1}^n \frac{\varepsilon_{t_i}}{Y_{t_{i-1}}} \right)^2 \\
 &+ 2(1-\gamma)^2 \sigma \left(\alpha - \frac{1}{2} \gamma \sigma^2 \right) \Delta^{\frac{3}{2}} \sum_{i=1}^n \frac{\varepsilon_{t_i}}{Y_{t_{i-1}}} \sum_{i=1}^n \frac{1}{Y_{t_{i-1}}^2}.
 \end{aligned}$$

Substituting (9) and (10) into the expression of $\widehat{\sigma^2}$, it is checked that

$$(11) \quad \widehat{\sigma^2} = \frac{\sigma^2 \sum_{i=1}^n \frac{1}{Y_{t_{i-1}}^2} \sum_{i=1}^n \varepsilon_{t_i}^2 - \sigma^2 \left(\sum_{i=1}^n \frac{\varepsilon_{t_i}}{Y_{t_{i-1}}} \right)^2}{n \sum_{i=1}^n \frac{1}{Y_{t_{i-1}}^2}}.$$

Thus, the error of estimation is

$$(12) \quad \widehat{\sigma^2} - \sigma^2 = \sigma^2 \left(\frac{1}{n} \sum_{i=1}^n \varepsilon_{t_i}^2 - 1 \right) - \frac{\sigma^2 \left(\frac{1}{n} \sum_{i=1}^n \frac{\varepsilon_{t_i}}{Y_{t_{i-1}}} \right)^2}{\frac{1}{n} \sum_{i=1}^n \frac{1}{Y_{t_{i-1}}^2}}.$$

Since ε_{t_i} is a i.i.d. $N(0,1)$ sequence, $\varepsilon_{t_i}^2$ is also a i.i.d. sequence and $\mathbb{E}[\varepsilon_{t_i}^2] = 1$. According to the strong law of large numbers, one has

$$(13) \quad \frac{1}{n} \sum_{i=1}^n \varepsilon_{t_i}^2 - 1 \xrightarrow{a.s.} 0.$$

Now we will prove that $\sum_{i=1}^n \frac{\varepsilon_{t_i}}{Y_{t_{i-1}}}$ is a martingale with zero mean with respect to the σ -algebra $\mathcal{F}_{n-1} = \sigma(\frac{1}{Y_{t_j}}, \varepsilon_{t_j}; 0 \leq j \leq n-1)$.

Since

$$\mathbb{E} \left[\sum_{i=1}^n \frac{\varepsilon_{t_i}}{Y_{t_{i-1}}} / \mathcal{F}_{n-1} \right] = \sum_{i=1}^{n-1} \frac{\varepsilon_{t_i}}{Y_{t_{i-1}}} + \mathbb{E} \left[\frac{\varepsilon_{t_n}}{Y_{t_{n-1}}} / \mathcal{F}_{n-1} \right] = \sum_{i=1}^{n-1} \frac{\varepsilon_{t_i}}{Y_{t_{i-1}}},$$

and

$$\mathbb{E} \left[\sum_{i=1}^n \frac{\varepsilon_{t_i}}{Y_{t_{i-1}}} \right] = \sum_{i=1}^n \mathbb{E} \left[\frac{\varepsilon_{t_i}}{Y_{t_{i-1}}} \right] = 0,$$

it follows that $\sum_{i=1}^n \frac{\varepsilon_{t_i}}{Y_{t_{i-1}}}$ is a martingale with zero mean with respect to the σ -algebra $\mathcal{F}_{n-1} = \sigma(\frac{1}{Y_{t_j}}, \varepsilon_{t_j}; 0 \leq j \leq n-1)$. As $\mathbb{E}[(\frac{\varepsilon_{t_i}}{Y_{t_{i-1}}})^2] = \mathbb{E}[\frac{1}{Y_{t_{i-1}}^2}]$ is bounded, from the law of large numbers for martingales, we obtain that

$$(14) \quad \frac{1}{n} \sum_{i=1}^n \frac{\varepsilon_{t_i}}{Y_{t_{i-1}}} \xrightarrow{a.s.} 0, (n \rightarrow \infty).$$

Thus, it can be checked that

$$(15) \quad \left(\frac{1}{n} \sum_{i=1}^n \frac{\varepsilon_{t_i}}{Y_{t_{i-1}}}\right)^2 \xrightarrow{a.s.} 0, (n \rightarrow \infty).$$

Let

$$(16) \quad Y_M = \sup_{0 \leq t_{i-1} < \infty} \{Y_{t_{i-1}}\}, \quad Y_N = \inf_{0 \leq t_{i-1} < \infty} \{Y_{t_{i-1}}\},$$

then we obtain that

$$(17) \quad \frac{1}{\frac{1}{n} \sum_{i=1}^n \frac{1}{Y_{t_{i-1}}^2}} \leq Y_M^2.$$

From above results, it is checked that

$$(18) \quad \widehat{\sigma^2} - \sigma^2 \xrightarrow{a.s.} 0, (n \rightarrow \infty).$$

Next we will prove that $\widehat{\alpha} - \alpha \xrightarrow{a.s.} 0, (n \rightarrow \infty)$.

From (3), one has

$$(19) \quad \sum_{i=1}^n \frac{Y_{t_i} - Y_{t_{i-1}}}{Y_{t_{i-1}}} = (1 - \gamma)(\alpha - \frac{1}{2}\gamma\sigma^2)\Delta \sum_{i=1}^n \frac{1}{Y_{t_{i-1}}^2} + (1 - \gamma)\sigma\sqrt{\Delta} \sum_{i=1}^n \frac{\varepsilon_{t_i}}{Y_{t_{i-1}}}.$$

Substituting (19) into the expression of $\widehat{\alpha}$, we obtain that

$$\begin{aligned} \widehat{\alpha} &= \frac{\gamma\widehat{\sigma^2}}{2} + \frac{(1 - \gamma)(\alpha - \frac{1}{2}\gamma\sigma^2)\Delta \sum_{i=1}^n \frac{1}{Y_{t_{i-1}}^2} + (1 - \gamma)\sigma\sqrt{\Delta} \sum_{i=1}^n \frac{\varepsilon_{t_i}}{Y_{t_{i-1}}}}{(1 - \gamma)\Delta \sum_{i=1}^n \frac{1}{Y_{t_{i-1}}^2}} \\ &= \frac{\gamma\widehat{\sigma^2}}{2} + \alpha - \frac{\gamma\sigma^2}{2} + \frac{\sigma \sum_{i=1}^n \frac{\varepsilon_{t_i}}{Y_{t_{i-1}}}}{\sqrt{\Delta} \sum_{i=1}^n \frac{1}{Y_{t_{i-1}}^2}}. \end{aligned}$$

Therefore, it can be checked that

$$(20) \quad \widehat{\alpha} - \alpha = \frac{\gamma}{2}(\widehat{\sigma^2} - \sigma^2) + \frac{\sigma \frac{1}{n} \sum_{i=1}^n \frac{\varepsilon_{t_i}}{Y_{t_{i-1}}}}{\sqrt{\Delta} \frac{1}{n} \sum_{i=1}^n \frac{1}{Y_{t_{i-1}}^2}}.$$

Since $\frac{1}{n} \sum_{i=1}^n \frac{\varepsilon_{t_i}}{Y_{t_{i-1}}} \xrightarrow{a.s.} 0$ and $\frac{1}{\frac{1}{n} \sum_{i=1}^n \frac{1}{Y_{t_{i-1}}^2}} \leq Y_M^2$, one has

$$(21) \quad \widehat{\alpha} - \alpha \xrightarrow{a.s.} 0, (n \rightarrow \infty).$$

Thus, $\widehat{\sigma^2}$ and $\widehat{\alpha}$ are strongly consistent.

The proof is complete. □

In the following theorem, the asymptotic normality of the error of estimation is proved by applying the law of large numbers for martingales, the strong law of large numbers and central-limit theorem.

Theorem 2. *Under the Assumptions (1) and (2), $\sqrt{n}(\widehat{\sigma^2} - \sigma^2) \xrightarrow{d} N(0, 2\sigma^4)$.*

Proof. From the expression of $\widehat{\sigma^2} - \sigma^2$, one has

$$(22) \quad \sqrt{n}(\widehat{\sigma^2} - \sigma^2) = \sigma^2 \sqrt{n} \left(\frac{1}{n} \sum_{i=1}^n \varepsilon_{t_i}^2 - 1 \right) - \frac{\sigma^2 \sqrt{n} \frac{1}{n^2} \left(\sum_{i=1}^n \frac{\varepsilon_{t_i}}{Y_{t_{i-1}}} \right)^2}{\frac{1}{n} \sum_{i=1}^n \frac{1}{Y_{t_{i-1}}^2}}.$$

Since $\mathbb{E}[\frac{1}{n} \sum_{i=1}^n (\varepsilon_{t_i}^2 - 1)] = 0$ and $\text{var}[\frac{1}{n} \sum_{i=1}^n (\varepsilon_{t_i}^2 - 1)] = \frac{2}{n}$, from the central-limit theorem, it is checked that

$$(23) \quad \sigma^2 \sqrt{n} \left(\frac{1}{n} \sum_{i=1}^n \varepsilon_{t_i}^2 - 1 \right) \xrightarrow{d} N(0, 2\sigma^4).$$

Since

$$\mathbb{E} \left[\sqrt{n} \frac{1}{n^2} \left(\sum_{i=1}^n \frac{\varepsilon_{t_i}}{Y_{t_{i-1}}} \right)^2 \right] = \mathbb{E} \left[\sqrt{n} \frac{1}{n^2} \sum_{i=1}^n \frac{\varepsilon_{t_i}^2}{Y_{t_{i-1}}^2} + \sqrt{n} \frac{1}{n^2} \sum_{i \neq j} \frac{\varepsilon_{t_i} \varepsilon_{t_j}}{Y_{t_{i-1}} Y_{t_{j-1}}} \right] \rightarrow 0.$$

According to Chebyshev inequality, one has

$$(24) \quad \sqrt{n} \frac{1}{n^2} \left(\sum_{i=1}^n \frac{\varepsilon_{t_i}}{Y_{t_{i-1}}} \right)^2 \xrightarrow{P} 0.$$

Thus, we obtain that

$$(25) \quad \frac{\sqrt{n} \frac{1}{n^2} \left(\sum_{i=1}^n \frac{\varepsilon_{t_i}}{Y_{t_{i-1}}} \right)^2}{\frac{1}{n} \sum_{i=1}^n \frac{1}{Y_{t_{i-1}}^2}} \xrightarrow{P} 0.$$

From above results, one has

$$(26) \quad \sqrt{n}(\widehat{\sigma^2} - \sigma^2) \xrightarrow{d} N(0, 2\sigma^4).$$

The proof is complete. $\square$

4. Hypothesis testing

In this section, we will introduce an example in the case of $\gamma = \frac{3}{2}$ in (1), which is described by the following stochastic differential equation:

$$(27) \quad \begin{cases} dX_t = \alpha X_t^2 dt + \sigma X_t^{\frac{3}{2}} dB_t \\ X_0 = x_0. \end{cases}$$

This is a economic model called Constantinides-Ingersoll model. We make the hypothesis testing for this example below.

We consider the testing problem as follows:

$$H_0 : \sigma^2 = \sigma_0^2, \quad H_1 : \sigma^2 \neq \sigma_0^2,$$

and the limit distribution of the test statistic based on this estimation.

Define:

$$(28) \quad Z_n = \frac{\sqrt{n}}{\sqrt{2}\sigma^2}(\widehat{\sigma^2} - \sigma^2),$$

then

$$(29) \quad Z_n \xrightarrow{d} N(0, 1).$$

As the significance level is 0.05, if $|Z_n| \geq Z_{0.975}$, we refuse the original hypothesis.

When the confidence level is 0.95, the confidence interval of σ^2 is

$$(30) \quad \left[\frac{\widehat{\sigma^2}}{1 + \frac{\sqrt{2}}{\sqrt{n}}Z_{0.975}}, \frac{\widehat{\sigma^2}}{1 - \frac{\sqrt{2}}{\sqrt{n}}Z_{0.975}} \right].$$

We assume that $\{\varepsilon_{t_i}\} \sim N(0, 1)$. For every given true value of the parameters- α and σ^2 , the size of the sample is represented as “Size n” and given in the first column of the table. In Table 1, $\Delta = 1$, the size of the sample is increasing from 100 to 500. In Table 2, $\Delta = 0.01$, the size is increasing from 10000 to 50000. These two tables list the value of “ $\alpha - MLE$ ”, “ $\sigma^2 - MLE$ ” and the absolute error of MLE(Maximum Likelihood Estimator). Table 1 and Table 2 illustrate that the absolute error of α and σ^2 depend on the size of given value of α and σ^2 . But under the hypothesis of normal distribution, there is no obvious difference between estimators and true value, estimators- $\widehat{\alpha}$, $\widehat{\sigma^2}$ are good.

Table 1: MLE simulation results of α and σ^2 $\Delta = 1$

True	Aver		AE		
(σ^2, α)	Size n	$\widehat{\sigma^2}$	$\widehat{\alpha}$	σ^2	α
(0.1,-0.1)	100	0.1116	-0.0928	0.0116	0.0072
	200	0.1059	-0.0975	0.0059	0.0025
	500	0.0998	-0.1010	0.0002	0.0010
(0.3,-0.2)	100	0.3208	-0.1799	0.0208	0.0201
	200	0.3165	-0.1894	0.0165	0.0106
	500	0.2992	-0.2032	0.0008	0.0032
(0.5,-0.4)	100	0.5206	-0.3836	0.0206	0.0164
	200	0.5138	-0.3892	0.0138	0.0108
	500	0.4990	-0.4039	0.0010	0.0039
(0.7,-0.6)	100	0.6832	-0.6186	0.0168	0.0186
	200	0.6884	-0.6108	0.0116	0.0108
	500	0.6993	-0.6027	0.0007	0.0027

Table 2: MLE simulation results of α and σ^2 $\Delta = 0.01$

True	Aver		AE		
(σ^2, α)	Size n	$\widehat{\sigma^2}$	$\widehat{\alpha}$	σ^2	α
(0.1,-0.1)	10000	0.1006	-0.0944	0.0006	0.0056
	20000	0.0996	-0.0932	0.0004	0.0068
	50000	0.0999	-0.0957	0.0001	0.0043
(0.3,-0.2)	10000	0.3010	-0.1973	0.0010	0.0027
	20000	0.2995	-0.1984	0.0005	0.0016
	50000	0.2998	-0.1998	0.0002	0.0002
(0.5,-0.4)	10000	0.5012	-0.3978	0.0012	0.0022
	20000	0.4996	-0.3994	0.0004	0.0006
	50000	0.4998	-0.4010	0.0002	0.0010
(0.7,-0.6)	10000	0.7025	-0.6008	0.0025	0.0008
	20000	0.6986	-0.6026	0.0014	0.0026
	50000	0.6995	-0.6035	0.0005	0.0035

Next we give some simulation results of the confidence interval of σ^2 under 0.95 confidence level. In Table 3, we suppose that $\{\varepsilon_i\} \sim N(0, 1)$. For every given true value of σ^2 , let $\Delta = 0.1$, the size of the sample is increasing from 1000 to 10000. This table lists the value of $\widehat{\sigma^2}$ and in the last column of the table lists the confidence interval of σ^2 . Table 3 illustrates that the length of the confidence interval is becoming small when the size of the sample is increasing.

Table 3: simulation results of confidence interval of σ^2 $\Delta = 0.1$

True	Aver			0.95
(σ^2, α)	Size n	$\widehat{\sigma^2}$	$\widehat{\alpha}$	
(0.3,-0.2)	1000	0.3054	-0.2036	[0.2786,0.3324]
	2000	0.3032	-0.2025	[0.2875,0.3246]
	5000	0.3019	-0.2018	[0.2904,0.3148]
	10000	0.3008	-0.2006	[0.2931,0.3092]
(0.5,-0.4)	1000	0.5062	-0.4053	[0.4651, 0.5548]
	2000	0.5043	-0.4046	[0.4792, 0.5423]
	5000	0.5028	-0.4031	[0.4845,0.5247]
	10000	0.5016	-0.4017	[0.4883,0.5162]
(0.7,-0.6)	1000	0.7068	-0.6061	[0.6513,0.7768]
	2000	0.7051	-0.6052	[0.6704,0.7592]
	5000	0.7036	-0.6033	[0.6784,0.7334]
	10000	0.7019	-0.6010	[0.6832, 0.7221]
(0.9,-0.8)	1000	0.9157	-0.8103	[0.8378,0.9988]
	2000	0.9112	-0.8066	[0.8623,0.9763]
	5000	0.9064	-0.8048	[0.8722,0.9434]
	10000	0.9030	-0.8007	[0.8786,0.9287]

5. Conclusion

The aim of this paper is to estimate the parameters for a class of nonlinear stochastic differential equation. The likelihood function has been given by using Euler method, the explicit expressions of estimators and the error of estimation have been obtained. The strong consistency of the estimators and asymptotic normality of the error of estimation have been proved by using the law of large numbers for martingales, the strong law of large numbers, Chebyshev inequality and central-limit theorem. Further topics will consider the parameter estimation for commonly nonlinear stochastic differential equation and stochastic differential equation driven by Lévy noises.

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MORE PROPERTIES OF AN OPERATION ON SEMI-GENERALIZED OPEN SETS

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Abstract. The paper continues studying properties of an operation on τ_{sg} . The notions of $sg\gamma$ -generalized closed sets and some of its properties are investigated. It also introduces $sg\gamma$ - $T_{\frac{1}{2}}$ space via $sg\gamma$ -generalized closed set and $sg\gamma$ -closed set. Some basic characterization of $sg(\gamma, \beta)$ -irresolute functions with $sg\beta$ -closed graphs have been obtained. It studies the concept of $sg\gamma_0$ -closed space. Finally, it gives some properties of $sg\gamma^*$ -regular and $sg\gamma^*$ -normal spaces by using sg -open and sg -closed sets.

Keywords: $sg\gamma$ -open sets, $sg\gamma g$ -closed sets, $sg\gamma$ - T_i spaces ($i \in \{0, \frac{1}{2}, 1, 2\}$), $sg(\gamma, \beta)$ -irresolute functions, $sg\beta$ -closed graphs, $sg\gamma_0$ -closed space, $sg\gamma^*$ -regular and $sg\gamma^*$ -normal spaces.

1. Introduction

Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real Analysis concerns the various modified forms of continuity, separation axioms, compactness etc. by utilizing general-

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ized open sets. Levine [9] introduced the concept of semi-open sets and semi-continuity in topological spaces. In 1987, Battacharyya and Lahiri [4] used semi-open sets to define the notion of semi-generalized closed sets.

Kasahara [10] introduced the notion of an α operation approaches on a class τ of sets and studied the concept of α -continuous functions with α -closed graphs and α -compact spaces. After this, Jankovic [8] introduced the concept of α -closure of a set in X via α -operation and investigated further characterizations of function with α -closed graph. Later, Ogata [13] defined and studied the concept of γ -open sets, and applied it to investigate operation-functions and operation-separation axioms.

Recently, several researchers developed many concepts of operation γ in a space (X, τ) . Krishnan, Ganster and Balachandran [11] introduced and studied the concept of the operation γ on the class of all semi-open sets of (X, τ) , and defined the notion of semi γ -open sets and investigated some of their properties. An, Cuong and Maki [1] defined and investigated an operation γ on the class of all preopen sets of (X, τ) and introduced the notion of pre- γ -open sets, and developed some of their properties. Tahiliani [14] defined an operation γ on the class of all β -open sets of (X, τ) , and described the notion of β - γ -open sets. Carpintero, Rajesh and Rosas [5] studied the operation γ on the class of all b -open sets of (X, τ) , and defined the notion of b - γ -open sets. Asaad [2] defined the notion of an operation γ on the class of all generalized open sets in (X, τ) and study some of its applications. Asaad and Ahmad [3] introduced the concept of an operation γ on the collection of all semi-generalized open sets (i.e. τ_{sg}) in (X, τ) . By using this operation, they defined the concept of sg - γ -open sets and studied some of their properties. Also, they introduced and investigated sg - γ - T_i spaces for $i \in \{0, 1, 2\}$.

The aim of this study is to introduce the concept of $sg\gamma$ -generalized closed sets by utilizing the operation γ on τ_{sg} and then investigate some of its properties. In addition, sg - γ - $T_{\frac{1}{2}}$ spaces are introduced and investigated. Some basic properties of sg -(γ, β)-irresolute functions with sg - β -closed graphs have been obtained. We study the concept of sg - γ_0 -closed space and some of its properties. Finally, we give some spaces called sg - γ^* -regular and sg - γ^* -normal by using sg -open and sg -closed sets and study some of their properties.

2. Preliminaries

In this study, the spaces (X, τ) and (Y, σ) (or simply X and Y) represent non-empty spaces on which no separation axioms are assumed, unless otherwise mentioned, and they are simply written as X and Y , respectively, when no confusion arises. The closure and the interior of a set S of a space X are denoted by $Cl(S)$ and $Int(S)$, respectively. A subset S of a space X is said to be semi-open [9] if $S \subseteq Cl(Int(S))$. The complement of a semi-open set is said to be semi-closed [6]. We denote by $SO(X)$ the set of all semi-open sets in (X, τ) . The semi-closure of S is defined as the intersection of all semi-closed sets

containing S and it is denoted by $sCl(S)$ [6]. A subset S of a space (X, τ) is said to be semi-generalized closed (in short sg -closed) [4] if $sCl(S) \subseteq U$ whenever $S \subseteq U$ and U is a semi-open set in X . The complement of an sg -closed set of X is sg -open. The family of all sg -open subsets of a space (X, τ) is denoted by τ_{sg} . In general, every semi-closed set of a space X is sg -closed. A space (X, τ) is semi- $T_{\frac{1}{2}}$ [4] if every sg -closed subset of X is semi-closed.

An operation γ on $SO(X)$ on X is a mapping $\gamma: SO(X) \rightarrow P(X)$ such that $U \subseteq \gamma(U)$ for each $U \in SO(X)$, where $P(X)$ is the power set of X and $\gamma(U)$ denotes the value of γ at U . A non-empty subset S of a space (X, τ) with an operation γ on $SO(X)$ is said to be semi γ -open [11] if for each $x \in S$, there exists a semi-open set U containing x such that $\gamma(U) \subseteq S$. The complement of a semi γ -open subset of a space X as semi γ -closed. The family of all semi γ -open sets of a space (X, τ) is denoted by $SO(X)_{\gamma}$. A point $x \in X$ is in the semi γ -closure [11] of a set $S \subseteq X$ if $\gamma(U) \cap S \neq \emptyset$ for each semi-open set U containing x . The set of all semi γ -closure points of S is called semi γ -closure of S and is denoted by $sCl_{\gamma}(S)$. A subset S of (X, τ) with an operation γ on $SO(X)$ is said to be semi γ - g -closed [11] if $sCl_{\gamma}(S) \subseteq U$ whenever $S \subseteq U$ and U is a semi γ -open set in (X, τ) .

Definition 2.1 ([3]). An operation γ on τ_{sg} is a mapping $\gamma: \tau_{sg} \rightarrow P(X)$ such that $U \subseteq \gamma(U)$ for every $U \in \tau_{sg}$. From this, for any operation $\gamma: \tau_{sg} \rightarrow P(X)$, we have $\gamma(X) = X$. A non-empty set S of X is said to be sg - γ -open if for each $x \in S$, there exists an sg -open set U such that $x \in U$ and $\gamma(U) \subseteq S$. The complement of an sg - γ -open set of X is sg - γ -closed. Assume that the empty set $\emptyset$ is also sg - γ -open set for any operation $\gamma: \tau_{sg} \rightarrow P(X)$. The family of all sg - γ -open subsets of a space (X, τ) is denoted by $\tau_{sg\gamma}$.

The union of any collection of sg - γ -open sets in a topological space X is sg - γ -open. While, the intersection of any two sg - γ -open sets in (X, τ) is generally not an sg - γ -open set. The relation between the concept of sg -open set and sg - γ -open set are independent [3].

Definition 2.2. Let S be any subset of a space (X, τ) . Then the class of all sg - γ -open sets in S is defined in a natural way as:

$$\tau_{sg\gamma_S} = \{G \cap S : \text{for all } G \in \tau_{sg\gamma}\}$$

That is H is sg - γ -open in S if and only if $H = G \cap S$, where $G \in \tau_{sg\gamma}$.

Definition 2.3 ([3]). The point $x \in X$ is in the sg -closure $_{\gamma}$ of a set S if $\gamma(U) \cap S \neq \emptyset$ for each sg -open set U containing x . The set of all sg -closure $_{\gamma}$ points of S is called sg -closure $_{\gamma}$ of S and is denoted by $sgCl_{\gamma}(S)$.

Definition 2.4 ([3]). Let S be any subset of a topological space (X, τ) and γ be an operation on τ_{sg} . The sg - γ -closure of S is defined as the intersection of all sg - γ -closed sets of X containing S and it is denoted by $sg_{\gamma}Cl(S)$. That is,

$$sg_{\gamma}Cl(S) = \bigcap \{F : S \subseteq F, X \setminus F \in \tau_{sg\gamma}\}.$$

Theorem 2.5 ([3]). *Let S be any subset of a topological space (X, τ) and γ be an operation on τ_{sg} . Then $x \in sg_\gamma Cl(S)$ if and only if $S \cap U \neq \emptyset$ for every $sg\text{-}\gamma$ -open set U of X containing x .*

Lemma 2.6 ([3]). *The following statements are true for any subsets S and T of a topological space (X, τ) with an operation γ on τ_{sg} .*

1. $sg_\gamma Cl(S)$ is $sg\text{-}\gamma$ -closed set in X and $sgCl_\gamma(S)$ is sg -closed set in X .
2. $S \subseteq sgCl_\gamma(S) \subseteq sg_\gamma Cl(S)$.
3. (a) S is $sg\text{-}\gamma$ -closed if and only if $sg_\gamma Cl(S) = S$ and,
(b) S is $sg\text{-}\gamma$ -closed if and only if $sgCl_\gamma(S) = S$.
4. If $S \subseteq T$, then $sg_\gamma Cl(S) \subseteq sg_\gamma Cl(T)$ and $sgCl_\gamma(S) \subseteq sgCl_\gamma(T)$.
5. (a) $sg_\gamma Cl(S \cap T) \subseteq sg_\gamma Cl(S) \cap sg_\gamma Cl(T)$ and,
(b) $sgCl_\gamma(S \cap T) \subseteq sgCl_\gamma(S) \cap sgCl_\gamma(T)$.
6. (a) $sg_\gamma Cl(S) \cup sg_\gamma Cl(T) \subseteq sg_\gamma Cl(S \cup T)$ and,
(b) $sgCl_\gamma(S) \cup sgCl_\gamma(T) \subseteq sgCl_\gamma(S \cup T)$.
7. $sg_\gamma Cl(sg_\gamma Cl(S)) = sg_\gamma Cl(S)$.

Theorem 2.7 ([3]). *Let S be any subset of a topological space (X, τ) and γ be an operation on τ_{sg} . Then the following statements are equivalent:*

1. S is $sg\text{-}\gamma$ -open set.
2. $sgCl_\gamma(X \setminus S) = X \setminus S$.
3. $sg_\gamma Cl(X \setminus S) = X \setminus S$.
4. $X \setminus S$ is $sg\text{-}\gamma$ -closed set.

Theorem 2.8 ([4]). *A topological space (X, τ) is semi- $T_{\frac{1}{2}}$ if and only if $\tau_{sg} = SO(X)$.*

Lemma 2.9 ([3]). *If the space (X, τ) is semi- $T_{\frac{1}{2}}$, then $\tau_{sg\gamma} = SO(X)_\gamma$.*

3. $sg\gamma$ -generalized closed sets and $sg\text{-}\gamma\text{-}T_{\frac{1}{2}}$ spaces

Definition 3.1. A subset S of a topological space (X, τ) with an operation γ on τ_{sg} is said to be $sg\gamma$ -generalized closed (in short $sg\gamma g$ -closed) if $sgCl_\gamma(S) \subseteq U$ whenever $S \subseteq U$ and U is an $sg\text{-}\gamma$ -open set in X .

Lemma 3.2. *Let (X, τ) be a topological space and γ be an operation on τ_{sg} . A set S in (X, τ) is $sg\gamma g$ -closed if and only if $S \cap sg_\gamma Cl(\{x\}) \neq \emptyset$ for every $x \in sgCl_\gamma(S)$.*

Proof. Suppose S is $sg\gamma g$ -closed set in X and suppose (if possible) that there exists an element $x \in sgCl_\gamma(S)$ such that $S \cap sg_\gamma Cl(\{x\}) = \phi$. This follows that $S \subseteq X \setminus sg_\gamma Cl(\{x\})$. Since $sg_\gamma Cl(\{x\})$ is sg - γ -closed implies $X \setminus sg_\gamma Cl(\{x\})$ is sg - γ -open and S is $sg\gamma g$ -closed set in X . Then, we have that $sgCl_\gamma(S) \subseteq X \setminus sg_\gamma Cl(\{x\})$. This means that $x \notin sgCl_\gamma(S)$. This is a contradiction. Hence $S \cap sg_\gamma Cl(\{x\}) \neq \phi$.

Conversely, let $U \in \tau_{sg\gamma}$ such that $S \subseteq U$. To show that $sgCl_\gamma(S) \subseteq U$. Let $x \in sgCl_\gamma(S)$. Then by hypothesis, $S \cap sg_\gamma Cl(\{x\}) \neq \phi$. So there exists an element $y \in S \cap sg_\gamma Cl(\{x\})$. Thus $y \in S \subseteq U$ and $y \in sg_\gamma Cl(\{x\})$. By Theorem 2.5, $\{x\} \cap U \neq \phi$. Hence $x \in U$ and so $sgCl_\gamma(S) \subseteq U$. Therefore, S is $sg\gamma g$ -closed set in (X, τ) . $\square$

Theorem 3.3. *Let S be a subset of topological space (X, τ) and γ be an operation on τ_{sg} . If S is $sg\gamma g$ -closed, then $sgCl_\gamma(S) \setminus S$ does not contain any non-empty sg - γ -closed set.*

Proof. Let F be a non-empty sg - γ -closed set in X such that $F \subseteq sgCl_\gamma(S) \setminus S$. Then $F \subseteq X \setminus S$ implies $S \subseteq X \setminus F$. Since $X \setminus F$ is sg - γ -open set and S is $sg\gamma g$ -closed set, then $sgCl_\gamma(S) \subseteq X \setminus F$. That is $F \subseteq X \setminus sgCl_\gamma(S)$. Hence $F \subseteq X \setminus sgCl_\gamma(S) \cap sgCl_\gamma(S) \setminus S \subseteq X \setminus sgCl_\gamma(S) \cap sgCl_\gamma(S) = \phi$. This shows that $F = \phi$. This is contradiction. Therefore, $F \not\subseteq sgCl_\gamma(S) \setminus S$. $\square$

Recall that an operation γ on τ_{sg} is said to be sg -open [3] if for each $x \in X$ and for every sg -open set U containing x , there exists an sg - γ -open set W containing x such that $W \subseteq \gamma(U)$.

Theorem 3.4 ([3]). *Let S be any subset of a topological space (X, τ) . If γ is an sg -open operation on τ_{sg} , then $sgCl_\gamma(S) = sg_\gamma Cl(S)$, $sgCl_\gamma(sgCl_\gamma(S)) = sgCl_\gamma(S)$ and $sgCl_\gamma(S)$ is sg - γ -closed set in X .*

Theorem 3.5. *If $\gamma: \tau_{sg} \rightarrow P(X)$ is an sg -open operation, then the converse of the Theorem 3.3 is true.*

Proof. Let U be an sg - γ -open set in (X, τ) such that $S \subseteq U$. Since $\gamma: \tau_{sg} \rightarrow P(X)$ is an sg -open operation, then by Theorem 3.4, $sgCl_\gamma(S)$ is sg - γ -closed set in X . Thus, we have $sgCl_\gamma(S) \cap X \setminus U$ is an sg - γ -closed set in (X, τ) . Since $X \setminus U \subseteq X \setminus S$, $sgCl_\gamma(S) \cap X \setminus U \subseteq sgCl_\gamma(S) \setminus S$. Using the assumption of the converse of the Theorem 3.3, $sgCl_\gamma(S) \subseteq U$. Therefore, S is $sg\gamma g$ -closed set in (X, τ) . $\square$

Corollary 3.6. *Let S be an $sg\gamma g$ -closed subset of topological space (X, τ) and let γ be an operation on τ_{sg} . Then S is sg - γ -closed if and only if $sgCl_\gamma(S) \setminus S$ is sg - γ -closed set.*

Proof. Let S be an sg - γ -closed set in (X, τ) . Then by Lemma 2.6 (3b), $sgCl_\gamma(S) = S$ and hence $sgCl_\gamma(S) \setminus S = \phi$ which is sg - γ -closed set.

Conversely, suppose $sgCl_\gamma(S) \setminus S$ is sg - γ -closed and S is $sg\gamma g$ -closed. Then by Theorem 3.3, $sgCl_\gamma(S) \setminus S$ does not contain any non-empty sg - γ -closed set and since $sgCl_\gamma(S) \setminus S$ is sg - γ -closed subset of itself, then $sgCl_\gamma(S) \setminus S = \phi$ implies $sgCl_\gamma(S) \cap X \setminus S = \phi$. Hence $sgCl_\gamma(S) = S$. This follows from Lemma 2.6 (3b) that S is sg - γ -closed set in (X, τ) . $\square$

Theorem 3.7. *Let (X, τ) be a topological space and γ be an operation on τ_{sg} . If a subset S of X is $sg\gamma g$ -closed and sg - γ -open, then S is sg - γ -closed.*

Proof. Since S is $sg\gamma g$ -closed and sg - γ -open set in X , then $sgCl_\gamma(S) \subseteq S$ and hence by Lemma 2.6 (3b), S is sg - γ -closed. $\square$

Theorem 3.8. *In any topological space (X, τ) with an operation γ on τ_{sg} . For an element $x \in X$, the set $X \setminus \{x\}$ is $sg\gamma g$ -closed or sg - γ -open.*

Proof. Suppose that $X \setminus \{x\}$ is not sg - γ -open. Then X is the only sg - γ -open set containing $X \setminus \{x\}$. This implies that $sgCl_\gamma(X \setminus \{x\}) \subseteq X$. Thus $X \setminus \{x\}$ is an $sg\gamma g$ -closed set in X . $\square$

Corollary 3.9. *In any topological space (X, τ) with an operation γ on τ_{sg} . For an element $x \in X$, either the set $\{x\}$ is sg - γ -closed or the set $X \setminus \{x\}$ is $sg\gamma g$ -closed.*

Proof. Suppose $\{x\}$ is not sg - γ -closed, then $X \setminus \{x\}$ is not sg - γ -open. Hence by Theorem 3.8, $X \setminus \{x\}$ is $sg\gamma g$ -closed set in X . $\square$

Definition 3.10. Let S be any subset of a topological space (X, τ) and γ be an operation on τ_{sg} . Then the $\tau_{sg\gamma}$ -kernel of S is denoted by $\tau_{sg\gamma}\text{-ker}(S)$ and is defined as follows:

$$\tau_{sg\gamma}\text{-ker}(S) = \cap \{U : S \subseteq U \text{ and } U \in \tau_{sg\gamma}\}$$

In other words, $\tau_{sg\gamma}\text{-ker}(S)$ is the intersection of all sg - γ -open sets of (X, τ) containing S .

Theorem 3.11. *Let $S \subseteq (X, \tau)$ and γ be an operation on τ_{sg} . Then S is $sg\gamma g$ -closed if and only if $sgCl_\gamma(S) \subseteq \tau_{sg\gamma}\text{-ker}(S)$.*

Proof. Suppose that S is $sg\gamma g$ -closed. Then $sgCl_\gamma(S) \subseteq U$, whenever $S \subseteq U$ and U is sg - γ -open. Let $x \in sgCl_\gamma(S)$. Then by Lemma 3.2, $S \cap sg_\gamma Cl(\{x\}) \neq \phi$. So there exists a point z in X such that $z \in S \cap sg_\gamma Cl(\{x\})$ implies that $z \in S \subseteq U$ and $z \in sg_\gamma Cl(\{x\})$. By Theorem 2.5, $\{x\} \cap U \neq \phi$. Hence we show that $x \in \tau_{sg\gamma}\text{-ker}(S)$. Therefore, $sgCl_\gamma(S) \subseteq \tau_{sg\gamma}\text{-ker}(S)$. Conversely, let $sgCl_\gamma(S) \subseteq \tau_{sg\gamma}\text{-ker}(S)$. Let U be any sg - γ -open set containing S . Let x be a point in X such that $x \in sgCl_\gamma(S)$. Then $x \in \tau_{sg\gamma}\text{-ker}(S)$. Namely, we have $x \in U$, because $S \subseteq U$ and $U \in \tau_{sg\gamma}$. That is $sgCl_\gamma(S) \subseteq \tau_{sg\gamma}\text{-ker}(S) \subseteq U$. Therefore, S is $sg\gamma g$ -closed set in X . $\square$

Definition 3.12. A topological space (X, τ) is said to be:

1. $sg\text{-}\gamma\text{-}T_0$ [3] (resp., semi $\gamma\text{-}T_0$ [11]) if for any two distinct points x, y in X , there exists an sg -open (resp., a semi-open) set U such that $x \in U$ and $y \notin \gamma(U)$ or $y \in U$ and $x \notin \gamma(U)$.
2. $sg\text{-}\gamma\text{-}T_1$ [3] (resp., semi $\gamma\text{-}T_1$ [11]) if for any two distinct points x, y in X , there exist two sg -open (resp., semi-open) sets U and V containing x and y respectively such that $y \notin \gamma(U)$ and $x \notin \gamma(V)$.
3. $sg\text{-}\gamma\text{-}T_2$ [3] (resp., semi $\gamma\text{-}T_2$ [11]) if for any two distinct points x, y in X , there exist two sg -open (resp., semi-open) sets U and V containing x and y respectively such that $\gamma(U) \cap \gamma(V) = \phi$.
4. semi $\gamma\text{-}T_{\frac{1}{2}}$ [11] if every semi $\gamma\text{-}g$ -closed set in X is semi γ -closed.

Definition 3.13. A topological space (X, τ) with an operation γ on τ_{sg} is said to be $sg\text{-}\gamma\text{-}T_{\frac{1}{2}}$ if every $sg\gamma g$ -closed set in X is $sg\text{-}\gamma$ -closed set.

Theorem 3.14. For any topological space (X, τ) with an operation γ on τ_{sg} . Then (X, τ) is $sg\text{-}\gamma\text{-}T_{\frac{1}{2}}$ if and only if for each element $x \in X$, the set $\{x\}$ is $sg\text{-}\gamma$ -closed or $sg\text{-}\gamma$ -open.

Proof. Let X be an $sg\text{-}\gamma\text{-}T_{\frac{1}{2}}$ space and let $\{x\}$ is not $sg\text{-}\gamma$ -closed set in (X, τ) . By Corollary 3.9, $X \setminus \{x\}$ is $sg\gamma g$ -closed. Since (X, τ) is $sg\text{-}\gamma\text{-}T_{\frac{1}{2}}$, then $X \setminus \{x\}$ is $sg\text{-}\gamma$ -closed set which means that $\{x\}$ is $sg\text{-}\gamma$ -open set in X .

Conversely, let F be any $sg\gamma g$ -closed set in the space (X, τ) . We have to show that F is $sg\text{-}\gamma$ -closed (that is $sgCl_{\gamma}(F) = F$ (by Lemma 2.6 (3b))). It is sufficient to show that $sgCl_{\gamma}(F) \subseteq F$. Let $x \in sgCl_{\gamma}(F)$. By hypothesis $\{x\}$ is $sg\text{-}\gamma$ -closed or $sg\text{-}\gamma$ -open for each $x \in X$. So we have two cases:

Case (1): If $\{x\}$ is $sg\text{-}\gamma$ -closed set. Suppose $x \notin F$, then $x \in sgCl_{\gamma}(F) \setminus F$ contains a non-empty $sg\text{-}\gamma$ -closed set $\{x\}$. A contradiction since F is $sg\gamma g$ -closed set and according to the Theorem 3.3. Hence $x \in F$. This follows that $sgCl_{\gamma}(F) \subseteq F$ and hence $sgCl_{\gamma}(F) = F$. This means from by Lemma 2.6 (3b) that F is $sg\text{-}\gamma$ -closed set in (X, τ) . Thus (X, τ) is $sg\text{-}\gamma\text{-}T_{\frac{1}{2}}$ space.

Case (2): If $\{x\}$ is $sg\text{-}\gamma$ -open set. Then by Theorem 2.5, $F \cap \{x\} \neq \phi$ which implies that $x \in F$. So $sgCl_{\gamma}(F) \subseteq F$. Thus by Lemma 2.6 (3b), F is $sg\text{-}\gamma$ -closed. Therefore, (X, τ) is $sg\text{-}\gamma\text{-}T_{\frac{1}{2}}$ space. $\square$

Theorem 3.15. For any topological space (X, τ) and any operation γ on τ_{sg} , the following properties hold.

1. Every $sg\text{-}\gamma\text{-}T_2$ space is $sg\text{-}\gamma\text{-}T_1$, and every $sg\text{-}\gamma\text{-}T_1$ space is $sg\text{-}\gamma\text{-}T_0$ [3].
2. Every $sg\text{-}\gamma\text{-}T_1$ space is $sg\text{-}\gamma\text{-}T_{\frac{1}{2}}$.
3. Every $sg\text{-}\gamma\text{-}T_{\frac{1}{2}}$ space is $sg\text{-}\gamma\text{-}T_0$.

Remark 3.16. The following diagram of implications follows directly from Theorem 3.15, Remark 3.5 in [11] and Remark 4.12 in [11], we obtain the following diagram of implications.

$$\begin{array}{ccccccc}
 sg - \gamma - T_2 & \longrightarrow & sg - \gamma - T_1 & \longrightarrow & sg - \gamma - T_{\frac{1}{2}} & \longrightarrow & sg - \gamma - T_0 \\
 \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 \text{semi}\gamma - T_2 & \longrightarrow & \text{semi}\gamma - T_1 & \longrightarrow & \text{semi}\gamma - T_{\frac{1}{2}} & \longrightarrow & \text{semi}\gamma - T_0
 \end{array}$$

Where $S \rightarrow T$ represents S implies T .

In the sequel, we shall show that none of the implications that concerning $sg\text{-}\gamma\text{-}T_{\frac{1}{2}}$ space in the above diagram is reversible.

Example 3.17. Consider the space (X, τ) as in Example 3.6 in [3]. Then the space (X, τ) is $sg\text{-}\gamma\text{-}T_{\frac{1}{2}}$, but (X, τ) is not $\text{semi}\gamma\text{-}T_{\frac{1}{2}}$.

Example 3.18. Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}, \{b, c\}\} = SO(X)$. Then $\tau_{sg} = P(X)$. Let $\gamma: \tau_{sg} \rightarrow P(X)$ be an operation on τ_{sg} defined as follows:
For every set $S \in \tau_{sg}$

$$\gamma(S) = \begin{cases} S, & \text{if } b \in S \\ Cl(S), & \text{if } b \notin S \end{cases}$$

Thus, $\tau_{sg\gamma} = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$, and the sets $\{a\}$, $\{c\}$, $\{a, b\}$, $\{a, c\}$, $\{b, c\}$ are $sg\gamma g$ -closed. Then the space (X, τ) is $sg\text{-}\gamma\text{-}T_0$, but it is not $sg\text{-}\gamma\text{-}T_{\frac{1}{2}}$. Since $\{a, b\}$ is $sg\gamma g$ -closed set in (X, τ) , but $\{a, b\}$ is not $sg\text{-}\gamma$ -closed set in (X, τ) .

Example 3.19. Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$. Then $\tau_{sg} = \tau$. Let $\gamma: \tau_{sg} \rightarrow P(X)$ be an operation on τ_{sg} defined as follows:
For every set $S \in \tau_{sg}$

$$\gamma(S) = \begin{cases} S, & \text{if } a \in S \\ Cl(S), & \text{if } a \notin S \end{cases}$$

Thus, $\tau_{sg\gamma} = \tau$. Therefore, the space (X, τ) is $sg\text{-}\gamma\text{-}T_{\frac{1}{2}}$, but it is not $sg\text{-}\gamma\text{-}T_1$.

4. $sg\text{-}(\gamma, \beta)$ -irresolute functions with $sg\text{-}\beta$ -closed graphs

Let (X, τ) and (Y, σ) be two topological spaces and let $\gamma: \tau_{sg} \rightarrow P(X)$ and $\beta: \sigma_{sg} \rightarrow P(Y)$ be operations on τ_{sg} and σ_{sg} respectively. In this section, we introduce a new class of functions called $sg\text{-}(\gamma, \beta)$ -irresolute. Some characterizations and properties of this function are investigated.

Definition 4.1. A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is said to be $sg\text{-}(\gamma, \beta)$ -irresolute if for each $x \in X$ and each sg -open set V containing $f(x)$, there exists an sg -open set U containing x such that $f(\gamma(U)) \subseteq \beta(V)$.

Theorem 4.2. Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be an $sg\text{-}(\gamma, \beta)$ -irresolute function, then,

1. $f(\text{sgCl}_\gamma(S)) \subseteq \text{sgCl}_\beta(f(S))$, for every $S \subseteq (X, \tau)$.
2. $f^{-1}(F)$ is $\text{sg-}\gamma$ -closed set in (X, τ) , for every $\text{sg-}\beta$ -closed set F of (Y, σ) .

Proof. (1) Let $y \in f(\text{sgCl}_\gamma(S))$ and V be any sg -open set containing y . Then by hypothesis, there exists $x \in X$ and sg -open set U containing x such that $f(x) = y$ and $f(\gamma(U)) \subseteq \beta(V)$. Since $x \in \text{sgCl}_\gamma(S)$, we have $\gamma(U) \cap S \neq \emptyset$. Hence $\emptyset \neq f(\gamma(U) \cap S) \subseteq f(\gamma(U)) \cap f(S) \subseteq \beta(V) \cap f(S)$. This implies that $y \in \text{sgCl}_\beta(f(S))$. Therefore, $f(\text{sgCl}_\gamma(S)) \subseteq \text{sgCl}_\beta(f(S))$.

(2) Let F be any $\text{sg-}\beta$ -closed set of (Y, σ) . By using (1), we have

$$f(\text{sgCl}_\gamma(f^{-1}(F))) \subseteq \text{sgCl}_\beta(F) = F.$$

Therefore, $\text{sgCl}_\gamma(f^{-1}(F)) = f^{-1}(F)$. Hence $f^{-1}(F)$ is $\text{sg-}\gamma$ -closed set in (X, τ) . $\square$

Recall that a topological space (X, τ) with an operation γ on τ_{sg} is said to be $\text{sg-}\gamma$ -regular [3] if for each $x \in X$ and for each sg -open set U containing x , there exists an sg -open set W such that $x \in W$ and $\gamma(W) \subseteq U$. The space (X, τ) is an $\text{sg-}\gamma$ -regular if and only if $\tau_{\text{sg}} \subseteq \tau_{\text{sg}\gamma}$ [3].

Theorem 4.3. *In Theorem 4.2, the properties of $\text{sg-}(\gamma, \beta)$ -irresoluteness of f , (1) and (2) are equivalent to each other if either the space (Y, σ) is $\text{sg-}\beta$ -regular or the operation β is sg -open.*

Proof. It follows from the proof of Theorem 4.2 that we know the following implications: " $\text{sg-}(\gamma, \beta)$ -irresoluteness of f " $\Rightarrow$ (1) $\Rightarrow$ (2). Thus, when the space (Y, σ) is $\text{sg-}\beta$ -regular, we prove the implication: (2) $\Rightarrow$ $\text{sg-}(\gamma, \beta)$ -irresoluteness of f . Let $x \in X$ and let $V \in \sigma_{\text{sg}}$ such that $f(x) \in V$. Since (Y, σ) is an $\text{sg-}\beta$ -regular space, then $V \in \sigma_{\text{g}\beta}$. By using (2) of Theorem 4.2, $f^{-1}(V) \in \tau_{\text{sg}\gamma}$ such that $x \in f^{-1}(V)$. So there exists an sg -open set U such that $x \in U$ and $\gamma(U) \subseteq f^{-1}(V)$. This implies that $f(\gamma(U)) \subseteq V \subseteq \beta(V)$. Therefore, f is $\text{sg-}(\gamma, \beta)$ -irresolute.

Now, when β is an sg -open operation, we show the implication: (2) $\Rightarrow$ $\text{sg-}(\gamma, \beta)$ -irresoluteness of f . Let $x \in X$ and let $V \in \sigma_{\text{sg}}$ such that $f(x) \in V$. Since β is an sg -open operation, then there exists $W \in \sigma_{\text{g}\beta}$ such that $f(x) \in W$ and $W \subseteq \beta(V)$. By using (2) of Theorem 4.2, $f^{-1}(W) \in \tau_{\text{sg}\gamma}$ such that $x \in f^{-1}(W)$. So there exists an sg -open set U such that $x \in U$ and $\gamma(U) \subseteq f^{-1}(W) \subseteq f^{-1}(\beta(V))$. This implies that $f(\gamma(U)) \subseteq \beta(V)$. Hence f is $\text{sg-}(\gamma, \beta)$ -irresolute. $\square$

Definition 4.4. A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is said to be

1. $\text{sg-}(\gamma, \beta)$ -closed if the image of each $\text{sg-}\gamma$ -closed set of X is $\text{sg-}\beta$ -closed in Y .
2. $\text{sg-}\beta$ -closed if the image of each sg -closed set of X is $\text{sg-}\beta$ -closed in Y .

Theorem 4.5. *Suppose that a function $f: (X, \tau) \rightarrow (Y, \sigma)$ is both $sg-(\gamma, \beta)$ -irresolute and $sg-\beta$ -closed, then:*

1. *For every $sg\gamma g$ -closed set S of (X, τ) , the image $f(S)$ is $sg\beta g$ -closed in (Y, σ) .*
2. *If (X, τ) is semi- $T_{\frac{1}{2}}$, then the inverse set $f^{-1}(T)$ is $sg\gamma g$ -closed in (X, τ) , for every $sg\beta g$ -closed set T of (Y, σ) .*

Proof. (1) Let G be any $sg-\beta$ -open set in (Y, σ) such that $f(S) \subseteq G$. Since f is $sg-(\gamma, \beta)$ -irresolute function, then by using Theorem 4.2 (2), $f^{-1}(G)$ is $sg-\gamma$ -open set in (X, τ) . Since S is $sg\gamma g$ -closed and $S \subseteq f^{-1}(G)$, we have $sgCl_{\gamma}(S) \subseteq f^{-1}(G)$, and hence $f(sgCl_{\gamma}(S)) \subseteq G$. Thus, by Lemma 2.6 (1), $sgCl_{\gamma}(S)$ is sg -closed set and since f is $sg-\beta$ -closed, then $f(sgCl_{\gamma}(S))$ is $sg-\beta$ -closed set in Y . Therefore, $sgCl_{\beta}(f(S)) \subseteq sgCl_{\beta}(f(sgCl_{\gamma}(S))) = f(sgCl_{\gamma}(S)) \subseteq G$. This implies that $f(S)$ is $sg\beta g$ -closed in (Y, σ) .

(2) Let H be any $sg-\gamma$ -open set of a semi- $T_{\frac{1}{2}}$ space (X, τ) such that $f^{-1}(T) \subseteq H$. Let $C = sgCl_{\gamma}(f^{-1}(T)) \cap (X \setminus H)$, then by Theorem 2.7, $C = sgCl_{\gamma}(f^{-1}(T)) \cap sgCl_{\gamma}(X \setminus H)$ and hence by Lemma 2.6 (1) and Theorem 2.8, C is sg -closed set in (X, τ) . Since f is $sg-\beta$ -closed function. Then $f(C)$ is $sg-\beta$ -closed in (Y, σ) . Since f is $sg-(\gamma, \beta)$ -irresolute function, then by using Theorem 4.2 (1), we have $f(C) = f(sgCl_{\gamma}(f^{-1}(T))) \cap f(X \setminus H) \subseteq sgCl_{\beta}(T) \cap f(X \setminus H) \subseteq sgCl_{\beta}(T) \cap (Y \setminus T) = sgCl_{\beta}(T) \setminus T$. Since T is an $sg\beta g$ -closed set of (Y, σ) . Thus, this implies from Theorem 3.3 that $f(C) = \phi$, and hence $C = \phi$. So $sgCl_{\gamma}(f^{-1}(T)) \subseteq H$. Therefore, $f^{-1}(T)$ is $sg\gamma g$ -closed in (X, τ) . $\square$

Theorem 4.6. *Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be injection, $sg-(\gamma, \beta)$ -irresolute and $sg-\beta$ -closed function. If (Y, σ) is $sg-\beta-T_{\frac{1}{2}}$, then (X, τ) is $sg-\gamma-T_{\frac{1}{2}}$.*

Proof. Let G be any $sg\gamma g$ -closed set of (X, τ) . Since f is $sg-(\gamma, \beta)$ -irresolute and $sg-\beta$ -closed function. Then by Theorem 4.5 (1), $f(G)$ is $sg\beta g$ -closed in (Y, σ) . Since (Y, σ) is $sg-\beta-T_{\frac{1}{2}}$, then $f(G)$ is $sg-\beta$ -closed in Y . Again, since f is $sg-(\gamma, \beta)$ -irresolute, then by Theorem 4.2 (2), $f^{-1}(f(G))$ is $sg-\gamma$ -closed in X . Hence G is $sg-\gamma$ -closed in X since f is injection. Therefore, (X, τ) is an $sg-\gamma-T_{\frac{1}{2}}$ space. $\square$

Theorem 4.7. *Let a function $f: (X, \tau) \rightarrow (Y, \sigma)$ be surjection, $sg-(\gamma, \beta)$ -irresolute and $sg-\beta$ -closed. If (X, τ) is $sg-\gamma-T_{\frac{1}{2}}$, then (Y, σ) is $sg-\beta-T_{\frac{1}{2}}$.*

Proof. Let H be an $sg\beta g$ -closed set of (Y, σ) . Since f is $sg-(\gamma, \beta)$ -irresolute and $sg-\beta$ -closed function. Then by Theorem 4.5 (2), $f^{-1}(H)$ is $sg\gamma g$ -closed in (X, τ) . Since (X, τ) is $sg-\gamma-T_{\frac{1}{2}}$, then we have, $f^{-1}(H)$ is $sg-\gamma$ -closed set in X . Again, since f is $sg-\beta$ -closed function, then $f(f^{-1}(H))$ is $sg-\beta$ -closed in Y . Therefore, H is $sg-\beta$ -closed in Y since f is surjection. Hence (Y, σ) is $sg-\beta-T_{\frac{1}{2}}$ space. $\square$

Theorem 4.8. *If $f: (X, \tau) \rightarrow (Y, \sigma)$ is injection $sg-(\gamma, \beta)$ -irresolute function and the space (Y, σ) is $sg-\beta-T_2$, then the space (X, τ) is $sg-\gamma-T_2$.*

Proof. Let x_1 and x_2 be any distinct points of a space (X, τ) . Since f is injection function and (Y, σ) is $sg-\beta-T_2$. Then there exist two sg -open sets U_1 and U_2 in Y such that $f(x_1) \in U_1$, $f(x_2) \in U_2$ and $\beta(U_1) \cap \beta(U_2) = \phi$. Since f is $sg-(\gamma, \beta)$ -irresolute, there exist sg -open sets V_1 and V_2 in X such that $x_1 \in V_1$, $x_2 \in V_2$, $f(\gamma(V_1)) \subseteq \beta(U_1)$ and $f(\gamma(V_2)) \subseteq \beta(U_2)$. Therefore $\beta(U_1) \cap \beta(U_2) = \phi$. Hence (X, τ) is $sg-\gamma-T_2$. $\square$

Theorem 4.9. *If $f: (X, \tau) \rightarrow (Y, \sigma)$ is injection $sg-(\gamma, \beta)$ -irresolute function and the space (Y, σ) is $sg-\beta-T_i$, then the space (X, τ) is $sg-\gamma-T_i$ for $i \in \{0, 1\}$.*

Proof. The proof is similar to Theorem 4.8. $\square$

Definition 4.10. A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is said to be $sg-(\gamma, \beta)$ -homeomorphism if f is bijection, $sg-(\gamma, \beta)$ -irresolute and f^{-1} is $sg-(\beta, \gamma)$ -irresolute.

Theorem 4.11. *Suppose that the function $f: (X, \tau) \rightarrow (Y, \sigma)$ is bijection $sg-(\gamma, \beta)$ -irresolute and β be an sg -open operation. Then f is $sg-(\gamma, \beta)$ -open (resp., $sg-(\gamma, \beta)$ -closed) if and only if f^{-1} is $sg-(\beta, \gamma)$ -irresolute.*

Proof. Obvious. $\square$

Theorem 4.12. *Assume that a function $f: (X, \tau) \rightarrow (Y, \sigma)$ is $sg-(\gamma, \beta)$ -homeomorphism. If (X, τ) is $sg-\gamma-T_{\frac{1}{2}}$, then (Y, σ) is $sg-\beta-T_{\frac{1}{2}}$.*

Proof. Let $\{y\}$ be any singleton set of (Y, σ) . Then there exists an element x of X such that $y = f(x)$. So by hypothesis and Theorem 3.14, we have $\{x\}$ is $sg-\gamma$ -closed or $sg-\gamma$ -open set in X . By using Theorem 4.2, $\{y\}$ is $sg-\beta$ -closed or $sg-\beta$ -open set. Hence the space by Theorem 3.14, (Y, σ) is $sg-\beta-T_{\frac{1}{2}}$. $\square$

For a function $f: (X, \tau) \rightarrow (Y, \sigma)$, the subset $\{(x, f(x)) : x \in X\}$ of the product space $(X \times Y, \tau \times \sigma)$ is called the graph of f and is denoted by $G(f)$ [7]. In this section, we further investigate general operator approaches of closed graphs of functions. Let $\lambda: (\tau \times \sigma)_{sg} \rightarrow P(X \times Y)$ be an operation on $(\tau \times \sigma)_{sg}$.

Definition 4.13. The graph $G(f)$ of $f: (X, \tau) \rightarrow (Y, \sigma)$ is called $sg-\beta$ -closed if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exist sg -open sets $U \subseteq X$ and $V \subseteq Y$ containing x and y , respectively, such that $(U \times \beta(V)) \cap G(f) = \phi$.

The proof of the following lemma follows directly from the above definition.

Lemma 4.14. *A function $f: (X, \tau) \rightarrow (Y, \sigma)$ has $sg-\beta$ -closed graph if and only if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exist $U \in \tau_{sg}$ containing x and $V \in \sigma_{sg}$ containing y such that $f(U) \cap \beta(V) = \phi$.*

Definition 4.15. An operation $\lambda: (\tau \times \sigma)_{sg} \rightarrow P(X \times Y)$ is said to be *sg-associated* with γ and β if $\lambda(U \times V) = \gamma(U) \times \beta(V)$ holds for each $U \in \tau_{sg}$ and $V \in \sigma_{sg}$.

Recall that an operation γ on τ_{sg} is said to be *sg-regular* [3] if for each $x \in X$ and for every pair of *sg-open* sets U_1 and U_2 such that both containing x , there exists an *sg-open* set W containing x such that $\gamma(W) \subseteq \gamma(U_1) \cap \gamma(U_2)$.

Definition 4.16. The operation $\lambda: (\tau \times \sigma)_{sg} \rightarrow P(X \times Y)$ is said to be *sg-regular* with respect to γ and β if for each $(x, y) \in X \times Y$ and each *sg-open* set W containing (x, y) , there exist *sg-open* sets U in X and V in Y such that $x \in U$, $y \in V$ and $\gamma(U) \times \beta(V) \subseteq \lambda(W)$.

Theorem 4.17. Let $\lambda: (\tau \times \tau)_{sg} \rightarrow P(X \times X)$ be an *sg-associated* operation with γ and γ . If $f: (X, \tau) \rightarrow (Y, \sigma)$ is an *sg- (γ, β) -irresolute* function and (Y, σ) is an *sg- β - T_2* space, then the set $S = \{(x, y) \in X \times X : f(x) = f(y)\}$ is an *sg- λ -closed* set of $(X \times X, \tau \times \tau)$.

Proof. We want to prove that $sgCl_\lambda(S) \subseteq S$. Let $(x, y) \in (X \times X) \setminus S$. Since (Y, σ) is *sg- β - T_2* . Then there exist two *sg-open* sets U and V in (Y, σ) such that $f(x) \in U$, $f(y) \in V$ and $\beta(U) \cap \beta(V) = \phi$. Moreover, for U and V there exist *sg-open* sets G and H in (X, τ) such that $x \in G$, $y \in H$ and $f(\gamma(G)) \subseteq \beta(U)$ and $f(\gamma(H)) \subseteq \beta(V)$ since f is *sg- (γ, β) -irresolute*. Therefore we have $(x, y) \in \gamma(G) \times \gamma(H) = \lambda(G \times H) \cap S = \phi$ because $G \times H \in (\tau \times \tau)_{sg}$. This shows that $(x, y) \notin sgCl_\lambda(S)$. $\square$

Corollary 4.18. Suppose $\lambda: (\tau \times \tau)_{sg} \rightarrow P(X \times X)$ is *sg-associated* operation with γ and γ , and it is *sg-regular* with γ and γ . A space (X, τ) is *sg- γ - T_2* if and only if the diagonal set $\Delta = \{(x, x) : x \in X\}$ is *sg- λ -closed* of $(X \times X, \tau \times \tau)$.

Theorem 4.19. Let $\lambda: (\tau \times \sigma)_{sg} \rightarrow P(X \times Y)$ be an *sg-associated* operation with γ and β . If $f: (X, \tau) \rightarrow (Y, \sigma)$ is *sg- (γ, β) -irresolute* and (Y, σ) is *sg- β - T_2* , then the graph of f , $G(f) = \{(x, f(x)) \in X \times Y\}$ is an *sg- λ -closed* set of $(X \times Y, \tau \times \sigma)$.

Proof. The proof is similar to Theorem 4.17. $\square$

Definition 4.20. Let (X, τ) be a topological space and γ be an operation on τ_{sg} . A subset S of X is said to be *sg- γ -compact* if for every *sg-open* cover $\{U_i, i \in \mathbb{N}\}$ of S , there exists a finite subfamily $\{U_1, U_2, \dots, U_n\}$ such that $S \subseteq \gamma(U_1) \cup \gamma(U_2) \cup \dots \cup \gamma(U_n)$.

Theorem 4.21. Suppose that γ is *sg-regular* and $\lambda: (\tau \times \sigma)_{sg} \rightarrow P(X \times Y)$ is *sg-regular* with respect to γ and β . Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be a function whose graph $G(f)$ is *sg- λ -closed* in $(X \times Y, \tau \times \sigma)$. If a subset S is *sg- β -closed* in (Y, σ) , then $f^{-1}(S)$ is *sg- γ -closed* in (X, τ) .

Proof. Suppose that $f^{-1}(S)$ is not $sg\text{-}\gamma$ -closed then there exist a point x such that $x \in sgCl_\gamma(f^{-1}(S))$ and $x \notin f^{-1}(S)$. Since $(x, s) \notin G(f)$ and each $s \in S$ and $sgCl_\lambda(G(f)) \subseteq G(f)$, there exists an sg -open set W of $(X \times Y, \tau \times \sigma)$ such that $(x, s) \in W$ and $\beta(W) \cap G(f) = \phi$. By sg -regularity of λ , for each $s \in S$ we can take two sg -open sets $U(s)$ and $V(s)$ in (Y, σ) such that $x \in U(s)$, $s \in V(s)$ and $\gamma(U(s)) \times \beta(V(s)) \subseteq \lambda(W)$. Then we have $f(\gamma(U(s))) \cap \beta(V(s)) = \phi$. Since $\{V(s) : s \in S\}$ is sg -open cover of S , then by $sg\text{-}\gamma$ -compactness there exists a finite number $s_1, s_2, \dots, s_n \in S$ such that $S \subseteq \beta(V(s_1)) \cup \beta(V(s_2)) \cup \dots \cup \beta(V(s_n))$. By the sg -regularity of γ , there exist an sg -open set U such that $x \in U$, $\gamma(U) \subseteq \gamma(U(s_1)) \cap \gamma(U(s_2)) \cap \dots \cap \gamma(U(s_n))$. Therefore, we have $\gamma(U) \cap f^{-1}(S) \subseteq U(s_i) \cap f^{-1}(\beta(V(s_i))) = \phi$. This shows that $x \notin sgCl_\gamma(f^{-1}(S))$. This is a contradiction. Therefore, $f^{-1}(S)$ is $sg\text{-}\gamma$ -closed. $\square$

Theorem 4.22. Suppose that the following condition hold:

1. $\gamma: \tau_{sg} \rightarrow P(X)$ is sg -open
2. $\beta: \sigma_{sg} \rightarrow P(Y)$ is sg -regular, and
3. $\lambda: (\tau \times \sigma)_{sg} \rightarrow P(X \times Y)$ is associated with γ and β , and λ is sg -regular with respect to γ and β .

Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be a function whose graph $G(f)$ is $sg\text{-}\lambda$ -closed in $(X \times Y, \tau \times \sigma)$. If every cover of S by $sg\text{-}\gamma$ -open sets of (X, τ) has finite sub cover, then $f(S)$ is $sg\text{-}\beta$ -closed in (Y, σ) .

Proof. Similar to Theorem 4.21. $\square$

5. $sg\text{-}\gamma_0$ -closed spaces

Definition 5.1. A space X is said to be $sg\text{-}\gamma_0$ -closed if for every cover $\{U_i : i \in I\}$ of X by $sg\text{-}\gamma$ -open sets of X , there exists a finite subset I_0 of I such that $X = \bigcup_{i \in I_0} sgCl_\gamma(U_i)$.

Theorem 5.2. A space X is $sg\text{-}\gamma_0$ -closed if and only if every class of $sg\text{-}\gamma$ -open and $sg\text{-}\gamma$ -closed sets with empty intersection has a finite subclass with empty intersection.

Proof. Obvious. $\square$

Lemma 5.3. Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be a function and β be an sg -open operation. Then f is $sg\text{-}(\gamma, \beta)$ -irresolute if and only if $f^{-1}(V)$ is $sg\text{-}\gamma$ -open set in X , for every $sg\text{-}\beta$ -open set V of Y .

Proof. Obvious. $\square$

Theorem 5.4. Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be an $sg\text{-}(\gamma, \beta)$ -irresolute function from an $sg\text{-}\gamma_0$ -closed X onto a space Y and β be an sg -open operation. Then Y is $sg\text{-}\beta_0$ -closed.

Proof. Let $\{V_i : i \in I\}$ be a cover of Y by sg - β -open sets of Y . Since f is sg - (γ, β) -irresolute and β is an sg -open operation. Then by Lemma 5.3, the cover $\{U_i : U_i = f^{-1}(V_i) : i \in I\}$ of X is an sg - γ -open sets of X . Since X is sg - γ_0 -closed, then there exists a finite subset I_0 of I such that $X = \bigcup_{i \in I_0} sgCl_\gamma(U_i)$. This gives that

$$\begin{aligned} Y = f(X) &= f\left(\bigcup_{i \in I_0} sgCl_\gamma(U_i)\right) \\ &= \bigcup_{i \in I_0} f(sgCl_\gamma(U_i)) \text{ (by Theorem 4.2)} \\ &\subseteq \bigcup_{i \in I_0} sgCl_\beta(f(U_i)) = \bigcup_{i \in I_0} sgCl_\beta(V_i). \end{aligned}$$

Therefore, the space Y is sg - β_0 -closed. $\square$

Definition 5.5. A subset S of a topological space (X, τ) is said to be sg - γ_0 -closed if for every cover $\{U_i : i \in I\}$ of S by sg - γ -open sets of X , there exists a finite subset I_0 of I such that $S \subseteq \bigcup_{i \in I_0} sgCl_\gamma(U_i)$.

Theorem 5.6. Suppose that $f : (X, \tau) \rightarrow (Y, \sigma)$ is injection sg - (γ, β) -irresolute function and the operation β is sg -open. If S is an sg - γ_0 -closed subspace of a space X . Then $f(S)$ is sg - β_0 -closed subspace of a space Y .

Proof. Let $\{V_i : i \in I\}$ be a cover of $f(S)$ by sg - β -open sets of Y . Then $V_i = f(S) \cap U_i$, where U_i is an sg - β -open sets in Y . Since f is sg - (γ, β) -irresolute and β is an sg -open operation. Then by Lemma 5.3, $f^{-1}(U_i)$ is sg - γ -open sets in X . Therefore, we have $f^{-1}(V_i) = S \cap f^{-1}(U_i)$ and hence the cover $\{f^{-1}(V_i) : i \in I\}$ of S is an sg - γ -open sets of X . Since S is sg - γ_0 -closed, then there exists a finite subset I_0 of I such that $S \subseteq \bigcup_{i \in I_0} sgCl_\gamma(f^{-1}(V_i))$. This gives that

$$\begin{aligned} f(S) &\subseteq f\left(\bigcup_{i \in I_0} sgCl_\gamma(f^{-1}(V_i))\right) \\ &= \bigcup_{i \in I_0} f(sgCl_\gamma(f^{-1}(V_i))) \subseteq \bigcup_{i \in I_0} sgCl_\beta(f(f^{-1}(V_i))) \text{ (by Theorem 4.2)} \\ &= \bigcup_{i \in I_0} sgCl_\beta(V_i). \end{aligned}$$

Hence, $f(S)$ is sg - β_0 -closed subspace of a space Y . $\square$

Theorem 5.7. If γ is an sg -regular operation. Then each sg - γ -closed subset of an sg - γ_0 -closed space X is sg - γ_0 -closed.

Proof. Let F be an sg - γ -closed subset of an sg - γ_0 -closed space X , and let $\{V_i : i \in I\}$ be a cover of F by sg - γ -open sets of X . Then for sg - γ -open set V_i in F , we have $V_i = F \cap U_i$, where U_i is an sg - γ -open sets in X . Since X is sg - γ_0 -closed

space, then there exists a finite subset I_0 of I such that $X = \bigcup_{i \in I_0} sgCl_\gamma(U_i)$. Since γ is an sg -regular operation. Therefore, we have

$$\begin{aligned} F &= F \cap X \\ &= sgCl_\gamma(F) \cap \bigcup_{i \in I_0} sgCl_\gamma(U_i) \text{ (by Lemma 2.6)} \\ &= \bigcup_{i \in I_0} (sgCl_\gamma(F) \cap sgCl_\gamma(U_i)) \\ &\subseteq \bigcup_{i \in I_0} (sgCl_\gamma(F \cap U_i)) \\ &\subseteq \bigcup_{i \in I_0} (sgCl_\gamma(V_i)). \end{aligned}$$

Thus, the set F is sg - γ_0 -closed. $\square$

Definition 5.8. An operation γ on τ_{sg} is said to be sg - γ -open if $\gamma(U)$ is sg - γ -open for each sg - γ -open set U .

Theorem 5.9. Let γ be both sg -regular and sg - γ -open operation, and let X be an sg - γ - T_2 space. Suppose that F is an sg - γ_0 -closed subset of X and $x \in X \setminus F$. Then there are sg -open sets U_x and V_x in X such that $x \in \gamma(U_x)$, $F \subseteq \gamma(V_x)$ and $\gamma(U_x) \cap \gamma(V_x) = \phi$.

Proof. Suppose that F is an sg - γ_0 -closed subset of X and $x \in X \setminus F$. For every $y \in F$, $y \neq x$. Since the space X is sg - γ - T_2 , there exist sg -open sets U_x and V_y containing x and y respectively such that $\gamma(U_x) \cap \gamma(V_y) = \phi$. Now, let $\{\gamma(V_y) \cap F : y \in F\}$ be a cover of F by sg - γ -open sets of X . Therefore, this cover has a finite subset $\{\gamma(V_{y_1}) \cap F, \gamma(V_{y_2}) \cap F, \dots, \gamma(V_{y_n}) \cap F\}$ such that $F = \bigcup_{i=1}^n sgCl_\gamma(\gamma(V_{y_i}) \cap F)$. Let $\gamma(U_{y_1}), \gamma(U_{y_2}), \dots, \gamma(U_{y_n})$ be the corresponding sg - γ -open sets containing x . Take

$$\gamma(U_x) = \bigcap_{i=1}^n sgCl_\gamma(\gamma(U_{y_i})) \text{ and } \gamma(V_x) = \bigcup_{i=1}^n sgCl_\gamma(\gamma(V_{y_i}))$$

Then $x \in \gamma(U_x)$ and $F \subseteq \gamma(V_x)$, where $\gamma(U_x)$ and $\gamma(V_x)$ are sg - γ -closed since γ is sg -regular. Also,

$$\begin{aligned} \gamma(U_x) \cap \gamma(V_x) &= \bigcap_{i=1}^n sgCl_\gamma(\gamma(U_{y_i})) \cap \bigcup_{i=1}^n sgCl_\gamma(\gamma(V_{y_i})) \\ &= \bigcap_{i=1}^n \bigcup_{i=1}^n sgCl_\gamma(\gamma(U_{y_i})) \cap sgCl_\gamma(\gamma(V_{y_i})) \\ &= \bigcap_{i=1}^n \bigcup_{i=1}^n sgCl_\gamma[\gamma(U_{y_i}) \cap \gamma(V_{y_i})] \text{ (\gamma is sg-regular)} \\ &= \bigcap_{i=1}^n \bigcup_{i=1}^n sgCl_\gamma(\phi) = \phi. \end{aligned}$$

This completes the proof. $\square$

Theorem 5.10. *Let γ be both sg -regular and sg - γ -open operation, and let X be an sg - γ - T_2 space. Then each sg - γ_0 -closed subset of X is sg - γ -closed.*

Proof. The proof is similar to Theorem 5.9. $\square$

Theorem 5.11. *Let the operations γ be sg -regular, and β be sg -regular, sg - β -open and sg -open. If the function $f: (X, \tau) \rightarrow (Y, \sigma)$ is injection sg - (γ, β) -irresolute from an sg - γ_0 -closed X into an sg - β - T_2 space Y . Then it is sg - (γ, β) -closed.*

Proof. Let X be an sg - γ_0 -closed space and S be an sg - γ -closed subset of X . Since γ is sg -regular, then by Theorem 5.7, S is sg - γ_0 -closed subset of X . Since β is sg -open and f is injection sg - (γ, β) -irresolute, then by Theorem 5.6, $f(S)$ is sg - β_0 -closed subset of Y . Since Y is sg - β - T_2 , and β is sg -regular and sg - β -open. Thus, by Theorem 5.10, $f(S)$ is sg - β -closed in Y . Therefore, f is sg - (γ, β) -closed. $\square$

The proof of the following theorem is immediate:

Theorem 5.12. *Let the operations γ be sg -regular, and β be sg -regular, sg - β -open and sg -open. If the function $f: (X, \tau) \rightarrow (Y, \sigma)$ is bijection sg - (γ, β) -irresolute from an sg - γ_0 -closed space X onto an sg - β - T_2 space Y . Then it is sg - (γ, β) -homeomorphism.*

Theorem 5.13. *Let γ be sg -regular and sg -open operation. If S and T are sg - γ_0 -closed subsets of X such that $X = sgCl_\gamma(S) \cup sgCl_\gamma(T)$, then X is sg - γ_0 -closed.*

Proof. The proof is easy and hence it is omitted. $\square$

6. sg - γ^* -regular and sg - γ^* -normal spaces

Definition 6.1. A space (X, τ) is said to be sg - γ^* -regular if for each sg -closed set F of X not containing $x \in X$, there exist sg -open sets G and H such that $x \in G$, $F \subseteq H$ and $\gamma(G) \cap \gamma(H) = \phi$.

Example 6.2. Consider the space $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}, \{b, c\}\}$. Then $\tau_{sg} = P(X)$. Define an operation $\gamma: \tau_{sg} \rightarrow P(X)$ by $\gamma(S) = S$ for all $S \in \tau_{sg}$. Hence it is easy to show that X is sg - γ^* -regular space.

Example 6.3. Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}\{b, c\}, X\} = SO(X)$. Then $\tau_{sg} = \tau$. Let $\gamma: \tau_{sg} \rightarrow P(X)$ be an operation on τ_{sg} defined as follows:
For every set $S \in \tau_{sg}$

$$\gamma(S) = \begin{cases} S, & \text{if } b \in S \\ Cl(S), & \text{if } b \notin S \end{cases}$$

Clearly, the space X is not sg - γ^* -regular.

Theorem 6.4. *If γ is sg -regular, then every subspace of sg - γ^* -regular space X is sg - γ^* -regular.*

Proof. Let S be a subspace of an sg - γ^* -regular space X . We show that S is sg - γ^* -regular. Suppose F is sg -closed set in S and $a \in S$ such that $a \notin F$. Then $F = E \cap S$, where E is sg -closed in X . Then $a \notin E$. Since X is sg - γ^* -regular space, then there exist sg -open sets G, H in X such that $a \in G$, $E \subseteq H$ and $\gamma(G) \cap \gamma(H) = \phi$. Then $G \cap S$ and $H \cap S$ are sg -open sets in S containing a and F respectively, also since γ is sg -regular, so we have

$$\begin{aligned}\gamma(G \cap S) \cap \gamma(H \cap S) &\subseteq (\gamma(G) \cap \gamma(S)) \cap (\gamma(H) \cap \gamma(S)) \\ &= (\gamma(G) \cap \gamma(H)) \cap \gamma(S) = \phi \cap \gamma(S) = \phi.\end{aligned}$$

Therefore, S is sg - γ^* -regular. $\square$

Definition 6.5. A space (X, τ) is said to be sg - γ^* -normal if for each disjoint sg -closed sets E, F of X , there exist sg -open sets G and H such that $E \subseteq G$, $F \subseteq H$ and $\gamma(G) \cap \gamma(H) = \phi$.

Example 6.6. It is obvious from Example 6.3 that the space (X, τ) is sg - γ^* -normal.

Example 6.7. Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, X\}$. Then $SO(X) = \{\phi, \{a\}, \{a, b\}, \{a, c\}, X\} = \tau_{sg}$. Let $\gamma: \tau_{sg} \rightarrow P(X)$ be an operation on τ_{sg} defined as follows:

For every set $S \in \tau_{sg}$

$$\gamma(S) = \begin{cases} S, & \text{if } a \in S \\ Cl(S), & \text{if } a \notin S \end{cases}$$

Obviously, the space X is not sg - γ^* -normal.

Theorem 6.8. *If γ is sg -regular, then every sg -closed subspace of sg - γ^* -normal space X is sg - γ^* -normal.*

Proof. Let S be an sg -closed subspace of an sg - γ^* -normal space X . We show that S is sg - γ^* -normal space. Suppose E_1 and E_2 are disjoint sg -closed sets of S . Then there exist sg - γ -closed sets F_1 and F_2 in X such that $E_1 = F_1 \cap S$ and $E_2 = F_2 \cap S$. Since S is sg -closed in X , thus E_1 and E_2 are sg -closed sets in X . Since X is sg - γ^* -normal space, then there exist sg -open sets G, H in X such that $E_1 \subseteq G$, $E_2 \subseteq H$ and $\gamma(G) \cap \gamma(H) = \phi$. Then $G \cap S$ and $H \cap S$ are sg -open sets in S such that $E_1 \subseteq G \cap S$ and $E_2 \subseteq H \cap S$, also since γ is sg -regular, therefore we have

$$\begin{aligned}\gamma(G \cap S) \cap \gamma(H \cap S) &\subseteq (\gamma(G) \cap \gamma(S)) \cap (\gamma(H) \cap \gamma(S)) \\ &= (\gamma(G) \cap \gamma(H)) \cap \gamma(S) = \phi \cap \gamma(S) = \phi.\end{aligned}$$

This means that, S is sg - γ^* -normal. $\square$

Theorem 6.9. *Let γ be both sg -regular and sg - γ -open operation. Then each sg - γ_0 -closed and sg - γ - T_2 space is sg - γ^* -normal.*

Proof. Let X be an sg - γ_0 -closed and sg - γ - T_2 space and let E and F be disjoint sg -closed sets of X . Then E and F are sg - γ -closed sets because γ is sg -regular. Since X is sg - γ_0 -closed space, then by Theorem 5.7, E and F are sg - γ_0 -closed. Since X is sg - γ - T_2 , then by Theorem 5.9, for sg - γ_0 -closed F and $x \in X \setminus F$. Then there are sg -open sets U_x and V_x in X such that $x \in \gamma(U_x)$, $F \subseteq \gamma(V_x)$ and $\gamma(U_x) \cap \gamma(V_x) = \phi$. Let $\{\gamma(U_x) : x \in E\}$ be a cover of E by sg - γ -open and sg - γ -closed sets of X (since by Theorem 5.10, E is sg - γ -closed). Therefore, there are finite number of elements $x_1, x_2, \dots, x_n$ such that $E \subseteq \bigcup_{i=1}^n sgCl_\gamma(\gamma(U_{x_i})) = \bigcup_{i=1}^n \gamma(U_{x_i})$. Hence

$$\gamma\left(\bigcup_{i=1}^n \gamma(U_{x_i}) \cap \bigcap_{i=1}^n \gamma(V_{x_i})\right) = \gamma(\phi) = \phi.$$

Which implies that X is sg - γ^* -normal. $\square$

Definition 6.10. An operation γ on τ_{sg} is said to be strongly sg -regular if for each $x \in X$ and for every pair of sg -open sets U_1 and U_2 such that both containing x , there exists an sg -open set W containing x such that $\gamma(W) = \gamma(U_1) \cap \gamma(U_2)$.

Theorem 6.11. *Let γ be both strongly sg -regular and sg - γ -open operation. If for each sg -closed set F in X and each sg -open set G containing F , there exists an sg -open set H containing F such that $sgCl_\gamma(\gamma(H)) \subseteq \gamma(G)$, then X is sg - γ^* -normal.*

Proof. Let E and F be two sg -closed sets in X such that $E \cap F = \phi$. Then $F \subseteq X \setminus E$, where $X \setminus E$ is sg -open set in X . By hypothesis, there exists an sg -open set H containing F such that $sgCl_\gamma(\gamma(H)) \subseteq \gamma(X \setminus E)$. This gives that $\gamma(E) \subseteq \gamma(X \setminus sgCl_\gamma(\gamma(H)))$ and $H \cap X \setminus sgCl_\gamma(\gamma(H)) = \phi$. Thus, $F \subseteq H$, $E \subseteq X \setminus sgCl_\gamma(\gamma(H))$ and $\gamma(H) \cap \gamma(X \setminus sgCl_\gamma(\gamma(H))) = \phi$. Consequently, X is sg - γ^* -normal space. $\square$

Definition 6.12 ([12]). A topological space (X, τ) is said to be semi generalized- T_1 (in short sg - T_1) if for any two distinct points x, y in X , there exist two sg -open sets, one containing x but not y , and the other containing y but not x .

Theorem 6.13 ([12]). *The space (X, τ) is sg - T_1 if and only if for every point $x \in X$, $\{x\}$ is an sg -closed set.*

Now we prove the following theorem.

Theorem 6.14. *Every sg - γ^* -normal and sg - T_1 space (X, τ) is sg - γ^* -regular.*

Proof. Let F be an sg -closed set in X and $x \in X$ does not belong to F . Since X is sg - T_1 . Then by Theorem 6.13, $\{x\}$ is sg -closed. So $\{x\}$ and F are two disjoint sg -closed sets of X . Since X is sg - γ^* -normal, then there exist sg -open sets G and H such that $x \in \{x\} \subseteq G$, $F \subseteq H$ and $\gamma(G) \cap \gamma(H) = \phi$. Hence X is sg - γ^* -regular. $\square$

7. Conclusion

This paper continues studied properties of an operation on τ_{sg} . The notions of $sg\gamma$ -generalized closed sets and some of its properties have been investigated. It has been introduced $sg\text{-}\gamma\text{-}T_{\frac{1}{2}}$ space via $sg\gamma$ -generalized closed set and $sg\text{-}\gamma$ -closed set. Some basic characterization of $sg\text{-}(\gamma, \beta)$ -irresolute functions with $sg\text{-}\beta$ -closed graphs have been obtained. It has been studied the concept of $sg\text{-}\gamma_0$ -closed space. Finally, some properties of $sg\text{-}\gamma^*$ -regular and $sg\text{-}\gamma^*$ -normal spaces by using sg -open and sg -closed sets have been given.

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NORMAL EDGE-TRANSITIVE CAYLEY GRAPHS WHOSE ORDER ARE A PRODUCT OF THREE PRIMES

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Abstract. The Cayley graph $X = \text{Cay}(G, S)$ on group G with respect to connection set S is normal edge-transitive, if $N_{\text{Aut}(X)}(R(G))$ acts transitively on edge set. In this paper we determine the structure of automorphism group of all tetravalent normal edge-transitive Cayley graphs of order pqr .

Keywords: wreath product, normal edge-transitive Cayley graph, automorphism group.

1. Introduction

All graphs considered here are finite, simple, undirected and connected. The vertex set and the edge set of graph Γ are denoted by $V(\Gamma)$ and $E(\Gamma)$, respectively. An automorphism of graph Γ is a mapping from the vertices of Γ back to vertices of Γ such that the resulting graph is isomorphic with Γ . The set of all automorphisms of Γ under the composition of mappings forms a group known as the automorphism group denoted by $\text{Aut}(\Gamma)$. Among all graphs, determining the automorphism group of Cayley graphs is a very difficult task. Thus, the majority of this research has been focused on normal edge-transitive Cayley graphs. Due to the complexity, this article only briefly examines properties of Cayley graphs of order pqr , where p, q, r are prime numbers.

Here, in the next section, we give the necessary definitions and some preliminary results. Section three contains main results of this paper and the automorphism group of Cayley graphs of order pqr is verified in this section.

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2. Definitions and preliminaries

The non-empty set $S \subseteq G$ is symmetric connecting set if S generates the group, $1 \notin S$ and $S = S^{-1}$. The Cayley graph $X = \text{Cay}(G, S)$ has the vertex set $V(X) = G$ and edge set $E(X) = \{(g, sg) | g \in G, s \in S\}$. Let S be a generating set of G , it is a well-known fact that X is connected.

Let $R(G) = \{\rho_g : G \rightarrow G, \rho_g(x) = xg, \forall x \in G\}$ and $\text{Aut}(G, S) = \{\alpha \in \text{Aut}(G), \alpha(S) = S\}$, then $\text{Aut}(G, S)$ is a subgroup of $\text{Aut}(G)$ which fixes the subset S . Let $A = \text{Aut}(X)$, the normalizer of $R(G)$ in A is equal to

$$N_A(R(G)) = R(G) \rtimes \text{Aut}(G, S),$$

where $\rtimes$ denotes the semidirect product of two groups. The Cayley graph X is called normal edge-transitive or normal arc-transitive if $N_A(R(G))$ acts transitively on the set of edges or arcs of X , respectively. The Cayley graph $X = \text{Cay}(G, S)$ is said to be normal if $R(G)$ is normal in $\text{Aut}(X)$, for more details on normal Cayley graphs see [2, 4, 6, 12, 14, 15, 16].

In [2, 3] authors studied respectively normal edge-transitivity of Cayley graphs on abelian groups of valency at most five and edge-transitivity of Cayley graphs of valency four on non-abelian simple groups. In [5, 13] authors obtained all normal edge-transitive Cayley graphs of valency four and in [12] the authors classified all connected tetravalent non-normal arc-transitive Cayley graphs on dihedral groups. Darafsheh et al. in [4] studied the normal edge-transitive Cayley graphs on non-abelian groups of order $4p$, where p is a prime number and Ghorbani et al. in [8] studied hexavalent normal edge-transitive Cayley graphs on groups of order pqr . The main results of this paper are based on three following fundamental results:

Proposition 2.1 ([2, 14]). *Let $\Gamma = \text{Cay}(G, S)$ be a connected Cayley graph. Then Γ is normal edge-transitive if and only if $\text{Aut}(G, S)$ is either transitive on S or has two orbits in S in the form of T and T^{-1} , where T is a non-empty subset of S and $S = T \cup T^{-1}$.*

Corollary 2.2. *Let $\Gamma = \text{Cay}(G, S)$ and H be the subset of all involutions of the group G . If $\langle H \rangle \neq G$ and Γ is connected normal edge-transitive, then its valency is even.*

Corollary 2.3. *Let Γ be connected normal edge-transitive Cayley graph, then all elements of S have the same order.*

Let $p > q > r$ be three primes greater than 2. Due to Corollary 2.2, we consider the tetravalent normal edge-transitive Cayley graphs of order pqr .

3. Main results

Here, for easily a normal edge-transitive Cayley graph is denoted by *NET* Cayley graph. Let $p > q$ be two prime numbers such that $q|p-1$. A non-abelian

group of order pq with the following presentation is called a Frobenius group:

$$F_{p,q} = \langle x, y : x^p = y^q = 1, y^{-1}xy = x^u \rangle,$$

where $u^q \equiv 1 \pmod{p}$, see [10]. Let $p > q > r$ be prime numbers, in [9] all groups of order pqr are determined and in [7] the following structures are given

$$\begin{aligned} G_1 &= \mathbb{Z}_{pqr}, G_2 = F_{p,qr}(qr|p-1), G_3 = \mathbb{Z}_r \times F_{p,q}(q|p-1), \\ G_4 &= \mathbb{Z}_p \times F_{q,r}(r|q-1), G_5 = \mathbb{Z}_q \times F_{p,r}(r|p-1), \\ G_{d+5} &= \langle x, y, z : x^p = y^q = z^r = 1, xy = yx, c^{-1}yz = y^u, z^{-1}xz = x^v \rangle, \end{aligned}$$

where $r|p-1, q-1, u^r \equiv 1 \pmod{q}$ and $v^r \equiv 1 \pmod{p}$ ($1 \leq d \leq r-1$).

For graph Γ , let X be a subgroup of $\text{Aut}(\Gamma)$, Γ is called X -vertex-transitive or X -edge-transitive, if X is transitive on the set of vertices or the set of edges, respectively.

Theorem 3.1. *Let $\Gamma = \text{Cay}(G, S)$ be a tetravalent X -normal edge-transitive Cayley graph, where G is a cyclic group and $|G| = pqr$ (p, q, r are odd prime numbers). Let 1 be the vertex corresponding to the identity element. Then*

- i) Γ is $(X, 1)$ -transitive and $S = \{g, g^\rho, g^{\rho^2}, g^{\rho^3}\}$,
- ii) $X_1 \leq \mathbb{Z}_2 \times \mathbb{Z}_2$ and $S = \{g, g^\tau, g^{-1}, (g^{-1})^\tau\}$, where $o(\tau) = 2$.

Proof. i) Let $l = pqr$ and k be an integer, where $k^2 \equiv -1 \pmod{pqr}$. Then G has an automorphism ρ with $\rho(g) = g^k$ and $\rho^2(g) = g^{k^2}$. Suppose $S = \{g, g^k, g^{k^2}, g^{k^3}\}$, thus $S = \{g, g^k, g^{-1}, g^{-k}\}$ and so $X = G \rtimes \mathbb{Z}_4$. This means that Γ is $(X, 1)$ -transitive.

ii) Let l has two odd factors. Let $\tau \in \text{Aut}(G)$ and $\tau(g) = g^{-1}$. Then $\text{Aut}(G)$ contains an automorphism $\sigma \in \text{Aut}(G) \setminus \{\tau\}$ such that $\sigma\tau = \tau\sigma$. Let $\sigma(g) = g^k$ where $k^2 \equiv 1 \pmod{l}$. Let $S = \{g, g^{-1}, g^k, g^{-k}\}$, then $X = G \rtimes \langle \sigma, \tau \rangle \cong \mathbb{Z}_l \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$ and so Γ is X -NET Cayley graph. $\square$

Proposition 3.2 ([8]). *Let G is a finite group and S be a subset of G where $G = \langle S \rangle$ and $|S| = 4$. Then $\text{Aut}(G, S)$ is a subgroup of D_8 .*

Let H_n be the hyper-cube graph of dimension n . The graph $\Gamma = H_n^+$ can be defined as follows:

$$H_n^+ = H_n + E', \text{ where } E' = \{\{x, y\}; x, y \in V(H_n), d(x, y) = n\}.$$

For given group G and positive integer d , by G^d we mean the direct product group $G \times \cdots \times G$ (d times). Let n, m, k and t be positive integers with $m|n, n = mk, n \geq 3, m > 1, (t, k) = 1$ and $0 \leq t \leq k-1$. Let $G = \mathbb{Z}_n \times \mathbb{Z}_m = \langle a \rangle \times \langle b \rangle$ and $S_t = \{a; a^{-1}; a^tb; a^{-t}b^{-1}\}$. We denote the Cayley graph $\text{Cay}(G; S_t)$ by $AC(n, m, t)$ which is a tetravalent graph. The wreath product of two groups G and H is also denoted by $G \wr H$.

Theorem 3.3 ([1]). *Let $\Gamma = \text{Cay}(G, S)$ be a Cayley graph on non-cyclic abelian group G with regarding to connecting set S of valency at most 5. Then Γ is normal edge-transitive if one of the following happens:*

1. $G = \langle a_1, a_2, \dots, a_d \rangle \cong Z_2^d, S = \{a_1, a_2, \dots, a_d\}, \Gamma = H_d, \text{Aut}(\Gamma) \cong S_2 \wr S_d = S_2^d : S_d$, for $d = 2, 3$, and 4.
2. $G = \langle a_1, a_2, \dots, a_d \rangle \cong Z_2^d, S = \{a_1, a_2, \dots, a_d, a_1, a_2, \dots, a_d\}, \Gamma = H_d^+$, $\text{Aut}(\Gamma) \cong S_2 \wr S_{d+1}$, for $d = 2, 3$, and 4.
3. $G = \langle a_1, a_2, \dots, a_d \rangle \cong Z_n^d$, where $n \geq 3$, $S = \{a_1, a_1^{-1}, a_2, a_2^{-1}, \dots, a_d, a_d^{-1}\}$, $\Gamma = AC(n, n, 0)$, for $d = 2$. Also if $n = 4$, then Γ is non-normal and for $n \geq 3$, $n \neq 4$, Γ is normal.
4. $\Gamma = AC(2m, m, 1)$ for $m \geq 3$.
5. $\Gamma = AC(n, m, w)$ for $k \geq 3$ and $w^2 \equiv \pm 1 \pmod{k}$.
6. $G = \mathbb{Z}_m \times \mathbb{Z}_2 = \langle a \rangle \times \langle b \rangle$, $m \geq 3$, $m \neq 4$, m is even, $S = \{a, ab, a^{-1}, a^{-1}b\}$, $\Gamma = AC(m, 2, \pm 1) = C_m[2K_1]$, $\text{Aut}(\Gamma) = \mathbb{Z}_2 \wr D_{2m}$.

Due to Theorems 3.1 and 3.3, we focus only on non-abelian groups.

Lemma 3.4 ([8]). *Let p is a prime number and $q \mid p - 1$, then $\text{Aut}(F_{p,q}) \cong F_{p,p-1}$.*

Theorem 3.5 ([8]). *Let $G = F_{p,q}$ and $S = \{b^i a^m, b^i a^n, (b^i a^m)^{-1}, (b^i a^n)^{-1}\}$. If the Cayley graph $\Gamma = \text{Cay}(G, S)$ is NET, then $\text{Aut}(G, S) \cong \mathbb{Z}_2$.*

Theorem 3.6 ([8]). *Let $S = \{cba^m, (cba^m)^{-1}, cba^n, (cba^n)^{-1}\}$. If $\text{Cay}(G_3, S)$ is NET Cayley graph, then $\text{Aut}(G_3, S) \cong \mathbb{Z}_2$.*

A similar discussion shows that $\text{Aut}(G_4, S) \cong \text{Aut}(G_5, S) \cong \mathbb{Z}_2$. Consider the group G_6 and put $S = \{c, c^{-1}, cb^l a^s, (cb^l a^s)^{-1}\}$, if $\beta \in \text{Aut}(G_6)$ is an automorphism such that $\beta(c) = cb^l a^s$ and $\beta(cb^l a^s) = c$, then necessarily $\beta(a) = a^{-s}, \beta(b) = b^{-l}$ and $\beta(c) = cb^l a^s$ ($1 \leq l \leq q, 1 \leq s \leq p$). This means that $o(\beta) = 2$. On the other hand, an automorphism of G_6 maps c to c^{-1} or $(cb^l a^s)^{-1}$. Hence $\text{Aut}(G_6, S) \cong \langle \beta \rangle$. Therefore, we proved the following theorem:

Theorem 3.7. *Let $\Gamma = \text{Cay}(G_6, S)$ be NET Cayley graph with the connecting set $S = \{c, c^{-1}, cb^l a^s, (cb^l a^s)^{-1}\}$, then $\text{Aut}(G_6, S) \cong \mathbb{Z}_2$.*

Corollary 3.8. *Let G be a non-cyclic group of order pqr and $\Gamma = \text{Cay}(G, S)$ be tetravalent NET Cayley graph, then $\text{Aut}(G, S) \cong \mathbb{Z}_2$.*

3.1 Groups of order pq^2

In [9] it is proved that there are five groups of order pq^2 ($p > q$) with following structures:

$$\begin{aligned} H_1 &= \mathbb{Z}_p \times \mathbb{Z}_{q^2}, H_2 = \mathbb{Z}_p \times (\mathbb{Z}_q \times \mathbb{Z}_q), H_3 = \mathbb{Z}_q \times F_{p,q} \ (q|p-1), \\ H_4 &= F_{p,q^2} \ (q^2|p-1), H_5 = \langle a, b : a^p = b^{q^2} = 1, b^{-1}ab = a^r, r^q \equiv 1 \pmod{p} \rangle. \end{aligned}$$

By using Theorems 3.1 and 3.2, one can compute easily $\text{Aut}(H, S)$, where $H \cong H_1$ or $H \cong H_2$. For non-abelian group H_3 , we have the following theorem.

Theorem 3.9. *Let $S = \{c, c^{-1}, abc, (abc)^{-1}\}$ be a connecting set of group H_3 , then Cayley graph $\Gamma = \text{Cay}(H_3, S)$ is NET and $\text{Aut}(H_3, S) \cong \mathbb{Z}_2$.*

Proof. Let $S = \{c, c^{-1}, abc, (abc)^{-1}\}$, we show that $\text{Aut}(H_3, S)$ has two orbits on S . Let $\alpha \in \text{Aut}(H_3, S)$ and $\alpha(a) = a^i, \alpha(b) = b^j$ and $\alpha(c) = a^k b^l c$ ($1 \leq i \leq p-1, 1 \leq j \leq q-1, 0 \leq k \leq p-1, 0 \leq l \leq q-1$). It is not difficult to see that if $\alpha(c^{-1}ac) = \alpha(a^t)$, where $o(t) = q$ in $\mathbb{Z}_p^*$, then $a^{it^k} = a^{ti}$ and so $k = 1$. If $l = 1$ then $\alpha(c) = abc$. On the other hand, suppose $\alpha(abc) = c^{-1}$ then $ca^{kt+i}b^{j+l} = c^{-1}$, a contradiction. Also, it is obvious that $\alpha(c) \neq (abc)^{-1}$ and so $\text{Aut}(H_3, S)$ on S has two orbits. This completes the proof. $\square$

Theorem 3.10. *Let $\Gamma = \text{Cay}(H_3, S)$ be tetravalent NET Cayley graph, then $S = \{a^m b^n c^h, (a^m b^n c^h)^{-1}, a^r b^n c^h, (a^r b^n c^h)^{-1}\}$.*

Proof. Let $S = \{a^m b^n c^h, (a^m b^n c^h)^{-1}, a^r b^n c^f, (a^r b^n c^f)^{-1}\}$ and $\alpha \in \text{Aut}(H_3, S)$, then α is as given in Theorem 3.9.

Clearly, $\alpha(a^m b^n c^h) = a^r b^n c^f$ and thus $a^{mi+k-r}c^h a^{k(t^{h-1}+\dots+t)}b^{nj+lh} = c^f b^s$.

By a similar method, if $\alpha(a^r b^n c^f) = a^m b^n c^h$, then we have $nj+lh \equiv s \pmod{q}$ and $sj+lf \equiv n \pmod{q}$. So $(n-s)(j+1) + l(h-f) \equiv 0 \pmod{q}$. On the other hand, $q-h \equiv -f \pmod{q}$ and $q-f \equiv -h \pmod{q}$. Then $h = f$ and $q|(n-s)(j+1)$. It yields that $n \equiv s \pmod{q}$ or $j \equiv -1 \pmod{q}$. This completes the proof. $\square$

Theorem 3.11. *The Cayley graph $\Gamma = \text{Cay}(H_4, S)$ is not NET, where $S = \{a^i b^{qk_1+1}, (a^i b^{qk_1+1})^{-1}, a^j b^{qk_2+1}, (a^j b^{qk_2+1})^{-1}\}$, $k_1 \neq k_2$.*

Proof. Let $\alpha(a) = a^l, \alpha(a) = a^k b^s$, where $1 \leq l, k \leq p-1$ and $0 \leq s \leq q^2-1$. Similar to the proof of Theorem 3.10, one can see that $s = 1$. We show that there is no $\alpha \in \text{Aut}(H_4, S)$ such that $o(\alpha) \neq 1$. Let $\alpha(a^i b^{qk_1+1}) = a^j b^{qk_2+1}$. Hence, $a^{il+k(r^{-1}+\dots+r^{-qk_1-1})}b^{qk_1+1} = a^j b^{qk_2+1}$, where $o(r) = q^2$. It can be verified that $il+k(r^{-1}+\dots+r^{-qk_1-1})b^{qk_1+1} \equiv j \pmod{p}$ and $qk_1+1 \equiv qk_2+1 \pmod{q^2}$ which is a contradiction. Now, assume that $\alpha(a^i b^{qk_1+1}) = (a^i b^{qk_1+1})^{-1}$, thus $a^{il+k(r^{-1}+\dots+r^{-qk_1-1})}b^{qk_1+1} = a^{-ir^{qk_1+1}}b^{-qk_1-1}$, a contradiction. By a similar method, if $\alpha(a^j b^{qk_2+1}) = (a^j b^{qk_2+1})^{-1}$, then

$$a^{il+k(r^{-1}+\dots+r^{-qk_1-1})}b^{qk_1+1} = a^{-jr^{qk_2+1}}b^{-qk_2-1}$$

and so $q^2|q(k_1+k_2)+2$, a contradiction. This means that $\text{Aut}(H_4, S) \cong \text{id}$. $\square$

Theorem 3.12. $\Gamma = \text{Cay}(H_4, S)$ is tetravalent NET if and only if $S = \{a^m b^n, (a^m b^n)^{-1}, a^r b^n, (a^r b^n)^{-1}\}$.

Proof. If $\alpha \in \text{Aut}(H_4)$ then necessarily $\alpha(a) = a^i$ and $\alpha(b) = a^j b$, where $1 \leq i \leq p-1$ and $1 \leq j \leq p$. In particular, there is no automorphism of H_4 where $\alpha(b) = b^{-1}$. Now assume that $\alpha(a^m b^n) = b^{-n} a^{-m}$, then $b^n a^{(mi+j)s^n + j(s^{n-1} + \dots + s)} = b^{-n} a^{-m}$, where $o(s) = q^2$ in $\mathbb{Z}_p^*$. It yields $2n \equiv 0 \pmod{q^2}$, contradiction. Finally, if $\alpha(a^m b^n) = a^r b^t$, then $b^{n-t} a^{(mi+j)s^n + j(s^{n-1} + \dots + s)} = b^{-t} a^r b^t$. Hence, $n - t \equiv 0 \pmod{q^2}$. By a similar method, if $\alpha(a^r b^t) = b^m a^n$, then

$$\begin{cases} (mi + j)s^n + j(s^n - 1) - rs^t \equiv 0 \pmod{p} \\ (ri + j)s^n + j(s^n - 1) - ms^n \equiv 0 \pmod{p} \end{cases}.$$

So $s^n(m - r)(i + 1) \equiv 0 \pmod{p}$. Hence $i \equiv -1 \pmod{p}$. In this case, $\text{Aut}(H_4, S)$ has two orbits on S and the proof is completed. $\square$

Theorem 3.13. The Cayley graph $\Gamma = \text{Cay}(H_5, S)$ is NET, if $S = \{a^i b^{kq+1}, a^{-ir} b^{-kq-1}, a^j b^{kq+1}, a^{-jr} b^{-kq-1}\}$.

Proof. Let $S = \{a^i b^{kq+1}, a^{-ir} b^{-kq-1}, a^j b^{kq+1}, a^{-jr} b^{-kq-1}\}$, it is not difficult to show that $b^q \in Z(H_5)$ and $a^i b^j$ ($0 \leq i \leq p-1, 1 \leq j \leq q^2-1$) is of order q^2 . Let $\alpha \in \text{Aut}(H_5, S)$ and $\alpha(a) = a^l, \alpha(b) = a^s b^{qt+1}$ where $1 \leq l, s \leq p-1$ and $1 \leq t \leq q-1$. We claim that there is no $\alpha \in \text{Aut}(H_5, S)$ where $\alpha(a^i b^{kq+1}) = a^{-ir} b^{-kq-1}$, otherwise $\alpha(a^i b^{kq+1}) = a^{il+s} b^{q(k+t)+1}$, then $a^{il+s} b^{q(k+t)+1} = a^{-ir} b^{-kq-1}$. So $il + s \equiv -ir \pmod{p}$ and $q(k+t)+1 \equiv -kq-1 \pmod{q^2}$. Similarly, $\alpha(a^{-ir} b^{-kq-1}) = a^{-ir-l-s} b^{-qt-qk+1}$ and so $-irl-sr \equiv i \pmod{p}$ and $-qt-qk+1 \equiv qk+1 \pmod{q^2}$. But in this case, $2 \equiv 0 \pmod{q^2}$, a contradiction. It can be shown there is $\alpha \in \text{Aut}(H_5, S)$ where $\alpha(a^i b^{kq+1}) = a^{-jr} b^{-kq-1}$. Hence $\text{Aut}(H_5, S)$ is not transitive on S . Let $\alpha(a^i b^{kq+1}) = a^j b^{kq+1}$, then $a^{il+s} b^{q(k+t)+1} = a^j b^{kq+1}$ and so $t = 0$. On the other hand, $\alpha(a^{il+s} b^{kq+1}) = a^{il^2+sl} (a^s b)^{qk+1}$ which means $a^{il^2+sl+s} b^{qk+1} = a^i b^{kq+1}$ and by a similar method, if $\alpha(a^j b^{kq+1}) = a^i b^{kq+1}$, then we have $il + s \equiv j \pmod{p}, il^2 + sl + s \equiv i \pmod{p}$, and $jl + s \equiv i \pmod{p}$. It follows that $l(i-j) + (i-j) \equiv 0 \pmod{p}$ and thus $l \equiv (p-1) \pmod{p}, i+j = s$. This completes the proof. $\square$

Corollary 3.14. Let G be a non-abelian group of order pq^2 and $\Gamma = \text{Cay}(G, S)$ is NET tetravalent Cayley graph, then $\text{Aut}(G, S) \cong \mathbb{Z}_2$.

3.2 Groups of order p^3

Let $X = \text{Cay}(G, S)$ be a connected tetravalent NET Cayley graph on non-abelian group G of order p^3 , where $G = \langle S \rangle, S^{-1} = S$ and $|S| = 4$. By [5, Corollary 3.2], X is normal, hence $\text{Aut}(X)_1 = \text{Aut}(G, S)$ and by Proposition 3.2, $\text{Aut}(G, S) \leq D_8$. Consequently 2 or 4 divides $|\text{Aut}(G, S)|$. By the elementary

group theory, there are two non-abelian groups of order p^3

$$K_1(p^3) = \langle a, b | a^{p^2} = b^p = 1, b^{-1}ab = a^{p+1} \rangle,$$

$$K_2(p^3) = \langle a, b, c | a^p = b^p = c^p = 1, [a, b] = c, [a, c] = [b, c] = 1 \rangle.$$

Theorem 3.15. *Let $\Gamma = \text{Cay}(K_1, S)$ and $S = \{a^f b^l, (a^f b^l)^{-1}, a^r b^s, (a^r b^s)^{-1}\}$ then $\text{Aut}(K_1, S) \cong \text{id}$.*

Proof. Let $\text{Aut}(K_1, S) \not\cong \text{id}$, it is not difficult to see that the following map is an automorphism of $\text{Aut}(K_1)$: $\alpha(a) = a^i b^j$ $0 \leq j \leq p-1$, $(i, p^2) = 1$ and $\alpha(b) = a^{pk} b$, $0 \leq k \leq p-1$. Since α is an automorphism, $\alpha(b^{-1}ab) = \alpha(a^{p+1}) = (a^i b^j)^{p+1} = a^{i(p+1)} b^j$. On the other hand, $\alpha(b^{-1}ab) = \alpha(b^{-1})\alpha(a)\alpha(b)$. It is easy to see that $\alpha(b^{-1}ab) = a^{i(p+1)} b^j$. If $p \equiv 3 \pmod{4}$, then $4 \nmid |\text{Aut}(K_1)|$, because $|\text{Aut}(K_1)| = p^3(p-1)$. If $p \equiv 1 \pmod{4}$, then we show there is no an element of order four in $\text{Aut}(K_1)$, too. To do this assume $S = \{a^f b^l, (a^f b^l)^{-1}, a^r b^s, (a^r b^s)^{-1}\}$.

Let $\alpha(a^r b^s) = \alpha((a^r b^s)^{-1})$, then $b^{rj+s} a^{r(p+1)i+ip+r} = a^{-r}$, a contradiction. Also, if $\alpha(a^f b^l) = \alpha(a^r b^s)$ then $(a^i b^j)^f a^{pkl} b^{l-s} = a^r$ and so $b^{fj} a^{k(p+1)i+ip+pkl} b^l = a^r b^s$. Hence we have $b^{fj+l-s} a^{k(p+1)i+ip+pkl} = a^r$. Similarly, let $\alpha(a^r b^s) = \alpha(a^f b^l)$, then $i = 0$, a contradiction, since $o(a) \neq o(b^j)$, ($o(x)$ denotes the order of x in K_1). This means that $\text{Aut}(K_1, S)$ has no element of order two. Consequently it has no element of order four which yields that $\text{Aut}(K_1, S) \cong \text{id}$. $\square$

Let $G = K_2 = \langle x, y \rangle$, where $o(x) = o(y) = p$. Since $[x, y] \in Z(K_2) = \langle c \rangle$, we have $\beta(a) = x, \beta(b) = y$ and $\beta(c) = [x, y]$, where $\beta \in \text{Aut}(K_2)$. Two following maps are elements of $\text{Aut}(K_2)$: $\alpha_1(a) = b, \alpha_1(b) = a$ and $\alpha_1(c) = c$ and $\alpha_2(a) = b, \alpha_2(b) = a^{-1}, \alpha_2(c) = c$. Clearly, $\alpha_1, \alpha_2 \in \text{Aut}(K_2, S)$ and $\langle \alpha_1, \alpha_2 \rangle \cong D_8$. On the other hand, $\text{Aut}(K_2, S)$ is a subgroup of dihedral group D_8 and so $\text{Aut}(K_2, S) \cong D_8$. So we proved the following theorem.

Theorem 3.16. *Let $X = \text{Cay}(K_2, S)$ is NET Cayley graph. Then $\text{Aut}(K_2, S)$ is isomorphic with group D_8 , where $S = \{a^f b^l, (a^f b^l)^{-1}, a^r b^s, (a^r b^s)^{-1}\}$.*

Theorem 3.17. *Let G be a group of order pqr and $X = \text{Cay}(G, S)$ is a tetravalent NET Cayley graph. Then*

- i) if $p > q > r$ are three primes or $p = q > r$, then $\text{Aut}(G, S) \cong \mathbb{Z}_2$,
- ii) if $p = q = r$, then $\text{Aut}(G, S) \cong \text{id}$ or D_8 .

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A BAYESIAN METHOD TO FIT AN ARMA MODEL

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Abstract. The method of time series analysis is widely used in many fields of science, engineering, finance and economics etc, and fitting a time series model accurately is the important basis of time series analysis. Based on the Bayesian statistical theory, this paper presents a Bayesian method which can identify an ARMA (autoregressive moving-average) model and estimate the model parameters simultaneously. Firstly, in order to determine the orders of the ARMA model, an identification model with the recognition variables is constructed. Moreover, the problem of determining the orders of the ARMA model is transformed into two sets of hypothesis tests. By the principle of Bayesian hypothesis testing, it is suggested to solve the above hypothesis test problems by calculating the posterior probabilities of the hypotheses. However, due to the large number of the unknown parameters in the identification model, this paper proposes to obtain the samples by Gibbs sampling and then calculate the posterior probabilities of the hypotheses, the AR coefficients, the MA coefficients and the variance of the random errors to fit the ARMA model. Finally, in order to illustrate the good performance of

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the method proposed in this article, we design three simulation examples and compare the results of our method with two existing methods: RJMCMC method and EACF method. It can be found clearly that the method proposed in this paper has more accurate results for fitting an ARMA model.

Keywords: time series, ARMA model, model fitting, Bayesian statistic, Gibbs sampling.

1. Introduction

Time series analysis is a kind of method of dynamic data analysis which is widely used in various fields, like science, engineering, finance and economics etc [1-3]. Using the time series analysis method, we must know the internal mechanism of generating the time series data, establish the accurate time series model to forecast the future values. Therefore, the establishment of an accurate time series model on the basis of the time series data is the most fundamental and critical part of the time series analysis. For the identification of the time series ARMA model, many domestic and foreign scholars have done a lot of works and got a wealth of research results. The identification methods of the time series ARMA model can be divided into two categories. One is the non-Bayesian method, for example, Tsay and Tiao (1984) determined the orders of the ARMA model by using the EACF (extended autocorrelation function) method [4]; In 1985, Tsay and Tiao proposed a canonical correlation approach for determining the orders of the ARMA model [5], but these two methods cannot determine the orders of the ARMA model accurately and do not think about the estimation of the model parameters. The other is the Bayesian method. Such as Schwarz (1978) putted forward a BIC criteria to identify the ARMA model [6], however this method is cumbersome; Ong et al. (2005) suggested to use the genetic algorithm to identify the ARIMA model [7], and this method includes the BIC theory and it is also cumbersome; Ehlers and Brooks (2004) illustrated a RJ (reversible jump) MCMC approach to select the orders of the ARIMA model and estimate the parameters simultaneously [8], but the method has an imprecise result. Therefore, it is necessary to establish a more accurate method of identifying the ARMA model. Based on the Bayesian statistical theory, this paper presents a Bayesian method which can identify the ARMA model and estimate the model parameters simultaneously for fitting the ARMA model exactly. The rest of the paper is organized as follows. In section 2, an identification model based on the recognition variables is established for fitting the ARMA model, and the problem of the ARMA model identification is transformed into two sets of hypothesis tests. What's more, we proposed to solve the problems of hypothesis tests by the Bayesian statistical theory. Section 3 provides the conditional posterior distributions of the unknown parameters to calculate the posterior probabilities of the hypotheses and estimate the AR coefficients, the MA coefficients and the variance of the random errors based on the Gibbs sampling. In section 4, a Bayesian method of fitting an ARMA model on the basis of the Gibbs sampling

is presented. Section 5 shows the better performances of the method proposed in this article comparing with the other existing approaches by some simulating examples. Finally, some conclusions are given in section 6.

2. The criterion of ARMA model identification

In general, the ARMA (p, q) model [1-3] is:

$$(2.1) \quad \begin{cases} \phi(B)z_t = \theta(B)\varepsilon_t \\ \varepsilon_t \text{ i.i.d } N(0, \sigma^2) \end{cases}$$

where, $\{x_t\}$ is the time series data, which can be recorded as $X^* = (x_1, x_2, \dots, x_n)^T$, $\phi(B) = I - \phi_1 B - \phi_2 B^2 - \dots - \phi_p B^p$, $\theta(B) = I - \theta_1 B - \theta_2 B^2 - \dots - \theta_q B^q$, B is a backshift operator such that $B^k x_t = x_{t-k}$, p and q are the autoregressive order and the moving-average order of the model respectively, $\Phi = (\phi_1, \phi_2, \dots, \phi_p)^T$ and $\Theta = (\theta_1, \theta_2, \dots, \theta_q)^T$ are the autoregressive coefficients and the moving-average coefficients of the model, respectively. $\{\varepsilon_t\}$ is a sequence of the independent random errors identically distributed $N(0, \sigma^2)$. To ensure the ARMA (p,q) model being stationary and invertible, assume that all of the zeros of $\phi(B) = I - \phi_1 B - \phi_2 B^2 - \dots - \phi_p B^p$ and $\theta(B) = I - \theta_1 B - \theta_2 B^2 - \dots - \theta_q B^q$ are on or outside the unite circle.

When we use the ARMA model to fit the time series data $\{x_t\}$, the model orders p, q , the unknown parameters $\Phi = (\phi_1, \phi_2, \dots, \phi_p)^T$, $\Theta = (\theta_1, \theta_2, \dots, \theta_q)^T$ and the variance σ^2 of the random errors need to be determined.

Firstly, assume a larger family M of the ARMA models, that is $M = \{ARMA(a, b), a = 0, 1, \dots, e; b = 0, 1, \dots, f\}$. By the recognition variables [12], an identification model is proposed as follows:

$$(2.2) \quad \begin{cases} x_t = k_1 \phi_1 x_{t-1} + k_2 \phi_2 x_{t-1} + \dots + k_e \phi_e x_{t-e} \\ + \varepsilon_t - l_1 \theta_1 \varepsilon_{t-1} - l_2 \theta_2 \varepsilon_{t-2} - \dots - l_f \theta_f \varepsilon_{t-f} \\ \varepsilon_t \text{ i.i.d } N(0, \sigma^2) \end{cases}$$

to determine the orders of the ARMA model, where $K = (k_1, k_2, \dots, k_e)$ are the recognition variables for the autoregressive items, $L = (l_1, l_2, \dots, l_f)$ are the recognition variables for the moving-average items, and the values of $k_i, i = 1, 2, \dots, e$ and $l_j, j = 1, 2, \dots, f$ only can be 0 or 1. If $k_i = 1 (i = 1, 2, \dots, e)$, the ARMA model includes the i -th autoregressive item $\phi_i x_{t-i}$, otherwise, the ARMA model does not include the i -th autoregressive item $\phi_i x_{t-i}$; if $l_j = 1 (j = 1, 2, \dots, e)$, the ARMA model includes the j -th moving-average item $\phi_j \varepsilon_{t-j}$, otherwise, the ARMA model does not include the j -th moving-average item $\phi_j \varepsilon_{t-j}$.

To determine the orders of the ARMA model, we construct the following two sets of hypothesis test questions:

$$(2.3) \quad H_{1,i}^A : k_i = 1, \quad H_{2,i}^A : k_i = 0 \quad (i = 1, 2, \dots, e)$$

$$(2.4) \quad H_{1,j}^M : l_j = 1, \quad H_{2,j}^M : l_j = 0 \quad (j = 1, 2, \dots, f)$$

For each of the above hypothesis test questions, based on the Bayesian statistical theory [13-17], after $X = (x_{m+1}, x_{m+2}, \dots, x_n)^T$ ($m = \max(e, f), m < n$) is got, we can calculate the posterior probabilities of the hypotheses:

$$P(H_{1,i}^A|X), P(H_{2,i}^A|X), P(H_{1,j}^M|X) \text{ and } P(H_{2,j}^M|X).$$

If $P(H_{1,i}^A|X) \geq P(H_{2,i}^A|X)$, the hypothesis $H_{1,i}^A$ is accepted, otherwise the hypothesis $H_{2,i}^A$ is accepted; Similarly, if $P(H_{1,j}^M|X) \geq P(H_{2,j}^M|X)$, the hypothesis $H_{1,j}^M$ is accepted, otherwise the hypothesis $H_{2,j}^M$ is accepted.

3. Caculate the conditional posterior distributions of the unknown parameters

Because the autoregressive coefficients, the moving-average coefficients and the variance of the random errors in the identification model (2.2) are unknown, it is difficult to calculate directly the posterior probabilities of the hypotheses. Therefore we need to use the Gibbs sampling method [18-20] to get the samples based on the condition posterior distributions of the unknown parameters in the identification model, and then calculate the posterior probabilities of the hypotheses so as to determine the orders of the ARMA model and estimate the unknown parameters.

3.1 Determine the prior distributions of the unknown parameters

When calculating the conditional posterior distributions of the unknown parameters, the priori distributions of the unknown parameters are required. By the selection methods of the prior distribution [12], the prior distributions of the unknown parameters are given as follows.

Because the values of $k_i, i = 1, 2, \dots, e$ and $l_j, j = 1, 2, \dots, f$ only can be 0 or 1, we take the Bernoulli distribution as the prior distributions of them, $k_i \sim b(1, \alpha_1), i = 1, 2, \dots, e, l_j \sim b(1, \alpha_2), j = 1, 2, \dots, f$, where α_1 and α_2 are the hyper parameters that are determined in advance.

To the autoregressive coefficients $\Phi = (\phi_1, \phi_2, \dots, \phi_e)^T$ and the moving-average coefficients $\Theta = (\theta_1, \theta_2, \dots, \theta_f)^T$, we take the multiple normal distribution as the prior distributions of them, $\Phi \sim N_e(\Phi_0, \Sigma_1), \Theta \sim N_f(\Theta_0, \Sigma_2)$, where $\Phi_0, \Sigma_1, \Theta_0$ and Σ_2 are the hyper parameters that are determined in advance, and Φ_0 is an e -dimensional column vector, Θ_0 is an f -dimensional column vector, Σ_1 is an $e \times e$ nonnegative definite matrix, Σ_2 is an $f \times f$ nonnegative definite matrix.

To the variance σ^2 of the random errors, we take the inverted gamma distribution as the prior distribution of it, $\sigma^2 \sim IG(v, \lambda)$, where v and λ are the hyper parameters that are determined in advance.

3.2 Calculate the conditional posterior distributions of the unknown parameters

The joint probability density function of $X = (x_{m+1}, x_{m+2}, \dots, x_n)^T$ ($m = \max(e, f)$) has to be calculated before we calculate the conditional posterior distribution of every unknown parameter, and it is

$$p(X|\Phi, \Theta, K, L) \propto \exp\left\{-\frac{1}{2\sigma^2} \sum_{t=m+1}^n (x_t - k_1\phi_1x_{t-1} - \dots - k_e\phi_ex_{t-e} + l_1\theta_1\varepsilon_{t-1} + \dots + l_f\theta_f\varepsilon_{t-f})^2\right\}$$

By the Bayesian formula proposed earlier by J. O. Berger (1985) and P. E. Rossi and G. M. Allenby (2005), the conditional posterior distribution of every unknown parameter is calculated as follows:

(1) The conditional posterior distribution of $k_i, i = 1, 2, \dots, e$ is

$$(3.1) \quad k_i|X, K_{(-i)}, L, \Phi, \Theta, \sigma^2 \sim b(1, \beta_i^A)$$

where, $K_{(-i)} = (k_1, k_2, \dots, k_{i-1}, k_{i+1}, \dots, k_e)$, $\beta_i^A = \frac{p_{2,i}^A}{p_{1,i}^A + p_{2,i}^A}$,

$$\begin{aligned} p_{1,i}^A &= p(k_i = 0|X, K_{(-i)}, L, \Phi, \Theta) \\ &= (1 - \alpha_1) \exp\left\{-\frac{1}{2\sigma^2} \sum_{t=m+1}^n (x_t - k_1\phi_1x_{t-1} - \dots - k_e\phi_ex_{t-e} + l_1\theta_1\varepsilon_{t-1} + \dots + l_f\theta_f\varepsilon_{t-f})^2\right\} \\ p_{2,i}^A &= p(k_i = 1|X, K_{(-i)}, L, \Phi, \Theta) \\ &= \alpha_1 \exp\left\{-\frac{1}{2\sigma^2} \sum_{t=m+1}^n (x_t - k_1\phi_1x_{t-1} - \dots - k_e\phi_ex_{t-e} + l_1\theta_1\varepsilon_{t-1} + \dots + l_f\theta_f\varepsilon_{t-f})^2\right\} \end{aligned}$$

(2) The conditional posterior distribution of $l_j, j = 1, 2, \dots, f$ is

$$(3.2) \quad l_j|X, K, L_{(-j)}, \Phi, \Theta, \sigma^2 \sim b(1, \beta_j^M)$$

where, $L_{(-j)} = (l_1, l_2, \dots, l_{j-1}, l_{j+1}, \dots, l_f)$, $\beta_j^M = \frac{p_{2,j}^M}{p_{1,j}^M + p_{2,j}^M}$,

$$\begin{aligned} p_{1,j}^M &= p(l_j = 0|X, K, L_{(-j)}, \Phi, \Theta) \\ &= (1 - \alpha_2) \exp\left\{-\frac{1}{2\sigma^2} \sum_{t=m+1}^n (x_t - k_1\phi_1x_{t-1} - \dots - k_e\phi_ex_{t-e} + l_1\theta_1\varepsilon_{t-1} + \dots + l_f\theta_f\varepsilon_{t-f})^2\right\}, \end{aligned}$$

$$\begin{aligned}
p_{2,j}^M &= p(l_j = 1 | X, K, L_{(-j)}, \Phi, \Theta) \\
&= \alpha_2 \exp \left\{ -\frac{1}{2\sigma^2} \sum_{t=m+1}^n (x_t - k_1 \phi_1 x_{t-1} - \cdots - k_e \phi_e x_{t-e} \right. \\
&\quad \left. + l_1 \theta_1 \varepsilon_{t-1} + \cdots + l_f \theta_f \varepsilon_{t-f})^2 \right\}
\end{aligned}$$

(3) After $K = (k_1, k_2, \dots, k_e)$ and $L = (l_1, l_2, \dots, l_f)$ are determined, the orders p and q of the ARMA model also can be determined. And then we can calculate the conditional posterior distribution of Φ_p which consists of the first p components in Φ ,

$$(3.3) \quad \Phi_p | X, K, L, \Theta, \sigma^2 \sim N_p(\hat{\Phi}_0, \hat{\Sigma}_1)$$

where,

$$\begin{aligned}
\hat{\Sigma}_1 &= \left[\frac{1}{\sigma^2} \sum_{t=m+1}^n X_{(-t)} X_{(-t)}^T + \Sigma_1^* \right]^{-1}, \\
\hat{\Phi}_0 &= \hat{\Sigma}_1 \left[\frac{1}{\sigma^2} \sum_{t=m+1}^n X_{(-t)} (x_t + \Theta^T \varepsilon_{(-t)}) + (\Sigma_1^*)^{-1} \Phi_0^* \right], \\
X_{(-t)} &= (x_{(t-1)}, x_{(t-2)}, \dots, x_{(t-p)})^T
\end{aligned}$$

and $\varepsilon_{(-t)} = (\varepsilon_{(t-1)}, \varepsilon_{(t-2)}, \dots, \varepsilon_{(t-q)})^T$. Φ_0^* is a p -dimensional column vector which consists of the first p components in Φ_0 , Φ_0 is an e -dimensional column vector as described previously, and $\Phi_0 = (\varphi_1^{(0)}, \varphi_2^{(0)}, \dots, \varphi_p^{(0)}, \varphi_{p+1}^{(0)}, \dots, \varphi_e^{(0)})^T$, $\Phi_0^* = (\varphi_1^{(0)}, \varphi_2^{(0)}, \dots, \varphi_p^{(0)})^T$. What's more, Σ_1^* is a $p \times p$ nonnegative definite matrix which consists of the top p rows and top p columns components in Σ_1 , Σ_1 is an $e \times e$ nonnegative definite matrix as described previously, and

$$\begin{aligned}
\Sigma_1 &= \begin{pmatrix} \sigma_{1,1}^1 & \sigma_{1,2}^1 & \cdots & \sigma_{1,p}^1 & \sigma_{1,p+1}^1 & \cdots & \sigma_{1,e}^1 \\ \sigma_{2,1}^1 & \sigma_{2,2}^1 & \cdots & \sigma_{2,p}^1 & \sigma_{2,p+1}^1 & \cdots & \sigma_{2,e}^1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{p,1}^1 & \sigma_{p,2}^1 & \cdots & \sigma_{p,p}^1 & \sigma_{p,p+1}^1 & \cdots & \sigma_{p,e}^1 \\ \sigma_{p+1,1}^1 & \sigma_{p+1,2}^1 & \cdots & \sigma_{p+1,p}^1 & \sigma_{p+1,p+1}^1 & \cdots & \sigma_{p+1,e}^1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{e,1}^{(1)} & \sigma_{e,2}^{(1)} & \cdots & \sigma_{e,p}^{(1)} & \sigma_{e,p+1}^{(1)} & \cdots & \sigma_{e,e}^{(1)} \end{pmatrix}, \\
\Sigma_1^* &= \begin{pmatrix} \sigma_{1,1}^{(1)} & \sigma_{1,2}^{(1)} & \cdots & \sigma_{1,p}^{(1)} \\ \sigma_{2,1}^{(1)} & \sigma_{2,2}^{(1)} & \cdots & \sigma_{2,p}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{p,1}^{(1)} & \sigma_{p,2}^{(1)} & \cdots & \sigma_{p,p}^{(1)} \end{pmatrix}
\end{aligned}$$

(4) After the orders p and q of the ARMA model be determined, we can calculate the conditional posterior distribution of Θ_q which consists of the first q components in Θ ,

$$(3.4) \quad \Theta_q | X, K, L, \Phi, \sigma^2 \sim N_q(\hat{\Theta}_0, \hat{\Sigma}_2)$$

where, $\hat{\Sigma}_2 = [\frac{1}{\sigma^2} \sum_{t=m+1}^n \varepsilon_{(-t)} \varepsilon_{(-t)}^T + \Sigma_2^*]^{-1}$, $\hat{\Phi}_0 = \hat{\Sigma}_2 [(\Sigma_2^*)^{-1} \Theta_0^* - \frac{1}{\sigma^2} \sum_{t=m+1}^n \varepsilon_{(-t)} (x_t - \Phi^T x_{(-t)})]$. Θ_0^* is a q -dimensional column vector which consists of the first q components in Θ_0 , Θ_0 is an f -dimensional column vector as described previously, and $\Theta_0 = (\theta_1^{(0)}, \theta_2^{(0)}, \dots, \theta_q^{(0)}, \theta_{q+1}^{(0)}, \dots, \theta_f^{(0)})^T$, $\Theta_0^* = (\theta_1^{(0)}, \theta_2^{(0)}, \dots, \theta_q^{(0)})^T$. What is more, Σ_2^* is a $q \times q$ nonnegative definite matrix which consists of the top q rows and top q columns components in Σ_2 , Σ_2 is an $f \times f$ nonnegative definite matrix as described previously, and

$$\Sigma_2 = \begin{pmatrix} \sigma_{1,1}^{(2)} & \sigma_{1,2}^{(2)} & \cdots & \sigma_{1,q}^{(2)} & \sigma_{1,q+1}^{(2)} & \cdots & \sigma_{1,f}^{(2)} \\ \sigma_{2,1}^{(2)} & \sigma_{2,2}^{(2)} & \cdots & \sigma_{2,q}^{(2)} & \sigma_{2,q+1}^{(2)} & \cdots & \sigma_{2,f}^{(2)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{q,1}^{(2)} & \sigma_{q,2}^{(2)} & \cdots & \sigma_{q,q}^{(2)} & \sigma_{q,q+1}^{(2)} & \cdots & \sigma_{q,f}^{(2)} \\ \sigma_{q+1,1}^{(2)} & \sigma_{q+1,2}^{(2)} & \cdots & \sigma_{q+1,q}^{(2)} & \sigma_{q+1,q+1}^{(2)} & \cdots & \sigma_{q+1,f}^{(2)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{f,1}^{(2)} & \sigma_{f,2}^{(2)} & \cdots & \sigma_{f,q}^{(2)} & \sigma_{f,q+1}^{(2)} & \cdots & \sigma_{f,f}^{(2)} \end{pmatrix},$$

$$\Sigma_2^* = \begin{pmatrix} \sigma_{1,1}^{(2)} & \sigma_{1,2}^{(2)} & \cdots & \sigma_{1,q}^{(2)} \\ \sigma_{2,1}^{(2)} & \sigma_{2,2}^{(2)} & \cdots & \sigma_{2,q}^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{q,1}^{(2)} & \sigma_{q,2}^{(2)} & \cdots & \sigma_{q,q}^{(2)} \end{pmatrix}$$

(5) The conditional posterior distribution of σ^2 can be calculated as follows:

$$(3.5) \quad \sigma^2 | X, K, L, \Phi_p, \Theta_q \sim IG(\hat{\nu}, \hat{\lambda})$$

where, $\hat{\nu} = \nu - \frac{n-m}{2}$, $\hat{\lambda} = \lambda + \frac{1}{2} \sum_{t=m+1}^n (x_t - \Phi_p^T X_{(-t)} + \Theta_q^T \varepsilon_{(-t)})^2$. The orders of $\Phi_p, \Theta_q, X_{(-t)}$ and $\varepsilon_{(-t)}$ change with p and q .

4. The implementation of the Bayesian method of fitting an ARMA model

The implementation of the Bayesian method of fitting an ARMA model is given as follows:

Step 1: Choose the hyper parameters $\alpha_1, \alpha_2, \Phi_0, \Sigma_1, \Theta_0, \Sigma_2, \nu, \lambda$.

Step 2: Choose the initial values $K^{(0)}, L^{(0)}, \Phi^{(0)}, \Theta^{(0)}, (\sigma^2)^{(0)}$.

Step 3: Implement the Gibbs sampling as follows and get the Gibbs samples. Suppose that the $(r-1)$ -th sample $(K^{(r-1)}, L^{(r-1)}, \Phi^{(r-1)}, \Theta^{(r-1)}, (\sigma^2)^{(r-1)})$ has been acquired. Then, the r -th sample can be obtained by the following procedure:

Step 3.1: Determine p and q based on $K^{(r-1)}$ and $L^{(r-1)}$.

Step 3.2: Obtain $\Phi_p^{(r)}$ from $\Phi_p|X, K^{(r-1)}, L^{(r-1)}, \Theta^{(r-1)}, (\sigma^2)^{(r-1)}$, and make the previous p components of $\Phi^{(r)}$ equal to the corresponding components in $\Phi_p^{(r)}$, and make the last $e-p$ components of $\Phi^{(r)}$ equal to the corresponding components in $\Phi^{(r-1)}$.

Step 3.3: Obtain $\Theta_q^{(r)}$ from $\Theta_q|X, K^{(r-1)}, L^{(r-1)}, \Phi^{(r)}, (\sigma^2)^{(r-1)}$, and make the previous q components of $\Theta^{(r)}$ equal to the corresponding components in $\Theta_q^{(r)}$, and make the last $f-q$ components of $\Theta^{(r)}$ equal to the corresponding components in $\Theta^{(r-1)}$.

Step 3.4: Obtain $(\sigma^2)^{(r)}$ from $\sigma^2|X, K^{(r-1)}, L^{(r-1)}, \Phi^{(r)}, \Theta^{(r)}$.

Step 3.5: Calculate $(\beta_i^A)^{(r)}$ and obtain $k_i^{(r)}$ from $k_i|X, K_{(-i)}^{(r, r-1)}, L^{(r-1)}, \Phi^{(r)}, \Theta^{(r)}, (\sigma^2)^{(r)}$, where $K_{(-i)}^{(r, r-1)} = (k_1^{(r)}, k_2^{(r)}, \dots, k_{i-1}^{(r)}, k_{i+1}^{(r-1)}, \dots, k_e^{(r-1)})$.

Step 3.6: Calculate $(\beta_j^M)^{(r)}$ and obtain $l_j^{(r)}$ from $l_j|X, L_{(-j)}^{(r, r-1)}, K^{(r)}, \Phi^{(r)}, \Theta^{(r)}, (\sigma^2)^{(r)}$, where $L_{(-j)}^{(r, r-1)} = (l_1^{(r)}, l_2^{(r)}, \dots, l_{j-1}^{(r)}, l_{j+1}^{(r-1)}, \dots, l_f^{(r-1)})$.

Implement and end the iterative procedure after the Gibbs sampling is convergent.

Step 4: Make the Bayesian inference and identify the ARMA model. Supposing that N samples are acquired and the Gibbs sampling is convergent after acquiring the M -th sample, we use the last $N-M$ samples to make the following Bayesian inference.

Step 4.1. Determine the model orders p and q . By the above Gibbs sampling, we use the following formulas to calculate the posterior probabilities of the hypotheses

$$(4.1) \quad P(H_{1,i}^A) = p(k_i = 1|X) = \frac{1}{N-M} \sum_{r=M+1}^N (\beta_i^A)^{(r)};$$

$$(4.2) \quad P(H_{2,i}^A) = p(k_i = 0|X) = \frac{1}{N-M} \sum_{r=M+1}^N [1 - (\beta_i^A)^{(r)}];$$

$$(4.3) \quad P(H_{1,j}^M) = p(l_j = 1|X) = \frac{1}{N-M} \sum_{r=M+1}^N (\beta_j^M)^{(r)};$$

$$(4.4) \quad P(H_{2,j}^M) = p(l_j = 0|X) = \frac{1}{N-M} \sum_{r=M+1}^N [1 - (\beta_j^M)^{(r)}].$$

If $P(H_{1,i}^A) \geq P(H_{2,i}^A)$, the hypothesis $H_{1,i}^A$ will be accepted, otherwise the hypothesis $H_{2,i}^A$ is accepted; If $P(H_{1,j}^M) \geq P(H_{2,j}^M)$, the hypothesis $H_{1,j}^M$ will be

accepted, otherwise the hypothesis $H_{2,j}^M$ is accepted. And we can determine the model orders by $p = \max\{i, k_i = 1\}$ and $q = \max\{j, l_j = 1\}$.

Step 4.2. Estimate the unknown parameters Φ, Θ and σ^2 . After the orders p and q of the ARMA model are determined, we use the following formulas to estimate the unknown parameters Φ, Θ and σ^2 .

$$(4.5) \quad \Phi = \frac{1}{N-M} \sum_{r=M+1}^N \Phi^{(r)},$$

$$(4.6) \quad \Theta = \frac{1}{N-M} \sum_{r=M+1}^N \Theta^{(r)},$$

$$(4.7) \quad \sigma^2 = \frac{1}{N-M} \sum_{r=M+1}^N (\sigma^2)^{(r)}.$$

5. Examples and analysis

In order to illustrate the performance of the Bayesian approach for fitting the ARMA model, three examples are designed as follows.

Example 1. Get 100 observations from the model ARMA(2,2):

$$\begin{cases} x_t = 0.5x_{t-1} + 0.4x_{t-2} + \varepsilon_t - 0.3\varepsilon_{t-1} \\ \quad - 0.2\varepsilon_{t-2} \\ \varepsilon_t \text{ i.i.d. } N(0, 1), \end{cases}$$

use the Bayesian method proposed in this article to fit the model and compare with the existing RJMCMC method [8] and EACF method [4]. The results are shown in Table 1 and Table 2.

It can be found clearly that the model is determined as ARMA(2,2) by the Bayesian method proposed in this article, and the AR coefficients are estimated as $\hat{\phi}_1 = 0.5183$ and $\hat{\phi}_2 = 0.4307$, the MA coefficients are estimated as $\hat{\theta}_1 = 0.3135$ and $\hat{\theta}_2 = 0.2515$, and the variance σ^2 of the random errors is estimated as $\hat{\sigma}^2 = 0.8926$. So, by the method proposed in this article, the model can be identified correctly and the model parameters can be estimated accurately.

However, when we use the RJMCMC method to identify the model, it is identified as ARMA (3,2), so the RJMCMC method has an imprecise result of model identification. Similarly, the EACF method also has an imprecise result of model identification because the model is identified as ARMA (1,2) by it.

Example 2. Get three groups of data from ARMA(2,1) and ARMA(3,2), respectively, and the sizes of three groups of data are 25, 50 and 100, and then we use the Bayesian method proposed in this article, the RJMCMC method and the EACF method to identify the model. Repeat 100 times of the above experiments and the results are shown in Table 3.

From the experimental results, we can find that the Bayesian method proposed in this article has a higher correct recognition rate to model than two existing methods in most cases. Whats more, the correct recognition rate to model is increase with the size of data.

Example 3. To show the effect of estimating parameters, we design some experiments as follows: Get two groups of data from ARMA(1,1), ARMA(2,2) and ARMA(3,2) respectively, and the sizes of two groups of data are 50 and 100, and then we use the Bayesian method proposed in this paper and the RJMCMC method to identify the model and estimate the model parameters. The comparative results of estimating parameters are shown in Table 4. From the experimental results, we can find that the Bayesian method proposed in this article has a higher accuracy for estimating parameters than the RJMCMC method. In particularly, the model parameters could be estimated precisely by the Bayesian method based on the data whose size is 100.

6. Conclusions

Firstly, an identification model for fitting the ARMA model is established based on the method of recognition variable, and the ARMA model identification problem is reduced to two sets of hypothesis test questions for recognition variables.

TABLE 1: Comparison of two methods.

The method proposed in this article.		RJMCMC method.	
Lag items.	The posterior probabilities.	Lag items.	The posterior probabilities.
x_{t-1}	0.9354	x_{t-1}	0.8470
x_{t-2}	0.8610	x_{t-2}	0.7885
x_{t-3}	0.1588	x_{t-3}	0.5799
x_{t-4}	0.0942	x_{t-4}	0.2518
ε_{t-1}	0.8668	ε_{t-1}	0.3799
ε_{t-2}	0.7780	ε_{t-2}	0.6856
ε_{t-3}	0.1200	ε_{t-3}	0.0925
ε_{t-4}	0.0002	ε_{t-4}	0.1021

Then, based on the Bayesian statistical theory, this paper proposes to solve the above hypothesis test by calculating the posterior probabilities of the hy-

TABLE 2: The results of model identification by the EACF method.

MA \ AR	0	1	2	3	4	5	6
0	X	X	X	X	X	X	X
1	X	X	O	O	O	O	O
2	X	X	O	O	O	O	O
3	X	O	O	O	O	O	O
4	X	O	O	O	O	O	O
5	X	X	O	O	O	O	O
6	X	O	O	O	O	O	O

TABLE 3: The correct recognition rate of three methods.

Model	n	Bayesian method	RJMCMC method	EACF method
ARMA (2,1)	25	68%	46%	52%
	50	55%	51%	56%
	100	76%	67%	72%
ARMA (3,2)	25	30%	18%	26%
	50	35%	32%	33%
	100	43%	38%	40%

TABLE 4: Comparison of estimating parameters.

Model and parameters	n	Bayesian method	RJMCMC method
ARMA (1,1): $\phi_1 = 0.5,$ $\theta_1 = 0.3$	50	$\phi_1 = 0.436,$ $\theta_1 = 0.386$	$\phi_1 = 0.602,$ $\theta_1 = 0.212$
	100	$\phi_1 = 0.503,$ $\theta_1 = 0.291$	$\phi_1 = 0.552,$ $\theta_1 = 0.366$
ARMA (2,2): $\phi_1 = 0.5,$ $\phi_2 = 0.3,$ $\theta_1 = 0.3,$ $\theta_2 = 0.2$	50	$\phi_1 = 0.565, \phi_2 = 0.352,$ $\theta_1 = 0.336, \theta_2 = 0.326$	$\phi_1 = 0.683, \phi_2 = 0.366,$ $\theta_1 = 0.236, \theta_2 = 0.469$
	100	$\phi_1 = 0.523, \phi_2 = 0.313,$ $\theta_1 = 0.306, \theta_2 = 0.236$	$\phi_1 = 0.556, \phi_2 = 0.353$ $\theta_1 = 0.312, \theta_2 = 0.252$
ARMA (3,2): $\phi_1 = 0.5,$ $\phi_2 = 0.3,$ $\phi_3 = 0.1,$ $\theta_1 = 0.3,$ $\theta_2 = 0.2$	50	$\phi_1 = 0.566, \phi_2 = 0.386,$ $\phi_3 = 0.165$ $\theta_1 = 0.396, \theta_2 = 0.232$	$\phi_1 = 0.623, \phi_2 = 0.529,$ $\phi_3 = 0.003$ $\theta_1 = 0.231, \theta_2 = 0.168$
	100	$\phi_1 = 0.523, \phi_2 = 0.333,$ $\phi_3 = 0.113$ $\theta_1 = 0.328, \theta_2 = 0.201$	$\phi_1 = 0.596, \phi_2 = 0.389,$ $\phi_3 = 0.018$ $\theta_1 = 0.263, \theta_2 = 0.187$

potheses. However, since the autoregressive coefficients, the moving-average coefficients and the variance of the random errors in the identification model are unknown, the posterior probabilities of the hypotheses cannot be calculated

directly. Therefore, this paper proposes to use the Gibbs sampling method to get the samples from the conditional posterior distribution of the unknown parameters, and then calculate the posterior probabilities of the hypotheses to determine the model orders and estimate the unknown parameters in the model. Finally, in order to show the good performance of Bayesian method proposed in this article, we design three simulation examples and compare the Bayesian method proposed in this paper with the two existing methods: RJM-CMC method and EACF method. And some advantages of the Bayesian method proposed in this article can be found as follows: (i) The Bayesian method proposed in this article can identify the ARMA model more accurately than the existing methods. And it is obvious that our method has a larger correct recognition rate of the ARMA model than these existing methods. (ii) The Bayesian method proposed in this article can estimate the autoregressive coefficients, the moving-average coefficients and the variance of the random errors in the model accurately at the same time as identifying the ARMA model.

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ROUGH SOFT BCK -ALGEBRAS AND THEIR DECISION MAKING

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Abstract. Rough sets and soft sets are important tools to deal with uncertainties. In this paper, we apply rough soft set theory to BCK -algebras. The lower and upper rough soft BCK -algebras (ideals) are discussed. Finally, we establish a kind of decision making method for rough soft BCK -algebras.

Keywords: rough soft set, subalgebra (ideals), BCK -algebra, decision making.

1. Introduction

In 1982, Pawlak [23] introduced the concept of rough sets as an important tool to discuss imprecision, vagueness and uncertainties. Since then, this subject has been investigated in many studies, for examples, see [2, 24, 28, 29, 27]. It soon invoked a natural question concerning a possible connection between rough sets and algebraic systems. Biswas [4] introduced the concepts of rough groups and rough subgroups. Kuroki [17] studied the properties of rough ideals in semigroups. In particular, Davvaz [8] dealt with a relationship between rough sets and rings with respect to an ideal of rings.

In 1999, Molodtsov [22] put forward the concept of soft sets as a new mathematical tool for dealing with uncertainties. At present, research on the soft set theory is progressing rapidly. Maji [19] defined some basic operations on soft sets. In 2009, Ali [3] gave some new operations on soft sets. In particular, Çağman and Maji [5, 6, 20] applied soft set theory to decision making. At the same time, some soft algebras were also discussed, such as [1, 14, 16, 18, 25, 26]. Chen [7] presented a new concept of soft set parameterization reduction, and compared this concept with the related concept of attributes reduction in rough set theory. In particular, Feng [10, 11] proposed rough soft sets by combining

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Pawlak rough sets and soft sets, rough sets can be regarded as a collection of rough sets sharing a common Pawlak approximation space.

As is well known, *BCK* and *BCI*-algebras [12] are two classes of algebras of logic. They have been extensively investigated by many researchers, for examples, see [15, 26]. Dudek and Jun applied rough set theory to *BCK* and *BCI*-algebras [9, 13], respectively. In 2008, Jun [14, 16] applied soft set theory to *BCI*-algebras.

In the present paper, we apply rough soft set theory to *BCK*-algebras. Some new basic theory on rough soft *BCK*-algebras are obtained. Finally, we put forward a kind of decision making for rough soft *BCK*-algebras.

2. Preliminaries

Recall that a *BCK*-algebra is an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the following:

- (1) $((x * y) * (x * z)) * (x * y) = 0$,
- (2) $(x * (x * y)) * y = 0$,
- (3) $x * x = 0$,
- (4) $0 * x = 0$,
- (5) $x * y = 0$ and $y * x = 0$ imply $x = y$.

For any *BCK*-algebra X , the relation $\leq$ defined by $x \leq y$ if and only if $x * y = 0$ is a partial order on X .

A non-empty subset S of a *BCK*-algebra X is called a subalgebra of X if $x * y \in S$ whenever $x, y \in S$. A non-empty subset I of X is called an ideal of X , denoted by $I \triangleleft X$, if it satisfies: (1) $0 \in I$; (2) $x * y \in I$ and $y \in I$ imply $x \in I$ for all $x, y \in X$. Note that every ideal of a *BCK*-algebra X is a subalgebra of X .

Throughout this paper, X is always a *BCK*-algebra.

Definition 2.1 ([22]). A pair $\mathfrak{S} = (F, A)$ is called a soft set over U , where $A \subseteq E$ and $F : A \rightarrow \mathcal{P}(U)$ is a set-valued mapping.

Definition 2.2 ([14]). Let (F, A) be a soft set over X . Then

(1) (F, A) is called a soft *BCK*-algebra over X if $F(x)$ is a subalgebra of X for all $x \in \text{Supp}(F, A)$.

(2) (F, A) is called an idealistic soft *BCK*-algebra if $F(x)$ is an ideal of X for all $x \in \text{Supp}(F, A)$, where $\text{Supp}(F, A) = \{x \in A \mid F(x) \neq \emptyset\}$ is called a soft support of the soft set (F, A) .

Definition 2.3 ([3]). Let (F, A) and (G, B) be two soft sets over a common universe U .

(1) The restricted intersection of (F, A) and (G, B) , denoted by $(F, A) \pitchfork (G, B)$, is defined as the soft set (H, C) , where $C = A \cap B$, and $H(c) = F(c) \cap G(c)$ for all $c \in C$,

(2) The extended intersection of (F, A) and (G, B) , denoted by $(F, A) \sqcap_{\mathcal{E}} (G, B)$, is defined as the soft set (H, C) , where $C = A \cup B$, and $\forall e \in C$,

$$H(e) = \begin{cases} F(e), & \text{if } e \in A - B, \\ G(e), & \text{if } e \in B - A, \\ F(e) \cap G(e), & \text{if } e \in A \cap B. \end{cases}$$

(3) The restricted union of (F, A) and (G, B) , denoted by $(F, A) \cup_{\mathcal{R}} (G, B)$, is defined as the soft set (H, C) , where $C = A \cap B$, and $H(c) = F(c) \cup G(c)$ for all $c \in C$,

(4) The extended union of (F, A) and (G, B) , denoted by $(F, A) \widetilde{\cup} (G, B)$, is defined as the soft set (H, C) , where $C = A \cup B$, and $\forall e \in C$,

$$H(e) = \begin{cases} F(e), & \text{if } e \in A - B, \\ G(e), & \text{if } e \in B - A, \\ F(e) \cup G(e), & \text{if } e \in A \cap B. \end{cases}$$

Definition 2.4 ([23]). Let ρ be an equivalence relation on the universe X , (X, ρ) be a Pawlak approximation space. A subset $A \subseteq X$ is called definable if $\underline{\rho}(A) = \overline{\rho}(A)$, otherwise, X is a rough set, where

$$\underline{\rho}(A) = \{x \in X : [x]_{\rho} \subseteq A\},$$

and

$$\overline{\rho}(A) = \{x \in X : [x]_{\rho} \cap A \neq \emptyset\}.$$

3. Lower and upper soft approximations

Let $I \triangleleft X$. Define a relation $\equiv_I$ on X as follows:

$$x \equiv_I y \iff x * y \in I \text{ and } y * x \in I.$$

Then $\equiv_I$ is an equivalence on x with respect to (briefly, w.r.t.) I . Moreover, $\equiv_I$ satisfies $x \equiv_I y$ and $u \equiv_I v \implies x * u \equiv_I y * v$. Hence $\equiv_I$ is a congruence on X .

Let $[x]_I$ denote the equivalence class of X w.r.t. I and X/I denote the set of all equivalence classes, that is, $X/I = \{[x]_I | x \in X\}$. Define an operation $*$ on X/I by $[x]_I * [y]_I = [x * y]_I$, then it is clear that $(X/I, *, I)$ is a BCK-algebra.

Definition 3.1. Let $I \triangleleft X$ and $\mathfrak{S} = (F, A)$ a soft set over X . The lower and upper soft approximation of $\mathfrak{S} = (F, A)$ w.r.t. I are denoted by:

$\underline{Apr}_I(\mathfrak{S}) = (\underline{F}_I, A)$ and $\overline{Apr}_I(\mathfrak{S}) = (\overline{F}_I, A)$, which are soft sets over X with $\underline{F}_I(x) = \underline{Apr}_I(F(x)) = \{y \in X | [y]_I \subseteq F(x)\}$ and $\overline{F}_I(x) = \overline{Apr}_I(F(x)) = \{y \in X | [y]_I \cap F(x) \neq \emptyset\}$, for all $x \in A$.

(i) $\underline{Apr}_I(\mathfrak{S}) = \overline{Apr}_I(\mathfrak{S})$, $\mathfrak{S}$ is called definable;

(ii) $\underline{Apr}_I(\mathfrak{S}) \neq \overline{Apr}_I(\mathfrak{S})$, $\underline{Apr}_I(\mathfrak{S})$ ($\overline{Apr}_I(\mathfrak{S})$) is called a lower (upper) rough soft set. Moreover, $\mathfrak{S}$ is called a rough soft set.

Therefore, when $U = X$ and ρ is the induced relation by an ideal I , then we use the pair (X, I) instead of the approximation space (U, ρ) .

Similar to Theorem 4 in [10], we have

Theorem 3.2. Let $I \triangleleft X$, (X, I) a Pawlak approximation space and $\mathfrak{S} = (F, A)$ and $\mathfrak{T} = (G, B)$ two soft sets over X . Then

- (1) $\underline{Apr}_I(\mathfrak{S} \cap \mathfrak{T}) = \underline{Apr}_I(\mathfrak{S}) \cap \underline{Apr}_I(\mathfrak{T})$;
- (2) $\underline{Apr}_I(\mathfrak{S} \cap_{\mathcal{E}} \mathfrak{T}) = \underline{Apr}_I(\mathfrak{S}) \cap_{\mathcal{E}} \underline{Apr}_I(\mathfrak{T})$;
- (3) $\overline{Apr}_I(\mathfrak{S} \cap \mathfrak{T}) \subseteq \overline{Apr}_I(\mathfrak{S}) \cap \overline{Apr}_I(\mathfrak{T})$;
- (4) $\overline{Apr}_I(\mathfrak{S} \cap_{\mathcal{E}} \mathfrak{T}) \subseteq \overline{Apr}_I(\mathfrak{S}) \cap_{\mathcal{E}} \overline{Apr}_I(\mathfrak{T})$;
- (5) $\underline{Apr}_I(\mathfrak{S} \cup_{\mathcal{R}} \mathfrak{T}) \supseteq \underline{Apr}_I(\mathfrak{S}) \cup_{\mathcal{R}} \underline{Apr}_I(\mathfrak{T})$;
- (6) $\underline{Apr}_I(\mathfrak{S} \widetilde{\cup} \mathfrak{T}) \supseteq \underline{Apr}_I(\mathfrak{S}) \widetilde{\cup} \underline{Apr}_I(\mathfrak{T})$;
- (7) $\overline{Apr}_I(\mathfrak{S} \cup_{\mathcal{R}} \mathfrak{T}) = \overline{Apr}_I(\mathfrak{S}) \cup_{\mathcal{R}} \overline{Apr}_I(\mathfrak{T})$;
- (8) $\overline{Apr}_I(\mathfrak{S} \widetilde{\cup} \mathfrak{T}) = \overline{Apr}_I(\mathfrak{S}) \widetilde{\cup} \overline{Apr}_I(\mathfrak{T})$;
- (9) $\mathfrak{S} \subseteq \mathfrak{T} \Rightarrow \underline{Apr}_I(\mathfrak{S}) \subseteq \underline{Apr}_I(\mathfrak{T}), \overline{Apr}_I(\mathfrak{S}) \subseteq \overline{Apr}_I(\mathfrak{T})$.

Theorem 3.3. Let (X, I) be an approximation space and $\mathfrak{S} = (F, A)$ a soft set over X . If $I = \{0\}$, then $\mathfrak{S}$ is definable.

Proof. For all $x \in A$, we have $[x]_I = \{y \in X | x * y, y * x = 0\} = \{x\}$. Hence $\underline{F}_I(x) = \{y \in X | [y]_I \subseteq F(x)\} = \{y \in X | \{y\} \subseteq F(x)\} = F(x)$ and $\overline{F}_I(x) = \{y \in X | [y]_I \cap F(x) \neq \emptyset\} = \{y \in X | \{y\} \cap F(x) \neq \emptyset\} = F(x)$. Thus, for all $x \in A$, $\underline{F}_I(x) = \overline{F}_I(x)$. Thus means that $\underline{Apr}_I(\mathfrak{S}) = \overline{Apr}_I(\mathfrak{S})$, which implies, $\mathfrak{S}$ is definable. $\square$

Let $A, B \subseteq X$, we denote $A * B = \{x * y | \forall x \in A, y \in B\}$.

Definition 3.4. Let $\mathfrak{S} = (F, A)$ and $\mathfrak{T} = (G, B)$ be two soft sets over X , then we denote $\mathfrak{S} * \mathfrak{T}$ by $\mathfrak{S} * \mathfrak{T} = (F, A) * (G, B) = (H, A * B)$, where $H(x, y) = F(x) * G(y)$ for all $x \in A, y \in B$.

Example 3.5. Let $X = \{0, 1, 2, 3, 4\}$ be a BCK-algebra with the following Cayley table:

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	1	1
2	2	2	0	2	2
3	3	3	3	0	3
4	4	4	4	4	0

Let $I = \{0, 1, 2\} \triangleleft X$, then $[0]_I = [1]_I = [2]_I = \{0, 1, 2\}$, $[3]_I = \{3\}$ and $[4]_I = \{4\}$.

Define two soft sets $\mathfrak{S} = (F, A)$ and $\mathfrak{T} = (G, B)$ over X , where $A = \{0, 1, 3\}$ and $B = \{0, 3, 4\}$ by $F(0) = \{0, 2\}$, $F(1) = \{0, 1\}$, $F(3) = \{0, 4\}$ and $G(0) = \{0, 1, 4\}$, $G(3) = \{0, 2\}$, $G(4) = \{0, 1\}$.

Thus, $H(0, 0) = \{0, 2\}$, $H(0, 3) = \{0, 2\}$, $H(0, 4) = \{0, 2\}$, $H(1, 0) = \{0, 1\}$, $H(1, 3) = \{0, 1\}$, $H(1, 4) = \{0, 1\}$, $H(3, 0) = \{0, 4\}$, $H(3, 3) = \{0, 4\}$, $H(3, 4) = \{0, 4\}$.

Theorem 3.6. Let $I \triangleleft X$ and (X, I) a Pawlak approximation space. Suppose that $\mathfrak{S} = (F, A)$ and $\mathfrak{T} = (G, B)$ are two soft sets over X . Then

$$\overline{Apr}_I(\mathfrak{S}) * \overline{Apr}_I(\mathfrak{T}) \subseteq \overline{Apr}_I(\mathfrak{S} * \mathfrak{T}).$$

Proof. Let $z \in \underline{F}_I(x) * \overline{G}_I(y)$, then there exist $u \in \underline{F}_I(x)$ and $v \in \overline{G}_I(y)$ such that $z = u * v$, and so $[u]_I \cap F(x) \neq \emptyset$ and $[v]_I \cap G(y) \neq \emptyset$. Thus, there exist $a \in F(x)$ and $b \in G(y)$ such that $a \in [u]_I, b \in [v]_I$, and so $a * b \in [u]_I * [v]_I = [u * v]_I$, which implies $[u * v]_I \cap (F(x) * G(y)) \neq \emptyset$. This means that $z \in \overline{Apr}_I(\mathfrak{S} * \mathfrak{T})$. Therefore, $\overline{Apr}_I(\mathfrak{S}) * \overline{Apr}_I(\mathfrak{T}) \subseteq \overline{Apr}_I(\mathfrak{S} * \mathfrak{T})$. $\square$

The following example shows that the inclusion symbol “ $\subseteq$ ” in above theorem may not be replaced by an equal sign.

Example 3.7. Let $X = \{0, 1, 2, 3, 4\}$ be a *BCK*-algebra with the following Cayley table:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	3	4	1	0

Let $I = \{0, 1, 2\} \triangleleft X$, then $[0]_I = [1]_I = [2]_I = \{0, 1, 2\}$ and $[3]_I = [4]_I = \{3, 4\}$.

Define two soft sets $\mathfrak{S} = (F, A)$ and $\mathfrak{T} = (G, B)$ over X , where $A = \{1\}$ and $B = \{3\}$ by $F(1) = \{4\}$ and $G(3) = \{0, 4\}$. By calculation, we have $\overline{F}_I(1) = \{3, 4\}$ and $\overline{G}_I(3) = \{0, 1, 2, 3, 4\}$. Thus, $\overline{F}_I(1) \cdot \overline{G}_I(3) = \{0, 1, 3, 4\}$ and $\overline{Apr}_I(F(1) \cdot G(3)) = \overline{Apr}_I(\{0, 4\}) = \{0, 1, 2, 3, 4\}$, which implies, $\overline{Apr}_I(\mathfrak{S}) * \overline{Apr}_I(\mathfrak{T}) \subsetneq \overline{Apr}_I(\mathfrak{S} * \mathfrak{T})$.

4. Rough soft ideals

In this section, we introduce the concepts of rough soft subalgebras (ideals) of *BCK*-algebras and obtain some related properties.

Definition 4.1. Let $I \triangleleft X$, (X, I) a Pawlak approximation space and $\mathfrak{S} = (F, A)$ a soft set over X . Then $\overline{Apr}_I(\mathfrak{S})$ ($\overline{Apr}_I(\mathfrak{S})$) is called a lower (upper) rough soft *BCK*-algebras (resp., ideal) w.r.t. I over X if $\underline{F}_I(x)$ ($\overline{F}_I(x)$) is a subalgebra (resp., ideal) of X for all $x \in \text{Supp}(F, A)$. Moreover, $\mathfrak{S}$ is called a rough soft *BCK*-algebra (resp., rough soft ideal) w.r.t. I over X if $\underline{F}_I(x)$ and $\overline{F}_I(x)$ are subalgebras (resp., ideals) of X for all $x \in \text{Supp}(F, A)$.

Example 4.2. Let $X = \{0, 1, 2, 3\}$ be a *BCK*-algebra with the Cayley table as follows:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	2	0	2
3	3	3	3	0

Let $I = \{0, 1\} \triangleleft X$, then $[0]_I = [1]_I = \{0, 1\}$, $[2]_I = \{2\}$ and $[3]_I = \{3\}$.

Let $A = X$ and $F : A \rightarrow \mathcal{P}(X)$ be a set-valued function define by $F(0) = F(1) = X$, $F(2) = \{0, 1, 3\}$ and $F(3) = \{0, 1, 2\}$.

By calculations, $\underline{F}_I(0) = \underline{F}_I(1) = X$, $\underline{F}_I(2) = \underline{F}_I(3) = \{0, 1\} \triangleleft X$ and $\overline{F}_I(0) = \overline{F}_I(1) = X$, $\overline{F}_I(2) = \{0, 1, 3\} \triangleleft X$ and $\overline{F}_I(3) = \{0, 1, 2\} \triangleleft X$.

This means that, $\mathfrak{S}$ is a rough soft ideal w.r.t. I over X .

Since any ideal of a BCK -algebra is a subalgebra of X [21], we can obtain the following:

Proposition 4.3. Any rough soft ideal of X is a rough soft BCK -algebra.

The converse of the above proposition may not be true as shown in the following example:

Example 4.4. Let $X = \{0, 1, 2, 3, 4\}$ be a BCK -algebra with the following Cayley table:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	2	0
3	3	1	3	0	3
4	4	4	4	4	0

Let $I = \{0, 2\} \triangleleft X$, then $[0]_I = [2]_I = \{0, 2\}$, $[1]_I = \{1\}$, $[3]_I = \{3\}$ and $[4]_I = \{4\}$.

Define a soft set $\mathfrak{S} = (F, A)$ over X , where $A = \{2\}$ by $F(2) = \{0, 3\}$. By calculations, $\underline{F}_I(2) = \emptyset$ and $\overline{F}_I(2) = \{0, 2, 3\}$. Then $\mathfrak{S}$ is a rough soft BCK -algebra w.r.t. I over X , but it is not a rough soft ideal w.r.t. I over X since $\overline{F}_I(2) = \{0, 2, 3\}$ is not an ideal of X .

Now, we give some operations of rough soft ideals.

Theorem 4.5. Let $I \triangleleft X$ and (X, I) a Pawlak approximation space. Assume that $\underline{Apr}_I(\mathfrak{S}) = (\underline{F}_I, A)$ and $\underline{Apr}_I(\mathfrak{T}) = (\underline{G}_I, B)$ are lower rough soft ideals w.r.t. I over X . If $\mathfrak{S} \cap \mathfrak{T}$ is a non-null soft set, then $\underline{Apr}_I(\mathfrak{S} \cap \mathfrak{T})$ and $\underline{Apr}_I(\mathfrak{S} \cap_{\mathcal{E}} \mathfrak{T})$ are lower rough soft ideals over X .

Proof. By the hypothesis, $\forall x \in \text{Supp}(F, A)$, $y \in \text{Supp}(G, B)$, $\underline{F}_I(x)$ and $\underline{G}_I(y)$ are ideals of X . Since $\mathfrak{S} \cap \mathfrak{T}$ is non-empty, $\forall x' \in A \cap B$, $\underline{F}_I(x') \cap \underline{G}_I(x')$ is an ideal of X . By Theorem 3.2, $\underline{Apr}_I(\mathfrak{S} \cap \mathfrak{T})$ is a lower rough soft ideal over X . Similarly, we can prove $\underline{Apr}_I(\mathfrak{S} \cap_{\mathcal{E}} \mathfrak{T})$ is also a lower rough soft ideal over X . $\square$

Remark 4.6. In general, $\overline{Apr}_I(\mathfrak{S} \cap \mathfrak{T})$ and $\overline{Apr}_I(\mathfrak{S} \cap_{\mathcal{E}} \mathfrak{T})$ are not upper rough soft ideals over X if $\overline{Apr}_I(\mathfrak{S}) = (\overline{F}_I, A)$ and $\overline{Apr}_I(\mathfrak{T}) = (\overline{G}_I, A)$ are both upper rough soft ideals over X as shown in the following example:

Example 4.7. Let $X = \{0, 1, 2, 3, 4\}$ be a BCK-algebra with the Cayley table as follows:

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	2	0	2	0
3	3	3	3	0	0
4	4	4	4	4	0

Let $I = \{0, 1, 2\} \triangleleft X$, then $[0]_I = [1]_I = [2]_I = \{0, 1, 2\}$, $[3]_I = \{3\}$ and $[4]_I = \{4\}$. Define two soft sets $\mathfrak{S} = (F, A)$ and $\mathfrak{T} = (G, B)$ over X , where $A = \{0, 1, 3\}$ and $B = \{3, 4\}$ by $F(0) = \{0, 2\}$, $F(1) = \{0, 1\}$, $F(3) = \{0, 3\}$ and $G(3) = \{0, 3, 4\}$, $G(4) = \{0, 4\}$.

By calculations, $\overline{F}_I(3) = \{0, 1, 2, 3\} \triangleleft X$ and $\overline{G}_I(3) = \{0, 1, 2, 3, 4\} \triangleleft X$.

Since $A \cap B = \{3\}$ and $F(3) \cap G(3) = \{3\}$, but $\overline{Apr}_I(\mathfrak{S} \cap \mathfrak{T}) = \overline{Apr}_I(\{3\}) = \{3\} \not\triangleleft X$.

Theorem 4.8. Let $I \triangleleft X$ and $\mathfrak{S} = (F, A)$ a soft BCK-algebra over X . If $\underline{Apr}_I(\mathfrak{S}) \neq \emptyset$, then $\mathfrak{S}$ is a lower rough soft BCK-algebra w.r.t. I over X .

Proof. $\forall a, b \in \underline{F}_I(x)$, then $[a]_I \subseteq F(x)$ and $[b]_I \subseteq F(x)$. Since $F(x)$ is a subalgebra of X for all $x \in A$, then $F(x) * F(x) \subseteq F(x)$, $[a * b]_I = [a]_I * [b]_I \subseteq F(x) * F(x) \subseteq F(x)$, which implies, $a * b \in \underline{F}_I(x)$. Thus, $\mathfrak{S}$ is a lower rough soft BCK-algebra w.r.t. I over X . $\square$

Theorem 4.9. Let $I \triangleleft X$ and $\mathfrak{S} = (F, A)$ a soft BCK-algebra over X , then $\mathfrak{S}$ is an upper rough soft BCK-algebra w.r.t. I over X .

Proof. $\forall a, b \in \overline{F}_I(x)$, then $[a]_I \cap F(x) \neq \emptyset$ and $[b]_I \cap F(x) \neq \emptyset$, and so there exist $y, z \in F(x)$ such that $y \in [a]_I$ and $z \in [b]_I$. Since $F(x)$ is a subalgebra of X , then $y * z \in F(x)$. Moreover, $y * z \in [a]_I * [b]_I = [a * b]_I$. This implies that $y * z \in F(x) \cap [a * b]_I$, that is, $F(x) \cap [a * b]_I \neq \emptyset$. Thus, $a * b \in \overline{F}_I(x)$, and so $\mathfrak{S}$ is an upper rough soft BCK-algebra w.r.t. I over X . $\square$

Corollary 4.10. Let $I \triangleleft X$ and $\mathfrak{S} = (F, A)$ a soft BCK-algebra over X . If $\underline{Apr}_I(\mathfrak{S}) \neq \emptyset$, then $\mathfrak{S}$ is a rough soft BCK-algebra w.r.t. I over X .

Theorem 4.11. Let $I \triangleleft X$ and $\mathfrak{S} = (F, A)$ an idealistic soft BCK-algebra over X . If $\underline{Apr}_I(\mathfrak{S}) \neq \emptyset$, then $\mathfrak{S}$ is a lower rough soft ideal w.r.t. I over X .

Proof. Since $\mathfrak{S}$ is an idealistic soft BCK-algebra over X , then for all $x \in A$, $F(x)$ is an ideal of X . Let $y \in [0]_I$, then $y = y * 0 \in I \subseteq F(x)$, which implies, $[0]_I \subseteq F(x)$, that is, $0 \in \underline{F}_I(x)$. Now, let $a, b \in X$ be such that $b \in \underline{F}_I(x)$ and

$a * b \in \underline{F}_I(x)$, then $[b]_I \subseteq F(x)$ and $[a]_I * [b]_I = [a * b]_I \subseteq F(x)$. Suppose $y \in [a]_I$ and $z \in [b]_I$, then $y * z \in [a]_I * [b]_I = [a * b]_I \subseteq F(x)$. Since $z \in [b]_I \subseteq F(x)$ and $F(x)$ is an ideal of X , $y \in F(x)$, that is, $[a]_I \subseteq F(x)$. This means that $a \in \underline{F}_I(x)$, which implies, $\underline{F}_I(x)$ is an ideal of X . Hence $\mathfrak{S}$ is a lower rough soft ideal w.r.t. I over X . $\square$

In general, we need to add a condition on upper rough soft ideals as shown in the following:

Theorem 4.12. Let $I \triangleleft X$ and $\mathfrak{S} = (F, A)$ an idealistic soft BCK -algebra over X with $I \subseteq F(x)$ for all $x \in A$, then $\mathfrak{S}$ is an upper rough soft ideal w.r.t. I over X .

Proof. Clearly, $0 \in \overline{F}_I(x)$. Let $y, z \in X$ be such that $z \in \overline{F}_I(x)$ and $y * z \in \overline{F}_I(x)$, then $[z]_I \cap F(x) \neq \emptyset$ and $[y * z]_I \cap F(x) \neq \emptyset$, and so there exist $a, b \in F(x)$ such that $a \in [z]_I$ and $b \in [y * z]_I$. Thus, $z * a \in I \subseteq F(x)$ and $(y * z) * b \in I \subseteq F(x)$. Since $F(x)$ is an ideal of X , then $z \in F(x)$ and $y * z \in F(x)$, and so $y \in F(x)$. This means that $[y]_I \cap F(x) \neq \emptyset$, and so, $y \in \overline{F}_I(x)$. Hence $\overline{F}_I(x)$ is an ideal of X . Thus, $\mathfrak{S}$ is an upper rough soft ideal w.r.t. I over X . $\square$

Corollary 4.13. Let $I \triangleleft X$ and $\mathfrak{S} = (F, A)$ an idealistic soft BCK -algebra over X with $I \subseteq F(x)$ for all $x \in A$. If $\underline{Apr}_I(\mathfrak{S}) \neq \emptyset$, then $\mathfrak{S}$ is a rough soft ideal w.r.t. I over X .

5. Applications of rough soft BCK -algebras in decision making

In this section, we illustrate a kind of new decision making method for rough soft sets on BCK -algebras.

Decision making method:

We will put forward the new method to find which is the best parameter e of a given soft set $\mathfrak{S} = (F, A)$. In other words, $F(e)$ is the nearest accurate BCK -algebra on $\mathfrak{S}$ w.r.t. an ideal of BCK -algebra.

Let X be a BCK -algebra and E a set of related parameters. Let $A = \{e_1, e_2, \dots, e_m\} \subseteq E$ and $\mathfrak{S} = (F, A)$ be an original description soft set over X . Let $I \triangleleft X$ and (X, I) be a Pawlak approximation space. Then we present the decision algorithm for rough soft BCK -algebras as follows:

Step 1. Input the original description BCK -algebra X , soft set $\mathfrak{S}$ and Pawlak approximation space (X, I) , where $I \triangleleft X$.

Step 2. Compute the lower and upper rough soft approximation operators $\underline{Apr}_I(\mathfrak{S})$ and $\overline{Apr}_I(\mathfrak{S})$ on $\mathfrak{S}$, respectively.

Step 3. Compute the different values of $\|F(e_i)\|$, where

$$\|F(e_i)\| = \frac{|\overline{F}_I(e_i)| - |\underline{F}_I(e_i)|}{|F(e_i)|}.$$

Step 4. Find the minimum value $\|F(e_k)\|$ of $\|F(e_i)\|$, where $\|F(e_k)\| = \min_i \|F(e_i)\|$.

Step 5. The decision is $F(e_k)$.

Example 5.1. Assume that we want to find the nearest accurate *BCK*-algebra on a soft set $\mathfrak{S}$. Let a *BCK*-algebra as in Example 4.7, $I = \{0, 1, 2\} \triangleleft X$. Define a soft set $\mathfrak{S} = (F, A)$ over X , where $A = \{e_1, e_2, e_3, e_4\}$. The tabular representation of the soft set $\mathfrak{S}$ is given in Table 1.

Table 1 table for soft set $\mathfrak{S}$

	0	1	2	3	4
e_1	0	1	0	1	0
e_2	0	0	1	0	1
e_3	1	1	0	1	1
e_4	1	0	1	0	0

Now, the tabular representations of two soft sets $\underline{Apr}_I(\mathfrak{S})$ and $\overline{Apr}_I(\mathfrak{S})$ over X are given by Tables 2 and 3, respectively.

Table 2 table for soft set $\underline{Apr}_I(\mathfrak{S})$

	0	1	2	3	4
e_1	0	0	0	1	0
e_2	0	0	1	0	1
e_3	0	0	0	1	1
e_4	0	0	0	0	0

Table 3 table for soft set $\overline{Apr}_I(\mathfrak{S})$

	0	1	2	3	4
e_1	1	1	1	1	0
e_2	1	1	1	0	1
e_3	1	1	1	1	1
e_4	1	1	1	0	0

Then, we can calculate $\|F(e_1)\| = 1.5$, $\|F(e_2)\| = 1$, $\|F(e_3)\| = 0.75$, $\|F(e_4)\| = 1.5$. This means the minimum value for $\|F(e_i)\|$ is $\|F(e_3)\| = 0.75$. That is $F(e_3)$ is the closest accurate *BCK*-algebra on $\mathfrak{S}$.

6. Conclusion

Recently, some researchers established some decision making methods based on soft sets [20, 5] and fuzzy soft sets. In the present paper, we first put forward a kind of new decision making method based on rough soft sets. We apply rough soft set theory to *BCK*-algebras and investigate some related results. We hope it would be served as a foundation of rough soft sets and other decision making methods in different areas, such as theoretical computation sciences, information sciences and intelligent systems, and so on.

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ISO-ARRAYS AND CONDITIONAL COMMUNICATION ON P SYSTEM

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Abstract. Among the theoretical applications of formal language theory, membrane computing is a new theoretical model. It is a cell-like structure in which regions are separated by membranes. It is introduced by G.H. Paun, called P system. This computational system can be used for generating string languages, two dimensional picture languages, tiling patterns and tessellations. In this paper we introduced triangular tile pasting P system with conditional communication and explained with an example. Comparison result between triangular tile pasting system and triangular tile pasting P system has been given. The results on generating powers of iso-array grammars and triangular tile pasting P system are also examined in this paper.

Keywords: iso-triangular tiles, tile pasting P System, iso-array grammars, permitting and forbidden conditions.

1. Introduction

Membrane computing is a new theoretical model to computer researchers and non-computer researches. G.H. Paun introduced a new computing model called P system in [7]. It has a membrane structure. In this structure membranes are working as living cells in biological structure model. P system is a distributed and parallel computing model. P system has merely a biological background and mathematical formalism. The evolution rules and evolving objects are presented in the regions of membranes. The rules are applied non-deterministically and the computation starts by applying the rules to the tiles presented in the region in parallel mechanism. P system with multi rewriting rules, tissue like P system and neural like P system are some types of P system. Square tiles [11] are considered in the tile pasting P system to generate some two dimensional picture languages. And the computational power of this P system is examined with the

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computational powers of other existing P systems. By relating P system and array grammars, Ceterchi et al. [5] began a study on linking these two areas, which were not discussed in early research. Connecting P system and array re-writing grammars, array-rewriting P system is introduced to generate two dimensional picture languages. Iso-triangular tile pasting system is introduced by Kalyani et.al. [9] which is an interesting tile pasting system, generating iso-picture patterns and iso-picture languages. Connecting P system and iso-triangular tile pasting system, Triangular tile pasting P system is defined in [1].

In recent years many different types of P system have been introduced, among the variants of P system, conditional communication [8] is a new study, in which the communication to the membranes are controlled by the tiles or tiling not by the pasting rules. To control the communication, permitting and forbidden conditions are defined in the tiles or sub tiles which exists in a given picture. Tissue-like P system is a type of P system to generate two dimensional pictures. In this P system membranes are considered as elementary membranes and it is differentiated from usual P system in such a way that it has no skin membranes and environment. But the computational result is collected only in the output membrane. It is not necessary that the membranes should connect or communicate to each other directly and the communication of membranes are permitted only if the membranes are designed in the system. The communication of membranes is represented graphically by using the synapses (syn). Ceterchi et.al. [6] defined a type of tissue-like P system in which instead of considering a hierarchical structure, membranes are placed at the nodes of a graph.

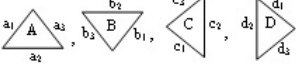
By linking the two models, conditional communication P system and Tissue-like P system, Robinson et.al. introduced tissue-like P system with conditional communication in membranes for the tiles by means of tile pasting. It is noticed that the evaluation rules are applied only in maximally parallel way to generate the pictures. Puzzle grammar and context-free grammars have been studied in [10]. By connecting P system and iso-array grammar rules, iso-array rewriting P system with context free iso-array rules is introduced in [2]. Contextual iso-array rewriting P system [3] is introduced to generate the triangular picture languages. Also we have examined the generating powers of the triangular tile pasting P system and array generating petrinets in [4]. Inspired by these studies, in this paper conditional communication on triangular tile pasting P system is introduced to generate the triangular picture languages. The computational powers of triangular tile pasting P system and iso-array grammars are examined here.

2. Preliminaries

Tiling is an art and it is a well-known theory in the applications of pattern generation. In this section we recollect the notion of iso-triangular tiles and tile

pasting system. A tile is a topological disc with closed boundary in the XOY plane, whose edges are glueable. A tiling is a family of countable tiles with no gaps or overlaps that covers the Euclidean plane.

Definition 2.1. Consider the labeled iso-triangular tiles

 , whose horizontal (Vertical) and side edges are of length 1 unit and $1/\sqrt{2}$ unit respectively.

Definition 2.2. Generally a pasting rule is a pair (x, y) of labeled tiles with distinct edges (not necessarily distinct).

For example, the triangular tile  and tile  are joined by the edges (z, t) , which means that the edge z of tile A is glued with the edge t of tile B , we get the pattern . Note that the edges are of same length. The set of all edge labels is called an edge set denoted by E .

Tile pasting rules of the tiles A, B, C, D are given below:

1. Tile A can be glued with tile B by the pasting rules $\{(a_1, b_1), (a_2, b_2), (a_3, b_3)\}$ with tile C by the rule $\{(a_3, c_1)\}$ and with tile D by the rule $\{(a_1, d_3)\}$
2. Tile B can be glued with tile A by the pasting rules $\{(b_1, a_1), (b_2, a_2), (b_3, a_3)\}$ with tile C by the rule $\{(b_1, c_3)\}$ and with tile D by the rule $\{(b_3, d_1)\}$
3. Tile C can be glued with tile A by the pasting rule $\{(c_1, a_3)\}$ with tile B by $\{(c_3, b_1)\}$ and with the tile D by the pasting rules $\{(c_1, d_1), (c_2, d_2), (c_3, d_3)\}$
4. Tile D can be glued with tile A by the pasting rule $\{(d_3, a_1)\}$ with tile B by $\{(d_1, b_3)\}$ and with the tile C by the pasting rules $\{(d_1, c_1), (d_2, c_2), (d_3, c_3)\}$.

Notations:

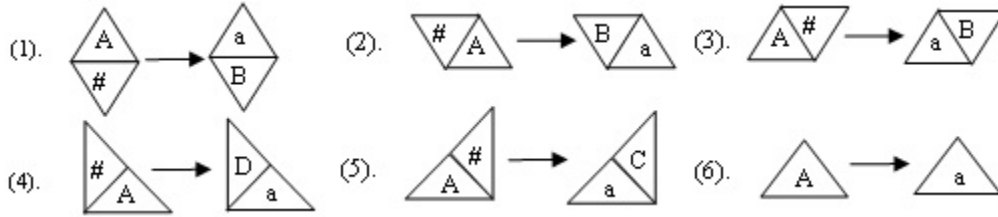
1. Iso-triangular tiles = $\left\{ \begin{array}{c} \triangle_{a_1 a_2 a_3}^A, \triangle_{a_{11} a_{12} a_{13}}^{A_1}, \triangle_{b_1 b_2 b_3}^B, \triangle_{b_{11} b_{12} b_{13}}^{B_1}, \triangle_{c_1 c_2 c_3}^C, \triangle_{d_1 d_2 d_3}^D \end{array} \right\}$
2. Non-terminal symbols $N = \left\{ \triangle_A, \triangle_B, \triangle_C, \triangle_D \right\}$
3. Terminal symbols $T = \left\{ \triangle_a, \triangle_b, \triangle_c, \triangle_d \right\}$

The edge set $E = \{a_1, a_2, a_3, a_{11}, a_{12}, a_{13}, b_1, b_2, b_3, b_{11}, b_{12}, b_{13}, c_1, c_2, c_3, d_1, d_2, d_3\}$. Note that in all further results the tiles are mentioned by their labels to avoid taking much space.

Definition 2.3. A Triangular tile pasting system (TTPS) is $S = (\Sigma, P, t_0)$, where Σ is a finite set of labeled triangular tiles, P is a finite set of pasting rules and t_0 is the axiom of pattern.

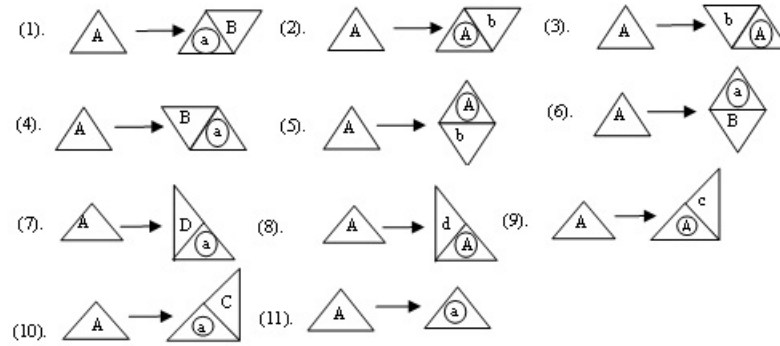
A pattern p_2 is generated from a pattern p_1 by applying the pasting rules in parallel manner to the edges of the pattern p_1 , where pasting is possible. Note that the labels of pasted edges in a pattern are ignored once the tiles are pasted. The set of all patterns generated from the axiom t_0 constitutes the picture language $L(TTPS)$.

Definition 2.4. A regular iso-array grammar (RIAG) is a structure $G = (N, T, P, S)$ where $N = \{A, B, C, D\}$, and $T = \{a, b, c, d\}$ are finite sets of symbols (isosceles right angled triangular tiles), $N \cap T = \emptyset$, Elements of N and T are called non-terminals and terminals respectively. $S \in N$, S is the start symbol of the axiom, P consists of rules of the following forms:



Similar rules can be given for the other tiles B, C and D . The set of all languages generated by RIAG is $RIAL$. By this grammar rules one can rewrite a non-terminal iso-triangular tile on the left hand side by a terminal iso-triangular tile on the right hand side. The non-terminal iso-triangular tile associated with the terminal on the right hand side of the rule can be admitted to the empty place.

Definition 2.5. Basic Puzzle Iso-Array Grammar is a structure $G = (N, T, P, S)$, where $N = \{A, B, C, D\}$ and $T = \{a, b, c, d\}$ are finite sets of symbols. The non terminals are in N and the terminal symbols are in T . $S \in N$ is the start symbol, P consists of the rules of the following forms:



Similar rules can be given for the other tiles B, C, D . The language generated by BPIAG is $L(BPIAG)$. The set of all languages generated by BPIAG is $BPIAL$. In BPIAG the non-terminal iso-triangular tile of left hand side of a rule can be replaced by a circled iso-triangular tile of right hand side.

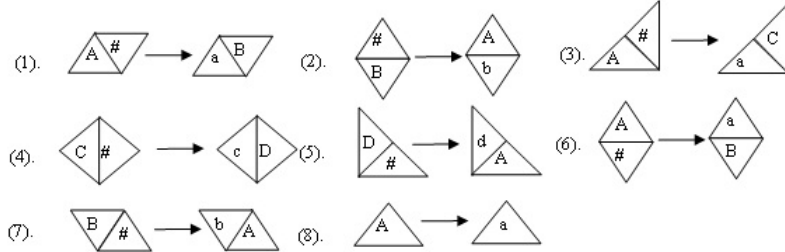
3. Triangular tile pasting P system

Definition 3.1. A triangular tile pasting P system (TTPPS) is defined as $\Pi = (\Sigma, \mu, F_1, F_2, \dots, F_m, R_1, R_2, \dots, R_m, i_0)$, where Σ is a finite set of iso-triangular tiles. μ is a membrane structure. In μ , the membranes are labeled in a one-one manner with the labels $1, 2, \dots, m$. $F_1, F_2, \dots, F_m$ are finite sets of picture over the tiles of Σ associated with m regions of the membranes. $R_1, R_2, \dots, R_m$ are finite sets of rules of the type rules $(t_i, (x_i, y_i), 1 \leq i \leq n)$ associated with m regions of μ and i_0 is the output membrane, which is an elementary membrane. An evaluation in TTPPS is defined in such a way that, to each picture pattern in each region of the system, a pasting rule could be applied. The picture pattern is moved (retained) in to another region (in the same region) by the target indication associated with the pasting rule.

A computation is successful only if the computation is stopped, that is if there is no further rule can be applied. The result of halting picture pattern is composed only by the pasting rules and the pattern halts in the membrane labeled by i_0 ; the pattern halts in the membrane labeled with i_0 . The set of all picture pattern computed by a TTPPS is denoted by $TTPPL(\Pi)$. The set of all such languages $TTPPL(\Pi)$ is generated by the system Π is denoted by $TTPPL_m$.

Example 3.1. A class of language of arrow heads can be generated by a RIAG with 4 distinct labeled iso-triangular tiles.

Consider the RIAG $G = (N, T, P, S)$ where $N = \{A, B, C, D\}$ and $T = \{a, b, c, d\}$ are finite sets of non-terminal and terminal symbols respectively. S is the labeled tile A , start symbol and P consists of the following rules;



The picture language is shown with three members in Figure 1

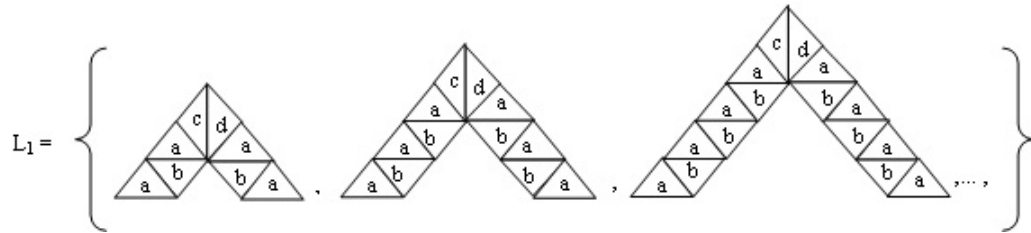
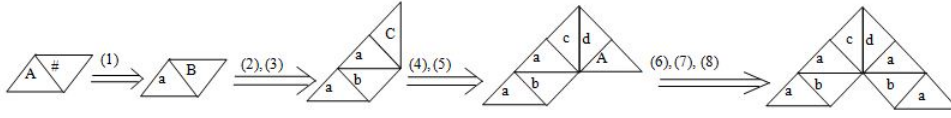


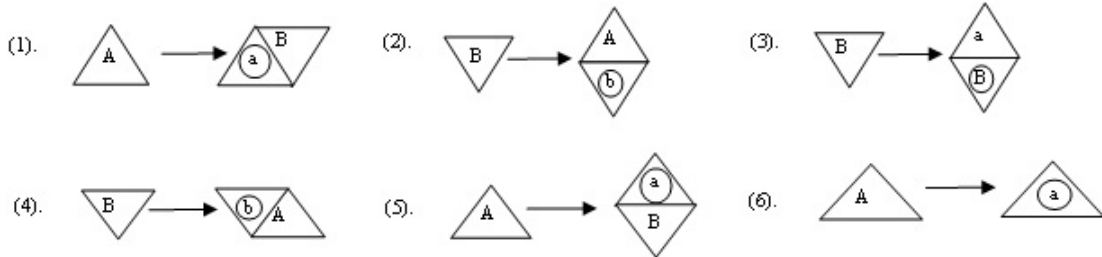
Figure 1: Language of arrow heads with the tiles “c” and “d” on the top

The first member of the picture language is shown with the derivation steps.



In the derivation of second member of the language the rules are taken in the sequence 1, 2, 1, 2, 3, 4, 5, 6, 7, 6, 7, 8. Note that only arrow heads can be generated by this RIAG.

Example 3.2. Consider the BPIAG $G = (N, T, P, S)$, where $N = \{A, B\}$, $T = \{a, b\}$, $S = A$ is the start symbol and P consists of the rules;



The picture language consists of arrow heads with a tile on the top is generated by BPIAG. The picture language is shown in figure 2.

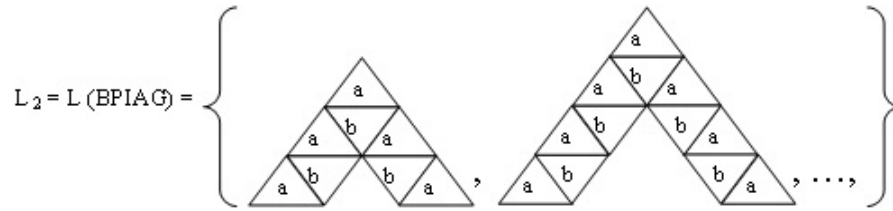


Figure 2: The language of arrow heads with tile “ a ” on the top

The first member of the languages can be derived with the sequence of rules 1, 2, 1, 3, 4, 5, 4, 6.

Example 3.3. The triangular tile pasting P system $\Pi_1 = (\Sigma, [1[2]2[3]3]_1, F_1, F_2, F_3, R_1, R_2, R_3, 1)$ generates the family of two dimensional iso-triangular picture languages $\mathcal{L}_1$ and $\mathcal{L}_2$. Here $\Sigma = \{A, A_1, B, B_1\}$, $F_2 = A$, $F_1 = \phi$, $F_3 = \phi$, $R_1 = \{(A, (a_3, b_3), in_2), (B_1, (b_{11}, a_1), here)\}$, $R_2 = \{(B, (b_2, a_{12}), in_3), (B, (b_1, a_{11}), in_3), (A_1, (a_{13}, b_{13}), in_3)\}$, $R_3 = \{(B, (b_2, a_2), in_2), (B_1, (b_{12}, a_{12}), out), (B, (b_1, a_1), out)\}$ and 1 is the output region.

The rules of R_1 , R_2 and R_3 are applied with the target indications which belongs to the set $tar \in \{here, in, out\}$ and then the resultant iso-triangular picture

pattern is collected in the output region one finally. Two members of the Picture language $\mathcal{L}_1$ is shown below with derivation steps and $\mathcal{L}_1$ is shown in figure 3.

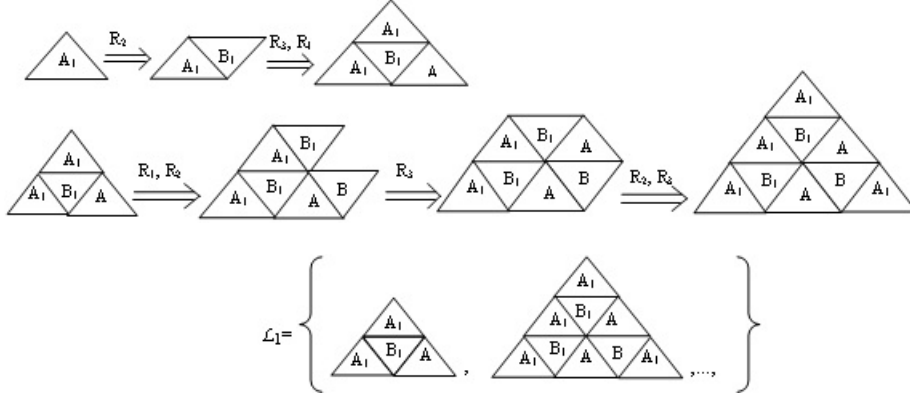


Figure 3: Language consists of iso-triangular arrays

On the other hand, the computation starts from the region R_2 . The first member of the language “adjoin of iso-triangles” is collected in the region 1. Again the generation is started in the same region one the 2nd member of the language is generated and it is collected in the region one. The computation is continued in this way, the members of the language are collected in the region 1.

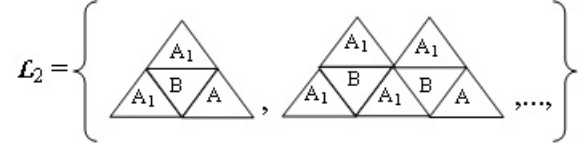


Figure 4: The family of languages of adjoin of iso-triangles

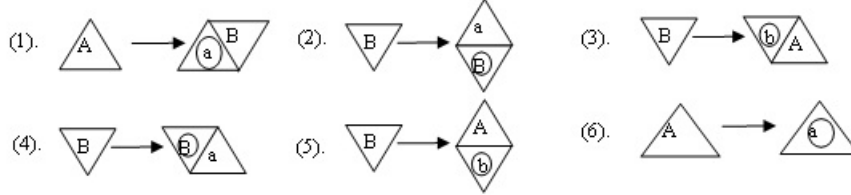
Theorem 3.1. $L(BPIAG) - L(RIAG) \neq \phi$

Proof. From the Definitions 2.4 and 2.5 all the rules of BPIAG contains the rules of RIAG. Hence it is clear that RIAG is the proper subset of BPIAG. In BPIAG the non-terminal symbol in the left hand side of the rules are replaced by the circled symbol (The circled symbol being either non-terminal or terminal). But in RIAG the non-terminal symbols of left hand side of the rules are replaced only by terminal symbols. The picture language L_2 shown in Example 3.2 cannot be generated by any RIAG. In L_2 the top of each member has iso-triangular tile a and in L_1 top of each member has iso-triangular tiles c and d . Here the shapes of L_1 and L_2 are same but they are not equal. $\square$

Theorem 3.2. $TTPPL_3 - BPIAL \neq \phi$

Proof. The language $\mathcal{L}_1$ is generated by the TTPPS Π_1 explained in example 3.3 with the rules $R_1 = \{(A, (a_3, b_3), in_2), (B, (b_{11}, a_1), here)\}$, $R_2 = \{(B, (b_2, a_{12}),$

in_3), $(B, (b_1, a_{11}), in_3)$, $(A_1, (a_{13}, b_{13}), in_3)$ }, $R_3 = \{(B, (b_2, a_2), out), (B_1, (b_{12}, a_{12}), out), (B, (b_1, a_1), out)\}$ (refer Example 3.3) cannot be generated by any BPIAG. Since in BPIAG, the non-terminal symbols of left hand side of each rule is replaced only by the circled symbol (circled symbol may be non-terminal or terminal). By the rules of BPIAG the resultant iso-array shown in Figure 3 cannot be generated. For example, consider the BPIAG $G_1 = (N, T, P, S)$, where $N = \{A, B\}$, $T = \{a, b\}$, $S = A$ and P consists of the rules;



The sequence of rules 1, 4, 5, 6 and the sequence of rules 1, 2, 3, 6 of BPIAG generate the languages L_3 and L_4 respectively shown in Figures 5 and 6. $\square$

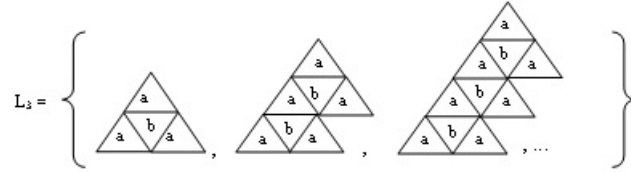


Figure 5: Language consists of step model of iso-triangles

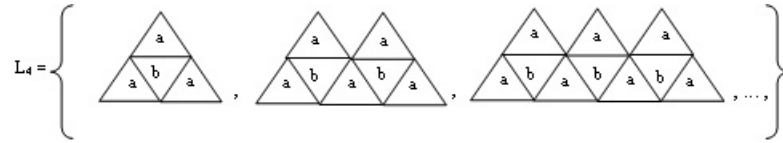


Figure 6: Language consists of adjoin of iso-triangles

Theorem 3.3. $TTPPL_3 - RIAL \neq \phi$

Proof. The family of languages $\mathcal{L}_3$ is generated by $TTPPS \Pi_2 = (\Sigma, [_1[_2[_3[_3]_1]_1, F_1, F_2, F_3, R_1, R_2, R_3, 3)$, where $\Sigma = \{A, B, B_1\}$, $F_1 = A$, $F_2 = \phi$, $F_3 = \phi$, $R_1 = \{(A, (a_3, b_3), here), (B, (b_2, a_2), here), (A, (a_3, c_1), in_2)\}$, $R_2 = \{(C, (c_2, d_2), here), (D, (d_3, a_1), in_3)\}$, $R_3 = \{(A, (a_2, b_{12}), here), (B_1, (b_{11}, a_1), here)\}$ and 3 is the output region. The family of language $\mathcal{L}_3$ generated by the above rules are collected in the region three of the membrane 3. It is shown in the Figure 7. This family of language cannot be generated by any RIAG. (refer Example 3.3). $\square$

Theorem 3.4. $TTPPL_3 \cap L(TTPS) \neq \phi$

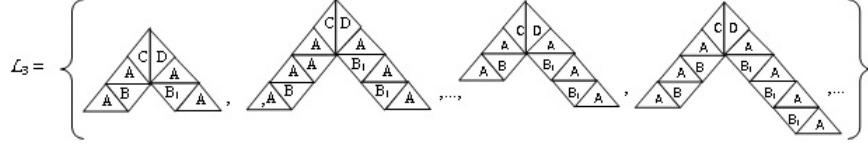


Figure 7: Language consists of arrow heads and arrow heads with long tail on the right

Proof. The triangular tile pasting system $G = (\Sigma, P, t_0)$ generates the picture language L_5 , where $\Sigma = \{A, B, B_1\}$, $t_0 = A$ and $P = \{(a_3, b_3), (b_2, a_2), (a_3, c_1), (c_2, d_2), (d_3, a_1), (a_2, b_{12}), (b_{11}, a_1)\}$. The computation starts with the rule (a_3, b_3) with the tile A . Then the rules given in P are applied one by one. The resultant language is $\mathcal{L}_3$. It is shown in Figure 7.

On the other hand if the computation starts by the rule (a_2, b_{12}) , the language L_5 can be generated and which is given in the Figure 8. Thus the TTPS generate the families of languages $\mathcal{L}_3$ and L_5 .

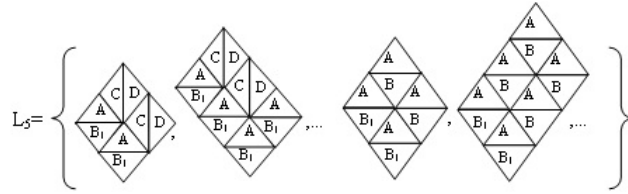


Figure 8: A language of overlaps of Rhombuses

TTPPS defined in Theorem 3.3 generates the picture language $\mathcal{L}_3$ shown in the Figure 7 (refer Theorem 3.3). Hence the intersection of families of languages generated by TTPPS and TTPS is non-empty. $\square$

4. Conditional communication on triangular tile pasting P system

Definition 4.1. Conditional communication on triangular tile pasting P system (CCTTPPS) is defined in a way such that $\Pi = (V, \mu, (t_0)_i, syn, (R_{(i,j)}, P_{(i,j)}, F_{(i,j)}), i_0)$ where V is the set of all labeled iso-triangular tiles, μ is the membrane structure contains finite no of membranes, which are labeled from the set $\{1, 2, 3, \dots, n\}$, t_0 is the axiom of picture pattern, present in the membrane i ($1 \leq i \leq m$), $Syn = \{(i, j)/i, j \in k \ (1 \leq k \leq m)\}$ is the set of links of the membranes, $R_{(i,j)}$ is the finite set of evolution rules over the set of iso-triangular tiles V present in the regions of the membrane system. An evolution rule is a pair $\{(\alpha_i, \beta_i)/\alpha_i, \beta_i \in E\}$, $P_{(i,j)}$ is the set of permitting conditions and $F_{(i,j)}$ is the set of forbidden conditions associated with region (i, j) where $1 \leq i, j \leq m$. The conditions can be defined as follows;

1. **Empty** There is no restriction or condition given on generating the pattern and the pattern transferred from one membrane to another membrane freely. We denote the permitting and forbidden conditions are empty. An empty permitting condition is denoted by $(true, in_j)$ and an empty forbidden condition by $(false, in_j)$.

2. **Labeled Tile Checking** $P_{(i,j)}$ present in each region is a set of pair (α, in_j) for $\alpha \in V$ and in_j is the target indication which transfer the pattern to the membrane j and $F_{(i,j)}$ present in each region of membrane is a pair $(\beta, not\ in_j)$ for $\beta \in V$. A pattern can be sent from a membrane to another membrane if the pair $(\alpha, in_j) \in P_{(i,j)}$ and if the pair $(\beta, not\ in_j) \in F_{(i,j)}$, β does not belongs to $P_{(i,j)}$.

3. **Sub Pattern Checking** Each P_i is a set of pair (u, in_j) for $u \in V^+$ and each F_i is a set of pair $(v, not\ in_j)$ for $v \in V$. A pattern p can go to another membrane only if there is a pair $(u, in_j) \in P_i$ with $u \in sub\ pattern(p)$ and for each pair $(v, not\ in_j) \in F_i$ with $v \notin sub\ pattern(p)$.

The evolution method is explained below.

The tile pasting rules present in each region is applied non-deterministically. The pattern obtained in this way is checked through by the conditions in $P_{(i,j)}$ and $F_{(i,j)}$ presented in the corresponding region. If the condition $P_{(i,j)}$ is satisfied in the region then pattern sent to the j^{th} membrane. If the pattern fulfills the forbidden condition $F_{(i,j)}$ then the pattern retained in the same region. If the pattern is not satisfied by both the conditions then the picture pattern retained in the same region of the membrane. The computational process continued in this manner and then the resultant picture pattern is collected in the output membrane. If the permitting condition $P_{(i,j)}$ is never satisfied by the pattern, then the computation is an unsuccessful one.

The set of all such picture patterns generated by this way is denoted by $L(CCTTPPS)$. The family of all such languages $L(CCTTPPS)$ generated by the system $CCTTPPS(\Pi)$ is denoted by $L_m(CCTTPPS)$.

Example 4.1. Consider $\Pi = (V, [1[2]2[3]3]_1, t_0, syn, (R_i, P_i, F_i), i_0 = 1)$,

where $V = \{A, A_1, B, B_1, C, D\}$, $t_0 =$ 

$F_1 = \phi$, $F_2 = (\triangle_{A_1}^C, out)$, $F_3 = \phi$. $F_3 = \phi$, $F_3 = \phi$

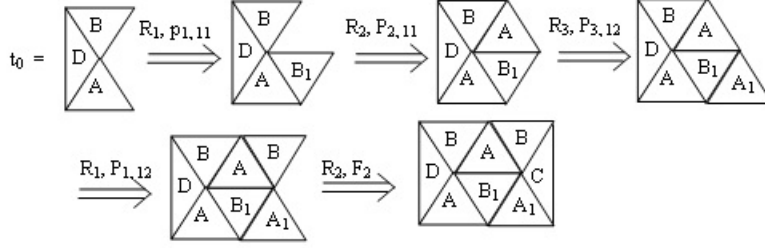
$R_1 = \{(a_3, b_{13}), (a_3, b_3), (b_{12}, a_2)\}$, $P_1 = \{(P_{1,11}, in_2), (P_{1,12}, in_2)\}$,

$R_2 = \{(b_1, a_1), (a_{13}, c_1), (a_{13}, b_{13})\}$, $P_2 = \{(P_{2,11}, in_3), (P_{2,12}, in_3)\}$,

$R_3 = \{(b_{11}, a_{11})\}$, $P_3 = \{(P_{3,11}, in_1), (P_{3,12}, in_2)\}$,

$P_{1,11} =$ , $P_{1,12} =$ , $P_{2,11} =$ , $P_{2,12} =$ ,
 $P_{3,11} =$ , $P_{3,12} =$ 

The first member of the picture language is shown with the derivation steps below.



The language is shown with three members in Figure 9.

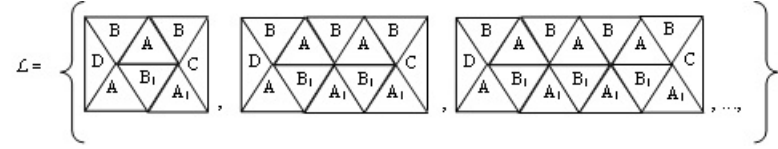


Figure 9: Language of rectangles with size of $4(n + 1)$ iso-triangular tiles

Each member of the language consists of $4(n + 1)$ iso-triangular tiles and n is the number of the member of the languages.

Theorem 4.1. $\mathcal{L}(TTPS)$ and $\mathcal{L}(CCTTPPS)$ are not disjoint

It is true from the Definitions 2.3 and 4.1

Theorem 4.2. $\mathcal{L}(TTPS) - \mathcal{L}(CCTTPPS) \neq \phi$

It is true from the Definition 3.1 and 4.1

5. Conclusion

In this paper we have defined the triangular tile pasting P system with conditional communication and explained with suitable example. The generating powers of regular iso array grammar and basic puzzle iso-array grammar have been studied. The computational powers of the triangular tile pasting system, the triangular tile pasting P system and conditional communicational triangular tile pasting P system are examined.

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A HIGH-ORDER ACCURACY EXPLICIT DIFFERENCE SCHEME WITH BRANCHING STABILITY FOR SOLVING FOUR-DIMENSIONAL PARABOLIC EQUATIONS

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Abstract. A high-order explicit difference scheme for solving four-dimensional parabolic equations is given. The scheme is constructed by the method of undetermined coefficients, and appropriate parameter is chosen to endow the truncation error of schemes is $O(\Delta t^4 + \Delta x^4)$. And the new difference scheme is proved to be stable if $r \leq \frac{1}{12}$ with the Fourier analysis method. Finally, the numerical experiment shows the numerical solutions of difference scheme and the exact solutions are matched and the difference scheme is effective.

Keywords: four-dimensional parabolic equations, explicit difference scheme, truncation error.

1. Introduction

Many mathematical models of modern science, technology and engineering can be described by partial differential equation and lots of basic equations themselves in natural science are partial differential equations. People have been always using differential equation to describe, explain and predict all kinds of natural phenomena, which turns out to be successful. However, many partial differential equations don't have analytical solutions that people can only figure out its numerical solution through all kinds of methods. Up till now, differential equations main numerical computation methods are finite difference method and finite element method, meanwhile, there are boundary element, mixed finite element, spectral method and finite volume method etc, among which, finite difference method is still the more effective method to solve the numerical solution of partial differential equation.

The difference method of parabolic equation is a classical problem in numerical solution of partial differential equation. Equations described the motion law of underground flow are mostly parabolic equations, in particular, the motion of underground petroleum and natural gas are the typical examples of using parabolic equation to describe. We can also encounter the following multidimen-

sional parabolic equations in the fields of seepage, diffusion, heat conduction and so on

$$(1) \quad \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial w^2}, 0 < x, y, z, w < 1, t > 0,$$

$$(2) \quad u(x, y, z, w, 0) = \varphi(x, y, z, w), 0 \leq x, y, z, w \leq 1,$$

$$(3) \quad \begin{aligned} u(0, y, z, w, t) &= f_1(y, z, w, t), u(1, y, z, w, t) \\ &= f_2(y, z, w, t), 0 \leq y, z, w \leq 1, 0 \leq t \leq T, \end{aligned}$$

$$(4) \quad \begin{aligned} u(x, 0, z, w, t) &= g_1(x, z, w, t), u(x, 1, z, w, t) \\ &= g_2(x, z, w, t), 0 \leq x, z, w \leq 1, 0 \leq t \leq T, \end{aligned}$$

$$(5) \quad \begin{aligned} u(x, y, 0, w, t) &= h_1(x, y, w, t), u(x, y, 1, w, t) \\ &= h_2(x, y, w, t), 0 \leq x, y, w \leq 1, 0 \leq t \leq T, \end{aligned}$$

$$(6) \quad \begin{aligned} u(x, y, z, 0, t) &= k_1(x, y, z, t), u(x, y, z, 1, t) \\ &= k_2(x, y, z, t), 0 \leq x, y, z \leq 1, 0 \leq t \leq T. \end{aligned}$$

Where $\varphi, f_1, f_2, g_1, g_2, h_1, h_2, k_1$ and k_2 are sufficiently smooth functions.

Various numerical finite difference schemes have been proposed to solve parabolic problems approximately. For multidimensional problems, the explicit difference scheme and implicit difference scheme are the common finite difference schemes. The implicit difference scheme has the advantage of good stability, but it is needed to solve different linear equations on each time layer which will cost to big computation. The alternating-direction implicit (ADI) difference scheme can overcome these disadvantages.

As we known, the ADI scheme is unconditional stable and only need to solve a sequence of tridiagonal linear systems ^[1-2]. In recent years, there are many new methods which use ADI scheme to solve the multidimensional parabolic equations ^[3-9], some of them have the accuracy of $O(\Delta t^2 + \Delta x^4)$ ^[3,6,7,8,9].

The explicit difference scheme has worse stability than the implicit difference scheme, but it has the advantage of smaller amount of calculation. The general explicit scheme is the classical explicit scheme with the stability condition of $r \leq \frac{1}{8}$, where $r = \frac{\Delta t}{\Delta x^2}$ is the mesh spacing ratio. Its deficiency is that the accuracy is not high, and its truncation error is $O(\Delta t + \Delta x^2)$. Recently, there has been a interest in the development and application of explicit difference schemes for the numerical solution of multidimensional parabolic equations ^[10-12], but the schemes only have the truncation error of $O(\Delta t^2 + \Delta x^4)$. For four-dimensional situation, in reference [12], Ma constructed an explicit scheme with the truncation error of $O(\Delta t^2 + \Delta x^4)$ and the stability condition is $r < \frac{3}{8}$, the scheme's accuracy is not high enough. This paper presents an explicit scheme for solving Eq.(1), the stability condition is $r \leq \frac{1}{12}$, and the truncation error is $O(\Delta t^4 + \Delta x^4)$, the scheme has higher accuracy than the above schemes.

The remainder of this paper is organized as follows. In Section 2, we construct a three-layer explicit difference scheme with the accuracy of $O(\Delta t^4 + \Delta x^4)$;

In Section 3, by using the Fourier analysis method, it is proved that the difference scheme is stable when $r \leq \frac{1}{12}$. In Section 4, though choosing the proper parameter θ , we obtain a three-level explicit scheme with branching stability. In Section 5, we compare the difference of exact solution and the scheme constructed in this paper with that in the reference [12], and compare the computational efficiency of the two difference schemes with the classical explicit scheme. The results shows that the difference scheme in this paper is effective.

2. Construction of the difference scheme

Let Δt denote the step length of time and $\Delta x = \Delta y = \Delta z = \Delta w$ be the step length of space in the direction of x, y, z, w respectively. We approximate the Eq.(1) using the following difference equation with parameters

$$(7) \quad \begin{aligned} & \Delta_t u_{ijkl}^n + \theta_1 \Delta_t u_{ijkl}^{n-1} + \theta_2 \Delta_t \left(\frac{1}{3} \diamond \right) u_{ijkl}^{n-1} + \theta_3 \Delta_t \left(\frac{1}{12} \square \right) u_{ijkl}^{n-1} \\ &= \frac{1}{\Delta x^2} \left[\left(\frac{\theta_4}{12} \square + \frac{\theta_5}{3} \diamond \right) u_{ijkl}^n + \left(\frac{\theta_6}{12} \square + \frac{\theta_7}{3} \diamond \right) u_{ijkl}^{n-1} \right], \end{aligned}$$

where u_{ijkl}^n denotes the value of u at node $(i\Delta x, j\Delta y, k\Delta z, l\Delta w, n\Delta t)$, $\Delta_t u_{ijkl}^n = \frac{u_{ijkl}^{n+1} - u_{ijkl}^n}{\Delta t}$, and

$$\begin{aligned} \square u_{ijkl}^n &= (x\square + y\square + z\square + w\square) u_{ijkl}^n, \\ \diamond u_{ijkl}^n &= (x\diamond + y\diamond + z\diamond + w\diamond) u_{ijkl}^n, \\ x\square u_{ijkl}^n &= u_{i,j+1,k+1,l+1}^n + u_{i,j-1,k+1,l+1}^n \\ &\quad + u_{i,j+1,k-1,l+1}^n + u_{i,j-1,k-1,l+1}^n + u_{i,j+1,k+1,l-1}^n \\ &\quad + u_{i,j-1,k+1,l-1}^n + u_{i,j+1,k-1,l-1}^n + u_{i,j-1,k-1,l-1}^n - 8u_{ijkl}^n, \\ x\diamond u_{ijkl}^n &= u_{i,j+1,k,l}^n + u_{i,j-1,k,l}^n + u_{i,j,k+1,l}^n \\ &\quad + u_{i,j,k-1,l}^n + u_{i,j,k,l+1}^n + u_{i,j,k,l-1}^n - 6u_{ijkl}^n, \end{aligned}$$

the rest can be inferred by analogy. $\theta_1 - \theta_7$ are parameters to be determined. A proper choice of undetermined parameters $\theta_1 - \theta_7$ can make difference equation (7) approach Eq.(1), and not only has truncation error with order as high as possible, but also has higher stability.

When the solution of Eq.(1) is smooth enough, we can get the following relation:

$$(8) \quad \frac{\partial^n}{\partial t^n} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial w^2} \right)^m u = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial w^2} \right)^{m+2n} u.$$

Using Taylor's expansion of u at node $(i\Delta x, j\Delta y, k\Delta z, l\Delta w, n\Delta t)$, we have

$$\begin{aligned}\Delta_t u_{ijkl}^n &= \frac{\partial u}{\partial x} + \frac{\Delta t}{2} \frac{\partial^2 u}{\partial x^2} + \frac{\Delta t^2}{6} \frac{\partial^3 u}{\partial x^3} + O(\Delta t^3) \\ \Delta_t u_{ijkl}^{n-1} &= \frac{\partial u}{\partial x} - \frac{\Delta t}{2} \frac{\partial^2 u}{\partial x^2} + \frac{\Delta t^2}{6} \frac{\partial^3 u}{\partial x^3} + O(\Delta t^3) \\ \diamond u_{ijkl}^n &= 3\Delta x^2 \frac{\partial u}{\partial t} + \frac{\Delta x^4}{4} \frac{\partial^2 u}{\partial t^2} - \frac{\Delta x^4}{2} \left(\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial x^2 \partial z^2} \right. \\ &\quad \left. + \frac{\partial^4 u}{\partial x^2 \partial w^2} + \frac{\partial^4 u}{\partial y^2 \partial z^2} + \frac{\partial^4 u}{\partial y^2 \partial w^2} + \frac{\partial^4 u}{\partial z^2 \partial w^2} \right) + O(\Delta x^6) \\ \square u_{ijkl}^n &= 12\Delta x^2 \frac{\partial u}{\partial t} + \Delta x^4 \frac{\partial^2 u}{\partial t^2} + 2\Delta x^4 \left(\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial x^2 \partial z^2} \right. \\ &\quad \left. + \frac{\partial^4 u}{\partial x^2 \partial w^2} + \frac{\partial^4 u}{\partial y^2 \partial z^2} + \frac{\partial^4 u}{\partial y^2 \partial w^2} + \frac{\partial^4 u}{\partial z^2 \partial w^2} \right) + O(\Delta x^6).\end{aligned}$$

Substituting the above Taylor expansions into (7) and using relation (8), we can obtain

$$\begin{aligned}& (1 + \theta_1) \frac{\partial u}{\partial t} + \frac{\Delta t}{2} (1 - \theta_1) \frac{\partial^2 u}{\partial t^2} + \frac{\Delta t^2}{6} (1 + \theta_1) \frac{\partial^3 u}{\partial t^3} + \frac{\Delta t^3}{24} (1 - \theta_1) \frac{\partial^4 u}{\partial t^4} \\ & + \Delta x^2 (\theta_2 + \theta_3) \frac{\partial^2 u}{\partial t^2} - \frac{\Delta t \Delta x^2}{2} (\theta_2 + \theta_3) \frac{\partial^3 u}{\partial t^3} + \frac{\Delta t^2 \Delta x^2}{6} (\theta_2 + \theta_3) \frac{\partial^4 u}{\partial t^4} \\ & = (\theta_4 + \theta_5 + \theta_6 + \theta_7) \frac{\partial u}{\partial t} + \frac{\Delta x^2}{12} (\theta_4 + \theta_5 + \theta_6 + \theta_7) \frac{\partial^2 u}{\partial t^2} - \Delta t (\theta_6 + \theta_7) \frac{\partial^2 u}{\partial t^2} \\ & + \frac{\Delta x^2}{6} [(\theta_4 + \theta_6) - (\theta_5 + \theta_7)] \left(\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial x^2 \partial z^2} + \frac{\partial^4 u}{\partial x^2 \partial w^2} \right. \\ & \quad \left. + \frac{\partial^4 u}{\partial y^2 \partial z^2} + \frac{\partial^4 u}{\partial y^2 \partial w^2} + \frac{\partial^4 u}{\partial z^2 \partial w^2} \right) \\ & + \frac{\Delta t^2}{2} (\theta_6 + \theta_7) \frac{\partial^3 u}{\partial t^3} - \frac{\Delta t \Delta x^2}{12} (\theta_6 + \theta_7) \frac{\partial^3 u}{\partial t^3} + \frac{\Delta t^2 \Delta x^2}{24} (\theta_6 + \theta_7) \frac{\partial^4 u}{\partial t^4} \\ & - \frac{\Delta t \Delta x^2}{6} (\theta_6 - \theta_7) \left(\frac{\partial^5 u}{\partial x^2 \partial y^2 \partial t} + \frac{\partial^5 u}{\partial x^2 \partial z^2 \partial t} + \frac{\partial^5 u}{\partial x^2 \partial w^2 \partial t} \right. \\ & \quad \left. + \frac{\partial^5 u}{\partial y^2 \partial z^2 \partial t} + \frac{\partial^5 u}{\partial y^2 \partial w^2 \partial t} + \frac{\partial^5 u}{\partial z^2 \partial w^2 \partial t} \right) \\ & + \frac{\Delta t^2 \Delta x^2}{12} (\theta_6 - \theta_7) \left(\frac{\partial^6 u}{\partial x^2 \partial y^2 \partial t^2} + \frac{\partial^6 u}{\partial x^2 \partial z^2 \partial t^2} + \frac{\partial^6 u}{\partial x^2 \partial w^2 \partial t^2} \right. \\ & \quad \left. + \frac{\partial^6 u}{\partial y^2 \partial z^2 \partial t^2} + \frac{\partial^6 u}{\partial y^2 \partial w^2 \partial t^2} + \frac{\partial^6 u}{\partial z^2 \partial w^2 \partial t^2} \right) \\ & - \frac{\Delta t^3}{6} (\theta_6 + \theta_7) \frac{\partial^4 u}{\partial t^4} + O(\Delta t^4 + \Delta x^4).\end{aligned}$$

In order to make the truncation error of scheme (7) getting to $O(\Delta t^4 + \Delta x^4)$, the following equation system should be available.

$$(9) \quad \begin{cases} 1 + \theta_1 = \theta_4 + \theta_5 + \theta_6 + \theta_7 \\ \frac{r}{2}(1 - \theta_1) + \theta_2 + \theta_3 = \frac{1}{12}(\theta_4 + \theta_5 + \theta_6 + \theta_7) - r(\theta_6 + \theta_7) \\ \theta_4 + \theta_6 - \theta_5 - \theta_7 = 0 \\ \frac{r^2}{6}(1 + \theta_1) - \frac{r}{2}(\theta_2 + \theta_3) = \frac{r^2}{2}(\theta_6 + \theta_7) - \frac{r}{12}(\theta_6 + \theta_7) \\ \frac{r^3}{24}(1 - \theta_1) + \frac{r^2}{6}(\theta_2 + \theta_3) = \frac{r^2}{24}(\theta_6 + \theta_7) - \frac{r^3}{6}(\theta_6 + \theta_7) \\ \theta_6 - \theta_7 = 0. \end{cases}$$

Where $r = \frac{\Delta t}{\Delta x^2}$. Let $\theta_3 = \theta$, the solution of the above equation system is:

$$\theta_1 = 24r - 1; \theta_2 = -24r^3 + 6r^2 + r - \theta; \theta_4 = \theta_5 = 9r - 12r^2; \theta_6 = \theta_7 = 12r^2 + 3r.$$

Substituting the above values into (7), we obtain the following single parameter three-level explicit difference scheme with its truncation error getting to $O(\Delta t^4 + \Delta x^4)$.

$$(10) \quad \begin{aligned} 12u_{ijkl}^{n+1} &= [24(1 - 2r) + (9r^2 - 12r^3 - \theta)\square + (48r^3 + 12r^2 - 4r + 4\theta)\diamondsuit]u_{ijkl}^n \\ &+ [12(24r - 1) + (12r^3 + 3r^2 + \theta)\square + (-48r^3 + 36r^2 + 4r - 4\theta)\diamondsuit]u_{ijkl}^{n-1}. \end{aligned}$$

3. Analysis of stability

According to Fourier method for analyzing stability. The two-level equation system equivalent to (10) is

$$(11) \quad \begin{cases} u_{ijkl}^{n+1} = \frac{24(1-2r) + (9r^2 - 12r^3 - \theta)\square + (48r^3 + 12r^2 - 4r + 4\theta)\diamondsuit}{12} u_{ijkl}^n \\ \quad + \frac{12(24r-1) + (12r^3 + 3r^2 + \theta)\square + (-48r^3 + 36r^2 + 4r - 4\theta)\diamondsuit}{12} u_{ijkl}^{n-1} \\ v_{ijkl}^{n+1} = u_{ijkl}^n. \end{cases}$$

Let

$$(12) \quad u_{ijkl}^n = U^n e^{I(i\theta + j\varphi + k\psi + l\zeta)}, v_{ijkl}^n = V^n e^{I(i\theta + j\varphi + k\psi + l\zeta)},$$

where $I = \sqrt{-1}$. And through simple calculation, we have

$$(13) \quad \square u_{ijkl}^n = -4s_2 u_{ijkl}^n, \diamondsuit u_{ijkl}^n = -12s_1 u_{ijkl}^n,$$

where

$$\begin{aligned} s_1 &= \sin^2 \frac{\theta}{2} + \sin^2 \frac{\varphi}{2} + \sin^2 \frac{\psi}{2} + \sin^2 \frac{\zeta}{2} \in [0, 4] \\ s_2 &= \sin^2 \frac{\theta + \varphi + \psi}{2} + \sin^2 \frac{\theta - \varphi + \psi}{2} + \sin^2 \frac{\theta + \varphi - \psi}{2} \\ &\quad + \sin^2 \frac{\theta - \varphi - \psi}{2} + \sin^2 \frac{\theta + \varphi + \zeta}{2} + \sin^2 \frac{\theta + \varphi - \zeta}{2} + \sin^2 \frac{\theta - \varphi + \zeta}{2} \\ &\quad + \sin^2 \frac{\theta - \varphi - \zeta}{2} + \sin^2 \frac{\theta + \psi + \zeta}{2} + \sin^2 \frac{\theta + \psi - \zeta}{2} + \sin^2 \frac{\theta - \psi + \zeta}{2} + \sin^2 \frac{\theta - \psi - \zeta}{2} \\ &\quad + \sin^2 \frac{\varphi + \psi + \zeta}{2} + \sin^2 \frac{\varphi + \psi - \zeta}{2} + \sin^2 \frac{\varphi - \psi + \zeta}{2} + \sin^2 \frac{\varphi - \psi - \zeta}{2} \in [0, 16]. \end{aligned}$$

Substituting (12) into (11) and using (13) we obtain

$$\begin{bmatrix} U^{k+1} \\ V^{k+1} \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} U^k \\ V^k \end{bmatrix} = \mathbf{G}(\mathbf{s}_1, \mathbf{s}_2) \begin{bmatrix} U^k \\ V^k \end{bmatrix}$$

where

$$\begin{aligned} g_{11} &= 2(1 - 12r) - \left(3r^2 - 4r^3 - \frac{\theta}{3}\right) s_2 - 4(12r^3 + 3r^2 - r + \theta) s_1, \\ g_{12} &= 24r - 1 - \left(4r^3 + r^2 + \frac{\theta}{3}\right) s_2 - 4(-12r^3 + 9r^2 + r - \theta) s_1, \quad g_{21} = 1, \quad g_{22} = 0. \end{aligned}$$

The characteristic equation of propagation matrix $\mathbf{G}(\mathbf{s}_1, \mathbf{s}_2)$ is

$$(14) \quad \lambda^2 - g_{11}\lambda - g_{12} = 0.$$

Lemma 1 ([13]). *The two roots of real coefficient quadratic Eq.(14) are less than or equal to 1 in norm if and only if*

$$(15) \quad |g_{11}| \leq 1 - g_{12} \leq 2.$$

Lemma 2 ([13]). *The difference scheme (10) is stable, i.e., the family of matrices $\mathbf{G}^n(\mathbf{s}_1, \mathbf{s}_2)$ ($(s_1, s_2) \in [0, 4] \times [0, 16]$, $n = 1, 2, \dots$) is uniformly bounded if and only if*

- (1) $|\lambda_{1,2}| \leq 1$ ($\lambda_{1,2}$ are roots of (14));
- (2) (s_1, s_2) which assures $1 - g_{11}^2/4 = g_{11}^2 + 4g_{12} = 0$ is not existent or not in the region of $[0, 4] \times [0, 16]$.

Theorem 1. *A sufficient condition for scheme (10) being stable is*

$$(16) \quad r \leq \frac{1}{12}, \max \left\{ -12r^3 + 3r^2, -12r^3 + 15r^2 - \frac{3}{4}r \right\} \leq \theta \leq -12r^3 + 3r^2 + r.$$

Proof. If $g_{12} \neq -1, 1 - g_{11}^2/4 = g_{11}^2 + 4g_{12} = 0$ do not hold for any (s_1, s_2) . By Lemma 1 and Lemma 2, the stability conditions of scheme (10) become

$$-1 + g_{12} \leq g_{11} \leq 1 - g_{12} < 2.$$

From $g_{11} \leq 1 - g_{12}$, we have

$$(17) \quad -4r^2s_2 - 48r^2s_1 \leq 0.$$

it is hold unconditionally.

Because $1 - g_{12} < 2$, we have

$$(18) \quad 24r - \left(4r^3 + r^2 + \frac{\theta}{3}\right)s_2 - 4(-12r^3 + 9r^2 + r - \theta)s_1 > 0.$$

A sufficient condition which assures the above inequality hold is

$$(19) \quad 4r^3 + r^2 + \frac{\theta}{3} \geq 0,$$

$$(20) \quad -12r^3 + 9r^2 + r - \theta \geq 0,$$

$$(21) \quad 24r - 16\left(4r^3 + r^2 + \frac{\theta}{3}\right) - 16(-12r^3 + 9r^2 + r - \theta) > 0,$$

it is equivalent to

$$(22) \quad -12r^3 - 3r^2 \leq \theta \leq -12r^3 + 9r^2 + r,$$

$$(23) \quad \theta > -12r^3 + 15r^2 - \frac{3}{4}r.$$

Using $-1 + g_{12} \leq g_{11}$, we obtain

$$(24) \quad 3(16r - 1) - \left(8r^3 - 2r^2 + \frac{2}{3}\theta\right)s_2 - 4(-24r^3 + 6r^2 + 2r - 2\theta)s_1 \leq 1.$$

A sufficient condition which assures the above inequality hold is

$$(25) \quad 8r^3 - 2r^2 + \frac{2}{3}\theta \geq 0,$$

$$(26) \quad -24r^3 + 6r^2 + 2r - 2\theta \geq 0,$$

$$(27) \quad 3(16r - 1) \leq 1,$$

it is equivalent to

$$(28) \quad -12r^3 + 3r^2 \leq \theta \leq -12r^3 + 3r^2 + r,$$

$$(29) \quad r \leq \frac{1}{12}.$$

By combining the inequalities (22),(23),(28) and (29), we complete the proof.

4. Choice of parameter and determination of difference scheme

We should choose parameters θ such that (16) is satisfied. Provide methods as follows:

If $-12r^3 + 3r^2 \geq -12r^3 + 15r^2 - \frac{3}{4}r$, we have $r \leq \frac{1}{16}$, now $-12r^3 + 3r^2 \leq \theta \leq -12r^3 + 3r^2 + r$. In particular, take $\theta = -12r^3 + 3r^2$, we obtain a three-level explicit scheme as follow:

$$(30) \quad \begin{aligned} 6u_{ijkl}^{n+1} &= [12(1-2r) + 3r^2\Box + (12r^2 - 2r)\Diamond] u_{ijkl}^n \\ &+ [6(24r-1) + 3r^2\Box + (12r^2 + 2r)\Diamond] u_{ijkl}^{n-1}. \end{aligned}$$

The above scheme is stable if $r \leq \frac{1}{16}$, and the truncation error is $O(\Delta t^4 + \Delta x^4)$.

If $-12r^3 + 3r^2 \leq -12r^3 + 15r^2 - \frac{3}{4}r$, we have $r \geq \frac{1}{16}$. Consider the inequality (29), we obtain $\frac{1}{16} \leq r \leq \frac{1}{12}$. And when $\frac{1}{16} \leq r \leq \frac{1}{12}$, the inequality $-12r^3 + 15r^2 - \frac{3}{4}r \leq -12r^3 + 3r^2 + r$ holds. Now $-12r^3 + 15r^2 - \frac{3}{4}r \leq \theta \leq -12r^3 + 3r^2 + r$. In particular, take $\theta = -12r^3 + 15r^2 - \frac{3}{4}r$, we obtain a three-level explicit scheme as follow:

$$(31) \quad \begin{aligned} 12u_{ijkl}^{n+1} &= \left[24(1-2r) + \left(-6r^2 + \frac{3}{4}r \right) \Box + (72r^2 - 7r)\Diamond \right] u_{ijkl}^n \\ &+ \left[12(24r-1) + \left(18r^2 - \frac{3}{4}r \right) \Box + (-24r^2 + 7r)\Diamond \right] u_{ijkl}^{n-1}. \end{aligned}$$

The above scheme is stable if $\frac{1}{16} \leq r \leq \frac{1}{12}$, and the truncation is $O(\Delta t^4 + \Delta x^4)$. If we use schemes (30) and (31) simultaneously, a three-level explicit scheme is constructed which is stable for arbitrary $0 < r \leq \frac{1}{12}$ and its truncation error is $O(\Delta t^4 + \Delta x^4)$. Since the above two schemes are the same when $r = \frac{1}{16}$, we call them an explicit difference scheme with branching stability.

5. Numerical experiment

Consider initial and boundary value problem as follows:

$$(32) \quad \left\{ \begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial w^2}, (0 < x, y, z, w < 1, t > 0) \\ u(x, y, z, w, 0) &= \sin(x + y + z + w), (0 \leq x, y, z, w \leq 1) \\ u(0, y, z, w, t) &= e^{-4t} \sin(y + z + w), u(1, y, z, w, t) \\ &= e^{-4t} \sin(1 + y + z + w), (0 \leq y, z, w \leq 1, t \geq 0) \\ u(x, 0, z, w, t) &= e^{-4t} \sin(x + z + w), u(x, 1, z, w, t) \\ &= e^{-4t} \sin(x + 1 + z + w), (0 \leq x, z, w \leq 1, t \geq 0) \\ u(x, y, 0, w, t) &= e^{-4t} \sin(x + y + w), u(x, y, 1, w, t) \\ &= e^{-4t} \sin(x + y + 1 + w), (0 \leq x, y, w \leq 1, t \geq 0) \\ u(x, y, z, 0, t) &= e^{-4t} \sin(x + y + z), u(x, y, z, 1, t) \\ &= e^{-4t} \sin(x + y + z + 1), (0 \leq x, y, z \leq 1, t \geq 0). \end{aligned} \right.$$

Taking $\Delta x = \Delta y = \Delta z = \Delta w = 0.1, \Delta t = r\Delta x^2 = r/100, r = 1/18, 1/16, 1/15, 1/12$. For convenience, we use the exact solution of (32) $u(x, y, z, w, t) = e^{-4t} \sin(x + y + z + w)$ to calculate the value of the first level u_{ijkl}^1 .

Table 1 shows the comparison of the exact solutions and the scheme constructed in this paper and that in the reference [12] at the time strata $n=100$. From Table 1, one can easily see that numerical results of schemes (30) and (31) are completely identical with theoretical analysis, it has higher accuracy than the scheme in reference [12]. Table 2 shows the comparison of the efficiency of the scheme constructed in this paper and the classical scheme one and that in the reference [12]. The result is tested by the unit of second. From table 2, we can see that the computational efficiency of three schemes are similar. Among these three schemes, the classical explicit scheme has the highest computational efficiency, the second one is the scheme in this paper, and the lowest is the one constructed in the reference [12]. From the result of these two tables, we can see that the scheme constructed in this paper is a high accuracy and efficiency difference scheme.

Table 1 Comparison of calculating results among difference schemes with exact solution

r	result	(x, y, z, w)			
		(0.1, 0.1, 0.1, 0.1)	(0.3, 0.3, 0.3, 0.3)	(0.5, 0.5, 0.5, 0.5)	(0.7, 0.7, 0.7, 0.7)
$\frac{1}{18}$	exact solution	0.311 821 832	0.746 318 557	0.728 108 460	0.268 237 541
	scheme (30)	0.311 821 790	0.746 318 060	0.728 107 769	0.268 237 293
	reference [12]	0.311 821 775	0.746 317 889	0.728 107 532	0.268 237 208
$\frac{1}{16}$	exact solution	0.303 279 309	0.725 872 770	0.708 161 548	0.260 889 033
	scheme (30)	0.303 279 269	0.725 872 281	0.708 160 857	0.260 888 784
	reference [12]	0.303 279 256	0.725 872 117	0.708 160 626	0.260 888 701
$\frac{1}{15}$	exact solution	0.298 266 543	0.713 875 148	0.696 456 667	0.256 576 917
	scheme (31)	0.298 266 507	0.713 874 702	0.696 456 031	0.256 576 687
	reference [12]	0.298 266 491	0.713 874 508	0.696 455 756	0.256 576 588
$\frac{1}{12}$	exact solution	0.279 030 435	0.667 835 187	0.651 540 076	0.240 029 498
	scheme (31)	0.279 030 413	0.667 834 911	0.651 539 673	0.240 029 351
	reference [12]	0.279 030 390	0.667 834 619	0.651 539 249	0.240 029 197

Table 2 Comparison of the calculation efficiency among three kinds of difference schemes (unit:s)

difference scheme	$r = \frac{1}{16}, n=20$	$r = \frac{1}{12}, n=20$	$r = \frac{1}{16}, n=50$	$r = \frac{1}{12}, n=50$
classical explicit scheme	74.382 486	75.544 137	184.658 025	181.476 046
schemes (30) and (31)	76.473 674	76.003 696	184.740 697	182.145 808
reference [12] scheme	77.470 928	76.269 907	185.713 634	183.290 512

6. Conclusions

In this paper, we proposed a high-order accuracy explicit difference scheme with branching stability for solving four-dimensional parabolic problems. The stable character of the scheme is which has been verified by a discrete Fourier analysis. The scheme which proposed in this paper is fourth-order accurate in space and fourth-order accurate in time and allows a considerable saving in computing time. Numerical examples are given to test its high accuracy and to show its superiority over some other schemes in terms of accuracy and computational costs.

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HERMITE-HADAMARD TYPE INEQUALITIES FOR GENERALIZED (s, m, φ) -PREINVEX GODUNOVA-LEVIN FUNCTIONS

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Abstract. In the present paper, the notion of (m, φ) -invex set and generalized (s, m, φ) -preinvex Godunova-Levin function of second kind are introduced and some new integral inequalities involving generalized (s, m, φ) -preinvex Godunova-Levin function of second kind along with beta function are given. By using new identities for fractional integrals some new estimates on generalizations of Hermite-Hadamard type inequalities for generalized (s, m, φ) -preinvex Godunova-Levin functions of second kind via Riemann-Liouville fractional integral are established. At the end, some applications to special means are given.

Keywords: Hermite-Hadamard inequality, Hölder's inequality, Minkowski's inequality, power mean inequality, Riemann-Liouville fractional integral, m -invex, P -function.

1. Introduction and preliminaries

The following notation are used throughout this paper. We use I to denote an interval on the real line $\mathbb{R} = (-\infty, +\infty)$ and I° to denote the interior of I . For any subset $K \subseteq \mathbb{R}^n$, K° is used to denote the interior of K . $\mathbb{R}^n$ is used to denote a n -dimensional vector space. The nonnegative real numbers are denoted by $\mathbb{R}_0 = [0, +\infty)$. The set of integrable functions on the interval $[a, b]$ is denoted by $L_1[a, b]$.

The following inequality, named Hermite-Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

Theorem 1.1. *Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex function on I and $a, b \in I$ with $a < b$. Then the following inequality holds:*

$$(1.1) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

In recent years, various generalizations, extensions and variants of such inequalities have been obtained. For other recent results concerning Hermite-Hadamard type inequalities through various classes of convex functions, (see [18]) and the references cited therein, also (see [17]) and the references cited therein.

Fractional calculus (see [18]) and the references cited therein, was introduced at the end of the nineteenth century by Liouville and Riemann, the subject of which has become a rapidly growing area and has found applications in diverse fields ranging from physical sciences and engineering to biological sciences and economics.

Definition 1.2. *Let $f \in L_1[a, b]$. The Riemann-Liouville integrals $J_{a+}^\alpha f$ and $J_{b-}^\alpha f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

and

$$J_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad b > x,$$

where $\Gamma(\alpha) = \int_0^{+\infty} e^{-u} u^{\alpha-1} du$. Here $J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x)$.

In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

Due to the wide application of fractional integrals, some authors extended to study fractional Hermite-Hadamard type inequalities for functions of different classes (see [18]-[19]) and the references cited therein.

Definition 1.3 (see [2]). *A nonnegative function $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}_0$ is said to be P -function or P -convex, if*

$$f(tx + (1-t)y) \leq f(x) + f(y), \quad \forall x, y \in I, t \in [0, 1].$$

Definition 1.4 (see [3]). *A function $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}_0$ is said to be a Godunova-Levin function or $f \in Q(I)$, if f is nonnegative and for all $x, y \in I, t \in (0, 1)$, we have that*

$$f(tx + (1-t)y) \leq \frac{f(x)}{t} + \frac{f(y)}{1-t}.$$

The class $Q(I)$ was firstly described in (see [3]) by Godunova and Levin. Some further properties of it are given in (see [2],[4],[5]). Among others, it is noted that nonnegative monotone and nonnegative convex functions belong to this class of functions.

Definition 1.5 (see [6]). A function $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}_o$ is said to be (s, m) -Godunova-Levin functions of first kind or $f \in Q_{(s,m)}^1$, if $\forall s, m \in (0, 1]$, we have

$$f(tx + m(1-t)y) \leq \frac{1}{t^s} f(x) + m \left(\frac{1}{1-t^s} \right) f(y), \quad \forall x, y \in I, t \in (0, 1).$$

We would like to mention that Definition 1.5 is also introduced and studied by Li et al. (see [7]) independently. For $m = 1$ in Definition 1.5 we have the definition of s -Godunova-Levin functions of first kind, which is introduced and investigated by Noor et al. (see [8]).

Definition 1.6 (see [6]). A function $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}_o$ is said to be (s, m) -Godunova-Levin functions of second kind or $f \in Q_{(s,m)}^2$, if $s \in [0, 1]$, $m \in (0, 1]$, we have

$$f(tx + m(1-t)y) \leq \frac{1}{t^s} f(x) + m \left(\frac{1}{(1-t)^s} \right) f(y), \quad \forall x, y \in I, t \in (0, 1).$$

It is obvious that for $s = 0, m = 1$, (s, m) -Godunova-Levin functions of second kind reduces to Definition 1.3 of P -functions. If $s = 1, m = 1$, it then reduces to Godunova-Levin functions. For $m = 1$, we have the definition of s -Godunova-Levin function of second kind introduced and studied by Dragomir (see [9],[10]).

Definition 1.7 (see [14]). A set $K \subseteq \mathbb{R}^n$ is said to be invex with respect to the mapping $\eta : K \times K \rightarrow \mathbb{R}^n$, if $x + t\eta(y, x) \in K$ for every $x, y \in K$ and $t \in [0, 1]$.

Notice that every convex set is invex with respect to the mapping $\eta(y, x) = y - x$, but the converse is not necessarily true (see [14],[15]) and the references therein.

Definition 1.8 (see [16]). The function f defined on the invex set $K \subseteq \mathbb{R}^n$ is said to be preinvex with respect η , if for every $x, y \in K$ and $t \in [0, 1]$, we have that

$$f(x + t\eta(y, x)) \leq (1-t)f(x) + tf(y).$$

The concept of preinvexity is more general than convexity since every convex function is preinvex with respect to the mapping $\eta(y, x) = y - x$, but the converse is not true.

The Gauss-Jacobi type quadrature formula has the following

$$(1.2) \quad \int_a^b (x-a)^p (b-x)^q f(x) dx = \sum_{k=0}^{+\infty} B_{m,k} f(\gamma_k) + R_m^* |f|,$$

for certain $B_{m,k}, \gamma_k$ and rest $R_m^* |f|$ (see [11]).

Recently, Liu (see [12]) obtained several integral inequalities for the left hand side of (1.2) under the Definition 1.3 of P -function.

Also in (see [13]), Özdemir et al. established several integral inequalities concerning the left-hand side of (1.2) via some kinds of convexity.

Motivated by these results, the aim of this paper is to establish some generalizations of Hermite-Hadamard type inequalities using new identities given in Section 3 for generalized (s, m, φ) -preinvex Godunova-Levin functions of second kind via Riemann-Liouville fractional integral.

The paper is organized as follows: In Section 2, the notion of (m, φ) -invex set is introduced and an interesting property is derived. Also the notion of generalized (s, m, φ) -preinvex Godunova-Levin functions of second kind is introduced and some new integral inequalities involving generalized (s, m, φ) -preinvex Godunova-Levin functions of second kind along with beta function are given. In Section 3, some generalized integral inequalities of Hermite-Hadamard via fractional integrals for generalized (s, m, φ) -preinvex Godunova-Levin functions of second kind are given. In Section 4, some applications to special means are given. These results provide new estimates on these Hermite-Hadamard types.

2. New integral inequalities

Definition 2.1 (see [1]). A set $K \subseteq \mathbb{R}^n$ is said to be m -invex with respect to the mapping $\eta : K \times K \times (0, 1] \rightarrow \mathbb{R}^n$ for some fixed $m \in (0, 1]$, if $mx + t\eta(y, mx) \in K$ holds for each $x, y \in K$ and any $t \in [0, 1]$.

Remark 2.2. In Definition 2.1, under certain conditions, the mapping $\eta(y, mx)$ could reduce to $\eta(y, x)$. For example when $m = 1$, then the m -invex set degenerates an invex set on K .

First we give new definition, to be referred as (m, φ) -invex set.

Definition 2.3. Let $\varphi : I \rightarrow K$ be an arbitrary function. A set $K \subseteq \mathbb{R}^n$ is said to be (m, φ) -invex with respect to the mapping $\eta : K \times K \times (0, 1] \rightarrow \mathbb{R}^n$ for some fixed $m \in (0, 1]$, if $m\varphi(y) + t\eta(\varphi(x), \varphi(y), m) \in K$ holds for each $x, y \in I$ and any $t \in [0, 1]$.

Example 2.4. Let $m = \frac{1}{4}$ and $X = (0, \frac{\pi}{2}] \cup (\pi, \frac{3\pi}{2}]$

$$\eta(x, y, m) = \begin{cases} m \sin(y - x), & \text{if } x \in (0, \frac{\pi}{2}], y \in (0, \frac{\pi}{2}]; \\ -m \sin(y - x), & \text{if } x \in (\pi, \frac{3\pi}{2}], y \in (\pi, \frac{3\pi}{2}]; \\ m \sin x, & \text{if } x \in (0, \frac{\pi}{2}], y \in (\pi, \frac{3\pi}{2}]; \\ -m \sin x, & \text{if } x \in (\pi, \frac{3\pi}{2}], y \in (0, \frac{\pi}{2}]. \end{cases}$$

Then X is an (m, φ) -invex set with respect to η for function $\varphi(x) = x$, $\forall x \in I$, $\forall t \in [0, 1]$ and $m = \frac{1}{4}$. It is obvious that X is not a convex set.

According to the above definition, we derive an interesting property of the (m, φ) -invex set. The proof of the following proposition 2.5 is straightforward.

Proposition 2.5. *If $K_i, i \in I = \{1, 2, \dots, n\}$ is a family of (m, φ) -invex sets in $\mathbb{R}^n$ with respect to the same $\eta : \mathbb{R}^n \times \mathbb{R}^n \times (0, 1] \rightarrow \mathbb{R}^n$ for some fixed $m \in (0, 1]$ and same arbitrary function φ , then the intersection $\bigcap_{i \in I} K_i$ is an (m, φ) -invex set.*

We next give new definition, to be referred as generalized (s, m, φ) -preinvex Godunova-Levin function of second kind.

Definition 2.6. *Let $K \subseteq \mathbb{R}$ be an open nonempty m -invex set with respect to $\eta : K \times K \times (0, 1] \rightarrow \mathbb{R}$ and $\varphi : I \rightarrow K$ a continuous function. For $f : K \rightarrow \mathbb{R}$, any fixed $s \in [0, 1]$ and some fixed $m \in (0, 1]$, if*

$$(2.1) \quad f(m\varphi(y) + t\eta(\varphi(x), \varphi(y), m)) \leq \frac{f(\varphi(x))}{t^s} + m \frac{f(\varphi(y))}{(1-t)^s},$$

is valid for all $x, y \in I, t \in (0, 1)$, then we say that $f(x)$ is a generalized (s, m, φ) -preinvex Godunova-Levin function of second kind with respect to η or $f \in Q_{(s, m, \varphi)}^{*2}$.

Remark 2.7. In Definition 2.6, it is worthwhile to note that generalized (s, m, φ) -preinvex Godunova-Levin function of second kind is an (s, m) -Godunova-Levin functions of second kind on $K = I$ with respect to $\eta(\varphi(x), \varphi(y), m) = \varphi(x) - m\varphi(y)$ and $\varphi(x) = x$, for all $x, y \in I$.

In this section, in order to prove our main results regarding some new integral inequalities involving generalized (s, m, φ) -preinvex Godunova-Levin functions of second kind along with beta function, we need the following new lemma:

Lemma 2.8. *Let $\varphi : I \rightarrow K$ be a continuous function and $\eta : K \times K \times (0, 1] \rightarrow \mathbb{R}$. Assume that $f : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \rightarrow \mathbb{R}$ be a continuous function on K° and $\eta(\varphi(b), \varphi(a), m) > 0$. Then for some fixed $m \in (0, 1]$ and any fixed $p, q > 0$, we have*

$$\begin{aligned} & \int_{m\varphi(a)}^{m\varphi(a) + \eta(\varphi(b), \varphi(a), m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x) dx \\ &= \eta^{p+q+1}(\varphi(b), \varphi(a), m) \int_0^1 t^p (1-t)^q f(m\varphi(a) + t\eta(\varphi(b), \varphi(a), m)) dt. \end{aligned}$$

Proof. It is easy to observe that

$$\begin{aligned} & \int_{m\varphi(a)}^{m\varphi(a) + \eta(\varphi(b), \varphi(a), m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x) dx \\ &= \eta(\varphi(b), \varphi(a), m) \int_0^1 (m\varphi(a) + t\eta(\varphi(b), \varphi(a), m) - m\varphi(a))^p \\ & \quad \times (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - m\varphi(a) - t\eta(\varphi(b), \varphi(a), m))^q \\ & \quad \times f(m\varphi(a) + t\eta(\varphi(b), \varphi(a), m)) dt \end{aligned}$$

$$= \eta^{p+q+1}(\varphi(b), \varphi(a), m) \int_0^1 t^p (1-t)^q f(m\varphi(a) + t\eta(\varphi(b), \varphi(a), m)) dt.$$

□

The following definition will be used in the sequel.

Definition 2.9. The Euler beta function is defined for $x, y > 0$ as

$$\beta(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Theorem 2.10. Let $\varphi : I \rightarrow K$ be a continuous function and $\eta : K \times K \times (0, 1] \rightarrow \mathbb{R}$. Assume that $f : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \rightarrow \mathbb{R}$ be a continuous function on K° and $\eta(\varphi(b), \varphi(a), m) > 0$. If $k > 1$ and $|f|^{\frac{k}{k-1}}$ is a generalized (s, m, φ) -preinvex Godunova-Levin function of second kind on K for some fixed $m \in (0, 1]$ and any fixed $s \in [0, 1)$, then for any fixed $p, q > 0$,

$$\begin{aligned} & \int_{m\varphi(a)}^{m\varphi(a) + \eta(\varphi(b), \varphi(a), m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x) dx \\ & \leq \frac{\eta^{p+q+1}(\varphi(b), \varphi(a), m)}{(1-s)^{\frac{k-1}{k}}} \left[\beta(kp+1, kq+1) \right]^{\frac{1}{k}} \left(m|f(\varphi(a))|^{\frac{k}{k-1}} + |f(\varphi(b))|^{\frac{k}{k-1}} \right)^{\frac{k-1}{k}}. \end{aligned}$$

Proof. Since $|f|^{\frac{k}{k-1}}$ is a generalized (s, m, φ) -preinvex Godunova-Levin function of second kind on K , combining with Lemma 2.8, Definition 2.9, Hölder inequality and properties of the modulus, we get

$$\begin{aligned} & \int_{m\varphi(a)}^{m\varphi(a) + \eta(\varphi(b), \varphi(a), m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x) dx \\ & \leq |\eta(\varphi(b), \varphi(a), m)|^{p+q+1} \left[\int_0^1 t^{kp} (1-t)^{kq} dt \right]^{\frac{1}{k}} \\ & \quad \times \left[\int_0^1 |f(m\varphi(a) + t\eta(\varphi(b), \varphi(a), m))|^{\frac{k}{k-1}} dt \right]^{\frac{k-1}{k}} \\ & \leq \eta^{p+q+1}(\varphi(b), \varphi(a), m) \left[\beta(kp+1, kq+1) \right]^{\frac{1}{k}} \\ & \quad \times \left[\int_0^1 \left(\frac{m|f(\varphi(a))|^{\frac{k}{k-1}}}{(1-t)^s} + \frac{|f(\varphi(b))|^{\frac{k}{k-1}}}{t^s} \right) dt \right]^{\frac{k-1}{k}} \\ & = \frac{\eta^{p+q+1}(\varphi(b), \varphi(a), m)}{(1-s)^{\frac{k-1}{k}}} \left[\beta(kp+1, kq+1) \right]^{\frac{1}{k}} \left(m|f(\varphi(a))|^{\frac{k}{k-1}} + |f(\varphi(b))|^{\frac{k}{k-1}} \right)^{\frac{k-1}{k}}. \end{aligned}$$

The proof of Theorem 2.10 is completed. □

Theorem 2.11. Let $\varphi : I \longrightarrow K$ be a continuous function and $\eta : K \times K \times (0, 1] \longrightarrow \mathbb{R}$. Assume that $f : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \longrightarrow \mathbb{R}$ be a continuous function on K° and $\eta(\varphi(b), \varphi(a), m) > 0$. If $l \geq 1$ and $|f|^l$ is a generalized (s, m, φ) -preinvex Godunova-Levin function of second kind on K for some fixed $m \in (0, 1]$ and any fixed $s \in [0, 1]$, then for any fixed $p, q > 0$,

$$\begin{aligned} & \int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b), \varphi(a), m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x) dx \\ & \leq \eta^{p+q+1}(\varphi(b), \varphi(a), m) \left[\beta(p+1, q+1) \right]^{\frac{l-1}{l}} \\ & \quad \times \left[m|f(\varphi(a))|^l \beta(p+1, q-s+1) + |f(\varphi(b))|^l \beta(p-s+1, q+1) \right]^{\frac{1}{l}}. \end{aligned}$$

Proof. Since $|f|^l$ is a generalized (s, m, φ) -preinvex Godunova-Levin function of second kind on K , combining with Lemma 2.8, Definition 2.9, the well-known power mean inequality and properties of the modulus, we get

$$\begin{aligned} & \int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b), \varphi(a), m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x) dx \\ & = \eta^{p+q+1}(\varphi(b), \varphi(a), m) \\ & \quad \times \int_0^1 \left[t^p (1-t)^q \right]^{\frac{l-1}{l}} \left[t^p (1-t)^q \right]^{\frac{1}{l}} f(m\varphi(a) + t\eta(\varphi(b), \varphi(a), m)) dt \\ & \leq |\eta(\varphi(b), \varphi(a), m)|^{p+q+1} \left[\int_0^1 t^p (1-t)^q dt \right]^{\frac{l-1}{l}} \\ & \quad \times \left[\int_0^1 t^p (1-t)^q |f(m\varphi(a) + t\eta(\varphi(b), \varphi(a), m))|^l dt \right]^{\frac{1}{l}} \\ & \leq \eta^{p+q+1}(\varphi(b), \varphi(a), m) \left[\beta(p+1, q+1) \right]^{\frac{l-1}{l}} \\ & \quad \times \left[\int_0^1 t^p (1-t)^q \left(\frac{m|f(\varphi(a))|^l}{(1-t)^s} + \frac{|f(\varphi(b))|^l}{t^s} \right) dt \right]^{\frac{1}{l}} \\ & = \eta^{p+q+1}(\varphi(b), \varphi(a), m) \left[\beta(p+1, q+1) \right]^{\frac{l-1}{l}} \\ & \quad \times \left[m|f(\varphi(a))|^l \beta(p+1, q-s+1) + |f(\varphi(b))|^l \beta(p-s+1, q+1) \right]^{\frac{1}{l}}. \end{aligned}$$

The proof of Theorem 2.11 is completed. $\square$

3. Generalized integral inequalities via fractional integrals

In this section, in order to prove our main results regarding some Hermite-Hadamard type inequalities for generalized (s, m, φ) -preinvex Godunova-Levin

function of second kind via fractional integrals, we need the following two new lemmas:

Lemma 3.1. *Let $\varphi : I \rightarrow K$ be a continuous function. Suppose $K \subseteq \mathbb{R}$ be an open nonempty m -invex subset with respect to $\eta : K \times K \times (0, 1] \rightarrow \mathbb{R}$ for some fixed $m \in (0, 1]$ and let $a < b$ with $\eta(\varphi(a), \varphi(b), m) > 0$. Assume that $f : K \rightarrow \mathbb{R}$ is a differentiable function on K° and $f' \in L_1[m\varphi(b), m\varphi(b) + \eta(\varphi(a), \varphi(b), m)]$. Then, for each $x \in [m\varphi(b), m\varphi(b) + \eta(\varphi(a), \varphi(b), m)]$ and $\alpha, \lambda \in (0, 1]$, we have*

$$\begin{aligned}
 & \frac{(1 + \alpha(1 - \lambda))f(m\varphi(b) + \eta(\varphi(a), \varphi(b), m)) + (1 - \alpha(1 - \lambda))f(m\varphi(b))}{2} \\
 & - \frac{\Gamma(\alpha + 1)}{2\eta(\varphi(a), \varphi(b), m)^\alpha} \\
 (3.1) \quad & \times \left[J_{(m\varphi(b))^+}^\alpha f(m\varphi(b) + \eta(\varphi(a), \varphi(b), m)) \right. \\
 & \left. + J_{(m\varphi(b) + \eta(\varphi(a), \varphi(b), m))^-}^\alpha f(m\varphi(b)) \right] \\
 & = \frac{\eta(\varphi(a), \varphi(b), m)}{2} \int_0^1 (t^\alpha + \alpha(1 - \lambda) \\
 & - (1 - t)^\alpha) f'(m\varphi(b) + t\eta(\varphi(a), \varphi(b), m)) dt.
 \end{aligned}$$

Proof. A simple proof of the equality (3.1) can be done by performing an integration by parts in the integrals from the right side and changing the variable. The details are left to the interested reader. $\square$

Lemma 3.2. *Let $\varphi : I \rightarrow K$ be a continuous function. Suppose $K \subseteq \mathbb{R}$ be an open nonempty m -invex subset with respect to $\eta : K \times K \times (0, 1] \rightarrow \mathbb{R}$ for some fixed $m \in (0, 1]$ and let $a < b$ with $\eta(\varphi(a), \varphi(b), m) > 0$ where $\eta(\varphi(a), \varphi(b), m) \neq 0$. Assume that $f : K \rightarrow \mathbb{R}$ be a twice differentiable function on K° and $f'' \in L_1[m\varphi(b), m\varphi(b) + \eta(\varphi(a), \varphi(b), m)]$. Then, for each $x \in [m\varphi(b), m\varphi(b) + \eta(\varphi(a), \varphi(b), m)]$ and $\alpha \in (0, 1]$, we have*

$$\begin{aligned}
 & \frac{f(m\varphi(b) + \eta(\varphi(a), \varphi(b), m)) + f(m\varphi(b))}{2} - \frac{\Gamma(\alpha + 1)}{2\eta(\varphi(a), \varphi(b), m)^\alpha} \\
 & \times \left[J_{(m\varphi(b))^+}^\alpha f(m\varphi(b) + \eta(\varphi(a), \varphi(b), m)) \right. \\
 (3.2) \quad & \left. + J_{(m\varphi(b) + \eta(\varphi(a), \varphi(b), m))^-}^\alpha f(m\varphi(b)) \right] \\
 & = \frac{\eta^2(\varphi(a), \varphi(b), m)}{2(\alpha + 1)} \int_0^1 (1 - (1 - t)^{\alpha+1} - t^{\alpha+1}) \\
 & \cdot f''(m\varphi(b) + t\eta(\varphi(a), \varphi(b), m)) dt.
 \end{aligned}$$

Proof. A simple proof of the equality (3.2) can be done by performing an integration by parts in the integrals from the right side and using Lemma 3.1 with $\lambda = 1$. The details are left to the interested reader. $\square$

Using Lemma 3.1, the following results can be obtained for the corresponding version for power of the absolute value of the first derivative.

Theorem 3.3. *Let $\varphi : I \rightarrow A$ be a continuous function. Suppose $A \subseteq \mathbb{R}$ be an open nonempty m -invex subset with respect to $\eta : A \times A \times (0, 1] \rightarrow \mathbb{R}$ for any fixed $s \in [0, 1]$, some fixed $m \in (0, 1]$ and let $a < b$ with $\eta(\varphi(a), \varphi(b), m) > 0$. Assume that $f : A \rightarrow \mathbb{R}$ is a differentiable function on A° . If $|f'|^q$ is a generalized (s, m, φ) -preinvex Godunova-Levin function of second kind on $[m\varphi(b), m\varphi(b) + \eta(\varphi(a), \varphi(b), m)]$, $q > 1$, $p^{-1} + q^{-1} = 1$, then for each $x \in [m\varphi(b), m\varphi(b) + \eta(\varphi(a), \varphi(b), m)]$ with $\alpha, \lambda \in (0, 1]$, we have*

$$\begin{aligned}
 & \left| \frac{(1 + \alpha(1 - \lambda))f(m\varphi(b) + \eta(\varphi(a), \varphi(b), m)) + (1 - \alpha(1 - \lambda))f(m\varphi(b))}{2} \right. \\
 & - \frac{\Gamma(\alpha + 1)}{2\eta(\varphi(a), \varphi(b), m)^\alpha} \\
 & \times \left[J_{(m\varphi(b))^+}^\alpha f(m\varphi(b) + \eta(\varphi(a), \varphi(b), m)) \right. \\
 (3.3) \quad & \left. + J_{(m\varphi(b) + \eta(\varphi(a), \varphi(b), m))^-}^\alpha f(m\varphi(b)) \right] \left| \right. \\
 & \leq \frac{\eta(\varphi(a), \varphi(b), m)}{2} \left[\alpha(1 - \lambda) + 2 \left(\frac{1}{p\alpha + 1} \right)^{\frac{1}{p}} \right] \\
 & \cdot \left(\frac{|f'(\varphi(a))|^q + m|f'(\varphi(b))|^q}{1 - s} \right)^{\frac{1}{q}}.
 \end{aligned}$$

Proof. Denote

$$\begin{aligned}
 S_{f, \eta, \varphi}(\alpha, \lambda, m, a, b) &= \frac{\eta(\varphi(a), \varphi(b), m)}{2} \\
 (3.4) \quad & \times \int_0^1 (t^\alpha + \alpha(1 - \lambda) - (1 - t)^\alpha) f'(m\varphi(b) + t\eta(\varphi(a), \varphi(b), m)) dt.
 \end{aligned}$$

Suppose that $q > 1$. Using Lemma 3.1, relation (3.4), the fact that $|f'|^q$ is a generalized (s, m, φ) -preinvex Godunova-Levin function of second kind, property of the modulus, Hölder's inequality, Minkowski's inequality and properties of the modulus, we have

$$\begin{aligned}
 & |S_{f, \eta, \varphi}(\alpha, \lambda, m, a, b)| \\
 & \leq \frac{|\eta(\varphi(a), \varphi(b), m)|}{2} \left[\int_0^1 (t^\alpha + \alpha(1 - \lambda)) |f'(m\varphi(b) + t\eta(\varphi(a), \varphi(b), m))| dt \right. \\
 & \left. + \int_0^1 (1 - t)^\alpha |f'(m\varphi(b) + t\eta(\varphi(a), \varphi(b), m))| dt \right] \\
 & \leq \frac{\eta(\varphi(a), \varphi(b), m)}{2}
 \end{aligned}$$

$$\begin{aligned}
& \times \left[\left(\int_0^1 (t^\alpha + \alpha(1-\lambda))^p dt \right)^{\frac{1}{p}} \left(\int_0^1 |f'(m\varphi(b) + t\eta(\varphi(a), \varphi(b), m))|^q dt \right)^{\frac{1}{q}} \right. \\
& \left. + \left(\int_0^1 (1-t)^{p\alpha} dt \right)^{\frac{1}{p}} \left(\int_0^1 |f'(m\varphi(b) + t\eta(\varphi(a), \varphi(b), m))|^q dt \right)^{\frac{1}{q}} \right] \\
& \leq \frac{\eta(\varphi(a), \varphi(b), m)}{2} \left[\left(\int_0^1 t^{p\alpha} dt \right)^{\frac{1}{p}} + \left(\int_0^1 \alpha^p (1-\lambda)^p dt \right)^{\frac{1}{p}} + \left(\int_0^1 (1-t)^{p\alpha} dt \right)^{\frac{1}{p}} \right] \\
& \times \left(\int_0^1 \left(\frac{|f'(\varphi(a))|^q}{t^s} + \frac{m|f'(\varphi(b))|^q}{(1-t)^s} \right) dt \right)^{\frac{1}{q}} \\
& = \frac{\eta(\varphi(a), \varphi(b), m)}{2} \left[\alpha(1-\lambda) + 2 \left(\frac{1}{p\alpha+1} \right)^{\frac{1}{p}} \right] \left(\frac{|f'(\varphi(a))|^q + m|f'(\varphi(b))|^q}{1-s} \right)^{\frac{1}{q}}.
\end{aligned}$$

The proof of Theorem 3.3 is completed. $\square$

Corollary 3.4. *Under the conditions of Theorem 3.3, if we choose $\lambda = m = 1$ and $\eta(\varphi(a), \varphi(b), 1) = \varphi(a) - \varphi(b)$, we get the following generalized Hermite-Hadamard type inequality for fractional integrals:*

$$\begin{aligned}
& \left| \frac{f(\varphi(a)) + f(\varphi(b))}{2} - \frac{\Gamma(\alpha+1)}{2(\varphi(a) - \varphi(b))^\alpha} \left[J_{\varphi(b)+}^\alpha f(\varphi(a)) + J_{\varphi(a)-}^\alpha f(\varphi(b)) \right] \right| \\
(3.5) \quad & \leq (\varphi(b) - \varphi(a)) \left(\frac{1}{p\alpha+1} \right)^{\frac{1}{p}} \left(\frac{|f'(\varphi(a))|^q + |f'(\varphi(b))|^q}{1-s} \right)^{\frac{1}{q}}.
\end{aligned}$$

Theorem 3.5. *Let $\varphi : I \rightarrow A$ be a continuous function. Suppose $A \subseteq \mathbb{R}$ be an open nonempty m -invex subset with respect to $\eta : A \times A \times (0, 1] \rightarrow \mathbb{R}$ for any fixed $s \in [0, 1)$, some fixed $m \in (0, 1]$ and let $a < b$ with $\eta(\varphi(a), \varphi(b), m) > 0$. Assume that $f : A \rightarrow \mathbb{R}$ is a differentiable function on A° . If $|f'|^q$ is a generalized (s, m, φ) -preinvex Godunova-Levin function of second kind on $[m\varphi(b), m\varphi(b) + \eta(\varphi(a), \varphi(b), m)]$, $q \geq 1$, then for each $x \in [m\varphi(b), m\varphi(b) + \eta(\varphi(a), \varphi(b), m)]$ with $\alpha, \lambda \in (0, 1]$, we have*

$$\begin{aligned}
& \left| \frac{(1 + \alpha(1-\lambda))f(m\varphi(b) + \eta(\varphi(a), \varphi(b), m)) + (1 - \alpha(1-\lambda))f(m\varphi(b))}{2} \right. \\
& \left. - \frac{\Gamma(\alpha+1)}{2\eta(\varphi(a), \varphi(b), m)^\alpha} \right. \\
& \times \left[J_{(m\varphi(b))+}^\alpha f(m\varphi(b) + \eta(\varphi(a), \varphi(b), m)) + J_{(m\varphi(b)+\eta(\varphi(a), \varphi(b), m))-}^\alpha f(m\varphi(b)) \right] \Big| \\
& \leq \frac{\eta(\varphi(a), \varphi(b), m)}{2}
\end{aligned}$$

$$\begin{aligned}
& \times \left[\left(\frac{1}{\alpha+1} + \alpha(1-\lambda) \right)^{1-\frac{1}{q}} \left[|f'(\varphi(a))|^q \left(\frac{1}{\alpha-s+1} + \frac{\alpha(1-\lambda)}{1-s} \right) \right. \right. \\
(3.6) \quad & \left. \left. + |f'(\varphi(b))|^q \left(m\beta(\alpha+1, 1-s) + \frac{m\alpha(1-\lambda)}{1-s} \right) \right]^{\frac{1}{q}} \right. \\
& \left. + \left(\frac{1}{\alpha+1} \right)^{1-\frac{1}{q}} \left[|f'(\varphi(a))|^q \beta(\alpha+1, 1-s) + \frac{m|f'(\varphi(b))|^q}{\alpha-s+1} \right]^{\frac{1}{q}} \right].
\end{aligned}$$

Proof. Suppose that $q \geq 1$. Using Lemma 3.1, relation (3.4), the fact that $|f'|^q$ is a generalized (s, m, φ) -preinvex Godunova-Levin function of second kind, the well-known power mean inequality and properties of the modulus, we have

$$\begin{aligned}
& |S_{f,\eta,\varphi}(\alpha, \lambda, m, a, b)| \\
& \leq \frac{|\eta(\varphi(a), \varphi(b), m)|}{2} \left[\int_0^1 (t^\alpha + \alpha(1-\lambda)) |f'(m\varphi(b) + t\eta(\varphi(a), \varphi(b), m))| dt \right. \\
& \quad \left. + \int_0^1 (1-t)^\alpha |f'(m\varphi(b) + t\eta(\varphi(a), \varphi(b), m))| dt \right] \\
& \leq \frac{\eta(\varphi(a), \varphi(b), m)}{2} \left[\left(\int_0^1 (t^\alpha + \alpha(1-\lambda)) dt \right)^{1-\frac{1}{q}} \right. \\
& \quad \times \left(\int_0^1 (t^\alpha + \alpha(1-\lambda)) |f'(m\varphi(b) + t\eta(\varphi(a), \varphi(b), m))|^q dt \right)^{\frac{1}{q}} \\
& \quad \left. + \left(\int_0^1 (1-t)^\alpha dt \right)^{1-\frac{1}{q}} \left(\int_0^1 (1-t)^\alpha |f'(m\varphi(b) + t\eta(\varphi(a), \varphi(b), m))|^q dt \right)^{\frac{1}{q}} \right] \\
& \leq \frac{\eta(\varphi(a), \varphi(b), m)}{2} \left[\left(\frac{1}{\alpha+1} + \alpha(1-\lambda) \right)^{1-\frac{1}{q}} \right. \\
& \quad \times \left(\int_0^1 (t^\alpha + \alpha(1-\lambda)) \left(\frac{|f'(\varphi(a))|^q}{t^s} + \frac{m|f'(\varphi(b))|^q}{(1-t)^s} \right) dt \right)^{\frac{1}{q}} \\
& \quad \left. + \left(\frac{1}{\alpha+1} \right)^{1-\frac{1}{q}} \left(\int_0^1 (1-t)^\alpha \left(\frac{|f'(\varphi(a))|^q}{t^s} + \frac{m|f'(\varphi(b))|^q}{(1-t)^s} \right) dt \right)^{\frac{1}{q}} \right] \\
& = \frac{\eta(\varphi(a), \varphi(b), m)}{2} \left[\left(\frac{1}{\alpha+1} + \alpha(1-\lambda) \right)^{1-\frac{1}{q}} \left[|f'(\varphi(a))|^q \left(\frac{1}{\alpha-s+1} + \frac{\alpha(1-\lambda)}{1-s} \right) \right. \right. \\
& \quad \left. \left. + |f'(\varphi(b))|^q \left(m\beta(\alpha+1, 1-s) + \frac{m\alpha(1-\lambda)}{1-s} \right) \right]^{\frac{1}{q}} \right.
\end{aligned}$$

$$+ \left(\frac{1}{\alpha+1} \right)^{1-\frac{1}{q}} \left[|f'(\varphi(a))|^q \beta(\alpha+1, 1-s) + \frac{m|f'(\varphi(b))|^q}{\alpha-s+1} \right]^{\frac{1}{q}}.$$

The proof of Theorem 3.5 is completed. $\square$

Corollary 3.6. *Under the conditions of Theorem 3.5, if we choose $\lambda = m = 1$ and $\eta(\varphi(a), \varphi(b), 1) = \varphi(a) - \varphi(b)$, we get the following generalized Hermite-Hadamard type inequality for fractional integrals:*

$$\begin{aligned} & \left| \frac{f(\varphi(a)) + f(\varphi(b))}{2} - \frac{\Gamma(\alpha+1)}{2(\varphi(a) - \varphi(b))^\alpha} \left[J_{\varphi(b)+}^\alpha f(\varphi(a)) + J_{\varphi(a)-}^\alpha f(\varphi(b)) \right] \right| \\ (3.7) \quad & \leq \frac{(\varphi(b) - \varphi(a))}{2} \left(\frac{1}{\alpha+1} \right)^{1-\frac{1}{q}} \left[\left(\frac{|f'(\varphi(a))|^q}{\alpha-s+1} + |f'(\varphi(b))|^q \beta(\alpha+1, 1-s) \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(|f'(\varphi(a))|^q \beta(\alpha+1, 1-s) + \frac{|f'(\varphi(b))|^q}{\alpha-s+1} \right)^{\frac{1}{q}} \right]. \end{aligned}$$

Using Lemma 3.2, the following results can be obtained for the corresponding version for power of the absolute value of the second derivative.

Theorem 3.7. *Let $\varphi : I \rightarrow A$ be a continuous function. Suppose $A \subseteq \mathbb{R}$ be an open nonempty m -invex subset with respect to $\eta : A \times A \times (0, 1] \rightarrow \mathbb{R}$ for any fixed $s \in [0, 1)$, some fixed $m \in (0, 1]$ and let $a < b$ with $\eta(\varphi(a), \varphi(b), m) > 0$. Assume that $f : A \rightarrow \mathbb{R}$ be a twice differentiable function on A° . If $|f''|^q$ is a generalized (s, m, φ) -preinvex Godunova-Levin function of second kind on $[m\varphi(b), m\varphi(b) + \eta(\varphi(a), \varphi(b), m)]$, $q > 1$, $p^{-1} + q^{-1} = 1$, then for each $x \in [m\varphi(b), m\varphi(b) + \eta(\varphi(a), \varphi(b), m)]$ and $\alpha \in (0, 1]$, we have*

$$\begin{aligned} & \left| \frac{f(m\varphi(b) + \eta(\varphi(a), \varphi(b), m)) + f(m\varphi(b))}{2} - \frac{\Gamma(\alpha+1)}{2\eta(\varphi(a), \varphi(b), m)^\alpha} \right. \\ & \quad \times \left[J_{(m\varphi(b)) +}^\alpha f(m\varphi(b) + \eta(\varphi(a), \varphi(b), m)) \right. \\ (3.8) \quad & \quad \left. \left. + J_{(m\varphi(b) + \eta(\varphi(a), \varphi(b), m)) -}^\alpha f(m\varphi(b)) \right] \right| \\ & \leq \frac{\eta^2(\varphi(a), \varphi(b), m)}{2(\alpha+1)} \left(\frac{p(\alpha+1) - 1}{p(\alpha+1) + 1} \right)^{\frac{1}{p}} \left(\frac{|f''(\varphi(a))|^q + m|f''(\varphi(b))|^q}{1-s} \right)^{\frac{1}{q}}. \end{aligned}$$

Proof. Denote

$$\begin{aligned} R_{f, \eta, \varphi}(\alpha, m, a, b) &= \frac{\eta^2(\varphi(a), \varphi(b), m)}{2(\alpha+1)} \\ (3.9) \quad & \times \int_0^1 (1 - (1-t)^{\alpha+1} - t^{\alpha+1}) f''(m\varphi(b) + t\eta(\varphi(a), \varphi(b), m)) dt. \end{aligned}$$

Suppose that $q > 1$. Using Lemma 3.2, relation (3.9), the fact that $|f''|^q$ is a generalized (s, m, φ) -preinvex Godunova-Levin function of second kind, Hölder's inequality and properties of the modulus, we have

$$\begin{aligned}
 |R_{f,\eta,\varphi}(\alpha, m, a, b)| &\leq \frac{\eta^2(\varphi(a), \varphi(b), m)}{2(\alpha+1)} \\
 &\times \int_0^1 (1 - (1-t)^{\alpha+1} - t^{\alpha+1}) |f''(m\varphi(b) + t\eta(\varphi(a), \varphi(b), m))| dt \\
 &\leq \frac{\eta^2(\varphi(a), \varphi(b), m)}{2(\alpha+1)} \left(\int_0^1 (1 - (1-t)^{\alpha+1} - t^{\alpha+1})^p dt \right)^{\frac{1}{p}} \\
 &\times \left(\int_0^1 |f''(m\varphi(b) + t\eta(\varphi(a), \varphi(b), m))|^q dt \right)^{\frac{1}{q}} \\
 &\leq \frac{\eta^2(\varphi(a), \varphi(b), m)}{2(\alpha+1)} \left(\int_0^1 (1 - (1-t)^{p(\alpha+1)} - t^{p(\alpha+1)}) dt \right)^{\frac{1}{p}} \\
 &\times \left(\int_0^1 \left(\frac{|f''(\varphi(a))|^q}{t^s} + \frac{m|f''(\varphi(b))|^q}{(1-t)^s} \right) dt \right)^{\frac{1}{q}} \\
 &= \frac{\eta^2(\varphi(a), \varphi(b), m)}{2(\alpha+1)} \left(\frac{p(\alpha+1)-1}{p(\alpha+1)+1} \right)^{\frac{1}{p}} \left(\frac{|f''(\varphi(a))|^q + m|f''(\varphi(b))|^q}{1-s} \right)^{\frac{1}{q}}.
 \end{aligned}$$

The proof of Theorem 3.7 is completed. $\square$

Corollary 3.8. *Under the conditions of Theorem 3.7, if we choose $m = 1$ and $\eta(\varphi(a), \varphi(b), 1) = \varphi(a) - \varphi(b)$, we get the following generalized Hermite-Hadamard type inequality for fractional integrals:*

$$\begin{aligned}
 &\left| \frac{f(\varphi(a)) + f(\varphi(b))}{2} - \frac{\Gamma(\alpha+1)}{2(\varphi(a) - \varphi(b))^\alpha} \left[J_{\varphi(b)+}^\alpha f(\varphi(a)) + J_{\varphi(a)-}^\alpha f(\varphi(b)) \right] \right| \\
 (3.10) \quad &\leq \frac{(\varphi(b) - \varphi(a))^2}{2(\alpha+1)} \left(\frac{p(\alpha+1)-1}{p(\alpha+1)+1} \right)^{\frac{1}{p}} \left(\frac{|f''(\varphi(a))|^q + |f''(\varphi(b))|^q}{1-s} \right)^{\frac{1}{q}}.
 \end{aligned}$$

Theorem 3.9. *Let $\varphi : I \rightarrow A$ be a continuous function. Suppose $A \subseteq \mathbb{R}$ be an open nonempty m -invex subset with respect to $\eta : A \times A \times (0, 1] \rightarrow \mathbb{R}$ for any fixed $s \in [0, 1)$, some fixed $m \in (0, 1]$ and let $a < b$ with $\eta(\varphi(a), \varphi(b), m) > 0$. Assume that $f : A \rightarrow \mathbb{R}$ be a twice differentiable function on A° . If $|f''|^q$ is a generalized (s, m, φ) -preinvex Godunova-Levin function of second kind on $[m\varphi(b), m\varphi(b) + \eta(\varphi(a), \varphi(b), m)]$, $q \geq 1$, then for each $x \in [m\varphi(b), m\varphi(b) + \eta(\varphi(a), \varphi(b), m)]$ and $\alpha \in (0, 1]$, we have*

$$\begin{aligned}
 &\left| \frac{f(m\varphi(b) + \eta(\varphi(a), \varphi(b), m)) + f(m\varphi(b))}{2} - \frac{\Gamma(\alpha+1)}{2\eta(\varphi(a), \varphi(b), m)^\alpha} \right. \\
 &\times \left. \left[J_{(m\varphi(b))}^\alpha f(m\varphi(b) + \eta(\varphi(a), \varphi(b), m)) + J_{(m\varphi(b) + \eta(\varphi(a), \varphi(b), m))}^\alpha f(m\varphi(b)) \right] \right|
 \end{aligned}$$

$$\begin{aligned}
 (3.11) \quad &\leq \frac{\eta^2(\varphi(a), \varphi(b), m)}{2(\alpha+1)} \left(\frac{\alpha}{\alpha+2} \right)^{1-\frac{1}{q}} \\
 &\times \left(\frac{\alpha+1}{(1-s)(\alpha-s+2)} - \beta(\alpha+2, 1-s) \right)^{\frac{1}{q}} \left[|f''(\varphi(a))|^q + m |f''(\varphi(b))|^q \right]^{\frac{1}{q}}.
 \end{aligned}$$

Proof. Suppose that $q \geq 1$. Using Lemma 3.2, relation (3.9), the fact that $|f''|^q$ is a generalized (s, m, φ) -preinvex Godunova-Levin function of second kind, the well-known power mean inequality and properties of the modulus, we have

$$\begin{aligned}
 |R_{f, \eta, \varphi}(\alpha, m, a, b)| &\leq \frac{\eta^2(\varphi(a), \varphi(b), m)}{2(\alpha+1)} \\
 &\times \int_0^1 (1 - (1-t)^{\alpha+1} - t^{\alpha+1}) |f''(m\varphi(b) + t\eta(\varphi(a), \varphi(b), m))| dt \\
 &\leq \frac{\eta^2(\varphi(a), \varphi(b), m)}{2(\alpha+1)} \left(\int_0^1 (1 - (1-t)^{\alpha+1} - t^{\alpha+1}) dt \right)^{1-\frac{1}{q}} \\
 &\times \left(\int_0^1 (1 - (1-t)^{\alpha+1} - t^{\alpha+1}) |f''(m\varphi(b) + t\eta(\varphi(a), \varphi(b), m))|^q dt \right)^{\frac{1}{q}} \\
 &\leq \frac{\eta^2(\varphi(a), \varphi(b), m)}{2(\alpha+1)} \left(\frac{\alpha}{\alpha+2} \right)^{1-\frac{1}{q}} \\
 &\times \left(\int_0^1 (1 - (1-t)^{\alpha+1} - t^{\alpha+1}) \left(\frac{|f''(\varphi(a))|^q}{t^s} + \frac{m |f''(\varphi(b))|^q}{(1-t)^s} \right) dt \right)^{\frac{1}{q}} \\
 &= \frac{\eta^2(\varphi(a), \varphi(b), m)}{2(\alpha+1)} \left(\frac{\alpha}{\alpha+2} \right)^{1-\frac{1}{q}} \\
 &\times \left(\frac{\alpha+1}{(1-s)(\alpha-s+2)} - \beta(\alpha+2, 1-s) \right)^{\frac{1}{q}} \left[|f''(\varphi(a))|^q + m |f''(\varphi(b))|^q \right]^{\frac{1}{q}}.
 \end{aligned}$$

□

Corollary 3.10. *Under the conditions of Theorem 3.9, if we choose $m = 1$ and $\eta(\varphi(a), \varphi(b), 1) = \varphi(a) - \varphi(b)$, we get the following generalized Hermite-Hadamard type inequality for fractional integrals:*

$$\begin{aligned}
 (3.12) \quad &\left| \frac{f(\varphi(a)) + f(\varphi(b))}{2} - \frac{\Gamma(\alpha+1)}{2(\varphi(a) - \varphi(b))^\alpha} \left[J_{\varphi(b)+}^\alpha f(\varphi(a)) + J_{\varphi(a)-}^\alpha f(\varphi(b)) \right] \right| \\
 &\leq \frac{(\varphi(b) - \varphi(a))^2}{2(\alpha+1)} \left(\frac{\alpha}{\alpha+2} \right)^{1-\frac{1}{q}} \\
 &\times \left(\frac{\alpha+1}{(1-s)(\alpha-s+2)} - \beta(\alpha+2, 1-s) \right)^{\frac{1}{q}} \left[|f''(\varphi(a))|^q + |f''(\varphi(b))|^q \right]^{\frac{1}{q}}.
 \end{aligned}$$

4. Applications to special means

Consider the following special means for arbitrary real numbers α, β ($\alpha \neq \beta$) as follows:

1. The arithmetic mean:

$$A := A(\alpha, \beta) = \frac{\alpha + \beta}{2},$$

2. The harmonic mean:

$$H := H(\alpha, \beta) = \frac{2}{\frac{1}{\alpha} + \frac{1}{\beta}}, \quad \alpha, \beta \in \mathbb{R} \setminus \{0\},$$

3. The logarithmic mean:

$$L := L(\alpha, \beta) = \frac{\beta - \alpha}{\ln |\beta| - \ln |\alpha|}; \quad |\alpha| \neq |\beta|, \quad \alpha\beta \neq 0,$$

4. The generalized log-mean:

$$L_n := L_n(\alpha, \beta) = \left[\frac{\beta^{n+1} - \alpha^{n+1}}{(n+1)(\beta - \alpha)} \right]^{\frac{1}{n}}; \quad n \in \mathbb{Z} \setminus \{-1, 0\}, \quad \alpha \neq \beta.$$

It is well known that L_n is monotonic nondecreasing over $n \in \mathbb{R}$ with $L_{-1} := L$. In particular, we have the following inequality $H \leq L \leq A$. Now, using the theory results in Section 3, we give some applications to special means of real numbers.

Theorem 4.1. *Let $\varphi : I \rightarrow A$ be a continuous function. Suppose $A \subseteq \mathbb{R}$ be an open nonempty 1-convex subset with respect to $\eta : A \times A \times (0, 1] \rightarrow \mathbb{R}$ for any fixed $s \in [0, 1)$. Then the following inequality for fractional integrals holds:*

$$(4.1) \quad \left| \frac{1}{H(\varphi(a), \varphi(b))} - \frac{1}{L(\varphi(a), \varphi(b))} \right| \leq (\varphi(b) - \varphi(a)) \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left(\frac{\varphi^{2q}(a) + \varphi^{2q}(b)}{\varphi^{2q}(a)\varphi^{2q}(b)(1-s)} \right)^{\frac{1}{q}},$$

where $q > 1$, $p^{-1} + q^{-1} = 1$.

Proof. Applying Theorem 3.3 for $f(x) = \frac{1}{x}$, $\eta(\varphi(a), \varphi(b), 1) = \varphi(a) - \varphi(b)$, $\alpha = \lambda = 1$ one can obtain the result immediately. $\square$

Theorem 4.2. Let $\varphi : I \rightarrow A$ be a continuous function. Suppose $A \subseteq \mathbb{R}$ be an open nonempty 1-inver subset with respect to $\eta : A \times A \times (0, 1] \rightarrow \mathbb{R}$ for any fixed $s \in [0, 1)$ and $n \in \mathbb{Z} \setminus \{-1, 0\}$. Then the following inequality for fractional integrals holds:

$$(4.2) \quad \left| A(\varphi^n(a), \varphi^n(b)) - L_n^n(\varphi(a), \varphi(b)) \right| \leq 2^{\frac{1}{q}-2} n(\varphi(b) - \varphi(a)) \\ \times \left[\left(\frac{\varphi^{(n-1)q}(a)}{2-s} + \frac{\varphi^{(n-1)q}(b)}{(s-1)(s-2)} \right)^{\frac{1}{q}} + \left(\frac{\varphi^{(n-1)q}(a)}{(s-1)(s-2)} + \frac{\varphi^{(n-1)q}(b)}{2-s} \right)^{\frac{1}{q}} \right],$$

where $q \geq 1$.

Proof. Applying Theorem 3.5 for $f(x) = x^n$, $\eta(\varphi(a), \varphi(b), 1) = \varphi(a) - \varphi(b)$, $\alpha = \lambda = 1$ one can obtain the result immediately. $\square$

Theorem 4.3. Let $\varphi : I \rightarrow A$ be a continuous function. Suppose $A \subseteq \mathbb{R}$ be an open nonempty 1-inver subset with respect to $\eta : A \times A \times (0, 1] \rightarrow \mathbb{R}$ for any fixed $s \in [0, 1)$. Then the following inequality for fractional integrals holds:

$$(4.3) \quad \left| \frac{1}{H(\varphi(a), \varphi(b))} - \frac{1}{L(\varphi(a), \varphi(b))} \right| \\ \leq 2^{\frac{1}{q}-2} (\varphi(b) - \varphi(a))^2 \left(\frac{2p-1}{2p+1} \right)^{\frac{1}{p}} \left(\frac{\varphi^{3q}(a) + \varphi^{3q}(b)}{\varphi^{3q}(a)\varphi^{3q}(b)(1-s)} \right)^{\frac{1}{q}},$$

where $q > 1$, $p^{-1} + q^{-1} = 1$.

Proof. Applying Theorem 3.7 for $f(x) = \frac{1}{x}$, $\eta(\varphi(a), \varphi(b), 1) = \varphi(a) - \varphi(b)$, $\alpha = 1$ one can obtain the result immediately. $\square$

Theorem 4.4. Let $\varphi : I \rightarrow A$ be a continuous function. Suppose $A \subseteq \mathbb{R}$ be an open nonempty 1-inver subset with respect to $\eta : A \times A \times (0, 1] \rightarrow \mathbb{R}$ for any fixed $s \in [0, 1)$ and $n \in \mathbb{Z} \setminus \{-1, 0\}$. Then the following inequality for fractional integrals holds:

$$(4.4) \quad \left| A(\varphi^n(a), \varphi^n(b)) - L_n^n(\varphi(a), \varphi(b)) \right| \leq \frac{2^{\frac{1}{q}-2}}{3^{1-\frac{1}{q}}} n(n-1)(\varphi(b) - \varphi(a))^2 \\ \times \left(\frac{1}{(s-1)(s-3)} + \frac{1}{(s-1)(s-2)(s-3)} \right)^{\frac{1}{q}} \left[\varphi^{(n-2)q}(a) + \varphi^{(n-2)q}(b) \right]^{\frac{1}{q}},$$

where $q \geq 1$.

Proof. Applying Theorem 3.9 for $f(x) = x^n$, $\eta(\varphi(a), \varphi(b), 1) = \varphi(a) - \varphi(b)$, $\alpha = 1$ one can obtain the result immediately. $\square$

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ALGEBRAIC HYPERSTRUCTURES AND SOCIAL RELATIONS

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Abstract. The relations between the people of a certain set of individuals can be described, from a static point of view, through the arrays of Moreno. From a dynamic point of view, however, account must be taken of the coalitions formed between people. These coalitions can be specified or one or more alternatives can be selected while making decisions involving more decision-makers; further there are presented coordination strategies for more people in order to ensure the maximum utility in social problems modelling using cooperative games. This paper presents how algebraic hyperstructures can be a useful mathematical tool both for the study of social relations from a static point of view and for the study of the social dynamics leading to the formation of coalitions. It shows that, surprisingly, many properties deemed significant only from an algebraic or geometric viewpoints in a set of individuals, deep meanings from the social point of view.

Keywords: social relations, social aggregations, hyperoperations, multiperson decision making, cooperative games.

1. Introduction

The relations between the people of a certain set of individuals can be described, from a “static” point of view, through the socio-matrices of Moreno. The Moreno technique was developed by Moreno in 1946, published in [24, 25]. The line of Moreno research was developed and used by various authors. The mathematical tool used by Moreno et al. is the *theory of relations*. We consider a set $U = \{x_1, x_2, \dots, x_n\}$ of individuals that we assume forms a “social group” (e.g. the students in a school, the professors in a university, etc.). A relation R on U is represented by a matrix $M_R = (m_{rs}), r, s \in \{1, 2, \dots, n\}$, where $m_{rs} = 1$ if

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$x_r R x_s$ and $m_{rs} = 0$ if x_r is not in the relation R with x_s . We use also the notation $x_r R x_s = m_{rs}$, i.e. $x_r R x_s = 1$ means that x_r is in relation with x_s , and $x_r R x_s = 0$ that x_r is not in relation with x_s .

The objective of the Moreno et al. research is to study, from the relation R , as, and from some particular points of view, social subgroups can be formed, using suitable algorithms based on statistical theories.

In this paper, in addition to above mentioned Moreno research, we propose the construction of social subgroups with algebraic algorithms based on the theory of algebraic hyperstructures while looking for and attributing social meanings to hyperoperations considered. In particular, a hyperproduct is seen as the result of the interaction of the components of an ordered pair of elements of U .

From a “dynamic” point of view, coalitions formed between people should be considered. These coalitions can be specified or one or more alternatives can be selected while making decisions involving more decision-makers; further there are presented coordination strategies for more people in order to ensure the maximum utility in social problems modelling using cooperative games.

In this paper we show how algebraic hyperstructures can be useful mathematical tool, both for the study of social relations from a static point of view and for the study of the social dynamics leading to the formation of coalitions. Our research consists in interpreting, in some social contexts, ideas and results obtained by theories on multivalued operations, and to develop, from these, algorithms for the construction or the recognition of social groups, that are often latent, i.e. not detectable by traditional survey tools.

2. A “social approach” to the algebraic hyperstructures

Let us recall some fundamental concepts on algebraic hyperstructures (see e.g. [1-5, 10, 29]). We denote with U a given nonempty set (e.g. of numbers, individuals, media, etc.), with $P(U)$ the family of subsets of U . Moreover let $P^*(U) = P(U) - \{\emptyset\}$.

Definition 2.1. A *partial hyperoperation* (or *multivalued operation*) on U is a function $\alpha : U \times U \rightarrow P(U)$ that to every ordered pair (x, y) of elements of U associates a subset of U , denoted with $x\alpha y$. The pair (U, α) is called a *partial hypergroupoid*.

If $x\alpha y \neq \emptyset, \forall x, y \in U$, then α is said to be a *hyperoperation* on U and (U, α) is said *hypergroupoid*.

Usually, if there is no possibility of misunderstanding, it is used multiplicative notation and $x\alpha y$ is called “*hyperproduct* of x and y ”

The partial hyperoperation α is said to be:

- *commutative* if $x\alpha y = y\alpha x, \forall x, y \in U$;
- *weak commutative* if $x\alpha y \neq \emptyset \Rightarrow x\alpha y \cap y\alpha x \neq \emptyset, \forall x, y \in U$ ([29]).

Social interpretation 2.2. Let U be a social group. Then the hyper-product $x\alpha y$ can be interpreted, in a suitable social context, as the set of the individuals chosen *for a particular activity* or *from a suitable viewpoint* by x and y , with the condition that first expresses his/her opinion x and then y . The condition $x\alpha y \neq \emptyset, \forall x, y \in U$ means that at least one individual must be selected; the commutativity means that it is not important who is expressed first; the weak commutativity means that you can agree on a compromise, whatever the individual who expresses the ideas first. In other words, the weak commutativity can be considered a condition for activating mediation or consensus strategies.

Definition 2.3. Let (U, α) be a *partial hypergroupoid*. For every A, B , subsets of U , we define $A\alpha B$ as the union of the sets $x\alpha y$, with $x \in A, y \in B$. Moreover we write $x\alpha B$ and $A\alpha y$ to denote $\{x\}\alpha B$ and $A\alpha\{y\}$, respectively.

Definition 2.4. A partial hypergroupoid (U, α) is said to be:

- *associative* (or *partial semi-hypergroup*) if $(x\alpha y)\alpha z = x\alpha(y\alpha z), \forall x, y, z \in U$;
- *weak associative* if

$$[(x\alpha y)\alpha z \neq \emptyset \text{ or } x\alpha(y\alpha z) \neq \emptyset] \Rightarrow (x\alpha y)\alpha z \cap x\alpha(y\alpha z) \neq \emptyset, \forall x, y, z \in U;$$

- *strictly weak associative* if, for every non empty subsets A, B, C of U :

$$(2.1) \quad [(A\alpha B)\alpha C \neq \emptyset \text{ or } A\alpha(B\alpha C) \neq \emptyset] \Rightarrow (A\alpha B)\alpha C \cap A\alpha(B\alpha C) \neq \emptyset.$$

Social interpretation 2.5. Let U be a set of individuals. Then, for every nonempty subsets A, B , of U , $A\alpha B$ is the set of the individuals chosen for a particular activity or from a suitable viewpoint by at least an x in A and a y in B , with the condition that first expresses his/her opinion x and then y . The associativity means that, if three social groups A, B, C , in this order, aggregate their opinions, is not important if aggregate first A and B or B and C . The strictly weak associativity means that you can agree on a compromise, whatever the social groups aggregate first. In other words, the strictly weak associativity can be considered a condition for activating mediation or consensus among groups ([12, 15]).

Unfortunately, in a political or social context, associativity, and always also the weak associativity, usually does not apply. The success of political action is often determined by the order in which the various groups aggregate to put forward their opinions and interests.

Definition 2.6. A partial hypergroupoid (U, α) is said to be a *partial quasi-hypergroup* if, for every x in U , $x\alpha U = U = U\alpha x$.

Social interpretation 2.7. Let U be a set of individuals. In a social context let us say that x marginalizes y as first decision making if $y \notin x\alpha U$. Similarly, we say that y marginalizes x as second decision maker if $x \notin U\alpha y$.

If (U, α) is a partial quasi-hypergroup we have as social interpretation that no individual can be marginalized by another, whatever the order in which the opinions are expressed.

Definition 2.8. A hypergroupoid (U, α) is said to be a *hypergroup* if it is a quasi-hypergroup and a semi-hypergroup.

Social interpretation 2.9. Let U be a set of individuals. If (U, α) is a hypergroup, then:

- any ordered pair (x, y) of $U \times U$, choose at least an individual belonging to U ;
- no individual can be marginalized by another;
- the choices made by the three groups A, B, C, of individuals, in order, that aggregate their opinions, do not depend on the fact that are aggregated before A and B or B and C before;
- if (U, α) is commutative, the result of the aggregation of three groups A, B, C, (and, in general, also n groups) is not dependent on the order in which the groups are aggregated ([13, 14]).

So, if you have a commutative hypergroup, the choices are, in a sense, objective, democratic and cannot be altered significantly by political skills.

Social example 2.10. In [17, 28] has been described a research on *self-organizing socialization* carried out in schools of Abruzzo and were built socio-matrices, addressing students of various schools questions like:

- Q1 = "Who would you invite to a party?",
- Q2 = "Who do you think can help you in case of trouble ?",
- Q3 = "Who do you think can be invited, as a friend, for his skills?",
- Q4 = "With whom would you go on vacation?".

Each question Qj induces a relationship Rj in the set $U = \{x_1, x_2, \dots, x_n\}$ of students (for a class or school) interviewed, placing $xRjy$ if y is one of the students indicated by x in question Qj . So each question Qj corresponds a Moreno socio-matrix $Mj = (mj_{rs})$, with n rows and n columns.

Using an approach based on algebraic hyperstructures the same questions should be targeted to the ordered pairs (x, y) of students making sure first x expresses his/her opinion, then y and finally you find an agreement between x and y . The questions to be put to pairs of students would be:

- D1 = "Who do you agree on to invite to a party?",
- D2 = "Who do you both think can help you in case of difficulties (troubles)?",
- D3 = "Who would you consider to be your friends for his/her skills?",
- D4 = "With whom do you both think is pleasant to go on vacation?".

From this point of view any Moreno socio-matrix $M_{Rj} = (mj(x, z)), x, z \in U$, is replaced by a "hyperoperation socio-matrices" $Hj = (hj((x, y), z)), (x, y) \in U \times U, z \in U$, with n^2 rows and n columns.

Obtain directly the hyperoperation socio-matrices Hj from the interviews Moreno can be very difficult for the time needed to interview the couples. There are, however, many criteria on which these matrices are obtained as a processing of those of Moreno [24, 25].

Two simple criteria are the following "or" and "and" criteria.

Criterion "or": we assume, for every $(x, y) \in U \times U, z \in U$:

$$(2.2) \quad hj^{or}((x, y), z) = \max\{mj(x, z), mj(y, z)\};$$

Criterion "and": we assume, for every $(x, y) \in U \times U, z \in U$:

$$(2.3) \quad hj^{and}((x, y), z) = \min\{mj(x, z), mj(y, z)\}.$$

The applications:

$$- \alpha_{or} : (x, y) \in U \times U \rightarrow \{z : hj^{or}((x, y), z) = 1\};$$

$$- \alpha_{and} : (x, y) \in U \times U \rightarrow \{z : hj^{and}((x, y), z) = 1\};$$

are two algebraic partial hyperoperations, associated to the relation Rj , and, for every $(x, y) \in U \times U$, $\alpha_{and}(x, y) \subseteq \alpha_{or}(x, y)$.

Some other partial hyperoperations, associated to the relation Rj , are:

$$- \alpha_{nor} : (x, y) \in U \times U \rightarrow \{z : hj^{or}((x, y), z) = 0\};$$

$$- \alpha_{nand} : (x, y) \in U \times U \rightarrow \{z : hj^{and}((x, y), z) = 0\};$$

$$- \alpha_d : (x, y) \in U \times U \rightarrow \{z : hj^{or}((x, y), z) - hj^{and}((x, y), z) = 1\}.$$

3. A "geometric and social" approach to coalition formation

Let us assume the following definition:

Definition 3.1. A hypergroupoid (U, α) is said to be a *coalition forming structure* if, for every x, y in U , $\{x, y\} \subseteq x\alpha y$.

Social interpretation 3.2. Let U be a set of individuals. If (U, α) is a *coalition forming structure*, then individuals who decide who can add to a given task or a given job fit always even themselves. That is, for successive aggregations, they are formed coalitions ever more numerous, able to achieve certain goals. This can be very important in the case of decisions made by most decision-makers, in which it is crucial to reach a majority or in cooperative games ([15, 27]).

Geometric interpretation 3.3. From a geometric point of view the individuals of U can be regarded as points of a particular geometric space. U is represented by a "point cloud" and the hyperproducts $x\alpha y$ are particular *clusters* of such cloud. Two particular cases are obtained from the socio-matrices of Moreno and from multi-criteria analysis.

A geometric representation of Moreno matrices. Let $U = \{x_1, x_2, \dots, x_n\}$. If $M_R = (m_{rs}), r, s \in \{1, 2, \dots, n\}$ is the socio-matrix of a relation R on

U , we can introduce two geometric (Euclidean) representations of U , in the Euclidean n -dimensional space E^n . The first is considering x_i as the point P_i of E^n with coordinates m_{is} , let us call the “*active or direct representation*” with respect to R ; the second is considering x_i as the point Q_i of E^n with coordinates m_{ri} , let us call the “*passive or inverse representation*” with respect to R . Then we obtain the “*active cloud*”, $\{P_i, i = 1, 2, \dots, n\}$ and the “*passive cloud*” $\{Q_i, i = 1, 2, \dots, n\}$.

In each of these clouds we can introduce a metric, such as the Euclidean metric d_E or the Hamming distance d_H , on which to base the formation of coalitions. For example, we can introduce a partial hyperoperation α setting a positive number ϵ and placing:

$$(3.1) \quad \forall x, y \in U, x\alpha y = \{z \in U : d_E(x, z) \diamond d_E(y, z)\} < \epsilon,$$

where “ $\diamond$ ” is a suitable algebraic operation in the set of nonnegative real numbers, e.g. addition, multiplication, maximum, minimum, etc.

A geometric representation of multi-criteria analysis. We analyze the elements of U on the basis of a set of criteria $K = \{C_1, C_2, \dots, C_m\}$, aimed at a certain objective. We attribute to each individual x_i a score s_{ij} against each criterion C_j . This score is a positive real number, usually belonging to $[0, 1]$ which measures the extent to which individual satisfies the criterion. In such a case each individual x_i is represented, in the Euclidean space E^m by the point $P_i = (s_{i1}, s_{i2}, \dots, s_{im})$. Also in this case we can introduce partial hyperoperations according to the above formula (3.1).

4. Conclusions and perspective of research

We have shown that the use of partial hyperstructures can be a useful tool for analyzing social relationships, and allows us to discover latent social subgroups, evaluate the formation of coalitions, and see aspects of social groups that do not appear in the classical analysis. In addition, each of the algebraic properties of hyperstructures can be translated to properties and behaviours of the social group; therefore the social analysis with the algebraic hyperstructures looks as a very powerful analysing tool.

One line of research is the application of the theories presented and hyperstructures introduced to case studies, statistical analyses to be made, or case studies already analysed by a statistical point of view, for which you want to obtain further processing.

A second research direction is achieved by replacing the relations with fuzzy relations, obtaining fuzzy Moreno socio-matrices. In this case, we can consider many possible hyperstructures associated with the α -cuts of the fuzzy sets obtained. Another generalization of the Moreno models is obtained by considering fuzzy relations, that appear to be more adequate than the crisp one to represent

human perceptions and communication [9, 16, 26, 30, 31], for statistical inferences with imprecise data [17-20], and for decision making [22]. The connections between hyperstructures and fuzzy sets are described in [6-8, 14, 21, 23].

Every fuzzy relation $R^\sim$ in the set $U = \{x_1, x_2, \dots, x_n\}$ of individuals is represented by a fuzzy socio-matrix $M_{R^\sim} = (m_{rs})$ where $m_{rs} \in [0, 1]$ is the degree in which the fuzzy relation $R^\sim$ holds. We write $x_r R^\sim x_s = m_{rs}$.

The sociological results known for the crisp socio-matrices can be, in many ways, extended to fuzzy socio-matrices. This approach is described in [11].

Finally, many results can be obtained considering hyperstructures associated to relations composed by those presented, in particular the powers of these relations.

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NOTE ON RELATIONS AMONG MULTIPLE ZETA(-STAR) VALUES

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Abstract. In the present paper, we shall show that various relations among multiple zeta(-star) values and their multivariable extensions can be derived from the hypergeometric identities of G. E. Andrews, C. Krattenthaler and T. Rivoal. The results in the present paper give us various identities for multiple Hurwitz zeta values also.

Keywords: Multiple zeta value, multiple zeta-star value, hypergeometric series, multiple Hurwitz zeta value.

1. Introduction

The multiple zeta value $\zeta(k_1, \dots, k_n)$ (MZV for short) and the multiple zeta-star value $\zeta^*(k_1, \dots, k_n)$ (MZSV for short) are defined by the multiple series

$$\begin{aligned}\zeta(k_1, \dots, k_n) &:= \sum_{0 < m_1 < \dots < m_n < \infty} \frac{1}{m_1^{k_1} \dots m_n^{k_n}}, \\ \zeta^*(k_1, \dots, k_n) &:= \sum_{1 \leq m_1 \leq \dots \leq m_n < \infty} \frac{1}{m_1^{k_1} \dots m_n^{k_n}},\end{aligned}$$

where $n \in \mathbb{Z}_{\geq 1}$, $k_1, \dots, k_{n-1} \in \mathbb{Z}_{\geq 1}$, $k_n \in \mathbb{Z}_{\geq 2}$ (Euler [11], Hoffman [15], Zagier [30]). MZVs can be expressed by $\mathbb{Z}$ -linear combinations of MZSVs and vice versa. In recent years, many researchers have been studying MZ(S)V and the study of MZ(S)V has become one of the active research areas of mathematics. One of the interesting properties of MZ(S)V is that MZ(S)V satisfy various relations. In fact, there are many results on this property. In the present paper, we shall also study relations among MZ(S)V.

We recall the definition of the generalized hypergeometric series:

$${}_{p+1}F_p \left(\begin{matrix} a_1, \dots, a_{p+1} \\ b_1, \dots, b_p \end{matrix} ; z \right) := \sum_{m=0}^{\infty} \frac{(a_1)_m \dots (a_{p+1})_m}{(b_1)_m \dots (b_p)_m} \frac{z^m}{m!},$$

where $p \in \mathbb{Z}_{\geq 1}$, $z, a_i \in \mathbb{C}$, $b_i \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ and $(a)_m$ denotes the Pochhammer symbol defined by $(a)_m := a(a+1) \dots (a+m-1)$ ($m \in \mathbb{Z}_{\geq 1}$), $(a)_0 := 1$. The

Pochhammer symbol $(a)_m$ can be expressed as $(a)_m = \Gamma(a+m)/\Gamma(a)$ by using the gamma function $\Gamma(z)$. The above power series converges absolutely for all $z \in \mathbb{C}$ such that $|z| = 1$ provided $\operatorname{Re}(\sum_{i=1}^p b_i - \sum_{i=1}^{p+1} a_i) > 0$.

In [24], Krattenthaler and Rivoal proved the following hypergeometric identities by taking a limit in the basic hypergeometric identity of Andrews [1, Theorem 4]:

Theorem A (Krattenthaler and Rivoal [24, Proposition 1 (i) and (ii)]). (i) *Let s be a positive integer, and let a, b_i, c_i ($i = 1, \dots, s+1$) be complex numbers. Suppose that the complex numbers a, b_i, c_i ($i = 1, \dots, s+1$) satisfy the conditions $1+a-b_i, 1+a-c_i \notin \mathbb{Z}_{\leq 0}$ ($i = 1, \dots, s+1$),*

$$\begin{aligned} \operatorname{Re} \left((2s+1)(a+1) - 2 \sum_{i=1}^{s+1} (b_i + c_i) \right) &> 0, \\ \operatorname{Re} \left(\sum_{i=r}^{s+1} A_i (1+a-b_i-c_i) \right) &> 0 \quad (r = 2, \dots, s+1) \end{aligned}$$

for all possible choices of $A_i = 1$ or 2 ($i = 2, \dots, s$), $A_{s+1} = 1$. (For the details of the choices of A_i , see [24].) Then the following identity holds:

$$\begin{aligned} & {}_{2s+4}F_{2s+3} \left(\begin{matrix} a, \frac{a}{2} + 1, b_1, c_1, \dots, b_{s+1}, c_{s+1} \\ \frac{a}{2}, 1+a-b_1, 1+a-c_1, \dots, 1+a-b_{s+1}, 1+a-c_{s+1} \end{matrix} ; -1 \right) \\ &= \frac{\Gamma(1+a-b_{s+1})\Gamma(1+a-c_{s+1})}{\Gamma(1+a)\Gamma(1+a-b_{s+1}-c_{s+1})} \\ &\times \sum_{l_1, \dots, l_s=0}^{\infty} \prod_{i=1}^s \frac{(1+a-b_i-c_i)_{l_i} (b_{i+1})_{l_1+\dots+l_i} (c_{i+1})_{l_1+\dots+l_i}}{l_i! (1+a-b_i)_{l_1+\dots+l_i} (1+a-c_i)_{l_1+\dots+l_i}}. \end{aligned}$$

(ii) *Let s be a positive integer, and let a, b_i, c_i ($i = 1, \dots, s$), c_0 be complex numbers. Suppose that the complex numbers a, b_i, c_i ($i = 1, \dots, s$), c_0 satisfy the conditions $1+a-b_i, 1+a-c_j \notin \mathbb{Z}_{\leq 0}$ ($i = 1, \dots, s; j = 0, 1, \dots, s$),*

$$\begin{aligned} \operatorname{Re} \left(2s(a+1) - 2c_0 - 2 \sum_{i=1}^s (b_i + c_i) \right) &> 0, \\ \operatorname{Re} \left(-c_0 + \sum_{i=1}^s A_i (1+a-b_i-c_i) \right) &> 0, \\ \operatorname{Re} \left(\sum_{i=r}^s A_i (1+a-b_i-c_i) \right) &> 0 \quad (r = 2, \dots, s) \end{aligned}$$

for all possible choices of $A_i = 1$ or 2 ($i = 1, \dots, s-1$), $A_s = 1$. (For the details of the choices of A_i , see [24].) Then the following identity holds:

$$\begin{aligned} & {}_{2s+3}F_{2s+2} \left(\begin{matrix} a, \frac{a}{2} + 1, c_0, b_1, c_1, \dots, b_s, c_s \\ \frac{a}{2}, 1 + a - c_0, 1 + a - b_1, 1 + a - c_1, \dots, 1 + a - b_s, 1 + a - c_s \end{matrix} ; 1 \right) \\ &= \frac{\Gamma(1 + a - b_s) \Gamma(1 + a - c_s)}{\Gamma(1 + a) \Gamma(1 + a - b_s - c_s)} \\ &\times \sum_{l_1, \dots, l_s=0}^{\infty} \frac{(b_1)_{l_1} (c_1)_{l_1}}{l_1! (1 + a - c_0)_{l_1}} \left\{ \prod_{i=2}^s \frac{(1 + a - b_{i-1} - c_{i-1})_{l_i} (b_i)_{l_1 + \dots + l_i} (c_i)_{l_1 + \dots + l_i}}{l_i! (1 + a - b_{i-1})_{l_1 + \dots + l_i} (1 + a - c_{i-1})_{l_1 + \dots + l_i}} \right\}. \end{aligned}$$

In [24], Krattenthaler and Rivoal used the above hypergeometric identities to give an alternative proof of the identity of Zudilin [33, Theorem 5], which is an identity between very-well-poised hypergeometric series and multiple integrals related to the construction of $\mathbb{Q}$ -linear forms in the Riemann zeta values. The hypergeometric identities in Theorem A are limiting cases of the basic hypergeometric identity of Andrews [1, Theorem 4], which is a multiple series generalization of Watson's q -analogue of Whipple's hypergeometric identity (see Watson [29]). In [2], Andrews proved a further multiple series generalization of the Watson's q -analogue also (see [2, Theorem 1]). In [1], [2] and [3], Andrews derived various multiple q -series identities from his basic hypergeometric identities [1, Theorem 4], [2, Theorem 1].

In the present paper, we show that various relations among $\text{MZ}(\text{S})\text{Vs}$ and their multivariable extensions can be derived from the hypergeometric identities in Theorem A by taking a specialization and further consideration. See also Remark 1 (i) below. By virtue of the multivariable extensions, most of the results in the present paper can be regarded as relations among the same multiple series. The hypergeometric identities of Andrews, Krattenthaler and Rivoal were useful for us to find multivariable extensions of $\text{MZ}(\text{S})\text{Vs}$ which satisfy various relations. In fact, by studying Theorem A, we found some multivariable extensions of $\text{MZ}(\text{S})\text{Vs}$ and their relations (see Section 2). Other hypergeometric identities (e.g. Andrews [2, Theorem 1]) may also be useful for this kind of study in extensions of $\text{MZ}(\text{S})\text{Vs}$. In the present paper, to prove the relations, we express the partial derivatives of the hypergeometric series by linear combinations of multiple zetas. This is done by using some calculational techniques for the Pochhammer symbols $(a)_m$, which are based on the techniques used in [20]. In particular, we avoid calculating the products of finite multiple harmonic sums by appropriate calculations for the Pochhammer symbols $(a)_m$. We remark that the calculational techniques used in the present paper can be applied to deriving relations among multiple zetas from other hypergeometric identities. For the study of the multiple zeta expression for hypergeometric series, see also [19, Remark 2] and the references therein. We hope that the results in the present paper are useful for the study of relations among multiple zetas and the study of the interaction between multiple zetas and (multiple) hypergeometric series. In the present paper, we prove various relations among the multiple series (1), (22)

and (41) below, which are the main objects of study in [21] and in the present paper also. We remark that many of those relations are written in compact forms. The compactness comes from appropriate calculations for the Pochhammer symbols $(a)_m$ and appropriate choices of special cases of the multiple series (1), (22) and (41) below. From the results in the present paper and their proofs, we think that to make further research on relations among the multiple series (1), (22) and (41) below is interesting and the hypergeometric method is useful for the research.

Remark 1. (i) By examining the proof of [24, Proposition 1], we see that the results in the present paper can be derived from the identity of Andrews [1, Theorem 4]; therefore various relations among multiple zetas can be derived from “one” hypergeometric identity. This fact also means that the present research is an application of the basic hypergeometric identity of Andrews [1, Theorem 4] to the study of relations among multiple zetas.

(ii) Many of the results in the present paper can be regarded as generalizations of the following two identities for MZSVs:

$$\begin{aligned}\zeta^*(\underbrace{2, \dots, 2}_s) &= 2 \sum_{m=0}^{\infty} (-1)^m (m+1)^{-2s} = 2(1 - 2^{1-2s})\zeta(2s), \\ \zeta^*(1, \underbrace{2, \dots, 2}_s) &= 2\zeta(2s+1)\end{aligned}$$

for all $s \in \mathbb{Z}_{\geq 1}$ (Aoki–Ohno [4, Theorem 1], Vasil’ev [28, Theorem], Zlobin [32]). These two identities can also be derived from the identity of Andrews [1, Theorem 4] by following the proof of [24, Proposition 1].

The present research is motivated by [17, Remarks 2.6 and 2.7]. The present paper is an expanded version of [19], [20], and a revised version of [21]. I proved the results in the present paper in 2013–2016 and modified several of them in 2016–2017. (I proved (29) and (30) below in 2011.)

2. Applications of the hypergeometric identities of Andrews, Krattenthaler and Rivoal

2.1 Notations and definitions

We use the notations

$$\begin{aligned}\{a\}^n &:= \underbrace{a, \dots, a}_n, \quad \{a_{i1}, \dots, a_{in}\}_{i=1}^m := a_{11}, \dots, a_{1n}, \dots, a_{m1}, \dots, a_{mn}, \\ \sum^{(k,l)} &:= \sum_{i=0}^{k+l} \sum_{\substack{k_1+\dots+k_{i+1}=k+1 \\ l_1+\dots+l_{i+1}=l+1 \\ k_j, l_j \in \mathbb{Z}_{\geq 0}, k_j+l_j \geq 1 \\ (j=1, \dots, i); k_{i+1}, l_{i+1} \in \mathbb{Z}_{\geq 1}}}\end{aligned}$$

$(k, l \in \mathbb{Z}_{\geq 0}, m, n \in \mathbb{Z}_{\geq 1})$. We regard $\{a\}^0, \{a_{i1}, \dots, a_{in}\}_{i=1}^0$ as the empty set $\emptyset$. The summation $\sum^{(k,l)}$ has the symmetry $\sum^{(k,l)} = \sum^{(l,k)}$. For brevity, we frequently use the following notations for the partial differential operators:

$$\partial^{(k)}(x) \Big|_{x=\alpha} := \frac{1}{k!} \frac{\partial^k}{\partial x^k} \Big|_{x=\alpha}, \quad \partial^{(k,l)}(x, y) \Big|_{\substack{x=\alpha \\ y=\beta}} := \frac{1}{k!l!} \frac{\partial^{k+l}}{\partial x^k \partial y^l} \Big|_{\substack{x=\alpha \\ y=\beta}}$$

$(k, l \in \mathbb{Z}_{\geq 0}, \alpha, \beta \in \mathbb{C})$. We define the symbol $e(n)$ by

$$e(n) = \begin{cases} 1, & \text{if } n \geq 1, \\ 0, & \text{if } n = 0. \end{cases}$$

The symbols $\lesssim, \lesssim_i$ ($i \in \mathbb{Z}_{\geq 0}$) mean $<$ or $\leq$ (cf. Fischler–Rivoal [12]). We consider the following special value of the multiple Hurwitz zeta function:

$$\begin{aligned} & \zeta_{n,\pm}^{\lesssim}(\{a_{i1}\}_{i=1}^n | \dots | \{a_{ik}\}_{i=1}^n; (\{\alpha_j\}_{j=1}^k)) \\ &:= \sum_{0 \leq m_1 \lesssim \dots \lesssim m_n < \infty} (\pm 1)^{m_n} \left\{ \prod_{i=1}^n \prod_{j=1}^k \frac{1}{(m_i + \alpha_j)^{a_{ij}}} \right\}, \end{aligned}$$

where $k, n \in \mathbb{Z}_{\geq 1}, \alpha_j \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}, a_{ij} \in \mathbb{Z}$ such that $\sum_{j=1}^k a_{ij} \geq 1$ ($i = 1, \dots, n-1$), $\sum_{j=1}^k a_{nj} \geq 2$. We call this special value the multiple Hurwitz zeta value (MHZV for short). For brevity, we put

$$\zeta_{-}(\{k_i\}_{i=1}^n) := \zeta_{n,-}^{\lesssim}(\{a_{i1}\}_{i=1}^n | \dots | \{a_{ik}\}_{i=1}^n; (\{1\}^k)),$$

where $k_1, \dots, k_{n-1} \in \mathbb{Z}_{\geq 1}, k_n \in \mathbb{Z}_{\geq 2}$. This alternating multiple series is a special value of multiple polylogarithms. (For the study of special values of multiple polylogarithms, see e.g. [5], [6], [7].) The results in the present paper give us various identities for MHZVs.

2.2 Application 1

We consider the multiple series

$$\begin{aligned} (1) \quad & \sum_{\substack{0=m_0 \leq \lesssim_0 m_1 \lesssim_1 \dots \lesssim_{p-1} m_p \\ m_p \lesssim_p \dots \lesssim_{p+q-1} m_{p+q} \lesssim_{p+q} < \infty}} \frac{(\alpha)_{m_p}}{m_p!} \frac{(\beta)_{m_p}}{(\gamma)_{m_p}} \frac{m_{p+q}!}{(\alpha)_{m_{p+q}}} \frac{(\gamma)_{m_{p+q}}}{(\beta)_{m_{p+q}}} \\ & \times \left\{ \prod_{i=1}^{p+q} \frac{(2m_i + \alpha + \beta)^{s_i} (2m_i + \gamma + \delta)^{t_i}}{(m_i + \alpha)^{a_i} (m_i + \beta)^{b_i} (m_i + \gamma)^{c_i} (m_i + \delta)^{d_i}} \right\}, \end{aligned}$$

where $p, q \in \mathbb{Z}_{\geq 0}$ such that $p+q \geq 1$; $a_i, b_i, c_i, d_i, s_i, t_i \in \mathbb{Z}$ such that $a_i + b_i + c_i + d_i - s_i - t_i \geq 1$ ($i = 1, \dots, p+q-1$), $a_{p+q} + b_{p+q} + c_{p+q} + d_{p+q} - s_{p+q} - t_{p+q} \geq 2$; $\alpha, \beta, \gamma, \delta \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ such that $\operatorname{Re}(\alpha + \beta - \gamma) > 0$ and $(\alpha + \beta)/2, (\gamma + \delta)/2 \notin \mathbb{Z}_{\leq 0}$ in case of $s_i, t_i < 0$ for some i . The symbol $(a)_m$ is the Pochhammer symbol.

The case $\alpha = \beta = \gamma = \delta = 1$, $s_i = t_i = 0$ ($i = 1, \dots, p+q$) of (1) is the extension of MZV considered by Fischler and Rivoal [12], Kawashima [23]. By considering the multiple series with the summation in (1), many of the identities in the present paper can be written in compact forms. The case $q = 0$ of (1) is a MHZV; therefore the multiple series (1) gives us an extension of MHZV. In this subsection, we study relations among the multiple series (1) by using Theorem A.

We first prove the following identity, which is frequently used in the present paper:

Lemma 2.1. *The identity*

$$\begin{aligned} \frac{(X)_m}{(Z)_{m+1}} \frac{(Y)_m}{(W)_{m+1}} &= \frac{(a)_m}{(c)_m} \frac{(b)_m}{(d)_m} \frac{1}{(Z+m)(W+m)} \\ &\times \sum_{i=0}^m \sum_{0 \leq m_1 < \dots < m_i < m} \prod_{j=1}^i \left(\frac{(X-a)(Y+m_j)(c+m_j)(d+m_j)}{(a+m_j)(b+m_j)(Z+m_j)(W+m_j)} \right. \\ &\quad \left. + \frac{(Y-b)(c+m_j)(d+m_j)}{(b+m_j)(Z+m_j)(W+m_j)} - \frac{(Z-c)(d+m_j)}{(Z+m_j)(W+m_j)} - \frac{W-d}{W+m_j} \right) \end{aligned}$$

holds for all $m \in \mathbb{Z}_{\geq 0}$, $X, Y \in \mathbb{C}$, $a, b, c, d, Z, W \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$.

Proof. By direct calculation, we get the identities

$$\begin{aligned} \frac{(X)_m}{(Z)_{m+1}} \frac{(Y)_m}{(W)_{m+1}} &= \frac{(X)_m}{(Z)_{m+1}} \frac{(Y)_m}{(W)_{m+1}} \frac{(c)_m}{(a)_m} \frac{(d)_m}{(b)_m} \frac{(a)_m}{(c)_m} \frac{(b)_m}{(d)_m} \\ &= \frac{(a)_m}{(c)_m} \frac{(b)_m}{(d)_m} \frac{1}{(Z+m)(W+m)} \prod_{i=0}^{m-1} \frac{(X+i)(Y+i)(c+i)(d+i)}{(a+i)(b+i)(Z+i)(W+i)} \\ &= \frac{(a)_m}{(c)_m} \frac{(b)_m}{(d)_m} \frac{1}{(Z+m)(W+m)} \prod_{i=0}^{m-1} \left(1 + \frac{(X-a)(Y+i)(c+i)(d+i)}{(a+i)(b+i)(Z+i)(W+i)} \right. \\ &\quad \left. + \frac{(Y-b)(c+i)(d+i)}{(b+i)(Z+i)(W+i)} - \frac{(Z-c)(d+i)}{(Z+i)(W+i)} - \frac{W-d}{W+i} \right). \end{aligned}$$

Expanding the last product, we get Lemma 2.1. □

We consider the following special case of (1):

$$\begin{aligned} &\Phi_{(\{r_i\}_{i=1}^{p+q})}^{(p,q)}(\{\{k_{ij}\}_{j=1}^{r_i}, K_i\}_{i=1}^{p+q}; (\alpha, \beta, \gamma)) \\ &:= \sum_{\substack{0=m_0 \leq m_{11} < \dots < m_{1r_1} < m_1 \\ \vdots \\ m_{p-1} \leq m_{p1} < \dots < m_{pr_p} < m_p \\ \vdots \\ m_{p+q-1} \leq m_{p+q1} < \dots < m_{p+qr_{p+q}} < m_{p+q} < \infty}} \frac{(\alpha)_{m_p}}{m_p!} \frac{(\beta)_{m_p}}{(\gamma)_{m_p}} \frac{m_{p+q}!}{(\alpha)_{m_{p+q}}} \frac{(\gamma)_{m_{p+q}}}{(\beta)_{m_{p+q}}} \end{aligned}$$

$$\begin{aligned} & \times \left\{ \prod_{i=1}^p \frac{1}{(m_i + \gamma)^{K_i}} \prod_{j=1}^{r_i} \frac{1}{(m_{ij} + \gamma)^{k_{ij}}} \right\} \\ & \times \left\{ \prod_{i=p+1}^{p+q} \frac{1}{(m_i + \alpha)(m_i + \beta)(m_i + \gamma)^{K_i-2}} \prod_{j=1}^{r_i} \frac{1}{(m_{ij} + \gamma)^{k_{ij}}} \right\}, \end{aligned}$$

where $p, q \in \mathbb{Z}_{\geq 0}$ such that $p+q \geq 1$; $r_i \in \mathbb{Z}_{\geq 0}$, $k_{ij} \in \mathbb{Z}_{\geq 1}$; $K_1, \dots, K_{p+q-1} \in \mathbb{Z}_{\geq 1}$, $K_{p+q} \in \mathbb{Z}_{\geq 2}$.

Theorem 2.2. *The identity*

$$\begin{aligned} & \sum_{\substack{r_1 + \dots + r_{k+s} = l \\ r_i \in \mathbb{Z}_{\geq 0}}} \Phi_{(\{r_i\}_{i=1}^{k+s})}^{(k+1, s-1)}(\{1\}^{\sum_{i=1}^{k+1} (r_i+1)}, \{\{1\}^{r_i}, 2\}_{i=k+2}^{k+s}; (\alpha, \beta, \gamma)) \\ (2) \quad & = \sum^{(k, l)} \phi_{i+1} \left(\begin{array}{c} \{e(k_j) + e(l_j)\}_{j=1}^i, 1 \\ \{k_j + e(l_j)\}_{j=1}^i, k_{i+1} | \{l_j + e(k_j)\}_{j=1}^i, l_{i+1} | \{0\}^i, s-1 | \{0\}^i, s-1 \\ (\gamma, \alpha + \beta - \gamma, \alpha, \beta) \end{array} \right) \end{aligned}$$

holds for all $k, l \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}_{\geq 2}$, $\alpha, \beta, \gamma \in \mathbb{C}$ such that $\operatorname{Re} \alpha, \operatorname{Re} \beta > 0$, $\operatorname{Re}(\alpha + \beta) > \operatorname{Re} \gamma > 0$, where

$$\begin{aligned} & \phi_p \left(\begin{array}{c} \{s_i\}_{i=1}^p \\ \{a_i\}_{i=1}^p | \{b_i\}_{i=1}^p | \{c_i\}_{i=1}^p | \{d_i\}_{i=1}^p \\ (x, y, z, w) \end{array} \right) \\ & := \sum_{0 \leq m_1 < \dots < m_p < \infty} \prod_{i=1}^p \frac{(2m_i + x + y)^{s_i}}{(m_i + x)^{a_i} (m_i + y)^{b_i} (m_i + z)^{c_i} (m_i + w)^{d_i}}, \end{aligned}$$

which is also a special case of (1).

Proof. Taking $a = \alpha + \beta$, $c_0 = \alpha + \beta - z$, $b_i = \alpha$, $c_i = \beta$ ($i = 1, \dots, s-1$), $b_s = 1$, $c_s = w$, where $s \in \mathbb{Z}_{\geq 1}$, $\alpha, \beta, z, w \in \mathbb{C}$ such that $\operatorname{Re} \alpha, \operatorname{Re} \beta > 0$, $\operatorname{Re}(\alpha + \beta) > \operatorname{Re} z > 0$, $\operatorname{Re}(\alpha + \beta) > \operatorname{Re} w > 0$, $s-1 > \operatorname{Re}(w-z)$, in Theorem A (ii) and multiplying both sides of the result by z^{-1} , we get the identity

$$\begin{aligned} (3) \quad & \sum_{0 \leq m_1 \leq \dots \leq m_s < \infty} \frac{(\alpha)_{m_1}}{m_1!} \frac{(\beta)_{m_1}}{(z)_{m_1+1}} \frac{m_s!}{(\alpha)_{m_s}} \frac{(w)_{m_s}}{(\beta)_{m_s}} \left\{ \prod_{i=2}^s \frac{1}{(m_i + \alpha)(m_i + \beta)} \right\} \\ & = \sum_{m=0}^{\infty} \frac{(\alpha + \beta - z)_m}{(z)_{m+1}} \frac{(w)_m}{(\alpha + \beta - w)_{m+1}} \frac{2m + \alpha + \beta}{(m + \alpha)^{s-1} (m + \beta)^{s-1}}. \end{aligned}$$

Using the identity

$$(4) \quad \frac{(-1)^r}{r!} \frac{d^r}{dz^r} \left(\frac{1}{(z)_{m+1}} \right) = \frac{1}{(z)_{m+1}} \sum_{0 \leq m_1 \leq \dots \leq m_r \leq m} \prod_{i=1}^r \frac{1}{m_i + z}$$

$(m, r \in \mathbb{Z}_{\geq 0})$, we get the identity

$$\begin{aligned}
 & \frac{(-1)^k}{k!} \frac{\partial^k}{\partial z^k} \left(\sum_{0 \leq m_1 \leq \dots \leq m_s < \infty} \frac{(\alpha)_{m_1}}{m_1!} \frac{(\beta)_{m_1}}{(z)_{m_1+1}} \frac{m_s!}{(\alpha)_{m_s}} \frac{(w)_{m_s}}{(\beta)_{m_s}} \right. \\
 & \quad \left. \times \left\{ \prod_{i=2}^s \frac{1}{(m_i + \alpha)(m_i + \beta)} \right\} \right) \\
 (5) \quad &= \sum_{\substack{0 \leq m_1 \leq \dots \leq m_{k+1} \\ m_{k+1} \leq \dots \leq m_{k+s} < \infty}} \left\{ \prod_{i=1}^{k+1} \frac{1}{m_i + z} \right\} \frac{(\alpha)_{m_{k+1}}}{m_{k+1}!} \frac{(\beta)_{m_{k+1}}}{(z)_{m_{k+1}}} \frac{m_{k+s}!}{(\alpha)_{m_{k+s}}} \frac{(w)_{m_{k+s}}}{(\beta)_{m_{k+s}}} \\
 & \quad \times \left\{ \prod_{i=k+2}^{k+s} \frac{1}{(m_i + \alpha)(m_i + \beta)} \right\}
 \end{aligned}$$

$(k \in \mathbb{Z}_{\geq 0})$. Using the identity

$$(w)_{m_{k+s}} = \prod_{i=1}^{k+s} \frac{(w)_{m_i}}{(w)_{m_{i-1}}},$$

where we regard m_0 as 0, we get the identities

$$\begin{aligned}
 & \frac{1}{l!} \frac{d^l}{dw^l} (w)_{m_{k+s}} = \sum_{\substack{r_1 + \dots + r_{k+s} = l \\ r_i \in \mathbb{Z}_{\geq 0}}} \prod_{i=1}^{k+s} \frac{(w)_{m_i}}{(w)_{m_{i-1}}} \sum_{m_{i-1} \leq m_{i1} < \dots < m_{ir_i} < m_i} \prod_{j=1}^{r_i} \frac{1}{m_{ij} + w} \\
 (6) \quad &= (w)_{m_{k+s}} \sum_{\substack{r_1 + \dots + r_{k+s} = l \\ r_i \in \mathbb{Z}_{\geq 0}}} \sum_{\substack{0 = m_0 \leq m_{11} < \dots < m_{1r_1} < m_1 \\ \vdots \\ m_{k+s-1} \leq m_{k+s1} < \dots < m_{k+s r_{k+s}} < m_{k+s}}} \prod_{i=1}^{k+s} \prod_{j=1}^{r_i} \frac{1}{m_{ij} + w}
 \end{aligned}$$

for $k, l \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}_{\geq 1}$ and $m_1, \dots, m_{k+s} \in \mathbb{Z}$ such that $0 \leq m_1 \leq \dots \leq m_{k+s}$. Thus, applying the operator $(-1)^k \partial^{(l,k)}(w, z)|_{z=\gamma}^{w=\gamma}$ ($k, l \in \mathbb{Z}_{\geq 0}$; $\gamma \in \mathbb{C}$ such that $\operatorname{Re}(\alpha + \beta) > \operatorname{Re} \gamma > 0$, $\operatorname{Re} \alpha, \operatorname{Re} \beta > 0$) to the left-hand side of (3) with $s \in \mathbb{Z}_{\geq 2}$ and using (5) and (6), we get the left-hand side of (2). The right-hand side of (2) can be proved as follows: Taking $a = \alpha + \beta - \gamma$, $b = \delta$, $c = \gamma$, $d = \alpha + \beta - \delta$, $X = \alpha + \beta - z$, $Y = w$, $Z = z$, $W = \alpha + \beta - w$ in Lemma 2.1 and slightly modifying the result, we get the identity

$$\begin{aligned}
 & \frac{(\alpha + \beta - z)_m}{(z)_{m+1}} \frac{(w)_m}{(\alpha + \beta - w)_{m+1}} \\
 &= \frac{(\alpha + \beta - \gamma)_m}{(\gamma)_m} \frac{(\delta)_m}{(\alpha + \beta - \delta)_m} \frac{1}{(z + m)(\alpha + \beta - w + m)}
 \end{aligned}$$

$$\begin{aligned} & \times \sum_{i=0}^m \sum_{0 \leq m_1 < \dots < m_i < m} \prod_{j=1}^i \left(-\frac{(z-\gamma)(w-\delta)(\gamma+m_j)(\alpha+\beta-\delta+m_j)}{(z+m_j)(\alpha+\beta-w+m_j)(\alpha+\beta-\gamma+m_j)(\delta+m_j)} \right. \\ & - \frac{(z-\gamma)(\alpha+\beta-\delta+m_j)(2m_j+\alpha+\beta)}{(z+m_j)(\alpha+\beta-w+m_j)(\alpha+\beta-\gamma+m_j)} + \frac{(w-\delta)(\gamma+m_j)(\alpha+\beta-\delta+m_j)}{(\alpha+\beta-w+m_j)(z+m_j)(\delta+m_j)} \\ & \left. + \frac{w-\delta}{\alpha+\beta-w+m_j} \right). \end{aligned}$$

Using this identity, we get the identity

$$\begin{aligned} & (-1)^k \partial^{(l,k)}(w, z) \left(\frac{(\alpha+\beta-z)_m}{(z)_{m+1}} \frac{(w)_m}{(\alpha+\beta-w)_{m+1}} \right) \Big|_{\substack{w=\delta \\ z=\gamma}} \\ & = \sum_{i=0}^m \sum_{\substack{k_1+\dots+k_{i+1}=k+1 \\ l_1+\dots+l_{i+1}=l+1 \\ k_j, l_j \in \mathbb{Z}_{\geq 0}, k_j+l_j \geq 1 \\ (j=1, \dots, i); k_{i+1}, l_{i+1} \in \mathbb{Z}_{\geq 1}}} \sum_{0 \leq m_1 < \dots < m_i < m} \\ (7) \quad & \left\{ \prod_{j=1}^i \frac{(2m_j+\alpha+\beta)^{e(k_j)} (2m_j+\alpha+\beta)^{e(l_j)}}{(m_j+\gamma)^{k_j} (m_j+\alpha+\beta-\gamma)^{e(k_j)} (m_j+\alpha+\beta-\delta)^{l_j} (m_j+\delta)^{e(l_j)}} \right\} \\ & \times \frac{1}{(m+\gamma)^{k_{i+1}} (m+\alpha+\beta-\delta)^{l_{i+1}}} \frac{(\alpha+\beta-\gamma)_m}{(\gamma)_m} \frac{(\delta)_m}{(\alpha+\beta-\delta)_m} \end{aligned}$$

for $k, l, m \in \mathbb{Z}_{\geq 0}$, $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ such that $\gamma, \delta, \alpha+\beta-\gamma, \alpha+\beta-\delta \notin \mathbb{Z}_{\leq 0}$. Thus, applying $(-1)^k \partial^{(l,k)}(w, z) \Big|_{\substack{w=\delta \\ z=\gamma}}$ to the right-hand side of (3) with $s \in \mathbb{Z}_{\geq 2}$ and using the case $\gamma = \delta$ of (7), we get the right-hand side of (2). $\square$

Here we put

$$\Phi_{p,q}^*(\{K_i\}_{i=1}^{p+q}; (\alpha, \beta, \gamma)) := \Phi_{(\{0\}^{p+q})}^{(p,q)}(\{K_i\}_{i=1}^{p+q}; (\alpha, \beta, \gamma)).$$

Using (2), we can prove the following identities:

Corollary 2.3. *The following two identities hold:*

(i)

$$\begin{aligned} & \Phi_{k+1, s-1}^*(\{1\}^{k+1}, \{2\}^{s-1}; (\alpha, \beta, \gamma)) \\ (8) \quad & = \sum^{(k,0)} \phi_{i+1} \left(\frac{\{1\}^{i+1}}{\{k_j\}_{j=1}^{i+1} |\{1\}^{i+1} | \{0\}^i, s-1 | \{0\}^i, s-1} ; (\gamma, \alpha+\beta-\gamma, \alpha, \beta) \right) \end{aligned}$$

for all $k \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}_{\geq 2}$, $\alpha, \beta, \gamma \in \mathbb{C}$ such that $\operatorname{Re} \alpha, \operatorname{Re} \beta > 0$, $\operatorname{Re}(\alpha+\beta) > \operatorname{Re} \gamma > 0$.

(ii)

$$\begin{aligned}
& \sum_{\substack{r_1+\dots+r_{k+s}=l \\ r_i \in \mathbb{Z}_{\geq 0}}} \Phi_{(\{r_i\}_{i=1}^{k+s})}^{(k+1, s-1)}(\{1\}^{\sum_{i=1}^{k+1} (r_i+1)}, \{\{1\}^{r_i}, 2\}_{i=k+2}^{k+s}; (\alpha, \beta, \gamma)) \\
(9) \quad &= \sum_{\substack{r_1+\dots+r_{l+s}=k \\ r_i \in \mathbb{Z}_{\geq 0}}} \Phi_{(\{r_i\}_{i=1}^{l+s})}^{(l+1, s-1)}(\{1\}^{\sum_{i=1}^{l+1} (r_i+1)}, \{\{1\}^{r_i}, 2\}_{i=l+2}^{l+s}; (\alpha, \beta, \alpha + \beta - \gamma))
\end{aligned}$$

for all $k, l \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}_{\geq 2}$, $\alpha, \beta, \gamma \in \mathbb{C}$ such that $\operatorname{Re} \alpha, \operatorname{Re} \beta > 0$, $\operatorname{Re}(\alpha + \beta) > \operatorname{Re} \gamma > 0$.

Proof. Taking $l = 0$ in (2), we get (8). The identity (9) can be proved as follows: By the replacements $k \leftrightarrow l$ and $\gamma \leftrightarrow \alpha + \beta - \gamma$ in the right-hand side of (2), we see that it has a symmetry with respect to $k, l, \gamma, \alpha + \beta - \gamma$. By this fact and (2), we get (9). $\square$

Remark 2. Taking $l = 0$ in (9), we get the following another expression for the multiple series on the left-hand side of (8):

$$\begin{aligned}
& \Phi_{k+1, s-1}^*(\{1\}^{k+1}, \{2\}^{s-1}; (\alpha, \beta, \gamma)) \\
(10) \quad &= \sum_{\substack{r_1+\dots+r_s=k \\ r_i \in \mathbb{Z}_{\geq 0}}} \Phi_{(\{r_i\}_{i=1}^s)}^{(1, s-1)}(\{1\}^{r_1+1}, \{\{1\}^{r_i}, 2\}_{i=2}^s; (\alpha, \beta, \alpha + \beta - \gamma))
\end{aligned}$$

for all $k \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}_{\geq 2}$, $\alpha, \beta, \gamma \in \mathbb{C}$ such that $\operatorname{Re} \alpha, \operatorname{Re} \beta > 0$, $\operatorname{Re}(\alpha + \beta) > \operatorname{Re} \gamma > 0$. The symmetry of (9) with respect to α, β, γ comes from the symmetry of the left-hand side of the identity in Theorem A (ii) with respect to $b_1, \dots, b_s, c_0, \dots, c_s$.

Here we consider the multiple series

$$\begin{aligned}
& \Phi_{A^{(p,q)}, B^{(p,q)}}^{(p,q), \lesssim (p+q-1)} \left(\begin{array}{c} \{s_i\}_{i=1}^{p+q} \\ \{a_i\}_{i=1}^{p+q} | \{b_i\}_{i=1}^{p+q} | \{c_i\}_{i=1}^{p+q} \end{array} ; (\alpha, \beta, \gamma, \delta) \right) \\
(11) \quad &:= \sum_{\substack{0=m_0 \leq \lesssim_0 m_1 \lesssim_1 \dots \lesssim_{p-1} m_p \\ m_p \lesssim_p \dots \lesssim_{p+q-1} m_{p+q} \lesssim_{p+q} \dots < \infty}} \frac{(\alpha)_{m_p} (\beta)_{m_p} m_{p+q}! (\gamma)_{m_{p+q}}}{m_p! (\gamma)_{m_p} (\alpha)_{m_{p+q}} (\beta)_{m_{p+q}}} \\
& \times \left\{ \prod_{i=1}^p \frac{A_i(m_i + \gamma)^{s_i} + B_i(m_i + \delta)^{s_i}}{(m_i + \gamma)^{a_i} (m_i + \delta)^{b_i} (2m_i + \gamma + \delta)^{c_i}} \right\} \\
& \times \left\{ \prod_{i=p+1}^{p+q} \frac{A_i(m_i + \alpha)^{s_i} + B_i(m_i + \beta)^{s_i}}{(m_i + \alpha)^{a_i} (m_i + \beta)^{b_i} (2m_i + \alpha + \beta)^{c_i}} \right\},
\end{aligned}$$

where $p, q, \alpha, \beta, \gamma, \delta$ are the same as those in the definition of (1); $a_i, b_i, c_i, s_i \in \mathbb{Z}$ such that $a_i + b_i + c_i - s_i \geq 1$ ($i = 1, \dots, p+q-1$), $a_{p+q} + b_{p+q} + c_{p+q} - s_{p+q} \geq 2$;

$A_i, B_i \in \mathbb{C}$; $A^{(p+q)} := \{A_i\}_{i=1}^{p+q}$, $B^{(p+q)} := \{B_i\}_{i=1}^{p+q}$; $\lesssim^{(p+q-1)} := \{\lesssim_i\}_{i=1}^{p+q-1}$. The multiple series (11) is a one-multiple series expression for a linear combination of the multiple series (1). We denote the case $q = 0$, $\lesssim_1 = \dots = \lesssim_{p-1} = \lesssim$ ($\in \{<, \leq\}$) of (11) by

$$(12) \quad \phi_{A^{(p)}, B^{(p)}}^{p, \lesssim} \left(\begin{array}{c} \{s_i\}_{i=1}^p \\ \{a_i\}_{i=1}^p | \{b_i\}_{i=1}^p | \{c_i\}_{i=1}^p \end{array} ; (\gamma, \delta) \right).$$

Differentiating both sides of (8) with respect to α , β and γ , we can derive the following identities:

Corollary 2.4. *The following identities hold:*

(i)

$$(13) \quad \begin{aligned} & \sum_{\substack{r_1 + \dots + r_{s-1} = r \\ r_i \in \mathbb{Z}_{\geq 0}}} \Phi^{(k+1, r+s-1), \leq}(\{1\}^{k+r+s} | \{0\}^{k+1}, \{\{0\}^{r_i}, 1\}_{i=1}^{s-1}; (\alpha, \beta, \gamma)) \\ &= \sum^{(k, 0)} \sum_{\substack{r_1 + \dots + r_{i+2} = r \\ r_j \in \mathbb{Z}_{\geq 0}}} \binom{s-2+r_{i+2}}{r_{i+2}} \\ & \times \phi_{i+1} \left(\begin{array}{c} \{1-e(r_j)\}_{j=1}^{i+1} \\ \{k_j - e(r_j)\}_{j=1}^{i+1} | \{1+r_j\}_{j=1}^{i+1} | \{0\}^i, s-1+r_{i+2} | \{0\}^i, s-1 \end{array} ; \right. \\ & \quad \left. (\gamma, \alpha + \beta - \gamma, \alpha, \beta) \right) \end{aligned}$$

for all $k, r \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}_{\geq 2}$, $\alpha, \beta, \gamma \in \mathbb{C}$ such that $\operatorname{Re} \alpha, \operatorname{Re} \beta > 0$, $\operatorname{Re}(\alpha + \beta) > \operatorname{Re} \gamma > 0$, where $\Phi^{(p, q), \leq}(\{a_i\}_{i=1}^{p+q} | \{b_i\}_{i=1}^{p+q}; (\alpha, \beta, \gamma))$ denotes the case $\delta = \gamma$, $\lesssim_i = \leq$ ($i = 1, \dots, p+q-1$), $c_i = s_i = A_i = B_i = 1$ ($i = 1, \dots, p+q$) of (11).

(ii)

$$(14) \quad \begin{aligned} & \sum_{\substack{r_1 + \dots + r_{k+s} = r \\ r_i \in \mathbb{Z}_{\geq 0}}} \Phi_{k+1, s-1}^*(\{r_i + 1\}_{i=1}^{k+1}, \{r_i + 2\}_{i=k+2}^{k+s}; (\alpha, \beta, \beta)) \\ &= \sum^{(k, 0)} \sum_{\substack{r_1 + \dots + r_{i+1} = r \\ r_j \in \mathbb{Z}_{\geq 0}}} \\ & \phi_{A^{(i+1)}, B^{(i+1)}}^{i+1, <} \left(\begin{array}{c} \{1\}^{i+1} \\ \{1\}^i, s | \{k_j + r_j\}_{j=1}^i, k_{i+1} + r_{i+1} + s - 1 | \{0\}^{i+1} \end{array} ; \right. \\ & \quad \left. (\alpha, \beta) \right) \end{aligned}$$

for all $k, r \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}_{\geq 2}$, $\alpha, \beta \in \mathbb{C}$ such that $\operatorname{Re} \alpha > 0$, $\beta \notin \mathbb{Z}_{\leq 0}$, where $A^{(i+1)} = \left\{ \binom{k_j + r_j - 1}{r_j} \right\}_{j=1}^i, \binom{k_{i+1} + r_{i+1} + s - 2}{r_{i+1}}$; $B^{(i+1)} = \left\{ \binom{k_j + r_j - 2}{r_j} \right\}_{j=1}^i, \binom{k_{i+1} + r_{i+1} + s - 3}{r_{i+1}}$.

The right-hand side of (14) can be written as a linear combination of $\zeta_{i+1,+}^{<}(\{a_j\}_{j=1}^{i+1}|\{b_j\}_{j=1}^{i+1};(\alpha,\beta))$, explicitly.

(iii)

$$\begin{aligned}
 & \sum_{\substack{u_1+\dots+u_{k+1} \\ +r_1+\dots+r_{s-1}=r \\ r_i, u_i \in \mathbb{Z}_{\geq 0}}} (-1)^{\sum_{i=1}^{k+1} u_i} \Phi_{(\{0\}^{k+1}, \{r_i\}_{i=1}^{s-1})}^{(k+1, s-1)}(\{1+u_i\}_{i=1}^{k+1}, \{\{1\}^{r_i}, 2\}_{i=1}^{s-1}; (\alpha, \beta, \gamma)) \\
 (15) \quad &= \sum^{(k,0)} \sum_{\substack{u_1+\dots+u_{i+1} \\ +r_1+\dots+r_{i+1}=r \\ r_j, u_j \in \mathbb{Z}_{\geq 0}}} \left\{ \prod_{j=1}^{i+1} (-1)^{u_j} \binom{k_j-1+u_j}{u_j} \right\} \\
 &\times \phi_{i+1} \left(\begin{array}{c} \{1\}^{i+1} \\ \{k_j+u_j\}_{j=1}^{i+1} | \{1+r_j\}_{j=1}^{i+1} | \{0\}^i, s-1 | \{0\}^i, s-1 \end{array} ; \right. \\
 &\quad \left. (\gamma, \alpha + \beta - \gamma, \alpha, \beta) \right)
 \end{aligned}$$

for all $k, r \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}_{\geq 2}$, $\alpha, \beta, \gamma \in \mathbb{C}$ such that $\operatorname{Re} \alpha, \operatorname{Re} \beta > 0$, $\operatorname{Re}(\alpha + \beta) > \operatorname{Re} \gamma > 0$. The right-hand side of (15) can be written as a linear combination of $\zeta_{i+1,+}^{<}(\{a_j\}_{j=1}^{i+1}|\{b_j\}_{j=1}^{i+1}|\{0\}^i, s-1|\{0\}^i, s-1; (\gamma, \alpha + \beta - \gamma, \alpha, \beta))$, explicitly. In particular, the right-hand side of the case $k=0$ of (15) can be written as

$$(16) \quad \zeta_{1,+}^{<}(r+1|s-1|s-1; (\alpha+\beta-\gamma, \alpha, \beta)) + (-1)^r \zeta_{1,+}^{<}(r+1|s-1|s-1; (\gamma, \alpha, \beta)).$$

Proof. Using the identity

$$\frac{(\alpha)_{m_p}}{(\alpha)_{m_{p+q}}} \left(\prod_{i=1}^{p+q} \frac{1}{(m_i + \alpha)^{a_i}} \right) = \left(\prod_{i=1}^p \frac{1}{(m_i + \alpha)^{a_i}} \right) \left(\prod_{i=p+1}^{p+q} \frac{(\alpha)_{m_{i-1}}}{(\alpha)_{m_i+1}} \frac{1}{(m_i + \alpha)^{a_i-1}} \right),$$

we get the identities

$$\begin{aligned}
 & \frac{(-1)^r}{r!} \frac{d^r}{d\alpha^r} \left\{ \frac{(\alpha)_{m_p}}{(\alpha)_{m_{p+q}}} \left(\prod_{i=1}^{p+q} \frac{1}{(m_i + \alpha)^{a_i}} \right) \right\} \\
 (17) \quad &= \sum_{\substack{u_1+\dots+u_{p+q} \\ +r_1+\dots+r_q=r \\ r_i, u_i \in \mathbb{Z}_{\geq 0}}} \left\{ \prod_{i=1}^p \binom{a_i-1+u_i}{u_i} \frac{1}{(m_i + \alpha)^{a_i+u_i}} \right\} \\
 &\times \left\{ \prod_{i=p+1}^{p+q} \frac{(\alpha)_{m_{i-1}}}{(\alpha)_{m_i+1}} \binom{a_i-2+u_i}{u_i} \frac{1}{(m_i + \alpha)^{a_i-1+u_i}} \right. \\
 &\quad \left. \times \sum_{m_{i-1} \leq n_{i1} \leq \dots \leq n_{ir_{i-p}} \leq m_i} \prod_{j=1}^{r_{i-p}} \frac{1}{n_{ij} + \alpha} \right\}
 \end{aligned}$$

$$\begin{aligned}
&= \frac{(\alpha)_{m_p}}{(\alpha)_{m_{p+q}}} \sum_{\substack{u_1+\dots+u_{p+q} \\ +r_1+\dots+r_q=r \\ r_i, u_i \in \mathbb{Z}_{\geq 0}}} \left\{ \prod_{i=1}^p \binom{a_i-1+u_i}{u_i} \frac{1}{(m_i+\alpha)^{a_i+u_i}} \right\} \\
&\times \left\{ \prod_{i=p+1}^{p+q} \binom{a_i-2+u_i}{u_i} \frac{1}{(m_i+\alpha)^{a_i+u_i}} \right\} \\
&\times \sum_{\substack{m_p \leq m_{11} \leq \dots \leq m_{1r_1} \leq m_{p+1} \\ \vdots \\ m_{p+q-1} \leq m_{q1} \leq \dots \leq m_{qr_q} \leq m_{p+q}}} \prod_{i=1}^q \prod_{j=1}^{r_i} \frac{1}{m_{ij} + \alpha}
\end{aligned}$$

for $p, q, r \in \mathbb{Z}_{\geq 0}$ and $m_p, \dots, m_{p+q} \in \mathbb{Z}$ such that $0 \leq m_p \leq \dots \leq m_{p+q}$, where we regard m_0 as 0. Differentiating the left-hand side of (8) r times with respect to α and using the case $p = k+1$, $q = s-1$, $a_i = 0$ ($i = 1, \dots, k+1$), $a_i = 1$ ($i = k+2, \dots, k+s$) of (17), we get the left-hand side of (13). On the other hand, using the expression

$$\begin{aligned}
&\phi_n \left(\{a_i\}_{i=1}^n | \{b_i\}_{i=1}^n | \{0\}^{n-1}, c | \{0\}^{n-1}, d ; (x, y, z, w) \right) \\
(18) \quad &= \sum_{0 \leq m_1 < \dots < m_n < \infty} \left\{ \prod_{i=1}^n \left(\frac{1}{(m_i+x)^{a_i-1}(m_i+y)^{b_i}} + \frac{1}{(m_i+x)^{a_i}(m_i+y)^{b_i-1}} \right) \right\} \\
&\times \frac{1}{(m_n+z)^c(m_n+w)^d}
\end{aligned}$$

and differentiating the right-hand side of (8) r times with respect to α , we can get the right-hand side of (13).

The identity (14) can be proved as follows: Using (18) and differentiating both sides of the case $\gamma = \beta$ of (8) r times with respect to β , we get the identity

$$\begin{aligned}
&\sum_{\substack{r_1+\dots+r_{k+s}=r \\ r_i \in \mathbb{Z}_{\geq 0}}} \Phi_{k+1, s-1}^* (\{r_i+1\}_{i=1}^{k+1}, \{r_i+2\}_{i=k+2}^{k+s}; (\alpha, \beta, \beta)) \\
(19) \quad &= \sum^{(k,0)} \sum_{\substack{r_1+\dots+r_{i+1}=r \\ r_j \in \mathbb{Z}_{\geq 0}}} \sum_{0 \leq m_1 < \dots < m_{i+1} < \infty} \\
&\left\{ \prod_{j=1}^i \left(\frac{\binom{k_j+r_j-1}{r_j}}{(m_j+\beta)^{k_j+r_j}} + \frac{\binom{k_j+r_j-2}{r_j}}{(m_j+\beta)^{k_j+r_j-1}(m_j+\alpha)} \right) \right\} \\
&\times \left(\frac{\binom{k_{i+1}+r_{i+1}+s-2}{r_{i+1}}}{(m_{i+1}+\beta)^{k_{i+1}+r_{i+1}+s-1}(m_{i+1}+\alpha)^{s-1}} \right. \\
&\left. + \frac{\binom{k_{i+1}+r_{i+1}+s-3}{r_{i+1}}}{(m_{i+1}+\beta)^{k_{i+1}+r_{i+1}+s-2}(m_{i+1}+\alpha)^s} \right)
\end{aligned}$$

for all $k, r \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}_{\geq 2}$, $\alpha, \beta \in \mathbb{C}$ such that $\operatorname{Re} \alpha > 0$, $\beta \notin \mathbb{Z}_{\leq 0}$. The right-hand side of (19) is the same as that of (14). Further, applying the identity

$$(20) \quad \prod_{i=1}^n (X_i + Y_i) = \sum_{\substack{s_i+t_i=1 \\ s_i, t_i \in \{0,1\} \\ (i=1, \dots, n)}} \prod_{i=1}^n X_i^{s_i} Y_i^{t_i}$$

($n \in \mathbb{Z}_{\geq 1}$) to the products on the right-hand side of (19), we can write the right-hand side of (19) as a linear combination of $\zeta_{i+1,+}^{<}(\{a_j\}_{j=1}^{i+1} | \{b_j\}_{j=1}^{i+1}; (\alpha, \beta))$, explicitly.

The identity (15) can be proved in a way similar to that of proving (13). Indeed, using the identity

$$\frac{(\gamma)_{m_{p+q}}}{(\gamma)_{m_p}} = \prod_{i=p}^{p+q-1} \frac{(\gamma)_{m_{i+1}}}{(\gamma)_{m_i}},$$

we get the identities

$$(21) \quad \begin{aligned} & \frac{1}{r!} \frac{d^r}{d\gamma^r} \left\{ \frac{(\gamma)_{m_{p+q}}}{(\gamma)_{m_p}} \left(\prod_{i=1}^{p+q} \frac{1}{(m_i + \gamma)^{c_i}} \right) \right\} \\ &= \sum_{\substack{u_1 + \dots + u_{p+q} \\ + r_1 + \dots + r_q = r \\ r_i, u_i \in \mathbb{Z}_{\geq 0}}} \left\{ \prod_{i=1}^{p+q} \binom{c_i - 1 + u_i}{u_i} \frac{(-1)^{u_i}}{(m_i + \gamma)^{c_i + u_i}} \right\} \\ & \times \left\{ \prod_{i=p}^{p+q-1} \frac{(\gamma)_{m_{i+1}}}{(\gamma)_{m_i}} \sum_{m_i \leq n_{i1} < \dots < n_{ir_i - p+1} < m_{i+1}} \prod_{j=1}^{r_i - p+1} \frac{1}{n_{ij} + \gamma} \right\} \\ &= \frac{(\gamma)_{m_{p+q}}}{(\gamma)_{m_p}} \sum_{\substack{u_1 + \dots + u_{p+q} \\ + r_1 + \dots + r_q = r \\ r_i, u_i \in \mathbb{Z}_{\geq 0}}} \left\{ \prod_{i=1}^{p+q} \binom{c_i - 1 + u_i}{u_i} \frac{(-1)^{u_i}}{(m_i + \gamma)^{c_i + u_i}} \right\} \\ & \times \sum_{\substack{m_p \leq m_{11} < \dots < m_{1r_1} < m_{p+1} \\ \vdots \\ m_{p+q-1} \leq m_{q1} < \dots < m_{qr_q} < m_{p+q}}} \prod_{i=1}^q \prod_{j=1}^{r_i} \frac{1}{m_{ij} + \gamma} \end{aligned}$$

for $p, q, r \in \mathbb{Z}_{\geq 0}$ and $m_p, \dots, m_{p+q} \in \mathbb{Z}$ such that $0 \leq m_p \leq \dots \leq m_{p+q}$, where we regard m_0 as 0. Differentiating both sides of (8) r times with respect to γ and using the case $p = k + 1$, $q = s - 1$, $c_i = 1$ ($i = 1, \dots, k + 1$), $c_i = 0$ ($i = k + 2, \dots, k + s$) of (21), we get (15). Using (18) and (20), we can write the right-hand side of (15) as a linear combination of $\zeta_{i+1,+}^{<}(\{a_j\}_{j=1}^{i+1} | \{b_j\}_{j=1}^{i+1} | \{0\}^i, s - 1 | \{0\}^i, s - 1; (\gamma, \alpha + \beta - \gamma, \alpha, \beta))$, explicitly. The expression (16) can be proved

by differentiating both sides of the case $n = 1$, $a_1 = b_1 = 1$, $c = d = s - 1$, $x = \gamma$, $y = \alpha + \beta - \gamma$, $z = \alpha$, $w = \beta$ of (18) r times with respect to γ . $\square$

Here we consider the multiple series

$$(22) \quad \sum_{0 \leq m_1 \lesssim \dots \lesssim m_n < \infty} (\pm 1)^{m_n} \frac{(y)_{m_n}}{(x)_{m_n}} \left\{ \prod_{i=1}^n \frac{(2m_i + x + y)^{s_i}}{(m_i + x)^{a_i} (m_i + y)^{b_i} (m_i + z)^{c_i} (m_i + w)^{d_i}} \right\},$$

where $n \in \mathbb{Z}_{\geq 1}$, $x, y, z, w \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, $a_i, b_i, c_i, d_i, s_i \in \mathbb{Z}$ such that $a_i + b_i + c_i + d_i - s_i \geq 1$ ($i = 1, \dots, n-1$), $a_n + b_n + c_n + d_n - s_n + \operatorname{Re}(x - y) > 1$ and $(x + y)/2 \notin \mathbb{Z}_{\leq 0}$ in case of $s_i < 0$ for some i . The symbol $(a)_m$ is the Pochhammer symbol. We denote the case $\lesssim = <$ of (22) by

$$\psi_{n, \pm} \left(\begin{matrix} \{s_i\}_{i=1}^n \\ \{a_i\}_{i=1}^n | \{b_i\}_{i=1}^n | \{c_i\}_{i=1}^n | \{d_i\}_{i=1}^n \end{matrix} ; (x, y, z, w) \right).$$

Remark 3. Coppo [8], Coppo and Candelpergher [9], Émery [10], Hasse [14] studied relations between special values of the multiple series

$$(23) \quad \sum_{0 \leq m_1 \lesssim \dots \lesssim m_n < \infty} z^{m_n} \frac{m_n!}{(\alpha)_{m_n}} \left\{ \prod_{i=1}^n \frac{1}{(m_i + \alpha)^{a_i} (m_i + 1)^{b_i}} \right\}$$

and special values of the Hurwitz–Lerch zeta function $\sum_{m=0}^{\infty} z^{m+1} (m + \alpha)^{-s}$. See also the part **(R3)** of Section 3, Subsection 2.3, and [16]. The multiple series (22) and (41) below give us extensions of the multiple series (23) with $z = \pm 1$.

We can prove the following identity, which is similar to (2) with $\gamma = \alpha + \beta - 1$:

Theorem 2.5. *The identity*

$$(24) \quad \sum_{\substack{r_1 + \dots + r_s = l \\ r_i \in \mathbb{Z}_{\geq 0}}} \Phi_{(\{r_i\}_{i=1}^s)}^{(1, s-1)}(\{1\}^{r_1}, k + 2, \{\{1\}^{r_i}, 2\}_{i=2}^s; (\alpha, \beta, \alpha + \beta - 1)) \\ = \sum^{(k, l)} \psi_{i+1, -} \left(\begin{matrix} \{e(k_j) + e(l_j)\}_{j=1}^i, 1 \\ \{k_j + e(l_j)\}_{j=1}^i, k_{i+1} + 1 | \{l_j + e(k_j)\}_{j=1}^i, l_{i+1} | \{0\}^i, s - 1 | \{0\}^i, s - 1 \\ (\alpha + \beta - 1, 1, \alpha, \beta) \end{matrix} \right);$$

holds for all $k, l \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}_{\geq 1}$, $\alpha, \beta \in \mathbb{C}$ such that $\operatorname{Re} \alpha, \operatorname{Re} \beta > 0$, $\operatorname{Re}(\alpha + \beta) > 1$.

Proof. Taking $a = \alpha + \beta$, $b_1 = \alpha + \beta - z$, $b_i = \alpha$ ($i = 2, \dots, s$), $b_{s+1} = w$, $c_1 = 1$, $c_i = \beta$ ($i = 2, \dots, s$), $c_{s+1} = 1$, where $s \in \mathbb{Z}_{\geq 1}$, $\alpha, \beta, z, w \in \mathbb{C}$ such that

$\operatorname{Re} \alpha, \operatorname{Re} \beta > 0$, $\operatorname{Re}(\alpha + \beta) > 1$, $\operatorname{Re}(\alpha + \beta) > \operatorname{Re} z > 0$, $\operatorname{Re}(\alpha + \beta) > \operatorname{Re} w > 0$, $2s - 3 > \operatorname{Re}(2(w - z) - \alpha - \beta)$, in Theorem A (i) and multiplying both sides of the result by $(\alpha + \beta - 1)^{-1}$, we get the identity

$$\begin{aligned}
 (25) \quad & \sum_{0 \leq m_1 \leq \dots \leq m_s < \infty} \frac{(\alpha)_{m_1}}{m_1!} \frac{(\beta)_{m_1}}{(\alpha + \beta - 1)_{m_1}} \frac{m_s!}{(\alpha)_{m_s}} \frac{(w)_{m_s}}{(\beta)_{m_s}} \\
 & \times \frac{1}{(m_1 + \alpha + \beta - 1)(m_1 + z)} \left\{ \prod_{i=2}^s \frac{1}{(m_i + \alpha)(m_i + \beta)} \right\} \\
 & = \sum_{m=0}^{\infty} \frac{(\alpha + \beta - z)_m}{(z)_{m+1}} \frac{(w)_m}{(\alpha + \beta - w)_{m+1}} \\
 & \times \frac{m!}{(\alpha + \beta - 1)_{m+1}} \frac{2m + \alpha + \beta}{(m + \alpha)^{s-1}(m + \beta)^{s-1}} (-1)^m.
 \end{aligned}$$

Applying the operator $(-1)^k \partial^{(l,k)}(w, z)|_{z=\alpha+\beta-1}^{w=\alpha+\beta-1}$ ($k, l \in \mathbb{Z}_{\geq 0}$; $\alpha, \beta \in \mathbb{C}$ such that $\operatorname{Re} \alpha, \operatorname{Re} \beta > 0$, $\operatorname{Re}(\alpha + \beta) > 1$) to the left-hand side of (25) and using (6), we get the left-hand side of (24). The right-hand side of (24) can be proved by applying $(-1)^k \partial^{(l,k)}(w, z)|_{z=\alpha+\beta-1}^{w=\alpha+\beta-1}$ to the right-hand side of (25) and using the case $\gamma = \delta = \alpha + \beta - 1$ of (7). $\square$

We can derive the following identities from (24):

Corollary 2.6. *The following identities hold:*

(i)

$$\begin{aligned}
 (26) \quad & \Phi_{1,s-1}^*(k+2, \{2\}^{s-1}; (\alpha, \beta, \alpha + \beta - 1)) \\
 & = \sum^{(k,0)} \psi_{i+1,-} \left(\begin{array}{c} \{1\}^{i+1} \\ \{k_j\}_{j=1}^i, k_{i+1} + 1 | \{1\}^{i+1} | \{0\}^i, s-1 | \{0\}^i, s-1 \\ (\alpha + \beta - 1, 1, \alpha, \beta) \end{array} \right)
 \end{aligned}$$

for all $k \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}_{\geq 1}$, $\alpha, \beta \in \mathbb{C}$ such that $\operatorname{Re} \alpha, \operatorname{Re} \beta > 0$, $\operatorname{Re}(\alpha + \beta) > 1$.

(ii)

$$\begin{aligned}
 (27) \quad & \zeta_{l+1,+}^{<}(\{1\}^l, k+2; \alpha) \\
 & = \sum^{(k,l)} \psi_{i+1,-} \left(\begin{array}{c} \{e(k_j) + e(l_j)\}_{j=1}^i, 1 \\ \{k_j + e(l_j)\}_{j=1}^i, k_{i+1} + 1 | \{l_j + e(k_j)\}_{j=1}^i, l_{i+1} \\ (\alpha, 1) \end{array} \right)
 \end{aligned}$$

for all $k, l \in \mathbb{Z}_{\geq 0}$, $\alpha \in \mathbb{C}$ with $\operatorname{Re} \alpha > 0$, where

$$\psi_{n,-} \left(\begin{array}{c} \{s_i\}_{i=1}^n \\ \{a_i\}_{i=1}^n | \{b_i\}_{i=1}^n \\ (x, y) \end{array} \right) := \psi_{n,-} \left(\begin{array}{c} \{s_i\}_{i=1}^n \\ \{a_i\}_{i=1}^n | \{b_i\}_{i=1}^n | \{0\}^n | \{0\}^n \\ (x, y, z, w) \end{array} \right).$$

(iii)

$$(28) \quad \zeta(\{1\}^l, k+2) = \sum^{(k,l)} 2^{\sum_{j=1}^i \{e(k_j) + e(l_j)\} + 1} \zeta_{-}(\{k_j + l_j\}_{j=1}^{i+1})$$

for all $k, l \in \mathbb{Z}_{\geq 0}$.

Proof. Taking $l = 0$ in (24), we get (26). Taking $\beta = s = 1$ in (24) and $\alpha = \beta = s = 1$ in (24), we get (27) and (28), respectively. $\square$

Taking $\alpha = \beta = \gamma = 1$ in (8) and (26), we get (29) and (30) below, respectively:

Corollary 2.7. *The following two identities hold:*

(i)

$$(29) \quad \zeta^*(\{1\}^{r+1}, \{2\}^{s-1}) = \sum_{i=0}^r \sum_{\substack{r_1+\dots+r_{i+1}=r+1 \\ r_j \in \mathbb{Z}_{\geq 1}}} 2^{i+1} \zeta(\{r_j\}_{j=1}^i, r_{i+1} + 2s - 2)$$

for all $r \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}_{\geq 2}$.

(ii)

$$(30) \quad \zeta^*(r+2, \{2\}^{s-1}) = \sum_{i=0}^r \sum_{\substack{r_1+\dots+r_{i+1}=r+1 \\ r_j \in \mathbb{Z}_{\geq 1}}} 2^{i+1} \zeta_-(\{r_j\}_{j=1}^i, r_{i+1} + 2s - 1)$$

for all $r \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}_{\geq 1}$.

Remark 4. The identities (29) and (30) were proved in [19]. The identity (29) is the two-one formula for $\zeta^*(\{1\}^{s_1}, \{2\}^{s_2})$ ($s_1, s_2 \in \mathbb{Z}_{\geq 1}$). The two-one formula for MZSVs was conjectured by Ohno and Zudilin in [26, pp. 327–328]. The general case of the two-one formula was proved by Zhao in [31, Theorem 1.1]. The identity (30) seems to give a solution to the project of Ohno and Zudilin stated in [26, p. 326]. In [19], I proved (30) to solve my observation for $\zeta^*(s_1 + 2, \{2\}^{s_2})$ ($s_1, s_2 \in \mathbb{Z}_{\geq 0}$) stated in [19, (A4)]. I remark that the observation for $\zeta^*(s_1 + 2, \{2\}^{s_2})$ comes from the similarity between (31) and (32) below, and my observation for the right-hand side of (31) below stated in [19, (A3)]:

$$(31) \quad \zeta^*(\{1\}^{r+1}, \{2\}^s) = 2 \sum_{m=0}^{\infty} \frac{1}{(m+1)^{2s+1}} \left(\sum_{i=0}^r S_m^<(\{1\}^{r-i}) S_m^<(\{1\}^i) \right),$$

$$(32) \quad \zeta^*(r+2, \{2\}^{s-1}) = 2 \sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1)^{2s}} \left(\sum_{i=0}^r S_m^<(\{1\}^{r-i}) S_m^<(\{1\}^i) \right)$$

for all $r \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}_{\geq 1}$, where

$$S_m^<(\{k_i\}_{i=1}^n) := \sum_{0 \leq m_1 \lesssim \dots \lesssim m_n \lesssim m} \prod_{i=1}^n \frac{1}{(m_i + 1)^{k_i}}$$

($n \in \mathbb{Z}_{\geq 1}$; $k_1, \dots, k_n \in \mathbb{Z}_{\geq 1}$), $S_m^<(\emptyset) := 1$. The identities (31) and (32) were also derived in [19] by using Theorem A.

We can prove the following two identities for (11) also:

Theorem 2.8. *The following two identities hold:*

(i)

$$\begin{aligned}
 & \sum_{i=0}^k \sum_{\substack{k_1+\dots+k_{i+1}=k+1 \\ k_j \in \mathbb{Z}_{\geq 1} (j=1, \dots, i+1)}} \\
 (33) \quad & \Phi_{A^{(i+s)}, B^{(i+s)}}^{(i+1, s-1), \lesssim^{(i+s-1)}} \left(\begin{array}{c} \{k_j + 1\}_{j=1}^i, k_{i+1}, \{1\}^{s-1} \\ \{k_j\}_{j=1}^{i+1}, \{1\}^{s-1} | \{k_j\}_{j=1}^{i+1}, \{1\}^{s-1} | \{1\}^{i+s} ; \\ (\alpha, \beta, \alpha + \beta - 1, 1) \end{array} \right) \\
 & = \psi_{1,-}(1|k+1|s-1|s-1; (1, \alpha + \beta - 1, \alpha, \beta)) \\
 & \quad + (-1)^k \psi_{1,-}(k+2|0|s-1|s-1; (1, \alpha + \beta - 1, \alpha, \beta))
 \end{aligned}$$

for all $k \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}_{\geq 1}$, $\alpha, \beta \in \mathbb{C}$ such that $\operatorname{Re} \alpha, \operatorname{Re} \beta > 0$, $2s+1 > \operatorname{Re}(\alpha+\beta) > 1$, where $A^{(i+s)} = \{(-1)^{k_j}\}_{j=1}^i, (-1)^{k_{i+1}-1}, \{1\}^{s-1}$; $B^{(i+s)} = \{1\}^{i+s}$; $\lesssim^{(i+s-1)} = \{<\}^i, \{\leq\}^{s-1}$ and

$$\begin{aligned}
 & \psi_{n,\pm}(\{a_i\}_{i=1}^n | \{b_i\}_{i=1}^n | \{c_i\}_{i=1}^n | \{d_i\}_{i=1}^n; (x, y, z, w)) \\
 & := \psi_{n,\pm} \left(\begin{array}{c} \{0\}^n \\ \{a_i\}_{i=1}^n | \{b_i\}_{i=1}^n | \{c_i\}_{i=1}^n | \{d_i\}_{i=1}^n ; (x, y, z, w) \end{array} \right).
 \end{aligned}$$

(ii)

$$\begin{aligned}
 & \sum_{i=0}^k \sum_{\substack{k_1+\dots+k_{i+1}=k+1 \\ k_j \in \mathbb{Z}_{\geq 1} (j=1, \dots, i+1)}} \\
 (34) \quad & \Phi_{A^{(i+s)}, B^{(i+s)}}^{(1, i+s-1), \lesssim^{(i+s-1)}} \left(\begin{array}{c} 1, \{k_j + 1\}_{j=1}^i, k_{i+1}, \{1\}^{s-2} \\ 1, \{k_j\}_{j=1}^{i+1}, \{1\}^{s-2} | 0, \{k_j\}_{j=1}^{i+1}, \{1\}^{s-2} | \{1\}^{i+s} ; \\ (\alpha, \beta, \gamma, \gamma) \end{array} \right) \\
 & = \zeta_{1,+}^{<}(k+s-1|s-2|1|1; (\alpha, \beta, \gamma, \alpha + \beta - \gamma)) \\
 & \quad + (-1)^k \zeta_{1,+}^{<}(s-2|k+s-1|1|1; (\alpha, \beta, \gamma, \alpha + \beta - \gamma))
 \end{aligned}$$

for all $k \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}_{\geq 2}$, $\alpha, \beta, \gamma \in \mathbb{C}$ such that $\operatorname{Re} \alpha, \operatorname{Re} \beta > 0$, $\operatorname{Re}(\alpha + \beta) > \operatorname{Re} \gamma > 0$, where $A^{(i+s)} = 1, \{(-1)^{k_j}\}_{j=1}^i, (-1)^{k_{i+1}-1}, \{1\}^{s-2}$; $B^{(i+s)} = \{1\}^{i+s}$; $\lesssim^{(i+s-1)} = \leq, \{<\}^i, \{\leq\}^{s-2}$.

Proof. Taking $a = \alpha + \beta$, $b_1 = \alpha + \beta - z$, $c_1 = z$, $b_i = \alpha$, $c_i = \beta$ ($i = 2, \dots, s$), $b_{s+1} = 1$, $c_{s+1} = \alpha + \beta - 1$, where $s \in \mathbb{Z}_{\geq 1}$, $\alpha, \beta, z \in \mathbb{C}$ such that $\operatorname{Re} \alpha, \operatorname{Re} \beta > 0$,

$2s + 1 > \operatorname{Re}(\alpha + \beta) > 1$, $\operatorname{Re}(\alpha + \beta) > \operatorname{Re} z > 0$, in Theorem A (i), we get the identity

$$\begin{aligned}
 (35) \quad & \sum_{0 \leq m_1 \leq \dots \leq m_s < \infty} \frac{m_1!}{(z)_{m_1+1}} \frac{(\alpha + \beta - 1)_{m_1}}{(\alpha + \beta - z)_{m_1+1}} \frac{(\alpha)_{m_1}}{m_1!} \frac{(\beta)_{m_1}}{(\alpha + \beta - 1)_{m_1}} \\
 & \times \frac{m_s!}{(\alpha)_{m_s}} \frac{(\alpha + \beta - 1)_{m_s}}{(\beta)_{m_s}} \left\{ \prod_{i=2}^s \frac{1}{(m_i + \alpha)(m_i + \beta)} \right\} \\
 & = \sum_{m=0}^{\infty} \frac{(\alpha + \beta - 1)_m}{(m+1)!} \frac{(-1)^m}{(m + \alpha)^{s-1}(m + \beta)^{s-1}} \left(\frac{1}{m + z} + \frac{1}{m + \alpha + \beta - z} \right).
 \end{aligned}$$

Taking $a = X$, $b = Y$, $c = 1$, $d = \alpha + \beta - 1$, $Z = \alpha + \beta - z$, $W = z$ in Lemma 2.1 and slightly modifying the result, we get the identity

$$\begin{aligned}
 & \frac{m_1!}{(z)_{m_1+1}} \frac{(\alpha + \beta - 1)_{m_1}}{(\alpha + \beta - z)_{m_1+1}} \\
 & = \left(\frac{1}{(z + m_1)(\alpha + \beta + 2m_1)} + \frac{1}{(\alpha + \beta - z + m_1)(\alpha + \beta + 2m_1)} \right) \\
 & \times \sum_{i=0}^{m_1} \sum_{0 \leq n_1 < \dots < n_i < m_1} \prod_{j=1}^i \left(-\frac{(z - \alpha - \beta + 1)(1 + n_j)}{(z + n_j)(\alpha + \beta + 2n_j)} \right. \\
 & \left. + \frac{(z - \alpha - \beta + 1)(\alpha + \beta - 1 + n_j)}{(\alpha + \beta - z + n_j)(\alpha + \beta + 2n_j)} \right).
 \end{aligned}$$

Using this identity, we get the identity

$$\begin{aligned}
 (36) \quad & (-1)^k \partial^{(k)}(z) \left(\frac{m_1!}{(z)_{m_1+1}} \frac{(\alpha + \beta - 1)_{m_1}}{(\alpha + \beta - z)_{m_1+1}} \right) \Big|_{z=\alpha+\beta-1} \\
 & = \sum_{i=0}^{m_1} \sum_{\substack{k_1 + \dots + k_{i+1} = k \\ k_j \in \mathbb{Z}_{\geq 1} (j=1, \dots, i); \\ k_{i+1} \in \mathbb{Z}_{\geq 0}}} \sum_{0 \leq n_1 < \dots < n_i < m_1} \\
 & \left\{ \prod_{j=1}^i \left(\frac{1 + n_j}{(\alpha + \beta - 1 + n_j)^{k_j}(\alpha + \beta + 2n_j)} + \frac{(-1)^{k_j}(\alpha + \beta - 1 + n_j)}{(1 + n_j)^{k_j}(\alpha + \beta + 2n_j)} \right) \right\} \\
 & \times \left(\frac{1}{(\alpha + \beta - 1 + m_1)^{k_{i+1}+1}(\alpha + \beta + 2m_1)} + \frac{(-1)^{k_{i+1}}}{(1 + m_1)^{k_{i+1}+1}(\alpha + \beta + 2m_1)} \right)
 \end{aligned}$$

for $k, m_1 \in \mathbb{Z}_{\geq 0}$, $\alpha, \beta \in \mathbb{C}$ such that $(\alpha + \beta)/2, \alpha + \beta - 1 \notin \mathbb{Z}_{\leq 0}$. Thus, applying the operator $(-1)^k \partial^{(k)}(z)|_{z=\alpha+\beta-1}$ ($k \in \mathbb{Z}_{\geq 0}$; $\alpha, \beta \in \mathbb{C}$ such that $\operatorname{Re} \alpha, \operatorname{Re} \beta > 0$, $2s + 1 > \operatorname{Re}(\alpha + \beta) > 1$) to the left-hand side of (35) and using (36), we get the left-hand side of (33). The right-hand side of (33) can be proved by applying $(-1)^k \partial^{(k)}(z)|_{z=\alpha+\beta-1}$ to the right-hand side of (35).

The identity (34) can be proved in a way similar to the above. Indeed, taking $a = \alpha + \beta$, $c_0 = \alpha + \beta - \gamma$, $b_1 = z$, $c_1 = \alpha + \beta - z$, $b_i = \alpha$, $c_i = \beta$

($i = 2, \dots, s-1$), $b_s = 1$, $c_s = \gamma$, where $s \in \mathbb{Z}_{\geq 2}$, $\alpha, \beta, \gamma, z \in \mathbb{C}$ such that $\operatorname{Re} \alpha, \operatorname{Re} \beta > 0$, $\operatorname{Re}(\alpha + \beta) > \operatorname{Re} \gamma > 0$, $\operatorname{Re}(\alpha + \beta) > \operatorname{Re} z > 0$, in Theorem A (ii), we get the identity

$$\begin{aligned}
 (37) \quad & \sum_{0 \leq m_1 \leq \dots \leq m_s < \infty} \frac{(z)_{m_1}}{(z)_{m_2+1}} \frac{(\alpha + \beta - z)_{m_1}}{(\alpha + \beta - z)_{m_2+1}} \frac{(\alpha)_{m_2}}{(\alpha)_{m_1}} \frac{(\beta)_{m_2}}{(\beta)_{m_1}} \\
 & \times \frac{(\alpha)_{m_1}}{m_1!} \frac{(\beta)_{m_1}}{(\gamma)_{m_1+1}} \frac{m_s!}{(\alpha)_{m_s}} \frac{(\gamma)_{m_s}}{(\beta)_{m_s}} \left\{ \prod_{i=3}^s \frac{1}{(m_i + \alpha)(m_i + \beta)} \right\} \\
 & = \sum_{m=0}^{\infty} \frac{1}{(m + \alpha)^{s-2} (m + \beta)^{s-2} (m + \gamma) (m + \alpha + \beta - \gamma)} \\
 & \times \left(\frac{1}{m + z} + \frac{1}{m + \alpha + \beta - z} \right).
 \end{aligned}$$

Using the case $m = m_2 - m_1$, $a = c = X = \beta + m_1$, $b = Y$, $d = \alpha + m_1$, $Z = \alpha + \beta - z + m_1$, $W = z + m_1$ of Lemma 2.1, where $m_1, m_2 \in \mathbb{Z}$ such that $0 \leq m_1 \leq m_2$, we get the identities

$$\begin{aligned}
 (38) \quad & \frac{(z)_{m_1}}{(z)_{m_2+1}} \frac{(\alpha + \beta - z)_{m_1}}{(\alpha + \beta - z)_{m_2+1}} \frac{(\alpha)_{m_2}}{(\alpha)_{m_1}} \frac{(\beta)_{m_2}}{(\beta)_{m_1}} \\
 & = \frac{(\alpha + m_1)_{m_2-m_1}}{(z + m_1)_{m_2-m_1+1}} \frac{(\beta + m_1)_{m_2-m_1}}{(\alpha + \beta - z + m_1)_{m_2-m_1+1}} \\
 & = \left(\frac{1}{(z + m_2)(\alpha + \beta + 2m_2)} + \frac{1}{(\alpha + \beta - z + m_2)(\alpha + \beta + 2m_2)} \right) \\
 & \times \sum_{i=0}^{m_2-m_1} \sum_{m_1 \leq n_1 < \dots < n_i < m_2} \prod_{j=1}^i \left(-\frac{(z - \alpha)(\beta + n_j)}{(z + n_j)(\alpha + \beta + 2n_j)} \right. \\
 & \left. + \frac{(z - \alpha)(\alpha + n_j)}{(\alpha + \beta - z + n_j)(\alpha + \beta + 2n_j)} \right).
 \end{aligned}$$

Using (38), we get the identity

$$\begin{aligned}
 (39) \quad & (-1)^k \partial^{(k)}(z) \left(\frac{(z)_{m_1}}{(z)_{m_2+1}} \frac{(\alpha + \beta - z)_{m_1}}{(\alpha + \beta - z)_{m_2+1}} \frac{(\alpha)_{m_2}}{(\alpha)_{m_1}} \frac{(\beta)_{m_2}}{(\beta)_{m_1}} \right) \Big|_{z=\alpha} \\
 & = \sum_{i=0}^{m_2-m_1} \sum_{\substack{k_1 + \dots + k_{i+1} = k \\ k_j \in \mathbb{Z}_{\geq 1} (j=1, \dots, i); \\ k_{i+1} \in \mathbb{Z}_{\geq 0}}} \sum_{m_1 \leq n_1 < \dots < n_i < m_2} \\
 & \left\{ \prod_{j=1}^i \left(\frac{\beta + n_j}{(\alpha + n_j)^{k_j} (\alpha + \beta + 2n_j)} + \frac{(-1)^{k_j} (\alpha + n_j)}{(\beta + n_j)^{k_j} (\alpha + \beta + 2n_j)} \right) \right\} \\
 & \times \left(\frac{1}{(\alpha + m_2)^{k_{i+1}+1} (\alpha + \beta + 2m_2)} + \frac{(-1)^{k_{i+1}}}{(\beta + m_2)^{k_{i+1}+1} (\alpha + \beta + 2m_2)} \right)
 \end{aligned}$$

for $k, m_1, m_2 \in \mathbb{Z}$, $\alpha, \beta \in \mathbb{C}$ such that $0 \leq m_1 \leq m_2$ and $\alpha, \beta, (\alpha + \beta)/2 \notin \mathbb{Z}_{\leq 0}$. Thus, applying the operator $(-1)^k \partial^{(k)}(z)|_{z=\alpha}$ ($k \in \mathbb{Z}_{\geq 0}$; $\alpha \in \mathbb{C}$ such that $\operatorname{Re} \alpha, \operatorname{Re} \beta > 0$, $\operatorname{Re}(\alpha + \beta) > \operatorname{Re} \gamma > 0$) to the left-hand side of (37) and using (39), we get the left-hand side of (34). The right-hand side of (34) can be proved by applying $(-1)^k \partial^{(k)}(z)|_{z=\alpha}$ to the right-hand side of (37). $\square$

Remark 5. We consider the following special case of (1):

$$\begin{aligned} & \Phi_{\mathbf{2}, (\{r_i\}_{i=1}^{p+q})}^{(p,q)}(\{\{a_{ij}\}_{j=1}^{r_i}, A_i\}_{i=1}^{p+q} | \{\{b_{ij}\}_{j=1}^{r_i}, B_i\}_{i=1}^{p+q}; (\alpha, \beta, \gamma, \delta)) \\ := & \sum_{\substack{0=m_0 \leq m_{11} < \dots < m_{1r_1} < m_1 \\ \vdots \\ m_{p-1} \leq m_{p1} < \dots < m_{pr_p} < m_p \\ \vdots \\ m_{p+q-1} \leq m_{p+q1} < \dots < m_{p+qr_{p+q}} < m_{p+q} < \infty}} \frac{(\alpha)_{m_p}}{m_p!} \frac{(\beta)_{m_p}}{(\gamma)_{m_p}} \frac{m_{p+q}!}{(\alpha)_{m_{p+q}}} \frac{(\gamma)_{m_{p+q}}}{(\beta)_{m_{p+q}}} \\ & \times \left\{ \prod_{i=1}^p \frac{1}{(m_i + \gamma)^{A_i} (m_i + \delta)^{B_i}} \prod_{j=1}^{r_i} \frac{1}{(m_{ij} + \gamma)^{a_{ij}} (m_{ij} + \delta)^{b_{ij}}} \right\} \\ & \times \left\{ \prod_{i=p+1}^{p+q} \frac{1}{(m_i + \alpha)^{A_i} (m_i + \beta)^{B_i}} \prod_{j=1}^{r_i} \frac{1}{(m_{ij} + \gamma)^{a_{ij}} (m_{ij} + \delta)^{b_{ij}}} \right\}, \end{aligned}$$

where $p, q \in \mathbb{Z}_{\geq 0}$ such that $p + q \geq 1$; $r_i \in \mathbb{Z}_{\geq 0}$, $a_{ij}, b_{ij}, A_i, B_i \in \mathbb{Z}$. We remark that the left-hand sides of the identities (2), (13)–(15), (24) can be rewritten in this multiple series. Using Theorem A (i), we can prove the following identity also:

$$\begin{aligned} (40) \quad & \sum_{i=0}^k \sum_{\substack{k_1 + \dots + k_{i+1} = k+1 \\ l_1 + \dots + l_{i+1} = l \\ k_j \in \mathbb{Z}_{\geq 1}, l_j \in \mathbb{Z}_{\geq 0} \\ (j=1, \dots, i+1)}} (-1)^i (\alpha + \beta - 2)^{\sum_{j=1}^i \{1 - e(l_j)\}} \\ & \times \Phi_{\mathbf{2}, (i, \{0\}^{s-1})}^{(1, s-1)}(\{k_j - e(l_j)\}_{j=1}^i, k_{i+1}, \{1\}^{s-1} | \{l_j + 1\}_{j=1}^{i+1}, \{1\}^{s-1}; \\ & \quad (\alpha, \beta, \alpha + \beta - 1, 1)) \\ & = \sum_{i=0}^{k+l} \sum_{\substack{k_1 + \dots + k_{i+1} = k+1 \\ l_1 + \dots + l_i + r_2 + \dots + r_s = l \\ k_j \in \mathbb{Z}_{\geq 0}, l_j \in \{0, 1\}, k_j + l_j \geq 1 \\ (j=1, \dots, i); k_{i+1} \in \mathbb{Z}_{\geq 1}, \\ r_2, \dots, r_s \in \mathbb{Z}_{\geq 0}}} (2 - \alpha - \beta)^{\sum_{j=1}^i e(k_j)} \\ & \times \Phi_{\mathbf{2}, (i, \{r_j\}_{j=2}^s)}^{(1, s-1)}(\{k_j + e(l_j)\}_{j=1}^i, k_{i+1}, \{1\}^{\sum_{j=2}^s (r_j + 1)} | \\ & \quad \{e(k_j)\}_{j=1}^{i+1}, \{\{0\}^{r_j}, 1\}_{j=2}^s; (\alpha, \beta, \alpha + \beta - 1, 1)) \end{aligned}$$

for all $k, l \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}_{\geq 1}$, $\alpha, \beta \in \mathbb{C}$ such that $\operatorname{Re} \alpha, \operatorname{Re} \beta > 0$, $2s + 1 > \operatorname{Re}(\alpha + \beta) > 1$. The identity (40) can be proved in a way similar to the proof of Theorem 2.15 below. Taking $k = 0$ in (40), we get the following identity, which is similar to (10):

$$\begin{aligned} & \Phi_{\mathbf{2},(\{0\}^s)}^{(1,s-1)}(\{1\}^s | l + 1, \{1\}^{s-1}; (\alpha, \beta, \alpha + \beta - 1, 1)) \\ &= \sum_{\substack{r_1 + \dots + r_s = l \\ r_j \in \mathbb{Z}_{\geq 0}}} \Phi_{\mathbf{2},(\{r_j\}_{j=1}^s)}^{(1,s-1)}(\{1\}^{\sum_{j=1}^s (r_j+1)} | \{\{0\}^{r_j}, 1\}_{j=1}^s; (\alpha, \beta, \alpha + \beta - 1, 1)) \end{aligned}$$

for all $l \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}_{\geq 1}$, $\alpha, \beta \in \mathbb{C}$ such that $\operatorname{Re} \alpha, \operatorname{Re} \beta > 0$, $2s + 1 > \operatorname{Re}(\alpha + \beta) > 1$.

2.3 Application 2

We consider the multiple series

$$(41) \quad \sum_{0 \leq m_1 \lesssim m_2 \lesssim \dots \lesssim m_n \lesssim \infty} \frac{(y)_{m_n}}{(x)_{m_n}} \frac{(w)_{m_n}}{(z)_{m_n}} \times \left\{ \prod_{i=1}^n \frac{(2m_i + x + y)^{s_i} (2m_i + z + w)^{t_i}}{(m_i + x)^{a_i} (m_i + y)^{b_i} (m_i + z)^{c_i} (m_i + w)^{d_i}} \right\},$$

where $n \in \mathbb{Z}_{\geq 1}$, $x, y, z, w \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, $a_i, b_i, c_i, d_i, s_i, t_i \in \mathbb{Z}$ such that $a_i + b_i + c_i + d_i - s_i - t_i \geq 1$ ($i = 1, \dots, n-1$), $a_n + b_n + c_n + d_n - s_n - t_n + \operatorname{Re}(x - y + z - w) > 1$ and $(x + y)/2, (z + w)/2 \notin \mathbb{Z}_{\leq 0}$ in case of $s_i, t_i < 0$ for some i . The symbol $(a)_m$ is the Pochhammer symbol. The multiple series (41) also gives us extensions of MZV and MHZV. In this subsection, we study relations among the multiple series (41) by using Theorem A.

We consider the following special case of (41):

$$(42) \quad \sum_{\substack{0 = m_0 \leq m_{11} < \dots < m_{1r_1} < m_1 \\ \vdots \\ m_{p-1} \leq m_{p1} < \dots < m_{pr_p} < m_p \\ \vdots \\ m_{p+q-1} \leq m_{p+q1} < \dots < m_{p+qr_{p+q}} < m_{p+q} < \infty}} \frac{(y)_{m_{p+q}}}{(x)_{m_{p+q}}} \frac{m_{p+q}!}{(w)_{m_{p+q}}} \times \left\{ \prod_{i=1}^p \frac{1}{(m_i + x)(m_i + y)^{K_i-1}} \prod_{j=1}^{r_i} \frac{1}{(m_{ij} + y)^{k_{ij}}} \right\} \\ \times \left\{ \prod_{i=p+1}^{p+q} \frac{1}{(m_i + x)(m_i + y)^{K_i-2}(m_i + z)} \prod_{j=1}^{r_i} \frac{1}{(m_{ij} + y)^{k_{ij}}} \right\},$$

where $p, q \in \mathbb{Z}_{\geq 0}$ such that $p + q \geq 1$; $r_i \in \mathbb{Z}_{\geq 0}$, $k_{ij}, K_i \in \mathbb{Z}_{\geq 1}$. We denote the case $w = 1$ of (42) by $\Psi_{(\{r_i\}_{i=1}^{p+q})}^{(p,q)}(\{\{k_{ij}\}_{j=1}^{r_i}\}_{i=1}^{p+q}, K_i)_{i=1}^{p+q}; (x, y, z)$. We can prove the following two identities, which are similar to (2) and (24):

Theorem 2.9. *The following two identities hold:*

(i)

$$\begin{aligned}
 & \sum_{\substack{r_1+\dots+r_{k+s}=l \\ r_i \in \mathbb{Z}_{\geq 0}}} \Psi_{(\{r_i\}_{i=1}^{k+s})}^{(k+1, s-1)}(\{1\}^{\sum_{i=1}^{k+1} (r_i+1)}, \{\{1\}^{r_i}, 2\}_{i=k+2}^{k+s}; (\alpha, \beta, 1)) \\
 (43) \quad &= \sum^{(k, l)} \Psi_{i+1} \left(\begin{array}{c} \{e(k_j)\}_{j=1}^i | \{e(l_j)\}_{j=1}^i, 0 \\ \{k_j\}_{j=1}^i, k_{i+1} + s - 1 | \{e(k_j)\}_{j=1}^i, s - 1 | \{l_j\}_{j=1}^{i+1} | \{e(l_j)\}_{j=1}^i, 0 \\ (\alpha, 1, \alpha - \beta + 1, \beta) \end{array} ; \right)
 \end{aligned}$$

for all $k, l \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}_{\geq 1}$, $\alpha, \beta \in \mathbb{C}$ such that $\operatorname{Re} \alpha, \operatorname{Re} \beta > 0$, $\operatorname{Re}(\alpha - \beta) > \max\{-1, -s + 1\}$, where

$$\Psi_n \left(\begin{array}{c} \{s_i\}_{i=1}^n | \{t_i\}_{i=1}^n \\ \{a_i\}_{i=1}^n | \{b_i\}_{i=1}^n | \{c_i\}_{i=1}^n | \{d_i\}_{i=1}^n \end{array} ; (x, y, z, w) \right)$$

denotes the case $\lesssim_i = <$ ($i = 1, \dots, n-1$) of (41).

(ii)

$$\begin{aligned}
 & \sum_{\substack{r_1+\dots+r_s=l \\ r_i \in \mathbb{Z}_{\geq 0}}} \Psi_{(\{r_i\}_{i=1}^s)}^{(1, s-1)}(\{1\}^{r_1}, k+2, \{\{1\}^{r_i}, 2\}_{i=2}^s; (\alpha, \beta, 1)) \\
 (44) \quad &= \sum^{(k, l)} \psi_{i+1, -} \left(\begin{array}{c} \{e(k_j) + e(l_j)\}_{j=1}^i, 1 \\ \{0\}^i, s | \{0\}^i, s - 1 | \{k_j + e(l_j)\}_{j=1}^i, k_{i+1} | \{l_j + e(k_j)\}_{j=1}^i, l_{i+1} \\ (\alpha, 1, \beta, \alpha - \beta + 1) \end{array} ; \right)
 \end{aligned}$$

for all $k, l \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}_{\geq 1}$, $\alpha, \beta \in \mathbb{C}$ such that $\operatorname{Re} \alpha > 0$, $\operatorname{Re} \alpha + 1 > \operatorname{Re} \beta > 0$.

Proof. The proofs of (43) and (44) are similar to those of (2) and (24), respectively. Indeed the identity (43) can be proved by applying the operator $(-1)^k \partial^{(l, k)}(w, z)|_{z=\alpha}^{w=\beta}$ ($k, l \in \mathbb{Z}_{\geq 0}$; $\alpha, \beta \in \mathbb{C}$ such that $\operatorname{Re} \alpha, \operatorname{Re} \beta > 0$, $\operatorname{Re}(\alpha - \beta) > \max\{-1, -s + 1\}$, $s \in \mathbb{Z}_{\geq 1}$) to both sides of the case $\beta = 1$ of (3) and using (5), (6) and the case $\beta = 1$, $\gamma = \alpha$, $\delta = \beta$ of (7). The identity (44) can be proved by applying $(-1)^k \partial^{(l, k)}(w, z)|_{z=\beta}^{w=\beta}$ ($k, l \in \mathbb{Z}_{\geq 0}$; $\beta \in \mathbb{C}$ such that $\operatorname{Re} \alpha + 1 > \operatorname{Re} \beta > 0$, $\operatorname{Re} \alpha > 0$) to both sides of the case $\beta = 1$ of (25) and using (6) and the case $\beta = 1$, $\gamma = \delta = \beta$ of (7). $\square$

Here we put

$$\Psi_{p, q}^*(\{K_i\}_{i=1}^{p+q}, (x, y, z)) := \Psi_{(\{0\}_{i=1}^{p+q})}^{(p, q)}(\{K_i\}_{i=1}^{p+q}, (x, y, z)).$$

Taking $l = 0$ in (43) and $s = 1$ in (44), we get (45) and (46) below, respectively:

Corollary 2.10. *The following two identities hold:*

(i)

$$(45) \quad \begin{aligned} & \Psi_{k+1,s-1}^* (\{1\}^{k+1}, \{2\}^{s-1}; (\alpha, \beta, 1)) \\ &= \sum^{(k,0)} \Psi_{i+1} \left(\begin{array}{c} \{1\}^{i+1} |\{0\}^{i+1} \\ \{k_j\}_{j=1}^i, k_{i+1} + s - 1 | \{1\}^i, s - 1 | \{0\}^i, 1 | \{0\}^{i+1} ; \\ (\alpha, 1, \alpha - \beta + 1, \beta) \end{array} \right) \end{aligned}$$

for all $k \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}_{\geq 1}$, $\alpha, \beta \in \mathbb{C}$ such that $\operatorname{Re} \alpha, \operatorname{Re} \beta > 0$, $\operatorname{Re}(\alpha - \beta) > \max\{-1, -s + 1\}$.

(ii)

$$(46) \quad \begin{aligned} & \Psi_{l+1}(\{1\}^l, k + 2; (\alpha, \beta)) \\ &= \sum^{(k,l)} \psi_{i+1,-} \left(\begin{array}{c} \{e(k_j) + e(l_j)\}_{j=1}^i, 1 \\ \{0\}^i, 1 | \{0\}^{i+1} | \{k_j + e(l_j)\}_{j=1}^i, k_{i+1} | \{l_j + e(k_j)\}_{j=1}^i, l_{i+1} ; \\ (\alpha, 1, \beta, \alpha - \beta + 1) \end{array} \right) \end{aligned}$$

for all $k, l \in \mathbb{Z}_{\geq 0}$, $\alpha, \beta \in \mathbb{C}$ such that $\operatorname{Re} \alpha > 0$, $\operatorname{Re} \alpha + 1 > \operatorname{Re} \beta > 0$, where

$$\Psi_{r_1+1}(\{k_{1j}\}_{j=1}^{r_1}, K_1; (x, y)) := \Psi_{(r_1)}^{(1,0)}(\{k_{1j}\}_{j=1}^{r_1}, K_1; (x, y, z)).$$

We consider the following special case of (41):

$$(47) \quad \begin{aligned} & \sum_{\substack{0=m_0 \leq m_{11} \leq \dots \leq m_{1r_1} \leq m_1 \\ m_{p-1} \leq m_{p1} \leq \dots \leq m_{pr_p} \leq m_p \\ \vdots \\ m_{p+q-1} \leq m_{p+q1} \leq \dots \leq m_{p+qr_{p+q}} \leq m_{p+q} < \infty}} \frac{(y)_{m_{p+q}}}{(x)_{m_{p+q}}} \frac{m_{p+q}!}{(w)_{m_{p+q}}} \\ & \times \left\{ \prod_{i=1}^p \frac{1}{(m_i + x)(m_i + y)^{K_i-1}} \prod_{j=1}^{r_i} \frac{1}{(m_{ij} + x)^{k_{ij}}} \right\} \\ & \times \left\{ \prod_{i=p+1}^{p+q} \frac{1}{(m_i + x)(m_i + y)^{K_i-2}(m_i + z)} \prod_{j=1}^{r_i} \frac{1}{(m_{ij} + x)^{k_{ij}}} \right\}, \end{aligned}$$

where p, q, r_i, k_{ij}, K_i are the same as those for (42). We denote the case $w = 1$ of (47) by $\Psi_{(\{r_i\}_{i=1}^{p+q})}^{(p,q), \leq}(\{\{k_{ij}\}_{j=1}^{r_i}, K_i\}_{i=1}^{p+q}; (x, y, z))$. We can prove the following identity:

Theorem 2.11. *The identity*

$$\begin{aligned}
 & \sum_{\substack{r_1+\dots+r_s=l \\ r_i \in \mathbb{Z}_{\geq 0}}} \Psi_{(\{r_i\}_{i=1}^s)}^{(1,s-1),\leq}(\{1\}^{r_1}, k+2, \{\{1\}^{r_i}, 2\}_{i=2}^s; (\alpha, \beta, 1)) \\
 (48) \quad &= \sum^{(k,l)} \sum_{\substack{0 \leq n_j \leq e(k_j)(l_j+1) \\ (j=1, \dots, i); \\ 0 \leq n_{i+1} \leq l_{i+1}}} \binom{s-1+l_{i+1}-n_{i+1}}{s-1} \left(\frac{l_{i+1}}{l_{i+1}+s-1} \right)^{1-e(n_{i+1})} \\
 & \quad \times \psi_{i+1,-}(\{l_j-n_j+e(n_j)\}_{j=1}^i, l_{i+1}-n_{i+1}+e(n_{i+1})+s-1|\{0\}^i, s-1| \\
 & \quad \{k_j-e(n_j)\}_{j=1}^{i+1}|\{n_j\}_{j=1}^{i+1}; (\alpha, 1, \beta, \alpha-\beta+1))
 \end{aligned}$$

holds for all $k, l \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}_{\geq 1}$, $\alpha, \beta \in \mathbb{C}$ such that $\operatorname{Re} \alpha > 0$, $\operatorname{Re} \alpha + 1 > \operatorname{Re} \beta > 0$, where $\psi_{n,-}(\{a_i\}_{i=1}^n|\{b_i\}_{i=1}^n|\{c_i\}_{i=1}^n|\{d_i\}_{i=1}^n; (x, y, z, w))$ is the same as that in Theorem 2.8 (i).

Proof. Taking $\alpha = 1$ in (25) and making the replacement $\beta \leftrightarrow w$ in the result, we get the identity

$$\begin{aligned}
 (49) \quad & \sum_{0 \leq m_1 \leq \dots \leq m_s < \infty} \frac{(\beta)_{m_s}}{(w)_{m_s}} \frac{1}{(m_1+z)(m_1+w)} \left\{ \prod_{i=2}^s \frac{1}{(m_i+w)(m_i+1)} \right\} \\
 &= \sum_{m=0}^{\infty} (-1)^m \frac{m!}{(z)_{m+1}} \frac{(\beta)_m}{(w)_{m+1}} \frac{(w+1-z)_m}{(w+1-\beta)_{m+1}} \frac{w+2m+1}{(m+w)^{s-1}(m+1)^{s-1}}
 \end{aligned}$$

for all $s \in \mathbb{Z}_{\geq 1}$, $\beta, z, w \in \mathbb{C}$ such that $\operatorname{Re} w > 0$, $\operatorname{Re} w + 1 > \operatorname{Re} z > 0$, $\operatorname{Re} w + 1 > \operatorname{Re} \beta > 0$, $2s-2 > \operatorname{Re}(2(\beta-z)-w)$. Taking $a = w+1-\beta$, $b = \delta+1-\gamma$, $c = \gamma$, $d = \delta$, $X = \delta+1-\beta$, $Y = w+1-z$, $Z = z$, $W = w$ in Lemma 2.1 and slightly modifying the result, we get the identity

$$\begin{aligned}
 (50) \quad & \frac{m!}{(z)_{m+1}} \frac{(\beta)_m}{(w)_{m+1}} \frac{(w+1-z)_m}{(w+1-\beta)_{m+1}} \frac{w+2m+1}{(w+m)^{s-1}(1+m)^{s-1}} \\
 &= \frac{m!}{(\gamma)_m} \frac{(\beta)_m}{(\delta)_m} \frac{(\delta+1-\gamma)_m}{(\delta+1-\beta)_m} \frac{w+2m+1}{(z+m)(w+m)^s(w+1-\beta+m)(1+m)^{s-1}} \\
 & \quad \times \sum_{i=0}^m \sum_{0 \leq m_1 < \dots < m_i < m} \prod_{j=1}^i \left(- \frac{(w-\delta)(\beta-\gamma)(\gamma+m_j)(\delta+m_j)}{(w+m_j)(w+1-\beta+m_j)(z+m_j)(\delta+1-\gamma+m_j)} \right. \\
 & \quad \left. - \frac{(z-\gamma)(\gamma+m_j)(\delta+m_j)(\delta+1-\beta+m_j)}{(z+m_j)(w+m_j)(w+1-\beta+m_j)(\delta+1-\gamma+m_j)} \right. \\
 & \quad \left. - \frac{(z-\gamma)(\delta+m_j)}{(z+m_j)(w+m_j)} - \frac{w-\delta}{w+m_j} \right).
 \end{aligned}$$

We can calculate the partial differential coefficients of the factors on the right-hand side of (50) as follows:

$$\begin{aligned}
 & (-1)^{k+l} \partial^{(k,l)}(z, w) \left(\frac{w+2m+1}{(z+m)(w+m)^s(w+1-\beta+m)(1+m)^{s-1}} \right) \Big|_{\substack{z=\gamma \\ w=\delta}} \\
 (51) \quad &= \sum_{n=0}^{l+1} \binom{s+l-n}{s-1} \left(\frac{l+1}{l+s} \right)^{1-e(n)} \\
 & \quad \times \frac{(\beta+m)^{e(n)}}{(\gamma+m)^{k+1}(\delta+m)^{l-n+e(n)+s}(\delta+1-\beta+m)^n(1+m)^{s-1}}
 \end{aligned}$$

$(k, l, m \in \mathbb{Z}_{\geq 0}, s \in \mathbb{Z}_{\geq 1}, \gamma, \delta, \delta+1-\beta \notin \mathbb{Z}_{\leq 0})$ and

$$\begin{aligned}
 & (-1)^{k+l} \partial^{(k,l)}(z, w) \left(-\frac{(w-\delta)(\beta-\gamma)(\gamma+m)(\delta+m)}{(w+m)(w+1-\beta+m)(z+m)(\delta+1-\gamma+m)} \right. \\
 & \quad - \frac{(z-\gamma)(\gamma+m)(\delta+m)(\delta+1-\beta+m)}{(z+m)(w+m)(w+1-\beta+m)(\delta+1-\gamma+m)} \\
 (52) \quad & \quad \left. - \frac{(z-\gamma)(\delta+m)}{(z+m)(w+m)} - \frac{w-\delta}{w+m} \right) \Big|_{\substack{z=\gamma \\ w=\delta}} \\
 &= \begin{cases} 0, & \text{if } k=l=0, \\ \sum_{n=0}^{l+1} \frac{(\beta-\gamma)^{\{1-e(k)\}e(n)}(\beta+m)^{e(k)e(n)}}{(\gamma+m)^k(\delta+m)^{l-n+e(n)}(\delta+1-\beta+m)^{n-1}(\delta+1-\gamma+m)}, & \text{if } k, l \in \mathbb{Z}_{\geq 0}, k+l \geq 1 \end{cases}
 \end{aligned}$$

$(m \in \mathbb{Z}_{\geq 0}, \gamma, \delta, \delta+1-\beta, \delta+1-\gamma \notin \mathbb{Z}_{\leq 0})$. Using (50), (51) and (52), we get the identity

$$\begin{aligned}
 (53) \quad & (-1)^{k+l} \partial^{(k,l)}(z, w) \left(\frac{m!}{(z)_{m+1}} \frac{(\beta)_m}{(w)_{m+1}} \frac{(w+1-z)_m}{(w+1-\beta)_{m+1}} \frac{w+2m+1}{(m+w)^{s-1}(m+1)^{s-1}} \right) \Big|_{\substack{z=\gamma \\ w=\delta}} \\
 &= \sum_{i=0}^m \sum_{\substack{k_1+\dots+k_{i+1}=k+1 \\ l_1+\dots+l_{i+1}=l+1 \\ k_j, l_j \in \mathbb{Z}_{\geq 0}, k_j+l_j \geq 1 \\ (j=1, \dots, i); k_{i+1}, l_{i+1} \in \mathbb{Z}_{\geq 1}}} \sum_{\substack{0 \leq n_j \leq l_j+1 \\ (j=1, \dots, i); \\ 0 \leq n_{i+1} \leq l_{i+1}}} \binom{s-1+l_{i+1}-n_{i+1}}{s-1} \left(\frac{l_{i+1}}{l_{i+1}+s-1} \right)^{1-e(n_{i+1})} \\
 & \quad \times \sum_{0 \leq m_1 < \dots < m_i < m} \left\{ \prod_{j=1}^i \frac{(\beta-\gamma)^{\{1-e(k_j)\}e(n_j)}(\beta+m_j)^{e(k_j)e(n_j)}}{(\gamma+m_j)^{k_j}(\delta+m_j)^{l_j-n_j+e(n_j)}(\delta+1-\beta+m_j)^{n_j-1}(\delta+1-\gamma+m_j)} \right\} \\
 & \quad \times \frac{(\beta+m)^{e(n_{i+1})}}{(\gamma+m)^{k_{i+1}}(\delta+m)^{l_{i+1}-n_{i+1}+e(n_{i+1})+s-1}(\delta+1-\beta+m)^{n_{i+1}}(1+m)^{s-1}} \\
 & \quad \times \frac{m!}{(\gamma)_m} \frac{(\beta)_m}{(\delta)_m} \frac{(\delta+1-\gamma)_m}{(\delta+1-\beta)_m}
 \end{aligned}$$

for $k, l, m \in \mathbb{Z}_{\geq 0}, s \in \mathbb{Z}_{\geq 1}, \beta, \gamma, \delta \in \mathbb{C}$ such that $\gamma, \delta, \delta+1-\beta, \delta+1-\gamma \notin \mathbb{Z}_{\leq 0}$. We see that the right-hand side of the case $\gamma = \beta, \delta = \alpha, k, l \in \mathbb{Z}_{\geq 0}, k+l \geq 1$

of (52) can be written as

$$\sum_{n=0}^{e(k)(l+1)} \frac{1}{(\alpha+m)^{l-n+e(n)}(\beta+m)^{k-e(n)}(\alpha-\beta+1+m)^n}.$$

Thus, applying the operator $(-1)^{k+l}\partial^{(k,l)}(z,w)|_{w=\alpha}^{z=\beta}$ ($k, l \in \mathbb{Z}_{\geq 0}$; $\alpha, \beta \in \mathbb{C}$ such that $\operatorname{Re} \alpha > 0$, $\operatorname{Re} \alpha + 1 > \operatorname{Re} \beta > 0$) to the right-hand side of (49) and using the case $\gamma = \beta$, $\delta = \alpha$ of (53), we get the right-hand side of (48). The left-hand side of (48) can be proved by applying $(-1)^{k+l}\partial^{(k,l)}(z,w)|_{w=\alpha}^{z=\beta}$ to the left-hand side of (49) and using the case $p = 0$, $q = s$, $a_i = 1$ ($i = 1, \dots, s$) of (17):

$$(54) \quad \frac{(-1)^l}{l!} \frac{d^l}{dw^l} \left(\frac{1}{(w)_{m_s}} \prod_{i=1}^s \frac{1}{m_i + w} \right) \\ = \frac{1}{(w)_{m_s}} \left(\prod_{i=1}^s \frac{1}{m_i + w} \right) \sum_{\substack{r_1+\dots+r_s=l \\ r_i \in \mathbb{Z}_{\geq 0}}} \sum_{\substack{0=m_0 \leq m_{11} \leq \dots \leq m_{1r_1} \leq m_1 \\ \vdots \\ m_{s-1} \leq m_{s1} \leq \dots \leq m_{sr_s} \leq m_s}} \prod_{i=1}^s \prod_{j=1}^{r_i} \frac{1}{m_{ij} + w}$$

for $l \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}_{\geq 1}$ and $m_1, \dots, m_s \in \mathbb{Z}$ such that $0 \leq m_1 \leq \dots \leq m_s$. $\square$

Using (48), we can derive the following identities, which are similar to (26), (27) and (28):

Corollary 2.12. *The following identities hold:*

(i)

$$(55) \quad \Psi_{1,s-1}^*(k+2, \{2\}^{s-1}; (\alpha, \beta, 1)) \\ = \sum^{(k,0)} \psi_{i+1,-} \left(\begin{array}{c} \{1\}^{i+1} \\ \{0\}^i, s | \{0\}^i, s-1 | \{k_j\}_{j=1}^{i+1} | \{1\}^{i+1} \end{array} ; \right. \\ \left. (\alpha, 1, \beta, \alpha - \beta + 1) \right)$$

for all $k, l \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}_{\geq 1}$, $\alpha, \beta \in \mathbb{C}$ such that $\operatorname{Re} \alpha > 0$, $\operatorname{Re} \alpha + 1 > \operatorname{Re} \beta > 0$.

(ii)

$$(56) \quad \Psi_{l,1}^*(\{1\}^l, k+2; (\alpha, \beta, \beta)) \\ = \sum^{(k,l)} \sum_{\substack{0 \leq n_j \leq e(k_j)(l_j+1) \\ (j=1, \dots, i); \\ 0 \leq n_{i+1} \leq l_{i+1}}} \psi_{i+1,-}(\{l_j - n_j + e(n_j)\}_{j=1}^{i+1} | \{0\}^{i+1} | \\ \{k_j - e(n_j)\}_{j=1}^{i+1} | \{n_j\}_{j=1}^{i+1}; (\alpha, 1, \beta, \alpha - \beta + 1))$$

for all $k, l \in \mathbb{Z}_{\geq 0}$, $\alpha, \beta \in \mathbb{C}$ such that $\operatorname{Re} \alpha > 0$, $\operatorname{Re} \alpha + 1 > \operatorname{Re} \beta > 0$.

(iii)

$$(57) \quad \zeta_{l+1,+}^{\leq}(\{1\}^l, k+2; \alpha) = \sum^{(k,l)} \sum_{\substack{0 \leq n_j \leq e(k_j)(l_j+1) \\ (j=1, \dots, i); \\ 0 \leq n_{i+1} \leq l_{i+1}}} \psi_{i+1,-}(\{k_j + l_j - n_j\}_{j=1}^{i+1} | \{n_j\}_{j=1}^{i+1}; (\alpha, 1))$$

for all $k, l \in \mathbb{Z}_{\geq 0}$, $\alpha \in \mathbb{C}$ with $\operatorname{Re} \alpha > 0$, where

$$\psi_{n,\pm}(\{a_i\}_{i=1}^n | \{b_i\}_{i=1}^n; (x, y)) := \psi_{n,\pm} \left(\begin{array}{c} \{0\}^n \\ \{a_i\}_{i=1}^n | \{b_i\}_{i=1}^n | \{0\}^n | \{0\}^n \end{array} ; (x, y, z, w) \right).$$

(iv)

$$(58) \quad \zeta^*(\{1\}^l, k+2) = \sum^{(k,l)} \left\{ \prod_{j=1}^i (l_j + 2)^{e(k_j)} \right\} (l_{i+1} + 1) \zeta_{-}(\{k_j + l_j\}_{j=1}^{i+1})$$

for all $k, l \in \mathbb{Z}_{\geq 0}$.

Proof. For $n_j = 0, 1$ ($j = 1, \dots, i+1$), we see that

$$e(n_j) = n_j, \quad \binom{s - n_{i+1}}{s - 1} s^{e(n_{i+1})-1} = 1.$$

By these facts and using (20), we get the expressions

$$(59) \quad \begin{aligned} & \sum_{\substack{0 \leq n_j \leq 1 \\ (j=1, \dots, i+1)}} \binom{s - n_{i+1}}{s - 1} s^{e(n_{i+1})-1} \\ & \times \left\{ \prod_{j=1}^{i+1} \frac{1}{(\alpha + m_j)^{-n_j + e(n_j)} (\beta + m_j)^{k_j - e(n_j)} (\alpha - \beta + 1 + m_j)^{n_j}} \right\} \\ & = \sum_{\substack{0 \leq n_j \leq 1 \\ (j=1, \dots, i+1)}} \prod_{j=1}^{i+1} \frac{1}{(\beta + m_j)^{k_j - n_j} (\alpha - \beta + 1 + m_j)^{n_j}} \\ & = \prod_{j=1}^{i+1} \left(\frac{1}{(\beta + m_j)^{k_j - 1} (\alpha - \beta + 1 + m_j)} + \frac{1}{(\beta + m_j)^{k_j}} \right) \\ & = \prod_{j=1}^{i+1} \frac{2m_j + \alpha + 1}{(\beta + m_j)^{k_j} (\alpha - \beta + 1 + m_j)}. \end{aligned}$$

Thus, taking $l = 0$ in (48) and using (59), we get (55). Taking $s = 1$ in (48) and $s = 1$, $\alpha = \beta$ in (48), we get (56) and (57), respectively. The identity (58) can

be proved by taking $s = \alpha = \beta = 1$ in (48) and using the identities

$$\begin{aligned} \sum_{n=0}^{e(k)(l+1)} \frac{1}{(m+1)^n(m+1)^{l-n+e(n)}(m+1)^{k-e(n)}} &= \left(\sum_{n=0}^{e(k)(l+1)} 1 \right) \frac{1}{(m+1)^{k+l}} \\ &= (e(k)(l+1) + 1) \frac{1}{(m+1)^{k+l}} = (l+2)^{e(k)} \frac{1}{(m+1)^{k+l}} \end{aligned}$$

$(k, l, m \in \mathbb{Z}_{\geq 0})$: the last identity comes from the definition of the symbol $e(k)$. $\square$

We denote the cases $w = z$ of (42) and (47) by $\Psi_{\mathbf{2}, (\{r_i\}_{i=1}^{p+q})}^{(p,q)}(\{\{k_{ij}\}_{j=1}^{r_i}, K_i\}_{i=1}^{p+q}; (x, y, z))$ and $\Psi_{\mathbf{2}, (\{r_i\}_{i=1}^{p+q})}^{(p,q), \leq}(\{\{k_{ij}\}_{j=1}^{r_i}, K_i\}_{i=1}^{p+q}; (x, y, z))$, respectively. We consider the following special case of (41) also:

$$\Psi_p^*(\{k_i\}_{i=1}^p; (x, y)) := \sum_{0 \leq m_1 \leq \dots \leq m_p < \infty} \frac{m_p!}{(x)_{m_p}} \left\{ \prod_{i=1}^p \frac{1}{(m_i + x)(m_i + y)^{k_i-1}} \right\},$$

where $p \in \mathbb{Z}_{\geq 1}$, $k_1, \dots, k_{p-1} \in \mathbb{Z}_{\geq 1}$, $k_p \in \mathbb{Z}$, $x, y \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ such that $k_p + \operatorname{Re} x > 2$.

Theorem 2.13. *The following identities hold:*

(i)

$$\begin{aligned} &\sum_{\substack{r_1 + \dots + r_s = l \\ r_i \in \mathbb{Z}_{\geq 0}}} \Psi_{\mathbf{2}, (\{r_i\}_{i=1}^s)}^{(0,s)}(\{\{1\}^{r_i}, 2\}_{i=1}^s; (\alpha, \gamma, \beta)) \\ (60) \quad &= \sum_{i=0}^l \sum_{\substack{l_1 + \dots + l_{i+1} = l+1 \\ l_j \in \mathbb{Z}_{\geq 1}}} \psi_{i+1, -} \left(\begin{array}{c} \{1\}^{i+1} \\ \{l_j\}_{j=1}^{i+1} | \{1\}^i, 0 | \{0\}^i, s | \{0\}^i, s \end{array} ; \right. \\ &\quad \left. (\alpha + \beta - \gamma, \gamma, \alpha, \beta) \right) \end{aligned}$$

for all $l \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}_{\geq 1}$, $\alpha, \beta, \gamma \in \mathbb{C}$ such that $\operatorname{Re} \alpha, \operatorname{Re} \beta > 0$, $\operatorname{Re}(\alpha + \beta) > \operatorname{Re} \gamma > 0$, $2s - 1 > \operatorname{Re}(2\gamma - \alpha - \beta)$.

(ii)

$$\begin{aligned} &\sum_{\substack{r_1 + \dots + r_s = r \\ r_i \in \mathbb{Z}_{\geq 0}}} \Psi_{\mathbf{2}, (\{r_i\}_{i=1}^s)}^{(0,s), \leq}(\{\{1\}^{r_i}, 2\}_{i=1}^s; (\alpha, \gamma, \beta)) \\ (61) \quad &= \sum_{i=0}^r \binom{s-1+i}{i} \psi_{r-i+1, -}^{\leq}(\{1\}^{r-i+1}; s+i | \{0\}^{r-i+1}; s-1; \\ &\quad (\alpha + \beta - \gamma, \gamma, \alpha, \beta)) \\ &\quad + \sum_{i=0}^r \binom{s-2+i}{i} \psi_{r-i+1, -}^{\leq}(\{1\}^{r-i+1}; s-1+i | \{0\}^{r-i+1}; s; \\ &\quad (\alpha + \beta - \gamma, \gamma, \alpha, \beta)) \end{aligned}$$

for all $r \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}_{\geq 1}$, $\alpha, \beta, \gamma \in \mathbb{C}$ such that $\operatorname{Re} \alpha, \operatorname{Re} \beta > 0$, $\operatorname{Re}(\alpha + \beta) > \operatorname{Re} \gamma > 0$, $2s - 1 > \operatorname{Re}(2\gamma - \alpha - \beta)$, where

$$\begin{aligned} & \psi_{n,-}^{\leq}(\{a_i\}_{i=1}^n; c|\{b_i\}_{i=1}^n; d; (x, y, z, w)) \\ &:= \sum_{0 \leq m_1 \leq \dots \leq m_n < \infty} (-1)^{m_n} \frac{(y)_{m_n}}{(x)_{m_n}} \left\{ \prod_{i=1}^n \frac{1}{(m_i + x)^{a_i} (m_i + y)^{b_i}} \right\} \frac{1}{(m_n + z)^c (m_n + w)^d}. \end{aligned}$$

(iii)

$$\begin{aligned} & \sum_{\substack{r_1 + \dots + r_s = r \\ r_i \in \mathbb{Z}_{\geq 0}}} \Psi_s^*(\{r_i + 2\}_{i=1}^s; (\alpha, \beta)) \\ (62) \quad &= \sum_{i=0}^r (-1)^{r-i} \binom{s-1+i}{i} \psi_{r-i+1,-}(\{0\}^{r-i}, s|\{1\}^{r-i}, s+i; (\alpha, \beta)) \\ &+ \sum_{i=0}^r (-1)^{r-i} \binom{s-2+i}{i} \psi_{r-i+1,-}(\{0\}^{r-i}, s+1|\{1\}^{r-i}, s-1+i; (\alpha, \beta)) \end{aligned}$$

for all $r \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}_{\geq 1}$, $\alpha, \beta \in \mathbb{C}$ such that $\operatorname{Re} \alpha, \operatorname{Re} \beta > 0$, $2s - 1 > \operatorname{Re}(\beta - \alpha)$, where $\psi_{n,-}(\{a_i\}_{i=1}^n|\{b_i\}_{i=1}^n; (x, y))$ is the same as that in Corollary 2.12 (iii).

(iv)

$$(63) \quad \sum_{\substack{p_1 + \dots + p_s + q_1 + \dots + q_s = r \\ p_i, q_i \in \mathbb{Z}_{\geq 0}}} \Psi_{\sum_{i=1}^s p_i + s}^*(\{\{1\}^{p_i}, q_i + 2\}_{i=1}^s; (\alpha, \alpha)) = 2 \binom{2s+r-1}{r} \zeta_{1,-}^{\leq}(2s+r; \alpha)$$

for all $r \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}_{\geq 1}$, $\alpha \in \mathbb{C}$ with $\operatorname{Re} \alpha > 0$.

Proof. Taking $a = \alpha + \beta$, $b_i = \alpha$, $c_i = \beta$ ($i = 1, \dots, s$), $b_{s+1} = w$, $c_{s+1} = 1$, where $s \in \mathbb{Z}_{\geq 1}$, $\alpha, \beta, w \in \mathbb{C}$ such that $\operatorname{Re} \alpha, \operatorname{Re} \beta > 0$, $\operatorname{Re}(\alpha + \beta) > \operatorname{Re} w > 0$, $2s - 1 > \operatorname{Re}(2w - \alpha - \beta)$, in Theorem A (i), we get the identity

$$\begin{aligned} (64) \quad & \sum_{0 \leq m_1 \leq \dots \leq m_s < \infty} \frac{(w)_{m_s}}{(\alpha)_{m_s}} \frac{m_s!}{(\beta)_{m_s}} \left\{ \prod_{i=1}^s \frac{1}{(m_i + \alpha)(m_i + \beta)} \right\} \\ &= \sum_{m=0}^{\infty} \frac{(w)_m}{(\alpha + \beta - w)_{m+1}} \frac{2m + \alpha + \beta}{(m + \alpha)^s (m + \beta)^s} (-1)^m. \end{aligned}$$

The identity (60) can be proved by applying the operator $\partial^{(l)}(w)|_{w=\gamma}$ ($l \in \mathbb{Z}_{\geq 0}$; $\gamma \in \mathbb{C}$ such that $\operatorname{Re} \alpha, \operatorname{Re} \beta > 0$, $\operatorname{Re}(\alpha + \beta) > \operatorname{Re} \gamma > 0$, $2s - 1 > \operatorname{Re}(2\gamma - \alpha - \beta)$) to both sides of (64) and using (6) and the case $k = 0$, $\delta = \gamma$ of (7). The identity (61) can be proved by differentiating both sides of (64) r times with respect to α and using (4) and (54). The identity (62) can be proved by differentiating

both sides of the case $w = \beta$ of (64) r times with respect to β and using the identity

$$(65) \quad \frac{1}{r!} \frac{d^r}{dw^r} (w)_m = (w)_m \sum_{0 \leq m_1 < \dots < m_r < m} \prod_{i=1}^r \frac{1}{m_i + w}$$

($m, r \in \mathbb{Z}_{\geq 0}$; see Hoffman [15, Proof of Corollary 4.2]). The identity (63) can be proved by differentiating both sides of the case $\alpha = \beta = w$ of (64) r times with respect to α and using (54). $\square$

Remark 6. (i) The identity (63) and the cases $\alpha = \beta = \gamma$ of the identities (60), (61), (62) give relations between the MHZVs $\zeta_{n,-}^<(\{a_i\}_{i=1}^n; \alpha)$, $\zeta_{n,-}^{\leq}(\{a_i\}_{i=1}^n; \alpha)$ and the multiple series (23) with $z = 1$. The cases $\beta = \gamma = 1$ of the identities (60), (61), (62) also give relations between the multiple series (23) with $z = \pm 1$.

(ii) Taking $\alpha = 1$ in (63) and using the identity $\zeta_{1,-}^<(a; 1) = \sum_{m=0}^{\infty} (-1)^m (m+1)^{-a} = (1 - 2^{1-a})\zeta(a)$, we get the relation among MZSVs of Aoki and Ohno [4, Theorem 1]. We see that the sums on the left-hand sides of the cases $\alpha = \beta = \gamma$ of the identities (61) and (62) are partial sums of the sum on the left-hand side of (63).

We remark that the multiple series (42) and (47) are special cases of the multiple series (1) with $p = 0$; therefore the results in this subsection can be regarded as identities for the multiple series (1).

2.4 Application 3

In this subsection, we derive several identities for MHZVs by using Theorem A (i).

Theorem 2.14. *The following two identities hold:*

$$(i) \quad \begin{aligned} & \zeta_{l+1,+}^<(\{1\}^{l+1}|\{0\}^l, k+1; (\alpha, \beta)) \\ (66) \quad &= \sum^{(k,l)} \psi_{i+1,-} \left(\begin{array}{c} \{e(k_j) + e(l_j)\}_{j=1}^i, 1 \\ \{k_j\}_{j=1}^{i+1} | \{e(k_j)\}_{j=1}^i, 0 | \{e(l_j)\}_{j=1}^i, 1 | \{l_j\}_{j=1}^{i+1} \\ (\beta, \alpha - \beta + 1, \alpha, 1) \end{array} \right) \end{aligned}$$

for all $k, l \in \mathbb{Z}_{\geq 0}$, $\alpha, \beta \in \mathbb{C}$ such that $\operatorname{Re} \alpha + 1 > \operatorname{Re} \beta > \operatorname{Re} \alpha/2 > 0$.

(ii)

$$\begin{aligned}
& \zeta_{l+1,+}^{\leq}(\{1\}^{l+1}|\{0\}^l, k+1; (\alpha, \beta)) \\
&= \sum^{(k,l)} \sum_{\substack{0 \leq n_j \leq l_j+1 \\ (j=1, \dots, i); \\ 0 \leq n_{i+1} \leq l_{i+1}}} (\alpha - \beta)^{\sum_{j=1}^i \mathbf{e}(k_j, n_j)} \psi_{i+1,-}(\{k_j\}_{j=1}^{i+1}|\{1\}^i, 0| \\
& \quad \{l_j - n_j + \mathbf{e}(k_j, n_j)\}_{j=1}^i, l_{i+1} - n_{i+1}| \\
& \quad \{n_j - 1\}_{j=1}^i, n_{i+1}; (\beta, \alpha - \beta + 1, \alpha, 1))
\end{aligned}
\tag{67}$$

for all $k, l \in \mathbb{Z}_{\geq 0}$, $\alpha, \beta \in \mathbb{C}$ such that $\operatorname{Re} \alpha + 1 > \operatorname{Re} \beta > \operatorname{Re} \alpha/2 > 0$, where

$$\mathbf{e}(k, n) := \{1 - e(k)\}e(n) = \begin{cases} 1, & \text{if } k = 0 \text{ and } n \geq 1, \\ 0, & \text{otherwise,} \end{cases}$$

and $\psi_{n,-}(\{a_i\}_{i=1}^n|\{b_i\}_{i=1}^n|\{c_i\}_{i=1}^n|\{d_i\}_{i=1}^n; (x, y, z, w))$ is the same as that in Theorem 2.8 (i).

Proof. Taking $s = \beta = 1$ in (25), we get the identity

$$\begin{aligned}
& \sum_{m=0}^{\infty} \frac{(w)_m}{(\alpha)_m} \frac{1}{(m+\alpha)(m+z)} \\
&= \sum_{m=0}^{\infty} \frac{(\alpha+1-z)_m}{(z)_{m+1}} \frac{(w)_m}{(\alpha+1-w)_{m+1}} \frac{m!}{(\alpha)_{m+1}} (2m+\alpha+1)(-1)^m
\end{aligned}
\tag{68}$$

for all $\alpha, z, w \in \mathbb{C}$ such that $\operatorname{Re} \alpha > 0$, $\operatorname{Re} \alpha + 1 > \operatorname{Re} z > 0$, $\operatorname{Re} \alpha + 1 > \operatorname{Re} w > 0$, $\operatorname{Re}(\alpha + 2(z-w)) > 0$. Applying the operator $(-1)^k \partial^{(l,k)}(w, z)|_{z=\beta}^{w=\alpha}$ ($k, l \in \mathbb{Z}_{\geq 0}$; $\alpha, \beta \in \mathbb{C}$ such that $\operatorname{Re} \alpha + 1 > \operatorname{Re} \beta > \operatorname{Re} \alpha/2 > 0$) to the right-hand side of (68) and using the case $\beta = 1$, $\gamma = \beta$, $\delta = \alpha$ of (7), we get the right-hand side of (66). The left-hand side of (66) can be proved by applying $(-1)^k \partial^{(l,k)}(w, z)|_{z=\beta}^{w=\alpha}$ to the left-hand side of (68) and using (65).

The identity (67) can be proved in a way similar to the above. Indeed, making the replacement $\alpha \leftrightarrow w$ in (68), we get the identity

$$\begin{aligned}
& \sum_{m=0}^{\infty} \frac{(\alpha)_m}{(w)_m} \frac{1}{(m+z)(m+w)} \\
&= \sum_{m=0}^{\infty} \frac{m!}{(z)_{m+1}} \frac{(\alpha)_m}{(w)_{m+1}} \frac{(w+1-z)_m}{(w+1-\alpha)_{m+1}} (2m+w+1)(-1)^m
\end{aligned}
\tag{69}$$

for all $\alpha, z, w \in \mathbb{C}$ such that $\operatorname{Re} w > 0$, $\operatorname{Re} w + 1 > \operatorname{Re} z > 0$, $\operatorname{Re} w + 1 > \operatorname{Re} \alpha > 0$, $\operatorname{Re}(w + 2(z-\alpha)) > 0$. Applying the operator $(-1)^{k+l} \partial^{(k,l)}(z, w)|_{w=\alpha}^{z=\beta}$ ($k, l \in \mathbb{Z}_{\geq 0}$; $\alpha, \beta \in \mathbb{C}$ such that $\operatorname{Re} \alpha + 1 > \operatorname{Re} \beta > \operatorname{Re} \alpha/2 > 0$) to the right-hand side of (69) and using the case $s = 1$, $\beta = \delta = \alpha$, $\gamma = \beta$ of (53), we get the right-hand side of (67). The left-hand side of (67) can be proved by applying $(-1)^{k+l} \partial^{(k,l)}(z, w)|_{w=\alpha}^{z=\beta}$ to the left-hand side of (69) and using the identity (4). $\square$

We can prove the following expressions for the MHZVs on the left-hand sides of the identities (66) and (67) also:

Theorem 2.15. *The following two identities hold:*

(i)

$$\begin{aligned}
 & \zeta_{l+1,+}^{<}(\{1\}^{l+1}|\{0\}^l, k+1; (\alpha, \beta)) \\
 (70) \quad &= \sum_{i=k}^{k+l} (-1)^i (\alpha - \beta)^{\sum_{j=1}^i \{1-e(k_j)\}} (\alpha - 1)^{\sum_{j=1}^i \{1-e(l_j)\}} \\
 & \quad \times \psi_{i+1,+}(\{k_j\}_{j=1}^{i+1}|\{1\}^i, 0|\{1-e(k_j)-e(l_j)\}_{j=1}^i, 0|\{l_j\}_{j=1}^{i+1}; \\
 & \quad (\beta, \beta - \alpha + 1, \alpha, 1))
 \end{aligned}$$

for all $k, l \in \mathbb{Z}_{\geq 0}$, $\alpha, \beta \in \mathbb{C}$ such that $\operatorname{Re} \alpha, \operatorname{Re} \beta > 0$, $\operatorname{Re} \alpha + 1 > \operatorname{Re} \beta > \max\{\operatorname{Re} \alpha/2, \operatorname{Re} \alpha - 1\}$, where $\psi_{n,+}(\{a_i\}_{i=1}^n|\{b_i\}_{i=1}^n|\{c_i\}_{i=1}^n|\{d_i\}_{i=1}^n; (x, y, z, w))$ is the same as that in Theorem 2.8 (i).

(ii)

$$\begin{aligned}
 & \zeta_{l+1,+}^{\leq}(\{1\}^{l+1}|\{0\}^l, k+1; (\alpha, \beta)) \\
 (71) \quad &= \sum_{i=k}^{k+l} \sum_{\substack{k_1+\dots+k_i=k \\ l_1+\dots+l_{i+2}=l+2 \\ k_j \in \{0,1\}, l_j \in \mathbb{Z}_{\geq 0}, k_j+l_j \geq 1 \\ (j=1,\dots,i); l_{i+1}, l_{i+2} \in \mathbb{Z}_{\geq 1}}} (1 - \beta)^{\sum_{j=1}^i \{1-e(k_j)\}} \psi_{i+1,+}(\{l_j\}_{j=1}^{i+1}|\{1\}^i, 0|\{0\}^i, l_{i+2}; \\
 & \quad (\alpha, \alpha - \beta + 1, 1))
 \end{aligned}$$

for all $k, l \in \mathbb{Z}_{\geq 0}$, $\alpha, \beta \in \mathbb{C}$ such that $\operatorname{Re} \alpha + 1 > \operatorname{Re} \beta > \operatorname{Re} \alpha/2 > 0$, where

$$\begin{aligned}
 & \psi_{n,+}(\{a_i\}_{i=1}^n|\{b_i\}_{i=1}^n|\{c_i\}_{i=1}^n; (x, y, z)) \\
 &:= \psi_{n,+} \left(\begin{array}{c} \{0\}^n \\ \{a_i\}_{i=1}^n|\{b_i\}_{i=1}^n|\{c_i\}_{i=1}^n|\{0\}^n \end{array} ; (x, y, z, w) \right).
 \end{aligned}$$

Proof. Taking $s = 1$, $a = \alpha + 1$, $b_1 = \alpha + 1 - z$, $c_1 = w$, $b_2 = c_2 = 1$, where $\alpha, z, w \in \mathbb{C}$ such that $\operatorname{Re} \alpha > 0$, $\operatorname{Re} \alpha + 1 > \operatorname{Re} z > 0$, $\operatorname{Re} \alpha + 1 > \operatorname{Re} w > 0$, $\operatorname{Re}(\alpha + 2(z - w)) > 0$, $\operatorname{Re}(z - w + 1) > 0$, in Theorem A (i) and multiplying both sides of the result by $\{z(\alpha + 1 - w)\}^{-1}$, we get the identity

$$\begin{aligned}
 & \sum_{m=0}^{\infty} \frac{(z - w + 1)_m}{(z)_{m+1}} \frac{m!}{(\alpha + 1 - w)_{m+1}} \\
 (72) \quad &= \sum_{m=0}^{\infty} \frac{(\alpha + 1 - z)_m}{(z)_{m+1}} \frac{(w)_m}{(\alpha + 1 - w)_{m+1}} \frac{m!}{(\alpha)_{m+1}} (2m + \alpha + 1)(-1)^m.
 \end{aligned}$$

Taking $a = b = c = \beta - \alpha + 1$, $d = \beta$, $X = z - w + 1$, $Y = 1$, $Z = \alpha + 1 - w$, $W = z$ in Lemma 2.1 and slightly modifying the result, we get the identity

$$\begin{aligned} & \frac{(z - w + 1)_m}{(z)_{m+1}} \frac{m!}{(\alpha + 1 - w)_{m+1}} \\ &= \frac{(\beta - \alpha + 1)_m}{(\beta)_m} \frac{1}{(z + m)(\alpha + 1 - w + m)} \\ & \times \sum_{i=0}^m \sum_{0 \leq m_1 < \dots < m_i < m} \prod_{j=1}^i \left(-\frac{z - \beta}{z + m_j} + \frac{(z - \beta)(1 + m_j)(\beta + m_j)}{(z + m_j)(\alpha + 1 - w + m_j)(\beta - \alpha + 1 + m_j)} \right. \\ & \left. + \frac{(w - \alpha)(\beta - \alpha)(\beta + m_j)}{(z + m_j)(\alpha + 1 - w + m_j)(\beta - \alpha + 1 + m_j)} \right). \end{aligned}$$

Using this identity, we get the identity

$$\begin{aligned} (73) \quad & (-1)^k \partial^{(l,k)}(w, z) \left(\frac{(z - w + 1)_m}{(z)_{m+1}} \frac{m!}{(\alpha + 1 - w)_{m+1}} \right) \Big|_{\substack{w=\alpha \\ z=\beta}} \\ &= \sum_{i=0}^m \sum_{\substack{k_1 + \dots + k_{i+1} = k+1 \\ l_1 + \dots + l_{i+1} = l+1 \\ k_j, l_j \in \mathbb{Z}_{\geq 0}, k_j + l_j \geq 1 \\ (j=1, \dots, i); k_{i+1}, l_{i+1} \in \mathbb{Z}_{\geq 1}}} (-1)^i (\alpha - \beta)^{\sum_{j=1}^i \{1 - e(k_j)\}} (\alpha - 1)^{\sum_{j=1}^i \{1 - e(l_j)\}} \\ & \times \sum_{0 \leq m_1 < \dots < m_i < m} \left\{ \prod_{j=1}^i \frac{1}{(\beta + m_j)^{k_j} (\beta - \alpha + 1 + m_j)(\alpha + m_j)^{1 - e(k_j) - e(l_j)} (1 + m_j)^{l_j}} \right\} \\ & \times \frac{1}{(\beta + m)^{k_{i+1}} (1 + m)^{l_{i+1}}} \frac{(\beta - \alpha + 1)_m}{(\beta)_m} \end{aligned}$$

for $k, l, m \in \mathbb{Z}_{\geq 0}$, $\alpha, \beta \in \mathbb{C}$ such that $\alpha, \beta, \beta - \alpha + 1 \notin \mathbb{Z}_{\leq 0}$. Thus, applying the operator $(-1)^k \partial^{(l,k)}(w, z) \Big|_{\substack{w=\alpha \\ z=\beta}}$ ($k, l \in \mathbb{Z}_{\geq 0}$; $\alpha, \beta \in \mathbb{C}$ such that $\operatorname{Re} \alpha, \operatorname{Re} \beta > 0$, $\operatorname{Re} \alpha + 1 > \operatorname{Re} \beta > \max\{\operatorname{Re} \alpha/2, \operatorname{Re} \alpha - 1\}$) to the left-hand side of (72) and using (73), we get the right-hand side of (70). We note that the right-hand side of (72) is completely the same as that of (68). By this fact and using (68) and (72), we get (70).

The identity (71) can also be proved in a way similar to the above. Indeed, taking $s = 1$, $a = w + 1$, $b_1 = \alpha$, $b_2 = w + 1 - z$, $c_1 = c_2 = 1$, where $\alpha, z, w \in \mathbb{C}$ such that $\operatorname{Re} w > 0$, $\operatorname{Re} w + 1 > \operatorname{Re} z > 0$, $\operatorname{Re} w + 1 > \operatorname{Re} \alpha > 0$, $\operatorname{Re}(w + 2(z - \alpha)) > 0$, in Theorem A (i) and multiplying both sides of the result by w^{-1} , we get the identity

$$\begin{aligned} (74) \quad & \sum_{m=0}^{\infty} \frac{(w + 1 - z)_m}{(w)_{m+1}} \frac{1}{w + 1 - \alpha + m} \\ &= \sum_{m=0}^{\infty} \frac{m!}{(z)_{m+1}} \frac{(\alpha)_m}{(w)_{m+1}} \frac{(w + 1 - z)_m}{(w + 1 - \alpha)_{m+1}} (2m + w + 1)(-1)^m. \end{aligned}$$

Taking $a = X$, $b = Y$, $c = w + 1 - z$, $d = \alpha$, $Z = \alpha - \beta + 1$, $W = w$ in Lemma 2.1 and slightly modifying the result, we get the identity

$$\begin{aligned}
 & \frac{(w+1-z)_m}{(w)_{m+1}} \frac{1}{w+1-\alpha+m} \\
 &= \frac{(\alpha-\beta+1)_m}{(\alpha)_m} \frac{1}{(w+m)(w+1-\alpha+m)} \\
 (75) \quad & \times \sum_{i=0}^m \sum_{0 \leq m_1 < \dots < m_i < m} \prod_{j=1}^i \left(-\frac{(z-\beta)(\alpha+m_j)}{(w+m_j)(\alpha-\beta+1+m_j)} \right. \\
 & \left. + \frac{(w-\alpha)(\beta-1)}{(w+m_j)(\alpha-\beta+1+m_j)} \right).
 \end{aligned}$$

Using this identity, we get the identity

$$\begin{aligned}
 & (-1)^{k+l} \partial^{(k,l)}(z, w) \left(\frac{(w+1-z)_m}{(w)_{m+1}} \frac{1}{w+1-\alpha+m} \right) \Big|_{\substack{z=\beta \\ w=\alpha}} \\
 &= \sum_{i=0}^m \sum_{\substack{k_1+\dots+k_i=k \\ l_1+\dots+l_{i+2}=l+2 \\ k_j \in \{0,1\}, l_j \in \mathbb{Z}_{\geq 0}, k_j+l_j \geq 1 \\ (j=1,\dots,i); l_{i+1}, l_{i+2} \in \mathbb{Z}_{\geq 1}}} (1-\beta)^{\sum_{j=1}^i \{1-e(k_j)\}} \frac{(\alpha-\beta+1)_m}{(\alpha)_m} \\
 (76) \quad & \times \sum_{0 \leq m_1 < \dots < m_i < m} \left\{ \prod_{j=1}^i \frac{1}{(\alpha+m_j)^{l_j}(\alpha-\beta+1+m_j)} \right\} \\
 & \times \frac{1}{(\alpha+m)^{l_{i+1}}(1+m)^{l_{i+2}}}
 \end{aligned}$$

for $k, l, m \in \mathbb{Z}_{\geq 0}$, $\alpha, \beta \in \mathbb{C}$ such that $\alpha, \alpha - \beta + 1 \notin \mathbb{Z}_{\leq 0}$. Since the right-hand side of (75) is a polynomial in terms of z , the summation $\sum_{i=0}^m$ on the right-hand side of (76) can be changed into $\sum_{i=k}^m$. Thus, applying the operator $(-1)^{k+l} \partial^{(k,l)}(z, w) \Big|_{\substack{z=\beta \\ w=\alpha}}$ ($k, l \in \mathbb{Z}_{\geq 0}$; $\alpha, \beta \in \mathbb{C}$ such that $\operatorname{Re} \alpha + 1 > \operatorname{Re} \beta > \operatorname{Re} \alpha/2 > 0$) to the left-hand side of (74) and using (76), we get the right-hand side of (71). We note that the right-hand side of (74) is completely the same as that of (69). By this fact and using (69) and (74), we get (71). $\square$

Remark 7. (i) Taking $\alpha = \beta = 1$ in (70), we get the following identity for MZVs, which is similar to (28):

$$(77) \quad \zeta(\{1\}^l, k+2) = \sum_{i=0}^{\min\{k,l\}} \sum_{\substack{k_1+\dots+k_{i+1}=k+1 \\ l_1+\dots+l_{i+1}=l+1 \\ k_j, l_j \in \mathbb{Z}_{\geq 1}}} (-1)^i \zeta(\{k_j+l_j\}_{j=1}^{i+1})$$

for all $k, l \in \mathbb{Z}_{\geq 0}$. By the replacement $k \leftrightarrow l$ in the right-hand sides of the identities (28) and (77), we see that they have a symmetry with respect to k and

l . That symmetry gives the identity $\zeta(\{1\}^l, k+2) = \zeta(\{1\}^k, l+2)$ ($k, l \in \mathbb{Z}_{\geq 0}$) (Hoffman [15, Theorem 4.4]). Taking $\alpha = \beta = 1$ in (71), we get the expression for $\zeta^*(\{1\}^l, k+2)$ ($k, l \in \mathbb{Z}_{\geq 0}$) in terms of MZVs in Ohno [25, Theorem 2 and its proof].

(ii) Taking $\beta = (\alpha+1)/2$ in (66) and (67), $\alpha = 1$ in (70), and $\beta = 1$ in (71), we get relations among the MHZVs $\zeta_{n,\pm}^{\leq}(\{a_i\}_{i=1}^n | \{b_i\}_{i=1}^n | \{c_i\}_{i=1}^n; ((x+1)/2, x, 1))$.

(iii) Making the replacement $\alpha \leftrightarrow w$ in (72) and using the result of the replacement, the identity (69), the case $a = b = c = \beta - \alpha + 1$, $d = \beta$, $X = z - \alpha + 1$, $Y = 1$, $Z = w + 1 - \alpha$, $W = z$ of Lemma 2.1, a calculation and a way like those of proving (70), we can prove the following modification of (71) also, which is similar to (70):

$$\begin{aligned} & \zeta_{l+1,+}^{\leq}(\{1\}^{l+1} | \{0\}^l, k+1; (\alpha, \beta)) \\ &= \sum^{(k,l)} (1-\alpha)^{\sum_{j=1}^i e(k_j)} \psi_{i+1,+}(\{k_j\}_{j=1}^{i+1} | \{e(k_j)\}_{j=1}^i, 0 | \{l_j\}_{j=1}^{i+1}; (\beta, \beta - \alpha + 1, 1)) \end{aligned}$$

for all $k, l \in \mathbb{Z}_{\geq 0}$, $\alpha, \beta \in \mathbb{C}$ such that $\operatorname{Re} \alpha, \operatorname{Re} \beta > 0$, $\operatorname{Re} \alpha + 1 > \operatorname{Re} \beta > \max\{\operatorname{Re} \alpha/2, \operatorname{Re} \alpha - 1\}$, where $\psi_{n,+}(\{a_i\}_{i=1}^n | \{b_i\}_{i=1}^n | \{c_i\}_{i=1}^n; (x, y, z))$ is the same as that in Theorem 2.15 (ii).

Theorem 2.16. *The identity*

$$\begin{aligned} & \sum_{i=0}^k 2^{k-i} \binom{i+r}{i} \sum_{\substack{k_1+\dots+k_s=k-i \\ k_j \in \mathbb{Z}_{\geq 0}}} \zeta_{s,+}^{\leq}(i+r+1, \{0\}^{s-1} | \{k_j+1\}_{j=1}^s | \\ (78) \quad & 0, \{1\}^{s-1}; ((\alpha+1)/2, \alpha, 1)) = \sum_{i=0}^r \sum_{\substack{k_1+\dots+k_{i+2}=k \\ r_1+\dots+r_{i+1}=r+1 \\ k_j \in \mathbb{Z}_{\geq 0}, r_j \in \mathbb{Z}_{\geq 1}}} 2^{i+1+k_{i+2}} \\ & \times \left\{ \prod_{j=1}^i \binom{k_j+r_j-1}{k_j} \right\} \binom{k_{i+1}+r_{i+1}-2}{k_{i+1}} \binom{k_{i+2}+s-1}{k_{i+2}} \\ & \times \zeta_{i+1,-}^{\leq}(\{k_j+r_j\}_{j=1}^i, k_{i+1}+r_{i+1}-1 | \{0\}^i, k_{i+2}+s | \{0\}^i, s; ((\alpha+1)/2, \alpha, 1)) \end{aligned}$$

holds for all $k, r \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}_{\geq 1}$, $\alpha \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$.

Proof. Applying the operator $(-1)^r \partial^{(r)}(z)|_{z=(\alpha+1)/2}$ ($r \in \mathbb{Z}_{\geq 0}$; $\alpha \in \mathbb{C}$ with $\operatorname{Re} \alpha > 0$) to both sides of the case $\beta = 1$, $w = \alpha$ of (25) and using the case

$l = 0$, $\beta = 1$, $\gamma = (\alpha + 1)/2$, $\delta = \alpha$ of (7), we get the identity

$$(79) \quad \sum_{0 \leq m_1 \leq \dots \leq m_s < \infty} \frac{1}{(m_1 + \alpha)(m_1 + (\alpha + 1)/2)^{r+1}} \left\{ \prod_{i=2}^s \frac{1}{(m_i + \alpha)(m_i + 1)} \right\} \\ = \sum_{i=0}^r \sum_{\substack{r_1 + \dots + r_{i+1} = r+1 \\ r_j \in \mathbb{Z}_{\geq 1}}} 2^{i+1} \sum_{0 \leq m_1 < \dots < m_{i+1} < \infty} \left\{ \prod_{j=1}^i \frac{1}{(m_j + (\alpha + 1)/2)^{r_j}} \right\} \\ \times \frac{1}{(m_{i+1} + (\alpha + 1)/2)^{r_{i+1}-1}} \frac{(-1)^{m_{i+1}}}{(m_{i+1} + \alpha)^s (m_{i+1} + 1)^s}$$

for all $r \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}_{\geq 1}$, $\alpha \in \mathbb{C}$ with $\operatorname{Re} \alpha > 0$. Further, differentiating both sides of (79) k times with respect to α and multiplying both sides of the result by 2^k , we get (78) under the condition $\operatorname{Re} \alpha > 0$. This condition can be changed into $\alpha \notin \mathbb{Z}_{\leq 0}$, because both sides of (78) are holomorphic in $\mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ as functions of α . $\square$

Remark 8. Taking $\alpha = 1$ in (78), we get relations between MZSVs and $\zeta_{-}(\{k_i\}_{i=1}^n)$. For example, taking $\alpha = 1$, $r = 0$ in (78) and using the identity $\zeta_{1,-}^{<}(a; 1) = \sum_{m=0}^{\infty} (-1)^m (m+1)^{-a} = (1 - 2^{1-a})\zeta(a)$, we get the identity

$$\sum_{i=0}^k 2^{k-i} \sum_{\substack{k_1 + \dots + k_s = k-i \\ k_j \in \mathbb{Z}_{\geq 0}}} \zeta^{\star}(k_1 + i + 2, \{k_j + 2\}_{j=2}^s) \\ = 2^{1+k} \binom{k+s-1}{k} (1 - 2^{1-k-2s}) \zeta(k+2s)$$

for all $k \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}_{\geq 1}$. Taking $k = 1$ in this identity, we get the identity for $\zeta(2s+1)$ of Ihara, Kajikawa, Ohno and Okuda [22, Theorem 2].

Here we put

$$P(n; (\alpha, \beta)) := \frac{1}{n!} \frac{\partial^n}{\partial \gamma^n} (\psi(\alpha + \beta - \gamma) - \psi(\gamma)) \Big|_{\gamma=\beta} \\ = \begin{cases} (\alpha - \beta) \zeta_{1,+}^{<}(1|1; (\alpha, \beta)), & \text{if } n = 0, \\ -\zeta_{1,+}^{<}(n+1; \alpha) + (-1)^n \zeta_{1,+}^{<}(n+1; \beta), & \text{if } n \in \mathbb{Z}_{\geq 1}, \end{cases}$$

where $\alpha, \beta \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ and $\psi(z) := \Gamma'(z)/\Gamma(z)$ is the digamma function. The second equality for $P(n; (\alpha, \beta))$ follows from the following two properties of the digamma function: $\psi(z) - \psi(w) = (z-w) \sum_{m=0}^{\infty} (m+z)^{-1} (m+w)^{-1}$ ($z, w \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$) and $\psi^{(n)}(z) = (-1)^{n+1} n! \zeta_{1,+}^{<}(n+1; z)$ ($n \in \mathbb{Z}_{\geq 1}$), where $\psi^{(n)}(z)$ is the n -th derivative of $\psi(z)$. (For the property of the digamma function, see e.g. [27, Subsection 1.2].) We define the symbol $G(n; (\alpha, \beta))$ by

$$G(n; (\alpha, \beta)) := \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \sum_{i=1}^n \sum_{0=l_0 < l_1 < \dots < l_{i-1} < l_i = n} \prod_{j=1}^i l_j^{-1} P(l_j - l_{j-1} - 1; (\alpha, \beta))$$

($n \in \mathbb{Z}_{\geq 1}$) and $G(0; (\alpha, \beta)) := \Gamma(\alpha + \beta) / (\Gamma(\alpha)\Gamma(\beta))$, where $\alpha, \beta \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ such that $\alpha + \beta \notin \mathbb{Z}_{\leq 0}$. We consider the series

$$\psi_{1,-}^{\#}(a|b; (x, y)) := \sum_{m=0}^{\infty} \frac{(x+y)_m}{m!} \frac{2m+x+y}{(m+x)^a(m+y)^b} (-1)^m,$$

where $a, b \in \mathbb{Z}$, $x, y \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ such that $a+b - \operatorname{Re}(x+y) > 1$. We can prove the following evaluation for the multiple series (12) in terms of the single series $\psi_{1,-}^{\#}(a|b; (x, y))$, $\zeta_{1,+}^{<}(a|b; (x, y))$:

Theorem 2.17. *The identity*

$$\begin{aligned} & \sum_{\substack{r_1+\dots+r_s=r \\ r_i \in \mathbb{Z}_{\geq 0}}} \phi_{A^{(s)}, B^{(s)}}^{s, \leq} \left(\begin{array}{c} \{r_i+1\}_{i=1}^s \\ \{r_i+1\}_{i=1}^s | \{r_i+1\}_{i=1}^s | \{1\}^s \end{array}; (\alpha, \beta) \right) \\ (80) \quad &= \sum_{\substack{r_1+r_2+r_3=r \\ r_i \in \mathbb{Z}_{\geq 0}}} (-1)^{r_3} \binom{r_2+s}{r_2} \binom{r_3+s}{r_3} G(r_1; (\alpha, \beta)) \\ & \times \psi_{1,-}^{\#}(r_2+s+1|r_3+s+1; (\alpha, \beta)) \end{aligned}$$

holds for all $r \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}_{\geq 1}$, $\alpha, \beta \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ such that $2s+1 > \operatorname{Re}(\alpha+\beta) > 0$, where $A^{(s)} = \{(-1)^{r_i}\}_{i=1}^s$, $B^{(s)} = \{1\}^s$. The left-hand side of (80) can be written as a linear combination of $\zeta_{s,+}^{<}(\{a_i\}_{i=1}^s | \{b_i\}_{i=1}^s | \{1\}^s; (\alpha, \beta, (\alpha+\beta)/2))$, explicitly.

Remark 9. We see that

$$\begin{aligned} \psi_{1,-}^{\#}(a|b; (2-\beta, \beta)) &= 2\zeta_{1,-}^{<}(a|b-2; (2-\beta, \beta)) + 4(1-\beta)\zeta_{1,-}^{<}(a|b-1; (2-\beta, \beta)) \\ & \quad + 2(1-\beta)^2\zeta_{1,-}^{<}(a|b; (2-\beta, \beta)). \end{aligned}$$

By this fact, the left-hand side of the case $\alpha = 2-\beta$ of (80) can be evaluated by only the single series $\zeta_{1,\pm}^{<}(a|b; (2-\beta, \beta))$.

To prove Theorem 2.17, we prove the following lemma:

Lemma 2.18. *The identity*

$$\begin{aligned} & \frac{\Gamma(\alpha)\Gamma(\beta)}{r!} \frac{\partial^r}{\partial \gamma^r} \left(\frac{1}{\Gamma(\alpha+\beta-\gamma)\Gamma(\gamma)} \right) \Big|_{\gamma=\beta} \\ (81) \quad &= \sum_{i=1}^r \sum_{0=l_0 < l_1 < \dots < l_{i-1} < l_i=r} \prod_{j=1}^i l_j^{-1} P(l_j - l_{j-1} - 1; (\alpha, \beta)) \end{aligned}$$

holds for all $r \in \mathbb{Z}_{\geq 1}$, $\alpha, \beta \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$.

Proof. The left-hand side of (81) can be calculated as follows:

$$\begin{aligned}
 & \frac{\Gamma(\alpha)\Gamma(\beta)}{r!} \frac{\partial^r}{\partial \gamma^r} \left(\frac{1}{\Gamma(\alpha + \beta - \gamma)\Gamma(\gamma)} \right) \Big|_{\gamma=\beta} \\
 (82) \quad &= \frac{\Gamma(\alpha)\Gamma(\beta)}{r!} \frac{\partial^{r-1}}{\partial \gamma^{r-1}} \left(\frac{1}{\Gamma(\alpha + \beta - \gamma)\Gamma(\gamma)} \{ \psi(\alpha + \beta - \gamma) - \psi(\gamma) \} \right) \Big|_{\gamma=\beta} \\
 &= \frac{1}{r} \sum_{i=0}^{r-1} \frac{\Gamma(\alpha)\Gamma(\beta)}{i!} \frac{\partial^i}{\partial \gamma^i} \left(\frac{1}{\Gamma(\alpha + \beta - \gamma)\Gamma(\gamma)} \right) \Big|_{\gamma=\beta} P(r-1-i; (\alpha, \beta))
 \end{aligned}$$

for $r \in \mathbb{Z}_{\geq 1}$, $\alpha, \beta \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$. By using (82) and the induction on $r \in \mathbb{Z}_{\geq 1}$, we prove (81). Taking $r = 1$ in (82), we get the case $r = 1$ of (81). Now we suppose that the identity (81) holds for any $r \in \mathbb{Z}_{\geq 1}$. Then, using (82) and the inductive hypothesis, we get the following:

$$\begin{aligned}
 & \frac{\Gamma(\alpha)\Gamma(\beta)}{(r+1)!} \frac{\partial^{r+1}}{\partial \gamma^{r+1}} \left(\frac{1}{\Gamma(\alpha + \beta - \gamma)\Gamma(\gamma)} \right) \Big|_{\gamma=\beta} \\
 &= \frac{1}{r+1} \sum_{i=0}^r \frac{\Gamma(\alpha)\Gamma(\beta)}{i!} \frac{\partial^i}{\partial \gamma^i} \left(\frac{1}{\Gamma(\alpha + \beta - \gamma)\Gamma(\gamma)} \right) \Big|_{\gamma=\beta} P(r-i; (\alpha, \beta)) \\
 &= (r+1)^{-1} P(r; (\alpha, \beta)) \\
 &+ \sum_{i=1}^r \left(\sum_{j=1}^i \sum_{0=l_0 < l_1 < \dots < l_{j-1} < l_j=i} \prod_{k=1}^j l_k^{-1} P(l_k - l_{k-1} - 1; (\alpha, \beta)) \right) \\
 &\times (r+1)^{-1} P(r-i; (\alpha, \beta)) \\
 &= (r+1)^{-1} P(r; (\alpha, \beta)) \\
 &+ \sum_{j=1}^r \sum_{i=j}^r \sum_{0=l_0 < l_1 < \dots < l_{j-1} < l_j=i} \left(\prod_{k=1}^j l_k^{-1} P(l_k - l_{k-1} - 1; (\alpha, \beta)) \right) \\
 &\times (r+1)^{-1} P(r-i; (\alpha, \beta)) \\
 &= \sum_{j=0}^r \sum_{0=l_0 < l_1 < \dots < l_j < l_{j+1}=r+1} \prod_{k=1}^{j+1} l_k^{-1} P(l_k - l_{k-1} - 1; (\alpha, \beta)).
 \end{aligned}$$

This completes the proof of (81). $\square$

Now we prove Theorem 2.17.

Proof of Theorem 2.17. Taking $a = \alpha + \beta$, $b_i = \alpha + \beta - \gamma$, $c_i = \gamma$ ($i = 1, \dots, s+1$), where $s \in \mathbb{Z}_{\geq 1}$, $\alpha, \beta, \gamma \in \mathbb{C}$ such that $2s+1 > \operatorname{Re}(\alpha + \beta) > 0$, $\alpha + \beta - \gamma, \gamma \notin \mathbb{Z}_{\leq 0}$, in Theorem A (i), we get the identity

$$\begin{aligned}
 (83) \quad & \sum_{0 \leq m_1 \leq \dots \leq m_s < \infty} \prod_{i=1}^s \frac{1}{(m_i + \alpha + \beta - \gamma)(m_i + \gamma)} \\
 &= \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha + \beta - \gamma)\Gamma(\gamma)} \sum_{m=0}^{\infty} \frac{(\alpha + \beta)_m}{m!} \frac{2m + \alpha + \beta}{(m + \alpha + \beta - \gamma)^{s+1}(m + \gamma)^{s+1}} (-1)^m.
 \end{aligned}$$

By a direct calculation, we get the identities

$$\begin{aligned}
 & \frac{1}{r!} \frac{\partial^r}{\partial \gamma^r} \left(\prod_{i=1}^s \frac{1}{(m_i + \alpha + \beta - \gamma)(m_i + \gamma)} \right) \Big|_{\gamma=\beta} \\
 &= \frac{1}{r!} \frac{\partial^r}{\partial \gamma^r} \left(\prod_{i=1}^s \left(\frac{1}{(m_i + \alpha + \beta - \gamma)(2m_i + \alpha + \beta)} \right. \right. \\
 (84) \quad & \left. \left. + \frac{1}{(m_i + \gamma)(2m_i + \alpha + \beta)} \right) \right) \Big|_{\gamma=\beta} \\
 &= \sum_{\substack{r_1 + \dots + r_s = r \\ r_i \in \mathbb{Z}_{\geq 0}}} \prod_{i=1}^s \left(\frac{1}{(m_i + \alpha)^{r_i+1}(2m_i + \alpha + \beta)} \right. \\
 & \left. + \frac{(-1)^{r_i}}{(m_i + \beta)^{r_i+1}(2m_i + \alpha + \beta)} \right)
 \end{aligned}$$

for $r \in \mathbb{Z}_{\geq 0}$. Thus, applying the operator $\partial^{(r)}(\gamma)|_{\gamma=\beta}$ ($r \in \mathbb{Z}_{\geq 0}$; $\beta \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ such that $2s+1 > \operatorname{Re}(\alpha + \beta) > 0$, $\alpha \notin \mathbb{Z}_{\leq 0}$) to the left-hand side of (83) and using (84), we get the left-hand side of (80). The right-hand side of (80) can be proved by applying $\partial^{(r)}(\gamma)|_{\gamma=\beta}$ to the right-hand side of (83) and using Lemma 2.18. This completes the proof of (80). Applying (20) to the most right-hand side of (84), we can write the left-hand side of (80) as a linear combination of $\zeta_{s,+}^{\leq}(\{a_i\}_{i=1}^s | \{b_i\}_{i=1}^s | \{1\}^s; (\alpha, \beta, (\alpha + \beta)/2))$, explicitly. $\square$

From (80), we can get the following identity:

Corollary 2.19. *The identity*

$$\begin{aligned}
 & \sum_{\substack{r_1 + \dots + r_s = r \\ r_i \in \mathbb{Z}_{\geq 0} (i=1, \dots, s): \\ \text{even}}} \zeta_{s,+}^{\leq}(\{r_i + 2\}_{i=1}^s; \alpha) \\
 (85) \quad &= 2 \sum_{i=0}^{\lfloor \frac{r}{2} \rfloor} \binom{i+s}{i} G(r-2i; (\alpha, \alpha)) \psi_{1,-}^{\sharp}(2i+2s+1; \alpha)
 \end{aligned}$$

holds for all $r \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}_{\geq 1}$, $\alpha \in \mathbb{C}$ with $s+1/2 > \operatorname{Re} \alpha > 0$, where $[x]$ denotes the greatest integer not greater than x and

$$\psi_{1,-}^{\sharp}(a+b-1; \alpha) := \sum_{m=0}^{\infty} \frac{(2\alpha)_m}{m!} \frac{(-1)^m}{(m+\alpha)^{a+b-1}} = 2^{-1} \psi_{1,-}^{\sharp}(a|b; (\alpha, \alpha)).$$

Proof. Taking $\alpha = \beta$ in (80) and using the identity

$$\prod_{i=1}^s (1 + (-1)^{r_i}) = \begin{cases} 2^s, & \text{if } r_i \text{ is even for all } i, \\ 0, & \text{if } r_i \text{ is odd for some } i, \end{cases}$$

we get the left-hand side of (85). The right-hand side of the case $\alpha = \beta$ of (80) can be rewritten as

$$\begin{aligned}
 & 2 \sum_{\substack{r_1+r_2+r_3=r \\ r_i \in \mathbb{Z}_{\geq 0}}} (-1)^{r_3} \binom{r_2+s}{r_2} \binom{r_3+s}{r_3} \\
 & \times G(r_1; (\alpha, \alpha)) \psi_{1,-}^{\sharp}(r_2+r_3+2s+1; \alpha) \\
 (86) \quad & = 2 \sum_{r_1=0}^r G(r-r_1; (\alpha, \alpha)) \psi_{1,-}^{\sharp}(r_1+2s+1; \alpha) \\
 & \times \sum_{\substack{r_2+r_3=r_1 \\ r_2, r_3 \in \mathbb{Z}_{\geq 0}}} (-1)^{r_3} \binom{r_2+s}{r_2} \binom{r_3+s}{r_3}.
 \end{aligned}$$

Applying the identity

$$\sum_{\substack{r_2+r_3=r_1 \\ r_2, r_3 \in \mathbb{Z}_{\geq 0}}} (-1)^{r_3} \binom{r_2+s}{r_2} \binom{r_3+s}{r_3} = \begin{cases} \binom{r_1/2+s}{r_1/2}, & \text{if } r_1 \text{ is even,} \\ 0, & \text{if } r_1 \text{ is odd} \end{cases}$$

($r_1 \in \mathbb{Z}_{\geq 0}$) to the right-hand side of (86), we get the right-hand side of (85). We remark that the above binomial sum identity follows from $(1-x)^{-s-1}(1+x)^{-s-1} = (1-x^2)^{-s-1}$. $\square$

3. Remarks

We give some remarks related to the contents of Section 2:

(R1) By the contents of Subsection 2.2, it seems to be interesting to study multiple series of the form

$$\begin{aligned}
 & \sum_{\substack{0 \leq \lesssim m_1 \lesssim \dots \lesssim_{P_1-1} m_{P_1} \\ \lesssim_{P_1} m_{P_1+1} \lesssim_{P_1+1} \dots \lesssim_{P_2-1} m_{P_2} \\ \vdots \\ \lesssim_{P_{n-1}} m_{P_{n-1}+1} \lesssim_{P_{n-1}+1} \dots \lesssim_{P_n-1} m_{P_n} \lesssim_{P_n} < \infty}} \prod_{i=0}^{n-1} \frac{(\alpha)_{m_{P_i+1}} (\beta)_{m_{P_i+1}}}{m_{P_i+1}!} \frac{m_{P_{i+1}}!}{(\gamma)_{m_{P_i+1}}} \frac{(\gamma)_{m_{P_{i+1}}}}{(\alpha)_{m_{P_{i+1}}} (\beta)_{m_{P_{i+1}}}} \\
 (87) \quad & \times \left\{ \prod_{j=P_i+1}^{P_{i+1}} \frac{1}{(m_j + \alpha)^{a_j} (m_j + \beta)^{b_j} (m_j + \gamma)^{c_j} (m_j + \delta)^{d_j}} \right\},
 \end{aligned}$$

where $n \in \mathbb{Z}_{\geq 1}$; $p_1 \in \mathbb{Z}_{\geq 1}$, $p_2, \dots, p_n \in \mathbb{Z}_{\geq 0}$; $P_0 := 0$, $P_i := p_1 + \dots + p_i$ ($i = 1, \dots, n$); $a_j, b_j, c_j, d_j \in \mathbb{Z}$ such that $a_j + b_j + c_j + d_j \geq 1$ ($j = 1, \dots, P_n - 1$), $a_{P_n} + b_{P_n} + c_{P_n} + d_{P_n} \geq 2$; $\alpha, \beta, \gamma, \delta \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ such that $\operatorname{Re}(\alpha + \beta - \gamma) > 0$. By Theorems 2.2, 2.5 and Corollaries 2.3, 2.4, 2.6, we think that the case $a_j = b_j = 0$ ($j = P_i + 1$; $i = 0, \dots, n-1$), $a_j = b_j = 1$ ($j = P_i + 2, \dots, P_{i+1}$; $i = 0, \dots, n-1$),

$d_j = 0$ ($j = 1, \dots, P_n$) of (87) is particularly interesting. We hope that the multiple series (87) is useful for the study of relations among MHZVs. (The case $P_i = i$ ($i = 1, \dots, n$) of (87) is a MHZV.) We remark that some identities for (87) can be derived from Theorem A. It seems to be also interesting to consider whether the multiple series (87) with $\alpha = \beta = \gamma = \delta$, $\lesssim_{i=\leq}$ ($i = 1, \dots, P_n - 1$) satisfy the same relation as the two-one formula for MZSVs. This problem was deduced from my consideration for the case $\alpha = \beta = \gamma$ of (8) written in [21, (R2)].

(R2) We consider the multiple series

$$Z_n^{\lesssim}(\{k_i\}_{i=1}^n; (\alpha, \beta)) \\ := \sum_{0 \leq m_1 \lesssim \dots \lesssim m_n < \infty} \frac{(\alpha)_{m_1}}{m_1!} \frac{m_n!}{(\alpha)_{m_n+1}} \left\{ \prod_{i=1}^{n-1} \frac{1}{(m_i + \beta)^{k_i}} \right\} \frac{1}{(m_n + \beta)^{k_n-1}},$$

where $n \in \mathbb{Z}_{\geq 1}$, $k_1, \dots, k_{n-1} \in \mathbb{Z}_{\geq 1}$, $k_n \in \mathbb{Z}_{\geq 2}$, $\alpha, \beta \in \mathbb{C}$ such that $\operatorname{Re} \alpha > 0$, $\beta \notin \mathbb{Z}_{\leq 0}$. The multiple series $Z_n^{\lesssim}$ satisfy a sum formula, which generalizes the sum formula for MZVs (see (91) below). A sum formula for $Z_n^{\lesssim}$ can be proved by using the sum formula for $Z_n^{\leq}$ and Hoffman's argument in [15, p. 283] (see (93) below). The proof of the sum formula for $Z_n^{\leq}$ is based on a multiple integral representation of $Z_n^{\leq}$ (see [16], [18, Lemma 2.2]). We remark that these sum formulas can also be proved by using Theorem A (ii) and in a way similar to that of proving Corollary 2.4 (i) and (iii). Indeed, taking $z = w = \gamma$, $s = 2$ in (3), we get the identity

$$(88) \quad \sum_{0 \leq m_1 \leq m_2 < \infty} \frac{(\alpha)_{m_1}}{m_1!} \frac{m_2!}{(\alpha)_{m_2+1}} \frac{(\beta)_{m_1}}{(\beta)_{m_2+1}} \frac{(\gamma)_{m_2}}{(\gamma)_{m_1+1}} \\ = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)(m+\beta)(m+\alpha+\beta-\gamma)} + \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)(m+\beta)(m+\gamma)}$$

for $\alpha, \beta, \gamma \in \mathbb{C}$ such that $\operatorname{Re} \alpha, \operatorname{Re} \beta > 0$, $\operatorname{Re}(\alpha + \beta) > \operatorname{Re} \gamma > 0$. The identity (88) can be rewritten as

$$(89) \quad \sum_{0 \leq m_1 < m_2 < \infty} \frac{(\alpha)_{m_1}}{m_1!} \frac{m_2!}{(\alpha)_{m_2+1}} \frac{(\beta)_{m_1}}{(\beta)_{m_2+1}} \frac{(\gamma)_{m_2}}{(\gamma)_{m_1+1}} \\ = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)(m+\beta)(m+\alpha+\beta-\gamma)}.$$

Differentiating both sides of (89) n times with respect to γ at $\gamma = \beta$ and using the case $m = m_2 - m_1 - 1$, $w = \gamma + m_1 + 1$ of (65), we get the identity

$$(90) \quad \sum_{0 \leq m_1 < l_1 < \dots < l_n < m_2 < \infty} \frac{(\alpha)_{m_1}}{m_1!} \frac{m_2!}{(\alpha)_{m_2+1}} \frac{1}{(m_1 + \beta)(m_2 + \beta)} \left\{ \prod_{i=1}^n \frac{1}{l_i + \beta} \right\} \\ = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^{n+2}(m+\beta)}$$

for $n \in \mathbb{Z}_{\geq 0}$, $\alpha, \beta \in \mathbb{C}$ with $\operatorname{Re} \alpha, \operatorname{Re} \beta > 0$. Further, differentiating both sides of (90) k (≥ 0) times with respect to β , we get the sum formula for $Z_n^<$ ([16], [18, Remark 2.4]):

$$(91) \quad \sum_{\substack{k_1 + \dots + k_n = k+n \\ k_i \in \mathbb{Z}_{\geq 1} (i=1, \dots, n-1), \\ k_n \in \mathbb{Z}_{\geq 2}}} Z_n^<(\{k_i\}_{i=1}^n; (\alpha, \beta)) = \zeta_{1,+}^<(n|k; (\alpha, \beta))$$

for all $k, n \in \mathbb{Z}_{\geq 1}$, $\alpha, \beta \in \mathbb{C}$ such that $\operatorname{Re} \alpha > 0$, $\beta \notin \mathbb{Z}_{\leq 0}$. Similarly, differentiating both sides of (88) n times with respect to β at $\beta = \gamma$ and using the case $m = m_2 - m_1$, $z = \beta + m_1$ of (4), we get the identity

$$(92) \quad \sum_{0 \leq m_1 \leq l_1 \leq \dots \leq l_n \leq m_2 < \infty} \frac{(\alpha)_{m_1}}{m_1!} \frac{m_2!}{(\alpha)_{m_2+1}} \frac{1}{(m_1 + \gamma)(m_2 + \gamma)} \left\{ \prod_{i=1}^n \frac{1}{l_i + \gamma} \right\} \\ = \sum_{i=0}^{n+1} \sum_{m=0}^{\infty} \frac{1}{(m + \alpha)^{n-i+2} (m + \gamma)^{i+1}}$$

for $n \in \mathbb{Z}_{\geq 0}$, $\alpha, \gamma \in \mathbb{C}$ with $\operatorname{Re} \alpha, \operatorname{Re} \gamma > 0$. Further, differentiating both sides of (92) k (≥ 0) times with respect to γ , we get the sum formula for $Z_n^{\leq}$ also:

$$(93) \quad \sum_{\substack{k_1 + \dots + k_n = k+n \\ k_i \in \mathbb{Z}_{\geq 1} (i=1, \dots, n-1), \\ k_n \in \mathbb{Z}_{\geq 2}}} Z_n^{\leq}(\{k_i\}_{i=1}^n; (\alpha, \gamma)) = \sum_{i=0}^{n-1} \binom{k-1+i}{i} \zeta_{1,+}^<(n-i|k+i; (\alpha, \gamma))$$

for all $k, n \in \mathbb{Z}_{\geq 1}$, $\alpha, \gamma \in \mathbb{C}$ such that $\operatorname{Re} \alpha > 0$, $\gamma \notin \mathbb{Z}_{\leq 0}$. Taking $\alpha = \beta = \gamma = 1$ in (91) and (93), we get the sum formulas for MZVs and MZSVs, respectively (Granville [13], Hoffman [15], Zagier); therefore the sum formulas for MZ(S)V s can also be derived by using the basic hypergeometric identity of Andrews [1, Theorem 4].

(R3) In [16], I proved the identity

$$(94) \quad \sum_{\substack{k_1 + \dots + k_n = k \\ k_i \in \mathbb{Z}_{\geq 1} (i=1, \dots, n-1), \\ k_n \in \mathbb{Z}_{\geq 2}}} \zeta_{n,+}^<(\{k_i\}_{i=1}^n; \alpha) \\ = \frac{1}{(k-n-1)!} \sum_{m=0}^{\infty} \frac{1}{(m+1)^n} \frac{\partial^{k-n-1}}{\partial x^{k-n-1}} \left(\frac{(1-x)_m}{(\alpha-x)_{m+1}} \right) \Big|_{x=0}$$

for all $k, n \in \mathbb{Z}$ such that $0 < n < k$, $\alpha \in \mathbb{C}$ with $\operatorname{Re} \alpha > 0$. I remark that the identity (94) gives a relation between the MHZVs $\zeta_{n,+}^<(\{a_i\}_{i=1}^n; \alpha)$ and the multiple series (23) with $z = 1$. (See also [16, Examples].) Indeed, taking $a = c = X = 1 - x$, $b = d = Y = \alpha$, $Z = \alpha - x$, $W = 1$ in Lemma 2.1, we get

the identity

$$\begin{aligned} & \frac{(1-x)_m}{(\alpha-x)_{m+1}} \frac{(\alpha)_m}{(m+1)!} \\ &= \frac{1}{(\alpha-x+m)(1+m)} \sum_{i=0}^m \sum_{0 \leq m_1 < \dots < m_i < m} \prod_{j=1}^i \frac{x(1-\alpha)}{(\alpha-x+m_j)(1+m_j)}. \end{aligned}$$

Using this identity, we get the identity

$$\begin{aligned} & \frac{1}{r!} \frac{\partial^r}{\partial x^r} \left(\frac{(1-x)_m}{(\alpha-x)_{m+1}} \frac{(\alpha)_m}{(m+1)!} \right) \Big|_{x=0} \\ (95) \quad &= \sum_{i=0}^m \sum_{\substack{k_1+\dots+k_{i+1}=r+1 \\ k_j \in \mathbb{Z}_{\geq 1}}} (1-\alpha)^i \sum_{0 \leq m_1 < \dots < m_i < m_{i+1}=m} \prod_{j=1}^{i+1} \frac{1}{(\alpha+m_j)^{k_j}(1+m_j)} \end{aligned}$$

for $m, r \in \mathbb{Z}_{\geq 0}$, $\alpha \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$. By using (95), the right-hand side of (94) can be rewritten as

$$\sum_{i=0}^{k-n-1} \sum_{\substack{k_1+\dots+k_{i+1}=k-n \\ k_j \in \mathbb{Z}_{\geq 1}}} (1-\alpha)^i H_{i+1,+}^{<}(\{k_j\}_{j=1}^{i+1} | \{1\}^i, n; \alpha)$$

($\alpha \in \mathbb{C}$ with $\operatorname{Re} \alpha > 0$), where $H_{n,+}^{<}(\{a_i\}_{i=1}^n | \{b_i\}_{i=1}^n; \alpha)$ denotes the case $z = 1$, $\lesseqgtr$ of the multiple series (23).

I also remark that the identity (94) can also be proved by applying the product of operators $(-1)^{k-n-1} \partial^{(k-n-1)}(w)|_{w=z} \partial^{(n-1)}(\alpha)|_{\alpha=w}$ to the left-hand sides of the identities (69) and (74); the right-hand sides are completely the same. Using (94) and Hoffman's argument in [15, p. 283], I can also get a sum formula for $\zeta_{n,+}^{\leq}(\{a_i\}_{i=1}^n; \alpha)$ similar to (94).

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LINEAR MAPS ON $M_n(\mathbb{C})$ PRESERVING INNER LOCAL SPECTRAL RADIUS ZERO

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Abstract. Let $M_n(\mathbb{C})$ be the algebra of all complex $n \times n$ matrices. Let x_0 be a nonzero vector in $\mathbb{C}^n$. We show that a linear unital map $\phi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ preserves the inner local spectral radius zero at x_0 , if and only if there exists an invertible matrix $A \in M_n(\mathbb{C})$ such that $\phi(T) = ATA^{-1}$.

Keywords: inner local spectral radius, linear preserver.

1. Introduction

Let $\mathcal{B}(X)$ be the algebra of all linear bounded operators on a complex Banach space X , and let $M_n(\mathbb{C})$ be the algebra of $n \times n$ complex matrices. The local resolvent set of an operator $T \in \mathcal{B}(X)$ at a point $x \in X$, denoted by $\rho_T(x)$, is the union of all open subsets $U \subseteq \mathbb{C}$ for which there exists an analytic function $f : U \rightarrow X$ such that $(T - \lambda)f(\lambda) = x$ for all $\lambda \in U$. The local spectrum of T at x is defined by $\sigma_T(x) := \mathbb{C} \setminus \rho_T(x)$ and is a (possibly empty) closed subset of $\sigma(T)$, the usual spectrum of $T \in \mathcal{B}(X)$. The local spectral radius of T at x is defined by :

$$r_T(x) := \limsup_{n \rightarrow \infty} \|T^n x\|^{\frac{1}{n}}.$$

An operator $T \in \mathcal{B}(X)$ is said to have the single-valued extension property (abbreviated SVEP) if for every open subset U of $\mathbb{C}$, the equation $(\lambda - T)f(\lambda) = 0$, $(\lambda \in U)$, has no nontrivial X -valued analytic solution f on U . Note that T has SVEP provided that its point spectrum has an empty interior. In particular, every matrix $T \in M_n(\mathbb{C})$ has SVEP and thus $\sigma_T(x) \neq \emptyset$ for all $T \in M_n(\mathbb{C})$ and all nonzero vectors $x \in \mathbb{C}^n$.

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For a closed subset F of $\mathbb{C}$, the global spectral subspace $\mathcal{X}_T(F)$ of T associated with F is defined by

$$\mathcal{X}_T(F) = \{x \in X : (\lambda - T)f(\lambda) = x \text{ has an analytic solution } f \text{ on } \mathbb{C} \setminus F\}.$$

The inner local spectral radius, $\iota_T(x)$, of T at $x \in X$ is defined by

$$\iota_T(x) := \sup\{r \geq 0 : x \in \mathcal{X}_T(\mathbb{C} \setminus D(0, r))\},$$

where $D(0, r)$ is the open disc centered at the origin with the radius r . The inner local spectral radius coincide with the minimum modulus of $\sigma_T(x)$ provided that T has the single-valued extension property. For further information on spectral and local spectral theory we refer the reader to [1, 12].

An active research topic in matrix theory is the linear preserver problems (LPP) that deal with the characterization of linear operators on matrix spaces with some special properties such as leaving certain functions, subsets or relations invariant. Recently many authors have devoted their studies to maps that preserve local spectral quantities; see [1, 3, 4, 5, 6, 8, 9, 10].

In [3], Bourhim characterized continuous surjective linear maps on $\mathcal{B}(X)$ preserving the inner local spectral radius at a fixed nonzero vector in $\mathbb{C}^n$.

recently, M. Ech-chrif El Kettani and H. Benbouziane, in [7], described surjective additive maps on $\mathcal{B}(X)$ which preserve the operators of inner local spectral radius zero at all points of X . They showed that a surjective additive maps $\varphi : \mathcal{B}(X) \rightarrow \mathcal{B}(X)$ satisfying

$$\iota_{\varphi(T)}(x) = 0 \Leftrightarrow \iota_T(x) = 0 \text{ for all } x \in X \text{ and } T \in \mathcal{B}(X),$$

if and only if there is a nonzero scalar $c \in \mathbb{C}$ such that $\varphi(T) = cT$, for all $T \in \mathcal{B}(X)$. This result has been extended in [11] to the non additive setting where T. Jari gave the form all surjective not necessarily linear or additive $\varphi : \mathcal{B}(X) \rightarrow \mathcal{B}(X)$ satisfying $\iota_{\varphi(T)-\varphi(S)}(x) = 0 \Leftrightarrow \iota_{T-S}(x) = 0$ for all $T, S \in \mathcal{B}(X)$ and $x \in X$.

The aim of this paper is to study the linear unital map from $M_n(\mathbb{C})$ into its self which preserve operators of inner local spectral radius zero a fixed nonzero vector x_0 in $\mathbb{C}^n$. We show that a unitaly linear map $\phi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ satisfies

$$(1.1) \quad \iota_T(x_0) = 0 \iff \iota_{\phi(T)}(x_0) = 0, (T \in M_n(\mathbb{C})),$$

if and only if there exists an invertible matrix $A \in M_n(\mathbb{C})$ such that $Ax_0 = x_0$ and $\phi(T) = ATA^{-1}$, for all $T \in M_n(\mathbb{C})$.

2. Notations and preliminaries

In this section, we introduce some notations and preliminary results which will be used to prove our main results in the next section. The duality between the

Banach space X and its dual, X^* , will be denoted by $\langle \cdot, \cdot \rangle$. For an $x \in X$ and $f \in X^*$, we denote as usual by $x \otimes f$ the rank one operator on X given by $z \mapsto \langle z, f \rangle x$.

The following lemma summarizes some basic properties of the local spectrum.

Lemma 2.1. *Let $T \in \mathcal{B}(X)$.*

- (1) $\sigma_{su}(T) = \bigcup_{x \in X} \sigma_T(x)$.
- (2) *If $x = y + z$, then $\sigma_T(x) \subseteq \sigma_T(y) \cup \sigma_T(z)$.*

Proof. See [1, 12]. □

An operator $T \in \mathcal{B}(X)$ is said to have the single-valued extension property at $\lambda_0 \in \mathbb{C}$, abbreviated T has the SVEP at λ_0 , if for every neighbourhood U of λ_0 the only analytic function $f : U \rightarrow X$ which satisfies the equation

$$(\lambda - T)f(\lambda) = 0 \quad (\lambda \in U)$$

is the constant function $f \equiv 0$. The operator T is said to have the SVEP if T has the SVEP at every $\lambda \in \mathbb{C}$. Evidently, T has SVEP at every $\lambda \in \mathbb{C} \setminus \text{int}(\sigma_p(T))$, where $\text{int}(\sigma_p(T))$ denotes the interior of the point spectrum of T . In particular, if $\sigma_p(T)$ has empty interior, then T has SVEP. For example, every finite rank operator has SVEP.

In the finite dimensional case, every T belonging to $M_n(\mathbb{C})$ has the single-valued extension property. we can see, for example, the books [1, 12] for further information on the local spectral theory.

We shall make extensive use of the following result.

Lemma 2.2 ([1, Theorem 2.22]). *Let $T \in \mathcal{B}(X)$, let $\lambda \in \mathbb{C}$ and let $x \in \ker(T - \lambda)$. then $\sigma_T(x) \subset \{\lambda\}$. If, further, $x \neq 0$ and T has SVEP at λ , then $\sigma_T(x) = \{\lambda\}$.*

Given an operator $T \in \mathcal{B}(X)$, for every closed subset Ω of $\mathbb{C}$, the global spectral subspace $\mathcal{X}_T(\Omega)$ of T associated with Ω is defined by

$$\mathcal{X}_T(\Omega) = \{x \in X : (\lambda - T)f(\lambda) = x \text{ has an analytic solution } f \text{ on } \mathbb{C} \setminus \Omega\}.$$

It is a T -hyperinvariant subspace but not necessarily closed. Recall that the local spectral radius of T at x coincides with $\inf\{r \geq 0 : x \in \mathcal{X}_T(\overline{D}(0, r))\}$; see [12, Proposition 3.3.13]. Here, $\overline{D}(0, r)$ (resp. $D(0, r)$) denotes the closed (resp. the open) disc centered at the origin with radius r . Analogously, the *inner local spectral radius*, $\iota_T(x)$, of T at $x \in X$ is defined by

$$\iota_T(x) := \sup\{r \geq 0 : x \in \mathcal{X}_T(\mathbb{C} \setminus D(0, r))\},$$

and coincides with the minimum modulus of $\sigma_T(x)$ provided that T has the single-valued extension property; see [13]. For more details on the local spectral theory, we refer the reader to the remarkable books [1, 12].

Next lemma, quoted from [13], summarizes some elementary properties of the inner local spectral radius.

Lemma 2.3. *For an operator $T \in \mathcal{B}(X)$ and a vector $x \in X$, the following hold.*

- (a) $x = 0$ is the only vector for which $\iota_T(x) = \infty$.
- (b) $\iota_T(x) = 0$ if and only if $0 \in \sigma_T(x)$.
- (c) $\iota_T(x) \leq \min\{|\lambda| : \lambda \in \sigma_T(x)\}$. The equality holds if T has SVEP.
- (d) For every bijective operator $A \in \mathcal{B}(X, Y)$, we have $\iota_{ATA^{-1}}(Ax) = \iota_T(x)$.

Let $\mathcal{K}(C)$ denote the set of all nonempty compact subsets of $\mathbb{C}$, endowed with the Hausdorff metric. The following lemma proved by Augetit [2].

Lemma 2.4. *Let X be a finite dimensional normed space. Then the map*

$$\sigma : T \in \mathcal{B}(X) \rightarrow \sigma(T) \in \mathcal{K}(C)$$

is continuous.

Proof. It is a consequence of [2, Corollary 3.4.5]. $\square$

Let x_0 be a nonzero fixed vector in $\mathbb{C}^n$, we denote by $\mathcal{M}_{x_0}$ the set given by $\mathcal{M}_{x_0} := \{T \in M_n(\mathbb{C}) : \{Tx_0, T^2x_0, \dots, T^nx_0\} \text{ is a basis of } \mathbb{C}^n \text{ and } |\sigma(T)| = n\}$, where $|\sigma(T)|$ is the number of elements of $\sigma(T)$.

The following lemma that summarizes some important properties of $\mathcal{M}_{x_0}$, which plays an important role in the proof of the main theorem.

Lemma 2.5. *Let x_0 be a nonzero fixed vector in $\mathbb{C}^n$. The set $\mathcal{M}_{x_0}$ is dense in $M_n(\mathbb{C})$ and $\sigma(T) = \sigma_T(x_0)$ for all $T \in \mathcal{M}_{x_0}$.*

Proof. See [5, Lemma 2.5]. $\square$

We close this section with this lemma in which we give a characterization of the zero matrix of $M_n(\mathbb{C})$ in terms of inner local spectral radius.

Lemma 2.6. *Let x_0 be a nonzero fixed vector in $\mathbb{C}^n$. Let A be a matrix in $M_n(\mathbb{C})$. The following conditions are equivalent:*

- (1) $A = 0$.
- (2) $\iota_{A+T}(x_0) = 0$ for every nilpotent $T \in M_n(\mathbb{C})$.

Proof. We only need to prove (2) $\Rightarrow$ (1). indeed, let T be a nilpotent operator in $M_n(\mathbb{C})$ such that $\iota_{A+T}(x_0) = 0$. By Lemma 2.3 we have $0 \in \sigma_{A+T}(x_0) \subseteq \sigma(A+T)$, which implies, by [14, Proposition 5.2], that $A = 0$; as desired. $\square$

3. Main results

Theorem 3.1. *Fix a nonzero vector fixed x_0 in $\mathbb{C}^n$. A unitaly linear map $\phi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ satisfies*

$$(3.1) \quad \iota_T(x_0) = 0 \iff \iota_{\phi(T)}(x_0) = 0, \text{ for all } T \in M_n(\mathbb{C}),$$

if and only if there exists an invertible matrix $A \in M_n(\mathbb{C})$ such that $Ax_0 = x_0$ and $\phi(T) = ATA^{-1}$, for all $T \in M_n(\mathbb{C})$.

Proof. The "if" part is obvious by Lemma 2.3, then we only need to check the necessity condition. So, assume that ϕ is a unitaly linear map from $M_n(\mathbb{C})$ into its self satisfying (3.1). Let us first show that ϕ is bijective. To do so, let $R \in \mathcal{M}_n$ such that $\phi(R) = 0$. For every nilpotent operator $T \in M_n(\mathbb{C})$, we have $\iota_T(x_0) = 0$ and $\iota_{\phi(T)}(x_0) = 0$. On the other hand $\iota_{\phi(T)+\phi(R)}(x_0) = \iota_{\phi(T)}(x_0) = 0$ therefore $\iota_{T+R}(x_0) = 0$ for every nilpotent operator T . By [14, Propositio 5.2], we have $R = 0$ which implies that ϕ is injective. It is in fact a bijective map since $M_n(\mathbb{C})$ is a finite dimensional space.

Next, we prove that

$$(3.2) \quad \sigma(T) \subseteq \sigma(\phi(T)), \text{ for all } T \in M_n(\mathbb{C}).$$

Let $T \in \mathcal{M}_{x_0}$. By Lemma 2.1 there exists x such that $\sigma_{su}(T) = \sigma_T(x)$ and $x = \alpha_1 x_0 + \dots + \alpha_{n-1} T^{n-1} x_0$. It follows from Lemma 2.5 that

$$\sigma(T) = \sigma_T(x) \subseteq \sigma_T(x_0) \cup \dots \cup \sigma_T(T^{n-1} x_0) \subseteq \sigma_T(x_0).$$

Therefore $\sigma(T) = \sigma_T(x_0)$.

Let $\lambda \in \sigma(T) = \sigma_T(x_0)$ we have $0 \in \sigma_{T-\lambda}(x_0)$, thus $\iota_{T-\lambda}(x_0) = 0$. By hypothesis 3.1, $\iota_{\phi(T)-\lambda}(x_0) = 0$ and $0 \in \sigma_{\phi(T)-\lambda}(x_0)$. This implies that $\lambda \in \sigma(\phi(T))$ and therefore

$$\sigma(T) \subset \sigma(\phi(T)), \text{ for all } T \in \mathcal{M}_{x_0}.$$

Now, let $T \in M_n(\mathbb{C})$, by Lemma 2.5, there exists a sequence (T_p) in $\mathcal{M}_{x_0}$ such that $T_p \rightarrow T$. The continuity of ϕ and the continuity of the spectrum on $M_n(\mathbb{C})$ (Lemma 2.5) imply that

$$\sigma(T) = \lim \sigma(T_p) \subseteq \lim \sigma(\phi(T_p)) = \sigma(\phi(T)), \text{ for all } T \in M_n(\mathbb{C}).$$

Since ϕ is bijective and ϕ^{-1} satisfies 3.1, with similar reasoning, we get the reverse inclusion. And therefore

$$\sigma(\phi(T)) = \sigma(T), \text{ for all } T \in M_n(\mathbb{C}).$$

By [15, Theorem 2], there exists an invertible matrix $A \in M_n(\mathbb{C})$ such that either

$$(3.3) \quad \phi(T) = ATA^{-1}, \text{ (} T \in M_n(\mathbb{C}) \text{)}$$

or

$$(3.4) \quad \phi(T) = AT^t A^{-1}, \quad (T \in M_n(\mathbb{C})),$$

where T^t denotes, as usual, the transpose of $T \in M_n(\mathbb{C})$.

The second form 3.4 is not possible. Indeed, assume for sake of contradiction that ϕ takes a such form. Let y be a non zero vector such that x_0 and y are linearly independent and $\langle y, A^{-1}x_0 \rangle = 1$, and let f be a linear functional on $\mathbb{C}^n$ such that $f(x_0) = 0$ and $f(y) = 1$. Set $T := y \otimes f$. Note that $Tx_0 = 0$ and thus $i_T(x_0) = 0$. On the other hand $T^t(A^{-1}x_0) = f$ and $T^t f = f$. Then

$$\sigma_{T^t}(A^{-1}x_0) = \sigma_{T^t}(T^t A^{-1}x_0) = \sigma_{T^t}(f) = \{1\}.$$

Which proves, by Lemma 2.3, that

$$i_{\phi(T)}(x_0) = i_{AT^t A^{-1}}(x_0) = i_{T^t}(A^{-1}x_0) \neq 0.$$

This contradiction shows that φ takes the form (3.3).

It remains to prove that $Ax_0 = x_0$. Assume on the contrary that $A^{-1}x_0$ and x_0 are linearly independent. We could find a linear functional $f : \mathbb{C}^n \rightarrow \mathbb{C}$ such that $f(A^{-1}x_0) = 0$ and $f(x_0) = 1$. Then for $T := x_0 \otimes f$ we have $Tx_0 = x_0$. Thus, $i_T(x_0) = 1$. On the other hand $ATA^{-1}x_0 = 0$ then $i_{\phi(T)}(x_0) = 0$, arriving to a contradiction. We conclude that $Ax_0 = \mu x_0$ for some nonzero scalar μ . Dividing A by μ if necessary, we may assume that $Ax_0 = x_0$. This finishes the proof. $\square$

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AN EQUIVALENT DEFINITION OF A $\mathcal{C}$ -GROUP

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Abstract. In 1974, Fattahi [2] classified finite non-nilpotent groups in which every subgroup is either normal or abnormal. Surely, it is an interesting topic to define and classify a bigger class of groups containing those classified by Fattahi. In 2012, the first author etc defined a $\mathcal{C}$ -group. A finite group G is said to be a $\mathcal{C}$ -group if for each divisor d of the order of G , G always contains a subgroup H of order d such that H is either normal or abnormal in G . The class of $\mathcal{C}$ -groups is really a class of groups bigger than those classified by Fattahi. During the first author etc investigate the structure of $\mathcal{C}$ -groups, they found that some property cannot hold if 'normal' is changed into 'subnormal'. So the authors of this paper is motivated to find an equivalent definition of a $\mathcal{C}$ -group, which can be described by 'subnormal' and 'abnormal'. A $\mathcal{C}_1$ -group is defined, some good properties of a $\mathcal{C}_1$ -group are given, then it is proved that a $\mathcal{C}_1$ -group is equivalent to a $\mathcal{C}$ -group. At last, as an example of effectiveness of a $\mathcal{C}_1$ -group, the necessary and sufficient conditions of a semi-product of a p -group and a p' -group to be a $\mathcal{C}$ -group is given.

Keywords: normal, abnormal, subnormal, a $\mathcal{C}$ -group, a $\mathcal{C}_1$ -group.

1. Introduction

All groups considered in this paper are finite. Our notation and terminology are standard, see, for example, Robinson [6].

In 1974, Fattahi [2] classified finite non-nilpotent groups in which every subgroup is either normal or abnormal. Surely, it is an interesting topic to define and classify a bigger class of groups containing those classified by Fattahi. In 2012, Liu, Li and He [5] was motivated by this and introduced the following definition: A group G is called a $\mathcal{C}$ -group if for each divisor d of the order of G , G contains a subgroup H of order d such that H is either normal or abnormal in G . Of course, a finite group with every subgroup is either normal or abnormal is a $\mathcal{C}$ -group. So the class of $\mathcal{C}$ -groups is really a class of groups bigger than those classified by Fattahi. In [5], it is investigated the structure of finite groups

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based on the assumption that some subgroups are $\mathcal{C}$ -groups and the following equivalence is proved:

Theorem 1.1 ([5, Corollary 3.2]). *The following statements for a group G are equivalent:*

- (1) G is a $\mathcal{C}$ -group.
- (2) For each subgroup B of G , there exists a subgroup H of G of order $|B|$ in G such that H is either normal or abnormal in G .

Notice that the above theorem doesn't hold if "normal" is replaced by "subnormal" in Statement (2). For example, the alternating group of degree 4 is a counterexample. This motivates us to investigate if there exists a kind of equivalent definition of $\mathcal{C}$ -group, which is described just by using "subnormal" and "abnormal". To reach this target, we give the following definition:

Definition 1.2. *A group G is called a $\mathcal{C}_1$ -group if for each subgroup B of a group G , there exists a subgroup H of order $|G : B|$ in G such that H is either subnormal or abnormal in G .*

By using the definition of a $\mathcal{C}_1$ -group, some good properties of a $\mathcal{C}_1$ -group are given as follows:

Theorem 1.3. *The following statements for a group G are equivalent:*

- (a) G is a $\mathcal{C}_1$ -group.
- (b) *Either G is nilpotent or G satisfies the following three conditions:*
 - (b-1) G is supersolvable.
 - (b-2) $G/F(G)$ is cyclic of order p , where p is the smallest prime dividing the order of G .
 - (b-3) $G/O_p(G)$ is a Frobenius group with Frobenius complement $P/O_p(G)$, where P is a Sylow p -subgroup of G and $P/O_p(G)$ is cyclic of order p .

Then, we get the following equivalence by using above theorem:

Theorem 1.4. *A group G is a $\mathcal{C}$ -group if and only if G is a $\mathcal{C}_1$ -group.*

Although a $\mathcal{C}_1$ -group and a $\mathcal{C}$ -group are equivalent, we find that it is easier to get structure of a $\mathcal{C}$ -group by using definition of $\mathcal{C}_1$ -group. At last, as an example of effectiveness of a $\mathcal{C}_1$ -group, the necessary and sufficient conditions of a semi-product of a p -group and a p' -group to be a $\mathcal{C}_1$ -group is given as follows:

Theorem 1.5. *Let P be a p -group that acts non-trivially on a p' -group M . Then the semidirect product $P \ltimes M$ is a $\mathcal{C}$ -group if the following conditions hold:*

- (1) *The automorphism group induced by P on M is a cyclic group $\langle \alpha \rangle$ of order p .*
- (2) *α is a fixed-point-free automorphism.*
- (3) *α is a semi-power-automorphism.*

The converse is true for a non-nilpotent $\mathcal{C}_1$ -group.

2. Preliminaries

In this section we present some lemmas which are required in Section 3. A subgroup H of a group G is said to be abnormal in G if $g \in \langle H, H^g \rangle$ for all g in G . A subgroup H is said to be pronormal in a group G if for all g in G the subgroups H and H^g are conjugate in $\langle H, H^g \rangle$.

Lemma 2.1. *Let H be a subgroup of a group G . Then the following statements are true:*

- (a) *If H is abnormal in G and $H \leq K \leq G$, then K is also abnormal in G .*
- (b) *H is abnormal in G if and only if H is self-normalizing and pronormal in G .*
- (c) *Let $N \trianglelefteq G$ and $N \leq H$. Then H is abnormal in G if and only if H/N is abnormal in G/N .*
- (d) *If $A \leq B < G$ and B is subnormal in G , then A is not abnormal in G .*
- (e) *A subnormal Sylow p -subgroup is normal and in a solvable group a subnormal Hall subgroup is normal.*

Proof. Statements (a) and (b) hold by [1, Chapter 1, 6.20 and 6.21]. Statement (c) follows by directly verifying the definition. Statement (d) also follows from Statements (a) and (b). $\square$

Lemma 2.2. *Let $G = H \ltimes M$ be the semidirect product of the non-trivial subgroup H with the normal subgroup M . Then the following statements are equivalent:*

- (a) *G is a Frobenius group with Frobenius kernel M .*
- (b) *H is a group of fixed-point-free automorphisms of M .*

Proof. See [4, V, 8.5 Satz]. $\square$

Lemma 2.3 ([5, Theorem 3.1]). *The following statements for a group G are equivalent:*

- (a) *G is a $\mathcal{C}$ -group.*

- (b) *Either G is nilpotent or G satisfies the following three conditions:*
- (b-1) *G is supersolvable,*
 - (b-2) *$G/F(G)$ is cyclic of order p , where p is the smallest prime divisor of the order of G , and*
 - (b-3) *$G/O_p(G)$ is a Frobenius group whose Frobenius complement $P/O_p(G)$ is cyclic of order p , where P is a Sylow p -subgroup of G .*

Lemma 2.4 ([3, Theorem 1]). *A group G is nilpotent if and only if for each divisor d of the order of G there exists a normal subgroup of order d .*

Note that the above lemma doesn't hold if the word "nilpotent" is replaced by "supersolvable". But we have:

Lemma 2.5 ([5, Lemma 2.5]). *Let M be a normal nilpotent subgroup of a group G . If G is supersolvable, then for every divisor d of the order of M there exists a subgroup H of M of order d such that H is normal in G .*

Let α be an automorphism of a group G . Then we say that α is a semi-power-automorphism of G if there exist elements $a_1, a_2, \dots, a_n$ which generate G such that α maps a_i to a power of a_i for all $i \in \{1, 2, \dots, n\}$ (see [5, Definition 1.4]).

Lemma 2.6 ([5, Lemma 2.2]). *The following statements for a group G are true:*

- (a) *Suppose that $G = H\langle x \rangle$, where H is a normal nilpotent subgroup of G . If x induces a semi-power-automorphism on $H/\Phi(H)$, then G is supersolvable.*
- (b) *If N is a normal nilpotent subgroup of a supersolvable group G , then each element x of G induces a semi-power-automorphism on $N/\Phi(N)$.*

3. Proofs of theorems

In this section, we prove Theorems 1.3, 1.4, and 1.5:

The proof of the Theorem 1.3: (a) $\Rightarrow$ (b): Suppose that G is not nilpotent. Take the primary decomposition of $|G|$ as

$$|G| = p_1^{m_1} p_2^{m_2} \cdots p_s^{m_s}, m_i \geq 1.$$

where $p_1 < p_2 < \cdots < p_s$. We split the proof into the following steps:

- (1) G is solvable.

Let P_i be a Sylow p_i -subgroup of G , where $i \in \{1, \dots, s\}$. Then from our assumption there exists a subgroup H of G with order $|G : P_i|$. This implies that H is a Hall p_i' -subgroup of G . By a well-known theorem of P. Hall [6, 9.1.8], we can see that G is solvable.

(2) G is p_1 -nilpotent and the normal p_1 -complement is nilpotent.

Since G has a subgroup of order p_1 , G has a subgroup K of index p_1 which is either subnormal or abnormal. By using the permutation representation of G on K , we see that G/K_G is isomorphic to a subgroup of the symmetric group S_{p_1} of degree p_1 . The minimality of p_1 forces $K = K_G$. It follows that K is normal in G . Let $q \in \{p_2, \dots, p_s\}$ and H be a Hall q' -subgroup of G . Then, by assumption, there exists a Sylow q -subgroup S of G which is either abnormal or subnormal in G . Since K is normal in G , Lemma 2.1 (d) implies that S is subnormal. Since S is a Sylow q -subgroup of G , Lemma 2.1 (e) gives S is normal. Hence the product of all the Sylow q -subgroups, $q \in \{p_2, \dots, p_s\}$ is a normal p_1 -complement and this normal p -complement is nilpotent.

(3) Let P be a Sylow p_1 -subgroup of G , then P is abnormal in G .

Let B be a Hall p_1' -subgroup of G . Then from our assumption there exists a subgroup H of order $|G : B|$ in G such that H is either subnormal or abnormal in G . If H is subnormal in G , then it follows from Lemma 2.1 (e) that H is normal in G , whence $H = P$. By Statement (2), we have that G is nilpotent. This contradiction shows that H is abnormal in G . By Lemma 2.1, we obtain that P is abnormal in G .

(4) $|G/F(G)| = |P/O_{p_1}(G)| = p_1$.

If $m_1 = 1$, then Statement (4) follows from Statements (2) and (3). Hence we may assume that $m_1 \geq 2$. Let B_1 be a subgroup of G of order p_1 and M a normal p_1 -complement of G . Then B_1M is a subgroup of G . Set $B = B_1M$. By hypothesis, there exists a subgroup H of G of order $|G : B| = p_1^{m_1-1}$ such that H is either subnormal or abnormal in G . By Sylow's theorem, we have that $H < P^x$ for some $x \in G$, whence $H < N_G(H)$. It follows from Lemma 2.1 (b) that H is subnormal in G and so $H \leq O_{p_1}(G)$. On the other hand, by Statement (3), P is abnormal in G , which means that $O_{p_1}(G) < P$. Therefore $H = O_{p_1}(G)$ by comparing the orders and hence $P/O_{p_1}(G)$ is a cyclic group of order p_1 . The above proofs and Statement (2) are combined to give that $G/F(G)$ is also a cyclic group of order p_1 .

(5) $G/O_{p_1}(G)$ is a Frobenius group with Frobenius complement $P/O_{p_1}(G)$.

Set $\bar{G} = G/O_{p_1}(G)$. Then $\bar{G} = \bar{P} \ltimes F(\bar{G})$. If $\bar{G}$ is not a Frobenius group, then there is a non-trivial element $\bar{y}$ in $\bar{P}$ such that $C_{F(\bar{G})}(\bar{y}) \neq \bar{1}$. We can select a non-trivial element $\bar{x} \in C_{F(\bar{G})}(\bar{y})$, then $[\bar{x}, \bar{y}] = \bar{1}$ and so $[\bar{x}, \bar{P}] = \bar{1}$ as $|\bar{P}| = p_1$. On the other hand, by Statement (3) and Lemma 2.1 (c), we can see that $\bar{P}$ is abnormal in $\bar{G}$. This contradiction shows that $C_{F(\bar{G})}(\bar{y}) = \bar{1}$ and hence $\bar{P}$ induces a fixed-point-free automorphism on $F(\bar{G})$. It follows from Lemma 2.2 that $\bar{G}$ is a Frobenius group with Frobenius complement $\bar{P}$.

(6) G is supersolvable.

In view of Statement (2), we can see that $G/M \cong P$, where M is the normal p_1 -complement of G . Hence we only need to show that every chief factor of G below M is cyclic in order to conclude that G is supersolvable. Now, let K/L be a chief factor of G such that $K \leq M$. Then K/L is an elementary

abelian q -group for some prime $q \in \pi(M)$. By [4, VI, 5.4 Satz], we can see that $F(G) \leq C_G(K/L)$. Then the automorphism group induced by G on K/L is of order p_1 . We may assume without loss generality that $q = p_2$. By Statement (2), M is nilpotent and so M contains a subgroup B of order $p_2^{m_2-1} p_3^{m_3} \cdots p_s^{m_s}$. By hypothesis, there exists a subgroup H of order $|G : B| = p_1^{m_1} p_2 = p_1^{m_1} q$ in G such that H is either subnormal or abnormal in G . Then H contains a Sylow p_1 -subgroup of G , say P . Applying Statement (3), P is abnormal in G and hence H is abnormal in G by Lemma 2.1 (a) too. Let Q be a Sylow q -subgroup of H . Then Q is of order q and is contained in M by Statement (2). It follows that $Q = H \cap M \trianglelefteq H$. If $C_H(Q) = H$, then Q centralizes P , in contradiction to the fact that P is abnormal in G . Thus $C_H(Q) < H$. Since $[O_{p_1}(G), Q] = 1$, we obtain that $H/C_H(Q)$ is cyclic of order p_1 . It follows that $H/C_H(Q)$ is isomorphic to a subgroup of $\text{Aut}(Q)$, a cyclic group of order dividing $q-1$. This means that $p_1 | (q-1)$. According to [7, Chapter 1, Lemma 1.3], we conclude that K/L is cyclic of order q , as desired.

(b) $\Rightarrow$ (a): If G is nilpotent, then G is a $\mathcal{C}_1$ -group by Lemma 2.4. Hence we may assume that G is not nilpotent. Let B be a subgroup of G of index p_1^{mt} , where $(p_1, t) = 1$ and $m \geq 0$. We proceed in two cases:

Case 1: $m < m_1$.

By Statement (b-2), we can see that p_1^{mt} is a divisor of the order of $F(G)$. In view of Statement (b-1) and Lemma 2.5, there exists a subgroup of order p_1^{mt} in $F(G)$ such that it is normal in G , as desired.

Case 2: $m = m_1$.

It is clear that t is a divisor of the order of $F(G)$. By (b-1) and Lemma 2.5, there exists a subgroup H of order t in $F(G)$ such that H is normal in G . Then the subgroup PH is of order p_1^{mt} . Set $\bar{G} = G/O_{p_1}(G)$, then $\bar{G} = \bar{P} \rtimes F(\bar{G})$. If $\bar{P}$ is not self-normalizing, then there is a non-trivial element $\bar{x}$ in $N_{\bar{G}}(\bar{P})$ but not in $\bar{P}$. It follows from the theorem of Burnside [4, IV, 2.6 Hauptsatz and 2.7 Satz] that $\langle \bar{x} \rangle \bar{P}$ is p -nilpotent and so $[\bar{x}, \bar{P}] = \bar{1}$. The Statement (b-3) provides a contradiction and thus $N_{\bar{G}}(\bar{P}) = \bar{P}$. We have that $\bar{P}$ is pronormal in $\bar{G}$ since $\bar{P}$ is a Sylow subgroup of $\bar{G}$. By Lemma 2.1, we obtain that $\bar{P}$ is abnormal in $\bar{G}$. According to Lemma 2.1 (c), we can see that P is abnormal in G and so is PH by Lemma 2.1 (a), as desired.

The Statement (a) is now completely proved by means of the previous cases.

This completes the proof of the theorem. $\square$

The proof of the Theorem 1.4: This is transparent from Theorem 1.3 and Lemma 2.3. $\square$

The proof of the Theorem 1.5: Let $G = P \rtimes M$. We assume that G is a $\mathcal{C}_1$ -group. We can assume that G is not nilpotent and G satisfies Statements (b-1), (b-2) and (b-3) of Theorem 1.3. By (b-2), we have $M \leq F(G)$ and so M is nilpotent.

Now, Statement (b-1) and Lemma 2.6 imply Statements (1) and (3). Statement (2) follows from Statement (b-3).

Conversely, by the theorem of Thompson [6, Theorem 10.5.4], we can see that M is nilpotent. Let $K = \langle x \in P \mid a^x = a, a \in M \rangle$. Then P/K is of order p by hypothesis and so $K = O_p(G)$. Thus $F(G) = KM$ and $G/F(G)$ is cyclic of order p . It follows that $G/O_p(G)$ is a Frobenius group, whose Frobenius complement $P/O_p(G)$ is cyclic of order p . Applying Lemma 2.6, we can see that G is supersolvable. Therefore G is a $\mathcal{C}_1$ -group by Theorem 1.3. $\square$

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STUDY OF EXTENDED WEYL k -FRACTIONAL INTEGRAL VIA CHEBYSHEV INEQUALITIES

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Abstract. In this paper, we introduce the extended Weyl k -fractional integral and then present its results and some inequalities. These results and inequalities hold true for: (i) k -Weyl fractional integral when $s \rightarrow 0$; (ii) Weyl fractional integral when $s \rightarrow 0$ and $k \rightarrow 1$.

Keywords: extended Weyl k -fractional integral, Mellin transform, Chebyshev inequalities.

1. Introduction

Considerable work has been done in recent years on fractional integrals because of their applications in many fields of science and technology. The fractional integrals are powerful tools in applied mathematics to solve many problems

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from different fields of science and engineering. By the help of fractional integrals many useful results are found in mathematical physics, thermodynamics, control theory, hydrology, astrophysics and bioengineering.

The objective and motivation of this work is to find the generalized results of the Weyl fractional integrals and to highlight their applications. Many authors had studied fractional integrals [1-21].

Diaz and Pariguan [7] paved the way for the extensions in fractional calculus by introducing the k -gamma and k -beta functions. Now, many new results are introduced which are based on these functions. They defined k -gamma function as

$$(1) \quad \Gamma_k(u) = \int_0^\infty e^{-\frac{x^k}{k}} x^{u-1} dx = k^{k-1} \Gamma\left(\frac{u}{k}\right), \operatorname{Re}(u) > 0,$$

They defined the k -beta function as

$$(2) \quad B_k(u, v) = \frac{1}{k} \int_0^1 z^{\frac{u}{k}-1} (1-z)^{\frac{v}{k}-1} dz, \operatorname{Re}(u) > 0, \operatorname{Re}(v) > 0,$$

From (1) and (2), we have

$$(3) \quad \Gamma(u) = \lim_{k \rightarrow 1} \Gamma_k(u), \quad \Gamma_k(k) = 1, \quad \Gamma_k(u+k) = u \Gamma_k(u).$$

and

$$(4) \quad B_k(c, d) = \frac{\Gamma_k(c) \Gamma_k(d)}{\Gamma_k(c+d)} = \frac{1}{k} B\left(\frac{c}{k}, \frac{d}{k}\right).$$

Romero and Luque [15] have used the idea of Diaz and Pariguan to define the k -Weyl fractional integral as

$$(5) \quad W_k^\alpha[f(x)] = \frac{1}{k \Gamma_k(\alpha)} \int_x^\infty (t-x)^{\frac{\alpha}{k}-1} f(t) dt, \quad t > 0.$$

Here, we introduce the extended weyl k -fractional integral [(k, s) - weyl fractional integral] of order α . Let f be continuous on $[0, \infty)$ and $\alpha \in (0, 1)$, $k \in (0, \infty)$, $s > -1$. Then $\forall x \geq 0$, the extended k -weyl fractional integral is defined by

$$(6) \quad {}^s_k W^\alpha[f(x)] = \frac{(s+1)^{1-\frac{\alpha}{k}}}{k \Gamma_k(\alpha)} \int_x^\infty (t^{s+1} - x^{s+1})^{\frac{\alpha}{k}-1} t^s f(t) dt, \quad t > 0.$$

Note: (i) For $s \rightarrow 0$, k -Weyl fractional integral is recaptured.

(ii) For $s \rightarrow 0$ and $k \rightarrow 1$, usual Weyl fractional integral is recaptured.

For a real scalar function g , the Mellin transform is dened by

$$(7) \quad g^*(s_1) = M[g(x)] = \int_0^\infty x^{s_1-1} g(x) dx, \operatorname{Re}(s_1) > 0.$$

whenever the integral exists and s_1 is parameter.

The introduction of the extended Weyl k -fractional integral and the study of its properties and results will help many researchers in solving the differential as well as integral problems. These results will be very useful in solving the problems as existing in the paper texture enhancement for medical images based on fractional differential masks [12].

2. Main results and discussion

Theorem 2.1 (semi group property and commutative law). *Let f be continuous on $[0, \infty)$ and $\alpha, \beta \in (0, 1), k \in (0, \infty), s > -1$. Then $\forall x \geq 0$*

$$(8) \quad {}^s_k W^\alpha \left[{}^s_k W^\beta f(x) \right] = {}^s_k W^{\alpha+\beta} f(x) = {}^s_k W^\beta \left[{}^s_k W^\alpha f(x) \right].$$

Proof. Using the result (6) in the LHS of the equation (8), we have

$$(9) \quad {}^s_k W^\alpha \left[{}^s_k W^\beta f(x) \right] = \frac{(s+1)^{2-\frac{\alpha}{k}-\frac{\beta}{k}}}{k^2 \Gamma_k(\alpha) \Gamma_k(\beta)} \int_x^\infty (t^{s+1} - x^{s+1})^{\frac{\alpha}{k}-1} t^s \cdot \left[\int_t^\infty (u^{s+1} - t^{s+1})^{\frac{\beta}{k}-1} u^s f(u) du \right] dt,$$

Using Fubini's theorem, we get

$$(10) \quad {}^s_k W^\alpha \left[{}^s_k W^\beta f(x) \right] = \frac{(s+1)^{2-\frac{\alpha}{k}-\frac{\beta}{k}}}{k^2 \Gamma_k(\alpha) \Gamma_k(\beta)} \int_x^\infty u^s f(u) \cdot \left[\int_x^u (t^{s+1} - x^{s+1})^{\frac{\alpha}{k}-1} (u^{s+1} - t^{s+1})^{\frac{\beta}{k}-1} t^s dt \right] du,$$

Substituting $z = \frac{u^{s+1}-t^{s+1}}{u^{s+1}-x^{s+1}}$

$$(11) \quad {}^s_k W^\alpha \left[{}^s_k W^\beta f(x) \right] = \frac{(s+1)^{1-\frac{\alpha}{k}-\frac{\beta}{k}}}{k^2 \Gamma_k(\alpha) \Gamma_k(\beta)} \int_x^\infty (u^{s+1} - x^{s+1})^{\frac{\alpha+\beta}{k}-1} u^s f(u) \cdot \left[\int_0^1 (1-z)^{\frac{\alpha}{k}-1} (z)^{\frac{\beta}{k}-1} dz \right] du,$$

Using (2), (4) and (6), we get

$$(12) \quad {}^s_k W^\alpha \left[{}^s_k W^\beta f(x) \right] = {}^s_k W^{\alpha+\beta} f(x),$$

Similarly, we can prove

$${}^s_k W^\beta \left[{}^s_k W^\alpha f(x) \right] = {}^s_k W^{\alpha+\beta} f(x)$$

and hence we get (8). □

Theorem 2.2. Let $\alpha \in (0, 1)$, $\beta, k \in (0, \infty)$, $s > -1$. Then $\forall x \geq 0$,

$$(14) \quad {}^s_k W^\alpha [(x^{s+1})^{-\frac{\beta}{k}}] = \frac{\Gamma_k(\beta - \alpha)(x^{s+1})^{\frac{\alpha}{k} - \frac{\beta}{k}}}{(s+1)^{\frac{\alpha}{k}} \Gamma_k(\beta)}.$$

Proof. Using the result (6) in the LHS of the equation (14), we have

$$(15) \quad {}^s_k W^\alpha [(x^{s+1})^{-\frac{\beta}{k}}] = \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_x^\infty (t^{s+1} - x^{s+1})^{\frac{\alpha}{k}-1} t^s [(t^{s+1})^{-\frac{\beta}{k}}] dt,$$

Substituting $y = \frac{x^{s+1}}{t^{s+1}}$

$$(16) \quad {}^s_k W^\alpha [(x^{s+1})^{-\frac{\beta}{k}}] = \frac{(s+1)^{-\frac{\alpha}{k}} (x^{s+1})^{\frac{\alpha}{k} - \frac{\beta}{k}}}{k\Gamma_k(\alpha)} \left[\int_0^1 (1-y)^{\frac{\alpha}{k}-1} (y)^{\frac{(\beta-\alpha)}{k}-1} dy \right].$$

Using the result (2), we get (14). $\square$

Theorem 2.3. Let $\alpha \in (0, 1)$, $\mu, k \in (0, \infty)$, $s > -1$. Then $\forall x \geq 0$,

$$(17) \quad {}^s_k W^\alpha (e^{-\mu x^{s+1}}) = \frac{e^{-\mu x^{s+1}}}{[\mu k(s+1)]^{\frac{\alpha}{k}}}.$$

Proof. Using the result (6) in the LHS of the equation (17), we have

$$(18) \quad {}^s_k W^\alpha (e^{-\mu x^{s+1}}) = \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_x^\infty (t^{s+1} - x^{s+1})^{\frac{\alpha}{k}-1} t^s [e^{-\mu t^{s+1}}] dt,$$

Substituting $t^{s+1} - x^{s+1} = z^{s+1}$

$$(19) \quad {}^s_k W^\alpha (e^{-\mu x^{s+1}}) = \frac{(s+1)^{1-\frac{\alpha}{k}} e^{-\mu x^{s+1}}}{k\Gamma_k(\alpha)} \int_0^\infty (z^{s+1})^{\frac{\alpha}{k}-1} z^s e^{-\mu z^{s+1}} dz.$$

Substituting $\mu z^{s+1} = y$

$$(20) \quad {}^s_k W^\alpha (e^{-\mu x^{s+1}}) = \frac{(s+1)^{-\frac{\alpha}{k}} e^{-\mu x^{s+1}}}{\mu^{\frac{\alpha}{k}} k\Gamma_k(\alpha)} \int_0^\infty (y)^{\frac{\alpha}{k}-1} e^{-y} dy.$$

Using (1), we get (17). $\square$

Theorem 2.4. Let h, f be continuous on $[0, \infty)$ and let $\alpha \in (0, 1)$, $k \in (0, \infty)$, $s > -1$. Then $\forall x \geq 0$,

$$(21) \quad {}^s_k W^\alpha [h(x)] = {}^s_k W^\alpha [f(x)] \Rightarrow h = f.$$

Proof. Using (6) and linearity of integrals, we get

$$(22) \quad \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_x^\infty (t^{s+1} - x^{s+1})^{\frac{\alpha}{k}-1} t^s [h(t) - f(t)] dt = 0, \quad t > 0.$$

By using theorem 2.12.5 of [3], we get

$$(23) \quad h(t) - f(t) = 0 \Rightarrow h = f.$$

□

Corollary 2.5. Let $\alpha \in (0, 1)$, $\lambda, \mu, \beta, k \in (0, \infty)$, $s > -1$. Then $\forall x \geq 0$,

$$(24) \quad {}^s_k W^\alpha [\mu(x+a)^\beta + \lambda(x^{s+1})^{-\frac{\beta}{k}}] = \mu_k^s W^\alpha (x+a)^\beta + \lambda_k^s W^\alpha (x^{s+1})^{-\frac{\beta}{k}}.$$

Theorem 2.6. Let g be continuous on $[0, \infty)$ and let $\alpha \in (0, 1)$, $k \in (0, \infty)$, $s > -1$. Then $\forall x \geq 0$,

$$(25) \quad M \left[{}^s_k W^\alpha (x^{s+1})^{-\frac{\alpha}{k}} g(x) \right] = (s+1)^{-\frac{\alpha}{k}} \frac{\Gamma_k \left(\frac{s_1 k}{(s+1)} \right)}{\Gamma_k \left(\frac{s_1 k}{(s+1)} + \alpha \right)} g^*(s_1), \quad \operatorname{Re}(s_1) > 0.$$

Proof. Using the results (6) and (7)

$$(26) \quad M \left[{}^s_k W^\alpha (x^{s+1})^{-\frac{\alpha}{k}} g(x) \right] = \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_0^\infty x^{s_1-1} \cdot \left[\int_x^\infty (t^{s+1} - x^{s+1})^{\frac{\alpha}{k}-1} (t^{s+1})^{-\frac{\alpha}{k}} t^s g(t) dt \right] dx.$$

By Fubini's theorem

$$(27) \quad M \left[{}^s_k W^\alpha (x^{s+1})^{-\frac{\alpha}{k}} g(x) \right] = \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_0^\infty (t^{s+1})^{-\frac{\alpha}{k}} g(t) \cdot \left[\int_0^t (t^{s+1} - x^{s+1})^{\frac{\alpha}{k}-1} x^{s_1-1} t^s dx \right] dt.$$

Substituting $x^{s+1} = t^{s+1}y$

$$(28) \quad M \left[{}^s_k W^\alpha (x^{s+1})^{-\frac{\alpha}{k}} g(x) \right] = \frac{(s+1)^{-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_0^\infty t^{s_1-1} g(t) \cdot \left[\int_0^1 (1-y)^{\frac{\alpha}{k}-1} y^{\frac{s_1 k}{(s+1)k}-1} dy \right] dt.$$

(2) and (6) leads to (25).

□

Example 2.7. Let $\alpha \in (0, 1)$, $k \in (0, \infty)$, $s > -1$. Then $\forall x \geq 0$, using theorem 2.6, we can get

$$(29) \quad M \left[{}^s W_k^\alpha (x^{s+1})^{-\frac{\alpha}{k}} (e^{-x}) \right] = (s+1)^{-\frac{\alpha}{k}} \frac{\Gamma_k \left(\frac{s_1 k}{(s+1)} \right)}{\Gamma_k \left(\frac{s_1 k}{(s+1)} + \alpha \right)} \Gamma(s_1), \operatorname{Re}(s_1) > 0.$$

Theorem 2.8. Let f be continuous on $[0, \infty)$ and $\alpha \in (0, 1)$, $k \in (0, \infty)$, $s > -1$. Then $\forall x \geq 0$,

$$(30) \quad M \left[{}^s W_k^\alpha f(x) \right] = (s+1)^{-\frac{\alpha}{k}} \frac{\Gamma_k \left(\frac{s_1 k}{(s+1)} \right)}{\Gamma_k \left(\frac{s_1 k}{(s+1)} + \alpha \right)} f^* \left(s_1 + \frac{(s+1)\alpha}{k} \right), \operatorname{Re}(s_1) > 0.$$

Proof. Using the results (6) and (7)

$$(31) \quad M \left[{}^s W_k^\alpha f(x) \right] = \frac{(s+1)^{1-\frac{\alpha}{k}}}{k \Gamma_k(\alpha)} \int_0^\infty x^{s_1-1} \left[\int_x^\infty (t^{s+1} - x^{s+1})^{\frac{\alpha}{k}-1} t^s f(t) dt \right] dx.$$

By Fubini's theorem

$$(32) \quad M \left[{}^s W_k^\alpha f(x) \right] = \frac{(s+1)^{1-\frac{\alpha}{k}}}{k \Gamma_k(\alpha)} \int_0^\infty f(t) \left[\int_0^t (t^{s+1} - x^{s+1})^{\frac{\alpha}{k}-1} x^{s_1-1} t^s dx \right] dt.$$

Substituting $x^{s+1} = t^{s+1}y$

$$(33) \quad M \left[{}^s W_k^\alpha f(x) \right] = \frac{(s+1)^{-\frac{\alpha}{k}}}{k \Gamma_k(\alpha)} \int_0^\infty t^{\frac{k s_1 + (s+1)\alpha}{k} - 1} f(t) \cdot \left[\int_0^1 (1-y)^{\frac{\alpha}{k}-1} y^{\frac{s_1 k}{(s+1)k} - 1} dy \right] dt.$$

(2) and (7) leads to (30). $\square$

Example 2.9. Let $\alpha \in (0, 1)$, $k \in (0, \infty)$, $s > -1$. Then $\forall x \geq 0$, using theorem 2.8, we can get

$$(34) \quad M \left[{}^s W_k^\alpha \ln x f(x) \right] = (s+1)^{-\frac{\alpha}{k}} \frac{\Gamma_k \left(\frac{s_1 k}{(s+1)} \right)}{\Gamma_k \left(\frac{s_1 k}{(s+1)} + \alpha \right)} \frac{d}{ds} \left[f^* \left(s_1 + \frac{(s+1)\alpha}{k} \right) \right],$$

$\operatorname{Re}(s_1) > 0$.

Some inequalities of extended k -Weyl fractional integral:

Theorem 2.10. Let h, f are two synchronous on $[0, \infty)$ and $\alpha, \beta \in (0, 1)$, $k > 0$, $s > -1$. Then $\forall t \geq 0$, we have

$$(35) \quad {}^s W_k^\alpha (1) {}^s W_k^\alpha (hf(t)) \geq {}^s W_k^\alpha (h(t)) {}^s W_k^\alpha (f(t)).$$

$$(36) \quad {}^s W_k^\alpha (hf(t)) {}^s W_k^\beta (1) + {}^s W_k^\beta (hf(t)) {}^s W_k^\alpha (1) \geq {}^s W_k^\alpha (h(t)) {}^s W_k^\beta (f(t)) + {}^s W_k^\beta (h(t)) {}^s W_k^\alpha (f(t)).$$

Proof. Since h, f are two synchronous on $[0, \infty)$, then $\forall v, w \geq 0$, we have

$$(37) \quad [h(v) - h(w)][f(v) - f(w)] \geq 0.$$

Therefore,

$$(38) \quad h(v)f(v) + h(w)f(w) \geq h(v)f(w) + h(w)f(v).$$

Multiplying this by $\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}(v^{s+1} - t^{s+1})^{\frac{\alpha}{k}-1}v^s$ and taking integral w.r.t. v over (t, ∞) , we get

$$(39) \quad \begin{aligned} & \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_t^\infty (v^{s+1} - t^{s+1})^{\frac{\alpha}{k}-1} v^s h(v) f(v) dv \\ & + \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_t^\infty (v^{s+1} - t^{s+1})^{\frac{\alpha}{k}-1} v^s h(w) f(w) dv \\ & \geq \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_t^\infty (v^{s+1} - t^{s+1})^{\frac{\alpha}{k}-1} v^s h(v) f(w) dv \\ & + \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_t^\infty (v^{s+1} - t^{s+1})^{\frac{\alpha}{k}-1} v^s h(w) f(v) dv. \end{aligned}$$

Using (6), we get

$$(40) \quad {}^s_k W^\alpha(hf(t)) + h(w)f(w){}_k^s W^\alpha(1) \geq f(w){}_k^s W^\alpha(h(t)) + h(w){}_k^s W^\alpha(f(t)).$$

Multiplying this by $\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}(w^{s+1} - t^{s+1})^{\frac{\alpha}{k}-1}w^s$ and taking integral w.r.t. w over (t, ∞) , we get

$$(41) \quad \begin{aligned} & \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \left[\int_t^\infty (w^{s+1} - t^{s+1})^{\frac{\alpha}{k}-1} w^s {}^s_k W^\alpha(hf(t)) dw \right. \\ & + \int_t^\infty (w^{s+1} - t^{s+1})^{\frac{\alpha}{k}-1} w^s h(w) f(w) {}_k^s W^\alpha(1) dw \\ & \geq \int_t^\infty (w^{s+1} - t^{s+1})^{\frac{\alpha}{k}-1} w^s f(w) {}^s_k W^\alpha(h(t)) dw \\ & \left. + \int_t^\infty (w^{s+1} - t^{s+1})^{\frac{\alpha}{k}-1} w^s h(w) {}_k^s W^\alpha(f(t)) dw \right]. \end{aligned}$$

Using (6) and simplifying we obtained (35).

Again multiplying (40) by $\frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)}(w^{s+1} - t^{s+1})^{\frac{\beta}{k}-1}w^s$ and taking integral w.r.t. w over (t, ∞) , then using (6) and simplifying, we obtained (36). $\square$

Theorem 2.11. Let h, f are two synchronous on $[0, \infty)$, $g \geq 0$ and $\alpha, \beta \in (0, 1), k > 0, s > -1$. Then $\forall z \geq 0$, we have

$$\begin{aligned}
 (42) \quad & {}^s_k W^\alpha h f g(z) {}^s_k W^\beta (1) + {}^s_k W^\alpha (1) {}^s_k W^\beta h f g(z) \\
 & \geq {}^s_k W^\alpha h g(z) {}^s_k W^\beta f(z) + {}^s_k W^\alpha f g(z) {}^s_k W^\beta h(z) \\
 & - {}^s_k W^\alpha g(z) {}^s_k W^\beta h f(z) - {}^s_k W^\alpha h f(z) {}^s_k W^\beta g(z) \\
 & + {}^s_k W^\alpha h(z) {}^s_k W^\beta f g(z) + {}^s_k W^\alpha f(z) {}^s_k W^\beta h g(z).
 \end{aligned}$$

Proof. Since h, f are synchronous on $[0, \infty)$, then $\forall r, t \geq 0$,

$$(43) \quad [h(r) - h(t)][f(r) - f(t)][g(r) + g(t)] \geq 0.$$

Multiplying and arranging, we get

$$\begin{aligned}
 (44) \quad & h(r)f(r)g(r) + h(t)f(t)g(t) \geq h(r)f(t)g(r) + h(t)f(r)g(r) \\
 & - h(t)f(t)g(r) - h(r)f(r)g(t) + h(r)f(t)g(t) + h(t)f(r)g(t).
 \end{aligned}$$

Multiplying by $\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}(r^{s+1} - z^{s+1})^{\frac{\alpha}{k}-1}r^s$ and taking integral w.r.t. r over (z, ∞)

$$\begin{aligned}
 (45) \quad & \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \left[\int_z^\infty (r^{s+1} - z^{s+1})^{\frac{\alpha}{k}-1} r^s h(r) f(r) g(r) dr \right. \\
 & + h(t) f(t) g(t) \int_z^\infty (r^{s+1} - z^{s+1})^{\frac{\alpha}{k}-1} r^s dr \\
 & \geq f(t) \int_z^\infty (r^{s+1} - z^{s+1})^{\frac{\alpha}{k}-1} r^s h(r) g(r) dr \\
 & + h(t) \int_z^\infty (r^{s+1} - z^{s+1})^{\frac{\alpha}{k}-1} r^s f(r) g(r) dr \\
 & - h(t) f(t) \int_z^\infty (r^{s+1} - z^{s+1})^{\frac{\alpha}{k}-1} r^s g(r) dr \\
 & - g(t) \int_z^\infty (r^{s+1} - z^{s+1})^{\frac{\alpha}{k}-1} r^s h(r) f(r) dr \\
 & + f(t) g(t) \int_z^\infty (r^{s+1} - z^{s+1})^{\frac{\alpha}{k}-1} r^s h(r) dr \\
 & \left. + h(t) g(t) \int_z^\infty (r^{s+1} - z^{s+1})^{\frac{\alpha}{k}-1} r^s f(r) dr \right].
 \end{aligned}$$

Using (6), we get

$$\begin{aligned}
 (46) \quad & {}^s_k W^\alpha h f g(z) + h(t) f(t) g(t) {}^s_k W^\alpha (1) \\
 & \geq f(t) {}^s_k W^\alpha h g(z) + h(t) {}^s_k W^\alpha f g(z) \\
 & - h(t) f(t) {}^s_k W^\alpha g(z) - g(t) {}^s_k W^\alpha h f(z) \\
 & + f(t) g(t) {}^s_k W^\alpha h(z) + h(t) g(t) {}^s_k W^\alpha f(z).
 \end{aligned}$$

Multiplying by $\frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)}(t^{s+1} - z^{s+1})^{\frac{\beta}{k}-1}t^s$ and taking integral w.r.t. t over (z, ∞) , then using (6), we get the required result. $\square$

Corollary 2.12. *Let h, f are two synchronous on $[0, \infty)$, $g \geq 0$ and $\alpha \in (0, 1), k > 0, s > -1$. Then $\forall t \geq 0$, we have*

$$(47) \quad \begin{aligned} {}^s_k W^\alpha f g h(t) {}^s_k W^\alpha(1) &\geq {}^s_k W^\alpha f h(t) {}^s_k W^\alpha g(t) \\ &+ {}^s_k W^\alpha g h(t) {}^s_k W^\alpha f(t) - {}^s_k W^\alpha h(t) {}^s_k W^\alpha f g(t). \end{aligned}$$

Theorem 2.13. *Let h, f are two synchronous on $(0, \infty)$, $g \geq 0$ and $\alpha, \beta \in (0, 1), k > 0, s > -1$. Then $\forall t \geq 0$, we have*

$$(48) \quad \begin{aligned} &{}^s_k W^\alpha f g h(t) {}^s_k W^\beta(1) - {}^s_k W^\alpha(1) {}^s_k W^\beta f g h(t) \\ &\geq {}^s_k W^\alpha f h(t) {}^s_k W^\beta g(t) + {}^s_k W^\alpha g h(t) {}^s_k W^\beta f(t) \\ &- {}^s_k W^\alpha h(t) {}^s_k W^\beta f g(t) + {}^s_k W^\alpha f g(t) {}^s_k W^\beta h(t) \\ &- {}^s_k W^\alpha f(t) {}^s_k W^\beta g h(t) - {}^s_k W^\alpha g(t) {}^s_k W^\beta f h(t). \end{aligned}$$

Proof. This can be proved by using the steps as done in Theorem 2.11. $\square$

Theorem 2.14. *Let h, f are two synchronous on $[0, \infty)$ and $\alpha, \beta \in (0, 1), k > 0, s > -1$. Then $\forall z \geq 0$, we have*

$$(49) \quad {}^s_k W^\alpha (h^2(z)) {}^s_k W^\beta(1) + {}^s_k W^\beta (f^2(z)) {}^s_k W^\alpha(1) \geq 2 {}^s_k W^\alpha (h(z)) {}^s_k W^\beta (f(z)).$$

and

$$(50) \quad {}^s_k W^\alpha h^2(z) {}^s_k W^\beta f^2(z) + {}^s_k W^\beta h^2(z) {}^s_k W^\alpha f^2(z) \geq 2 {}^s_k W^\alpha h f(z) {}^s_k W^\beta h f(z).$$

Proof. Since h, f are synchronous on $(0, \infty)$ then $\forall r, t \geq 0$,

$$(51) \quad [h(r) - f(t)]^2 \geq 0.$$

Therefore,

$$(52) \quad h^2(r) + f^2(t) \geq 2h(r)f(t).$$

Multiply by $\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}(r^{s+1} - z^{s+1})^{\frac{\alpha}{k}-1}r^s$ and taking integral w.r.t. r over (z, ∞)

$$(53) \quad \begin{aligned} &\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_z^\infty (r^{s+1} - z^{s+1})^{\frac{\alpha}{k}-1} r^s h^2(r) dr \\ &+ \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_z^\infty (r^{s+1} - z^{s+1})^{\frac{\alpha}{k}-1} r^s f^2(t) dr \\ &\geq \frac{2(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_z^\infty (r^{s+1} - z^{s+1})^{\frac{\alpha}{k}-1} r^s h(r) f(t) dr. \end{aligned}$$

Using (6), we get

$$(54) \quad {}^s_k W^\alpha (h^2(z)) + f^2(t) {}^s_k W^\alpha (1) \geq 2f(t) {}^s_k W^\alpha (h(z)).$$

Multiply by $\frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)}(t^{s+1} - z^{s+1})^{\frac{\beta}{k}-1}t^s$ and taking integral w.r.t. t over (z, ∞)

$$(55) \quad \begin{aligned} & \frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)} \int_z^\infty (t^{s+1} - z^{s+1})^{\frac{\beta}{k}-1} t^s {}^s_k W^\alpha (h^2(z)) dt \\ & + \frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)} \int_z^\infty (t^{s+1} - z^{s+1})^{\frac{\beta}{k}-1} t^s f^2(t) {}^s_k W^\alpha (1) dt \\ & \geq \frac{2(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)} \int_z^\infty (t^{s+1} - z^{s+1})^{\frac{\beta}{k}-1} t^s f(t) {}^s_k W^\alpha (h(z)) dt. \end{aligned}$$

Using the result (6), we get (49).

For (50), we take

$$(56) \quad [h(r)f(t) - h(t)f(r)]^2 \geq 0.$$

and get result as done in (49). $\square$

Corollary 2.15. *Let h, f are two synchronous on $[0, \infty)$ and $\alpha, \beta \in (0, 1), k > 0, s > -1$. Then $\forall y \geq 0$, we have*

$$(57) \quad {}^s_k W^\alpha h(1) [{}^s_k W^\alpha h^2(y) + {}^s_k W^\alpha f^2(y)] \geq 2 {}^s_k W^\alpha h(y) {}^s_k W^\alpha f(y),$$

and

$$(58) \quad {}^s_k W^\alpha h^2(y) {}^s_k W^\alpha f^2(y) \geq [{}^s_k W^\alpha hf(y)]^2.$$

Theorem 2.16. *Let $g : (-\infty, \infty) \rightarrow (-\infty, \infty)$ and we also have $\bar{g}(x) = \int_x^\infty g(u)u^s du$, Then $\alpha \in (0, 1), k > 0, s > -1$,*

$$(59) \quad {}^s_k W^\alpha \bar{g}(x) = k {}^s_k W^{\alpha+k} g(x).$$

Proof. Using the result (6) in the LHS of the equation (59), we have

$$(60) \quad {}^s_k W^\alpha \bar{g}(x) = \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_x^\infty (t^{s+1} - x^{s+1})^{\frac{\alpha}{k}-1} t^s \left[\int_x^\infty g(u)u^s du \right] dt.$$

Substituting the value of $\bar{g}(x)$

$$(61) \quad {}^s_k W^\alpha \bar{g}(x) = \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_x^\infty (t^{s+1} - x^{s+1})^{\frac{\alpha}{k}-1} t^s \left[\int_x^\infty g(u)u^s du \right] dt.$$

By Fubini's theorem

$$(62) \quad {}^s_k W^\alpha \bar{g}(x) = \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_x^\infty g(u)u^s \left[\int_x^u (t^{s+1} - x^{s+1})^{\frac{\alpha}{k}-1} t^s dt \right] du.$$

Integrating and using (6), we get (59). $\square$

Example 2.17. Let $\alpha \in (0, 1)$, $k \in (0, \infty)$, $s > -1$. Then $\forall x \geq 0$ and $f(x) = e^{-\mu x^{s+1}}$, using the theorem 2.16, we can get

$$(63) \quad {}^s_k W^\alpha \bar{f}(x) = \frac{k e^{-\mu x^{s+1}}}{[\mu k(s+1)]^{\frac{\alpha}{k}+1}}.$$

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MODIFICATION OF ACCURACY ESTIMATION USING STOCK MARKET DATA

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Abstract. It is well known that the simplest way of estimation of statistical parameters is the method of least squares using linear functions. However, the problem with this method is in how to find out a linear observations. estimation accuracy is very important concept in many field such that; medicine, humanities, engineering, industry, economics and others since an unbiased vision is crucial in order to support the industry for decision making, also suitable data to obtain is, how estimates are completed, what factors encourage the choice of estimation methods and the current level of estimation accuracy. Therefore, this article purposes a novel technique in field of improving inference about population characteristic estimation, mathematical models were implemented in content of stock market data are collected from Amman stock exchange (ASE). Estimation accuracy directly will be implemented and Daubechies Wavelet transform (DWT) combined with interval estimation accuracy will be calculated also. As a result, the DWT combines with traditional estimation accuracy is better than traditional estimation accuracy directly. The results are implemented using (SPSS) and MATLAB.

Keywords: estimations accuracy, wavelet transform, amman stock exchange.

1. Introduction

Inferential statistics focused on sketch correct inferences about population characteristic using sample information. drawn a sample from a population , evaluated sample statistics on variable (X) and then make inference about variable (X) in the population from witch the sample drawn ([10]).

Stock market sample is drowning to realize the performance of the entire population. For example, the familiar stock market averages are samples designed to show the broader stock market and indicate its performance return. For the domestic publicly-traded stock market, populated with at least 10,000 or more companies, the Dow Jones Industrial Average (DJIA) has just 30 representatives; the S&P 500 has 500. Yet these samples are taken as valid indicators of the broader population ([3, 4]).

Recently, very important to comprehend the mechanics of sample estimation, especially with stock market data and have the insight to analysis the quality of research derived from sampling efforts. Therefore, recently the estimation

accuracy and processes become very important topics. Consequently, in this article the estimation accuracy will be improved using one of the most popular spectral analysis functions which are called by Daubechies Wavelet Transform (DWT) by reducing the bound of error.

This study attempts to employ the proposed method to the daily stock market data from ASE. Selected estimation models are used in the proposed method comparison to assess its performance. Experimental results show that the modified method which is interval estimation with DWT is superior to existing method in terms of some accuracy estimation error measure. Section 2 introduces the literature of necessary used term. In section 3 the research methodology and its mathematical models will be presented. In section 4, the dataset, results and discussion will be presented. Finally, in section 6 the conclusion will be presented.

2. Literature review

In [1] discussed the efficiency of Islamic banks in the Middle East and North Africa region. For adjusting the estimation and estimating confidence intervals for the estimated efficiency scores at desired levels of significance ([2]). Has proved that the point estimate can be confusing, therefore, the interval estimation is better than the point estimation and they discussed maximum likelihood point estimates and confidence intervals depended on delta. The researcher improve model studies using both positive factor analysis and standard errors models ([3]) have found that the interval estimates can suitable for the irritation parameters in content of artificial price time series. [4] Explain confidence intervals for the single coefficient of variation and the variance of coefficients of variation using exponential distributions ([5]). Have found that the realistic likelihood process to give suggestion on the bivariate subsistence job of paired failure times by estimating the subsistence job of cut time with the Kaplan–Meier estimator ([6]). Have discussed the Analytical tools that for the parameter estimates using residual analysis and the Cook space for worldwide influence ([7]). Have focus on joint maximum estimation and semi-parametric estimation of copula parameters in a bivariate t-copula.

One of the most important paper is Bruzda in 2015 ([8]) since he discussed nonparametric estimator of random signals established on the Wavelet Transform (WT), he discussed stochastic signals rooted in white noise and abstractions with wavelet de-noising procedures using the non-decimated discrete wavelet transform and the awareness of wavelet scaling. He assesses properties of these estimators through extensive computer simulations and partially also analytically. WT estimator strong benefits over parametric maximum likelihood approaches as far as computational subjects are concerned.

Has transformed regression models to multiple linear regression models by discrete wavelet transformation ([9]). In case the number of analytical curves is huge. They also apply correlation established sparse regression technique to

the caused high dimensional regression model. The original feature of sparse technique is the researcher execute sparsely consequence on the way of the estimate of the coefficient course in its place of the estimate itself and only the direction of the estimate is determined by an optimization problem. Comparing method with both functional regression methods and other WT grounded sparse regression approaches on together simulated data and four real data sets has evaluated, with the cases of single and multiple predictive curves. The results indicate that sparse wavelet regression methods are enhanced in removing local features and method.

WT is a well-known method in estimation, forecasting and other fields ([16, 17]). However, still some caps in the WT' literature is not fill up until today. Therefore, after intensive review in the estimation literature especially with WT, the researcher has found no paper has improved the estimation accuracy using DWT in content of stock market data. Therefore, this research is different from others because the researchers gathered stock market data from website of (ASE) using the closed price dataset from 1992 until 2017 in order to draw right inferences about population characteristic (mean, variance, stander deviation) through computing confidence interval using standard formula directly then computing and comparing the results confidence interval combined with DWT.

3. Methodology and mathematical models

3.1 Research framework

DWT is a good model in decomposing and it can notice much real fluctuation. Therefore, it will be used in this article to improve the accuracy of estimation process. The following figure will be presented the flowchart diagram for this article with its methodology:

Referring to the upper research framework, the methodology of this article can be summarizing as follows:

1. Decomposing the stock market data based on DWT.
2. Using the details coefficients in order to detect the fluctuations and outlier values from the data used.
3. Combining the smoothed coefficients with the standard confidence interval estimation to improve the estimation accuracy.
4. Comparing the results that generates from DWT with standard interval estimation with the results that generates from the standard interval estimation directly.
5. Selecting the best method that reduces the error term.

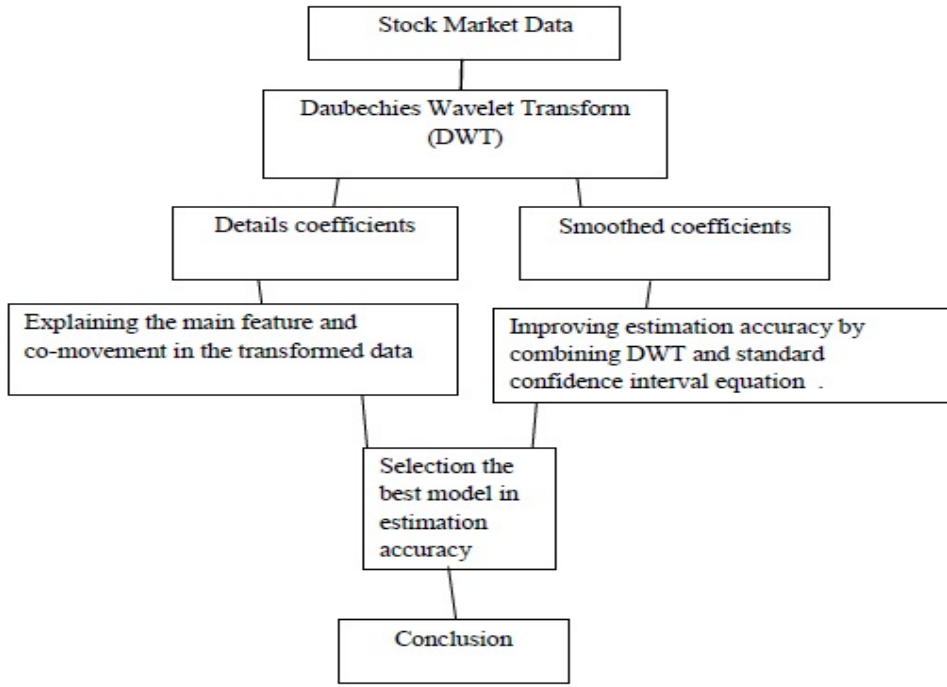


Figure 1: Research framework.

3.2 Interval estimation

One Population Mean can be estimated as $(\bar{x} \mp z_{\alpha,2} \sqrt{\frac{s}{n}})$ the interval estimate offered an estimation range where in which true population parameter might fall in statistics, interval estimation is the habit of sample data to compute an interval of likely values of an unidentified population parameter, in difference to point estimation, which is a single number. For more details about the interval estimation please refer to [1].

3.3 Wavelet transform

WT is used in many fields for prediction and estimation. The key aspect of DWT for financial analysis is decomposition by time scale. Wavelet transform (WT) is based on Fourier Transform (FT), which characterizes any function as the sum of the sine and cosine functions. WT is function of time t that obeys a basic condition ([11, 12, 13, 14, 15]):

$$C_{\varphi} = \int_0^{\infty} \frac{|\varphi(f)|}{f} df < \infty.$$

Where $\varphi(f)$ are the FT and a function of frequency f . WT is a mathematical tool that which has much application, such as image analysis and signal processing. WT has good dealing with non-stationary signals, or when dealing with

signals that are localized in time, space, or frequency. There are two types of WT within a given family. Father wavelet defines the smooth and low-frequency parts of a signal, and mother wavelet defines the detailed and high-frequency components. In the following equations, (2a) represents the father wavelet and (2b) represents the mother wavelet, with $j = 1 \dots J$ in the J -level wavelet decomposition ([14],[15])

$$\phi_{j,k} = 2^{-j/2} \phi(t - 2^j k / 2^j) \quad \varphi_{j,k} = 2^{-j/2} \varphi(t - 2^j k / 2^j).$$

Where: J denotes the maximum scale sustainable by the number of data points. WT should stratify the following conditions: $\int \phi(t) dt = 1$ and $\int \varphi(t) dt = 0$

Time series data, i.e., function $f(t)$, is an input represented by WT, and can be constructed as a sequence of projections onto father and mother wavelets indexed by both $\{k\}$, $k = \{0, 1, 2, \dots\}$ and by $\{S\}=2^j$, $\{j=1,2,3, \dots J\}$. Mathematically, it is convenient to use a dyadic expansion. The expansion coefficients are given by the projections [15]:

$$(3.1) \quad S_{j,k} = \int \phi_{j,k} f(t) dt, \quad d_{j,k} = \int \varphi_{j,k} f(t) dt.$$

The WT series approximation to $f(t)$ is defined by:

$$(3.2) \quad \begin{aligned} F(t) = & \sum S_{j,k} \phi_{j,k}(t) + \sum d_{j,k} \varphi_{j,k}(t) \\ & + \sum d_{j-1,k} \varphi_{j-1,k}(t) + \dots + \sum d_{1,k} \varphi_{1,k}(t) \end{aligned}$$

$$(3.3) \quad \begin{aligned} S_j(t) &= \sum S_{j,k} \phi_{j,k}(t), \\ D_j(t) &= \sum d_{j,k} \varphi_{j,k}(t). \end{aligned}$$

The WT is used to evaluate the approximation and details coefficients based on equation 3.3.

The HWT was improved and developed the frequency-domain characteristics by DWT. However, there is no specific formula for this method of wavelet transform. Thus, we tend to use the square gain function of their scaling filter, it is defined as:

$$(3.4) \quad g(f) = 2 \cos^l(\pi f) \sum_{l=0}^{\frac{l}{2}-1} \binom{\frac{l}{2}-1+l}{l} \sin^{2l}(\pi f).$$

Where l : Positive number and represents the length of the filter ([11]).

4. Dataset, results and discussion

In order to illustrate the effectiveness of WT in estimation the Daubechies function is applied for daily closed price for the time period from 1992 until 2017

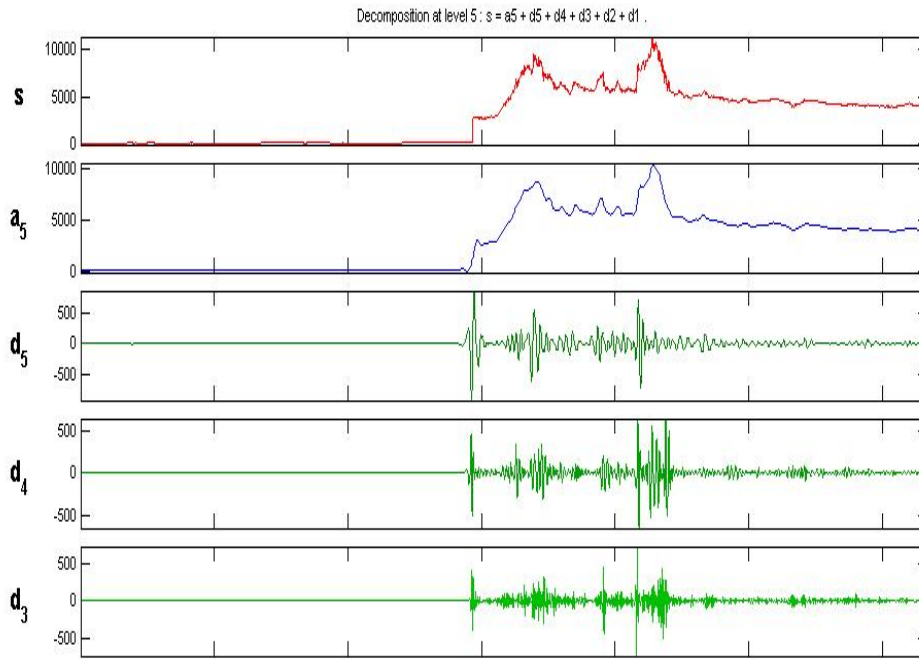


Figure 2: DWT analysis until level3.

were selected from ASE. The analysis with its behavior for the dataset using the mentioned models will be presented.

Referring to the up Figure, then notice the following:

1. In this article the decomposition for level 3 has used. However, the author can use any other level since the smoothed data only is used estimation accuracy. Therefore, the level of decomposition is unjustified.
2. DWT function is used in this article since it is well known that Daubechies function is the best function in the WT field.

The application of the DWT to the historical data decomposes them into a variety of resolution levels that expose their essential structure and it generates detail coefficients at each one of the three decomposition levels. In DWT, the three levels of decomposition can be carried out by the DWT using the following equation: $S = a_3 + d_3 + d_2 + d_1$. S refers to the original signal which is represented in the topmost part of Figure 1. Then the next part consists of one approximation level (a_3), a_3 which shows the plot of the approximation coefficients for the transformed data using DWT. The following parts of d_1 , d_2 and d_3 represent the details levels, whereby, d_1 is the plot of the first level of the details coefficients, d_2 is the plot of the second level of the details coefficients and d_3 is the plot of the third level of the details coefficients. Any of these three levels (d_1 , d_2 and d_3) can be adopted for explaining the data. Starting with d_1 which is

the first details level (see Figure 4.3), the transformed data is filtered from d1 until d3 through the details levels. As can be seen the data becomes smoother in d2, since the amount of data will be reduced automatically in the hope of obtaining a suitable level for detecting the stock market behavior. In this regard, we notice that at level d3 most of the fluctuations and high frequencies appear after the observation number 3000, which is from 2004 onwards. The reason of these fluctuations can be refer to the following events:

1. In 2003: Publication of annual book “Emerging Jordan 2003” by Oxford Business Group. Also High Performance of public shareholding companies and The California Public Employees’ Retirement System in USA selected Jordan together with 13 other countries as recipients of funds for the development of emerging markets.
2. In 2004, it was a golden year for all ASE performance indicators. We notice that in d3 Fig. 1 there was a spike between the observations 3000 and 3500 which indicates a high volume of trading occurred in this year. Some of the important events that contributed to this were: Inviting foreign and local investors to give some attention to ASE by participating in the specialist meeting “Enhancing the Capacity of Financial Markets to Promote Intra Investment among IDB Member Countries”.
3. In 2005, ASE participated in ‘Workshop on Exchange Technology’ in the USA to discuss some issues related to technologies used in stock exchanges also A report on the most attractive 27 emerging stock markets placed the ASE as 10th. And The ASE attracted a lot of visitors and interested people; Arabs and foreign ministers, which motivates the ASE to increase the number of investors.
4. In 2006 the fluctuations are still very high as a result of some events such as: Securities Depository Center in the ASE initiated new services for investments and new instructions for the purchase of share of public shareholding companies. These instructions enhanced the general investment climate. New instructions for dealing with subscription rights in order to preserve the rights of investors; to protect the investors; to provide cash for investors, and to enhance the level of its performance.
5. From 2007 to 2009: Between 2007- 2009, many events occurred in the ASE which effected the stability of the stock market such as: Dow Jones Composite Index currently includes component stocks of 10 of the 32 member states of the Federation of Euro-Asian Stock Exchanges. The exchanges included ASE and other stock markets. The International Organization of Securities Commission selected the Executive Chairman of the ASE as a member of a special committee.

After the year 2009 and referring to the level d3, then the fluctuations become more stable. However, there are some event such that;

6. 2009- 2017 Signing an Memorandum of Understanding between Amman Stock Exchange and the Egyptian Exchange's has finished stage of preparing the new website. New Version of the Electronic Trading System. The meetings of the Working Committee of the Federation of Euro-Asian Stock Exchanges (FEAS).S&P Indices and Arab Federation of Exchanges Create S&P AFE 40 Index and ASE receives an economic delegation from the French Embassy.

In this article the results of estimation accuracy which was comparing between confidence intervals use standard formula directly and DWT with Confidence intervals using standard formula estimation. The researcher found that the estimation using confidence interval directly gives less information about the population and not very significant While Confidence intervals with DWT representation more exact about population parameter characteristics; the following table shows the results about the estimation process.

Std. Error	interval estimation	Wavelet Transform
Std. Error of mean for one sample	2.561	0.892
Std. Deviation for one sample	1.002	0.148

Table 1: represent the standard error for data used using the method used

Refer to Table 1 then it is noticeable that the estimation has improved and the modified method is better than estimation directly this is return to the ability of DWT since the standard error has been reduced from 2.561 to 0.892 and the Std. Deviation also reduced from 1.002 to 0.148 this results implies to the ability of the DWT in improving the estimation accuracy.

5. Conclusion

The overall objective of carrying out this article is to investigate the comparative and contrastive efficiency and accuracy and hence suitability of DWT in the decomposition, processing and estimation of stock market data based on a sample dataset taken from the ASE. Based on the findings of the experiments, the significant contributions of this study can be summarized as follows:

- The experiments have shown that the level of estimation accuracy is improved in ASE when DWT is combined with a suitable standard estimation model compared to using standard estimation model directly.
- The experiments in this study have shown that the DWT is a good model in decomposing the data in content of ASE and detecting the events from

the dataset. Therefore, it would be to its advantage to implement DWT applications to assist in solving some of its financial issues such as uncertainties, volatility and structure breaks which will be suitable for investment sector.

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HYPER QUASI-MV ALGEBRAS AND IDEALS

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Abstract. In this paper, we introduce hyper quasi-MV algebras as the generalizations of hyper MV-algebras and quasi-MV algebras. First we give the definition of hyper quasi-MV algebras and investigate some basic properties of hyper quasi-MV algebras. Second we introduce ideals and weak ideals in a hyper quasi-MV algebra. Especially, we study two types of (weak) ideals and discuss the relationship between them. We also present the dual notions of ideals and weak ideals in this paper. Finally, we show the properties of ideals and weak ideals under the homomorphism of hyper quasi-MV algebras.

Keywords: hyper MV-algebras, hyper quasi-MV-algebras, homomorphisms, ideals.

1. Introduction

The theory of hyper algebraic structures was firstly introduced by Marty in 1934 [15]. Unlike the classical algebraic structures, the composition of two elements in the hyper algebraic structure is a subset. Since the theory was introduced, various hyper structures have been studied such as hyper K-algebras [3], hyper BCK-algebras [13], hyper MV-algebras [8], hyper pseudo MV-algebras [1], hyper effect algebras [6] and so on. Among them, the notion of hyper MV-algebras as a generalization of MV-algebras was paid more attentions and several related results were obtained. In [9, 10], authors discussed quotient structures and category theory of hyper MV-algebras. Hyper MV-ideals of hyper MV-algebras were investigated by Torkzadeh and Ahadpanah in [17]. Rasouli and Davvaz also studied hyper MV-ideals and introduced homomorphism, dual homomorphism and strong homomorphism of hyper MV-algebras in [20]. Meanwhile,

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they showed the properties of hyper MV-ideals under these mappings. Jun et al. introduced some new types of hyper MV-deductive systems in hyper MV-algebras and discussed their relationships [11, 12]. In addition, many other properties of hyper MV-algebras can be seen in [2, 7, 16, 18, 19].

Quasi-MV algebras [14] arising from quantum computational logic can be regarded as a generalization of MV-algebras. In this paper, we want to define hyper quasi-MV algebras as a hyper structural generalization of quasi-MV algebras and a generalization of hyper MV-algebras and study ideals of hyper quasi-MV algebras. The paper is organized as follows. In Section 2, we recall some definitions and results which will be used in the following. In Section 3, we define hyper quasi-MV algebras and investigate the related properties. In Section 4, we discuss two types of ideals and weak ideals and discuss the relationship between them. We also present the dual notions of ideals and weak ideals in this section. In Section 5, we show the properties of ideals and weak ideals under the homomorphism of hyper quasi-MV algebras.

2. Preliminary

In this section, we recall some definitions and show more results which will be used in the subsequent sections.

In [14], Ledda et al. introduced the quasi-MV algebra as a first significant step towards an algebraic characterization of the quantum computational logic. A *quasi-MV algebra* is an algebra $\mathbf{A} = \langle A; \oplus, ', 0, 1 \rangle$ of type $\langle 2, 1, 0, 0 \rangle$ satisfying the following identities for any $x, y, z \in A$:

- (Q1) $x \oplus (y \oplus z) = (x \oplus z) \oplus y$;
- (Q2) $x'' = x$;
- (Q3) $x \oplus 1 = 1$;
- (Q4) $(x' \oplus y)' \oplus y = (y' \oplus x)' \oplus x$;
- (Q5) $(x \oplus 0)' = x' \oplus 0$;
- (Q6) $(x \oplus y) \oplus 0 = x \oplus y$;
- (Q7) $0' = 1$.

Obviously, any MV-algebra is a quasi-MV algebra. Conversely, any quasi-MV algebra with $x \oplus 0 = x$ is an MV-algebra. On any quasi-MV algebra, we can define some operations: $x \odot y = (x' \oplus y')'$, $x \vee y = (x' \oplus y)' \oplus y$ and $x \wedge y = (x' \vee y')'$. We can also define a relation $x \leq y$ if and only if $x \vee y = y \oplus 0$. Below we list some elementary properties of them.

Proposition 2.1 ([14]). *Let $\mathbf{A}$ be a quasi-MV algebra. Then for any $x, y, z \in A$, we have*

- (1) $x \vee y = y \vee x$ and $x \wedge y = y \wedge x$;
- (2) $(x \vee y) \vee z = x \vee (y \vee z)$ and $(x \wedge y) \wedge z = x \wedge (y \wedge z)$;
- (3) $x \wedge y \leq x, y$ and $x, y \leq x \vee y$;
- (4) $x \leq x \wedge x$ and $x \wedge x \leq x$;
- (5) $x \leq x \oplus 0$ and $x \oplus 0 \leq x$;

- (6) $x \odot y \leq z$ if and only if $x \leq y' \oplus z$;
- (7) if $x \leq y$, then $y' \leq x'$;
- (8) $0 \leq x \leq 1$.

Let $\mathbf{A}$ be a quasi-MV algebra and I be a non-empty subset of A . We say that I is an *ideal* of $\mathbf{A}$ if, for any $x, y \in I$, the following conditions are satisfied: (I1) $0 \in I$; (I2) if $x, y \in I$, then $x \oplus y \in I$; (I3) if $x \in I$ and $y \in A$ with $y \leq x$, then $y \in I$. On the other hand, I is called a *weak ideal* of $\mathbf{A}$ if, for any $x, y \in I$, the following conditions are satisfied: (WI1) $0 \in I$; (WI2) if $x, y \in I$, then $x \oplus y \in I$; (WI3) if $x \in I$ and $y \in A$, then $x \odot y \in I$. In [14], authors proved that I is an ideal of $\mathbf{A}$ if and only if (1) I is a weak ideal of $\mathbf{A}$ and (2) $x \in I \Leftrightarrow x \oplus 0 \in I$.

Hyper MV-algebras were introduced in [8] as a generalization of MV-algebras. In a hyper MV-algebra M , the composition of x and y is a subset of M .

Definition 2.1 ([8]). A hyper MV-algebra is a non-empty set M endowed with a binary hyper operation $\oplus$, a unary operation $*$ and a constant 0 satisfying the following conditions for any $x, y, z \in M$:

- (hmv 1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
- (hmv 2) $x \oplus y = y \oplus x$;
- (hmv 3) $(x^*)^* = x$;
- (hmv 4) $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$;
- (hmv 5) $0^* \in x \oplus 0^*$;
- (hmv 6) $0^* \in x \oplus x^*$;
- (hmv 7) if $x \ll y$ and $y \ll x$, then $x = y$, where $x \ll y$ is defined as $0^* \in x^* \oplus y$.

For every non-empty subset A, B of M , we define $A \oplus B = \cup\{x \oplus y | x \in A, y \in B\}$. If A has just only one element x , then we write $x \oplus B = \{x\} \oplus B$. In addition, $A \ll B$ if and only if there exist $x \in A$ and $y \in B$ such that $x \ll y$.

Proposition 2.2 ([8]). Let M be a hyper MV-algebra. Then for any $x, y, z \in M$ and for any non-empty subset A, B and C of M the following hold

- (1) $(A \oplus B) \oplus C = A \oplus (B \oplus C)$;
- (2) $0 \ll x$;
- (3) $x \ll x$;
- (4) if $x \ll y$, then $y^* \ll x^*$ and $A \ll B$ implies $B^* \ll A^*$;
- (5) $x \ll 1$;
- (6) $A \ll A$;
- (7) $A \subseteq B$ implies $A \ll B$;
- (8) $x \ll x \oplus y$, $A \ll A \oplus B$;
- (9) $0 \oplus 0 = \{0\}$;
- (10) $x \in x \oplus 0$;
- (11) if $y \in x \oplus 0$, then $y \ll x$;
- (12) if $x \oplus 0 = y \oplus 0$, then $x = y$.

For hyper MV-algebras, ideals and filters are defined by many authors. Please see the following. However, all these definitions have similar yet slightly different formats. In Section 4, we present the relationship between these notions.

Definition 2.2. Let M be a hyper MV-algebra and I be a non-empty subset of M . Then

(HF) [8] I is called a *hyper MV-filter* of M , if (i) $1 \in I$; (ii) if $I \ll x^* \oplus y$ and $x \in I$, then $y \in I$.

(HI1) [20] I is called a *hyper MV-ideal* of M , if (i) $x, y \in I$ imply $x \oplus y \subseteq I$; (ii) if $x \ominus y \ll I$ and $y \in I$, then $x \in I$, where $x \ominus y = (x^* \oplus y)^*$.

(HI2) [17] I is called a *hyper MV-ideal* of M , if (i) $x, y \in I$ imply $x \oplus y \subseteq I$; (ii) if $y \ll x$ and $x \in I$, then $y \in I$.

(HDS) [11] I is called a *hyper MV-deductive system* of M , if (i) $0 \in I$; (ii) if $(x^* \oplus y)^* \ll I$ and $y \in I$, then $x \in I$.

3. Hyper quasi-MV algebras

In this section, we give the definition of hyper quasi-MV algebras and list some properties of them.

Definition 3.1. A *hyper quasi-MV algebra* is a non-empty set M endowed with a binary hyper operation $\oplus$, a unary operation $*$ and a constant 0 satisfying the following conditions for any $x, y, z \in M$:

- (hqm v 1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
- (hqm v 2) $x \oplus y = y \oplus x$;
- (hqm v 3) $(x^*)^* = x$;
- (hqm v 4) $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$;
- (hqm v 5) $0^* \in x \oplus 0^*$;
- (hqm v 6) $0^* \in x \oplus x^*$;
- (hqm v 7) $x \oplus y \subseteq x \oplus y \oplus 0$;
- (hqm v 8) $0 \in 0 \oplus 0$;
- (hqm v 9) if $x \ll y$ and $y \ll x$, then $x \oplus 0 = y \oplus 0$, where $x \ll y$ is defined as $0^* \in x^* \oplus y$.

Denote $0^* = 1$. If $0 \neq 1$, then we say that hyper quasi-MV algebra is *non-trivial* or *non-flat*¹. In what follows, let M denote a non-trivial hyper quasi-MV algebra unless otherwise specified.

Example 3.1. Any hyper MV-algebra is a hyper quasi-MV algebra.

Indeed, let M be a hyper MV-algebra. Following from Proposition 3.7 of [8], we have $x \in x \oplus 0$, so $0 \in 0 \oplus 0$ and $x \oplus y \subseteq x \oplus y \oplus 0$. If $x \ll y$ and $y \ll x$, then $x = y$, it turns out that $x \oplus 0 = y \oplus 0$. Hence M is a hyper quasi-MV algebra.

1. A quasi-MV algebra with $0 = 1$ is called *flat*.

Example 3.2. Any quasi-MV algebra is a hyper quasi-MV algebra.

Let $\langle M, \oplus, ', 0 \rangle$ be a quasi-MV algebra. We define $x \oplus^H y = \{x \oplus y\}$ and $x^* = x'$. Then it is easy to verify that $\langle M, \oplus^H, *, 0 \rangle$ is a hyper quasi-MV algebra.

Example 3.3. Let $M = \{0, a, b, 1\}$. Consider the following tables:

$\oplus$	0	a	b	1
0	$\{0\}$	$\{0, b\}$	$\{0, b\}$	M
a	$\{0, b\}$	M	M	M
b	$\{0, b\}$	M	M	M
1	M	M	M	M

*	0	a	b	1
	1	a	b	0

Then $\langle M, \oplus, *, 0 \rangle$ is a hyper quasi-MV algebra. Moreover, since $a \notin a \oplus 0$, we have that M is not a hyper MV-algebra.

Example 3.4. Let $M = [0, 1] \times [0, 1]$. Define the operations as follows: $\langle a, b \rangle \oplus \langle c, d \rangle = [0, \min\{1, a + c\}] \times [0, 1]$; $\langle a, b \rangle^* = \langle 1 - a, 1 - b \rangle$; $0 = \langle 0, \frac{1}{2} \rangle$. Then $\langle M, \oplus, *, 0 \rangle$ is a hyper quasi-MV algebra.

The following proofs are similar to the case of hyper MV-algebra in [8, 17].

Proposition 3.1. Let $\langle M_1, \oplus_1, *, 0_1 \rangle$ and $\langle M_2, \oplus_2, *, 0_2 \rangle$ be hyper quasi-MV algebras and $M = M_1 \times M_2$. We define a hyper operation $\oplus$ on M as follows: $\langle a_1, b_1 \rangle \oplus \langle a_2, b_2 \rangle = \langle a_1 \oplus_1 a_2, b_1 \oplus_2 b_2 \rangle$, a unary operation $*$ on M as follows: $\langle a, b \rangle^* = \langle a^{*1}, b^{*2} \rangle$, and $0 = \langle 0_1, 0_2 \rangle$. Then

- (1) $0^* \in \langle a_1, b_1 \rangle^* \oplus \langle a_2, b_2 \rangle$ if and only if $0_1^{*1} \in a_1^{*1} \oplus_1 a_2$ and $0_2^{*2} \in b_1^{*1} \oplus_2 b_2$.
- (2) $\langle M, \oplus, *, 0 \rangle$ is a hyper quasi-MV algebra.

Proposition 3.2. Let M be a hyper quasi-MV algebra. Then for any $x, y, z \in M$ and for any non-empty subset A, B and C of M the following hold

- (hq1) $(A \oplus B) \oplus C = A \oplus (B \oplus C)$;
- (hq2) $0 \ll x$;
- (hq3) $x \ll x$;
- (hq4) if $x \ll y$, then $y^* \ll x^*$ and $A \ll B$ implies $B^* \ll A^*$;
- (hq5) $x \ll 1$;
- (hq6) $A \ll A$;
- (hq7) $A \subseteq B$ implies $A \ll B$;
- (hq8) $x \ll x \oplus y$, especially, $x \ll x \oplus 0$.

According to (hp3), we note that the relation $\ll$ is reflexive. However, unlike hyper MV-algebras, it is not antisymmetric. It is easy to see that if the relation $\ll$ is antisymmetric, any hyper quasi-MV algebra is a hyper MV-algebra. Meanwhile, it is not transitive either. Hence if $\ll$ is transitive, then M is called a *transitive hyper quasi-MV algebra*.

On any hyper quasi-MV algebra M , we can define some hyper operations as follows:

$$x \odot y = (x^* \oplus y^*)^*;$$

$$\begin{aligned}x \vee y &= (x^* \oplus y)^* \oplus y; \\x \wedge y &= (x^* \vee y^*)^*.\end{aligned}$$

Below we will see some properties of these operations.

Lemma 3.1. *Let M be a hyper MV-algebra. For any $x, y \in M$ the following conditions are equivalent:*

- (1) $1 \in x^* \oplus y$;
- (2) $0 \in x \odot y^*$.

Proposition 3.3. *Let M be a hyper quasi-MV algebra. Then for any $x, y, z \in M$ the following hold:*

- (hq9) $x \odot (y \odot z) = x \odot (y \odot z)$;
- (hq10) $x \odot y = y \odot x$;
- (hq11) $0 \in x \odot x^*$;
- (hq12) $0 \in x \odot 0$;
- (hq13) $x \odot y \ll x, y$, especially, $x \odot 1 \ll x$;
- (hq14) $x \odot y \subseteq x \odot y \odot 1$;
- (hq15) $x \vee y = y \vee x$ and $x \wedge y = y \wedge x$;
- (hq16) $x \oplus y \subseteq (x \vee y) \oplus (x \wedge y)$;
- (hq17) if $x \in x \oplus y$, then $0 \in x^* \wedge y$;
- (hq18) if $x \in x \odot y$, then $1 \in x^* \vee y$;
- (hq19) $x \odot y \ll x \wedge y \ll x, y$;
- (hq20) $x, y \ll x \vee y \ll x \oplus y$;
- (hq21) if $x \in x \oplus x$, then $x \ll x \odot x$, if $x \in x \odot x$, then $x \ll x \oplus x$;
- (hq22) $x \ll y$ implies $x \wedge z \ll y \wedge z$ and $x \vee z \ll y \vee z$;
- (hq23) $x \ll y$ implies $x \in x \wedge y$ and $y \in x \vee y$;
- (hq24) if $x \ll y$, then $x \oplus z \ll y \oplus z$ and $x \odot z \ll y \odot z$;
- (hq25) $z \odot x \ll y$ if and only if $z \ll x^* \oplus y$.

4. Ideals, weak ideals and duality

In this section, we introduce some types of (weak) ideals for hyper quasi-MV algebras and investigate related properties of them. Let M be a hyper quasi-MV algebra and I be a non-empty subset of M . Denote

$$\begin{aligned}\mathbb{I}_1(M) &= \{I \subseteq M \mid \text{if } x \in I \text{ and } y \in M \text{ such that } y \odot x^* \ll I, \text{ then } y \in I\}; \\ \mathbb{I}_2(M) &= \{I \subseteq M \mid \text{if } x \in I \text{ and } y \in M \text{ such that } y \ll x, \text{ then } y \in I\}; \\ \mathbb{I}_3(M) &= \{I \subseteq M \mid \text{if } x \in I \text{ and } y \in M \text{ such that } (y \odot x^*) \cap I \neq \emptyset, \text{ then } y \in I\}.\end{aligned}$$

Proposition 4.1. *Let M be a hyper quasi-MV algebra and I be a non-empty subset of M . If $I \in \mathbb{I}_1(M)$, then we have*

- (1) $0 \in I$;
- (2) if $x, y \in I$, then $x \oplus y \ll I$;
- (3) if $x, y \in I$, then $(x \oplus y) \cap I \neq \emptyset$;
- (4) $I \in \mathbb{I}_2(M)$;
- (5) $I \in \mathbb{I}_3(M)$.

Proof. (1) Since I is a non-empty set, we can suppose that $x \in I$. Then by (hqm v 5) and Lemma 3.1, we have $0 \in 0 \odot x^*$. Because $0 \ll x$, we have $0 \odot x^* \ll I$, it follows that $0 \in I$.

(2) Let $x, y \in I$. Then by (hq 11), $0 \in (x \oplus y) \odot (x \oplus y)^* = (x \oplus y) \odot (x^* \odot y^*) = ((x \oplus y) \odot x^*) \odot y^*$, it turns out that there exists $a \in (x \oplus y) \odot x^*$ such that $0 \in a \odot y^*$. By (1), $0 \in I$, we have $a \odot y^* \ll I$, so $a \in I$ and $(x \oplus y) \odot x^* \ll I$. Similarly, there exists $b \in x \oplus y$ such that $b \odot x^* \ll I$. Using the hypothesis again, we have $b \in I$. Hence $x \oplus y \ll I$.

(3) Let $x, y \in I$. By (2), we have $x \oplus y \ll I$, so there exist $a \in x \oplus y$ and $b \in I$ such that $a \ll b$, it turns out that $0 \in a \odot b^*$, so $a \odot b^* \ll I$, we have $a \in I$, thus $(x \oplus y) \cap I \neq \emptyset$.

(4) Let $x \in I$ and $y \in M$ such that $y \ll x$. Then $0 \in y \odot x^*$, so by (1), $y \odot x^* \ll I$, we have $y \in I$. Hence $I \in \mathbb{I}_2(M)$.

(5) Let $x \in I$ and $y \in M$ such that $(y \odot x^*) \cap I \neq \emptyset$, we suppose that $a \in (y \odot x^*) \cap I$, then $a \in y \odot x^*$ and $a \in I$, so $y \odot x^* \ll I$, it follows that $y \in I$. Hence $I \in \mathbb{I}_3(M)$. $\square$

Remark 4.1. Let $I \in \mathbb{I}_1(M)$. Then $x, y \in I$ may not imply $x \oplus y \subseteq I$ in general. See the following example.

Example 4.1. Let $M = \{0, a, b, 1\}$ and the operations are given by the following tables:

$\oplus$	0	a	b	1
0	{0}	{b}	{b}	{1}
a	{b}	{1}	{1}	{1}
b	{b}	{1}	{1}	{1}
1	{1}	{1}	{1}	{1}

$*$	0	a	b	1
0	1	a	b	0

Then $\langle M, \oplus, *, 0 \rangle$ is a hyper quasi-MV algebra. Denote $I = \{0, a\}$. Then $I \in \mathbb{I}_1(M)$. However, $a \oplus a = \{1\}$, and $1 \notin I$.

Lemma 4.1. Let M be a hyper quasi-MV algebra and I be a non-empty subset of M . If $I \in \mathbb{I}_3(M)$ and $0 \in I$, then $I \in \mathbb{I}_1(M)$.

Proof. Let $x \in I$ and $y \in M$ such that $y \odot x^* \ll I$. Then there exist $a \in y \odot x^*$ and $b \in I$ such that $a \ll b$, so $0 \in a \odot b^*$. Since $0 \in I$, we have $(a \odot b^*) \cap I \neq \emptyset$, it follows that $a \in I$. Hence $(y \odot x^*) \cap I \neq \emptyset$ and we have $y \in I$. $\square$

Remark 4.2. The condition “ $0 \in I$ ” is necessary in Lemma 4.1. See the following example.

Example 4.2. Let $M = \{0, a, b, a^*, 1\}$ and the operations are given by the following tables:

$\oplus$	0	a	b	a	1	$*$	
0	$\{0\}$	$\{0\}$	$\{b\}$	$\{1\}$	$\{1\}$	0	1
a	$\{0\}$	$\{0\}$	$\{b\}$	$\{1\}$	$\{1\}$	a	a^*
b	$\{b\}$	$\{b\}$	$\{1\}$	$\{1\}$	$\{1\}$	b	b
a^*	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	a^*	a
1	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	1	0

Then $\langle M, \oplus, *, 0 \rangle$ is a hyper quasi-MV algebra. Denote $I = \{a^*, 1\}$. Then $I \in \mathbb{I}_3(M)$. However, $I \notin \mathbb{I}_1(M)$. Indeed, $0 \odot a^{**} \ll I$ but $0 \notin I$.

Proposition 4.2. *Let M be a hyper quasi-MV algebra and I be a non-empty subset of M . Then the following conditions are equivalent*

- (1) $I \in \mathbb{I}_1(M)$ and if $x, y \in I$, then $x \oplus y \subseteq I$;
- (2) $I \in \mathbb{I}_2(M)$ and if $x, y \in I$, then $x \oplus y \subseteq I$;
- (3) $I \in \mathbb{I}_3(M)$ and $0 \in I$ and if $x, y \in I$, then $x \oplus y \subseteq I$.

Proof. (1) $\Rightarrow$ (2) By Proposition 4.1.

(2) $\Rightarrow$ (1) Let $x \in I$ and $y \in M$ with $y \odot x^* \ll I$. Then there exist $a \in y \odot x^*$ and $b \in I$ such that $0 \in a \odot b^*$, it follows that $0 \in (y \odot x^*) \odot b^* = y \odot (x^* \odot b^*) = y \odot (x \oplus b)^*$, so there exists $m \in x \oplus b$ such that $0 \in y \odot m^*$, we have $y \ll m$. Note that $x, b \in I$, we have $x \oplus b \subseteq I$, thus $m \in I$. Since $I \in \mathbb{I}_2(M)$, we get $y \in I$.

(1) $\Leftrightarrow$ (3) By Proposition 4.1 and Lemma 4.1. $\square$

Now, we give the definition of ideals in hyper quasi-MV algebras.

Definition 4.1. Let M be a hyper quasi-MV algebra and I be a non-empty subset of M . Then I is called a *hyper quasi-MV type-1 ideal* (*type-1 ideal*, for short), if it satisfies any one of the equivalent conditions in Proposition 4.2.

Proposition 4.3. *Let M be a hyper quasi-MV algebra and $\{I_i | i \in S\}$ be a family of type-1 ideals of M . Then $\bigcap_{i \in S} I_i$ is also a type-1 ideal of M .*

Proof. Since $0 \in \bigcap_{i \in S} I_i$, we have that $\bigcap_{i \in S} I_i$ is a non-empty subset of M . Let $x, y \in \bigcap_{i \in S} I_i$. Then for any $i \in S$, we have $x \in I_i$ and $y \in I_i$, it follows that $x \oplus y \subseteq I_i$, so $x \oplus y \subseteq \bigcap_{i \in S} I_i$. Now, suppose that $x \in \bigcap_{i \in S} I_i$ and $y \in M$ such that $y \odot x^* \ll \bigcap_{i \in S} I_i$. Then for any $i \in S$, $x \in I_i$ and $y \odot x^* \ll I_i$, we have $y \in I_i$ for any $i \in S$, so $y \in \bigcap_{i \in S} I_i$. Hence $\bigcap_{i \in S} I_i$ is a type-1 ideal of M . $\square$

Proposition 4.4. *Let M be a hyper quasi-MV algebra with $0 \oplus 0 = \{0\}$. Then $[0] = \{x \in M | x \ll 0\}$ is a type-1 ideal of M .*

Proof. Suppose that $x, y \in [0]$. Then $x \ll 0$ and $y \ll 0$, it follows that $x \oplus 0 = 0 \oplus 0 = \{0\}$ and $y \oplus 0 = 0 \oplus 0 = \{0\}$. We have $x \oplus y \subseteq x \oplus y \oplus 0 \subseteq x \oplus 0 \oplus y \oplus 0 = 0 \oplus 0 \oplus 0 \oplus 0 = \{0\}$, so for any $a \in x \oplus y$, then $a = 0$, we have $a \ll 0$. Hence $x \oplus y \subseteq [0]$. Now, let $x \in [0]$ and $y \in M$ such that $y \ll x$. Then $1 \in y^* \oplus x \subseteq y^* \oplus x \oplus 0 = y^* \oplus 0 \oplus 0 = y^* \oplus 0$, it turns out that $y \ll 0$, so $y \in [0]$. Hence $[0] = \{x \in M | x \ll 0\}$ is a type-1 ideal of M . $\square$

According to Proposition 4.1, if $I \in \mathbb{I}_1(M)$, then $x \oplus y \ll I$ for any $x, y \in I$. However, we need to point out that if $I \in \mathbb{I}_2(M)$, then for any $x, y \in I$, the result $x \oplus y \ll I$ may be not true.

Example 4.3. Let $M = \{0, b, 1\}$ and the operations are given by the following tables:

$\oplus$	0	b	1				
0	$\{0\}$	$\{0, b\}$	M	*	0	b	1
b	$\{0, b\}$	$\{1\}$	M		1	b	0
1	M	M	M				

Then $\langle M, \oplus, *, 0 \rangle$ is a hyper MV-algebra [17] and so it is a hyper quasi-MV algebra. Denote $I = \{0, b\}$. Then $I \in \mathbb{I}_2(M)$, however, $b \oplus b = \{1\} \not\ll I$.

Proposition 4.5. Let M be a hyper quasi-MV algebra and I be a non-empty subset of M . Then the following conditions are equivalent

- (1) $I \in \mathbb{I}_2(M)$ and if $x, y \in I$, then $x \oplus y \ll I$;
- (2) $I \in \mathbb{I}_2(M)$ and if $x, y \in I$, then $(x \oplus y) \cap I \neq \emptyset$.

Proof. (1) $\Rightarrow$ (2) Let $x, y \in I$ and $x \oplus y \ll I$. Then there exist $a \in x \oplus y$ and $b \in I$ such that $a \ll b$. Note that $I \in \mathbb{I}_2(M)$, we have $a \in I$. Hence $(x \oplus y) \cap I \neq \emptyset$.

(2) $\Rightarrow$ (1) Suppose that $x, y \in I$ and $(x \oplus y) \cap I \neq \emptyset$. Then we can suppose that $a \in (x \oplus y) \cap I$, it turns out that $a \in x \oplus y$ and $a \in I$, so $x \oplus y \ll I$. $\square$

Definition 4.2. Let M be a hyper quasi-MV algebra and I be a non-empty subset of M . Then I is called a *hyper quasi-MV type-1 weak ideal* (*type-1 weak ideal*, for short), if it satisfies any one of the equivalent conditions in Proposition 4.5.

Proposition 4.6. Let M be a hyper quasi-MV algebra. Then any type-1 ideal of M is a type-1 weak ideal of M .

Proof. Follows from Proposition 4.2 and Proposition 3.2(hq7). $\square$

Below we give another type of ideals in hyper quasi-MV algebras.

Definition 4.3. Let M be a hyper quasi-MV algebra and I be a non-empty subset of M . Then I is called a *hyper quasi-MV type-2 ideal* of M (*type-2 ideal*, for short), if it satisfies the following conditions:

- (1) $0 \in I$;
- (2) if $x, y \in I$, then $x \oplus y \subseteq I$;
- (3) if $x \in I$ and $y \in M$, then $(x \odot y) \cap I \neq \emptyset$.

Proposition 4.7. Let M be a hyper quasi-MV algebra and $\{I_i | i \in S\}$ be a family of type-2 ideals of M . Then $\bigcap_{i \in S} I_i$ is also a type-2 ideal of M .

Proof. Since $0 \in \bigcap_{i \in S} I_i$, we have that $\bigcap_{i \in S} I_i$ is a non-empty subset of M . Let $x, y \in \bigcap_{i \in S} I_i$. Then for any $i \in S$, we have $x \in I_i$ and $y \in I_i$, it follows that $x \oplus y \subseteq I_i$, so $x \oplus y \subseteq \bigcap_{i \in S} I_i$. Now, suppose that $x \in \bigcap_{i \in S} I_i$ and $y \in M$. Then for any $i \in S$, $x \in I_i$ and $(x \odot y) \cap I_i \neq \emptyset$, so $(x \odot y) \cap (\bigcap_{i \in S} I_i) \neq \emptyset$. Hence $\bigcap_{i \in S} I_i$ is a type-2 ideal of M . $\square$

Proposition 4.8. *Let M be a hyper quasi-MV algebra. Then any type-1 ideal of M is a type-2 ideal of M .*

Proof. Let I be a type-1 ideal of M and $x \in I$. Since $0 \ll x$, we have $0 \in I$. Now, suppose that $x \in I$ and $y \in M$. Since $x \odot y \ll x$ by (hq13), there exists $a \in x \odot y$ such that $a \ll x$, it follows that $a \in I$. Hence $(x \odot y) \cap I \neq \emptyset$. $\square$

Proposition 4.9. *Let M be a hyper quasi-MV algebra. Then I is a type-1 ideal of M if and only if (1) I is a type-2 ideal of M and (2) $y \in I \Leftrightarrow y \odot 1 \ll I$.*

Proof. Let I be a type-1 ideal of M and $x \in I$. Then I is a type-2 ideal of M by Proposition 4.7. Moreover, if $y \in I$, then $y \odot 1 \ll y$ by (hq13), so $y \odot 1 \ll I$. Conversely, if $y \odot 1 \ll I$, then $y \odot 0^* \ll I$. Since $0 \in I$ and I is a type-1 ideal, it follows that $y \in I$.

Conversely, suppose that I is a type-2 ideal of M and let $x \in I$ and $y \in M$ such that $y \ll x$. We want to prove $y \in I$. By (2), we only show that $y \odot 1 \ll I$. Because $1 \in y^* \oplus x \subseteq y^* \oplus x \oplus 0 = (y^* \oplus 0) \oplus x = (y \odot 1)^* \oplus x$, there exists $a \in y \odot 1$ such that $1 \in a^* \oplus x$, it follows that $a \ll x$. Note that $x \in I$, we get $y \odot 1 \ll I$. $\square$

Definition 4.4. Let M be a hyper quasi-MV algebra and I be a non-empty subset of M . Then I is called a *hyper quasi-MV type-2 weak ideal* of M (*type-2 weak ideal*, for short), if it satisfies the following conditions

- (1) $0 \in I$;
- (2) if $x, y \in I$, then $x \oplus y \ll I$;
- (3) if $x \in I$ and $y \in M$, then $(x \odot y) \cap I \neq \emptyset$.

Proposition 4.10. *Let M be a hyper quasi-MV algebra. Then $\{0\}$ is a type-2 weak ideal of M .*

Proof. Suppose that $x, y \in \{0\}$. Then $x \oplus y = 0 \oplus 0$. Since $0 \in 0 \oplus 0$, we have $x \oplus y \ll \{0\}$. Let $x \in \{0\}$ and $y \in M$. Then we have $x \odot y = 0 \odot y$. And by (hq12), we have $0 \in 0 \odot y$, so $(x \odot y) \cap \{0\} \neq \emptyset$. $\square$

Proposition 4.11. *Let M be a hyper quasi-MV algebra. Then any type-2 ideal of M is a type-2 weak ideal of M .*

Proof. It is obvious to get the result by (hq7). $\square$

Proposition 4.12. *Let M be a hyper quasi-MV algebra. Then any type-1 weak ideal of M is a type-2 weak ideal of M .*

Proof. The proof is similar to Proposition 4.8. $\square$

Proposition 4.13. *Let M be a hyper quasi-MV algebra and I be a non-empty subset of M . Then I is a type-1 weak ideal of M if and only if (1) I is a type-2 weak ideal of M and (2) $y \in I \Leftrightarrow y \odot 1 \ll I$.*

Proof. The proof is similar to Proposition 4.9. $\square$

Theorem 4.1. *Let M_1 and M_2 be hyper quasi-MV algebras and consider the hyper quasi-MV algebra $M_1 \times M_2$. Then*

(1) *If I_1 and I_2 are two type-1/type-2 ideals (type-1/type-2 weak ideals) of M_1, M_2 , respectively, then $I_1 \times I_2$ is a type-1/type-2 ideal (type-1/type-2 weak ideal) of $M_1 \times M_2$.*

(2) *If I is a type-1/type-2 ideal (type-1/type-2 weak ideal) of $M_1 \times M_2$, then there are two unique type-1/type-2 ideals (type-1/type-2 weak ideals) I_1 and I_2 of M_1 and M_2 , respectively, such that $I = I_1 \times I_2$.*

Proof. The proof is based on Proposition 3.1 and the definitions of ideals and weak ideals. $\square$

Recall that in a hyper MV-algebra, Ghorbani introduced the notions of hyper-MV filters and weak hyper-MV filters in [8]. Furthermore, Davvaz et al. defined hyper-MV ideals in [20] and clarified the connection between hyper-MV filters and hyper-MV ideals. In the following, we generalize these results in a hyper quasi-MV algebra.

Definition 4.5. Let M be a hyper quasi-MV algebra and F be a non-empty subset of M . Then

(1) F is called a *hyper quasi-MV type-1 filter* (type-1 filter, for short) of M , if it satisfies the following conditions:

(hf11) if $x, y \in F$, then $x \odot y \subseteq F$;

(hf12) if $x \in F$ and $y \in M$ such that $F \ll x^* \oplus y$, then $y \in F$.

(2) F is called a *hyper quasi-MV type-1 weak filter* (type-1 weak filter, for short) of M , if it satisfies the following conditions:

(hwf11) if $x, y \in F$, then $x \odot y \ll F$;

(hwf12) if $x \in F$ and $y \in M$ such that $x \ll y$, then $y \in F$.

(3) F is called a *hyper quasi-MV type-2 filter* (type-2 filter, for short) of M , if it satisfies the following conditions:

(hf21) $1 \in F$;

(hf22) if $x, y \in F$, then $x \odot y \subseteq F$;

(hf23) if $x \in F$ and $y \in M$, then $(x \oplus y) \cap F \neq \emptyset$.

(4) F is called a *hyper quasi-MV type-2 weak filter* (type-2 weak filter, for short) of M , if it satisfies the following conditions:

(hwf21) $1 \in F$;

(hwf22) if $x, y \in F$, then $x \odot y \ll F$;

(hwf23) if $x \in F$ and $y \in M$, then $(x \oplus y) \cap F \neq \emptyset$.

Proposition 4.14. *Let M be a hyper quasi-MV algebra and I be a non-empty subset of M . Then*

- (1) *I is a type-1 ideal of M if and only if I^* is a type-1 filter of M ;*
- (2) *I is a type-1 weak ideal of M if and only if I^* is a type-1 weak filter of M ;*
- (3) *I is a type-2 ideal of M if and only if I^* is a type-2 filter of M ;*
- (4) *I is a type-2 weak ideal of M if and only if I^* is a type-2 weak filter of M .*

Proof. We only prove (1) and the others can be proved similarly. Let I be a type-1 ideal of M . Then for any $x^*, y^* \in I^*$, we have $x \odot y = (x^* \oplus y^*)^*$. Since $x^*, y^* \in I$ and $x^* \oplus y^* \subseteq I$, we have $x \odot y \subseteq I^*$. Suppose that $x \in I^*$ and $y \in M$ such that $I^* \ll x^* \oplus y$. Then $(x^* \oplus y)^* \ll I$, i.e., $(x^*)^* \odot y^* \ll I$. Since $x^* \in I$ and $y^* \in M$, we have $y^* \in I$, so $y \in I^*$. The converse can be proved similarly. $\square$

Remark 4.3. Let M be a hyper MV-algebra and I be a non-empty subset of M . Following from our results, I is a hyper MV-filter of M defined in [8] if and only if I^* is a hyper MV-deductive system of M defined in [11]. Meanwhile, the conditions of these two definitions are non-independent. The condition (i) in hyper MV-filters can be implied by condition (ii). On the other hand, hyper MV-ideals defined in [20] are same as hyper MV-ideals defined in [17] and they are hyper MV-deductive systems.

In the end of this section, we present some properties of subalgebras for hyper quasi-MV algebras.

Definition 4.6. Let M be a hyper quasi-MV algebra and S be a non-empty subset of M . If S is a hyper quasi-MV algebra with respect to the hyper operation $\oplus$ and the unary operation $*$ of M , then S is called a *hyper quasi-MV subalgebra* of M .

Proposition 4.15. *Let M be a hyper quasi-MV algebra and S be a non-empty subset of M . Then S is a hyper quasi-MV subalgebra of M if and only if $x^* \in S$ and $x \oplus y \subseteq S$ for any $x, y \in S$.*

Proof. It is similar to the case of hyper MV-algebras in [8]. $\square$

Corollary 4.1. *Let M be a hyper quasi-MV algebra and S be a non-empty subset of M . If S is a hyper quasi-MV subalgebra of M , then $0 \in S$ and $x^* \oplus y \subseteq S$.*

Proposition 4.16. *Let M be a hyper quasi-MV algebra and I be a proper type-2 ideal of M . Then it is not a subalgebra of M .*

Proof. If not, then I is a subalgebra of M , so by Proposition 4.15, we have $x^* \in I$, it turns out that $1 \in x \oplus x^* \subseteq I$, this is a contradiction with I proper. $\square$

5. Homomorphisms

In this section, we introduce the homomorphism, weak homomorphism and dual weak homomorphism of hyper quasi-MV algebras and investigate some properties of ideals and weak ideals under these mappings.

Definition 5.1. Let M_1 and M_2 be hyper quasi-MV algebras.

A mapping $f : M_1 \rightarrow M_2$ is called a *homomorphism*, if for any $x, y \in M_1$, we have

- (1) $f(x \oplus y) = f(x) \oplus f(y)$;
- (2) $f(x^*) = f(x)^*$;
- (3) $f(0) = 0$.

A mapping f is called a *weak homomorphism*, if for any $x, y \in M_1$, we have

- (1) $f(x \oplus y) \subseteq f(x) \oplus f(y)$;
- (2) $f(x^*) = f(x)^*$;
- (3) $f(0) = 0$.

A mapping f is called a *dual weak homomorphism*, if for any $x, y \in M_1$, we have

- (1) $f(x \oplus y) \supseteq f(x) \oplus f(y)$;
- (2) $f(x^*) = f(x)^*$;
- (3) $f(0) = 0$.

Obviously, any homomorphism is a (dual) weak homomorphism. Moreover, if f is a (dual) weak homomorphism, then $f(1) = 1$.

Proposition 5.1. Let M_1 and M_2 be hyper quasi-MV algebras and $f : M_1 \rightarrow M_2$ be a homomorphism. Then

- (1) $f(x \odot y) = f(x) \odot f(y)$;
- (2) $f(x \vee y) = f(x) \vee f(y)$;
- (3) $f(x \wedge y) = f(x) \wedge f(y)$.

The following results can be proved similarly as hyper MV-algebras [20].

Proposition 5.2. Let M_1 and M_2 be hyper quasi-MV algebras, $f : M_1 \rightarrow M_2$ be a weak homomorphism and $g : M_1 \rightarrow M_2$ be a dual weak homomorphism. Then the following properties hold:

- (1) for any $x, y \in M_1$, if $x \ll y$, then $f(x) \ll f(y)$.
- (2) if A is a subalgebra of M_1 , then $g(A)$ is a subalgebra of M_2 and if B is a subalgebra of M_2 , then $f^{-1}(B)$ is a subalgebra of M_1 .
- (3) if $h : M_1 \rightarrow M_2$ is a homomorphism, $x \ll y$, A and B are subalgebras of M_1 and M_2 , respectively, then $h(x) \ll h(y)$, $h(A)$ is a subalgebra of M_2 and $h^{-1}(B)$ is a subalgebra of M_1 .

Proposition 5.3. Let M_1 and M_2 be hyper quasi-MV algebras, $f : M_1 \rightarrow M_2$ be a mapping. Then

- (1) if M_2 is a quasi-MV algebra and M_1 is a hyper quasi-MV algebra, then f is a weak homomorphism if and only if f is a homomorphism.

(2) if M_1 is a quasi-MV algebra and M_2 is a hyper quasi-MV algebra, then f is a dual weak homomorphism if and only if f is a homomorphism.

(3) if M_1 and M_2 are quasi-MV algebras, then f is a weak homomorphism (dual weak or a homomorphism) if and only if f is the usual homomorphism of quasi-MV algebras.

Proposition 5.4. *Let M_1 , M_2 and M_3 be hyper quasi-MV algebras and $f : M_1 \rightarrow M_2$, $g : M_2 \rightarrow M_3$ be weak homomorphisms (dual weak homomorphism) while $g \circ f$ be a dual weak homomorphism (weak homomorphism). Then*

- (1) *if f is onto, then g is a homomorphism.*
- (2) *if g is one-to-one, then f is a homomorphism.*

Below we will see the properties of ideals and weak ideals under homomorphisms.

Theorem 5.1. *Let M_1 and M_2 be hyper quasi-MV algebras and $f : M_1 \rightarrow M_2$ be a homomorphism. Then*

- (1) *if I is a type-1 (weak) ideal of M_2 , then $f^{-1}(I)$ is a type-1 (weak) ideal of M_1 ; if I is a type-2 (weak) ideal of M_2 , then $f^{-1}(I)$ is a type-2 (weak) ideal of M_1 .*
- (2) *Denote $\ker(f) = \{x \in M_1 | f(x) = 0\}$. then $\ker(f)$ is a type-2 weak ideal of M_1 .*
- (3) *if f is one-to-one, then $\ker(f) = \{0\}$.*
- (4) *if f is onto and I is a type-2 (weak) ideal of M_1 , then $f(I)$ is a type-2 (weak) ideal of M_2 ; if f is onto and I is a type-1 (weak) ideal of M_1 which contains $\ker(f)$, then $f(I)$ is a type-1 (weak) ideal of M_2 .*

Proof. (1) Suppose that $x, y \in f^{-1}(I)$. Then there exist $a, b \in I$ such that $f(x) = a$ and $f(y) = b$, we have $f(x \oplus y) = f(x) \oplus f(y) = a \oplus b \subseteq I$, thus $x \oplus y \subseteq f^{-1}(I)$. Let $x \in f^{-1}(I)$ and $y \in M_1$ such that $y \ll x$. Then $f(y) \ll f(x)$ by Proposition 5.2. Since $f(x) \in I$ and I is a type-1 ideal of M_2 , we have $f(y) \in I$, so $y \in f^{-1}(I)$. Hence $f^{-1}(I)$ is a type-1 ideal of M_1 . The case of type-1 weak ideal can be proved similarly.

Now, we prove the case of type-2 ideal. Obviously, $0 \in f^{-1}(I)$. Suppose that $x, y \in f^{-1}(I)$. Then $x \oplus y \subseteq f^{-1}(I)$ can be proved as above. Let $x \in f^{-1}(I)$ and $y \in M$. Then by Proposition 5.1, $f(x \odot y) \cap I = (f(x) \odot f(y)) \cap I \neq \emptyset$, so there exists $a \in f(x \odot y) \cap I$, we have $a \in f(x \odot y)$ and $a \in I$, it turns out that there exists $m \in x \odot y$ such that $a = f(m) \in I$, so $m \in f^{-1}(I)$ and $(x \odot y) \cap f^{-1}(I) \neq \emptyset$. The case of type-2 weak ideal can be proved similarly.

(2) By (1) and Proposition 4.10.

(3) Obviously, $\{0\} \subseteq \ker(f)$. For any $x \in \ker(f)$, then $f(x) = 0 = f(0)$. Since f is one-to-one, we have $x = 0$.

(4) Since $0 = f(0)$ and $0 \in I$, we have $0 \in f(I)$. Let $x, y \in f(I)$. Then there exist $a, b \in I$ such that $f(a) = x$ and $f(b) = y$. We have $x \oplus y = f(a) \oplus f(b) = f(a \oplus b)$. Because $a \oplus b \subseteq I$, it follows that $x \oplus y \subseteq f(I)$. Let $x \in f(I)$ and

$y \in M_2$. Then there exist $a \in I$ and $b \in M_1$ such that $x = f(a)$ and $y = f(b)$. We have $(x \odot y) \cap f(I) = (f(a) \odot f(b)) \cap f(I) = f((a \odot b) \cap I)$. Since $(a \odot b) \cap I \neq \emptyset$, it turns out that $(x \odot y) \cap f(I) \neq \emptyset$. The case of type-2 weak ideal can be proved similarly.

Now, let $x \in f(I)$ and $y \in M_2$ such that $y \ll x$. Then there exist $a \in I$ and $b \in M_1$ such that $f(a) = x$, $f(b) = y$ and $f(b) \ll f(a)$, it turns out that $0 \in f(b) \odot f(a)^* = f(b \odot a^*)$, so there exist $m \in b \odot a^*$ such that $0 = f(m)$, we have $m \in \ker(f) \subseteq I$, thus $b \odot a^* \ll I$. Since I is type-1 ideal of M_1 , we get $b \in I$ and so $y \in f(I)$. The case of type-1 weak ideal can be proved similarly. $\square$

Definition 5.2. Let M be a hyper quasi-MV algebra and I be a proper (weak) ideal of M . Then I is called *maximal*, if $I \subseteq J \subseteq M$ for some (weak) ideal J of M , then $J = I$ or $J = M$.

Theorem 5.2. Let M_1 and M_2 be hyper quasi-MV algebras and $f : M_1 \rightarrow M_2$ be an epimorphism. Then

- (1) if I is a maximal type-1 (weak) ideal of M_2 , then $f^{-1}(I)$ is a maximal type-1 (weak) ideal of M_1 which contains $\ker(f)$.
- (2) if I is a maximal type-1 (weak) ideal of M_1 which contains $\ker(f)$, then $f(I)$ is a maximal type-1 (weak) ideal of M_2 .
- (3) the mapping $I \mapsto f(I)$ is bijective corresponding between the maximal type-1 (weak) ideals of M_1 containing $\ker(f)$ and maximal type-1 (weak) ideals of M_2 .

Proof. (1) By Theorem 5.1 (1), we have that $f^{-1}(I)$ is a type-1 (weak) ideal of M_1 . Now, let J be a type-1 (weak) ideal of M_1 such that $f^{-1}(I) \subsetneq J \subseteq M_1$. Then $I \subsetneq f(J)$. Since I is a maximal type-1 (weak) ideal of M_2 , we have $f(J) = M_2$, so $J = M_1$. If not, there exists $x \in M_1 \setminus J$ and $f(x) \in M_2$, so there exists $a \in J$ such that $f(a) = f(x)$, we have $0 \in f(x) \odot f(a)^* = f(x \odot a^*)$, so there is $m \in x \odot a^*$ such that $0 = f(m)$, it turns out that $m \in \ker(f) \subseteq f^{-1}(I) \subsetneq J$, thus $x \odot a^* \ll J$ and then $x \in J$, this is a contradiction. Hence $f^{-1}(I)$ which contains $\ker(f)$ is a maximal type-1 (weak) ideal of M_1 .

(2) By Theorem 5.1 (4), we have that $f(I)$ is a type-1 (weak) ideal of M_2 . Suppose that J be a type-1 (weak) ideal of M_2 such that $f(I) \subsetneq J \subseteq M_2 = f(M_1)$. Then $J = f(J')$ and $I \subsetneq J' \subseteq M_1$ where J' is a type-1 (weak) ideal of M_1 . Since I is a maximal ideal of M_1 , we have $J' = M_1$, so $J = M_2$.

(3) Based on (1) and (2). $\square$

6. Conclusion

In this paper, we introduce hyper quasi-MV algebras and mainly study ideals and weak ideals of hyper quasi-MV algebras. In the future work, on the one hand, we will focus on the quotient algebra of hyper quasi-MV algebra. On the other hand, the non-commutative generalization of quasi-MV algebras had been

introduced in [4, 5], it is natural to study the corresponding hyper structure as the generalization of hyper quasi-MV algebras and pseudo-quasi-MV algebras.

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ON AN EXTENSION OF THE DUBINS CONDITIONAL PROBABILITY AXIOMATIC TO COHERENT PROBABILITY OF FUZZY EVENTS

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Abstract. An approach to the concept of fuzzy event as an extension of conditional event is introduced. The probability of fuzzy events is presented as an extension of the de Finetti's probability of conditional events and depends on a score function subjectively assigned by an expert. It is shown that the introduced fuzzy probability extends in a fuzzy ambit the conditions considered by Dubins for finitely additive conditional probability. Possible applications for decision-making under uncertainty are sketched.

Keywords: conditional events, finitely additive conditional probability, fuzzy events, fuzzy probability, decision-making under uncertainty.

1. Introduction

According to the de Finetti's theory (1970), a conditional event $A|B$ is considered as a logical entity with three truth values: true, false, and a third truth value that can be: undeterminate or empty or undecidable or other term, depending occurring $A \cap B, A^c \cap B, B^c$. Many authors have studied the three-valued logic (Reichenbach, 1944: 269-297; Gentilhomme, 1968: 54; De Finetti, 1970:685-686; Fadini, 1979:41; Nguyen et al., 2003: 1061), attributing different terms and meanings to the third truth value. However, the logical operations do not depend on the term which is used. Following the literature regarding the three value logic (Gentilhomme, 1968; Fadini, 1979:45; Lane, 1999; Nguyen et al., 2003: 1061; Negarestani, 2012: 54), we represent the three logical values "true", "false", and "third truth value" with "1", "0", and "1/2", respectively.

The formalization of conditional probability of de Finetti has been largely studied in the literature (de Finetti, 1970; Dubins, 1975; Scozzafava, 1993, 2001; Yexin et al., 2005; Mundici, 2006; Montagna et al., 2013; Flaminio et al., 2015a; 2015b). In many of these studies the assessment of conditional probability is obtained by the concept of coherent bet. Various authors (e.g. Zadeh, 1968; Yager, 1999; Maturo, 2000; Mundici, 2006; Maturo and Doria, 2008) have considered fuzzy events and fuzzy probability. In this paper, we follow a different perspective; specifically, we consider the concept of fuzzy event as an extension of conditional event, propose a possible approach to the idea of probability of

fuzzy events, and study their mathematical properties. Moreover, we introduce some possible meanings of fuzzy event and fuzzy probability (which are defined following our approach) in the context of decisional processes. Furthermore, we aim to discover formulas for the probability of fuzzy events, which satisfy, under broad conditions, the mathematical properties of conditional events, in particular the axiomatic conditions of Dubin (1976) for the finitely additive conditional probability.

2. Functional and algebraic representations of conditional events

In the subjective approach to the calculus of probability (e.g. de Finetti, 1970; Dubins, 1975; Scozzafava, 1993, 2001), a conditional event $A|B$, defined if B is a non-impossible event, is a statement assuming the following values: true if the intersection $A \cap B$ is true, false if the intersection $A^c \cap B$ is true (where A^c is the contrary of A), and undetermined if B is false. In this paper, for a better algebraic treatment, we also consider the conditional event $A|B$ with $B = \emptyset$, called the totally undetermined conditional event.

To show how fuzzy events, as will be defined in this paper, can be seen as extensions of conditional events, let's introduce an alternative (but logically equivalent) definition of conditional event $A|B$. Then we give the following definition.

Definition 2.1. Let Π be a partition of the certain event. We define *conditional event in functional form (briefly FFCE)*, with *domain* Π , every function $X : \Pi \rightarrow \{0, 1/2, 1\}$. The union of the elements $a \in \Pi$ such that $X(a) = 1$ (resp. $X(a) = 0, X(a) = 1/2$) is said to be the *true part* of X , denoted by T_X (resp. the *false part* of X , denoted by F_X , and the *undetermined part* of X , indicated with U_X). The event $D_X = T_X \cup F_X$ is called *the determined part* of X .

Definition 2.2. If X and Y are FFCE, we say that X is equivalent to Y , we write $X \sim Y$ if $(T_X, U_X, F_X) = (T_Y, U_Y, F_Y)$.

Evidently, a *FFCEX*, and all the *FFCEY* equivalent to X , individuate the conditional event in normal form $T_X|D_X$ and vice versa, every conditional event $A|B$ represents the equivalence class of the *FFCE* with $T_X = A \cap B$, $U_X = B^c$, $F_X = A^c \cap B$. In other words, an *FFCF* equivalence class is a random number with codomain $\{0, 1/2, 1\}$.

The contrary of X is the conditional event $X^c : \Pi \rightarrow \{0, 1/2, 1\}$ having the same domain and determined part of X , but with false part equal to the true part of X .

The triplet (T_X, U_X, F_X) is said to be the algebraic representation of X . For indicating that X is a conditional event with algebraic representation (T_X, U_X, F_X) , we write $X = (T_X, U_X, F_X)$.

Definition 2.3. Two conditional events X and Y are:

- disjoint, if $T_X \cap T_Y = \emptyset$;

- exhaustive, if $F_X \cap F_Y = \emptyset$;
- homogeneous, if $D_X = D_Y$.

Clearly, X and Y are contrary, if and only if, they are disjoint, exhaustive, and homogeneous.

In the subjective probability framework, a conditional event X is interpreted as a bet in which a decision-maker D wins if the true part occurs, loses if the false part happens, and the bet is canceled if the undetermined part is obtained.

Given two partitions Π_1 and Π_2 of the certain event, the product $\Pi_1 \bullet \Pi_2$ is the set of the not impossible intersections $a \cap b$, with $a \in \Pi_1$, $b \in \Pi_2$.

Definition 2.4. Let $*$ be an operation on $\{0, 1/2, 1\}$. For all conditional events $X : \Pi_X \rightarrow \{0, 1/2, 1\}$, $Y : \Pi_Y \rightarrow \{0, 1/2, 1\}$, the product of X and Y induced by $*$, or “*-product”, is the conditional event $X * Y : \Pi_X \bullet \Pi_Y \rightarrow \{0, 1/2, 1\}$ such that:

$$(z \in \Pi_X \bullet \Pi_Y, z = a \cap b, a \in \Pi_X, b \in \Pi_Y) \Rightarrow (X * Y)(z) = X(a) * Y(b).$$

Important operations in $\{0, 1/2, 1\}$ are:

- the union (or disjunction) $\cup$, defined by: $a \cup b = \max\{a, b\}$;
- the intersection (or conjunction) $\cap$, defined by: $a \cap b = \min\{a, b\}$.

3. Fuzzy event as an extension of the concept of conditional event

We assume the following definition of fuzzy event as extension of conditional event.

Definition 3.1. Let Π be a partition of the certain event. A *fuzzy event* with *domain* Π is a function $X : \Pi \rightarrow [0, 1]$. The union of the elements $a \in \Pi$ such that $X(a) > 1/2$ (resp. $X(a) < 1/2$, $X(a) = 1/2$) is said to be the *true part* of X , denoted with T_X (resp. *false part* of X , denoted with F_X , and *undetermined part* of X , denoted with U_X). The event $D_X = T_X \cup F_X$ is the *determined part* of X .

The *contrary* of X is the fuzzy event $X^c : \Pi \rightarrow [0, 1]$ such that, $\forall a \in \Pi$, $X^c(a) = 1 - X(a)$.

The conditional event X_0 , with domain Π and such that $X_0(a) = 1, 1/2, 0$ depending on which $X(a) > 1/2, X(a) = 1/2, X(a) < 1/2$, is said to be the *crisp approximation* of X . In algebraic form $X_0 = (T_X, U_X, F_X)$.

Let V_X be the set of all the values assumed by the fuzzy event $X : \Pi \rightarrow [0, 1]$. For all $v \in V_X$, the union of the $a \in \Pi$ such that $X(a) = v$ is called *part at level* v of X , denoted X_v . The fuzzy event $X^* : \{X_v, v \in V_X\} \rightarrow [0, 1]$ that associates to X_v the number v is said to be the *normal form* of X . Using a slight modification of the notations introduced by Zadeh (Zadeh, 1965, 1968, 1975a, 1975b; Klir, Yuan, 1995; Zadeh et al., 1996), we write $X = \{v/X_v, v \in V_X\}$.

Two fuzzy events X and Y will be said equivalent if they have the same normal form, i.e. $V_X = V_Y$ and, $\forall v \in V_X$, $X_v = Y_v$.

With this notation, a $FFCEX = (T_X, U_X, F_X)$ is a fuzzy event such that $V_X \subseteq \{1, 1/2, 0\}$. Using the fuzzy notation, we can write $X = \{1/T_X, 0.5/U_X, 0/F_X\}$.

Extending to fuzzy events the basic ideas of subjective setting of probability theory, a fuzzy event X is interpreted as a bet in which a decision-maker D partially or totally wins if the true part occurs, partially or totally loses if the false part happens, and the bet is canceled if the undetermined part is obtained. Unlike the conditional event, the amount of winning (if $v > 1/2$) or loss (if $v < 1/2$) depends on the truth-value v , i.e. it decreases if v approaches $1/2$, increases when v approaches the extremes of the interval $[0, 1]$, and is null if $v = 1/2$.

The concepts expressed by the definition 2.3 can be extended in various ways to fuzzy events. For this purpose, we introduce the following definition:

Definition 3.2. Two fuzzy events X and Y are called:

- quasi-disjoint, if $T_X \cap T_Y = \emptyset$;
- quasi-exhaustive, if $F_X \cap F_Y = \emptyset$;
- quasi-homogenous, if $D_X = D_Y$.

In general, for each fuzzy event X , we use the prefix "quasi" for indicating a property that holds for the crisp approximation X_0 of X .

We can extend the definition 2.4 to fuzzy events as follows:

Definition 3.3. Let $*$ be an operation in $[0, 1]$. If $X : \Pi_X \rightarrow [0, 1]$ and $Y : \Pi_Y \rightarrow [0, 1]$ are two fuzzy events, we define product of X and Y induced by $*$, or " $*$ - product", the fuzzy event $X * Y : \Pi_X \bullet \Pi_Y \rightarrow [0, 1]$ such that:

$$(z \in \Pi_X \bullet \Pi_Y, z = a \cap b, a \in \Pi_X, b \in \Pi_Y) \Rightarrow (X * Y)(z) = X(a) * Y(b).$$

The most important operations in $[0, 1]$ we will use during the work are the following:

- the union (or disjunction) $\cup$, defined by: $a \cup b = \max\{a, b\}$;
- the intersection (or conjunction) $\cap$, defined by: $a \cap b = \min\{a, b\}$;

4. An approach to subjective probability of fuzzy events

Let Π be a finite partition of the certain event and let $X : \Pi \rightarrow \{0, 1/2, 1\}$ be a conditional event. The subjective conditional probability $p = p(X)$ of X is defined from a bet, with stake S and possible winning $V \geq S$. Three circumstances may happen:

- if T_X occurs, he receives the winning V ;
- if U_X occurs, he gets back the stake S ;
- if F_X happens, he loses the stake S .

The subjective conditional probability p is defined as the ratio $p = S/V$. Therefore, the gain of D is the random number:

$$(4.1) \quad G = [(1 - p)|T_X| - p|F_X|]V,$$

where, for any event E , we denote by $|E|$ its indicator.

The bet is coherent with respect a probability $p_1 : \Pi \rightarrow [0, 1]$, if and only if the prevision of G is null, i.e.

$$(4.2) \quad [p_1(T_X) + p_1(F_X)]p = p_1(T_X),$$

where $p_1(T_X)$ and $p_1(F_X)$ are, respectively, the probabilities of the true part and the false part of X , and $p_1(T_X) + p_1(F_X) = p_1(D_X)$.

If $p_1(D_X) \neq 0$, we get a single coherent value for p . Precisely, if we put $Q_X^1 = p_1(D_X) = p_1(T_X) + p_1(F_X)$, $Q_X^{+1} = p_1(T_X)$, $Q_X^{-1} = p_1(F_X)$, we have:

$$(4.3) \quad p(X) = Q_X^{+1}/Q_X^1, p(X^c) = Q_X^{-1}/Q_X^1.$$

If $p_1(D_X) = 0$, but $D_X \neq \emptyset$, we can consider the set $\Pi_1 = \{E \in \Pi : p_1(E) = 0\}$ and we can assign (de Finetti, 1970) a subjective conditional probability $p_2(E) = p(E/\Pi_1)$ for every $E \in \Pi$ with the condition

$$(4.4) \quad \sum \{p_2(E) : E \in \Pi_1\} = 1.$$

The bet is coherent with respect the conditional probability $p_2 : \Pi \rightarrow [0, 1]$, if and only if the prevision of G is null, i.e.

$$(4.5) \quad [p_2(T_X) + p_2(F_X)]p = p_2(T_X).$$

If $p_2(D_X) \neq 0$, we get a single coherent value for p . Precisely, if we put $Q_X^2 = p_2(D_X) = p_2(T_X) + p_2(F_X)$, $Q_X^{+2} = p_1(T_X)$, $Q_X^{-2} = p_1(F_X)$, we have:

$$(4.6) \quad p(X) = Q_X^{+2}/Q_X^2, p(X^c) = Q_X^{-2}/Q_X^2.$$

Let us call 1st order probability and 2nd order probability, the probabilities given by (4.3) and (4.6), respectively. Proceeding for recurrence, as Π is finite, if $D_X \neq \emptyset$, we obtain an integer j and a j^{th} order probability for X and X^c given by formulae:

$$(4.7) \quad p(X) = Q_X^{+j}/Q_X^j, p(X^c) = Q_X^{-j}/Q_X^j.$$

where $Q_X^j = p_j(D_X) = p_j(T_X) + p_j(F_X)$, $Q_X^{+j} = p_j(T_X)$, $Q_X^{-j} = p_j(F_X)$.

We can get an extension of the subjective conditional probability to fuzzy events in various ways. We propose a possible approach, which is based on a suitable assessment of a "score function". The decision maker D assigns a "score function" $f : [0, 1] \rightarrow [0, 1]$, decreasing in $[0, 1/2]$, increasing in $[1/2, 1]$, and satisfying the conditions $f(0) = 1$, $f(1/2) = 0$, $f(1) = 1$.

Some desirable additional conditions are:

- (a) $f(x) = f(1 - x)$ (symmetry);
- (b) $f(x) \neq 0$ for $x \neq 1/2$ (positivity);
- (c) $f(x)$ is strictly decreasing in $[0, 1/2]$ and strictly increasing in $[1/2, 1]$;

(d) $f(x)$ is continuous.

The condition of symmetry has the following interesting implications:

(1) the number $2|x - 1/2|$ can be interpreted as the degree of occurrence or non-occurrence of an event of the partition with value x ;

(2) the meaning of $f(x)$ is that of a function that increases with the increase of the degree of occurrence or non-occurrence”.

The other three conditions can be useful in many contexts. Hereafter, we assume that the conditions of symmetry and positivity hold.

Suppose, from now on, that the domains of fuzzy events considered are finite.

Let $X : \Pi = \{A_1, A_2, \dots, A_n\} \rightarrow [0, 1]$ be a fuzzy event, $X(A_i) = x_i$. Let us define the gain of the bettor D as the random number:

$$(4.8) \quad G = [(1 - p)\Sigma\{f(x_i)|A_i| : x_i > 1/2\} - p\Sigma\{f(x_i)|A_i| : x_i < 1/2\}]V,$$

where $p = p(X)$ is the unknown probability of the fuzzy event X .

We assume that the bet is coherent with respect a probability $p_1 : \Pi \rightarrow [0, 1]$ if and only if the prevision of G is null. Then the coherence condition implies the formula:

$$(4.9) \quad [\Sigma\{f(x_i)p_1(A_i) : x_i \neq 1/2\}]p = \Sigma\{f(x_i)p_1(A_i) : x_i > 1/2\}.$$

If $p_1(D_X) \neq 0$, because of the condition of positivity, we obtain a single coherent value for p . Precisely, if we put

$$(4.10) \quad Q_X^1 = \Sigma\{f(x_i)p_1(A_i) : x_i \neq 1/2\},$$

$$(4.11) \quad Q_X^{+1} = \Sigma\{f(x_i)p_1(A_i) : x_i > 1/2\}, Q_X^{-1} = \Sigma\{f(x_i)p_1(A_i) : x_i < 1/2\},$$

we have formulae:

$$(4.12) \quad p(X) = Q_X^{+1}/Q_X^1, p(X^c) = Q_X^{-1}/Q_X^1,$$

that extends to fuzzy events formulae (4.3). If X is a conditional event, then p , defined by (4.12), reduces to the conditional probability.

If $p_1(D_X) = 0$, but $D_X \neq \emptyset$, we can consider the set $\Pi_1 = \{E \in \Pi : p_1(E) = 0\}$ and we can assign (de Finetti, 1970) a subjective conditional probability $p_2(E) = p(E/\Pi_1)$ for every $E \in \Pi$ with the condition $\Sigma\{p_2(E) : E \in \Pi_1\} = 1$.

Formulae (4.10), (4.11), and (4.12) are replaced by:

$$(4.13) \quad Q_X^2 = \Sigma\{f(x_i)p_2(A_i) : x_i \neq 1/2\},$$

$$(4.14) \quad Q_X^{+2} = \Sigma\{f(x_i)p_2(A_i) : x_i > 1/2\}, Q_X^{-2} = \Sigma\{f(x_i)p_2(A_i) : x_i < 1/2\},$$

and we have formulae:

$$(4.15) \quad p(X) = Q_X^{+2}/Q_X^2, p(X^c) = Q_X^{-2}/Q_X^2.$$

Proceeding for recurrence, as Π is finite, if $D_X \neq \emptyset$, we obtain an integer j and a j th order probability for X and X^c given by formulae:

$$(4.16) \quad p(X) = Q_X^{+j}/Q_X^j, p(X^c) = Q_X^{-j}/Q_X^j,$$

where

$$(4.17) \quad Q_X^j = \Sigma\{f(x_i)p_j(A_i) : x_i \neq 1/2\},$$

$$(4.18) \quad Q_X^{+j} = \Sigma\{f(x_i)p_j(A_i) : x_i > 1/2\}, Q_X^{-j} = \Sigma\{f(x_i)p_j(A_i) : x_i < 1/2\}.$$

Let us call “*subjective fuzzy probability (briefly SFP) associated to the function f and the probability p_j* ” the probability p given by (4.16).

In the theory of decisions under uncertainty, a fuzzy event X represents an alternative compared with a given alternative X_0 , called the null alternative, Q_X^{+j} represents the positive change (improvement) compared to the X_0 , Q_X^{-j} is the negative change (worsening), and Q_X^j is the total variation. Then, the fuzzy probability $p(X)$ is the ratio of the positive change to the total change, and can represent a significant index of the validity of the alternative X . In particular, if $p(X) > 1/2$, we can assume that X is better than the null strategy, while it is worst whether $p(X) < 1/2$.

The applications of fuzzy events to decision making may be particularly significant in multiagent decision-making (see, e.g., Maturo and Ventre, 2009) or in fuzzy statistical decisions related to, for example, fuzzy regression (see, e.g., Maturo, 2016; Maturo and Hošková-Mayerova, 2017; Maturo and Ventre, 2018).

An approximation of $p(X)$ is obtained by the probability of the crisp approximation X_0 of X . Precisely, if we put $Q_{X_0}^j = p_j(D_X^j)$, $Q_{X_0}^{+j} = p_j(T_X^j)$, $Q_{X_0}^{-j} = p_j(F_X^j)$, we have:

$$(4.19) \quad p(X_0) = Q_{X_0}^{+j}/Q_{X_0}^j, p(X_0^c) = Q_{X_0}^{-j}/Q_{X_0}^j.$$

Formula (4.16) reduces to (4.19) if $f(x) = 1$ for $x \neq 1/2$ and $f(1/2) = 0$. Such a function satisfies the conditions of symmetry and positivity but not the other conditions. Let us call it “the crisp score function”.

5. A fuzzy extension of the axiomatic definition by Dubins

In (Dubins, 1975) the following axiomatic definition of finitely additive conditional probability is assumed.

Definition 5.1. Let $\mathbf{E}$ be an algebra of events. A finitely additive conditional probability on E is a function $p : (A, B) \in (\mathbf{E} \times (\mathbf{E} - \{\emptyset\})) \rightarrow p(A|B) \in [0, 1]$ such that:

- (PC1) $\forall B \in (\mathbf{E} - \{\emptyset\})$, the partial function $p_B : A \in \mathbf{E} \rightarrow p(A|B)$ is a finitely additive probability on $\mathbf{E}$;
- (PC2) $\forall B \in (\mathbf{E} - \{\emptyset\})$, $p(B|B) = 1$;
- (PC3) $\forall A \in \mathbf{E}, \forall B, C \in (\mathbf{E} - \{\emptyset\}), A \subseteq B \subseteq C \Rightarrow p(A|B)p(B|C) = p(A|C)$.

To extend the definition 5.1 to fuzzy events, it is necessary to rewrite it in the fuzzy (functional) notation, which has been introduced in Sec 3.

Let us denote with Ω (called the totally true event), $\emptyset$ (totally false), and U (totally undetermined), the fuzzy events with $X(\Pi) = \{1\}$, $X(\Pi) = \{0\}$, and $X(\Pi) = \{1/2\}$, respectively. The definition 5.1 is equivalent to the following one in functional notation:

Definition 5.2. Let C_E be the family of the conditional events $X : \Pi \rightarrow \{0, 1/2, 1\}$ with the domain Π contained in an algebra of events $\mathbf{E}$. A finitely additive conditional probability on C_E is a function $p : X \in C_E^* = C_E - \{U\} \rightarrow p(X) \in [0, 1]$ such that:

$$(PA1) \quad \forall X, Y \in C_E^*, (T_{X \cap Y} = \emptyset, D_X = D_Y) \Rightarrow p(X \cup Y) = p(X) + p(Y);$$

$$(PA2) \quad \forall X \in C_E^*, (F_X = \emptyset) \Rightarrow p(X) = 1;$$

$$(PA3) \quad \forall X, Y, Z \in C_E^*, (T_X = T_Z, D_X = T_Y, D_Y = D_Z) \Rightarrow p(X)p(Y) = p(Z).$$

Let us extend to fuzzy events the notions of disjoint, exhaustive, and homogeneous events. Without loss of generality, considering eventually refinements of the domains of the fuzzy events, we can assume that each pair of considered fuzzy events has the same domain.

Definition 5.3. Let $X : \Pi = \{E_1, E_2, \dots, E_n\} \rightarrow [0, 1], Y : \Pi = \{E_1, E_2, \dots, E_n\} \rightarrow [0, 1]$, be two fuzzy events, $X(E_i) = x_i, Y(E_i) = y_i$. We say that X and Y are:

- homogeneous, if $x_i = y_i$ or $x_i = 1 - y_i$;
- disjoint, if $x_i + y_i \leq 1$;
- exhaustive, if $x_i + y_i \geq 1$.

The homogeneity is equivalent to the fact that the values x_i and y_i are equidistant from $1/2$. In particular $(X \text{ and } Y \text{ homogeneous}) \Rightarrow (X \text{ and } Y \text{ quasi-homogeneous, i.e. } D_X = D_Y)$. If X and Y are conditional events, then we have also $(X \text{ and } Y \text{ quasi-homogeneous}) \Rightarrow (X \text{ and } Y \text{ homogeneous})$.

By previous definition, it follows:

Proposition 5.1. *Let X and Y be two fuzzy events. If they are homogeneous, then also $X^c, Y^c, X \cup Y, X \cap Y$ are homogeneous with X and Y .*

Moreover, the following properties are equivalent:

- X and Y are homogeneous;
- $X \cup X^c = Y \cup Y^c$;
- $X \cap X^c = Y \cap Y^c$.

For each fuzzy event $X : \Pi \rightarrow [0, 1]$ with set of values $V_X, X = \{v/X_v, v \in V_X\}$, we put $X^+ = \{v/X_v, v \in V_X, v > 1/2\}, X^- = \{v/X_v, v \in V_X, v < 1/2\}$.

We note that, for conditional events, the condition $(T_X \cap T_Y = \emptyset, D_X = D_Y)$ is equivalent to $(X \text{ and } Y \text{ are disjoint and homogeneous})$; moreover, the condition $(T_X = T_Z, D_X = T_Y, D_Y = D_Z)$ is equivalent to $(X^+ = Z^+, (X \cup$

$X^c)^+ = Y^+, Y \cup Y^c = Z \cup Z^c$). Then the definition 5.2 can be extended to fuzzy events in the following way.

Definition 5.4. Let $F_{\mathbf{E}}$ be the family of fuzzy events $X : \Pi \rightarrow [0, 1]$, with the domain Π contained in an algebra of events $\mathbf{E}$. We define finitely additive fuzzy probability on $F_{\mathbf{E}}$ every function $p : X \in F_{\mathbf{E}}^* = F_{\mathbf{E}} - \{U\} \rightarrow p(X) \in [0, 1]$ such that:

- (PF1) $\forall X, Y \in F_{\mathbf{E}}^*, (T_{X \cap Y} = \emptyset, X \cup X^c = Y \cup Y^c) \Rightarrow p(X \cup Y) = p(X) + p(Y);$
 (PF2) $\forall X \in F_{\mathbf{E}}^*, (F_X = \emptyset) \Rightarrow p(X) = 1;$
 (PF3) $\forall X, Y, Z \in F_{\mathbf{E}}^*, (X^+ = Z^+, (X \cup X^c)^+ = Y^+, Y \cup Y^c = Z \cup Z^c) \Rightarrow p(X)p(Y) = p(Z).$

6. An extension theorem to subjective fuzzy probability

We prove that the fuzzy probability given by (4.16) satisfies the conditions (PF1), (PF2), and (PF3) of the definition 5.4, and then, according to this definition, it is a finitely additive fuzzy probability on F_E .

Theorem 6.1 (Extension theorem). Let F_E be the family of fuzzy events $X : \Pi \rightarrow [0, 1]$ with the domain Π contained in a finite algebra of events $\mathbf{E}$. If p is a fuzzy probability associated to a function f that satisfies the conditions of symmetry and positivity, then p is a finitely additive fuzzy probability on F_E .

Proof. Let X and Y be two fuzzy events of $F_{\mathbf{E}}^*$. Considering possibly a refinement, we can always assume that X and Y have the same domain $\Pi = \{A_1, A_2, \dots, A_n\}$. Let $X(A_i) = x_i, Y(A_i) = y_i$.

(PF1) Let $T_{X \cap Y} = \emptyset, X \cup X^c = Y \cup Y^c$. Then we have $x_i = y_i$, or $x_i = 1 - y_i$ and $\min\{x_i, y_i\} \leq 1/2, D_X = D_Y$. Using the notations of Section 4, if $D_X \neq \emptyset$, we have, for every j :

$$Q_X^j = \Sigma\{f(x_i)p_j(A_i) : x_i \neq 1/2\}; Q_Y^j = \Sigma\{f(y_i)p_j(A_i) : y_i \neq 1/2\}.$$

Since $f(x) = f(1 - x)$, it follows $Q_X^j = Q_Y^j = Q_{(X \cup Y)}^j$.

Then from the procedure of Section 4 we obtain an integer j and the j^{th} order probability for X, Y and $X \cup Y$ given by formulae $p(X) = Q_X^{+j}/Q_X^j, p(Y) = Q_Y^{+j}/Q_Y^j, p(X \cup Y) = Q_{(X \cup Y)}^{+j}/Q_{(X \cup Y)}^j$, where denominators are equal and not null and

$$Q_X^{+j} = \Sigma\{f(x_i)p_j(A_i) : x_i > 1/2\}; Q_Y^{+j} = \Sigma\{f(y_i)p_j(A_i) : y_i > 1/2\}.$$

Since $T_{X \cap Y} = \emptyset$, it follows $Q_{(X \cup Y)}^{+j} = Q_X^{+j} + Q_Y^{+j}$. Therefore, we have:

$$p(X \cup Y) = Q_{(X \cup Y)}^{+j}/Q_{(X \cup Y)}^j = Q_X^{+j}/Q_X^j + Q_Y^{+j}/Q_Y^j = p(X) + p(Y).$$

Thus, (PF1) is verified.

(PF2) If $F_X = \emptyset$ then $Q_X^j = Q_X^{+j} \neq 0$, and we have $p(X) = Q_X^{+j}/Q_X^j = 1$ and so (PF2) holds.

(PF3) Let $X, Y, Z \in F_{\mathbf{E}}^*$, such that $X^+ = Z^+$, $(X \cup X^c)^+ = Y^+$, $Y \cup Y^c = Z \cup Z^c$. From formula (4.16), there exist integers i, j, t such that:

$$p(X) = Q_X^{+i}/Q_X^i, p(Y) = Q_Y^{+j}/Q_Y^j, p(Z) = Q_Z^{+t}/Q_Z^t.$$

Since $Y \cup Y^c = Z \cup Z^c$, $(X \cup X^c)^+ = Y^+$, we have $j = t$, $i \geq j$.

If $i > j$ then $Q_Y^{+j} = Q_X^{+j} = 0$, $Q_Z^{+t} = Q_X^{+t} = 0$ and equality (PF3) reduces to $0 = 0$.

If $i = j$ then $Q_X^i = Q_Y^{+i}$, $Q_X^{+i} = Q_Z^{+i}$, $Q_Y^i = Q_Z^i$, and so $p(X)p(Y) = (Q_X^{+i}/Q_X^i)(Q_Y^{+i}/Q_Y^i) = Q_X^{+i}/Q_Y^i = Q_Z^{+i}/Q_Z^i = p(Z)$. Therefore, (PF3) is verified.

In conclusion, the theorem 6.1 extends the relationship between the subjective probability of de Finetti and finitely additive conditional probability of Dubins, to the more general relation between the fuzzy probability and finitely additive fuzzy probability, which have been introduced in the previous sections.

7. Conclusions and perspective of research

In this study, we have shown how the fuzzy probability introduced in Sec. 4 as an extension of the conditional probability can be significant in the case of decision-making under uncertainty.

This probability depends on a score function and appears to have good mathematical properties if this function satisfies at least the conditions of positivity and symmetry. In particular, we have shown that it is possible to obtain a fuzzy extension of Dubins conditions for finitely additive conditional probabilities.

An extension of the theory introduced can be made by replacing the crisp partitions of the certain event with fuzzy partitions. This may be useful for classification problems based on a given objective function.

In particular contexts, the imposition of suitable conditions to the score function can lead to very significant results both from the mathematical point of view and from the point of view of applications.

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SANDWICH SETS AND CONGRUENCES IN COMPLETELY INVERSE AG^{**} -GROUPOIDS

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Abstract. In this paper, we investigate sandwich set in a completely inverse AG^{**} -groupoid and make use of it for congruence pairs of a completely inverse AG^{**} -groupoid. We discuss completely inverse AG^{**} -groupoids in terms of partial order and show that the natural partial order is an equality relation in an AG -group. Further, we study idempotent-separating and idempotent-pure congruences on completely inverse AG^{**} -groupoids and show that the quotient structure for a maximum idempotent-separating congruence on an AG^{**} -groupoid is fundamental. Further, we characterize E -unitary completely inverse AG^{**} -groupoids and provide a condition for a compatibility relation to be transitive.

Keywords: Completely inverse AG^{**} -groupoid, AG -group, partial order relation, trace of congruence, kernel of congruence, idempotent-separating, idempotent-pure, compatibility relation, E -unitary inverse AG^{**} -groupoid.

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1. Introduction

An *Abel-Grassmann's* groupoid (abbreviated as AG-groupoid) or Left Almost Semigroup (briefly LA-semigroup) is a groupoid S satisfying the left invertive law $(ab)c = (cb)a$ for all $a, b, c \in S$. An AG-groupoid satisfying the identities $a(bc) = b(ac)$ and $(ab)(cd) = (dc)(ba)$ for all $a, b, c, d \in S$ is called an AG**-groupoid (cf. [5, 6, 8]).

The inverse AG**-groupoids and completely inverse AG**-groupoids constitute the most important class of AG-groupoids. The structural properties, characterizations and congruences on inverse AG**-groupoids via kernel-normal system and kernel-trace approaches can be found in literature (see for example [1, 4, 9, 11, 12]). In [3], study on congruences and lattices of congruences in completely inverse AG**-groupoids has been carried out.

In this paper, our purpose is to study some congruences in completely inverse AG**-groupoids, the natural partial order relation on completely inverse AG**-groupoids and E -unitary completely inverse AG**-groupoids. In the second section, some preliminaries and basic results on completely inverse AG**-groupoids are recalled. We also introduce sandwich set for a completely inverse AG**-groupoid. In section 3, we introduce completely left inverse AG**-groupoids and investigate some basic congruences using the congruence pairs. We show that if ρ is a congruence on a completely left inverse AG**-groupoid, then $(\ker \rho, \text{tr} \rho)$ is a congruence. In section 4, the natural order relation and its relation with the AG-group is investigated. We further show that the set of all permissible subsets of a completely inverse AG**-groupoid is an inverse AG**-groupoid. In section 5, we investigate minimum and maximum congruences on completely inverse AG**-groupoids. The relation between a smallest congruence on an AG-group is established in this section. Idempotent-separating and idempotent-pure congruences are also studied in this section. We show that if μ is the maximum idempotent-separating congruence, then S/μ is fundamental. Finally, in section 6, E -unitary completely inverse AG**-groupoids and their different characterizations are provided. We show that if the compatibility relation is transitive, then S is E -unitary and conversely.

In the following section, we recall some necessary basic notions and then we present a few auxiliary results that will be used throughout the paper.

2. Preliminaries

An AG-groupoid S is regular if $a \in (aS)a$ for all $a \in S$. If for $a \in S$, there exists an element a' such that $a = (aa')a$ and $a' = (a'a)a'$, then we say that a' is inverse of a . In addition, if inverses commute, that is $a'a = aa'$, then S is called completely regular. If $a \in S$, then

$$V(a) = \{a' \in S : a = aa' \cdot a \text{ and } a' = a'a \cdot a'\}$$

is called the set of all inverses of $a \in S$. Note that if $a' \in V(a)$ and $b' \in V(b)$, then $a \in V(a')$ and $a'b' \in V(ab)$. AG-groupoid S in which every element has

an unique inverse is called inverse AG-groupoid. If a^{-1} is the unique inverse of $a \in S$, then a groupoid satisfying the following identities is called a completely inverse AG^{**}-groupoid, that is for all $a, b, c \in S$

$$(ab)c = (cb)a, \quad a(bc) = b(ac) \\ a = (aa^{-1})a, \quad a^{-1} = (a^{-1}a)a^{-1} \text{ and } aa^{-1} = a^{-1}a.$$

If S is a completely inverse AG^{**}-groupoid, then $a^{-1}a \in E_S$, where E_S is the set of idempotents of S . If S is a completely inverse AG^{**}-groupoid, then E_S is either empty or a semilattice. If S is an AG-groupoid such that for all $x \in S, x^2 = x$ and for all $a, b \in S, a = (ab)a$, then we say that S is a rectangular AG-band. For any idempotent $e \in E_S, e^{-1} = e$. Moreover, the set E_S of an AG-groupoid S is a rectangular AG-band, that is for all $e, f \in E_S$ and $e = (ef)e$. For further concepts and results, the reader is referred to [3].

The sandwich set $S(e, f)$ of $e, f \in E_S$ is defined as below:

$$S(e, f) = \{g \in E_S : eg \cdot f = ef\}.$$

Proposition 2.1 ([3, Proposition 4.1]). *Let S be a completely inverse AG^{**}-groupoid and let $a, b \in S$ such that $ab \in E_S$. Then $ab = ba$.*

Theorem 2.2. *Let S be a completely inverse AG^{**}-groupoid and $a, b \in S$. If $g \in S(aa^{-1}, bb^{-1})$, then $a^{-1} \cdot gb^{-1} \in V(ab)$.*

Proof. Let $a, b \in S, a^{-1} \in V(a), b^{-1} \in V(b)$ and $g \in S(aa^{-1}, bb^{-1})$. Then

$$\begin{aligned} \{(ab)(a^{-1} \cdot gb^{-1})\}(ab) &= \{(aa^{-1})(b \cdot gb^{-1})\}(ab) \\ &= \{(aa^{-1})(g \cdot bb^{-1})\}(ab) \\ &= \{(aa^{-1} \cdot bb^{-1})\}(ab) && (\text{since } g \in S(aa^{-1}, bb^{-1})) \\ &= (aa^{-1} \cdot a)(bb^{-1} \cdot b) \\ &= ab \end{aligned}$$

and

$$\begin{aligned} \{(a^{-1} \cdot gb^{-1})(ab)\}(a^{-1} \cdot gb^{-1}) &= \{(a^{-1}a)(gb^{-1} \cdot b)\}(a^{-1} \cdot gb^{-1}) \\ &= \{(a^{-1}a)(g \cdot b^{-1}b)\}(a^{-1} \cdot gb^{-1}) \\ &= (a^{-1}a \cdot b^{-1}b)(a^{-1} \cdot gb^{-1}) \\ &&& (\text{since } g \in S(aa^{-1}, bb^{-1})) \\ &= (a^{-1}a \cdot a^{-1})(g(b^{-1}b \cdot b^{-1})) \\ &= a^{-1} \cdot gb^{-1}. \end{aligned}$$

Moreover,

$$\begin{aligned} (a^{-1} \cdot gb^{-1})(ab) &= (a^{-1}a)(gb^{-1} \cdot b) = (a^{-1}a)(bb^{-1} \cdot g) = (g \cdot bb^{-1})(aa^{-1}) \\ &= (aa^{-1} \cdot bb^{-1})g = g(aa^{-1} \cdot bb^{-1}) = (aa^{-1})(g \cdot bb^{-1}) \\ &= (ab)(a^{-1} \cdot gb^{-1}). \end{aligned}$$

This completes the proof. □

3. Congruences in completely inverse AG**-groupoids

A congruence pair for a completely inverse AG**-groupoid which is based on a pair of a normal AG**-groupoid N and a congruence ζ has been established in this section. We will make use of the sandwich set to prove the main theorem.

Let ρ be a congruence on a completely inverse AG**-groupoid S and E_S be the set of idempotents of S . The restriction of ρ on E_S , that is, $\rho|_{E_S}$ is the trace of ρ denoted by $\text{tr}\rho$. The subset $\ker\rho = \{a \in S : (\exists e \in E_S) (a, e) \in \rho\}$ is the kernel of ρ .

Definition 3.1 ([10, Definition 4.1]). A nonempty subset N of a completely inverse AG**-groupoid S is said to be normal ($N \triangleleft S$) if

- (1) $E_S \subseteq N$,
- (2) for every $a \in S$, $a \cdot Na^{-1} \subseteq N$,
- (3) for every $a \in N$, $a^{-1} \in N$.

Definition 3.2 ([10, Definition 4.2]). Let N be normal subgroupoid of a completely inverse AG**-groupoid S and ζ be a congruence on E_S such that for every $a \in S$ and $e \in E_S$, $ea \in N$ and $(e, a^{-1}a) \in \zeta$ implies $a \in N$. Then the pair (N, ζ) is a congruence pair for S .

If (N, ζ) is a congruence pair where $N \triangleleft S$ and ζ is a congruence, then $\rho_{(N, \zeta)}$ defined as $(a, b) \in \rho_{(N, \zeta)}$ if and only if $a^{-1}b \in N$, $(aa^{-1}, bb^{-1}) \in \zeta$ is a relation on S .

Notice that if $a^{-1}b \in N \triangleleft S$, then $ab^{-1} \in V(a^{-1}b) \in N$. In the light of Proposition 2.1, the condition $ab^{-1} \in N$ is equivalent to $b^{-1}a \in N$.

Lemma 3.3. *If (N, ζ) is a congruence pair on a completely inverse AG**-groupoid S and $(a, b) \in \rho_{(N, \zeta)}$, then*

$$(aa^{-1}, aa^{-1} \cdot bb^{-1}) \in \zeta \text{ and } (bb^{-1}, bb^{-1} \cdot aa^{-1}) \in \zeta.$$

Proof. Let $(a, b) \in \rho_{(N, \zeta)}$. Then $aa^{-1} = (aa^{-1})(aa^{-1}) \equiv_{\zeta} (aa^{-1})(bb^{-1})$ (since $(aa^{-1}, bb^{-1}) \in \zeta$).

In similar way, replacing a and b , we get $bb^{-1} \equiv_{\zeta} bb^{-1} \cdot aa^{-1}$. □

Lemma 3.4. *Let (N, ζ) be a congruence pair on a completely inverse AG**-groupoid S . Then for $a, b \in S$ and $e \in E_S$, if $ab \in N$ and $(e, aa^{-1}) \in \zeta$, then $a \cdot eb \in N$.*

Proof. Proof is straight forward. □

Lemma 3.5. *Let (N, ζ) be a congruence pair on a completely inverse AG**-groupoid S and $g \in S(aa^{-1}, cc^{-1})$ and $h \in S(bb^{-1}, cc^{-1})$. If $(aa^{-1}, bb^{-1}) \in \zeta$ and $a^{-1}b \in N$, then $(a \cdot ga^{-1}, b \cdot hb^{-1}) \in \zeta$.*

Proof. Let (N, ζ) be a congruence pair. Let $a, b \in S$ such that $(aa^{-1}, bb^{-1}) \in \zeta$ and $a^{-1}b \in N$. Suppose $x \in S(aa^{-1}, bb^{-1})$. Then $a \cdot xb^{-1} \in V(a^{-1}b)$. Also if $t \in S((a^{-1}b)(a \cdot xb^{-1}), g)$, then $(a^{-1}b)(tg) \in V((a \cdot xb^{-1})g)$. Since $(aa^{-1}, bb^{-1}) \in \zeta$, then

$$(1) \quad (a^{-1}b)(t(a \cdot xb^{-1})) = t((a^{-1}a)(b \cdot xb^{-1})) = t((a^{-1}a \cdot x)(bb^{-1})) = t(b \cdot xb^{-1}).$$

Again $(aa^{-1}, bb^{-1}) \in \zeta$ and $x \in S(aa^{-1}, bb^{-1})$, then

$$(bb^{-1})(x \cdot bb^{-1}) \equiv_{\zeta} (aa^{-1})(x \cdot bb^{-1}) \equiv_{\zeta} (aa^{-1})(bb^{-1}) \equiv_{\zeta} (bb^{-1})(bb^{-1}) \equiv_{\zeta} bb^{-1}.$$

On the other hand $(bb^{-1})(x \cdot bb^{-1}) = x(bb^{-1} \cdot bb^{-1}) = b \cdot xb^{-1}$ and because $b \cdot xb^{-1}$ is an idempotent,

$$(2) \quad (b \cdot xb^{-1}, bb^{-1}) \in \zeta.$$

Since ζ is compatible, it follows from (2) that

$$(3) \quad (t(b \cdot xb^{-1}), t \cdot bb^{-1}) \in \zeta.$$

Using (1) and (3) with transitivity of ζ , we have

$$(4) \quad ((a^{-1}b)(t(a \cdot xb^{-1})), t \cdot bb^{-1}) \in \zeta.$$

Similarly, since $x \in S(aa^{-1}, bb^{-1})$,

$$(aa^{-1})(x \cdot aa^{-1}) \equiv_{\zeta} (aa^{-1})(x \cdot bb^{-1}) \equiv_{\zeta} (aa^{-1})(bb^{-1}) \equiv_{\zeta} (aa^{-1})(aa^{-1}) \equiv_{\zeta} aa^{-1}.$$

Thus, we have

$$(5) \quad (a \cdot xa^{-1}, aa^{-1}) \in \zeta.$$

From the compatibility of ζ , we get

$$\begin{aligned} b(((a \cdot xb^{-1})(t \cdot a^{-1}b))b^{-1}) &= ((a \cdot xb^{-1})(t \cdot a^{-1}b))(bb^{-1}) \\ &= (t((aa^{-1})(x \cdot bb^{-1}))(b^{-1}b)) \\ &= (t((aa^{-1})(bb^{-1}))(b^{-1}b)) \quad (\text{since } x \in S(aa^{-1}, bb^{-1})) \\ &= t \cdot aa^{-1} \quad (\text{since } (aa^{-1}, bb^{-1}) \in \zeta) \end{aligned}$$

and since $t \cdot aa^{-1} \in E_S$, it follows

$$(6) \quad (b(((a \cdot xb^{-1})(t \cdot a^{-1}b))b^{-1}), t \cdot aa^{-1}) \in \zeta.$$

Further, we show that $(a \cdot ta^{-1}, a \cdot ga^{-1}) \in \zeta$.

Since $x \in S(aa^{-1}, bb^{-1})$ and $(aa^{-1}, bb^{-1}) \in \zeta$, then $(aa^{-1} \cdot x)(tg) = (aa^{-1}(x \cdot bb^{-1}))(tg) = (aa^{-1} \cdot bb^{-1})(tg) \equiv_{\zeta} (aa^{-1} \cdot aa^{-1})(tg) = aa^{-1} \cdot tg$. Because of the

compatibility of ζ , (5) gives $((a \cdot xa^{-1})(tg), (aa^{-1})(tg)) \in \zeta \implies ((aa^{-1} \cdot tg), a \cdot ga^{-1}) \in \zeta$ which further shows that

$$(7) \quad (a \cdot ta^{-1}, a \cdot ga^{-1}) \in \zeta.$$

Finally, we prove that $(b \cdot tb^{-1}, b \cdot hb^{-1}) \in \zeta$. Since $b \cdot tb^{-1} \in E_S$ and $g \in S(aa^{-1}, cc^{-1})$. Thus

$$\begin{aligned} b \cdot tb^{-1} &\equiv_{\zeta} (t \cdot aa^{-1})(bb^{-1}) \\ &\equiv_{\zeta} (g \cdot aa^{-1})(bb^{-1}) && \text{(it follows from (7))} \\ &\equiv_{\zeta} g \cdot bb^{-1} && \text{(since } (aa^{-1}, bb^{-1}) \in \zeta) \\ &\equiv_{\zeta} g(bb^{-1} \cdot cc^{-1}) && \text{(using Lemma 3.3)} \\ &\equiv_{\zeta} aa^{-1} \cdot cc^{-1} && \text{(since } (aa^{-1}, bb^{-1}) \in \zeta \text{ and } g \in S(aa^{-1}, cc^{-1})) \\ &\equiv_{\zeta} (bb^{-1} \cdot h)(cc^{-1}) && \text{(since } h \in S(bb^{-1}, cc^{-1})) \\ &\equiv_{\zeta} b \cdot hb^{-1} && \text{(using Lemma 3.3)} \end{aligned}$$

thus, we have

$$(8) \quad (b \cdot tb^{-1}, b \cdot hb^{-1}) \in \zeta.$$

Combining (4) and (8) with the compatibility of ζ and since both $b \cdot hb^{-1}$ and $b(((a \cdot xb^{-1})(t \cdot a^{-1}b))b^{-1})$ are in E_S , we have

$$(9) \quad (b(((a \cdot xb^{-1})(t \cdot a^{-1}b))b^{-1}), b \cdot hb^{-1}) \in \zeta.$$

Using (6) and (7) and the transitivity of ζ , we get

$$(10) \quad (b(((a \cdot xb^{-1})(t \cdot a^{-1}b))b^{-1}), a \cdot ga^{-1}) \in \zeta.$$

Finally, (9) and (10) with the transitivity of ζ , completes the proof as $(a \cdot ga^{-1}, b \cdot hb^{-1}) \in \zeta$. $\square$

Theorem 3.6. *If (N, ζ) is a congruence pair of a completely inverse AG**-groupoid S , then $\rho_{(N, \zeta)}$ defined as*

$$(a, b) \in \rho_{(N, \zeta)} \text{ if and only if } a^{-1}b \in N, (aa^{-1}, bb^{-1}) \in \zeta$$

is congruence on S with kernel N and trace ζ . Conversely, if ρ is a congruence on S , then $(\ker \rho, \text{tr} \rho)$ is a congruence pair and $\rho = \rho_{(\ker \rho, \text{tr} \rho)}$.

Proof. Let $\rho_{(N, \zeta)}$ be a relation with congruence pair (N, ζ) . First, we show that $\rho_{(N, \zeta)}$ is a congruence. $\rho_{(N, \zeta)}$ is reflexive. Symmetry is evident because ζ is symmetric and $(bb^{-1}, aa^{-1}) \in \rho_{(N, \zeta)}$. Since $a^{-1}b \in N$, then by Proposition 2.1, $b^{-1}a \in N$. For transitivity, let $(a, b), (b, c) \in \rho_{(N, \zeta)}$. Then $(aa^{-1}, bb^{-1}) \in \zeta$, $a^{-1}b \in N$ and $(bb^{-1}, cc^{-1}) \in \zeta$, $b^{-1}c \in N$. But ζ is transitive, thus $(aa^{-1}, cc^{-1}) \in \rho_{(N, \zeta)}$. Since $b^{-1}a, bc^{-1} \in N$ and N is subgroupoid of S , $b^{-1}a \cdot$

$bc^{-1} \in N$. We need to show $ac^{-1} \in N$. Let us take g, h such that $g \in S(bb^{-1}, cc^{-1})$ and $h \in (aa^{-1}, cc^{-1})$. Then by Lemma 3.3, we have

$$N \ni b^{-1}a \cdot bc^{-1} = a(bb^{-1} \cdot c^{-1}) = a((bb^{-1} \cdot c^{-1}c)c^{-1}) = a(((b^{-1}c)(b \cdot gc^{-1}))c^{-1})$$

because $b^{-1}c \in V(b \cdot gc^{-1})$. Therefore $(b^{-1}c)(b \cdot gc^{-1}) \in E_S$. We have

$$cc^{-1} \equiv_{\zeta} cc^{-1} \cdot cc^{-1} \equiv_{\zeta} bb^{-1} \cdot cc^{-1} \equiv_{\zeta} bb^{-1}(g \cdot cc^{-1}) \equiv_{\zeta} cc^{-1}(g \cdot cc^{-1}) = g \cdot cc^{-1}$$

that is, $(cc^{-1}, c \cdot gc^{-1}) \in \zeta$. In the same manner, as $h \in S(aa^{-1}, cc^{-1})$ and $(aa^{-1}, cc^{-1}) \in \zeta$, we have $(cc^{-1}, c \cdot hc^{-1}) \in \zeta$. So

$$(b^{-1}c)(b \cdot gc^{-1}) = c((bb^{-1} \cdot g)c^{-1}) \equiv_{\zeta} c \cdot gc^{-1} = cc^{-1} \equiv_{\zeta} c \cdot hc^{-1} = (a^{-1}c)(a \cdot hc^{-1}).$$

Thus $((b^{-1}c)(b \cdot gc^{-1}), (a^{-1}c)(a \cdot hc^{-1})) \in \zeta$. From (10)-(11), $ac^{-1} \in N$ and by Definition 3.1(3), $a^{-1}c \in N$. Thus $\rho_{(N, \zeta)}$ is an equivalence relation. To show that $\rho_{(N, \zeta)}$ is compatible, let $g \in S(aa^{-1}, cc^{-1})$ and $h \in S(cc^{-1}, bb^{-1})$. Then

$$a^{-1} \cdot gc^{-1} \in V(ac) \text{ and } b^{-1} \cdot hc^{-1} \in V(bc)$$

$(ac)(a^{-1} \cdot gc^{-1}) = (aa^{-1} \cdot g)(cc^{-1}) = c \cdot gc^{-1}$ and $(bc)(b^{-1} \cdot hc^{-1}) = (bb^{-1} \cdot h)(cc^{-1}) = c \cdot hc^{-1}$. Since $g \in S(aa^{-1}, cc^{-1})$ and $h \in S(cc^{-1}, bb^{-1})$, it follows

$$\begin{aligned} c \cdot gc^{-1} &= (aa^{-1} \cdot g)(cc^{-1}) = (aa^{-1})(cc^{-1}) \equiv_{\zeta} (bb^{-1})(cc^{-1}) \\ &= (bb^{-1})(h \cdot cc^{-1}) \equiv_{\zeta} (cc^{-1})(h \cdot cc^{-1}) = c \cdot hc^{-1}. \end{aligned}$$

Thus $((ac)(a^{-1} \cdot gc^{-1}), (bc)(b^{-1} \cdot hc^{-1})) = (c \cdot gc^{-1}, c \cdot hc^{-1}) \in \zeta$. Also $(a^{-1} \cdot gc^{-1})(bc) = (a^{-1}b)(cc^{-1} \cdot g) \in N$ because $a^{-1}b \in N$ and $cc^{-1} \cdot g \in E_S$. Therefore $(ac, bc) \in \rho_{(N, \zeta)}$.

For $(ca, cb) \in \rho_{(N, \zeta)}$, let us suppose again that $g \in S(aa^{-1}, cc^{-1})$ and $h \in S(cc^{-1}, bb^{-1})$. Then $(c^{-1} \cdot ga^{-1}) \in V(ca)$, $(c^{-1} \cdot hb^{-1}) \in V(cb)$ and

$$\begin{aligned} (ca)(c^{-1} \cdot ga^{-1}) &= (aa^{-1} \cdot g)(cc^{-1}) \equiv_{\zeta} (c \cdot gc^{-1}), \\ (cb)(c^{-1} \cdot hb^{-1}) &= (cc^{-1})(h \cdot bb^{-1}) \equiv_{\zeta} (c \cdot hc^{-1}). \end{aligned}$$

Since $c \cdot gc^{-1}$ and $c \cdot hc^{-1} \in E_S$, then $((ca)(c^{-1} \cdot ga^{-1}), (cb)(c^{-1} \cdot hb^{-1})) = (c \cdot gc^{-1}, c \cdot hc^{-1}) \in \zeta$ and since $a^{-1}b \in N$, we have $(c^{-1} \cdot ga^{-1})(cb) = (c \cdot gc^{-1})(a^{-1}b) \in N$. Hence $(ca, cb) \in \rho_{(N, \zeta)}$.

We further show that $N = \ker \rho_{(N, \zeta)}$ and $\zeta = \text{tr} \rho_{(N, \zeta)}$. Suppose that $a \in \ker \rho_{(N, \zeta)}$. Then there exists $e \in E_S$ such that $(e, a) \rho_{(N, \zeta)}$. Using the definition of $\rho_{(N, \zeta)}$, it is clear that $(ee, aa^{-1}) \in \zeta$, $ea \in N$ i.e. $a \in N$. Hence $\ker \rho_{(N, \zeta)} \subseteq N$.

Conversely suppose that $a \in N$. Because N is normal subgroupoid, then $(a^{-1}a, aa^{-1} \cdot a^{-1}a) \in \zeta$ and $a^{-1} \cdot aa^{-1} \in N$, that is, $(a, a^{-1}a) \in \rho_{(N, \zeta)} \implies a \in \ker \rho_{(N, \zeta)}$. Thus $N \subseteq \ker \rho_{(N, \zeta)}$. Hence $\ker \rho_{(N, \zeta)} = N$.

Suppose $a, b \in E_S$ such that $(a, b) \in \rho_{(N, \zeta)}$. Then $(aa^{-1}, bb^{-1}) \in \zeta$ and $a^{-1}b \in N$. Since ζ is a congruence on E_S , it follows that $(a)_{\zeta} = (a)_{\zeta}(a^{-1})_{\zeta} = (b)_{\zeta}(b^{-1})_{\zeta} = (b)_{\zeta}$. Thus $(a, b) \in \zeta$. Hence $\text{tr} \rho_{(N, \zeta)} \subseteq \zeta$.

Conversely, $(a, b) \in \zeta$ for $a, b \in E_S$. Then $(a^{-1})_\zeta, (b^{-1})_\zeta \in E_{S/\zeta}$. Because $(a, b) \in \zeta$, then $(a)_\zeta(a^{-1})_\zeta = (b)_\zeta(b^{-1})_\zeta$. That is, $(aa^{-1}, bb^{-1}) \in \zeta$, $a^{-1}b \in N$. Thus $(a, b) \in \rho_{(N, \zeta)}$. Hence $\text{tr}\rho_{(N, \zeta)} = \zeta$.

To prove the converse of the theorem, first we show that $\ker\rho \triangleleft S$. Let $a, b \in \ker\rho$, then $(a)_\rho, (b)_\rho \in E_{S/\rho}$ and $(a)_\rho(b)_\rho = (ab)_\rho \in E_{S/\rho}$. We show that $\ker\rho$ is self-conjugate. Let $a \in S$ and $x \in \ker\rho$, then for $(x)_\rho \in E_{S/\rho}$, we have $(a \cdot xa^{-1})_\rho = (a)_\rho \cdot (x)_\rho(a^{-1})_\rho \in E_{S/\rho}$. Also, if $a \in \ker\rho$, then $(a)_\rho \in E_{S/\rho}$. Thus $(a^{-1})_\rho \in E_{S/\rho} \subseteq \ker\rho$ showing $\ker\rho$ is normal. Hence by Definition 3.2, $(\ker\rho, \text{tr}\rho)$ is a congruence pair.

Finally, for any $a, b \in S$, let $(a, b) \in \rho$. Then $(a^{-1}, b^{-1}) \in \rho$, $(aa^{-1}, bb^{-1}) \in \rho$ and $(ab^{-1}, bb^{-1}) \in \rho$. Since $aa^{-1}, bb^{-1} \in E_S$, $(aa^{-1}, bb^{-1}) \in \text{tr}\rho$ and $a^{-1}b \in \ker\rho$, it follows that $\rho \subseteq \rho_{(\ker\rho, \text{tr}\rho)}$.

Conversely, to show $\rho_{(\ker\rho, \text{tr}\rho)} \subseteq \rho$. Let $(a, b) \in \rho_{(\ker\rho, \text{tr}\rho)}$ such that $(aa^{-1}, bb^{-1}) \in \text{tr}\rho$ and $a^{-1}b \in \ker\rho$. Since $ab^{-1} \in V(a^{-1}b) \in \ker\rho$, it follows that for some $(e)_\rho \in E_{S/\rho}$, $(ab^{-1})_\rho = (e)_\rho = (e)_\rho(e^{-1})_\rho = (ab^{-1})_\rho((ab^{-1})^{-1})_\rho$. Then

$$\begin{aligned} (a)_\rho &\equiv_\rho (aa^{-1} \cdot a)_\rho \equiv_\rho (ab^{-1})_\rho \cdot (b)_\rho \equiv_\rho ((ab^{-1})(ab^{-1})^{-1})_\rho (b)_\rho \\ &\equiv_\rho (aa^{-1} \cdot bb^{-1})_\rho (b)_\rho \equiv_\rho (b^{-1}b)_\rho \cdot (b)_\rho \equiv_\rho (b)_\rho. \end{aligned}$$

Hence, $\rho_{(\ker\rho, \text{tr}\rho)} = \rho$. □

4. Natural partial order relation

In this section, we discuss the natural partial order relation on completely inverse AG**-groupoids which indeed constitute AG-groups containing exactly one idempotent. We show that the natural partial order is the equality relation if and only if S is an AG-group.

Definition 4.1. Let S be a completely inverse AG**-groupoid. Then the relation

$$a \leq b \text{ if and only if } a \in E_S b.$$

on S is a natural partial order relation.

Lemma 4.2. Let S be a completely inverse AG**-groupoid. Then the following are equivalent:

- (1) $a \leq b$.
- (2) $a = fb$ for some $f \in E_S$.
- (3) $a^{-1} \leq b^{-1}$.
- (4) $a = aa^{-1} \cdot b$.

Proof. Let $a, b \in S$, $a^{-1} \in V(a)$. Then

(1) $\Rightarrow$ (2): It is clear from Definition 4.1.

(2) $\Rightarrow$ (3): Since $a = fb$, it follows that $a^{-1} = (fb)^{-1} = fb^{-1} \leq b^{-1}$.

(3) $\Rightarrow$ (4): By assumption, we have $a^{-1} = ea^{-1}$ for some $e \in E_S$. Then $a = eb$ and $ea = eb = a$ and thus $aa^{-1} \cdot b = (ea \cdot a^{-1})b = (be)(a^{-1}a) = aa^{-1} \cdot a = a$.

(4) $\Rightarrow$ (1): This is obvious because $a^{-1}a$ is an idempotent. □

Proposition 4.3. *Let S be a completely inverse AG^{**} -groupoid. Then the following statements hold:*

- (1) *The relation $\leq$ is a partial order on S .*
- (2) *For $e, f \in E_S$, $e \leq f$ if and only if $e = ef = fe$.*
- (3) *If $a \leq b$ and $c \leq d$, then $ab \leq cd$.*
- (4) *If $a \leq b$, then $a^{-1}a \leq b^{-1}b$.*

Proof. (1) The relation $\leq$ is reflexive, since $a = (aa^{-1})a \leq a$. Let $a \leq b$ and $b \leq a$, then $a = (aa^{-1})b$ and $b = (bb^{-1})a$. We have $a = (aa^{-1})b = (aa^{-1})(bb^{-1} \cdot a) = (bb^{-1})(aa^{-1} \cdot a) = (bb^{-1})a = b$. Further, let $a \leq b$ and $b \leq c$. Then there exist idempotents e, f such that $a = eb$ and $b = fc$. It follows that $a = eb = e(fc) = (ef)c \leq c$.

(2) If for $e, f \in E_S$, $e \leq f$, then $e = if$ for some $i \in E_S$. Now $fe = f(if) = if = e$. Similarly, $ef = (if)f = if = e$. The converse is trivial.

(3) Suppose that $a \leq b$ and $c \leq d$ then for idempotents e, f we have $a = eb$ and $c = fd$. Hence $ac = (eb)(fd) = (ef)(bd) \leq bd$.

(4) This follows from Lemma 4.2(3) and (3) above. $\square$

Note that a completely inverse AG^{**} -groupoid including only one idempotent is an AG -group, that is if $E_S = \{e\}$, then $ea = (a^{-1}a)a = (aa^{-1})a = a$ for $e = a^{-1}a = aa^{-1}$. A connection between an AG -group which simply contains an idempotent and the natural partial order relation is established in the following proposition.

Proposition 4.4. *The natural partial order on a completely inverse AG^{**} -groupoid S is an equality relation if and only if S is an AG -group.*

Proof. Suppose the natural partial order relation is the equality relation. Then for idempotents $e, f \in S$, $ef \leq e, f$. Thus $e = ef = f$. Hence S contains exactly one idempotent. The converse is trivial. $\square$

Definition 4.5. Let S be a completely inverse AG^{**} -groupoid. For all $a, b \in S$ the compatibility relation is defined as

$$a \sim b \text{ if and only if } ab^{-1}, a^{-1}b \in E_S.$$

The above mentioned relation is definitely reflexive. It is symmetric because

$$\begin{aligned} b^{-1}a &= b^{-1}(aa^{-1} \cdot a) = (aa^{-1} \cdot b^{-1}a) = (b^{-1}a \cdot aa^{-1}) = (aa^{-1} \cdot a)b^{-1} = ab^{-1} \\ ba^{-1} &= (bb^{-1} \cdot b)a^{-1} = (a^{-1}b \cdot bb^{-1}) = (bb^{-1} \cdot a^{-1}b) = a^{-1}(bb^{-1} \cdot b) = a^{-1}b. \end{aligned}$$

For A, B nonempty subsets of a completely inverse AG^{**} -groupoid S , we define the set $A^{-1} = \{a^{-1} : a \in A\}$ and $AB = \{ab : a \in A, b \in B\}$. A nonempty subset A of a completely inverse AG^{**} -groupoid S is compatible if the elements of A are compatible, that is $AA^{-1} \subset E_S$ and $A^{-1}A \subset E_S$.

Definition 4.6. A nonempty subset A of a completely inverse AG**-groupoid S is an order ideal if it satisfies the following property:

$$a \in A, x \leq a \implies x \in A.$$

Definition 4.7. A subset A of a completely inverse AG**-groupoid S is said to be permissible if it is a compatible order ideal. The set of all permissible subsets of S is denoted by $C(S)$.

The set $A^{-1} = \{a^{-1} : a \in A\}$ for a permissible subset A of S is also permissible and is the inverse of A .

Proposition 4.8. *The set $C(S)$ of a completely inverse AG**-groupoid S is a completely inverse AG**-groupoid under the multiplication of subsets of S . The mapping $\iota : S \rightarrow C(S)$, defined by $x \mapsto [x]$, where $[x]$ represents the permissible subset of $x \in S$, is an injective morphism.*

Proof. To show that $AB = \{ab : a \in A, b \in B\}$ is a compatible order ideal, let $A, B \in C(S)$. Also for $a \in A, b \in B$, let $x \leq ab$. Then by Lemma 4.2(4)

$$x = (xx^{-1})(ab) = a(xx^{-1} \cdot b) = ab^*.$$

Since, B is an order ideal, $b^* = xx^{-1} \cdot b \in B$. Hence $ab^* \in AB$ and AB is order ideal. Further, to show that AB is compatible subset of S , let $ab, cd \in AB$. Then

$$(ab)^{-1}(cd) = (a^{-1}b^{-1} \cdot cd) = (a^{-1}c \cdot b^{-1}d) \in E_S.$$

Since $a \sim c$, $b \sim d$, and $a^{-1}c, b^{-1}d \in E_S$, similarly, $(ab)(cd)^{-1} \in E_S$. Thus $ab \sim cd$. Hence AB is a compatible subset of S and $AB \in C(S)$. To show that any $A \subseteq C(S)$ is completely inverse, let we consider $ab^{-1} \cdot c$ for $a, b, c \in A$. Also consider that $s = ab^{-1} \cdot b$. Then $s = bb^{-1} \cdot a$ which shows that $s \leq a$ and since A is an order ideal, then $s \in A$. Moreover, since $a \sim b$, then $ab^{-1} \in E_S$. Thus we have

$$\begin{aligned} ss^{-1} &= (ab^{-1} \cdot b)(ab^{-1} \cdot b)^{-1} = (ab^{-1} \cdot b)(a^{-1}b \cdot b^{-1}) = (ab^{-1} \cdot a^{-1}b)(bb^{-1}) \\ &= (ab^{-1} \cdot (ab^{-1})^{-1})(bb^{-1}) = (b^{-1}b)(ab^{-1}) = a(b^{-1}b \cdot b^{-1}) = ab^{-1}. \end{aligned}$$

So we can write $ab^{-1} \cdot c = ss^{-1} \cdot c \leq c$. But since A is an order ideal, then $ab^{-1} \cdot c \in A$. Hence $AA^{-1} \cdot A \subseteq A$. For the inverse inclusion it is clear that $A \subseteq AA^{-1} \cdot A$. Thus $A = AA^{-1} \cdot A$. The condition $AA^{-1} = A^{-1}A$ is clear.

Suppose that $[x]$ represents the permissible subset of $x \in S$. Then it is an order ideal. For compatibility, let $a, b \leq x$. Then $a = ex$, $b = fx$ for some $e, f \in E_S$. Now $a^{-1}b = (ex^{-1})(fx) = (fx \cdot x^{-1})e = (ef)(x^{-1}x) \in E_S$ and $ab^{-1} = (ex)(fx^{-1}) = (ef)(xx^{-1}) \in E_S$. Finally, let $x, t \in S$. Then $[x], [t] \in C(S)$ and $[x] \cdot [t] = [xt]$. Thus ι is a morphism. If $[x] = [t]$, then $x = t$. Hence ι is one-one. $\square$

5. Minimum and maximum congruences

In this section, we study minimum congruence and maximum congruence. Minimum congruence is important in the sense that it develops a connection between an AG-group and a completely inverse AG^{**}-groupoid.

Definition 5.1. Let S be a completely inverse AG^{**}-groupoid and let $a, b \in S$. We define a relation σ by

$$a\sigma b \iff \exists x \in S : x \leq a, b.$$

From the fact that the natural partial order is compatible with the multiplication, we conclude that σ is compatible, that is, if $c \in S$, then

$$a\sigma b \implies \exists x : x \leq a, b \implies xc \leq ac, bc \text{ and } cx \leq ca, cb.$$

In the following main result of this section, we show that σ is the smallest congruence containing the compatibility relation $\sim$ and itself is contained in any congruence ρ such that S/ρ is an AG-group.

Theorem 5.2. Let S be a completely inverse AG^{**}-groupoid. Then

- (1) σ is the smallest congruence on S containing the compatibility relation $\sim$.
- (2) S/σ is an AG-group.
- (3) If ρ is any congruence such that S/ρ is an AG-group, then $\sigma \subseteq \rho$.

Proof. (1) Reflexivity and symmetry are clear. For transitivity, suppose that $a\sigma b$ and $b\sigma c$, where $a, b, c \in S$. Then there exist r, s such that $s \leq a, b$ and $r \leq b, c$, which further give $s, r \leq b$. Consequently, $s, r \in [b]$. But $[b]$ is compatible subset of S , thus $s \sim r$. Consider the quantity $ss^{-1} \cdot r$ which gives $ss^{-1} \cdot r \leq r$ and $ss^{-1} \cdot r = rs^{-1} \cdot s \leq s$, because $s \sim r$ and $rs^{-1} \in E_S$. This further implies $ss^{-1} \cdot r \leq a$ and $ss^{-1} \cdot r \leq r \leq c$. Thus $a \sim c$. Since σ is compatible with multiplication, then σ is a congruence.

Let $a \sim b$. Then $a^{-1}b \cdot a \leq a$ and $a^{-1}b \cdot a = (a^{-1}b)(aa^{-1} \cdot a) = (a^{-1}b \cdot aa^{-1})a = (a^{-1}a \cdot ba^{-1})a = (ba^{-1})a = aa^{-1} \cdot b$. Thus $a^{-1}b \cdot a \leq a, b$ and so $a \equiv_{\sigma} b$. Hence $\sim \subseteq \sigma$.

Let ρ be any congruence such that $\sim \subseteq \rho$. For $a \equiv_{\sigma} b$ there is $s \in S$ such that $s \leq a, b$ and $s = ea, s = fb$ for some $e, f \in E_S$. Then we have

$$\begin{aligned} s^{-1}a &= (ea)^{-1}a = (aa^{-1})e \in E_S \iff s \sim a \\ sa^{-1} &= (ea)a^{-1} = a^{-1}a \cdot e \in E_S \iff s \sim a. \end{aligned}$$

In the same manner, $s^{-1}b, sb^{-1} \iff s \sim b$. Thus $s \sim a, b$ and by assumption $a \equiv_{\rho} s$ and $s \equiv_{\rho} b$. But ρ is a congruence, so $a \equiv_{\rho} b$. Hence $\sigma \subseteq \rho$.

(2) Since σ is a congruence on S , then S/σ is a completely inverse AG^{**}-groupoid. Suppose that $(a)_{\sigma} \in E_{S/\sigma}$, then

$$(a)_{\sigma} = ((a)_{\sigma})((a^{-1})_{\sigma}) = (aa^{-1})_{\sigma} = (a^{-1}a)_{\sigma} = (e)_{\sigma} \text{ for some } a^{-1}a = e \in E_S.$$

Hence for every $(a)_\sigma \in E_{S/\sigma}$, there is $(e)_\sigma$ such that $a_\sigma = (e)_\sigma$, that is, $(a, e) \in \sigma$. Also $(ea)_\sigma = (aa^{-1} \cdot a)_\sigma = (a)_\sigma$. For $e, f \in E_S$, it is clear that $ef \leq e, f$, that is $ef \sim e$ and $ef \sim f$ and thus $e \sim f$. Since σ is a congruence, then $e \sim f$ implies $e \equiv_\sigma f$, that is, all idempotents are contained in a single σ -class. Thus S/σ containing a single σ -class of idempotents is an AG-group.

(3) Suppose that ρ is a congruence such that S/ρ is an AG-group. Suppose $a \equiv_\sigma b$. Then for some $s \in S$, $s \leq a, b$. Therefore there exists $e, f \in E_S$ such that $ea = s = fb$. Hence

$$(s)_\rho = (ea)_\rho = (e)_\rho(a)_\rho, \quad (s)_\rho = (fb)_\rho = (f)_\rho(b)_\rho,$$

where $(e)_\rho, (f)_\rho \in E_{S/\rho}$. Since S/ρ is an AG-group, $(e)_\rho = (f)_\rho$, that is $(e, f) \in \rho$. Thus $(a)_\rho = (s)_\rho = (b)_\rho$ and $a \equiv_\rho b$. Therefore $\sigma \subseteq \rho$. $\square$

Lemma 5.3. *Let S be a completely left inverse AG**-groupoid. Then $a \equiv_\sigma b$ if and only if there exists an idempotent i such that $ia = ib$.*

Proof. Suppose $a\sigma b$. Then for some $x \in S$ and $e, f \in E_S$, we have $ea = x = fb$. Thus $(ef)a = e(fa) = (ef)(ea) = (ef)b$ where ef is the required element. The converse is trivial. $\square$

An idempotent-separating congruence μ is a congruence on a completely inverse AG**-groupoid in which distinct idempotents lie in distinct congruence classes, that is $(e)_\mu = (f)_\mu \implies e = f$ for $e, f \in E_S$. In the following proposition, we investigate a maximum idempotent-separating congruence of a completely inverse AG**-groupoid S .

Proposition 5.4. *Let S be a completely inverse AG**-groupoid. Then the relation*

$$(a, b) \in \mu \text{ if and only if } a^{-1} \cdot ea = b^{-1} \cdot eb \text{ for all } e \in E_S$$

is the maximum idempotent-separating congruence.

Proof. The relation μ indeed is an equivalence relation. To show that μ is a congruence relation let $(a, b), (c, d) \in \mu$ and $e \in E_S$. Then

$$\begin{aligned} (ac)^{-1}(e(ac)) &= (a^{-1}c^{-1})(e(ac)) = (a^{-1}e)(c^{-1}(ac)) \\ &= c^{-1}((a \cdot ea^{-1})c) = d^{-1}((b \cdot eb^{-1})d) = (bd)^{-1}(e(bd)) \end{aligned}$$

which shows that μ is a congruence. For idempotent-separating, choose idempotents e, f such that $(e, f) \in \mu$. Then

$$e = e^{-1} \cdot ee = f^{-1} \cdot ef = ef = e^{-1} \cdot ef = e^{-1} \cdot fe = f^{-1} \cdot ff = f.$$

Hence μ is an idempotent-separating congruence. Further suppose ρ be another idempotent-separating congruence. Then for $(a, b) \in \rho$ and any idempotent e , both (a^{-1}, b^{-1}) and $(a^{-1} \cdot ea, b^{-1} \cdot eb)$ are in ρ . But since ρ is idempotent-separating, we have $a^{-1} \cdot ea = b^{-1} \cdot eb$. Hence $(a, b) \in \mu$. $\square$

Green's relations are the most important tools for understanding semigroups. Green's relations for AG-groupoids are defined the same way as in semigroup theory [2]. There are five Green's relations denoted by $\mathcal{L}, \mathcal{R}, \mathcal{J}, \mathcal{H} = \mathcal{L} \cap \mathcal{R}$ and $\mathcal{D} = \mathcal{L} \vee \mathcal{R}$. It is important to be noted that every $\mathcal{H}$ -class contains exactly one idempotent. In the following proposition, the largest congruence μ that is contained in $\mathcal{H}$ is considered.

Proposition 5.5. *Let S be a completely inverse AG^{**} -groupoid. Then μ is the largest congruence contained in $\mathcal{H}$.*

Proof. For $\mu \subseteq \mathcal{H}$, it is enough to show that $a \mathcal{L} b$. Let $(a, b) \in \mu$, then for some $bb^{-1} \in E_S$, $a^{-1}(bb^{-1} \cdot a) = b^{-1}(bb^{-1} \cdot b)$. But $b^{-1}(bb^{-1} \cdot b) = bb^{-1} \cdot b^{-1}b = b^{-1}(bb^{-1} \cdot b) = b^{-1}b$ and then $b = bb^{-1} \cdot b = (a^{-1}(bb^{-1} \cdot a))b = (b(bb^{-1} \cdot a^{-1}))a \in Sa$. Similarly, $a \in Sb$ and $a \mathcal{L} b$. Thus $a \mathcal{H} b$. Suppose ν be another congruence such that $(a, b) \in \nu \subseteq \mathcal{H}$. Then $(a^{-1}, b^{-1}) \in \nu$ and for some $e \in E_S$, $(a^{-1} \cdot ea, b^{-1} \cdot eb) \in \nu$. Then $a^{-1} \cdot ea = b^{-1} \cdot eb$, since every H -class contains only one idempotent and thus $(a^{-1} \cdot ea, b^{-1} \cdot eb) \in \mu$. Hence μ contains an arbitrary congruence $\nu \subseteq \mathcal{H}$. $\square$

A completely inverse AG^{**} -groupoid is fundamental if μ is the equality relation. Every completely inverse AG^{**} -groupoid can be made *fundamental* if the factor AG^{**} -groupoid is kept with respect to μ . The following proposition is a direct connection between the fundamental completely inverse AG^{**} -groupoid and maximum idempotent-separating congruence.

Proposition 5.6. *Let μ be the maximum idempotent-separating congruence on a completely inverse AG^{**} -groupoid S . Then S/μ is fundamental and it has the semilattice of idempotents isomorphic to E_S .*

Proof. Consider the morphism $\natural : S \rightarrow S/\mu$. Then for any $e \in E_S$ idempotent in S/μ it can be written as $(e)_\mu$. Suppose that $((a)_\mu, (b)_\mu) \in \mu_{S/\mu}$. Then $(a^{-1} \cdot ea)_\mu = (a^{-1})_\mu \cdot (e)_\mu(a)_\mu = ((a)_\mu)^{-1} \cdot (e)_\mu(a)_\mu \mu_{S/\mu}((b)_\mu)^{-1} \cdot (e)_\mu(b)_\mu = (b^{-1})_\mu \cdot (e)_\mu(b)_\mu = (b^{-1} \cdot eb)_\mu$. Thus $((a^{-1} \cdot ea)_\mu, (b^{-1} \cdot eb)_\mu) \in \mu_{S/\mu}$. But $\mu_{S/\mu}$ is idempotent-separating, so we have $(a^{-1} \cdot ea)_\mu = (b^{-1} \cdot eb)_\mu$. Since μ is idempotent-separating, it follows that $a^{-1} \cdot ea = b^{-1} \cdot eb$. Hence $(a)_\mu = (b)_\mu$.

For each $(e)_\mu \in E_{S/\mu}$, there exists $e \in E_S$ such that $e \rightarrow (e)_\mu$. Hence $E_S \cong E_{S/\mu}$. $\square$

Theorem 5.7. *Let S be a completely inverse AG^{**} -groupoid on S . Then*

$$\varrho = \{(a, b) \in S \times S : ea^2 = b^2f \text{ for some } e, f \in E_S\}$$

is a congruence relation on S .

Proof. Reflexivity and symmetry are immediate. We show that ϱ is transitive and compatible. Let $(a, b) \in \varrho$ and $(b, c) \in \varrho$, then for some idempotents e, f, g, h , we have $ea^2 = b^2f$ and $gb^2 = c^2h$. Since $ge \in E_S$,

$$(ge)a^2 = g(ea^2) = g(b^2f) = (c^2h)f = c^2(fh)$$

where $fh \in E_S$. Further, suppose $(a, b) \in \varrho$. Then for some $c \in S$,

$$e(ca)^2 = c^2(ea^2) = c^2(b^2f) = (fb^2)c^2 = (cb)^2f.$$

Similarly, if $(a, b) \in \varrho$, then for $c \in S$,

$$e(ac)^2 = (ea^2)c^2 = (b^2f)c^2 = c^2(fb^2) = f(bc)^2.$$

Hence ϱ is a congruence on S . $\square$

A congruence ρ on a completely inverse AG^{**} -groupoid S is idempotent-pure if for $e \in E_S$, $a \in S$ and $(a, e) \in \rho \implies a \in E_S$.

Proposition 5.8. *Let S be a completely inverse AG^{**} -groupoid. Then a congruence ρ is idempotent-pure if and only if $\rho \subseteq \sim$.*

Proof. Suppose that $(a, b) \in \rho$. Then $(ab^{-1}, bb^{-1}) \in \rho$. Clearly ab^{-1} is an idempotent because bb^{-1} is idempotent and ρ is idempotent-pure. Similarly, $a^{-1}b$ is an idempotent and hence $a \sim b$.

Conversely, suppose that the congruence ρ is contained in $\sim$. Let for an idempotent e , $(a, e) \in \rho$. Then $(e^{-1}, a^{-1}) \in \rho$. Thus $e = e^{-1}e = a^{-1}a$. Since ρ is a congruence, it is clear that $a \equiv_{\rho} a^{-1}a$. By assumption $\rho \subseteq \sim$, thus we have $a \sim a^{-1}a$. Then $a = aa^{-1} \cdot a = a^{-1}a \cdot a = (aa^{-1})^{-1}a \in E_S$. Hence ρ is idempotent-pure. $\square$

6. E-unitary completely inverse AG^{**} -groupoid

In this section, we characterize E -unitary completely inverse AG^{**} -groupoid and investigate idempotent-pure congruence with a connection in compatibility relation containing the idempotent-pure congruence.

A subgroupoid A of an AG -groupoid S is called left (right) unitary if for $a \in A$, $s \in S$ and $as \in A$ ($sa \in A$), imply that $s \in A$. A is unitary if it is both left and right unitary. An AG -groupoid S is E -unitary if E_S is unitary.

Proposition 6.1 ([3, Proposition 4.4]). *Let S be an AG^{**} -groupoid. Then the following are equivalent:*

- (1) E_S is right unitary.
- (2) E_S is left unitary.
- (3) If $e \in E_S$ and $e \leq a$, then $a \in E_S$.

Proof. (1) $\Rightarrow$ (2) : Suppose $e, ea \in E_S$. Then we have $E_S \ni af = ea$ for some idempotent f . Thus E_S is left unitary.

(2) $\Rightarrow$ (3) : Suppose E_S is left unitary and $e \leq a$ for $e \in E_S$. Then $e = ea$ and hence $a \in E_S$.

(3) $\Rightarrow$ (1) : Suppose $ea = f \in E_S$. Then $f \leq a$ and thus E_S is right unitary. $\square$

Proposition 6.2. *The compatibility relation on a completely inverse AG^{**}-groupoid S is transitive if and only if S is E -unitary.*

Proof. Let the compatibility relation $\sim$ be transitive and for $e \in E_S$, $e \leq a$. Clearly, $e^{-1}a = ea = e$ and $ea^{-1} \leq aa^{-1}$ implies that ea^{-1} is an idempotent because $aa^{-1} \in E_S$. Hence $a \sim e$. Since e and aa^{-1} both are idempotents, therefore $e \sim aa^{-1}$. But $\sim$ is transitive and hence $a \sim aa^{-1}$. It is clear that a is an idempotent, since $a = aa^{-1} \cdot a = a^{-1}a \cdot a = (aa^{-1})^{-1}a \in E_S$ and thus E_S is E -unitary.

Conversely, suppose that E_S is E -unitary and $a \sim b$ and $b \sim c$. Then by assumption $a^{-1}b$, ab^{-1} and $b^{-1}c$, bc^{-1} are idempotents. Thus we have

$$\begin{aligned} a^{-1}b \cdot b^{-1}c &= (a^{-1}b)((b^{-1}b \cdot b^{-1})c) = (a^{-1}b)(cb^{-1} \cdot b^{-1}b) \\ &= (a^{-1}b \cdot cb^{-1})(b^{-1}b) \\ &= (a^{-1}c \cdot bb^{-1})(b^{-1}b) = (bb^{-1} \cdot b^{-1}b)(a^{-1}c) \leq a^{-1}c. \end{aligned}$$

In the same manner,

$$\begin{aligned} ab^{-1} \cdot bc^{-1} &= (ab^{-1})((bb^{-1} \cdot b)c^{-1}) = (ab^{-1})(c^{-1}b \cdot b^{-1}b) \\ &= (ab^{-1} \cdot c^{-1}b)(b^{-1}b) = (ac^{-1} \cdot bb^{-1})(b^{-1}b) = (bb^{-1} \cdot b^{-1}b)(ac^{-1}) \\ &\leq ac^{-1}. \end{aligned}$$

Since S is E -unitary, $a^{-1}c$ and ac^{-1} are idempotents. Hence $a \sim c$. $\square$

Corollary 6.3. Let S be E -unitary completely inverse AG^{**}-groupoid. Then the compatibility relation $\sim$ is a congruence.

Lemma 6.4. *Let ρ be an idempotent-pure congruence on a completely inverse AG^{**}-groupoid S . Then S is E -unitary if and only if S/ρ is E -unitary.*

Proof. Suppose S is E -unitary. Let $a, b \in S$ such that $(a)_\rho, (b)_\rho \in S/\rho$ and $(b)_\rho(a)_\rho \in E_{S/\rho}$. Then there is $e \in E_S$ such that $(b)_\rho = (e)_\rho$, that is $(b, e) \in \rho$. Since ρ is idempotent-pure, $ea \in E_S$. But S is E -unitary, we have $a \in E_S$. Thus $(a)_\rho \in E_{S/\rho}$. The converse is simple. $\square$

Theorem 6.5. *Let S be a completely inverse AG^{**}-groupoid. Then the following are equivalent:*

- (1) S is E -unitary.
- (2) $\sim = \sigma$.
- (3) σ is idempotent-pure.
- (4) for all $e \in E_S$, $\sigma(e) = E_S$.
- (5) AA^{-1} and $A^{-1}A$ are ideals of E_S for every σ -class.
- (6) $A^{-1} \cdot eA$ is an ideal of E_S for every $e \in E_S$ and every σ -class A .

Proof. (1) $\Rightarrow$ (2) : Suppose S is E -unitary. It is clear from Theorem 5.2 that $\sim \subseteq \sigma$. Let $(a, b) \in \sigma$ then $s \leq a, b$ for some $s \in S$. Therefore for idempotents e, f , we have $ea = s = fb$. It follows that

$$ss^{-1} = (ea)(fb)^{-1} = (ef)(ab^{-1}) \leq ab^{-1}$$

and similarly $s^{-1}s \leq a^{-1}b$. But S is E -unitary so ab^{-1} and $a^{-1}b$ are idempotents. Thus a and b are compatible.

(2) $\Rightarrow$ (3) : From Proposition 5.8, since $\sigma \subseteq \sim$, it follows immediately that σ is idempotent-pure.

(3) $\Rightarrow$ (4) : It follows from the definition of idempotent-pure congruence.

(4) $\Rightarrow$ (5) : We only show AA^{-1} is an ideal. If $(a, b) \in \sigma$, then $ab^{-1} \in E_S$. For $e \in E_S$,

$$(ab^{-1})e = e(ab^{-1}) = a(eb)^{-1} \in AA^{-1}.$$

Hence AA^{-1} is an ideal of E_S .

(5) $\Rightarrow$ (6) : Let $a, b \in A$. Then for any idempotent e , we have $a^{-1} \cdot eb = e \cdot a^{-1}b \in E_S$, because by hypothesis $a^{-1}b \in E_S$. Thus $A^{-1} \cdot eA$ is contained in E_S . For every $f \in E_S$,

$$f(a^{-1} \cdot eb) = (a^{-1} \cdot eb)f = (fe \cdot a^{-1}b) = (fa^{-1} \cdot eb) = (fa)^{-1}(eb) \in A^{-1} \cdot eA.$$

Hence $A^{-1} \cdot eA$ is an ideal of E_S .

(6) $\Rightarrow$ (1) : Suppose $ea, e \in E_S$. Then $(a, ea) \in \sigma$. Also

$$ea = ee \cdot a = e \cdot ea = ea \cdot e \leq e$$

which implies that $(ea, e) \in \sigma$. Thus $(a, e) \in \sigma$. Moreover, $(aa^{-1}, ee^{-1}) \in \sigma$. Letting $A = (a)_\sigma$ we have

$$a = (aa^{-1})a = (aa^{-1})(aa^{-1} \cdot a) \in ((a)_\sigma)^{-1} \cdot (aa^{-1})((a)_\sigma) \in A^{-1} \cdot (aa^{-1})A \subseteq E_S.$$

Hence it follows that S is E -unitary. $\square$

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SOME GENERALIZATION OF SUBPULLBACK FLAT

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Abstract. Bulman-Fleming et al. in [*Pullbacks and flatness properties of acts II*, Comm. Algebra, 29(2) (2001), 851-878] initiated the study of three flatness properties of right S -act A_S that can be described by means of when the functor $A_S \otimes -$ preserves certain types of pullbacks. The present paper is extended these results to S -posets and is presented equivalences of them. Moreover, we show that flatness properties of S -posets that mentioned in this paper and in [A. Golchin, L. Nouri, *Subpullbacks and Po-flatness Properties of S-posets*, J. Sci. Islam. Repub. Iran, 25(4) (2014), 369-377], can be transferred to their coproducts, and vice versa.

Keywords: S -poset, subpullback diagram, surjectivity, (weak) po-surjectivity, order embedding.

1. Introduction

Let S be a partially ordered monoid, or briefly, pomonoid with identity element 1. A nonempty poset A is called a right S -poset, usually denoted A_S , if S acts on A from the right, that is there exists a mapping $A \times S \rightarrow A$, $(a, s) \mapsto as$, which satisfies the conditions: (i) the action is monotonic in each variable, (ii) $(as)t = a(st)$ and $a1 = a$, for all $a \in A$ and all $s, t \in S$. Left S -poset ${}_S B$ is defined analogously, and $\Theta_S = \{\theta\}$ is the one-element right S -poset. All left (resp., right) S -posets form a category, denoted **S-POS** (resp., **POS-S**), in which the

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morphisms are the functions that preserve both the action and the order (see [3]). The partially ordered sets form a category, denoted **POS**, in which the morphisms are the functions that preserve the order. In these categories the monomorphisms and epimorphisms are the injective and surjective morphisms, respectively, and a morphism $g : A \rightarrow B$ is called *order embedding* if $g(a) \leq g(a')$ implies $a \leq a'$, for all $a, a' \in A$. A surjective order embedding is called an *order isomorphism*.

A nonempty subset I of a pomonoid S , is called an *ordered right ideal* of S if (i) $IS \subseteq I$ and (ii) $a \leq b \in I$ implies $a \in I$, for all $a, b \in S$. An S -subposet B_S of a right S -poset A_S is called *strongly convex* if $a \leq b$ implies $a \in B_S$, for any $a \in A_S$ and $b \in B_S$. Clearly, if I is an ordered right ideal of a pomonoid S , then I is a strongly convex S -subposet of the S -poset I .

In the 1980s, preliminary work on flatness properties of S -posets was done by Fakhruddin (see [5, 6]). Recently in this area some new articles appeared, such as [3, 1, 4, 13].

In [9] the notation $P(M, N, f, g, Q)$ was introduced to denote the pullback diagram of homomorphisms $f : {}_S M \rightarrow {}_S Q$ and $g : {}_S N \rightarrow {}_S Q$ in the category of left S -acts, where S is a monoid. Tensoring such a diagram by A_S produces a diagram (in the category of sets) that may or may not be a pullback diagram, depending on whether or not the mapping φ , obtained via the universal property of pullbacks in the category of sets, is bijective. It was shown that, if we require either bijectivity or surjectivity of φ for pullback diagrams of certain types, we not only recover most of the well-known forms of flatness, but also obtain Conditions (WP) and (PWP) as well. In [8] Golchin and Rezaei extended the results from [9] to S -posets and two new Conditions $(WP)_w$ and $(PWP)_w$ were introduced. In [7] Golchin and Nouri described po-flatness properties of S -posets over pomonoids by po-surjectivity of φ corresponding to certain subpullback diagrams. Then they introduced three new Conditions (P_{sw}) , $(WP)_{sw}$ and $(PWP)_{sw}$ and describe these properties by po-surjectivity of φ corresponding to certain subpullback diagrams and described Conditions (P_w) , $(WP)_w$ and $(PWP)_w$ by weak po-surjectivity of φ corresponding to certain subpullback diagrams.

For basic definitions and terminology relating to S -posets, we refer the reader to [4].

We recall from [3] that in the categories of S -posets and posets the order relation on morphism sets is defined pointwise (i.e. $f \leq g$ for $f, g : A \rightarrow B$ if and only if $f(a) \leq g(a)$ for every $a \in A$). In such categories, a diagram

$$\begin{array}{ccc}
 P & \xrightarrow{p_1} & M \\
 p_2 \downarrow & & \downarrow f \\
 N & \xrightarrow{g} & Q
 \end{array}$$

(P₁)

is subcommutative if $fp_1 \leq gp_2$. If $fp_1 \leq gp_2$ and for every left S -poset P' and all homomorphisms $p'_1 : P' \rightarrow M$ and $p'_2 : P' \rightarrow N$ such that $fp'_1 \leq gp'_2$, there exists a unique homomorphism $h : P' \rightarrow P$ such that $p_1h = p'_1$ and $p_2h = p'_2$, then diagram (P₁) will be called *subpullback diagram* for f and g . In the category of S -posets or the category of posets, P may in fact be realized as

$$P = \{(m, n) \in M \times N \mid f(m) \leq g(n)\}$$

where p_1 and p_2 are the first and second coordinate projections (Notice that P is possibly empty). The subpullback diagram (P₁) will be denoted by $P(M, N, f, g, Q)$. Tensoring the subpullback diagram $P(M, N, f, g, Q)$ by any right S -poset A one gets the subcommutative diagram

$$\begin{array}{ccc}
 A \otimes P & \xrightarrow{id_A \otimes p_1} & A \otimes M \\
 id_A \otimes p_2 \downarrow & & \downarrow id_A \otimes f \\
 A \otimes N & \xrightarrow{id_A \otimes g} & A \otimes Q
 \end{array}$$

in the category of posets. For the subpullback of mappings $id_A \otimes f$ and $id_A \otimes g$ in the category of posets we may take

$$P' = \{(a \otimes m, a' \otimes n) \in (A \otimes M) \times (A \otimes N) \mid a \otimes f(m) \leq a' \otimes g(n)\}$$

with p'_1 and p'_2 the restrictions of the projections. It follows from the definition of subpullbacks that there exists a unique monotone mapping $\varphi : A \otimes_S P \rightarrow P'$

such that, in the diagram

$$\begin{array}{ccccc}
 A \otimes P & & & & \\
 \searrow \varphi & \searrow id_A \otimes p_1 & & & \\
 & P' & \xrightarrow{p'_1} & A \otimes M & \\
 \searrow id_A \otimes p_2 & \downarrow p'_2 & & \downarrow id_A \otimes f & \\
 & A \otimes N & \xrightarrow{id_A \otimes g} & A \otimes Q & \\
 & & (P_2) & &
 \end{array}$$

$p'_i \varphi = id_A \otimes p_i$, for $i = 1, 2$. We shall call this mapping *the φ corresponding to the subpullback diagram $P(M, N, f, g, Q)$* . It can be seen that the mapping φ in diagram (P_2) is given by

$$\varphi(a \otimes (m, n)) = (a \otimes m, a \otimes n)$$

for all $a \in A_S$ and $(m, n) \in {}_S P$.

Notice that surjectivity of φ means

$$\begin{aligned}
 (\forall a, a' \in A_S)(\forall m \in {}_S M)(\forall n \in {}_S N)[a \otimes f(m) \leq a' \otimes g(n) \Rightarrow \\
 (\exists a'' \in A_S)(\exists m' \in {}_S M)(n' \in {}_S N) \\
 (f(m') \leq g(n') \wedge a \otimes m = a'' \otimes m' \wedge a' \otimes n = a'' \otimes n')],
 \end{aligned}$$

po-surjectivity of φ means

$$\begin{aligned}
 (\forall a, a' \in A_S)(\forall m \in {}_S M)(\forall n \in {}_S N)[a \otimes f(m) \leq a' \otimes g(n) \Rightarrow \\
 (\exists a'' \in A_S)(\exists m' \in {}_S M)(n' \in {}_S N) \\
 (f(m') \leq g(n') \wedge a \otimes m = a'' \otimes m' \wedge a'' \otimes n' \leq a' \otimes n)],
 \end{aligned}$$

and weak po-surjectivity of φ means

$$\begin{aligned}
 (\forall a, a' \in A_S)(\forall m \in {}_S M)(\forall n \in {}_S N)[a \otimes f(m) \leq a' \otimes g(n) \Rightarrow \\
 (\exists a'' \in A_S)(\exists m' \in {}_S M)(n' \in {}_S N) \\
 (f(m') \leq g(n') \wedge a \otimes m \leq a'' \otimes m' \wedge a'' \otimes n' \leq a' \otimes n)].
 \end{aligned}$$

Obviously, surjectivity $\Rightarrow$ po-surjectivity $\Rightarrow$ weak po-surjectivity.

Also injectivity of φ means that

$$\begin{aligned} & (\forall a, a' \in A_S)(\forall m, m' \in {}_S M)(\forall n, n' \in {}_S N) \\ & [f(m) \leq g(n) \wedge f(m') \leq g(n') \wedge a \otimes m = a' \otimes m' \wedge a \otimes n = a' \otimes n' \Rightarrow \\ & \quad a \otimes (m, n) = a' \otimes (m', n')], \end{aligned}$$

and φ is order embedding if

$$\begin{aligned} & (\forall a, a' \in A_S)(\forall m, m' \in {}_S M)(\forall n, n' \in {}_S N) \\ & [f(m) \leq g(n) \wedge f(m') \leq g(n') \wedge a \otimes m \leq a' \otimes m' \wedge a \otimes n \leq a' \otimes n' \Rightarrow \\ & \quad a \otimes (m, n) \leq a' \otimes (m', n')]. \end{aligned}$$

Similar to S -acts, coproduct of S -posets are disjoint unions, with S -action and order defined componentwise. If $A = \dot{\cup}_{i \in I} A_i$, where A_i , $i \in I$ is a strongly convex right S -poset, then by the mapping corresponding to the subpullback diagram $P(M, N, f, g, Q)$, for A_i , $i \in I$, we mean the unique mapping φ_i which makes the diagram

$$\begin{array}{ccccc} A_i \otimes P & & & & \\ & \searrow \varphi_i & \nearrow id_{A_i} \otimes p_1 & & \\ & P'_i & \xrightarrow{p'_{1i}} & A_i \otimes M & \\ & \downarrow p'_{2i} & & \downarrow id_{A_i} \otimes f & \\ id_{A_i} \otimes p_2 \swarrow & A_i \otimes N & \xrightarrow{id_{A_i} \otimes g} & A_i \otimes Q & \\ & & (P_{2i}) & & \end{array}$$

commutative, where

$$P'_i = \{(a \otimes m, a' \otimes n) \in (A_i \otimes M) \times (A_i \otimes N) \mid a \otimes f(m) \leq a' \otimes g(n)\},$$

p'_{1i} and p'_{2i} are restrictions of projections to P'_i .

A po-surjective order embedding is called a *po-order isomorphism* and a weak po-surjective order embedding is called a *weak po-order isomorphism*. This paper continues the investigation of the class of right S -posets A_S over S , for which the functor $A_S \otimes -$ has certain subpullback preservation properties. The variations of the types of subpullbacks considered in [7, 10] and in this paper are of the following types.

Here the abbreviations stand for the following properties:

OI= order isomorphism,

POI=po-order isomorphism,

WPOI=weak po-order isomorphism,

WPS=weak po-surjective.

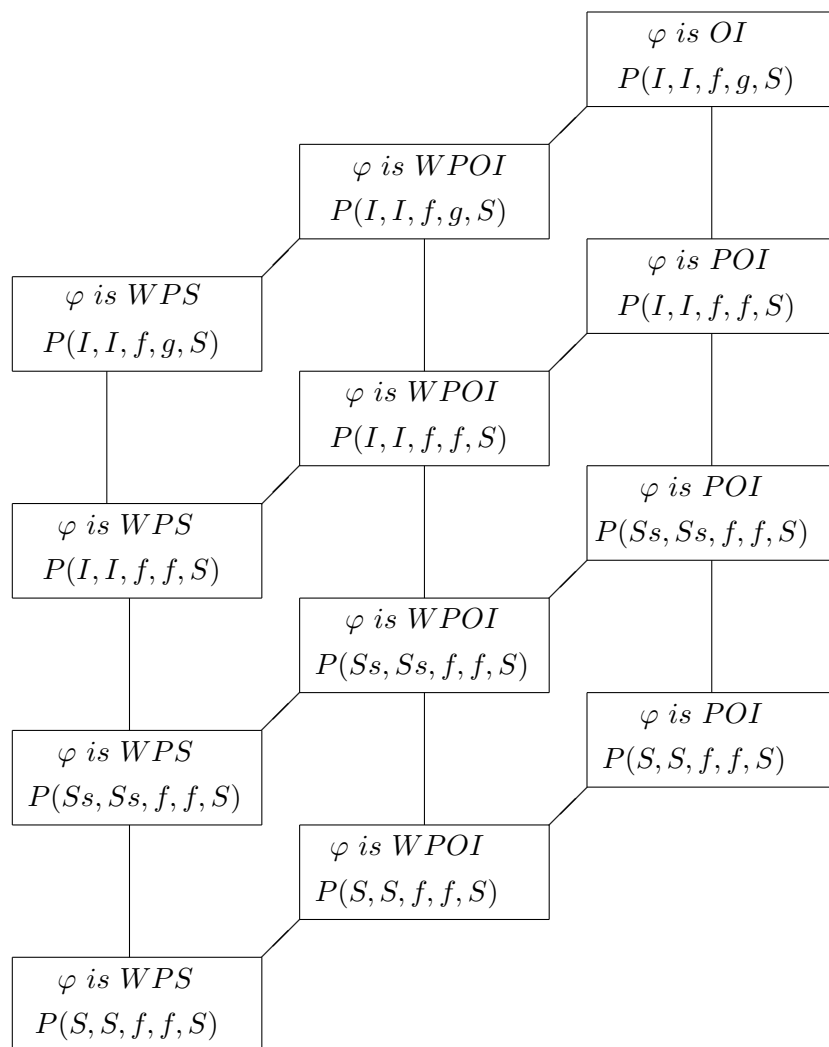


Figure 1.

2. Basic results

In this section, we discuss the classes of right S -poset A_S corresponding to the two and three columns of Figure 1. Also we give equivalences for them. Finally we will show that flatness properties of S -posets that mentioned in this paper and [7], can be transferred to their coproducts, and vice versa.

First we consider the following lemma that formulated in [13].

Lemma 2.1. *Let A_S be a right S -posets and ${}_S B$ be a left S -posets. $a \otimes b \leq a' \otimes b'$ in $A_S \otimes {}_S B$, for $a, a' \in A_S$ and $b, b' \in {}_S B$ if and only if there exist $a_1, a_2, \dots, a_n \in A_S$, $b_2, \dots, b_n \in {}_S B$, $s_1, t_1, \dots, s_n, t_n \in S$, such that*

$$\begin{aligned} a &\leq a_1 s_1 \\ a_1 t_1 &\leq a_2 s_2 & s_1 b &\leq t_1 b_2 \\ a_2 t_2 &\leq a_3 s_3 & s_2 b_2 &\leq t_2 b_3 \\ &\dots & &\dots \\ a_n t_n &\leq a' & s_n b_n &\leq t_n b'. \end{aligned}$$

Definition 2.2. *A right S -poset A_S is called:*

- (1) *weakly subpullback po-flat, if the corresponding φ is order isomorphism, for every subpullback diagram $P(I, I, f, g, S)$.*
- (2) *weakly subpullback flat, if the corresponding φ is weak po-order isomorphism, for every subpullback diagram $P(I, I, f, g, S)$.*
- (3) $(WKPF)_{sw}$, *if the corresponding φ is po-order isomorphism, for every subpullback diagram $P(I, I, f, f, S)$.*
- (4) $(PWKPF)_{sw}$, *if the corresponding φ is po-order isomorphism, for every subpullback diagram $P(Ss, Ss, f, f, S)$.*
- (5) $(TKPF)_{sw}$, *if the corresponding φ is po-order isomorphism, for every subpullback diagram $P(S, S, f, f, S)$.*
- (6) $(WKPF)_w$, *if the corresponding φ is weak po-order isomorphism, for every subpullback diagram $P(I, I, f, f, S)$.*
- (7) $(PWKPF)_w$, *if the corresponding φ is weak po-order isomorphism, for every subpullback diagram $P(Ss, Ss, f, f, S)$.*
- (8) $(TKPF)_w$, *if the corresponding φ is weak po-order isomorphism, for every subpullback diagram $P(S, S, f, f, S)$.*

An S -poset A_S is called *subpullback flat* if the functor $A_S \otimes -$ takes subpullbacks in **S-POS** to subpullbacks in **POS**. Clearly A_S is subpullback flat if and

only if the corresponding φ is order isomorphic, for every subpullback diagram $P(M, N, f, g, Q)$.

Recall from [3] that an S -poset A_S satisfies *Condition (P)*, if for all $a, a' \in A_S$ and $s, s' \in S$, $as \leq a's'$ implies $a = a''u$, $a' = a''v$, for some $a'' \in A_S$ and $u, v \in S$, such that $us \leq vs'$. The S -poset A_S satisfies *Condition (E)*, if for all $a \in A_S$ and $s, s' \in S$, $as \leq as'$ implies $a = a''u$, for some $a'' \in A_S$ and $u \in S$, such that $us \leq us'$.

Also recall from [12] that an S -poset A_S satisfies *Condition (PF)*, if for all $a, a' \in A_S$ and $s, s', t, t' \in S$, $as \leq a's'$ and $at \leq at'$ imply $a = a''u$, $a' = a''v$, for some $a'' \in A_S$ and $u, v \in S$, such that $us \leq vs'$ and $ut \leq vt'$.

Now we give equivalences for subpullback flat.

Theorem 2.3. *For any S -poset A_S , the following statements are equivalent:*

- (1) A_S is subpullback flat;
- (2) A_S satisfies *Condition (PF)*
- (3) A_S satisfies *Conditions (P) and (E)*;

Proof. (2) $\Leftrightarrow$ (3). It follows from [12, Proposition 2.3].

(1) $\Rightarrow$ (2). It is obvious that A_S satisfies *Condition (P)*, by [8, Theorem 3.2]. Let $as \leq a's'$ and $at \leq a't'$, for $a, a' \in A_S$ and $s, s', t, t' \in S$. Now it is clear that

$$\begin{array}{ccc} {}_S S \amalg {}_S S & \xrightarrow{p_1} & {}_S S \\ p_2 \downarrow & & \downarrow c_\theta \\ {}_S S & \xrightarrow{c_\theta} & {}_S \Theta \end{array}$$

is the subpullback diagram in $\mathbf{S}\text{-Act}$, where p_1 and p_2 are the projections. Since A_S is subpullback flat, the diagram

$$\begin{array}{ccc} A_S \otimes ({}_S S \amalg {}_S S) & \xrightarrow{id_A \otimes p_1} & A_S \otimes {}_S S \\ id_A \otimes p_2 \downarrow & & \downarrow id_A \otimes c_\theta \\ A_S \otimes {}_S S & \xrightarrow{id_A \otimes c_\theta} & A_S \otimes {}_S \Theta \end{array}$$

is a subpullback diagram in **Set**. Applying the canonical $A_S \otimes_S S \cong A_S$, (see [12, Lemma 3.2]) we have the diagram

$$\begin{array}{ccc} P & \xrightarrow{p'_1} & A_S \\ p'_2 \downarrow & & \downarrow id_A \otimes c_\theta \\ A_S & \xrightarrow{id_A \otimes c_\theta} & A_S \otimes_S \Theta \end{array}$$

is the subpullback diagram in **S-Act**. By assumption the mapping $\varphi : A_S \otimes ({}_S S \amalg {}_S S) \rightarrow P$, with $\varphi(\bar{a} \otimes (x, y)) = (\bar{a}x, \bar{a}y)$, for any $\bar{a} \in A_S$ and $x, y \in S$ is order isomorphic. From $as \leq a's'$ and $at \leq a't'$ follow that $\varphi(a \otimes (s, t)) \leq \varphi(a' \otimes (s', t'))$. Since φ is order embedding, thus $a \otimes (s, t) \leq a' \otimes (s', t')$. Now since A_S satisfies Condition (P), thus by [8, Lemma 3.1], there exist $a'' \in A_S$ and $u, v \in S$, such that $a = a''u$, $a' = a''v$ and $u(s, t) \leq v(s', t')$. Therefore $us \leq vs'$ and $ut \leq vt'$, and so A_S satisfies Condition (PF), as required.

(1) $\Rightarrow$ (2). Since Condition (PF) implies Condition (P), thus the corresponding φ is surjective, for every subpullback diagram $P(M, N, f, g, Q)$, by [8, Theorem 3.2]. Now we show that φ is order embedding. Let

$$f(m) \leq g(n) \wedge f(m') \leq g(n') \wedge a \otimes m \leq a' \otimes m' \wedge a \otimes n \leq a' \otimes n',$$

for $a, a' \in A_S$, $m, m' \in {}_S M$ and $n, n' \in {}_S N$. Then Condition (P) implies that there exist $a_1, a_2 \in A_S$, $u_1, u_2, v_1, v_2 \in S$, such that $a = a_1 u_1 = a_2 u_2$, $a' = a_1 v_1 = a_2 v_2$, $u_1 m \leq v_1 m'$ and $u_2 n \leq v_2 n'$. Now applying Condition (BF) to the equalities $a_1 u_1 = a_2 u_2$ and $a_1 v_1 = a_2 v_2$ implies that there exist $a'' \in A_S$ and $u, v \in S$ such that $a_1 = a''u$, $a_2 = a''v$, $uu_1 \leq vv_2$ and $uv_1 \leq vv_2$. Thus

$$\begin{aligned} (m, n) &= a_1 u_1 \otimes (m, n) = a'' u u_1 \otimes (m, n) = a'' \otimes u u_1 (m, n) \\ &= a'' \otimes (u u_1 m, u u_1 n) \leq a'' \otimes (u v_1 m', u u_1 n) \leq a'' \otimes (u v_1 m', v u_2 n) \\ &= a'' \otimes (v v_2 m', v u_2 n) \leq a'' \otimes (v v_2 m', v v_2 n') = a'' \otimes v v_2 (m', n') \\ &= a'' v v_2 \otimes (m', n') = a' \otimes (m', n'), \end{aligned}$$

and so φ is order embedding, as required. $\square$

Recall from [11] that a pomonoid S is called *left collapsible*, if for any $s, s' \in S$, there exists $u \in S$, such that $us = us'$. A pomonoid S is called *weakly left collapsible*, if for any $s, s', z \in S$, $sz = s'z$ implies that there exists $u \in S$, such that $us = us'$.

Definition 2.4. A pomonoid S is called *strongly weakly left collapsible*, if for any $s, s', z, k \in S$, $sz \leq k$ and $s'z \leq k$ implies that there exists $u \in S$, such that $us = us'$.

It is obvious that every strongly weakly left collapsible pomonoid is weakly left collapsible, but in general, the converse is not true. See the following example.

Example 2.5. Let $S = \mathbb{N}$ be natural numbers, with natural order $\leq$. Obviously S is weakly left collapsible, but S is not strongly weakly left collapsible, since $2.5 < 20$, $3.5 < 20$ and for every $u \in S$, $u.2 \neq u.3$

Recall from [11] that an S -poset A_S satisfies *Condition (E')*, if for all $a \in A_S$, $s, s', z \in S$, $as \leq as'$ and $sz = s'z$ imply $a = a'u$, for some $a' \in A_S$ and $u \in S$ with $us \leq us'$. An S -poset A_S is called *weakly subpullback flat* if it satisfies Conditions (P) and (E'). Now we give the following definition.

Definition 2.6. An S -poset A_S satisfies *Condition (E')_s*, if for all $a \in A_S$, $s, s', z, k \in S$, $as \leq as'$, $sz \leq k$ and $s'z \leq k$ imply $a = a'u$, for some $a' \in A_S$ and $u \in S$ with $us \leq us'$.

The proof of the following theorem is similar to that of [1, Theorem 1(3)].

Theorem 2.7. For any pomonoid S , Θ_S satisfies *Condition (E')_s*, if and only if S is strongly weakly left collapsible.

Obviously *Condition (E')_s* implies *Condition (E')*, but Example 2.5 shows that the converse is not true.

Definition 2.8. An S -poset A_S satisfies *Condition (PF')*, if for all $a, a' \in A_S$ and $s, s', t, t', z, w \in S$, $as \leq a's'$, $at \leq a't'$, $sz \leq tw$ and $s'z \leq t'w$ imply $a = a''u$, $a' = a''v$, for some $a'' \in A_S$ and $u, v \in S$, such that $us \leq vs'$ and $ut \leq vt'$.

Now we give equivalences for weakly subpullback po-flat.

Theorem 2.9. For any S -poset A_S , the following statements are equivalent:

- (1) A_S is weakly subpullback po-flat;
- (2) the corresponding φ is order isomorphic, for every subpullback diagram $P(Ss, Ss, f, g, S)$;
- (3) the corresponding φ is order isomorphic, for every subpullback diagram $P(S, S, f, g, S)$;
- (4) A_S satisfies *Condition (PF')*;
- (5) A_S satisfies *Conditions (P) and (E')_s*.

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (4). It is obvious that A_S satisfies *Condition (P)*, by [8, Theorem 3.2]. Now let $as \leq a's'$, $at \leq a't'$, $sz \leq tw$ and $s'z \leq t'w$, for $a, a' \in A_S$ and

$s, s', t, t', z, w \in S$. Define $f, g : {}_S S \rightarrow {}_S S$, such that $f(u) = uz$ and $g(u) = uw$, for every $u \in S$. Then $f(s) \leq g(t)$, $f(s') \leq g(t')$, $a \otimes s \leq a' \otimes s'$ and $a \otimes t \leq a' \otimes t'$, and so $\varphi(a \otimes (s, t)) \leq \varphi(a' \otimes (s', t'))$. By assumption, since φ is order embedding, thus $a \otimes (s, t) \leq a' \otimes (s', t')$. Now by [8, Lemma 3.12], there exist $a'' \in A_S$ and $u, v \in S$, such that $a = a''u$, $a' = a''v$ and $u(s, t) \leq v(s', t')$. The last inequality implies that $us \leq vs'$ and $ut \leq vt'$, and so A_S satisfies Condition (PF') , as required.

(4) $\Rightarrow$ (5). Let $as \leq a's'$, for $a, a' \in A_S$ and $s, s' \in S$. By putting $t = s$, $t' = s'$ and $z = w = 1$ in the definition of Condition (PF') , at once Condition (P) is followed. Now let $as \leq as'$, $sz \leq k$ and $s'z \leq k$, for $a, a' \in A_S$ and $s, s', k \in S$. By putting $t = t' = 1$ and $w = k$, Condition (PF') implies that there exist $a'' \in A_S$ and $u, v \in S$ such that $a = a''u = a''v$ and $us \leq vs'$, and so A_S satisfies Condition $(E')_s$.

(5) $\Rightarrow$ (1). Since A_S satisfies Condition (P) , thus the corresponding φ is surjective, for every subpullback diagram $P(I, I, f, g, S)$. Now let $\varphi(a \otimes (s, t)) \leq \varphi(a' \otimes (s', t'))$, for $a, a' \in A_S$ and $s, s', t, t' \in I$. Then $f(s) \leq g(t)$, $f(s') \leq g(t')$, $a \otimes s \leq a' \otimes s'$ and $a \otimes t \leq a' \otimes t'$. Now $a \otimes s \leq a' \otimes s'$ and $a \otimes t \leq a' \otimes t'$ imply that $as \leq a's'$ and $at \leq a't'$, respectively. By utilization Condition (P) for inequality $at \leq a't'$, there exist $a_1 \in A_S$ and $u_1, v_1 \in S$ such that $a = a_1u_1$, $a' = a_1v_1$ and $u_1t \leq v_1t'$. Then $a_1(u_1s) \leq a_1(v_1s')$,

$$u_1sf(1) = u_1f(s) \leq u_1g(t) = utg(1) \leq v_1t'g(1)$$

and

$$v_1s'f(1) = v_1f(s') \leq v_1g(t') = v_1t'g(1).$$

Condition $(E')_s$ implies that there exist $a'' \in A_S$ and $v \in S$ such that $a_1 = a''v$ and $vu_1s \leq vv_1s'$. Now

$$\begin{aligned} a \otimes (s, t) &= a_1u_1 \otimes (s, t) = a''uu_1 \otimes (s, t) = a'' \otimes uu_1(s, t) \\ &= a'' \otimes (uu_1ms, uu_1t) \leq a'' \otimes (uv_1s', uv_1t') \leq a'' \otimes uv_1(s', t') \\ &= a''uv_1 \otimes (s', t') = a' \otimes (s', t'), \end{aligned}$$

and so φ is order embedding, as required. $\square$

Definition 2.10. An S -poset A_S satisfies Condition $(PF')_{sw}$, if for all $a, a' \in A_S$ and $s, s', t, t', z, w \in S$, $as \leq a's'$, $at \leq a't'$, $sz \leq tw$ and $s'z \leq t'w$ imply $a = a''u$, $a''v \leq a'$, for some $a'' \in A_S$ and $u, v \in S$, such that $us \leq vs'$ and $ut \leq vt'$.

Definition 2.11. An S -poset A_S satisfies Condition $(PF')_w$, if for all $a, a' \in A_S$ and $s, s', t, t', z, w \in S$, $as \leq a's'$, $at \leq a't'$, $sz \leq tw$ and $s'z \leq t'w$ imply $a \leq a''u$, $a''v \leq a'$, for some $a'' \in A_S$ and $u, v \in S$, such that $us \leq vs'$ and $ut \leq vt'$.

Recall from [12] that an S -poset A_S satisfies *Condition* (P_w) , if for all $a, a' \in A_S$ and $s, s' \in S$, $as \leq a's'$ implies $a \leq a''u$, $a''v \leq a'$, for some $a'' \in A_S$ and $u, v \in S$, such that $us \leq vs'$. Now by a similar argument as in the proof of Theorem 2.9, we can show the following Theorem.

Theorem 2.12. *For any S -poset A_S , the following statements are equivalent:*

- (1) A_S is weakly subpullback flat;
- (2) the corresponding φ is weak po-order isomorphism, for every subpullback diagram $P(Ss, Ss, f, g, S)$;
- (3) the corresponding φ is weak po-order isomorphism, for every subpullback diagram $P(S, S, f, g, S)$;
- (4) A_S satisfies *Condition* $(PF')_w$;
- (5) A_S satisfies *Conditions* (P_w) and $(E')_s$.

Now we show that flatness properties of S -posets that mentioned in this paper and [7], can be transferred to their coproducts.

Lemma 2.13 ([10]). *Let $A = \bigcup_{i \in I} A_i$, where $A_i, i \in I$ is a right S -poset. Let ${}_S B$ be a left S -poset. If $a \otimes b \leq a' \otimes b'$ in $A_i \otimes B, i \in I$, then $a \otimes b \leq a' \otimes b'$ in $A \otimes B$.*

Lemma 2.14 ([10]). *Let $A = \bigcup_{i \in I} A_i$, where $A_i, i \in I$ is a right strongly convex S -poset. Let ${}_S B$ be a left S -poset and suppose that $a \otimes b \leq a' \otimes b'$ in $A \otimes B$. Then $a \in A_i$, for some $i \in I$, if and only if $a' \in A_i$.*

Corollary 2.15 ([10]). *Let $A = \bigcup_{i \in I} A_i$, where $A_i, i \in I$ is a right strongly convex S -poset. Let ${}_S B$ be a left S -poset. If $a \in A_i$, then $a \otimes b \leq a' \otimes b'$ in $A \otimes B$, if and only if $a \otimes b \leq a' \otimes b'$ in $A_i \otimes B$.*

Lemma 2.16 ([10]). *Let $A = \bigcup_{i \in I} A_i$, where $A_i, i \in I$ is a right strongly convex S -poset. Let ${}_S B$ be a left S -poset and $a \otimes b \in A \otimes B$, for $a \in A$ and $b \in B$. If $a \in A_i$, then $a \otimes b \in A_i \otimes B$.*

Corollary 2.17 ([10]). *Let $A = \bigcup_{i \in I} A_i$, where $A_i, i \in I$ is a right strongly convex S -poset. Let $\varphi : A \otimes P \rightarrow P'$ be the mapping corresponding to the subpullback diagram $P(M, N, f, g, Q)$, for A . If $\varphi_i = \varphi|_{A_i \otimes P}$, then $\varphi_i : A_i \otimes P \rightarrow P'_i$.*

Lemma 2.18 ([10]). *Let $A = \bigcup_{i \in I} A_i$, where $A_i, i \in I$ is a right strongly convex S -poset. Let $\varphi : A \otimes P \rightarrow P'$ be the mapping and suppose that $\varphi_i = \varphi|_{A_i \otimes P}$. Then φ is the mapping corresponding to the subpullback diagram $P(M, N, f, g, Q)$, for A if and only if φ_i is the mapping corresponding to the subpullback diagram $P(M, N, f, g, Q)$, for $A_i, i \in I$.*

Theorem 2.19 ([10]). *Let φ be the mapping corresponding to the subpullback diagram $P(M, N, f, g, Q)$ for A_S , and let $\varphi_i, i \in I$, be as in Lemma 2.18. Then φ is surjective if and only if φ_i is surjective, for every $i \in I$.*

By a similar argument as in the proof of above theorem, we can show the following Theorems.

Theorem 2.20. *Let φ be the mapping corresponding to the subpullback diagram $P(M, N, f, g, Q)$ for A_S , and let $\varphi_i, i \in I$, be as in Lemma 2.18. Then φ is po-surjective if and only if φ_i is po-surjective, for every $i \in I$.*

Theorem 2.21. *Let φ be the mapping corresponding to the subpullback diagram $P(M, N, f, g, Q)$ for A_S , and let $\varphi_i, i \in I$, be as in Lemma 2.18. Then φ is weak po-surjective if and only if φ_i is weak po-surjective, for every $i \in I$.*

Theorem 2.22 ([10]). *Let φ be the mapping corresponding to the subpullback diagram $P(M, N, f, g, Q)$ for A_S , and let $\varphi_i, i \in I$, be as in Lemma 2.18. Then φ is order embedding if and only if φ_i is order embedding, for every $i \in I$.*

In [7], some flatness properties of S -posets, in the following theorem, investigated by property of φ , for subpullback diagrams of certain types.

Theorem 2.23. *Let S be a pomonoid and $A = \bigcup_{i \in I} A_i$, where $A_i, i \in I$ is a right strongly convex S -poset. Then A is weakly subpullback po-flat, weakly subpullback flat, $(WKPF)_{sw}$, $(PWKPF)_{sw}$, $(TKPF)_{sw}$, $(WKPF)_w$, $(PWKPF)_w$, $(TKPF)_w$, po-flat, weakly po-flat, principally weakly po-flat, po-torsion free, and satisfies Conditions (P_{sw}) , (P_w) , $(WP)_{sw}$, $(WP)_w$, $(PWP)_{sw}$, $(PWP)_w$ if and only if A_i has these properties, for every $i \in I$.*

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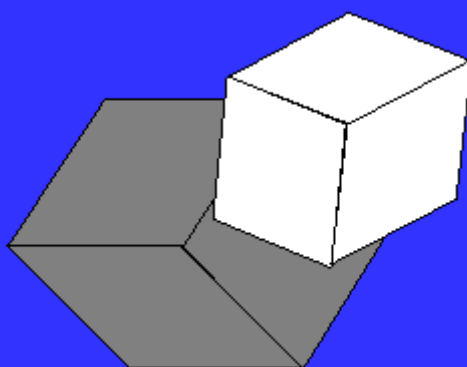
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