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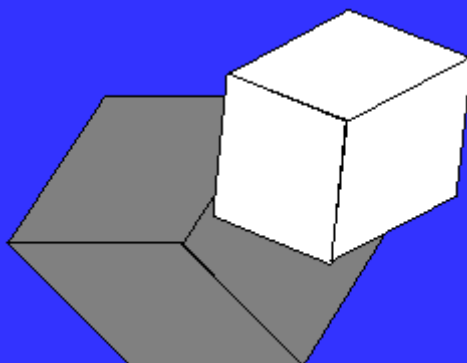
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83. IJMSI - Iranian Journal of Mathematical Sciences & Informatics, Tarbiat Modares University, Tehran	IR
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91. IRMAR - Inst. of Math. de Rennes - Rennes	F
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107. Kumamoto Journal of Mathematics - Kumamoto	J
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111. Le Matematiche - Università di Catania	I
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157. Periodica Matematica Hungarica - Budapest	H
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190. Rendiconti di Matematica e delle sue Applicazioni - Roma	I
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211. Saitama Mathematical Journal - Saitama University	J
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PRESERVING INJECTIVE PROPERTIES OF ACTS OVER MONOIDS UNDER LIMITS AND THEIR TRANSFER FROM COLIMITS TO THE COMPONENTS

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Abstract. This paper is devoted to the preservation of injective properties under limits and their transfer from colimits to the components. We prove that an injective property α is preserved under limits if and only if all acts satisfy property α . Besides we prove that an injective property α is transferred from colimits to their components if and only if all acts satisfy property α .

Keywords: Limit, colimit, product, coproduct, injective act.

1. Introduction and preliminaries

Throughout this paper, unless stated otherwise, S stands for a monoid and 1 denotes its identity element. A set A together with a mapping $A \times S \rightarrow A$, $(a, s) \rightsquigarrow as$, is called a right S -act or simply an act (and is denoted by A_S) if $a(st) = (as)t$ and $a1 = a$ for all $a \in A$, $s, t \in S$. For right S -acts A_S and B_S a mapping from A to B preserving S -action is called an act homomorphism. The class of right S -acts together with act homomorphisms as morphisms form a category denoted by **Act- S** . We mean by $A \sqcup B$ the disjoint union of sets A and B . The one element act is called zero act and is denoted by $\Theta_S = \{\theta\}$. A right S -act A_S is called decomposable provided that there exist subacts $B_S, C_S \subseteq A_S$ such that $A_S = B_S \cup C_S$ and $B_S \cap C_S = \emptyset$. In this case $A_S = B_S \cup C_S$ is called a decomposition of A_S . Otherwise A_S is called indecomposable. It is well-known that every S -act A_S has a unique decompo-

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sition into indecomposable subacts. Indeed, indecomposable components of A_S are precisely the equivalence classes of the relation $\sim$ on A_S defined by $a \sim b$ if there exist $s_1, s_2, \dots, s_n, t_1, t_2, \dots, t_n \in S$, $a_1, a_2, \dots, a_n \in A_S$ such that

$$a = a_1 s_1, a_1 t_1 = a_2 s_2, a_2 t_2 = a_3 s_3, \dots, a_n t_n = b$$

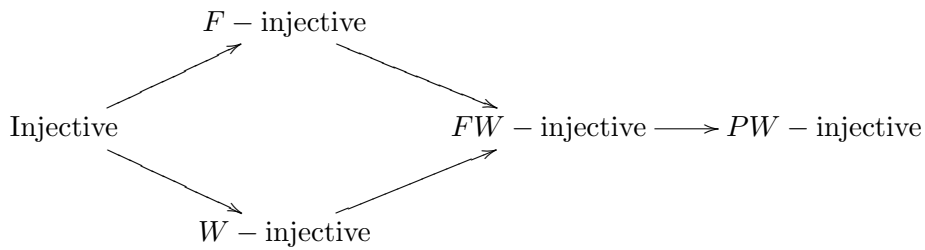
(see [6]). Therefore, elements $a, b \in A_S$ are in the same indecomposable component if and only if there exists a sequence of equalities of length n as above connecting a to b . We refer the reader to [1] and [5] for more details on the concepts mentioned in this paper.

A great deal of works has been done on the concept of injectivity relative to various classes of monomorphisms in the category of S -acts for a monoid S such as [2, 3, 4, 7, 8]. Hereby, this work mostly concentrates on the preservation of the notions C -injectivity and CC -injectivity under limits and colimits and their transfer from limits and colimits to the components in this category.

Recall that a right S -act A is called injective if for any S -act N , any subact M of N , and any homomorphism $f \in \text{Hom}(M, A)$, there exists a homomorphism $g \in \text{Hom}(N, A)$ making the following diagram commutative, i.e., $g|_M = f$,

$$\begin{array}{ccc} M & \xrightarrow{\subseteq} & N \\ f \downarrow & \swarrow g & \\ A & & \end{array}$$

In the diagram if M is finitely generated then A is called F -injective, if N is S and M is a (principal, finitely generated) right ideal of N then A is called W -injective (PW -injective, FW -injective), if M is cyclic then A is called C -injective and if both M and N are cyclic then A is called CC -injective. Indeed for a class $\mathfrak{M}$ of monomorphisms in $\mathbf{Act}\text{-}S$, A is called $\mathfrak{M}$ -injective if in the above diagram the inclusion mapping $\subseteq$, can be replaced by all monomorphisms in $\mathfrak{M}$. Note that in this paper we mean by injective properties all the mentioned properties. Visualizing relations between these notions we have the following strict implications:



2. Limits of injective properties

In this section we investigate conditions under which limit of a family of acts, satisfying an injective property, satisfies the same injective property. Recall that

a category $\mathbf{A}$ is said to be complete (cocomplete) if for each small diagram in $\mathbf{A}$ there exists a limit (colimit). The next well-known theorem in Category Theory presents a crucial setting on limit and colimit preservation (see for example [1], p. 211):

Theorem 2.1. *For each category $\mathbf{A}$ the following conditions are equivalent:*

- i) $\mathbf{A}$ is complete,
- ii) $\mathbf{A}$ has products and equalizers,
- iii) $\mathbf{A}$ has products and finite intersections.

In order to employ the proceeding theorem for a monoid S , we wish to consider the category $\overline{\mathbf{Act}} - S$ consisting of all right S -acts together with the empty set as an object for ensuring the existence of equalizer of parallel morphisms as

$$A_S \begin{matrix} \xrightarrow{f} \\ \xRightarrow{g} \end{matrix} B_S.$$

Hereby, to reach our target we engage in the problem of preservation of injective properties under products, equalizers and their duals.

Injective properties for which acts satisfying them posses a zero element are transferred from products to their components. Indeed if $\{A_i \mid i \in I\}$ is a family of right S -acts for which each A_i , $i \in I$ has a zero element then A_i is a retract of $\prod_{i \in I} A_i$. Note that CC -injective acts do not have necessarily zero elements. Therefore the problem of transferring CC -injectivity from products to their components remains open.

Regarding the problem of injective properties preservation under products, an adaption of [5, Proposition 3.1.12] yields the following theorem.

Theorem 2.2. *All injective properties are preserved under products.*

In the next theorem we give conditions under which equalizers of parallel morphisms of acts satisfying injective properties are injective.

Proposition 2.3. *Suppose that α stands for an injective property. Equalizers of parallel morphisms of acts satisfying property α satisfy property α if and only if all right S -acts satisfy property α .*

Proof. We just need to prove the necessity part. Let A_S be a right S -act and let $E(A_S)$ be its injective envelope. Take the parallel morphisms

$$E(A_S) \begin{matrix} \xrightarrow{f} \\ \xRightarrow{g} \end{matrix} E(E(A_S) \prod_{A_S} E(A_S))$$

where

$$E(A_S) \prod_{A_S} E(A_S) = \{(x, 1) \mid x \in E(A_S) \setminus A_S\} \cup A_S \cup \{(x, 2) \mid x \in E(A_S) \setminus A_S\},$$

with the operation of S defined by

$$(x, n)s = \begin{cases} (xs, n), & \text{if } xs \notin A_S \\ xs, & \text{otherwise} \end{cases}$$

for every $x \in E(A_S) \setminus A_S$, $s \in S$ and $n \in \{1, 2\}$,

$$f(x) = \begin{cases} (x, 1), & \text{if } x \in E(A_S) \setminus A_S \\ x, & \text{if } x \in A_S \end{cases}$$

and

$$g(x) = \begin{cases} (x, 2), & \text{if } x \in E(A_S) \setminus A_S \\ x, & \text{if } x \in A_S. \end{cases}$$

Now it is clear that A_S together with the inclusion map is the equalizer of the parallel pair (f, g) and hence is injective. $\square$

The next theorem is a result of Theorems 2.1, 2.2 and Proposition 2.3.

Theorem 2.4. *For a monoid S an injective property α is preserved under limits if and only if all right S -acts satisfy property α .*

3. Colimits of injective properties

In this section we provide conditions under which injective properties are preserved under colimits. To establish the goal we need to concentrate on coproducts and coequalizers of injective properties.

A monoid S is called *left reversible* if every two right ideals of S have a non-empty intersection, that is, $aS \cap bS \neq \emptyset$, for each $a, b \in S$.

It is known that coproducts of injective (F -injective, W -injective, FW -injective, PW -injective) acts are injective (F -injective, W -injective, FW -injective, PW -injective) if and only if S is left reversible.

The preservation of the above injective properties under coequalizers remains an open problem, though it is clear that preservation of such injective properties under quotient is a sufficient condition for the problem.

The case of C -injectivity. It can be routinely checked that any C -injective act contains a zero element and retracts of C -injective acts are C -injective. The next proposition is a straight forward result of the definition of C -injectivity and CC -injectivity.

Proposition 3.1. *Coproducts of C -injective (CC -injective) acts are C -injective (CC -injective).*

In light of the dual of Theorem 2.1 the following theorem is obtained.

Theorem 3.2. *Colimits of C -injective (CC -injective) acts are C -injective (CC -injective) if and only if coequalizer of any parallel morphisms as $A_S \rightrightarrows B_S$ is C -injective (CC -injective) for C -injective (CC -injective) acts A_S and B_S .*

Our next target is providing equivalent conditions on monoids for transferring C -injective and CC -injective properties from coproducts to their components. We need some ingredients to establish the results.

Proposition 3.3. *Let S be a left reversible monoid. Then a right S -act A_S is indecomposable if and only if for any $a, a' \in A_S$ there exist $s, s' \in S$ such that $as = a's'$.*

Proof. Let S be a left reversible monoid. Suppose that A_S is indecomposable and $a, a' \in A_S$. So there exists a sequence of equalities, connecting a to a' , of the form

$$a = a_1 s_1, a_1 t_1 = a_2 s_2, a_2 t_2 = a_3 s_3, \dots, a_n t_n = a',$$

for $a_i \in A_S$, $s_i, t_i \in S$, $1 \leq i \leq n$. Left reversibility of S provides $u_1, u_2 \in S$ such that $s_1 u_1 = t_1 v_1$ and in consequence $au_1 = a_1 s_1 u_1 = a_1 t_1 v_1 = a_2 s_2 v_1$. Proceeding inductively, we get $u, v \in S$ providing $au = a_n t_n v = a'v$, as desired.

Conversely, it is obvious. $\square$

Proposition 3.4. *For a monoid S all subacts of indecomposable right S -acts are indecomposable if and only if S is left reversible.*

Proof. Necessity. Let $a, b \in S$. Since S is indecomposable, our assumption implies that $aS \cup bS$ is indecomposable and therefore $aS \cap bS \neq \emptyset$.

Sufficiency. This is a straightforward application of the Proposition 3.3. $\square$

Recall that for a nonempty set I , I^S is an $|I|$ -cofree right S -act where fs for $f \in I^S$, $s \in S$ is defined by $fs(t) = f(st)$ for every $t \in S$. It should be mentioned that the 1-cofree object or terminal object in **Act**- S is the one element act which is indecomposable. The next proposition characterizes monoids over which non-zero cofree acts are decomposable.

Proposition 3.5. *For a monoid S the following are equivalent:*

- i) *all non-zero cofree S -acts are decomposable,*
- ii) *there exists a decomposable cofree right S -act,*
- iii) *S is left reversible.*

Proof. $i \implies ii$ is clear. $ii \implies iii$. By way of contradiction suppose that $aS \cap bS = \emptyset$ for some $a, b \in S$. Let X^S be a decomposable $|X|$ -cofree act and $f, g \in X^S$. Let $h \in X^S$ be given by

$$h(x) = \begin{cases} f(x), & \text{if } x \in aS, \\ g(x), & \text{otherwise.} \end{cases}$$

So we get the sequence $f = f.1$, $fa = ha$, $hb = gb$, $g.1 = g$, which implies that f and g are in the same indecomposable component. Therefore X^S is indecomposable a contradiction.

iii $\implies$ i. Let S be a left reversible monoid and X^S be a non-zero cofree S -act. Take constant functions $f = c_{x_1}$ and $g = c_{x_2}$ in X^S for different elements x_1 and x_2 in X . Then f and g are zero elements of X^S (note that zero elements of X^S are the same constant functions). If f and g are in the same indecomposable component, in light of Proposition 3.3, there exist $a, b \in S$ such that $fa = gb$ and by this we obtain $f = g$, a contradiction. $\square$

Proposition 3.6. *Let S be a monoid. The C -injective property is transferred from coproducts to their components if and only if S is not left reversible or contains a left zero.*

Proof. *Necessity.* Let S be a left reversible monoid for which C -injective property is transferred from coproducts to their components. Since S is left reversible using Proposition 3.5 the cofree right S -act S^S is decomposable. Suppose $\bigsqcup_{i \in I} Q_i$ is its unique decomposition into indecomposable acts. Let $\text{id}_S \in Q_{i_0}$, for some $i_0 \in I$. Our assumption necessitates the existence of a zero element in Q_{i_0} namely f . Then in account of Proposition 3.4 there exists $s, t \in S$ such that $\text{id}_S t = fs = f$. So $\text{id}_S t$ is a zero element and hence for each $x \in S$, $\text{id}_S t(x) = \text{id}_S(t)$, which implies $tx = t$, for each $x \in S$. So t is a left zero in S , as desired.

Sufficiency: Suppose that $\{Q_i \mid i \in I\}$ is a family of right S -acts for which $Q_S = \bigsqcup_{i \in I} Q_i$ is C -injective. If S contains a left zero then all right S -acts have zeros element and hence regarding [7, Proposition 8] Q_i is injective for any $i \in I$. If S is not left reversible suppose that $aS \cap bS = \emptyset$ for some $a, b \in S$. For $i \in I$ and $q \in Q_i$ consider the homomorphism $f : aS \rightarrow Q_S$ given by $f(as) = qas$ for each $s \in S$. Taking aS as a subact of the Rees factor act S/bS , f can be extended to a morphism from S/bS and hence Q_i contains a zero element. Thus Q_i is a retract of the C -injective act Q_S and is consequently C -injective. $\square$

Note that the above proposition is a generalized version of Proposition 8 in [7].

Considering the fact that homomorphic images of indecomposable acts are indecomposable we have the next proposition.

Proposition 3.7. *Let S be a monoid. The CC -injective property is transferred from coproducts to their components.*

Proposition 3.8. *Let S be a monoid and α be an injective property. The following are equivalent:*

- i) For any parallel morphisms $aS \xrightleftharpoons[g]{f} BS$, if their coequalizer satisfies property α then AS and BS satisfy property α ,

ii) *All right S -acts satisfy property α .*

Proof. Clearly we just need to prove the necessity part. Let A_S be a right S -act. Take the parallel morphisms as $A_S \begin{smallmatrix} \subseteq \\ \rightrightarrows \\ \subseteq \end{smallmatrix} E(A_S)$ whereas $E(A_S)$ is injective envelope of A_S and $\subseteq$ is the inclusion homomorphism from A_S to $E(A_S)$. Therefore their coequalizer is $E(A_S)/\rho$ together with the canonical homomorphism $\pi_\rho : E(A_S) \rightarrow E(A_S)/\rho$ where ρ is the right congruence on A_S generated by all the pairs (a, a) , $a \in A$ which yields $\rho = \Delta_{E(A_S)}$ and hence $E(A_S)/\rho$ is isomorphic to $E(A_S)$. Since $E(A_S)$ is injective then it satisfies property α and by assumption A_S satisfies property α . $\square$

The next theorem is a result of the above proposition and Theorem 2.1.

Theorem 3.9. *For a monoid S an injective property α transferred from colimits to their components if and only if all right S -acts satisfy property α .*

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ON DERIVED OF SOME POLYGROUPS AND GENERALIZED ALTERNATIVE POLYGROUPS

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Abstract. In this paper, we investigate the derived of some polygroups and we give some results on them. Also, we define generalized alternative polygroups which is the derived of generalized symmetric polygroups.

Keywords: Polygroups, derived polygroup, alternative polygroup.

1. Introduction

Hyperstructure theory was born in 1934 at the 8th congress of Scandinavian Mathematicians, when Marty [11] introduced the hypergroup notion as a generalization of groups and later, he proved its utility in solving some problems of groups, algebraic functions and rational fractions. Surveys of the theory can be found in the books of Corsini [3], Davvaz and Leoreanu-Fotea [6], Corsini and Leoreanu [4] and Vougiouklis [12]. One of the important classes of hypergroups is polygroups which their properties are close to groups. There are many mathematicians that interest to work on this class. The authors introduced the notion of derived of polygroups in [1], in this paper we investigate the derived of $\mathcal{A}[\mathcal{B}]$, the extension of $\mathcal{A}$ by $\mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$, are two polygroups and introduced by Comer [2]. Also, we define generalized alternative polygroup by the derived of generalized symmetric polygroups.

2. Preliminaries

In this section, first we recall some basic notions of polygroup theory and we give some results on the derived polygroups that we introduced in [1].

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Let H be a non-empty set and $\mathcal{P}^*(H)$ be the set of all non-empty subsets of H . Let $\cdot$ be a *hyperoperation* (or *join operation*) on H , that is, $\cdot$ is a function from $H \times H$ into $\mathcal{P}^*(H)$. If $(a, b) \in H \times H$, its image under $\cdot$ in $\mathcal{P}^*(H)$ is denoted by $a \cdot b$. The join operation is extended to subsets of H in a natural way, that is, for non-empty subsets A, B of H , $A \cdot B = \cup \{a \cdot b \mid a \in A, b \in B\}$. The notation $a \cdot A$ is used for $\{a\} \cdot A$ and $A \cdot a$ for $A \cdot \{a\}$. Generally, the singleton $\{a\}$ is identified with its member a . The structure $(H, \cdot)$ is called a *semihypergroup* if $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all $a, b, c \in H$, which means that

$$\bigcup_{u \in x \cdot y} u \cdot z = \bigcup_{v \in y \cdot z} x \cdot v,$$

A semihypergroup is a *hypergroup* if $a \cdot H = H \cdot a = H$ for all $a \in H$. A non-empty subset K of a hypergroup $(H, \cdot)$ is called a *subhypergroup* if it is a hypergroup. The subhypergroup K is called *invertible on the left* (*on the right*) if for all $(x, y) \in H^2$ from $x \in K \cdot y$ (resp. $x \in y \cdot K$), it follows that $y \in K \cdot x$ (resp. $y \in x \cdot K$). We say K is *invertible* if it is invertible to the left and to the right. An element e of H is called an *identity* if, for all $x \in H$, $x \in x \cdot e \cap e \cdot x$ and $a' \in H$ is called an *inverse* of a in H if $e \in a \cdot a' \cap a' \cdot a$. Suppose that $(H, \cdot)$ and $(H', \circ)$ are two semihypergroups. A function $f : H \longrightarrow H'$ is called a *homomorphism* if $f(a \cdot b) \subseteq f(a) \circ f(b)$ for all a and b in H . We say that f is a *good homomorphism* if for all a and b in H , $f(a \cdot b) = f(a) \circ f(b)$. If $(H, \cdot)$ is a hypergroup and $\rho \subseteq H \times H$ is an equivalence relation, then for all non-empty subsets A, B of H we set

$$A \stackrel{\bar{\rho}}{=} B \Leftrightarrow a \rho b, \text{ for all } a \in A, b \in B.$$

The relation ρ is called *strongly regular on the left* (*on the right*) if $x \rho y \Rightarrow a \cdot x \stackrel{\bar{\rho}}{=} a \cdot y$ ($x \rho y \Rightarrow x \cdot a \stackrel{\bar{\rho}}{=} y \cdot a$, respectively), for all $(x, y, a) \in H^3$. Moreover, ρ is called *strongly regular* if it is strongly regular on the right and on the left.

Theorem 2.1 (Theorem 31, [3]). *If $(H, \cdot)$ is a semihypergroup (hypergroup) and ρ is a strongly regular relation on H , then the quotient H/ρ is a semigroup (group) under the operation :*

$$\rho(x) \otimes \rho(y) = \rho(z), \text{ for all } z \in x \cdot y.$$

We denote $\rho(x)$ by $\bar{x}$ and instead of $\bar{x} \otimes \bar{y}$ we write $\bar{x}\bar{y}$.

For all $n > 1$, we define the relation β_n on a semihypergroup H , as follows :

$$a \beta_n b \Leftrightarrow \exists (x_1, \dots, x_n) \in H^n : \{a, b\} \subseteq \prod_{i=1}^n x_i,$$

and $\beta_H = \bigcup_{i=1}^n \beta_n$, where $\beta_1 = \{(x, x) \mid x \in H\}$ is the diagonal relation on H . This relation was introduced by Koskas [10] and studied mainly by Corsini [3]. Suppose that β_H^* is the transitive closure of β . The relation β_H^* is a strongly regular relation [3]. Also, we have :

Theorem 2.2 (Freni, [9]). *If H is hypergroup, then $\beta_H = \beta_H^*$.*

Note that, in general, for a semihypergroup may be $\beta_H \neq \beta_H^*$. The relation β_H^* is the least equivalence relation on a hypergroup H , such that the quotient H/β_H^* is a group. The *heart* ω_H of a hypergroup H is the set of all elements x of H , for which the equivalence class $\beta_H^*(x)$ is the identity of the group H/β_H^* .

A hypergroup P is called *polygroup* and is denoted by $\langle P, \cdot, e, {}^{-1} \rangle$ if the following conditions hold :

- (1) P has a scalar identity e (i.e., $e \cdot x = x \cdot e = x$, for every $x \in P$);
- (2) every element x of P has a unique inverse x^{-1} in P ;
- (3) $x \in y \cdot z$ implies $y \in x \cdot z^{-1}$ and $z \in y^{-1} \cdot x$.

In the following we recall some properties of derived subpolygroups from [1].

Definition 2.3. Let P be a polygroup. We define

- (1) $[x, y]_r = \{h \in P \mid x \cdot y \cap y \cdot x \cdot h \neq \emptyset\}$;
- (2) $[x, y]_l = \{h \in P \mid x \cdot y \cap h \cdot y \cdot x \neq \emptyset\}$;
- (3) $[x, y] = [x, y]_r \cup [x, y]_l$.

We call $[x, y]_r$, $[x, y]_l$ and $[x, y]$ *right commutator x and y* , *left commutator x and y* and *commutator x and y* , respectively. Also, we will denote $[P, P]_r$, $[P, P]_l$ and $[P, P]$ the set of all right commutators, left commutators and commutators, respectively.

A non-empty subset K of a polygroup $\langle P, \cdot, e, {}^{-1} \rangle$ is a *subpolygroup* of P if $x, y \in K$ implies $x \cdot y \in K$, and $x \in K$ implies $x^{-1} \in K$.

Let X be a nonempty subset of a polygroup $\langle P, \cdot, e, {}^{-1} \rangle$. Let $\{A_i \mid i \in J\}$ be the family of all subpolygroups of P in which contain X . Then, $\cap_{i \in J} A_i$ is called the subpolygroup generated by X . This subpolygroup is denoted by $\langle X \rangle$ and we have $\langle X \rangle = \cup \{x_1^{\varepsilon_1} \cdot \dots \cdot x_k^{\varepsilon_k} \mid x_i \in X, k \in \mathbb{N}, \varepsilon_i \in \{-1, 1\}\}$. If $X = \{x_1, x_2, \dots, x_n\}$, then the subpolygroup $\langle X \rangle$ is denoted $\langle x_1, x_2, \dots, x_n \rangle$. In a special case $\langle [P, P]_r \rangle$, $\langle [P, P]_l \rangle$ and $\langle [P, P] \rangle$ are shown by P'_r , P'_l and P' , respectively.

Proposition 2.4. *Let $\langle P, \cdot, e, {}^{-1} \rangle$ be a polygroup $(x, y) \in P^2$. Then,*

- (1) $[x, y]_r = [x^{-1}, y^{-1}]_l$;
- (2) $P' = P'_r = P'_l$;
- (3) $x \in P' \Rightarrow x^{-1} \in P'$.

Corollary 2.5. *If $\langle P, \cdot, e, {}^{-1} \rangle$ is a polygroup, then P' is a subpolygroup of P .*

From now on we call P' the *derived subpolygroup* of P .

Proposition 2.6. *Let $\langle P, \cdot, e, {}^{-1} \rangle$ be a polygroup. Then, $P' = \{e\}$ if and only if P be an abelian group.*

A subpolygroup N of a polygroup $\langle P, \cdot, e, {}^{-1} \rangle$ is *normal* in P if $x^{-1} \cdot N \cdot x \subseteq N$, for all $x \in P$.

Let K and N be subpolygroups of a polygroup P with N normal in P . Then,

- (1) $Na = aN$, for all $a \in P$;
- (2) $(Na)(Nb) = Nab$, for all $a, b \in P$;
- (3) $Na = Nb$, for all $b \in Na$;
- (4) $N \cap K$ is a normal subpolygroup of K ;
- (5) $NK = KN$, is a subpolygroup of P ;
- (6) N is a normal subpolygroup of NK .

Proposition 2.7. *If N is a normal subpolygroup of P , then $\langle P/N, \circ, N, -I \rangle$ is a polygroup, where $Nx \circ Ny = \{Nz | z \in xy\}$ and $(Nx)^{-I} = Nx^{-1}$.*

Proposition 2.8. *If N is a normal subpolygroup of P , then $(P/N)' = NP'/N$*

Proof. Suppose that $(x, y) \in P^2$. From the equations

$$[xN, yN] = xNyNx^{-1}Ny^{-1}N = \{zN | z \in [x, y]\}$$

and

$$NP'/N = \{yN | y \in NP'\} = \{yN | y \in nz, n \in N, z \in P'\} = \{zN | z \in P'\},$$

we obtain $(P/N)' = [P/N, P/N] = NP'/N$. $\square$

3. On extension polygroups

In this section, we investigate extension polygroups and we give some new results on this class of polygroups.

Suppose that $\mathcal{A} = \langle A, \cdot, e, {}^{-1} \rangle$ and $\mathcal{B} = \langle B, \cdot, e, {}^{-1} \rangle$ are two polygroups whose elements have been renamed so that $A \cap B = \{e\}$. A new system $\mathcal{A}[\mathcal{B}] = \langle M, *, e, I \rangle$ called the extension of $\mathcal{A}$ by $\mathcal{B}$ is formed in the following way : Set $M = A \cup B$ and let $e^I = e, x^I = x^{-1}, x * e = e * x = x$ for all $x \in M$, and for all $x, y \in M - \{e\}$,

$$x * y = \begin{cases} x \cdot y, & \text{if } x, y \in A, \\ x, & \text{if } x \in B, y \in A \\ y, & \text{if } x \in A, y \in B \\ x \cdot y, & \text{if } x, y \in B, y \neq x^{-1} \\ x \cdot y \cup A, & \text{if } x, y \in B, y = x^{-1}. \end{cases}$$

In this case, $\mathcal{A}[\mathcal{B}]$ is a polygroup which is called the extension of $\mathcal{A}$ by $\mathcal{B}$ [2].

Remark 3.1. Notice that if $\mathcal{A} = \{e\}$, then $\mathcal{A}[\mathcal{B}] = \mathcal{B}$.

Lemma 3.2. Let $\mathcal{A} = \langle A, \cdot, e, {}^{-1} \rangle$ and $\mathcal{B} = \langle B, \cdot, e, {}^{-1} \rangle$ be two polygroups, where $A \neq \{e\}$ and $\mathcal{A}[\mathcal{B}] = \langle M, *, e, I \rangle$ be a extension of $\mathcal{A}$ by $\mathcal{B}$. Then, for all $(x_1, x_2, \dots, x_n) \in M^n$ we have

- (i) If $x_i \in A$, for all $1 \leq i \leq n$, then $*\prod_{i=1}^n x_i \subseteq A$, where $*\prod_{i=1}^n x_i = x_1 * x_2 * \dots * x_n$.
- (ii) If there exists j such that $x_j \in B$, then $*\prod_{i=1}^n x_i \subseteq B$ or $A \subseteq *\prod_{i=1}^n x_i$.
- (iii) If $A \subseteq *\prod_{i=1}^n x_i$, then there exists j such that for all $k \geq j$, $x_k = x_{k-1}^{-1}$ or $x_k^{-1} \in \cdot\prod_{i=1}^n x_i$.
- (iv) If $A \subseteq *\prod_{i=1}^n x_i$, then $*\prod_{i=1}^n x_i = \cdot\prod_{j=1}^m t_j \cup A$ and $e \in \cdot\prod_{j=1}^m t_j$, where $t_j \in \{x_1, x_2, \dots, x_n\} \cap B \subseteq M$, for all $1 \leq j \leq m$.

Proof. The part (i) is obvious. (ii) Suppose that there exists j , such that $x_j \in B$. It is easy to see that $*\prod_{i=1}^n x_i = *\prod_{j=1}^m y_j$, where $y_j \in \{x_1, x_2, \dots, x_n\} \cap B$. Hence $*\prod_{i=1}^n x_i = *\prod_{j=1}^m y_j \subseteq B$ or $A \subseteq *\prod_{i=1}^n x_i = *\prod_{j=1}^m y_j$. (iii) Suppose that (iii) is not true therefore $x_k \neq x_{k-1}^{-1}$ and $x_k^{-1} \notin \cdot\prod_{i=1}^n x_i$, for all $1 \leq k \leq n$. Thus $*\prod_{i=1}^n x_i = \cdot\prod_{i=1}^n x_i \subseteq B$, which is a contradiction. (iv) It is easy to see that $*\prod_{i=1}^n x_i = *\prod_{i=1}^m t_i$, where $t_i \in \{x_1, x_2, \dots, x_m\} \cap B$. According to (iii) there exists j such that for all $k \geq j$, $t_k = t_{k-1}^{-1}$ or $t_k^{-1} \in \cdot\prod_{i=1}^m t_i$. Suppose that j is the smallest number with the mentioned condition hence $*\prod_{i=1}^m t_i = (\cdot\prod_{i=1}^{j-1} t_i) * t_j * (*\prod_{i=j+1}^m t_i) = (\cdot\prod_{i=1}^j t_i \cup A) * (*\prod_{i=j+1}^m t_i) = (\cdot\prod_{i=1}^j t_i) * (*\prod_{i=j+1}^m t_i) \cup *\prod_{i=j+1}^m t_i = \cdot\prod_{i=1}^m t_i \cup A$. Moreover $e \in \cdot\prod_{i=1}^m t_i$. $\square$

Example 3.3. Let $\mathcal{A}$ and $\mathcal{B}$ be the following polygroups

$\cdot$	1	2
1	1	2
2	2	$\{1, 2\}$

$\cdot$	1	3	4
1	1	3	4
3	3	$\{1, 4\}$	$\{3, 4\}$
4	4	$\{3, 4\}$	$\{1, 3\}$

The hyperoperation extension $\mathcal{A}$ by $\mathcal{B}$, $\mathcal{A}[\mathcal{B}] = M$ as follows :

$*$	1	2	3	4
1	1	2	3	4
2	2	$\{1, 2\}$	3	4
3	3	3	$\{1, 2, 4\}$	$\{3, 4\}$
4	4	4	$\{3, 4\}$	$\{1, 2, 3\}$

Notice that if $(x_1, x_2, \dots, x_n) \in M^n$ and $A \subseteq *\prod_{i=1}^n x_i$ we have $*\prod_{i=1}^n x_i = \{\{e, a\}, \{e, a, b\}, \{e, a, c\}, \{e, a, b, c\}\}$. Hence $*\prod_{i=1}^n x_i = e \cdot e \cup A$, or $c \cdot c \cup A$, or $b \cdot b \cup A$ and or $b \cdot b \cdot c \cup A$, where $e \in e \cdot e \cap c \cdot c \cap b \cdot b \cap b \cdot b \cdot c$.

Theorem 3.4. Let $\mathcal{A} = \langle A, \cdot, e, {}^{-1} \rangle$ and $\mathcal{B} = \langle B, \cdot, e, {}^{-1} \rangle$ be two polygroups and $\mathcal{A}[\mathcal{B}]$ be an extension of $\mathcal{A}$ by $\mathcal{B}$. Then,

$$\mathcal{A}[\mathcal{B}]/\beta_{\mathcal{A}[\mathcal{B}]}^* \cong \mathcal{B}/\beta_{\mathcal{B}}^*.$$

Proof. Suppose that $\varphi : \mathcal{A}[\mathcal{B}]/\beta_{\mathcal{A}[\mathcal{B}]}^* \longrightarrow \mathcal{B}/\beta_{\mathcal{B}}^*$, such that

$$\varphi([x]) = \begin{cases} \bar{e}, & \text{if } x \in A \\ \bar{x}, & \text{if } x \in B - A, \end{cases}$$

where $[x] = \beta_{\mathcal{A}[\mathcal{B}]}^*(x)$ and $\bar{x} = \beta_{\mathcal{B}}^*(x)$. First of all, we prove φ is well defined map. Suppose that $x, y \in M$ and $[x] = [y]$, so there exists $(z_1, z_2, \dots, z_n) \in M^n$ such that $x, y \in * \prod_{i=1}^n z_i$. By Lemma 3.2, we have the following cases:

Case 1. If $z_i \in A$, for all $1 \leq i \leq n$, then $x, y \in A$ and so $\varphi([x]) = \varphi([y]) = \bar{e}$.

Case 2. If there exists j such that $z_j \in B$ and $* \prod_{i=1}^n z_i \subseteq B$, we have $* \prod_{i=1}^n z_i = * \prod_{j=1}^m t_j$, where $t_j \in \{z_1, z_2, \dots, z_n\} \cap B$ and hence $x, y \in \cdot \prod_{j=1}^m t_j$ and so $\bar{x} = \bar{y}$ which means that $\varphi([x]) = \varphi([y])$.

Case 3. If there exists j , $z_j \in B$ and $A \subseteq * \prod_{i=1}^n z_i$. In this case, we have $* \prod_{i=1}^n z_i = \cdot \prod_{j=1}^m t_j \cup A$, where $t_j \in \{z_1, z_2, \dots, z_n\} \cap B$ and $e \in \cdot \prod_{j=1}^m t_j$. Since $\{x, y, e\} \subseteq * \prod_{i=1}^n z_i$, one of the following statements happen :

- (i) $\{x, y\} \subseteq \cdot \prod_{j=1}^m t_j$;
- (ii) $\{x, y\} \subseteq A$;
- (iii) $x \in \cdot \prod_{j=1}^m t_j$ and $y \in A$;
- (iv) $y \in \cdot \prod_{j=1}^m t_j$ and $x \in A$.

In all of the above cases, we have $\varphi([x]) = \varphi([y])$. Therefore, φ is well defined. Suppose that $x, y \in M$. In order to prove that φ is a good homomorphism we need to consider the following steps :

- (1) $(x, y) \in A^2$, $\varphi([x] \otimes [y]) = \varphi([z])$, where $z \in x * y = x \cdot y$. So, $z \in A$ and hence $\varphi([z]) = \bar{e} = \bar{e}\bar{e} = \varphi([x])\varphi([y])$.
- (2) $(x, y) \in B^2$, $\varphi([x] \otimes [y]) = \varphi([z])$, where $z \in x * y = x \cdot y$ or $x * y = x \cdot y \cup A$ and $y = x^{-1}$. If $z \in x * y = x \cdot y$, then $\varphi([z]) = \bar{z} = \bar{x} \cdot \bar{y} = \varphi([x])\varphi([y])$. If $z \in x * y = x \cdot y \cup A$ and $y = x^{-1}$, then $z \in A$ or $z \in x \cdot y$, if $z \in A$ we have $\varphi([z]) = \bar{e}$ and $\varphi([x])\varphi([y]) = \varphi([x])\varphi([x^{-1}]) = \bar{x}\bar{x}^{-1} = \bar{x}\bar{x}^{-1} = \bar{e}$. Consequently, $\varphi([x] \otimes [y]) = \varphi([x])\varphi([y])$. If $z \in x \cdot y = x \cdot x^{-1}$, then $\bar{z} = \bar{x} \cdot \bar{y} = \bar{x} \cdot \bar{x}^{-1} = \bar{e}$. Thus, $\bar{e} = \varphi([z]) = \varphi([x])\varphi([y])$.
- (3) $x \in A$ and $y \in B$, $\varphi([x] \otimes [y]) = \varphi([z])$, where $z \in x * y = y$. So, $z = y$ and hence $\varphi([x] \otimes [y]) = \varphi([y]) = \bar{y} = \bar{e} \cdot \bar{y} = \varphi([x])\varphi([y])$.
- (4) $x \in B$ and $y \in A$, this step is similar to (3).

It is easy to see that φ is one to one and onto and the proof is completed. $\square$

Corollary 3.5. *Let $\mathcal{A} = \langle A, \cdot, e, {}^{-1} \rangle$ and $\mathcal{B} = \langle B, \cdot, e, {}^{-1} \rangle$ be two polygroups and $\mathcal{A}[\mathcal{B}]$ be an extension of $\mathcal{A}$ by $\mathcal{B}$. Then, $\mathcal{A}[\mathcal{B}]$ is special (in the sense of Definition 4.8) if and only if $\mathcal{B}$ is special.*

4. Derived of extension polygroups and generalized symmetric polygroups

In this section first we investigate the derived of $\mathcal{A}[\mathcal{B}]$, the extension of the $\mathcal{A}$ by $\mathcal{B}$, and then we define generalized alternative polygroup by the derived of generalized symmetric polygroups. Also, we give some properties of alternative polygroups.

Proposition 4.1. *Let $\mathcal{A} = \langle A, \cdot, e, {}^{-1} \rangle$ and $\mathcal{B} = \langle B, \cdot, e, {}^{-1} \rangle$ be two polygroups and $\mathcal{A}[\mathcal{B}] = \langle M, *, e, I \rangle$ be a extension of $\mathcal{A}$ by $\mathcal{B}$. Then, for all $x, y \in M$,*

$$[x, e]_M = [e, x]_M = \begin{cases} [e, x]_A, & \text{if } x \in A \\ [e, x]_B \cup A, & \text{if } x \in B. \end{cases}$$

$$[x, y]_M = \begin{cases} [x, y]_A, & \text{if } x, y \in A \\ [x, y]_B \cup A, & \text{if } x, y \in B, y \neq x^{-1}, xy \cap yx \neq \emptyset \text{ or } x, y \in B, y = x^{-1} \\ [x, y]_B, & \text{if } x, y \in B, y \neq x^{-1}, xy \cap yx = \emptyset \\ [x, e]_B \cup A, & \text{if } x \in B, y \in A \\ [e, y]_B \cup A, & \text{if } x \in A, y \in B. \end{cases}$$

Proof. Suppose that $(x, y) \in M^2$. If $(x, y) \in B^2$, then

$$\begin{aligned} [x, y]_M &= (x * y) * (y * x)^{-1} \\ &= \begin{cases} (x \cdot y) * (y \cdot x)^{-1}, & \text{if } x \neq y^{-1} \\ (x \cdot y \cup A) * (y \cdot x \cup A)^{-1}, & \text{if } x = y^{-1} \end{cases} \\ &= \begin{cases} (x \cdot y) * (y \cdot x)^{-1}, & \text{if } x \neq y^{-1} \\ (x \cdot y) * (y \cdot x)^{-1} \cup x \cdot y \cup (y \cdot x)^{-1} \cup A, & \text{if } x = y^{-1}. \end{cases} \end{aligned}$$

Since $x = y^{-1}$, we conclude that $e \in x \cdot y$ and so $x \cdot y \cup (y \cdot x)^{-1} \subseteq (x \cdot y) \cdot (y \cdot x)^{-1}$. Hence, we have

$$[x, y]_M = \begin{cases} [x, y]_B \cup A, & \text{if } x, y \in B, x \neq y^{-1}, xy \cap yx \neq \emptyset \text{ or } x, y \in B, x = y^{-1}, \\ [x, y]_B, & \text{if } x, y \in B, x \neq y^{-1}, xy \cap yx = \emptyset. \end{cases}$$

It is not difficult to see that the other cases also hold. $\square$

Corollary 4.2. *Let $\mathcal{A} = \langle A, \cdot, e, {}^{-1} \rangle$ and $\mathcal{B} = \langle B, \cdot, e, {}^{-1} \rangle$ be two polygroups and $\mathcal{A}[\mathcal{B}] = \langle M, *, e, I \rangle$ be an extension of $\mathcal{A}$ by $\mathcal{B}$. Then, $\mathcal{A} \cup \mathcal{B}' \subseteq (\mathcal{A}[\mathcal{B}])'$.*

In [7], Davvaz studied some aspects of polygroups. By using the concept of generalized permutation, he defined permutation polygroups and some concepts related to it. He obtained a generalization of Cayley's theorem, too. In the following, we recall the definition.

Definition 4.3. Let Ω be a non-empty set. A map $f : \Omega \longrightarrow P^*(\Omega)$ is called a generalized permutation on Ω if

$$\bigcup_{\omega \in \Omega} f(\omega) = f(\Omega) = \Omega,$$

where $P^*(\Omega)$ is the set of all the non-empty subsets of Ω . We write $f = \begin{pmatrix} x \\ f(x) \end{pmatrix}$ for the generalized permutation f . Denote M_Ω the set of all the generalized permutations on Ω .

Definition 4.4. Let $M = \langle P, \cdot, e, {}^{-1} \rangle$ be a polygroup and Ω be a non-empty set. A map $f : \Omega \times P \longrightarrow P^*(\Omega)$ is called an action P on Ω if the following axioms hold :

- (1) $f(\omega, e) = \{\omega\} = \omega$, for all $\omega \in \Omega$;
- (2) $f(f(\omega, g), h) = \bigcup_{\omega \in g \cdot h} f(\omega, \alpha)$, for all $g, h \in P$ and $\omega \in \Omega$;
- (3) $\bigcup_{\omega \in \Omega} f(\omega, g) = \Omega$, for all $g \in P$;
- (4) For all $g \in P$, $\alpha \in f(\beta, g) \implies \beta \in f(\alpha, g^{-1})$.

From the second condition, we obtain $\bigcup_{\omega_0 \in f(\omega, g)} f(\omega_0, h) = \bigcup_{\alpha \in g \cdot h} f(\omega, \alpha)$. For $\omega \in \Omega$, we write $\omega^g := f(\omega, g)$. In this case, we say that P is a permutation polygroup on a set Ω and it is said that P acts on Ω .

Theorem 4.5 (Generalization of Cayley's Theorem). *Let P be a polygroup acting on a nonempty finite set Ω . Then, there is a subset of M_Ω which is a polygroup under the induced action of P and is isomorphism to P .*

Proposition 4.6 ([7]). *Let $S_\Omega = \left\{ \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{|\Omega|} \\ \alpha_1^g & \alpha_2^g & \cdots & \alpha_{|\Omega|}^g \end{pmatrix} \mid g \in P \right\}$. Then, $\langle S_\Omega, \circ, i, {}^{-I} \rangle$ is a polygroup, where*

$$\begin{aligned} & \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{|\Omega|} \\ \alpha_1^g & \alpha_2^g & \cdots & \alpha_{|\Omega|}^g \end{pmatrix} \circ \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{|\Omega|} \\ \alpha_1^h & \alpha_2^h & \cdots & \alpha_{|\Omega|}^h \end{pmatrix} \\ &= \left\{ \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{|\Omega|} \\ \alpha_1^f & \alpha_2^f & \cdots & \alpha_{|\Omega|}^f \end{pmatrix} \mid f \in g \cdot h \right\} \end{aligned}$$

and

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{|\Omega|} \\ \alpha_1^g & \alpha_2^g & \cdots & \alpha_{|\Omega|}^g \end{pmatrix} {}^{-I} = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{|\Omega|} \\ \alpha_1^{g^{-1}} & \alpha_2^{g^{-1}} & \cdots & \alpha_{|\Omega|}^{g^{-1}} \end{pmatrix}.$$

Example 4.7. Let $\Omega = \{1, 2, 3, \dots, n\}$, $n \in \mathbb{N}$, $P = S_n$ (the symmetric group of order n). Then, $f : \Omega \times P \longrightarrow P^*(\Omega)$, such that $f(k, \sigma) = \{\sigma(k)\}$ is an action, which we call it trivial action P on Ω .

Definition 4.8. A polygroup P is called special if $P/\beta^* \cong S_n$.

Example 4.9. Let $P = \{1, 2, 3, 4, 5, 6, 7\}$. Consider the polygroup $\langle P, \bullet, 1, {}^{-1} \rangle$, where $\bullet$ is defined on P as follows :

$\bullet$	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7
2	2	$\{1, 2\}$	3	4	5	6	7
3	3	3	$\{1, 2\}$	7	6	5	4
4	4	4	6	$\{1, 2\}$	7	3	5
5	5	5	7	6	$\{1, 2\}$	4	3
6	6	6	4	5	3	7	$\{1, 2\}$
7	7	7	5	3	4	$\{1, 2\}$	6

It is easy to see that P is a special polygroup.

Definition 4.10. Let P be a special polygroup and f be an action of P on $\Omega = \{1, 2, 3, \dots, n\}$, we call S_Ω *generalized symmetric polygroup* and $S'_\Omega = A_\Omega$ is called the *generalized alternative polygroup*.

Example 4.11. Let P be the special polygroup of Example 4.9 and $\Omega = \{1, 2, 3, \dots, 7\}$. Consider the action $k \cdot g = k \bullet g$. In this case $S_\Omega = P$ and $A_\Omega = P'$, where $P' = \langle 1, 2, 6, 7 \rangle = \{1, 2, 6, 7\} \trianglelefteq P$.

Lemma 4.12. $A_\Omega = \left\{ \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{|\Omega|} \\ \alpha_1^g & \alpha_2^g & \cdots & \alpha_{|\Omega|}^g \end{pmatrix} \mid g \in P' \right\}$.

Proof. We have

$$\begin{aligned} & \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{|\Omega|} \\ \alpha_1^g & \alpha_2^g & \cdots & \alpha_{|\Omega|}^g \end{pmatrix} \circ \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{|\Omega|} \\ \alpha_1^h & \alpha_2^h & \cdots & \alpha_{|\Omega|}^h \end{pmatrix} \circ \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{|\Omega|} \\ \alpha_1^g & \alpha_2^g & \cdots & \alpha_{|\Omega|}^g \end{pmatrix}^{-I} \\ & \circ \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{|\Omega|} \\ \alpha_1^h & \alpha_2^h & \cdots & \alpha_{|\Omega|}^h \end{pmatrix}^{-I} = \left\{ \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{|\Omega|} \\ \alpha_1^f & \alpha_2^f & \cdots & \alpha_{|\Omega|}^f \end{pmatrix} \mid f \in g \cdot h \cdot g^{-1} \cdot h^{-1} \right\}. \end{aligned}$$

It is easy to see that $A_\Omega = \left\{ \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{|\Omega|} \\ \alpha_1^g & \alpha_2^g & \cdots & \alpha_{|\Omega|}^g \end{pmatrix} \mid g \in P' \right\}$ □

Theorem 4.13. $A_\Omega \trianglelefteq S_\Omega$.

Proof. Suppose that

$$x = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{|\Omega|} \\ \alpha_1^a & \alpha_2^a & \cdots & \alpha_{|\Omega|}^a \end{pmatrix} \in S_\Omega \text{ and } y = \begin{pmatrix} \beta_1 & \beta_2 & \cdots & \beta_{|\Omega|} \\ \beta_1^b & \beta_2^b & \cdots & \beta_{|\Omega|}^b \end{pmatrix} \in A_\Omega, \text{ where } a \in$$

$$P \text{ and } b \in P'. \text{ We have } x^{-1}yx = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{|\Omega|} \\ \alpha_1^{a^{-1}} & \alpha_2^{a^{-1}} & \cdots & \alpha_{|\Omega|}^{a^{-1}} \end{pmatrix} \circ \begin{pmatrix} \beta_1 & \beta_2 & \cdots & \beta_{|\Omega|} \\ \beta_1^b & \beta_2^b & \cdots & \beta_{|\Omega|}^b \end{pmatrix} \circ$$

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{|\Omega|} \\ \alpha_1^a & \alpha_2^a & \cdots & \alpha_{|\Omega|}^a \end{pmatrix} = \left\{ \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{|\Omega|} \\ \alpha_1^g & \alpha_2^g & \cdots & \alpha_{|\Omega|}^g \end{pmatrix} \mid g \in a^{-1}ba \right\}.$$

Since $a^{-1}ba \subseteq bb^{-1}a^{-1}ba \subseteq P'$, hence $x^{-1}yx \subseteq A_\Omega$. Therefore, $x^{-1}A_\Omega x \subseteq A_\Omega$. $\square$

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n -EDGE-DISTANCE-BALANCED GRAPHS**M. Faghani***

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Abstract. Throughout this paper, we present a new class of graphs so-called n -edge-distance-balanced graphs inspired by the concept of edge-distance-balanced property initially introduced by Tavakoli et al. [Tavakoli M., Yousefi-Azari H., Ashrafi A.R., Note on edge distance-balanced graphs, Trans. Combin. 1 (1) (2012), 1-6]. Moreover, we propose some characteristic results to recognize 2-edge-distance-balanced graphs by using the lack of 2-connectivity in graphs. Some examples are provided in order to illustrate the obtained conclusions.

Keywords: graph theory, edge-distance-balanced, regularity.

1. Introduction and preliminaries

It is well known that the graph theory is a crucial tool to utilize the modeling of the phenomena and is extensively used in a series of investigations for a few decades. In order to use better and more beneficial from the graph theory, all graphs are usually classified based on their distinguishing quality. Recently, as a new tool using in optimization a class of graphs so-called distance-balanced graphs has been introduced by Jerebic et al. [5] and then studied in some recent papers, see ([1],[2],[4],[6]-[10]) and the reference therein.

Let G be a finite, undirected and connected graph with diameter d , and let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G , respectively. For $u, v \in V(G)$, we let $d(u, v) = d_G(u, v)$ denote the minimal path-length distance between u and v . For a pair of adjacent vertices u, v of G we denote

$$W_{uv}^G = \{x \in V(G) | d(x, u) < d(x, v)\}.$$

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Similarly, we can define W_{vu}^G . Also, consider the notion

$${}_uW_v^G = \{x \in V(G) | d(x, u) = d(x, v)\}.$$

which all the sets as above make a partition of $V(G)$ and moreover we have the following definition.

Definition 1.1 ([5]). *We say that G is distance-balanced whenever for an arbitrary pair of adjacent vertices u and v of G there exists a positive integer γ_{uv} , such that*

$$|W_{uv}^G| = |W_{vu}^G| = \gamma_{uv}.$$

Very recently, the authors [3] generalized this concept and introduced n -distance-balanced graphs (denoted by n -DB) and characterized n -DB graphs for $n = 2, 3$. Furthermore, they studied the invariance of this property on the product of graphs.

In 2012, Tavakoli et al. [11] presented a new type of distance-balanced property in the edge sense as follows.

Definition 1.2 ([11]). *Let G be a graph, $e = uv \in E(G)$, $m_u^G(e)$ denotes the number of edges lying closer to the vertex u than the vertex v , and $m_v^G(e)$ is defined analogously. Then G is called edge-distance-balanced (as we denote it by EDB), if $m_a^G(e) = m_b^G(e)$ holds for each edge $e = ab \in E(G)$.*

The purpose of this paper is to introduce a new class of graphs based on the distance-balanced property in edge sense. The structure of the paper is as follows. In the next section we define the notion of n -edge-distance-balanced graph and then present some results related to either regularity or n -locally regularity. We also create a series of n -locally regular graphs with diameter n , n -edge-distance-balanced property and without i -edge-distance-balanced property for $i = 1, 2, \dots, n - 1$. In Section 3, using the concept of 2-connectivity of graph we propose some results to characterize 2-edge-distance-balanced graphs by presenting some necessary and sufficient conditions.

2. EDB and n -EDB graphs and locally regularity

In this section, inspired by the concept of edge-distance-balanced graph, we initially define and introduce a class of graphs including these graphs which are called n -edge-distance-balanced graphs and then we conclude some results and examples related to this concept.

Definition 2.1. *A connected graph G is called n -edge-distance-balanced (n -EDB for short) if for each $a, b \in V(G)$ with $d(a, b) = n$ we have $|E_{anb}^G| = |E_{bna}^G|$ where*

$$E_{anb}^G = \{e \in E(G) \mid e \text{ is closer to the vertex } a \text{ than the vertex } b\}.$$

Similarly, we can define E_{bna}^G . Also, consider the notion

$${}_n E_b^G = \{e \in E(G) \mid \text{the distance of } e \text{ to both vertices } a \text{ and } b \text{ is the same}\}.$$

Similar to the recent proposed concepts (DB, EDB and n -DB) one can observe that all three sets as above form a partition for $E(G)$. Moreover, 1-edge-distance-balanced property coincides to edge-distance-balanced property and so the collection of all n -EDB graphs are larger than the set of all EDB graphs. In the following, denoted by E_{ab}^G we mean E_{a1b}^G . Otherwise specified, we consider all the notations without superscript G .

As the first result, in the following we present a characterization for the graphs with EDB and n -EDB properties inspired by the virtue of proof of similar result in [5]. We notice that the proof would be slightly different. For convenience of notation, we let for a vertex x of a connected graph G and $k \geq 0$, that

$$\begin{aligned} N_k(x) &= \{y \in V(G) \mid d(x, y) = k\}, & N_k[x] &= \{y \in V(G) \mid d(x, y) \leq k\}, \\ M_k(x) &= \{e \in E(G) \mid d(x, e) = k\}, & M_k[x] &= \{e \in E(G) \mid d(x, e) \leq k\}. \end{aligned}$$

and for $k = 1$, these symbols are used while the indices are removed. Here, denoted by $d(x, e)$ we mean the length of the shortest path between the vertex x and the edge e , i.e., the number of edges lying between the vertex x and the edge e in the shortest path. Following the notations as above we see that

$$|M_k(x)| \leq |N_k(x)| \leq |M_{k-1}(x)|, \quad \forall x \in V(G),$$

and also replacing $[x]$ by (x) we have the same conclusion.

Proposition 2.2. *A graph G of diameter d is EDB if and only if*

$$(2.1) \quad \sum_{k=0}^{d-1} |M_k(a) \setminus M_k[b]| = \sum_{k=0}^{d-1} |M_k(b) \setminus M_k[a]|,$$

or equivalently,

$$\sum_{k=0}^{d-1} |M_k(a) \cap M_{k+1}(b)| = \sum_{k=0}^{d-1} |M_k(b) \cap M_{k+1}(a)|.$$

holds for any edge $e = ab \in E(G)$.

Proof. Considering $W_k(ab) = \{e \in E(G) \mid d(e, a) = k, d(e, b) = k + 1\}$, $k \geq 0$, we get

$$E_{ab} = \bigcup_{k=0}^{d-1} W_k(ab).$$

Using the fact that $W_k(ab) = M_k(a) \setminus M_k[b] = M_k(a) \cap M_{k+1}(b)$ we obtain that G is EDB if and only if one of the following equivalent equalities hold:

$$\sum_{k=0}^{d-1} |M_k(a) \setminus M_k[b]| = \sum_{k=0}^{d-1} |M_k(b) \setminus M_k[a]|,$$

$$\sum_{k=0}^{d-1} |M_k(a) \cap M_{k+1}(b)| = \sum_{k=0}^{d-1} |M_k(b) \cap M_{k+1}(a)|.$$

Corollary 2.3. *Let G be a regular graph of diameter d . Then G is EDB if and only if*

$$\sum_{k=1}^{d-1} |M_k(a) \setminus M_k[b]| = \sum_{k=1}^{d-1} |M_k(b) \setminus M_k[a]|$$

holds for any edge $e = ab \in E(G)$.

Corollary 2.4. *Suppose that G is a graph with diameter $d = 2$ and any circle C_5 (if there is) in G has no path ae_1ue_2b where $e_1, e_2 \notin E(C_5)$, $u \notin V(C_5)$ and $a, b \in V(C_5)$ with $d(a, b) = 2$ and $d(u, x) = 2$ for all $x \in V(C_5) - \{a, b\}$ (see Figure 1). Then G is an EDB graph if and only if G is regular.*

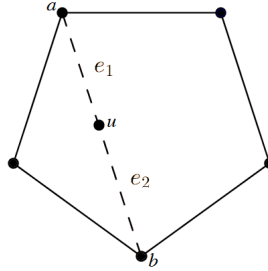


Figure 1: Circle C_5 in graph G not satisfying in hypotheses of Corollary 2.4.

Proof. Suppose that G is regular. Since $d = 2$, for any $ab \in E(G)$ we have

$$\forall e = uv \in E(G) \text{ s.t. } d(e, a) = 0 \rightarrow d(e, b) = 1,$$

$$\forall e = uv \in E(G) \text{ s.t. } d(e, b) = 0 \rightarrow d(e, a) = 1.$$

Now, considering the fact that $\deg(a) = \deg(b)$ we obtain that

$$(2.2) \quad |M_0(a) \setminus M_0[b]| = |M_0(b) \setminus M_0[a]|.$$

On the other hand, let

$$e_1 = uv \in E(G) \text{ s.t. } d(a, u) = 1, \quad d(e_1, a) = 1, \quad d(e_1, b) = 2,$$

where $p_w : vwb$ is the shortest path connecting v to b . This together with Figure 2 shows that for any edge $e_1 \in M(a) \setminus M[b]$, we easily find that $e_2 = vw \in M(b) \setminus M[a]$. Note that since e_1 and e_2 are located in C_5 , then following the hypotheses, there exists no edge $v\hat{u}$ in the path $v\hat{u}b$. It means that in the way as above for any edge e_1 , the edge e_2 is chosen, uniquely. Therefore,

$$|M(a) \setminus M[b]| \leq |M(b) \setminus M[a]|.$$

Similarly, the converse can be proved. Now, applying Proposition 2.2 and

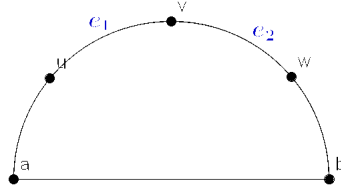


Figure 2: A section of graph G with edges ab , uv and vw .

relation (2.2) with equality

$$(2.3) \quad |M(a) \setminus M[b]| = |M(b) \setminus M[a]|$$

we conclude that G is an EDB graph. In order to prove the converse, i.e. equality (2.2), we only need to show the recent relation. To do this, one can easily see that conclusion (2.3) is only related to the fact that G has no subgraph C_5 with specified property and this completes the proof.

Remark 2.5. In Corollary 2.4, we notice that the mentioned condition for C_5 in G is a sufficient assumption to apply for characterization of all the EDB graphs with diameter 2. As illustration, Corollary 2.4 can be utilized for Peterson graph, Hoffman-Singleton graph, complete bipartite graph $K_{n,n}(n \neq 1)$, circles C_4 and C_5 and the following 3-regular EDB graph with $d = 2$ which also contains four circles C_5 not similar as depicted in Figure 1.

Remark 2.6. Following Corollary 2.4 and its proof, note that the vertex u generally can be adjacent to any other vertex in C_5 and so the hypotheses could be weakened, however, this additional condition on u makes the recent graph and similar ones included in the class of graphs compatible with conditions of Corollary 2.4.

Now, using the symbols $M_k[x]$ and $M_k(x)$ we develop our study to n -EDB graphs as follows.

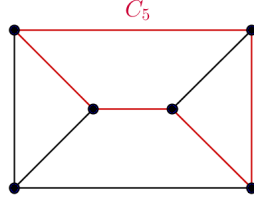


Figure 3: A 3-regular EDB graph with four vertices C_5 not as form shown in Figure 1.

Proposition 2.7. *A graph G of diameter d is n -EDB if and only if*

$$\sum_{k=0}^{d-1} |M_k(a) \setminus M_k[b]| = \sum_{k=0}^{d-1} |M_k(b) \setminus M_k[a]|,$$

holds for any vertices $a, b \in V(G)$ with $d(a, b) = n$.

Proof. Suppose that $W_j^i(ab) = \{e \in E(G) \mid d(e, a) = j, d(e, b) = i + j\}$, for $i = 1, 2, \dots, n$ and $0 \leq j \leq d - 1$. Then we get

$$E_{anb} = \bigcup_{i=1}^n \bigcup_{j=0}^{d-1} W_j^i(ab).$$

On the other hand, since $M_k[a] \setminus M_k[b] = \bigcup_{i=1}^n \bigcup_{j=0}^k W_j^i(ab)$ and $M_k(a) \setminus M_k[b] = \bigcup_{i=1}^n W_k^i(ab)$, then G is n -EDB if and only if for all vertices $a, b \in V(G)$ with $d(a, b) = n$ we have

$$\sum_{k=0}^{d-1} |M_k(a) \setminus M_k[b]| = \sum_{k=0}^{d-1} |M_k(b) \setminus M_k[a]|,$$

which is as same as (2.1) in the previous proposition.

Remark 2.8. By the virtue of proof of Proposition 2.7 we see that graph G with diameter d is n -EDB if and only if

$$|M_{d-1}[a] \setminus M_{d-1}[b]| = |M_{d-1}[b] \setminus M_{d-1}[a]|,$$

for any vertices $a, b \in V(G)$ with $d(a, b) = n$.

Following Propositions 2.2 and 2.7 and in order to detect n -EDB graphs it only needs to establish (2.1) for any pair of vertices with distance n .

Definition 2.9 ([3]). *The graph G is called locally regular with respect to n (for short n -locally regular) if we have*

$$(2.4) \quad \forall a, b \in V(G), \quad d(a, b) = n \implies \deg(a) = \deg(b).$$

As the authors mentioned in [3], this kind of property is obviously weaker than the usual regularity. In the following we introduce a class of 2-locally regular graphs with 2-EDB property and diameter 2.

Suppose that $G := \vee_m K_n$ is a graph formed by joining complete graph K_n to $m - 1$ copies of itself in a unique edge. Then G is 2-locally regular graph with diameter 2 and 2-EDB property (without EDB property), and has two central vertices. For example see the non-regular graphs shown in Figure 4. Similar to

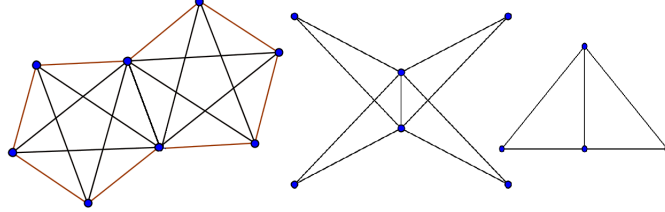


Figure 4: 2-locally regular graphs $\vee_2 K_5$, $\vee_4 K_3$ and $\vee_2 K_3$.

the technique as above, let $G := K_n^1 \vee_m K_n^2 \vee_m \cdots \vee_m K_n^k$ ($n \geq 4$) be a chain of k number of K_n where each K_n has unique common edge to a copy of itself. Then G is a k -locally regular graph with diameter k , k -EDB property and without i -EDB property for $i = 1, 2, \dots, k - 1$. A 3-locally regular 3-EDB graph with diameter 3 and a 4-locally regular 4-EDB graph with diameter 4 are shown in Figures 5 and 6, respectively. The common edges in each one are indicated by the black bullets.

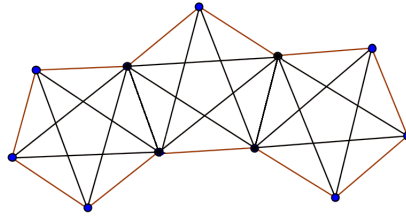


Figure 5: $K_5^1 \vee_1 K_5^2 \vee_1 K_5^3$.

Remark 2.10. We note that the structure of the graph given in definition as above can be modified and one can consider the chain in closed form, i.e., circular form, and obtain similar properties (for example see Figure 7).

3. 2-EDB graphs

In this section, inspired by the concept of graph joint in [3] and 2-connectivity we find necessary and sufficient conditions for 2-EDB graphs.

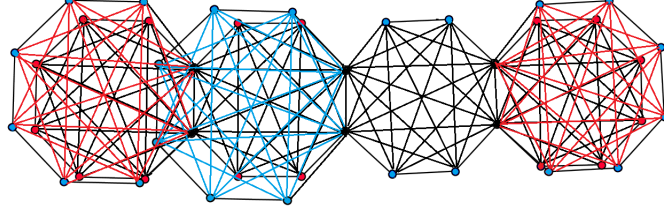


Figure 6: $K_8^1 \vee_3 K_8^2 \vee_3 K_8^3 \vee_3 K_8^4$ with three common edges and three plies of K_8 on each one.

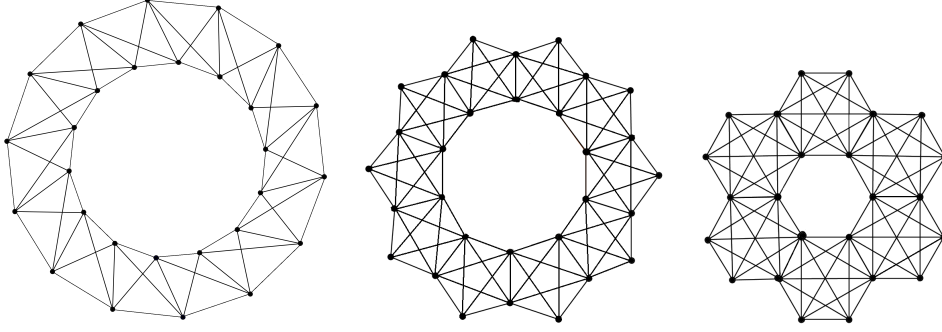


Figure 7: Closed forms of $K_4^1 \vee_1 K_4^2 \vee_1 \cdots \vee_1 K_4^{14}$, $K_5^1 \vee_1 K_5^2 \vee_1 \cdots \vee_1 K_5^{10}$ and $K_6^1 \vee_1 K_6^2 \vee_1 \cdots \vee_1 K_6^6$.

Definition 3.1 ([3]). Let G be an arbitrary non-complete graph and $K_1 = \{u\}$ be an external vertex not belonging to $V(G)$. Then graph joint $G \vee K_1$ of graphs G and K_1 is a graph with

$$\begin{cases} V(G \vee K_1) = V(G) \cup \{u\}, \\ E(G \vee K_1) = E(G) \cup \{uv \mid v \in V(G)\}. \end{cases}$$

Clearly, $G \vee K_1$ is connected and $\text{diam}(G \vee K_1) = 2$. Moreover, G is connected if and only if $G \vee K_1$ is 2-connected, i.e., it remains connected whenever any arbitrary vertex is removed.

In the following we give a condition which will be needed in our next results.

(A) Suppose that G is an arbitrary connected graph and there is no induced subgraphs H_1 and H_2 shown in Figure 8 and obtained by removing finite number of vertices and their adjacent edges.

Theorem 3.2. Suppose that G is a non-complete disconnected regular graph, then graph $G \vee K_1$ satisfying the condition (A) is 2-EDB and not 2-disconnected.

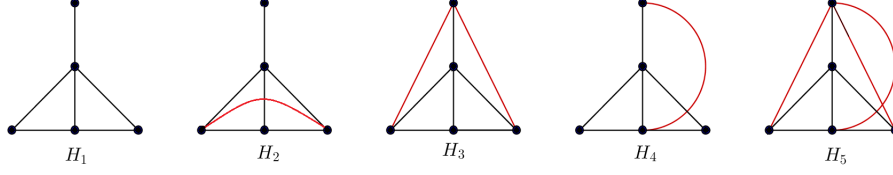


Figure 8: Impermissible subgraphs H_1 and H_2 together with some allowed subgraphs

Proof. Let G be a regular graph with valency k and $G \vee K_1$ be a graph constructed as above where $K_1 = \{u\}$ for an arbitrary fixed vertex $u \notin V(G)$. So u is adjacent to any vertex in $V(G)$. Assume $G_1, G_2, \dots, G_n$ are all the connected components of G for some $n \geq 2$. All essentially different types of vertices a, b with $d(a, b) = 2$ in $G \vee K_1$ are either both from $V(G_i)$ or one from $V(G_i)$, the other from $V(G_j)$ for some $1 \leq i \neq j \leq n$. First, suppose that $a, b \in V(G_i)$ such that $d(a, b) = 2$ and $1 \leq i \leq n$. Then the fact that $\text{diam}(G \vee K_1) = 2$ yields

$$(3.5) \quad E_{a2b}^{G \vee K_1} = M_0^{(G_i)}(a) \cup M_1^{(G_i)}(a) \cup \{au\}.$$

where $M_j^{(G_i)}(a)$ means $M_j(a)$ limited to graph G_i . Based on (3.5) we observe that

$$(3.6) \quad |E_{a2b}^{G \vee K_1}| = |M_0^{(G_i)}(a)| + |M_1^{(G_i)}(a)| + 1.$$

Similarly, we have

$$(3.7) \quad |E_{b2a}^{G \vee K_1}| = |M_0^{(G_i)}(b)| + |M_1^{(G_i)}(b)| + 1.$$

Now, if $e \in M_1^{(G_i)}(a)$, then $e \in E(G_i)$ and $d(e, a) = 1$. This together with the fact that G has at least two components implies that G has a subgraph as form of either H_1 or H_2 shown in Figure 8 which is a contradiction. So $M_1^{(G_i)}(a) = M_1^{(G_i)}(b) = \emptyset$. Applying this equality together with (3.6), (3.7) and regularity of G we conclude that $|E_{a2b}^{G \vee K_1}| = |E_{b2a}^{G \vee K_1}|$ and the claim is proven for the first case. For the second case, let us consider $a \in V(G_i)$ and $b \in V(G_j)$ arbitrarily, for some $1 \leq i \neq j \leq n$. Then we have the similar conclusions as follows.

$$(3.8) \quad \begin{aligned} E_{a2b}^{G \vee K_1} &= M_0^{(G_i)}(a) \cup M_1^{(G_i)}(a) \cup \{au\}, \\ E_{b2a}^{G \vee K_1} &= M_0^{(G_j)}(b) \cup M_1^{(G_j)}(b) \cup \{bu\}. \end{aligned}$$

Again the regularity of G together with (3.8) and the hypothesis (A) implies that $|E_{a2b}^{G \vee K_1}| = |E_{b2a}^{G \vee K_1}| = k + 1$ where k is the valency of G . Therefore, $G \vee K_1$ is 2-EDB and this completes the proof.

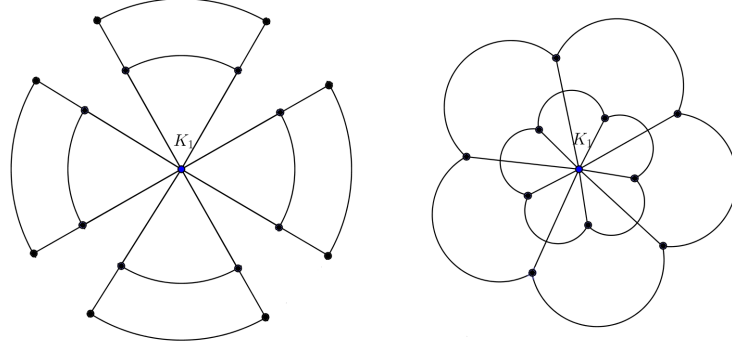


Figure 9: 2-EDB graphs $(C_4 \cup C_4 \cup C_4 \cup C_4) \vee K_1$ and $(C_5 \cup C_5) \vee K_1$

In Figure 9 two graphs satisfying the hypotheses of Theorem 3.2 are given. Now, we present the following result for the converse of the previous theorem which is more complicated and it needs some additional conditions.

Proposition 3.3. *Suppose that G is a connected 2-EDB graph with $|V(G)| < 2n + 1$ for some integer $n \geq 3$ and G satisfies the condition (A). Moreover, assume that G is not 2-connected and $G - x$ has n nontrivial connected components for some $x \in V(G)$. Then $G \cong H \vee K_1$ for some regular graph H which is not connected and $K_1 = \{v\}$ for some $v \in V(G)$.*

Proof. Let G be a connected 2-EDB graph that is not 2-connected and u a cut vertex in G such that $G - u$ has at least three connected components in G . If we exclude the vertex u and the related edges, we obtain a subgraph H with connected components $H_1, H_2, \dots, H_n$ for some $n \geq 3$. We prove that u is adjacent to every vertex in H_k for some $k \in \{1, 2, \dots, n\}$. On the contrary, suppose that for arbitrary H_i there exists $a_i \in V(H_i)$ such that $d(a_i, u) = 2$. From the connectivity of H_i we can find $b_i \in V(H_i)$ with $d(b_i, u) = d(a_i, b_i) = 1$. Similarly, one can find such vertices for the other components with the properties as above. On the other hand, following the definition of 2-EDB we get

$$\left\{ \begin{array}{l} E_{u2a_i}^G \supseteq \{b_i u\} \cup \{e_j \mid \text{there exists path } p_{c_j} := (u \cdots e_j \cdots c_j) \text{ with vertices in} \\ \quad H_j, c_j \in V(H_j)\} \\ \\ \cup \{e_k \mid \text{there exists path } p_{c_k} := (u \cdots e_k \cdots c_k) \text{ with vertices in} \\ \quad H_k, c_k \in V(H_k)\} \\ \\ \implies 1 + |V(H_j)| + |V(H_k)| \leq |E_{u2a_i}^G|, \\ \\ E_{a_i 2u}^G \subseteq E(H_i) \implies |E_{a_i 2u}^G| \leq \frac{|V(H_i)|(|V(H_i)| - 1)}{2}, \end{array} \right.$$

which together with the fact that G is 2-EDB implies that

$$(3.9) \quad 1 + |V(H_j)| + |V(H_k)| \leq \frac{|V(H_i)|(|V(H_i)| - 1)}{2}.$$

Following the process as above, one can obtain similar conclusions for compo-

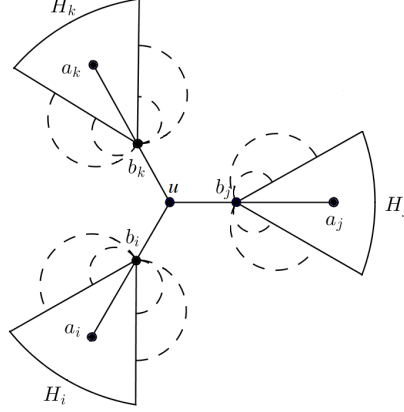


Figure 10: Cut vertex u with some connected components of $G - u$.

nents H_j , H_k and the others as same as (3.9). Therefore,

$$\begin{aligned} (n-1)A + n &\leq \frac{1}{2} \left(|V(H_1)|(|V(H_1)| - 1) + |V(H_2)|(|V(H_2)| - 1) \right. \\ &\quad \left. + \cdots + |V(H_n)|(|V(H_n)| - 1) \right) \\ &\leq \frac{1}{2}(A^2 - A) \end{aligned}$$

where $A := |V(G)| - 1$. Hence $|V(G)| \geq 2n + 1$ which is a contradiction.

Hence, u is adjacent to every vertex in H_k for at least one $k \in \{1, 2, \dots, n\}$. Assume that u is adjacent to every vertex in $V(H_1)$. We prove that the induced subgraph H_1 is regular. Let us take an arbitrary $w \in V(H) \setminus V(H_1)$ adjacent to u . Obviously, $d(v, w) = 2$ for every $v \in V(H_1)$. On the other hand, for vertex $x \in V(H_1)$ if $e \in E_{w2x}^G \cap E(H_1)$ then

$$d(e, w) = d(e, u) + d(u, w) = 2 < d(x, e),$$

which is a contradiction followed by the fact that

$$d(x, e) \leq d(x, u) + d(u, e) = 2.$$

Hence we get $E_{w2x}^G \subseteq E(G) \setminus E(H_1)$. If we choose $e \in E_{w2x}^G$, then by the recent inclusion we have

$$d(x, e) = d(x, u) + d(u, e) = 1 + d(u, e) = d(y, e)$$

for any vertex $y \in V(H_1)$. This implies that $|E_{w2x}^G| = |E_{w2y}^G|$. Next, since $E_{x2w}^G \subseteq E(H_1)$ then

$$E_{x2w}^G = M_0^{(H_1)}(x) \cup M_1^{(G)}(x) \cup \{ux\}.$$

Thus, by the fact that G is 2-EDB we get

$$\begin{aligned} |E_{x2w}^G| &= |M_0^{(H_1)}(x)| + |M_1^{(G)}(x)| + 1, \\ |E_{x2w}^G| &= |E_{w2x}^G| = |E_{w2y}^G| = |E_{y2w}^G|, \end{aligned}$$

and so

$$\begin{aligned} |M_0^{(H_1)}(x)| + |M_1^{(G)}(x)| + 1 &= |M_0^{(H_1)}(y)| + |M_1^{(G)}(y)| + 1 \\ \implies \deg(x) + |M_1^{(G)}(x)| &= \deg(y) + |M_1^{(G)}(y)|. \end{aligned}$$

Now, using the assumption (A) and the fact that $u \in V(G) \setminus V(H_1)$ we conclude that $M_1^{(G)}(x) \cap E(H_1) = \emptyset$ and $M_1^{(G)}(x) = \{uv \mid v \in V(H_1) \setminus \{x\}\}$ for any vertex $x \in V(H_1)$. It means that $\deg(x) = \deg(y)$ and so H_1 is regular.

Suppose that H_1 has a valency k . We show that u is adjacent to every vertex in $V(H)$. Suppose that this statement is not true. Then without loss of generality, we can assume that there is $v_2 \in V(H_2)$ such that $d(v_2, u) = 2$. Hence one can find $v_1 \in V(H_2)$ such that $d(v_1, u) = d(v_1, v_2) = 1$. As we know that $|M_0^{(G)}(w)| = k + 1$ for arbitrary $w \in V(H_1)$, then using the hypothesis (A) we conclude

$$\begin{aligned} E_{w2v_1}^G &= M_0^{(G)}(w) \cup M_1^{(H_1)}(w) \implies |E_{w2v_1}^G| = |M_1^{(H_1)}(w)| + k + 1 = k + 1 \\ E_{u2v_2}^G &\supseteq E(H_1) \cup \{uw \mid w \in V(H_1)\} \implies |E_{u2v_2}^G| \geq |E(H_1)| + |V(H_1)| \end{aligned}$$

Now we define $D = \bigcup_{i=1}^d (W_i^0(v_1u) \cup W_{i-1}^1(v_1u))$ where W_i^0 and W_i^1 are given in the previous section. We easily observe that $E_{v_22u}^G \subseteq D \subseteq E_{v_12w}^G$ which means that

$$\begin{aligned} |E(H_1)| + k + 1 &\leq |E(H_1)| + |V(H_1)| \leq |E_{u2v_2}^G| = |E_{v_22u}^G| \leq |E_{v_12w}^G| \\ &= |E_{w2v_1}^G| = k + 1, \end{aligned}$$

which implies that H_1 is trivial and $V(H_1) = \{w\}$. This is a contradiction. So u is adjacent to every vertex in $V(H)$. Finally, we prove that graph H is regular with valency k . In order to do this, it suffices to show that H_2 is regular with valency k . Let us consider an arbitrary $v \in V(H_2)$ and $w \in V(H_1)$ which has already been found that $d(v, w) = 2$. Therefore,

$$\begin{aligned} E_{v2w}^G &= M_0^{(G)}(v) \cup M_1^{(H_1)}(v) \cup \{uv\}, \\ E_{w2v}^G &= M_0^{(G)}(w) \cup M_1^{(H_1)}(w) \cup \{uw\}. \end{aligned}$$

Hence we get

$$(3.10) \quad |M_0^{(G)}(v)| + |M_1^{(H_1)}(v)| = |M_0^{(G)}(w)| + |M_1^{(H_1)}(w)|.$$

On the other hand, using the property (A) we have

$$|M_1^{(H_1)}(v)| = |M_1^{(H_1)}(w)| = 0.$$

We notice that

$$\begin{aligned} M_1^{(G)}(v) &= M_1^{(H_1)}(v) \cup \{xu \mid x \in V(H_i), i = 1, 2, \dots, n\} \setminus \{uv\}, \\ M_1^{(G)}(w) &= M_1^{(H_1)}(w) \cup \{xu \mid x \in V(H_i), i = 1, 2, \dots, n\} \setminus \{wv\}, \end{aligned}$$

Now, relation (3.10) implies that

$$\deg(v) = |M_0^{(G)}(v)| = |M_0^{(G)}(w)| = k + 1 \quad \text{for all } v \in V(H_2).$$

Hence, H is regular with valency k and the consequence follows.

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A CHARACTERIZATION OF SOME PROJECTIVE SPECIAL LINEAR GROUPS

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Abstract. In this paper, we show that projective special linear groups $S := L_3(q)$ with q less than 100 are uniquely determined by their orders and degree patterns of their prime graphs. Indeed, we prove that if G is a finite group whose order and degree pattern of its prime graph is the same as the order and the degree pattern of S , then G is isomorphic to S .

Keywords: Projective special linear groups; Prime graph; Degree pattern.

1. Introduction

For a positive integer n , the set of all primes dividing n is denoted by $\pi(n)$. Let G be a finite group. Set $\pi(G) := \pi(|G|)$, say, $\pi(G) = \{p_1, p_2, \dots, p_k\}$. The *prime graph* $\Gamma(G)$ of G is a simple graph whose vertex set is $\pi(G)$ and two distinct primes p_i and p_j in $\pi(G)$ are adjacent if and only if there exists an element of order $p_i p_j$ in G . If p_i is adjacent to p_j , we sometime write $p_i \sim p_j$. For $p_i \in \pi(G)$, the *degree* of p_i is the number $\deg_G(p_i) := |\{p_j \in \pi(G) \mid p_i \sim p_j\}|$. The *degree pattern* $D(G)$ of G is the k -tuple $(\deg_G(p_1), \deg_G(p_2), \dots, \deg_G(p_k))$ in which $p_1 < p_2 < \dots < p_k$. A group G is said to be *OD-characterizable* if there exists exactly one isomorphic class of finite groups with the same order and degree pattern as G .

Darafsheh et al in [6] studied the quantitative structure of finite groups using their degree patterns and proved that if $|\pi((q^2 + q + 1)/d)| = 1$, where $d = (3, q - 1)$ and $q \geq 5$, then $L_3(q)$ is OD-characterizable. In [9], it is shown

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that $L_3(25)$ is OD-characterizable. Also the authors in [8] proved that the simple group $L_3(2^n)$ with $n \in \{4, 5, 6, 7, 8, 10, 12\}$ is OD-characterizable. Note in passing that all finite simple groups whose orders are less than 10^8 are OD-characterizable, see [13]. In this paper, we show that $L_3(q)$, where q is a prime power less than 100, is uniquely determined by its order and the degree pattern of its prime graph, that is to say,

Theorem 1.1. *Let G be a finite group, and let q be a prime power less than 100. If $|G| = |L_3(q)|$ and $D(G) = D(L_3(q))$, then $G \cong L_3(q)$.*

In order to prove Theorem 1.1, by [6, 8, 9], we only need to show that $L_3(q)$ is OD-characterisable for $q \in \{11, 23, 29, 37, 47, 49, 53, 61, 67, 79, 81, 83\}$.

Throughout this article all groups are finite. The *spectrum* $\omega(G)$ of a group G is the set of orders of its elements, and $\mu(G)$ is the set of elements of $\omega(G)$ that are maximal with respect to divisibility relation. Let $\pi_i := \pi_i(G)$ be *connected components* of $\Gamma(G)$, for $i = 1, \dots, t(G)$. When $|G|$ is even, we always assume that $2 \in \pi_1$. Let $|G| = m_1 \cdots m_{t(G)}$ with $\pi(m_i) = \pi_i$. Then each m_i is called an *order component* of G . If m_i is odd (even), then m_i is called an *odd (even) order component* of G . The p -part of a positive integer n is denoted by n_p , that is to say, $n_p = p^\alpha$ if $p^\alpha \mid n$ but $p^{\alpha+1} \nmid n$. All further definitions and notation are standard and can be found in [1, 2].

2. Preliminaries

In this section, we mention some useful results to be used in proof of Theorem 1.1.

Lemma 2.1 ([4, Theorem 1]). *Let G be a finite solvable group all of whose elements are of prime power order. Then $|\pi(G)| \leq 2$.*

Lemma 2.2. *Let G be a Frobenius group with kernel K and complement H . Then*

- (a) K is a nilpotent group.
- (b) $|H|$ divides $|K| - 1$.
- (c) Every subgroup of H of order pq with p and q primes (not necessarily distinct), is cyclic.
- (d) Every Sylow subgroup of H of odd order is cyclic and a Sylow 2-subgroup of H is either cyclic, or a generalized quaternion group.
- (e) If H is non-solvable group, then H has a subgroup of index at most 2 isomorphic to $SL(2, 5) \times M$, where M has cyclic Sylow p -subgroups and $(|M|, 30) = 1$.

Proof. The proof follows from [2, Theorem 10.3.1] and [7, Theorem 18.6]. $\square$

A group G is said to be a *2-Frobenius* group if there exists a normal series $1 \trianglelefteq H \triangleleft K \trianglelefteq G$ such that K and G/H are Frobenius groups with kernel H and K/H , respectively.

Lemma 2.3 ([4, Theorem 2]). *Let G be 2-Frobenius group of even order. Then*

- (a) $t(G) = 2$, $\pi_1(G) = \pi(H) \cup \pi(G/K)$ and $\pi_2(G) = \pi(K/H)$.
- (b) G/K and K/H are cyclic, $|G/K| \mid |\text{Aut}(K/H)|$, and $(|G/K|, |K/H|) = 1$.
- (c) H is nilpotent and G is solvable group.

Lemma 2.4 ([11]). *Let G be a finite group with $t(G) \geq 2$ and $2 \in \pi_1$. Then G is one of the following groups:*

- (a) G is a Frobenius or 2-Frobenius group.
- (b) G has a normal series $1 \trianglelefteq H \triangleleft K \trianglelefteq G$ such that H is nilpotent π_1 -group, G/K is a π_1 -group and K/H is non-Abelian finite simple group such that $|G/H| \mid |\text{Aut}(K/H)|$. Moreover, any odd order component of G is also an odd order component of K/H .

Lemma 2.5 ([5, Lemma 2.8]). *Let S be a finite non-abelian simple group, and let p be the largest prime divisor of $|S|$ with $|S|_p = p$. Then $p \nmid |\text{Out}(S)|$.*

An *independent set* $\Delta(\Gamma)$ of a graph Γ is a set of vertices of Γ no two of which are adjacent. The *independence number* $\alpha(\Gamma)$ of Γ is the maximum cardinality of an independent set among all independent sets of Γ . For convenience, if G is a group, we set $\Delta(G) := \Delta(\Gamma(G))$ and $\alpha(G) := \alpha(\Gamma(G))$. Moreover, for a vertex $r \in \pi(G)$, let $\alpha(r, G)$ denote the maximal number of vertices in independent sets of $\Gamma(G)$ containing r .

Lemma 2.6 ([10, Theorem 1]). *Let G be a finite group with $\alpha(G) \geq 3$ and $\alpha(2, G) \geq 2$, and let K be the maximal normal solvable subgroup of G . Then the quotient group G/K is an almost simple group, that is, there exists a non-Abelian finite simple group S such that $S \leq G/K \leq \text{Aut}(S)$.*

Lemma 2.7. *Let G be a finite group of even order with $\alpha(G) \geq 3$. Then G is non-solvable, and so it is not a 2-Frobenius group. If, moreover, $|G|_3 \neq 3$ or $|G|_5 \neq 5$, then G is not a Frobenius group.*

Proof. Suppose that $\alpha(G) \geq 3$. If G were solvable, then it would have a Hall $\{p, q, r\}$ -subgroup T with $\{p, q, r\}$ an independent subset of $\Gamma(G)$. Then p, q and r are not pairwise adjacent in $\Gamma(G)$, and so each element of T is of prime power order. Since T is solvable, it follows from Lemma 2.1 that $|\pi(T)| \leq 2$, which is a contradiction. Therefore, G is non-solvable, and so by Lemma 2.3, G is not a 2-Frobenius. Let now G be a Frobenius group with complement H and kernel K . Since G is non-solvable, it follows from Lemma 2.2 that H has a normal subgroup H_0 with $|H : H_0| \leq 2$ such that $H_0 = SL(2, 5) \times Z$, where $(|Z|, 30) = 1$. Then $|H| = 2^a \cdot 3 \cdot 5 \cdot |Z|$ with $a = 3, 4$. Therefore, $|G|_3 = 3$ and $|G|_5 = 5$. $\square$

Lemma 2.8. *Let $\Gamma(G)$ be the prime graph of a group G with exactly two vertices of degree 1. Then $\alpha(G) \geq 3$ if one of the following holds:*

- (a) $|\pi(G)| = 6$ and $\Gamma(G)$ has at least two vertices of degree 2;
- (b) $|\pi(G)| \geq 7$ and $\Gamma(G)$ has at least two vertices of degree 3.

Proof. Suppose that p_1 and p_2 are two distinct vertices of $\Gamma(G)$ with $\deg(p_i) = 1$, for $i = 1, 2$.

Let first p_1 be adjacent to p_2 . If $|\pi(G)| = 6$, there are four vertices which are not adjacent to p_1 and p_2 . Since in this case there are at least two vertices of degree 2, there exist two non-adjacent vertices p_3 and p_4 in $\Gamma(G)$. Therefore, $\{p_1, p_3, p_4\}$ is an independent set of $\Gamma(G)$ which implies that $\alpha(G) \geq 3$. Similarly, in the case where $|\pi(G)| \geq 7$, there are at least five vertices which are not adjacent to p_1 and p_2 , and since we have at least two vertices of degree 3, we can find two non-adjacent vertices p_3 and p_4 in $\Gamma(G)$, and hence $\{p_1, p_3, p_4\}$ is an independent set of $\Gamma(G)$, consequently, $\alpha(G) \geq 3$.

Let now p_1 and p_2 be non-adjacent. Since $|\pi(G)| \geq 6$ and both p_1 and p_2 are of degree 1, there exists a vertex p_3 which is not adjacent to p_1 and p_2 . Thus $\{p_1, p_2, p_3\}$ is an independent set of $\Gamma(G)$, and hence $\alpha(G) \geq 3$. $\square$

3. Proof of main result

In this section, we prove Theorem 1.1. For convenience, in Table 1, we list the order, spectrum and degree pattern of $S := L_3(q)$, where $q \in \{11, 23, 29, 37, 47, 49, 53, 61, 67, 79, 81, 83\}$. In order to determine the degree pattern of S as in the first column of Table 1, we use $\mu(S)$ (see [3, Theorem 9]):

$$\mu(S) = \left\{ \frac{q^2 + q + 1}{(3, q - 1)}, \frac{q^2 - 1}{(3, q - 1)}, q - 1, \frac{p(q - 1)}{(3, q - 1)} \right\}.$$

Note that if $(3, q - 1) = 1$, then $\mu(S) = \{q^2 + q + 1, q^2 - 1, p(q - 1)\}$. We also note that the order of S is

$$|L_3(q)| = \frac{1}{(3, q - 1)} q^3 (q^2 - 1) (q^3 - 1).$$

In what follows, we assume that G is a finite group with $|G| = |S|$ and $D(G) = D(S)$, see Table 1 below.

Proposition 3.1. *If $|G| = |L_3(11)|$ and $D(G) = D(L_3(11))$, then $G \cong L_3(11)$.*

Proof. By Table 1, we have that $|G| = 2^4 \cdot 3 \cdot 5^2 \cdot 7 \cdot 11^3 \cdot 19$ and $D(G) = (3, 2, 3, 1, 2, 1)$. Then Lemma 2.8 implies that $\alpha(G) \geq 3$. Furthermore, $\alpha(2, G) \geq 2$ as $\deg(2) = 3$ and $|\pi(G)| = 6$. By Lemma 2.6, there is a non-Abelian finite simple group S such that $S \leq G/K \leq \text{Aut}(S)$, where K is a maximal normal solvable subgroup of G .

Table 1: The order, spectrum and degree pattern of $L_3(q)$, for $q \in \{11, 23, 29, 37, 47, 49, 53, 61, 64, 67, 79, 81, 83\}$.

S	$ S $	$\mu(S)$	$D(S)$
$L_3(11)$	$2^4 \cdot 3 \cdot 5^2 \cdot 7 \cdot 11^3 \cdot 19$	$\{7 \cdot 19, 2^3 \cdot 3 \cdot 5, 2 \cdot 5 \cdot 11\}$	$(3, 2, 3, 1, 2, 1)$
$L_3(23)$	$2^5 \cdot 3 \cdot 7 \cdot 11^2 \cdot 23^3 \cdot 79$	$\{7 \cdot 79, 2^4 \cdot 3 \cdot 11, 2 \cdot 11 \cdot 23\}$	$(3, 2, 1, 3, 2, 1)$
$L_3(29)$	$2^5 \cdot 3 \cdot 5 \cdot 7^2 \cdot 13 \cdot 29^3 \cdot 67$	$\{13 \cdot 67, 2^3 \cdot 3 \cdot 5 \cdot 7, 2^2 \cdot 7 \cdot 29\}$	$(4, 3, 3, 4, 1, 2, 1)$
$L_3(37)$	$2^5 \cdot 3^4 \cdot 7 \cdot 19 \cdot 37^3 \cdot 67$	$\{7 \cdot 67, 2^3 \cdot 3 \cdot 19, 2^2 \cdot 3^2 \cdot 2^2 \cdot 3 \cdot 37\}$	$(3, 3, 1, 2, 2, 1)$
$L_3(47)$	$2^6 \cdot 3 \cdot 23^2 \cdot 37 \cdot 47^3 \cdot 61$	$\{37 \cdot 61, 2^5 \cdot 3 \cdot 23, 2 \cdot 23 \cdot 47\}$	$(3, 2, 3, 1, 2, 1)$
$L_3(49)$	$2^9 \cdot 3^2 \cdot 5^2 \cdot 7^6 \cdot 19 \cdot 43$	$\{19 \cdot 43, 2^5 \cdot 5^2 \cdot 2^4 \cdot 3 \cdot 2^4 \cdot 7\}$	$(3, 1, 1, 1, 1, 1)$
$L_3(53)$	$2^5 \cdot 3^3 \cdot 7 \cdot 13^2 \cdot 53^3 \cdot 409$	$\{7 \cdot 409, 2^3 \cdot 3^3 \cdot 13, 2^2 \cdot 13 \cdot 53\}$	$(3, 2, 1, 3, 2, 1)$
$L_3(61)$	$2^5 \cdot 3^2 \cdot 5^2 \cdot 13 \cdot 31 \cdot 61^3 \cdot 97$	$\{13 \cdot 97, 2^3 \cdot 5 \cdot 31, 2^2 \cdot 3 \cdot 5 \cdot 2^2 \cdot 5 \cdot 61\}$	$(4, 2, 4, 1, 2, 2, 1)$
$L_3(67)$	$2^4 \cdot 3^2 \cdot 7^2 \cdot 11^2 \cdot 17 \cdot 31 \cdot 67^3$	$\{7^2 \cdot 31, 2^3 \cdot 11 \cdot 17, 2 \cdot 3 \cdot 11 \cdot 2 \cdot 11 \cdot 67\}$	$(4, 2, 1, 4, 2, 1, 2)$
$L_3(79)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 13^2 \cdot 43 \cdot 79^3$	$\{7^2 \cdot 43, 2^5 \cdot 5 \cdot 13, 2 \cdot 3 \cdot 13 \cdot 2 \cdot 13 \cdot 79\}$	$(4, 2, 2, 1, 4, 1, 2)$
$L_3(81)$	$2^9 \cdot 3^{12} \cdot 5^2 \cdot 7 \cdot 13 \cdot 41 \cdot 73$	$\{7 \cdot 13 \cdot 73, 2^5 \cdot 5 \cdot 41, 2^4 \cdot 3 \cdot 5\}$	$(3, 2, 3, 2, 2, 2, 2)$
$L_3(83)$	$2^4 \cdot 3 \cdot 7 \cdot 19 \cdot 41^2 \cdot 83^3 \cdot 367$	$\{19 \cdot 367, 2^3 \cdot 3 \cdot 7 \cdot 41, 2 \cdot 41 \cdot 83\}$	$(4, 3, 3, 1, 4, 2, 1)$

We show that $\pi(K) \subseteq \{2, 3, 5, 11\}$. Assume the contrary. Then $19 \in \pi(K)$. We show that p is adjacent to 19, where $(p, a) \in \{(5, 1), (5, 2), (7, 1)\}$. If $p \in \pi(K)$, then K contains an Abelian Hall subgroup of order $p^a \cdot 19$, and so p is adjacent to 19. If $p \notin \pi(K)$, then by Frattini argument $G = K\mathbf{N}_G(P)$, where P is a Sylow 19-subgroup of K . Thus $\mathbf{N}_G(P)$ contains an element of order p , say x . So $P\langle x \rangle$ is a cyclic subgroup of G of order $p \cdot 19$ concluding that p is adjacent to 19. Therefore, both 5 and 7 are adjacent to 19, and hence 19 is of degree at least 2, which is a contradiction. Similarly, we can show that $7 \notin \pi(K)$. Hence $\pi(K) \subseteq \{2, 3, 5, 11\}$.

We now prove that S is isomorphic to $L_3(11)$. Note by Lemma 2.5 that $19 \notin \pi(\text{Out}(S))$. Then $19 \notin \pi(K) \cup \pi(\text{Out}(S))$, and so $19 \in \pi(S)$. Now by [12, Table 1], S is isomorphic to one of the simple groups J_1 and $L_3(11)$.

If S were isomorphic to J_1 , then $|S|$ would be $2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$, and since $\text{Out}(S) = 1$, we must have $|K| = 2 \cdot 5 \cdot 11^2$. Let $P \in \text{Syl}_{11}(K)$, and let $r \in \{7, 19\}$. By Frattini argument, $G = K\mathbf{N}_G(P)$, and so $\mathbf{N}_G(P)$ contains an element of order r , say x . Since P is normal in K and $P \cap \langle x \rangle = 1$, $L := P\langle x \rangle$ is a subgroup of K of order $r \cdot 11^2$. Since also L is Abelian, it has an element of order $r \cdot 11$. This shows that both 7 and 19 are adjacent to 11. Note that the degree of 11 is two, and 7 and 19 are of degree one. Thus 2 can not be adjacent to none of 7, 11 and 19. Since $|\pi(G)| = 6$, the degree of 2 is at most 2, which is a contradiction.

Therefore, S is isomorphic to $L_3(11)$, and hence $L_3(11) \leq G/K \leq \text{Aut}(L_3(11))$. Note that $|G| = |L_3(11)|$. Thus $K = 1$, and hence G is isomorphic to $L_3(11)$. $\square$

Proposition 3.2. *If $|G| = |L_3(23)|$ and $D(G) = D(L_3(23))$, then $G \cong L_3(23)$.*

Proof. According to Table 1, we have that $|G| = 2^5 \cdot 3 \cdot 7 \cdot 11^2 \cdot 23^3 \cdot 79$ and $D(G) = (3, 2, 1, 3, 2, 1)$. Then by Lemma 2.8, we conclude that $\alpha(G) \geq 3$. Since $\deg(2) = 3$ and $|\pi(G)| = 6$, $\alpha(2, G) \geq 2$. Therefore, by Lemma 2.6, there is a non-Abelian finite simple group S such that $S \leq G/K \leq \text{Aut}(S)$, where K is a maximal normal solvable subgroup of G .

We claim that $79 \notin \pi(K)$. Assume the contrary. We show that p is adjacent to 79, where $(p, a) \in \{(7, 1), (11, 1), (11, 2)\}$. If $p \in \pi(K)$, then K contains an

abelain Hall subgroup of order $p^a \cdot 79$ which implies that p is adjacent to 79. If $p \notin \pi(K)$, then by Frattini argument $G = K\mathbf{N}_G(P)$, where P is a Sylow 79-subgroup of K , and so $\mathbf{N}_G(P)$ contains an element x of order p . Note that $P\langle x \rangle$ is a cyclic subgroup of G of order $p \cdot 79$. Then p is adjacent to 79. Hence, both 7 and 11 are adjacent to 79, and consequently, degree of 79 is at least 2, which is a contradiction. Similarly, we can show that $7 \notin \pi(K)$. Therefore $\pi(K) \subseteq \{2, 3, 7, 11, 23\}$, by Lemma 2.5, we have that $79 \notin \pi(\text{Out}(S))$. Then $79 \notin \pi(K) \cup \pi(\text{Out}(S))$, and so $79 \in \pi(S)$. Therefore by [12, Table 1], S is isomorphic to $L_3(23)$. Since $|G| = |L_3(23)|$, we must have $K = 1$, and hence G is isomorphic to $L_3(23)$. $\square$

Proposition 3.3. *If $|G| = |L_3(29)|$ and $D(G) = D(L_3(29))$, then $G \cong L_3(29)$.*

Proof. It follows from Table 1 that $|G| = 2^5 \cdot 3 \cdot 5 \cdot 7^2 \cdot 13 \cdot 29^3 \cdot 67$ and $D(G) = (4, 3, 3, 4, 1, 2, 1)$. Then Lemma 2.8 implies that $\alpha(G) \geq 3$. Furthermore, $\alpha(2, G) \geq 2$ as $\deg(2) = 4$ and $|\pi(G)| = 7$. Therefore, Lemma 2.6 implies that there is a non-Abelian finite simple group S such that $S \leq G/K \leq \text{Aut}(S)$, where K is a maximal normal solvable subgroup of G .

We show that $67 \notin \pi(K)$. Assume the contrary. Then K has element of order 67. We show that p is adjacent to 67 for all $p \in \{5, 13\}$. If $p \in \pi(K)$, then we consider a cyclic Hall subgroup of order $p \cdot 67$ of K , and so p and 73 are adjacent. If $p \notin \pi(K)$, then we apply Frattini argument and have that $G = K\mathbf{N}_G(P)$, where P is a Sylow 67-subgroup of K . Thus $\mathbf{N}_G(P)$ contains an element x of order p . Now $P\langle x \rangle$ is a cyclic subgroup of G of order $p \cdot 67$ which again implies that p and 67 are adjacent. Therefore, both 5 and 13 are adjacent to 67, and hence the degree of 67 must be at least 2, which is a contradiction. Then $\pi(K) \subseteq \{2, 3, 5, 7, 13, 29\}$. By Lemma 2.5, $67 \notin \pi(\text{Out}(S))$, then $67 \notin \pi(K) \cup \pi(\text{Out}(S))$, and so $67 \in \pi(S)$. Using [12, Table 1] we observe that S is isomorphic to $L_3(29)$. Since $|G| = |L_3(29)|$, we conclude that $K = 1$, and hence G is isomorphic to $L_3(29)$. $\square$

Proposition 3.4. *If $|G| = |L_3(37)|$ and $D(G) = D(L_3(37))$, then $G \cong L_3(37)$.*

Proof. Note by Table 1 that $|G| = 2^5 \cdot 3^4 \cdot 7 \cdot 19 \cdot 37^3 \cdot 67$ and $D(G) = (3, 3, 1, 2, 2, 1)$. Then by Lemma 2.8, we must have $\alpha(G) \geq 3$. Furthermore, $\alpha(2, G) \geq 2$ since $\deg(2) = 3$ and $|\pi(G)| = 6$. By Lemma 2.6, there is a non-Abelian finite simple group S such that $S \leq G/K \leq \text{Aut}(S)$, where K is a maximal normal solvable subgroup of G . We show that $67 \notin \pi(K)$. Assume the contrary. $67 \in \pi(K)$. We prove that p would be adjacent to 67, where $p \in \{7, 19\}$. If $p \in \pi(K)$, then K contains a cyclic Hall subgroup of order $p \cdot 67$, and so p is adjacent to 67. If $p \notin \pi(K)$, then it follows from Frattini argument that $G = K\mathbf{N}_G(P)$, where P is a Sylow 67-subgroup of K , and so $\mathbf{N}_G(P)$ has an element x of order p . Thus $P\langle x \rangle$ is a cyclic subgroup of G of order $p \cdot 67$. Therefore both 7 and 19 are adjacent to 67 which contradicts the fact that the degree of 67 is 1. Therefore, $67 \notin \pi(K)$, and hence $\pi(K) \subseteq \{2, 3, 7, 19, 37\}$. Now we prove that $S \cong L_3(37)$.

By Lemma 2.5, $67 \notin \pi(\text{Out}(S))$, then $67 \notin \pi(K) \cup \pi(\text{Out}(S))$, and so $67 \in \pi(S)$. Therefore by [12, Table 1], S is isomorphic to $L_3(37)$ as claimed. Since now $|G| = |L_3(37)|$, we must have $K = 1$, and hence G is isomorphic to $L_3(37)$. $\square$

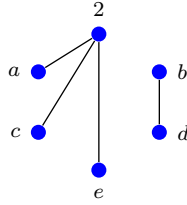
Proposition 3.5. *If $|G| = |L_3(47)|$ and $D(G) = D(L_3(47))$, then $G \cong L_3(47)$.*

Proof. By Table 1, we have that $|G| = 2^6 \cdot 3 \cdot 23^2 \cdot 37 \cdot 47^3 \cdot 61$ and $D(G) = (3, 2, 3, 1, 2, 1)$. It follows from Lemma 2.8 that $\alpha(G) \geq 3$. Note that $\deg(2) = 3$ and $|\pi(G)| = 6$. Then $\alpha(2, G) \geq 2$, and so by Lemma 2.6, there is a non-Abelian finite simple group S such that $S \leq G/K \leq \text{Aut}(S)$, where K is a maximal normal solvable subgroup of G . We claim that $61 \notin \pi(K)$. Assume the contrary. Then $19 \in \pi(K)$. We show that p is adjacent to 61, where $(p, a) \in \{(23, 1), (23, 2), (37, 1)\}$. If $p \in \pi(K)$, then K contains an abelian Hall subgroup of order $p^a \cdot 61$, and so p is adjacent to 61. If $p \notin \pi(K)$, then by Frattini argument $G = K\mathbf{N}_G(P)$, where P is a Sylow 61-subgroup of K , and so $\mathbf{N}_G(P)$ contains an element x of order p . Note that $P\langle x \rangle$ is a cyclic subgroup of G of order $p \cdot 61$. Then p is adjacent to 61. Therefore, 61 is adjacent to both 23 and 37 in $\Gamma(G)$, and hence the degree of 61 is at least 2, which is a contradiction. Hence $61 \notin \pi(K)$. Therefore, $\pi(K) \subseteq \{2, 3, 23, 37, 47\}$, and hence $61 \notin \pi(\text{Out}(S))$ by Lemma 2.5. Then $61 \notin \pi(K) \cup \pi(\text{Out}(S))$, and so $61 \in \pi(S)$. Now by [12, Table 1], S is isomorphic to $L_3(47)$, and so $L_3(47) \leq G/K \leq \text{Aut}(L_3(47))$. Note that $|G| = |L_3(47)|$. Then $K = 1$, and hence G is isomorphic to $L_3(47)$. $\square$

Proposition 3.6. *If $|G| = |L_3(49)|$ and $D(G) = D(L_3(49))$, then $G \cong L_3(49)$.*

Proof. According to Table 1, $|G| = 2^9 \cdot 3^2 \cdot 5^2 \cdot 7^6 \cdot 19 \cdot 43$ and $D(G) = (3, 1, 1, 1, 1, 1)$. Then we observe that $\Gamma(G)$ is the graph as in Figure 1 in which $\{a, b, c, d, e\} = \{3, 5, 7, 19, 43\}$. We also observe that $t(G) = 2$ and $\{a, b, e\}$ is an

Figure 1: Possibilities for the prime graph of G in Proposition 3.6.



independent set. Thus $\alpha(G) \geq 3$. It is also easily seen that $\alpha(2, G) \geq 2$. Then by Lemma 2.7, G is neither Frobenius, nor 2-Frobenius, and so Lemma 2.4 implies that G has a normal series $1 \trianglelefteq H \triangleleft K \trianglelefteq G$ such that K/H is a non-Abelian finite simple group. Since $|K/H|$ divides $|K|$, it divides $|G|$, and so by [12, Table 1], the factor group K/H is isomorphic to one of the simple groups S as in the first column of Table 2 below.

Table 2: Non-Abelian finite simple groups S whose order divides $|L_3(49)|$

S	$ S $	$ \text{Out}(S) $	Primes in $\pi(H)$
$L_2(4)$	$2^2 \cdot 3 \cdot 5$	2	7, 19, 43
$L_2(9)$	$2^3 \cdot 3^2 \cdot 5$	4	7, 19, 43
$L_2(7)$	$2^3 \cdot 3 \cdot 7$	2	7, 19, 43
$L_2(8)$	$2^3 \cdot 3^2 \cdot 7$	3	7, 19, 43
A_7	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	2	7, 19, 43
$L_2(49)$	$2^4 \cdot 3 \cdot 5^2 \cdot 7^2$	4	7, 19, 43
$L_3(4)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	12	7, 19, 43
$L_4(2)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	2	7, 19, 43
$S_4(7)$	$2^8 \cdot 3^2 \cdot 5^2 \cdot 7^4$	2	7, 19, 43
$L_2(19)$	$2^2 \cdot 3^2 \cdot 5 \cdot 19$	2	2, 7, 43
$L_3(7)$	$2^5 \cdot 3^2 \cdot 7^3 \cdot 19$	6	2, 7, 43
$U_3(7)$	$2^7 \cdot 3 \cdot 7^3 \cdot 43$	2	2, 5, 7
$L_2(7^3)$	$2^3 \cdot 3^2 \cdot 7^3 \cdot 19 \cdot 43$	6	2, 5, 7
$L_3(49)$	$2^9 \cdot 3^2 \cdot 5^2 \cdot 7^6 \cdot 19 \cdot 43$	12	-

If K/H is isomorphic to one of the groups S listed in the first column of Table 2 except $L_3(49)$, then $\pi(H)$ consists of three primes as in the forth column of the same table. Since H is nilpotent, it follows that the prime graph of G has a triangle, which is a contradiction. Therefore, $K/H \cong L_3(49)$. As $L_3(49) \leq G/H \leq \text{Aut}(L_3(49))$ and $|G| = |L_3(49)|$, we conclude that $|H| = 1$, and hence G is isomorphic to $L_3(49)$. $\square$

Proposition 3.7. *If $|G| = |L_3(53)|$ and $D(G) = D(L_3(53))$, then $G \cong L_3(53)$.*

Proof. By Table 1, $|G| = 2^5 \cdot 3^3 \cdot 7 \cdot 13^2 \cdot 53^3 \cdot 409$ and $D(G) = (3, 2, 1, 3, 2, 1)$. Now by applying Lemma 2.8, we must have $\alpha(G) \geq 3$. Furthermore, $\alpha(2, G) \geq 2$ since $\deg(2) = 3$ and $|\pi(G)| = 6$. So by Lemma 2.6, there is a non-Abelian finite simple group S such that $S \leq G/K \leq \text{Aut}(S)$, where K is a maximal normal solvable subgroup of G . We claim that $\pi(K)$ does not contain 409. Assume the contrary. Then $409 \in \pi(K)$. We show that p is adjacent to 409, where $(p, a) \in \{(7, 1), (13, 1), (13, 2)\}$. If $p \in \pi(K)$, then K contains an abelian Hall subgroup of order $p^a \cdot 409$, so p and 409 are adjacent. If $p \notin \pi(K)$, then we apply Frattini argument and have that $G = K\mathbf{N}_G(P)$, where P is a Sylow 409-subgroup of K , and so $\mathbf{N}_G(P)$ has an element x of order p . Now $P\langle x \rangle$ is a cyclic subgroup of G of order $p \cdot 409$ concluding that p and 409 are adjacent. Thus both 7 and 13 are adjacent to 409 in $\Gamma(G)$, and hence the degree of 409 is at least 2, which is a contradiction. Therefore, $409 \notin \pi(K)$, and hence it follows from Lemma 2.5 that $409 \notin \pi(\text{Out}(S))$. Then $409 \notin \pi(K) \cup \pi(\text{Out}(S))$, and so $409 \in \pi(S)$. Therefore by [12, Table 1], S is isomorphic to $L_3(53)$ and $L_3(53) \leq G/K \leq \text{Aut}(L_3(53))$. Moreover, since $|G| = |L_3(53)|$, it follows that $K = 1$, and hence $G \cong L_3(53)$. $\square$

Proposition 3.8. *If $|G| = |L_3(61)|$ and $D(G) = D(L_3(61))$, then $G \cong L_3(61)$.*

Table 3: Non-Abelian finite simple groups S whose order divides $|L_3(67)|$

S	$ S $	$ \text{Out}(S) $
$L_2(7)$	$2^3 \cdot 3 \cdot 7$	2
$L_2(8)$	$2^3 \cdot 3^2 \cdot 7$	3
$L_2(17)$	$2^4 \cdot 3^2 \cdot 17$	2
$L_2(67)$	$2^2 \cdot 3 \cdot 11 \cdot 17 \cdot 67$	2
$L_3(67)$	$2^4 \cdot 3^2 \cdot 7^2 \cdot 11^2 \cdot 17 \cdot 31 \cdot 67^3$	6

Proof. It follows from Table 1 that $|G| = 2^5 \cdot 3^2 \cdot 5^2 \cdot 13 \cdot 31 \cdot 61^3 \cdot 97$ and $D(G) = (4, 2, 4, 1, 2, 2, 1)$. Then by Lemma 2.8, $\alpha(G) \geq 3$. Moreover, $\alpha(2, G) \geq 2$ as $\deg(2) = 4$ and $|\pi(G)| = 7$. By Lemma 2.6, there is a non-Abelian finite simple group S such that $S \leq G/K \leq \text{Aut}(S)$, where K is a maximal normal solvable subgroup of G . We show that $97 \notin \pi(K)$. Assume the contrary. Then $97 \in \pi(K)$. Let $p \in \{13, 31\}$. If $p \in \pi(K)$, then K contains a cyclic Hall subgroup of order $p \cdot 97$, and so p is adjacent to 97. If $p \notin \pi(K)$, then by Frattini argument $G = K\mathbf{N}_G(P)$, where P is a Sylow 97-subgroup of K , and so $\mathbf{N}_G(P)$ has an element x of order p . Now $P\langle x \rangle$ is a cyclic subgroup of G of order $p \cdot 97$ which implies that p is adjacent to 97. Therefore, 97 is adjacent to 13 and 31 which is a contradiction as 97 is of degree 1. Thus $97 \notin \pi(K)$. Now by Lemma 2.5, $97 \notin \pi(\text{Out}(S))$, then $97 \notin \pi(K) \cup \pi(\text{Out}(S))$, and so $97 \in \pi(S)$. Therefore by [12, Table 1], S is isomorphic to $L_3(61)$ and $L_3(61) \leq G/K \leq \text{Aut}(L_3(61))$. Moreover, since $|G| = |L_3(61)|$, we have that $K = 1$, and hence G is isomorphic to $L_3(61)$. $\square$

Proposition 3.9. *If $|G| = |L_3(67)|$ and $D(G) = D(L_3(67))$, then $G \cong L_3(67)$.*

Proof. By Table 1, we have $|G| = 2^4 \cdot 3^2 \cdot 7^2 \cdot 11^2 \cdot 17 \cdot 31 \cdot 67^3$ and $D(G) = (4, 2, 1, 4, 2, 1, 2)$. It follows from Lemma 2.8 that $\alpha(G) \geq 3$. Note that $\deg(2) = 4$ and $|\pi(G)| = 7$. Then $\alpha(2, G) \geq 2$. It follows from Lemma 2.6 that there is a non-Abelian finite simple group S such that $S \leq G/K \leq \text{Aut}(S)$, where K is a maximal normal solvable subgroup of G . Since $|S|$ divides $|G/K|$, so does $|G|$, and so by [12, Table 1], S is isomorphic to one of groups in the first column of Table 3.

If S is isomorphic to one of the groups $L_2(7)$, $L_2(8)$, $L_2(17)$ and $L_2(67)$, then $7, 11, 31 \in \pi(K)$. As K is solvable, we can consider a Hall $\{7, 31\}$ -subgroup $K_1 := P_7 P_{31}$ and a Hall $\{11, 31\}$ -subgroup $K_2 := P_{11} P_{31}$. Then $|K_1| = 7^2 \cdot 31$ and $|K_2| = 11^2 \cdot 31$, and consequently K_i , for $i = 1, 2$, is Abelian which implies that 31 is adjacent to both 7 and 11, and so $\deg(31) \geq 2$, which is a contradiction. Thus $S \cong L_3(67)$. Since $L_3(67) \leq G/K \leq \text{Aut}(L_3(67))$ and $|G| = |L_3(67)|$, we conclude that $|K| = 1$, and hence G is isomorphic to $L_3(67)$. $\square$

Proposition 3.10. *If $|G| = |L_3(79)|$ and $D(G) = D(L_3(79))$, then $G \cong L_3(79)$.*

Table 4: Non-Abelian finite simple groups S whose order divides $|L_3(79)|$.

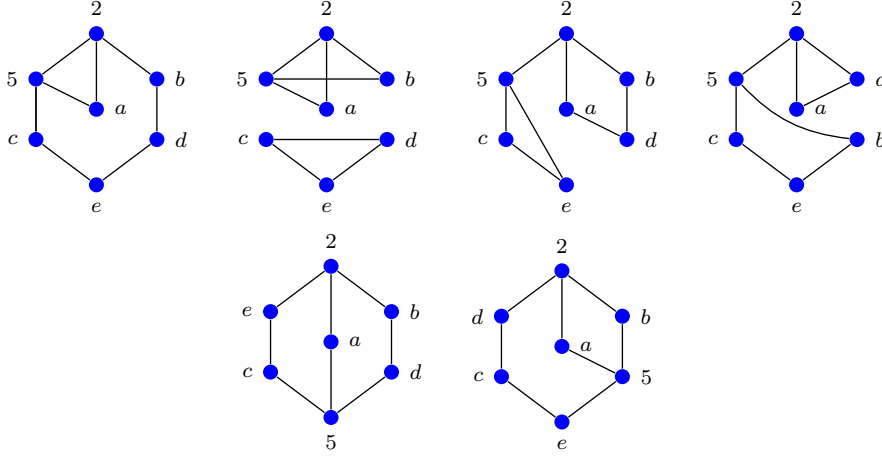
S	$ S $	$ \text{Out}(S) $
$L_2(4)$	$2^2 \cdot 3 \cdot 5$	2
$L_2(9)$	$2^3 \cdot 3^2 \cdot 5$	4
$L_2(7)$	$2^3 \cdot 3 \cdot 7$	2
$L_2(8)$	$2^3 \cdot 3^2 \cdot 7$	3
A_7	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	2
$L_3(4)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	12
$L_4(2)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	2
$L_2(13)$	$2^2 \cdot 3 \cdot 7 \cdot 13$	2
$Sz(8)$	$2^6 \cdot 5 \cdot 7 \cdot 13$	3
$L_2(64)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$	6
$L_2(79)$	$2^4 \cdot 3 \cdot 5 \cdot 7 \cdot 13 \cdot 79$	2
$L_3(79)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 13^2 \cdot 43 \cdot 79^3$	6

Proof. By Table 1, $|G| = 2^6 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 13^2 \cdot 43 \cdot 79^3$ and $D(G) = (4, 2, 2, 1, 4, 1, 2)$. Then by applying Lemma 2.8, we conclude that $\alpha(G) \geq 3$. Furthermore, $\alpha(2, G) \geq 2$ since $\deg(2) = 4$ and $|\pi(G)| = 7$. By Lemma 2.6, there is a non-Abelian finite simple group S such that $S \leq G/K \leq \text{Aut}(S)$, where K is a maximal normal solvable subgroup of G . We show that $\pi(K)$ does not contain 43. Assume the contrary. Then $43 \in \pi(K)$. Let $(p, a) \in \{(5, 1), (13, 1), (13, 2)\}$. If $p \in \pi(K)$, then K contains an abelian Hall subgroup of order $p \cdot 43$, and so p and 43 are adjacent. If $p \notin \pi(K)$, then by Frattini argument $G = K\mathbf{N}_G(P)$, where P is a Sylow 43-subgroup of K . This shows that $\mathbf{N}_G(P)$ contains an element x of order p , and so $P\langle x \rangle$ is a cyclic subgroup of G of order $p \cdot 43$ concluding that p is adjacent to 43. Therefore, 5 and 13 are adjacent to 43, and so 43 has degree at least 2, which is a contradiction. Therefore, $43 \notin \pi(K)$. We prove that $S \cong L_3(79)$. Since $|S|$ divides G/K , so does $|G|$, and so by [12, Table 1], S is isomorphic to one of groups in Table 4 below.

If S is isomorphic to a simple group listed in the first column of Table 4 except $L_3(79)$, then $43 \in K$, which is a contradiction. Therefore, $S \cong L_3(79)$ and $L_3(79) \leq G/K \leq \text{Aut}(L_3(79))$. Moreover, since $|G| = |L_3(79)|$, it follows that $|K| = 1$, and hence $G \cong L_3(79)$. $\square$

Proposition 3.11. *If $|G| = |L_3(81)|$ and $D(G) = D(L_3(81))$, then $G \cong L_3(81)$.*

Proof. According to Table 1, $|G| = 2^9 \cdot 3^{12} \cdot 5^2 \cdot 7 \cdot 13 \cdot 41 \cdot 73$ and $D(G) = (3, 2, 3, 2, 2, 2, 2)$. Then the only possible graphs for $\Gamma(G)$ are as in Figure 2. In each case, we observe that $\Delta = \{a, b, c\}$ forms an independent set of $\Gamma(G)$, and so $\alpha(G) \geq 3$. Note that $\deg(2) = 3$ and $|\pi(G)| = 7$. Then $\alpha(2, G) \geq 2$, and so by Lemma 2.6, there is a non-Abelian finite simple group S such that $S \leq G/K \leq \text{Aut}(S)$, where K is a maximal normal solvable subgroup of G . We show that $73 \notin \pi(K)$. Assume the contrary. Then $73 \in \pi(K)$. Suppose

Figure 2: Possibilities for the prime graph of G in Proposition 3.11.

$p \in \{7, 13, 41\}$. If $p \in \pi(K)$, then K has a cyclic Hall subgroup of order $p \cdot 73$, and so p is adjacent to 73, for all $p \in \{7, 13, 41\}$. If $p \notin \pi(K)$, then by Frattini argument $G = K\mathbf{N}_G(P)$, where P is a Sylow 73-subgroup of K . Hence $\mathbf{N}_G(P)$ contains an element x of order p . Now $P\langle x \rangle$ is a cyclic subgroup of order $p \cdot 73$. This implies that p and 73 are adjacent. Therefore, 73 is adjacent to p , for all $p \in \{7, 13, 41\}$, which is a contradiction as the degree of 73 is 2. Therefore, $73 \notin \pi(K)$. We now prove that $S \cong L_3(81)$. By Lemma 2.5, we must have $73 \notin \pi(\text{Out}(S))$, and so $73 \notin \pi(K) \cup \pi(\text{Out}(S))$. This implies that $73 \in \pi(S)$. Now by [12, Table 1], S is isomorphic to one of the groups in Table 5 below. If S is isomorphic to one of the groups $U_3(9)$, $L_2(3^6)$ and $G_2(9)$, then $41 \in \pi(K)$, which is a contradiction. Therefore $S \cong L_3(81)$, and since $L_3(81) \leq G/K \leq \text{Aut}(L_3(81))$ and $|G| = |L_3(81)|$, it follows that $|K| = 1$, and hence G is isomorphic to $L_3(81)$. $\square$

Table 5: Non-Abelian finite simple groups S whose order divides $|L_3(81)|$.

S	$ S $	$ \text{Out}(S) $
$U_3(9)$	$2^5 \cdot 3^6 \cdot 5^2 \cdot 73$	2
$L_2(3^6)$	$2^3 \cdot 3^6 \cdot 5 \cdot 7 \cdot 13 \cdot 73$	9
$G_2(9)$	$2^8 \cdot 3^{12} \cdot 5^2 \cdot 7 \cdot 13 \cdot 73$	6
$L_3(81)$	$2^9 \cdot 3^{12} \cdot 5^2 \cdot 7 \cdot 13 \cdot 41 \cdot 73$	8

Proposition 3.12. *If $|G| = |L_3(83)|$ and $D(G) = D(L_3(83))$, then $G \cong L_3(83)$.*

Proof. According to Table 1, $|G| = 2^4 \cdot 3 \cdot 7 \cdot 19 \cdot 41^2 \cdot 83^3 \cdot 367$ and $D(G) = (4, 3, 3, 1, 4, 2, 1)$. By Lemma 2.8, we have that $\alpha(G) \geq 3$. Furthermore, $\alpha(2, G) \geq 2$ as $\deg(2) = 4$ and $|\pi(G)| = 7$. By Lemma 2.6, there is a non-Abelian finite

simple group S such that $S \leq G/K \leq \text{Aut}(S)$, where K is a maximal normal solvable subgroup of G . We show that $\pi(K)$ does not contain 367. Assume the contrary. Then $367 \in \pi(K)$. Suppose $p \in \{7, 19\}$. If $p \in \pi(K)$, then K contains a cyclic Hall subgroup of order $p \cdot 367$, and so p is adjacent to 367. If $p \notin \pi(K)$, then by Frattini argument $G = K\mathbf{N}_G(P)$, where P is a Sylow 367-subgroup of K , and so $\mathbf{N}_G(P)$ has an element x of order p . Note that $P\langle x \rangle$ is a cyclic subgroup of G of order $p \cdot 367$. Then p is adjacent to 367. Therefore, both 7 and 19 and 367 are adjacent in $\Gamma(G)$, which is a contradiction. We prove that $S \cong L_3(83)$. By Lemma 2.5, $367 \notin \pi(\text{Out}(S))$, then $367 \notin \pi(K) \cup \pi(\text{Out}(S))$, and so $367 \in \pi(S)$. Therefore by [12, Table 1], S is isomorphic to $L_3(83)$, and so $L_3(83) \leq G/K \leq \text{Aut}(L_3(83))$. Moreover, since $|G| = |L_3(83)|$, it follows that $|K| = 1$, and hence G is isomorphic to $L_3(83)$. $\square$

Proof of Theorem 1.1. The proof of Theorem 1.1 follows immediately from [6, 8, 9] and Propositions 3.1-3.12.

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ZERO KNOWLEDGE UNDENIABLE SIGNATURE SCHEME OVER SEMIGROUP ACTION PROBLEM

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Abstract. The concept of Semigroup Action Problem (SAP) was introduced by C. Monico in 2002. He defined Diffie-Hellman key exchange and ElGamal cryptosystem taking SAP as an underlying problem.

The aim of this paper is to define the application of SAP in designing a zero knowledge undeniable signature scheme. We also discuss the security analysis of the proposed scheme.

Keywords: Group action, semigroup action problem, undeniable signature scheme, zero knowledge proof systems.

1. Introduction

In 1976, Diffie-Hellman proposed the idea of public-key cryptography and digital signatures in [4]. Since then many ideas have been proposed to cover the different security aspects of open communication channel. One such idea in the direction of achieving authenticity, was proposed by David Chaum and V. Antwerpen in [2]. They introduced the concept of undeniable signature scheme which provides authenticity, data integrity and non-repudiation like digital signatures but not publicly verifiable. The aim of introducing these signature schemes is to achieve

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the security in the scenarios where signer wants that only authorized parties can verify his/her signatures. For example, if a software company launches a software and want that only the payable customers can use it then instead of applying digital signatures on software, the company will use undeniable signatures which will be verified by the verifier only if the company found that the involving verifier is payable.

After proposing undeniable signature schemes in [2], a zero knowledge undeniable signature scheme is proposed in [3], which is considered more efficient than the undeniable signature scheme. The advantage of using zero knowledge proof in designing of cryptographic schemes is that it assures the validity of an assertion without revealing any secret information and it also force two communicating parties to follow a protocol properly. Some undeniable signature schemes and zero knowledge undeniable signature schemes have also been proposed in [3, 14, 5] whose security relies on different mathematical hard problems like discrete logarithm problem, factorisation problem, elliptic curve discrete logarithm problem [10] etc.

Many other such mathematical hard problems have been presented in literature on which public-key cryptographic protocols can be designed efficiently. One such mathematical problem was introduced by C. Monico in 2002, which is named as the Semigroup Action Problem (SAP) which is generalised form of the Discrete Logarithm Problem (DLP). Taking SAP as computational hard problem he defined Diffie-Hellman key exchange and ElGamal cryptosystem. After the proposal of SAP, different key-exchange protocols were designed taking different algebraic structures. But the designing was confined to key-exchange protocols and ElGamal cryptosystems only.

Our contribution: The aim of this paper is to design a zero-knowledge undeniable signature scheme whose security relies on the hardness of the SAP. The proposed zero knowledge undeniable signature scheme is proved to satisfy the completeness and soundness property.

The paper is organize in following manner. In section 2, we give the basic preliminaries for the understanding of paper. In section 3, we define zero knowledge undeniable signature scheme over SAP. In section 4, security of the proposed scheme is analysed with respect to completeness and soundness . Finally in section 5, we conclude the paper.

2. Preliminaries

In this section, we discuss some basic definitions which will be useful for understanding of the paper.

Definition 2.1 (Group action). Let $(G, \cdot)$ be a group and S be a non empty set then G is said to act on S if there exist a function

$$\varphi : G \times S \mapsto S, \varphi(a, x) = a * x \text{ for } a \in G, x \in S$$

where, $*$ is the operation between elements of G and S and satisfies following properties:

$$a * (b * x) = (a \cdot b) * x \quad \text{and} \quad e * x = x \quad \forall a, b \in G \quad \text{and} \quad x \in S.$$

Here, e is the identity element of G .

Definition 2.2 (Semi-group action). Let $(G, \cdot)$ be a semi-group and S be a non-empty set then the semi-group action is defined as the mapping $\varphi : G \times S \mapsto S$ such that $\varphi(g, s) = g * s$ satisfying $g * (h * s) = (g \cdot h) * s \quad \forall g, h \in G$ and $s \in S$.

Definition 2.3 (Zero knowledge undeniable signature scheme). Zero knowledge undeniable signature scheme consists of following steps:

Set-up. A security parameter κ is given as an input to the algorithm and it outputs the system parameters.

Key-gen. This algorithm takes system parameters as an input generated by set-up algorithm. Then it returns the key-pairs (sk_S, pk_S) and (sk_V, pk_V) of signer and verifier respectively.

Sign-gen. This algorithm takes sk_S and hash of the message $m \in \mathcal{M}$ as an input and generate the signature over it.

Verification protocol. After getting the signature, verifier interacts with the signer for the verification of signature. If the signature is correct verifier accepts the signature otherwise proceed the disavowal protocol. If the signature is invalid then the probability that a signer is able to convince the verifier that the corresponding signature belongs to his/her public key is negligible.

Disavowal protocol. If the signature is found as an invalid signature, then verifier interacts with the signer in disavowal protocol. With the help of disavowal protocol signer is able to prove that the corresponding signature do not belongs to his/her public key. But if the signer tried to show dishonesty i.e., if he/she tried to convince the verifier for accepting a valid signature as fraud signature then the probability that the signer succeeds in doing so is negligible.

3. Semigroup action problem and its security analysis

In this section we explain the semigroup action problem proposed by Chris Monico in [7].

Definition 3.1 (Semigroup Action Problem(SAP)). Let $(G, \cdot)$ be a commutative semigroup and S be a set. Then to find $a \in G$ in the equation $y = a * s$ for given $y \in S$ and $s \in S$ is known as SAP where $*$ is the operation between the elements of G and S .

The SAP can be considered as a generalised form of discrete logarithm problem in groups. For example, let $G = \mathbb{Z}$ be a the set of positive integers, S be a group and φ be the action of $(\mathbb{Z}, \cdot)$ over S i.e.,

$$\varphi : \mathbb{Z} \times S \mapsto S \text{ defined as}$$

$$\varphi(l, s) = s^l.$$

Then in this particular example the semigroup action problem of finding l for given (a, s^l) is equivalent to the DLP in group. Thus, the DLP can be considered as special case of the SAP.

3.1 Security analysis of SAP

The attacks like Pollard's rho attack and square root attack which are applicable to the DLP cannot be applied directly to the SAP. As the algebraic structure used to define SAP does not possess invertible elements. Now, we examine the Brute force complexity of SAP. Let η be the cardinality of G . To break the SAP, adversary needs to find an $a \in G$ such that $y = as$. For this adversary will calculate $y_i = a_i s$ where $a_i \in G, 1 \leq i \leq \eta$ and compare this with y . The complexity of applying Brute force attack over the SAP is explained in algorithm 1.

Algorithm 1: Exhaustive search algorithm to solve SAP

Input: $y, s \in S$ such that $y = as$

Output: Secret parameter $a \in G$

```

for  $i \leftarrow 1$  to  $\eta$  do
     $a_i \leftarrow a$ ;
     $y_i \leftarrow a_i s$ ;
    Compare  $y = y_i$ ;
    if  $y = y_i$ ;
        return  $a_i$  & exit;
    else
        go to next step;
     $i \leftarrow i + 1$ ;
return  $a$ 

```

The number of steps used in the above algorithm are at most η times. Therefore, the brute force complexity of solving SAP is proportional to $O(\eta)$.

4. The scheme

The zero knowledge undeniable signature scheme over SAP is defined as below:

Set-up. The security parameter κ is given to the algorithm as an input and it returns system parameters $(R, G, S, \hbar, \varphi)$, where $(R, \cdot)$ is a commutative semi-group, G and S are commutative sub-semigroups of R such that $G \cap S = \{\Phi\}$, $\hbar$ is the hash function defined as, $\hbar : \{0, 1\}^* \mapsto S$ and φ is the semigroup action defined as $\varphi : G \times S \rightarrow S$ such that $\varphi(a, x) = ax \in S$ for $a \in G$ and $x \in S$.

Key-gen. The system parameters $(R, G, S, \hbar, \varphi)$ are given as an input to this algorithm then algorithm returns the secret key $sk_S = b$ and public key $pk_S = y = bs$ of signer, where $b \in G$ and $y, s \in S$.

Sign-gen. This algorithm takes the message $m \in \{0, 1\}^*$ to be signed and secret key $sk_S = b \in G$. After this, it generates the signature on the message and returns the signature $\sigma = b\hbar(m) \in S$, where c is randomly chosen from G .

Verification protocol. After getting signature (m, σ) , verifier interacts with the signer and follow the verification protocol to check validity of the signature. The complete protocol is depicted in table 1. At the end, if the verifier gets $A_1 = (ra)(\hbar(m)y)$ and $A_2 = (ra)(\sigma s)$ then the signature will be accepted otherwise the verifier will switch to follow the disavowal protocol with the signer.

Signer		Verifier
		$a \leftarrow G$ $v = a(\hbar(m)s)$
$r \leftarrow G$ $A_1 = (rb)v \in S$ $A_2 = cA_1 \in S$	$\xleftarrow{v}$	
	$\xrightarrow{A_1, A_2}$	
	$\xleftarrow{a}$	
$v \stackrel{?}{=} a(\hbar(m)s)$	$\xrightarrow{r}$	$A_1 \stackrel{?}{=} ra(y\hbar(m)) \in S$ $A_2 \stackrel{?}{=} (ra)(\sigma s) \in S$

Table 1: Verification protocol

Disavowal protocol. After finding the signature as an invalid in verification protocol, the verifier interacts with the signer in disavowal protocol. The complete protocol is depicted in table 2. At the end of protocol if the verifier gets $k = k'$ the verifier convinced that the signature were forged.

5. Security analysis of the scheme

In this section we will prove that the scheme is secure with respect to completeness, soundness and existential unforgeability.

Completeness. According to completeness property which the signer is able to convince the verifier for accepting a valid statement. The proposed zero

Signer		Verifier
$\text{find } k' \text{ s.t.,}$ $b^2(C_1 \sigma^{k'} s) = b(C_2 h(m)^{k'})$ $C_1 \stackrel{?}{=} \alpha(h(m))^{k'}$ $C_2 \stackrel{?}{=} \alpha(y \sigma^{k'})$	$\xleftarrow{(C_1, C_2)}$ $\xrightarrow[\alpha]{\text{commit}(k')}$ $\xrightarrow{\text{reveal}(k')}$	$k \leftarrow \mathbb{Z}_n$ $\alpha \leftarrow G$ $C_1 = \alpha h(m)^k \in S$ $C_2 = \alpha(y \sigma^k) \in S$ $k \stackrel{?}{=} k'$

In the protocol, $\text{commit}(k')$ (k' is a blob[1]) denotes the commitment of k' made by signer and $\text{reveal}(k')$ denotes that signer reveals k' .

Table 2: Disavowal protocol

knowledge undeniable signature scheme is complete because both the verification and disavowal protocols are complete as shown by following theorems.

Theorem 5.1 (Completeness of verification protocol). *The verification protocol is said to be complete if the signer always gets $v = a(h(m)s)$ and the verifier gets $A_1 = (ra)(y h(m))$ and $A_2 = (ra)(\sigma s)$ where both of them follow the verification protocol properly.*

Proof. When the verifier sends a to the signer, then the signer checks the equality $v = a(h(m)s)$ and this will hold if the correct value of a is sent by the verifier.

Similarly the signer sends r to the verifier. Then the verifier checks whether $A_1 = (rb)v$ and $A_2 = cA_1$, using signature and public key of the signer. For this, the verifier calculates $A_1 = ra(y h(m))$ and $A_2 = (ra)(\sigma s)$ because value of these terms will be equal to $(rb)v$ and cA_1 respectively as explained below:

$$\begin{aligned}
A_1 &= (ra)(y h(m)) = (ra)(b s h(m)) \\
&= (rab)(h(m)s) = (rba)(h(m)s) \\
&= (rb)(a(h(m)s)) = (rb)v.
\end{aligned}$$

and

$$\begin{aligned}
A_2 &= (ra)(\sigma s) = (ra)((b c h(m))s) \\
&= (rabc)(h(m)s) \\
&= (crb)(a(h(m)s)) = c((rb)v) = cA_1.
\end{aligned}$$

□

Theorem 5.2 (Completeness of disavowal protocol). *The disavowal protocol is said to be complete if signer always gets $C_1 = \alpha(h(m))^{k'}$, $C_2 = \alpha(y\sigma^{k'})$ and verifier gets $k = k'$ when signer and verifier follow the disavowal protocol properly.*

Proof. On receiving α , the signer calculates C_1, C_2 using his/her public-key, signature, hash of the message $h(m)$, and k' . The value of C_1, C_2 calculated by the signer will be equal to the value of C_1, C_2 respectively send by the verifier if the signer finds correct value of k' and the verifier sends correct value of α .

Similarly on receiving k' , the verifier checks the equality $k = k'$? The equality will hold if the signer is able to find correct value of k . $\square$

Soundness. The proposed zero knowledge undeniable signature scheme is said to satisfy soundness property, if the probability that the dishonest signer will be able to convince the verifier for accepting an inaccurate result of the communication is negligible.

Theorem 5.3 (Soundness of verification protocol). *The probability that the dishonest signer convince the verifier for accepting invalid signature is not greater than maximum of $(\frac{1}{\eta}, \frac{1}{\rho^2})$.*

Proof. On receiving v from the verifier, the signer will try to guess a such that $A_1 = (ra)(yh(m))$ and $A_2 = (ra)(\sigma s)$ or the signer will pick (A_1, A_2) such that the equalities $A_1 = (ra)(yh(m))$ and $A_2 = (ra)(\sigma s)$ holds.

The probability of choosing such $a \in G$ is not greater than $\frac{1}{\eta}$, where η is order of G and the probability of choosing $(A_1, A_2) \in S$ is not greater than $\frac{1}{\rho^2}$, where ρ is the cardinality of S . Thus the probability that the dishonest signer can convince the verifier for accepting invalid signature is not greater than maximum of $(\frac{1}{\eta}, \frac{1}{\rho^2})$ and this will be negligible if the size of G and S is chosen appropriately. $\square$

Theorem 5.4 (Soundness of disavowal protocol). *The probability that the dishonest signer convince the verifier for accepting a valid signature as fraud signature is not greater than $\frac{1}{n}$.*

Proof. Let $\sigma = bch(m)$ be a valid signature of signer on the message m . Suppose dishonest signer tries to convince the verifier for accepting a valid signature as a fraud signature. To achieve this, signer should guess the correct value of $k \in \mathbb{Z}_n$.

The probability of guessing the correct value of $k \in \mathbb{Z}_n$ is not greater than $\frac{1}{n}$. $\square$

Example 5.5 (Example for defining SAP based protocols). Let R be a semiring and $Mat_m(R)$ be the set of all $m \times m$ matrices with entries in semiring R i.e.,

$$(1) \quad Mat_m(R) = \left\{ \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mm} \end{pmatrix} \text{ such that } a_{ij} \in R \right\}$$

Let $C \subset R$ be the center of R i.e., the subset of R consisting of elements that commutes with any other element. Then C forms a commutative semiring and $C[t]$, the polynomial semiring in the indeterminant t also forms commutative semiring over C . If $A \in \text{Mat}_{m \times m}(R)$ then $C[A]$ forms a commutative semiring [15] of the matrix semiring $\text{Mat}_{m \times m}(R)$.

If $p(t) = u_0 + u_1t + u_2t^2 + \cdots + u_nt^n \in C[t]$ then,

$$p(A) = u_0 + u_1A + u_2A^2 + \cdots + u_nA^n.$$

Now, consider the semiring,

$$G = C[A] = \{p(A) \mid p(t) \in C[t]\}$$

and

$$M = C_1[A] = \{p(A) \mid p(t) \in C[t] \text{ where } u_0 = u_{2k+1} = 0 \text{ for } n, k = 0, 1, 2, \dots\}$$

and $S = C_2[A] = \{q(A) \mid q(t) \in C[t] \text{ where } u_i \neq 0 \text{ for any } i = 0, 1, 2, \dots\}$ then $S \cap M = \{\Phi\}$.

Now, φ define a semigroup action of M over S i.e., $\varphi : M \times S \rightarrow S$ such that $p(A)q(A) = t(A) \in S$.

This example can be taken to design the above proposed zero knowledge undeniable signature scheme.

6. Conclusion

In this paper we proposed a zero knowledge undeniable signature scheme whose security relies on the hardness of SAP. We also proved that the proposed scheme satisfies the completeness and soundness property. In future, we will try to give the security proof of scheme in random oracle and the appropriate size of parameters to achieve better security. We will also try to design other cryptographic protocols like digital signature scheme, authentication scheme, sign-cryption scheme over SAP.

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ON c -NORMAL AND S -QUASINORMALLY EMBEDDED SUBGROUPS OF A FINITE GROUP

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Abstract. If P is a p -group for some prime p we shall write $\mathcal{M}(P)$ to denote the set of all maximal subgroups of P and $\mathcal{M}_d(P) = \{P_1, \dots, P_d\}$ to denote any set of maximal subgroups of P such that $\bigcap_{i=1}^d P_i = \Phi(P)$ and d is as small as possible. In this paper, the structure of a finite group G under some assumptions on the c -normal or S -quasinormally embedded subgroups in $\mathcal{M}_d(P)$, for each prime p , and Sylow p -subgroups P of G is researched. Some known results are generalized.

Keywords: c -normal subgroup, S -quasinormally embedded subgroup, supersolvable groups.

1. Introduction

All groups considered in this paper are finite. Let G be a group and let $\mathcal{M}(G)$ be the set of all maximal subgroups of all Sylow subgroups of G . A interesting topic in group theory is to study the influence of the elements of $\mathcal{M}(G)$ on the structure of G . A typical result in this direction is due to Srinivasan [1]. He proved that G is supersolvable provided that every member of $\mathcal{M}(G)$ is normal in G . This result has been widely generalized.

A subgroup H of G is called s -quasinormal in G provided H permutes with all Sylow subgroups of G , i.e, $HP = PH$ for any Sylow subgroup P of G . This concept was introduced by Kegel in [2] and has been studied extensively by Deskins [3] and Schmidt [4]. More recently, Ballester-Bolinches and Pedraza-Aguilera [5] generalized s -quasinormal subgroups to s -quasinormally embedded subgroups. A subgroup H of G is said to be s -quasinormally embedded in G provided every Sylow subgroup of H is a Sylow subgroup of some s -quasinormal subgroup of G . In [5], Ballester-Bolinches and Pedraza-Aguilera showed that, if every subgroup in $\mathcal{M}(G)$ is s -quasinormally embedded in G , then G is supersolvable. Assad and Heliel [6] showed that G is p -nilpotent for the smallest prime p dividing $|G|$ if and only if all members of $\mathcal{M}(P)$ are s -quasinormally embedded in G , where P is a Sylow p -subgroup of G .

As another generalization of the normality, Wang [7] introduced the following concept: A subgroup H of G is called c -normal in G if there is a normal subgroup K such that $G = HK$ and $H \cap K \leq H_G$, where H_G is the normal core of H in G . In [7], Wang showed that G is supersolvable if every member of $\mathcal{M}(G)$

is c -normal. Wang's result has been generalized by some authors (see [8-12], etc). For example, Guo and Shum showed in [8] the following result. Let p be the smallest prime dividing the order of G and let P be a Sylow p -subgroup of G . If every member of $\mathcal{M}(P)$ is c -normal, then G is p -nilpotent. The research on c -normal subgroups has formed a series, which is similar to the series of s -quasinormal subgroups. However, the two series are independent of each other. The aim of this article is to unify and improve the results of [1], [5], [7] and some of [8].

If P is a p -group for some prime p we shall write $\mathcal{M}(P)$ to denote the set of all maximal subgroups of P and $\mathcal{M}_d(P) = \{P_1, \dots, P_d\}$ to denote any set of maximal subgroups of P such that $\bigcap_{i=1}^d P_i = \Phi(P)$ and d is as small as possible.

Such subset $\mathcal{M}_d(P)$ is not unique for a fixed P in general. We know that

$$|\mathcal{M}(P)| = \frac{p^d - 1}{p - 1}, |\mathcal{M}_d(P)| = d, \lim_{d \rightarrow \infty} \frac{p^d - 1}{(p - 1)d} = \infty,$$

so $|\mathcal{M}(P)| \gg |\mathcal{M}_d(P)|$.

In this paper, we study the influence of the members of some fixed $\mathcal{M}_d(P)$ on the structure of group G . Our results are more general.

2. Basic definitions and preliminary results

In this section, we give some results that are needed in this paper.

Lemma 2.1 ([5]). *Suppose that U is an s -quasinormally embedded subgroup of G and that K is a normal subgroup of G . Then:*

- (a) U is s -quasinormally embedded in H whenever $U \leq H \leq G$;
- (b) UK is s -quasinormally embedded in G and UK/K is s -quasinormally embedded in G/K .

Lemma 2.2 ([7]). *Let $X \leq H \leq G$ and $N \trianglelefteq G$. Then:*

- (a) If X is c -normal in G , then X is also c -normal in H ;
- (b) If X is c -normal in G , then XN/N is c -normal in G/N .

Lemma 2.3 ([3]). *If H is an s -quasinormal subgroup of the group G , then H/H_G is nilpotent.*

Lemma 2.4 ([4]). *For a nilpotent subgroup H of G , the following two statements are equivalent:*

- (a) H is s -quasinormal in G ;
- (b) The Sylow subgroups of H are s -quasinormal in G .

Lemma 2.5 ([6]). *Let G be a group and let P_0 be a maximal subgroup of P . Then the following two statements are equivalent:*

- (a) P_0 is normal in G ;
- (b) P_0 is s -quasinormal in G .

Lemma 2.6 ([13]). *If P is a Sylow p -subgroup of G and $N \trianglelefteq G$ such that $P \cap N \leq \Phi(P)$, then N is p -nilpotent.*

Lemma 2.7 ([14]). *A group G is supersolvable if and only if there exists a subgroup of order dividing $|H|$ for every subgroup H of G .*

Lemma 2.8 ([15]). *Let p_1 be the minimal prime dividing $|G|$ and p_s the maximal prime dividing $|G|$. If G possesses two supersolvable subgroups H and K with $|G : H| = p_1$ and $|G : K| = p_s$, then G is supersolvable.*

Lemma 2.9 ([16]). *Let H be a p -subgroup of G . Then, the following statements are equivalent:*

- (a) H is s -quasinormal in G ;
- (b) $H \leq O_p(G)$, and H is s -quasinormally embedded in G .

3. Main results

Theorem 3.1. *Let p be the smallest prime dividing the order of a group G and P be a Sylow p -subgroup of G . Assume that every member of some fixed $\mathcal{M}_d(P)$ is either c -normal or s -quasinormally embedded in G . Then G is p -nilpotent.*

Proof. Assume that the result is not true and let G be a counterexample of minimal order. Let $\mathcal{M}_d(P) = \{P_1, \dots, P_d\}$. By hypothesis, each P_i is either c -normal or s -quasinormally embedded in G . Without loss of generality, let I_1 be the subset of $\{1, \dots, d\}$ such that every $P_i (i \in I_1)$ is c -normal in G and I_2 is the subset such that every $P_i (i \in I_2)$ is s -quasinormally embedded in G . We prove the theorem by the following claims:

- (1) $O_{p'}(G) = 1$.

Set $N = O_{p'}(G)$. Consider the quotient group G/N . We know that PN/N is a Sylow p -subgroup of G/N , $N_{G/N}(PN/N) = N_G(P)N/N$ and $\mathcal{M}(PN/N) = \{P_1N/N, \dots, P_mN/N\}$. Now, by Lemma 2.1 and Lemma 2.2, we see easily that G/N satisfies the condition. If $O_{p'}(G) > 1$, then $G/O_{p'}(G)$ is p -nilpotent and hence G itself is p -nilpotent, a contradiction. Thus claim (1) holds.

- (2) G/P_{iG} is p -nilpotent for all $i \in I_1$, where P_{iG} is the core of P_i in G .

In this case, P_i is a c -normal subgroup of G . We know that there exists a normal subgroup K_i of G such that $G = P_iK_i$ and $P_i \cap K_i = P_{iG}$. Hence,

$$G/P_{iG} = P_i/P_{iG} \cdot K_i/P_{iG}, P_i \cap K_i = P_{iG}.$$

Therefore,

$$|K_i/P_{iG}|_p = |G : P_i|_p = |P : P_i| = p.$$

As p is the smallest prime dividing $|G|$, we know that K_i/P_{iG} is p -nilpotent by Burnside's theorem. Therefore, K_i/P_{iG} has a normal Hall p' -subgroup H/P_{iG} . We see that H/P_{iG} is also a normal Hall p' -subgroup of G/P_{iG} because K_i/P_{iG} is normal in G/P_{iG} . It follows that G/P_{iG} is p -nilpotent for all $i \in I_1$.

For every $P_i (i \in I_2)$, there exists an S -quasinormal subgroup H_i of G such that P_i is a Sylow p -subgroup of H_i .

(3) G/H_{iG} is p -nilpotent for all $i \in I_2$, where H_{iG} is the core of H_i in G .

In fact, As H_i is an S -quasinormal subgroup of G and P_i is a Sylow p -subgroup of H_i , it follows that H_i/H_{iG} is S -quasinormal in G/H_{iG} , and the Lemma of 2.3 asserts that H_i/H_{iG} is nilpotent. Hence, H_i/H_{iG} is an S -quasinormal nilpotent subgroup of G/H_{iG} . By Lemma 2.4, every Sylow subgroup of H_i/H_{iG} is S -quasinormal in G/H_{iG} . Since $P_i H_i/H_{iG}$ is a Sylow p -subgroup of H_i/H_{iG} , it follows that $P_i H_i/H_{iG}$ is S -quasinormal in G/H_{iG} . Thus, Lemma 2.5 indicates that $P_i H_i/H_{iG}$ is normal in G/H_{iG} . Therefore, $P_i H_{iG} \trianglelefteq G$. Noting that P_i is a Sylow p -subgroup of H_i , we have $P_i \leq H_{iG}$. Therefore, $|G/H_{iG}|_p = p$. Now, as p is the smallest prime dividing $|G|$, by Burnside's theorem, we see that G/H_{iG} is p -nilpotent for each $i \in I_2$, which proves (3).

Let

$$N = \left(\bigcap_{i \in I_1} P_{iG} \right) \cap \left(\bigcap_{i \in I_2} H_{iG} \right).$$

(4) N is p -nilpotent.

First, as all P_{iG} and H_{iG} are normal in G , we get $N \trianglelefteq G$. Second, we consider the subgroup $P \cap N$. Recall that P_i is a Sylow p -subgroup of H_{iG} and $P_i \leq P$, so $P \cap H_{iG} \leq P_i$. Moreover, $P_i \leq P \cap H_{iG}$. We have $P \cap H_{iG} = P_i$. Therefore,

$$P \cap N = \left(\bigcap_{i \in I_1} P_{iG} \right) \cap \left(\bigcap_{i \in I_2} H_{iG} \cap P \right) = \left(\bigcap_{i \in I_1} P_{iG} \right) \cap \left(\bigcap_{i \in I_2} P_i \right) = \Phi(P).$$

Applying Lemma 2.6, we know that N is p -nilpotent.

(5) Final contradiction.

Now, N possesses a Hall p' -normal subgroup $N_{p'}$ such that $N = N_p N_{p'}$, where N_p is a Sylow p -subgroup of N . Then, $N_{p'} \text{ char } N \trianglelefteq G$, so $N_{p'}$ is normal in G , and hence, $N_{p'} \leq O_{p'}(G)$. It follows by $O_{p'}(G) = 1$ that $N_{p'} = 1$. Consequently, N is a normal p -subgroup of G , and so, $N = P \cap N = \Phi(P)$. Also, note that the class of p -nilpotent groups is a formation, by steps (2) and (3), we have G/N must be p -nilpotent. It follows that $G/\Phi(P)$ is p -nilpotent. Moreover, by III, 3.3 Hils-Satz in [17], $\Phi(P) \leq \Phi(G)$, so $G/\Phi(G)$ is p -nilpotent. It follows that G would be p -nilpotent, contrary to the choice of G .

The following corollaries are immediate from Theorem 3.1.

Corollary 3.2 ([8]). *Let p be the smallest prime dividing the order of G and let P be a Sylow p -subgroup of G . If every member of $\mathcal{M}(P)$ is c -normal, then G is p -nilpotent.*

Corollary 3.3 ([6]). *Let p be the smallest prime dividing the order of G and let P be a Sylow p -subgroup of G . If every member of $\mathcal{M}(P)$ is s -quasinormally embedded in G , then G is p -nilpotent.*

Theorem 3.4. *Let G be a group. If there exists a normal subgroup H of G such that G/H is supersolvable, and for each Sylow subgroup P of H , every member in some fixed $\mathcal{M}_d(P)$ is either c -normal or s -quasinormally embedded in G , then G is supersolvable.*

Proof. Suppose that the theorem is false so that there exists a counterexample G of minimal order. We shall finish the proof by the following claims.

(1) H is a q -group for some prime q .

By hypothesis and Theorem 3.1, we have that H has a Sylow tower of supersolvable type. Let q be the largest prime dividing $|H|$, and let Q be a Sylow q -subgroup of H . The property that H possesses an order Sylow tower property implies that Q is normal in H . Now, $Q \text{ char } H$ and $H \trianglelefteq G$, so $Q \trianglelefteq G$. Furthermore, $(G/Q)/(H/Q) \cong G/H$, and Lemmas 2.1 and 2.2 show that G/Q satisfies the condition of the theorem, by the choice of G , G/Q is supersolvable. Hence, $H = Q$ by the choice of H .

(2) Q is a Sylow q -subgroup of G .

Suppose that Q is not a Sylow q -subgroup of G . Let p be the smallest prime dividing $|G/Q|$ and r the largest prime dividing $|G/Q|$. By (1), G/Q is supersolvable. By Lemma 2.7, G/Q contains two subgroups M_1/Q and M_2/Q with $|G : M_1| = p$ and $|G : M_2| = r$. By Lemmas 2.1 and 2.2, $(M_i, Q)(i = 1, 2)$ satisfy the condition of the theorem. By the choice of G , M_1 and M_2 are supersolvable. Now, by Lemma 2.8, G would be supersolvable, which is a contradiction. Thus, (2) holds.

(3) $\Phi(Q) = 1$.

Otherwise, by Lemmas 2.1 and 2.2, $G/\Phi(Q)$ satisfies the hypothesis, applying induction, we have $G/\Phi(Q)$ is supersolvable. Furthermore, $\Phi(Q) \leq \Phi(G)$ by III, 3.3 Hils-satz in [17], so $G/\Phi(G)$ is supersolvable. It follows that G is supersolvable, which is a contradiction.

(4) Q is a minimal normal subgroup of G .

Let N be a minimal normal subgroup of G contained in Q . Clearly the quotient group $(G/N, Q/N)$ satisfies the condition, so G/N is supersolvable. As the class of supersolvable groups is a formation, N must be the unique minimal normal subgroup of G which is contained in Q and $N \not\leq (G)$. So there is a maximal subgroup M of G such that $G = NM$ and $N \cap M = 1$. Thus $Q = N(Q \cap M)$. As $G = QM$ and Q is normal abelian in G , we know that $Q \cap M$ is normal in G . If $Q \cap M > 1$, let N_1 be a minimal normal subgroup of G such that $N_1 \leq Q \cap M$, then $N_1 \leq Q$ and $N \neq N_1$, this is a contradiction. Hence $Q \cap M = 1$, which implies $Q = N$.

(5) Every $Q_i \in \mathcal{M}_d(Q) = \{Q_1, \dots, Q_d\}$ is S -quasinormally embedded in G .

Assume that there is a Q_i in $\mathcal{M}_d(Q)$ such that Q_i is c -normal in G . By definition, there is a normal subgroup K_i of G such that $G = Q_i K_i$ and $Q_i \cap K_i = Q_{iG}$ is a normal subgroup of G . By (4), $Q_i \cap K_i = 1$ or Q . If $Q_i \cap K_i = Q$, then $Q_i = Q$, a contradiction. If $Q_i \cap K_i = 1$, then $Q = Q_i(Q \cap K_i)$. But then $Q \cap K_i$ is a normal subgroup of order q of G . So $Q = Q \cap K_i$ by (4). As the class of supersolvable groups is a formation, thus G is supersolvable, contrary to the choice of G .

(6) $Q_i (i = 1, 2, \dots, d)$ are normal subgroups of G .

Lemmas 2.5 and 2.9 imply that $Q_i (i = 1, 2, \dots, d)$ are normal subgroups of G .

(7) The final contradiction.

Now,

$$(G/Q_i)/(Q/Q_i) \cong G/Q,$$

by (1), G/Q is supersolvable. As Q/Q_i is cyclic of order q , it follows that G/Q_i is supersolvable. Set

$$N = \bigcap_{i=1}^d Q_i.$$

By the definition of $\mathcal{M}_d(Q)$,

$$\bigcap_{i=1}^d Q_i = \Phi(Q),$$

so $N = \Phi(Q)$. Now, by the class of supersolvable groups is a formation, $G/\Phi(Q)$ is supersolvable. It follows that $G/\Phi(G)$ is supersolvable, and hence, G is supersolvable. which is a final contradiction. The proof is now completed.

The following corollaries are immediate from Theorem 3.4.

Corollary 3.5 ([7]). *Let G be a group. If every member of $\mathcal{M}(G)$ is c -normal, then G is supersolvable.*

Corollary 3.6 ([5]). *Let G be a group. If every subgroup in $\mathcal{M}(G)$ is s -quasinormally embedded in G , then G is supersolvable.*

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STUDY OF INTEGRAL TRANSFORMS ASSOCIATED WITH GENERALIZED BESSEL FUNCTION

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Abstract. Integral transforms involving Bessel functions play a crucial role in problems related to many diverse field of mathematical physics. Due to the importance of such type of transforms, in this paper, we present (presumably) a new (potentially) useful integral transform involving the product of Whitaker and generalized Bessel functions, which is expressed in terms of Kampé de Fériet functions. Some more results as special cases of our main integral transform are also considered.

Keywords: Generalized Bessel function, Kampé de Fériet function, Whitaker function, Laplace Transform.

1. Introduction and definition

The generalized Bessel function $\omega_{\nu,c}^b(z)$ of the first kind is defined for $z \in C \setminus \{0\}$ and $b, c, \nu \in C$ with $\Re(\nu) > -1$ by the following series [12], (see also, eg., [4, p.10]), for recent works (see also [1,2,3] and [13, p.2]):

$$(1.1) \quad \omega_{\nu,c}^b(z) = \sum_{k=0}^{\infty} \frac{(-1)^k c^k \left(\frac{z}{2}\right)^{\nu+2k}}{k! \Gamma(\nu + k + \frac{1+b}{2})},$$

where C denotes the set of complex numbers, $\Gamma(z)$ is the familiar Gamma function (see, eg., [11, Section 1.1]), and $\omega_{\nu,c}^b(0) = 0$

It is well known that

$$(1.2) \quad \omega_{\nu,1}^1(z) = J_{\nu}(z),$$

*. Corresponding author

where $J_\nu(z)$ is the Bessel function of first kind [6] and

$$(1.3) \quad \omega_{\nu,-1}^1(z) = I_\nu(z),$$

where $I_\nu(z)$ is the Modified Bessel function of first kind [6].

$\omega_{\nu,c}^b(z)$ also have the following relations with sine and cosine functions ([12], see also [13])

$$(1.4) \quad \omega_{-\frac{b}{2},c^2}^b(z) = \left(\frac{2}{z}\right)^{\frac{b}{2}} \frac{\cos cz}{\sqrt{\pi}},$$

$$(1.5) \quad \omega_{-\frac{b}{2},-c^2}^b(z) = \left(\frac{2}{z}\right)^{\frac{b}{2}} \frac{\cosh cz}{\sqrt{\pi}},$$

$$(1.6) \quad \omega_{1-\frac{b}{2},c^2}^b(z) = \left(\frac{2}{z}\right)^{\frac{b}{2}} \frac{\sin cz}{\sqrt{\pi}}$$

and

$$(1.7). \quad \omega_{1-\frac{b}{2},-c^2}^b(z) = \left(\frac{2}{z}\right)^{\frac{b}{2}} \frac{\sinh cz}{\sqrt{\pi}}.$$

The Whitaker function of second kind $W_{k,\mu}(z)$ is defined as [10, p,39, eq.(24)], (see also [7],[8] and [9])

$$(1.8) \quad W_{k,\mu}(z) = z^{\mu+\frac{1}{2}} \exp\left(-\frac{z}{2}\right) \Psi\left(\mu - k + \frac{1}{2}, 2\mu + 1; z\right),$$

which have the following relations with other special functions

$$(1.9) \quad W_{\frac{\alpha}{2}+\frac{1}{2}+n,\frac{\alpha}{2}}(z) = (-1)^n n! e^{-\frac{z}{2}} z^{\frac{\alpha}{2}+\frac{1}{2}} L_n^\alpha(z),$$

where $L_n^\alpha(z)$ is a Laguerre polynomial [6],

$$(1.10) \quad W_{\frac{1}{4}+n,\frac{1}{4}}(z^2) = 2^{-n} e^{-\frac{z^2}{2}} \sqrt{z} H_n(z),$$

where $H_n(z)$ is a Hermite polynomial [6],

$$(1.11) \quad W_{0,\mu}(2z) = \sqrt{\frac{2z}{\pi}} K_\mu(z),$$

where $K_\mu(z)$ is the Modified Bessel function of second kind [6].

2. Main transformation

We establish an integral of the form

$$\begin{aligned}
 & \int_0^\infty t^{\lambda-\frac{1}{2}} e^{-pt} W_{k,\mu}(\alpha t) \omega_{\nu,c}^b(\beta t) dt \\
 &= \left(\frac{\beta}{2}\right)^\nu \alpha^{\mu+\frac{1}{2}} \frac{\Gamma(\lambda+\nu-\mu+1)}{\Gamma(\nu+\frac{1+b}{2})(p+\frac{\alpha}{2})^{\mu+\lambda+\nu+1}} \left\{ \frac{\Gamma(\mu+\lambda+\nu+1)}{\Gamma(\lambda+\nu-k+\frac{3}{2})} \right. \\
 & \times F_{2:1;1}^{2:2;2} \left[\begin{matrix} \frac{\mu+\lambda+\nu+1}{2}, \frac{\mu+\lambda+\nu+2}{2} : \frac{\lambda+\nu-\mu+1}{2}, \frac{\lambda+\nu-\mu+2}{2} ; \frac{\mu-k+\frac{1}{2}}{2}, \frac{\mu-k+\frac{3}{2}}{2} ; \\ \frac{\lambda+\nu-k+\frac{3}{2}}{2}, \frac{\lambda+\nu-k+\frac{5}{2}}{2} : \nu+\frac{1+b}{2} ; \frac{1}{2} ; \end{matrix} \right. \\
 & \quad \left. + \left(\frac{2p-\alpha}{2p+\alpha}\right) \frac{\Gamma(\mu+\lambda+\nu+2)\Gamma(\mu-k+\frac{3}{2})}{\Gamma(\mu-k+\frac{1}{2})\Gamma(\lambda+\nu-k+\frac{5}{2})} \right. \\
 & \left. \times F_{2:1;1}^{2:2;2} \left[\begin{matrix} \frac{\mu+\lambda+\nu+2}{2}, \frac{\mu+\lambda+\nu+3}{2} : \frac{\lambda+\nu-\mu+1}{2}, \frac{\lambda+\nu-\mu+2}{2} ; \frac{\mu-k+\frac{3}{2}}{2}, \frac{\mu-k+\frac{5}{2}}{2} ; \\ \frac{\lambda+\nu-k+\frac{5}{2}}{2}, \frac{\lambda+\nu-k+\frac{7}{2}}{2} : \nu+\frac{1+b}{2} ; \frac{3}{2} ; \end{matrix} \right. \right. \\
 & \quad \left. \left. -c \left(\frac{2\beta}{2p+\alpha}\right)^2, \left(\frac{2p-\alpha}{2p+\alpha}\right)^2 \right] \right\},
 \end{aligned}
 \tag{2.1}$$

where $F_{l:m;n}^{p:q:k}$ denotes the Kampé de Fériet function [10; p.63, Eq.(16)].

Proof. In order to derive our main result (2.1), we denote the left-hand side of (2.1) by I, expressing $\omega_{\nu,c}^b(\beta t)$ as a series with the help of (1.1) and integrating term by term with the help of the result [5; p.216(16)], we get

$$\begin{aligned}
 I &= \left(\frac{\beta}{2}\right)^\nu \alpha^{\mu+\frac{1}{2}} \sum_{m=0}^{\infty} \frac{(-1)^m c^m \left(\frac{\beta}{2}\right)^{2m}}{m! \Gamma(\nu+m+\frac{1+b}{2})} \\
 & \frac{\Gamma(\mu+\lambda+\nu+2m+1)\Gamma(\lambda+\nu+2m-\mu+1)}{\Gamma(\lambda+\nu+2m-k+\frac{3}{2})(p+\frac{1}{2}\alpha)^{\mu+\lambda+\nu+2m+1}} \\
 & \times {}_2F_1 \left[\begin{matrix} \mu+\lambda+\nu+2m+1, \lambda+\nu+2m-\mu+1 ; \\ \lambda+\nu+2m-k+\frac{3}{2} ; \end{matrix} \frac{2p-\alpha}{2p+\alpha} \right],
 \end{aligned}
 \tag{2.2}$$

where $Re(\lambda+\nu+\frac{1}{2}\pm\mu) > -\frac{1}{2}$.

On expanding ${}_2F_1$ in its defining series, separating this series into its even and odd terms, and then by arranging the resulting expression into Kampé de Fériet function [10; p.63, Eq.(16)], we get our required result (2.1). $\square$

3. Special cases

(i) On taking $b=c=1$ in (2.1) and then by using (1.2), we get

$$\int_0^\infty t^{\lambda-\frac{1}{2}} e^{-pt} W_{k,\mu}(\alpha t) J_\nu(\beta t) dt$$

$$\begin{aligned}
&= \left(\frac{\beta}{2}\right)^\nu \alpha^{\mu+\frac{1}{2}} \frac{\Gamma(\lambda+\nu-\mu+1)}{\Gamma(\nu+1)(p+\frac{\alpha}{2})^{\mu+\lambda+\nu+1}} \left\{ \frac{\Gamma(\mu+\lambda+\nu+1)}{\Gamma(\lambda+\nu-k+\frac{3}{2})} \right. \\
&\times F_{2:1;1}^{2:2;2} \left[\begin{matrix} \frac{\mu+\lambda+\nu+1}{2}, \frac{\mu+\lambda+\nu+2}{2} : & \frac{\lambda+\nu-\mu+1}{2}, \frac{\lambda+\nu-\mu+2}{2} & ; & \frac{\mu-k+\frac{1}{2}}{2}, \frac{\mu-k+\frac{3}{2}}{2} & ; \\ \frac{\lambda+\nu-k+\frac{3}{2}}{2}, \frac{\lambda+\nu-k+\frac{5}{2}}{2} : & & \nu+1 & ; & \frac{1}{2} & ; \end{matrix} \right. \\
&\quad \left. + \left(\frac{2p-\alpha}{2p+\alpha}\right) \frac{\Gamma(\mu+\lambda+\nu+2)\Gamma(\mu-k+\frac{3}{2})}{\Gamma(\mu-k+\frac{1}{2})\Gamma(\lambda+\nu-k+\frac{5}{2})} \right. \\
&\quad \left. \left. - \left(\frac{2\beta}{2p+\alpha}\right)^2, \left(\frac{2p-\alpha}{2p+\alpha}\right)^2 \right] \right\} \\
&\quad \times F_{2:1;1}^{2:2;2} \left[\begin{matrix} \frac{\mu+\lambda+\nu+2}{2}, \frac{\mu+\lambda+\nu+3}{2} : & \frac{\lambda+\nu-\mu+1}{2}, \frac{\lambda+\nu-\mu+2}{2} & ; & \frac{\mu-k+\frac{3}{2}}{2}, \frac{\mu-k+\frac{5}{2}}{2} & ; \\ \frac{\lambda+\nu-k+\frac{5}{2}}{2}, \frac{\lambda+\nu-k+\frac{7}{2}}{2} : & & \nu+1 & ; & \frac{3}{2} & ; \end{matrix} \right. \\
&\quad \left. \left. - \left(\frac{2\beta}{2p+\alpha}\right)^2, \left(\frac{2p-\alpha}{2p+\alpha}\right)^2 \right] \right\}.
\end{aligned}
\tag{3.1}$$

(ii) On taking $b = 1, c = -1$ in (2.1) and then by using (1.3), we get

$$\begin{aligned}
&\int_0^\infty t^{\lambda-\frac{1}{2}} e^{-pt} W_{k,\mu}(\alpha t) I_\nu(\beta t) dt \\
&= \left(\frac{\beta}{2}\right)^\nu \alpha^{\mu+\frac{1}{2}} \frac{\Gamma(\lambda+\nu-\mu+1)}{\Gamma(\nu+1)(p+\frac{\alpha}{2})^{\mu+\lambda+\nu+1}} \left\{ \frac{\Gamma(\mu+\lambda+\nu+1)}{\Gamma(\lambda+\nu-k+\frac{3}{2})} \right. \\
&\times F_{2:1;1}^{2:2;2} \left[\begin{matrix} \frac{\mu+\lambda+\nu+1}{2}, \frac{\mu+\lambda+\nu+2}{2} : & \frac{\lambda+\nu-\mu+1}{2}, \frac{\lambda+\nu-\mu+2}{2} & ; & \frac{\mu-k+\frac{1}{2}}{2}, \frac{\mu-k+\frac{3}{2}}{2} & ; \\ \frac{\lambda+\nu-k+\frac{3}{2}}{2}, \frac{\lambda+\nu-k+\frac{5}{2}}{2} : & & \nu+1 & ; & \frac{1}{2} & ; \end{matrix} \right. \\
&\quad \left. + \left(\frac{2p-\alpha}{2p+\alpha}\right) \frac{\Gamma(\mu+\lambda+\nu+2)\Gamma(\mu-k+\frac{3}{2})}{\Gamma(\mu-k+\frac{1}{2})\Gamma(\lambda+\nu-k+\frac{5}{2})} \right. \\
&\quad \left. \left. \left(\frac{2\beta}{2p+\alpha}\right)^2, \left(\frac{2p-\alpha}{2p+\alpha}\right)^2 \right] \right\} \\
&\quad \times F_{2:1;1}^{2:2;2} \left[\begin{matrix} \frac{\mu+\lambda+\nu+2}{2}, \frac{\mu+\lambda+\nu+3}{2} : & \frac{\lambda+\nu-\mu+1}{2}, \frac{\lambda+\nu-\mu+2}{2} & ; & \frac{\mu-k+\frac{3}{2}}{2}, \frac{\mu-k+\frac{5}{2}}{2} & ; \\ \frac{\lambda+\nu-k+\frac{5}{2}}{2}, \frac{\lambda+\nu-k+\frac{7}{2}}{2} : & & \nu+1 & ; & \frac{3}{2} & ; \end{matrix} \right. \\
&\quad \left. \left. \left(\frac{2\beta}{2p+\alpha}\right)^2, \left(\frac{2p-\alpha}{2p+\alpha}\right)^2 \right] \right\},
\end{aligned}
\tag{3.2}$$

where $I_\nu(\beta t)$ is the Modified Bessel function of first kind.

(iii) On taking $k = 0$ in (2.1) and then by using (1.11), we get

$$\begin{aligned}
&\int_0^\infty t^\lambda e^{-pt} K_\mu\left(\frac{\alpha}{2}t\right) \omega_{\nu,c}^b(\beta t) dt \\
&= \sqrt{\pi} \left(\frac{\beta}{2}\right)^\nu \alpha^\mu \frac{\Gamma(\lambda+\nu-\mu+1)}{\Gamma(\nu+\frac{1+b}{2})(p+\frac{\alpha}{2})^{\mu+\lambda+\nu+1}} \left\{ \frac{\Gamma(\mu+\lambda+\nu+1)}{\Gamma(\lambda+\nu+\frac{3}{2})} \right. \\
&\times F_{2:1;1}^{2:2;2} \left[\begin{matrix} \frac{\mu+\lambda+\nu+1}{2}, \frac{\mu+\lambda+\nu+2}{2} : & \frac{\lambda+\nu-\mu+1}{2}, \frac{\lambda+\nu-\mu+2}{2} & ; & \frac{\mu+\frac{1}{2}}{2}, \frac{\mu+\frac{3}{2}}{2} & ; \\ \frac{\lambda+\nu+\frac{3}{2}}{2}, \frac{\lambda+\nu+\frac{5}{2}}{2} : & & \nu+\frac{1+b}{2} & ; & \frac{1}{2} & ; \end{matrix} \right. \\
&\quad \left. -c \left(\frac{2\beta}{2p+\alpha}\right)^2, \left(\frac{2p-\alpha}{2p+\alpha}\right)^2 \right] \\
&\quad + \left(\frac{2p-\alpha}{2p+\alpha}\right) \frac{\Gamma(\mu+\lambda+\nu+2)\Gamma(\mu+\frac{3}{2})}{\Gamma(\mu+\frac{1}{2})\Gamma(\lambda+\nu+\frac{5}{2})}
\end{aligned}$$

$$(3.3) \quad \times F_{2:1;1}^{2:2;2} \left[\begin{array}{c} \frac{\mu+\lambda+\nu+2}{2}, \frac{\mu+\lambda+\nu+3}{2} : \frac{\lambda+\nu-\mu+1}{2}, \frac{\lambda+\nu-\mu+2}{2} ; \frac{\mu+\frac{3}{2}}{2}, \frac{\mu+\frac{5}{2}}{2} ; \\ \frac{\lambda+\nu+\frac{5}{2}}{2}, \frac{\lambda+\nu+\frac{7}{2}}{2} : \nu + \frac{1+b}{2} ; \frac{3}{2} ; \end{array} ; -c \left(\frac{2\beta}{2p+\alpha} \right)^2, \left(\frac{2p-\alpha}{2p+\alpha} \right)^2 \right],$$

where $K_\nu(\frac{\alpha}{2}t)$ is the Modified Bessel function of second kind.

(iv) On taking $\nu = -\frac{b}{2}$, replacing c by $-c^2$ in (2.1) and then on using (1.5), we get

$$\begin{aligned} & \int_0^\infty t^{\lambda-\frac{b}{2}-\frac{1}{2}} e^{-pt} W_{k,\mu}(\alpha t) \cosh(c\beta t) dt \\ &= \alpha^{\mu+\frac{1}{2}} \frac{\Gamma(\lambda-\frac{b}{2}-\mu+1)}{(p+\frac{\alpha}{2})^{\mu+\lambda-\frac{b}{2}+1}} \left\{ \frac{\Gamma(\mu+\lambda-\frac{b}{2}+1)}{\Gamma(\lambda-\frac{b}{2}-k+\frac{3}{2})} \right. \\ & \times F_{2:1;1}^{2:2;2} \left[\begin{array}{c} \frac{\mu+\lambda-\frac{b}{2}+1}{2}, \frac{\mu+\lambda-\frac{b}{2}+2}{2} : \frac{\lambda-\frac{b}{2}-\mu+1}{2}, \frac{\lambda-\frac{b}{2}-\mu+2}{2} ; \frac{\mu-k+\frac{1}{2}}{2}, \frac{\mu-k+\frac{3}{2}}{2} ; \\ \frac{\lambda-\frac{b}{2}-k+\frac{3}{2}}{2}, \frac{\lambda-\frac{b}{2}-k+\frac{5}{2}}{2} : \frac{1}{2} ; \frac{1}{2} ; \end{array} ; \left(\frac{2c\beta}{2p+\alpha} \right)^2, \left(\frac{2p-\alpha}{2p+\alpha} \right)^2 \right] \\ & \left. + \left(\frac{2p-\alpha}{2p+\alpha} \right) \frac{\Gamma(\mu+\lambda-\frac{b}{2}+2)\Gamma(\mu-k+\frac{3}{2})}{\Gamma(\mu-k+\frac{1}{2})\Gamma(\lambda-\frac{b}{2}-k+\frac{5}{2})} \right\} \end{aligned}$$

$$(3.4) \quad \times F_{2:1;1}^{2:2;2} \left[\begin{array}{c} \frac{\mu+\lambda-\frac{b}{2}+2}{2}, \frac{\mu+\lambda-\frac{b}{2}+3}{2} : \frac{\lambda-\frac{b}{2}-\mu+1}{2}, \frac{\lambda-\frac{b}{2}-\mu+2}{2} ; \frac{\mu-k+\frac{3}{2}}{2}, \frac{\mu-k+\frac{5}{2}}{2} ; \\ \frac{\lambda-\frac{b}{2}-k+\frac{5}{2}}{2}, \frac{\lambda-\frac{b}{2}-k+\frac{7}{2}}{2} : \frac{1}{2} ; \frac{3}{2} ; \end{array} ; \left(\frac{2c\beta}{2p+\alpha} \right)^2, \left(\frac{2p-\alpha}{2p+\alpha} \right)^2 \right] \Bigg\}.$$

(v) On taking $\nu = 1 - \frac{b}{2}$, replacing c by c^2 in (2.1) and then on using (1.6), we get

$$\begin{aligned} & \int_0^\infty t^{\lambda-\frac{b}{2}-\frac{1}{2}} e^{-pt} W_{k,\mu}(\alpha t) \sin(c\beta t) dt \\ &= \alpha^{\mu+\frac{1}{2}} \beta \frac{\Gamma(\lambda-\frac{b}{2}-\mu+2)}{(p+\frac{\alpha}{2})^{\mu+\lambda-\frac{b}{2}+2}} \left\{ \frac{\Gamma(\mu+\lambda-\frac{b}{2}+2)}{\Gamma(\lambda-\frac{b}{2}-k+\frac{5}{2})} \right. \\ & \times F_{2:1;1}^{2:2;2} \left[\begin{array}{c} \frac{\mu+\lambda-\frac{b}{2}+2}{2}, \frac{\mu+\lambda-\frac{b}{2}+3}{2} : \frac{\lambda-\frac{b}{2}-\mu+2}{2}, \frac{\lambda-\frac{b}{2}-\mu+3}{2} ; \frac{\mu-k+\frac{1}{2}}{2}, \frac{\mu-k+\frac{3}{2}}{2} ; \\ \frac{\lambda-\frac{b}{2}-k+\frac{5}{2}}{2}, \frac{\lambda-\frac{b}{2}-k+\frac{7}{2}}{2} : \frac{3}{2} ; \frac{1}{2} ; \end{array} ; -\left(\frac{2c\beta}{2p+\alpha} \right)^2, \left(\frac{2p-\alpha}{2p+\alpha} \right)^2 \right] \\ & \left. + \left(\frac{2p-\alpha}{2p+\alpha} \right) \frac{\Gamma(\mu+\lambda-\frac{b}{2}+3)\Gamma(\mu-k+\frac{3}{2})}{\Gamma(\mu-k+\frac{1}{2})\Gamma(\lambda-\frac{b}{2}-k+\frac{7}{2})} \right\} \end{aligned}$$

$$(3.5) \quad \times F_{2:1;1}^{2:2;2} \left[\begin{array}{c} \frac{\mu+\lambda-\frac{b}{2}+3}{2}, \frac{\mu+\lambda-\frac{b}{2}+4}{2} : \frac{\lambda-\frac{b}{2}-\mu+2}{2}, \frac{\lambda-\frac{b}{2}-\mu+3}{2} ; \frac{\mu-k+\frac{1}{2}}{2}, \frac{\mu-k+\frac{3}{2}}{2} ; \\ \frac{\lambda-\frac{b}{2}-k+\frac{7}{2}}{2}, \frac{\lambda-\frac{b}{2}-k+\frac{9}{2}}{2} : \frac{3}{2} ; \frac{3}{2} ; \end{array} ; -\left(\frac{2c\beta}{2p+\alpha} \right)^2, \left(\frac{2p-\alpha}{2p+\alpha} \right)^2 \right] \Bigg\}.$$

(vi) On taking $\nu = 1 - \frac{b}{2}$, replacing c by $-c^2$ in (2.1) and then by using (1.7), we get

$$\int_0^\infty t^{\lambda-\frac{b}{2}-\frac{1}{2}} e^{-pt} W_{k,\mu}(\alpha t) \sinh(c\beta t) dt$$

$$\begin{aligned}
&= \alpha^{\mu+\frac{1}{2}} \beta \frac{\Gamma(\lambda - \frac{b}{2} - \mu + 2)}{(p + \frac{\alpha}{2})^{\mu+\lambda-\frac{b}{2}+2}} \left\{ \frac{\Gamma(\mu + \lambda - \frac{b}{2} + 2)}{\Gamma(\lambda - \frac{b}{2} - k + \frac{5}{2})} \right. \\
&\times F_{2:1;1}^{2:2;2} \left[\begin{matrix} \frac{\mu+\lambda-\frac{b}{2}+2}{2}, \frac{\mu+\lambda-\frac{b}{2}+3}{2} : & \frac{\lambda-\frac{b}{2}-\mu+2}{2}, \frac{\lambda-\frac{b}{2}-\mu+3}{2} & ; & \frac{\mu-k+\frac{1}{2}}{2}, \frac{\mu-k+\frac{3}{2}}{2} & ; \\ \frac{\lambda-\frac{b}{2}-k+\frac{5}{2}}{2}, \frac{\lambda-\frac{b}{2}-k+\frac{7}{2}}{2} : & \frac{3}{2} & ; & \frac{1}{2} & ; \end{matrix} \right. \\
&\quad \left. + \left(\frac{2p-\alpha}{2p+\alpha} \right) \frac{\Gamma(\mu + \lambda - \frac{b}{2} + 3) \Gamma(\mu - k + \frac{3}{2})}{\Gamma(\mu - k + \frac{1}{2}) \Gamma(\lambda - \frac{b}{2} - k + \frac{7}{2})} \right. \\
&\quad \left. \times F_{2:1;1}^{2:2;2} \left[\begin{matrix} \frac{\mu+\lambda-\frac{b}{2}+3}{2}, \frac{\mu+\lambda-\frac{b}{2}+4}{2} : & \frac{\lambda-\frac{b}{2}-\mu+2}{2}, \frac{\lambda-\frac{b}{2}-\mu+3}{2} & ; & \frac{\mu-k+\frac{1}{2}}{2}, \frac{\mu-k+\frac{3}{2}}{2} & ; \\ \frac{\lambda-\frac{b}{2}-k+\frac{7}{2}}{2}, \frac{\lambda-\frac{b}{2}-k+\frac{9}{2}}{2} : & \frac{3}{2} & ; & \frac{3}{2} & ; \end{matrix} \right. \right. \\
&\quad \left. \left. \left(\frac{2c\beta}{2p+\alpha} \right)^2, \left(\frac{2p-\alpha}{2p+\alpha} \right)^2 \right] \right\}.
\end{aligned}
\tag{3.6}$$

(vii) On taking $k = \frac{\delta}{2} + \frac{1}{2} + q$, $\mu = \frac{\delta}{2}$ in (2.1) and then by using (1.9), we obtain

$$\begin{aligned}
&\int_0^\infty t^{\lambda+\frac{\delta}{2}+\frac{1}{2}-\frac{1}{2}} e^{-(p+\frac{\alpha}{2})t} L_q^\delta(\alpha t) \omega_{\nu,c}^b(\beta t) dt \\
&= \frac{1}{(-1)^q q!} \left(\frac{\beta}{2} \right)^\nu \frac{\Gamma(\lambda + \nu - \frac{\delta}{2} + 1)}{\Gamma(\nu + \frac{1+b}{2})(p + \frac{\alpha}{2})^{\frac{\delta}{2}+\lambda+\nu+1}} \left\{ \frac{\Gamma(\frac{\delta}{2} + \lambda + \nu + 1)}{\Gamma(\lambda + \nu - \frac{\delta}{2} - q + 1)} \right. \\
&\times F_{2:1;1}^{2:2;2} \left[\begin{matrix} \frac{\frac{\delta}{2}+\lambda+\nu+1}{2}, \frac{\frac{\delta}{2}+\lambda+\nu+2}{2} : & \frac{\lambda+\nu-\frac{\delta}{2}+1}{2}, \frac{\lambda+\nu-\frac{\delta}{2}+2}{2} & ; & \frac{-q}{2}, \frac{-q+1}{2} & ; \\ \frac{\lambda+\nu-\frac{\delta}{2}-q+1}{2}, \frac{\lambda+\nu-\frac{\delta}{2}-q+2}{2} : & \nu + \frac{1+b}{2} & ; & \frac{1}{2} & ; \end{matrix} \right. \\
&\quad \left. + \left(\frac{2p-\alpha}{2p+\alpha} \right) \frac{\Gamma(\frac{\delta}{2} + \lambda + \nu + 2) \Gamma(-q + 1)}{\Gamma(-q) \Gamma(\lambda + \nu - \frac{\delta}{2} - q + 2)} \right. \\
&\quad \times F_{2:1;1}^{2:2;2} \left[\begin{matrix} \frac{\frac{\delta}{2}+\lambda+\nu+2}{2}, \frac{\frac{\delta}{2}+\lambda+\nu+3}{2} : & \frac{\lambda+\nu-\frac{\delta}{2}+1}{2}, \frac{\lambda+\nu-\frac{\delta}{2}+2}{2} & ; & \frac{-q+1}{2}, \frac{-q+2}{2} & ; \\ \frac{\lambda+\nu-\frac{\delta}{2}-q+2}{2}, \frac{\lambda+\nu-\frac{\delta}{2}-q+3}{2} : & \nu + \frac{1+b}{2} & ; & \frac{3}{2} & ; \end{matrix} \right. \\
&\quad \left. \left. -c \left(\frac{2\beta}{2p+\alpha} \right)^2, \left(\frac{2p-\alpha}{2p+\alpha} \right)^2 \right] \right\},
\end{aligned}
\tag{3.7}$$

where $L_q^\delta(\alpha t)$ is the Laguerre polynomial.

(viii) On taking $k = \frac{1}{4} + q$, $\mu = \frac{1}{4}$ in (2.1) and then by using (1.10), we get

$$\begin{aligned}
&\int_0^\infty t^{\lambda+\frac{1}{4}-\frac{1}{2}} e^{-(p+\frac{\alpha}{2})t} H_q(\sqrt{\alpha}t) \omega_{\nu,c}^b(\beta t) dt \\
&= 2^q \left(\frac{\beta}{2} \right)^\nu \sqrt{\alpha} \frac{\Gamma(\lambda + \nu + \frac{3}{4})}{\Gamma(\nu + \frac{1+b}{2})(p + \frac{\alpha}{2})^{\lambda+\nu+\frac{5}{4}}} \left\{ \frac{\Gamma(\lambda + \nu + \frac{5}{4})}{\Gamma(\lambda + \nu - q + \frac{5}{4})} \right. \\
&\times F_{2:1;1}^{2:2;2} \left[\begin{matrix} \frac{\lambda+\nu+\frac{5}{4}}{2}, \frac{\lambda+\nu+\frac{9}{4}}{2} : & \frac{\lambda+\nu+\frac{3}{4}}{2}, \frac{\lambda+\nu+\frac{7}{4}}{2} & ; & \frac{-q+\frac{1}{2}}{2}, \frac{-q+\frac{3}{2}}{2} & ; \\ \frac{\lambda+\nu-q+\frac{5}{4}}{2}, \frac{\lambda+\nu-q+\frac{9}{4}}{2} : & \nu + \frac{1+b}{2} & ; & \frac{1}{2} & ; \end{matrix} \right. \\
&\quad \left. -c \left(\frac{2\beta}{2p+\alpha} \right)^2, \left(\frac{2p-\alpha}{2p+\alpha} \right)^2 \right] \right\}
\end{aligned}$$

$$\begin{aligned}
& + \left(\frac{2p-\alpha}{2p+\alpha} \right) \frac{\Gamma(\lambda+\nu+\frac{9}{4})\Gamma(-q+\frac{3}{2})}{\Gamma(-q+\frac{1}{2})\Gamma(\lambda+\nu-q+\frac{9}{4})} \\
(3.8) \quad & \times F_{2:1;1}^{2:2;2} \left[\begin{array}{c} \frac{\lambda+\nu+\frac{9}{4}}{2}, \frac{\lambda+\nu+\frac{13}{4}}{2} : \quad \frac{\lambda+\nu+\frac{3}{4}}{2}, \frac{\lambda+\nu+\frac{7}{4}}{2} \quad ; \quad \frac{-q+\frac{3}{2}}{2}, \frac{-q+\frac{5}{2}}{2} \quad ; \\ \frac{\lambda+\nu-q+\frac{9}{4}}{2}, \frac{\lambda+\nu-q+\frac{13}{4}}{2} : \quad \nu + \frac{1+b}{2} \quad ; \quad \frac{3}{2} \quad ; \end{array} \quad -c \left(\frac{2\beta}{2p+\alpha} \right)^2, \left(\frac{2p-\alpha}{2p+\alpha} \right)^2 \right] \Bigg\},
\end{aligned}$$

where $H_q(\sqrt{\alpha}t)$ is the Hermite polynomial.

(ix) On taking $\nu = -\frac{b}{2}$, replacing c by c^2 in (2.1) and then by using (1.4), we get

$$\begin{aligned}
& \int_0^\infty t^{\lambda-\frac{b}{2}-\frac{1}{2}} e^{-pt} W_{k,\mu}(\alpha t) \cos(c\beta t) dt \\
& = \alpha^{\mu+\frac{1}{2}} \frac{\Gamma(\lambda-\frac{b}{2}-\mu+1)}{(p+\frac{\alpha}{2})^{\mu+\lambda-\frac{b}{2}+1}} \left\{ \frac{\Gamma(\mu+\lambda-\frac{b}{2}+1)}{\Gamma(\lambda-\frac{b}{2}-k+\frac{3}{2})} \right. \\
& \times F_{2:1;1}^{2:2;2} \left[\begin{array}{c} \frac{\mu+\lambda-\frac{b}{2}+1}{2}, \frac{\mu+\lambda-\frac{b}{2}+2}{2} : \quad \frac{\lambda-\frac{b}{2}-\mu+1}{2}, \frac{\lambda-\frac{b}{2}-\mu+2}{2} \quad ; \quad \frac{\mu-k+\frac{1}{2}}{2}, \frac{\mu-k+\frac{3}{2}}{2} \quad ; \\ \frac{\lambda-\frac{b}{2}-k+\frac{3}{2}}{2}, \frac{\lambda-\frac{b}{2}-k+\frac{5}{2}}{2} : \quad \frac{1}{2} \quad ; \quad \frac{1}{2} \quad ; \end{array} \quad -\left(\frac{2c\beta}{2p+\alpha} \right)^2, \left(\frac{2p-\alpha}{2p+\alpha} \right)^2 \right] \\
& \quad + \left(\frac{2p-\alpha}{2p+\alpha} \right) \frac{\Gamma(\mu+\lambda-\frac{b}{2}+2)\Gamma(\mu-k+\frac{3}{2})}{\Gamma(\mu-k+\frac{1}{2})\Gamma(\lambda-\frac{b}{2}-k+\frac{5}{2})} \\
(3.9) \quad & \times F_{2:1;1}^{2:2;2} \left[\begin{array}{c} \frac{\mu+\lambda-\frac{b}{2}+2}{2}, \frac{\mu+\lambda-\frac{b}{2}+3}{2} : \quad \frac{\lambda-\frac{b}{2}-\mu+1}{2}, \frac{\lambda-\frac{b}{2}-\mu+2}{2} \quad ; \quad \frac{\mu-k+\frac{3}{2}}{2}, \frac{\mu-k+\frac{5}{2}}{2} \quad ; \\ \frac{\lambda-\frac{b}{2}-k+\frac{5}{2}}{2}, \frac{\lambda-\frac{b}{2}-k+\frac{7}{2}}{2} : \quad \frac{1}{2} \quad ; \quad \frac{3}{2} \quad ; \end{array} \quad -\left(\frac{2c\beta}{2p+\alpha} \right)^2, \left(\frac{2p-\alpha}{2p+\alpha} \right)^2 \right] \Bigg\}.
\end{aligned}$$

Remark. In a similar way, for some parametric replacement of b, c, ν, k and μ , we can easily establish some other integral transforms involving the product of Laguerre polynomial, Hermite polynomial and Modified Bessel function of second kind with Bessel function of first kind, Modified Bessel function of first kind, sine function and cosine function.

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ENTIRE FUNCTIONS SHARING TWO SMALLER ORDER ENTIRE FUNCTIONS WITH THEIR DIFFERENCE OPERATORS

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Abstract. In this paper we mainly study the uniqueness of entire functions with finite order sharing two smaller order entire functions with their difference operators. Our results improve some recent theorems due to Liu and Mao, Zhang and Liao.

Keywords: Difference Nevanlinna theory, entire function, sharing value, uniqueness.

1. Introduction and main results

In this paper, a meromorphic function always means it is meromorphic in the whole complex plane. We assume that the reader is familiar with Nevanlinna's value distribution theory (see [5, 7, 13, 14]) and its associated standard notions, such as $T(r, f)$, $m(r, f)$, $N(r, f)$, $\bar{N}(r, f)$, $\dots$. $S(r, f)$ denotes any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \rightarrow \infty$ outside some exceptional set of finite measure, not necessarily the same at each occurrence. A meromorphic function $a(z)$ is said to be a small function with respect to $f(z)$ iff $T(r, a) = S(r, f)$.

Let f and g be two nonconstant meromorphic functions, and let a be a value in the extended plane. We say that f and g share the value a IM, provided that f and g have the same a -points ignoring multiplicities. Moreover, we say that f and g share the value a CM, provided that f and g have the same a -points with the same multiplicities; see [13]. Suppose that b is a meromorphic function. If $f - b$ and $g - b$ share 0 CM, we say that f and g share b CM. If $f - b$ and $g - b$ share 0 IM, we say that f and g share b IM. In addition, we denote by $\mu(f)$, $\sigma(f)$ and $\lambda(f)$ the lower order of f , the order of f and the exponent of convergence of zeros of f respectively. If $\mu(f) = \sigma(f)$, we say that f is of regular growth.

1977, Rubel and Yang [12] proved

Theorem A. *Let f be a nonconstant entire function. If f and f' share two distinct finite values CM, then $f \equiv f'$.*

Later, Zheng and Wang [17] improved that the conclusion still holds if f and f' share two distinct non-infinite small functions $a(z)$ and $b(z)$ CM.

We define the difference operator $\Delta f = f(z + \eta) - f(z)$ as usual, where η is a nonzero complex constant. Recently, Nevanlinna theory for the difference operator and the difference analogue of logarithmic derivative lemma have been established; see [3, 4]. These new theories bring about a lot of papers (for example [1, 2, 8, 15]) focusing on the uniqueness of meromorphic functions sharing some values with their difference operators (or shifts). It is well known that Δf can be considered as the difference counterpart of f' in certain sense. For example, Chen and Yi [2] considered the problem that f and Δf sharing some values CM under the hypothesis that f has a finite Borel exceptional value. As an improvement, Zhang and Liao [16] proved the following result, which means the conclusion in Theorem A is still valid when f' is replaced by Δf .

Theorem B. *Let $f(z)$ be a transcendental entire function of finite order and a, b be two distinct constants. If f and $\Delta f (\not\equiv 0)$ share a, b CM, then $\Delta f \equiv f$. Furthermore, $f(z)$ must be of the following form $f(z) = 2^z h(z)$, where $h(z)$ is a periodic entire function with period 1.*

Regarding this theorem, a natural question is that what can be said if entire function f with finite order shares some smaller order entire functions with Δf ? This question has been studied in some recent papers [9, 10, 11]. Liu and Mao [11] obtained the following two results.

Theorem C. *Let $f(z)$ be a nonconstant entire function of finite order and $a(z)$ be an entire function of $\sigma(a) < \sigma(f)$. If f and Δf share $0, a(z)$ CM, then $\Delta f \equiv f$.*

Theorem D. *Let $f(z)$ be a nonconstant entire function of finite order and $a(z) (\not\equiv 0)$ be an entire function of $\sigma(a) < \sigma(f)$ and $\lambda(f - a) < \sigma(f)$. If f and Δf share $a(z)$ CM, then $\sigma(f) = 1$.*

In the present paper, we shall study the general case of Theorem C, that is, if f and Δf share two distinct smaller order entire functions $a(z), b(z)$ CM, does the conclusion still hold? In fact, we have

Theorem 1.1. *Let $f(z)$ be a nonconstant entire function of finite order and $a(z), b(z)$ be two entire functions with $a(z) \not\equiv b(z)$ and $\max\{\sigma(a), \sigma(b)\} < \sigma(f)$. If f and Δf share $a(z), b(z)$ CM, then $\Delta f \equiv f$.*

As to Theorem D, we get the following improvement.

Theorem 1.2. *Let $f(z)$ be a nonconstant entire function of finite order and $a(z)$ be an entire function with $\sigma(a) < \sigma(f)$ and $\lambda(f - a) < \sigma(f)$. If f and Δf share $a(z)$ CM, then $a(z) \equiv 0$ and $f(z)$ must be of the form $f(z) = e^{mz+n}$, where $m (\neq 0), n$ are two constants.*

2. Preliminary lemmas

Lemma 2.1 ([7]). *Let $f(z)$ be a nonconstant meromorphic function. Then*

$$m\left(r, \frac{f'}{f}\right) = O(\log r), \quad r \rightarrow \infty,$$

if f is of finite order, and

$$m\left(r, \frac{f'}{f}\right) = O(\log(rT(r, f))), \quad r \rightarrow \infty,$$

possibly outside a set E of r with finite linear measure if $f(z)$ is of infinite order.

Lemma 2.2. ([3]). *Let $f(z)$ be a meromorphic function in the complex plane with order $\sigma = \sigma(f) < \infty$, and let c be a fixed nonzero complex constant. Then, for each $\varepsilon > 0$, we have*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = O(r^{\sigma-1+\varepsilon}).$$

Remark 2.1 ([16, Remark 2.2]). The equation

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = O(r^{\sigma-1+\varepsilon}),$$

where σ is the finite order of f and $\varepsilon > 0$, implies

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = S(r, f)$$

possibly outside a set of finite logarithmic measure.

Lemma 2.3 ([3]). *Let $f(z)$ be a meromorphic function in the complex plane with order $\sigma = \sigma(f) < \infty$, and let η be a fixed nonzero complex constant. Then, for each $\varepsilon > 0$, we have*

$$T(r, f(z+\eta)) = T(r, f(z)) + O(r^{\sigma-1+\varepsilon}) + O(\log r), \text{ i.e.,}$$

$$T(r, f(z+\eta)) = T(r, f(z)) + S(r, f)$$

possibly outside a set of finite logarithmic measure.

It is evident that $S(r, f(z+\eta)) = S(r, f)$ from Lemma 2.3.

Lemma 2.4 ([13]). *Let $f(z)$ be a non-constant meromorphic function in the complex plane and*

$$R(f) = \frac{P(f)}{Q(f)},$$

where $P(f) = \sum_{k=0}^p a_k f^k$ and $Q(f) = \sum_{j=0}^q b_j f^j$ are two mutually prime polynomials in $f(z)$. If the coefficients a_k, b_j are small functions of $f(z)$ and $a_p \not\equiv 0, b_q \not\equiv 0$, then

$$T(r, R(f)) = \max\{p, q\}T(r, f) + S(r, f).$$

Lemma 2.5 ([13]). *Let $f_j(z) (j = 1, 2, \dots, n) (n \geq 2)$ be meromorphic functions and $g_j(z) (j = 1, 2, \dots, n)$ be entire functions such that*

- (1) $\sum_{j=1}^n f_j(z) \exp\{g_j(z)\} \equiv 0$;
- (2) when $1 \leq j < k \leq n$, $g_j(z) - g_k(z)$ is not constant;
- (3) when $1 \leq j \leq n, 1 \leq h < k \leq n$, $T(r, f_j) = o\{T(r, \exp\{g_h - g_k\})\}$, $(r \rightarrow \infty, r \notin E)$, where $E \subset (1, \infty)$ is of finite linear measure or finite logarithmic measure. Then $f_j(z) \equiv 0$ ($j = 1, \dots, n$).

Lemma 2.6 ([13, Theorem 2.11]). *Let f be a transcendental meromorphic function in the complex plane such that $\sigma(f) > 0$. If f has two distinct Borel exceptional values in the extended complex plane, then $\mu(f) = \sigma(f)$ and $\sigma(f)$ is a positive integer or ∞ .*

Lemma 2.7 ([6]). *Let $\varphi(r)$ be a nondecreasing, continuous function on $\mathbb{R}^+$, and let $0 < \rho < \limsup_{r \rightarrow \infty} \frac{\log \varphi(r)}{\log r}$ and $H = \{r \in \mathbb{R}^+ : |\varphi(r)| \geq r^\rho\}$. Then*

$$\overline{\log dens} H = \limsup_{r \rightarrow \infty} \frac{\int_{H \cup [1, r]} \frac{1}{t} dt}{\log r} > 0.$$

It's known that for two entire functions $f(z)$ and $a(z)$, $\sigma(f) > \sigma(a)$ can not guarantee $T(r, a) = o(T(r, f)) = S(r, f)$. However, we have the following result.

Lemma 2.8. *Let $f(z)$ and $a(z)$ be two entire functions, and $\sigma(f) > \sigma(a)$. Then $T(r, a) = o(T(r, f))$ holds for $r \in H$, which satisfies $\overline{\log dens} H > 0$.*

Proof. By the definition of the order entire function f , we know that

$$\limsup_{r \rightarrow \infty} \frac{\log T(r, f)}{\log r} = \sigma(f).$$

Set $\sigma(f) > \rho_1 > \rho_2 > \sigma(a)$ and applying Lemma 2.7 to $T(r, f)$, we get $T(r, f) \geq r^{\rho_1}$ holds for $r \in H$, which satisfies $\overline{\log dens} H > 0$. Moreover, by the definition of order of entire function $a(z)$, we have $T(r, a) < r^{\rho_2}$. Thus, $T(r, a) < r^{\rho_2} < r^{\rho_1} \leq T(r, f)$ holds when $r \in H$. Then we complete the proof. $\square$

Remark 2.2. We know the Nevanlinna characteristic function is non-decreasing function of r , then for any two entire functions $a(z)$ and $b(z)$, one of following three cases may occur: 1. $\lim_{r \rightarrow \infty} \frac{T(r, a(z))}{T(r, b(z))} = 0$, that is, $T(r, a(z)) = o(T(r, b(z)))$; 2. $\lim_{r \rightarrow \infty} \frac{T(r, a(z))}{T(r, b(z))} = \alpha$, $0 < \alpha < +\infty$; 3. $\lim_{r \rightarrow \infty} \frac{T(r, a(z))}{T(r, b(z))} = +\infty$, $T(r, b(z)) = o(T(r, a(z)))$.

3. Proof of Theorems

Proof of Theorem 1.1. Since f is entire function with finite order and shares two nonequivalent smaller order entire functions $a(z), b(z)$ CM with Δf , then there exists two polynomial $p(z)$ and $q(z)$ such that

$$(3.1) \quad \Delta f - a(z) = [f(z) - a(z)]e^{p(z)}$$

and

$$(3.2) \quad \Delta f - b(z) = [f(z) - b(z)]e^{q(z)}.$$

Since $\max\{\sigma(a), \sigma(b)\} < \sigma(f)$ and $\sigma(\Delta f) \leq \sigma(f)$, by (3.1) and (3.2) we have

$$(3.3) \quad \deg p(z) = \sigma(e^{p(z)}) \leq \sigma(f), \deg q(z) = \sigma(e^{q(z)}) \leq \sigma(f).$$

If $e^{p(z)} \equiv e^{q(z)}$, then we can deduce $f \equiv \Delta f$ easily by (3.1) and (3.2).

By solving f and Δf from (3.1) and (3.2), we get

$$(3.4) \quad f(z) = \frac{b(z) - a(z) + a(z)e^{p(z)} - b(z)e^{q(z)}}{e^{p(z)} - e^{q(z)}}$$

and

$$(3.5) \quad \Delta f = \frac{b(z)e^{p(z)} - a(z)e^{q(z)} + (a(z) - b(z))e^{p(z)+q(z)}}{e^{p(z)} - e^{q(z)}}.$$

If $e^{p(z)} \not\equiv e^{q(z)}$ and $p(z), q(z)$ are two distinct constants, then by (3.4) we have $\sigma(f) \leq \max\{\sigma(a), \sigma(b)\}$, which contradicts the assumption. If $e^{p(z)} \not\equiv e^{q(z)}$ and only one of $p(z), q(z)$ is constant. Without loss of generality, we assume $q(z)$ is a constant q . From (3.4), if z_0 is a zero of $e^{p(z)} - e^q$, then $a(z_0) - b(z_0) = 0$ or $e^{p(z_0)} = 1$. We claim $e^q = 1$. Otherwise, all the zeros of $e^{p(z)} - e^q$ must be zeros of $a(z) - b(z)$, that is,

$$(3.6) \quad N\left(r, \frac{1}{e^{p(z)} - e^q}\right) \leq N\left(r, \frac{1}{a(z) - b(z)}\right).$$

By the second main theorem of Nevanlinna theory, we get

$$(3.7) \quad \begin{aligned} T(r, e^{p(z)}) &\leq N\left(r, \frac{1}{e^{p(z)} - e^q}\right) + S(r, e^{p(z)}) \leq N\left(r, \frac{1}{a(z) - b(z)}\right) + S(r, e^{p(z)}) \\ &\leq 2 \max\{T(r, a(z)), T(r, b(z))\} + S(r, e^{p(z)}). \end{aligned}$$

Note that $q(z) = q$ is a constant, combining (3.4) and (3.7) we have $\sigma(f) < \max\{\sigma(a(z)), \sigma(b(z))\}$, which contradicts the assumption. Thus, $e^q = 1$, then by (3.2), we have $f \equiv \Delta f$. In the following, we just need to consider the remaining case that $e^{p(z)} \not\equiv e^{q(z)}$ and $p(z), q(z)$ both are nonconstant polynomials. If one

of $a(z), b(z)$ is zero, it reduces to Theorem C. Thus, we assume $a(z), b(z)$ are both nonzero entire functions.

Firstly, we set

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$$

and

$$q(z) = b_m z^m + b_{m-1} z^{m-1} + \cdots + b_0,$$

where $a_n (\neq 0), a_{n-1}, \dots, a_0$ and $b_m (\neq 0), b_{m-1}, \dots, b_0$ are constants. We claim

$$(3.8) \quad m = n \quad \text{and} \quad |a_n| = |b_m|.$$

Set $g(z) := f(z) - a(z)$, then (3.1) becomes

$$(3.9) \quad \Delta g + a(z + \eta) - 2a(z) = g(z)e^{p(z)}.$$

By differentiating (3.9) and eliminating $e^{p(z)}$, we obtain

$$(3.10) \quad A(z)g(z) + B(z) + [2a(z) - a(z + \eta)] \frac{g'(z)}{g(z)} = 0,$$

where

$$A(z) := \frac{g'(z + \eta)}{g(z)} - \frac{g'(z)g(z + \eta)}{g(z)^2} - p'(z) \frac{g(z + \eta)}{g(z)} + p'(z),$$

$$B(z) := a'(z + \eta) - 2a'(z) - a(z + \eta)p'(z) + 2a(z)p'(z).$$

If $A(z) \equiv 0$, then (3.10) becomes

$$(3.11) \quad B(z) + [2a(z) - a(z + \eta)] \frac{g'(z)}{g(z)} = 0.$$

Solving it, we get the solutions

$$(3.12) \quad g(z) = c[a(z + \eta) - 2a(z)]e^{-p(z)},$$

where c is a nonzero constant. Substituting (3.12) into (3.9) we get

$$(3.13) \quad \frac{a(z + 2\eta) - 2a(z + \eta)}{a(z + \eta) - 2a(z)} e^{-p(z+\eta)} = e^{-p(z)} + \frac{c-1}{c}.$$

If $c \neq 1$, then by (3.13) we get

$$(3.14) \quad N\left(r, \frac{1}{e^{-p(z)} - \frac{c-1}{c}}\right) \leq N\left(r, \frac{1}{a(z + 2\eta) - 2a(z + \eta)}\right).$$

By the first and second main theorem of Nevanlinna theory and Lemma 2.3, we have

$$\begin{aligned}
 T(r, e^{p(z)}) = T(r, e^{-p(z)}) + O(1) &\leq N\left(r, \frac{1}{e^{-p(z)} - \frac{c-1}{c}}\right) + S(r, e^{p(z)}) \\
 &\leq N\left(r, \frac{1}{a(z+2\eta) - 2a(z+\eta)}\right) + S(r, e^{p(z)}) \\
 (3.15) \qquad &\leq 3T(r, a(z)) + S(r, a(z)) + S(r, e^{p(z)})
 \end{aligned}$$

Thus, we have

$$(3.16) \qquad \deg p(z) = \sigma(e^{p(z)}) \leq \sigma(a).$$

Combining (3.12) and (3.16) we have $\sigma(g) \leq \sigma(a)$. Since $\sigma(f) > \sigma(a)$ and $g(z) = f(z) - a(z)$, we have $\sigma(g) = \sigma(f)$. Thus, we obtain a contradiction.

Then $c = 1$. By (3.13), we have $\Delta g \equiv 0$, that is, $\Delta f \equiv \Delta a$. Since $|\eta|$ can be sufficiently small and $f(z), a(z)$ are entire functions, by the definition of derivatives, we have $f'(z) = a'(z)$. Recalling $\sigma(f) = \sigma(f'), \sigma(a) = \sigma(a')$ and $\sigma(f) > \sigma(a)$, we obtain a contradiction again.

Thus, we have $A(z) \not\equiv 0$. By Lemma 2.8 and Remark 2.2, we declare in advance that the following equalities and inequalities concerning $S(r, f)$ hold for $r \in H$, which satisfies $\log dens H > 0$. By Lemma 2.1 and Lemma 2.2, we obtain

$$(3.17) \qquad m(r, A) = S(r, f).$$

By the definition of $A(z)$, we see

$$(3.18) \qquad N(r, A) \leq \overline{N}\left(r, \frac{1}{g}\right) + N\left(r, \frac{1}{g}\right).$$

By (3.10), (3.18), $\sigma(f) > \sigma(a)$ and the first main theorem of Nevanlinna theory, we have

$$\begin{aligned}
 m(r, g) \leq m\left(r, \frac{1}{A}\right) + S(r, f) &\leq T(r, A) + S(r, f) = N(r, A) + S(r, f) \\
 (3.19) \qquad &\leq \overline{N}\left(r, \frac{1}{g}\right) + N\left(r, \frac{1}{g}\right) + S(r, f).
 \end{aligned}$$

Since $g(z)$ is entire function, $N(r, g) = 0$. So, we have

$$(3.20) \quad m(r, g) = T(r, g) = T\left(r, \frac{1}{g}\right) + O(1) = m\left(r, \frac{1}{g}\right) + N\left(r, \frac{1}{g}\right) + O(1).$$

Thus, combining (3.19) and (3.20) we obtain

$$(3.21) \qquad m\left(r, \frac{1}{g}\right) \leq \overline{N}\left(r, \frac{1}{g}\right) + S(r, f).$$

Rewrite (3.9) as

$$(3.22) \quad \frac{g(z+\eta)}{g(z)} + \frac{a(z+\eta) - 2a(z)}{g(z)} = e^{p(z)} + 1.$$

By (3.22), we get

$$(3.23) \quad m(r, e^{p(z)}) \leq m\left(r, \frac{1}{g}\right) + S(r, f).$$

Since $e^{q(z)}$ is entire function, by the second main theorem of Nevanlinna theory, we can easily get

$$(3.24) \quad \overline{N}\left(r, \frac{1}{e^{q(z)} - 1}\right) = m(r, e^{q(z)}) + S(r, e^{q(z)}).$$

By (3.4), we deduce that

$$(3.25) \quad \begin{aligned} \overline{N}\left(r, \frac{1}{f-a}\right) &\leq \overline{N}\left(r, \frac{1}{e^{q(z)} - 1}\right) + \overline{N}\left(r, \frac{1}{a(z) - b(z)}\right) \\ &\leq \overline{N}\left(r, \frac{1}{e^{q(z)} - 1}\right) + S(r, f). \end{aligned}$$

Combining (3.21), (3.23), (3.24) and (3.25), we see

$$(3.26) \quad m(r, e^{p(z)}) \leq m(r, e^{q(z)}) + S(r, e^{q(z)}) + S(r, f).$$

Similarly, we have

$$(3.27) \quad m(r, e^{q(z)}) \leq m(r, e^{p(z)}) + S(r, e^{p(z)}) + S(r, f).$$

From (3.1) and Lemma 2.3, 2.4 we have

$$(3.28) \quad T(r, e^{p(z)}) \leq T(r, f) + S(r, f).$$

By (3.4) we obtain

$$(3.29) \quad T(r, f) \leq \max\{T(r, e^{p(z)}), T(r, e^{q(z)})\} + S(r, f).$$

Without loss of generality, suppose $\max\{T(r, e^{p(z)}), T(r, e^{q(z)})\} = T(r, e^{p(z)})$. Thus, we have

$$(3.30) \quad T(r, e^{p(z)}) = (1 + o(1))T(r, f), \quad S(r, e^{p(z)}) = S(r, f).$$

From (3.2) and Lemma 2.3, 2.4, we know that

$$(3.31) \quad T(r, e^{q(z)}) \leq T(r, f) + S(r, f).$$

Thus, $S(r, e^{q(z)}) = S(r, f)$. Combining (3.26) and (3.27), we have

$$(3.32) \quad m(r, e^{p(z)}) \leq m(r, e^{q(z)}) + S(r, f) \leq m(r, e^{p(z)}) + S(r, f)$$

that is,

$$(3.33) \quad T(r, e^{q(z)}) = T(r, e^{p(z)}) + S(r, f).$$

Combining (3.30) and (3.33), we deduce

$$(3.34) \quad S(r, f) = S(r, e^{p(z)}) = S(r, e^{q(z)})$$

and

$$(3.35) \quad m(r, e^{p(z)}) = m(r, e^{q(z)}) + S(r, e^{q(z)}) = m(r, e^{q(z)}) + S(r, e^{p(z)}).$$

Noting the fact that $m(r, e^{a_n z^n}) = \frac{|a_n| r^n}{\pi}$, we have, for all $r \in (0, +\infty)$,

$$(3.36) \quad m(r, e^{p(z)}) = \frac{|a_n| r^n}{\pi} (1 + o(1)), \quad m(r, e^{q(z)}) = \frac{|b_m| r^m}{\pi} (1 + o(1)).$$

Thus, from (3.35) and (3.36) we proved (3.8).

For simplicity, denote $\bar{a} := a(z + \eta)$, $a := a(z)$, so do $\bar{b}, \bar{p}, \bar{q}$. By (3.4) we have

$$(3.37) \quad \Delta f = \frac{\bar{b} - \bar{a} + \bar{a}e^{\bar{p}} - \bar{b}e^{\bar{q}}}{e^{\bar{p}} - e^{\bar{q}}} - \frac{b - a + ae^p - be^q}{e^p - e^q}.$$

Combining the above identity with (3.5), we obtain that

$$(3.38) \quad C_1 e^p + C_2 e^q + C_3 e^{2p} + C_4 e^{2q} + C_5 e^{2p+q} + C_6 e^{p+2q} + C_7 e^{p+q} = 0,$$

where

$$\begin{aligned} C_1 &:= \bar{b} - \bar{a} + (a - b)e^{\Delta p}; \\ C_2 &:= \bar{a} - \bar{b} + (b - a)e^{\Delta q}; \\ C_3 &:= (\bar{a} - a - b)e^{\Delta p}; \\ C_4 &:= (\bar{b} - a - b)e^{\Delta q}; \\ C_5 &:= (b - a)e^{\Delta p}; \\ C_6 &:= (a - b)e^{\Delta q}; \\ C_7 &:= (a + b - \bar{a})e^{\Delta p} + (a + b - \bar{b})e^{\Delta q}. \end{aligned}$$

Our key point is applying Lemma 2.5 to (3.38). In order to do this, we need to verify (3.38) satisfies the conditions of this lemma. We need two steps.

Step 1, we shall show the polynomials $p, q, 2p, 2q, 2p+q, p+2q, p+q$, which means $g_j(z)$ in Lemma 2.5, have the same degree n as the polynomial p . Since $m = n$ and $|a_n| = |b_m|$, we just need to show that the degree of $p+q$ is n .

Suppose that the degree of $p + q$ is $k < n$, then it must have $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$ and $q(z) = -a_n z^n + b_{n-1} z^{n-1} + \cdots + b_0$. Substituting them into (3.38), we deduce $C_3 e^{A_3} e^{4a_n z^n} + (C_1 e^{A_1} + C_5 e^{A_5}) e^{3a_n z^n} + C_7 e^{A_7} e^{2a_n z^n} + (C_2 e^{A_2} + C_6 e^{A_6}) e^{a_n z^n} + C_4 e^{A_4} = 0$, where A_j ($j = 1, 2, \dots, 7$) are polynomials with degree at most $n - 1$. Denote $H := e^{a_n z^n}$, we have

$$(3.39) \quad \begin{aligned} & C_3 e^{A_3} H^4 + (C_1 e^{A_1} + C_5 e^{A_5}) H^3 + C_7 e^{A_7} H^2 \\ & + (C_2 e^{A_2} + C_6 e^{A_6}) H + C_4 e^{A_4} = 0. \end{aligned}$$

Applying Lemma 2.5 to (3.39), we get $C_3 \equiv C_4 \equiv 0$, that is, $\bar{a} \equiv \bar{b}$. For arbitrary z , let $\eta \rightarrow 0$ we have $a(z) \equiv b(z)$ by the continuous of functions $a(z)$ and $b(z)$. This is impossible. Therefore, $\deg(p + q) = n$.

Step 2, we shall show that $p, q, 2p, 2q, p - q, p + q, 2p + q, p + 2q, \dots$, which mean $g_j - g_k$ in Lemma 2.5, are polynomials with degree n . From step 1 and $m = n, |a_n| = |b_m|$, we just need to consider polynomial $p - q$. Suppose that the degree of $p - q$ is $k < n$, it must be $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$ and $q(z) = a_n z^n + b_{n-1} z^{n-1} + \cdots + b_0$. Denote $r(z) := p(z) - q(z)$, then $\deg r(z) = k < n$. Substituting $r(z)$ into (3.38), we get

$$(3.40) \quad C_1 e^r e^q + C_2 e^q + C_3 e^{2r} e^{2q} + C_4 e^{2q} + C_5 e^{2r} e^{3q} + C_6 e^r e^{3q} + C_7 e^r e^{2q} = 0,$$

that is,

$$(3.41) \quad (C_1 e^r + C_2) e^q + (C_4 + C_7 e^r + C_3 e^{2r}) e^{2q} + (C_5 e^{2r} + C_6 e^r) e^{3q} = 0.$$

By Lemma 2.5 and (3.41), we obtain

$$(3.42) \quad C_1 e^r + C_2 \equiv C_5 e^{2r} + C_6 e^r \equiv 0.$$

Since $a(z) \not\equiv b(z)$, we obtain $e^{r(z)} \equiv 1$, i.e., $e^{p(z)} \equiv e^{q(z)}$, which is a contradiction. So, $\deg(p - q) = n$.

Finally, applying Lemma 2.5 to (3.38), we see

$$(3.43) \quad C_j \equiv 0, \quad (j = 1, 2, \dots, 7),$$

which means $a(z) \equiv b(z)$. It obviously contradicts the assumption. Thus, we complete the proof.

Proof of Theorem 1.2. Suppose that $a(z) \not\equiv 0$, then by Theorem D, we have $\sigma(f) = 1$, then $\sigma(a) < \sigma(f) = 1$. By the assumption and as in the proof of Theorem 1.1, there exists a polynomial $p(z)$ such that

$$(3.44) \quad \Delta f - a(z) = [f(z) - a(z)] e^{p(z)}$$

and $\deg p(z) = \sigma(e^{p(z)}) \leq \sigma(f)$. Since $\lambda(f - a) < \sigma(f)$ and the fact $\sigma(f - a) = \sigma(f) = 1$, by Lemma 2.6 and Hadamard factorization theorem, there exists an entire function $H(z)$ and a polynomial $h(z)$ such that

$$(3.45) \quad f(z) - a(z) = H(z) e^{h(z)}$$

and $\sigma(H) = \lambda(H) = \lambda(f - a) < \sigma(f) = 1$. Thus, we have $\deg h = \sigma(f) = 1$. We set $h(z) := sz + t$, where $s(\neq 0), t$ are constants. By (3.45) we can obtain

$$(3.46) \quad e^{h(z)}[H(z + \eta)e^{s\eta} - H(z)] + a(z + \eta) - 2a(z) = H(z)e^{p(z)+h(z)}.$$

If $a(z + \eta) - 2a(z) \not\equiv 0$, from (3.46) and $\sigma(H) < 1, \sigma(a(z)) < 1$, we have

$$(3.47) \quad N\left(r, \frac{1}{e^{h(z)} + \frac{a(z+\eta)-2a(z)}{w(z)}}\right) = N\left(r, \frac{1}{H(z)}\right) \leq T(r, H(z)) = S(r, e^{h(z)}),$$

where $w(z) := H(z + \eta)e^{s\eta} - H(z)$. Moreover, by the second main theorem of Nevanlinna theory to small functions [13, Theorem 1.36], we have

$$(3.48) \quad (1 - \varepsilon)T(r, e^{h(z)}) \leq N\left(r, \frac{1}{e^{h(z)} - \frac{a(z+\eta)-2a(z)}{w(z)}}\right) + S(r, e^{h(z)}),$$

where ε is any positive constant. From (3.47), we know it is impossible.

If $a(z + \eta) - 2a(z) \equiv 0$, i.e., $a(z + \eta) \equiv 2a(z)$. Suppose $a(z)$ is nonconstant function, since $a(z)$ is entire and $\sigma(a) < 1$, we can set z_0 is a zero of it, then $z_0 + \eta, z_0 + 2\eta, z_0 + 3\eta, \dots$ are zeros of $a(z)$. Thus,

$$(3.49) \quad T(r, a(z)) \geq N\left(r, \frac{1}{a(z)}\right) \geq cr,$$

where c is a positive constant. By (3.49), we get $\sigma(a(z)) \geq 1$, this is also impossible. Therefore, $a(z)$ is a constant. We get $a = 0$ immediately, which contradicts the assumption in the beginning.

So $a(z) \equiv 0$. Similar as the arguments in case 2, proof of Theorem 1.3 in [16], we get $f(z)$ must be the form of $f(z) = e^{mz+n}$, where $m(\neq 0), n$ are constants.

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SOME VERTEX-DEGREE-BASED TOPOLOGICAL INDICES UNDER EDGE CORONA PRODUCT

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Abstract. A topological index is called vertex-degree-based if it can be defined by vertex degrees. The harmonic, atom-bond connectivity and Randić indices are three important examples of such topological indices. The aim of this paper is to find lower and upper bounds for Randić, harmonic and atom-bond connectivity indices of edge corona product of graphs. Some closed formulas are obtained when the factors are regular graphs.

Keywords: Edge corona product, Randić index, harmonic index, atom-bond connectivity index.

1. Introduction and preliminaries

Suppose G is a simple graph with vertex set $V(G)$ and edge set $E(G)$. The Randić index of G , $R(G)$, is defined as the sum of $\frac{1}{\sqrt{\deg(v_i)\deg(v_j)}}$ over all edges $uv \in E(G)$, where $\deg(x)$, as a short $d(x)$, denotes the degree of a vertex x in G [10]. This parameter, sometimes referred to as connectivity index, has been used to characterize the degree of branching of organic compounds. As an example, this number successfully explained the occurrence of critical alloy compositions in 18 different binary alloys [9]. The higher order Randić indices are also of interest in chemical graph theory. For $h \geq 1$, the h -th order Randić

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index $R_h(G)$ of G is the sum of the term $\frac{1}{\sqrt{\deg(v_{i_1})\deg(v_{i_2})\dots\deg(v_{i_{h+1}})}}$ overall paths $v_{i_1}, v_{i_2}, \dots, v_{i_{h+1}}$ of length h contained as a subgraph in G [5, 6]. The case that $h = 1$ is ordinary Randić index. We encourage the reader to consult [7, 8, 15] and references therein for more information on this topic. The Harmonic index of a graph G , $H(G)$, is defined as $H(G) = \sum_{uv \in E(G)} \frac{2}{d_G(u) + d_G(v)}$. As far as we know, this graph invariant first appeared in [3]. The atom-bond connectivity index (ABC index for short) was introduced by Ernesto Estrada et al. for studying the stability of alkanes and the strain energy of cycloalkanes [2]. This index can be defined as $ABC(G) = \sum_{uv \in E(G)} \sqrt{\frac{d(u) + d(v) - 2}{d(u)d(v)}}$.

Shwetha Shetty [11] obtained exact formulas for the Harary index of join, corona product, Cartesian product, composition and symmetric difference of graphs. Zhong [16] obtained the minimum and maximum values of the harmonic index for simple connected graphs and trees. He also characterized the corresponding extremal graphs. Xu [12] established some relationships between harmonic, Randić and ABC indices of graphs.

The edge corona product of two graphs G and H , $G \diamond H$, is a graph obtained by taking a copy of G and $|E(G)|$ copies of H and joining each end vertex of i -th edge of G to every vertex in the i -th copy of H [1, 4]. Following Yan et al. [13], the graph $R(G)$ is obtained from G by adding a new vertex corresponding to each edge of G , then joining each new vertex to the end vertices of the corresponding edge. Another way to describe $R(G)$ is to replace each edge of G by a triangle. It is clear that if G is a graph and H is a trivial graph, then $G \diamond H \cong R(G)$. In [14], Yero et al. studied the Randić index of corona product of graphs. In this paper, we continue this work by computing the Randić index of edge corona product of graphs.

Throughout this paper our notation is standard. A k -regular graph is a graph in which degree of each vertex is equal to k . In a graph G with at least one cycle, the length of a longest cycle is called its circumference and the length of a shortest cycle its girth. Our other notions are standard and can be taken from the standard books on graph theory.

2. Main results

The aim of this section is computing the Randić, atom bond connectivity and harmonic indices of edge corona product of graphs. The following simple lemma is crucial in our results.

Lemma 2.1. *By definition of edge corona we have:*

*If $x \in V(G_1)$ then $d_{G_1 \diamond G_2}(x) = d_{G_1}(x)(n_2 + 1)$,
If $x \in V(G_2)$ then $d_{G_1 \diamond G_2}(x) = d_{G_2}(x) + 2$.*

Theorem 2.2. *Let G_1 and G_2 be graphs. Thus, we have*

$$R_1(G_1 \diamond G_2) \geq \frac{m_1}{\Delta_1(1 + n_2)} + \frac{m_1 m_2}{\Delta_2 + 2} + \frac{2m_1 n_2}{\sqrt{\Delta_1(\Delta_2 + 2)(1 + n_2)}},$$

$$R_1(G_1 \diamond G_2) \leq \frac{m_1}{\delta_1(1+n_2)} + \frac{m_1 m_2}{\delta_2 + 2} + \frac{2m_1 n_2}{\sqrt{\delta_1(\delta_2 + 2)(1+n_2)}}.$$

Proof. Let

$$\begin{aligned} A_1 &= \sum_{ab \in E(G_1)} \frac{1}{\sqrt{d(a)d(b)(n_2+1)^2}} \geq \frac{m_1}{\Delta_1(1+n_2)}. \\ A_2 &= \sum_{uv \in E(G_2)} \frac{1}{\sqrt{(d(u)+2)(d(v)+2)}} \geq \frac{m_1 m_2}{\Delta_2 + 2}. \\ A_3 &= \sum_{a \in V(G_1), u \in V(G_2)} \frac{1}{\sqrt{d(a)(n_2+1)(d(u)+2)}} \geq \frac{2m_1 n_2}{\sqrt{\Delta_1(\Delta_2 + 2)(1+n_2)}}. \end{aligned}$$

By summation of A_1 , A_2 and A_3 , the result can be proved.

Corollary 2.3. For $i \in \{1, 2\}$, if G_i be δ_i -regular graph of order n_i , then

$$R_1(G_1 \diamond G_2) = \frac{m_1}{\delta_1(1+n_2)} + \frac{m_1 m_2}{\delta_2 + 2} + \frac{2m_1 n_2}{\sqrt{\delta_1(\delta_2 + 2)(1+n_2)}}.$$

Theorem 2.4. Let G_1 and G_2 be graphs. Then

$$\begin{aligned} R_2(G_1 \diamond G_2) &\geq \frac{n_1}{\Delta_2 + 2} \sqrt{\frac{\delta_1}{n_2 + 1}} \left(\frac{n_2(n_2 - 1)}{2} + 2m_2 + \frac{(\delta_1 - 1)n_2^2}{2} \right) \\ &\quad + \frac{m_1 n_2}{(n_2 + 1)\Delta_1 \sqrt{\Delta_2 + 2}} (2\delta_1 + 1) \\ &\quad + \frac{1}{2(n_2 + 1)^{3/2} \Delta_1} \left(\sum_{d_{G_1}(v_i) \geq 2} \frac{d_{G_1}(v_i)(d_{G_1}(v_i) - 1)}{\sqrt{d_{G_1}(v_i)}} \right) \\ &\quad + \frac{1}{2(\Delta_2 + 2)} \left(\sum_{d_{G_2}(u_i) \geq 2} \frac{d_{G_2}(u_i)(d_{G_2}(u_i) - 1)}{\sqrt{d_{G_2}(u_i) + 2}} \right). \\ R_2(G_1 \diamond G_2) &\leq \frac{n_1}{\delta_2 + 2} \sqrt{\frac{\Delta_1}{n_2 + 1}} \left(\frac{n_2(n_2 - 1)}{2} + 2m_2 + \frac{(\Delta_1 - 1)n_2^2}{2} \right) \\ &\quad + \frac{m_1 n_2}{(n_2 + 1)\delta_1 \sqrt{\delta_2 + 2}} (2\Delta_1 + 1) \\ &\quad + \frac{1}{2(n_2 + 1)^{3/2} \delta_1} \left(\sum_{d_{G_1}(v_i) \geq 2} \frac{d_{G_1}(v_i)(d_{G_1}(v_i) - 1)}{\sqrt{d_{G_1}(v_i)}} \right) \\ &\quad + \frac{1}{2(\delta_2 + 2)} \left(\sum_{d_{G_2}(u_i) \geq 2} \frac{d_{G_2}(u_i)(d_{G_2}(u_i) - 1)}{\sqrt{d_{G_2}(u_i) + 2}} \right). \end{aligned}$$

Proof. Suppose $V_1 = \{v_1, v_2, \dots, v_{n_1}\}$ and $V_2 = \{u_1, u_2, \dots, u_{n_2}\}$ are the vertex set of G_1 and G_2 , respectively. For $v \in V_i$, $N_{G_i}(v)$ is the set of all adjacent vertices to v in G_i . Suppose $\rho_h(G)$ denotes the set of all paths of length h in G and Consider the following partition of paths of length two in $G_1 \diamond G_2$.

1. Paths $u_i v_j u_k, i \neq k$ where $u_i, u_k \in V_2$ and $v_j \in V_1$,
2. Paths $u_i v_j v_k, j \neq k$ where $v_j, v_k \in V_1$ and $u_i \in V_2$,
3. Paths $v_i u_j u_k, j \neq k$ where $u_j, u_k \in V_2$ and $v_i \in V_1$,
4. Paths $u_i v_j u_k$, where $u_i, u_k \in V_2$ (two different copies of G_2) and $v_j \in V_1$,
5. Paths $v_i u_j v_k, i \neq k$ where $u_j \in V_2, v_i, v_k \in V_1$ and $v_i v_k \in E(G_1)$,
6. Paths of length two belonging to G_1 ,
7. Paths of length two belonging to m_1 copies of G_2 .

Define T_i to be the set of all paths of type $i, 1 \leq i \leq 7$. Therefore, $R_2(G_1 \diamond G_2) = \sum_{i=1}^7 A_i$, where

$$\begin{aligned}
A_1 &= \sum_{u_{i_1} u_{i_2} u_{i_3} \in \{P \in \rho_2(G_1 \diamond G_2) \mid P \in T_1\}} \frac{1}{\sqrt{d(u_{i_1})d(u_{i_2})d(u_{i_3})}} \\
&= \sum_{u_i, u_k \in V_2, v_j \in V_1} \frac{1}{\sqrt{(d_{G_2}(u_i) + 2)(d_{G_2}(u_k) + 2)(n_2 + 1)d_{G_1}(v_j)}} \\
&= \sum_{j=1}^{n_1} \frac{d_{G_1}(v_j)}{\sqrt{d_{G_1}(v_j)(n_2 + 1)}} \cdot \sum_{i=1}^{n_2-1} \sum_{k=i+1}^{n_2} \frac{1}{\sqrt{(d_{G_2}(u_i) + 2)(d_{G_2}(u_k) + 2)}} \\
&\geq \frac{n_1 n_2 (n_2 - 1)}{2(\Delta_2 + 2)} \cdot \sqrt{\frac{\delta_1}{n_2 + 1}}, \\
A_2 &= \sum_{u_{i_1} u_{i_2} u_{i_3} \in \{P \in \rho_2(G_1 \diamond G_2) \mid P \in T_2\}} \frac{1}{\sqrt{d(u_{i_1})d(u_{i_2})d(u_{i_3})}} \\
&= \sum_{v_j, v_k \in V_1, u_i \in V_2} \frac{1}{\sqrt{(d_{G_2}(u_i) + 2)d_{G_1}(v_j)d_{G_1}(v_k)(n_2 + 1)^2}} \\
&= \frac{1}{n_2 + 1} \left(\sum_{i=1}^{n_2} \frac{1}{\sqrt{d_{G_2}(u_i) + 2}} \cdot \sum_{j=1}^{n_1} d_{G_1}(v_j) \sum_{v_k \in N_{G_1}(v_j)} \frac{1}{\sqrt{d_{G_1}(v_j)d_{G_1}(v_k)}} \right) \\
&\geq \frac{2\delta_1 m_1 n_2}{(n_2 + 1)\Delta_1 \sqrt{\Delta_2 + 2}}, \\
A_3 &= \sum_{u_{i_1} u_{i_2} u_{i_3} \in \{P \in \rho_2(G_1 \diamond G_2) \mid P \in T_3\}} \frac{1}{\sqrt{d(u_{i_1})d(u_{i_2})d(u_{i_3})}} \\
&= \sum_{u_j, u_k \in V_2, v_i \in V_1} \frac{1}{\sqrt{d_{G_1}(v_i)(n_2 + 1)(d_{G_2}(u_j) + 2)(d_{G_2}(u_k) + 2)}}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{i=1}^{n_1} \frac{d_{G_1}(v_i)}{\sqrt{d_{G_1}(v_i)(n_2+1)}} \cdot \sum_{j=1}^{n_2} \sum_{u_k \in N_{G_2}(u_j)} \frac{1}{\sqrt{(d_{G_2}(u_j)+2)(d_{G_2}(u_k)+2)}} \\
&\geq \frac{2n_1m_2\sqrt{\delta_1}}{(\Delta_2+2)\sqrt{1+n_2}}, \\
A_4 &= \sum_{u_{i_1}u_{i_2}u_{i_3} \in \{P \in \rho_2(G_1 \diamond G_2) \mid P \in T_4\}} \frac{1}{\sqrt{d(u_{i_1})d(u_{i_2})d(u_{i_3})}} \\
&= \sum_{u_i \in V_{2_t}, u_k \in V_{2_l} (l \neq t), v_j \in V_1} \frac{1}{\sqrt{(d_{G_2}(u_i)+2)(n_2+1)d_{G_1}(v_j)(d_{G_2}(u_k)+2)}} \\
&= \sum_{j=1}^{n_1} \frac{d_{G_1}(v_j)(d_{G_1}(v_j)-1)}{2\sqrt{d_{G_1}(v_j)(n_2+1)}} \cdot \sum_{i=1}^{n_2} \sum_{k=1}^{n_2} \frac{1}{\sqrt{(d_{G_2}(u_i)+2)(d_{G_2}(u_k)+2)}} \\
&\geq \frac{n_1n_2^2(\delta_1-1)}{2(\Delta_2+2)} \cdot \sqrt{\frac{\delta_1}{(n_2+1)}}, \\
A_5 &= \sum_{u_{i_1}u_{i_2}u_{i_3} \in \{P \in \rho_2(G_1 \diamond G_2) \mid P \in T_5\}} \frac{1}{\sqrt{d(u_{i_1})d(u_{i_2})d(u_{i_3})}} \\
&= \sum_{u_j \in V_2, v_i v_k \in E(G_1)} \frac{1}{\sqrt{d_{G_1}(v_i)d_{G_1}(v_k)(n_2+1)^2(d_{G_2}(u_j)+2)}} \\
&= \frac{1}{n_2+1} \sum_{v_i v_k \in E(G_1)} \left(\frac{1}{\sqrt{d_{G_1}(v_i)d_{G_1}(v_k)}} \cdot \sum_{j=1}^{n_2} \frac{1}{\sqrt{d_{G_2}(u_j)+2}} \right) \\
&\geq \frac{m_1n_2}{(n_2+1)\Delta_1\sqrt{\Delta_2+2}}, \\
A_6 &= \sum_{u_{i_1}u_{i_2}u_{i_3} \in \{P \in \rho_2(G_1 \diamond G_2) \mid P \in T_6\}} \frac{1}{\sqrt{d(u_{i_1})d(u_{i_2})d(u_{i_3})}} \\
&= \sum_{v_i v_j v_k \in \rho_2(G_1)} \frac{1}{\sqrt{d_{G_1}(v_i)d_{G_1}(v_j)d_{G_1}(v_k)(n_2+1)^3}} \\
&= \frac{1}{(n_2+1)^{3/2}} \sum_{v_i v_j v_k \in \rho_2(G_1)} \frac{1}{\sqrt{d_{G_1}(v_i)d_{G_1}(v_j)d_{G_1}(v_k)}} \\
&\geq \frac{1}{2(n_2+1)^{3/2}\Delta_1} \sum_{d_{G_1}(v_i) \geq 2} \frac{d_{G_1}(v_i)(d_{G_1}(v_i)-1)}{\sqrt{d_{G_1}(v_i)}}, \\
A_7 &= \sum_{u_{i_1}u_{i_2}u_{i_3} \in \{P \in \rho_2(G_1 \diamond G_2) \mid P \in T_7\}} \frac{1}{\sqrt{d(u_{i_1})d(u_{i_2})d(u_{i_3})}} \\
&= \sum_{u_i u_j u_k \in \rho_2(G_2)} \frac{1}{\sqrt{(d_{G_2}(u_i)+2)(d_{G_2}(u_j)+2)(d_{G_2}(u_k)+2)}}
\end{aligned}$$

$$\geq \frac{1}{2(\Delta_2 + 2)} \sum_{d_{G_2}(u_i) \geq 2} \frac{d_{G_2}(u_i)(d_{G_2}(u_i) - 1)}{\sqrt{d_{G_2}(u_i) + 2}}.$$

Now a simple calculations will complete the proof.

Corollary 2.5. *For $i \in \{1, 2\}$, if G_i be δ_i -regular graph of order n_i , then*

$$\begin{aligned} R_2(G_1 \diamond G_2) &= \frac{n_1}{\delta_2 + 2} \sqrt{\frac{\delta_1}{n_2 + 1}} \left(2m_2 + \frac{n_2(n_2 - 1)}{2} + \frac{(\delta_1 - 1)n_2^2}{2} \right) \\ &\quad + \frac{m_1 n_2}{(n_2 + 1)\delta_1 \sqrt{\delta_2 + 2}} (2\delta_1 + 1) \\ &\quad + \frac{1}{2(n_2 + 1)^{3/2} \delta_1} \sum_{d_{G_1}(v_i) \geq 2} \frac{d_{G_1}(v_i)(d_{G_1}(v_i) - 1)}{\sqrt{d_{G_1}(v_i)}} \\ &\quad + \frac{1}{2(\delta_2 + 2)} \sum_{d_{G_2}(u_i) \geq 2} \frac{d_{G_2}(u_i)(d_{G_2}(u_i) - 1)}{\sqrt{d_{G_2}(u_i) + 2}}. \end{aligned}$$

To prove our main result, we state an important result of [14].

Lemma 2.6 ([14]). *Let $G = (V, E)$ be a graph with girth $g(G)$. If $\delta \geq 2$ and $g(G) > h$, then the number of paths of length h in G is bounded by*

$$\frac{(\delta - 1)^{h-2}}{2} \sum_{u \in V} d(u)(d(u) - 1) \leq |\rho_h(G)| \leq \frac{(\Delta - 1)^{h-2}}{2} \sum_{u \in V} d(u)(d(u) - 1).$$

In what follows, let N_k denote the empty graph of order k .

Theorem 2.7. *Let $G = (V, E)$ be a graph with girth $g(G)$, minimum degree δ , and maximum degree Δ . If $\delta \geq 2$ and $g(G) > h \geq 3$, then*

$$\begin{aligned} R_h(G \diamond N_k) &\leq \left(\frac{(\Delta - 1)^{h-2}}{2\sqrt{\delta^{h-1}(k+1)^{h-1}}} \left(\frac{k^2}{2} + \frac{1}{\delta(k+1)} + \frac{2k(h+1)}{\sqrt{2(k+1)\delta}} \right) \right. \\ &\quad \left. + \frac{k(k-1)(\Delta - 1)^{h-3}}{8\sqrt{(k+1)^{h-1}\delta^{h-1}}} \right) \cdot \sum_{u \in V} d(u)(d(u) - 1), \\ R_h(G \diamond N_k) &\geq \left(\frac{(\delta - 1)^{h-2}}{2\sqrt{\Delta^{h-1}(k+1)^{h-1}}} \left(\frac{k^2}{2} + \frac{1}{\Delta(k+1)} + \frac{2k(h+1)}{\sqrt{2(k+1)\Delta}} \right) \right. \\ &\quad \left. + \frac{k(k-1)(\delta - 1)^{h-3}}{8\sqrt{(k+1)^{h-1}\Delta^{h-1}}} \right) \cdot \sum_{u \in V} d(u)(d(u) - 1). \end{aligned}$$

Proof. The paths of length h in G contribute to $R_h(G \diamond N_k)$ in

$$\sum_{v_{i_1} v_{i_2} \dots v_{i_{h+1}} \in \rho_h(G)} \frac{1}{\sqrt{(k+1)^{h+1} \prod_{l=1}^{h+1} d_G(v_{i_l})}}.$$

Moreover, each path of length h in G leads to $2k+k^2$ paths of length h in $G \diamond N_k$; thus, the paths of length h in G contribute to $R_h(G \diamond N_k)$ in

$$\sum_{v_{i_1} v_{i_2} \dots v_{i_{h+1}} \in \rho_h(G)} \left(\frac{k}{\sqrt{2(k+1)^h \prod_{l=1}^h d(v_{i_l})}} + \frac{k}{\sqrt{2(k+1)^h \prod_{l=2}^{h+1} d(v_{i_l})}} + \frac{k^2}{\sqrt{4(k+1)^{h-1} \prod_{l=2}^h d(v_{i_l})}} \right).$$

Furthermore, each cycle of length h in G leads to $2kh$ paths of length h in $G \diamond N_k$ and also each cycle of length $h-1$ in G leads to $(h-1)\binom{k}{2}$ paths of length h in $G \diamond N_k$; thus, the cycle of length h and $h-1$ in G contribute to $R_h(G \diamond N_k)$ in

$$\sum_{v_{i_1} v_{i_2} \dots v_{i_h} v_{i_1} \in \zeta_h(G)} \frac{2kh}{\sqrt{2(k+1)^h \prod_{l=1}^h d(v_{i_l})}},$$

$$\sum_{v_{i_1} v_{i_2} \dots v_{i_{h-1}} v_{i_1} \in \zeta_{h-1}(G)} \frac{(h-1)\binom{k}{2}}{\sqrt{4(k+1)^{h-1} \prod_{l=1}^{h-1} d(v_{i_l})}},$$

respectively, where $\zeta_h(G)$ denotes the set of cycles of length h contained (as subgraphs) in G . So,

$$\begin{aligned} & R_h(G \diamond N_k) \\ &= \sum_{v_{i_1} v_{i_2} \dots v_{i_{h+1}} \in \rho_h(G)} \left(\frac{1}{\sqrt{(k+1)^{h+1} \prod_{l=1}^{h+1} d(v_{i_l})}} + \frac{k}{\sqrt{2(k+1)^h \prod_{l=1}^h d(v_{i_l})}} \right. \\ & \quad \left. + \frac{k}{\sqrt{2(k+1)^h \prod_{l=2}^{h+1} d(v_{i_l})}} + \frac{k^2}{\sqrt{4(k+1)^{h-1} \prod_{l=2}^h d(v_{i_l})}} \right) \\ &+ \sum_{v_{i_1} v_{i_2} \dots v_{i_h} v_{i_1} \in \zeta_h(G)} \frac{2kh}{\sqrt{2(k+1)^h \prod_{l=1}^h d(v_{i_l})}} \\ &+ \sum_{v_{i_1} v_{i_2} \dots v_{i_{h-1}} v_{i_1} \in \zeta_{h-1}(G)} \frac{(h-1)\binom{k}{2}}{\sqrt{4(k+1)^{h-1} \prod_{l=1}^{h-1} d(v_{i_l})}} \\ &\leq |\rho_h(G)| \left(\frac{1}{\sqrt{(k+1)^{h+1} \delta^{h+1}}} + \frac{2k}{\sqrt{2(k+1)^h \delta^h}} + \frac{k^2}{\sqrt{4(k+1)^{h-1} \delta^{h-1}}} \right) \\ &+ |\zeta_h(G)| \frac{2kh}{\sqrt{2(k+1)^h \delta^h}} + |\zeta_{h-1}(G)| \frac{(h-1)\binom{k}{2}}{\sqrt{4(k+1)^{h-1} \delta^{h-1}}}. \end{aligned}$$

By taking into account that $|\zeta_h(G)| \leq |\rho_h(G)|$, $|\zeta_{h-1}(G)| \leq |\rho_{h-1}(G)|$ and Lemma 2.6 we obtain the upper bound and the lower bound.

Theorem 2.8. *Let G_1 and G_2 be graphs. Thus, we have*

$$\begin{aligned} H(G_1 \diamond G_2) &\geq \frac{1}{1+n_2} H(G_1) + \frac{m_1 m_2}{\Delta_2 + 2} + \frac{2m_1 n_2}{\Delta_1(n_2 + 1) + \Delta_2 + 2}, \\ H(G_1 \diamond G_2) &\leq \frac{1}{1+n_2} H(G_1) + \frac{m_1 m_2}{\delta_2 + 2} + \frac{2m_1 n_2}{\delta_1(n_2 + 1) + \delta_2 + 2}. \end{aligned}$$

Proof. The edges of $G_1 \diamond G_2$ are partitioned into three subsets E_1 , E_2 and E_3 as follows:

$$\begin{aligned} E_1 &= \{e \in E(G_1 \diamond G_2) | e \in E(G_1)\}, \\ E_2 &= \{e \in E(G_1 \diamond G_2) | e \in E(G_{2_i}), i = 1, 2, \dots, |E(G_1)|\}, \\ E_3 &= \{e \in E(G_1 \diamond G_2) | e = uv, u \in V(G_{2_i}), i = 1, 2, \dots, |E(G_1)|, v \in V(G_1)\}. \end{aligned}$$

Therefore,

$$H(G_1 \diamond G_2) = \sum_{uv \in E(G_1 \diamond G_2)} \frac{2}{d_{G_1 \diamond G_2}(u) + d_{G_1 \diamond G_2}(v)} = A_1 + A_2 + A_3.$$

Where

$$\begin{aligned} A_1 &= \sum_{uv \in E_1} \frac{2}{d(u) + d(v)} = \sum_{uv \in E_1} \frac{2}{(n_2 + 1)d_{G_1}(u) + (n_2 + 1)d_{G_1}(v)} = \frac{1}{(1 + n_2)} H(G_1), \\ A_2 &= \sum_{uv \in E_2} \frac{2}{d(u) + d(v)} = m_1 \sum_{uv \in E_2} \frac{2}{d_{G_2}(u) + 2 + d_{G_2}(v) + 2} \geq \frac{m_1 m_2}{\Delta_2 + 2}, \\ A_3 &= \sum_{uv \in E_3} \frac{2}{d(u) + d(v)} = \sum_{uv \in E_3} \frac{2}{d_{G_2}(u) + 2 + (n_2 + 1)d(v)} \geq \frac{2m_1 n_2}{\Delta_1(1 + n_2) + \Delta_2 + 2}. \end{aligned}$$

By summation of A_1 , A_2 and A_3 , the result can be proved. Also for the reverse bound we can do analogously.

Corollary 2.9. *For $i \in \{1, 2\}$, if G_i be δ_i -regular graph of order n_i , then*

$$H(G_1 \diamond G_2) = \frac{1}{1+n_2} H(G_1) + \frac{m_1 m_2}{\delta_2 + 2} + \frac{2m_1 n_2}{\delta_1(n_2 + 1) + \Delta_2 + 2}.$$

Corollary 2.10. *For a graph G of size m and an empty graph N_k of order k , we have*

$$\begin{aligned} H(G \diamond N_k) &\geq \frac{1}{1+k} H(G) + \frac{2mk}{\Delta_1(k+1) + 2}, \\ H(G \diamond N_k) &\leq \frac{1}{1+k} H(G) + \frac{2mk}{\delta_1(k+1) + 2}. \end{aligned}$$

Theorem 2.11. *Let G_1 and G_2 be graphs. Thus, we have*

$$\begin{aligned} ABC(G_1 \diamond G_2) &\geq \frac{m_1}{\Delta_1(1+n_2)} \sqrt{2(n_2+1)\delta_1-2} + \frac{m_1m_2}{\Delta_2+2} \sqrt{2\delta_2+2} \\ &\quad + 2m_1n_2 \sqrt{\frac{\delta_1(n_2+1)+\delta_2}{(n_2+1)(\Delta_2+2)\Delta_1}}, \\ ABC(G_1 \diamond G_2) &\leq \frac{m_1}{\delta_1(1+n_2)} \sqrt{2(n_2+1)\Delta_1-2} + \frac{m_1m_2}{\delta_2+2} \sqrt{2\Delta_2+2} \\ &\quad + 2m_1n_2 \sqrt{\frac{\Delta_1(n_2+1)+\Delta_2}{(n_2+1)(\delta_2+2)\delta_1}}. \end{aligned}$$

Proof. The edges of $G_1 \diamond G_2$ are partitioned into three subsets E_1 , E_2 and E_3 as follows:

$$\begin{aligned} E_1 &= \{e \in E(G_1 \diamond G_2) | e \in E(G_1)\}, \\ E_2 &= \{e \in E(G_1 \diamond G_2) | e \in E(G_{2_i}), i = 1, 2, \dots, |E(G_1)|\}, \\ E_3 &= \{e \in E(G_1 \diamond G_2) | e = uv, u \in V(G_{2_i}), i = 1, 2, \dots, |E(G_1)|, v \in V(G_1)\}. \end{aligned}$$

Therefore,

$$ABC(G_1 \diamond G_2) = \sum_{uv \in E(G_1 \diamond G_2)} \sqrt{\frac{d_{G_1 \diamond G_2}(u) + d_{G_1 \diamond G_2}(v) - 2}{d_{G_1 \diamond G_2}(u)d_{G_1 \diamond G_2}(v)}} = A_1 + A_2 + A_3.$$

Where

$$\begin{aligned} A_1 &= \sum_{uv \in E_1} \sqrt{\frac{(n_2+1)(d_{G_1}(u) + d_{G_1}(v)) - 2}{(n_2+1)^2 d_{G_1}(u)d_{G_1}(v)}} \geq \frac{m_1}{\Delta_1(1+n_2)} \sqrt{2(n_2+1)\delta_1-2}, \\ A_2 &= m_1 \sum_{uv \in E_2} \sqrt{\frac{d_{G_2}(u) + 2 + d_{G_2}(v) + 2 - 2}{(d_{G_1}(u) + 2)(d_{G_1}(v) + 2)}} \geq \frac{m_1m_2}{\Delta_2+2} \sqrt{2\delta_2+2}, \\ A_3 &= \sum_{uv \in E_3} \sqrt{\frac{(n_2+1)d_{G_1}(u) + d_{G_2}(v) + 2 - 2}{(n_2+1)d_{G_1}(u)(d_{G_2}(v) + 2)}} \geq 2m_1n_2 \sqrt{\frac{(n_2+1)\delta_1+\delta_2}{(n_2+1)(\Delta_2+2)\Delta_1}}. \end{aligned}$$

By summation of A_1 , A_2 and A_3 , the result can be proved. Also for the reverse bound we can do analogously.

Corollary 2.12. *For $i \in \{1, 2\}$, if G_i be δ_i -regular graph of order n_i , then*

$$\begin{aligned} ABC(G_1 \diamond G_2) &= \frac{m_1}{\delta_1(1+n_2)} \sqrt{2(n_2+1)\delta_1-2} + \frac{m_1m_2}{\delta_2+2} \sqrt{2\delta_2+2} \\ &\quad + 2m_1n_2 \sqrt{\frac{\delta_1(n_2+1)+\delta_2}{(n_2+1)(\delta_2+2)\delta_1}}. \end{aligned}$$

Corollary 2.13. *For a graph G of size m and an empty graph N_k of order k , we have*

$$ABC(G \diamond N_k) \geq \frac{m}{\Delta_1(1+k)} \sqrt{2(k+1)\delta_1 - 2} + 2mk \sqrt{\frac{\delta_1(k+1) + 2}{4(k+1)\Delta_1}},$$

$$ABC(G \diamond N_k) \leq \frac{m}{\delta_1(1+k)} \sqrt{2(k+1)\Delta_1 - 2} + 2mk \sqrt{\frac{\Delta_1(k+1) + 2}{4(k+1)\delta_1}}.$$

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ON AN EXTENSION TO KHAN'S FIXED POINT THEOREM

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Abstract. In this present paper, the fixed point theorem that was proved by Khan [2] is extended to sequences of self maps through rational expressions. The present theorem includes non-continuous maps also.

Keywords: Fixed point, 2-metric space, complete 2-metric space, self-map.

1. Introduction

The notion of 2-metric space was introduced by Gahler [1] in 1963 as a generalization of area function for Euclidean triangles. Many fixed point theorems were established by various authors like Khan[2], Rhoades [4], etc. A point $x \in X$ is said to be a fixed point of a self-map $f : X \rightarrow X$ if $f(x) = x$, where X is a non-empty set. Theorems concerning fixed points of self-maps are known as fixed point theorems. Most of the fixed point theorems were proved for contraction mappings. It is well known that every contraction on a metric space is continuous. The converse is not necessarily true. The identity mapping on $[0,1]$ simply serves the counter example.

In this present work, the Khans fixed point theorem [2] is extended to a more general form in 2-metric space setting. Our generalized theorem holds for non-continuous selfmaps also.

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2. Preliminaries

In this section, we present some basic definitions which are needed for the further study of this paper.

Definition 2.1. Let X be a non-empty set and $d : X \times X \times X \rightarrow \mathbb{R}$. For all x, y, z and u in X , if d satisfies the following conditions:

- (a) $d(x, y, z) = 0$ if at least two of x, y, z are equal
- (b) $d(x, y, z) = d(x, z, y) = d(y, z, x) = \dots$
- (c) $d(x, y, z) \leq d(x, y, u) + d(x, u, z) + d(u, y, z)$.

Then d is called a 2-metric on X and the pair (X, d) is called a 2-metric space.

Definition 2.2. Let (X, d) be a 2-metric space. A sequence $\{x_n\}$ in X is called a *Cauchy sequence*, if $d(x_m, x_n, a) \rightarrow 0$ as $m, n \rightarrow \infty$, for all $a \in X$.

Definition 2.3. Let (X, d) be a 2-metric space. A sequence $\{x_n\}$ is said to converge to a point x in X if $\lim_{n \rightarrow \infty} d(x_n, x, a) = 0$, for every a in X .

Definition 2.4. A 2-metric space (X, d) is said to be a complete 2-metric space if every Cauchy sequence in X converges in X .

3. Generalized fixed point theorem

The following Theorem 3.2 is an extension of Khan's Fixed point theorem for non-continuous maps. We present Khan's result as Theorem 3.1 for completeness.

In what follows X stands for a complete 2-metric space with 2-metric d .

Theorem 3.1. Suppose that $f : X \rightarrow X$,

$$d(f(x), f(y), a) \leq \alpha \frac{[d(x, f(x), a)]^{r+w} [d(y, f(y), a)]^{1-r}}{[d(x, y, a)]^w} + \beta [d(x, y, a)]^{1-r-w} [d(f(x), f(y), a)]^{r+w},$$

for all x, y, a in X and for some $\alpha, \beta, w \in [0, 1)$, $r \in (0, 1)$ such that $\alpha + \beta < 1$ and $2r + w = 1$ when $w \neq 0$. Then f has a unique fixed point in X .

Theorem 3.2. Let $\{f_n\}$ and $\{g_n\}$ be two sequences of self-maps on X satisfying

$$(1) \quad d(f_n^p(x), g_n^q(y), a) \leq \alpha \frac{[d(x, f_n^p(x), a)]^{r+w} [d(y, g_n^q(y), a)]^{1-r}}{[d(x, y, a)]^w} + \beta [d(x, y, a)]^{1-r-w} [d(f_n^p(x), g_n^q(y), a)]^{r+w},$$

for all x, y, a in X and for some $\alpha, \beta, w \in [0, 1)$, $r \in (0, 1)$ such that $\alpha + \beta < 1$ and $2r + w = 1$ when $w \neq 0$, p and q are positive integers then f_n and g_n have a unique common fixed point in X .

Proof. Let $x_0 \in X$. Define a sequence $\{x_n\}$ in X as follows $x_{2n-1} = f_n^p(x_{2n-2})$ and $x_{2n} = g_n^q(x_{2n-1})$ for $n = 1, 2, 3, \dots$. Then we have

$$\begin{aligned}
 d(x_{2n-1}, x_n, a) &= d(f_n^p(x_{2n-2}), g_n^q(x_{2n-1}), a) \\
 &\leq \alpha \frac{[d(x_{2n-2}, f_n^p(x_{2n-2}), a)]^{r+w} [d(x_{2n-1}, g_n^q(x_{2n-1}), a)]^{1-r}}{[d(x_{2n-2}, x_{2n-1}, a)]^w} \\
 &\quad + \beta [d(x_{2n-2}, x_{2n-1}, a)]^{1-r-w} [d(f_n^p(x_{2n-2}), g_n^q(x_{2n-1}), a)]^{r+w} \\
 &\quad \alpha \frac{[d(x_{2n-2}, x_{2n-1}, a)]^{r+w} [d(x_{2n-1}, x_{2n}, a)]^{1-r}}{[d(x_{2n-2}, x_{2n-1}, a)]^w} \\
 &= \beta [d(x_{2n-2}, x_{2n-1}, a)]^{1-r-w} [d(x_{2n-1}, x_{2n}, a)]^{r+w} \\
 &= \alpha [d(x_{2n-2}, x_{2n-1}, a)]^r [d(x_{2n-1}, x_{2n}, a)]^{1-r} \\
 &\quad + \beta [d(x_{2n-2}, x_{2n-1}, a)]^{1-r-w} [d(x_{2n-1}, x_{2n}, a)]^{r+w}
 \end{aligned}
 \tag{2}$$

or

$$\begin{aligned}
 d(x_{2n-1}, x_{2n}, a) &\leq \alpha [d(x_{2n-2}, x_{2n-1}, a)]^r [d(x_{2n-1}, x_{2n}, a)]^{1-r} \\
 &\quad + \beta [d(x_{2n-2}, x_{2n-1}, a)]^{1-r-w} [d(x_{2n-1}, x_{2n}, a)]^{r+w}.
 \end{aligned}
 \tag{3}$$

Now

$$\begin{aligned}
 d(x_{2n}, x_{2n+1}, a) &= d(g_n^q(x_{2n-1}), f_n^p(x_{2n}), a) \\
 &= d(f_n^p(x_{2n}), g_n^q(x_{2n-1}), a) \\
 &\leq \alpha \frac{[d(x_{2n}, f_n^p(x_{2n}), a)]^{r+w} [d(x_{2n-1}, g_n^q(x_{2n-1}), a)]^{1-r}}{[d(x_{2n}, x_{2n-1}, a)]^w} \\
 &\quad + \beta [d(x_{2n}, x_{2n-1}, a)]^{1-r-w} [d(f_n^p(x_{2n}), g_n^q(x_{2n-1}), a)]^{r+w} \\
 &= \alpha \frac{[d(x_{2n}, x_{2n+1}, a)]^{r+w} [d(x_{2n-1}, x_{2n}, a)]^{1-r}}{[d(x_{2n}, x_{2n-1}, a)]^w} \\
 &\quad + \beta [d(x_{2n-1}, x_{2n}, a)]^{1-r-w} [d(x_{2n+1}, x_{2n}, a)]^{r+w} \\
 &= \alpha [d(x_{2n}, x_{2n+1}, a)]^{r+w} [d(x_{2n-1}, x_{2n}, a)]^{1-r-w} \\
 &\quad + \beta [d(x_{2n-1}, x_{2n}, a)]^{1-r-w} [d(x_{2n}, x_{2n+1}, a)]^{r+w} \\
 &= (\alpha + \beta) [d(x_{2n-1}, x_{2n}, a)]^{1-r-w} [d(x_{2n}, x_{2n+1}, a)]^{r+w} \\
 &\Rightarrow d(x_{2n}, x_{2n+1}, a) \leq (\alpha + \beta) [d(x_{2n-1}, x_{2n}, a)]^{1-r-w} [d(x_{2n}, x_{2n+1}, a)]^{r+w} \\
 &\Rightarrow [d(x_{2n}, x_{2n+1}, a)]^{1-r-w} \leq (\alpha + \beta) [d(x_{2n-1}, x_{2n}, a)]^{1-r-w} \\
 &\Rightarrow d(x_{2n}, x_{2n+1}, a) \leq (\alpha + \beta)^{1/1-r-w} d(x_{2n-1}, x_{2n}, a).
 \end{aligned}
 \tag{4}$$

Case (i). When $w \neq 0$, we obtain

$$d(x_{2n}, x_{2n+1}, a) \leq (\alpha + \beta)^{1/r} d(x_{2n-1}, x_{2n}, a)$$

and from (2), we have

$$\begin{aligned}
 d(x_{2n-1}, x_{2n}, a) &\leq \alpha[d(x_{2n-2}, x_{2n-1}, a)]^r [d(x_{2n-1}, x_{2n}, a)]^{1-r} \\
 &\quad + \beta[d(x_{2n-2}, x_{2n-1}, a)]^r [d(x_{2n-1}, x_{2n}, a)]^{1-r} \\
 (5) \quad &= (\alpha + \beta)[d(x_{2n-2}, x_{2n-1}, a)]^r [d(x_{2n-1}, x_{2n}, a)]^{1-r} \\
 &\Rightarrow [d(x_{2n-1}, x_{2n}, a)]^r \leq (\alpha + \beta)[d(x_{2n-2}, x_{2n-1}, a)]^r \\
 &\Rightarrow d(x_{2n-1}, x_{2n}, a) \leq k_1 d(x_{2n-2}, x_{2n-1}, a)
 \end{aligned}$$

where $k_1 = (\alpha + \beta)^{1/r} < 1$. Therefore we have

$$\begin{aligned}
 d(x_{2n}, x_{2n+1}, a) &\leq k_1 d(x_{2n-1}, x_{2n}, a) \leq k_1^2 d(x_{2n-2}, x_{2n-1}, a) \\
 &\leq \dots \leq k_1^{2n} d(x_0, x_1, a) \\
 &\Rightarrow d(x_{2n}, x_{2n+1}, a) \leq k_1^{2n} d(x_0, x_1, a).
 \end{aligned}$$

Since $k_1 < 1$, $k_1^{2n} \rightarrow 0$ as $n \rightarrow \infty$. Hence $d(x_{2n}, x_{2n+1}, a) = 0$ as $n \rightarrow \infty$

Case (ii)

Suppose that $w = 0$

From (3) and (2), we have $d(x_{2n}, x_{2n+1}, a) \leq (\alpha + \beta)^{1/1-r} d(x_{2n-1}, x_{2n}, a)$

and

$$\begin{aligned}
 d(x_{2n-1}, x_{2n}, a) &\leq \alpha[d(x_{2n-2}, x_{2n-1}, a)]^r [d(x_{2n-1}, x_{2n}, a)]^{1-r} \\
 (6) \quad &\quad + \beta[d(x_{2n-2}, x_{2n-1}, a)]^{1-r} d[(x_{2n-1}, x_{2n}, a)]^r.
 \end{aligned}$$

We claim that $d(x_{2n-1}, x_{2n}, a) \leq d(x_{2n-2}, x_{2n-1}, a)$.

If it is not so, suppose that $d(x_{2n-1}, x_{2n}, a) > d(x_{2n-2}, x_{2n-1}, a)$. Then from (4), we have

$$\begin{aligned}
 d(x_{2n-1}, x_{2n}, a) &\leq \alpha[d(x_{2n-1}, x_{2n}, a)]^r [d(x_{2n-1}, x_{2n}, a)]^{1-r} \\
 &\quad + \beta[d(x_{2n-1}, x_{2n}, a)]^{1-r} [d(x_{2n-1}, x_{2n}, a)]^r \\
 &= (\alpha + \beta)d(x_{2n-1}, x_{2n}, a)
 \end{aligned}$$

or

$$d(x_{2n-1}, x_{2n}, a) \leq (\alpha + \beta)d(x_{2n-1}, x_{2n}, a)$$

Which is a contradiction, since $\alpha + \beta < 1$.

Therefore

$$\begin{aligned}
 d(x_{2n-1}, x_{2n}, a) &\leq d(x_{2n-2}, x_{2n-1}, a) \leq d(x_{2n-3}, x_{2n-2}, a) \\
 &\leq k_2^2 d(x_{2n-4}, x_{2n-3}, a) \leq \dots \leq k_2^{2n-2} d(x_0, x_1, a) \\
 &\Rightarrow d(x_{2n-1}, x_{2n}, a) \leq k_2^{2n-2} d(x_0, x_1, a).
 \end{aligned}$$

Where $k_2 = (\alpha + \beta)^{1/r} < 1$. Since $k_2 < 1$, $k_2^{2n-2} \rightarrow 0$ as $n \rightarrow \infty$.

Hence $d(x_{2n-1}, x_{2n}, 0) = 0$ as $n \rightarrow \infty$.

Hence $d(x_{2n}, x_{2n+1}, a) = 0$ as $n \rightarrow \infty$.

Thus in two cases $\{x_n\}$ is a Cauchy sequence in X . Since X is complete, $\{x_n\}$ converges to some point, say, u in X .

Now we prove that u is a unique common fixed point of f_n and g_n , ($n = 1, 2, 3, \dots$).

For this first we prove that $u = f_n^p(u) = g_n^q(u)$, $n = 1, 2, 3, \dots$

Now by the properties of 2-metric, we have

$$\begin{aligned} d(u, f_n^p(u), a) &\leq d(u, f_n^p(u), x_{2n}) + d(u, x_{2n}, a) + d(x_{2n}, f_n^p(u), a) \\ &= d(u, f_n^p(u), x_{2n}) + d(u, x_{2n}, a) + d(f_n^p(u), g_n^q(x_{2n-1}), a). \end{aligned}$$

Taking $x = u$ and $y = x_{2n-1}$ in (1), we get

$$\begin{aligned} d(f_n^p(u), g_n^q(x_{2n-1}), a) &\leq \alpha \frac{[d(u, f_n^p(u), a)]^{r+w} [d(x_{2n-1}, g_n^q(x_{2n-1}), a)]^{1-r}}{[d(u, x_{2n-1}, a)]^w} \\ &\quad + \beta [d(u, x_{2n-1}, a)]^{1-r-w} [d(f_n^p(u), g_n^q(x_{2n-1}), a)]^{r+w}. \end{aligned}$$

Hence

$$\begin{aligned} d(u, f_n^p(u), a) &\leq d(u, f_n^p(u), x_{2n}) + d(u, x_{2n}, a) \\ &\quad + \alpha \frac{[d(u, f_n^p(u), a)]^{r+w} [d(x_{2n-1}, g_n^q(x_{2n-1}), a)]^{1-r}}{[d(u, x_{2n-1}, a)]^w} \\ &\quad + \beta [d(u, x_{2n-1}, a)]^{1-r-w} [d(f_n^p(u), g_n^q(x_{2n-1}), a)]^{r+w}. \end{aligned}$$

When as $n \rightarrow \infty$, $d(u, f_n^p(u), a) \leq 0 + \alpha(0) + \beta(0)$, which implies that $d(u, f_n^p(u), a) = 0$. Hence $f_n^p(u) = u$. Similarly $g_n^q(u) = u$. Therefore u is a common fixed point of f_n^p and g_n^q . To show that u is unique, let v be the another common fixed point of f_n^p and $g_n^q \Rightarrow v = f_n^p(v) = g_n^q(v)$, $n = 1, 2, 3, \dots$. Then by given condition

$$\begin{aligned} d(f_n^p(u), g_n^q(v), a) &\leq \alpha \frac{[d(u, f_n^p(u), a)]^{r+w} [d(v, g_n^q(v), a)]^{1-r}}{[d(u, v, a)]^w} \\ &\quad + \beta [d(u, v, a)]^{1-r-w} [d(f_n^p(u), g_n^q(v), a)]^{r+w} \\ &= \alpha \frac{[d(u, u, a)]^{r+w} [d(v, v, a)]^{1-r}}{[d(u, v, a)]^w} + \beta [d(u, v, a)]^{1-r-w} [d(u, v, a)]^{r+w} \end{aligned}$$

or $(1 - \beta)d(u, v, a) \leq 0$, which implies that $d(u, v, a) = 0$, for every a in X .

$$\Rightarrow u = v$$

Hence u is a unique common fixed point of f_n^p and g_n^q .

Finally we show that u is the only common fixed point of f_n and g_n . For $f_n^p(f_n(u)) = f_n(f_n^p(u))$ gives $f_n^p(f_n(u)) = f_n(u)$. Hence $f_n(u)$ is the fixed point of f_n^p . Since u is the unique fixed point of f_n^p , $f_n(u) = u$.

Similarly $g_n(u) = u$. To show that u is unique common fixed point of f_n and g_n , let z be the another common fixed point of f_n and g_n . That is, $z = f_n(z) =$

$g_n(z)$. From the given condition, we have

$$\begin{aligned}
 d(u, z, a) &= d(f_n(u), g_n(z), a) \\
 &= d(f_n^p(u), g_n^q(z), a) \\
 &\leq \alpha \frac{[d(u, f_n^p(u), a)]^{r+q} [d(z, g_n^q(z), a)]^{1-r}}{[d(u, z, a)]^w} \\
 &\quad + \beta [d(u, z, a)]^{1-r-w} [d(f_n^p(u), g_n^q(z), a)]^{r+w} \\
 &= \alpha \frac{[d(u, u, a)]^{r+w} [d(z, z, a)]^{1-r}}{[d(u, z, a)]^w} + \beta [d(u, z, a)]^{1-r-w} [d(u, z, a)]^{r+w}
 \end{aligned}$$

or

$$(1 - b)d(u, z, a) \leq 0$$

which implies that $d(u, z, a) = 0$ for every a in X .

$$\Rightarrow u = z$$

Hence u is a unique common fixed point of f_n and g_n , ($n = 1, 2, 3, \dots$) $\square$

Remark 3.3. Theorem 3.1 can be easily deduced as a corollary from Theorem 3.2 by taking $f_n^p = g_n^q = f$. Hence Theorem 3.2 is a generalized version of Khan's theorem [2] in 2-metric space.

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SOFT ROUGH BCI -ALGEBRAS AND CORRESPONDING DECISION MAKING

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Abstract. In this paper, we study soft rough BCI -algebras with respect to MS -approximation spaces. Some new soft rough operations over BCI -algebras are explored. In particular, lower and upper soft rough BCI -algebras with another soft set are investigated. Finally, a kind of decision making method for soft rough BCI -algebras are originally investigated.

Keywords: Soft set, soft rough set, MSR -set, decision making method, BCI -algebra.

1. Introduction

The concept of rough sets was initiated by Pawlak [23] as an approach to copy with inexact and uncertain knowledge. As far as known that that an equivalence relation on a set partitions the set into disjoint classes and vice versa. We know that a subset can be written as union of these classes, which is called definable, otherwise it is not definable. In this case, it can be approximated by two definable subsets called lower and upper approximations of the set. Some general models can be found in [30–32]. Nowadays, this theory has been applied to many fields, such as patten recognition, intelligent systems, machines learning, cognitive science, image processing, signal analysis and so on. On the other hand, some researchers applied this theory to algebraic structures, for examples, see [5, 6, 11].

The concept of soft sets was initiated by Molodsov [22] as a new mathematical tool for dealing with uncertainties. It is free that soft set theory is free from the difficulties that have troubled the usual theoretical approaches. Nowadays, the research on soft sets is progressing rapidly. In 2003, Maji [20] proposed some basic operations. Further, Ali [1] revised some operations. Afterwards, a wide range of applications of soft sets have been studied in many different fields including game theory, operation researches, data analysis, mea-

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surement theory, decision making, forecasting, and so on. In recent years, there has been a rapid growth of interest in soft set theory and its applications, for examples, see [2–4, 15, 21, 25]. In particular, Zhan [29] reviewed on decision making methods based on (fuzzy) soft sets and rough soft sets. At the same time, many researchers applied this theory to algebraic structures, for examples, see [5, 6, 12, 13].

As far as known that the research of t -norm-based logical systems has become increasingly more important in the field of logic. It is well known that *BCK* and *BCI*-algebras are two classes of algebras of logic which were introduced by Imai and Iseki [9, 10]. We know that these two classes of logical algebras have been investigated by many researchers, see [14, 16, 18, 19, 26, 27]. Most of the algebras related to the t -norm based logic, such as *MTL*-algebras, *BL*-algebras, *MV*-algebras and Boolean algebras et al, are extensions of *BCK*-algebras. This shows that *BCK/BCI*-algebras are considerably general structures, which means that it is an important topic on these two kinds of logical algebras.

Recently, Feng [7] proposed rough soft sets by combining Pawlak rough sets and soft sets, rough sets can be regarded as a collection of rough sets sharing a common Pawlak approximation space. In [28], Zhan initiated rough soft set theory to algebraic structures—hemirings. On the other hand, Ma [17] put forth rough soft *BCI*-algebras by means of an ideal of the *BCI*-algebra. In 2011, Feng [8] proposed soft rough sets by combining soft sets with rough sets, which can be regarded as a kind of new rough set as a soft set instead of an equivalence relation. However, the soft set must be a full soft set in order to resolve theoretical and practical aspects. Recently, Shabir [24] pointed out that there exist two problems on Feng’s soft rough set as follows: (1) An upper approximation of a non-empty set may be empty; (2) The upper approximation of a subset X may not contain the set X . To resolve this shortcoming, Shabir modified this concept and proposed a class of revised soft rough set, which is called an *MSR*-set. The *MSR*-sets are not only no restrictions on the soft sets but also the underlying concepts are very similar to classical rough sets.

Based on the above idea, in the present paper, we apply this novel soft rough set theory to *BCI*-algebras. In section 2, we recall some basic concepts on rough sets, soft sets and *BCI*-algebras. In section 3, we study some operations with respect to *MS*-approximation spaces and some new soft rough operations over *BCI*-algebras are explored. Further, some lower and upper *MSR-BCI*-algebras are investigated in section 4. In particular, in section 5 we discuss soft rough *BCI*-algebras based another soft set. Finally, we initiate to put forth a kind of decision making method for soft rough *BCI*-algebras in section 6.

2. Preliminaries

For any *BCI*-algebra X , the relation $\leq$ defined by $x \leq y$ if and only if $x * y = 0$ is a partial order on X . Throughout this paper, X is always a *BCI*-algebra.

A non-empty subset S of X is called a *subalgebra* of X if $x * y \in S$ whenever $x, y \in S$. A non-empty subset I of X is called an *ideal* of X , denoted by $I \triangleleft X$, if it satisfies: (1) $0 \in I$; (2) $x * y \in I$ and $y \in I$ imply $x \in I$ for all $x, y \in X$.

Definition 2.1 ([22]). A pair $\mathfrak{S} = (F, A)$ is called a *soft set* over U , where $A \subseteq E$ and $F : A \rightarrow \mathcal{P}(U)$ is a set-valued mapping.

Definition 2.2 ([7]). A soft set $\mathfrak{S} = (F, A)$ over U is called a *full soft set* if $\bigcup_{a \in A} F(a) = U$.

Definition 2.3 ([12, 13]). Let (F, A) be a soft set over X . Then

(1) (F, A) is called a *soft BCI-algebra* over X if $F(x)$ is a subalgebra of X for all $x \in \text{Supp}(F, A)$,

(2) (F, A) is called a *soft ideal* if $F(x)$ is an ideal of X for all $x \in \text{Supp}(F, A)$, where $\text{Supp}(F, A) = \{x \in A \mid F(x) \neq \emptyset\}$ is called a soft support of the soft set (F, A) .

Definition 2.4 ([23]). Let R be an equivalence relation on the universe U , (U, R) be a Pawlak approximation space. A subset $X \subseteq U$ is called *definable* if $R_*X = R^*X$; in the opposite case, i.e., if $R_*X - R^*X \neq \emptyset$, X is said to be a *rough set*, where two operations are defined as:

$$R_*X = \{x \in U : [x]_R \subseteq X\},$$

$$R^*X = \{x \in U : [x]_R \cap X \neq \emptyset\}.$$

Definition 2.5 ([8]). Let $\mathfrak{S} = (F, A)$ be a soft set over U . Then the pair $P = (U, \mathfrak{S})$ is called a *soft approximation space*. Based on P , we define the following two operations:

$$\underline{\text{apr}}_P(X) = \{u \in U \mid \exists a \in A [u \in F(a) \subseteq X]\},$$

$$\overline{\text{apr}}_P(X) = \{u \in U \mid \exists a \in A [u \in F(a), F(a) \cap X \neq \emptyset]\},$$

assigning to every subset $X \subseteq U$, two sets $\underline{\text{apr}}_P(X)$ and $\overline{\text{apr}}_P(X)$ are called the *lower* and *upper soft rough approximations* of X in P , respectively. If $\underline{\text{apr}}_P(X) = \overline{\text{apr}}_P(X)$, X is said to be soft definable; otherwise X is called a *soft rough set*. In what follows, we call it *Feng-soft rough set*.

Definition 2.6 ([24]). Let (F, A) be a soft set over U and $\varphi : U \rightarrow \mathcal{P}(A)$ be a map defined as $\varphi(x) = \{a \mid x \in F(a)\}$. Then the pair (U, φ) is called an *MS-approximation space* and for any $X \subseteq U$, the lower *MSR*-approximation and upper *MSR*-approximation of X are denoted by $\underline{X}_\varphi$ and $\overline{X}_\varphi$ respectively, which two operations are defined as

$$\underline{X}_\varphi = \{x \in X \mid \varphi(x) \neq \varphi(y) \text{ for all } y \in X^c\}$$

and

$$\overline{X}_\varphi = \{x \in U \mid \varphi(x) = \varphi(y) \text{ for some } y \in X\}$$

If $\underline{X}_\varphi = \overline{X}_\varphi$, then the X is said to be *MS-definable*, otherwise X is said to be *MSR-set*.

3. Soft rough approximations

In this section, we investigate some operations and fundamental properties of modified soft rough sets over *BCI*-algebras. In order to illustrate the roughness in *BCI*-algebra X w.r.t. soft rough approximation spaces over *BCI*-algebras, we first introduce two special kinds of soft sets over *BCI*-algebras.

Definition 3.1. Let $\mathfrak{S} = (F, A)$ be a soft set over X and $\varphi : X \rightarrow \mathcal{P}(A)$ be a map defined as $\varphi(x) = \{a \mid x \in F(a)\}$, then $\mathfrak{S}$ is called a *C-soft set* over X if $\varphi(a) = \varphi(b)$ and $\varphi(c) = \varphi(d)$ imply $\varphi(a * c) = \varphi(b * d)$ for all $a, b, c, d \in X$.

Example 3.2. Let $X = \{0, a, b, c\}$ be a *BCI*-algebra with the following Cayley Table 1:

Table 1 table for *BCI*-algebra

*	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Define a soft set $\mathfrak{S} = (F, A)$ over X which is given by Table 2.

Table 2 table for soft set $\mathfrak{S}$

	0	a	b	c
e_1	0	0	0	0
e_2	1	1	1	1
e_3	1	1	0	0

Then the mapping $\varphi : X \rightarrow \mathcal{P}(A)$ of soft rough approximation space (X, φ) is given by $\varphi(0) = \varphi(a) = \{e_2, e_3\}$, $\varphi(b) = \varphi(c) = \{e_2\}$.

By calculations, $\mathfrak{S}$ is called a *C-soft set* over X .

Let Y, Z be any two non-empty subsets in any *BCI*-algebra X . Denote $Y * Z = \{y * z \mid \forall y \in Y, z \in Z\}$.

Theorem 3.3. Let $\mathfrak{S} = (F, A)$ be a *C-soft set* over X and (X, φ) an *MS*-approximation space. For any two non-empty subsets Y, Z in X . Then

$$\overline{Y}_\varphi * \overline{Z}_\varphi \subseteq \overline{Y * Z}_\varphi.$$

Proof. Let $c \in \overline{Y}_\varphi * \overline{Z}_\varphi$, then $c = a * b$, where $a \in \overline{Y}_\varphi$ and $b \in \overline{Z}_\varphi$, and so there exist $y \in Y$ and $z \in Z$ such that $\varphi(a) = \varphi(y)$ and $\varphi(b) = \varphi(z)$. Since $\mathfrak{S}$ is a *C-soft set*, $\varphi(a * b) = \varphi(y * z)$ for $y * z \in Y * Z$. Hence $c = a * b \in \overline{Y * Z}_\varphi$. That is, $\overline{Y}_\varphi * \overline{Z}_\varphi \subseteq \overline{Y * Z}_\varphi$. $\square$

The following example shows that the containment in Theorem 3.3 is proper.

Example 3.4. Let $X = \{0, a, b, c\}$ be a *BCI*-algebra with the following Cayley Table 3:

Table 3 table for BCI -algebra

$*$	0	a	b	c
0	0	0	0	c
a	a	0	a	c
b	b	b	0	c
c	c	c	c	0

Define a soft set $\mathfrak{S} = (F, A)$ over X which is given by Table 4.

Table 4 table for soft set $\mathfrak{S}$

	0	a	b	c
e_1	1	1	1	0
e_2	0	0	0	0
e_3	0	0	0	1
e_4	1	1	1	0

Then the mapping $\varphi : X \rightarrow \mathcal{P}(A)$ of soft rough approximation space (X, φ) is given by $\varphi(0) = \varphi(a) = \varphi(b) = \{e_1, e_4\}$, $\varphi(c) = \{e_3\}$. Then we can check that $\mathfrak{S}$ is a C -soft set over X .

If we take $Y = \{0, c\}$ and $Z = \{c\}$, then $Y * Z = \{0, c\}$, and so, $\overline{Y}_\varphi = \{0, a, b, c\}$ and $\overline{Z}_\varphi = \{c\}$, so $\overline{Y}_\varphi * \overline{Z}_\varphi = \{0, c\}$. Also we have $\overline{Y} * \overline{Z}_\varphi = \{0, c\}_\varphi = \{0, a, b, c\}$. Thus $\overline{Y}_\varphi * \overline{Z}_\varphi \subsetneq \overline{Y} * \overline{Z}_\varphi$.

Definition 3.5. Let $\mathfrak{S} = (F, A)$ be a C -soft set over X and (X, φ) an MS -approximation space, then $\mathfrak{S}$ is called a CC -soft set over X if for all $c \in X$, $\varphi(c) = \varphi(x * y)$, there exist $a, b \in X$, such that $\varphi(x) = \varphi(a)$ and $\varphi(y) = \varphi(b)$ satisfying $c = a * b$.

Remark 3.6. (1) $\mathfrak{S}$ in Example 3.4 is a C -soft set over X , but it is not a CC -soft set.

(2) $\mathfrak{S}$ in Example 3.2 is a CC -soft set over X .

If we strength the condition, we can obtain the following result:

Theorem 3.7. Let $\mathfrak{S} = (F, A)$ be a CC -soft set over X and (X, φ) an MS -approximation space. For any two non-empty subsets Y, Z in X . Then

$$\overline{Y}_\varphi \cdot \overline{Z}_\varphi = \overline{Y} \cdot \overline{Z}_\varphi.$$

Proof. By Theorem 3.3, we have $\overline{Y}_\varphi * \overline{Z}_\varphi \subseteq \overline{Y} * \overline{Z}_\varphi$. Now let $c \in \overline{Y} * \overline{Z}_\varphi$, so $\varphi(c) = \varphi(y * z)$ for some $y \in Y$ and $z \in Z$. Then there exist $a, b \in X$, such that $\varphi(a) = \varphi(y)$ and $\varphi(b) = \varphi(z)$ satisfying $c = a * b$ since $\mathfrak{S}$ is a CC -soft set over X . Thus $a \in \overline{Y}_\varphi$ and $b \in \overline{Z}_\varphi$. Hence $c \in \overline{Y}_\varphi * \overline{Z}_\varphi$. Summing up the above arguments, $\overline{Y}_\varphi * \overline{Z}_\varphi = \overline{Y} * \overline{Z}_\varphi$. $\square$

Next, we consider lower soft rough approximations over BCI -algebras.

Theorem 3.8. Let $\mathfrak{S} = (F, A)$ be a CC -soft set over X and (X, φ) an MS -approximation space. For any two non-empty subsets Y, Z in X . Then

$$\underline{Y}_\varphi * \underline{Z}_\varphi \subseteq \underline{Y} * \underline{Z}_\varphi.$$

Proof. Suppose that $\underline{Y}_\varphi * \underline{Z}_\varphi \subseteq \underline{Y} * \underline{Z}_\varphi$ does not hold, then there exists $c \in \underline{Y}_\varphi * \underline{Z}_\varphi$, but $c \notin \underline{Y} * \underline{Z}_\varphi$. Then $c = a * b$, where $a \in \underline{Y}_\varphi$ and $b \in \underline{Z}_\varphi$. This means that $\varphi(a) \neq \varphi(y)$ and $\varphi(b) \neq \varphi(y)$ for all $y \in Y^c$ and $z \in Z^c$. (Δ)

On the other hand, $c \notin \underline{Y} * \underline{Z}_\varphi$, then we may have the following two conditions:

- (i) $c \notin Y * Z$, which contradicts with $c \in \underline{Y}_\varphi * \underline{Z}_\varphi \subseteq Y * Z$;
- (ii) $c \in Y * Z$ and $\varphi(c) = \varphi(y' * z')$ for some $y' * z' \in (Y * Z)^c$. Thus $y' \in Y^c$ or $z' \in Z^c$. In fact, if $y' \notin Y^c$ and $z' \notin Z^c$, we have $y' * z' \in Y * Z$, a contradiction. Since $\mathfrak{S} = (F, A)$ is a *CC*-soft set over X , then there exist $a', b' \in X$ such that $\varphi(a') = \varphi(y')$ and $\varphi(b') = \varphi(z')$ satisfying $a' * b' = c$, for some $y' \in Y^c$ or $z' \in Z^c$. This is contradiction with (Δ). Hence $\underline{Y}_\varphi * \underline{Z}_\varphi \subseteq \underline{Y} * \underline{Z}_\varphi$. $\square$

The following example shows that Theorem 3.8 is not true if $\mathfrak{S}$ is not a *CC*-soft set over X .

Example 3.9. Let $X = \{0, a, b, c, d\}$ be a *BCI*-algebra with the following Cayley Table 5:

Table 5 table for *BCI*-algebra

*	0	a	b	c	d
0	0	0	0	c	c
a	a	0	a	c	c
b	b	b	0	c	c
c	c	c	c	0	0
d	d	c	d	a	c

Define a soft set $\mathfrak{S} = (F, A)$ over X which is given by Table 6.

Table 6 table for soft set $\mathfrak{S}$

	0	a	b	c	d
e_1	1	0	1	1	0
e_2	0	1	1	1	0
e_3	0	0	0	0	1
e_4	1	0	1	1	0

Clearly, $\mathfrak{S}$ is not a *CC*-soft set over X . If we take $Y = \{0, b\}$ and $Z = \{b, d\}$, then $\underline{Y}_\varphi = \{0\}$ and $\underline{Z}_\varphi = \{c\}$, so $\underline{Y}_\varphi * \underline{Z}_\varphi = \{c\}$. Also we have $\underline{Y} * \underline{Z}_\varphi = \{0\}$. This means that $\underline{Y}_\varphi * \underline{Z}_\varphi \not\subseteq \underline{Y} * \underline{Z}_\varphi$.

The following example shows that the containment in Theorem 3.8 is proper.

Example 3.10. Consider the *BCI*-algebra X and the soft set $\mathfrak{S} = (F, A)$ in Example 3.2. If we take $Y = \{0, a, b\}$ and $Z = \{0, b, c\}$, then $\underline{Y}_\varphi = \{0, a\}$ and $\underline{Z}_\varphi = \{b, c\}$, so $\underline{Y}_\varphi * \underline{Z}_\varphi = \{b, c\}$. On the other hand, $\underline{Y} * \underline{Z}_\varphi = \{0, a, b, c\}$. This means that $\underline{Y}_\varphi * \underline{Z}_\varphi \subsetneq \underline{Y} * \underline{Z}_\varphi$.

4. Soft rough BCI -algebras

In this section, we study the operations of lower and upper MSR -approximations of soft rough BCI -algebras.

Definition 4.1. Let $\mathfrak{S} = (F, A)$ be a soft set over X and (X, φ) an MS -approximation space. For any $Y \subseteq X$, the lower MSR -approximation and upper MSR -approximation of Y are denoted by $\underline{Y}_\varphi$ and $\overline{Y}_\varphi$, respectively, which two operations are defined as

$$\underline{Y}_\varphi = \{x \in Y \mid \varphi(x) \neq \varphi(y) \text{ for all } y \in Y^c\}$$

and

$$\overline{Y}_\varphi = \{x \in X \mid \varphi(x) = \varphi(y) \text{ for some } y \in Y\}$$

If $\underline{Y}_\varphi \neq \overline{Y}_\varphi$, then

- (i) Y is called a *lower (upper) soft rough BCI -algebra (resp., ideal)* over X , if $\underline{Y}_\varphi$ ($\overline{Y}_\varphi$) is a subalgebra (resp., ideal) of X ;
- (ii) Y is called a *soft rough BCI -algebra (resp., ideal)* over X , if $\underline{Y}_\varphi$ and $\overline{Y}_\varphi$ are subalgebras (resp., ideals) of X .

Example 4.2. Let $X = \{0, a, b, c, d\}$ be a BCI -algebra with the following Cayley Table 7:

Table 7 table for BCI -algebra

*	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	0
b	b	b	0	0	b
c	c	b	a	0	b
d	d	a	d	a	0

Define a soft set $\mathfrak{S} = (F, A)$ over X which is given by Table 8.

Table 8 table for soft set $\mathfrak{S}$

	0	a	b	c	d
e_1	1	0	0	0	0
e_2	0	1	0	1	0
e_3	1	1	1	1	1

Then the mapping $\varphi : X \rightarrow \mathcal{P}(A)$ of MS -approximation space (S, φ) is given by $\varphi(0) = \{e_1, e_3\}$, $\varphi(a) = \varphi(c) = \{e_2, e_3\}$ and $\varphi(b) = \varphi(d) = \{e_1\}$.

Let $Y = \{0, b, c, d\}$, then $\underline{Y}_\varphi = \{0, b, d\}$ and $\overline{Y}_\varphi = \{0, a, b, c, d\}$. This shows that $\underline{Y}_\varphi$ and $\overline{Y}_\varphi$ are subalgebras of X . In other words, Y is a soft rough BCI -algebra over X , but it is not a soft rough ideal over X since $\underline{Y}_\varphi$ is a subalgebra of X .

Example 4.3. Let $X = \{0, a, b, c, d\}$ be a BCI -algebra with the following Cayley Table 9:

Table 9 table for *BCI*-algebra

$*$	0	a	b	c
0	0	0	c	b
a	a	0	c	b
b	b	b	0	c
c	c	c	b	0

Define a soft set $\mathfrak{S} = (F, A)$ over X which is given by Table 10.

Table 10 table for soft set $\mathfrak{S}$

	0	a	b	c
e_1	1	1	0	0
e_2	1	1	1	1
e_3	0	0	1	1

Then the mapping $\varphi : X \rightarrow \mathcal{P}(A)$ of *MS*-approximation space (S, φ) is given by $\varphi(0) = \varphi(a) = \{e_1, e_2\}$, and $\varphi(b) = \varphi(c) = \{e_2, e_3\}$.

Let $Y = \{0, a, b\}$, then $\underline{Y}_\varphi = \{0, a\} \triangleleft X$ and $\overline{Y}_\varphi = \{0, a, b, c\} \triangleleft X$. This shows that Y is a soft rough ideal over X .

Proposition 4.4. Let (X, φ) be an *MS*-approximation space. If Y and Z are lower soft rough *BCI*-algebras (resp., ideals) over X , then so is $Y \cap Z$.

Proof. If Y and Z are lower soft rough *BCI*-algebras (resp., ideals) over X , then $\underline{Y}_\varphi$ and $\underline{Z}_\varphi$ are subalgebras (resp., ideals) of X , so $\underline{Y}_\varphi \cap \underline{Z}_\varphi$ is a subalgebra (resp., ideal) of X . By Theorem 3 in [24], we have $\underline{Y} \cap \underline{Z}_\varphi = \underline{Y}_\varphi \cap \underline{Z}_\varphi$ is also a subalgebra (resp., ideal) of X . Hence $X \cap Y$ is a lower soft rough *BCI*-algebra (resp., ideal) over X . $\square$

In general, $Y \cap Z$ is not an upper soft rough *BCI*-algebra (resp., ideal) over X , if Y and Z are upper soft rough *BCI*-algebras (resp., ideals) over X . Actually we have the following example.

Example 4.5. Consider the *BCI*-algebra X and the soft set $\mathfrak{S} = (F, A)$ in Example 4.3. Let $Y = \{0, c\}$ and $Z = \{a, c\}$, then $\overline{Y}_\varphi = \overline{Z}_\varphi = \{0, a, b, c\}$ are subalgebras of X . That is, Y and Z are upper soft rough *BCI*-algebras over X . But $\overline{Y} \cap \overline{Z}_\varphi = \overline{\{c\}}_\varphi = \{b, c\}$ is not subalgebra of X .

Finally, we study the upper and lower soft rough *BCI*-algebras.

Theorem 4.6. Let $\mathfrak{S} = (F, A)$ be a *CC*-soft set over X . If Y is a subalgebra of X , then Y is a soft rough *BCI*-algebra over X when $\underline{Y}_\varphi \neq \emptyset$.

Proof. (1) By Theorem 3.8, $\underline{Y}_\varphi * \underline{Y}_\varphi \subseteq \underline{Y} * \underline{Y}_\varphi$. Moreover, by Theorem 3 in [24], $\underline{Y} * \underline{Y}_\varphi \subseteq \underline{Y}_\varphi$ since $Y * Y \subseteq Y$. Hence $\underline{Y}_\varphi * \underline{Y}_\varphi \subseteq \underline{Y}_\varphi$. This means that Y is a lower soft rough *BCI*-algebra over X .

(2) Now, let $a, b \in \overline{Y}_\varphi$, then there exist $y, z \in Y$ such that $\varphi(a) = \varphi(y)$ and $\varphi(b) = \varphi(z)$. Since $\mathfrak{S}$ is a *C*-soft set over X , $\varphi(a * b) = \varphi(y * z)$. Also, $y * z \in Y$

since Y is a subalgebra of X , and so, Y is an upper soft rough BCI -algebra over X .

By (1) and (2), Y is a soft rough BCI -algebra over X . $\square$

Remark 4.7. The above theorem shows that any soft rough BCI -algebra is a generalization of a subalgebra of BCI -algebras.

Open Question: Let $\mathfrak{S} = (F, A)$ be a CC -soft set over X . If Y is an ideal of X , is it a soft rough ideal over X when $\underline{Y}_\varphi \neq \emptyset$?

5. Soft rough BCI -algebras w.r.t. another soft set

In this section, we investigate soft rough BCI -algebras based on another soft set.

Definition 5.1. Let $\mathfrak{S} = (F, A)$ be a soft set over X and (X, φ) an MS -approximation space. Let $\mathfrak{T} = (G, B)$ be another soft set defined over X . The lower MSR -approximation and upper MSR -approximation of $\mathfrak{T}$ w.r.t. $\mathfrak{S}$ are denoted by $\underline{(G, B)}_\varphi = (\underline{G}_\varphi, B)$ and $\overline{(G, B)}_\varphi = (\overline{G}_\varphi, B)$, respectively, which two operations are defined as

$$\underline{G(e)}_\varphi = \{x \in G(e) \mid \varphi(x) \neq \varphi(y) \text{ for all } y \in X - G(e)\}$$

and

$$\overline{G(e)}_\varphi = \{x \in X \mid \varphi(x) = \varphi(y) \text{ for some } y \in G(e)\},$$

for all $e \in B$.

(i) If $\underline{(G, B)}_\varphi = \overline{(G, B)}_\varphi$, then $\mathfrak{T}$ is called *definable*;

(ii) If $\underline{(G, B)}_\varphi \neq \overline{(G, B)}_\varphi$, then $\mathfrak{T}$ is called a *lower (upper) soft rough BCI -algebra (resp., ideal)* w.r.t. $\mathfrak{S}$ over X , if $\underline{G(e)}_\varphi$ ($\overline{G(e)}_\varphi$) is a subalgebra (resp., ideal) of X for all $e \in \text{Supp}(G, B)$;

Moreover, $\mathfrak{T}$ is called a *soft rough BCI -algebra (resp., ideal)* w.r.t. $\mathfrak{S}$ over X , if $\underline{G(e)}_\varphi$ and $\overline{G(e)}_\varphi$ are subalgebras (resp., ideals) of X for all $e \in \text{Supp}(G, B)$.

Example 5.2. Consider the BCI -algebra X and the soft set $\mathfrak{S} = (F, A)$ as in Example 3.2. Define another soft set $\mathfrak{T} = (G, B)$ as the following Table 11:

Table 11 table for soft set $\mathfrak{T}$

	0	a	b	c
e_1	1	0	0	0
e_2	1	1	0	1
e_3	1	0	1	0
e_4	1	0	0	1

By calculations, $\underline{G(e_1)}_\varphi = \emptyset$, $\overline{G(e_1)}_\varphi = \{0, a\}$, $\underline{G(e_2)}_\varphi = \{0, a\}$, $\overline{G(e_2)}_\varphi = \{0, a\}$, $\underline{G(e_3)}_\varphi = \emptyset$, $\overline{G(e_3)}_\varphi = \{0, a, b, c\}$, $\underline{G(e_4)}_\varphi = \emptyset$ and $\overline{G(e_4)}_\varphi = \{0, a, b, c\}$.

Thus, $\mathfrak{T}$ is both a soft rough *BCI*-algebra and a soft rough ideal w.r.t. $\mathfrak{S}$ over X .

Definition 5.3. Let $\mathfrak{S} = (F, A)$ and $\mathfrak{T} = (G, B)$ be two soft sets over X with $C = A \cap B \neq \emptyset$. The *product* $*$ is defined as $\mathfrak{S} * \mathfrak{T} = (F, A) * (G, B) = (K, C)$, where $K(c) = F(c) * G(c)$, for all $c \in C$.

Theorem 5.4. Let $\mathfrak{S} = (F, A)$ be a *C*-soft set over X and (X, φ) be an *MS*-approximation space. Let $\mathfrak{T}_1 = (G_1, B)$ and $\mathfrak{T}_2 = (G_2, C)$ be two soft sets over X with $D = B \cap C \neq \emptyset$. Then

$$\overline{(G_1, B)}_\varphi * \overline{(G_2, C)}_\varphi \subseteq \overline{(G_1 * G_2, D)}_\varphi.$$

Proof. For all $e \in \text{Supp}(G_1, B) \cap \text{Supp}(G_2, C)$, let $c \in \overline{G_1(e)}_\varphi * \overline{G_2(e)}_\varphi$, then $c = a * b$, where $a \in \overline{G_1(e)}_\varphi$ and $b \in \overline{G_2(e)}_\varphi$, and so there exist $y \in G_1(e)$ and $z \in G_2(e)$ such that $\varphi(a) = \varphi(y)$ and $\varphi(b) = \varphi(z)$. Since $\mathfrak{S}$ is a *C*-soft set, $\varphi(a * b) = \varphi(y * z)$ for $y * z \in G_1(e) * G_2(e)$. Hence $c = a * b \in \overline{G_1(e) * G_2(e)}_\varphi$. That is, $\overline{(G_1, B)}_\varphi * \overline{(G_2, C)}_\varphi \subseteq \overline{(G_1 * G_2, D)}_\varphi$. $\square$

The following example shows that the containment in Theorem 5.4 is proper.

Example 5.5. Let $X = \{0, a, b, c, d\}$ be a *BCI*-algebra with the following Cayley Table 12:

Table 12 table for *BCI*-algebra

*	0	a	b	c	d
0	0	0	0	c	c
a	a	0	0	c	c
b	b	a	0	c	c
c	c	c	c	0	0
d	d	c	c	a	0

Define a soft set $\mathfrak{S} = (F, A)$ over X which is given by Table 13.

Table 13 table for soft set $\mathfrak{S}$

	0	a	b	c	d
e_1	1	1	1	0	0
e_2	0	0	0	1	1
e_3	1	1	1	1	1
e_4	0	0	0	1	1

Then the mapping $\varphi : X \rightarrow \mathcal{P}(A)$ of soft rough approximation space (X, φ) is given by $\varphi(0) = \varphi(a) = \varphi(b) = \{e_1, e_3\}$, $\varphi(c) = \varphi(d) = \{e_2, e_3, e_4\}$. Then we can check that $\mathfrak{S}$ is a *C*-soft set over X , but it is not a *CC*-soft set over X .

Define two soft sets $\mathfrak{T}_1 = (G_1, B)$ and $\mathfrak{T}_2 = (G_2, C)$ over X , where $B = \{e_1, e_2\}$ and $C = \{e_2, e_3\}$ with $B \cap C = \{e_2\}$, by $G_1(e_2) = \{c\}$ and $G_2(e_2) = \{c\}$.

By calculations, $\overline{G_1(e_2)}_\varphi = \{c, d\}$ and $\overline{G_2(e_2)}_\varphi = \{c, d\}$, so $\overline{G_1(e_2)}_\varphi * \overline{G_2(e_2)}_\varphi = \{0, a\}$. But $G_1(e_2) * G_2(e_2) = \{0\}$, $\overline{G_1(e_2) * G_2(e_2)}_\varphi = \{0\}_\varphi = \{0, a, b\}$. Thus $\overline{(G_1, B)}_\varphi * \overline{(G_2, C)}_\varphi \subsetneq \overline{(G_1 * G_2, D)}_\varphi$.

If we strength the condition, we can obtain the following result:

Theorem 5.6. *Let $\mathfrak{S} = (F, A)$ be a CC -soft set over X and (X, φ) be an MS -approximation space. Let $\mathfrak{T}_1 = (G_1, B)$ and $\mathfrak{T}_2 = (G_2, C)$ be two soft sets over X with $D = B \cap C \neq \emptyset$. Then*

$$\overline{(G_1, B)}_\varphi * \overline{(G_2, C)}_\varphi = \overline{(G_1 * G_2, D)}_\varphi.$$

Proof. By Theorem 5.4, we have $\overline{(G_1, B)}_\varphi * \overline{(G_2, C)}_\varphi \subseteq \overline{(G_1 * G_2, D)}_\varphi$.

For all $e \in \text{Supp}(G_1, B) \cap \text{Supp}(G_2, C)$, let $x \in \overline{G_1(e)}_\varphi * \overline{G_2(e)}_\varphi$, so $\varphi(x) = \varphi(y * z)$ for some $y \in G_1(e)$ and $z \in G_2(e)$. Then there exist $a, b \in X$, such that $\varphi(a) = \varphi(y)$ and $\varphi(b) = \varphi(z)$ satisfying $x = a * b$ since $\mathfrak{S}$ is a CC -soft set over X . Thus $a \in \overline{G_1(e)}_\varphi$ and $b \in \overline{G_2(e)}_\varphi$. Hence $x \in \overline{G_1(e)}_\varphi * \overline{G_2(e)}_\varphi$. This shows that $\overline{(G_1 * G_2, D)}_\varphi \subseteq \overline{(G_1, B)}_\varphi * \overline{(G_2, C)}_\varphi$.

Summing up the above arguments, $\overline{(G_1, B)}_\varphi * \overline{(G_2, C)}_\varphi = \overline{(G_1 * G_2, D)}_\varphi$. $\square$

Combining Theorems 3.8 and 5.6, we can obtain the following result:

Theorem 5.7. *Let $\mathfrak{S} = (F, A)$ be a CC -soft set over X and (X, φ) be an MS -approximation space. Let $\mathfrak{T}_1 = (G_1, B)$ and $\mathfrak{T}_2 = (G_2, C)$ be two soft sets over X with $D = B \cap C \neq \emptyset$. Then*

$$\underline{(G_1, B)}_\varphi * \underline{(G_2, C)}_\varphi \subseteq \underline{(G_1 * G_2, D)}_\varphi.$$

Finally, we investigate the upper and lower soft rough BCI -algebras with respect to another soft set.

Theorem 5.8. *Let $\mathfrak{S} = (F, A)$ be a CC -soft set over X and (X, φ) be an MS -approximation space. If $\mathfrak{T} = (G, B)$ is a soft BCI -algebra over X , then $\mathfrak{T}$ is a soft rough BCI -algebra over X w.r.t. $\mathfrak{S}$ when $\underline{\mathfrak{T}}_\varphi \neq \emptyset$.*

Proof. (1) By Theorem 5.7, $\underline{(G, B)}_\varphi * \underline{(G, B)}_\varphi \subseteq \underline{(G * G, B)}_\varphi$. Since $\mathfrak{T} = (G, B)$ is a soft BCI -algebra over X , for any $e \in \text{Supp}(G, B)$, $G(e)$ is a subalgebra of X , then $G(e) * G(e) \subseteq G(e)$. By Theorem 9(14) in [24], $\underline{G(e) * G(e)}_\varphi \subseteq \underline{G(e)}_\varphi$, that is, $\underline{(G * G, B)}_\varphi \subseteq \underline{(G, B)}_\varphi$. Thus, $\underline{(G, B)}_\varphi * \underline{(G, B)}_\varphi \subseteq \underline{(G, B)}_\varphi$. This means that $\underline{(G, B)}_\varphi$ is a soft BCI -algebra over X , that is, for any $e \in \text{Supp}(G, B)$, $\underline{G(e)}_\varphi$ is a subalgebra of X . Hence $\mathfrak{T}$ is a lower soft rough BCI -algebra w.r.t. $\mathfrak{S}$ over X .

(2) For any $e \in \text{Supp}(G, B)$, let $a, b \in \overline{G(e)}_\varphi$, then there exist $y, z \in G(e)$ such that $\varphi(a) = \varphi(y)$ and $\varphi(b) = \varphi(z)$. Since $\mathfrak{S}$ is a C -soft set over X , $\varphi(a * b) = \varphi(y * z)$. Also, $y * z \in G(e) * G(e) \subseteq G(e)$ since $\mathfrak{T} = (G, B)$ is a soft BCI -algebra over X , and so, $a * b \in \overline{G(e)}_\varphi$, that is, $\overline{G(e)}_\varphi$ is a subalgebra of X . Hence, $\mathfrak{T}$ is an upper soft rough BCI -algebra w.r.t. $\mathfrak{S}$ over X .

By (1) and (2), $\mathfrak{T}$ is a soft rough BCI -algebra w.r.t. $\mathfrak{S}$ over X . $\square$

6. Soft rough *BCI*-algebras in decision making methods

In this section, we illustrate a kind of new decision making method for Shabir's soft rough sets to *BCI*-algebras.

We will put forth a new method to find which is the best parameter e of a given soft set $\mathfrak{S} = (F, A)$. In other words, $F(e)$ is the nearest accurate *BCI*-algebra on $\mathfrak{S}$ based on another soft set $\mathfrak{T}$ over *BCI*-algebras.

Decision making method:

Let X be a *BCI*-algebra and E a set of related parameters. Let $A = \{e_1, e_2, \dots, e_m\} \subseteq E$, $\mathfrak{S} = (F, A)$ be an original description soft set over X and (X, φ) be an *MS*-approximation space. Let $\mathfrak{S} = (G, B)$ be another soft set over X . Then we present the decision algorithm for soft rough *BCI*-algebras as follows:

Step 1 Input the original description *BCI*-algebra X , soft set $\mathfrak{S}$ and (X, φ) be an *MSR*-approximation space. Consider be another soft set $\mathfrak{S} = (G, B)$ over X .

Step 2 Compute the lower and upper rough soft approximation operators $\underline{(G, B)}_\varphi$ and $\overline{(G, B)}_\varphi$ w.r.t. $\mathfrak{S}$, respectively.

Step 3 Compute the different values of $\|G(e_i)\|$, where $\|G(e_i)\| = \frac{|\overline{(G, B)}_\varphi(e_i)| - |\underline{(G, B)}_\varphi(e_i)|}{|\overline{(G, B)}_\varphi(e_i)|}$.

Step 4 Find the minimum value $\|G(e_k)\|$ of $\|G(e_i)\|$, where $\|G(e_k)\| = \min_i \|G(e_i)\|$.

Step 5 The decision is $G(e_k)$.

Example 6.1. Assume that we want to find the nearest accurate *BCI*-algebra. Let $X = \{0, a, b, c, d\}$ be a *BCI*-algebra with the following Cayley Table 14:

Table 14 table for *BCI*-algebra

*	0	a	b	c	d
0	0	0	0	c	c
a	a	0	0	c	c
b	b	b	0	c	c
c	c	c	c	0	0
d	d	c	c	a	0

Define a soft set $\mathfrak{S} = (F, A)$ over X which is given by Table 15.

Table 15 table for soft set $\mathfrak{S}$

	0	a	b	c	d
e_1	1	1	1	0	0
e_2	1	1	1	1	1
e_3	1	1	1	1	1
e_4	0	0	0	0	0

Then the mapping $\varphi : X \rightarrow \mathcal{P}(A)$ of soft rough approximation space (X, φ) is given by $\varphi(0) = \varphi(a) = \varphi(b) = \{e_1, e_2, e_3\}$, $\varphi(c) = \varphi(d) = \{e_2, e_3\}$.

Define another soft set $\mathfrak{T} = (G, B)$ over X which is given by Table 16.

Table 16 table for soft set $\mathfrak{S}$

	0	a	b	c	d
e_1	1	1	0	0	0
e_2	1	1	0	1	1
e_3	0	0	1	1	1
e_4	1	0	1	1	0
e_5	1	1	1	1	0

That is, $G(e_1) = \{0, a\}$, $G(e_2) = \{0, a, c, d\}$, $G(e_3) = \{b, c, d\}$, $G(e_4) = \{0, b, c\}$ and $G(e_5) = \{0, a, b, c\}$.

By calculations, $\underline{G(e_1)}_\varphi = \emptyset$, $\overline{G(e_1)}_\varphi = \{0, a, b\}$, $\underline{G(e_2)}_\varphi = \{c, d\}$, $\overline{G(e_2)}_\varphi = \{0, a, b, c, d\}$, $\underline{G(e_3)}_\varphi = \{c, d\}$, $\overline{G(e_3)}_\varphi = \{c, d\}$, $\underline{G(e_4)}_\varphi = \emptyset$, $\overline{G(e_4)}_\varphi = \{0, a, b, c, d\}$, $\underline{G(e_5)}_\varphi = \{0, a, b\}$ and $\overline{G(e_5)}_\varphi = \{0, a, b, c, d\}$.

Then, we can calculate $\|G(e_1)\| = 1.5$, $\|G(e_2)\| = 0.75$, $\|G(e_3)\| = 1$, $\|G(e_4)\| = 1.67$ and $\|G(e_5)\| = 0.5$. This means the minimum value for $\|G(e_i)\|$ is $\|G(e_5)\| = 0.5$. That is, $G(e_5) = \{0, a, b, c\}$ is the closest accurate BCI -algebra.

Remark 6.2. (1) In [17], Ma applied rough soft set theory to BCI -algebra in order to find the nearest accurate BCI -algebra, but in the present paper, we try to find the nearest accurate BCI -algebra based on Shabir's soft rough set theory by means of two soft sets $\mathfrak{S}$ and $\mathfrak{T}$.

(2) Given a soft set $\mathfrak{S} = (F, A)$ over X , the decision maker can obtain different object by adjusting another soft set $\mathfrak{T} = (G, B)$. This means that by adjust different $\mathfrak{T} = (G, B)$, the decision maker can obtain the optimal one.

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ON INTUITIONISTIC PRODUCT FUZZY GRAPHS

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Abstract. In this paper, we provide three new operations on intuitionistic product fuzzy graphs; namely direct product, semi-strong product and strong product. We discuss which of these operations preserves the notions of strong and complete. We also give some new properties of balanced intuitionistic fuzzy graphs.

Keywords: Fuzzy product graph, fuzzy intuitionistic product graph, balanced intuitionistic product fuzzy graph.

1. Background

In 1736, Euler introduced the concept of graph theory while trying to find a solution to the well known Königsberg bridge problem. This subject is now considered as a branch of combinatorics. The theory of graph is an extremely useful tool in solving combinatorial problems in areas such as geometry, algebra, number theory, topology, operations research, optimization and computer science. The notion of fuzzy set was first introduced in [19] and the notion of fuzzy graph was first introduced in [4]. Since then, several authors explored this type of graphs. Since the notions of degree, complement, completeness, regularity, connectedness and many others play very important roles in the crisp graph case, the idea is to see what corresponds to these notions in the case of fuzzy graphs. Several authors introduced and studied what they called product fuzzy graphs, see for example [41]. AL-Hawary [32] introduced the concept of balanced fuzzy graphs. He defined three new operations on fuzzy graphs and explored what classes of fuzzy graphs are balanced. Since then, many authors have studied the idea of balanced on distinct kinds of fuzzy graphs, see for example [8, 9, 18, 24, 25, 27]. In 1983, Atanassov [16] introduced the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets [17]. Atanassov added a new component in the definition of fuzzy set. The fuzzy sets give the degree of

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membership of an element in a given set and the nonmembership degree equals one minus the degree of membership, while intuitionistic fuzzy sets give both a degree of membership and a degree of non-membership which are more-or-less independent from each other. The only requirement is that the sum of these two degrees is not greater than 1. Intuitionistic fuzzy sets have been applied in a wide variety of fields including computer science, engineering, mathematics, medicine, chemistry and economics [15, 28]. Parvathy and Karunambigai [29] introduced the concept of Intuitionistic fuzzy graph (IFG) elaborately and analyzed its components. Articles [16, 20, 22, 28] motivated us to analyze balanced IFGs and their properties. Several authors have studied balanced IFGs, see for example [24, 25] and balanced product IFGs (PIFGs) were studied by [27, 42]. The idea of balanced came from matroids, see [31, 33, 34, 35, 36, 37, 38, 40, 39].

This paper deals with the significant properties of balanced IFGs. The basic definition and theorems needed are discussed in section 2. In Section 3, we define three new operations on PIFGs and we discuss which of these operations preserves the notions of strong and complete. Section 4 is devoted to providing more new results on PIFGs. We remark that the results in this paper were done in Bayan Hourani masters thesis titled "On complete and balanced fuzzy graphs" under the supervision of Talal Al-Hawary at Yarmouk University in 2015.

2. Background

A fuzzy subset of a non-empty set V is a mapping $\sigma : V \rightarrow [0, 1]$ and a fuzzy relation μ on a fuzzy subset σ , is a fuzzy subset of $V \times V$. All throughout this paper, we assume that σ is reflexive, μ is symmetric and V is finite.

Definition 1 ([4]). A *fuzzy graph* with V as the underlying set is a pair $G : (\sigma, \mu)$ where $\sigma : V \rightarrow [0, 1]$ is a fuzzy subset and $\mu : V \times V \rightarrow [0, 1]$ is a fuzzy relation on σ such that $\mu(x, y) \leq \sigma(x) \wedge \sigma(y), \forall x, y \in V$, where $\wedge$ stands for minimum. The underlying crisp graph of G is denoted by $G^* : (\sigma^*, \mu^*)$ where $\sigma^* = \sup p(\sigma) = \{x \in V : \sigma(x) > 0\}$ and $\mu^* = \sup p(\mu) = \{(x, y) \in V \times V : \mu(x, y) > 0\}$. $H = (\sigma', \mu')$ is a *fuzzy subgraph* of G if there exists $X \subseteq V$ such that $\sigma' : X \rightarrow [0, 1]$ is a fuzzy subset and $\mu' : X \times X \rightarrow [0, 1]$ is a fuzzy relation on σ' such that $\mu(x, y) \leq \sigma(x) \wedge \sigma(y), \forall x, y \in X$.

Definition 2 ([41]). An intuitionistic fuzzy graph (*simply, IFG*) is of the form $G : (V, E)$ where

- (i) $V = \{\nu_0, \nu_1, \dots, \nu_n\}$ such that $\mu_1 : V \rightarrow [0, 1]$ and $\gamma_1 : V \rightarrow [0, 1]$, denotes the degree of membership and non-membership of the element $\nu_i \in V$, respectively and $0 \leq \mu_1(\nu_i) + \gamma_1(\nu_i) \leq 1, \forall \nu_i \in V, (i = 1, 2, \dots, n)$,

- (ii) $E \subseteq V \times V$ where $\mu_2 : V \times V \rightarrow [0, 1]$ and $\gamma_2 : V \times V \rightarrow [0, 1]$ are such that

$$\begin{aligned}\mu_2(\nu_i, \nu_j) &\leq \min[\mu_1(\nu_i), \mu_1(\nu_j)], \\ \gamma_2(\nu_i, \nu_j) &\leq \max[\gamma_1(\nu_i), \gamma_1(\nu_j)]\end{aligned}$$

and $0 \leq \mu_2(\nu_i, \nu_j) + \gamma_2(\nu_i, \nu_j) \leq 1$, for every $(\nu_i, \nu_j) \in E$, $(i, j = 1, 2, \dots, n)$.

Definition 3 ([41]). An IFG $H = (\acute{V}, \acute{E})$ is said to be an Intuitionistic fuzzy subgraph (IFSG) of the IFG $G : (\gamma_1; \mu_1, \gamma_2; \mu_2)$ if

- (i) $\acute{V} \subseteq V$, where $\acute{\mu}_{1i} = \mu_{1i}$, $\acute{\gamma}_{1i} = \gamma_{1i}$, $\forall \nu_i \in \acute{V}$, $i = 1, 2, 3, \dots, n$.
(ii) $\acute{E} \subseteq E$, where $\acute{\mu}_{2ij} = \mu_{2ij}$, $\acute{\gamma}_{2ij} \geq \gamma_{2ij}$, $\forall (\nu_i, \nu_j) \in \acute{E}$, $i, j = 1, 2, \dots, n$.

Corollary 1 ([30]). The complement of an IFG, $G : (V, E)$ is an IFG, $G^c : (V^c, E^c)$, where

- (i) $V^c = V$,
(ii) $\mu_{1i}^c = \mu_{1i}$ and $\gamma_{1i}^c = \gamma_{1i}$, $\forall i = 1, 2, \dots, n$,
(iii) $\mu_{2ij}^c = \min(\mu_{1i}, \mu_{1j}) - \mu_{2ij}$ and $\gamma_{2ij}^c = \max(\gamma_{1i}, \gamma_{1j}) - \gamma_{2ij}$, $\forall i, j = 1, 2, \dots, n$.

Definition 4 ([30]). An IFG, $G : (V, E)$ is said to be complete IFG if $\mu_{2ij} = \min(\mu_{1i}, \mu_{1j})$ and $\gamma_{2ij} = \max(\gamma_{1i}, \gamma_{1j})$, $\forall \nu_i, \nu_j \in V$.

Definition 5 ([30]). An IFG, $G : (V, E)$ is said to be strong IFG if $\mu_{2ij} = \min(\mu_{1i}, \mu_{1j})$ and $\gamma_{2ij} = \max(\gamma_{1i}, \gamma_{1j})$, $\forall \nu_i, \nu_j \in E$.

Definition 6 ([24]). Let $G_1 : (V_1, E_1)$ and $G_2 : (V_2, E_2)$ be two IFG's. An isomorphism between G_1 and G_2 (denoted by $G_1 \simeq G_2$) is a bijective map $h : V_1 \rightarrow V_2$ which satisfies

$$\mu_1(\nu_i) = \acute{\mu}_1(h(\nu_i)), \nu_1(\nu_i) = \acute{\nu}_1(h(\nu_i)), \mu_2(\nu_i, \nu_j) = \acute{\mu}_2(h(\nu_i), h(\nu_j))$$

and

$$\gamma_2(\nu_i, \nu_j) = \acute{\gamma}_2(h(\nu_i), h(\nu_j)), \forall \nu_i, \nu_j \in V.$$

Definition 7 ([24]). The density of an IFG, $G : (V, E)$ is

$$D(G) = (D_\mu(G), D_\gamma(G))$$

where

$$D_\mu(G) = \frac{2 \sum_{(\nu_i, \nu_j) \in V} (\mu_2(\nu_i, \nu_j))}{\sum_{(\nu_i, \nu_j) \in E} (\mu_1(\nu_i) \wedge \mu_1(\nu_j))}$$

and

$$D_\gamma(G) = \frac{2 \sum_{(\nu_i, \nu_j) \in V} (\gamma_2(\nu_i, \nu_j))}{\sum_{(\nu_i, \nu_j) \in E} (\gamma_1(\nu_i) \vee \gamma_1(\nu_j))}.$$

Definition 8 ([24]). An IFG, $G : (V, E)$ is balanced if $D(H) \leq D(G)$, that is $D_\mu(H) \leq D_\mu(G)$, $D_\gamma(H) \leq D_\gamma(G)$, $\forall$ non-empty subgraphs H of G .

Definition 9 ([25]). Let $G_1 : (V_1, E_1)$ and $G_2 : (V_2, E_2)$ be an IFG'S, where $V = V_1 \times V_2$ and

$$E = \{(x_1, y_1)(x_2, y_2) : (x_1, x_2) \in E_1, (y_1, y_2) \in E_2\}.$$

Then the direct product of G_1 and G_2 is an IFG denoted by $G_1 \sqcap G_2 = (V, E)$, where

$$\begin{aligned} (\mu_1 \sqcap \dot{\mu}_1)(x_1, y_1) &= \mu_1(x_1) \wedge \dot{\mu}_1(y_1), \quad \forall (x_1, y_1) \in V_1 \times V_2, \\ (\gamma_1 \sqcap \dot{\gamma}_1)(x_1, y_1) &= \gamma_1(x_1) \vee \dot{\gamma}_1(y_1), \quad \forall (x_1, y_1) \in V_1 \times V_2, \\ (\mu_2 \sqcap \dot{\mu}_2)(x_1, y_1)(x_2, y_2) &= \mu_2(x_1, x_2) \wedge \dot{\mu}_2(y_1, y_2), \quad \forall (x_1, x_2) \in E_1, (y_1, y_2) \in E_2, \\ (\gamma_2 \sqcap \dot{\gamma}_2)(x_1, y_1)(x_2, y_2) &= \gamma_2(x_1, x_2) \vee \dot{\gamma}_2(y_1, y_2), \quad \forall (x_1, x_2) \in E_1, (y_1, y_2) \in E_2. \end{aligned}$$

Definition 10 ([25]). Let $G_1 : (V_1, E_1)$ and $G_2 : (V_2, E_2)$ be an IFG'S, where $V = V_1 \times V_2$ and

$$\begin{aligned} E &= \{(x, x_2)(x, y_2) : x \in V_1, (x_2, y_2) \in E_2\} \\ &\cup \{(x_1, y_1)(x_2, y_2) : (x_1, x_2) \in E_1, (y_1, y_2) \in E_2\}. \end{aligned}$$

Then the semi-strong product of G_1 and G_2 is an IFG denoted by $G_1 \odot G_2 = (V, E)$, where

$$\begin{aligned} (\mu_1 \odot \dot{\mu}_1)(x_1, x_2) &= \mu_1(x_1) \wedge \dot{\mu}_1(x_2), \quad \forall (x_1, x_2) \in V_1 \times V_2, \\ (\gamma_1 \odot \dot{\gamma}_1)(x_1, x_2) &= \gamma_1(x_1) \vee \dot{\gamma}_1(x_2), \quad \forall (x_1, x_2) \in V_1 \times V_2, \\ (\mu_2 \odot \dot{\mu}_2)(x, x_2)(x, y_2) &= \mu_1(x) \wedge \dot{\mu}_2(x_2, y_2), \quad \forall x \in V_1, (x_2, y_2) \in E_2, \\ (\gamma_2 \odot \dot{\gamma}_2)(x, x_2)(x, y_2) &= \gamma_1(x) \vee \dot{\gamma}_2(x_2, y_2), \quad \forall x \in V_1, (x_2, y_2) \in E_2, \\ (\mu_2 \odot \dot{\mu}_2)(x_1, x_2)(y_1, y_2) &= \mu_2(x_1, x_2) \wedge \dot{\mu}_2(y_1, y_2), \quad \forall (x_1, x_2) \in E_1, (y_1, y_2) \in E_2, \\ (\gamma_2 \odot \dot{\gamma}_2)(x_1, x_2)(y_1, y_2) &= \gamma_2(x_1, x_2) \vee \dot{\gamma}_2(y_1, y_2), \quad \forall (x_1, x_2) \in E_1, (y_1, y_2) \in E_2. \end{aligned}$$

Definition 11 ([25]). Let $G_1 : (V_1, E_1)$ and $G_2 : (V_2, E_2)$ be an IFG'S, where $V = V_1 \times V_2$ and $E = \{(x, y_1)(x, y_2) : x \in V_1, (y_1, y_2) \in E_2\} \cup \{(x_1, y)(x_2, y) : y \in V_2, (x_1, x_2) \in E_1\} \cup \{(x_1, y_1)(x_2, y_2) : (x_1, x_2) \in E_1, (y_1, y_2) \in E_2\}$ Then the strong product of G_1 and G_2 is an IFG denoted by $G_1 * G_2 : (V, E)$, where

$$\begin{aligned} \mu_1 * \dot{\mu}_1(x_1, x_2) &= \mu_1(x_1) \wedge \dot{\mu}_1(x_2) \text{ forevery } (x_1, x_2) \in V_1 \times V_2, \\ (\gamma_1 * \dot{\gamma}_1)(x_1, x_2) &= \gamma_1(x_1) \vee \dot{\gamma}_1(x_2) \text{ forevery } (x_1, x_2) \in V_1 \times V_2, \\ (\mu_2 * \dot{\mu}_2)(x, x_2)(x, y_2) &= \mu_1(x) \wedge \dot{\mu}_2(x_2, y_2), \quad \forall x \in V_1, (x_2, y_2) \in E_2, \\ (\gamma_2 * \dot{\gamma}_2)(x, x_2)(x, y_2) &= \gamma_1(x) \vee \dot{\gamma}_2(x_2, y_2), \quad \forall x \in V_1, (x_2, y_2) \in E_2, \\ (\mu_2 * \dot{\mu}_2)(x_1, x_2)(y_1, y_2) &= \mu_2(x_1, x_2) \wedge \dot{\mu}_2(y_1, y_2), \quad \forall (x_1, x_2) \in E_1, (y_1, y_2) \in E_2, \\ (\gamma_2 * \dot{\gamma}_2)(x_1, x_2)(y_1, y_2) &= \gamma_2(x_1, x_2) \vee \dot{\gamma}_2(y_1, y_2), \quad \forall (x_1, x_2) \in E_1, (y_1, y_2) \in E_2, \\ (\mu_2 * \dot{\mu}_2)(x_1, y)(x_2, y) &= \mu_2(x_1, x_2) \wedge \dot{\mu}_2(y), \quad \forall (x_1, x_2) \in E_1, y \in V_2, \\ (\gamma_2 * \dot{\gamma}_2)(x_1, y)(x_2, y) &= \gamma_2(x_1, x_2) \vee \dot{\gamma}_2(y), \quad \forall (x_1, x_2) \in E_1, y \in V_2. \end{aligned}$$

3. Product intuitionistic fuzzy graph (PIFG)

The first definition of product intuitionistic fuzzy graph was introduced by Vinoth and Geetha in [27]. In this section, we recall some necessary definitions related to our work.

Definition 12 ([27]). Let $G : (V, E)$ be an IFG. If $\mu_2(\nu_i, \nu_j) \leq \mu_1(\nu_i)\mu_1(\nu_j)$ and $\gamma_2(\nu_i, \nu_j) \leq \gamma_1(\nu_i)\gamma_1(\nu_j)$, $\forall (\nu_i, \nu_j) \in V$, the intuitionistic fuzzy graph is called product intuitionistic fuzzy graph of G (simply, PIFG).

Definition 13 ([24]). A PIFG $G : (V, E)$ is said to be complete product if $\mu_2(\nu_i, \nu_j) = \mu_1(\nu_i)\mu_1(\nu_j)$ and

$$\gamma_2(\nu_i, \nu_j) = \gamma_1(\nu_i)\gamma_1(\nu_j), \forall \nu_i, \nu_j \in V.$$

Definition 14 ([24]). The complement of a PIFG $G : (V, E)$ is $G^c : (V^c, E^c)$ where $V^c = V$, $\mu_{1i}^c = \mu_{1i}$ and $\gamma_{1i}^c = \gamma_{1i}$, $\forall i = 1, 2, \dots, n$, $\mu_{2ij}^c = \mu_1(\nu_i)\mu_1(\nu_j) - \mu_2(\nu_i, \nu_j)$ and

$$\gamma_{2ij}^c = \gamma_1^c(\nu_i)\gamma_1^c(\nu_j) - \gamma_2^c(\nu_i, \nu_j), \forall i, j = 1, 2, \dots, n.$$

Definition 15 ([25]). The density of a PIFG $G : (V, E)$ is defined by $D(G) = (D_\mu(G), D_\gamma(G))$ where

$$D_\mu(G) = \frac{2 \sum_{\nu_i, \nu_j \in V} (\mu_2(\nu_i, \nu_j))}{\sum_{\nu_i, \nu_j \in E} (\mu_1(\nu_i)\mu_1(\nu_j))},$$

$$D_\gamma(G) = \frac{2 \sum_{\nu_i, \nu_j \in V} (\gamma_2(\nu_i, \nu_j))}{\sum_{\nu_i, \nu_j \in E} (\gamma_1(\nu_i)\gamma_1(\nu_j))}.$$

Definition 16 ([25]). A PIFG $G : (V, E)$ is balanced if $D(H) \leq D(G)$, that is, $D_\mu(H) \leq D_\mu(G)$, $D_\gamma(H) \leq D_\gamma(G)$, $\forall$ subgraph H of G .

4. Operations on PIFGs

In this section, we define the operations of direct product, semi-strong product and strong product on PIFG'S and we study some results related to balanced PIFG'S.

Definition 17. Let $G_1 : (V_1, E_1)$ and $G_2 : (V_2, E_2)$ be PIFG'S, where $V = V_1 \times V_2$ and $E = \{(u_1, v_1)(u_2, v_2) : (u_1, u_2) \in E_1, (v_1, v_2) \in E_2\}$. Then the direct product of G_1 and G_2 is a PIFG denoted by $G_1 \square G_2 : (V, E)$ where

$$\begin{aligned} (\mu_1 \square \dot{\mu}_1)(u_1, v_1) &= \mu_1(u_1)\dot{\mu}_1(v_1), \forall (u_1, v_1) \in V_1 \times V_2, \\ (\gamma_1 \square \dot{\gamma}_1)(u_1, v_1) &= \gamma_1(u_1)\dot{\gamma}_1(v_1), \forall (u_1, v_1) \in V_1 \times V_2, \\ (\mu_2 \square \dot{\mu}_2)(u_1, v_1)(u_2, v_2) &= \mu_2(u_1, u_2)\dot{\mu}_2(v_1, v_2), \forall (u_1, u_2) \in E_1, (v_1, v_2) \in E_2 \end{aligned}$$

and

$$(\gamma_2 \sqcap \dot{\gamma}_2)(u_1, v_1)(u_2, v_2) = \gamma_2(u_1, u_2)\dot{\mu}_2(v_1, v_2), \quad \forall (u_1, u_2) \in E_1, (v_1, v_2) \in E_2.$$

Definition 18. Let $G_1 : (V_1, E_1)$ and $G_2 : (V_2, E_2)$ be PIFG'S, where $V = V_1 \times V_2$ and

$$E = \{(u, u_2)(u, v_2) : u \in V_1, (u_2, v_2) \in E_2\} \\ \cup \{(u_1, v_1)(u_2, v_2) : (u_1, u_2) \in E_1, (v_1, v_2) \in E_2\}.$$

Then the semi-strong product of G_1 and G_2 is a PIFG denoted by $G_1 \odot G_2 = (V, E)$ where

$$\begin{aligned} (\mu_1 \odot \dot{\mu}_1)(u_1, u_2) &= \mu_1(u_1)\dot{\mu}_1(u_2), \quad \forall (u_1, u_2) \in V_1 \times V_2, \\ (\gamma_1 \odot \dot{\gamma}_1)(u_1, u_2) &= \gamma_1(u_1)\dot{\gamma}_1(u_2), \quad \forall (u_1, u_2) \in V_1 \times V_2, \\ (\mu_2 \odot \dot{\mu}_2)(u, u_2)(u, v_2) &= \mu_1^2(u)\dot{\mu}_2(u_2, v_2), \quad \forall u \in V_1, (u_2, v_2) \in E_2, \\ (\gamma_2 \odot \dot{\gamma}_2)(u, u_2)(u, v_2) &= \gamma_1^2(u)\dot{\gamma}_2(u_2, v_2), \quad \forall u \in V_1, (u_2, v_2) \in E_2, \\ (\mu_2 \odot \dot{\mu}_2)(u_1, u_2)(v_1, v_2) &= \mu_2(u_1, u_2)\dot{\mu}_2(v_1, v_2), \quad \forall (u_1, u_2) \in E_1, (v_1, v_2) \in E_2 \end{aligned}$$

and

$$(\gamma_2 \odot \dot{\gamma}_2)(u_1, u_2)(v_1, v_2) = \gamma_2(u_1, u_2)\dot{\gamma}_2(v_1, v_2), \quad \forall (u_1, u_2) \in E_1, (v_1, v_2) \in E_2.$$

Definition 19. Let $G_1 : (V_1, E_1)$ and $G_2 : (V_2, E_2)$ be PIFG'S, where $V = V_1 \times V_2$ and $E = \{(u, v_1)(u, v_2) : u \in V_1, (v_1, v_2) \in E_2\} \cup \{(u_1, v)(u_2, v) : v \in V_2, (u_1, u_2) \in E_1\} \cup \{(u_1, v_1)(u_2, v_2) : (u_1, u_2) \in E_1, (v_1, v_2) \in E_2\}$. Then the strong product of G_1 and G_2 is a PIFG denoted by $G_1 * G_2 = (V, E)$ where

$$\begin{aligned} (\mu_1 * \dot{\mu}_1)(u_1, u_2) &= \mu_1(u_1)\dot{\mu}_1(u_2) \quad \forall (u_1, u_2) \in V_1 \times V_2, \\ (\gamma_1 * \dot{\gamma}_1)(u_1, u_2) &= \gamma_1(u_1)\dot{\gamma}_1(u_2) \quad \forall (u_1, u_2) \in V_1 \times V_2, \\ (\mu_2 * \dot{\mu}_2)(u, u_2)(u, v_2) &= \mu_1^2(u)\dot{\mu}_2(u_2, v_2) \quad \forall u \in V_1, (u_2, v_2) \in E, \\ (\gamma_2 * \dot{\gamma}_2)(u, u_2)(u, v_2) &= \gamma_1^2(u)\dot{\gamma}_2(u_2, v_2) \quad \forall u \in V_1, (u_2, v_2) \in E_2, \\ (\mu_2 * \dot{\mu}_2)(u_1, u_2)(v_1, v_2) &= \mu_2(u_1, u_2)\dot{\mu}_2(v_1, v_2) \quad \forall (u_1, u_2) \in E_1, (v_1, v_2) \in E_2, \\ (\gamma_2 * \dot{\gamma}_2)(u_1, u_2)(v_1, v_2) &= \gamma_2(u_1, u_2)\dot{\gamma}_2(v_1, v_2) \quad \forall (u_1, u_2) \in E_1, (v_1, v_2) \in E_2, \\ (\mu_2 * \dot{\mu}_2)(u_1, v)(u_2, v) &= \mu_2(u_1, u_2)\dot{\mu}_2^2(v) \quad \forall (u_1, u_2) \in E_1, v \in V_2, \end{aligned}$$

and

$$(\gamma_2 * \dot{\gamma}_2)(u_1, v)(u_2, v) = \gamma_2(u_1, u_2)\dot{\gamma}_2^2(v) \quad \forall (u_1, u_2) \in E_1, v \in V_2.$$

Theorem 1. Let $G_1 : (V_1, E_1)$ and $G_2 : (V_1, E_1)$ be strong PIFG'S. Then $G_1 \sqcap G_2$ is strong.

Proof. If $(u_1, v_1)(u_2, v_2) \in E$, then since G_1 and G_2 are strong PIFG'S, then

$$\begin{aligned} (\mu_2 \sqcap \acute{\mu}_2)(u_1, v_1)(u_2, v_2) &= \mu_2(u_1, u_2)\acute{\mu}_2(v_1, v_2) \\ &= \mu_1(u_1)\mu_1(u_2)\acute{\mu}_1(v_1)\acute{\mu}_1(v_2) \\ &= (\mu_1 \sqcap \acute{\mu}_1)((u_1, v_1)(u_2, v_2)), \end{aligned}$$

and

$$\begin{aligned} (\gamma_2 \sqcap \acute{\gamma}_2)(u_1, v_1)(u_2, v_2) &= \gamma_2(u_1, u_2)\acute{\gamma}_2(v_1, v_2) \\ &= \gamma_1(u_1)\gamma_1(u_2)\acute{\gamma}_1(v_1)\acute{\gamma}_1(v_2) \\ &= (\gamma_1 \sqcap \acute{\gamma}_1)((u_1, v_1)(u_2, v_2)). \end{aligned}$$

Therefore, $G_1 \sqcap G_2$ is strong. $\square$

Theorem 2. Let $G_1 : (V_1, E_1)$ and $G_2 : (V_1, E_1)$ be strong PIFG's. Then $G_1 \odot G_2$ is strong.

Proof. If $(u, v_1)(u, v_2) \in E$, then since G_1 and G_2 are strong, then

$$\begin{aligned} (\mu_2 \sqcap \acute{\mu}_2)(u, v_1)(u, v_2) &= \mu_1(u)^2\acute{\mu}_2(v_1, v_2) \\ &= \mu_1(u)\mu_1(u)\acute{\mu}_1(v_1)\acute{\mu}_1(v_2) \\ &= (\mu_1 \sqcap \acute{\mu}_1)((u, v_1)(u, v_2)), \end{aligned}$$

and

$$\begin{aligned} (\gamma_2 \sqcap \acute{\gamma}_2)(u, v_1)(u, v_2) &= \gamma_1(u)^2\acute{\gamma}_2(v_1, v_2) \\ &= \gamma_1(u)\gamma_1(u)\acute{\gamma}_1(v_1)\acute{\gamma}_1(v_2) \\ &= (\gamma_1 \sqcap \acute{\gamma}_1)((u, v_1)(u, v_2)). \end{aligned}$$

If $(u_1, v_1)(u_2, v_2) \in E$, then since G_1 and G_2 are strong, then

$$\begin{aligned} (\mu_2 \sqcap \acute{\mu}_2)(u_1, v_1)(u_2, v_2) &= \mu_2(u_1, u_2)\acute{\mu}_2(v_1, v_2) \\ &= \mu_1(u_1)\mu_1(u_2)\acute{\mu}_1(v_1)\acute{\mu}_1(v_2) \\ &= (\mu_1 \sqcap \acute{\mu}_1)((u_1, v_1)(u_2, v_2)), \end{aligned}$$

and

$$\begin{aligned} (\gamma_2 \sqcap \acute{\gamma}_2)(u_1, v_1)(u_2, v_2) &= \gamma_2(u_1, u_2)\acute{\gamma}_2(v_1, v_2) \\ &= \gamma_1(u_1)\gamma_1(u_2)\acute{\gamma}_1(v_1)\acute{\gamma}_1(v_2) \\ &= (\gamma_1 \sqcap \acute{\gamma}_1)((u_1, v_1)(u_2, v_2)). \end{aligned}$$

Hence $G_1 \odot G_2$ is strong. $\square$

Theorem 3. Let $G_1 : (V_1, E_1)$ and $G_2 : (V_1, E_1)$ be strong PIFG'S. Then $G_1 \otimes G_2$ is strong.

Proof. If $(u, v_1)(u, v_2) \in E$, then since G_1 and G_2 are strong, then

$$\begin{aligned} (\mu_2 \sqcap \acute{\mu}_2)(u, v_1)(u, v_2) &= \mu_1(u)^2 \acute{\mu}_2(v_1, v_2) \\ &= \mu_1(u) \mu_1(u) \acute{\mu}_1(v_1) \acute{\mu}_1(v_2) \\ &= (\mu_1 \sqcap \acute{\mu}_2)((u, v_1)(u, v_2)), \end{aligned}$$

and

$$\begin{aligned} (\gamma_2 \sqcap \acute{\gamma}_2)(u, v_1)(u, v_2) &= \gamma_1(u)^2 \gamma_2(v_1, v_2) \\ &= \gamma_1(u) \gamma_1(u) \acute{\gamma}_1(v_1) \acute{\gamma}_1(v_2) \\ &= (\gamma_1 \sqcap \acute{\gamma}_1)((u, v_1)(u, v_2)). \end{aligned}$$

If $(u_1, v)(u_2, v) \in E$, then since G_1 and G_2 are strong, then

$$\begin{aligned} (\mu_2 \sqcap \acute{\mu}_2)(u_1, v)(u_2, v) &= \mu_2(u_1, u_2) \acute{\mu}(v)^2 \\ &= \mu_1(u_1) \mu_1(u_2) \acute{\mu}_1(v) \acute{\mu}_1(v) \\ &= (\mu_1 \sqcap \acute{\mu}_2)((u_1, v)(u_2, v)), \end{aligned}$$

and

$$\begin{aligned} (\gamma_2 \sqcap \acute{\gamma}_2)(u_1, v)(u_2, v) &= \gamma_1(u_1, u_2) \gamma_2(v, v) \\ &= \gamma_1(u_1) \gamma_1(u_2) \acute{\gamma}_1(v) \acute{\gamma}_1(v) \\ &= (\gamma_1 \sqcap \acute{\gamma}_1)((u_1, v)(u_2, v)). \end{aligned}$$

If $(u_1, v_1)(u_2, v_2) \in E$, then since G_1 and G_2 are strong, then

$$\begin{aligned} (\mu_2 \sqcap \acute{\mu}_2)(u_1, v_1)(u_2, v_2) &= \mu_2(u_1, u_2) \acute{\mu}_2(v_1, v_2) \\ &= \mu_1(u_1) \mu_1(u_2) \acute{\mu}_1(v_1) \acute{\mu}_1(v_2) \\ &= (\mu_1 \sqcap \acute{\mu}_2)((u_1, v_1)(u_2, v_2)), \end{aligned}$$

and

$$\begin{aligned} (\gamma_2 \sqcap \acute{\gamma}_2)(u_1, v_1)(u_2, v_2) &= \gamma_2(u_1, u_2) \gamma_2(v_1, v_2) \\ &= \gamma_1(u_1) \gamma_1(u_2) \acute{\gamma}_1(v_1) \acute{\gamma}_1(v_2) \\ &= (\gamma_1 \sqcap \acute{\gamma}_1)((u_1, v_1)(u_2, v_2)). \end{aligned}$$

Hence $G_1 \otimes G_2$ is strong. □

From Theorem 1 and Theorem 3, we get,

Corollary 2. *Let $G_1 : (V_1, E_1)$ and $G_2 : (V_1, E_1)$ be complete PIFG'S. Then $G_1 \sqcap G_2$ and $G_1 \odot G_2$ are strong.*

We remark that the above result need not be true for $G_1 \otimes G_2$ since $G_1 \otimes G_2$ is never complete by its definition.

5. Balanced PIFG'S

Balanced PIFGs were defined by Karunambigai, Sivasankar and Palanivel in [27] and some properties on PIFG were established. In this section, we examine many of the results that have not been studied.

Lemma 1. *Let $G : (V, E)$ be a self-complementary PIFG. Then*

$$\begin{aligned} \sum_{\nu_i, \nu_j \in V} \mu_2(\nu_i, \nu_j) &= \frac{1}{2} \sum_{\nu_i, \nu_j \in V} \mu_1(\nu_i) \mu_2(\nu_j), \\ \sum_{\nu_i, \nu_j \in V} \gamma_2(\nu_i, \nu_j) &= \frac{1}{2} \sum_{\nu_i, \nu_j \in V} \gamma_1(\nu_i) \gamma_1(\nu_j). \end{aligned}$$

Proof. Let $G : (V, E)$ be a self-complementary PIFG. Then by Definition of G^c , $V^c = V$, $\mu_{1i}^c = \mu_{1i}$ and $\gamma_{1i}^c = \gamma_{1i}$, $\forall i = 1, 2, \dots, n$, $\mu_2^c(h(\nu_i), h(\nu_j)) = (\mu_1(h(\nu_i), \mu_1(h(\nu_j)) - \mu_2(h(\nu_i), h(\nu_j))$ and $\gamma_2^c(h(\nu_i), h(\nu_j)) = (\gamma_1(h(\nu_i), \gamma_1(h(\nu_j)) - \gamma_2(h(\nu_i), h(\nu_j))), \forall i, j = 1, 2, \dots, n$, $\mu_2^c(h(\nu_i), h(\nu_j)) = \mu_2(\nu_i, \nu_j)$. Now

$$\sum_{\nu_i, \nu_j \in V} \mu_2(\nu_i, \nu_j) + \sum_{\nu_i, \nu_j \in V} \mu_2(h(\nu_i), h(\nu_j)) = \sum_{\nu_i, \nu_j \in V} \mu_1(\nu_i) \mu_1(\nu_j).$$

Thus

$$2 \sum_{\nu_i, \nu_j \in V} \mu_2(\nu_i, \nu_j) = \sum_{\nu_i, \nu_j \in V} \mu_1(\nu_i) \mu_1(\nu_j).$$

So

$$\sum_{\nu_i, \nu_j \in V} \mu_2(\nu_i, \nu_j) = \frac{1}{2} \sum_{\nu_i, \nu_j \in V} \mu_1(\nu_i) \mu_1(\nu_j).$$

Also,

$$\sum_{\nu_i, \nu_j \in V} \gamma_2(\nu_i, \nu_j) + \sum_{\nu_i, \nu_j \in V} \gamma_2(h(\nu_i), h(\nu_j)) = \sum_{\nu_i, \nu_j \in V} \gamma_1(\nu_i) \gamma_1(\nu_j).$$

Thus

$$2 \sum_{\nu_i, \nu_j \in V} \gamma_2(\nu_i, \nu_j) = \sum_{\nu_i, \nu_j \in V} \gamma_1(\nu_i) \gamma_1(\nu_j).$$

So

$$\sum_{\nu_i, \nu_j \in V} \gamma_2(\nu_i, \nu_j) = \frac{1}{2} \sum_{\nu_i, \nu_j \in V} \gamma_1(\nu_i) \gamma_1(\nu_j). \quad \square$$

Theorem 4. *Let $G : (V, E)$ be a PIFG and $G^c : (V^c, E^c)$ be its complement. Then $D(G) + D(G^c) \geq (2, 2)$.*

Proof. Let $G : (V, E)$ be a PIFG and $G^c : (V^c, E^c)$ be its complement. Let H be a non-empty fuzzy subgraph of G . Since G is PIFG

$$\mu_2^c(\nu_i, \nu_j) = \mu_1(\nu_i)\mu_1(\nu_j) - \mu_2(\nu_i, \nu_j),$$

Thus

$$\sum_{\nu_i, \nu_j \in V^C} \mu_2^c(\nu_i, \nu_j) + \sum_{\nu_i, \nu_j \in V} \mu_2(\nu_i, \nu_j) = \sum_{\nu_i, \nu_j \in V} \mu_1(\nu_i)\mu_1(\nu_j).$$

So

$$\frac{\sum_{\nu_i, \nu_j \in V^C} \mu_2^c(\nu_i, \nu_j)}{\sum_{\nu_i, \nu_j \in V^C} \mu_1(\nu_i)\mu_1(\nu_j)} + \frac{\sum_{\nu_i, \nu_j \in V} \mu_2(\nu_i, \nu_j)}{\sum_{\nu_i, \nu_j \in V} \mu_1(\nu_i)\mu_1(\nu_j)} = 1.$$

Hence

$$\frac{2 \sum_{\nu_i, \nu_j \in V^C} \mu_2^c(\nu_i, \nu_j)}{\sum_{\nu_i, \nu_j \in V^C} \mu_1(\nu_i)\mu_1(\nu_j)} + \frac{2 \sum_{\nu_i, \nu_j \in V} \mu_2(\nu_i, \nu_j)}{\sum_{\nu_i, \nu_j \in V} \mu_1(\nu_i)\mu_1(\nu_j)} = 2.$$

Therefore,

$$\begin{aligned} & \frac{2 \sum_{\nu_i, \nu_j \in V^C} \mu_2^c(\nu_i, \nu_j)}{\sum_{\nu_i, \nu_j \in V^C} \mu_1(\nu_i)\mu_1(\nu_j)} + \frac{2 \sum_{\nu_i, \nu_j \in V} \mu_2(\nu_i, \nu_j)}{\sum_{\nu_i, \nu_j \in V} \mu_1(\nu_i)\mu_1(\nu_j)} \\ & \leq \frac{2 \sum_{\nu_i, \nu_j \in V^C} \mu_2^c(\nu_i, \nu_j)}{\sum_{\nu_i, \nu_j \in E^C} \mu_1(\nu_i)\mu_1(\nu_j)} + \frac{2 \sum_{\nu_i, \nu_j \in V} \mu_2(\nu_i, \nu_j)}{\sum_{\nu_i, \nu_j \in E} \mu_1(\nu_i)\mu_1(\nu_j)}. \end{aligned}$$

Also

$$\begin{aligned} \gamma_2^c(\nu_i, \nu_j) &= \gamma_1(\nu_i)\gamma_1(\nu_j) - \gamma_2(\nu_i, \nu_j), \\ \mu_2^c(\nu_i, \nu_j) + \mu_2(\nu_i, \nu_j) &= \mu_1(\nu_i)\mu_1(\nu_j). \end{aligned}$$

So

$$\sum_{\nu_i, \nu_j \in V^C} \gamma_2^c(\nu_i, \nu_j) + \sum_{\nu_i, \nu_j \in V} \gamma_2(\nu_i, \nu_j) = \sum_{\nu_i, \nu_j \in V} \gamma_1(\nu_i)\gamma_1(\nu_j).$$

Hence

$$\frac{\sum_{\nu_i, \nu_j \in V^C} \gamma_2^c(\nu_i, \nu_j)}{\sum_{\nu_i, \nu_j \in V^C} \gamma_1(\nu_i)\gamma_1(\nu_j)} + \frac{\sum_{\nu_i, \nu_j \in V} \gamma_2(\nu_i, \nu_j)}{\sum_{\nu_i, \nu_j \in V} \gamma_1(\nu_i)\gamma_1(\nu_j)} = 1$$

and

$$\frac{2 \sum_{\nu_i, \nu_j \in V^C} \gamma_2^c(\nu_i, \nu_j)}{\sum_{\nu_i, \nu_j \in V^C} \gamma_1(\nu_i)\gamma_1(\nu_j)} + \frac{2 \sum_{\nu_i, \nu_j \in V} \gamma_2(\nu_i, \nu_j)}{\sum_{\nu_i, \nu_j \in V} \gamma_1(\nu_i)\gamma_1(\nu_j)} = 2$$

Therefore,

$$\begin{aligned} & \frac{2 \sum_{\nu_i, \nu_j \in V^C} \gamma_2^c(\nu_i, \nu_j)}{\sum_{\nu_i, \nu_j \in V^C} \gamma_1(\nu_i)\gamma_1(\nu_j)} + \frac{2 \sum_{\nu_i, \nu_j \in V} \gamma_2(\nu_i, \nu_j)}{\sum_{\nu_i, \nu_j \in V} \gamma_1(\nu_i)\gamma_1(\nu_j)} \\ & \leq \frac{2 \sum_{\nu_i, \nu_j \in V^C} \gamma_2^c(\nu_i, \nu_j)}{\sum_{\nu_i, \nu_j \in E^C} \gamma_1(\nu_i)\gamma_1(\nu_j)} + \frac{2 \sum_{\nu_i, \nu_j \in V} \gamma_2(\nu_i, \nu_j)}{\sum_{\nu_i, \nu_j \in E} \gamma_1(\nu_i)\gamma_1(\nu_j)}. \quad \square \end{aligned}$$

Theorem 5. Let $G : (V, E)$ be a self-complementary PIFG. Then $D(G) \geq (1, 1)$.

Proof.

$$\begin{aligned}
 D(G) &= (D_\mu(G), D_\gamma(G)) \\
 &= \left(\left(\frac{2 \sum_{\nu_i, \nu_j \in v} \mu_2(\nu_i, \nu_j)}{\sum_{\nu_i, \nu_j \in E} (\mu_1(\nu_i) \mu_2(\nu_j))}, \frac{2 \sum_{\nu_i, \nu_j \in v} \gamma_2(\nu_i, \nu_j)}{\sum_{\nu_i, \nu_j \in E} (\gamma_1(\nu_i) \gamma_1(\nu_j))} \right) \right) \\
 &\geq \left(\frac{2 \sum_{\nu_i, \nu_j \in v} \mu_2(\nu_i, \nu_j)}{\sum_{\nu_i, \nu_j \in v} (\mu_1(\nu_i) \mu_2(\nu_j))}, \frac{2 \sum_{\nu_i, \nu_j \in v} \gamma_2(\nu_i, \nu_j)}{\sum_{\nu_i, \nu_j \in v} (\gamma_1(\nu_i) \gamma_1(\nu_j))} \right).
 \end{aligned}$$

But by Lemma 1, we get

$$\begin{aligned}
 D(G) &\geq \left(\frac{2^{\frac{1}{2}} \sum_{\nu_i, \nu_j \in v} (\mu_1(\nu_i) \mu_2(\nu_j))}{\sum_{\nu_i, \nu_j \in v} (\mu_1(\nu_i) \mu_2(\nu_j))}, \frac{2^{\frac{1}{2}} \sum_{\nu_i, \nu_j \in v} (\gamma_1(\nu_i) \gamma_1(\nu_j))}{\sum_{\nu_i, \nu_j \in v} (\gamma_1(\nu_i) \gamma_1(\nu_j))} \right) \\
 &= (1, 1). \quad \square
 \end{aligned}$$

Theorem 6. Let $G_1 : (V_1, E_1)$ and $G_2 : (V_1, E_1)$ be two complete PIFG'S Then $D(G_i) \leq D(G_1 \sqcap G_2)$ for $i = 1, 2$ if and only if $D(G_1) = D(G_2) = D(G_1 \sqcap G_2)$.

Proof. If $D(G_i) \leq D(G_1 \sqcap G_2)$ for $i = 1, 2$, since G_1 and G_2 are complete PIFG'S,

$$D(G_1) = D(G_2) = 2.$$

and by Corollary 2, $G_1 \sqcap G_2$ is strong and hence $D(G_1 \sqcap G_2) = 2$. Thus $D(G_i) \geq D(G_1 \sqcap G_2)$ for $i = 1, 2$ and so $D(G_1) = D(G_2) = D(G_1 \sqcap G_2)$.

The converse is trivial. $\square$

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ON SOME SUBCLASSES OF MEROMORPHIC FUNCTIONS DEFINED BY FRACTIONAL DERIVATIVE OPERATOR

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Abstract. In this paper, we use fractional derivative operator to define new classes of meromorphic functions related to conic domains. Using convolution and differential subordination techniques, we prove some interesting properties of these newly defined classes.

Keywords: Fractional derivative operator, conic domain, bounded boundary rotation, coefficient bounds.

1. Introduction

Goodman [4] originated the idea of conic domains by introducing the classes UCV and US^*T . Further Ronning [19], Ma and Minda [10] gave a well ordered one variable characterization of these classes. Later on Kanas and Wisnniowska [6, 7] introduced k –uniformly convex and starlike functions. Similarly Rønning and several others [11, 12] studied the class of close to convex functions related

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with conic domains. Acu [1] extended this concept by using Salagean operator [21]. Lowner [9] introduced the class V_m of functions of bounded boundary rotations which was improved by Paatero [17]. Noor connected functions with bounded boundary rotations with conic domains, see for some details [13, 14, 15]. The class of alpha-quasi convex functions was introduced and studied by Noor et-al. [16]. Recently Haq et-al. [5] related this class with conic domains. We generalize this idea to the space of meromorphic functions and define some new classes of meromorphic functions by using fractional derivative of order α . We prove inclusion results, coefficient problems and some other interesting properties.

Let M denote the class of functions of the form

$$(1) \quad f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^{k-1},$$

which are analytic and univalent in $E^* = \{z : 0 < |z| < 1\} = E \setminus \{0\}$. Let $MS^*(\gamma)$, $MC(\gamma)$ be the subclasses of M that are meromorphic starlike and convex of order γ ($0 \leq \gamma < 1$) respectively. A function $f \in M$ is said to belong to the class $k - MS^*(\gamma)$ of meromorphic uniformly starlike of order γ , if

$$-\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > k \left| \frac{zf'(z)}{f(z)} - 1 \right| + \gamma, \quad k \geq 0, \quad 0 \leq \gamma < 1.$$

It is clear from the definition that for $k = 0$, we get the class $MS^*(\gamma)$. Next we define the meromorphic analogue of the class $k - K(\gamma, \beta)$ defined by Noor et-al. [11] as follows:

A function $f \in M$ is in the class $k - MK(\gamma, \beta)$, if

$$-\Re \left\{ \frac{zf'(z)}{g(z)} \right\} > k \left| \frac{zf'(z)}{g(z)} - 1 \right| + \beta, \quad \text{for some } g \in k - MS^*(\gamma),$$

$$k \geq 0, \quad 0 \leq \beta < 1.$$

Further an analytic function p with $p(0) = 1$ is said to belong to the class $P(p_{k,\gamma})$ if and only if $p(z)$ takes all values in the conic domain $\Lambda_{k,\gamma}$, $k \geq 0$, $\gamma \in [0, 1)$, such that

$$\Lambda_{k,\gamma} = (1 - \gamma) \Lambda_k + \gamma,$$

where

$$\Lambda_k = \left\{ u + iv : u > k \sqrt{(u-1)^2 + v^2} \right\}, \quad k \geq 0.$$

The extremal functions $p_{k,\gamma}(z)$ for $P(p_{k,\gamma})$ are given by

$$(2) \quad p_{k,\gamma}(z) = \begin{cases} \frac{1+(1-2\gamma)z}{1-z}, & k = 0, \\ 1 + \frac{2(1-\gamma)}{\pi^2} \left(\log \frac{1+\sqrt{z}}{1-\sqrt{z}} \right)^2, & k = 1, \\ \frac{1-\gamma}{1-k^2} \cosh \left\{ \left(\frac{2}{\pi} \arccos k \right) \log \frac{1+\sqrt{z}}{1-\sqrt{z}} \right\} - \frac{k^2-\gamma}{1-k^2}, & 0 < k < 1, \\ 1 + \frac{1-\gamma}{k^2-1} \sin \left(\frac{\pi}{2K(\kappa)} \int_0^{\frac{u(z)}{\sqrt{\kappa}}} \frac{dt}{\sqrt{1-t^2}\sqrt{1-\kappa^2 t^2}} \right) + \frac{k^2-\gamma}{k^2-1}, & k > 1. \end{cases}$$

where

$$u(z) = \frac{z - \sqrt{\kappa}}{1 - \sqrt{\kappa}z}, \quad z \in E,$$

and $\kappa \in (0, 1)$ is selected in such a way that $k = \cosh(\pi K'(\kappa)/(4K(\kappa)))$. Here $K(\kappa)$ is Legendre's complete elliptic integral of first kind and $K'(\kappa) = K(\sqrt{1 - \kappa^2})$ and $K'(t)$ is the complementary integral of $K(t)$.

Remark 1. If $p_{k,\gamma}(z)$, be given by

$$p_{k,\gamma}(z) = 1 + \delta_1 z + \delta_2 z^2 + \dots,$$

then

$$(3) \quad \delta_1 = \delta(k, \gamma) = \begin{cases} 8(1-\gamma)(\arccos k)^2, & 0 \leq k < 1 \\ \frac{8(1-\gamma)}{\pi^2}, & k = 1, \\ \frac{\pi^2(1-\gamma)}{4\sqrt{x(k^2-1)K^2(x)(x+1)}}, & k > 1. \end{cases}$$

The meromorphic analogue of the fractional derivative of order α , $0 \leq \alpha < 1$, is defined in [2] for a function $f(z)$ by

$$D_z^\alpha f(z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \left\{ z^{\alpha-1} \int_0^z (z-\xi)^{-\alpha} {}_2F_1 \left(1-\alpha, 1, 1-\alpha; 1-\frac{\xi}{z} \right) \xi^2 f(\xi) d\xi \right\},$$

where $f(z)$ is analytic function in a simply connected domain of the z -plane containing the origin and the multiplicity of $(z-\xi)^{-\alpha}$ is removed by requiring $\log(z-\xi)$ to be real when $(z-\xi) > 0$. Using $D_z^\alpha f(z)$, we define an operator $\Omega_z^\alpha f(z) : M \rightarrow M$, as follows:

$$(4) \quad \begin{aligned} \Omega_z^\alpha f(z) &= \frac{\Gamma(2-\alpha)}{\Gamma(2)} z D_z^\alpha f(z) \\ &= \frac{1}{z} + \sum_{k=1}^{\infty} \frac{(2)_k}{(2-\alpha)_k} a_k z^{k-1} \\ &= \phi(2, 2-\alpha; z) * f(z), \quad \alpha \neq 2, 3, 4, \dots, \end{aligned}$$

where

$$\phi(2, 2-\alpha; z) = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{(2)_k}{(2-\alpha)_k} z^{k-1}.$$

We now define the following classes of functions.

Definition 2. Let $f \in M$. Then $f \in k-MK(m, \gamma, \beta)$, $k \geq 0$, $\gamma, \beta \in (0, 1)$, $m \geq 2$ if and only if there exists $g \in k-MS^*(\gamma)$ such that

$$-\frac{zf'(z)}{g(z)} \in \mathcal{P}_m(p_{k,\beta}), \quad z \in E^*.$$

Also $f \in k - MK_\mu^\alpha(m, \gamma, \beta)$ if and only if

$$- \left\{ (1 - \mu) \Omega_z^\alpha f(z) + \mu \left(z (\Omega_z^\alpha f(z))' \right) \right\} \in k - MK(m, \gamma, \beta).$$

It can easily be seen that $f \in k - MUK_\mu^\alpha(m, \gamma, \beta)$ implies that

$$(5) \quad - \left\{ (1 - \mu) \frac{z (\Omega_z^\alpha f(z))'}{\Omega_z^\alpha g(z)} + \mu \frac{z \left(z (\Omega_z^\alpha f(z))' \right)'}{\Omega_z^\alpha g(z)} \right\} \in \mathcal{P}_m(p_{k,\beta}),$$

where $g \in k - MS^*(\gamma)$, $k \geq 0$, $\gamma, \beta \in (0, 1)$, $m \geq 2$, $\mu \in \mathbb{R}$ and $z \in E$.

Special cases

- i) For $\alpha = k = \gamma = \beta = 0$, $m = 2$, $\mu = 1$, the class $0 - MUK_1^0(2, 0, 0)$ reduces to the class MC^* of meromorphic quasi convex functions.
- ii) For $\alpha = k = \gamma = \beta = 0$, $m = 2$, $\mu = 0$, the class $0 - MUK_0^0(2, 0, 0)$ reduces to the class MK , see [8].

Remark 3. Note that if f is given by (1) then from (4), we can write

$$(1 - \mu) \Omega_z^\alpha f(z) + \mu z (\Omega_z^\alpha f(z))' = z + \sum_{j=2}^{\infty} \mathcal{A}_j z^j,$$

where

$$(6) \quad \mathcal{A}_j = \frac{\Gamma(j+1) \Gamma(2-\alpha)}{\Gamma(j+1-\alpha)} (1 + \mu(j-1)) a_j.$$

2. Preliminaries

Lemma 4 ([20]). Let f and g be convex and starlike univalent functions respectively. Then, for any analytic function F in E

$$\frac{f * F g}{f * g}(E) \subset \overline{co}(F(E)),$$

where $\overline{co}(F(E))$ denotes the close convex hull of $F(E)$.

Lemma 5 ([18]). Let $h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ be subordinate to $H(z) = 1 + \sum_{n=1}^{\infty} C_n z^n$ in E . If H is univalent and $H(E)$ is convex, then $|c_n| \leq |C_1|$, $n \geq 1$.

Now we give the following lemma concerning with the class $k - MS^*(\gamma)$. The proof is straight forward therefore, we omit it.

Lemma 6. Let $f \in k - MS^*(\gamma)$. Then

$$|b_j| \leq \frac{(|\delta(k, \gamma)|)_{j+1}}{(|\delta(k, \gamma)| + 1)(|\delta(k, \gamma)| + 2)(j+1)!}, \quad j \geq 2.$$

Proceeding in a similar way as [11], yields the following result.

Lemma 7. Let $\Omega_z^\alpha f(z) \in k - MS^*(\gamma)$. Then $f(z) \in k - MS^*(\gamma)$.

3. Main results

Here we shall investigate certain properties of above defined classes. We use techniques of convolution and differential subordination to study these properties. Throughout in this section, we assume $k \geq 0$, $m \geq 2$, $\alpha, \gamma, \beta \in [0, 1)$ and $z \in E$, unless otherwise stated.

Theorem 8. *Let*

$$\Re \left\{ \frac{z (\Omega_z^\alpha g(z))'}{\Omega_z^\alpha g(z)} \right\} < 2 - \rho, \quad 0 \leq \rho < 1.$$

Then

$$k - MK_\mu^\alpha(m, \gamma, \beta) \subset k - MUK_\mu^0(m, \gamma, \beta).$$

Proof. Let $f \in k - MUK_\mu^\alpha(m, \gamma, \beta)$. Then

$$- \left\{ (1 - \mu) \frac{z (\mathfrak{L}_\alpha f(z))'}{\mathfrak{L}_\alpha g(z)} + \mu \frac{z (z (\mathfrak{L}_\alpha f(z))')'}{\mathfrak{L}_\alpha g(z)} \right\} \in \mathcal{P}_m(p_{k,\beta}),$$

where

$$- \frac{z (\mathfrak{L}_\alpha g(z))'}{\mathfrak{L}_\alpha g(z)} \in \mathcal{P}(p_{k,\gamma}) \quad \text{in } E^*.$$

From Lemma 4, we have, $g \in k - MS^*(\gamma)$ whenever $\mathfrak{L}_\alpha g(z) \in k - MS^*(\gamma)$, in E^* . Now

$$\begin{aligned} & - \left\{ (1 - \mu) \frac{zf'(z)}{g(z)} + \mu \frac{z(zf'(z))'}{g(z)} \right\} \\ = & - \left\{ (1 - \mu) \frac{\varphi(2-\alpha, 2; z) * \varphi(2, 2-\alpha; z) * zf'(z)}{\varphi(2-\alpha, 2; z) * \varphi(2, 2-\alpha; z) * g(z)} \right. \\ & \left. + \mu \frac{\varphi(2-\alpha, 2; z) * \varphi(2, 2-\alpha; z) * z(zf'(z))'}{\varphi(2-\alpha, 2; z) * \varphi(2, 2-\alpha; z) * g(z)} \right\} \\ = & - \left\{ (1 - \mu) \frac{\varphi(2-\alpha, 2; z) * z(\Omega_z^\alpha f(z))'}{\varphi(2-\alpha, 2; z) * \Omega_z^\alpha g(z)} \right. \\ & \left. + \mu \frac{\varphi(2-\alpha, 2; z) * z(z(\Omega_z^\alpha f(z))')'}{\varphi(2-\alpha, 2; z) * \Omega_z^\alpha g(z)} \right\} \\ = & \frac{\varphi(2-\alpha, 2; z) * \left[- \left\{ (1 - \mu) \frac{z(\Omega_z^\alpha f(z))'}{\Omega_z^\alpha g(z)} + \mu \frac{z(z(\Omega_z^\alpha f(z))')'}{\Omega_z^\alpha g(z)} \right\} (\Omega_z^\alpha g) \right]}{\varphi(2-\alpha, 2; z) * \Omega_z^\alpha g(z)} \\ = & \frac{\varphi(2-\alpha, 2; z) * F(\Omega_z^\alpha g)}{\varphi(2-\alpha, 2; z) * \Omega_z^\alpha g(z)} \\ = & \left(\frac{m}{4} + \frac{1}{2} \right) \frac{\varphi(2-\alpha, 2; z) * F_1(\Omega_z^\alpha g)}{\varphi(2-\alpha, 2; z) * \Omega_z^\alpha g(z)} \\ & - \left(\frac{m}{4} - \frac{1}{2} \right) \frac{\varphi(2-\alpha, 2; z) * F_2(\Omega_z^\alpha g)}{\varphi(2-\alpha, 2; z) * \Omega_z^\alpha g(z)}, \end{aligned}$$

where $F_i \in \mathcal{P}(p_{k,\gamma})$, $i = 1, 2$ and by hypothesis of theorem, we get

$$\Re \left\{ \frac{z (\Omega_z^\alpha g(z))'}{\Omega_z^\alpha g(z)} \right\} < 2 - \rho, \quad 0 \leq \rho < 1.$$

From which we have

$$\Re \left\{ \frac{z (z^2 \Omega_z^\alpha g(z))'}{z^2 \Omega_z^\alpha g(z)} \right\} > \rho.$$

This implies $z^2 \Omega_z^\alpha g(z) \in S^*$, the class of usual starlike functions.

Now $z^2 \psi(2 - \alpha, 2; z) = \varphi(2 - \alpha, 2; z)$ is a convex function. Using Lemma 1, we have for $i = 1, 2$

$$\frac{z^2 \psi(2 - \alpha, 2; z) * G_i(z^2 \Omega_z^\alpha g)}{z^2 \psi(2 - \alpha, 2; z) * z^2 \Omega_z^\alpha g(z)} \subseteq \overline{co}(\mathcal{P}_{k,\beta}(E)).$$

This shows that

$$-\left\{ (1 - \mu) \frac{z f'(z)}{g(z)} + \mu \frac{z (z f'(z))'}{g(z)} \right\} \in \mathcal{P}_m(\mathcal{P}_{k,\beta}) \text{ in } E^*.$$

Thus $f \in k - MUK_\mu^0(m, \gamma, \beta)$. $\square$

The following theorem can be proved by using the similar arguments as above.

Theorem 9. *Let $0 \leq \alpha_1 < \alpha - 2 < 1$. Then for*

$$\begin{aligned} \Re \left\{ \frac{z (\Omega_z^\alpha g(z))'}{\Omega_z^\alpha g(z)} \right\} &< 2 - \rho, \quad 0 \leq \rho < 1, \\ k - MUK_\mu^{\alpha_2}(m, \gamma, \beta) &\subset k - MUK_\mu^{\alpha_1}(m, \gamma, \beta). \end{aligned}$$

Theorem 10. *Let $f \in k - MUK_\mu^\alpha(m, \gamma, \beta)$ and be given by (1). Then*

$$\begin{aligned} |a_j| \leq & \frac{\lceil(j+1-\alpha)\rceil}{(1+\mu(j-1))\lceil(j+1)\rceil\lceil(2-\alpha)\rceil} \left\{ \frac{2(\delta(k, \beta))_{j+1}}{(\delta(k, \beta)+1)(\delta(k, \beta)+2)j(j+1)!} \right. \\ & \left. + \frac{m}{2j} |\delta(k, \beta)| \left(1 + \sum_{l=1}^{j-1} \frac{2|\delta(k, \beta)|_{l+1}}{(\delta(k, \beta)+1)(\delta(k, \gamma)+2)(l+1)!} \right) \right\} \end{aligned}$$

Proof. Let $G(z) = \Omega_z^\alpha g(z) \in k - MS^*(\gamma)$ and write

$$\begin{aligned} G(z) &= \frac{1}{z} + \sum_{j=2}^{\infty} \mathcal{B}_j z^{j-1}, \\ g(z) &= \frac{1}{z} + \sum_{j=2}^{\infty} b_j z^{j-1}. \end{aligned}$$

Then

$$(7) \quad \mathcal{B}_j = \frac{\lceil(j+1)\rceil\lceil(2-\alpha)\rceil}{\lceil(j+1-\alpha)\rceil} b_j, \quad j \geq 2.$$

For $p \in P_m(p_{k,\beta})$ and $p(z) = 1 + c_1 z + c_2 z^2 + \dots$, let

$$p(z) = \left(\frac{m}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{m}{4} - \frac{1}{2}\right) p_2(z), \quad p_i(z) \prec p_{k,\beta}, \quad i = 1, 2,$$

writing

$$p_i(z) = 1 + d_1 z + d_2 z^2 + \dots, \quad j \geq 1,$$

we have

$$|d_j| \leq |\delta(k, \beta)|,$$

where $\delta(k, \beta)$ is given by (3) and we have used Lemma 2. Combining these facts, we have

$$(8) \quad |c_j| \leq \frac{m}{2} |\delta(k, \beta)|.$$

Now, using (4) and (5), we have

$$(9) \quad j\mathcal{A}_j = \mathcal{B}_j + \sum_{l=1}^{j-1} c_{j-l}\mathcal{B}_j, \quad j \geq 2.$$

From (7), (8), (9) and Lemma 3, it follows that

$$(10) \quad |\mathcal{A}_j| \leq \frac{(|\delta(k, \beta)|)_{j-1}}{j!} + \frac{m|\delta(k, \beta)|}{2j} \sum_{l=1}^{j-1} \frac{|\delta(k, \beta)|_{l-1}}{(l-1)!}.$$

We obtain the desire result from (4) and (10). $\square$

Theorem 11. Let $f \in k - MUK_\mu^\alpha(m, \gamma, \beta)$ and $h \in M$ such that $z^2 h$ be a convex univalent function. Then for $\Re\left\{\frac{z(\Omega_z^\alpha g(z))'}{\Omega_z^\alpha g(z)}\right\} < 2 - \rho$, $0 \leq \rho < 1$,

$$(f * h)(z) \in k - MUK_\mu^\alpha(m, \gamma, \beta), \quad z \in E^*.$$

Proof. Let $f \in k - MUK_\mu^\alpha(m, \gamma, \beta)$. Then

$$-\left\{(1-\mu)\frac{z(\Omega_z^\alpha f(z))'}{\Omega_z^\alpha g(z)} + \mu\frac{z(z(\Omega_z^\alpha f(z)))'}{\Omega_z^\alpha g(z)}\right\} \in \mathcal{P}_m(p_{k,\beta}),$$

where

$$\Omega_z^\alpha g(z) \in k - MS^*(\gamma) \subseteq MS^* \quad \text{in } E^*.$$

Now

$$\begin{aligned} & -\left\{(1-\mu)\frac{z(\Omega_z^\alpha (f * h)(z))'}{\Omega_z^\alpha (g * h)(z)} + \mu\frac{z(z(\Omega_z^\alpha (f * h)(z)))'}{\Omega_z^\alpha (g * h)(z)}\right\} \\ &= \frac{z^2 h * \left\{-\left\{(1-\mu)\frac{z(\Omega_z^\alpha f(z))'}{\Omega_z^\alpha g(z)} + \mu\frac{z(z(\Omega_z^\alpha f(z)))'}{\Omega_z^\alpha g(z)}\right\}\right\} z^2 \Omega_z^\alpha g}{z^2 h * z^2 \Omega_z^\alpha g} \\ &= \frac{z^2 h * F(z^2 \Omega_z^\alpha g)}{z^2 h * z^2 \Omega_z^\alpha g}. \end{aligned}$$

Now from the hypothesis of theorem and applying Lemma 1, we obtain our desire result. $\square$

The following operator was defined by Bajpai in [3]. For $\varepsilon \in \mathbb{C}$ and $\Re \varepsilon > 0$, we have $I_\varepsilon : M \rightarrow M$ as

$$(11) \quad I_\varepsilon G(z) = \frac{\varepsilon}{z^{\varepsilon+1}} \int_0^z G(t) t^\varepsilon dt.$$

Theorem 12. Let $G \in k - MUK_\mu^\alpha(m, \gamma, \beta)$ and let $f(z) = I_\varepsilon G(z)$, where I_ε is the integral operator defined by (11). Then $f \in k - MUK_\mu^\alpha(m, \gamma, \beta)$ for

$$\Re \left\{ \frac{z(\Omega_z^\alpha g(z))'}{\Omega_z^\alpha g(z)} \right\} < 2 - \rho, \quad 0 \leq \rho < 1$$

and $z \in E^*$.

Proof. As

$$I_\varepsilon G(z) = \Psi(z) * G(z),$$

where

$$\Psi_\varepsilon(z) = \frac{1}{z} + \sum_{j=0}^{\infty} \frac{\varepsilon}{1+j+\varepsilon} z^j, \quad \Re \{\varepsilon\} > 0.$$

Now $z^2 \Psi_\varepsilon(z) = \phi(z)$, with $\varepsilon = 1 + a$ is convex in E^* , see [20]. Proof follows immediately by applying Theorem 4, and hence $f \in k - MUK_\mu^\alpha(m, \gamma, \beta)$ for z in E^* . $\square$

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ON THE SUM OF THE SQUARES OF ALL DISTANCES IN SOME GRAPHS

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Abstract. Denote the sum of the squares of all distances between all pairs of vertices in G by $S(G)$. In this article, through the given vertices number of a graph and the chromatic, a lower bound of $S(G)$ is discussed. By giving the vertices number and the clique number of a graph, the upper and lower bounds of the $S(G)$ are discussed.

Keywords: chromatic number, clique number.

1. Introduction

In this paper, we only consider connected, simple and undirected graphs and assume that all graphs are connected, and refer to Bondy and Murty [2] for notation and terminologies used but not defined here.

Let $G = (V_G, E_G)$ be a graph with vertex set V_G and edge set E_G . $G - v, G - uv$ denote the graph obtained from G by deleting vertex $v \in V_G$ or edge $uv \in E_G$, respectively (this notation is naturally extended if more than one vertex or edge is deleted). Similarly, $G + uv$ is obtained from G by adding an edge $uv \notin E_G$. For $v \in V_G$, let $N_G(v)$ ($N(v)$ for short) denote the set of all the adjacent vertices of v in G and $d(v) = |N_G(v)|$, the degree of v in G .

Recall that G is called k -connected if $|G| > k$ and $G - Z$ is connected for every set $Z \subseteq V_G$ with $|Z| < k$. The greatest integer k such that G is k -connected is the connectivity $k(G)$ of G . Thus, $k(G) = 0$ if and only if G is disconnected or H_1 , and $k(H_1) = n - 1$ for all $n \geq 1$.

Analogously, if $|G| > 1$ and $G - N$ is connected for every set $N \subseteq E_G$ of fewer than l edges, then G is called l -edge-connected. The greatest integer l such that G is l -connected is the edge-connectivity $k'(G)$ of G . In particular, $k'(G) = 0$ if G is disconnected.

A bipartite graph G is a simple graph, whose vertex set V_G can be partitioned into two disjoint subsets V_1 and V_2 such that every edge of G joins a vertex of V_1 with a vertex of V_2 . A bipartite graph in which every two vertices from

different partition classes are adjacent is called complete, which is denoted by $H_{m,n}$, where $m = |V_1|, n = |V_2|$.

The distance $d(u, v)$ between vertices u and v in G is defined as the length of a shortest path between them. The diameter of G is the maximal distance between any two vertices of G . $L_G(u)$ denotes the sum of square of all distances from u in G .

Let $\mathcal{C}_n^s$ (resp. $\mathcal{D}_n^t$) be the class of all n -vertex bipartite graphs with connectivity s (resp. edge-connectivity t).

Let $S = S(G)$ be the sum of square of distances between all pairs of vertices of G , which is denoted by

$$S = S(G) = \sum_{u,v \in V_G} d_G^2(u, v) = \frac{1}{2} \sum_{v \in V_G} L_G(v).$$

This quantity was introduced by Mustapha Aouchich and Pierre Hansen in [1] and has been extensively studied in the monograph. Recently, $S(G)$ is applied to the research of distance spectral radius. Zhou and Trinajstić [17] proved an upper bound using the order n in addition to the sum of the squares of the distances $S(G)$, see [16, 18]. They also proved a lower bound on the distance spectral radius of a graph using only $S(G)$. As a continuance of it, in this paper, we determine sharp bounds on $S(G)$ for several classes of connected bipartite graphs. For surveys and some up-to-date papers related to Wiener index of trees and line graphs, see [5, 7, 9, 10, 11, 12, 13, 15] and [3, 4, 6, 8, 14], respectively.

In this paper we study the quantity S in the case of n -vertex bipartite graphs, which is an important class of graphs in graph theory. Based on the structure of bipartite graphs, sharp bounds on S among $\mathcal{C}_n^s$ (resp. $\mathcal{D}_n^t$) are determined. The corresponding extremal graphs are identified, respectively.

Further on we need the following lemma, which is the direct consequence of the definition of S .

Lemma 1.1. *Let G be a connected graph of order n and not isomorphic to S_n . Then for each edge $e \in \overline{G}$, $S(G) > S(G + e)$.*

In this article we only consider the simple, connected graphs. Let G be a graph with n vertices and $e(G)$ edges. The sum of distance between a vertex v and all other vertices is denoted by $d_G(v)$. The number of vertex pairs at distance k in G is denoted by $d(G, k)$. We use $\chi(G)$ and $C(G)$ to represent the chromatic number and clique number, respectively. K_n and P_n will denote the complement graph and the path on n vertices. Let $K_r \cdot P_n$ be the graph obtained from K_r and P_n by joining a vertex of K_r to one end vertex of P_n . We use the symbol $K_k(r_1, r_2, \dots, r_k)$ to denote the k -partite graph whose i th class contain exactly r_i vertices. The Turaán graph, $T_{m,n}$, is a complete m -partite graph on n vertices in which each part has either $\lceil n/k \rceil$ or $\lfloor n/k \rfloor$ vertices.

$S(G)$ is defined as the sum of square of distances between all pairs of vertices of G , which is denoted by

$$S(G) = \sum_{u,v \in V_G} d_G^2(u, v).$$

The upper and lower bound of a tree T_n , is given by the following lemma.

Lemma 1.2. *Let T_n be a tree on n vertices, then*

$$2n^2 - 5n + 3 \leq S(T_n) \leq \frac{n^4 - n^2}{12},$$

the upper bound is achieved if and only if $T_n \cong P_n$ and the lower bound is achieved if and only if $T_n \cong K_{1,n-1}$.

Proof. Let T^* be a graph with minimum $S(T)$ in the class of graphs with n vertices. It is obvious that T^* has a pendent vertex, say u . Let v be the pendent vertex of $K_{1,n-1}$. Then

$$\begin{aligned} S(T^*) &= S(T^* - u) + \sum_{p \in V_{T^*}} d^2(u, p) \\ S(K_{1,n-1}) &= S(K_{1,n-1} - v) + \sum_{p \in V_{K_{1,n-1}}} d^2(v, p). \end{aligned}$$

By the induction hypothesis,

$$S(T^* - u) \geq S(K_{1,n-1} - v).$$

It is easy to find that

$$\sum_{p \in V_{T^*}} d^2(u, p) \geq \sum_{p \in V_{K_{1,n-1}}} d^2(v, p)$$

thus,

$$S(T^*) \geq S(K_{1,n-1}).$$

Let $S(T')$ be the maximum value in the class of graphs. The pendent vertices of T' and P_n is u, v , respectively. Similar to the proof above, we can get that $S(T') \leq S(P_n)$. The computation of $K_{1,n-1}$ and P_n is following,

$$S(K_{1,n-1}) = \frac{[1 + 2^2(n-2)](n-1) + (n-1)}{2} = 2n^2 - 5n + 3.$$

Recall that

$$\begin{aligned} 1^2 + 2^2 + 3^2 + \cdots + n^2 &= \frac{n(n+1)(2n+1)}{6} \\ 1^3 + 2^3 + 3^3 + \cdots + n^3 &= \frac{n^2(n+1)^2}{4}. \end{aligned}$$

Then, we can compute the $S(P_n)$ by

$$\begin{aligned}
 S(P_n) &= \sum_{i=1}^{n-1} \frac{n(n+1)(2n+1)}{6} \\
 &= \sum_{i=1}^{n-1} \frac{2i^3 + 3i^2 + i}{6} \\
 &= \frac{1}{6} \left(\sum_{i=1}^{n-1} 2i^3 + \sum_{i=1}^{n-1} 3i^2 + \sum_{i=1}^{n-1} i \right) \\
 &= \frac{n^4 - n^2}{12}.
 \end{aligned}$$

Therefore,

$$2n^2 - 5n + 3 \leq T_n \leq \frac{n^4 - n^2}{12}$$

and the proof above implies that the upper bound is achieved if and only if $T_n \cong P_n$ and the lower bound is achieved if and only if $T_n \cong K_{1,n-1}$. $\square$

A upper bound of $S(G)$ with giving the chromatic number is shown in the second section. The upper and lower of $S(G)$ with giving the clique number are shown in the third section.

2. Chromatic number

Theorem 2.1. *Let G be a graph with $\chi(G) = k$. Then*

$$(2.1) \quad S(G) \geq 2n(n-1) - 3 \binom{n-t}{2} - 3(k-1) \binom{t+1}{2},$$

where $t = \lfloor n/k \rfloor$, and the equality holds if and only if $G \cong T_{k,n}$.

Proof. Assume G^* is a graph whose chromatic number is k and $S(G^*)$ is the minimum value in the class of graphs with n vertices. Then $V(G^*)$ can be partitioned into k classes such that no edges joins two vertices of the same class. Furthermore G^* contains all edges joining vertices in distinct classes. Otherwise, there exists two nonadjacent vertices, say x and y , in distinct class. Then the graph $G' + xy$ has chromatic number k and $S(G' + xy) \leq S(G^*)$, which is contradictory to the minimality of G^* . Thus G^* is a complete k -partite graph $K_k(r_1, r_2, \dots, r_k)$ with $r_1 + r_2 + \dots + r_k = n$.

We suppose that $G^* \cong T_{k,n}$. Otherwise, the class are not as equal as possible, that is to say the difference between the number of vertices in all classes are more than two, say $r_j \geq r_i + 2$, where $r_i(r_j)$ is the number of vertices in the i th(j th) class. Then by transferring one vertex from the j th class to the i th class, we would decrease the $S(G^*)$ by

$$[4(r_j - 1) + (n - r_j)] - [4r_i + (n - r_i - 1)] = 3[(r_j - r_i) - 1] \geq 1$$

which contradicts the minimality of G^* . Recall that

$$e(T_{k,n}) = \binom{n-t}{2} + (k-1)\binom{t+1}{2},$$

where $t = \lfloor n/k \rfloor$. so

$$\begin{aligned} S(G^*) &= S(T_{k,n}) \\ &= 4\binom{n}{2} - 3e(T_{k,n}) \\ &= 2n(n-1) - 3\binom{n-t}{2} - 3\binom{t+1}{2}. \quad \square \end{aligned}$$

The proof above implies that equality holds in (2.1) if and only if $G \cong T_{k,n}$.

3. Clique number

Theorem 3.1. *Let G be a graph on n vertices. If G contains no K_{m+1} , then $e(G) \leq e(T_{m,n})$. Moreover, $e(G) = e(T_{m,n})$ only if $G \cong T_{m,n}$.*

The demonstrate about this well-known theorem can refer to [3].

In the following theorem, we only consider the graph G on n vertices with $C(G) < n-1$. since for $C(G) = n$ or $n-1$, it is easy to get the lower bound and upper bound on $S(G)$.

Theorem 3.2. *Let G be a graph on n vertices with $C(G) = k < n-1$. then*

$$\begin{aligned} 2n(n-1) - \binom{n-t}{2} - 3(k-1)\binom{t+1}{2} &\leq S(G) \\ &\leq \binom{k}{2} + \frac{n^4 - n^2}{12} + (k-1)\frac{(n-k+1)(n-k+2)(2n-2k+3)}{6} \\ &\quad + \frac{(n-k+1)(n-k+2)(2n-2k+3)}{6}, \end{aligned}$$

where $t = \lfloor n/k \rfloor$. Moreover, the lower bound is achieved if and only if $G \cong T_{k,n}$ and the upper bound is achieved if and only if $G \cong K_k \cdot P_{n-k}$.

Proof. Let G be a graph on n vertices with $C(G) = k < n-1$. set $t = \lfloor n/k \rfloor$. suppose the diameter of G is l . then

$$\begin{aligned} S(G) &= e(G) + \sum_{i=2}^l i^2 d(G, i) \geq e(G) + 2^2 \sum_{i=2}^l d(G, i) \\ &\geq e(G) + 2^2 \sum_{i=2}^l d(G, i) \end{aligned}$$

$$\begin{aligned}
(3.1) \quad &= e(G) + 4 \left[\binom{n}{2} - e(G) \right] = 2n(n-1) - 3e(G) \\
&\geq 2n(n-1) - 3T_{k,n} \quad (\text{by Theorem 3.1}) \\
&= 2n(n-1) - 3 \binom{n-t}{2} - 3(k-1) \binom{t-1}{2}.
\end{aligned}$$

It is evident that equality in both (3.2) and (3.3) will hold if and only if $G \cong T_{k,n}$. Note that the Turán graph $T_{k,n}$ has clique number k . So

$$S(G) \geq 2n(n-1) - 3 \binom{n-t}{2} - 3 \binom{t+1}{2},$$

and the equality holds if and only if $G \cong T_{k,n}$.

Assume G^* is a graph whose clique number is k and $S(G^*)$ is the minimum value in the class of graphs with n vertices. It is obvious that G^* has a pendent vertex, say u . Let v be the pendent vertex of $K_k \cdot P_{n-k}$. Then

$$\begin{aligned}
S(G^*) &= S(G^* - u) + \sum_{p \in V_G^*} d^2(u, p) \\
S(K_k \cdot P_{n-k}) &= S(K_k \cdot P_{n-k} - v) + \sum_{p \in V_{K_k \cdot P_{n-k}}} d^2(v, p).
\end{aligned}$$

Since the graph $G^* - u$ has $n-1$ vertices and clique number k , by the induction hypothesis,

$$S(G^* - u) \leq S(K_k \cdot P_{n-k} - v).$$

It is easily checked that

$$\sum_{p \in V_G^*} d^2(u, p) \leq \sum_{p \in V_{K_k \cdot P_{n-k}}} d^2(v, p)$$

with the equality holding if and only if

$$G^* \cong K_k \cdot P_{n-k}.$$

So,

$$S(G^*) \leq S(K_k \cdot P_{n-k}).$$

A straightforward calculation gives that

$$\begin{aligned}
S(K_k \cdot P_{n-k}) &= \binom{k}{2} + \frac{n^4 - n^2}{12} + (k-1) \\
&\quad \cdot \left[\frac{(n-k+1)(n-k+2)(2n-2k+3)}{6} - 1 \right] \\
&\quad + \frac{(n-k)(n-k+1)}{2}.
\end{aligned}$$

Therefore,

$$S(G) \leq \binom{k}{2} + \frac{n^4 - n^2}{12} + (k-1) \cdot \left[\frac{(n-k+1)(n-k+2)(2n-2k+3)}{6} - 1 \right] + \frac{(n-k)(n-k+1)}{2}$$

and the proof above implies that the equality holds if and only if $G \cong K_k \cdot P_{n-k}$. $\square$

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QUALITATIVE STUDY OF A GENERALISED BRUSSELATOR TYPE EQUATION

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Abstract. In this paper we discuss a system of equations which is a generalisation of the Brusselator equations [13]. Such equations usually deal with some autocatalytic reactions. Some equations related to non-allosteric enzyme reactions which are similar to the Michaelis-Menten equations with regard to functional response term are also analysed.

Keywords: Autocatalytic reactions, Poincare compactification, enzyme reactions.

1. Introduction and Motivation

The classic tri-molecular Brusselator reaction model is known to display complex behaviour [4],[3],[6],[7]. In this paper we consider a multimolecular Brusselator type of reaction with a special emphasis on the trimolecular reaction. The generalised Brusselator type equations based on a multimolecular reaction could have a possible form

$$(1) \quad \frac{dx}{dt} = 1 - ax - x^p y^q, \quad \frac{dy}{dt} = b(x^p y^q - y),$$

where $x, y \geq 0$, integers $p, q \geq 0$ and parameters $a \geq 0, b \geq 0$ [10]. Several authors have investigated the cases where $a = 0$ (see for example [11]).

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Usually such reactions from which the differential equation (1) is derived are enzyme reactions. As with all such reactions this reaction follows the law of mass action which states that the rate of a reaction is proportional to the product of the concentrations of the reactants. This is a nonlinear phenomenon and several people have explained the mechanism [12]. This is not surprising since enzymes are biological catalysts. The Brusselator equation as is well-known represents an autocatalytic reaction (for example Belousov-Zhabotinsky reaction, see [13]). Enzymes alter the rates of reactions in cells without being changed themselves during the course of a reaction. Our interest in this paper is with (apart from other enzyme reactions) non-allosteric enzymes [1]. Non-allosteric enzymes are a part of enzymes that are involved in the control and regulation of biological processes.

The Michaelis Menten [12] enzyme reaction has been studied by several authors. We are interested in this model as well and we propose a new model where we introduce a functional response term of the Michaelis Menten type output into the generalised Brusselator equations. This is in Section 4.

We study the following equations in this paper

$$(2) \quad \frac{dx}{d\tau} = a - bx - x^p y^q, \quad \frac{dy}{d\tau} = x^p y^q - \frac{cy}{y+1}.$$

In this paper we discuss, amongst several cases the case where in equations (2), $q = 1, b = 0$. Equations (2) would then become

$$(3) \quad \frac{dx}{d\tau} = a - x^n y, \quad \frac{dy}{d\tau} = x^n y - \frac{cy}{y+1}.$$

2. The reaction diffusion system of the generalised Brusselator reaction

If the entities x and y represent chemicals then the equations (1) to (3) would be the rate of reactions corresponding to some reaction kinetics. Taking a hint from [13], the reaction diffusion equations can be written. Such equations would study the instability induced in a reaction (chemical) by diffusion [16]. Thus the corresponding equations with the inclusion of the diffusion term would be

$$\frac{\partial \tilde{x}}{\partial t} = D_1 \nabla^2 \tilde{x} + a - b\tilde{x} + \tilde{x}^p \tilde{y}^q, \quad \frac{\partial \tilde{y}}{\partial t} = D_2 \nabla^2 \tilde{y} + b(\tilde{x}^p \tilde{y}^q - \tilde{y}),$$

where $\nabla^2 \tilde{x}$ and $\nabla^2 \tilde{y}$ are the diffusion terms with D_1 and D_2 the diffusion coefficients, $a - b\tilde{x} + \tilde{x}^p \tilde{y}^q$ and $b(\tilde{x}^p \tilde{y}^q - \tilde{y})$ would be the reaction terms. We study the reaction diffusion reaction related to a Brusselator model as the Brusselator *is a perfectly acceptable model for the study of cooperative processes in chemical kinetics* (to quote Prigogine [13]). According to Prigogine the Brusselator model plays a somewhat similar role as the harmonic oscillator or the Heisenberg model in ferromagnetism which are studied to illustrate the basic laws of classical

and quantum mechanics. Prigogine stresses on the importance of trimolecular steps (shown by the x^2y and y^2x terms in the equations) because such cubic nonlinearities give rise to ‘cooperative’ behaviour.

3. Analysis of the generalised Brusselator reaction

Let us consider the system of equations (2).

Choose $b = 0$ and use cy in place of $\frac{cy}{y+1}$ (for the sake of some insight into a simpler system).

$$(4) \quad \frac{dx}{dt} = a - x^p y^q, \quad \frac{dy}{dt} = x^p y^q - cy.$$

This is a planar polynomial system. Solving these two equations for the equilibrium points, (by equating the derivatives on the left hand sides to zero), one of the equilibrium points is $x = a^{1/p}(\frac{c}{a})^{q/p}$, $y = \frac{a}{c}$.

Linearising the system about this equilibrium point would give the Jacobian matrix J as

$$J = \begin{bmatrix} \frac{1-q}{p} \frac{q}{c^p} & -qa^{1-q}c^q(\frac{a}{c})^{-1+q} \\ \frac{1-q}{p} \frac{q}{c^p} & -c + qa^{1-q}c^q(\frac{a}{c})^{-1+q} \end{bmatrix}.$$

Its eigenvalues are

$$\lambda_{41} = \frac{1}{2}[-c - a^{\frac{-1+q}{p}}(\frac{a}{c})^q c^{\frac{-q}{p}} a^{1-q}c^q p + (\frac{a}{c})^{-1+q} a^{1-q}c^q q] - \frac{1}{2}a^{\frac{-1-p}{p}}c^{\frac{-q}{p}} \times \\ \sqrt{-4a^{\frac{1+2p+q}{p}}(\frac{a}{c})^q c^{\frac{p+q}{p}} a^{1-q}c^q p + [a^{1+\frac{1}{p}}c^{\frac{p+q}{p}} + a^{\frac{p+q}{p}}(\frac{a}{c})^q a^{1-q}c^q p - a^{\frac{1}{p}}(\frac{a}{c})^q c^{\frac{p+q}{p}} a^{1-q}c^q q]^2}$$

and

$$\lambda_{42} = \frac{1}{2}[-c - a^{\frac{-1+q}{p}}(\frac{a}{c})^q c^{\frac{-q}{p}} a^{1-q}c^q p + (\frac{a}{c})^{-1+q} a^{1-q}c^q q] + \frac{1}{2}a^{\frac{-1-p}{p}}c^{\frac{-q}{p}} \times \\ \sqrt{-4a^{\frac{1+2p+q}{p}}(\frac{a}{c})^q c^{\frac{p+q}{p}} a^{1-q}c^q p + [a^{1+\frac{1}{p}}c^{\frac{p+q}{p}} + a^{\frac{p+q}{p}}(\frac{a}{c})^q a^{1-q}c^q p - a^{\frac{1}{p}}(\frac{a}{c})^q c^{\frac{p+q}{p}} a^{1-q}c^q q]^2}.$$

To study the special trimolecular case referred to earlier, let us choose $p+q = 3$ with $p = 2$ and $q = 1$. The system is

$$(5) \quad \frac{dx}{dt} = a - x^2 y, \quad \frac{dy}{dt} = x^2 y - cy.$$

Its equilibrium points are $(-\sqrt{c}, \frac{a}{c})$ and $(\sqrt{c}, \frac{a}{c})$.

The eigenvalues at $(-\sqrt{c}, \frac{a}{c})$ are

$$\lambda_{51} = \frac{a}{\sqrt{c}} - \frac{\sqrt{a^2 + 2ac^{3/2}}}{\sqrt{c}}, \quad \lambda_{52} = \frac{a}{\sqrt{c}} + \frac{\sqrt{a^2 + 2ac^{3/2}}}{\sqrt{c}}.$$

The eigenvalues at $(\sqrt{c}, \frac{a}{c})$ are

$$\lambda_{53} = \frac{-a}{\sqrt{c}} - \frac{\sqrt{a^2 - 2ac^{3/2}}}{\sqrt{c}}, \quad \lambda_{54} = \frac{-a}{\sqrt{c}} + \frac{\sqrt{a^2 - 2ac^{3/2}}}{\sqrt{c}}.$$

- Observing λ_{51} and λ_{52} it is apparent that they will not take on complex values for $c > 0$, $a > 0$ and hence the equilibrium point $(-\sqrt{c}, \frac{a}{c})$ is either a node or a saddle point (depending on the sign of the eigenvalues).
- λ_{53} and λ_{54} can take on complex values for $a^2 - 2ac^{3/2} < 0$ or $a < 2c^{3/2}$. In this case, since the real part of λ_{53} and λ_{54} is negative one would expect a spiral sink for the equilibrium point $(\sqrt{c}, \frac{a}{c})$.

This case is illustrated in Figure 1.

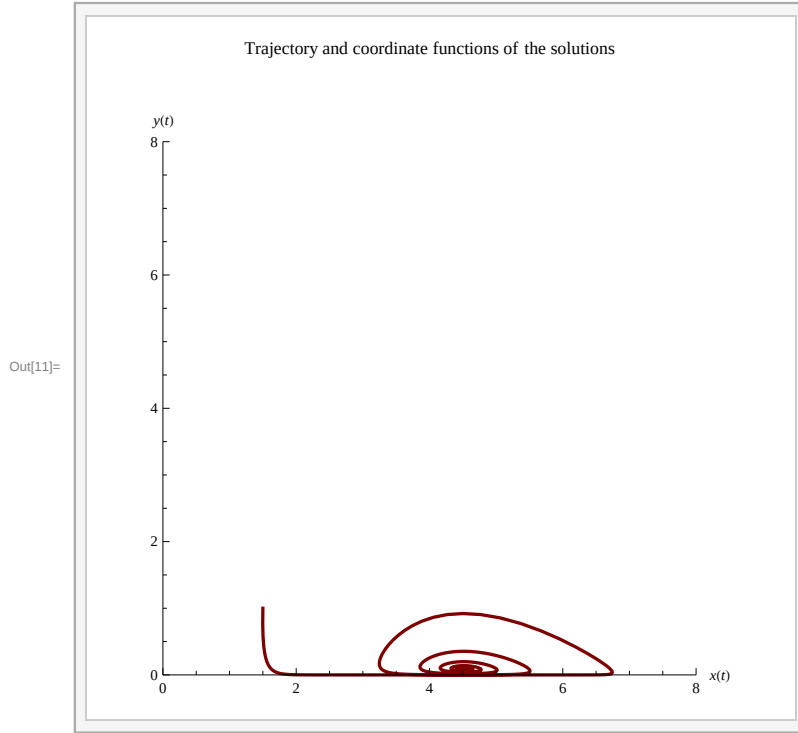


Figure 1: A spiral sink for the system (5) for some particular values $a = 1.8$, $c = 20.345$, $p = 2$, $q = 1$

In Figure 1 a spiral sink for the system (5) is shown for some particular values of a, b, c .

A simulation of equation (2)

We now consider a simulation of equation (2) by replacing the term cy by $\frac{cy}{y+1}$. It can be seen that a limit cycle appears for some values of parameters.

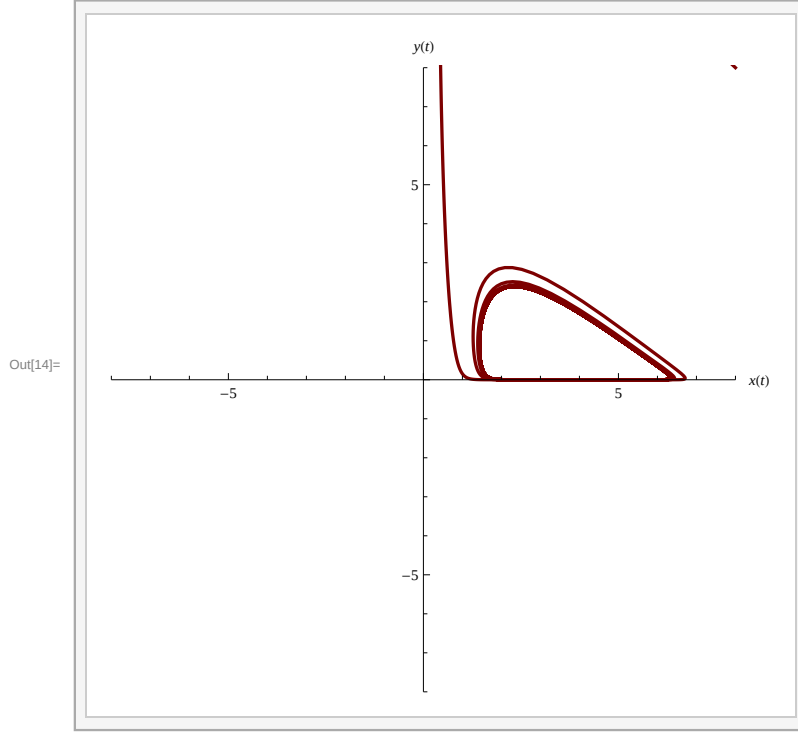


Figure 2: A Limit Cycle for the system (2) for some particular values $a = 1.8$, $b = 0$, $c = 4.8$, $p = 2$, $q = 1$

4. A generalised Brusselator type equation with a Michaelis-Menten functional response term

C.S.Holling studied the factors involved in the utilization of resources by predators. He described the changes in the feeding rate of organisms as “the functional response term”. He showed that there were three categories of functional response [15].

- Type 1. Refers to animals which consume food proportional to the rate of their encounter with food items.
- Type 2. Where the organisms take some time to eat and to capture their prey.

- Type 3. In this category the organism will not consume the prey if it is below a certain threshold density.

There is a remarkable parallel between enzyme reactions and the predator-prey Holling functional response [2]. The Michaelis-Menten enzyme reaction follows a type 2 functional response. Keeping this type of functional response in mind we propose the following model and analyse it.

$$(6) \quad \frac{dx}{dt} = a - bx - x^p y^q, \quad \frac{dy}{dt} = x^p y^q - \frac{cxy}{y+1}.$$

For the purpose of understanding and analysis, we take $b = 0, p = 2, q = 1$. Equations (6) reduce to

$$(7) \quad \frac{dx}{dt} = a - x^2 y, \quad \frac{dy}{dt} = x^2 y - \frac{cxy}{y+1}.$$

The equilibrium points of equation (7) are

$$\left(\frac{1}{2}[c - \sqrt{-4a + c^2}], \frac{-2a + c^2 + c\sqrt{-4a + c^2}}{2a} \right)$$

and

$$\left(\frac{1}{2}[c + \sqrt{-4a + c^2}], \frac{-2a + c^2 - c\sqrt{-4a + c^2}}{2a} \right).$$

Let (x_0, y_0) be an equilibrium point chosen from amongst the two equilibrium points of equation (7) so that $x_0 > 0, y_0 > 0$. This is possible if $c^2 > 4a$. Linearising the system (7) about its equilibrium point, the Jacobean matrix is

$$M = \begin{bmatrix} -2x_0 y_0 & -x_0^2 \\ 2x_0 y_0 - \frac{cy_0}{1+y_0} & x_0^2 + \frac{cx_0 y_0}{(1+y_0)^2} - \frac{cx_0}{1+y_0} \end{bmatrix},$$

where the elements in the linearised matrix are to be treated as the functions of the parameter c . The characteristic equation of this matrix has the form $\lambda^2 - s\lambda + D = 0$ where

$$s(c) = \text{the trace of } M = -2x_0 y_0 + x_0^2 + \frac{cx_0 y_0}{(1+y_0)^2} - \frac{cx_0}{1+y_0}$$

and $D = \text{determinant of } M = -\frac{2cx_0^2 y_0^2}{(1+y_0)^2} + \frac{cx_0^2 y_0}{1+y_0}$. Let the two eigenvalues of this matrix be λ_1 and λ_2 . These can be represented as a function of c .

$$\lambda_1, \lambda_2 = \frac{1}{2}[s(c) \pm \sqrt{s^2(c) - 4D(c)}].$$

We will now show that a Hopf bifurcation can occur in this equation for some values of the parameter c . A Hopf bifurcation condition would require that ([9], see page 91) the real part (Re) of the eigenvalues is equal to zero and

the imaginary part (Im) is nonzero $\Re = \theta(c) = \frac{1}{2}s(c)$ and $\Im = \nu(c) = \frac{1}{2}\sqrt{s^2(c) - 4D(c)}$ (see [8]). Solving for the parameter c after setting the trace to zero

$$(1) \quad x_0^2 - 2x_0y_0 + \frac{cx_0y_0}{(1+y_0)^2} - \frac{cx_0}{1+y_0} = 0 \Rightarrow c = (x_0 - 2y_0)(1+y_0)^2$$

$$(2) \quad \frac{d}{dc}(\text{Trace}) = \frac{x_0y_0}{(1+y_0)^2} - \frac{x_0}{1+y_0} \neq 0.$$

(1) is the non-hyperbolicity condition and (2) is the transversality condition. Thus showing the existence of a Hopf bifurcation for the parameter c . A simulation of the limit cycle is shown in Figure 3.

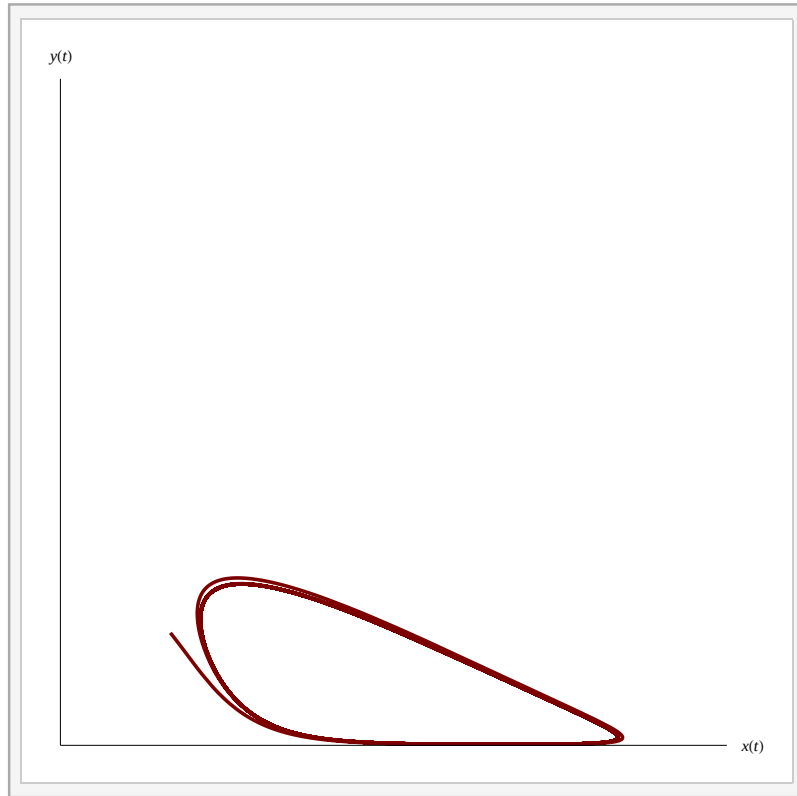


Figure 3: A limit cycle for the values of the parameter $a = 1.8$, $c = 1$, $p = 2$, $q = 1$

The generalised Brusselator equation (1) is a planar polynomial system. Usually in such systems one needs to study not just its finite equilibria but also the equilibria at infinity. When related to a chemical reaction this would mean that one studies the tendency of the concentrations of the chemicals over a large range. So for this it is appropriate to use a Poincaré Compactification.

5. A Brief Outline of Poincaré Compactification

In order to study the behaviour of the trajectories of a planar differential system near infinity we use a compactification. A good approach for studying the behaviour of trajectories near infinity is to use the Poincaré sphere, introduced by Poincaré [14]. It has the advantage that the singular points at infinity are spread out along the equator of the sphere. In order to draw the phase portrait of a vector field, we would have to work over the complete real plane $\mathbb{R}^2$, which is not very practical. If the functions defining the vector field are polynomials, we can apply Poincaré compactification, which will tell us how to draw it in a finite region. It controls the orbits which tend to or come from infinity. Here we use (x_1, x_2) as coordinates instead of (x, y) (in order to differentiate).

Let $X = P\partial/\partial x_1 + Q\partial/\partial x_2$ be a polynomial vector field (the functions P and Q are polynomials of arbitrary degree in the variables x_1 and x_2), or in other words:

$$\begin{aligned} \dot{x}_1 &= P(x_1, x_2) \\ \dot{x}_2 &= Q(x_1, x_2). \end{aligned}$$

The degree of X is represented as d where d is the maximum of the degrees of P and Q. Poincaré compactification works as follows: First we consider $\mathbb{R}^2$ as the plane in $\mathbb{R}^3$ defined by $(y_1, y_2, y_3) = (x_1, x_2, 1)$. We consider the sphere $\mathbb{S}^2 = \{y \in \mathbb{R}^3 : y_1^2 + y_2^2 + y_3^2 = 1\}$ which we will call here Poincaré sphere; it is tangent to $\mathbb{R}^2$ at the point $(0, 0, 1)$. We may divide this sphere into $H_+ = \{y \in \mathbb{S}^2 : y_3 > 0\}$ (the northern hemisphere), $H_- = \{y \in \mathbb{S}^2 : y_3 < 0\}$ (the southern hemisphere) and $\mathbb{S}^1 = \{y \in \mathbb{S}^2 : y_3 = 0\}$ (the equator). Now we consider the projection of the vector field X from $\mathbb{R}^2$ to $\mathbb{S}^2$ given by the central projections $f^+ : \mathbb{R}^2 \rightarrow \mathbb{S}^2$ and $f^- : \mathbb{R}^2 \rightarrow \mathbb{S}^2$. More precisely, $f^+(x)$ (respectively $f^-(x)$) is the intersection of the straight line passing through the point y and the origin with the northern (respectively, southern) hemisphere of $\mathbb{S}^2$.

$$\begin{aligned} f^+(x) &= \left(\frac{x_1}{\Delta(x)}, \frac{x_2}{\Delta(x)}, \frac{1}{\Delta(x)} \right), \\ f^-(x) &= \left(\frac{-x_1}{\Delta(x)}, \frac{-x_2}{\Delta(x)}, \frac{-1}{\Delta(x)} \right), \end{aligned}$$

where

$$\Delta(x) = \sqrt{(x_1)^2 + (x_2)^2 + 1}.$$

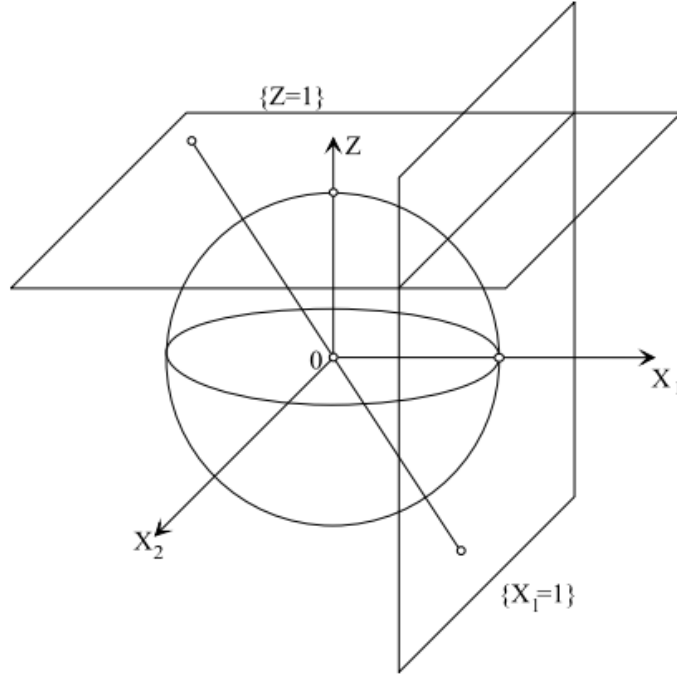


Figure 4: Poincaré Sphere

In this way we obtain induced vector fields in each hemisphere. (For more details see page 150 of [5]).

We now consider a Poincaré compactification of equation (2).

Let

$$\begin{aligned}\hat{P}(x, y) &= a - bx - x^p y^q, \\ \hat{Q}(x, y) &= x^p y^q - \frac{cy}{y+1}.\end{aligned}$$

For the local chart U_1 (see [5]), x is transformed to $\frac{1}{v}$ and y is transformed to $\frac{u}{v}$.

$$\begin{aligned}\hat{P}\left(\frac{1}{v}, \frac{u}{v}\right) &= a - \frac{b}{v} - \frac{u^q}{v^{p+q}}, \\ \hat{Q}\left(\frac{1}{v}, \frac{u}{v}\right) &= \frac{u^q}{v^{p+q}} - \frac{cu}{u+v}.\end{aligned}$$

The extended vector on $\mathbb{S}^2$ which is called the Poincaré compactification of the vector field X on $\mathbb{R}^2$ is denoted by $p(X)$. The expression for $p(X)$ in the local chart (U_1, ϕ_1) is given by

$$\begin{aligned}\dot{u} &= v^d [-u \hat{P}\left(\frac{1}{v}, \frac{u}{v}\right) + \hat{Q}\left(\frac{1}{v}, \frac{u}{v}\right)], \\ \dot{v} &= -v^{d+1} \hat{P}\left(\frac{1}{v}, \frac{u}{v}\right).\end{aligned}$$

where $d=(\text{maximum of the degrees of } \hat{P} \text{ and } \hat{Q})=p+q$. Therefore,

$$\begin{aligned}\dot{u} &= v^{p+q}[-au + \frac{bu}{v} + \frac{u^{q+1} + u^q}{v^{p+q}} - \frac{cu}{u+v}] \\ \dot{v} &= -v^{p+q+1}[a - \frac{b}{v} - \frac{u^q}{v^{p+q}}].\end{aligned}$$

For the local chart U_2 , x is transformed to $\frac{u}{v}$ and y is transformed to $\frac{1}{v}$. The expression for chart (U_2, ϕ_2) is given by

$$\begin{aligned}\dot{u} &= v^d[\hat{P}(\frac{u}{v}, \frac{1}{v}) - u\hat{Q}(\frac{u}{v}, \frac{1}{v})], \\ \dot{v} &= -v^{d+1}\hat{Q}(\frac{u}{v}, \frac{1}{v}).\end{aligned}$$

Therefore

$$\begin{aligned}\dot{u} &= v^{p+q}[a - \frac{bu}{v} - \frac{u^{p+1} + u^p}{v^{p+q}} - \frac{c}{v+1}], \\ \dot{v} &= -v^{p+q+1}[\frac{u^p}{v^{p+q}} - \frac{c}{v+1}].\end{aligned}$$

In order to illustrate this compactification process, using these equations we evaluate the local charts U_1 and U_2 . We choose some specific values for the various parameters in equation (2) as $a = 1, b = 0, c = 1, p = 2, q = 1$. The system is

$$\begin{aligned}\frac{dx}{dt} &= (1 - x^2y)(y + 1), \\ \frac{dy}{dt} &= x^2y(y + 1) - y.\end{aligned}$$

The expression for the local chart U_1 is

$$\begin{aligned}\dot{u} &= u(u+v)(1+u-v^3) - uv^3, \\ \dot{v} &= v(u+v)(u-v^3).\end{aligned}$$

For the local chart U_2

$$\begin{aligned}\dot{u} &= (1+v)(v^3 - u^2 - u^3) + uv^3, \\ \dot{v} &= v^4 - vu^2(1+v).\end{aligned}$$

In figure 5 there are many equilibrium points visible on the Poincaré sphere. One of them is an unstable node as can be seen from the trajectories moving away from the equilibrium point and a stable node where the trajectories move towards the equilibrium point. For the system

$$\begin{aligned}\frac{dx}{dt} &= (0.35 - x^2y)(y + 1), \\ \frac{dy}{dt} &= x^2y(y + 1) - 5y.\end{aligned}$$

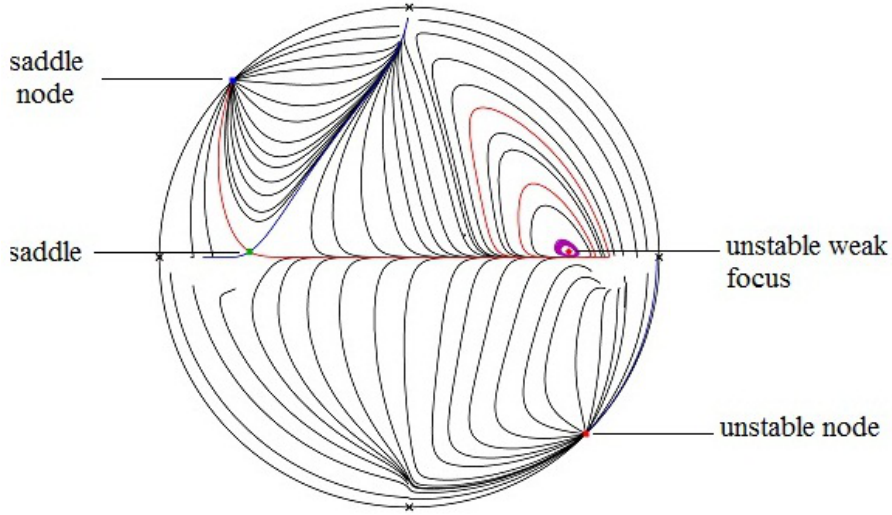


Figure 5: Phase portrait on the Poincaré sphere

The expression for the local chart U_1 is $\dot{u} = u(u + v)(1 + u - 0.35v^3) - 5uv^3$, $\dot{v} = v(u + v)(u - 0.35v^3)$.

For the local chart U_2 , is $\dot{u} = (1 + v)(0.35v^3 - u^2 - u^3) + 5uv^3$, $\dot{v} = 5v^4 - vu^2(1 + v)$.

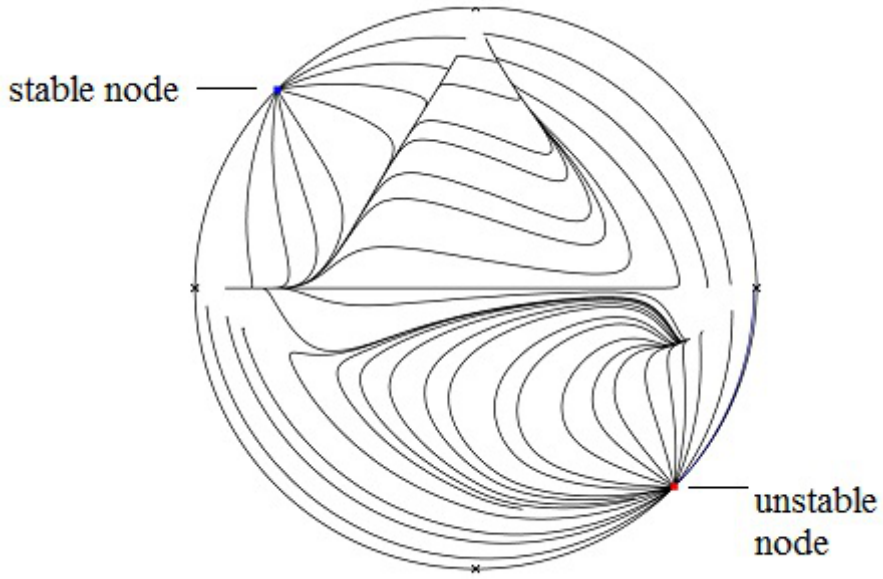


Figure 6: Phase portrait on the Poincaré sphere

With the change of the values of the parameters, the phase portrait on the Poincaré sphere has in addition to the stable and unstable nodes, a saddle point and a focus as can be seen from the figure 6.

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SOFT ROUGH GROUPS AND CORRESPONDING DECISION MAKING

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Abstract. In this article, we apply soft rough sets (briefly, *SR*-sets) to the special algebraic structure-group and give the concepts of soft rough groups (briefly, normal subgroups) (briefly, *SR*-groups, *SRN*-subgroups), which is an extended definition of rough groups. Further, we use the terminologies of *C*-soft sets and *CC*-soft sets to research soft rough algebraic structures. Moreover, the roughness in groups w.r.t. *MSR*-approximation spaces are investigated. At the same time, we study some soft rough operations over groups. Specially, upper and lower *SR*-groups (*SRN*-subgroups) are explored. Finally, we raise a kind of decision making method (DM-method) for *SR*-groups and give an actual example to illustrate.

Keywords: *SR*-set, *SR*-group, *SRN*-subgroup, *MSR*-set, decision making.

1. Introduction

The basic logical thinking methods of human understanding of things and the establishment of knowledge are the classification. A classification method in mathematics is a partition of objects. Each partition only corresponds to an equivalent relationship of the domain. In this sense, each equivalent relationship on the universe is a knowledge, each equivalence class is a basic concept of this knowledge, a family of equivalence relations are a knowledge base (briefly, KB). Therefore, the study of knowledge base will be translated into the study of equivalence relations. In the context of this logic, Pawlak [20] proposed rough set theory (briefly, RST) to deal with ambiguous uncertainty problems in 1982. Since then, research on RST has emerged in many fields. However, the equivalence relation can not be used effectively in many practical problems, thus restricts the application and development of RST to a certain extent. Therefore, the rough set models [24, 28, 29] based on general binary relations were proposed, which greatly enriched and developed the Pawlak RST. In particular, Zhang et al. [29] explored the constructed methods of RA-operations and multigranuation rough sets. At the same time, the combinations of rough sets and algebraic

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structures are also found in many literatures. For examples, Biswas and Nanda [5] applied RST to groups, proposed rough groups (subgroups) and obtained some related properties. Later on, Kuroki [13] put forth the concept of a rough ideal in semigroups and studied some properties of such notion. Ali [3] and Davvaz et al. [7] researched the roughness in rings and hemirings, respectively. Recently, Bağirmaz [4] investigated rough prime ideals on approximation spaces. In recent years, rough sets were widely used in DM problems.

In 1999, a novel theory such as soft set theory (briefly, SST) was put forth by Russian scholar–Molodtsov [18], which is mainly used to solve the uncertainty problem. Compared with the traditional mathematical approaches, for examples, probability theory, fuzzy analytic method and RST, This theory has the unique advantage in dealing with the problem of uncertainty. Maji et al. [16] defined equality of two soft sets and complement of a soft set with examples. We know that some soft binary operations like *AND*, *OR* and the operations of union and intersection were defined in [16]. Further, Ali [2] first pointed out that some assertions in [16] et al. are not true in general and gave some new operations of two soft sets. Based on this novel idea, the research on SST was wisely used in recent years. Several authors researched the combinations of soft sets and algebraic structures. In particular, Aktaş and N. Çağman [1] introduced the basic properties of soft sets and gave the notion of soft groups. Later on, Sezgin and A.O. Atagün [22] first corrected some of problematic cases in [1] and proposed the concepts of normalistic soft groups (briefly, NS-groups) and normalistic soft group homomorphism (briefly, NSG-homomorphisms), and studied some structures that are preserved under NSG-homomorphisms. In 2008, the definitions of some types of soft semirings, soft ideals and soft semiring homomorphism were given by Feng et al. [10]. In addition, SST in DM has become a hot topic. Firstly, Maji et al. [17] applied SST to solve a DM problem using SST. In particular, Çağman and N. Çağman [6] investigated the products of soft sets and constructed a uni-int DM-method by means of uni-int decision functions to solve the actual problems.

We know that uncertainties have many different performance, hence we can not catch hold of them by a single uncertain tool. Some researchers proposed some hybrid soft computing models, such as fuzzy soft sets (briefly, *FS*-sets), fuzzy rough sets (briefly, *FR*-sets), rough soft sets (briefly, *RS*-sets) and *SR*-sets, and so on. In 2010, Feng et al. [8] proposed *RS*-sets by means of RST and SST. Recently, based on the idea in [8], Zhan et al. [25] firstly applied *RS*-sets to hemirings, and investigated some vital properties of rough soft hemirings (briefly, *RS*-hemirings), and investigated some related properties. Moreover, Ma [14] and Pan and Zhan [19] applied *RS*-sets to *BCI*-algebras and groups, respectively. The most important criticism on RST is that it lacks parameterization tools and approaches. To tackle this problem, the notion of *SR*-sets was first proposed in [9]. Moreover, Qin et al. [21] researched *SR*-sets based on similarity measures. However, its shortcoming is that the soft set must be full in [9]. At the same time, Shabir et al. [23] showed clearly that there exist two problems on Feng's

approach in [9], such as, (1) an upper approximation of a given set may be empty; (2) the upper approximation of the given set may not contain itself on Feng's SR -sets. In view of this reason, they proposed the concept of MSR -sets to resolve this problem. Later on, Kumar and Inbarani[12] investigated modified SR -sets by means of ECG signal classification for cardiac arrhythmias. As a result, the combination of SST and RST is more effective when dealing with uncertain problems. In particular, Inbarani [11] studied SR -sets for heart value disease diagnosis. Based on the idea of [23], the MSR -set was applied to the algebraic structure. Zhan et al. [26] firstly applied SR -sets to hemirings. And so, the roughness in hemirings w.r.t. MSR -approximation spaces was studied. Further, Zhan and Zhu [27] proposed the concept of Z -SRF-sets and studied Z -SRF-ideals in hemirings by using three uncertain soft set models. Recently, Ma et al. [15] reviewed some types of DM-methods based on some kinds of hybrid soft set models.

Based on the above idea, in this present article, we apply the MSR -sets to groups. Referring to Shabir et al. [23] and Zhan et al. [26], we divide this paper into four parts. In section 2, we point out some basic terminologies, such as, rough sets and soft sets. In section 3, we give some operations of modified SR -sets in groups. In section 4, we investigate some characterizations of SR -groups and SRN -subgroups. Finally, an efficient approach for DM problem based on SR -groups is given in section 5.

2. Basic terminologies

Some useful terminologies about soft sets and rough sets are given.

Definition 2.1 ([18]). The $\mathfrak{S} = (\delta, A)$ is said to be a soft set over U , where $\delta : A \rightarrow P(U)$ is a set-valued mapping.

Definition 2.2 ([8]). A soft set $\mathfrak{S} = (\delta, A)$ over U is called full if $\bigcup_{a \in A} \delta(a) = U$.

Definition 2.3 ([1, 22]). (1) A soft set $\mathfrak{S} = (\delta, A)$ over group G is said to be a soft group over G if and only if $\delta(x) < G$ for all $x \in Supp(\delta, A)$,

(2) A soft set $\mathfrak{S} = (\delta, A)$ over group G is called a normalisitic soft group (briefly, NS-group) over G if $\delta(x)$ is a normal subgroup of G for all $x \in Supp(\delta, A)$.

Definition 2.4 ([9]). Let $\mathfrak{S} = (\delta, A)$ be a soft set over U . We define two basic operations:

$$\begin{aligned}\underline{apr}_P(X) &= \{u \in U | \exists a \in A : u \in \delta(a) \subseteq X\}, \\ \overline{apr}_P(X) &= \{u \in U | \exists a \in A : u \in \delta(a), \delta(a) \cap X \neq \emptyset\},\end{aligned}$$

assigning to every subset $X \subseteq U$, two sets $\underline{apr}_P(X)$ and $\overline{apr}_P(X)$ are called the lower and upper SR -approximations of X in P , respectively. If $\underline{apr}_P(X) = \overline{apr}_P(X)$, X is named as soft definable; if not X is named as a SR -set. In what follows, we name it Feng- SR -set.

Definition 2.5 ([23]). Put $\xi : U \rightarrow P(A)$ be a set-valued mapping shown as $\xi(x) = \{a | x \in \delta(a)\}$ and (U, ξ) be an MS -approximation space. For any $V \subseteq U$, the lower MSR -approximation and upper MSR -approximation of V are denoted by $\underline{V}_\xi$ and $\overline{V}_\xi$, resp., which are shown as

$$\underline{V}_\xi = \{x \in V | \xi(x) \neq \xi(z) \text{ for any } z \in V^c\}$$

and

$$\overline{V}_\xi = \{x \in U | \xi(x) = \xi(z) \text{ for some } z \in V\},$$

where $V^c = U - V$ is the complement of V .

Here, lower MSR -approximation can also be regarded as

$$\underline{V}_\xi = \{x \in U | \forall z \in V^c [\xi(x) \neq \xi(z)]\}.$$

If $\underline{V}_\xi = \overline{V}_\xi$, then V is called MS -definable, if not V is named as a Shabir-SR-set.

3. Soft rough approximations

Section 3 investigates some fundamental properties of modified soft rough sets over groups.

Definition 3.1. Assume that $\mathfrak{S} = (\delta, A)$ is a soft set over a group G and $\xi : G \rightarrow P(A)$ is a map defined as $\xi(x) = \{a \in A | x \in \delta(a)\}$, $\mathfrak{S}$ is named as a C -soft set over G if $\xi(a_1) = \xi(a_2)$ and $\xi(b_1) = \xi(b_2)$ imply $\xi(a_1 \cdot b_1) = \xi(a_2 \cdot b_2)$ for all $a_1, a_2, b_1, b_2 \in G$.

Example 3.2. We can consider that $G = \{\pm 1, \pm i\}$ is a group with $i^2 = -1$ and can define a soft set $\mathfrak{S} = (\delta, A)$ over G shown by Table 1.

Table 1 table for soft set $\mathfrak{S}$				
	1	-1	i	$-i$
e_1	0	0	1	1
e_2	1	1	1	1
e_3	0	0	0	0

Then $\xi : G \rightarrow P(A)$ of SRA-space (G, ξ) is shown by $\xi(1) = \xi(-1) = \{e_2\}$, $\xi(i) = \xi(-i) = \{e_1, e_2\}$.

It is easy to check that $\mathfrak{S}$ is a C -soft set over G .

Let H, K be any two non-empty subsets in any group, denote $H \cdot K = \{h \cdot k | \forall h \in H, k \in K\}$.

Theorem 3.3. We can suppose that $\mathfrak{S} = (\delta, A)$ is a C -soft set over G and (G, ξ) is an MS -approximation space. For any two non-empty subsets H, K in G . Then $\overline{H}_\xi \cdot \overline{K}_\xi \subseteq \overline{H \cdot K}_\xi$.

Proof. Let $x \in \overline{H}_\xi \cdot \overline{K}_\xi$, then $x = a \cdot b$, where $a \in \overline{H}_\xi$ and $b \in \overline{K}_\xi$, and so there exist $h \in H$ and $k \in K$.t. $\xi(a) = \xi(h)$ and $\xi(b) = \xi(k)$. By the hypothesis, $\mathfrak{S}$ is a C -soft set, so $\xi(a \cdot b) = \xi(h \cdot k)$ for $h \cdot k \in H \cdot K$. Hence $x = a \cdot b \in \overline{H} \cdot \overline{K}_\xi$. That is, $\overline{H}_\xi \cdot \overline{K}_\xi \subseteq \overline{H} \cdot \overline{K}_\xi$. $\square$

Example 3.4. Assume that $G = \{1, a, b, c\}$ is a group in the given Table 2:

Table 2 table for group G				
	1	a	b	c
1	1	a	b	c
a	a	1	c	b
b	b	c	1	a
c	c	b	1	a

Define $\mathfrak{S} = (\delta, A)$ over G which is given by Table 3.

Table 3 table for soft set $\mathfrak{S}$				
	1	a	b	c
e_1	0	0	0	1
e_2	1	1	1	0
e_3	0	0	0	0

Suppose that the mapping $\xi : G \rightarrow P(A)$ of SRA-space (G, ξ) is shown by $\xi(1) = \xi(a) = \xi(b) = \{e_2\}$, $\xi(c) = \{e_1\}$. Then it is clear that $\mathfrak{S}$ is a C -soft set over G .

Let $H = \{1, c\}$ and $K = \{c\}$, then $H \cdot K = \{1, c\}$, $\overline{H}_\xi = \{1, a, b, c\}$ and $\overline{K}_\xi = \{c\}$, so $\overline{H}_\xi \cdot \overline{K}_\xi = \{1, c\}$ and $\overline{H} \cdot \overline{K}_\xi = \{1, a, b, c\}$. Thus $\overline{H}_\xi \cdot \overline{K}_\xi \subset \overline{H} \cdot \overline{K}_\xi$.

Definition 3.5. Suppose that $\mathfrak{S} = (\delta, A)$ is a C -soft set over G and (G, ξ) is an MS -approximation space, then $\mathfrak{S}$ is named as a CC -soft set over G if for all $c \in G$, $\xi(c) = \xi(x \cdot y)$, $\exists a, b \in G$ s.t. $\xi(x) = \xi(a)$ and $\xi(y) = \xi(b)$ with $c = a \cdot b$.

Remark 3.6. From Definition 3.5, we can obtain that a CC -soft set over G is a C -soft set, but the converse do not hold in general.

Example 3.7. In Example 3.2, $\mathfrak{S}$ is a C -soft set over G , let $c = i \in G$ and $\xi(i) = \xi(1 \cdot i)$, from Table 1, we can see that $\xi(1) = \xi(-1)$ and $\xi(i) = \xi(i)$, but $c = i \neq (-i) = (-1) \cdot i$, so $\mathfrak{S}$ is not a CC -soft set over G .

Theorem 3.8. Consider $\mathfrak{S} = (\delta, A)$ be a CC -soft set over G and (G, ξ) an MS -approximation space. For any two non-empty subsets H, K in G . Then $\overline{H}_\xi \cdot \overline{K}_\xi = \overline{H} \cdot \overline{K}_\xi$.

Proof. By Theorem 3.3, we have $\overline{H}_\xi \cdot \overline{K}_\xi \subseteq \overline{H} \cdot \overline{K}_\xi$. So we only need to prove $\overline{H} \cdot \overline{K}_\xi \subseteq \overline{H}_\xi \cdot \overline{K}_\xi$.

Now let $c \in \overline{H} \cdot \overline{K}_\xi$, so $\xi(c) = \xi(x \cdot y)$ for some $x \in H$ and $y \in K$. Since $\mathfrak{S}$ is a CC -soft set over G , there exist $a, b \in G$ s.t. $\xi(a) = \xi(x)$ and $\xi(b) = \xi(y)$ with $c = a \cdot b$. Thus $a \in \overline{H}_\xi$ and $b \in \overline{K}_\xi$. Hence $c = a \cdot b \in \overline{H}_\xi \cdot \overline{K}_\xi$. That is $\overline{H} \cdot \overline{K}_\xi \subseteq \overline{H}_\xi \cdot \overline{K}_\xi$. So $\overline{H}_\xi \cdot \overline{K}_\xi = \overline{H} \cdot \overline{K}_\xi$. $\square$

Theorem 3.9. Consider $\mathfrak{S} = (\delta, A)$ be a CC -soft set over G and (G, ξ) an MS -approximation space. For any two non-empty subsets H, K in G . Then $\underline{H}_\xi \cdot \underline{K}_\xi \subseteq \underline{H \cdot K}_\xi$.

Proof. Suppose that $\underline{H}_\xi \cdot \underline{K}_\xi \subseteq \underline{H \cdot K}_\xi$ does not hold, then there exists $c \in \underline{H}_\xi \cdot \underline{K}_\xi$, but $c \notin \underline{H \cdot K}_\xi$. This means that $\xi(a) \neq \xi(x)$ and $\xi(b) \neq \xi(y)$ for all $x \in H^c$ and $y \in K^c$. ($\spadesuit$)

On the other hand, $c \notin \underline{H \cdot K}_\xi$, then we may have the following two conditions:

(1) $c \notin H \cdot K$, but $c \in \underline{H}_\xi \cdot \underline{K}_\xi \subseteq H \cdot K$. This is a contradiction.

(2) $c \in H \cdot K$ and $\xi(c) = \xi(x' \cdot y')$ for some $x' \cdot y' \in (H \cdot K)^c$. Thus $x' \in H^c$ or $y' \in K^c$. Indeed, if $x' \notin H^c$ and $y' \notin K^c$, we have $x' \cdot y' \in H \cdot K$, a contradiction. By the hypothesis, $\mathfrak{S}$ is a CC -soft set over G , so $\exists a', b' \in G$ such that $\xi(a') = \xi(x')$ and $\xi(b') = \xi(y')$ with $a' \cdot b' = c$ for some $x' \in H^c$ and $y' \in K^c$. Which contradicts to ($\spadesuit$). Hence $\underline{H}_\xi \cdot \underline{K}_\xi \subseteq \underline{H \cdot K}_\xi$. $\square$

If $\mathfrak{S} = (F, A)$ is not a CC -soft set over G , Theorem 3.9 is not true.

Example 3.10. Let $G = \{1, 3, 5, 7\} \subseteq \mathbb{Z}_8$ be a group where the operation is the ordinary multiplication.

We consider a soft set $\mathfrak{S} = (F, A)$ over G which is given by Table 4.

Table 4 table for soft set $\mathfrak{S}$				
	1	3	5	7
e_1	1	1	1	0
e_2	1	1	1	0
e_3	0	0	0	1

We can consider the mapping $\xi : G \rightarrow P(A)$ of SRA-space (G, ξ) is given by $\xi(1) = \xi(3) = \xi(5) = \{e_1, e_2\}$, $\xi(7) = \{e_3\}$.

Let $H = \{1, 3, 7\}$ and $K = \{7\}$. Then $\underline{H}_\xi = \{1, 3\}$ and $\underline{K}_\xi = \{7\}$, so $\underline{H}_\xi \cdot \underline{K}_\xi = \{5, 7\}$. Also we have $\underline{H \cdot K}_\xi = \{7\}$, so $\underline{H}_\xi \cdot \underline{K}_\xi \not\subseteq \underline{H \cdot K}_\xi$.

4. Characterizations of soft rough groups

Section 4 discusses the operations of lower and upper MSR -approximations of SR -groups. In order to investigate the roughness of group G with respect to soft rough approximation space over groups, we first give the notions of SR -groups and SRN -subgroups.

Definition 4.1. We can consider that $\mathfrak{S} = (\delta, A)$ is a soft set over G and (G, ξ) is an MS -approximation space. For any $X \subseteq G$, the lower and upper MSR -approximations of X are given by $\underline{X}_\xi$ and $\overline{X}_\xi$, resp., which two operations are defined as

$$\underline{X}_\xi = \{x \in X \mid \forall y \in X^c : \xi(x) \neq \xi(y)\}$$

and

$$\overline{X}_\xi = \{x \in G \mid \exists y \in X : \xi(x) = \xi(y)\}$$

If $\underline{X}_\xi \neq \overline{X}_\xi$, then

- (1) X is called a lower (upper) SR -group (resp., SRN -subgroup) over G , if $\underline{X}_\xi$ ($\overline{X}_\xi$) is a subgroup (resp., normal subgroup) of G ;
- (2) X is called an SR -group (resp., SRN -subgroup) over G , if $\underline{X}_\xi$ and $\overline{X}_\xi$ are subgroups (resp., normal subgroups) of G .

Example 4.2. Let $G = \{\pm 1, \pm i\}$ be a group with $i^2 = -1$ and define a soft set $\mathfrak{S} = (\delta, A)$ over G which is given by Table 5:

Table 5 table for soft set $\mathfrak{S}$				
	1	-1	i	$-i$
e_1	0	0	1	1
e_2	1	1	1	1
e_3	1	1	0	0

We can consider $\xi : G \rightarrow P(A)$ of SRA -space (G, ξ) is shown by $\xi(1) = \xi(-1) = \{e_2, e_3\}$ and $\xi(i) = \xi(-i) = \{e_1, e_2\}$.

Let $X = \{\pm 1, i\}$, then $\underline{X}_\xi = \{\pm 1\}$ and $\overline{X}_\xi = \{\pm 1, \pm i\} = G$. We can check that $\underline{X}_\xi$ and $\overline{X}_\xi$ are normal subgroups of G . This shows that X is an SRN -subgroup over G .

Proposition 4.3. We can consider that (G, ξ) is an MS -approximation space. If H and K are lower SR -groups (SRN -subgroups) over G , then so is $H \cap K$.

Proof. If H and K are lower SR -groups (SRN -subgroups) over G , then $\underline{H}_\xi$ and $\underline{K}_\xi$ are subgroups (normal subgroups) of G , so $\underline{H}_\xi \cap \underline{K}_\xi$ is a subgroup (normal subgroup) of G . By Definition 4.1, it is easy to verify $\underline{H \cap K}_\xi = \underline{H}_\xi \cap \underline{K}_\xi$. So $\underline{H \cap K}_\xi$ is also a subgroup (normal subgroup) of G . Hence $H \cap K$ is a lower SR -group (SRN -subgroup) of G . $\square$

In general, $H \cap K$ is not an upper SR -group of G , if H and K are upper SR -groups of G . Actually we give the following example to illustrate.

Example 4.4. Consider the group G and the soft set $\mathfrak{S} = (F, A)$ in Example 4.2. Now let $H = \{1, -i\}$ and $K = \{-1, -i\}$. By calculation, we have $\overline{H}_\xi = \overline{K}_\xi = \{\pm 1, \pm i\}$. It is obvious that $\overline{H}_\xi$ and $\overline{K}_\xi$ are subgroups of G , so H and K are upper SR -groups of G . But $\overline{H \cap K}_\xi = \overline{\{-i\}}_\xi = \{\pm i\}$ is not a subgroup of G , so $H \cap K$ is not an upper SR -groups of G .

Next, the example shows that $H \cup K$ is also not a lower (an upper) SR -group of G , if H and K are SR -groups of G .

Example 4.5. We can consider that $G = \{1, a, b, c\}$ is a set with a multiplication operation $(\cdot)$ as follow:

$\cdot$	1	a	b	c
1	1	a	b	c
a	a	1	c	b
b	b	c	1	a
c	c	b	a	1

Then G is a group. We can consider that $\mathfrak{S} = (\delta, A)$ is a soft set over G which is defined as Table 6.

Table 6 table for soft set $\mathfrak{S}$				
	1	a	b	c
e_1	0	0	1	1
e_2	1	1	0	0
e_3	1	1	0	1

We can consider $\xi : G \rightarrow P(A)$ of SRA-space (G, ξ) is given by $\xi(1) = \xi(a) = \{e_2, e_3\}$ and $\xi(b) = \{e_1\}$, $\xi(c) = \{e_1, e_3\}$.

If we take $H = \{1, a\}$ and $K = \{1, b\}$, then $\underline{H}_\xi = \{1, a\}$, $\underline{K}_\xi = \{1, b\}$, $\overline{H}_\xi = \{1, a\}$, $\overline{K}_\xi = \{1, b\}$, so H and K are SR -groups of G . Moreover, $\underline{H \cup K}_\xi = \{1, a, b\}_\xi = \{1, a, b\}$ and $\overline{H \cup K}_\xi = \{1, a, b\}_\xi = \{1, a, b\}$. That is, $H \cup K$ is not a lower (an upper) SR -group of G .

In the following, we investigate the upper and lower SR -groups.

Theorem 4.6. *We can consider $\mathfrak{S} = (\delta, A)$ be a C -soft set over G and H a subgroup of G . Then $\overline{H}_\xi$ is a subgroup of G .*

Proof. Since $H \subseteq \overline{H}_\xi$, then $e \in H \subseteq \overline{H}_\xi$. For all $a, b \in \overline{H}_\xi$, by Definition 4.1, $\xi(a) = \xi(x)$ and $\xi(b) = \xi(y)$ for some $x, y \in H$. By the hypothesis, $\mathfrak{S} = (F, A)$ is a C -soft set over G , so $\xi(a \cdot b) = \xi(x \cdot y)$ for $x \cdot y \in H \cdot H \subseteq H$, then $a \cdot b \in \overline{H}_\xi$. Since H is a subgroup of G , we have $a^{-1} \in H$ for all $a \in H$. That is, for all $a \in \overline{H}_\xi$, we have $a^{-1} \in \overline{H}_\xi$. Hence $\overline{H}_\xi$ is a subgroup of G . $\square$

Theorem 4.7. *We can consider $\mathfrak{S} = (\delta, A)$ be a CC -soft set over G and H a subgroup of G . Then $\underline{H}_\xi$ is a subgroup of G if $\underline{H}_\xi \neq \emptyset$.*

Proof. Let $\underline{H}_\xi \neq \emptyset$, for all $x, y \in \underline{H}_\xi$. Suppose that $x \cdot y \notin \underline{H}_\xi$. Then we have $\xi(x) \neq \xi(a)$ for all $a \in H^c$ and $\xi(y) \neq \xi(b)$ for all $b \in H^c$.

On the other hand, it may have the following two conditions, if $x \cdot y \notin \underline{H}_\xi$.

(1) $x \cdot y \notin H$, which contradicts with $x \cdot y \in \underline{H}_\xi \cdot \underline{H}_\xi \subseteq H \cdot H \subseteq H$;

(2) $x \cdot y \in H$ and $\xi(c) = \xi(m \cdot n)$ for some $c \in H^c$. By the hypothesis, $\mathfrak{S} = (F, A)$ is a CC -soft set over G , then $\exists m, n \in G$ such that $\xi(x) = \xi(m)$ and $\xi(y) = \xi(n)$ satisfying $m \cdot n = c \in H^c$. Thus, $m \in H^c$ or $n \in H^c$. If $m \notin H^c$ and $n \notin H^c$, we have $m \cdot n \in H \cdot H \subseteq H$, a contradiction. That is, $\exists m \in H^c$ s.t. $\xi(x) = \xi(m)$ or $n \in H^c$ with $\xi(y) = \xi(n)$. This is contradicts to $x \cdot y \notin \underline{H}_\xi$. Thus $x \cdot y \in \underline{H}_\xi$. Similarly, we have $x^{-1}, y^{-1} \in \underline{H}_\xi$. This implies $\underline{H}_\xi$ is a subgroup of G . $\square$

Theorem 4.8. *We can consider that $\mathfrak{S} = (F, A)$ is a C -soft set over G and H is a normal subgroup of G . Then $\overline{H}_\xi$ is a normal subgroup of G .*

Proof. Let H be a normal subgroup of G . By Theorem 4.6, we have $\overline{H}_\xi$ is a subgroup of G . If $x \in G$ and $y \in \overline{H}_\xi$, then $\xi(y) = \xi(a)$ for some $a \in H$, by the hypothesis, we have $x \cdot a \cdot x^{-1} \in H$. By the hypothesis, $\xi(x) = \xi(x)$, $\xi(x^{-1}) = \xi(x^{-1})$ and $\mathfrak{S} = (F, A)$ be a C -soft set, so $\xi(x \cdot y \cdot x^{-1}) = \xi(x \cdot a \cdot x^{-1})$ for some $x \cdot a \cdot x^{-1} \in H$, thus $x \cdot y \cdot x^{-1} \in \overline{H}_\xi$. Hence $\overline{H}_\xi$ is a normal subgroup of G . $\square$

Theorem 4.9. *We can consider that $\mathfrak{S} = (\delta, A)$ is a CC -soft set over G and H is a normal subgroup of G . Then $\underline{H}_\xi$ is a normal subgroup of G if $\underline{H}_\xi \neq \emptyset$.*

Proof. By Theorem 4.7 and the hypothesis, it is easy to verify. $\square$

The following example show that the converse of Theorems 4.8 and 4.9 are not true.

Example 4.10. Put the group G in Example 4.5. $\mathfrak{S} = (\delta, A)$ is a soft set over G shown by Table 7.

Table 7 table for soft set $\mathfrak{S}$				
	1	a	b	c
e_1	0	0	1	1
e_2	1	1	0	0
e_3	1	1	0	0

We can consider that $\xi : G \rightarrow P(A)$ of SRA-space (G, ξ) is given by $\xi(1) = \xi(a) = \{e_2, e_3\}$ and $\xi(b) = \xi(c) = \{e_1\}$. It is easy to check that $\mathfrak{S}$ is a CC -soft set over G .

Let $H = \{1, a, b\}$, then $\underline{H}_\xi = \{1, a\}$, $\overline{H}_\xi = \{1, a, b, c\}$. Thus we have $\underline{H}_\xi$ and $\overline{H}_\xi$ are normal subgroups of G , but H is not a normal subgroup of G .

5. Soft rough groups in decision making methods

In recent years, both SST and RST have been applied to tackle the problems in decision making. Due to the particularity of the environment and strategy of different forms, every decision making method has own benefit and drawback. So it is not possible to decide which method is the most appropriate.

In this section, we praise a kind of DM-method to choose the optimal parameter of $\mathfrak{S} = (F, A)$ which is given. That is, $F(e)$ is the nearest accurate groups on $\mathfrak{S}$ based on another soft set $\mathfrak{E}$ over groups.

We can consider G be a group and E a set of parameters. Let $A = \{e_1, e_2, \dots, e_n\} \subseteq E$, $\mathfrak{S} = (\delta, A)$ be an original description soft set over G and (G, ξ) be an MS -approximation space. Let $\mathfrak{E} = (X, B)$ be another soft set over G .

The algorithm of DM-method:

Step I Consider the group G , the given soft set $\mathfrak{S}$, the MSR -approximation space (G, ξ) and define another soft set $\mathfrak{E} = (X, B)$ over G .

Step II Reckon the lower (upper) SR -approximation operators $\underline{(X, B)}_\xi$ and $\overline{(X, B)}_\xi$ w.r.t $\mathfrak{S}$, respectively.

Step III Reckon $\|X(e_i)\|$, where $\|X(e_i)\| = \frac{|X(e_i)_\xi|}{|\overline{X(e_i)}_\xi|}$.

Step IV Find the maximum values $\|X(e_j)\|$ of $\|X(e_i)\|$, where $\|X(e_j)\| = \max_i \|X(e_i)\|$.

Step V The decision goal is $X(e_j)$.

Example 5.1. In order to find the nearest accurate group, we can consider $G = \{1, a, b, c\}$ be a group with the given Table 8:

Table 8 table for group G				
$\cdot$	1	a	b	c
1	1	a	b	c
a	a	1	c	b
b	b	c	1	a
c	c	b	1	a

Define $\mathfrak{S} = (\delta, A)$ over G which is given by Table 9.

Table 9 table for soft set $\mathfrak{S}$				
	1	a	b	c
e_1	1	1	1	0
e_2	1	1	0	1
e_3	0	0	0	1
e_4	c	0	1	1

We can consider $\xi : G \rightarrow P(A)$ of SRA-space (G, ξ) is given by $\xi(1) = \xi(a) = \{e_1, e_2\}$, $\xi(b) = \{e_1, e_4\}$, $\xi(c) = \{e_2, e_3, e_4\}$.

Define another soft set $\mathfrak{E} = (X, B)$ over G which is given by Table 10.

Table 10 table for soft set $\mathfrak{E}$				
	1	a	b	c
e_1	1	0	0	0
e_2	1	0	1	1
e_3	1	1	1	0
e_4	1	0	1	0
e_5	0	1	0	1

That is, $X(e_1) = \{1\}$, $X(e_2) = \{1, b, c\}$, $X(e_3) = \{1, a, b\}$, $X(e_4) = \{b, c\}$, $X(e_5) = \{a, c\}$.

By calculations, we have that $\underline{X}(e_1)_\xi = \emptyset$, $\overline{X}(e_1)_\xi = \{1, a\}$, $\underline{X}(e_2)_\xi = \{b, c\}$, $\overline{X}(e_2)_\xi = \{1, a, b, c\}$, $\underline{X}(e_3)_\xi = \{1, a, b\}$, $\overline{X}(e_3)_\xi = \{1, a, b, c\}$, $\underline{X}(e_4)_\xi = \{b\}$, $\overline{X}(e_4)_\xi = \{1, a, b\}$, $\underline{X}(e_5)_\xi = \{c\}$ and $\overline{X}(e_5)_\xi = \{1, a, c\}$.

Then, we can obtain that $\|X(e_1)\| = 0$, $\|X(e_2)\| = 0.5$, $\|X(e_3)\| = 0.75$, $\|X(e_4)\| = 0.33$ and $\|X(e_5)\| = 0.33$. So the maximum values $\|X(e_j)\|$ is $\|X(e_3)\| = 0.75$. This means that $X(e_3) = \{1, a, b\}$ is the closest accurate group.

Remark 5.2. (1) For a given soft set $\mathfrak{S} = (F, A)$ which is the original description over G , due to the flexibility of another soft set $\mathfrak{E} = (X, B)$, by adjusting soft set $\mathfrak{E}$, we can obtain different result. In other word, the decision maker can find the optimal object by changing the soft set $\mathfrak{E}$. In Example 5.1, if we change the soft set $\mathfrak{E}$ as follow Table 11:

Table 11 table for soft set $\mathfrak{E}$				
	1	a	b	c
e_1	1	0	0	0
e_2	1	0	1	1
e_3	1	1	0	0
e_4	1	0	1	0
e_5	0	1	0	1

So $X(e_1) = \{1\}$, $X(e_2) = \{1, b, c\}$, $X(e_3) = \{a, b\}$, $X(e_4) = \{b, c\}$, $X(e_5) = \{a, c\}$. By calculations, we find the nearest accurate group is $X(e_2) = \{1, b, c\}$;

(2) By Steps III and IV in the algorithm of the DM-method, we have that: the larger the value of $\|X(e_i)\|$ is, the closer $|\underline{X}(e_i)_\xi|$ and $|\overline{X}(e_i)_\xi|$ are. No matter how to change the soft set $\mathfrak{E}$, $0 \leq \|X(e_j)\| = \max_i \|X(e_i)\| \leq 1$.

6. Conclusion

As far as known that Shabir et al. [23] showed clearly that there exist two problems on Feng et al.'s approach in [9], such as, (1) an upper approximation of a given set may be empty; (2) the upper approximation of the given set may not contain itself on Feng et al.'s SR -sets. Moreover, in order to remove the limited condition that full soft sets are needed in Feng- SR -sets, Shabir et al. proposed the concept of MSR -sets. Based on the idea of [8, 9], we can find that RS -sets and SR -sets are different theories, RS -sets are soft sets while SR -sets are classical sets. For the development of two theories, rough soft algebras (RS -algebras) can be seen in many articles, but few authors investigated the soft rough algebras (SR -algebras). Until now, only Zhan et al. [26] applied SR -sets to hemirings and the roughness in hemirings w.r.t. MSR -approximation space was studied.

In the present paper, based on the ideas of Shabir et al. [23] and Zhan et al. [26], we apply the MSR -sets to groups. In section 3, we first investigate some

operations and fundamental properties of modified SR -set over groups which are different from the ordinary universe, it provides us the idea and theoretical basis for studying other algebraic structures. In section 4, by the characterization of groups, roughness such as lower and upper SR -groups (SRN -subgroups) in groups w.r.t. MSR -approximation space are explored. This can give us a correct idea for the subsequent studies on semigroups, n -ary groups n -ary semigroups and (m, n) -ary rings. In section 5, we put forward a kind of decision making method for SR -groups which are not the same to [14] and [26] and an actual example is given to illustrate the method.

In the future research, we can study the following topics:

- (1) Applying this novel SR -sets to other different algebras, such as semi-groups, n -ary groups, n -ary semigroups, (m, n) -ary rings and so on.
- (2) Studying soft rough fuzzy groups by exchanging groups for fuzzy groups.
- (3) Investigating DM-methods based on SRF-groups.
- (4) Applying this novel SR -sets to some applied areas.

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CRITICAL SEMIMODULES

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Abstract. In the paper, a particular class of semimodules (so called critical semimodules) typical for additively idempotent semirings possessing at least two right multiplicatively absorbing elements is investigated.

Keywords: semiring, semimodule, ideal, characteristic, critical.

The present note is a direct continuation of [1] and [2] and the reader is fully referred to [1], [2] as concerns notation, terminology and further references. Here, we introduce and study a certain type of (left) semimodules that are typical for additively idempotent semirings possessing at least two right multiplicatively absorbing elements.

1. Preliminaries

Let $A = A(*)$ be a groupoid. An element $a \in A$ is called *left (right) neutral* if $a * x = x$ ($x * a = x$) for all $x \in A$, and *left (right) absorbing* if $a * x = a$ ($x * a = a$) for all $x \in A$. If $A = A(+)$ then $0_A \in A$ ($o_A \in A$) means that 0_A (o_A) is (the unique) left and right neutral (absorbing) element of $A(+)$ and $0_A \notin A$ ($o_A \notin A$) denotes the fact that $A(+)$ has no (left and right) neutral (absorbing) element. Similarly, if $A = A(\cdot)$ then $1_A \in A$ means that 1_A is (the unique) left and right neutral element of $A(\cdot)$.

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A semiring is a non-empty set equipped with two associative binary operations that are usually written as addition and multiplication. The addition is commutative and the multiplication distributes over the addition. Given a semiring S , a (left S -)semimodule $({}_S M =) M$ is a commutative semigroup $M(+)$ together with a scalar multiplication $S \times M \rightarrow M$ such that $(a+b)x = ax + bx$, $a(x+y) = ax + ay$ and $a(bx) = (ab)x$ for all $a, b \in S$ and $x, y \in M$. If S is a semiring then $R = \underline{R}(S) = \{a \in S \mid Sa = \{a\}\}$ denotes the set of right multiplicatively absorbing elements. If $a \in \underline{R}(S)$ then $a + a = aa + aa = (a+a)a = a$ and $a(b+b) = ab + ab = ab$ for every $b \in S$. Consequently, the semiring S is additively idempotent, provided that the right semimodule $\underline{R}(S)_S$ is faithful, i.e., for all $a, b \in S$, $a \neq b$, there is at least one $x \in \underline{R}(S)$ with $xa \neq xb$.

Let S be a semiring. A non-empty subset I of S is a *left (right) ideal* if $SI \cup (I+I) \subseteq I$ ($IS \cup (I+I) \subseteq I$). A left (right) ideal I is called *minimal* if $|I| \geq 2$ and $J = I$ whenever J is a left (right) ideal with $|J| \geq 2$ and $J \subseteq I$. A non-empty subset I of S is an *ideal* if $SI \cup IS \cup (I+I) \subseteq I$ and it is a *bi-ideal* if $SI \cup IS \cup (I+S) \subseteq I$. In the latter case, the relation $(I \times I) \cup \text{id}_S$ is a congruence of the semiring S . Finally, S is called

- *simple* (more precisely: *congruence-simple*) if S has just two congruence relations (then these are id_S and $S \times S$ and $|S| \geq 2$);
- *(bi-)ideal-simple* if $S = I$ whenever I is an (bi-)ideal of S with $|I| \geq 2$.

Throughout the paper, all semirings and semimodules are assumed to be additively idempotent. It means that the respective additive semigroups $M(+)$ are semilattices, where the basic order relation is given by $\alpha \leq \beta$ iff $\alpha + \beta = \beta$.

2. Semirings possessing at least two right multiplicatively absorbing elements

In this section, let S be a semiring such that $|R| \geq 2$. Notice that the set R is an ideal of the semiring S and it is the smallest right ideal of S . The set $R+S$ is the smallest bi-ideal of S . Consequently, the semiring S is bi-ideal-simple if and only if $S = R+S$.

Lemma 2.1. *If S is simple then the right semimodule R_S is faithful.*

Proof. Easy to see. □

Lemma 2.2. (i) *If $0_S \in S$ then $0_R \in R$, $R0_S = \{0_R\}$ and $S0_S \leq 0_R$.*

(ii) *If $o_S \in S$ then $o_R \in R$, $Ro_S = \{o_R\}$ and $o_R \leq So_S$.*

Proof. (i) We have $a0_S + b = a0_S + ab = a(0_S + b) = ab = b$ for all $a \in S$ and $b \in R$.

(ii) We have $a0_S + b = ao_S + ab = a(o_S + b) = ao_S$ for all $a \in S$ and $b \in R$. □

Proposition 2.3. *Assume that either $S = R+S$ or the right semimodule R_S is faithful or the semiring S is simple. Then:*

- (i) If $0_S \in S$ then $0_S = 0_R \in R$.
- (ii) If $0_R \in R$ then $0_R = 0_S \in S$.

Proof. (i) Using 2.2(i), we have $ab0_S = 0_R = a0_S$ for all $a \in R$ and $b \in S$. Then $b0_S = 0_S$, provided that R_S is faithful. Furthermore, $0_R + S = 0_R + R + S$, provided that $R + S = S$. The rest is clear.

(ii) We have $b(a + 0_R) = ba + b0_R = ba + 0_R = ba$ for all $a \in S$ and $b \in R$. If R_S is faithful then $a + 0_R = a$. The rest is clear. $\square$

Proposition 2.4. Assume that either $R \cap (S + a) \neq \emptyset$ for every $a \in S$ or the right semimodule R_S is faithful or the semiring S is simple. Then:

- (i) If $o_S \in S$ then $o_S = o_R \in R$.
- (ii) If $o_R \in R$ then $o_R = o_S \in S$.

Proof. We proceed similarly as in the proof of 2.3. $\square$

Lemma 2.5. Let $e \in S$ be a right multiplicatively neutral element. Then:

- (i) $e \notin R$.
- (ii) If $e = a + b$, $a \in R$, $b \in S$, then $a = 0_S$.
- (iii) If $e + a \in R$ for some $a \in S$ then $e + a = o_S$.
- (iv) If the right semimodule S_S is faithful then $e = 1_S$ is multiplicatively neutral.

Proof. (i) Obvious.

- (ii) For every $c \in S$, $c = ce = ca + cb = a + cb$, so that $a + S = S$ and $a = 0_S$.
- (iii) We have $e + a = b(e + a) = be + ba = b + ba$ for every $b \in S$.
- (iv) We have $aeb = ab$ for all $a, b \in S$. $\square$

Corollary 2.6. Assume that the semiring S is simple. If $e \in S$ is right multiplicatively neutral then $e = 1_S$ is multiplicatively neutral and $0_S \in S$. If, moreover, $(1_S + S) \cap R \neq \emptyset$ then $o_S \in S$.

Lemma 2.7. Let $e \in S$ be left multiplicatively neutral. If the left semimodule ${}_S S$ is faithful then $e = 1_S$ is multiplicatively neutral.

Proof. Easy to see. $\square$

Corollary 2.8. Assume that the semiring S is simple. If $e \in S$ is left multiplicatively neutral then $0_S \in S$ and if, moreover, $|S| \geq 3$ then $e = 1_S$ is multiplicatively neutral.

Proposition 2.9. Assume that $o_S \in S$. The semiring S is simple if and only if $o_S \in R$, $S = R + S$ and the right semimodule R_S is faithful and simple.

Proof. First, assume that S is simple. Then S is bi-ideal-simple, and hence $R + S = S$. The right semimodule R_S is faithful by 2.1. If α is a congruence of R_S then σ is a congruence of S , where $(a, b) \in \sigma$ iff $(ca, cb) \in \alpha$ for every $c \in R$, and we have $\alpha = \sigma \cap (R \times R)$. Thus $\alpha = \text{id}_R$ or $\alpha = R \times R$.

Conversely, assume that $S = R + S$ and R_S is faithful and simple. Let $\varrho \neq \text{id}_S$ be a congruence of the semiring S and $(a, b) \in \varrho$, $a \neq b$. Since R_S is faithful, we have $ca \neq cb$ for at least one $c \in R$, and hence $\beta = \varrho \cap (R \times R) \neq \text{id}_R$. Clearly, β is a congruence of R_S , so that $\beta = R \times R$ and $R \times R \subseteq \varrho$. If $a \in S$ then $a = b + c$ for some $b \in S$, $c \in R$, $(c, o_S) \in \varrho$, and hence $(a, o_S) = (b + c, b + o_S) \in \varrho$. $\square$

Lemma 2.10. *Assume that $0_S \in S$ and the right semimodule R_S is faithful. Then $0_S \in R$ and the following conditions are equivalent:*

- (i) $\alpha_1 = (R_1 \times R_1) \cup \text{id}_R$, where $R_1 = R \setminus \{0_S\}$, is a congruence of R_S .
- (ii) $ab \neq 0_S$ for all $a \in R_1$ and $b \in S_1 = S \setminus \{0_S\}$.
- (iii) $cd \neq 0_S$ for all $c, d \in S_1$.
- (iv) S_1 is a subsemiring of S .

Proof. It is easy. $\square$

Lemma 2.11. *Assume that $o_S \in S$ and the right semimodule R_S is faithful. Then $o_S \in R$ and the following conditions are equivalent:*

- (i) $\alpha_2 = (R_2 \times R_2) \cup \text{id}_R$, where $R_2 = R \setminus \{o_S\}$, is a congruence of R_S .
- (ii) $a + b \neq o_S \neq ac$ for all $a, b \in R_2$ and $c \in S_2 = S \setminus \{o_S\}$.
- (iii) S_2 is a subsemiring of S .

Proof. It is easy. $\square$

Corollary 2.12. *Let the semiring S be simple and $|R| \geq 3$. Then:*

- (i) *If $0_S \in S$ then $ab = 0_S$ for some $a \in R \setminus \{0_S\}$ and $b \in S \setminus \{0_S\}$.*
- (ii) *If $o_S \in S$ then either $a + b = o_S$ or $ac = o_S$ for $a, b \in R \setminus \{o_S\}$ and $c \in S \setminus \{o_S\}$.*

Lemma 2.13. *Assume that $0_S, o_S \in R$. Then $\alpha_3 = (R_3 \times R_3) \cup \text{id}_R$, where $R_3 = R \setminus \{0_S, o_S\}$, is a congruence of R_S iff the following three conditions are satisfied:*

1. $R_3 + R_3 = R_3$ (equivalently, $R_2 + R_2 = R_2$).
2. If $ab = 0_S$ for some $a \in R_1$ and $b \in S_1$ then $R_2b = \{0_S\}$.
3. If $cd = o_S$ for some $c \in R_2$ and $d \in S_2$ then $R_1d = \{o_S\}$.

Proof. It is easy. $\square$

3. Critical semimodules (a)

In this section, let S be a non-trivial semiring and M be a *precharacteristic* (left S -)semimodule, i.e. $|M| \geq 2$, $0_M, o_M \in M$, $S0_M = \{0_M\}$ and $So_M = \{o_M\}$. Put $N = M \setminus \{o_M\}$ and $K = M \setminus \{0_M, o_M\}$.

Lemma 3.1. *Let $|M| \geq 4$. The following conditions are equivalent:*

- (i) $Sx = M$ for every $x \in K$.
- (ii) M has (at most) four subsemimodules (and these are $\{0_M\}$, $\{o_M\}$, $\{0_M, o_M\}$ and M).

Proof. (i) implies (ii). This implication is easy.

(ii) implies (i). Put $F = \{x \in M \mid Sx \subseteq \{0_M, o_M\}\}$. Then F is a subsemimodule of M , $\{0_M, o_M\} \subseteq F$ and (i) is clear, provided that $F = \{0_M, o_M\}$. On the other hand, if $F = M$ then the set $G_x = \{0_M, x, o_M\}$ is a proper subsemimodule of M for every $x \in K$, a contradiction. $\square$

A semimodule M will be called *almost minimal* if $|M| \geq 3$, M is precharacteristic and $Sx = M$ for every $x \in K$. Further, M will be called *almost critical* if it is almost minimal, faithful and simple.

Lemma 3.2. *Let M be almost minimal. Define a relation α on M by $(x, y) \in \alpha$ iff $\{a \in S \mid ax = 0_M\} = \{a \in S \mid ay = 0_M\}$. Then:*

- (i) α is a congruence of M and M/α is almost minimal.
- (ii) α is the (unique) greatest proper congruence of M and M/α is simple.
- (iii) $(x, y) \in \alpha$ iff $\{a \in S \mid ax = o_M\} = \{a \in S \mid ay = o_M\}$.

Proof. It is easy to see that α is a congruence of the semimodule M and $(0_M, o_M) \notin \alpha$, $(0_M, x) \notin \alpha$ and $(x, o_M) \notin \alpha$ for every $x \in K$. Thus $|M/\alpha| \geq 3$ and M/α is almost minimal. Now, let β be a congruence of M such that $\beta \not\subseteq \alpha$. If $(u, v) \in \beta \setminus \alpha$ and $a \in S$ is such that $au = 0_M$ and $w = av \neq 0_M$ then $(0_M, w) = (au, av) \in \beta$, $w \in M \setminus \{0_M\}$ and there is $b \in S$ with $bw = o_M$. Consequently, $(0_M, o_M) = (b0_M, bw) \in \beta$ and we conclude that $\beta = M \times M$. Thus we have proved assertions (i) and (ii).

Finally, define a relation γ on M by $(x, y) \in \gamma$ iff $\{a \in S \mid ax = o_M\} = \{a \in S \mid ay = o_M\}$. From (ii) it follows that $\gamma \subseteq \alpha$. On the other hand, if $(u, v) \in \alpha \setminus \gamma$ and $a \in S$ is such that $au = o_M \neq av$ then $(o_M, av) \in \alpha$, a contradiction. $\square$

Lemma 3.3. *Assume that M is almost minimal and S is simple. Then M is faithful.*

Proof. If M were not faithful then we get $ax = bx$ for all $a, b \in S$ and $x \in M$. But this is a contradiction with $Sx = M$ for $x \in K$. $\square$

Proposition 3.4. *Assume that the semiring S is simple. If there is at least one almost minimal semimodule then there is an almost critical semimodule.*

Proof. Combine 3.2 and 3.3. $\square$

In the remaining part of this section, assume that M is simple and $\{0_M, o_M\} \subseteq Sx$ for every $x \in K$. If I is an ideal of the semiring S and $y \in M$, we put $J_{I,y} = \{a \in I \mid ay = 0_M\}$.

Lemma 3.5. *If $w \in N$ then $J = J_{I,w}$ is a left ideal of S . If $a_0 = o_J \in J$ and $x, y \in M$ are such that $x \leq w$ and $y \not\leq w$ then $a_0x = 0_M$ and $a_0y = o_M$ (and hence $a_0 \in R$, provided that M is faithful).*

Proof. We have $0_M \in Sw$, so that $J \neq \emptyset$ and J is a left ideal of S . If $x \leq w$ then $a_0x \leq a_0w = 0_M$ and $a_0x = 0_M$. Now, let $y \not\leq w$. If $ay = o_M$ for some $a \in J$ then $o_M = ay \leq a_0y$ and $a_0y = o_M$. Assume, therefore, that $o_M \notin Jy$. Then $(w, w+y) \in \alpha$, where α is the relation defined on M by $(u, v) \in \alpha$ iff $\{0_M, o_M\} \not\subseteq \{au, av\}$ for every $a \in I$. Clearly, the relation α is reflexive, symmetric and if $(u, v) \in \alpha$ then $(bu, bv) \in \alpha$ for every $b \in S$. If, moreover, $c \in I$ and $z \in M$ are such that $c(u+z) = 0_M$ and $c(v+z) = o_M$ then $cu = 0_M = cz$ and $c(v+z) = o_M$, a contradiction with $(u, v) \in \alpha$. Thus $(u+z, v+z) \in \alpha$ and we see that β is a congruence of the semimodule M , where β denotes the transitive closure of α . Since $(w, w+y) \in \alpha \subseteq \beta$, we have $\beta \neq \text{id}_M$ and it follows that $\beta = M \times M$, the semimodule M being simple. In particular, $(0_M, t) \in \alpha$ for at least one $t \in M \setminus \{0_M\}$. It means that $\{0_M, o_M\} \not\subseteq \{a0_M, at\} = \{0_M, at\}$ for every $a \in I$ and $o_M \notin It$. But $o_M = dt$ for some $d \in S$, and if $e \in I$ then $edt = o_M$ and $ed \in I$, a contradiction. $\square$

Lemma 3.6. *If M is faithful and $a_0 = o_J \in J = J_{I,0_M}$ then $a_0 = o_S = o_I$.*

Proof. We have $J_{I,0_M} = I$, and hence $a_0 = o_I$. Now, $(a + a_0)y = ay + a_0y = ay + o_M = o_M = a_0y$ for all $a \in S$ and $y \in M \setminus \{0_M\}$. Of course, $(a + a_0)0_M = 0_M = a_0M$. Since M is faithful, we get $a + a_0 = a_0$. Thus $a_0 = o_S$. $\square$

Lemma 3.7. *Assume that $R \neq \emptyset$ and that the following condition is true:*

- (ε) *If $a_1 < a_2 < a_3 \dots$ is an infinite strictly increasing sequence of elements from R then for every $a \in R \setminus \{o_R\}$ there is $i \geq 1$ with $a \leq a_i$ (see [2, 4.18]).*

Let $w \in K$ be such that $o_J \notin J = J_{R,w}$. Then:

- (i) w is the smallest element of the set $M \setminus \{0_M\}$.
- (ii) $o_R \in R$, $o_R(M \setminus \{0_M\}) = \{o_M\}$ and $(R \setminus \{o_R\})w = \{0_M\}$.

Proof. The left ideal J has no greatest element, and hence it has no maximal element, either. Using (ε), we deduce that $R \setminus \{o_R\} \subseteq J$. Since $o_M \in Sw$, we have $o_M \in Rw$, and therefore $o_R \in R$ and $o_Rw = o_M$. Now, if $w' \in K$ is such that $J_{R,w'}$ has no greatest element then $J_{R,w} = R \setminus \{o_R\} = J_{R,w'}$ and

$aw = aw'$ for every $a \in R$. Thus $(w, w') \in \alpha$, where α is the relation defined on M by $(x, y) \in \alpha$ iff $ax = ay$ for every $a \in R$. Clearly, α is a congruence of the semimodule M and $(0_M, o_M) \notin \alpha$. Since M is simple, we get $\alpha = \text{id}_M$ and $w = w'$. It means that the element w is unique and it is a minimal element of K . Moreover, if $v \in K$ then $o_M \in Sv$, and hence $o_Rv = o_M$. If $v \neq w$ then the left ideal $J_{R,v}$ has the greatest element, say a_0 , and we have $a_0(w + v) = a_0w + 0_M = a_0w$. If $a_0w = 0_M$ then $w \leq v$ follows from 3.5. If $a_0w \neq 0_M$ then $a_0 = o_R$ and $0_M = a_0v = o_Rv = o_M$, a contradiction. $\square$

4. Critical semimodules (b)

Let S be a non-trivial semiring. A semimodule M will be called *characteristic* if M is faithful (i.e., for all $a, b \in S$, $a \neq b$, there is $x \in M$ with $ax \neq bx$), precharacteristic and there is a mapping $\varepsilon : N \rightarrow S$ such that $\varepsilon(x)y = 0_M$ and $\varepsilon(x)z = o_M$ for all $x, y, z \in M$, $y \leq x$, $z \not\leq x$. Further, M will be called *critical* if it is both almost critical and characteristic (see [2]).

In this section, we assume that $|R| \geq 2$ and $o_R \in R$. Assume also that the condition (ε) is satisfied in 4.1, ..., 4.4.

Proposition 4.1. *A precharacteristic semimodule M is characteristic if and only if the following three conditions are satisfied:*

1. M is simple and faithful.
2. $\{0_M, o_M\} \subseteq Sx$ for every $x \in K$.
3. If the set K has the smallest element then the set $R \setminus \{o_R\}$ has at least one maximal element (the greatest element, resp.).

Proof. If M is characteristic then our conditions follow from [2, 2.2, 2.3(i), 2.4(ii),(iii), 2.8, 2.9]. Now, let the three conditions be satisfied. For any $w \in N$, we have to find $\varepsilon(w) \in S$ such that $\varepsilon(w)x = 0_M$ and $\varepsilon(w)y = o_M$, whenever $x \leq w$, $y \not\leq w$. If $w = 0_M$ then $\varepsilon(w) = o_R = o_S$ (use 3.5, 3.6, (1) and (2)). If $w \in K$ and $o_J \in J = J_{r,w}$ then $\varepsilon(w) = o_J$ (use 3.5). If $o_J \notin J$ then w is the smallest element of K by 3.7(i), $J = R \setminus \{o_S\}$ by 3.7(ii), J is a left ideal and has no maximal element, a contradiction with (3). $\square$

Corollary 4.2. *An almost minimal semimodule M is critical if and only if M is faithful, simple and the condition 4.1(3) is true.*

Corollary 4.3. *An almost critical semimodule M is critical if and only if the condition 4.1(3) is true.*

Proposition 4.4. *Assume that the semiring S is simple, the set $R \setminus \{o_R\}$ has at least one maximal element and there is at least one almost minimal semimodule. Then there is a critical semimodule.*

Proof. By 3.4, there is an almost critical semimodule and it remains to use 4.3. $\square$

Proposition 4.5. *Assume that the semiring S is simple, $|S| \geq 3$ and the following condition is satisfied:*

- (γ) *If $a_1 > a_2 > a_3 > \dots$ is an infinite strictly decreasing sequence of elements from R then for every $a \in R \setminus \{0_R\}$ there is $i \geq 1$ with $a \geq a_i$.*

Let M be a characteristic semimodule that is not critical. Then:

- (i) *Both S and M are infinite.*
- (ii) *The set N has the greatest element w and $\underline{\varepsilon}(w) = 0_R = 0_S \in S$.*
- (iii) *$\underline{\varepsilon}(N) = R$ and the set $N \setminus \{w\}$ has no maximal element.*
- (iv) *The set $R \setminus \{0_S\}$ has no minimal element.*
- (v) *$G = M \setminus \{w\}$ is a subsemimodule of M .*
- (vi) *G is a critical semimodule, $o_M = o_G$ and $\underline{\varepsilon}(G \setminus \{o_M\}) = R \setminus \{0_S\}$.*

Proof. M is characteristic and almost critical, and it means that M is not almost minimal and $Sx_0 \neq M$ for at least one $x_0 \in K$. If $|M| = 3$ then $M = \{0_M, x_0, o_M\}$, $|S| = 3$, $1_S \in S$, a contradiction with $Sx_0 \neq M$. Thus $|M| \geq 4$ and, using 3.1, we see that M has a proper subsemimodule G such that $G \not\subseteq \{0_M, o_M\}$. By [2, 2.14, 2.15], G is a characteristic semimodule. By [2, 4.6], $w = o_N \in N$, $\underline{\varepsilon}(w) = 0_S \in S$, $\underline{\varepsilon}(N) = R$, $G = M \setminus \{w\}$ and $\underline{\varepsilon}(G \setminus \{o_M\}) = R \setminus \{0_S\}$. If a_0 is minimal in $R \setminus \{0_S\}$ then $0_S y = a_0 y$ for every $y \in G$, a contradiction with the fact that G is faithful. Thus $R \setminus \{0_S\}$ has no minimal element and $N \setminus \{w\}$ has no maximal element. By [2, 2.14, 4.6] and 3.1, the semimodule G is critical. $\square$

5. Summary

In this section, let S be a simple semiring such that $|S| \geq 3$, $|R| \geq 2$ and $o_S \in S$. Assume, furthermore, that every infinite strictly increasing (decreasing, resp.) sequence of elements from the set $R \setminus \{0_R, o_R\}$ is upwards (downwards, resp.) cofinal in that set (i.e., the conditions (ε) and (γ) are satisfied).

By 2.1, the right semimodule R_S is faithful. By 2.4(i), $o_S = o_R \in R$. By 2.3, if $0_S \in S$ ($0_R \in R$, resp.) then $0_S = 0_R \in R$ ($0_R = 0_S \in S$, resp.).

Theorem 5.1. *Assume that $0_S \notin S$. The following conditions are equivalent:*

- (i) *There is at least one critical semimodule.*
- (ii) *There is at least one critical semimodule M such that $\underline{\varepsilon}(M \setminus \{o_M\}) = R$.*
- (iii) *If the set $R_2 = R \setminus \{o_S\}$ has no maximal element and if $a \in R_2$, $b \in S$ are such that $R_2 b \leq a$ then $o_S b \leq a$ (and hence $Rb \leq a$).*

Proof. (i) implies (ii). Use [2, 4.2].

(ii) implies (iii). We have $R_2 \subseteq Q = \{c \in R \mid cb \leq a\}$. Thus $Q \neq \emptyset$ and, by [2, 3.3(ii)], we have $o_Q \in Q$. Now, if R_2 has no maximal element then $Q \neq R_2$, and hence $Q = R$.

(iii) implies (ii). Using (ε) , we see that $R \cup \{\omega\}$, where $\omega < R$, is a lattice. Now, let $a \in R$ and $b \in S$ be such that $Q = \{c \in R \mid cb \leq a\} \neq \emptyset$. If $o_Q \notin Q$ then $R_2 = Q$ follows from (ε) . Thus $R_2 b \leq a$ and $o_S b \not\leq a$, a contradiction. It means that $o_Q \in Q$ and it remains to use [2, 3.3] and 4.5. $\square$

Remark 5.2. If the conditions of 5.1 are satisfied then $M \cong {}^1_S R$ (see [1, 7.2]), and hence the critical semimodule ${}_S M$ is determined uniquely up to isomorphism.

Now, assume that 5.1(ii) is true and the set R_2 has no maximal element. Let $a \in R_2$ and $b \in S$ be such that $R_2 b \leq a$. We have $a = \underline{\varepsilon}(w)$ for some $w \in K = M \setminus \{0_M, o_M\}$. If $x \in K$ then $\underline{\varepsilon}(x) \in R_2$, $\underline{\varepsilon}(x)b \leq a$ and $\underline{\varepsilon}(x)bw \leq aw = 0_M$. Thus $bw \leq x$ and either $bw = 0_M$, or bw is the smallest element of K and $\underline{\varepsilon}(bw)$ is the greatest element of R_2 , a contradiction. Thus $bw = 0_M$, $(a+b)y = ay + by \leq aw + bw = 0_M = ay$ for $y \leq w$ and $(a+b)z = az + bz = o_M = az$ for $z \not\leq w$. Since ${}_S M$ is faithful, we get $a+b = a$ and $b \leq a$ (conversely, if $b \leq a$ then $o_S b \leq o_S a = a$ anyway).

Theorem 5.3. Assume that $|S| \geq 4$ and $0_S \in S$. The following conditions are equivalent:

- (i) There is at least one critical semimodule M such that $0_S \in \underline{\varepsilon}(M \setminus \{o_M\})$ or, equivalently, such that the set $M \setminus \{o_M\}$ has the greatest element.
- (ii) There is at least one critical semimodule M such that $\underline{\varepsilon}(M \setminus \{o_M\}) = R$.
- (iii) The condition 5.1(iii) holds and, besides, at least one of the following three conditions is satisfied:
 - (1) There are $a, b \in R_1 = R \setminus \{0_S\}$ such that for every $c \in R_1$ either $c \not\leq a$ or $c \not\leq b$.
 - (2) There are $a \in R_1$ and $c \in S$ such that $0_S c = 0_S$ and $bc \not\leq a$ for every $b \in R_1$.
 - (3) There is $c \in S$ such that $0_S c \neq 0_S$ and $0_S c \neq bc$ for every $b \in R_1$.

Proof. (i) implies (ii). Use [2, 4.2] and [2, 2.11].

(ii) implies (iii). The condition 5.1(iii) follows from [2, 3.3(ii)] (see the proof of 5.1). By [2, 2.11(iv)], $\underline{\varepsilon}(w) = 0_S$, where w is the greatest element of N . Since $|S| \geq 4$, we have $|M| \geq 4$ and $|N_1| \geq 2$, where $N_1 = N \setminus \{w\}$. Since M is almost minimal, the set $M \setminus \{w\}$ is not a subsemimodule of M (use 3.1).

Assume, first, that $N_1 + N_1 \not\subseteq N_1$. There are $u, v \in N_1$ such that $u + v = w$. Then $a = \underline{\varepsilon}(u) \in R_1$, $b = \underline{\varepsilon}(v) \in R_1$, and if $c \in R_1$ then $c = \underline{\varepsilon}(z)$ for some $z \in N_1$ and either $u \not\leq z$ and $c \not\leq a$, or $v \not\leq z$ and $c \not\leq b$. Thus (1) is true.

Assume, next, that $N_1 + N_1 \subseteq N_1$. Since $M \setminus \{w\}$ is not a subsemimodule, there are $e \in S$ and $x \in N_1$ such that $cx = w$. Then $x \neq 0_M$, $a = \underline{\varepsilon}(x) \in R_1$, $a \neq 0_S$ and $0_S cx = 0_S w = \underline{\varepsilon}(w)w = 0_M$. Consequently, $0_S c \leq a$ (see [2, 2.7]). If $b \in R_1$ then $b cx = bw = 0_M$ (see [2, 2.4, 2.7]), and so $bc \not\leq a$. Now, it is clear that either (2) or (3) is true.

(iii) implies (ii). Using [2, 3.3] and proceeding similarly as in the proof of 5.1, we find a characteristic semimodule M with $\underline{\varepsilon}(N) = R$. Again, $\underline{\varepsilon}(w) = 0_S$, where w is the greatest element of N . If M is not critical then $G = M \setminus \{w\}$ is a subsemimodule of M and G is critical by 4.5. If (1) is true and $a = \underline{\varepsilon}(u)$, $b = \underline{\varepsilon}(v)$, $u, v \in N_1 = N \setminus \{w\}$, then $\underline{\varepsilon}(u+v) \leq a$, $\underline{\varepsilon}(u+v) \leq b$, and hence $\underline{\varepsilon}(u+v) = 0_S$ and $u+v = w$, a contradiction. If (2) is true then $a = \underline{\varepsilon}(z)$, $z \in N_1$, $0_S cz = 0_S z = 0_M$, $cz \in G$, $b cz = 0_M$, $\underline{\varepsilon}(cz) = 0_S$ and $cz = w$, a contradiction. Finally, if (3) is true then $0_S \neq 0_S c < bc$ for every $b \in R_1$, $0_S c = \underline{\varepsilon}(t)$, $t \in N_1$, $0_S ct = 0_M$, $ct \in G$ and $b ct = 0_M$ for every $b \in R_1$ (see [2, 2.7]). Thus $\underline{\varepsilon}(ct) = 0_S$ and $ct = w$, a contradiction.

(ii) implies (i). It is clear. $\square$

Remark 5.4. If the conditions of 5.3 are satisfied then ${}_S M \cong {}^1_S R$ (see [1, 7.2]).

Remark 5.5. If $|S| = 3$ then the (left S -)semimodule ${}_S S$ is critical. In fact, $S = \{0_S, 1_S, o_S\}$, $R = \{0_S, o_S\}$, $\underline{\varepsilon}(0_S) = o_S$ and $\underline{\varepsilon}(1_S) = 0_S$. On the other hand, none of the conditions 5.3(1),(2),(3) is true.

Remark 5.6. Assume that $0_S \in$ and the set $R_1 = R \setminus \{0_S\}$ has at least one minimal element.

(i) If the set R_1 has at least two minimal elements then the condition 5.3(1) is true. Consequently, assume that R_1 has just one minimal element a_0 . Then a_0 is the smallest element of R_1 (use (γ)). Since S is simple and $|S| \geq 3$, the set $C = \{c \in S \mid 0_S c \neq a_0 c\}$ is non-empty. If $c \in C$ then $0_S c < a_0 c \leq bc$ for every $b \in R_1$. If $0_S c_0 \neq 0_S$ for at least one $c_0 \in C$ then the condition 5.3(3) is true. Assume, therefore, that $0_S C = \{0_S\}$. If $c \in C$ then $a_0 c \in R_1$, and if $a_0 c_1 \neq a_0$ for some $c_1 \in C$ then $b c_1 \not\leq a_0$ for every $b \in R_1$ and 5.3(2) is true. Assume, finally, that $a_0 C = \{a_0\}$. Now, it is easy to see that none of the conditions 5.3(1),(2),(3) is true.

Put $D_1 = \{d \in S \mid 0_S d = 0_S = a_0 d\}$, $D_2 = \{d \in S \mid 0_S d = a_0 = a_0 d\}$, $D_3 = \{d \in S \mid 0_S d = a_0 d > a_0\}$ and $D = D_1 \cup D_2 \cup D_3$. We get $S = C \cup D_1 \cup D_2 \cup D_3$ and this union is disjoint. Moreover, it is easy to check that $(C+C) \cup (C+D_1) = C$, $D_1 + D_1 = D_1$, $(C+D_2) \cup (D_1+D_2) \cup (D_2+D_2) = D_2$, $S + D_3 = D_3$, $CC \subseteq C$, $CD_1 \cup D_1 C \cup D_1 D_1 \cup D_2 D_1 \subseteq D_1$, $CD_2 \cup D_2 C \cup D_1 D_2 \cup D_2 D_2 \subseteq D_2$, $SD_3 \subseteq D_3$, $D_3 C \cup D_3 D_2 \subseteq D_2 \cup D_3$, $D_3 D_1 \subseteq D$ and $SD \cup DS \subseteq D$.

(ii) Assume that the condition 5.1(iii) is satisfied. By [2, 3.3], there is a characteristic semimodule M with $\underline{\varepsilon}(N) = R$ (see also the proof of 5.1). Let a_0 be a minimal element of R_1 . The set N has the greatest element w , $\underline{\varepsilon}(w) = 0_S$ and if $v \in N$ is such that $\underline{\varepsilon}(v) = a_0$ then v is a maximal element of $N \setminus \{w\}$. Now,

by 4.5, the semimodule M is critical, and hence at least one of the conditions 5.3(1),(2),(3) is satisfied due to 5.3 (cf. (i)).

Theorem 5.7. *Assume that $0_S \in S$. The following conditions are equivalent:*

- (i) *There is a critical semimodule M such that $0_S \notin \underline{\varepsilon}(M \setminus \{o_M\})$ (or, equivalently, such that the set $M \setminus \{o_M\}$ has no greatest element).*
- (ii) *There is a critical semimodule M such that $\underline{\varepsilon}(M \setminus \{o_M\}) = R \setminus \{0_S\}$ (and the set $M \setminus \{o_M\}$ has no maximal element).*
- (iii) *$|S| \geq 4$ (S is infinite, resp.) and the following four conditions are satisfied:*
 - (1) *If the set $R_2 = R \setminus \{o_S\}$ has no maximal element and if $a \in R_2 \setminus \{0_S\}$ and $b \in S$ are such that $R_2 b \leq a$ then $o_S b \geq a$.*
 - (2) *For all $a, b \in R_1 = R \setminus \{0_S\}$ there is $c \in R_1$ with $c \leq a$ $c \leq b$.*
 - (3) *For all $c \in S$ such that $0_S c = 0_S$ and $a \in R_1$ there is $b \in R_1$ with $bc \leq a$.*
 - (4) *For every $c \in S$ such that $0_S c \neq 0_S$ there is $b \in R_1$ with $0_S c = bc$.*

Proof. (i) implies (ii). Use [2, 4.2].

(ii) implies (iii). As concerns (1), we have $R_2 \subseteq Q = \{c \in R \mid cb \leq a\}$. Thus $Q \neq \emptyset$ and, by [2, 3.4(ii)], we get $o_Q \in Q$. If R_2 has no maximal element then $Q \neq R_2$, and hence $Q = R$. The conditions (2), (3) and (4) follow from [2, 3.4] (see [2, 3.2.3, 3.2.4, 3.2.5]). By [2, 2.11(viii)], the set $M \setminus \{o_M\}$ has no maximal element, and hence both M and S are infinite.

(iii) implies (ii). Since $0_S \in S$ and (ε) is satisfied, we see that the ordered set R is a lattice in fact. If $a \in R \setminus \{0_S\}$, $b \in S$ and $o_Q \notin Q = \{c \in R \mid cb \leq a\} \neq \emptyset$ then $R_2 = Q$ follows from (ε) . Thus $R_2 b \leq a$ and $o_S b \not\leq a$, a contradiction with (1). Now, according to [2, 3.4], we have a characteristic semimodule M such that $\underline{\varepsilon}(N) = R_1$. By 4.5, M contains a subsemimodule G such that G is critical, $o_M \in G$ and $\underline{\varepsilon}(G \setminus \{o_M\}) = R_1$. Since $\underline{\varepsilon}$ is injective, we get $G = M$. $\square$

Remark 5.8. If the conditions of 5.7 are satisfied then $M \cong \frac{2}{S}R$ (see [1, 7.3]).

Remark 5.9. If $|S| = 3$ then the conditions 5.7(1),(2),(3),(4) are satisfied trivially.

Remark 5.10. Notice that the condition 5.7(2) (5.7(3), 5.7(4), resp.) is just the negation of the condition 5.3(1) (5.3(2), 5.3(3), resp.)

Remark 5.11. According to [2, 5.2.4], under our assumptions the conditions 5.1(iii) and 5.7(1) are equivalent.

Remark 5.12. Assume that $0_S \in S$ and the set $R_1 = R \setminus \{0_S\}$ has at least one minimal element (cf. 5.6).

(i) If R_1 has at least two minimal elements then 5.7(2) is not true. On the other hand, if R_1 has just one minimal element, say a_0 , then a_0 is the smallest element of R_1 and 5.7(2) is satisfied trivially. Moreover, the condition 5.7(3) is satisfied iff, for any $c \in S$, $0_S c = 0_S$ implies $a_0 c \in \{0_S, a_0\}$. The condition 5.7(4) is satisfied iff, for any $c \in S$, $0_S c \neq 0_S$ implies $0_S c = a_0 c$.

(ii) Assume that the condition 5.1(iii) is satisfied. Then there is a characteristic semimodule M such that $\underline{\varepsilon}(N) = R$ and at least one of the conditions 5.7(2),(3),(4) is not true.

Theorem 5.13. *There is at most one critical semimodule (up to isomorphism).*

Proof. Let M and M' be critical semimodules. If $0_S \notin S$ then $M \cong \frac{1}{S}R \cong M'$ by 5.1 and 5.2. Assume, therefore, that $0_S \in S$. If $0_S \in \underline{\varepsilon}(M \setminus \{0_M\})$ then at least one of the conditions 5.3(1),(2),(3) is satisfied and it follows that at least one of the conditions 5.7(2),(3),(4) is not satisfied (see 5.10). Consequently, the (equivalent) conditions of 5.7 are not true, and hence $0_S \in \underline{\varepsilon}(M' \setminus \{0_{M'}\})$. Now, again, $M \cong \frac{1}{S}R \cong M'$ (see 5.3, 5.4). Finally, if $0_S \notin \underline{\varepsilon}(M \setminus \{0_M\})$ and $0_S \notin \underline{\varepsilon}(M' \setminus \{0_{M'}\})$ then $M \cong \frac{2}{S}R \cong M'$ (see 5.7, 5.8). $\square$

Theorem 5.14. *There is at least one critical semimodule if and only if either the set $R_2 = R \setminus \{0_S\}$ has a maximal element or $b \leq a$ whenever $a \in R_2$ and $b \in S$ are such that $R_2 b \leq a$.*

Proof. See 5.1, 5.2, 5.3, 5.7 and 5.11. $\square$

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SOLUTION OF STEADY-STATE HAMILTON-JACOBI EQUATION BASED ON ALTERNATING EVOLUTION METHOD

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Abstract. Hamilton-Jacobi equation is a kind of highly nonlinear partial differential equation which is difficult to be solved. The boundary value problem of steady-state Hamilton-Jacobi equation is supposed as $H(x, \nabla x\phi(x)) = 0$, $x \in \Omega/\Gamma$; $\phi(x) = q(x)$, $x \in \Gamma$ ($\Omega \in Rd$, d stands for the space dimensionality, Ω stands for a bounded open set with a boundary of Γ , and H stands for a given non-linear function, called Hamiltonian). Even though Hamiltonian function is smooth, the derivative of its solution may be disconnected at some cuspidal points. There are many ways to solve a steady-state Hamilton- Jacobi equation, among which, fast marching method (FMM) and fast sweeping method (FSM) are famous. This study solved Hamilton-Jacobi equation using alternating evolution method (AE). Firstly, an initial Hamilton-Jacobi equation was described using AE; then polynomials were constructed to approach the Hamilton-Jacobi equation and the equation was finally solved by selecting proper iterative methods and correct boundary conditions. An artificial parameter was generated in the process of construction of iterative format; the selection of the parameter could directly affect the stability and convergence of the iterative format. On account of this, the stability and convergence of the first-order AE algorithm was analyzed and the effectiveness and accuracy of the algorithm was proved by a numerical experiment.

Keywords: Hamilton-Jacobi equation, alternating evolution method, viscosity solution, convergence.

1. Introduction

Hamilton-Jacobi equation is a kind of highly non-linear hyperbolic partial differential equation which was applied in mechanical studies carried out by engineers and physicists at first and then extensively applied for optimum control and differential game. With the development of computer technology, mathematicians have paid more attentions to the solution of Hamilton-Jacobi equation using numerical calculation. ϕ is generally Lipschitz continuous, but not C^1 smooth; hence there is usually no classical solution for full nonlinear partial differential equation. The concept of weak solution is proposed when solving equations lacking of smoothness. Weak solution refers to a solution that satisfies equation

at points which can be derived and are continuous, but weak solution is not unique. To solve the non-uniqueness of weak solution, some experts proposed the definition of viscosity solution to illustrate the existence and uniqueness of viscosity solution.

There are many ways to solve Hamilton-Jacobi equations. This study explored the solution of steady-state Hamilton-Jacobi equation using alternating evolution

$$\phi = \phi^{SN} - \varepsilon H(\nabla_x \phi^{SN}).$$

In the formula, ε stands for an artificial parameter, whose selection should satisfy the condition of iterative stability. AE was proposed by Liu HL in 2008 and then applied in hyperbolic conservation equations. In 2011, Saran H, et al. applied AE to solve Hamilton-Jacobi equation containing time parameters and gained certain achievement. Hamilton-Jacobi equation containing time parameters was firstly converted into a new form using AE and then decomposed using discontinuous finite element; finally good numerical results could be obtained. The purpose of this study was to calculate the numerical solution of non-linear steady-state Hamilton-Jacobi equation using high-efficient and high-order AE algorithm.

2. Solution of first-order steady-state Hamilton-Jacobi equation based on AE

2.1 The construction of AE system

Before solving Hamilton-Jacobi equation containing time parameters based on AE, the following AE system was constructed:

$$(2.1) \quad H(x, \nabla_x v) = \frac{1}{\varepsilon}(v - u), \quad H(x, \nabla_x u) = \frac{1}{\varepsilon}(u - v).$$

Numerical solutions around grid points were updated based on the above equation and using the numerical solutions of points around grid points, shown as below

$$(2.2) \quad \phi(x)\phi(x)^{SN} - \varepsilon H(x, \nabla_x \phi(x)^{SN}).$$

Considering uniform partition $\{x_k, k \in Z\}$, its grid diameter was Δx . Suppose the real solution of the equation at grid point x_k as ϕ and the numerical value as E_k . r -order polynomial was constructed to approach ϕ^{SN} using $E_{k \pm l}$ and l depended on the number of orders of the polynomial. If the polynomial was supposed as $p_k[E](x)$, then E_k could be expressed as:

$$(2.3) \quad E_k = p_k^r[E](x) - \varepsilon H(x_k, \partial_x p_k^r[E](x)).$$

Next, $p_k^r[E](x)$ could be constructed using equation (2.3). If $I_k : [x_{k-1}, x_{k+1}]$, then non-oscillatory polynomial $p_k[E](x)$ was reconstructed on each grid point

x_k . We have:

$$(2.4) \quad p_k^r[E](x_{k\pm 1}) = E_{k\pm 1}.$$

Then first-order polynomial constructed based on $E_{k\pm 1}$ was:

$$(2.5) \quad p_k^1[E](x) = E_{k-1} + s_k(x - x_{k-1}), s_k = \frac{E_{k+1} - E_{k-1}}{2\Delta x}$$

s_k stands for the approximate value of first-order derivative $\partial_x \phi$. The following is a second-order AE polynomial constructed based on Newton divided difference interpolation mathematics

$$(2.6) \quad p_k^2[E](x) = p_k^1[E](x) + \frac{s'_k}{2}(x - x_{k-1})(x - x_{k+1})$$

s'_k stands for the approximate value of second-order derivative ϕ_{xx} . Based on the above AE form, second-order AE system was:

$$(2.7) \quad E_k = \frac{E_{k+1} + E_{k-1}}{2} - \frac{s'_k}{2}(\Delta x)^2 - \varepsilon H(x_k, s_k).$$

The major characteristic of essentially non-oscillatory (ENO) method is that it adopts self-adaptive template, which avoids offset and ensures the non-oscillatory property of AE format. A triangular unit was selected randomly from nonstructural grid, denoted as Δ_0 . Three vertexes of Δ_0 were $i(x_i, y_i)$, $j(x_j, y_j)$ and $k(x_k, y_k)$. ENO difference value polynomial of Δ_0 had 12 fundamental points. Those fundamental points could also be called the maximum template for numerical format construction, as shown in figure 1. When second-order derivative was calculated, the non-oscillatory property of viscosity solution of equation was

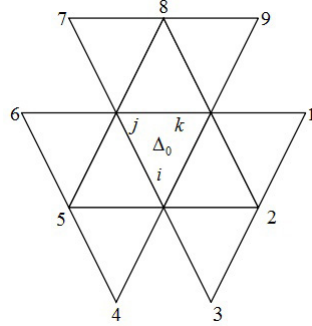


Figure 1: Template for the construction of ENO polynomial of Δ_0

calculated using ENO principle, shown as below

$$(2.8) \quad s'_k = \begin{cases} \frac{s_{k+2} - s_k}{2\Delta x}, & \text{if } \left| \frac{s_{k+2} - s_k}{2\Delta x} \right| \leq \left| \frac{s_k - s_{k-2}}{2\Delta x} \right| \\ \frac{s_k - s_{k-2}}{2\Delta x}, & \text{if } \left| \frac{s_{k+2} - s_k}{2\Delta x} \right| > \left| \frac{s_k - s_{k-2}}{2\Delta x} \right| \end{cases}.$$

Third-order AE system could be obtained similarly. Third-order AE polynomial was:

$$(2.9) \quad p_k^3[E](x) = p_k^2[E](x) + \frac{s_x''}{6}(x - x_{k+1})(x - x_{k+3}).$$

Third-order AE system was as follows:

$$(2.10) \quad \begin{aligned} E_k &= \frac{E_{k+1} + E_{k-1}}{2} - \frac{s_k'}{2}(\Delta x)^2 + \frac{s_k''}{6}(\Delta x)^2(x_k - x_{k+3}) \\ &- \varepsilon H(x_k, s_k - \frac{s_k''}{6}(\Delta x)^2). \end{aligned}$$

2.2 Hamilton-Jacobi equation

Hamilton-Jacobi equation has two formats, shown as below [15, 16].

$$(2.11) \quad \begin{cases} H(x, u, Du) = 0 \\ u = z \end{cases}.$$

The first format means within Ω and the second format means on $\partial\Omega$. The current Hamilton-Jacobi equation was a steady-state equation. The following equation is called Cauchy equation or developmental Hamilton-Jacobi equation

$$(2.12) \quad \begin{cases} u_t + H(x, t, u, Du) = 0 \\ u = z \\ u(x, 0) = u_0(x) \end{cases}.$$

The first format means within $\Omega \times [0, T]$, the second format means on $\partial\Omega \times [0, T]$, and the third format means within Ω . The current Ω stands for the open set of R^N , z and o stand for given boundary and initial value condition, and Du stands for the gradient of with regard to x . $H(x, u, Du)$ was a given function defined on $\Omega \times R \times R^N$ and $H(x, t, tu, Du)$ was a function defined on $H(x, t, tu, Du)$, both of them were Hamilton function.

3. Algorithm for solving steady-state Hamilton-Jacobi equation based on AE

The construction of algorithm includes initialization, iteration process and ending condition:

1. Initialization of algorithm: The initial value was supposed as E^0 , boundary points were assigned, non-boundary points were assigned with arbitrary initial values. The values were updated in the following iteration process.

2. E^{n+1} was calculated based on E^n iteration and it satisfied the following relation expression

$$(3.1) \quad E_k^{n+1} = p_k^r[E^n](x_k) - \varepsilon H(x_k, \partial_x p_k^r[E^n](x_k))$$

3. After the fixation of grid size, the ending condition for algorithm was:

$$(3.2) \quad \|E^{n+1} - E^n\| \leq \xi.$$

ξ was small enough and it showed different degrees of changes according to different precision. Simultaneous iteration could be divided into two parts, odd part and even part. The grid values of odds after $(n+1)$ times of iteration were calculated based on the even grid values obtained after n times of iteration; the rest could be deduced by analogy till the end.

When numerical solutions are calculated using AE format, mistakes may appear in the process of solution if proper values of calculation region boundary and values beyond the involved calculation region are not selected. The calculation region was supposed as a compact set, boundary as Γ , and some grids as even grids. If $\Gamma = \partial\Omega$, the exact solutions of boundary points could be used to calculate points inside region and extract boundary conditions were unnecessary to be given additionally, because the boundary values have been given.

4. Analysis of stability and convergence of AE

Suppose $E = (E_1, E_2, \dots, E_n)$, then first-order AE format was:

$$(4.1) \quad E = F(E)$$

$$(4.2) \quad F_k(E) = \frac{E_{k+1} + E_{k-1}}{2} - \varepsilon H \left(\frac{E_{k+1} - E_{k-1}}{2\Delta x} \right).$$

The following was a proof for the uniqueness of AE format.

Theorem 3.1. *If there was*

$$(4.3) \quad \frac{\varepsilon}{\Delta x} \max |H'(\cdot)| < 1$$

then $E = F(E)$ had at most only one solution.

Proof. Proof The following was a proof for theorem 1 using proof by contradiction. Suppose that equation (15) had two different solutions, i.e., ϕ^* and ϕ^x . The proof was considered tenable if $\zeta \equiv 0$ and $\zeta \equiv \phi^* - \phi^x$.

$$(4.4) \quad \zeta_k = \phi_k^* - \phi_k^x = F_k(\phi^*) - F_k(\phi^x) = \frac{\zeta_{k+1} + \zeta_{k-1}}{2} - \varepsilon H'(\theta_k) \frac{\zeta_{k+1} - \zeta_{k-1}}{2\Delta x}.$$

According to the boundary condition, we have:

$$(4.5) \quad \begin{aligned} \zeta_1 &= a_1 \zeta_2, \zeta_2 = \frac{1}{2} \left(1 - \frac{\varepsilon H'(\theta_k)}{\Delta x} \right) \zeta_{k+1} + \frac{1}{2} \left(1 + \frac{\varepsilon H'(\theta_x)}{\Delta x} \right) \zeta_{k-1} \\ \zeta_N &= b_N \zeta_{N-1}; 2 \leq k \leq N-1. \end{aligned}$$

Suppose $\|\zeta\|_\infty = C > 0$, then:

(1) $C = |\zeta_1|$, then $|\zeta_1| \geq |\zeta_2|$. But it was inconsistent with). Similarly, $C \neq |\zeta_N|$.

(2) $C = |\zeta_1|$ and $2 \leq l \leq N-1$, then $|\zeta_1| \geq |\zeta_{l\pm 1}|$. But the following two situations might occur:

Firstly, if $|\zeta_{l-1}| \neq |\zeta_{l+1}|$, then it was inconsistent with $|\zeta_l| < \max(|\zeta_{l-1}|, |\zeta_{l+1}|) \leq \|\zeta\|_\infty = C = |\zeta_l|$.

Secondly, if $|\zeta_{l-1}| = |\zeta_{l+1}|$, then $|\zeta_l| = |\zeta_{l\pm 1}|$. In this case, the above two steps were repeated on $C = |\zeta_{l\pm 1}|$ till the first situation appeared. Then $\|\zeta\|_\infty = 0$ was obtained, which suggested the uniqueness of the solution. $\square$

Theorem 3.2. Suppose E^n as the numerical solution of Hamilton-Jacobi equation $H(\partial_x \phi) = 0$ under the condition of first-order $AE E^{n+1} = F(E^n)$. If

$$(4.6) \quad \frac{\varepsilon}{\Delta x} \max |H'(\cdot)| < 1,$$

then $\{E^n\}$ was considered as convergent.

Proof. If

$$(4.7) \quad \zeta_k^{n+1} = E_k^{n+1} - E_k^n$$

the $E^{n+1} = F(E^n)$ could be transformed into:

$$(4.8) \quad \begin{aligned} \zeta_1^{n+1} &= a_1^n \zeta_2^n, \zeta_k^{n+1} = \frac{1}{2} \left(1 - \frac{\varepsilon H'(\theta_k^n)}{\Delta x} \right) \zeta_{k+1}^n + \frac{1}{2} \left(1 + \frac{\varepsilon H'(\theta_k^n)}{\Delta x} \right) \zeta_{k-1}^n, \\ \zeta_N^{n+1} &= b_N^n \zeta_{N-1}^n; 2 \leq k \leq N-1. \end{aligned}$$

If there was $\|\zeta^{n+1}\|_\infty = |\zeta_l^{n+1}|$ for $2 \leq l \leq N$, then there were two situations.

(1) For $l = 1$ or $l = N$, there was $\|\zeta^{n+1}\|_\infty = \alpha \|\zeta^n\|_\infty \alpha < 1$.

(2) For $2 \leq l \leq N-1$, there was $|\zeta_l^{n+1}| = |b_l^n \zeta_{l-1}^n + a_l^n \zeta_{l+1}^n| < \max(|\zeta_{l-1}^n|, |\zeta_{l+1}^n|) \leq \|\zeta^n\|_\infty$.

If $\zeta_{l-1}^n \neq \zeta_{l+1}^n$, then $\alpha < 1$; if $\zeta_{l-1}^n = \zeta_{l+1}^n$, then $\alpha \leq 1$. It could also be expressed as:

$$(4.9) \quad \|\zeta^n\|_\infty = \alpha_1 \|\zeta^{n-1}\|_\infty = \alpha_1 \alpha_2 \|\zeta^{n-2}\|_\infty = \dots = \alpha_1 \alpha_2 \dots \alpha_{n-1} \alpha_n \|\zeta^0\|_\infty.$$

If $\alpha_m = 1$, then $\alpha_s < 1 (s < m + N/2)$, we have:

$$(4.10) \quad \|\zeta^n\|_\infty \leq \alpha^{\frac{2n}{N}} \|\zeta^0\|_\infty.$$

The above testified the convergence of $\sum_{n=1}^\infty \zeta^n$ i.e., the convergence of $\{E^n\}$. $\square$

5. Analysis of example

The following was an example analysis based on two-dimensional AE system, which made the application of the solution of steady-state Hamilton-Jacobi equation using AE more extensive.

The numerical results obtained using second-order AE are shown in table 1. It could be known that, L_1 error gradually approached 2 with the decrease of grid size.

N	L_1		L_∞	
	Error	Order	Error	Order
$2^0 \times 10$	3.386×10^{-4}	-	3.348×10^{-3}	-
$2^1 \times 10$	6.606×10^{-5}	2.441	1.624×10^{-3}	1.968
$2^2 \times 10$	1.647×10^{-5}	2.012	4.014×10^{-4}	2.017
$2^3 \times 10$	4.135×10^{-6}	1.999	9.998×10^{-5}	2.009
$2^4 \times 10$	1.032×10^{-6}	1.999	2.486×10^{-5}	2.008
$2^5 \times 10$	2.574×10^{-7}	2.001	6.261×10^{-6}	1.988
$2^6 \times 10$	6.432×10^{-8}	2.001	1.639×10^{-6}	1.935
$2^7 \times 10$	1.610×10^{-8}	1.999	4.107×10^{-7}	1.997

Figure 2: The solution of example 1 $\frac{\varepsilon}{h} = \frac{1}{4}$ using second-order AE

If two-dimensional Eikonal equation was:

$$(5.1) \quad f(x, y) = \sqrt{\phi_x^2 + \phi_y^2} = \frac{\pi}{2} \sqrt{\sin^2\left(\frac{\pi x}{2}\right) + \sin^2\left(\frac{\pi y}{2}\right)}.$$

To obtain an accurate initial value, the boundary condition needed to be processed at first, because the boundary condition only involved one point. If boundary $\Gamma = \{(0,0)\}$, boundary condition was $q(x, y)|_\Gamma = 2$, and region $\Omega = [1, 1] \times [1, 1]$ was taken as numerical value computation region, then the exact solution of the problem was:

$$(5.2) \quad \phi(x, y) = \cos\left(\frac{\pi x}{2}\right) + \cos\left(\frac{\pi y}{2}\right).$$

Next $\varepsilon/h = 1/2$, was selected as the reference value of first-order system and $\varepsilon/h = 1/4$ as the reference value of first-order system. The numerical value results are shown in table 2. It could be seen from the data in the table that, the selected parameters satisfied the convergence condition.

6. Conclusions

This study mainly used AE to approach the viscosity solution of steady-state Hamilton-Jacobi. Firstly, proper parameters were selected by constructing polynomials; numerical solutions of surrounding grids were used to express the numerical solutions of iteration points, which avoided the problems encountered when implicit iterative expression was used. Then the stability and convergence

N	First-order AE system				Second-order AE system			
	L_1		L_{∞}		L_1		L_{∞}	
	Error	Order	Error	Order	Error	Order	Error	Order
$2^0 \times 10$	-	-	-	-	1.356×10^{-1}	-	3.265×10^{-1}	-
$2^1 \times 10$	1.907×10^{-1}	-	0.842	-	2.876×10^{-2}	2.243	7.112×10^{-2}	2.203
$2^2 \times 10$	6.962×10^{-2}	1.454	0.423	1.000	6.301×10^{-3}	2.196	1.603×10^{-2}	2.154
$2^3 \times 10$	2.474×10^{-2}	1.495	0.192	1.115	1.522×10^{-3}	2.075	3.826×10^{-3}	2.018
$2^4 \times 10$	9.108×10^{-3}	1.437	0.093	1.046	3.667×10^{-4}	2.034	9.682×10^{-4}	2.006
$2^5 \times 10$	3.326×10^{-3}	1.458	0.058	0.687	9.168×10^{-5}	2.002	2.418×10^{-5}	2.001
$2^6 \times 10$	1.222×10^{-3}	1.497	0.032	0.850	2.292×10^{-5}	2.000	6.046×10^{-5}	2.000
$2^7 \times 10$	4.337×10^{-4}	1.441	0.015	1.056	5.731×10^{-6}	2.001	1.512×10^{-5}	2.000

Figure 3: The numerical value results of first-order and second-order AE systems

of AE format were analyzed and an example analysis was made on AE format. It could be seen from the analysis results that, AE is effective in solving the viscosity solution of Hamilton-Jacobi equation and can achieve required precision.

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STOCHASTIC FINANCIAL MODEL BASED ON FRACTIONAL BROWN MOTION

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Abstract. Fractional brown motion (FBM) is featured by long-term memory and selfsimilarity compared to standard brown motion. Because of the properties, it can be used to describe the phenomenon (e.g. seasonal effect, scale effect and sharp peak and heavy tail) which cannot be described by some typical analytical methods in financial market. The fractal features of fractional brown motion make it a more suitable tool in financial studies. This study simulated the increment of FBM and the square of the increment using extended Maruyama symbols as well as the change path of underlying asset price and obtained the formula for European option pricing using insurance actuary pricing.

Keywords: frictional Brown motion, simulation, underlying asset price, share option.

1. Introduction

Fractional Brown motion (FBM) was first studied by Kolmogorov using Hilbert spatial framework and introducing the definition of Wiener helix in the 1940s. Mandelbrot deeply discussed FBM and formally proposed the concept of fractional Brown motion along with Van Ness in 1960s. They gave out the accurate definition of FBM and introduced its properties such as self-similarity, non-independence and differentiability [1] and used covariance function to express the correlation between increments of FBM. With the constant development of modern financial market, it has been found that, the initial assumption and actual situation of Black-Scholes model have difference [2] and the standard Brown motion motivation model has not been able to give reasonable explanations for more and more phenomena such as seasonal effect, scale effect and sharp peak and heavy tail [3]. Researchers also found that describing the change of asset price with FBM was more suitable than with standard Brown motion. The

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reasons for the above conclusion are as follows. Asset price will show the properties of Brown motion (martingale property, Markov property, etc.) when the standard Brown motion is used to describe the change of asset price, suggesting the price of underlying asset at a certain time point in the future is associated to the present rather than the past [4]. Obviously, the conclusion is impractical and disobeys the thought of people; compared to standard Brown motion, FBM does not have Markov property and semimartingale [5], thus it is more suitable for describing the change of asset price. Long-term memory and self-similarity are two properties of FBM, i.e., the price of underlying asset at some time point is associated to both the present and the past in words of financial language, which has been recognized by people.

The properties of FBM make FBM be able to describe the evolution process of asset price in financial market better, but FBM has its limitations [6]. In general sense of integral, FBM does not have martingale property, and moreover market has the phenomenon of interest arbitrage. Thus the studies on the application of FBM in financial system mainly have two orientations, the first is to study the memory of stock based on FBM (the memory of yield rate and fluctuation rate) and the second is to study the integral significance under which FBM has no interest arbitrage [7, 8] and its option pricing.

This study firstly introduced the definition and properties of FBM and then deduced that the model of $dS_t^\varepsilon = \mu S_t^\varepsilon dt + \sigma S_t^\varepsilon dB_t^\varepsilon$, $S_t^\varepsilon|_{t=0} = S_0$ was arbitrage-free for $\forall \varepsilon > 0$ based on the approach of FBM. Besides, the change path of underlying asset and European option pricing were studied.

2. The concept and properties of FBM

The concept of FBM

In 1905, Albert Einstein made a physical analysis on the motion of Brownian particles as the pioneer of the dynamic theory of Brown motion and proposed the mathematical model which was applicable to Brown motion. In 1923, Norbert Wiener proposed the concept of Wiener space based on the definition of measure and integral on Brown motion space and moreover gave a strict mathematical definition for Brown motion [9]. In 1968, Van Ness and Benoit Mandelbrot proposed the concept of FBM; FBM is the most extensively used model and its specific definition is as follows.

Suppose $B_H = \{B_H(t, \omega), t > 0\}$ as a random process. When $0 < H < 1$, is a set of random function values and $B_H(0, \omega) = b_0$ is an arbitrary real number, then the random process $\{B_H(t, \omega), t > 0\}$ is a FBM with Hurst parameter H :

$$B_H(t, \omega) - B_H = \frac{1}{\Gamma(1 + \alpha)} \left\{ \int_{-\infty}^0 [(t - s)^\alpha - (-s)^\alpha] dB(s, \omega) + \int_0^t (t - s)^\alpha dB(s, \omega) \right\}$$

in which, $\alpha = H - \frac{1}{2}$ and $\Gamma(1 + |\alpha|) = \int_0^\infty x^\alpha e^{-x}$; if $b_0 = 0$, $H = \frac{1}{2}$, then $B_H(t, \omega)$ is a standard Brown motion, reflected as random walk; FBM can be discussed by being divided into three sets, i.e., $0 < H < \frac{1}{2}$, $H = \frac{1}{2}$ and $\frac{1}{2} < H < 1$.

The property of FBM

When $0 < Hurstindex(H) < 1$, FBM has the following properties:

- (1) Hurst index is quite important for FBM as it determines the covariance in the past and future. Covariance function is $E[B_H(t)B_H(s)] = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t - s|^{2H})$;
- (2) FBM is a self-similar process.
- (3) The increments of FBM are not independent [10]. When $0 < H < \frac{1}{2}$, the correlation coefficient is negative and time sequence is anti-persistent; when $H = \frac{1}{2}$, time sequence is independent identically distributed random sequence, obeying standard normal distribution [11] and showing the feature of random walk; when $\frac{1}{2} < H < 1$, it has long-term memory.

3. The approach and increment of FBM

Approach

As FBM with $H=1/2$ is neither a semimartingale process nor a Markoff process, to make it adapt to financial theory application, Thao proposed a semimartingale process to approach FBM.

Lemma. For $\forall \varepsilon > 0$, $B_t^\varepsilon = \int_0^t (t - s + \varepsilon)^{H-\frac{1}{2}} ds$, $H \neq \frac{1}{2}$, $0 < H < 1$, was defined; then the process was a semimartingale and for $\forall t \in [0, T]$, B_t^ε uniformly converged to Bt in $L^2(\Omega)$ space when $\varepsilon \rightarrow 0$.

Lemma. For $\forall \varepsilon > 0$, the approximation FBM model $dS_t^\varepsilon = \mu S_t^\varepsilon dt + \sigma S_t^\varepsilon dB_t^\varepsilon$, $S_t^\varepsilon|_{t=0} = S_0$ is arbitrage-free.

Increment

Maruyama proposed to express the increment of the standard Brown motion with a symbol of $db(t) = \omega(t)(dt)^{1/2}$. In 2004, Guy Jumarie extended the symbol to H -order FBM derivatively defined by Riemann-Liouville using fractional-order Taylor expansion [12]. If $B(t, H)$ stands for H -order FBM, the increment of FBM can be expressed as $dB(t, H) = \omega(t)(dt)^H$, $0 < H < 1$, after the extension of $db(t) = \omega(t)(dt)^{1/2}$ proposed by Maruyama.

4. Simulation method

Because of the features of FBM, the model motivated by FBM can describe data flow more real compared to traditional models [13]. In the environment of FBM, queuing systems which are concerned more have already output some results;

however, how to simulate the process, for example, simulating the change of underlying asset price to obey a random differential equation model, should be known to solve practical problems. Usually, researchers simulate FBM firstly and then its increments. Simulation process can illustrate some simulation methods of FBM through exact and approach [14]. Exact methods used for simulating FBM mainly include Cholesky method, Hosking method, Davies method and Harte method. Approach methods for simulating FBM include random mid-point displacement, wavelet transform method, spectrum analysis method and random expression method. Random expression method and spectrum analysis method are frequently used [15].

Random expression method refers to directly discretize expression of FBM proposed by Mandelbrot

$$\frac{H}{\tilde{B}}(n) = C_H \left\{ \sum_{k=-b}^0 [(n-k)^\alpha - (-k)^\alpha] B_1(k) + \sum_{k=0}^n (n-k)^\alpha B_2(k) \right\} /.$$

The procedures of spectrum analysis are as follows. Firstly, time interval is divided to discretize the increment of FBM, i.e. $\frac{H}{\tilde{B}}(n) = C_H \left\{ \sum_{k=-b}^0 [(n-k)^\alpha - (-k)^\alpha] B_1(k) + \sum_{k=0}^n (n-k)^\alpha B_2(k) \right\}$, then variance function is calculated, followed by spectral density function and the increment of FBM.

5. Random simulation and European option pricing

The simulation of underlying asset price

The price of underlying asset under the condition of $1/2 < H < 1/2$ was simulated. Its price change satisfies fractional stochastic differential equation:

$$dS = \mu S dt + \sigma S dB_H + \lambda S (dB_H)^2$$

μ stands for the expected return rate of stock, σ stands for the fluctuation rate of stock yield, and λ stands for the disturbance term of stock.

Compared to the standard Brown motion motivated model, FBM model can describe the long-term memory of stock yield and its increment is not independent.

The simulation method used in this study was different from the methods mentioned above. The extended symbol $dB(t, H) = \omega(t)(dt)^H$ was used to simulate the increment of FBM. Hurst index H was specified as 0.42 and $S(t_j) = 60$. Firstly, the increment of FBM was simulated and then the change path of asset price was simulated using Monte Carlo simulation method. Figure 1 shows the flow of the simulation. (1) The simulation of FBM increments It has been mentioned above that, Hurst index H was specified as 0.42 and time interval $(0, T)$ was divided into N parts, for each part. For each equidistant interval, the increment of FBM was discretized, i.e., $\Delta B(t_j, H) = B(t_j + \Delta t, H) - B(t_j, H) = \omega(t)(\Delta t)^H$.

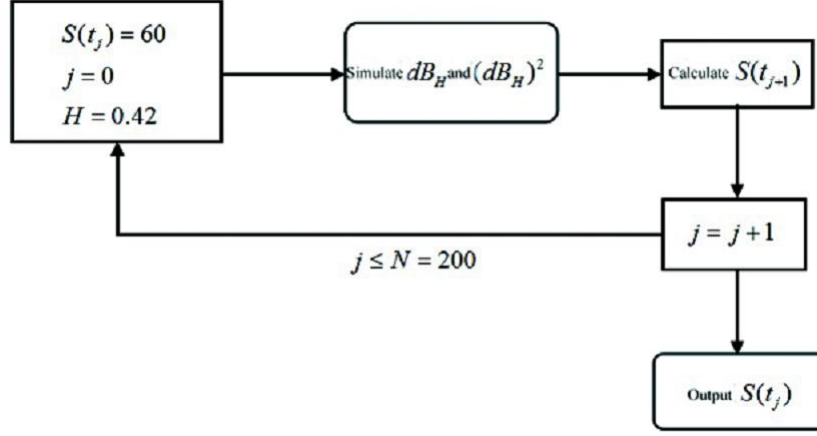


Figure 1: The construction process of decision tree

If the validity period $T = 1; N = 200$ and $t = 0:005$, the increment of FBM was simulated according to the above formula. (2) The simulation of underlying asset price

Suppose that the price change of underlying asset satisfies the following fractional stochastic differential equation:

$$(5.1) \quad dS_1 = \mu_1 S_1 dt + \sigma_1 S_1 dB_H + \lambda_1 S_1 (dB_H)^2, 1/3 < H < 1/2.$$

As $dB(t, H) = \omega_1(t)(dt)^H$ and $(dB(t, H))^2 = \omega_2(t)(dt)^{2H}$, the formula (1) could be written as:

$$(5.2) \quad dS_1 = \mu_1 S_1 dt + \sigma_1 S_1 \omega_1(t)(dt)^H + \lambda_1 S_1 \omega_2(t)(dt)^{2H}, 0 < H < 1$$

in which, $\omega_1(t)$ and $\omega_2(t)$ are independent, obeying standard normal distribution with an average value of 0 and a variance of 1.

The validity period of contract $[0, T]$ was divided into N small equidistant intervals, $t_0, t_1, t_2, t_3, \dots, t_N$ and the length of the interval was expressed as Δt . Then formula (2) was discretized, i.e.

$$(5.3) \quad dS_1 = \mu_1 S_1 dt + \sigma_1 S_1 \omega_1(t)(dt)^H + \lambda_1 S_1 \omega_2(t)(dt)^{2H},$$

$\Delta S_1(t_j) = S_1(t_{j+1}) - S_1(t_j)$. For each small interval, when $j = 1, 2, \dots, N$, there was

$$(5.4) \quad \begin{aligned} S_1(t_{j+1}) &= S_1(t_j) + \mu_1 S_1(t_j) \Delta t + \sigma_1 S_1(t_j) \omega_1(t) (\Delta t)^H \\ &\quad + \lambda_1 S_1(t_j) \omega_2(t) (\Delta t)^{2H}. \end{aligned}$$

Then Monte Carlo simulation method was used to simulate the price change path of underlying asset [16]. The values of relevant parameters were as follows:

$H = 0.42$, $T = 1$, S_0 (the initial price of underlying asset) = 60, μ_1 (the expected yield rate of stock) = 0.11, σ_1 (the fluctuation rate of stock) = 0.36 and λ_1 (the small disturbance term of stock yield) = 0.082.

Matlab software was used to perform Monte Carlo simulation on the change path of underlying asset price. Then a sample path ($H = 0.42$) was obtained. The sample path of the price of such underlying asset was simulated for several times. The approximate price of underlying asset at each time point, i.e., the change path of the price of underlying asset, could be obtained based on the average value of the price of underlying asset at each time point [17].

The actuarial method for European style option

Compared to the traditional option pricing method, the actuarial method is in no need of any assumptions about financial market while being used to study option pricing. Therefore, it is effective for any market (arbitrage, arbitrage-free, complete or incomplete markets) and it can also make option pricing be understood easily [18].

Definition 5.1. The expected yield rate of stock price $S(t)$ was $e^{\beta t} = \frac{ES(t)}{S(0)}$ in the time period of $[0, t]$.

Definition 5.2. European options would be exercised only when the difference between the present worth of stock which was discounted at due date according to the expected rate and the present worth of exercise price which was discounted according to risk-free interest rate was larger than 0, i.e.,

$$(5.5) \quad C(K, T) = E[(e^{-\beta T} S(T) - \exp[-\int_t^T r(s) ds] K) I_{\{e^{-\beta T} S(T) > \exp[-\int_t^T r(s) ds] K\}}].$$

Suppose that the pricing process of stock $\{S(t) : t \geq 0\}$ satisfied FBM and moreover $\sigma(t) = \sigma$ and $d\mu(t)$ were constants, then we had:

$$(5.6) \quad dS(t) = \mu S(t) dt + \sigma S(t) dB_H(s).$$

Theorem 5.3. For European call option whose due date was T and exercise price was K, when $r(t)$ was a non-random function and $\sigma(t) = \sigma$ was a constant, the price was:

$$(5.7) \quad C(K, T) = S(0)N(-d_1) - \exp\{-\int_t^T r(s) ds\} K N(-d_2)$$

and

$$(5.8) \quad \begin{aligned} d_1 &= \frac{\ln \frac{K}{S(0)} - \int_t^T r(s) ds - \frac{1}{2} \sigma^2 T^{2H}}{\sigma T^H}, \\ d_2 &= \frac{\ln \frac{K}{S(0)} - \int_t^T r(s) ds + \frac{1}{2} \sigma^2 T^{2H}}{\sigma T^H}. \end{aligned}$$

Proof. It could be known from $S(T) = S(0) \exp\{\mu t \frac{1}{2} \sigma^2 T^{2H} + \sigma B_H(T)\}$ that,

$$\begin{aligned}
 ES(T) &= \int_{-\infty}^{+\infty} S(0) \exp\{\mu T - \frac{1}{2} \sigma^2 T^{2H} + \sigma x\} \frac{1}{\sqrt{2\pi T^{2H}}} \exp\{-\frac{x^2}{2T^{2H}}\} dx \\
 &= S(0) \int_{-\infty}^{+\infty} \exp\{\mu T - \frac{1}{2} \sigma^2 T^{2H} + \sigma x - \frac{x^2}{2T^{2H}}\} \frac{1}{\sqrt{2\pi T^{2H}}} dx \\
 (5.9) \quad &= \frac{S(0)e^{\mu T}}{\sqrt{2\pi T^{2H}}} \int_{-\infty}^{\infty} \exp\{-\frac{1}{2} \sigma^2 T^{2H} + \sigma x - \frac{x^2}{2T^{2H}}\} dx \\
 &= \frac{S(0)e^{\mu T}}{\sqrt{2\pi T^{2H}}} \int_{-\infty}^{+\infty} \exp\{-\frac{x^2 + 2\sigma x T^{2H} - \sigma^2 T^{4H}}{2T^{2H}}\} dx \\
 &= \frac{S(0)e^{\mu T}}{\sqrt{2\pi T^{2H}}} \int_{-\infty}^{\infty} \exp\{-\frac{(x - \sigma T^{2H})^2}{2T^{2H}}\} dx.
 \end{aligned}$$

Suppose $y = \frac{x - \sigma T^{2H}}{T^H}$, then

$$(5.10) \quad ES(T) = \frac{S(0)e^{\mu T}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\{-\frac{y^2}{2}\} dy S(0), i.e.$$

$e^{\beta T} = \frac{ES(T)}{S(0)} = e^{\mu T}$. As

$$e^{-\beta T} S(T) > \exp\{-\int_t^T r(s) ds\} K,$$

then

$$(5.11) \quad B_H(t) > \frac{\ln \frac{K}{S(0) - \int_t^T r(s) ds + \frac{1}{2} \sigma^2 T^{2H}}}{\sigma}.$$

Suppose $y = \frac{x - \sigma T^{2H}}{T^H}$ and

$$d_1 = \frac{\ln \frac{K}{S(0)} - \int_t^T r(s) ds - \frac{1}{2} \sigma^2 T^{2H}}{\sigma T^H},$$

then

$$\begin{aligned}
 E[e^{-\beta T} S(T) I_{\{e^{-\beta T} S(T) > \exp[-\int_t^T r(s) ds] K\}}] &= S(0) \int_{d_1}^{+\infty} \exp\{-\frac{x^2}{2}\} \frac{1}{\sqrt{2\pi}} \\
 (5.12) \quad &= S(0) N(-d_1).
 \end{aligned}$$

Suppose $\frac{x}{T^H} = z$ and

$$d_2 = \frac{\ln \frac{K}{S(0)} - \int_t^T r(s) ds + \frac{1}{2} \sigma^2 T^{2H}}{\sigma T^H},$$

then

$$\begin{aligned}
 & E[\exp\{-\int_t^T r(s)ds\}KI_{\{e^{-\beta T}S(T) > \exp[-\int_t^T r(s)ds]K\}}] \\
 (5.13) \quad & = \exp[-\int_0^T r(s)ds]KN(-d_2).
 \end{aligned}$$

Thus

$$\begin{aligned}
 C(K, T) &= E[(e^{-\beta T}S(T) - \exp\{-\int_t^T r(s)ds\}K)I_{\{e^{-\beta T}S(T) > \exp[-\int_t^T r(s)ds]K\}}] \\
 (5.14) \quad &= S(0)N(-d_1) - \exp\{-\int_t^T r(s)ds\}KN(-d_2).
 \end{aligned}$$

The above is the proof process of the theorem. $\square$

6. Conclusions

The changing process of underlying asset price is featured by self-similarity and long-term memory. Thus researchers realize that describing the process using FBM is more in line with the practical situation of financial market. Through the approach of FBM, this study deduced that model $dS_t^\varepsilon = \mu S_t^\varepsilon dt + \sigma S_t^\varepsilon dB_t^\varepsilon$, $S_t^\varepsilon|_{t=0} = S_0$ was arbitrage free for $\forall \varepsilon > 0$. FBM considered both the past price and the present price while describing the change of underlying asset price; in some sense, it was more exact in describing objects. Through establishing financial model, simulating the increment of FBM and making a fitting analysis on sample paths, we obtained the change path of underlying asset price. Moreover, the pricing formula for European option was also obtained using actuarial approach.

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CERTAIN PROPERTIES ASSOCIATED WITH B-PREINVEX FUZZY MAPPINGS

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Abstract. We establish several new characterizations for B-preinvex fuzzy mappings. Under the condition of upper or lower semi-continuity and the well known Condition C introduced by Mohan and Neogy [J. Math. Anal. Appl., 189 (1995) 901-908], we obtain a sufficient condition for B-preinvex fuzzy mappings. Several necessary conditions for differentiable and twice differentiable B-preinvex fuzzy mappings are also presented and proved.

Keywords: Fuzzy numbers, B-preinvex fuzzy mappings, semi-continuity.

1. Introduction

Owing to the importance of the generalized convexity of fuzzy mappings and the generalized fuzzy convexity in the search for optimal conditions to solve the fuzzy optimization problems, many authors paid special attention to the research of fuzzy mappings, especially for generalized convex fuzzy mappings. For example, in earlier papers, the concept of fuzzy mappings was introduced by Chang and Zadeh (1972) [3]. Nanda and Kar (1992) [11] proposed the concept of convex fuzzy mappings and proved that a fuzzy mapping is convex if and only if its epigraph is a convex set. Noor(1994) [10] introduced the concept of fuzzy preinvex functions over the field of real numbers $\mathbb{R}$, and obtained some properties of fuzzy preinvex functions. Syau introduced concepts of all kinds of generalized convexity for fuzzy mappings of one variable such as convex and concave fuzzy mappings, preinvex fuzzy mapping (1999a) [16]. In the meantime, Syau also discussed many important properties of these generalized convex fuzzy mappings.

In recent ten years, Yan and Xu (2002) [21] discussed the convexity and quasiconvexity of fuzzy mappings by considering the concept of ordering. Mishra et al.(2006) [9] introduced concept of an explicitly B-preinvex fuzzy mapping

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and presented some properties of explicitly B-preinvex fuzzy mappings. Wu and Xu (2008) [20] introduced concepts of fuzzy pseudoconvex, fuzzy invex, fuzzy pseudoinvex, and fuzzy preinvex mapping from $\mathbb{R}^n$ to the set of fuzzy numbers based on the concept of differentiability of fuzzy mappings. Very recently, Li and Noor (2010) [7] studied the necessary and sufficient conditions for differentiable and twice differentiable preinvex fuzzy mapping by using the given equivalent condition of preinvex fuzzy mapping and established the semi-continuity of preinvex fuzzy mappings. Tang and Ding (2013) [19] introduced concepts of semilocally b -preinvex fuzzy mappings, semilocally quasi b -preinvex fuzzy mappings, semilocally pseudo b -preinvex fuzzy mappings and semilocally strongly pseudo b -preinvex fuzzy mappings, they also studied a fuzzy nonlinear programming which be considered involving generalized convex fuzzy mappings with η -semidifferentiability. Rufián-Lizana et al. (2014) [15] showed, by means of counterexamples, the characterizations given by Li et al. [7] were incomplete and provided valid characterizations. For more results on generalized fuzzy mappings, one can see the contributions [1, 2, 12, 13] and references therein.

Through the above researches, quite a number of contributions about preinvex fuzzy mappings was made, however, some new properties of B-preinvex fuzzy mappings should be worth studied. So we turn our attention to this new research.

Motivated by the works [6, 7, 9, 14, 15, 17, 19] going on in these areas, on the basis of the concept of parameterized triples of fuzzy numbers and by using the convexity, preinvexity and B-preinvexity, the purpose in the present paper is to study several important characterizations about B-preinvex fuzzy mappings. (i) We mainly study the sufficient conditions for B-preinvex fuzzy mappings under the semi-continuity conditions. (ii) We present and prove the necessary conditions for differentiable and twice differentiable B-preinvex fuzzy mappings. Compared with the results of Li et al. (2010) [7] and Rufián-Lizana et al. (2014) [15], the focus of our research is on the B-preinvex fuzzy mappings. That is to say, the result of B-preinvex fuzzy mappings in this paper can also be applied to preinvex fuzzy mappings, so our work has a generalized research significance.

The present paper is built up as follows. In Section 2, some preliminaries, including concepts of fuzzy numbers, preinvex fuzzy mappings, B-vex fuzzy mappings and differentiable fuzzy mappings of several variables are first reviewed. The generalized B-preinvex fuzzy mappings are then recalled, the basic properties of semi-continuity of B-preinvex fuzzy mappings are studied in Section 3. We provide and prove some results involving differentiable and twice differentiable B-preinvex fuzzy mappings in Section 4. Finally conclusions are given in Section 5.

2. Preliminaries

In this section, for convenience, several definitions and results with respect to fuzzy numbers and fuzzy mappings, which will be needed in sequel, from Diamond and Kloeden [4], Goetschel et al. [5], Noor [10] and Syau [16, 17] are summarized below.

We denote by $\mathbb{R}$ the set of all real numbers. A fuzzy number is a mapping $\mu : \mathbb{R} \rightarrow [0, 1]$ with the following properties:

- (1) μ is upper semi-continuous.
- (2) μ is normal, that is, there exists a $x \in \mathbb{R}$ such that $\mu(x) = 1$.
- (3) μ is convex, namely, $\mu(\lambda x + (1 - \lambda)y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in \mathbb{R}$ and $\lambda \in [0, 1]$.
- (4) the support of μ , $\text{supp}(\mu) = \{x \in \mathbb{R} : \mu(x) > 0\}$ and its closure $\text{cl}(\text{supp } \mu)$ is compact.

Let $\mathcal{F}_0$ denote the family of fuzzy numbers on $\mathbb{R}$. Since each $r \in \mathbb{R}$ can be considered as a fuzzy number $\bar{r}$ defined as

$$\bar{r} = \begin{cases} 1, & x = r, \\ 0, & x \neq r, \end{cases}$$

so $\mathbb{R}$ can be embedded in $\mathcal{F}_0$. It is well known that α -level set of a fuzzy set $\mu : \mathbb{R} \rightarrow [0, 1]$, $\alpha \in [0, 1]$, denoted by $[\mu]_\alpha$, is defined as

$$[\mu^*(\alpha), \mu_*(\alpha)] = [\mu]_\alpha = \begin{cases} \{x \in \mathbb{R} : \mu(x) > \alpha\}, & \text{if } 0 < \alpha \leq 1, \\ \text{cl}(\text{supp } \mu), & \text{if } \alpha = 0. \end{cases}$$

It can be seen easily that the α -level set of a fuzzy number is closed and bounded interval $[\mu^*(\alpha), \mu_*(\alpha)]$, where $\mu^*(\alpha)$ denotes the left-hand end point of $[\mu]_\alpha$ and $\mu_*(\alpha)$ denotes the right-hand end point of $[\mu]_\alpha$. Thus, a fuzzy number μ can be identified by a parameterized triples

$$\{(\mu^*(\alpha), \mu_*(\alpha), \alpha) : \alpha \in [0, 1]\}.$$

Let $\mu, \nu \in \mathcal{F}_0$ represented parametrical by

$$\{(\mu^*(\alpha), \mu_*(\alpha), \alpha) : \alpha \in [0, 1]\} \text{ and } \{(\nu^*(\alpha), \nu_*(\alpha), \alpha) : \alpha \in [0, 1]\},$$

respectively. It is said that $\mu \preceq \nu$, if for every $\alpha \in [0, 1]$, $\mu^*(\alpha) \leq \nu^*(\alpha)$ and $\mu_*(\alpha) \leq \nu_*(\alpha)$. It is said that $\mu \prec \nu$, if $\mu \preceq \nu$ and there exists a $\alpha_0 \in [0, 1]$ such that $\mu^*(\alpha_0) < \nu^*(\alpha_0)$ and $\mu_*(\alpha_0) < \nu_*(\alpha_0)$. It is said that $\mu = \nu$, if $\mu \preceq \nu$ and $\nu \preceq \mu$.

Note that $\preceq$ is a partial order on $\mathcal{F}_0$, and it is often convenient to write $\nu \succeq \mu$ instead of $\mu \preceq \nu$.

A subset S^* of $\mathcal{F}_0$ is said to be bounded above if there exists a fuzzy number $\mu \in \mathcal{F}_0$, called an upper bound of S^* , such that $\nu \preceq \mu$ for every $\nu \in S^*$. Further, a fuzzy number $\mu_0 \in \mathcal{F}_0$ is called the least upper bound (sup in short) for S^* if (i) μ_0 is an upper bound of S^* , and (ii) $\mu_0 \preceq \mu$ for every upper bound μ of S^* . A lower bound and the greatest lower bound (inf in short) are defined similarly.

For fuzzy numbers μ, ν and each nonnegative real number k , the addition $\mu \tilde{+} \nu$ and nonnegative scalar multiplication $k\mu$ are defined as follows:

$$(\mu \tilde{+} \nu)(x) = \sup_{y+z=x} \min\{\mu(y), \nu(z)\},$$

$$(k\mu)(x) = \begin{cases} \mu(k^{-1}x), & \text{if } k \neq 0, \\ 0, & \text{if } k = 0, \end{cases}$$

and

$$\begin{aligned} \mu \tilde{+} \nu &= \{(\mu^*(\alpha) + \nu^*(\alpha), \mu_*(\alpha) + \nu_*(\alpha), \alpha) : \alpha \in [0, 1]\}, \\ k\mu &= \{(k\mu^*(\alpha), k\mu_*(\alpha), \alpha) : \alpha \in [0, 1]\}. \end{aligned}$$

It is obvious that concepts of addition and the nonnegative scalar multiplication on $\mathcal{F}_0$ defined by the above two equations are equivalent to those derived from the universal extension criterion. And it is easy to see that $\mathcal{F}_0$ is closed under addition and nonnegative scalar multiplication. So it should be noted that $k\mu$ is not a fuzzy number for $k < 0$ and $\mu_0 \in \mathcal{F}_0$. The family of parametric representations of members of $\mathcal{F}_0$ and the parametric representations of their negative scalar multiplications form subsets of the vector space

$$\mathcal{U} = \{(\mu^*(\alpha), \mu_*(\alpha), \alpha) : \alpha \in [0, 1]\} : \mu^*, \mu_* : [0, 1] \rightarrow \mathbb{R}\},$$

where μ^*, μ_* are bounded functions. $\mathcal{U}$ is metricized by the metric as follows:

$$\begin{aligned} d\left(\{(\mu^*(\alpha), \mu_*(\alpha), \alpha) : \alpha \in [0, 1]\}, \{(\nu^*(\alpha), \nu_*(\alpha), \alpha) : \alpha \in [0, 1]\}\right) \\ = \sup \{ \max\{|\mu^*(\alpha) - \nu^*(\alpha)|, |\mu_*(\alpha) - \nu_*(\alpha)|\} : \alpha \in [0, 1] \}. \end{aligned}$$

Let

$$\mathcal{U}_0 = \{(\mu^*(\alpha), \mu_*(\alpha), \alpha) : \alpha \in [0, 1]\} : \mu^*, \mu_* : [0, 1] \rightarrow [0, +\infty)\},$$

where μ^*, μ_* are bounded functions. It is easy verified that $\mathcal{U}_0$ is a closed convex cone in the topological vector space $(\mathcal{U}_0, d)$.

We now turn to review definitions of preinvex fuzzy mappings and B-vex fuzzy mappings in $(\mathcal{U}_0, d)$.

Definition 2.1 ([16]). *A set $X \subseteq \mathbb{R}^n$ is said to be invex with respect to mapping $\eta : X \times X \rightarrow \mathbb{R}^n$, if for every $x, y \in X$, $y + \lambda\eta(x, y) \in X$ and $0 \leq \lambda \leq 1$.*

Definition 2.2 ([16]). Let $F : X \rightarrow \mathcal{F}_0$ be a fuzzy mapping defined on an invex set $X \subseteq \mathbb{R}^n$ and $X \neq \emptyset$, with respect to a mapping $\eta : X \times X \rightarrow \mathbb{R}^n$. F is said to be preinvex on X (with respect to η), if

$$(2.1) \quad \lambda F(x) + (1 - \lambda)F(y) - F(y + \lambda\eta(x, y)) \in \mathcal{U}_0,$$

for $\lambda \in [0, 1]$ and $x, y \in X$.

Definition 2.3 ([16]). At a point $x_0 \in S$, the fuzzy mapping $F : S \rightarrow \mathcal{F}_0$ is said to be:

(1) B -vex with respect to $b(x, x_0, \lambda)$ if, for all $x \in S$ and $\lambda \in [0, 1]$,

$$(2.2) \quad \lambda b(x, x_0, \lambda)F(x) + (1 - \lambda b(x, x_0, \lambda))F(y) - F(\lambda x + (1 - \lambda)x_0) \in \mathcal{U}_0,$$

(2) strictly B -vex, with respect to $b(x, x_0, \lambda)$ if, for all $x \in S$, $x \neq x_0$ and $\lambda \in (0, 1)$,

$$(2.3) \quad \lambda b(x, x_0, \lambda)F(x) + (1 - \lambda b(x, x_0, \lambda))F(y) - F(\lambda x + (1 - \lambda)x_0) \in \mathcal{U}_0 \setminus \{\tilde{0}\},$$

(3) B -linear with respect to $b(x, x_0, \lambda)$ if, for all $x \in S$ and $\lambda \in [0, 1]$,

$$(2.4) \quad \lambda b(x, x_0, \lambda)F(x) + (1 - \lambda b(x, x_0, \lambda))F(y) = F(\lambda x + (1 - \lambda)x_0).$$

To end this section, let us recall some concepts of differentiability of a fuzzy mapping.

Definition 2.4 ([7]). Let $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and $\mu = (\mu_1, \mu_2, \dots, \mu_n) \in \mathcal{F}^n$ be an n -dimensional real vector and an n -dimensional fuzzy vector, respectively. The product of a fuzzy vector is defined with a real vector as $\mu x^T = \sum_{i=1}^n \mu_i x_i$, which is a fuzzy number.

Let $F : T \rightarrow \mathcal{F}_0$ be a fuzzy mapping. For any $\alpha \in [0, 1]$, denote $[F(x)]^\alpha = [F^*(x, \alpha), F_*(x, \alpha)]$, where for each $\alpha \in [0, 1]$, $F^*(\cdot, \alpha)$ and $F_*(\cdot, \alpha) : T \rightarrow \mathbb{R}$ are upper and lower functions of F , respectively.

Definition 2.5 ([7]). Let $F : T \rightarrow \mathcal{F}_0$ be a fuzzy mapping, where $T \subset \mathbb{R}^n$ is an open set. Let $x = (x_1, x_2, \dots, x_n) \in T$, and D_{x_i} , $i = 1, 2, \dots, n$ stand for the partial differentiation with respect to the i th variable x_i . Assume that for all $\alpha \in [0, 1]$, $F^*(x, \alpha)$ and $F_*(x, \alpha)$ have continuous partial derivatives so that $D_{x_i}F^*(x, \alpha)$ and $D_{x_i}F_*(x, \alpha)$ are continuous. Define

$$[D_{x_i}F(x)]^\alpha = [D_{x_i}F^*(x, \alpha), D_{x_i}F_*(x, \alpha)],$$

for $i = 1, 2, \dots, n$ and $\alpha \in [0, 1]$. If for each i , $[D_{x_i}F(x)]^\alpha$ defines the α -level of a fuzzy number, then it is said that F is differentiable at x , and $\tilde{\nabla}F(x)$ is wrote by

$$\tilde{\nabla}F(x) = (D_{x_1}F(x, \alpha), D_{x_2}F(x, \alpha), \dots, D_{x_n}F(x, \alpha)).$$

The partial derivative $\tilde{\nabla}F(x)$ is said to be the gradient of the fuzzy mapping F at x .

Definition 2.6 ([7]). Let $F : T \rightarrow \mathcal{F}_0$ be a fuzzy mapping, where $T \subset \mathbb{R}^n$ is an open set. Let $x = (x_1, x_2, \dots, x_n) \in T$, and $D_{x_i x_j}$, $i, j = 1, 2, \dots, n$ stand for the second-order partial with respect to the i th variable x_i and j th variable x_j . Assume that $\tilde{\nabla} F(x)$ exists and for all $\alpha \in [0, 1]$, $F^*(x, \alpha)$ and $F_*(x, \alpha)$ have continuous second-order partial derivatives so that $D_{x_i x_j} F^*(x, \alpha)$ and $D_{x_i x_j} F_*(x, \alpha)$ are continuous. Define

$$[D_{x_i x_j} F(x)]^\alpha = [D_{x_i x_j} F^*(x, \alpha), D_{x_i x_j} F_*(x, \alpha)],$$

for $i, j = 1, 2, \dots, n$ and $\alpha \in [0, 1]$. If for each i, j , $[D_{x_i x_j} F(x)]^\alpha$ defines the α -level of a fuzzy number, then the Hessian of the fuzzy mapping (in the matrix notation) is defined as follows:

$$\tilde{\nabla}^2 F(x) = (D_{x_i x_j} F(x, \alpha))_{i,j=1,2,\dots,n}.$$

The fuzzy mapping F is said to be twice differentiable at x if the Hessian of the fuzzy mapping exists.

3. Properties about B-preinvex fuzzy mappings

Before approaching properties of B-preinvex fuzzy mappings, we first review some definitions of B-preinvex fuzzy mappings and fuzzy B-invex sets.

Definition 3.1 ([16]). Let $F : X \rightarrow \mathcal{F}_0$ be a fuzzy mapping defined on an invex set $X \subseteq \mathbb{R}^n$ and $X \neq \emptyset$, with respect to a mapping $\eta : X \times X \rightarrow \mathbb{R}^n$. F is said to be B-preinvex on X (with respect to η) with respect to a mapping $b : X \times X \times [0, 1] \rightarrow (0, 1]$, if

$$(3.1) \quad F(y + \lambda \eta(x, y)) \preceq \lambda b(x, y, \lambda) F(x) \tilde{+} (1 - \lambda b(x, y, \lambda)) F(y),$$

for $\lambda \in [0, 1]$ and $x, y \in X$; and strictly B-preinvex with respect to η and b , if

$$(3.2) \quad F(y + \lambda \eta(x, y)) \prec \lambda b(x, y, \lambda) F(x) \tilde{+} (1 - \lambda b(x, y, \lambda)) F(y),$$

for $\lambda \in (0, 1)$ and $x \neq y \in X$.

If (3.1) and (3.2) are reversed, then F is said to be B-preincave and strictly B-preincave on X with respect to η and b , respectively.

Definition 3.2 ([18]). Let $F : X \rightarrow \mathcal{F}_0$ be a fuzzy mapping, where $X \subset \mathbb{R}^n$ is an open set, parameterized by

$$F(x) = \left\{ (F^*(\alpha, x), F_*(\alpha, x), \alpha) : \alpha \in [0, 1] \right\}.$$

- (1) It is said that F is upper semi-continuous at $x_0 \in X$, if both $F^*(\alpha, x)$ and $F_*(\alpha, x)$ are upper semi-continuous at x_0 uniformly in $\alpha \in [0, 1]$. F is upper semi-continuous on X , if it is upper semi-continuous at each point of X .

- (2) It is said that F is lower semi-continuous at $x_0 \in X$, if both $F^*(\alpha, x)$ and $F_*(\alpha, x)$ are lower semi-continuous at x_0 uniformly in $\alpha \in [0, 1]$. F is lower semi-continuous on X , if it is lower semi-continuous at each point of X .

Lemma 3.1. Let X be a non-empty invex set in $\mathbb{R}^n$ with respect to $\eta : X \times X \rightarrow \mathbb{R}^n$, and $F : X \rightarrow \mathcal{F}_0$ be a fuzzy mapping parameterized by

$$F(x) = \left\{ (F^*(\alpha, x), F_*(\alpha, x), \alpha) : \alpha \in [0, 1] \right\}.$$

Then F is B -preinvex on X with respect to η and b , if and only if for any $\alpha \in [0, 1]$, $F^*(\alpha, x)$ and $F_*(\alpha, x)$ are B -preinvex on X with respect to η and b .

The demonstration of Lemma 3.1 is analogous to the proof of Lemma 3.1 which provided by Li and Noor (2010) [7].

Definition 3.3 ([16]). Given $S \subseteq \mathbb{R}^n \times \mathcal{F}_0$, S is said to be a fuzzy B -invex set with respect to mappings $\eta : X \times X \rightarrow \mathbb{R}^n$ and $b : X \times X \times [0, 1] \rightarrow (0, 1]$, if $(x, \mu), (y, \nu) \in S$ and $\lambda \in [0, 1]$ implies that

$$(3.3) \quad (y + \lambda\eta(x, y), \lambda b(x, y, \lambda)\mu \tilde{+} (1 - \lambda b(x, y, \lambda))\nu) \in S.$$

In the following, some basic results of B -preinvex fuzzy mappings are presented without proof.

Theorem 3.1. Let X be a non-empty invex subset of $\mathbb{R}^n$ with respect to $\eta : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, and $F_i : X \rightarrow \mathcal{F}_0$, $(i = 1, 2, \dots, n)$ be B -preinvex fuzzy mappings with respect to the same mappings η and b . Then the function $F : X \rightarrow \mathcal{F}_0$ defined by

$$F(x) = \sum_{i=1}^n a_i F_i(x), \quad a_i \geq 0$$

is a B -preinvex fuzzy mapping on X with respect to the same mappings η and b .

Theorem 3.2. Let X be a non-empty invex subset of $\mathbb{R}^n$ with respect to $\eta : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, and $F_i : X \rightarrow \mathcal{F}_0$, $(i \in I = \{1, 2, \dots, n\})$ be B -preinvex fuzzy mappings with respect to the same mappings η , b . Then the function $F : X \rightarrow \mathcal{F}_0$ defined by $F(x) = \sup_{i \in I} F_i(x)$ is a B -preinvex fuzzy mapping on X with respect to the same mappings η and b .

Theorem 3.3. Let X be a non-empty invex subset of $\mathbb{R}^n$ with respect to $\eta : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$. A fuzzy mapping $F : X \rightarrow \mathcal{F}_0$ is a B -preinvex fuzzy mapping on X with respect to mappings η , b , if and only if for all $x, y \in X$, $\mu, \nu \in \mathcal{F}_0$ and $\lambda \in (0, 1)$ such that $F(x) \prec \mu$, $F(y) \prec \nu$ and $F(y + \lambda\eta(x, y)) \preceq \lambda b(x, y, \lambda)\mu \tilde{+} (1 - \lambda b(x, y, \lambda))\nu$.

After the above results are presented, we now give the following conclusion, which a characterization of B -preinvex fuzzy mappings F in terms of its epigraph $\text{epi}(F)$ is given by $\text{epi}(F) = \{(x, \mu) : x \in X, \mu \in \mathcal{F}_0, F(x) \preceq \mu\}$.

Theorem 3.4. *Let X be a non-empty invex subset of $\mathbb{R}^n$ with respect to $\eta : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$. A fuzzy mapping $F : X \rightarrow \mathcal{F}_0$ is a B-preinvex fuzzy mapping on X with respect to mappings η and b , if and only if its epigraph $\text{epi}(F)$ is a fuzzy B-invex set in $\mathbb{R}^n \times \mathcal{F}_0$ with respect to η and b .*

Proof. Let $F : X \rightarrow \mathcal{F}_0$ be a B-preinvex fuzzy mapping and $(x, \mu), (y, \nu) \in \text{epi}(F)$ with $x, y \in X$ and $\mu, \nu \in \mathcal{F}_0$, then $F(x) \preceq \mu$ and $F(y) \preceq \nu$. Since F is B-preinvex on X and addition and nonnegative scalar multiplication preserve the order on $\mathcal{F}_0$, it follows that

$$\begin{aligned} F(y + \lambda\eta(x, y)) &\preceq \lambda b(x, y, \lambda)F(x) \tilde{+} (1 - \lambda b(x, y, \lambda))F(y) \\ &\preceq \lambda b(x, y, \lambda)\mu \tilde{+} (1 - \lambda b(x, y, \lambda))\nu, \end{aligned}$$

for $\lambda \in [0, 1]$, which implies that

$$(y + \lambda\eta(x, y), \lambda b(x, y, \lambda)\mu \tilde{+} (1 - \lambda b(x, y, \lambda))\nu) \in \text{epi}(F).$$

Thus, $\text{epi}(F)$ is a fuzzy B-invex set with respect to η, b .

Conversely, assume that $\text{epi}(F)$ is a fuzzy B-invex set and $x, y \in X$, so $(x, F(x)), (y, F(y)) \in \text{epi}(F)$. Meanwhile, $\text{epi}(F)$ is a fuzzy B-invex set with respect to η, b , it yields that

$$(y + \lambda\eta(x, y), \lambda b(x, y, \lambda)F(x) \tilde{+} (1 - \lambda b(x, y, \lambda))F(y)) \in \text{epi}(F).$$

which implies that, for $\lambda \in [0, 1]$,

$$F(y + \lambda\eta(x, y)) \preceq \lambda b(x, y, \lambda)F(x) \tilde{+} (1 - \lambda b(x, y, \lambda))F(y).$$

Hence, F is a B-preinvex fuzzy mapping with respect to η and b , which completes the proof.

Now, we begin to study the sufficient conditions of B-preinvex fuzzy mappings. To discuss this problem, we need the following well known Condition C which introduced by Mohan and Neogy (1995) [8].

Condition C: It is said that the function $\eta : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfied Condition C, if for any $x, y \in \mathbb{R}^n$,

$$\eta(y, y + \lambda\eta(x, y)) = -\lambda\eta(x, y), \quad \eta(x, y + \lambda\eta(x, y)) = (1 - \lambda)\eta(x, y)$$

are satisfied for $\lambda \in [0, 1]$.

Theorem 3.5. *Let X be a non-empty invex subset of $\mathbb{R}^n$ with respect to $\eta : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, where η satisfies Condition C. Assume that $F : X \rightarrow \mathcal{F}_0$ is an upper semi-continuous fuzzy mapping and satisfy $F(y + \eta(x, y)) \preceq b(x, y, \lambda)F(x)$ for $\forall x, y \in X$, and if there exists a $t \in (0, 1)$ such that*

$$F(y + t\eta(x, y)) \preceq tb(x, y, \lambda)F(x) \tilde{+} (1 - tb(x, y, \lambda))F(y),$$

for all $x, y \in X$, then F is a B-preinvex fuzzy mapping on X with respect to mappings η and b .

Proof. The proof is by contradiction. Suppose that F is not a B-preinvex fuzzy mapping, so there exist $x, y \in X$ and $\bar{\lambda} \in (0, 1)$ such that

$$(3.4) \quad F(y + \bar{\lambda}\eta(x, y)) \succ \bar{\lambda}b(x, y, \lambda)F(x) \dot{+} (1 - \bar{\lambda}b(x, y, \lambda))F(y),$$

i.e., there exist $x, y \in X$ and $\bar{\lambda} \in (0, 1)$ for some $\alpha_0 \in [0, 1]$ such that

$$(3.5) \quad F^*(\alpha_0, y + \bar{\lambda}\eta(x, y)) > \bar{\lambda}b(x, y, \lambda)F^*(\alpha_0, x) + (1 - \bar{\lambda}b(x, y, \lambda))F^*(\alpha_0, y),$$

or

$$(3.6) \quad F_*(\alpha_0, y + \bar{\lambda}\eta(x, y)) > \bar{\lambda}b(x, y, \lambda)F_*(\alpha_0, x) + (1 - \bar{\lambda}b(x, y, \lambda))F_*(\alpha_0, y).$$

Now let

$$g(\lambda) = F^*(\alpha_0, y + \lambda\eta(x, y)) - \lambda b(x, y, \lambda)F^*(\alpha_0, x) - (1 - \lambda b(x, y, \lambda))F^*(\alpha_0, y).$$

According to the Definition 3.2 and from the assumption F is an upper semi-continuous fuzzy mapping, $F^*(\alpha, x)$ is an upper semi-continuous real-valued function. Then, $g(\lambda)$ also is upper semi-continuous real-valued function in interval $[0, 1]$. Therefore, $g(\lambda)$ exists maximum $M_0 > 0$ (due to (3.5)) in interval $[0, 1]$. Let $\lambda_0 = \max\{\lambda \in [0, 1] : g(\lambda) = M_0\}$, it follows that

$$g(0) = 0,$$

$$g(1) = F^*(\alpha_0, y + \eta(x, y)) - b(x, y, 1)F^*(\alpha_0, x).$$

By the conditions $F(y + \eta(x, y)) \preceq b(x, y, \lambda)F(x)$ for $\forall x, y \in X, \lambda \in [0, 1]$ and maximum $M_0 > 0$. It can easily be shown that $g(1) \leq 0$ and $\lambda_0 \in (0, 1)$. For simplicity, let $b(x, y, \lambda) \triangleq b$. Choose a $\delta > 0$ such that

$$(b\lambda_0 - (1 - tb)\delta, b\lambda_0 + tb\delta) \subset (0, 1).$$

Let $b\lambda_2 = b\lambda_0 - (1 - tb)\delta$ and $b\lambda_1 = b\lambda_0 + tb\delta$, $\tilde{x} = y + \lambda_2\eta(x, y)$ and $\tilde{y} = y + \lambda_1\eta(x, y)$. Obviously, $\lambda_0 = tb\lambda_2 + (1 - tb)\lambda_1$, $\lambda_1 = \lambda_2 + \delta$ and $\lambda_1, \lambda_2 \neq \lambda_0$. By the Condition C, we have

$$\tilde{y} + t\eta(\tilde{x}, \tilde{y}) = y + \lambda_0\eta(x, y).$$

Combining Lemma 3.1, it is easy to see that $F^*(\alpha, x)$ is B-preinvex on X with respect to b . Thus,

$$\begin{aligned} M_0 &= g(\lambda_0) = F^*(\alpha_0, y + \lambda_0\eta(x, y)) - \lambda_0 b F^*(\alpha_0, x) - (1 - \lambda_0 b) F^*(\alpha_0, y) \\ &= F^*(\alpha_0, \tilde{y} + t\eta(\tilde{x}, \tilde{y})) - \lambda_0 b F^*(\alpha_0, x) - (1 - \lambda_0 b) F^*(\alpha_0, y) \\ &\leq tb F^*(\alpha_0, \tilde{x}) - (1 - \lambda_0 b) F^*(\alpha_0, \tilde{y}) - \lambda_0 b F^*(\alpha_0, x) - (1 - \lambda_0 b) F^*(\alpha_0, y) \\ &= tb F^*(\alpha_0, \tilde{x}) - (1 - tb) F^*(\alpha_0, \tilde{y}) \\ &\quad - (tb\lambda_2 + (1 - tb)\lambda_1) b F^*(\alpha_0, x) - (1 - (tb\lambda_2 + (1 - tb)\lambda_1) b) F^*(\alpha_0, y) \\ &= tb[F^*(\alpha_0, \tilde{x}) - \lambda_2 b F^*(\alpha_0, x) - (1 - \lambda_2 b) F^*(\alpha_0, y)] \\ &\quad + (1 - tb)[F^*(\alpha_0, \tilde{y}) - \lambda_1 b F^*(\alpha_0, x) - (1 - \lambda_1 b) F^*(\alpha_0, y)] \\ &= tb[F^*(\alpha_0, y + \lambda_2\eta(x, y)) - \lambda_2 b F^*(\alpha_0, x) - (1 - \lambda_2 b) F^*(\alpha_0, y)] \\ &\quad + (1 - tb)[F^*(\alpha_0, y + \lambda_1\eta(x, y)) - \lambda_1 b F^*(\alpha_0, x) - (1 - \lambda_1 b) F^*(\alpha_0, y)] \\ &= tb g(\lambda_2) + (1 - tb) g(\lambda_1) < tb M_0 + (1 - tb) M_0 = M_0. \end{aligned}$$

This contradiction proves the result that F is a B-preinvex fuzzy mapping on X with respect to mappings η and b .

By a similar way, using Lemma 3.1 and Definition 3.2, it is easy to deduce the following Theorem 3.6.

Theorem 3.6. *Let X be a non-empty invex subset of $\mathbb{R}^n$ with respect to $\eta : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, where η satisfies Condition C. Assume that $F : X \rightarrow \mathcal{F}_0$ is an lower semi-continuous fuzzy mapping and satisfy $F(y + \eta(x, y)) \preceq b(x, y, \lambda)F(x)$ for $\forall x, y \in X$. If there exists a $t \in (0, 1)$ such that*

$$F(y + t\eta(x, y)) \preceq tb(x, y, \lambda)F(x) + (1 - tb(x, y, \lambda))F(y),$$

for all $x, y \in X$, then F is a B-preinvex fuzzy mapping on X with respect to mappings η and b .

4. Main results

In this section, we will present and prove several necessary conditions for differentiable and twice differentiable of B-preinvex fuzzy mappings satisfied the famous condition C which introduced by Mohan and Neogy (1995) [8].

Theorem 4.1. *Let X be a non-empty open invex set in $\mathbb{R}^n$ with respect to $\eta : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, where $\eta(x, y) \geq 0$ for any $x, y \in X$, satisfies Condition C. Assume that $F : X \rightarrow \mathcal{F}_0$ is a differentiable B-preinvex fuzzy mapping, the mapping $b(x, y, \lambda)$ is continuous at $\lambda = 0$ for fixed $x, y \in X$, and $b(x, y) = \lim_{\lambda \rightarrow 0^+} b(x, y, \lambda)$. Then, for any $x, y \in X$,*

$$(4.1) \quad b(x, y)F(x) \succeq \tilde{\nabla} F(y)\eta(x, y)^T + b(x, y)F(y).$$

Proof. Since F is a B-preinvex fuzzy mapping with respect to η and $b(x, y, \lambda)$, then for all $x, y \in X$ and $\lambda \in [0, 1]$,

$$F(y + \lambda\eta(x, y)) \preceq \lambda b(x, y, \lambda)F(x) + (1 - \lambda b(x, y, \lambda))F(y),$$

i.e., for all $x, y \in X$, $\alpha \in [0, 1]$ and $\lambda \in [0, 1]$,

$$(4.2) \quad F^*(\alpha, y + \lambda\eta(x, y)) \leq \lambda b(x, y, \lambda)F^*(\alpha, x) + (1 - \lambda b(x, y, \lambda))F^*(\alpha, y)$$

and

$$(4.3) \quad F_*(\alpha, y + \lambda\eta(x, y)) \leq \lambda b(x, y, \lambda)F_*(\alpha, x) + (1 - \lambda b(x, y, \lambda))F_*(\alpha, y).$$

By the differentiability of F and according to the mean-valued theorem, we have

$$(4.4) \quad F^*(\alpha, y + \lambda\eta(x, y)) = F^*(\alpha, y) + \lambda \nabla F^*(\alpha, \varepsilon)(\eta(x, y))^T$$

and

$$(4.5) \quad F_*(\alpha, y + \lambda\eta(x, y)) = F_*(\alpha, y) + \lambda \nabla F_*(\alpha, \zeta)(\eta(x, y))^T,$$

where $\varepsilon = y + \theta_1 \lambda \eta(x, y)$, $\zeta = y + \theta_2 \lambda \eta(x, y)$ and $0 < \theta_1, \theta_2 < 1$. Combining the above inequalities (4.2) and (4.3) and the equalities (4.4) and (4.5), it follows that

$$(4.6) \quad \lambda b(x, y, \lambda) F^*(\alpha, x) \geq \lambda \nabla F^*(\alpha, y) (\eta(x, \varepsilon))^T + \lambda b(x, y, \lambda) F^*(\alpha, y)$$

and

$$(4.7) \quad \lambda b(x, y, \lambda) F_*(\alpha, x) \geq \lambda \nabla F_*(\alpha, y) (\eta(x, \zeta))^T + \lambda b(x, y, \lambda) F_*(\alpha, y).$$

Dividing the inequalities (4.6) and (4.7) by λ and taking $\lambda \rightarrow 0^+$, then $\theta_1 \rightarrow 0^+$, $\theta_2 \rightarrow 0^+$. It is easy to verify that

$$b(x, y) F^*(\alpha, x) \geq \nabla F^*(\alpha, y) (\eta(x, y))^T + b(x, y) F^*(\alpha, y)$$

and $b(x, y) F_*(\alpha, x) \geq \nabla F_*(\alpha, y) (\eta(x, y))^T + b(x, y) F_*(\alpha, y)$. That is, $b(x, y) F(x) \succeq \tilde{\nabla} F(y) \eta(x, y)^T + b(x, y) F(y)$ for any $x, y \in X$ and $\lambda \in [0, 1]$. Hence, the statement in Theorem 4.1 is proved.

Corollary 4.1. *Let X be a non-empty open invex set in $\mathbb{R}^n$ with respect to $\eta : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, where $\eta(x, y) \geq 0$ for any $x, y \in X$, and η satisfy Condition C. Suppose that $F : X \rightarrow \mathcal{F}_0$ is a differentiable B-preinvex fuzzy mapping. Then, for all $x, y \in X$,*

$$(4.8) \quad b(y, x) \tilde{\nabla} F(y) (\eta(x, y))^T + b(x, y) \tilde{\nabla} F(x) (\eta(y, x))^T \preceq \tilde{0}.$$

Proof. Since F is a differentiable B-preinvex fuzzy mapping, and by using Theorem 4.1, it is easy to show that, for all $x, y \in X$,

$$(4.9) \quad b(x, y) F^*(\alpha, x) \geq \nabla F^*(\alpha, y) (\eta(x, y))^T + b(x, y) F^*(\alpha, y),$$

$$(4.10) \quad b(x, y) F_*(\alpha, x) \geq \nabla F_*(\alpha, y) (\eta(x, y))^T + b(x, y) F_*(\alpha, y),$$

$$(4.11) \quad b(y, x) F^*(\alpha, y) \geq \nabla F^*(\alpha, x) (\eta(y, x))^T + b(y, x) F^*(\alpha, x),$$

and

$$(4.12) \quad b(y, x) F_*(\alpha, y) \geq \nabla F_*(\alpha, x) (\eta(y, x))^T + b(y, x) F_*(\alpha, x).$$

Multiplying inequality (4.9) by $b(y, x)$ and plus multiplying inequality (4.11) by $b(x, y)$, it follows that for $x, y \in X$,

$$(4.13) \quad b(y, x) \nabla F^*(\alpha, y) (\eta(x, y))^T + b(y, x) \nabla F^*(\alpha, x) (\eta(y, x))^T \leq 0.$$

By the same way, Multiplying inequality (4.10) by $b(y, x)$ and plus multiplying inequality (4.12) by $b(x, y)$, it yields that for $x, y \in X$,

$$(4.14) \quad b(y, x) \nabla F_*(\alpha, y) (\eta(x, y))^T + b(y, x) \nabla F_*(\alpha, x) (\eta(y, x))^T \leq 0.$$

Hence, from the above two inequalities (4.13) and (4.14), the desired conclusion (4.8) is obtained.

Theorem 4.2. *Let X be a non-empty open invex subset of $\mathbb{R}^n$ with respect to $\eta : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, where $\eta(x, y) \geq 0$ for any $x, y \in X$ and satisfies Condition C. Suppose that $F : X \rightarrow \mathcal{F}_0$ is a twice differentiable B-preinvex fuzzy mapping. Then, for any $x, y \in X$,*

$$(4.15) \quad \eta(x, y) \tilde{\nabla}^2 F(y) \eta(x, y)^T \succeq 0.$$

Proof. Suppose that F is a B-preinvex fuzzy mapping on X with respect to mappings η and b . Then, from Lemma 3.1, for any $\alpha \in [0, 1]$, $F^*(\alpha, x)$ and $F_*(\alpha, x)$ are B-preinvex on X with respect to η and b . Thus, the extreme functions $F^*(\alpha, x)$ and $F_*(\alpha, x)$ verify the relation (4.15). Replacing $\tilde{\nabla}$ by ∇ , for any $\alpha \in [0, 1]$ and $\eta(x, y) \geq 0$, F verify the relation (4.15). The proof is completed.

A necessary and sufficient conditions for B-preinvex fuzzy mappings is stated below.

Theorem 4.3. *Let X be a non-empty invex subset of $\mathbb{R}^n$ with respect to $\eta : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, where η satisfies Condition C. Assume that $F : X \rightarrow \mathcal{F}_0$ is a fuzzy mapping and satisfies $F(y + \eta(x, y)) \preceq F(x)$ for all $x, y \in X$. Then F is a B-preinvex fuzzy mapping, if and only if, for $\forall x, y \in X$,*

$$(4.16) \quad \varphi(\lambda) = F(y + \lambda\eta(x, y))$$

is a B-invex mapping on $[0, 1]$ with respect to b .

Proof. Suppose that F is a B-preinvex fuzzy mapping on X , for any fixed $x, y \in X$ and $\lambda, \alpha_1, \alpha_2 \in [0, 1]$, if $\alpha_1 = \alpha_2$, then the result is obvious.

If $\alpha_1 < \alpha_2$, then $\alpha_2 - \alpha_1 > 0$ and $\alpha_1 \neq 1$, thus we have $0 < (\alpha_2 - \alpha_1)/(1 - \alpha_1) < 1$. From Condition C, we have that, for any $x, y \in X$ and $\alpha \in [0, 1]$,

$$(4.17) \quad \eta(y + \alpha\eta(x, y), y) = -\eta(y, y + \alpha\eta(x, y)) = \alpha\eta(x, y).$$

Combining the equality (4.17) and Condition C, it follows that

$$(4.18) \quad \begin{aligned} \eta(y + \alpha_2\eta(x, y), y + \alpha_1\eta(x, y)) &= (\alpha_2 - \alpha_1)/(1 - \alpha_1)\eta(x, y + \alpha_1\eta(x, y)) \\ &= (\alpha_2 - \alpha_1)\eta(x, y). \end{aligned}$$

By the equality (4.18) and according to the B-preinvexity of F , thus for any fixed $x, y \in X$,

$$(4.19) \quad \begin{aligned} &\varphi(\alpha_1 + \lambda(\alpha_2 - \alpha_1)) \\ &= F(y + \alpha_1\eta(x, y) + \lambda(\alpha_2 - \alpha_1)\eta(x, y)) \\ &= F(y + \alpha_1\eta(x, y) + \lambda\eta(y + \alpha_2\eta(x, y), y + \alpha_1\eta(x, y))) \\ &\preceq \lambda b(x, y, \lambda)F(y + \alpha_2\eta(x, y)) \dot{+} (1 - \lambda b(x, y, \lambda))F(y + \alpha_1\eta(x, y)) \\ &= \lambda b(x, y, \lambda)\varphi(\alpha_2) + (1 - \lambda b(x, y, \lambda))\varphi(\alpha_1). \end{aligned}$$

By the same way, if $\alpha_1 > \alpha_2$, we have that, for any fixed $x, y \in X$,

$$(4.20) \quad \varphi(\alpha_2 + \lambda(\alpha_1 - \alpha_2)) \leq \lambda b(x, y, \lambda) \varphi(\alpha_1) + (1 - \lambda b(x, y, \lambda)) \varphi(\alpha_2).$$

Utilizing the inequalities (4.19) and (4.20), it yields that $\varphi(\lambda)$ is a B-invex function on $[0, 1]$.

Conversely, since $\varphi(\lambda) = F(y + \lambda\eta(x, y))$ is a B-invex function and $F(y + \eta(x, y)) \preceq F(x)$, then we have, for any $x, y \in X$ and $\lambda \in [0, 1]$,

$$\begin{aligned} F(y + \lambda\eta(x, y)) &= \varphi(\lambda) = \varphi(\lambda \cdot 1 + (1 - \lambda) \cdot 0) \\ &\leq \lambda b(x, y, \lambda) \varphi(1) + (1 - \lambda b(x, y, \lambda)) \varphi(0) \\ &= \lambda b(x, y, \lambda) F(y + \eta(x, y)) + (1 - \lambda b(x, y, \lambda)) F(y) \\ &\preceq b(x, y, \lambda) F(x) + (1 - \lambda b(x, y, \lambda)) F(y). \end{aligned}$$

Thus, F is a B-preinvex fuzzy mapping with respect to b and η . This ends the proof.

5. Conclusions

The first conclusion to be draw is that some results about sufficient conditions of B-preinvex fuzzy mappings, which be presented and proved under certain conditions. The second one is the necessary conditions for differentiable and twice differentiable B-preinvex fuzzy mappings. A point that should be stressed is that these results discussed here are on the B-preinvex fuzzy mappings. As a consequence, these results are the extension of results from Li and Noor (2010) in [7] and Rufián-Lizana et al. (2014) in [15].

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DYNAMICS OF ALMOST PERIODIC NICHOLSON'S BLOWFLIES MODEL WITH NONLINEAR DENSITY-DEPENDENT MORTALITY TERM

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Abstract. This paper deals with the dynamics of almost periodic Nicholson's blowflies model with nonlinear density-dependent mortality. Prior to the main results, we prove the boundedness and extinction of the solutions for the addressed model. By applying Shauder's fixed point theorem, we establish sufficient conditions for the existence of almost periodic positive solution. Under less restrictive assumptions, the exponential stability is derived by means of the Liapunov functional method. The reported results give an affirmative answer to the problem raised by L. Berezensky.

Keywords: Nicholson's blowflies model, density-dependent mortality, almost periodic solution, extinction, exponential stability.

1. Introduction

In [1], Gurney proposed the following delay differential equation

$$(1.1) \quad x'(t) = -ax(t) + px(t - \tau)e^{-\beta x(t-\tau)}$$

to describe the population of the Australian sheep-blowfly and to agree with the experimental data obtained by Nicholson in [2]. Equation (1.1) describes Nicholson's data of blowfly and thus it has been referred to as the Nicholson's blowflies model. The theory of this model has made a substantial progress during the last two decades [3-15]. Due to various seasonal effects of the environmental factors in real life situation (e.g., seasonal effects of weather, food supplies, mating habits, harvesting, etc.), researchers have found it rational and practical to study the population models under periodic assumptions. The recent years

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have witnessed the appearance of many papers that studied non autonomous differential equations with periodic coefficients of various versions of model (1.1); see for instance [4,5,7,9,12].

New studies indicated that the consideration of population models with density dependent mortality will be more accurate at low densities. In his remarkable paper [11], Berezensky has put forward an open problem about the dynamical behaviors of Nicholson's blowflies model with density-dependent mortality of the form

$$(1.2) \quad x'(t) = -M(x) + px(t - \tau)e^{-\beta x(t-\tau)},$$

where M denotes the mortality term that might be expressed in the form $\frac{ax}{b+x}$ or $a - be^{-x}$. Although the papers [16-22] have dealt with the permanence and periodicity of solutions, they have provided insufficient outcomes to answer the problem raised by Berezensky for model (1.2). The almost periodicity which is a natural generalization of periodicity has been the object of many researchers in the last years. Indeed, it has been encompassed to Nicholson's model and thus several results have been recently reported; see for instance the papers [23,24] and the monograph [28] for more details.

Motivated by the above discussions, we consider the non-autonomous almost periodic Nicholson's blowflies model with density-dependent mortality term of the form

$$(1.3) \quad x'(t) = -\frac{a(t)x(t)}{b(t) + x(t)} + p(t)x(t - \tau(t))e^{-\beta(t)x(t-\tau(t))},$$

where $a(t), b(t), \beta(t), p(t), \tau(t) \in C(R, R^+)$ and $a(t), b(t), \beta(t), p(t), \tau(t)$ are bounded almost periodic functions. Due to biological significance, we restrict our attention to positive solutions of equation (1.3). The initial condition of equation (1.2) is $x(t) = \phi(t) > 0$ for $t \in [-\bar{\tau}, 0]$, $\bar{\tau} = \sup_{t \in R} \tau(t)$, $\phi \in BC([-\bar{\tau}, 0], R^+)$, where $BC([-\bar{\tau}, 0], R^+) = \{\phi | \phi : [-\bar{\tau}, 0] \rightarrow R^+ \text{ is bounded continuous function} \}$.

In this paper, we provide sufficient conditions for the existence and exponential stability of almost periodic solution for model (1.3). Prior to the main results, we prove the boundedness and extinction for the addressed model. Unlike previously obtained results such as those given in [23,24], we utilize Schauder's fixed point theorem to prove the existence result. In addition to this, the exponential stability has been proved under less restrictive assumptions. To the best of our observation, no published paper has dealt with model (1.3) by the implementation of these two distinctive features.

2. Preliminaries

For any bounded function $f(t)$, we denote $\bar{f} = \sup_{t \in R} f(t)$ and $\underline{f} = \inf_{t \in R} f(t)$. Therefore, in the remaining part of the paper, we assume that the bounded almost periodic functions $a(t), b(t), \beta(t), p(t), \tau(t)$ satisfy $0 \leq a \leq a(t) \leq \bar{a}$,

$0 \leq b \leq b(t) \leq \bar{b}$, $0 \leq \underline{\beta} \leq \beta(t) \leq \bar{\beta}$, $0 < \underline{p} \leq p(t) \leq \bar{p}$ and $0 < \underline{\tau} \leq \tau(t) \leq \bar{\tau}$. In what follows, we set forth some assertions that will be used throughout the rest of the paper.

Definition 2.1 ([25]). Let $u(t) : \mathbb{R} \rightarrow \mathbb{R}^n$ be continuous in t , $u(t)$ is said to be almost periodic on $\mathbb{R}$ if, for any $\varepsilon > 0$, the set $T(u, \varepsilon) = \{\delta : |u(t + \delta) - u(t)| < \varepsilon, t \in \mathbb{R}\}$ is relatively dense, i.e., for any $\varepsilon > 0$, it is possible to find a real number $l = l(\varepsilon) > 0$, for any interval with length $l(\varepsilon)$, for which there exists a number $\delta = \delta(\varepsilon)$ in this interval such that $|u(t + \delta) - u(t)| < \varepsilon$, for all $t \in \mathbb{R}$.

Definition 2.2 ([25]). Let $x \in \mathbb{R}$ and $Q(t)$ be $n \times n$ continuous matrix defined on $\mathbb{R}$. The linear system

$$(2.1) \quad x'(t) = Q(t)x(t)$$

is said to admit an exponential dichotomy on $\mathbb{R}$ if there exist positive constants k, α , projection P and the fundamental solution matrix $X(t)$ of (2.1) satisfying

$$\begin{aligned} \|X(t)PX^{-1}(s)\| &\leq ke^{-\alpha(t-s)} \quad \text{for } t \geq s \\ \|X(t)(I - P)X^{-1}(s)\| &\leq ke^{-\alpha(s-t)} \quad \text{for } t \leq s. \end{aligned}$$

Definition 2.3 ([27]). Let $\tilde{x}(t)$ be an almost periodic solution of Eq. (1.3), $x(t)$ be another solution of Eq. (1.3). The solution $x(t)$ is said to be exponentially convergent to $\tilde{x}(t)$ as $t \rightarrow +\infty$ if there exist constants $\lambda > 0, K > 0$ such that $|x(t) - \tilde{x}(t)| \leq Ke^{-\lambda t}$, for all $t > 0$.

Lemma 2.4 ([25]). If the linear system (2.1) admits an exponential dichotomy, then the almost periodic system

$$(2.2) \quad x'(t) = Q(t)x(t) + g(t)$$

has a unique almost periodic solution $x(t)$, and

$$x(t) = \int_{-\infty}^t X(t)PX^{-1}(s)g(s)ds - \int_t^{+\infty} X(t)(I - P)X^{-1}(s)g(s)ds.$$

Lemma 2.5. Let $c_i(t)$ be almost periodic function on $\mathbb{R}$ and

$$M[c_i] = \lim_{T \rightarrow +\infty} \frac{1}{T} \int_t^{t+T} c_i(s)ds > 0, \quad i = 1, 2, \dots, n.$$

Then, the linear system

$$x'(t) = \text{diag}(-c_1(t), -c_2(t), \dots, -c_n(t))x(t)$$

admits an exponential dichotomy on $\mathbb{R}$.

Lemma 2.6. Every solution of equation (1.3) is positive.

Proof. Let $x(t)$ be any solution of equation (1.3) with initial condition $x(t) = \phi(t) > 0$ for $t \in [-\bar{\tau}, 0]$. We claim that

$$(2.3) \quad x(t) > 0, \text{ for all } t > 0.$$

Suppose the claim (2.3) is not true, then there must exist $t_1 \in (0, +\infty)$ such that $x(t_1) = 0$, $x'(t_1) \leq 0$ and $x(t) > 0$ for $t \in [-\bar{\tau}, t_1)$. From (1.3), we have

$$(2.4) \quad \begin{aligned} x'(t_1) &= -\frac{a(t_1)x(t_1)}{b(t_1) + x(t_1)} + p(t_1)x(t_1 - \tau(t_1))e^{-\beta(t_1)x(t_1 - \tau(t_1))} \\ &= p(t_1)x(t_1 - \tau(t_1))e^{-\beta(t_1)x(t_1 - \tau(t_1))}. \end{aligned}$$

Since $x(t_1 - \tau(t_1)) > 0$, then it follows from (2.4) that $x'(t_1) > 0$, which contradicts $x'(t_1) \leq 0$. Therefore, the claim (2.3) is true. The proof is complete. $\square$

Schauder fixed point theorem is an important tool in our proof.

Lemma 2.7 ([26]). (Schauder's fixed point theorem) Let Ω be a closed convex subset of Banach space X , $A : \Omega \rightarrow \Omega$ be a continuous operator such that $A\Omega$ is relatively compact. Then the operator A has at least one fixed point in Ω .

3. Boundedness and extinction of solutions

Let $m = \frac{\bar{p}}{\underline{\beta}e}$ and $H = \frac{m\bar{b}}{\underline{a}-m}$. We make the assumption:

$$(C_1) \quad \underline{a} > m.$$

Theorem 3.1. *Let (C_1) hold. Then, every solution of equation (1.3) is bounded.*

Proof. Let $x(t)$ be any solution of equation (1.3) with initial condition $x(t) = \phi(t) > 0$ for $t \in [-\bar{\tau}, 0]$. By Lemma 2.6, we know that $x(t) > 0$, for all $t > 0$.

Now, we prove that $x(t)$ is bounded.

Suppose $x(t)$ is unbounded, then there exists $t^* > 0$ such that $x(t') > H$, and there also exists $\bar{t} > 0$ satisfying $0 < \bar{t} < t^*$, $x(t) < x(t^*)$. From (1.3), we have

$$(3.1) \quad \begin{aligned} x'(t^*) &= -\frac{a(t^*)x(t^*)}{b(t^*)} + p(t^*)x(t^* - \tau(t^*))e^{-\beta(t^*)x(t^* - \tau(t^*))} \\ &\leq -\frac{\underline{a}x(t^*)}{\bar{b} + x(t^*)} + \bar{p}x(t^* - \tau(t^*))e^{-\underline{\beta}x(t^* - \tau(t^*))}. \end{aligned}$$

It is clear that the function $f(u) = ue^{-\underline{\beta}u}$, $u \in [0, +\infty)$ reaches its maximum $\frac{1}{\underline{\beta}e}$ at $u = \frac{1}{\underline{\beta}}$. Then, we get $x(t^* - \tau(t^*))e^{-\underline{\beta}x(t^* - \tau(t^*))} \leq \frac{1}{\underline{\beta}e}$. Thus, (3.1) implies that

$$(3.2) \quad x'(t^*) \leq -\frac{\underline{a}x(t^*)}{\bar{b} + x(t^*)} + \bar{p}\frac{1}{\underline{\beta}e}.$$

Note that the function $g(u) = \frac{au}{b+u}$ is strictly increasing on $u \in (0, +\infty)$. Since $x(t^*) > H$, then we have $g(x(t^*)) > g(H)$, that is

$$(3.3) \quad \frac{ax(t^*)}{\bar{b} + x(t^*)} > \frac{aH}{\bar{b} + H}.$$

By (3.2) and (3.3), we get

$$x'(t^*) < -\frac{aH}{\bar{b} + H} + \bar{p}\frac{1}{\underline{\beta}e} = 0.$$

Let $x(\hat{t}) = \max_{\tilde{t} \leq t \leq t^*}$. Since $x(\hat{t}) < x(t^*)$ and $x'(t^*) < 0$, then it follows that $\tilde{t} < \hat{t} < t^*$ and $x(\hat{t}) > x(t^*)$. Hence we have

$$(3.4) \quad x'(\hat{t}) = 0.$$

On the other hand, from (1.3), we get

$$\begin{aligned} x'(\hat{t}) &= -\frac{a(\hat{t})x(\hat{t})}{b(\hat{t}) + x(\hat{t})} + p(\hat{t})x(\hat{t} - \tau(\hat{t}))e^{-\beta(\hat{t})x(\hat{t})x(\hat{t}-\tau(\hat{t}))} \\ &\leq -\frac{ax(\hat{t})}{\bar{b} + x(\hat{t})} + \bar{p}x(\hat{t} - \tau(\hat{t}))e^{-\underline{\beta}x(\hat{t})x(\hat{t}-\tau(\hat{t}))} \\ &\leq -\frac{ax(\hat{t})}{\bar{b} + x(\hat{t})} + \bar{p}\frac{1}{\underline{\beta}e} \\ &< -\frac{ax(t^*)}{\bar{b} + x(t^*)} + \bar{p}\frac{1}{\underline{\beta}e} \\ &\quad - \frac{aH}{\bar{b} + H} + \bar{p}\frac{1}{\underline{\beta}e} = 0, \end{aligned}$$

which contradicts (3.4). Therefore, $x(t)$ is bounded. The proof is complete. $\square$

Let $L > H$, we define $U_L = \{\phi | \phi \in BC([-\bar{\tau}, 0], \mathbb{R}^+), 0 < \phi(t) < L, t \in [-\bar{\tau}, 0]\}$.

Theorem 3.2. *Let (C_1) hold. Then every solution $x(t)$ of equation (1.3) with the initial function $\phi \in U_L$ satisfies*

$$0 < x(t) < L, \text{ for all } t > 0.$$

Proof. Let $x(t)$ be any solution of equation (1.3) with initial function $\phi \in U_L$. For $t \in [-\bar{\tau}, 0]$, we have $0 < x(t) = \phi(t) < L$. By Lemma 2.6, we have $x(t) > 0$, for all $t > 0$. Now, we claim that

$$(3.5) \quad x(t) < L, \text{ for all } t > 0.$$

Suppose that (3.5) is not true. Then there must exist a $t_1 \in (0, +\infty)$ such that $x(t_1) = L$, $x'(t) \geq 0$ and $0 < x(t) < L$ for $t \in (0, t_1)$. From (1.3), we have

$$\begin{aligned} x'(t_1) &= -\frac{a(t_1)x(t_1)}{b(t_1) + x(t_1)} + p(t_1)x(t_1 - \tau(t_1))e^{\beta(t_1)x(t_1 - \tau(t_1))} \\ &= -\frac{a(t_1)L}{b(t_1) + L} + p(t_1)x(t_1)x(t_1 - \tau(t_1))e^{-\beta(t_1)x(t_1 - \tau(t_1))} \\ &\leq -\frac{\underline{a}L}{\bar{b} + L} + \bar{p}x(t_1 - \tau(t_1))e^{-\beta x(t_1 - \tau(t_1))} \\ &\leq -\frac{\underline{a}L}{\bar{b} + L} + \bar{p}\frac{1}{\underline{\beta}e} &< -\frac{\underline{a}H}{\bar{b} + H} + \bar{p}\frac{1}{\underline{\beta}e} = 0, \end{aligned}$$

which contradicts $x'(t_1) \geq 0$. So the claim (3.5) is true. The proof is complete. $\square$

Furthermore, we assume that

$$(C_2) \quad \frac{\underline{a} - \bar{p}\bar{b}}{\bar{p}} > H.$$

Theorem 3.3. *Assume that (C_1) and (C_2) hold. Let L be a positive constant satisfying $H < L < \frac{\underline{a} - \bar{p}\bar{b}}{\bar{p}}$. Then every solution $x(t)$ of equation (1.3) with initial function $\phi \in U_L$ satisfies*

$$x(t) \rightarrow 0 \quad \text{as} \quad t \rightarrow +\infty.$$

That is, every solution $x(t)$ with initial function $\phi \in U_L$ tends to extinction.

Proof. Let $x(t)$ be any solution of equation (1.3) with the initial function $\phi \in U_L$. For $t \in [-\bar{\tau}, 0]$, we have $0 < x(t) = \phi(t) < L$. By Theorem 2, we know that $0 < x(t) < L$ for all $t > 0$. From $H < L < \frac{\underline{a} - \bar{p}\bar{b}}{\bar{p}}$, it follows that $\underline{a} > \bar{p}\bar{b} + \bar{p}L$. Consider the function $F(x) = \bar{p}\bar{b}e^{\tau x} + \bar{b}x + Lx + \bar{p}L - \underline{a}$, $x \in [0, 1]$. Since $F(0) = \bar{p}\bar{b} + \bar{p}L - \underline{a} < 0$, then there exists a constant $\lambda \in (0, 1)$ such that $F(\lambda) < 0$. That is

$$(3.6) \quad \bar{p}\bar{b}e^{\bar{\lambda}\bar{\tau}} + \lambda\bar{b} + \lambda L + \bar{p}L - \underline{a} < 0$$

Let $\delta(t) = x(t)e^{\lambda t}$, then we have

$$\begin{aligned} \delta'(t) &= x'(t)e^{\lambda t} + \lambda x(t)e^{\lambda t} \\ (3.7) \quad &= \left[-\frac{a(t)x(t)}{b(t) + x(t)} + p(t)x(t - \tau(t))e^{-\beta(t)x(t - \tau(t))} + \lambda x(t) \right] e^{\lambda t}. \end{aligned}$$

Let $M = L + \sup_{-\bar{\tau} \leq t \leq 0} \phi(t)$. For all $t \in [-\bar{\tau}, 0]$, we have

$$0 < \delta(t) = x(t)e^{\lambda t} = \phi(t)e^{\lambda t} \leq \phi(t) \leq -\sup_{-\bar{\tau} \leq t \leq 0} \phi(t) < L + \sup_{-\bar{\tau} \leq t \leq 0} \phi(t) = M.$$

For all $t \in (0, +\infty)$, it is obvious that $\delta(t) = x(t)e^{\lambda t} > 0$. Now, we claim that

$$(3.8) \quad \delta(t) < M \text{ for all } t > 0.$$

Suppose that (3.8) is not true. Then, there must exist a $t^* > 0$, such that $\delta(t^*) = M$, $\delta'(t^*) \geq 0$ and $\delta(t) < M$ for $t < t^*$. It follows from (3.7) that

$$\begin{aligned}
 0 \leq \delta'(t^*) &= \left[-\frac{a(t^*)x(t^*)}{b(t^*) + x(t^*)} \right. \\
 &\quad \left. + p(t^*)x(t^* - \tau(t^*))e^{-\beta(t^*)x(t^* - \tau(t^*))} + \lambda x(t^*) \right] e^{\lambda t^*} \\
 &\leq \left[-\frac{\underline{a}x(t^*)}{\bar{b} + x(t^*)} + \bar{p}x(t^* - \tau(t^*)) + \lambda x(t^*) \right] e^{\lambda t^*} \\
 &\quad - \underline{a}x(t^*)e^{\lambda t^*} + \bar{p}\bar{b}x(t^* - \tau(t^*))e^{\lambda t^*} + \bar{p}x(t^* - \tau(t^*))x(t^*)e^{\lambda t^*} \\
 &\quad + \lambda \bar{b}x(t^*)e^{\lambda t^*} + \lambda x^2(t^*)e^{\lambda t^*} \\
 &= \frac{-\underline{a}x(t^*)e^{\lambda t^*} + \bar{p}\bar{b}x(t^* - \tau(t^*))e^{\lambda t^*} + \bar{p}x(t^* - \tau(t^*))x(t^*)e^{\lambda t^*} + \lambda \bar{b}x(t^*)e^{\lambda t^*} + \lambda x^2(t^*)e^{\lambda t^*}}{\bar{b} + x(t^*)} \\
 (3.9) \quad &= \frac{-\underline{a}\delta(t^*) + \bar{p}\bar{b}x(t^* - \tau(t^*))e^{\lambda(t^* - \tau(t^*))} + \bar{p}x(t^* - \tau(t^*))\delta(t^*) - \tau(t^*)\delta(t^*) + \lambda \bar{b}\delta(t^*) + \lambda x(t^*)\delta(t^*)}{\bar{b} + x(t^*)} \\
 &= \frac{-\underline{a}M + \bar{p}\bar{b}\delta(t^* - \tau(t^*))e^{\lambda\tau(t^*)} + \bar{p}x(t^* - \tau(t^*))M + \lambda \bar{b}M + \lambda x(t^*)M}{\bar{b} + x(t^*)} \\
 &< \frac{-\underline{a}M + \bar{p}\bar{b}Me^{\lambda\tau} + \bar{p}LM + \lambda \bar{b}M + \lambda LM}{\bar{b} + x(t^*)} \\
 &= \frac{M}{\bar{b} + x(t^*)} (-\underline{a} + \bar{p}\bar{b}e^{\lambda\tau} + \bar{p}L + \lambda \bar{b} + \lambda L)
 \end{aligned}$$

Thus, (3.9) implies $-\underline{a} + \bar{p}\bar{b}e^{\lambda\tau} + \bar{p}L + \lambda \bar{b} + \lambda L > 0$, which contradicts (3.6). Therefore (3.8) is true. Hence, $\delta(t) = x(t)e^{\lambda t} < M$, for all $t > 0$. That is

$$0 < x(t) < Me^{-\lambda t}, \text{ for all } t > 0.$$

which implies that $x(t) \rightarrow 0$ as $t \rightarrow +\infty$. The proof is complete. $\square$

4. Existence of almost periodic positive solution

It is assumed that (C_3) there exist two positive constants $L_2 > L_1 \geq \frac{1}{\underline{\beta}}$, and a bounded positive almost periodic function $\gamma(t) \in C(R, R^+)$ with $\underline{\gamma} > 0$, satisfying the following inequalities

$$\sup_{t \in R} \left\{ -\frac{a(t)L_1}{b(t) + L_1} + \gamma(t)L_2 + \frac{p(t)}{\beta(t)} \frac{l}{e} \right\} \leq \underline{\gamma}L_2$$

and

$$\inf_{t \in R, v \in [L_1, L_2]} \left\{ -\frac{a(t)L_2}{b(t) + L_2} + \gamma(t)L_1 + p(t)ve^{-\bar{\beta}v} \right\} \geq \bar{\gamma}L_1.$$

Let $X = \{w(t) \in C(\mathbb{R}, \mathbb{R}), w(t) \text{ is almost periodic function}\}$. For $w \in X$, we define $\|w\| = \sup_{t \in \mathbb{R}} |w(t)|$, then X is a Banach space.

We note that equation (1.3) is equivalent to

$$(4.1) \quad x'(t) = -\gamma(t)x(t) - \frac{a(t)x(t)}{b(t) + x(t)} + \gamma(t)x(t) + p(t)x(t - \tau(t))e^{-\beta(t)x(t - \tau(t))}.$$

For $w(t) \in X$, we consider the equation

$$(4.2) \quad \omega' = -\gamma(t)\omega(t) - \frac{a(t)\omega(t)}{b(t) + \omega(t)} + \gamma(t)\omega(t) + p(t)\omega(t - \tau(t))e^{-\beta(t)\omega(t - \tau(t))}.$$

Since $M[\gamma] > 0$, then from Lemma 2.5 we know that the linear equation $x' = -\gamma(t)x(t)$ admits exponential dichotomy on $\mathbb{R}$.

Hence, by Lemma 1, we know that equation (4.2) has exactly one almost periodic solution:

$$\begin{aligned} x_w(t) = & \int_{-\infty}^t e^{-\int_s^t \gamma(u)du} \left[-\frac{a(s)w(s)}{b(s) + w(s)} + \gamma(s)w(s) \right. \\ & \left. + p(s)w(s - \tau(s))e^{-\beta(s)w(s - \tau(s))} \right] ds. \end{aligned}$$

We define the operator $A : X \rightarrow X$,

$$\begin{aligned} (Aw)(t) = & \int_{-\infty}^t e^{-\int_s^t \gamma(u)du} \left[-\frac{a(s)w(s)}{b(s) + w(s)} \right. \\ & \left. + \gamma(s)w(s) + p(s)w(s - \tau(s))e^{-\beta(s)w(s - \tau(s))} \right] ds, \quad w \in X. \end{aligned}$$

Obviously, $w(t) \in C(\mathbb{R}, \mathbb{R})$ is the almost periodic solution of equation (4.1) if and only if w is the fixed point of operator A .

Theorem 4.1. *Let (C_3) hold. Then, equation (1.3) has at least one almost periodic positive solution.*

Proof. Define a closed convex subset Ω of X as follows

$$\Omega = \{w | w \in X, L_1 \leq w(t) \leq L_2, t \in \mathbb{R}\}.$$

Firstly, we prove that $A\Omega \subset \Omega$. For all $w \in \Omega$, we have

$$\begin{aligned} (Aw)(t) = & \int_{-\infty}^t e^{-\int_s^t \gamma(u)du} \left[-\frac{a(s)w(s)}{b(s) + w(s)} + \gamma(s)w(s) \right. \\ & \left. + p(s)w(s - \tau(s))e^{-\beta(s)w(s - \tau(s))} \right] ds \end{aligned}$$

$$\begin{aligned}
&\leq \int_{-\infty}^t e^{-\int_s^t \gamma(u)du} \left[-\frac{a(s)L_1}{b(s)+L_1} + g(s)L_2 \right. \\
&\quad \left. + \frac{p(s)}{\beta(s)} \beta(s)w(s-\tau(s))e^{-\beta(s)w(s-\tau(s))} \right] ds \\
(4.3) \quad &\leq \int_{-\infty}^t e^{-\int_s^t \underline{\gamma}(u)du} \left[-\frac{a(s)L_1}{b(s)+L_1} + \gamma(s)L_2 + \frac{p(s)}{\beta(s)} \frac{1}{e} \right] ds \\
&\quad \sup_{t \in \mathbb{R}} \left\{ -\frac{a(t)L_1}{b(t)+L_1} + \gamma(t)L_2 + \frac{p(t)}{\beta(t)} \frac{1}{e} \right\} \int_{-\infty}^t e^{-\underline{\gamma}(t-s)} ds \\
&= \sup_{t \in \mathbb{R}} \left\{ -\frac{a(t)L_1}{b(t)+L_1} + \gamma(t)L_2 + \frac{p(t)}{\beta(t)} \frac{1}{e} \right\} \frac{1}{\underline{\gamma}} \leq L_2.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
(Aw)(t) &= \int_{-\infty}^t e^{-\int_s^t \gamma(u)du} \left[\frac{a(s)w(s)}{b(s)+w(s)} + \gamma(s)w(s) \right. \\
&\quad \left. + p(s)w(s-\tau(s))e^{-\beta(s)w(s-\tau(s))} \right] ds \\
&\geq \int_{-\infty}^t e^{-\int_s^t \gamma(u)du} \left[\frac{a(s)L_2}{b(s)+L_2} + g(s)L_1 \right. \\
&\quad \left. + \frac{p(s)}{\beta(s)} \beta(s)w(s-\tau(s))e^{-\beta(s)w(s-\tau(s))} \right] ds \\
&\geq \int_{-\infty}^t e^{-\int_s^t \underline{\gamma}(u)du} \left[-\frac{a(s)L_2}{b(s)+L_2} + \gamma(s)L_1 \right. \\
(4.4) \quad &\quad \left. + p(s)w(s-\tau(s))e^{-\bar{\beta}w(s-\tau(s))} \right] ds \\
&\geq \inf_{t \in \mathbb{R}, v \in [L_1, L_2]} \left\{ -\frac{a(t)L_2}{b(t)+L_2} + \gamma(t)L_1 + p(t)ve^{-\bar{\beta}v} \right\} \int_{-\infty}^t e^{-\int_s^t \underline{\gamma}du} ds \\
&= \int_{t \in \mathbb{R}, v \in [L_1, L_2]} \left\{ -\frac{a(t)L_2}{b(t)+L_2} + \gamma(t)L_1 + p(t)ve^{-\bar{\beta}v} \right\} \frac{1}{\underline{\gamma}} \geq L_1.
\end{aligned}$$

Hence, (4.3) and (4.4) imply

$$(4.5) \quad L_1 \leq (Aw)(t) \leq L_2.$$

In addition, for all $w \in \Omega$, then $w(t)$ is almost periodic. By Lemma 1, we know that equation (4.2) has exactly one almost periodic solution:

$$x_w(t) = \int_{-\infty}^t e^{-\int_s^t \gamma(u)du} \left[-\frac{a(s)w(s)}{b(s)+w(s)} + \gamma(s)w(s) + p(s)w(s-\tau(s))e^{-\beta(s)w(s-\tau(s))} \right] ds.$$

Since $x_w(t)$ is almost periodic, then $(Aw)(t)$ is almost periodic. This, together with (4.5), imply that $Aw \in \Omega$. So we have $A\Omega \subset \Omega$.

Next, we prove that the operator $A : \Omega \rightarrow \Omega$ is continuous. Let $x_n = x_n(t) \in \Omega$ be such that $x_n \rightarrow x \in \Omega$ as $n \rightarrow +\infty$. Then, we have

$$\begin{aligned}
 \|Ax_n - Ax\| &= \sup_{t \in \mathbb{R}} |(Ax_n)(t) - (Ax)(t)| \\
 &= \sup_{t \in \mathbb{R}} \left| \int_{-\infty}^t e^{-\int_s^t \gamma(u) du} \left[-\frac{a(s)x_n(s)}{b(s) + x_n(s)} + \gamma(s)x_n(s) \right. \right. \\
 &\quad \left. \left. + p(s)x_n(s - \tau(s))e^{-\beta(s)x_n(s - \tau(s))} \right] ds \right. \\
 &\quad \left. - \int_{-\infty}^t e^{-\int_s^t \gamma(u) du} \left[-\frac{a(s)x(s)}{b(s) + x(s)} + \gamma(s)x(s) \right. \right. \\
 &\quad \left. \left. + p(s)x(s - \tau(s))e^{-\beta(s)x(s - \tau(s))} \right] ds \right| \\
 &= \sup_{t \in \mathbb{R}} \left| \int_{-\infty}^t e^{-\int_s^t \gamma(u) du} \left[\left(-\frac{a(s)x_n(s)}{b(s) + x_n(s)} \right. \right. \right. \\
 (4.6) \quad &\quad \left. \left. - \frac{a(s)x(s)}{b(s) + x(s)} \right) + (\gamma(s)x_n(s) - \gamma(s)x(s)) \right. \\
 &\quad \left. \left. + (p(s)x_n(s - \tau(s))e^{-\beta(s)x_n(s - \tau(s))} - p(s)x(s - \tau(s))e^{-\beta(s)x(s - \tau(s))}) \right] ds \right| \\
 &\leq \sup_{t \in \mathbb{R}} \int_{-\infty}^t e^{-\int_s^t \gamma(u) du} \left| \frac{a(s)x_n(s)}{b(s) + x_n(s)} - \frac{a(s)x(s)}{b(s) + x(s)} \right| \\
 &\quad + |\gamma(s)x_n(s) - \gamma(s)x(s)| \\
 &\quad + |p(s)x_n(s - \tau(s))e^{-\beta(s)x_n(s - \tau(s))} - p(s)x(s - \tau(s))e^{-\beta(s)x(s - \tau(s))}| ds \\
 &\leq \sup_{t \in \mathbb{R}} \int_{-\infty}^t e^{-\int_s^t \gamma(u) du} [\bar{a} \left| \frac{x_n(s)}{b(s) + x_n(s)} - \frac{x(s)}{b(s) + x(s)} \right| + \bar{\gamma}|x_n(s) - x(s)| \\
 &\quad + \bar{p}|x_n(s - \tau(s))e^{-\beta(s)x_n(s - \tau(s))} - x(\tau(s))e^{-\beta(s)x(s - \tau(s))}|] ds.
 \end{aligned}$$

Define the function $\Phi(x) = \frac{x}{x+1}$, $x \in (0, +\infty)$, then $\Phi'(x) = \frac{1}{(x+1)^2}$. By the mean value theorem, we then have

$$\begin{aligned}
 \left| \frac{x_n(s)}{b(s) + x_n(s)} - \frac{x(s)}{b(s) + x(s)} \right| &= \left| \frac{\frac{x_n(s)}{b(s)}}{1 + \frac{x_n(s)}{b(s)}} - \frac{\frac{x(s)}{b(s)}}{1 + \frac{x(s)}{b(s)}} \right| \\
 &= \left| \Phi\left(\frac{x_n(s)}{b(s)}\right) - \Phi\left(\frac{x(s)}{b(s)}\right) \right| \\
 (4.7) \quad &\leq \left| \Phi'(\xi_1) \left(\frac{x_n(s)}{b(s)} - \frac{x(s)}{b(s)} \right) \right| = \frac{1}{(1 + \xi_1)^2} \left| \frac{x_n(s)}{b(s)} - \frac{x(s)}{b(s)} \right| \\
 &\leq \left| \frac{x_n(s)}{b(s)} - \frac{x(s)}{b(s)} \right| = \frac{1}{b(s)} |x_n(s) - x(s)| \leq \frac{1}{\underline{b}} |x_n(s) - x(s)|,
 \end{aligned}$$

in which ξ_1 lies between $\frac{x_n(s)}{b(s)}$ and $\frac{x(s)}{b(s)}$.

We define the function $\Psi(x) = xe^{-x}$, then $\Psi'(x) = (1-x)e^{-x}$. Again by the mean value theorem, we get

$$\begin{aligned}
 & |x_n(s - \tau(s))e^{-\beta(s)x_n(s - \tau(s))} - x(s - \tau(s))e^{-\beta(s)x(s - \tau(s))}| \\
 &= \frac{1}{\beta(s)} |\beta(s)x_n(s - \tau(s))e^{-\beta(s)x_n(s - \tau(s))} - \beta(s)x(s - \tau(s))e^{-\beta(s)x(s - \tau(s))}| \\
 (4.8) \quad &= \frac{1}{\beta(s)} |\Psi(\beta(s)x_n(s - \tau(s))) - \Psi(\beta(s)x(s - \tau(s)))| \\
 &\leq \frac{1}{\beta(s)} |\Psi'(\xi_2)(\beta(s)x_n(s - \tau(s)) - \beta(s)x(s - \tau(s)))| \\
 &= \frac{1}{\beta(s)} |(1 - \xi_2)e^{-\xi_2}| |\beta(s)x_n(s - \tau(s)) - \beta(s)x(s - \tau(s))| \\
 &= |(1 - \xi_2)e^{-\xi_2}| |x_n(s - \tau(s)) - x(s - \tau(s))|,
 \end{aligned}$$

in which ξ_2 lies between $\beta(s)x_n(s - \tau(s))$ and $\beta(s)x(s - \tau(s))$. Since $x_n, x \in \Omega$, $L_1 \leq x_n(t) \leq L_2$ and $L_1 \leq x(t) \leq L_2$ for $t \in \mathbb{R}$, then we have

$$1 \leq \underline{\beta}L_1 \leq \beta(s)x_n(s - \tau(s)) \leq \bar{\beta}L_2 \text{ and } 1 \leq \underline{\beta}L_1 \leq \beta(s)x(s - \tau(s)) \leq \bar{\beta}L_2.$$

This implies

$$1 \leq \underline{\beta}L_1 < \xi_2 < \bar{\beta}L_2.$$

Note that the function $h(x) = |(1-x)e^{-x}|$, $x \in [1, +\infty)$ has maximum $h_{\max} = \frac{1}{e^2}$. Thus we have $h(\xi_2) = |(1 - \xi_2)e^{-\xi_2}| \leq \frac{1}{e^2}$. It follows from (4.8) that

$$\begin{aligned}
 & |x_n(s - \tau(s))e^{-\beta(s)x_n(s - \tau(s))} - x(s - \tau(s))e^{-\beta(s)x(s - \tau(s))}| \\
 (4.9) \quad & \leq \frac{1}{e^2} |x_n(s - \tau(s)) - x(s - \tau(s))|.
 \end{aligned}$$

From (4.6), (4.7) and (4.9), we get

$$\begin{aligned}
 \|Ax_n - Ax\| &\leq \sup_{t \in \mathbb{R}} \int_{-\infty}^t e^{-\int_s^t \gamma(u)du} \left(\frac{\bar{a}}{\underline{b}} |x_n(s) - x(s)| + \bar{\gamma} |x_n(s) - x_s| \right. \\
 &\quad \left. + \bar{p} \frac{1}{e^2} |x_n(s - \tau(s)) - x(s - \tau(s))| \right) ds \\
 &= \sup_{t \in \mathbb{R}} \int_{-\infty}^t e^{-\int_s^t \gamma(u)du} \left(\frac{\bar{a}}{\underline{b}} \|x_n - x\| + \bar{\gamma} \|x_n - x\| + \bar{p} \frac{1}{e^2} \|x_n - x\| \right) ds \\
 (4.10) \quad &= \sup_{t \in \mathbb{R}} \left\{ \left(\frac{\bar{a}}{\underline{b}} + \bar{\gamma} + \frac{\bar{p}}{e^2} \right) \|x_n - x\| \right\} \\
 &= \frac{1}{\underline{\gamma}} \left(\frac{\bar{a}}{\underline{b}} + \bar{\gamma} + \frac{\bar{p}}{e^2} \right) \|x_n - x\|.
 \end{aligned}$$

However, since $\|x_n - x\| \rightarrow 0$ as $n \rightarrow \infty$, then it follows from (4.10) that

$$\|Ax_n - Ax\| \rightarrow 0 \text{ as } n \rightarrow +\infty.$$

which means that the operator A is continuous. Finally, we show that $A\Omega$ is relatively compact. For all $w \in \Omega$, we have

$$\begin{aligned}
 |(Ax)(t)| &= \left| \int_{-\infty}^t e^{-\int_s^t \gamma(u)du} \left[-\frac{a(s)x(s)}{b(s) + x(s)} + \gamma(s)x(s) \right. \right. \\
 &\quad \left. \left. + p(s)x(s - \tau(s))e^{-\beta(s)x(s-\tau(s))} \right] ds \right| \\
 &\leq \int_{-\infty}^t e^{-\int_s^t \gamma(u)du} \left[\frac{a(s)x(s)}{b(s) + x(s)} + \gamma(s)x(s) \right. \\
 &\quad \left. + p(s)x(s - \tau(s))e^{-\beta(s)x(s-\tau(s))} \right] ds \\
 &\leq \int_{-\infty}^t e^{-\int_s^t \underline{\gamma} du} \left[\frac{\bar{a}x(s)}{\underline{b} + x(s)} + \bar{\gamma}L_2 + \bar{p}x(s - \tau(s))e^{-\underline{\beta}x(s-\tau(s))} \right] ds \\
 (4.11) \quad &\leq \int_{-\infty}^t e^{-\int_s^t \underline{\gamma} du} \left(\frac{\bar{a}L_2}{\underline{b} + L_2} + \bar{\gamma}L_2 + \bar{p}\frac{1}{\underline{\beta}e} \right) ds \\
 &= \left(\frac{\bar{a}L_2}{\underline{b} + L_2} + \bar{\gamma}L_2 + \bar{p}\frac{1}{\underline{\beta}e} \right) \int_{-\infty}^t e^{-\underline{\gamma}(t-s)ds} \\
 &= \frac{1}{\underline{\gamma}} \left(\frac{\bar{a}L_2}{\underline{b} + L_2} + \bar{\gamma}L_2 + \bar{p}\frac{1}{\underline{\beta}e} \right),
 \end{aligned}$$

which implies that $A\Omega \rightarrow \Omega$ is uniformly bounded. By calculating the derivative of operator A , we get

$$\frac{d}{dt}(Ax)(t) = -\gamma(t)(Ax)(t) - \frac{a(t)x(t)}{b(t) + x(t)} + \gamma(t)x(t) + p(t)x(t - \tau(t))e^{-\beta(t)x(t-\tau(t))}.$$

Hence we have

$$\begin{aligned}
 \left| \frac{d}{dt}(Ax)(t) \right| &= \left| -\gamma(t)(Ax)(t) - \frac{a(t)x(t)}{b(t) + x(t)} + \gamma(t)x(t) \right. \\
 &\quad \left. + p(t)x(t - \tau(t))e^{-\beta(t)x(t-\tau(t))} \right| \\
 &\leq \gamma(t)|(Ax)(t)| + \frac{a(t)x(t)}{b(t) + x(t)} + \gamma(t)x(t) + p(t)x(t - \tau(t))e^{-\beta(t)x(t-\tau(t))} \\
 &\leq \bar{\gamma}|(Ax)(t)| + \frac{\bar{a}x(t)}{\underline{b} + x(t)} + \bar{\gamma}x(t) + \bar{p}x(t - \tau(t))e^{-\underline{\beta}x(t-\tau(t))} \\
 &\leq \bar{\gamma}\frac{1}{\underline{\gamma}} \left(\frac{\bar{a}L_2}{\underline{b} + L_2} + \bar{\gamma}L_2 + \bar{p}\frac{1}{\underline{\beta}e} \right) + \frac{\bar{a}L_2}{\underline{b} + L_2} + \bar{\gamma}L_2 + \bar{p}\frac{1}{\underline{\beta}e} \\
 &= \left(1 + \frac{\bar{\gamma}}{\underline{\gamma}} \right) \left(\frac{\bar{a}L_2}{\underline{b} + L_2} + \bar{\gamma}L_2 + \bar{p}\frac{1}{\underline{\beta}e} \right),
 \end{aligned}$$

which implies that $A : \Omega \rightarrow \Omega$ is equicontinuous. Since $A : \Omega \rightarrow \Omega$ is uniformly bounded and equicontinuous, by the Ascoli-Arzelà theorem, we conclude that $A\Omega$ is relatively compact.

Thus, by Schauder's fixed point theorem, the operator A has at least one fixed point in Ω . This means that equation (1.3) has at least one almost periodic

positive solution $w^*(t)$ satisfying $L_1 \leq w^*(t) \leq L_2$. The proof of Theorem 4 is complete. $\square$

5. Exponential stability

To prove the main result of this section, we make the following assumptions:

$$(C_4) \quad L_2 > H,$$

$$(C_5) \quad \bar{p} < \frac{ab}{(\bar{b} + L_2)^2}.$$

$$\text{Let } U_{L_2} = \{\phi | \phi \in BC([-\bar{\tau}, 0], \mathbb{R}^*), < \phi(t) < L_2, t \in [-\bar{\tau}, 0]\}.$$

Theorem 5.1. *Assume that the conditions (C_1) , (C_3) , (C_4) and (C_5) hold. Then, every solution $x(t)$ of equation (1.3) with initial function $\phi \in U_{L_2}$ converges exponentially to $w^*(t)$ as $t \rightarrow +\infty$, where $w^*(t)$ is the almost periodic positive solution of equation (1.3) satisfying $L_1 \leq w^*(t) \leq L_2$.*

Proof. By Theorem 4 we know equation (1.3) has an almost periodic positive solution $w^*(t)$, and $L_1 \leq w^*(t) \leq L_2$. Assume the initial function of the almost periodic positive solution $w^*(t)$ is $w^*(t) = \psi(t) > 0$ for $-\bar{\tau} \leq t \leq 0$. Suppose $x(t)$ is arbitrary solution of equation (1.3) with initial function $\phi \in U_{L_2}$, here $0 < \phi(t) < L_2$ and $x(t) = \phi(t)$ for $-\bar{\tau} \leq t \leq 0$. By Theorem 3.2 we know $0 < x(t) < L_2$ for all $t > 0$.

$$\text{Consider function } G(x) = x - \frac{ab}{(\bar{b} + L_2)^2} + \bar{p}e^{\bar{\tau}x}, \quad x \in [0, 1].$$

Since $G(0) = x - \frac{ab}{(\bar{b} + L_2)^2} + \bar{p} < 0$, then there exists a constant $\lambda \in (0, 1)$ such that $G(\lambda) < 0$. That is

$$(5.1) \quad \lambda - \frac{ab}{(\bar{b} + L_2)^2} + \bar{p}e^{\lambda\bar{\tau}} < 0.$$

We define $V(t) = |x(t) - w^*(t)|e^{\lambda t}$, then it follows that

$$\begin{aligned} D^*V(t) &\leq [-a(t)|\frac{x(t)}{b(t) + x(t)} - \frac{w^*(t)}{b(t) + w^*(t)}| \\ &\quad + |p(t)x(t - \tau(t))e^{-\beta(t)x(t-\tau(t))} - p(t)w^*(t - \tau(t))e^{-\beta(t)w^*(t-\tau(t))}|]e^{\lambda t} \\ (5.2) \quad &+ \lambda|x(t) - w^*(t)|e^{\lambda t} \\ &= [-a(t)b(t)\frac{|x(t) - w^*(t)|}{(b(t) + x(t))(b(t) + w^*(t))} \\ &\quad + p(t)|x(t - \tau(t))e^{-\beta(t)x(t-\tau(t))} - w^*(t - \tau(t))e^{-\beta(t)w^*(t-\tau(t))}|]e^{\lambda t} \\ &\quad + \lambda|x(t) - w^*(t)|e^{\lambda t} \\ &\leq [-ab\frac{|x(t) - w^*(t)|}{(\bar{b} + L_2)^2}] + \bar{p}|x(t - \tau(t))e^{-\beta(t)x(t-\tau(t))} \\ &\quad - w^*(t - \tau(t))e^{-\beta(t)w^*(t-\tau(t))}| \\ &\quad + \lambda|x(t) - w^*(t)|e^{\lambda t}. \end{aligned}$$

Using the inequality $|xe^{-x} - ye^{-y}| \leq |x - y|$, for $x > 0, y > 0$, we get

$$\begin{aligned}
 & |x(t - \tau(t))|e^{-\beta(t)x(t - \tau(t))} - w^*(t - \tau(t))e^{-\beta(t)w^*(t - \tau(t))}| \\
 (5.3) \quad &= \frac{1}{\beta(t)} |\beta(t)x(t - \tau(t))e^{-\beta(t)x(t - \tau(t))} - \beta(t)w^*(t - \tau(t))e^{-\beta(t)w^*(t - \tau(t))}| \\
 &\leq \frac{1}{\beta(t)} |\beta(t)x(t - \tau(t)) - \beta(t)w^*(t - \tau(t))| \\
 &= |x(t - \tau(t)) - w^*(t - \tau(t))|.
 \end{aligned}$$

Hence, (5.2) and (5.3) imply that

$$\begin{aligned}
 D^+V(t) &\leq [-\frac{ab}{(\bar{b} + L_2)^2} |x(t) - w^*(t)| + \bar{p}|x(t - \tau(t)) \\
 &\quad - w^*(t - \tau(t))|]e^{\lambda t} + \lambda|x(t) - w^*(t)|e^{\lambda t} \\
 (5.4) \quad &= \lambda V(t) - \frac{ab}{(\bar{b} + L_2)^2} V(t) + \bar{p}|x(t - \tau(t)) - w^*(t - \tau(t))|e^{\lambda t}.
 \end{aligned}$$

Let $h = L_2 + \sup_{-\bar{\tau} \leq t \leq 0} |\phi(t) - \psi(t)|$. For all $t \in [-\bar{\tau}, 0]$, we get

$$\begin{aligned}
 V(t) &= |x(t) - w^*(t)|e^{\lambda t} \leq |x(t) - w^*(t)| = |\phi(t) - \psi(t)| \\
 &\leq \sup_{-\bar{\tau} \leq t \leq 0} |\phi(t) - \psi(t)| < L_2 + \sup_{-\bar{\tau} \leq t \leq 0} |\phi(t) - \psi(t)| = h.
 \end{aligned}$$

We claim that

$$(5.5) \quad V(t) < h, \text{ for all } t > 0.$$

Suppose the claim (5.5) is not true, then there must exist a $t^* > 0$, such that $V(t^*) = h$, $D^+V(t)|_{t=t^*} \geq 0$ and $V(t) < h$ for $t < t^*$. It follows from (5.4) that

$$\begin{aligned}
 0 &\leq D^+V(t)|_{t=t^*} \leq \lambda V(t^*) - \frac{ab}{(\bar{b} + L_2)^2} V(t^*) \\
 &\quad + \bar{p}|x(t^* - \tau(t^*)) - w^*(t^* - \tau(t^*))|e^{\lambda t^*} \\
 &= \lambda h - \frac{ab}{(\bar{b} + L_2)^2} h + \bar{p}|x(t^* - \tau(t^*)) - w^*(t^* - \tau(t^*))|e^{\lambda(t^* - \tau(t^*))}e^{\lambda \tau(t^*)} \\
 (5.6) \quad &= \lambda h - \frac{ab}{(\bar{b} + L_2)^2} h + \bar{p}V(t^* - \tau(t^*))e^{\lambda \tau(t^*)} \\
 &< \lambda h - \frac{\bar{a}\bar{b}}{(\bar{b} + L_2)^2} h + \bar{p}he^{\lambda \tau(t^*)} \\
 &\leq \lambda h - \frac{ab}{(\bar{b} + L_2)^2} h + \bar{p}he^{\lambda \bar{\tau}} \\
 &= (\lambda - \frac{ab}{(\bar{b} + L_2)^2} + \bar{p}e^{\lambda \bar{\tau}}).
 \end{aligned}$$

Thus, from (5.6) we have $\lambda - \frac{ab}{(b+L_2)^2} + \bar{p}e^{\lambda\bar{\tau}} > 0$, which contradicts (5.1).

Therefore, the claim (5.5) is true. Hence $V(t) = |x(t) - w^*(t)|e^{\lambda t} < h$, for all $t > 0$.

That is $|x(t) - w^*(t)|$, for all $t > 0$, which means $x(t)$ converges exponentially to $w^*(t)$ as $t \rightarrow +\infty$. The proof of Theorem 5 is complete. $\square$

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PERSONAL CREDIT SCORING MODEL RESEARCHBASED ON THE RF-GA-SVM MODEL

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Abstract. The importance measure of variables in the random forests algorithm is used to carry out a rank ordering to the importance of variables, so as to extract feature attributes on this basis. The feature attributes are regarded as inputs to conduct parameter optimization in order to support vector machine (SVM) model by using the genetic algorithm, building the classifier model by selecting the parameter with the highest accuracy of 5-fold cross-validation. The classifier model is utilized for empirical research, and the results show that the classifier is better than random forest classifier and support vector machine classifier in its higher classification accuracy.

Keywords: Genetic algorithm, random forest, support vector machine, data mining, credit scoring.

1. Introduction

The widespread use of credit cards has brought huge profits for credit card issuers, and also has brought huge risks. Through the personal measurement and control of credit risks, effectively avoiding risks and accurately dealing with the relationship between the benefits and risks is the key to success. Therefore, the measurement and control of personal credit risks has been always the important research topic in the development of personal banking business.

Along with the theory and technology of risk control of credit cards, especially with the rapid development of computer technology, more and more measures, such as statistics, operational research and other quantitative analysis tools, are applied to the field of credit scoring. Statistical methods mainly include linear regression, judgment analysis and Logistic regression. Operational research methods mainly include linear programming. Credit scoring model generally combines one or several of these methods for use. In recent years, some of the non-parametric statistical methods and artificial intelligence are gradually applied in the field of credit scoring [1-5], such as decision tree, neural network, genetic algorithm and support vector machine (SVM), etc. Currently, the common decision tree classification algorithms are divided into CLS algorithm,

C4.5 algorithm and CART algorithm as well as SPRINT algorithm, etc, and the common neural network algorithm is BP neural network algorithm.

However, although the decision tree algorithm possesses higher execution efficiency, the order of the attributes in the tree node is easy to influence performance. And even for small training sets, the decision tree also may be quite large, which affects the understanding of the decision tree. Neural network algorithm is easy to cause excessive fitting by excessive learning, so as to affect the prediction precision in practical application.

As a trainable machine learning method, SVM has been widely used in the field of data mining. In essence, SVM avoids the traditional processes from induction to deduction, efficiently implementing the transduction reasoning from training sample to forecast sample which, greatly simplifies the usual classification and regression problems. Final decision function of SVM is only determined by a few support vectors, which not only can help us to grasp the key sample, “eliminating” a large number of redundant samples, and the method is not only doomed to have simple algorithm, but also has good “robustness”. But due to that the SVM generally can not simplify the space dimensions of input vectors, it needs a quadratic programming to solve the support vectors. Hence, when the sample size is large, the input variables will largely cost a lot of machine memory and computing time. In order to improve the operation efficiency, this paper firstly conducts the sample space dimensionality reduction by use of the principle of attribute importance ordering in the random forests, making full use of the flexibly nonlinear modeling capability of SVM on this basis, and the superior global optimization search ability of GA (Genetic Algorithm), establishing k-fold cross-validation optimal SVM model, so as to realize the recognition for “good customer” and “bad” customer.

2. Relevant theory of RF-GA-SVM model

Genetic Algorithm

Genetic Algorithm (GA) is a calculation model simulating natural selection of Darwin biological evolution theory and genetic mechanism of the biological evolution process, which is a kind of method in support of searching optimal solutions by simulating the natural evolution process. Since 1975, it has been widely used in the field of artificial intelligence. Basic operation process of genetic algorithm randomly generates individuals as initial group, and then calculates the fitness of each individual in the community, and chooses excellent individual to propagate directly to the next generation or the intersection of other individuals generates new individuals for propagating again to the next generation according to the fitness, producing the next generation of group in the genetic process through selection, crossover and mutation. If the presupposed termination conditions have been met at that moment, the calculation should be stopped to output the optimal solution; otherwise the calculation will continue to choose, cross, mutate until the termination conditions are achieved.

Random Forest

When using data mining technology to carry out personal credit scoring, there is a vast amount of information. But not all of the information has larger influence on personal credit scoring. When SVM model is set up, many independent variable inputs obviously will affect the operational efficiency of the model. Hence, it is necessary to conduct knowledge reduction firstly, and that can be realized by making use of the gain rate ordering of various attributes in the random forest algorithm. Random forest (RF) is a kind of integrated machine learning method, which employs randomly resampling technology of bootstrap and node randomly splitting technology to build decision trees, so as to obtain the final classification result by the way of voting. RF possesses the ability of analyzing the classified characteristics of the complex interaction, which has good robustness for the noise data and data that exist missing values, and have faster learning speed. The measure on the variable importance can be used as a feature selection tool for high-dimensional data, which has been widely applied in all kinds of classification and prediction, feature selection and anomaly detection problems in recent years [6-9].

Feature selection algorithm can be divided into two major categories of the Filter and Wrapper [10] based on the characteristics of evaluation strategy that has adopted. Filter method is independent of the follow-up machine learning algorithms, which can quickly eliminate some of the non-key noise characteristics, reducing the searching range of optimal feature subsets. However, it can not guarantee selecting out a smaller optimal feature subset. In the process of screening characteristics, wrapper method directly uses the selected feature subsets to train a classifier, evaluating the pros and cons of the feature subset according to the performance of the classifier in the test sets. Filter method is better than the method in computational efficiency, but the size of the selected optimal feature subsets is relatively larger.

Based on taking random forest algorithm as the basic tool, the article researches Wrapper feature selection method by using the classification accuracy of random forest classifiers as feature separability criterion, utilizing the variable importance measure to conduct characteristic importance sorting based on the random forest algorithm itself, adopting the sequence backward selection method generalized sequence backward selection method to select featured subsets.

Support Vector Machine (SVM)

Support vector machine (SVM) is a classification technique based on statistical learning theory proposed in 1995 by AT&T Bell laboratories research team led by Vapnik, which put forward the duality theory in the traditional optimization problem, mainly including the maximum and minimum duality and Lagrangian duality. The key to the SVM lies in the kernel function. Vector sets in low dimensional space are usually difficult to divide, and the solution is

to map them to the high dimensional space. But the difficulty the method has brought is the increase of the computational complexity, and kernel function just ingeniously solves the problem. That is to say, as long as choosing the appropriate kernel function, the classification function in the higher dimensional space can be calculated. General common kernel functions include linear kernel function, polynomial kernel function, the RBF kernel function and Sigmoid kernel function [11-12]. Linear kernel function is a special case of kernel functions, which is used to find the linear classifier with optimal generalization. As a global kernel function, polynomial kernel function needs a large amount of calculations. Gaussian radial basis kernel function is the most widely used kernel function with powerful locality and very good learning ability, regardless of the sample sizes and high or low dimensions. When sigmoid kernel function is applied to SVM, it will build up the multi-layer perceptron neural networks to achieve the global optimization through learning. However but when applied to the classification, the conditions are quite harsh.

3. Design of RF-GA-SVM Algorithm

Step 1 the data sets shall be randomly arranged, and are divided into five parts. In order to guarantee the stability of the experimental results, this paper uses the 5-fold cross-validation methods. In each iteration, the four pieces of data are regarded as the training sets in support of building the random forest classifier, and the remaining one piece of data is regarded as the validation set data for validation.

Step 2 set the maximum classification accuracy $TGMaxAcc=0$ in validation sets, and the corresponding characteristic collection $FGSort$ is empty sets. Suppose the number of input attributes is $n; i1$.

Step 3: initialize the average classification accuracy $TLMeanAcc = 0$ of 5-fold cross validation, and the classification accuracy $TLAcc15 = 0$ in each iteration of cross validation. Run Random to create classifiers on data sets, and classify on the test sets. Compare the classification and the observed values and then calculate $TLAcc$.

Step 4: $TLMeanAcc = TLMeanAcc + TLAcc[i]/5$.

Step 5: if $TLMaxAcc \leq TLAcc[i]$, then $TLMaxAcc = TLAcc[i]$.

Step 6: measure the importance of each attribute according to the computation formula of gain rates of calculation information in the decision tree C4.5. The characteristics are arranged according to the variable importance and store this attribute set as $FSort$. If $(TGMaxAcc \leq TLMeanAcc)$, then $TGMaxAcc = TLMeanAcc, FGSort = FSort$.

Step 7: $i = i + 1$. If $i < n?1$ then execute Step 8 otherwise execute Step 9.

Step 8 take out a characteristic with the lowest score in its importance from $FSort$, and get the new data set, then perform Step 3.

Step 9 output the highest global classification accuracy $TGMaxAcc$, and the corresponding characteristics of collection $FGSort$.

Step 10 serve *FGSort* as the input data set, choose kernel function to build up SVM classification model.

Step 11 make use of the genetic algorithm c, g to determine the optimal parameters for searching the parameter space, perform 5-fold cross-validation on the training sets, build the classifier model through the selection of the parameter with the highest accuracy of cross-validation.

4. Positive analysis

Download from UCI data sets to Australia and Germanys related data of personal credit scoring, including Australia's personal credit scoring data set that contains 14 input attributes and a output attribute with a total of 690 data. There are 307 and 383 of "good credit" and "bad credit" samples respectively. German personal credit scoring data set contains 24 input attributes and a output attribute with a total of 1000 data, which contains 700 samples of "good credit", 300 samples of "bad credit".

Take x_i as the representative of the i input attributes in data set, y is used to represent the output properties of the data set. The methods in step 1 ~ step 9 are used to conduct attribute reduction on two data sets respectively, and Australia's data reduction result of credit cards is $\{x_1, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}\}$; while Germany's data reduction result of credit cards is $\{x_1, x_2, x_6, x_7, x_8, x_9, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}, x_{17}, x_{18}, x_{19}, x_{20}, x_{21}, x_{22}, x_{23}, x_{24}\}$.

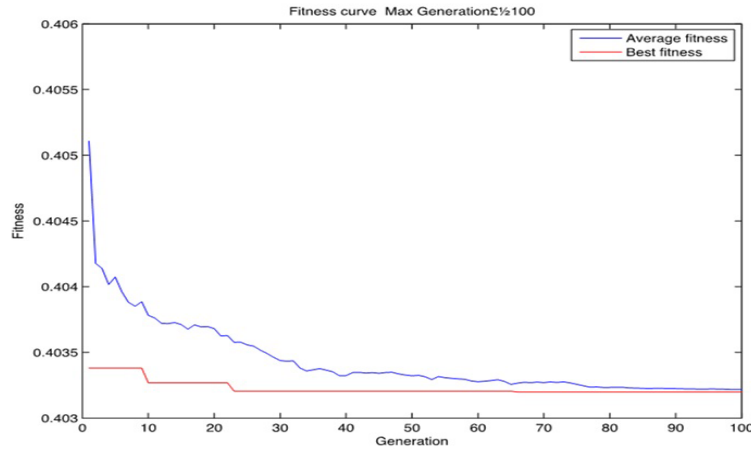


Figure 1: Fitness curve of Germany personal credit scoring data set based on RF - GA - SVM classifier

Take x_i as the representative of the i input attributes in data set, y is used to represent the output properties of the data set. The methods in step 1 ~ step 9 are used to conduct attribute reduction on two data sets respectively, and Australia's data reduction result of credit cards is $\{x_1, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10},$



Figure 2: Fitness curve of Australia personal credit scoring data set based on RF-GA-SVM classifier

Model	Classification accuracy of good customer	Classification accuracy of bad customer	Overall classification accuracy
RF	76.71%	85.88%	81.29%
SVM	44.00%	90.93%	67.41%
GA-SVM	86.88%	84.52%	84.71%
RF-GA-SVM	86.05%	86.90%	86.47%

Table 1: Classification accuracy of Australia's credit card data in all kinds of classification models

x_{11}, x_{12}, x_{13} }; while Germany's data reduction result of credit cards is $\{x_1, x_2, x_6, x_7, x_8, x_9, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}, x_{17}, x_{18}, x_{19}, x_{20}, x_{21}, x_{22}, x_{23}, x_{24}\}$.

Suppose the maximum iterative algebra in genetic algorithm is 100, the initial group population is 30, the crossover probability is 0.7 and the mutation probability is 0.1. The RBF kernel function is chosen to establish the SVM model, the Matlab programming is utilized to realize the algorithm designed in this article. The data after reduction are used to conduct 5-fold cross validation, so as to obtain the optimal parameters of the SVM classifier of the German credit cards $c = 54.8582, g = 99.6882$, and the optimal parameters of the SVM classifier of the Australian credit cards $c = 7.3611, g = 3.7405$.

Based on RF-GA-SVM classifier, Australia and Germany's credit card data are classified, and the optimal parameters that have achieved are used to test the data of validation set, which achieves satisfactory results. The results are shown in table 1 and table 2. As can be seen from the classification results, the classification accuracy rates of two data sets in the RF-GA-SVM model are very high, which are higher than single RF model, the SVM model and the GA - SVM model.

Model	Classification accuracy of good customer	Classification accuracy of bad customer	Overall classification accuracy
RF	86.66%	86.33%	86.50%
SVM	85.09%	84.85%	84.87%
GA-SVM	99.36%	85.98%	92.50%
RF-GA-SVM	100%	85.98%	92.81%

Table 2: Classification accuracy of Germany's credit card data in all kinds of classification models

5. Conclusions

This article constructs the credit scoring model based on random forest, genetic algorithm and support vector machine (SVM), and adopts the Australia and Germanys credit data in UCI data set for empirical testing, and the results show that:

(1) The use of random forests for data reduction can eliminate some noise properties, which not only reduce the SVM model input data and improve the efficiency of learning and prediction, as well as enhance the model prediction accuracy rates.

(2) The GA - the SVM model is built for the data after reduction, and this model is utilized to make a classification of bank credit card users in Australia and Germany, and the results show that the model established in this paper is efficient and precise, which possesses practical application prospect.

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OPTIMIZATION OF AGRICULTURAL PRODUCTION CONTROL BASED ON DATA PROCESSING TECHNOLOGY OF AGRICULTURAL INTERNET OF THINGS

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Abstract. Internet of things is an important part of a new generation of information technology and it is a big step towards modern informationization for modern agriculture for us to introduce and apply the Internet of Things technology in agriculture which will lead the development of intelligent agriculture. In order to optimize the agricultural production control ability, realize automatic production, optimization control and intelligent management, improve the crop yield and quality, this paper carries out a design study on the optimization of agricultural production and control on the basis of fully understanding the agricultural thing networking data processing technology.

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Firstly, a service framework was put forward for the analysis of different agricultural products. Secondly, an agricultural information forecasting model based on an output-input feedback mechanism was applied to effectively control the growth environment of crops. Finally, the service framework, forecasting system and other modules were integrated into an application platform, which was then tested and showed a good read and write rate and a precise retrieval accuracy rate, could effectively predict environmental factors and give feedbacks, realize real-time monitoring of the environmental parameters and control the agricultural production.

Keywords: Internet of things, agricultural production, data processing, prediction system.

1. Introduction

With the popularization and application of the computer network information technology, the importance of the Internet of Things [1] has become increasingly prominent. The application of the Internet of Things technology in agriculture takes its comprehensive perception, reliable transmission and intelligent processing technology as the support and means and is characterized by the ubiquity of the intellectualization of the whole industry and whole process of agriculture, with automated production, optimized control, intelligent management, systematic logistics and electronic trading as the main modes of production, which gradually changes the production mode of agriculture from the human-centered one which relies on machinery to the new information and software-centric one that applies electronic devices. GeWet al [2] held the view that Agricultural internet of things (Ag-IoT) is the highly integrated and comprehensive application of the new generation of information technology in agricultural field. He Qian [3] studied the theory and key technology of the structure and intelligent information processing of Internet of Things and put forward that data processing technology improved the efficiency of information processing and had excellent prospects for development. Paraforos D S et al [4] believed that agricultural production management is entering into a new era where every day farmers decisions are supported by highly sophisticated Farm Management Information Systems.

The application of the Internet of Things technology in agriculture is of great practical significance to crop production. The design of the data processing service structure of agricultural IOT can realize safer and more efficient data processing, and can then optimize the agricultural production control. The output-input feedback mechanism [5], monitoring information database module, monitoring data management module and forecasting expert module were applied for the establishment of the prediction model and the realization of the Ag-IoT application network system, which can predict environmental factors effectively, interact information data, send commands to the relevant equipment of the intelligent greenhouses and change the state of the control device. For example, if the soil moisture content is too low, the irrigation system will

automatically open its watering operations to ensure a good crop production environment.

2. Theoretical research on the data processing technology of agricultural IOT

2.1 Theoretical research on Ag-IoT

Internet of things is a network constructed based on the computer internet applying the RFID and wireless data communication technologies that covers everything in the world and can realize the automatic identification of goods and information interconnection and sharing without human intervention. Ag-IoT collects realtime environmental parameters through a variety of sensors and obtains the best conditions for crop growth through wireless network measurement, so as to realize remote monitoring, automatic control and management of the whole process to achieve the purpose of improving production and quality.

2.2 Theoretical research on data processing technology

Data processing technology is one of the core technologies of Internet of Things, which carries out effective transmission processing of large-scale real-time data and makes prompt feedback of the analysis results to the users so as to realize intelligent control. The difficulty of data processing is the large amount, timespace correlation and dynamic variability of the data of the Internet of Things and the heterogeneity of sampling data. At present, the main processing methods are hierarchical processing and dimension reduction processing.

3. Optimization design of agricultural production control

3.1 Design of data processing service architecture for agricultural Internet of Things

The service architecture of data processing for agricultural IOTS follows the principles of demand-orientation, integrated resources and innovative pragmatism. In this paper, taking the computer and computer network as the main equipment materials, combined with related technologies, such as agricultural forecasting and early warning technology, agricultural intelligent decision-making and control technology and agricultural inference technology [6, 7], a data processing service structure for agricultural IOT based on agricultural production control process was proposed, as shown in figure 1. As shown in figure 1, the process of the data classification processing business of different agricultural products is analyzed and the users service project is decomposed and grouped using the corresponding subservices and the obtained data results are finally returned to the user. Its content is close to the application and promotion of the Internet of Things technology, including agricultural production information, agricultural ecological environment information, agricultural market informa-

tion, agricultural resources information, which provides multiple information resources for the information service object.

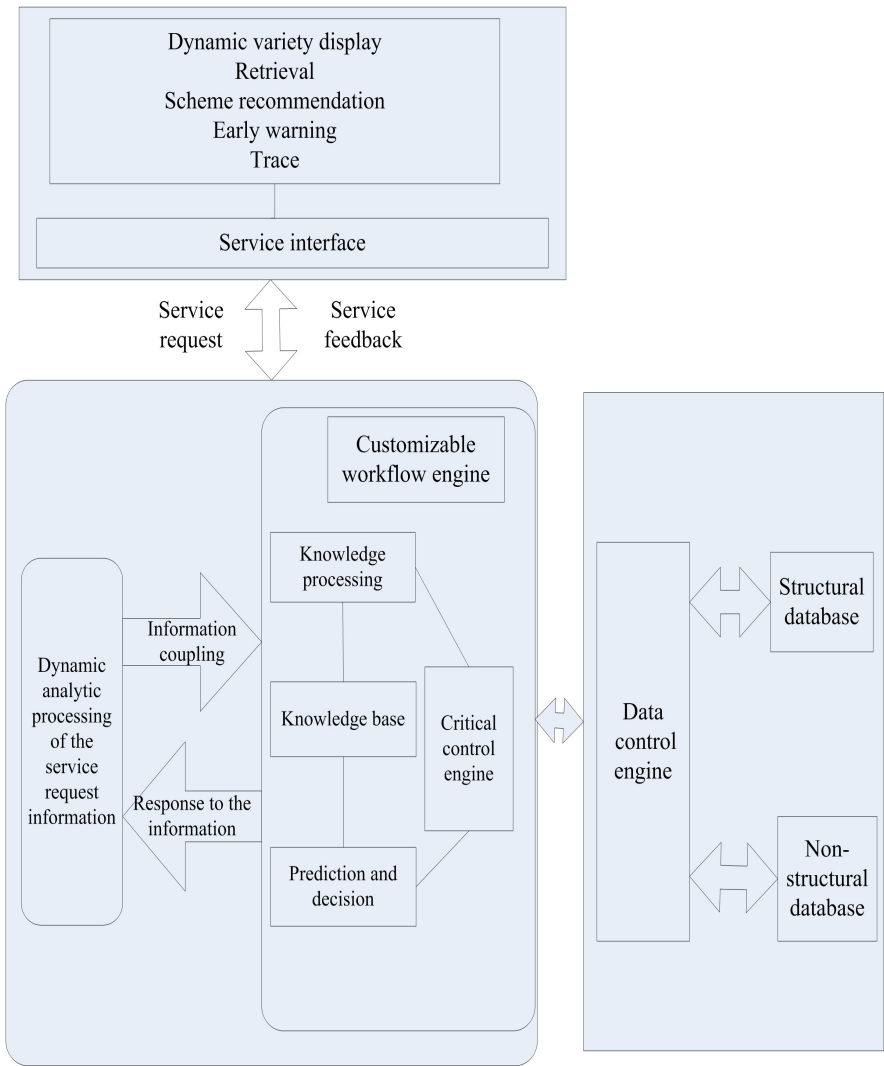


Figure 1: Data processing service architecture of agricultural IoT

3.2 Design of the agricultural information forecasting system

Agricultural information prediction system is mainly used to avoid adverse weather, pests and other environmental factors, with two function s of monitoring and early warning. Based on the output - input feedback mechanism of agricultural information forecasting system, the Elman neural network model was improved. Based on the feed-forward network [8], an additional layer is

added to the middle layer of the feed-forward network to make the system change in real time. It has storage and memory functions. Generally, there are four layers: input layer, intermediate layer or hidden layer, continuing layer and output layer. In the design, the feedback of the output layer and the period before the output layer is increased by using the computer networks, and the simulation is carried out by the MATLAB software [9]. Combining with the traditional dynamic theory, statistical theory and mathematical calculation technology as well as integrating the distributed, parallel-style features, the prediction system simulates human nervous system, enhances the sensitivity of the system to historical data, and thus improves the accuracy of prediction, effectively controls the growth environment of crops and improves crop yield and quality.

3.3 Design of application platform of agricultural Internet of Things

The construction of the application platform of agricultural IOT consists of service architecture, forecasting system and other functional modules mentioned above. It mainly includes six major applications: item information, system management, warehouse management, remote automatic control, real-time video monitoring and environmental monitoring. The real-time monitoring of environmental parameters is realized through installing wireless lighting, temperature and humidity sensors in intelligent greenhouses. Data acquisition, analysis, archiving and storage of the sensors are completed by the data software. The data is transmitted to the gateway through the ZigBee protocol [10]. The gateway then sends the sensor data to the data platform server via the data routing node or GPRS directly. The user can access the data platform to monitor real-time intelligent greenhouses onsite environmental parameters and control the field device through the integrated controller. The core of the platform is

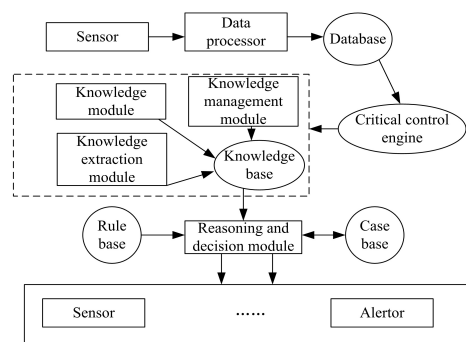


Figure 2: Structure chart of the agricultural knowledge processing and decision - making model

knowledge processing and decision-making, as shown in Figure 2, including data acquisition, knowledge processing, reasoning decision and practical application. Aiming at the key control points of the HACCP system and the GAP system,

classes, frameworks and production rules are defined as knowledge representation and storage strategy. Stereotyped keywords and quantitative keywords are used to express knowledge representation and storage. The K-Neighbor algorithm is used to calculate the similarity in the knowledge entry module to ensure the matching degree between the modules.

4. Test on the results of the optimization of agricultural production control

4.1 Test on the results of data processing service architecture of agricultural Internet of Things

In order to ensure the practical operability of the design results, the performance of the data processing service architecture for agricultural IOT is tested. The real-time monitoring test system data transmission stream is set to continue for 1000 minutes in order to test its accuracy of reading and writing and retrieval rate [11]. As shown in Fig. 3 and Fig. 4, the precision of the search decreases as the data transmission time increases. When the transmission time is about 500 min, the read / write time per second is the highest, and the data processing system has reliability. A reasonable service architecture ensures the read and write rate and retrieval accuracy rate, and the application of workflow engine can improve the versatility and reuse rate of the system.

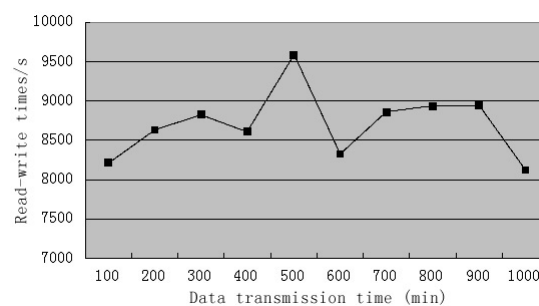


Figure 3: Read-write ability of the data processing system

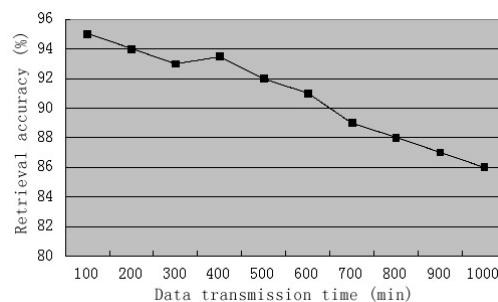


Figure 4: Retrieval accuracy of the data processing system

4.2 Test on the results of agricultural information prediction system

Taking the pest and disease prediction in agricultural production control as an example, the forecasting accuracy of the agricultural information forecasting system is tested. In this experiment, the related factors affecting the activity of rice pests were selected and the related data were collected. The six-month data from 2013 to 2015 were processed by Premnmx in the Matlab software.

Year	Month	Average temperature	Minimum temperature	Rainfall capacity	Illumination time	Degree of insect attack
2013	5	-0.4727	-0.6234	-0.5623	0.7823	Level 1
	6	0.8912	0.2315	-0.4865	0.89	Level 2
	7	0.5892	0.0000	0.0553	0.4582	Level 4
	8	0.6859	0.5968	-0.2969	0.2654	Level 4
	9	-0.0381	-0.1239	-0.3618	0.1688	Level 4
	10	-0.8265	-1.0056	-0.6214	-0.3236	Level 1
2014	5	0.1123	-0.2358	-0.5019	0.5599	Level 1
	6	0.2136	-0.0583	-0.5168	0.2458	Level 2
	7	0.5890	0.2368	0.3257	-0.1258	Level 2
	8	0.9812	0.5415	-0.0025	-0.0589	Level 4
	9	-0.0025	-0.1023	-0.5689	-0.4136	Level 3
	10	-0.6581	-0.8924	-0.1384	-0.1580	Level 3
2015	5	0.0059	-0.0036	-0.0856	-0.0985	Level 1
	6	0.2584	0.1254	0.9926	-0.5469	Level 2
	7	0.2154	0.0026	0.5812	-0.1528	Level 2
	8	-0.1235	-0.2588	-0.2589	0.5833	Level 4
	9	0.2589	-0.0254	-0.8521	-0.0539	Level 1
	10	-0.2158	-0.4586	-0.3265	-0.1247	Level 1

Table 1: Factors influencing the activities of pests in rice during the six month from 2013 to 2015

As shown in table 1, the data from 2013 to 2014 are taken as learning samples and the data in 2015 are taken as prediction samples. In the prediction system (using the IOIF-Elman neural network prediction model), there are 9 neurons in the middle layer and the S-type tangent function is taken as the transfer function; there are 4 neurons in the output layer and the S-type logarithm is taken as the transfer function. A network is built using the same data samples and simulation and comparison are made between the agricultural information prediction system, the Elman neural network and the BP network. Based on the data in table 2, the precision formula

$$p = 1 - \frac{\sum_{i=1}^n |y(x_i) - \hat{y}(x_i)|}{\sum_{i=1}^n y(x_i)}$$

is used to calculate the data results of the three network models. The results found that the prediction precision of agricultural information forecasting system is obviously higher than that of the other two systems, which shows that it is more powerful, stable and predictive to deal with dynamic information, and can forecast environmental factors effectively, and make feedback to remind users to adjust accordingly so as to improve the control on agricultural production.

4.3 Test on the results of the agricultural Internet of Things application platform

After the user enters the agricultural Internet of Things application platform and clicks the corresponding green house number, he or she can enter the green

Month	Degree of the actual plant disease and pests	BP prediction value	Elman prediction value	Prediction value of the agricultural prediction system	BP prediction precision	Elman prediction precision	Prediction precision of the agricultural prediction system
5	0	0.0000	0.0000	0.0000	0.9855	0.9714	0.9985
	0	0.0000	0.0206	0.0015			
	0	0.0145	0.0008	0.0000			
	1	1.0000	1.0000	1.0000			
6	0	0.0002	0.0012	0.0001	0.0023	0.6826	0.9362
	0	0.0000	0.1256	0.0035			
	1	0.9998	0.9986	0.9998			
	0	0.9957	0.1893	0.0600			
7	0	0.0000	0.0025	0.0000	0.2323	0.8953	0.9235
	0	0.0053	0.0045	0.0697			
	1	0.9997	0.9954	0.9932			
	0	0.7621	0.0931	0.0000			
8	1	0.9988	0.9888	0.9998	0.9923	0.9985	0.8532
	0	0.0000	0.0097	0.0156			
	0	0.0065	0.0000	0.1059			
	0	0.0000	0.0000	0.0251			
9	0	0.0000	0.1022	0.1084	0.6235	0.7265	0.7123
	0	0.01245	0.0244	0.0000			
	0	0.1240	0.0005	0.0412			
	1	0.8720	0.8536	0.8619			
10	0	0.0587	0.0000	0.0049	0.9236	0.9500	0.9887
	0	0.0000	0.0495	0.0059			
	0	0.0175	0.0001	0.0002			
	1	0.9998	0.9996	0.9997			
Prediction precision					0.62658	0.87072	0.90307

Table 2: simulation comparison between the agricultural prediction system, the Elman neural network and the BP network

house and observe the data. Besides, by clicking the view historical data item, the user can observe the change trends of the temperature, humidity, light, carbon dioxide concentration at different time points on various dates, which are presented in an Excel sheet by the program. As shown in figure 5, by clicking the the third green house item, the user can view the historical data of the green house. And by selecting the date of October 15, 2015, the change trend of the average temperature at each time point on that day can be viewed. Similarly, the data of the other sensors can be viewed. After testing, the application platform can monitor the environmental parameters in the greenhouses in real time, and has strong feasibility and practicability.

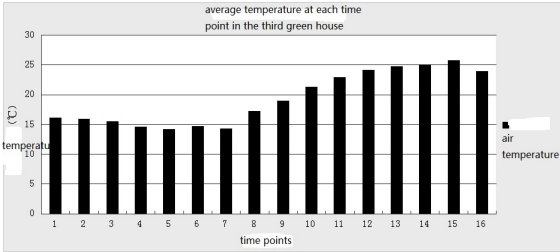


Figure 5: Average temperature at each time point in the third green house

5. Discussion

Due to the application of agricultural IoT technology, a large amount of data is generated in the process of agricultural production. If not taken advantage of, these data will go to waste. In order to solve the agricultural water and land resource use rate problem, domestic scholars F Li et al [12] designed a farmland intelligent irrigation management system and achieved the precise irrigation of farmland through the computer integrated management. Fuping et al [13] through the GPRS and ZigBee technology designed an intelligent irrigation monitoring system, which solved the problem of low utilization rate of agricultural water resources and low efficiency of land management in China.

In this design, the sensor data collected are transmitted to the application platform after data processing. When the data are obviously different from the preset values, the alarm will be triggered, which will make the prediction system send feedbacks of the information to the user and start the relevant equipment through remote control so as to ensure that the environmental factors are controlled within standard. RFID technology [14] will record the crop data at all stages, which is a combination of radio frequency technology and embedded technology that can trace the whole process of crop production. Nevertheless, the design has its own deficiencies. For example, many types of data storage management have not been covered; the initial programming can only complete the key functions of the platform, and many features need to be improved; the algorithm applied is simple and its cost, versatility, flexibility also need to be further improved.

6. Conclusions

In summary, through the design of the service architecture, forecasting system and application platform as well as the result test, it is found that the application platform of agricultural IOT can realize long-range monitoring, automatic control and intelligent management of agricultural products, improve production efficiency, save investment capital, further optimize agricultural production control, build intelligent agriculture, and produce enormous economic and social benefits.

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THE APPLICATION OF INFORMATION ENTROPY THEORY BASED DATA CLASSIFICATION ALGORITHM IN THE SELECTION OF TALENTS IN HOTELS

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Abstract. Background: With the rapid development of the society, some excellent enterprises are having a growing demand for talents. The recruitment and selection of talents is always concerned by the managers of enterprises. Materials and methods: This study analyzed the human resource of hotels and introduced C4.5 algorithm in decision tree algorithm and its implementation procedures. Talents were selected by investigating data of relevant ability of them and performing data analysis and data mining using C4.5 algorithm. Objective: Information entropy based data classi-

cation C4.5 algorithm was used to screen hotel talents. Results: Decision tree was obtained by calculating the comprehensive ability, working experience, random response capability and psychological quality of the enrolled employees. During the talents selection in hotels, the C4.5 algorithm can make the process simple and convenient. Decision tree algorithm can be used in the selection of talents in enterprises to preprocess data, construct data mining model of talent selection, and solve problems appearing in the recruitment and selection of talents.

Keywords: Data mining, decision tree, information entropy, human resource, C4.5 algorithm, information entropy theory, competition, data analysis, talents selection.

Significance statement

This study investigated the application of C4.5 in the talents selection in hotels. This study investigated decision tree classification algorithm, aiming to promote the application of data mining technology in the hotel industry. The paper has no conflicts of interest.

1. Introduction

In recent years, the market competition in hotel industry has showed a tendency of information high-technicalization and globalization. Among all resources, human resource has become one of the most important competition factors. Hotel talents investigation and selection using network algorithms is more competitive. Decision tree algorithm has been extensively applied in the fields such as machine learning, knowledge discovery and human resource analysis [1]. Data Mining (DM) as a kind of data analysis method and technique of finding potential information among massive data has become a hot spot in all circles [2,

3]. Chen J. [4] discussed the shortcomings of ID3 decision tree and proposed an improved algorithm integrating ID3 and association function which effectively made up the defects of ID3 decision tree and obtained more reasonable and effective rules. Wang Y. H. [5] applied Gini coefficient decision tree algorithm in human resource management system, constructed the decision trees by taking working tasks, working quality and working attitude as the decision attributes, established the enterprise performance evaluation model, and verified its feasibility. Bai W. J. [6] et al. investigated the recruitment and employment of university students, established human resource information database for screening professional talents through k-mean algorithm, and made classification and prediction on the employment data of previous graduates using C 4.5 algorithm.

In decision tree algorithm, the complexity and classification accuracy of decision tree are the two most important factors for consideration. The commonly used evaluation indexes include prediction accuracy, i.e., describing classification models and predicting new or unknown data class accurately, description conciseness (the more concise the description of model is, the easier to understand), computational complexity, and the robustness of model and processing scale [7]. This study analyzed data mining for the investigation of hotel talents by using C4.5 algorithm on the basis of information entropy theory and proposed to rapidly select and reasonably assign excellent talents using informatization.

2. Analysis of hotel human resource

Human resource as the most important core competition factor in hotel development and management has significant influence on the competition between hotels. The key for hotels with service nature to become the top-ranking star hotels lies on the working enthusiasm of staffs [8, 9]. The intensive competition between hotels, essentially, is the competition of talents. Rich human resource can greatly promote the development of hotels; however, many problems remain to be solved in the selection and management of talents, for example, how to attract and select servant talents who are needed by hotels, systematically screen talents, and arrange optimal positions for the existing staffs to realize the values of people and create more benefits for hotels. During the allocation of human resource, the screening and allocation of talents based on the reasonable use of information technology algorithm includes two parts [10]. The first part is the initial allocation of human resource, i.e., making a rough determination on the interviewees according to the provided data and rough impression in interview. The second part is the reallocation of human resource, i.e., investigating and selecting excellent talents using informatization algorithm and arranging staffs to suitable positions after assessment.

3. C4.5 Decision Tree Algorithm

3.1 The overview of decision tree

Decision tree is a process of classifying data through a series of rules, which is a tree structure that can classify data automatically and is the knowledge

representation of tree structure; it can be converted into decision rules directly [8-10]. Decision tree learning means comparing attribute values in the internal node adopting the top-down recursive method, determining the branches below the node according to different attribute values, and coming to conclusions in leaf nodes of a decision tree. The construction process of decision tree is shown in figure 1. As to data classification, decision tree is useful as it can construct a tree for modeling for classification problem and the tree can effectively classify problems and reach classification results. Therefore, two steps are needed in the classification using decision tree, i.e., establishing a classification model and applying data sets into the model for classification.

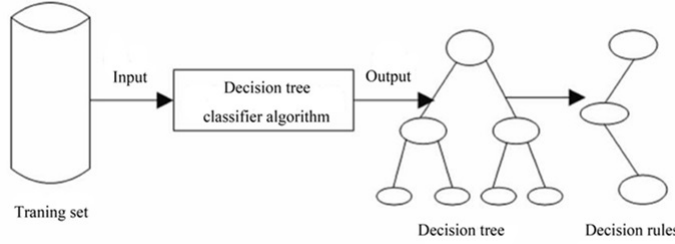


Figure 1: The construction process of decision tree

3.2 C4.5 Algorithm

The construction process of decision tree is a process of segmenting data uninterruptedly, and each segment is corresponding to a node. The C4.5 algorithm forms a decision tree recursively based on information entropy method.

(1) Definition of entropy measurement

Entropy is a metric that is widely used in information theory. Entropy with respect to Boolean classification is defined as:

$$(3.1) \quad Entropy(S) \equiv -P \log 2P - P \log 2P[1]$$

where P stands for the ratio of positive example in S and $-P$ stands for the ratio of counterexample in S . If the target attribute has c different values, then the entropy is defined as:

$$(3.2) \quad Entropy(S) \equiv \sum_{i=1}^c -P_i \log 2P_i$$

(2) Definition of information gain

Information gain metrics of training data of attributive classification can be defined by giving a definition for entropy:

$$(3.3) \quad Gain(S, A) \equiv Entropy(S) - \sum_{v \in Values(A)} \frac{|S_v|}{S} Entropy(S_v)$$

Values (A) are all the possible subsets of attribute A ; S_V is the subset of V value of attribute A in S . (3) Definition of split information and the ratio of gain Split information:

$$(3.4) \quad SplitInformation(S, A) \equiv - \sum_{i=1}^c \frac{|S_i|}{|S|} \log 2 \frac{|S_i|}{|S|}$$

Gain ratio:

$$(3.5) \quad GainRatio(S, A) \equiv \frac{Gain(S, A)}{SplitInformation(S, A)}.$$

The specific steps of the C4.5 algorithm are as follows: Step 1: Data source are preprocessed, and a training set of decision tree is formed by processing discretization for continuous attribute variables (ignore it if there is no continuous valued attributes) [12]. Firstly, the minimum value a_0 and maximum value $a_n + 1$ of the continuous attribute are searched on the basis of the original data; secondly, n values are inserted into the interval $[a; b]$ to divide it into $n + 1$ sections; thirdly, $a_i; i = 1, 2, \dots, n$ is taken as the segment point to divided interval $[a_0, a_{n+1}]$ into two subsections: $[a_0, a_i]$ and $[a_{i+1}, a_{n+1}]$ which are corresponding to two kinds of values of the continuous attribute variable. There are n division modes.

Step 2: The information gain and ratio of information gain ratio of each attribute is calculated. Firstly, the information gain $Gain(A)$ of attribute A is calculated. Secondly, the ratio of information gain $Gain Ratio(A)$ of attribute A is calculated.

The attribute with the largest information gain ratio was selected as the current attribute node to get the root node of decision tree.

Step 3: Each value of root node is corresponding to a subset, and the second step process is performed recursively for sample subset until the data has the same value in classification attributes in each subset. Finally, a decision tree is generated [13].

Step 4: New data sets are classified according to the extraction and classification rules of the constructed decision tree.

4. Application of decision tree in the selection of talents

With the continuous development of computer technology, C4.5 algorithm has been successfully applied in the business, financial, health care and other fields [11, 12]. Currently many universities and organizations are facing with the problem of talents selection. Colleges and universities will select some excellent talents to attend various national or provincial competitions; organizations tend to choose comprehensive talents. Whether a student is suitable for an organization or whether he can make achievements in a college or university needs to be determined. In order to solve such problems, this study introduced the data

classification algorithm - C4.5 algorithm [13] in the selection of talents based on information entropy theory by taking the selection of excellent talents in hostels as an example.

First of all, all staffs who achieved ranking in the selection of hotel talents were prepared (table 1), and the attributes that are suitable for digging were determined.

Name	Average score of comprehensive ability	Work experience	Reaction capability	Psychological quality	Whether ranking or not achieved
Li Yi	A	B	A	C	Yes
An Ji	C	A	B	A	Yes
Mao Zhenyu	B	B	C	B	No
...	...	...	...	...	...

Table 1: Relevant data of the staffs who achieved ranking

Value ranges of data attributes were as follows: Attribute (Average score of comprehensive ability) = $\{A, B, C\}$; attribute (Work experience) = $\{A, B, C\}$; attribute (Reaction capability) = $\{A, B, C\}$; attribute (Psychological quality in game) = $\{A, B, C\}$;

$$(4.1) \quad Entropy(S) \equiv -P \log 2P - P \log 2P \equiv 1.226$$

Information gain, split information and ratio of information gain of each attribute were obtained through calculation:

Gain(S, Average score of comprehensive ability) = 0.5099; gain (S, Work experience) 0.3061; gain (S, reaction capability) = 0.8835; gain (S, psychological quality in game) $\equiv$ 0.5594.

Gain Ratio (S, average score of computer major) $\equiv$ 0.236331; gain Ratio (S, experience in game) $\equiv$ 0.021109; gain Ratio (S, reaction capability) $\equiv$ 0.170 241; gain Ratio (S, psychological quality in game) $\equiv$ 0.105557.

According to information gain of four kinds of attributes, the ratio of information gain of average score attribute of comprehensive ability is the maximum, therefore; average score attribute of comprehensive ability was selected as the root node of decision tree. Then the following attributes were calculated: Gain Ratio (grade A, working experience) $\equiv$ 0.102021; Gain Ratio (grade A, reaction capability) $\equiv$ 0.104401; Gain Ratio (grade A, psychological quality) $\equiv$ 0.349101; Gain Ratio (grade B, work experience) 0.220584; Gain Ratio (grade B, reaction capability) $\equiv$ 0.404112; Gain Ratio (grade B, psychological quality) $\equiv$ 0.01201; therefore, psychological quality was regarded as the node under grade A and reaction capability as the node under grade B. Finally a decision tree was obtained.

Potential talents selection rules were digged out from the vast and clutter data using C4.5 algorithm based on demands, which omits the cumbersome

steps of human analysis and let the system analyze a large amount of data in the database automatically to obtain information [14, 15]. Due to the various existing data categories and large data size, human is very difficult to find useful information from those data. However, C4.5 algorithm that can perform dispersing processing on continuous data and reach highly-accurate results is suitable for analyzing that kind of data. Excellent servant talents could be efficiently screened through C4.5 algorithm to make managers understand the condition of human resource better. decision tree.

5. Discussion

With the rapid development of Chinese economy, the competition between hotels becomes more and more intense. Human resource as the core competition factor of hotels plays an important role in hotel competition. In talents recruitment and selection, the reasonable application of data informatization technology makes the process more convenient. Bouajaja S. et al. [16] proposed solving the problem of human resource optimal distribution using ant colony optimization which could improve the productivity and market competition of enterprises. Mukherjee A. N. et al. [17] studied applicant tracking system and proposed human resource information system model by applying applicant tracking system in human resource management. Based on the decision tree algorithm in data classification algorithm, this study analyzed relevant data of hotel staffs. Compared to other algorithms, C4.5 algorithm can process big data with continuous data discretization.

Evaluating the performance of the staffs with C4.5 algorithm could find out attributes that were suitable for mining and accurately deduce their values to the hotel. Then to achieve more benefits, the staffs were allocated and a correct human resource short-term plan was formulated according to the deduced information. There were some defects in this study. For example, the study did not involve all aspects, the classification of decision attributes was not comprehensive, and there was no comparison with other algorithms, which are expected to be improved in the future.

6. Conclusions

In conclusion, the informatization of human resource management can make the selection of talents more efficient and improve the competition of hotels. C4.5 algorithm, a data classification algorithm based on information entropy theory, was used to calculate different decision attributes in the process of talents selection. The algorithm makes talents selection convenient and creates beneficial prerequisite conditions for the success of hotels in market competition.

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ACTIVE FREQUENCY DRIFT ISLANDING DETECTION ALGORITHM FOR SINGLE-PHASE PHOTOVOLTAIC GRID-CONNECTED INVERTER

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Abstract. As the photovoltaic generation industry rises, the requirements for the reliability of photovoltaic power generation have become higher. The traditional active frequency drift islanding detection algorithm has advantages of simple operation, easy implementation, high speed and high detection rate of island; however, the large impact of the algorithm on the detection performance can bring harmonic pollution to power grid and even induce voltage flicker and instability of the network system. When multiple photovoltaic systems work in the form of grid connection, frequency drift appears and the effects cancel each other out, leading to the appearance of non-detection zones.

This study simulated a positive feedback islanding detection method using SIMULINK and proposed a novel active positive feedback active frequency drift detection method that could realize frequency drift by increasing disturbance with piecewise function based on the drawbacks of the active frequency drift islanding detection algorithm. Finally, the advantages of the system in resisting islanding were verified through computer simulation.

Keywords: inverter; enhancement; active frequency drift detection method; computer simulation software; islanding detection.

1. Introduction

Principle component analysis can process a problem of high-latitude space by transforming it into a problem of low-latitude space [1, 2], thus to make the problem simpler and more visualized. Moreover, in the process of principal component analysis, each principal component will generate its weight auto-

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matically [3], which resists the influence of human factors in evaluation process and makes evaluation results more objective. Principal component analysis can be divided into principal component analysis for population and principal component analysis for samples [4]. In the analysis of practical problems, population correlation coefficient matrix and covariance matrix are generally unknown; therefore, correlation coefficient matrix and sample covariance matrix need to be calculated for the evaluation or hypothesis testing of samples [5]. When practical problems are analyzed using principal component analysis, principal component which can cover population information is selected through changing variable values and moreover further analysis is made on this basis. But, the premise of principal component selection is to suppose that these principal components can cover population information [6]. Therefore, corresponding hypothesis testing is needed to determine whether the processing of follow-up question is feasible. The size of the non-detection zone (NDZ) is the evaluation criterion for islanding detection. Relevant theoretical analyses and experiments have found that, most of passive islanding detection methods have NDZs [4, 5]. Thus this study proposed a kind of active positive feedback active frequency drift detection method and a new algorithm which could reduce the NDZ, risks of current harmonic distortion and harmonic pollution to network system and accurately and efficiently realize islanding detection functions on the basis of the basic algorithm of active frequency detection.

2. A computer model for islanding detection

SIMULINK was selected as the platform for establishing the islanding detection model. Fig. 1 demonstrates an islanding detection model which is composed of S function module, inverter module, output control module, phase-locked loop (PLL) module, etc. The inputs were the voltage frequency and phase of common points. The signals which were output through the detection algorithm produced four pulses under control to command the output. In accordance with IEEE Std.32000-929/UL1741, the scope of frequency protection is 49.3 HZ 50.5 HZ, and the two seconds after network interruption is the time for islanding detection. The detection method will lose efficacy and a NDZ will appear if it is the time after network interruption is more than two seconds [6].

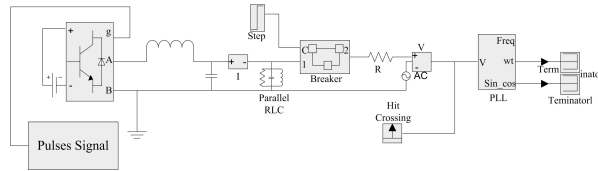


Figure 1: The Islanding Detection Model System

3. A novel positive feedback active frequency drift detection means

The detection method could control the current frequency of the output end of the inverter. The occurrence of islanding could be confirmed if the voltage frequency in the system changed alternately under the interference of disturbance signals. It operated by eliminating the balancing effect of disturbance signals on load under the condition of consistent frequency through adding frequency disturbance signals which were in positive and negative directions and were periodic and continuous into the output of the inverter.

The disturbance signals S_1 and S_2 in two directions were selected and assigned as $+5\%$ and -5% . They were added into the network system and turned to be $S_1 + \Delta S_1$ and $S_2 + \Delta S_2$. ΔS_1 and ΔS_2 were the differences between the output frequency of the inverter and the voltage frequency of the original power network. When detection started, S_1 and S_2 disturbed the output frequency of the inverter orderly and alternately, and the values of ΔS_1 and ΔS_2 were compared. When island appeared, the larger one between S_1 and S_2 was regarded as the standard of disturbance signals and added into positive feedback; finally, $S_{k+1} = S_k + \Delta S$ was obtained. ΔS refers to feedback quantity under such control. Due to the high speed of the output end frequency conversion of the inverter, the correlation between the inverter and the power network could be cut off by finding out the island and triggering the protective circuit in a short time [7].

Operation Process

To understand the operation process better, the flow of the positive-feedback active islanding detection is demonstrated in Fig. 2.

Suppose that the power network cut off suddenly, the detailed operation methods taking capacitive load as an example are as follows.

(1) When the disturbance signal $S_2(+5\%)$ disturbed the output frequency of the inverter, its frequency increased. But when the load was capacitive, the voltage became ahead of the current, and the output frequency was smaller than the frequency of the disturbance signals, i.e., $\Delta S_1 < S_1$.

(2) When the disturbance signal $S_2(-5\%)$ disturbed the output frequency of the inverter, the frequency of the inverter decreased. But when the load was capacitive, the voltage became ahead of the current, and the output frequency was larger than the frequency of the disturbance signals, i.e., $\Delta S_2 > S_2$.

(3) As $\Delta S_2 > S_1$, S_2 was selected to interfere the output frequency of the inverter as the criterion of the disturbance signal.

(4) After a series of actions above, the appearance of island could be determined if the specified frequency $f > f_2$, and at that moment, the protective circuit should be triggered immediately.

It could be noted that, the output frequency of the inverter changed obviously when there were island effects whether the load was inductive or capacitive, which eliminated effect offset in single direction. In the system, the disturbance

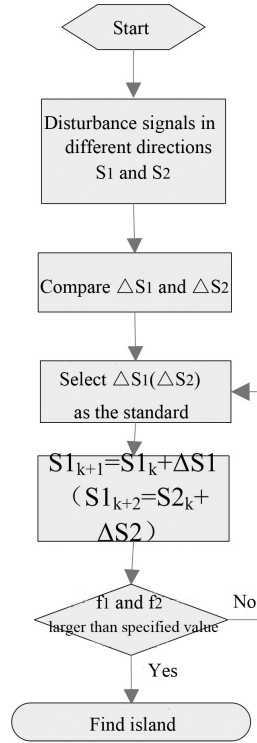


Figure 2: The Flow Diagram of the Islanding Detection Method

signals with large difference values in combination with positive feedback led to the larger variation scope of system frequency, which was beneficial to detect NDZs and accelerate detection; the protective circuit could be triggered to maintain the safety of the whole power network when islanding appeared [8-10].

4. New Algorithms for Detection Methods

The traditional active frequency drift method detects islanding by disturbing assigned frequency with voltage frequency at offset common points at the output end of the inverter, and the quantity of the offset is a fixed value. If power network is cut off, offset will result in unmatched power, inducing the changes in the frequency and amplitude of load voltage. Missing detection may happen if the value of the offset quantity was set too small; load power may be unmatched, thereby increasing the total harmonic distortion if the quantity of offset is too large [11].

To reduce the total harmonic distortion, the current of the inverter was set as segmented periodic current [12].

$$(4.1) \quad i_{(t)} = \begin{cases} I \sin(\omega t), & 0 \leq \omega t \leq \frac{\pi}{2} \\ I \sin(\omega t) - CI, & \frac{\pi}{2} \leq \omega t \leq \pi \\ I \sin(\omega t), & \pi \leq \omega t \leq \frac{3\pi}{2} \\ I \sin(\omega t) + CI, & \frac{3\pi}{2} \leq \omega t \leq 2\pi \end{cases}$$

where I refers to the current amplitude of the inverter and C refers to the frequency distortion factor. The waveform of the new current is demonstrated in Fig. 3. It could be noted from Fig. 3 that, the new current of the inverter had harmonic component. It needed to be processed by Frontier transform. The Frontier coefficient after transformation was:

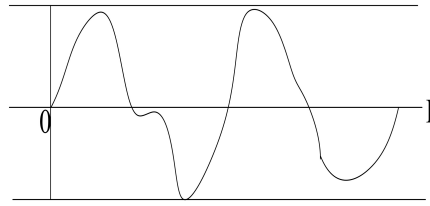


Figure 3: The New Current Waveform of the Inverter

$$(4.2) \quad \begin{aligned} a_n &= \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} I_{(t)} \cos n\omega t dt, \\ b_n &= \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} I_{(t)} \sin n\omega t dt, \\ A_n &= \sqrt{a_n^2 + b_n^2} \text{ and } \Phi_n = \arctan -\frac{b_n}{a_n}. \end{aligned}$$

After specific values were substituted into the above formulas, the Frontier coefficient of the current in the inverter could be obtained.

$$(4.3) \quad a_n = \frac{2CI}{\pi}, b_1 = I(1 - \frac{2C}{\pi}), A_1 = I\sqrt{1 + \frac{8C^2}{\pi^2} - \frac{4C}{\pi}} \text{ and } \Phi_1 = \arctan -\frac{b_1}{a_1}$$

A_1 and Φ_1 are the fundamental wave and phase angle of the new current of the inverter. Consequently, the total harmonic distortion, active power and reactive power of the grid-connected current were obtained

$$(4.4) \quad THD = \sqrt{\frac{C^2(\pi^2 - 8)}{\pi^2 - 4\pi C + 8C^2}}, Q = 2C \text{ and } P = \pi - 2C.$$

Table 1: THD and Q/P of the Two Algorithms

	The value of C	0.075	0.085	0.095
The traditional algorithm	THD and Q/P	5%	5.7%	6.44%
The traditional algorithm	THD and Q/P	5%	5.7%	6.44%
The new algorithm	THD	3.24%	3.90%	4.39%
	Q/P	5%	5.7%	6.44%

When the value of C was different, the aberration rate, active power and reactive power of the grid-connected current of the inverter all changed [13, 14]. Table 1 demonstrates the changes of THD, Q and P of the two algorithms when the value of C was different. It could be noted from Table 1 that, when the Q/P of the two algorithms was the same, the total harmonic distortion of the new algorithm was smaller than that of the traditional algorithm. The algorithm of the traditional detection method added dead time t at the end of two and a half cycles through controlling disturbance signals, while the algorithm of the detection method which was used in this study added dead time t at the end of one cycle to actively control the disturbance of the disturbance signals to the current frequency of the inverter. By doing this, the voltage frequency at the common points would become higher when islanding effect appeared. The property of the load also had impacts on the voltage frequency of the common points [15].

When the load was capacitive, the output end of the inverter selected S_2 as the standard disturbance signal by comparing ΔS_1 and ΔS_2 ; similarly, when the load was inductive, the output end of the inverter selected S_1 as the standard disturbance signal. Fig. 4 shows the comparison of the addition of dead time

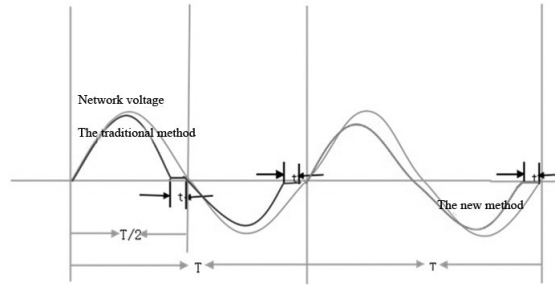


Figure 4: The Control Signals of the Inverter under the Two Algorithms

of two methods given by computer. The size of the NDZ is a vital evaluation criterion for islanding detection. Relevant theoretical analysis and experiments found that, NDZs existed in most of the passive islanding detection methods [16]. In the traditional detection method, the system parameter setting should satisfy $\Phi = \arctan[R(\omega C - \frac{1}{\omega L})] = \frac{\pi}{2}$; otherwise, overfrequency or underfrequency would happen, leading to NDZs. In the new detection method, the

disturbance signals were controlled in different segments according to the times of frequency changes, and feedback coefficient would not induce large harmonic distortion [17]. As mentioned in the criterion above, the frequency variation times of the common points within two seconds should be less than 10. When the variation times of the common points was more than 10 and the frequency was between 49.3 HZ and 50.5 HZ after power network cut off, the algorithm actively regulated disturbance signals S1 and S2, which could effectively avoid the failure of islanding detection caused by excessively fast frequency changes. With the intensity increase of disturbance signals, NDZs gradually got close to capacitive load interval; however, load is usually inductive in reality. Thus, the NDZ was greatly narrowed. Fig. 5 demonstrates the NDZ of the two algorithms. It could be seen from Fig. 5 that, the NDZ of the new method was smaller than that of the traditional method. The NDZ could be narrowed by correcting parameters, but could not be thoroughly eliminated.

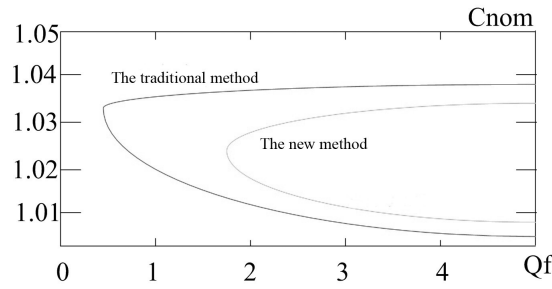


Figure 5: NDZs of the Two Algorithms

5. Experimental simulation of the new detection method

The parameters of the model were set according to the model and islanding detection criteria established above. The NDZ [18] could be the maximum when the output power of the inverter was equal to that of the load, resonant frequency of the Radio Link Control (RLC) parallel circuit reached 50HZ, or load quality factor $Q_f = 2.5$. The parameters at the time when the possibility of islanding detection failure was the highest could be calculated using the given value of Q_f

$$(5.1) \quad Q_f = \frac{2\pi(1/2CR^2I^2)}{\pi RI^2/\omega} = \omega RC = R \frac{1}{\omega L} = \frac{1}{P} \sqrt{Q_L Q_C} = 2.5$$

where R, L, C and I refer to the load resistance, inductance, capacitance and current value respectively, P stands for the load active power, and Q_L and Q_C stand for the inductive reactive power and capacitive reactive power respectively

$$(5.2) \quad P = \frac{U^2}{R}, \quad Q_L = \frac{U^2}{\omega L}, \quad Q_C = U^2 \omega C.$$

The values of the parameters could be obtained by combining the four formulas together

$$(5.3) \quad \begin{cases} R = 15.1\Omega, & L = 21.32mH, & C = 488.6\mu F \\ P = 3KW, & Q_L = 7.535Kvar, & Q_C = 7.535Kvar \end{cases}$$

The above parameters were input into the computer simulation software for the

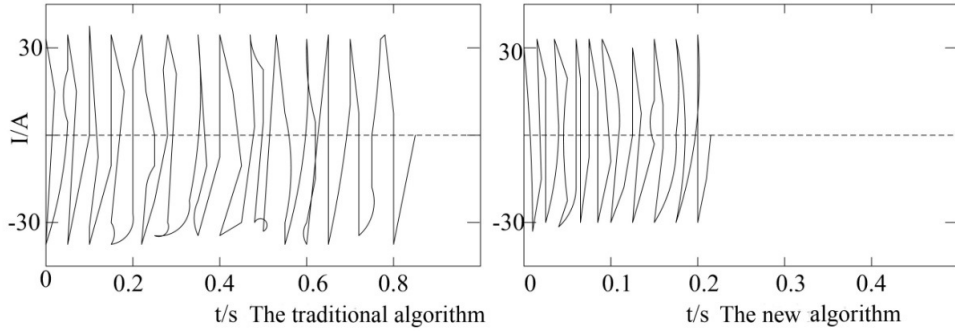


Figure 6: The Waveform of the Output Current of the Inverter under the Two Algorithms

output current simulation of the inverter. Fig. 6 demonstrates the simulation waveform of the output current under the two algorithms.

6. Conclusion

This study proposed a new active frequency drift islanding detection method with feedback on account of the active frequency drift islanding detection. The means could reduce the NDZ in islanding detection and eliminate effect offset in single direction with periodic disturbance signals, which could trigger the protective circuit and maintain the safety of the whole power network when islanding appeared. Moreover, an improved traditional algorithm was introduced; it was verified as qualified for islanding detection through simulation. But the experiment was simulated on the computer platform, lacking practical basis in practical production.

7. Acknowledgements

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IMPLEMENTATION OF PARALLELIZING MULTI-LAYER NEURAL NETWORKS BASED ON CLOUD COMPUTING

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Abstract. Background: Cloud computing, as a technology developed under the rapid development of modern network, is mainly used for processing large-scale data. The traditional data mining algorithms such as neural network algorithm are usually used for processing small-scale data. Therefore, the calculation of large-scale data using neural network algorithm must be based on cloud computing. Materials and Methods: Firstly a Hadoop system was established taking MapReduce as the programming framework. Then the parallelized traditional data mining algorithm was investigated based on cloud computing cluster to verify its feasibility in processing large-scale data. Finally, the speed and training precision of the algorithm were tested. Results: It was feasible to process large-scale data with cloud computing based parallelizing multi-layer neural network algorithm. The speed of parallel processing was faster if data size was larger, especially if the sample was in a size of more than 1 million. It was more superior to the serial back propagation network in training preciseness. Conclusion: Parallelizing multi-layer neural network based on cloud computing platform can process large-scale data effectively in the perspectives of time and quality.

Keywords: Cloud computing, neural networks, layered representation, parallelization, speed, training preciseness, data, feasibility.

Significance statement

This study realized multi-layer neural network parallelization based on cloud computing and applied it for processing large-scale data. Compared to the serial back propagation algorithm, the algorithm showed high operation speed and training precision in processing the sample in a size of more than 0.6 million.

1. Introduction

Since cloud computing technology first came into being in 2007, it has been developed greatly [1]. With the further development of the information society, data explosion has stepped into our life. As the conventional data mining algorithms cannot process such large-scale and complex data [2-3], large-scale data need to be processed with the help of distributed parallel computing technology.

Cloud computing is capable of processing the massive data, therefore, combining cloud computing with the conventional data mining algorithm is a preferable solution to the problem of data processing. So far, quite a few scholars have carried out relevant researches.

For instance, in 2012, Aizenberg et al. [4] performed a thorough research on low density parity check (LDPC), multinuclear parallel idea and artificial neural network; by combining parallel idea and neural network technology with LDPC decoding effectively, they designed and implemented LDPC parallel decoding algorithm, LDPC decoding algorithm based on general neural network and LDPC decoding algorithm based on multilayer perceptron. In 2014, Nunez et al. [5] applied the modified artificial neural network back propagation algorithm to the identification of physical parameters of multi-layer soil mass in deep foundation pit excavation engineering. Compared with the conventional back analysis method, this algorithm was easy to master and implement. In 2012, Krawczak et al. [6] put forward a new algorithm that integrated genetic algorithm with adaptive deformation gradient learning algorithm and could be applied in multilayer feedforward neural network training. Based on the experiment on Hadoop cloud computing platform, this paper presents a study on the parallelization of multi-layer neural network, suggesting that the parallelized network is advantageous in dealing with large-scale data for its high speed and high quality. Therefore, this study contributes to the development of this field to some extent.

2. Methodology

2.1 Parallelization of neural networks

For the parallelization of neural networks, there were two approaches. The first one was node parallelization, i.e., network nodes were distributed on different machine nodes so as to achieve parallelization. However, with this approach, the integrity of similar networks would be destroyed after node distribution [7, 8]. In cloud computing, programmers could not directly control the specific nodes; therefore, parallelization of this kind of nodes could not be achieved in cloud computing. The second one was data parallelization. With this approach, each compute node had a complete network at the same initial state.

2.2 Implementation of MapReduce

As an open-source distributed cloud computing system, Hadoop is established based on MapReduce programming framework and Google file system (GFS). Operating on Hadoop, users can avoid lowlevel details of parallel programming, such as automatic parallelization, load balancing and disaster management; in addition, users can process large-scale data using simple parallel programming framework. So far, Hadoop has become the most representative mainstream platform for distributed computing [9-11].

Hadoop distributed file system (HDFS), MapReduce and Common compose the Hadoop system. With clear structure, this kind of parallel programming framework is easy to use. On the large-scale cluster composed of thousands of nodes, as for the writing of application programs on the basis of this framework, parallel processing of data set can be implemented securely and reliably, and the fault tolerance is high [12-15].

As a functional calculation programming framework, MapReduce imitates the Map and Reduce functions of LIST Processing. It simplifies the parallel computing, and finally, the process is divided into mapping stage and reduction stage. Map refers to mapping stage: a group of input data is replaced with key value pairs [16]; according to the set rules, a list of new key value pairs are mapped and used as the input data of Reduce. Reduce refers to the reduction stage; according to the shared key groups, all the mapping is sorted, reduced and simplified; the final results are saved in Hadoop file system.

Cluster configuration of MapReduce The command "sudo vi/etc/profile" was executed, and the following information was added to the end:

```
export JAVA_HOME=/usr/hadoop/jdk1.7.0_25
```

```
export JRE_HOME=$JAVA_HOME/jre
```

```
export PATH=$JAVA_HOME/bin:$PATH
```

By executing java-version, it could be observed whether the configuration was successful.

Network configuration

In the network configuration files, static Internet protocol (IP), gateway and domain name server (DNS) were set.

The command "sudo vi /etc/network/interfaces" was implemented and the following content was added:

```
auto eth1
iface eth1 inet static
address 192.168.1.24
gateway 192.168.1.1
netmask 255.255.255.0
dns-nameservers 210.47.208.8
```

The command "sudo ifup eth1" was implemented; after restarting the network, the command "ifconfig" was implemented to check if the network configuration was correct. The command "sudo vi /etc/hostname" was implemented, so that the hostname was changed; for example, it could be renamed as "hadoop01". However, ubuntu of the original name had to be deleted; instead, only the set name remained.

Installation and configuration of Hadoop

Hadoop-1.1.2 was installed; Hadoop environment variables were added; “sudo vi /etc/profile” was implemented; the following content was added:

```
export HADOOP_HOME=/usr/hadoop/hadoop-1.1.2
export PATH=$PATH:$HADOOP_HOME
export PATH=$PATH:$HADOOP_HOME/bin
```

The file “hadoop-env.sh” was modified; the command “sudo vi /usr/hadoop/hadoop-1.1.2/conf / hadoop-env.sh” was implemented; the installation path of jdk was altered- export

```
JAVA\_HOME=
/usr/hadoop/j jdk1.7.0\_25; HADOOP_HOME
```

was added:

```
export
HADOOP_HOME=/usr/hadoop/hadoop-1.1.2/ export PATH=$PATH:$HADOOP_HOME/bin.
```

The configuration file “core-site.xml” was modified. The following command was implemented:

```
sudo vi /usr/hadoop/hadoop-1.1.2/conf / core-site.xml.
<configuration>
<property>
<name>hadoop.tmp.dir</name>
<value>/tmp/fs</value> /* the folder of hadoop file system was set */
</property>
<property>
<name>fs.default.name</name>
<value>hdfs://localhost:9000</value>
/* applied port number clamp of Namenode in HDFS*/
</property>
</configuration>
```

The file “hdfs-site.xml” was modified.

The following command was implemented:

```
sudo vi /usr/hadoop/hadoop-1.1.2/conf / hdfs-site.xml.
<configuration>
<property>
<name>dfs.replication</name>
<value>4</value> /* duplicate count of data block saved in HDFS */
</property>
</configuration>
```

The file “hdfs-site.xml” was modified.

The following command was implemented:

```

sudo vi /usr/hadoop/hadoop-1.1.2/conf / hdfs-site.xml.
<configuration>
<property>
<name>dfs.replication</name>
<value>4</value>
/* duplicate count of data block saved in HDFS */
</property>
</configuration>

```

The configuration file was modified: mapred-site.xml.

```

<configuration>
<property>
<name>mapred.job.tracker</name>
<value>localhost:9001</value> /* port number of JobTracker was set*/
</property>
</configuration>

```

The configuration file “masters” was modified; the name of the main node was set as “hadoop01”.

The configuration file “slaves” was modified; list of the node names was set: hadoop01, hadoop02 ...

Test of Hadoop cluster

First, HDFS was formatted on Namenode; “hadoop Namenode format” was implemented; then, all the daemons were started, and “start - all.sh” was implemented. The starting status of the process was checked; “jsp” was implemented.

3. Experiment on Hadoop cloud computing platform

3.1 Experiment environment

The experiment was performed with Hadoop as the cloud computing platform. The machine cluster on the platform consisted of seven machines, one of which was the node and the others were data node machines. Within the Hadoop cluster, machine hardware configuration was: Intel (R) Core (TM) i3- 3240 CPU 3.40 GHz, 8.00 GB of internal memory, 512 GB of hard drive; software environment was Linux operating system, JDK 1.6, Hadoop 0.20.1.

3.2 Experiment results and analysis

The purposes of the experiment included: validation of the feasibility of neural network algorithm based on cloud platform, test of the speed of the algorithm and test of training precision of the algorithm. The experiment involved two data sets Breast Cancer Wisconsin Data Set and KDD Cup 1999 data set. The former was used to test the classification effect. In the data set, there were 699

sample data. Removing the 16 data whose record was missing, the first 450 data were used as the data of training samples, while the other 233 data were used as test data. There were 11 fields in each record. Data field 1 was used to record id; data fields 2–10 were data input mode; data field 11 was used for classification of the records. Each record included two types: negative and positive. KDD Cup 1999 data set was used to detect abnormal network intrusion; each sample of the file contained 42 attribute values.

In the study, the training samples were arranged to multiple Mappers for training. In the experimental driving function, accuracy test on network classification was performed. Each experiment item was operated for eight times, and their average values were obtained. The results can be seen in Table 1 and Fig. 1.

	Amount of rounds	Error	Classification accuracy	Count of mappers
Serialback propagation(BP)	2	0.1701	0.7645	
Serial BP	8	0.1251	0.7813	
Serial BP	32	0.1302	0.9411	
MRBP	2	0.035	0.9926	3
MRBP	8	0.087	0.9803	3
MRBP	32	0.146	0.9827	3

Figure 1: Experimental results of multi-os remote booting protocol (MRBP)

Table 1 Experimental results of multi-os remote booting protocol (MRBP)

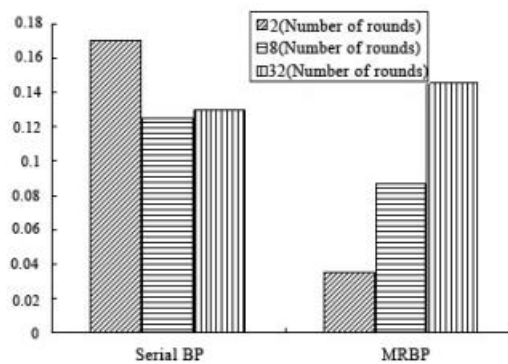


Figure 2: Comparison of errors of serial BP and MRBP

As shown in Table 1 and Figure 1, after two rounds of training on the training samples with MRBP, the effect was satisfactory and better than that of serial BP. The reason why MRBP was more effective was that the training samples were segmented properly and assigned to each Mapper; a reduction in sample size could accelerate the convergence rate on each Mapper, then the

entire training process was accelerated. However, as the segmented samples failed to cover the total samples effectively, multiple training was required.

Table 2 shows the running time of serial BP and MRBP in training samples when the amounts of samples were respectively: 240, 000; 600, 000; 1, 000, 000.

Sample amount	240,000	600,000	1,000,000
Running time of serial BP (s)	2459.1	4013.8	12351.9
Running time of MRBP (s)	2563.9	3601.5	7213.4

Figure 3: Running time of serial BP and MRBP

As shown in Table 2, when sample amount was 240,000, the parallel processing speed of serial BP was faster than that of data-parallel MRBP pattern classification algorithm, which was due to the influence of factors such as system overhead of Hadoop [17,18]. However, as sample amount increased to more than 600,000, there was parallel efficiency; and as the data size continued to increase, the parallel processing speed became faster and faster, and the speed-up ratio started to increase gradually, which was because parallelization and Hadoop distributed file storage could reduce data network transmission overhead and memory overhead. As the samples amounted to one million, the growth trend of speed-up ratio was more obvious, proving that on Hadoop platform, parallel MRBP combination classification algorithm was effective, feasible and superior in processing large-scale data.

Suppose T_a as the running time of Hadoop cluster and T_b as the running time of uniprocessor, then T_b/T_a could be defined as the speed-up ratio which was an important measuring standard of the experiment. Separately, with two kinds of data sets, six experiments were performed, and the results can be seen in Table 3.

Testing dataset	Number of cluster machines	T_a (s)	T_b (s)	T_b / T_a
KDD Cup 1999	2	3926	7264	1.85
	4	1859		3.91
	6	1628		4.46
Breast Cancer Wisconsin (Original) Data Set	2	4126	8016	1.94
	4	2413		3.32
	6	2203		3.64

Figure 4: Performance contrast of uniprocessor and Hadoop cluster

According to Table 3, with the use of these datasets, while the amount of cluster machines increased, speed-up ratio increased correspondingly. When

there were two machines in the cluster, the speed-up ratios were respectively up to 1.85 and 1.94, which were close to parallel amount 2; when the amount of cluster machines was 4, the speed-up ratios respectively raised to 3.91 and 3.32; when the amount of cluster machines increased to 6, the increase in speed-up ratios was not significant, indicating that a greater amount of machines did not necessarily mean the speed-up ratio would be higher. The more the machines in the cluster, the greater the overhead of communication between nodes [19]; if the processing data size did not increase correspondingly, the loss would outweigh the gain.

Therefore, it was suggested that a proper coherence point between machine amount and the processing data size should be determined where the speed-up ratio was the closest to the amount of cluster machines, so that cloud computing platform could be used the most effectively, which could be achieved by setting the parameters of MapReduce tasks and performing multiple experiments. In the experiment, the optimal condition was that there were six machines in the cluster when the running time of Hadoop cluster was respectively 4.46 and 3.64 times that of uniprocessor; still, there was a certain gap between the speed-up ratios and the parallel amount, which indicated that the parallel process of the algorithm stilled needed to be improved. To enhance the efficiency of parallel computing system, the applications should be optimized and their concurrency should be enhanced.

The number of cluster machines was increased to 8. $T_a(s)$ of KDD Cup 1999 was 1608; T_b/T_a was 4.84. $T_a(s)$ of Breast Cancer Wisconsin (Original) Data Set was 2147; T_b/T_a was 3.39. Compared to the situation when there were 6 cluster machines, the speed increase of $T_a(s)$ was not obvious, and the value of T_b/T_a was significantly different with the parallel number. Therefore, it could be concluded that, 6 cluster machines was the most suitable.

4. Discussion

BP neural network, a kind of multi-layer forward artificial neural network model simulating the learning and memory processes of the brains of creatures, has been extensively applied in many fields. With the increase of the accumulated data in various fields, the problems faced by BP neural network in practical application become more and more complicated. Ren C. et al. [20] predicted the optimal wind speed based on the optimal parameter selection of the particle swarm optimized BP neural network in wind power generation. In the medical field, Samanta S. et al. [21] distinguished normal retina from the retina which was affected by glaucoma using Haralick function, processed the extracted features with BP neural network, and successfully realized the classification of eyes with glaucoma, with an accuracy of 90%. Based on the previous studies on BP neural network, this study parallelized multi-layer neural network based on cloud computing and compared it with the serial BP neural network in the perspectives of the speed and preciseness in processing data set. The results

demonstrated that, the preciseness of MRBP was superior to that of the serial BP; the processing speed of the serial BP was larger than that of MRBP when the size of data set was smaller than 0.25 million; but the processing speed of MRBP was higher when the size of data set was larger than 0.6 million, and the speed was higher if data size became larger. Thus the advantage of the cloud computing based multi-layer neural network was that it could decompose the large-scale input data and effectively shorten operation time on the premise of ensuring the accuracy of the algorithm.

5. Conclusion

In this study, multi-layer neural network was parallelized based on cloud computing platform, and it achieved favorable effect in processing large-scale data, which performed well in time and quality. But due to the limited experimental conditions and resources, there were many defects in the experiments and analysis, which remain to be corrected and perfected in the future studies.

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DIGITAL INTEGRATION OF SERVICE MODES OF LIBRARIES BASED ON HYBRID METADATA

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Abstract. Public library is the largest repository of the information society. With the development of the network and the information technology, the demand for digital information is increasing, and library structure and service are beginning to develop in the direction of digital resources. With culture and civilization rooted in various kinds of books and resources, it is of great significance to strengthen the development and construction of cultural resources within libraries, so as to effectively strengthen cultural transmission and promote the integration of cultural resources as well as cultural construction. To explore the service mode of library resources in a digital environment from an integration perspective can effectively promote the construction of national culture. Based on a centralized, distributed and shared metadata management model, this paper designs a hybrid metadata management model suitable for the digital resource management of libraries, and designs a new library digital fusion service platform based on this. At the same time, this paper uses the fuzzy mathematics theory to evaluate the digital fusion service quality of libraries, which provides a new idea for activating the national cultural resources and inheriting civilization, which is of great significance for the promotion of the digital fusion service of libraries.

Keywords: Metadata, library, digital fusion service, resource sharing

Introduction

Libraries being the bearing ground of national civilized memories and playing an irreplaceable role in the real life of mankind, it is extremely important to

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optimize the library management and service model [1-3]. Due to the rapid development of computer network communication technology and other related high and new technologies, information storage, transmission and utilization has undergone tremendous changes. Affected by this, the library collection resources have also begun to change towards digitalization [4]. And the continuous growth of network information resources [5-6] makes it increasingly important to organize and manage these large amounts of information resources effectively. As the data information of data set [7], sound, image, text and other digital information, metadata has become an important tool for the organization and acquisition of digital resources in the modern Internet environment, which plays an important role in the optimization of library service model. For metadata research, the Metadata Consortium has proposed the OIM (Open Information Model) model [8], and the object management organization has proposed the CWM (Public Warehouse Model) model [9].

Zhengwei Sui, Yuan Tian et al. [10] proposed a distributed metadata management framework for mineral information resources with access control through summarizing and evaluating the three typical metadata management modes of centralized, distributed and shared models, taking into account the advantages and disadvantages and the management characteristics of the three models. Based on the advantages of three kinds of metadata, this paper constructs a hybrid metadata management method for the collection, management and dissemination of digital resources in libraries. Chinese scholar Wang Danyang concluded in her work [11] that the innovation of the library digital fusion service could better promote the dissemination and sharing of cultural resources. In this paper, a hybrid metadata management architecture was proposed, a library digital fusion service platform was constructed and the fuzzy mathematics theory was used to evaluate the service quality to make the management of libraries more orderly.

1. The construction of a hybrid metadata management system

Usually, the library service system has a meta-database to manage all the metadata in the library system. There are three kinds of metadata management models, namely, centralized, distributed, shared, which have the same basic features and functions, but also have their own advantages and disadvantages.

1.1 Overview of the three metadata management models

1.1.1 Centralized metadata management

Structure: All tools and database management systems do not require local storage or maintenance of metadata, but have direct access to a unified central knowledge base [12].

Advantages: little resource occupation, low cost, high system resource utilization rate, with only one object storage server in activities. Disadvantages:

When the cluster size is large, all the metadata in the system needs to go through the metadata server which is a centralized access path, and thus the impact on the system performance is large.

1.1.2 Distributed metadata management

Structure: All tools and database management systems are required to store and maintain metadata in their own meta-databases. Advantages: Both remote and local file access can be done through the same system call; the name or address of the server is not part of the file path, and changes to the file storage location do not cause the file path to change; distributed management allows parallel access to metadata to ensure access to metadata. Disadvantages: Due to the complexity of control, the design and implementation of distributed management system is difficult. To solve the problem of consistency maintenance of distributed management, complex protocols such as distributed lock technology are needed, which will cost a lot.

1.1.3 Shared metadata management

Structure: Each tool stores the local metadata in its own knowledge base, then defines the output metadata by defining the corresponding output scheme in the knowledge base, and finally combines the output schemes into a common metadata model. Advantage: Good autonomy, high efficiency of access, good synchronization and consistency, high utilization rate.

Disadvantage: it is difficult to realize in an environment where multiple libraries work together.

1.2 Mixed metadata management methods

Through the analysis, this paper takes the advantages of the above three methods and combines them into a mixed metadata structure to build a hybrid metadata management method, as shown in figure 1.

In the process of building, due to the poor practicality of the centralized model, it is added to the data access layer. In addition, a shared knowledge base is introduced to manage metadata associated with the whole situation and connected to specific components. In order to better control the flow of metadata, the building uses models, ETL and data access tools to better capture and record objects created during data analysis.

2. Construction and evaluation of digital fusion service platform for library resources The construction of the hybrid metadata management system and the aggregation of the library resources [12] laid the foundation for the construction of the library digital fusion service platform. The integration of a library's digital resource service models does not simply refers to the combination of the digital resources and service forms of the library but also includes the innovation of the service forms. Through the research of library digital fusion service model level,

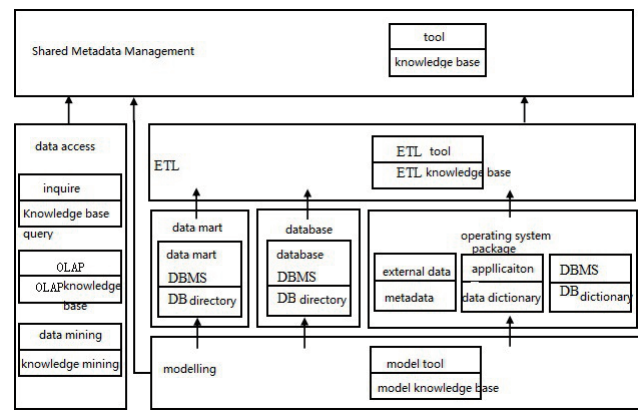


Figure 1: Mixed metadata management building results

the design of platform and the comprehensive evaluation of influence factors of fusion service based on user demand, this paper realizes the construction of the digital fusion service model of the whole library resources.

1.3 Digital resource fusion service platform construction

Through the construction of a unified portal platform, this study provides users with one-stop retrieval and evaluation services [13-1], with its basic framework divided into four levels, as shown in figure 2:

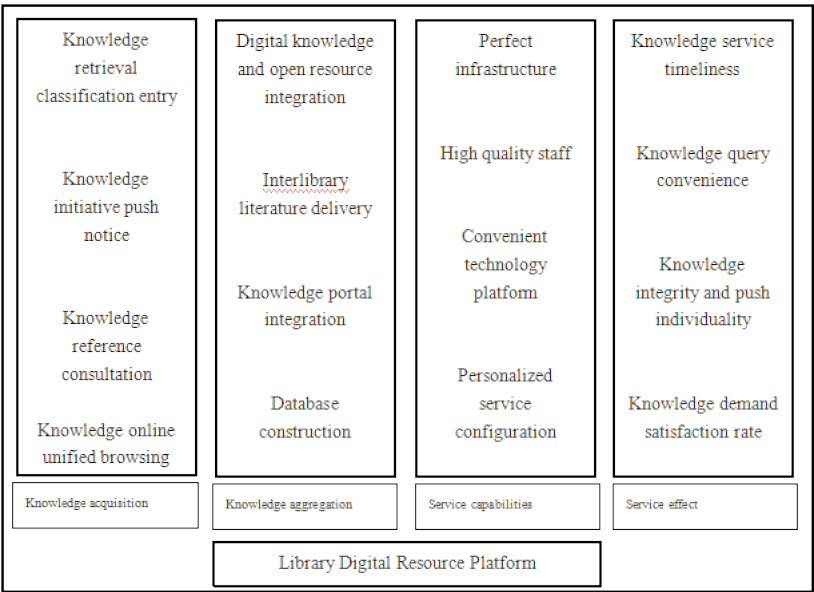


Figure 2: Library resources digital convergence service platform framework

The first layer is the knowledge acquisition layer. Metadata mapping, RDF or metadata program are used to achieve resource integration and build the linkage between the information so as to provide users with the most basic fusion search, online browsing and push services; the second layer is the knowledge aggregation layer. Through the associated data and other technologies, the library digital resources and portal resources are integrated to constitute a resource knowledge network; the third layer is the innovative service capability layer. With perfect infrastructure, high-quality staff, convenient technology platform and personalized service configuration, users are provided with the best services; the fourth layer is the service effect layer. The effectiveness of digital resources service effects are judged by knowledge service timeliness, knowledge query convenience, knowledge provision integrity and push individualization and knowledge requirements satisfaction rate.

1.4 Evaluation of library digital fusion service quality with fuzzy mathematical model

1.4.1 Establishment of evaluation model

In this study, V is set as the evaluation set, U as the factor set: $V = \{v_1, v_2, \dots, v_n\}$, $U = \{u_1, u_2, \dots, u_n\}$; u_i is the main factor affecting the integration services of the assessed digital library; v_j refers to rating levels. Since the membership degree of u_i for the evaluation set is a_{ij} , the result of the evaluation of the i th factor u_i is obtained and a single factor evaluation matrix A is constructed.

$$A = \begin{bmatrix} 1 & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & 1 \end{bmatrix}.$$

In essence, through the number in the $[0, 1]$ interval, A expresses the fuzzy relation of the factor set U to the evaluation set V , reflecting the degree of fuzzy relationship between the two. U, V, A forms the fuzzy comprehensive evaluation model for evaluating the quality of digital library integration services. We divided the reviews into five grades, namely:

$$V = \{\text{excellent}(v_1), \text{good}(v_2), \text{average}(v_3), \text{qualified}(v_4), \text{unqualified}(v_5)\}.$$

The digital library integration service quality evaluation (U) can be divided into four first-level indicators, which are knowledge acquisition ($S = u_1$), knowledge aggregation ($X = u_2$), service capabilities ($Y = u_3$), service effectiveness ($Z = u_4$).

Each first level indicator includes four secondary indicators. Knowledge acquisition (S) includes: knowledge retrieval and classification entries ($S1$), knowledge active push ($S2$), knowledge reference ($S3$), knowledge online browsing ($S4$). Knowledge integration (X) includes: digital knowledge and open resource

integration (X1), interlibrary literature transmission (X2), knowledge portal integration (X3), database construction (X4). Service capability (Y) includes: perfect infrastructure (Y1), high-quality staff (Y2), convenient technology platform (Y3), personalized service configuration (Y4). The service effect (Z) includes: knowledge service timeliness (Z1), knowledge query convenience (Z2), knowledge provision integrity and personalized push (Z3), knowledge demand satisfaction rate (Z4).

Since the focus of evaluation of each factor in U , i.e., u_i , is different, the focus of its weight is different. By determining the relative importance and weight of each factor, the weight vector $\omega = \{\omega_1, \omega_2, \dots, \omega_n\}$ is determined, where $0 \leq \omega_1 \leq 1$ and $\sum_{i=1}^n \omega_i = 1$. Then, the fuzzy comprehensive evaluation value B is determined, i.e., $B = \omega OA$, where O refers to a generalized synthetic operation.

1.4.2 Consistency test

The judgment matrix has the following properties:

$$\begin{aligned} a_{ii} &= 1; \\ a_{ij} &= 1/a_{ji}; \\ a_{ij} &= a_{ik}/a_{jk}; (i, j, k = 1, 2, \dots, n). \end{aligned}$$

As long as the a_{ij} in the judgment matrix satisfies the above three relations, the judgment matrix has complete consistency. When $a_{ij} > 0$, A is a positive reciprocal matrix.

The largest eigenvalue of A is a positive real number and the component of the corresponding eigenvector is the same number. If the unit eigenvector corresponding to the largest eigenvalue $\lambda_{\max}$ is $W = (w_1, \dots, w_n)^T$, then $a_{ij} = w_i/w_j (i, j = 1, 2, \dots, n)$, $\forall i, j = 1, 2, \dots, n$. $W = (w_1, \dots, w_n)^T$ is the weight vector of each indicator line for the superior indexes in the same subset.

However, since each factor in A is obtained through the pairwise comparison of indicators in the same indicator set, it is not necessarily credible to take the positive unit eigenvector corresponding to the maximum eigenvalue $\lambda_{\max}$ of matrix A_{as} the weight vector. Therefore, a consistency test is required.

The consistency test indicator used in this paper is $C.R. = \frac{C.I.}{R.I.}$, where $C.I. = \frac{\lambda_{\max} - n}{n-1}$, $\lambda_{\max}$ refers to the maximum eigenvalue of the judgment matrix and n is the order of the judgment matrix.

The consistency of the matrix is determined by the size of the value of C.I. . The greater the value of C.I., the greater the degree of deviation of the judgment matrix from complete consistency; the smaller the value of C.I., the closer the judgment matrix is to complete consistency. At the same time, the larger the order number n of the judgment matrix, the greater the value of the deviation from the complete consistency index C.I.; the smaller the order number n , the smaller the value of the deviation. When $n < 3$, the judgment matrix is always completely consistent. The ratio of the consistency index C.I. of the judgment

matrix to the average random consistency index R.I. of the same order is the random consistency ratio C.R..

1.4.3 Examples

$U = \{\text{knowledge acquisition } (u_1), \text{knowledge aggregation } (u_2), \text{service capability } (u_3), \text{service effect } (u_4)\}$, i.e., $U = \{u_1, u_2, u_3, u_4\}$. The result comparison matrix for each factor is as follows:

$$Q = \begin{bmatrix} 1 & 1/3 & 1/2 & 1/4 \\ 2 & 1 & 3 & 1 \\ 2 & 1/2 & 1 & 1/2 \\ 4 & 1 & 2 & 1 \end{bmatrix}$$

$$\begin{cases} Q\omega = \lambda_{\max}\omega \\ \sum_{i=1}^n \omega_i = 1. \end{cases}$$

Then, the weight vector $\omega = \{0.15, 0.29, 0.21, 0.34\}$ is obtained.

Through investigation, it is found that the proportion of excellent, good, average, qualified and unqualified is 30%, 20%, 30%, 10% and 10%. Hence, the single factor evaluation A_1 on u_1 level is obtained, as does the other factor evaluation. Afterwards, a fuzzy evaluation matrix A is obtained, as follows:

$$A = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix}, \text{ assume } A = \begin{bmatrix} 0.3 & 0.2 & 0.3 & 0.1 & 0.1 \\ 0.3 & 0.3 & 0.2 & 0.2 & 0 \\ 0.2 & 0.2 & 0.4 & 0.2 & 0.1 \\ 0.3 & 0.4 & 0.2 & 0.1 & 0 \end{bmatrix}.$$

Then, the following can be obtained:

$$\begin{aligned} B = \omega \circ A &= (0.15, 0.29, 0.21, 0.35) \circ \begin{bmatrix} 0.3 & 0.2 & 0.3 & 0.1 & 0.1 \\ 0. & 0.3 & 0.2 & 0.2 & 0 \\ 0.2 & 0.2 & 0.4 & 0.2 & 0.1 \\ 0.3 & 0.4 & 0.2 & 0.1 & 0 \end{bmatrix} \\ &= (0.2790, 0.2990, 0.2570, 0.1500, 0.0360). \end{aligned}$$

After normalization, $B' = 0.273, 0.293, 0.252, 0.147, 0.035$ can be obtained.

It can be concluded that 27.3% of users rated as excellent, 29.3% of users rated as good, 25.2% of user rated as average, 14.7% of users rated as qualified while 3.5% of users considered the quality poor, i.e., unqualified. The results of the evaluation applying fuzzy comprehensive evaluation method have strong credibility and comparability. The overall service quality of the digital library integration service platform is mostly excellent, good and average, and we can judge that the platform is satisfactory to customers, though, improvement exist.

2. Conclusion

With the advent of the large data era and the rise of Internet of things, cloud computing and mobile Internet, the development of digital library integration services is becoming an inevitable trend, with its construction becoming more and more challenging. The importance of book resources has led the country to gradually attach importance to and develop the library service management. This paper studied the digital fusion service model of libraries based on hybrid metadata in an environment with rapid information technological development and state's support to studies of library service models. Libraries should correctly establish their own service goals, and their digital integration services not only need to provide information and knowledge resources, but also need to explore the deep value of digital cultural resources. In this paper, a library digital fusion service platform was built and a comprehensive evaluation of library fusion service quality applying the fuzzy mathematical model was carried out, though, there are still shortcomings. Particularly, the validity detection of mixed metadata and the practicality detection of the metadata management scheme need to be further studied.

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REPRESENTATION OF GRAPHS USING m -POLAR FUZZY ENVIRONMENT

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Abstract. In this research article, we introduce the concepts of products in m -polar fuzzy graphs and investigate some of their interesting properties. We describe various properties of certain m -polar fuzzy graphs. We establish formulae of distance in complete m -polar fuzzy graphs and complete bipartite m -polar fuzzy graphs. We present an algorithm for computing the distance matrix, eccentricity of the vertices, radius and diameter in m -polar fuzzy graphs. We also discuss applications of m -polar fuzzy graphs in traveling and product manufacturing.

Keywords: m -polar fuzzy graphs, algorithm, eccentricity, central vertices, peripheral vertices, decision support systems.

1. Introduction

A fuzzy set [21] is an important mathematical structure to represent a collection of objects whose boundary is vague. Fuzzy models are becoming useful because of their aim in reducing the differences between the traditional numerical models used in engineering and sciences and the symbolic models used in expert systems. In 1994, Zhang [23] introduced the notion of bipolar fuzzy sets and relations. Bipolar fuzzy sets are extension of fuzzy sets whose membership degree ranges $[-1, 1]$. The membership degree $(0, 1]$ indicates that the object satisfies a certain property whereas the membership degree $[-1, 0)$ indicates that the object satisfies the counter property. Positive information represent what is considered to be possible and negative information represent what is granted to be impossible. Actually, a variety of decision making problems are based on two-sided bipolar thinking and judgements on a positive side and a negative side. Recently, Chen et al. [9] generalized the idea of bipolar fuzzy sets to m -polar fuzzy sets. In an m -polar fuzzy set, the membership value ranges over $[0, 1]^m$. In lots of real World problems, data are sometimes come from n agents ($n \geq 2$), that is, multipolar information exist which cannot be represented

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well by means of the existing graphs such as fuzzy graphs (correspond to single valued logic), bipolar fuzzy graphs (correspond to two valued logic), etc. Considering that graphic structures, m -polar fuzzy sets can be used to describe the relationship among several individuals. m -polar fuzzy sets have many applications in decision making problems when it is necessary to make judgements with a group of agreements. For example, weighted games, a country elects its leader or a group of friends wants to plan to visit a country.

Based on Zadeh's fuzzy relations [22] Kaufmann defined in [11] a fuzzy graph. The fuzzy relations between fuzzy sets were also considered by Rosenfeld [18] and he developed the structure of fuzzy graphs, obtaining analogs of several graph theoretical concepts. Later on, Bhattacharya [8] gave some remarks on fuzzy graphs, and some operations on fuzzy graphs were introduced by Mordeson and Peng [13]. Tom and Sunitha [20] introduced the concept of sum distance in fuzzy graphs and studied some of its properties. Akram *et al.* [1-7] introduced many new concepts, including bipolar fuzzy graphs, certain notions of m -polar fuzzy graphs, m -polar fuzzy competition graphs and m -polar fuzzy hypergraphs. In this research article, we introduce the concepts of products in m -polar fuzzy graphs and investigate some of their interesting properties. We establish formulae of distance in complete m -polar fuzzy graphs, complete bipartite m -polar fuzzy graphs and certain products of m -polar fuzzy graphs. We present an algorithm for computing the distance matrix, eccentricity of the vertices, radius and diameter in m -polar fuzzy graphs. We also discuss applications of m -polar fuzzy graphs in traveling and product manufacturing.

2. Representation of graphs using m -polar fuzzy environment

Definition 2.1. [6] Let C be an m -polar fuzzy set on a non-empty crisp set X . An m -polar fuzzy relation is an m -polar fuzzy subset $D = (P_1 \circ D, P_2 \circ D, \dots, P_m \circ D)$ of $X \times X$ such that $D(xy) \leq \inf\{C(x), C(y)\}$, for all $x, y \in X$, that is, for all $x, y \in X$ and for each $1 \leq i \leq m$, $P_i \circ D(xy) \leq \inf\{P_i \circ C(x), P_i \circ C(y)\}$, where $P_i \circ C(x)$ denotes the i th degree of membership of the vertex x and $P_i \circ D(xy)$ denotes the i th degree of membership of the edge xy .

Definition 2.2. [6, 9] An m -polar fuzzy graph on a non-empty X is a pair $G = (C, D)$ where, $C : X \rightarrow [0, 1]^m$ is an m -polar fuzzy set on the set of vertices X and $D : X \times X \rightarrow [0, 1]^m$ is an m -polar relation such that $D(xy) \leq \inf\{C(x), C(y)\}$, for all $xy \in E$, and $D(xy) = \mathbf{0}$, for all $xy \in X \times X - E$ where, $\mathbf{0} = (0, 0, \dots, 0)$ and $E \subseteq X \times X$ is the set of edges.

Throughout this paper, we use G^* as a crisp graph and G as an m -polar fuzzy graph.

Definition 2.3. An m -polar fuzzy walk in an m -polar fuzzy graph is an alternating sequence of vertices and edges $y_0, e_1, y_1, e_2, \dots, e_{n-1}, y_{n-1}$ such that $P_i \circ C(y_{j-1}) > 0$ and $P_i \circ D(e_j) > 0$, for all $1 \leq j \leq n$, for at least one i .

Definition 2.4. An m -polar fuzzy path in an m -polar fuzzy graph is a sequence of distinct vertices $x_1, x_2, \dots, x_n$ such that $P_i \circ D(x_j x_{j+1}) > 0$, for all $1 \leq j \leq n-1$, for at least one i . It is denoted by $\tilde{P}_n$. The graph of $\tilde{P}_5$ is shown in Figure 2.4.

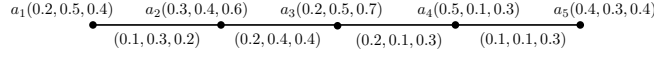


Figure 1: $\tilde{P}_5$

If $x_1 = x_n$, the m -polar fuzzy path is known as an m -polar fuzzy cycle, denoted by $\tilde{C}_n$.

Definition 2.5. The degree of a vertex x in an m -polar fuzzy graph $G = (C, D)$ is denoted by the m -tuple,

$$\deg(x) = (\deg^{(1)}(x), \deg^{(2)}(x), \dots, \deg^{(m)}(x)),$$

that is, $\deg(x) = (\sum_{xx_j \in E} P_1 \circ D(xx_j), \sum_{xx_j \in E} P_2 \circ D(xx_j), \dots, \sum_{xx_j \in E} P_m \circ D(xx_j))$.

If all vertices of G have same degree, then G is known as a *regular m -polar fuzzy graph*.

Definition 2.6. An m -polar fuzzy graph is known as a *complete m -polar fuzzy graph* if $P_i \circ D(xy) = P_i \circ C(x) \wedge P_i \circ C(y)$, for all $x, y \in X$, $1 \leq i \leq m$.

Definition 2.7. An m -polar fuzzy graph is known as *bipartite m -polar fuzzy graph* if the set of vertices X can be written as the union of two disjoint sets X_1 and X_2 such that, for some k and j ,

1. $P_i \circ D(x_k x_j) = 0$, if $x_k, x_j \in X_1$ or $x_k, x_j \in X_2$, for all $1 \leq i \leq m$,
2. $P_i \circ D(x_k x_j) > 0$, if $x_k \in X_1$ and $x_j \in X_2$ or $x_k \in X_2$ and $x_j \in X_1$, for at least one i .

Example 2.1. Let C be a 3-polar fuzzy set on $X = \{a_1, a_2, a_3\} \cup \{b_1, b_2, b_3\}$ and D be a 3-polar fuzzy relation in X . The bipartite 3-polar fuzzy graph G is shown in Fig. 2.1.

Definition 2.8. An m -polar fuzzy graph is called *complete bipartite m -polar fuzzy graph* if the set of vertices X can be written as the union of two disjoint sets X_1 and X_2 such that, for all k and j ,

1. $P_i \circ D(x_k x_j) = 0$, if $x_k, x_j \in X_1$ or $x_k, x_j \in X_2$, for all $1 \leq i \leq m$,
2. $P_i \circ D(x_k x_j) = P_i \circ C(x_k) \wedge P_i \circ C(x_j)$, if $x_k \in X_1$ and $x_j \in X_2$ or $x_k \in X_2$ and $x_j \in X_1$, for at least one i .

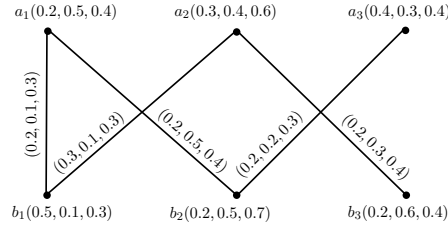
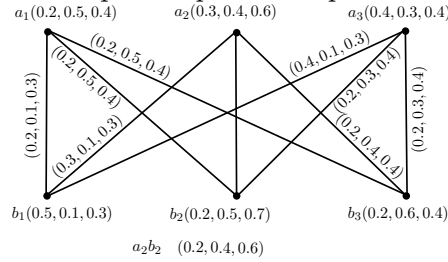


Figure 2: Bipartite 3-polar fuzzy graph

Figure 3: Complete bipartite 3-polar fuzzy graph



Example 2.2. An example of complete bipartite 3-polar fuzzy graph on the crisp graph $G^* = (X, E)$ where $X = \{a_1, a_2, a_3\} \cup \{b_1, b_2, b_3\}$, is shown in Fig. 3.

Definition 2.9. An m -polar fuzzy graph $H = (C', D')$ is called an m -polar fuzzy subgraph of m -polar fuzzy graph $G = (C, D)$ if, $C' \subseteq C$ and $D' \subseteq D$.

Definition 2.10. Let $G = (C, D)$ be an m -polar fuzzy graph. The P_i -strength of an m -polar fuzzy path $x_1 - x_2 - \dots - x_n$ is defined as, $S_{P_i}(x_1, x_n) = \inf\{P_i \circ D(x_k x_{k+1}) : 1 \leq k \leq n-1\}$. The strength of m -polar fuzzy path $x_1 - x_n$ is computed as, $S(x_1, x_n) = (S_{P_1}(x_1, x_n), S_{P_2}(x_1, x_n), \dots, S_{P_m}(x_1, x_n))$. A strongest path between any two vertices is the path with supremum strength. The strength of the strongest path $x - y$ is defined as the m -tuple $P^\infty(x, y) = (P_1^\infty(x, y), P_2^\infty(x, y), \dots, P_m^\infty(x, y))$, such that for all $x, y \in X$ and $1 \leq i \leq m$, $P_i^\infty(x, y) = \sup\{S_{P_i}(x, y), x - y \text{ is an } m\text{-polar fuzzy path in } G\}$. It is referred as *strength of connectedness* between x and y .

Example 2.3. Consider a 3-polar fuzzy graph as shown in Fig. 2.3. The strength of the path $b - a - c$ is $(0.3 \wedge 0.1, 0.2 \wedge 0.3, 0.1 \wedge 0.0) = (0.1, 0.2, 0)$ and that of $b - d - c$ is $(0.2 \wedge 0.2, 0.3 \wedge 0.4, 0.1 \wedge 0.1) = (0.2, 0.3, 0.1)$. The strength of connectedness between the vertices b and c is $(0.2, 0.3, 0.1)$.

Definition 2.11. For any m -polar fuzzy path, $R: x_1 - x_2 - \dots - x_n$, the P_i -length of R is defined as the sum of $P_i \circ D$ values of the edges, that is, $L_i(R) = \sum_{j=2}^n P_i \circ D(x_{j-1} x_j)$, for all $1 \leq i \leq m$. The length of m -polar fuzzy path R is represented by the m -tuple $L(R) = (L_1(R), L_2(R), \dots, L_m(R))$. For any two vertices x, y of

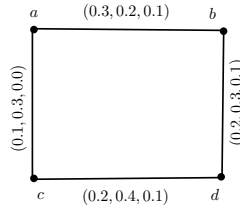


Figure 4: 3-polar fuzzy graph

G , let $\mathbb{R} = \{R_i, R_i \text{ is an } x-y \text{ } m\text{-polar fuzzy path, } i = 1, 2, 3, \dots\}$, be the set of all m -polar fuzzy paths from x to y . Then, P_i -distance of the path $x-y$, denoted by $d_i(x, y)$ and is defined as, $d_i(x, y) = \inf\{L_i(R_j) : R_j \in \mathbb{R}, j = 1, 2, 3, \dots\}$, for all $1 \leq i \leq m$. The distance of m -polar fuzzy path $x-y$, denoted by $d(x, y)$ or $d_G(x, y)$, is defined as the m -tuple $d(x, y) = (d_1(x, y), d_2(x, y), \dots, d_m(x, y))$ or $d_G(x, y) = (d_{1,G}(x, y), d_{2,G}(x, y), \dots, d_{m,G}(x, y))$.

Definition 2.12. Let $G_1 = (C_1, D_1)$ and $G_2 = (C_2, D_2)$ be two m -polar fuzzy graphs on X_1 and X_2 , respectively. The Cartesian product of G_1 and G_2 is denoted by $G_1 \square G_2$ and defined as a pair $(C_1 \square C_2, D_1 \square D_2)$, such that for each $1 \leq i \leq m$,

1. $P_i \circ (C_1 \square C_2)(x_1, x_2) = P_i \circ C_1(x_1) \wedge P_i \circ C_2(x_2)$, for all $(x_1, x_2) \in X_1 \times X_2$,
2. $P_i \circ (D_1 \square D_2)((x_1, x_2)(x_1, y_2)) = P_i \circ C_1(x_1) \wedge P_i \circ D_2(x_2 y_2)$, for all $x_1 \in X_1$ and $x_2 y_2 \in E_2$,
3. $P_i \circ (D_1 \square D_2)((x_1, x_2)(y_1, x_2)) = P_i \circ D_1(x_1 y_1) \wedge P_i \circ C_2(x_2)$, for all $x_2 \in X_2$ and $x_1 y_1 \in E_1$.

Example 2.4. The Cartesian product of two 3-polar fuzzy paths is shown in Fig. 2.4.

Theorem 2.1. Let $G_1 = (C_1, D_1)$ and $G_2 = (C_2, D_2)$ be two m -polar fuzzy graphs. If (x_1, x_2) and (y_1, y_2) are vertices of the Cartesian product $G_1 \square G_2$, then $d_{G_1 \square G_2}((x_1, x_2), (y_1, y_2)) \leq d_{G_1}(x_1, y_1) + d_{G_2}(x_2, y_2)$.

Proof. Assume that $d_{G_1}(x_1, y_1)$ and $d_{G_2}(x_2, y_2)$ are finite. Let $R_1, R_2, \dots, R_m$ be the m -polar fuzzy paths in G_1 and $Q_1, Q_2, \dots, Q_m$ are m -polar fuzzy paths in G_2 where, $R_i = x_{i1}, x_{i2}, \dots, x_{in} = y_1$ such that,

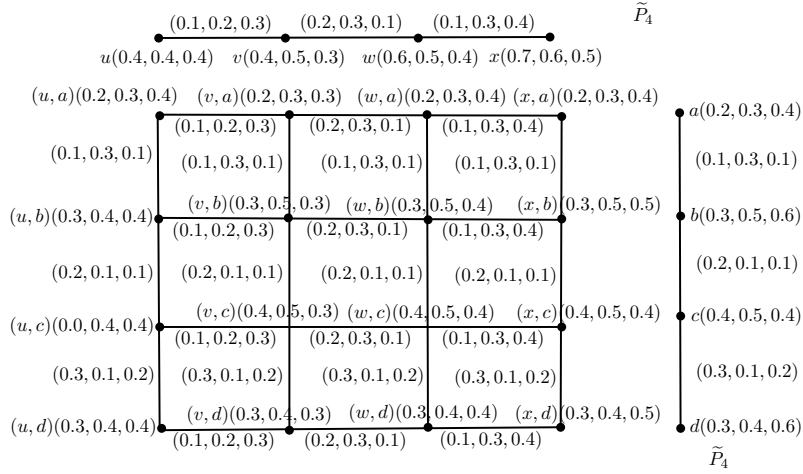
$$d_{G_1}(x_1, y_1) = (L_1(R_1), L_2(R_2), \dots, L_m(R_m)),$$

and $Q_i : x_2 = y_{i1}, y_{i2}, \dots, y_{in'} = y_2$ be an m -polar fuzzy path in G_2 such that,

$$d_{G_2}(x_2, y_2) = (L_1(Q_1), L_2(Q_2), \dots, L_m(Q_m)).$$

This establishes the following m -polar fuzzy paths in $G_1 \square G_2$,

$$\begin{aligned} R_i \times \{y_{i1}\} &= (x_{i1}, y_{i1}), (x_{i2}, y_{i1}), \dots, (x_{in}, y_{i1}) \\ \{x_{in}\} \times Q_i &= (x_{in}, y_{i1}), (x_{in}, y_{i2}), \dots, (x_{in}, y_{in'}), \quad 1 \leq i \leq m \end{aligned}$$

Figure 5: $\tilde{P}_4 \square P_4$.

whose join are the m -polar fuzzy paths of length $L_i(R_i \times \{x_2\}) + L_i(\{y_1\} \times Q_i)$, $1 \leq i \leq m$. It is clear that,

$$(2.1) \quad d_{i, G_1 \square G_2}((x_1, x_2), (y_1, y_2)) \leq L_i(R_i \times \{x_2\}) + L_i(\{y_1\} \times Q_i).$$

For each $1 \leq i \leq m$,

$$\begin{aligned} & L_i(R_i \times \{x_2\}) \\ &= P_i \circ D_1(x_{i1}x_{i2}) \wedge P_i \circ C_2(x_2) + P_i \circ D_1(x_{i2}x_{i3}) \\ & \wedge P_i \circ C_2(x_2) + \dots + P_i \circ D_1(x_{i, n-1}x_{in}) \wedge P_i \circ C_2(x_2), \\ & \leq P_i \circ D_1(x_{i1}x_{i2}) + P_i \circ D_1(x_{i2}x_{i3}) + \dots + P_i \circ D_1(x_{i, n-1}x_{in}), \\ &= L_i(R_i) \end{aligned}$$

$$(2.2) \quad L_i(R_i \times \{x_2\}) \leq L_i(R_i) = d_{i, G_1}(x_1, y_1),$$

$$(2.3) \quad \Rightarrow L_i(R_i \times \{x_2\}) \leq d_{i, G_1}(x_1, y_1).$$

By using similar argument, we can prove that

$$(2.4) \quad L_i(\{y_1\} \times Q_i) \leq L_i(Q_i) = d_{i, G_2}(x_2, y_2).$$

From Equation. (2.1), (2.2) and (2.4), we conclude that

$$\begin{aligned} & d_{i, G_1 \square G_2}((x_1, x_2), (y_1, y_2)) \leq d_{i, G_1}(x_1, y_1) + d_{i, G_2}(x_2, y_2), \quad 1 \leq i \leq m. \\ & \Rightarrow d_{G_1 \square G_2}((x_1, x_2), (y_1, y_2)) \leq d_{G_1}(x_1, y_1) + d_{G_2}(x_2, y_2). \quad \square \end{aligned}$$

Definition 2.13. Let $G_1^* \bullet G_2^* \bullet \dots \bullet G_k^*$ be any product of the graphs $G_1^*, G_2^*, \dots, G_k^*$ where, $\bullet$ represents any product, Cartesian product, direct product, strong product or lexicographic product. The mapping $f_{G_i} : G_1^* \bullet G_2^* \bullet \dots \bullet G_k^* \rightarrow G_i^*$, defined by

$$f_{G_i}(x_1, x_2, \dots, x_k) = x_i, \quad x_i \in V_i, 1 \leq i \leq k,$$

is called the projection of G_i^* onto $G_1^* \bullet G_2^* \bullet \dots \bullet G_k^*$.

Theorem 2.2. *Let S be an m -polar fuzzy path in $G_1 \square G_2$ and for all $1 \leq i, j \leq m$, $P_i \circ C_1 \geq P_j \circ D_2$ and $P_i \circ C_2 \geq P_j \circ D_1$ then, $L(S) = L(f_{G_1}(S)) + L(f_{G_2}(S))$.*

Proof. Let $P : x_1, x_1, \dots, x_n$ be an m -polar fuzzy path in G_1 and $Q : y_1, y_2, \dots, y_{n'}$ be an m -polar fuzzy path in G_2 . Let S be a path in $G_1 \square G_2$ which is established as follows,

$$S = (x_1, y_1), (x_2, y_1), \dots, (x_n, y_1), (x_n, y_1), (x_n, y_2), \dots, (x_n, y_{n'}),$$

Clearly, $f_{G_1}(S) = P$, $f_{G_2}(S) = Q$ and $L(S) = (L_1(S), L_2(S), \dots, L_m(S))$. It follows that,

$$\begin{aligned} L_i(S) &= P_i \circ D_1(x_1 x_2) \wedge P_i \circ C_2(y_1) + P_i \circ D_1(x_2 x_3) \wedge P_i \circ C_2(y_1) \\ &\quad + \dots + P_i \circ D_1(x_{n-1} x_n) \wedge P_i \circ C_2(y_1) + P_i \circ C_1(x_n) \wedge P_i \circ D_2(y_1 y_2) \\ &\quad + \dots + P_i \circ C_1(x_n) \wedge P_i \circ D_2(y_{n'-1} y_{n'}), \\ &= P_i \circ D_1(x_1 x_2) + P_i \circ D_1(x_2 x_3) + \dots + P_i \circ D_1(x_{n-1} x_n) \\ &\quad + P_i \circ D_2(y_1 y_2) + \dots + P_i \circ D_2(y_{n'-1} y_{n'}), \\ &= L(P) + L(Q) = L(f_{G_1}(S)) + L(f_{G_2}(S)). \end{aligned} \quad \square$$

Lemma 2.1. *Let G_1 and G_2 be two m -polar fuzzy graphs and (x_1, y_1) and (x_2, y_2) are vertices of a Cartesian product $G_1 \square G_2$. If for all $1 \leq i, j \leq m$, $P_i \circ C_1 \geq P_j \circ D_2$ and $P_i \circ C_2 \geq P_j \circ D_1$, then, $d_{G_1 \square G_2}((x_1, x_2), (y_1, y_2)) = d_{G_1}(x_1, y_1) + d_{G_2}(x_2, y_2)$.*

Proof. By Theorem. 2.1,

$$(2.5) \quad d_{G_1 \square G_2}((x_1, x_2), (y_1, y_2)) \leq d_{G_1}(x_1, y_1) + d_{G_2}(x_2, y_2).$$

Conversely, let $S_1, S_2, \dots, S_m$ be the shortest m -polar fuzzy paths between vertices (x_1, x_2) and (y_1, y_2) such that $d_{G_1 \square G_2}((x_1, x_2), (y_1, y_2)) = (L_1(S_1), L_2(S_2), \dots, L_m(S_m))$. The projections $f_{G_1}(S_i)$ and $f_{G_2}(S_i)$, $1 \leq i \leq m$, are the m -polar fuzzy paths between the vertices x_1 and y_1 in G_1 and x_2 and y_2 in G_2 . Consider,

$$\begin{aligned} &d_{G_1}(x_1, y_1) + d_{G_2}(x_2, y_2) \\ &\leq (L_1(f_{G_1}(S_1)), L_2(f_{G_1}(S_2)), \dots, L_m(f_{G_1}(S_m))) \\ (2.6) \quad &+ (L_1(f_{G_2}(S_1)), L_2(f_{G_2}(S_2)), \dots, L_m(f_{G_2}(S_m))), \\ &= (L_1(S_1), L_2(S_2), \dots, L_m(S_m)) = d_{G_1 \square G_2}((x_1, x_2), (y_1, y_2)). \end{aligned}$$

By combining Eqs. (2.5) and (2.6), required result is obtained. $\square$

Example 2.5. Consider the Cartesian product of two m -polar fuzzy paths in Fig. 2.4. It can be easily seen that, $d_{\tilde{P}_3 \square \tilde{P}_3}((u, a), (v, b)) = (0.2, 0.5, 0.4) = d_{\tilde{P}_3}(u, v) + d_{\tilde{P}_3}(a, b)$, $d_{\tilde{P}_3 \square \tilde{P}_3}((u, a), (u, c)) = (0.3, 0.4, 0.2) = d_{\tilde{P}_3}(u, u) + d_{\tilde{P}_3}(a, c)$. Similarly, for the other vertices.

Theorem 2.3. Let G_1 and G_2 be two m -polar fuzzy graphs. If $x_1 \in X_1$, $x_2 \in X_2$, and for all $1 \leq i, j \leq m$, $P_i \circ C_1 \geq P_j \circ D_2$ and $P_i \circ C_2 \geq P_j \circ D_1$, then, $\deg_{G_1 \square G_2}((x_1, x_2)) = \deg_{G_1}(x_1) + \deg_{G_2}(x_2)$.

Definition 2.14. The direct product of two m -polar fuzzy graphs $G_1 = (C_1, D_1)$ and $G_2 = (C_2, D_2)$ is denoted by $G_1 \times G_2$ and defined as a pair $(C_1 \times C_2, D_1 \times D_2)$, such that for each $1 \leq i \leq m$,

1. $P_i \circ (C_1 \times C_2)(x_1, x_2) = P_i \circ C_1(x_1) \wedge P_i \circ C_2(x_2)$, for all $(x_1, x_2) \in X_1 \times X_2$,
2. $P_i \circ (D_1 \times D_2)((x_1, x_2)(y_1, y_2)) = P_i \circ D_1(x_1 y_1) \wedge P_i \circ D_2(x_2 y_2)$, for all $x_1 y_1 \in E_1$ and $x_2 y_2 \in E_2$.

Example 2.6. The direct product of two 3-polar fuzzy paths is shown in Fig. 2.6.

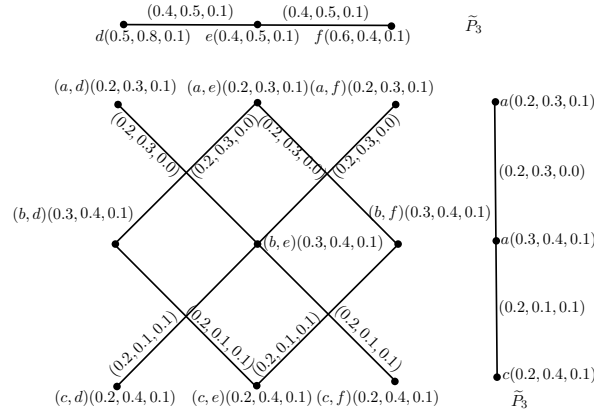


Figure 6: $\tilde{P}_3 \times \tilde{P}_3$

Theorem 2.4. The direct product $G_1 \times G_2$ of two m -polar fuzzy graphs G_1 and G_2 is an m -polar fuzzy graph.

Theorem 2.5. Let G_1 and G_2 be two m -polar fuzzy graphs. If $x_1, y_1 \in X_1$, $x_2, y_2 \in X_2$, and for each $1 \leq i, j \leq m$, $P_i \circ D_1 \leq P_j \circ D_2$. Let $R_1, R_2, \dots, R_m$ be the shortest m -polar paths between (x_1, x_2) and (y_1, y_2) such that, $d_{G_1 \times G_2}((x_1, x_2), (y_1, y_2)) = (L_1(R_1), L_2(R_2), \dots, L_m(R_m))$ then

$$d_{G_1 \times G_2}((x_1, x_2), (y_1, y_2)) = (L_1(f_{G_1}(R_1)), L_2(f_{G_1}(R_2)), \dots, L_m(f_{G_1}(R_m))).$$

Proof. Assume that for each $1 \leq i \leq m$, $R_i : (x_1, x_2) = (x_1^{(i1)}, x_2^{(i1)}), (x_1^{(i2)}, x_2^{(i2)}), \dots, (x_1^{(in)}, x_2^{(in)}) = (y_1, y_2)$ are the shortest m -polar fuzzy paths between (x_1, x_2) and (y_1, y_2) and,

$$(2.7) \quad d_{G_1 \times G_2}((x_1, x_2), (y_1, y_2)) = (L_1(R_1), L_2(R_2), \dots, L_m(R_m)).$$

If E is the set of edges in $G_1 \times G_2$ then for each $1 \leq i \leq m$,

$$\begin{aligned}
 & L_i(R_i) \\
 &= \sum_{(x_1^{(ij)}, x_2^{(ij)})(x_1^{(ij+1)}, x_2^{(ij+1)}) \in E} P_i \circ (D_1 \times D_2)(x_1^{(ij)}, x_2^{(ij)})(x_1^{(ij+1)}, x_2^{(ij+1)}), \\
 (2.8) \quad &= \sum_{x_1^{(ij)} x_1^{(ij+1)} \in E_1, x_2^{(ij)} x_2^{(ij+1)} \in E_2} P_i \circ D_1(x_1^{(ij)} x_1^{(ij+1)}) \wedge P_i \circ D_2(x_2^{(ij)} x_2^{(ij+1)}), \\
 &= \sum_{x_1^{(j)} x_1^{(j+1)} \in E_1} P_i \circ D_1(x_1^{(j)} x_1^{(j+1)}), \\
 &= L_i(f_{G_1}(R_i)).
 \end{aligned}$$

From Equation. (2.7) and (2.8),

$$d_{G_1 \times G_2}((x_1, x_2), (y_1, y_2)) = (L_1(f_{G_1}(R_1)), L_2(f_{G_1}(R_2)), \dots, L_m(f_{G_1}(R_m))). \square$$

Remark 2.1. If $P_i \circ D_2 \geq P_i \circ D_1$, and R_i , $1 \leq i \leq m$ are the shortest m -polar fuzzy paths between the vertices (x_1, x_2) and (y_1, y_2) of $G_1 \times G_2$ then,

$$d_{G_1 \times G_2}((x_1, x_2), (y_1, y_2)) = (L_1(f_{G_2}(R_1)), L_2(f_{G_2}(R_2)), \dots, L_m(f_{G_2}(R_m))).$$

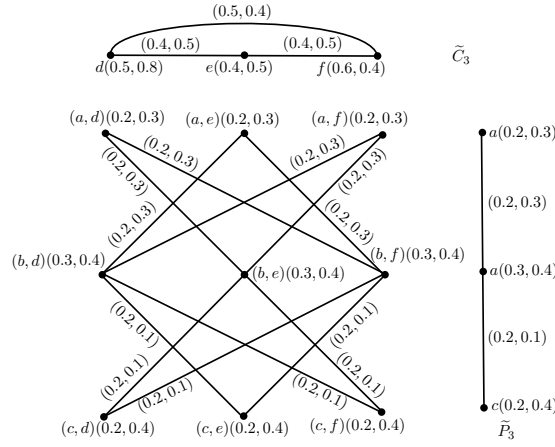
Example 2.7. In Fig. 2.6, the shortest 3-polar fuzzy path between the vertices (a, d) and (a, f) is $S : (a, d) - (b, e) - (a, f)$. $f_{\tilde{P}_3}(S) = a - b - a = P$. It can be easily seen that $d_{\tilde{P}_3 \times \tilde{P}_3}((a, d), (a, f)) = (0.4, 0.6, 0.0) = L(P)$. For the vertices (a, d) and (c, f) , the shortest 3-polar fuzzy path is $S : (a, d) - (b, e) - (c, f)$. Therefore, $d_{\tilde{P}_3 \times \tilde{P}_3}((a, d), (c, f)) = (0.4, 0.4, 0.1)$. The projection of S in $\tilde{P}_3$ is $a - b - c$ whose length is equal to $d_{\tilde{P}_3 \times \tilde{P}_3}((a, d), (c, f))$.

Theorem 2.6. Let G_1 and G_2 be two m -polar fuzzy graphs such that $P_i \circ D_1 \leq P_j \circ D_2$, for all $1 \leq i, j \leq m$. For any two vertices (x_1, x_2) and (y_1, y_2) of the direct product $G_1 \times G_2$, let k be a smallest positive integer such that G_1^* has a x_1, y_1 -walk of length k and G_2^* has a x_2, y_2 -walk of length k . The i th distance between (x_1, x_2) and (y_1, y_2) is the smallest P_i -length of any m -polar fuzzy walk between x_1 and y_1 whose length in the crisp graph G_1^* is k .

Example 2.8. The 2-polar fuzzy graph in Fig. 2.8 is the direct product of $\tilde{P}_3$ and $\tilde{C}_3$. Take the vertices (a, d) and (c, e) . The smallest $a - c$ and $d - e$ walks are of length 2. Therefore, the distance between (a, d) and (c, e) must be the smallest length of a 2-polar fuzzy walk in $\tilde{P}_3$ whose length in P_3 is 2. Such walk in $\tilde{P}_3 \times \tilde{C}_3$ is $(a, d) - (b, f) - (c, e)$. Hence, $d_{\tilde{P}_3 \times \tilde{C}_3}((a, d), (c, e)) = (0.4, 0.4)$.

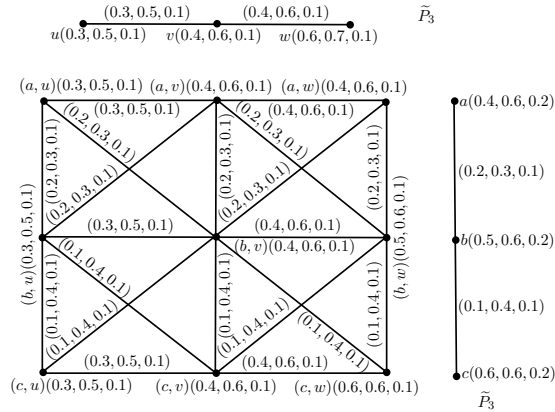
Theorem 2.7. Let G_1 and G_2 be two m -polar fuzzy graphs. If $x_1 \in X_1$ and $x_2 \in X_2$, and for all $1 \leq i, j \leq m$, $P_i \circ D_1 \geq P_j \circ D_2$, then, $\deg_{G_1 \times G_2}((x_1, x_2)) = (\text{number of vertices adjacent to } x_2) \deg_{G_1}(x_1)$. If $P_i \circ D_2 \geq P_i \circ D_1$, $\deg_{G_1 \times G_2}((x_1, x_2)) = (\text{number of vertices adjacent to } x_1) \deg_{G_2}(x_2)$.

Definition 2.15. The strong product of two m -polar fuzzy graphs $G_1 = (C_1, D_1)$ and $G_2 = (C_2, D_2)$, denoted by $G_1 \boxtimes G_2$ and defined as a pair $(C_1 \boxtimes C_2, D_1 \boxtimes D_2)$, such that for each $1 \leq i \leq m$,

Figure 7: $\tilde{P}_3 \times \tilde{C}_3$

1. $P_i \circ (C_1 \boxtimes C_2)(x_1, x_2) = P_i \circ C_1(x_1) \wedge P_i \circ C_2(x_2)$, for all $(x_1, x_2) \in X_1 \times X_2$,
2. $P_i \circ (D_1 \boxtimes D_2)((x_1, x_2)(x_1, y_2)) = P_i \circ C_1(x_1) \wedge P_i \circ D_2(x_2 y_2)$, for all $x_1 \in X_1$ and $x_2 y_2 \in E_2$,
3. $P_i \circ (D_1 \boxtimes D_2)((x_1, x_2)(y_1, x_2)) = P_i \circ D_1(x_1 y_1) \wedge P_i \circ C_2(x_2)$, for all $x_2 \in X_2$ and $x_1 y_1 \in E_1$,
4. $P_i \circ (D_1 \boxtimes D_2)((x_1, x_2)(y_1, y_2)) = P_i \circ D_1(x_1 y_1) \wedge P_i \circ D_2(x_2 y_2)$, for all $x_1 y_1 \in E_1$ and $x_2 y_2 \in E_2$.

Example 2.9. Fig. 2.9 is an example of strong product of two 3-polar fuzzy paths $\tilde{P}_3$ and $\tilde{P}_3$.

Figure 8: $\tilde{P}_3 \boxtimes \tilde{P}_3$

Theorem 2.8. The strong product $G_1 \boxtimes G_2$ of two m -polar fuzzy graphs G_1 and G_2 is an m -polar fuzzy graph.

Theorem 2.9. Let G_1 and G_2 be two m -polar fuzzy graphs such that for all $1 \leq i, j \leq m$, $P_i \circ C_1 \geq P_j \circ D_2$, $P_i \circ C_2 \geq P_j \circ D_1$ and $P_i \circ D_1 \leq P_j \circ D_2$, the following conditions are satisfied,

1. If $x_2 = y_2$, $d_{G_1 \boxtimes G_2}((x_1, x_2), (y_1, y_2)) = d_{G_1}(x_1, y_1)$.
2. If $x_1 = y_1$ and $x_2 \neq y_2$ or $x_1 \neq y_1$ and $x_2 \neq y_2$ then, $d_{i, G_1 \boxtimes G_2}((x_1, x_2), (y_1, y_2)) = L_i(W) \wedge L_i(T)$ where,
 - a) W is an m -polar fuzzy walk of smallest length in $G_1 \times G_2$ from (x_1, x_2) to (y_1, y_2) whose length in crisp direct product is the positive integer k such that $k = d_{G_1^*(x_1, y_1)} \vee d_{G_2^*(x_2, y_2)}$.
 - b) $L_i(T)$ is the smallest P_i -length of any m -polar fuzzy walk T , from x_1 to y_1 , in G_1 such that the length of T^* is greater than k .

Example 2.10. Consider the strong product in Fig. 2.9,

1. $d_{\tilde{P}_3 \boxtimes \tilde{P}_3}((a, u), (c, u)) = (0.3, 0.7, 0.2) = d_{\tilde{P}_3}(a, c)$.
2. $d_{\tilde{P}_3 \boxtimes \tilde{P}_3}((a, u), (a, w)) = (0.4, 0.6, 0.2) = L(W)$, $W : a - b - a$.
3. $d_{\tilde{P}_3 \boxtimes \tilde{P}_3}((a, u), (c, w)) = (0.3, 0.7, 0.2) = L(W)$, here $W = (a, u) - (b, v) - (c, w)$. It clear that $k = L(W^*) = d_{P_3}(a, c) \vee d_{P_3}(u, w)$.
4. $d_{\tilde{P}_3 \boxtimes \tilde{P}_3}((a, u), (b, w)) = (0.4, 0.9, 0.2) = (L_1(f_{\tilde{P}_3}(W_1)), L_2(W), L_3(W))$ where, $W = (a, u) - (b, v) - (b, w)$, $d_{P_3}(a, b) \vee d_{P_3}(u, w) = 2 = L(W^*)$ and $W_1 = (a, u) - (b, v) - (c, w) - (b, w)$. Clearly, $f_{\tilde{P}_3}(W_1) = a - b - c - b$ whose crisp length is 3 which is greater than k and $L_1(f_{\tilde{P}_3}(W_1)) < L(W)$.

Theorem 2.10. Let G_1 and G_2 be two m -polar fuzzy graphs. If $x_1 \in X_1$ and $x_2 \in X_2$, and for each $1 \leq i, j \leq m$, $P_i \circ C_1 \geq P_j \circ D_2$, $P_i \circ C_2 \geq P_j \circ D_1$ and $P_i \circ D_1 \leq P_j \circ D_2$. then, $\deg_{G_1 \boxtimes G_2}((x_1, x_2)) = \deg_{G_1}(x_1) + \deg_{G_2}(x_2) + r_2 \deg_{G_1}(x_1)$, where r_2 is the number of vertices adjacent to x_2 .

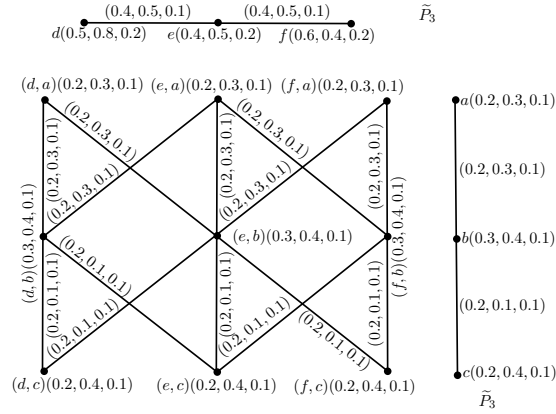
Definition 2.16. The *lexicographic product* of two m -polar fuzzy graphs $G_1 = (C_1, D_1)$ and $G_2 = (C_2, D_2)$, denoted by $G_1 \circ G_2$, is defined as a pair $(C_1 \circ C_2, D_1 \circ D_2)$, such that for each $1 \leq i \leq m$,

1. $P_i \circ (C_1 \circ C_2)(x_1, x_2) = P_i \circ C_1(x_1) \wedge P_i \circ C_2(x_2)$, for all $(x_1, x_2) \in X_1 \times X_2$,
2. $P_i \circ (D_1 \circ D_2)((x, x_2)(x, y_2)) = P_i \circ C_1(x) \wedge P_i \circ D_2(x_2 y_2)$ for all $x \in X_1$ and $x_2 y_2 \in E_2$,
3. $P_i \circ (D_1 \circ D_2)((x_1, x_2)(y_1, y_2)) = P_i \circ D_1(x_1 y_1) \wedge P_i \circ D_2(x_2 y_2)$, for all $x_1 y_1 \in E_1$ and $x_2 y_2 \in E_2$.

Example 2.11. The lexicographic product of $\tilde{P}_3$ and $\tilde{P}_3$ is given in Fig. 2.11.

Theorem 2.11. The lexicographic product $G_1 \circ G_2$ of two m -polar fuzzy graphs G_1 and G_2 is an m -polar fuzzy graph.

Theorem 2.12. Let G_1 and G_2 be two m -polar fuzzy graphs such that for each $1 \leq i \leq m$, $P_i \circ C_1 \geq P_i \circ D_2$, $P_i \circ C_2 \geq P_i \circ D_2$ and $P_i \circ D_1 \geq P_i \circ D_2$, the following conditions are satisfied.

Figure 9: $\tilde{P}_3 \circ \tilde{P}_3$

1. If $d_{G_1^*}(x_1, y_1) = k$, where k is even (or odd) and $d_{G_2^*}(x_2, y_2)$ is also even (or odd) then, $d_{i, G_1 \circ G_2}((x_1, x_2), (y_1, y_2)) = L_i(W)$, where W is an m -polar fuzzy walk of smallest P_i -length in G_2 such that W^* is a walk of length k in G_2^* .
2. If $d_{G_1^*}(x_1, y_1) = k$, where k is even (or odd) and $d_{G_2^*}(x_2, y_2)$ is odd (or even) then, $d_{i, G_1 \circ G_2}((x_1, x_2), (y_1, y_2)) = L_i(W)$, where W is an m -polar fuzzy walk of smallest P_i -length in G_2 such that W^* is a walk of length $k + 1$ in G_2^* .

Example 2.12. In Figure. 2.11,

1. $d_{\tilde{P}_3 \circ \tilde{P}_3}((d, a), (f, c)) = (0.4, 0.4, 0.2) = L(W)$, $W = a - b - c$. It clear that $L(W^*) = d_{P_3}(d, f)$ because both $d_{P_3}(d, f)$ and $d_{P_3}(a, c)$ are even.
2. $d_{\tilde{P}_3 \circ \tilde{P}_3}((d, a), (f, b)) = (0.6, 0.5, 0.2) = L(W)$, $W = a - b - c - b$. Here $L(W^*) = d_{P_3}(d, f) + 1$ because $d_{P_3}(d, f)$ is even and $d_{P_3}(a, b)$ is odd.

Theorem 2.13. Let G_1 and G_2 be two m -polar fuzzy graphs. If $x_1 \in X_1$ and $x_2 \in X_2$, and for each $1 \leq i, j \leq m$, $P_i \circ C_1 \geq P_j \circ D_2$, $P_i \circ C_2 \geq P_j \circ D_1$ and $P_i \circ D_1 \leq P_j \circ D_2$ then, $\deg_{G_1 \circ G_2}((x_1, x_2)) = \deg_{G_2}(x_2) + r_2 \deg_{G_1}(x_1)$, where r_2 is the number of vertices adjacent to x_2 .

We now define the concept of metric in m -polar fuzzy graphs.

Theorem 2.14. For an m -polar fuzzy graph $G = (C, D)$, $d = (d_1, d_2, \dots, d_m) : X \times X \rightarrow [0, 1]^m$ defines a metric on X , with the following conditions:

- (1) $d(x, y) \geq \mathbf{0}$,
- (2) $d(x, y) = \mathbf{0} \Leftrightarrow x = y$,
- (3) $d(x, y) = d(y, x)$,
- (4) $d(x, z) \leq d(x, y) + d(y, z)$, for all $x, y, z \in X$.

Proof. It is very clear from the definition of d_i , for each $1 \leq i \leq m$ $d_i(x, y) \geq 0$, for all $x, y \in X \Rightarrow d(x, y) \geq \mathbf{0}$. If $x = y$ then, $d(x, y) = d(x, x) = (d_1(x, x), d_2(x, x), \dots, d_m(x, x)) = (0, 0, \dots, 0)$. The inverse of any $x-y$ m -polar fuzzy path is a $y-x$ m -polar fuzzy path and vice versa of same distance. Let $P_1, P_2, \dots, P_m$ be $x-y$ m -polar fuzzy paths and $Q_1, Q_2, \dots, Q_m$ be $y-z$ m -polar fuzzy paths such that

$$\begin{aligned} d(x, y) &= (d_1(x, y), d_2(x, y), \dots, d_m(x, y)) = (L_1(P_1), L_2(P_2), \dots, L_m(P_m)), \\ d(y, z) &= (d_1(y, z), d_2(y, z), \dots, d_m(y, z)) = (L_1(Q_1), L_2(Q_2), \dots, L_m(Q_m)). \end{aligned}$$

P_1 followed by Q_1 , P_2 followed by Q_2 and so on P_m followed by Q_m are $x-z$ m -polar fuzzy walks, each of which contains only one m -polar fuzzy path whose length cannot exceed $d_1(x, y) + d_1(y, z)$, $d_2(x, y) + d_2(y, z)$ and so on $d_m(x, y) + d_m(y, z)$, respectively. Thus, we can write,

$$\begin{aligned} (d_1(x, z), d_2(x, z), \dots, d_m(x, z)) &\leq (d_1(x, y) + d_1(y, z), d_2(x, y) + d_2(y, z), \\ &\quad \dots, d_m(x, y) + d_m(y, z)) \\ (d_1(y, z), d_2(y, z), \dots, d_m(y, z)) &\leq (d_1(x, y), d_2(x, y), \dots, d_m(x, y)) \\ &\quad + (d_1(y, z), d_2(y, z), \dots, d_m(y, z)), \\ d(x, z) &\leq d(x, y) + d(y, z). \quad \square \end{aligned}$$

Definition 2.17. Let $G = (C, D)$ be an m -polar fuzzy graph. The eccentricity of a vertex x is denoted by the m -tuple $e(x) = (e_1(x), e_2(x), \dots, e_m(x))$ and defined as the distance to a vertex farthest from x , i.e.,

$$\begin{aligned} e_i(x) &= \max\{d_i(x, y) : y \in X\}, \quad 1 \leq i \leq m, \\ e(x) &= \max\{d(x, y) : y \in X\}. \end{aligned}$$

Definition 2.18. The radius of an m -polar fuzzy graph is the minimum of all the eccentricities of the vertices, i.e., $r(G) = \min\{e(x) : x \in X\}$.

Definition 2.19. The diameter of m -polar fuzzy graph is, denoted by $diam(G)$, defined as the maximum of all of the eccentricities of the vertices, i.e., $diam(G) = \max\{e(y) : y \in X\}$.

Definition 2.20. A vertex y at a distance $e(x)$ from x is called *eccentric vertex* of x .

Definition 2.21. A vertex x is called a *central vertex* if $e(x) = r(G)$.

Definition 2.22. The m -polar fuzzy subgraph induced by the central vertices is known as *center* of the m -polar fuzzy graph. If the center of G is G itself then G is called a *self-centered m -polar fuzzy graph*.

Definition 2.23. A vertex y is called a *peripheral vertex* if $e(y) = diam(G)$.

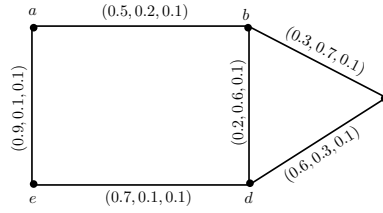


Figure 10: 3-polar fuzzy graph

Example 2.13. Consider an 3-polar fuzzy graph as shown in Fig. 2.13. Here, d is a central vertex and c is a peripheral vertex

Theorem 2.15. For any m -polar fuzzy graph G , radius and diameter satisfy the inequality, $r(G) \leq \text{diam}(G) \leq 2r(G)$

Proof. By Definition 2.18 and 2.19, we have $r_i(G) \leq \text{diam}_i(G)$, $1 \leq i \leq m$, $r(G) = (r_1(G), r_2(G), \dots, r_m(G))$, $\Rightarrow r(G) \leq (\text{diam}_1(G), \text{diam}_2(G), \dots, \text{diam}_m(G))$, $\Rightarrow r(G) \leq \text{diam}(G)$. Let u, v be central vertices of G , respectively then, $e(u) = r(G)$ and $e(v) = r(G)$. Let x be a peripheral vertex of G . Since d defines a metric, for some vertices $y_1, y_2, \dots, y_m \in X$,

$$\begin{aligned}
 \text{diam}(G) &= (\text{diam}_1(G), \text{diam}_2(G), \dots, \text{diam}_m(G)), \\
 &= (d_1(x, y_1), (d_2(x, y_2), \dots, d_m(x, y_m)), \\
 &\leq (d_1(x, u) + d_1(u, y_1), d_2(x, u) + d_2(u, y_2), \dots, d_m(x, u) + d_m(u, y_m)), \\
 &\quad \text{for some } u, v \in X \\
 &\leq (2r_1(G), 2r_2(G), \dots, 2r_m(G)), \\
 &= 2r(G).
 \end{aligned}$$

□

The following theorem gives an absolute difference between the eccentricities of any two adjacent vertices.

Theorem 2.16. For any two adjacent vertices x and y in an m -polar fuzzy graph G , $|e(x) - e(y)| \leq 1$.

By assuming any two arbitrary vertices x and y in Theorem 2.16, we obtain the following result.

Theorem 2.17. For any two vertices x and y in an m -polar fuzzy graph G , $|e(x) - e(y)| \leq d(x, y)$.

Theorem 2.18. For any two adjacent vertices x and y in an m -polar fuzzy graph G , $|d(x, z) - d(y, z)| \leq 1$, for every vertex z in G .

We now generalize Theorem. 2.18 for any two vertices x and y in the following theorem.

Theorem 2.19. For any two vertices x and y in an m -polar fuzzy graph G , $|d(x, z) - d(y, z)| \leq d(x, y)$.

Theorem 2.20. If G is a self centered m -polar fuzzy graph, then each vertex of G is an eccentric vertex.

Proof. Let y be an eccentric vertex of x then, $e(x) = d(x, y)$. Since G is self centered, $e(x) = e(y) = r(G)$, $e(y) = d(x, y)$. It shows that x is an eccentric vertex of y . Since x was taken to be arbitrary, theorem is true for all vertices. $\square$

Remark 2.2. The converse of Theorem. 2.20 is not true in general as it can be seen in the following example.

Example 2.14. Consider a 2-polar fuzzy graph as shown in the Fig. 2.14. $e(x) = (0.9, 0.5)$, $e(y) = (0.9, 0.5)$, $e(z) = (0.8, 0.5)$, $e(w) = (0.8, 0.5)$, $r(G) =$

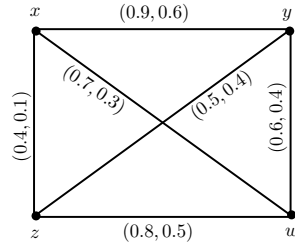
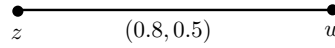


Figure 11: 2-polar fuzzy graph

$(0.8, 0.5)$, $\text{diam}(G) = (0.9, 0.5)$. All vertices of G are eccentric but G is not self-centered 2-polar fuzzy graph because the center of G is shown as,



Theorem 2.21. Let G be a self centered m -polar fuzzy graph. For every two vertices $x, y \in G$ if $x \in Y^*$, then $y \in X^*$, where X^* is the set of all eccentric vertices of x any Y^* is the set of all eccentric vertices of y .

Proof. Since x is a eccentric vertex of y , $e(x) = d(x, y) \Rightarrow x \in Y^*$. It is given that G is a self centered m -polar fuzzy graph, so $e(y) = e(x) = d(y, x)$, i.e., y is an eccentric vertex of x . Hence $y \in X^*$. $\square$

Remark 2.3. The converse of Theorem. 2.21 is not true in general. From example 2.14, $x \in Y^*$, $y \in X^*$, $z \in W^*$ and $w \in Z^*$ but G is not self-centered.

Theorem 2.22. Let G be an m -polar fuzzy graph then all peripheral vertices are eccentric vertices.

Proof. Let x be a peripheral vertex and y be its eccentric vertex then, $\text{diam}(G) = e(x) = d(x, y) = d(y, x)$. It is only possible if $\text{diam}(G) = d(y, x) = e(y)$. It shows that x is an eccentric vertex of y . Hence the proof. $\square$

Remark 2.4. The condition in Theorem. 2.22 is not sufficient. In example 2.14, z and w are eccentric vertices of each other but these are not peripheral vertices.

Theorem 2.23. Let G be a complete m -polar fuzzy graph such that $P_i \circ C(x_1) \leq P_i \circ C(x_2) \leq P_i \circ C(x_3) \leq \dots \leq P_i \circ C(x_n)$, for each $1 \leq i \leq m$, then the distance between any two vertices x_l and x_j is $(P_1 \circ D(x_l x_j) \wedge 2P_1 \circ C(x_1), P_m \circ D(x_l x_j) \wedge 2P_2 \circ C(x_1), \dots, P_m \circ D(x_l x_j) \wedge 2P_m \circ C(x_1))$.

Proof. Let x_l and x_j be any two vertices of G then for some $x_k^{(1)}, x_k^{(2)}, \dots, x_k^{(m)} \in X$, the distance between x_l and x_j can be defined as,

$$\begin{aligned}
 d(x_l, x_j) &= (d_1(x_l, x_j), d_2(x_l, x_j), \dots, d_m(x_l, x_j)), \\
 &= (\inf\{P_1 \circ D(x_l x_j), P_1 \circ D(x_l x_k^{(1)}) + P_1 \circ D(x_k^{(1)} x_j)\}, \\
 (2.9) \quad &\inf\{P_2 \circ D(x_l x_j), P_2 \circ D(x_l x_k^{(2)}) + P_2 \circ D(x_k^{(2)} x_j)\}, \\
 &\dots, \inf\{P_m \circ D(x_l x_j), P_m \circ D(x_l x_k^{(m)}) + P_m \circ D(x_k^{(m)} x_j)\}).
 \end{aligned}$$

Since G is a complete m -polar fuzzy graph, for each $1 \leq i \leq m$, $P_i \circ D(x_l x_k^{(i)}) = P_i \circ C(x_l) \wedge P_i \circ C(x_k^{(i)})$. Since, for each $1 \leq i \leq m$, $P_i \circ C(x_1) \leq P_i \circ C(x_2) \leq P_i \circ C(x_3) \leq \dots \leq P_i \circ C(x_n)$. Therefore, for $x_k^{(i)} = x_1$, $P_i \circ D(x_l x_k^{(i)}) = P_i \circ C(x_k^{(i)}) = P_i \circ C(x_1)$. Similarly, $P_i \circ D(x_k^{(i)} x_j) = P_i \circ C(x_k^{(i)}) = P_i \circ C(x_1)$. Equation. (2.10) takes the form as, $d(x_l, x_j) = (P_1 \circ D(x_l x_j) \wedge 2P_1 \circ C(x_1), P_2 \circ D(x_l x_j) \wedge 2P_2 \circ C(x_1), \dots, P_m \circ D(x_l x_j) \wedge 2P_m \circ C(x_1))$. $\square$

Theorem 2.24. Let G be a complete bipartite m -polar fuzzy graph where, $X_1 = \{x_1, x_2, \dots, x_n\}$ and $X_2 = \{y_1, y_2, \dots, y_{n'}\}$, such that for each $1 \leq i \leq m$, $P_i \circ C(x_1) \leq P_i \circ C(x_2) \leq P_i \circ C(x_3) \leq \dots \leq P_i \circ C(x_n)$, and $P_i \circ C(y_1) \leq P_i \circ C(y_2) \leq P_i \circ C(y_3) \leq \dots \leq P_i \circ C(y_{n'})$. If $P_i \circ C(x_1) \leq P_i \circ C(y_1)$ and $P_i \circ C(y_1) \leq P_i \circ C(x_j)$, for each $2 \leq j \leq n$, then,

$$d_i(u, v) = \begin{cases} 2P_i \circ C(x_1), & u, v \in X_2 \\ P_i \circ C(x_1) + P_i \circ C(y_1), & u, v \in X_1, u = x_1 \text{ or } v = x_1 \\ 2P_i \circ C(y_1), & u, v \in X_1, u \neq x_1, v \neq x_1 \\ P_1 \circ D(uv) \wedge (2P_1 \circ C(x_1) + P_1 \circ C(y_1)), & u \in X_1 \text{ and } v \in X_2. \end{cases}$$

If $P_i \circ C(y_1) \leq P_i \circ C(x_1)$ and $P_i \circ C(x_1) \leq P_i \circ C(y_j)$, for each $2 \leq j \leq n'$, then

$$d_i(u, v) = \begin{cases} 2P_i \circ C(y_1), & u, v \in X_1 \\ P_i \circ C(y_1) + P_i \circ C(x_1), & u, v \in X_2, u = y_1 \text{ or } v = y_1 \\ 2P_i \circ C(x_1), & u, v \in X_2, u \neq y_1, v \neq y_1 \\ P_1 \circ D(uv) \wedge (2P_1 \circ C(y_1) + P_1 \circ C(x_1)), & u \in X_1 \text{ and } v \in X_2. \end{cases}$$

Proof. Consider the case $P_i \circ C(x_1) \leq P_i \circ C(y_1)$ and $P_i \circ C(y_1) \leq P_i \circ C(x_j)$, for each $2 \leq j \leq n$. Let $u, v \in X_1$ be any two vertices of G then for some $y_k^{(1)}, y_k^{(2)}, \dots, y_k^{(m)} \in X$, the distance between u and v can be defined as:

$$\begin{aligned} d(u, v) &= (d_1(u, v), d_2(u, v), \dots, d_m(u, v)), \\ (2.10) \quad &= (P_1 \circ D(uy_k^{(1)}) + P_1 \circ D(y_k^{(1)}v), P_2 \circ D(uy_k^{(2)}) + P_2 \circ D(y_k^{(2)}v), \\ &\dots, P_m \circ D(uy_k^{(m)}) + P_m \circ D(y_k^{(m)}v)). \end{aligned}$$

Since, $P_i \circ D(uy_k^{(i)}) = P_i \circ C(u) \wedge P_i \circ C(y_k^{(i)})$. If $u = x_1$, then for $y_k^{(i)} = y_1$, $P_i \circ D(uy_k^{(i)}) = P_i \circ C(u) = P_i \circ C(x_1)$. Similarly, $P_i \circ D(y_k^{(i)}v) = P_i \circ C(y_k^{(i)}) = P_i \circ C(y_1)$. Equation. (2.10) takes the form as, $d(u, v) = (P_1 \circ C(x_1) + P_1 \circ C(y_1), P_2 \circ C(x_1) + P_2 \circ C(y_1), \dots, P_m \circ C(x_1) + P_m \circ C(y_1))$. If $u, v \neq x_1$, then for $y_k^{(i)} = y_1$, $P_i \circ D(uy_k^{(i)}) = P_i \circ C(y_k^{(i)}) = P_i \circ C(y_1)$. Similarly, $P_i \circ D(y_k^{(i)}v) = P_i \circ C(y_k^{(i)}) = P_i \circ C(y_1)$. Equation. (2.10) takes the form as, $d(u, v) = (2P_1 \circ C(y_1), 2P_2 \circ C(y_1), \dots, 2P_m \circ C(y_1))$. If $u, v \in X_2$, there exist some $x_k^{(1)}, x_k^{(2)}, \dots, x_k^{(m)} \in X_1$ such that,

$$\begin{aligned} d(u, v) &= (d_1(u, v), d_2(u, v), \dots, d_m(u, v)), \\ (2.11) \quad &= (P_1 \circ D(ux_k^{(1)}) + P_1 \circ D(x_k^{(1)}v), P_2 \circ D(ux_k^{(2)}) + P_2 \circ D(x_k^{(2)}v), \\ &\dots, P_m \circ D(ux_k^{(m)}) + P_m \circ D(x_k^{(m)}v)). \end{aligned}$$

Since $P_i \circ C(x_1) \leq P_i \circ C(y_1)$, for $x_k^{(i)} = x_1$, $P_i \circ D(ux_k^{(i)}) = P_i \circ C(x_k^{(i)}) = P_i \circ C(x_1)$. Similarly, $P_i \circ D(x_k^{(i)}v) = P_i \circ C(x_k^{(i)}) = P_i \circ C(x_1)$. Equation. (2.11) takes the form as, $d(x_l, x_j) = (2P_1 \circ C(x_1), 2P_2 \circ C(x_1), \dots, 2P_m \circ C(x_1))$. If $u \in X_1$ and $v \in X_2$,

$$\begin{aligned} d(u, v) &= (d_1(u, v), d_2(u, v), \dots, d_m(u, v)), \\ (2.12) \quad &= (P_1 \circ D(uv) \wedge (P_1 \circ D(uy_1) + P_1 \circ D(y_1x_1) + P_1 \circ D(x_1v)), \\ &P_2 \circ D(uv) \wedge (P_2 \circ D(uy_1) + P_2 \circ D(y_2x_2) + P_2 \circ D(x_1v))), \\ &\dots, P_m \circ D(uv) \wedge (P_m \circ D(uy_1) + P_1 \circ D(y_1x_1) + P_m \circ D(x_1v))). \end{aligned}$$

Equation (2.12) becomes, $d(u, v) = (P_1 \circ D(uv) \wedge (2P_1 \circ C(x_1) + P_1 \circ C(y_1)), P_2 \circ D(uv) \wedge (2P_2 \circ C(x_1) + P_2 \circ C(y_1)), \dots, P_m \circ D(uv) \wedge (2P_m \circ C(x_1) + P_1 \circ C(y_m)))$. Other cases can be proved similarly. $\square$

We now present an algorithm for computing distance, eccentricity of vertices, radius and diameter of any m -polar fuzzy graphs.

Description and Time Complexity: First the algorithm takes number of vertices v and pth , $1 \leq p \leq m$, adjacency matrix of membership values $a(i, j)$ as input. Starting from a vertex r , lines 1 – 5 find a vertex i adjacent to r such that ri has the least weight. Lines 6 – 14 calculate the distances of vertex r to

Algorithm

1. Input: Enter the number of vertices v and the i th adjacency matrix of membership values $a(i, j)$ row-wise.
 2. Output: The distance between all the vertices, eccentricity of vertices, radius and diameter.
 3. **do** r from 1 to v
 4. distance = ∞
 5. find a vertex i adjacent to r with minimum weight
 6. Take $u(n) = i$, $\text{sum}(r, i) = a(r, i)$ and $\text{sum}(r, k) = \infty$
 7. **do** k from 1 to v
 8. **if** ($\text{sum}(r, u(n)) + a(u(n), k) < \text{sum}(r, k)$)
 9. $\text{sum}(r, u(n)) + a(u(n), k) \leftarrow \text{sum}(r, k)$
 10. $u(n) = k$
 11. **else**
 12. $\text{sum}(r, k) \leftarrow \text{sum}(r, k)$
 13. **end if**
 14. $\min\{\text{distance}(r, k), \text{sum}(r, k)\} \leftarrow \text{distance}(r, k)$
 15. **end do**
 16. print of all the distances
 17. **end do**
 18. find eccentricities, radius and diameter.
-

all other vertices. The distances are printed in line 15. Line 17 calculates the eccentricities, radius and diameter. Repeat this algorithm m times to find the distances in m -polar fuzzy graph.

The running time complexity of lines 1 – 6, 15 – 17 is v and lines 7 – 14 is $v \times v$. Therefore, the net time complexity of the algorithm is $O(v^2)$, where v is the number of vertices of an m -polar fuzzy graph.

Applications of m -polar fuzzy graphs

We describe a pair of example applications of m -polar fuzzy graphs in decision support system.

A. m -polar fuzzy graphs in product manufacturing

A product can increase the profit of a company if it is sold in multiple areas. Before manufacturing a product, engineers and manufacturers test several important things in a product. Usually, graphical models are used in such decision making problems. m -polar fuzzy graphs are mostly used in decision making problems when it is necessary to gather a group of agreements. Suppose a multinational enterprise (MNE) has to decide to manufacture a product among four products P_1 , P_2 , P_3 and P_4 to market it in different countries. Every company consider the following points before manufacturing a product.

- Does the product follow the mass market demands?

- Is the product fast or time consuming to manufacture?
- Is the product sold at a high or low cost?
- Does the product appeal the people at global level?

We gather the above four points in a set as, $X = \{\text{Demand, Time, Cost, Appealing}\}$. Let the set of products is $P = \{P_1, P_2, P_3, P_4\}$. This phenomenon can be represented by a 4-polar fuzzy graph, taking P as the set of vertices. The membership value of each product represents the degree of demand, time consumption, sale price and attraction to people at a global level. Let $C(P_1) = (0.5, 0.8, 0.6, 0.4)$, $C(P_2) = (0.8, 0.2, 0.5, 0.9)$, $C(P_3) = (0.4, 0.6, 0.7, 0.4)$, $C(P_4) = (0.6, 0.8, 0.6, 0.4)$. That is, the degrees of P_1 corresponding to demand, time consumption, sale price and attraction to people are 0.5, 0.8, 0.6 and 0.4, respectively and similarly for other products. The edge between two products represents the degree of using common power equipments, materials, engineer employs and agencies involved for both of the products. Let $D(P_1P_2) = (0.5, 0.5, 0.7, 0.4)$, $D(P_1P_3) = (0.1, 0.3, 0.2, 0.4)$, $D(P_1P_4) = (0.2, 0.3, 0.1, 0.1)$, $D(P_2P_3) = (0.3, 0.4, 0.5, 0.4)$, $D(P_2P_4) = (0.6, 0.2, 0.5, 0.4)$. This means that P_1 and P_2 use 50% common equipments, 50% same materials, 70% common trained engineers and 40% same agencies. It can be easily verified that it is as 4-polar fuzzy graph as shown in Fig. 2. By observation it is easy to see that the production of P_2 has more

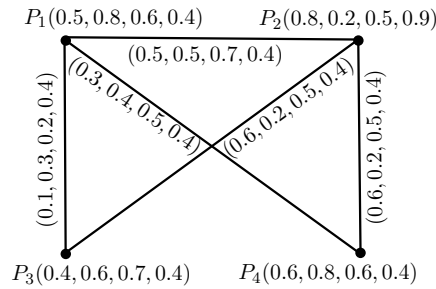


Figure 12: 4-polar fuzzy graph

demand and attraction, minimum time consumption and its price is such that it is in the range of all classes of people. Also it shares a lot of common things with other products. Consequently, it will give more profit to company as compared to other products.

B. Shortest path problem in m -polar fuzzy graphs

Graph are used as a common source to model the communication networks such as transportation, to find the shortest paths between any two points of the network. m -polar fuzzy graphs can be used to find the shortest paths when it is required to consider a group of consequences. An agency wants to deliver a secret envelope from Astana Kazakhstan to British Columbia through delivery. The delivery car can only travel 4000 kilometers before refilling the tank. The agency requires to deliver the envelope with minimum cost of fuel and safety.

From Astana to Columbia, there are seventeen fuel stations namely, Moscow Russia, London England, Barcelona Spain, Algiers Algeria, San Jaun, Boston, Houston Texas, Helena Montana, Beijing China, Delhi India, Tokyo, Singapore, Australia, Honolulu, Alaska and California. The agency demands to consider the following things.

- Refill the tank at a station having minimum fuel cost, minimum time consumption and less danger of robbery.
- Take into account the distance travelled(in kilometers) between two fuel stations.
- How much fuel is used from one station to another?
- Use the path in which there is less danger of robbery.

We represent the fuel stations by vertices. The membership value of each vertex represents the degree of fuel cost, time consumption and danger. The edge between two stations represent the degree of distance travelled, fuel used and danger. This situation can be represented by 3-polar fuzzy graph in Fig. 2. Since, in calculating the distance there is no use of membership value of vertices, we do not write these values in the 3-polar fuzzy graph. It is necessary to travel

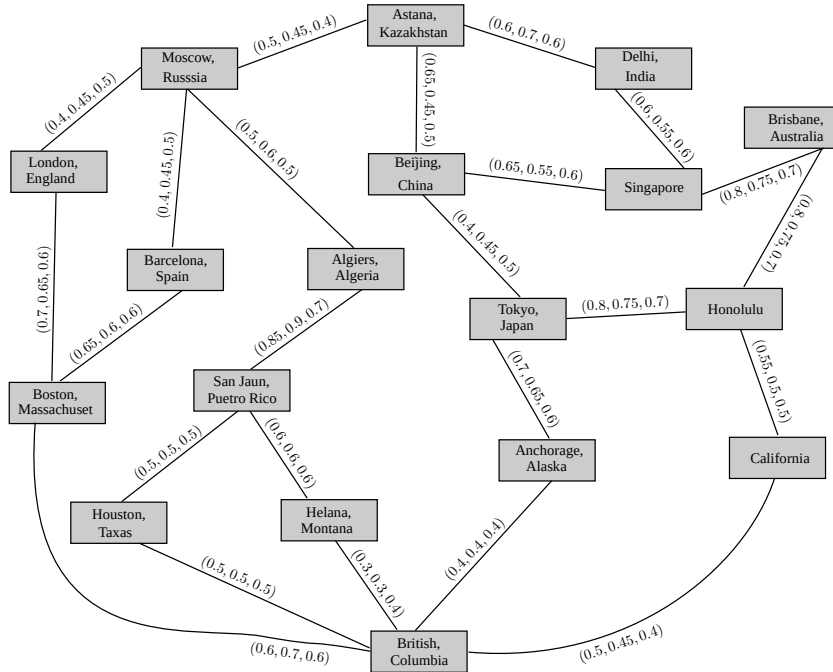


Figure 13: 3-polar fuzzy graph

through the path with least degree of “distance travelled, fuel used and danger of robbery”. For this, we calculate the length of all paths from Astana to British

Columbia and travel through a path with minimum length called distance. By routine calculations, it can be easily seen that the minimum distance between Astana and British Columbia is (2.15, 1.9, 2.0) which is obtained through the path Astana–Beijing–Tokyo–Alaska–British Columbia.

3. Conclusion and future work

An m -polar fuzzy model is a generalization of the bipolar fuzzy model. The m -polar fuzzy models give more precision, flexibility and compatibility to the system when more than one agreements are to be dealt with. In this paper, we have applied the concept of m -polar fuzzy model to graphs. We have presented an algorithm for computing the distance matrix, eccentricity of the vertices, radius and diameter in m -polar fuzzy graphs. We have also discussed applications of m -polar fuzzy graphs in traveling and product manufacturing. We are planing to extend our research work to (1) m -polar fuzzy soft hypergraphs, (2) Roughness in m -polar fuzzy hypergraphs, (3) m -polar fuzzy soft graphs.

Conflict of interest. The authors declare that they have no conflict of interest.

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A HYBRID EMD-MA FOR FORECASTING STOCK MARKET INDEX

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Abstract. Nowadays, stock market data forecasting has drawn a high attention in the field of nonstationary and nonlinear time series data with a high heteroscedasticity, since improving the forecasting accuracy is a hot topic for the researchers. Therefore, in this article the authors are proposed a new methodology via combining Empirical Mode decomposition and Moving Average model as a modified method to improve forecasting accuracy in content of stock market data. The strength of this proposed methodology lies in its ability to forecast nonlinear and non-stationary financial data without a need to use any transformation method. Moreover, this method provides a better model with sufficient forecasting accuracy. The daily stock market data of fourteen countries is applied to show the forecasting performance of the proposed method. Based on the five forecast accuracy measures, the results indicate that proposed forecasting method performance is superior to four selected forecasting techniques.

Keywords: Stock market index forecasting, Nonlinear and non-stationary time series, Empirical mode decomposition, Combined forecasting Model, Heteroscedasticity time series.

In financial time series analysis, one of the primary issues is modeling and forecasting financial time series data specifically stock market index. Usually, the transformation of a financial time series, rather than its original scale, is taken for describing its dynamics. Proper transformation is necessary to convert original non-stationary processes to stationary processes and subsequently to utilize mathematical and statistical properties for stationary processes. The hybrid models combine strengths of few traditional models to get a better forecasting accuracy. Recently, several hybrid models were applied EMD in the literature for time series forecasting. That by using EMD to decompose the non-stationary and non-linear time series data into Intrinsic Mode Functions (IMFs) and residual components. And then use forecasting model to forecast

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each component. Then all these forecasted values were aggregated to produce the final forecasted value of the original time series. Such as in [1] used a hybrid EMD-ARIMA (Autoregressive integrated moving average) to forecasting the monthly prices of rice data. Also, [14] also used the same methodology, but with wind speed data. A hybrid EMD-AR (Autoregressive) model was developed by coupling an AR model with the EMD technique in [8]. A hybrid EMD-LSSVR (least squares support vector regression) forecasting models has been applied on foreign exchange rate in [15]. While in [22] used a hybrid of EMD, LS-SVM (Least Squares-Support Vector Machines) and AR model with Kalman filter to predict wind speed data.

Therefore, the significant of this research article can be summarize as after intensive research in the financial forecasting literature, there are plenty research papers have been conducted in forecasting in content of stock market data such as [3], [2] and [12], also most of the articles have used the mentioned models directly without any combination such as [25].

With regard to all those literature reviews, this study attempts to employ the proposed method to forecast the daily stock market data of fourteen countries. Four selected forecasting models are used in the proposed method comparison to assess its performance of forecasting. Experimental results show that the proposed method is superior to existing method in terms of five accuracy forecasting measure. Section 2 introduces methods are used in methodology in this paper which are EMD, IMF and Moving Average Model. In this section introduces statistical techniques for consideration method. Section 3 presented the proposed methodology. Section 4 analyzes the daily stock market time series data of four countries with a discussion the result showing the capability of proposed forecasting method. Finally, in Section 5 some concluding remarks are addressed.

1. Methodology

In this section, the various steps for the implementation of the proposed forecasting method are described in detail. Which are Empirical Mode Decomposition, Moving Average Model and statistical techniques for consideration method.

1.1 Empirical mode decomposition (EMD)

EMD was described by [10], and this method has been modified by [16] and [13]. The main idea of *EMD* is the decomposing of nonlinear and non-stationary time series data into several of simple time series. And it analyzing time series with keeping the time domain of the signal. It supplies an strong and adaptive process to decompose a time series into a combination of time series that known as intrinsic mode functions (*IMF*) and residual. Later, the original signal can

be constructed back as the following:

$$(1.1) \quad x(t) = \sum_{i=1}^n IMF_i(t) + r(t)$$

where $x(t)$ represents the original time series, $r(t)$ represents the residue of the original time series data decomposition and IMF_i represent the i^{th} intrinsic mode function (IMF) series.

In order to estimate these $IMFs$, the following steps should be initiated and the process is called the sifting process of time series $x(t)$ [20] are shown below:

1. Start the first step by taking the original time series $x(t)$ for sifting process and assuming the iteration index value is $i = 1$.
2. Then, evaluate all of local extrema values of the time series $x(t)$.
3. After that, form the local maxima (local upper) envelope function $u(t)$ by connecting all local maxima values using a cubic spline line. In a similar way, form the local minimum (local lower) envelope function $l(t)$, and then form the mean function $m(t)$ by using this following

$$(1.2) \quad m(t) = \frac{u(t) + l(t)}{2}$$

4. Next, define a new function $h(t)$ using the mean envelope $m(t)$ and the signal $x(t)$ on this formula

$$(1.3) \quad h(t) = x(t) - m(t)$$

Check the function $h(t)$ is an IMF , according to IMF conditions (shown in the second part of this section). If the function $h(t)$ has satisfied IMF conditions, then go to step 5. If not, go back to step 2 and renew the value of $x(t)$ such that became $h(t)$, repeat steps 2 again until 4.

5. In step 5, firstly save the result of the IMF obtain from the last step. Secondly, renew the iteration index value such that became $i = i + 1$. Thirdly attain the residue function $r(t)$ using the IMF and the signal $x(t)$ on the formula

$$(1.4) \quad IMF_i(t) = h(t) \Rightarrow r_{i+1}(t) = x(t) - IMF_i(t).$$

6. Finally, make a decision whether the residue function $r(t)$ that acquire from step 5 is a monotonic or constant function. Then, save the residue and all the $IMFs$ obtained. If the residue is not monotonic or constant function, return to step 2.

The steps 1 to 6 which were discussed above allow the sifting process (EMD algorithm) to separate time-altering signal features.

1.2 Intrinsic Mode Function (IMF)

Based on the EMD algorithm presented in the previous section, the IMF produces by the sifting process need to satisfy two conditions [20] which are

$$(1.5) \quad |Num[extreme] - Num[cross - zero]| < 1.$$

where Num.extreme represents the number of local extreme points (all local maxima and all local minima), also Num[cross-zero] represent the number of cross-zero points

$$(1.6) \quad |m(t)| = \left| \frac{u(t) + l(t)}{2} \right| < \varepsilon,$$

where $u(t)$ represents the envelope function generated by using cubic spline line on all local maxima, $l(t)$ represents the envelope function generated by using a cube spline line on all local minima, $m(t)$ represents the mean function that it was obtained by evaluating the mean of $u(t)$ and $l(t)$, and ε is a very small positive number that close to zero, sometime equal zero.

1.3 Moving Average Model (MA)

A moving-average model is a model of linear regression uses past errors of the time series to describe the present observation. The moving average model of order q is denoted by $MA(q)$. Mathematically; $MA(q)$ of time series X_t is formally given by [7]:

$$(1.7) \quad X_t = \mu + \varepsilon_t + \sum_{j=1}^q \theta_j \varepsilon_{t-j} = \mu + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q}$$

where μ is the mean of the series, and $\theta_1, \dots, \theta_q$ are the parameters may be positive or negative. And the $\varepsilon_t, \varepsilon_{t-1}, \dots, \varepsilon_{t-q}$, are white noise error terms. The value of q is called the order of the MA model. This can be written using the backshift operator B as

$$(1.8) \quad X_t = \mu + (1 + \theta_1 B + \cdots + \theta_q B^q) \varepsilon_t.$$

The maximum of order q is selected by using the autocorrelation function (ACF) (see [23]). And the parameters $\theta_1, \dots, \theta_q$ are selected by using the Corrected Akaike's information criterion ($AICC$) by [11].

1.4 Statistical techniques for consideration method

In this study the proposed forecasting method is compered with four methods. Moving Average model, Holt-winter model was presented in [9] and [24], ARIMA models were presented in [18] and Random Walk method are used in order to validate the forecasting performance of proposed forecasting model.

These statistical methods were selected based on their performance in forecasting competitions and other empirical applications, as well as on their ability to capture salient features of the data. ARIMA (autoregressive integrated moving average) or Box-Jenkins models are generally denoted $ARIMA(p, d, q)$. Where parameters p , d , and q are non-negative integers, p is the order of the Autoregressive model, d is the degree of differencing, and q is the order of the Moving-average model. Recently ARIMA is employed in several of studies in forecasting financial time series such as [4]. He applied ARIMA to forecast the cultivated area and production of maize in Nigeria.

A random walk (RW) is a process where the current value of a variable is composed of the past value with adding an error term defined as a white noise. It was first studied several hundred years ago as models for games of chance. Recently, In [17] have been shown that the random walk model turns out to be a hard to beat benchmark in forecasting the CEE exchange rates. And in [21] was applied a variable drift term with the random walk process. This was estimated using a Kalman filter. This simple statistical process was shown to perform better than all the three models that he was selected in out of-sample forecasts.

2. Propose methodology and data

In this section, the various steps for the implementation of the proposed forecasting method are described in detail. Which are Empirical Mode Decomposition and Moving Average Model.

2.1 Data

In this study, nonlinear and non-stationary time series data from the daily stock market of fourteen countries are used. These countries are Australia, Denmark, Estonia, Finland, France, India, Lithuania, Malaysia, Netherlands, Norwegian, Slovenia, Switzerland, Thailand and UK. While Figure 4 shows the time series plot of these countries. Table 1 presents these countries with the Basic statistics for each country, where S.Deviation is Standard Deviation, N_{IMF} the number of *IMFs* and N is Number of observations. Moreover, Table 1 presents the p-value of KPSS (Kwiatkowski-Phillips-Schmidt-Shin by [5]), RESET (Ramsey Regression Equation Specification Error Test by [19]), BP (Breusch-Pagan test by [6]) for nonstationary, nonlinear and heteroscedasticity, respectively. According to this value all stock market are significantly nonlinear, nonstationary with high heteroscedasticity. The data are extracted from the Yahoo finance website. The daily closing prices are used as a general measure of the stock market over the past six years. The whole data set - for each country - covers the period from 9 February 2010 to 7 January 2016. The data set is divided into two parts. The first part (m observations) is used to determine the specifications of the models and parameters. The second part, on the other hand, (h observations)

is reserved for out-of-sample evaluation. This part is used comparison of performances among various forecasting models. Malaysia stock market data are taken as example. Where the number observation is $N=1459$, the first part is $m=1458, 1457, 1456, 1455, 1494$ and 1493 and the second part is $h=1, 2, 3, 4, 5$ and 6 respectively, are used.

2.2 Propose Methodology

The proposed methodology consists of four stages. Before began the proposed method, the time series data were divided into two parts. The first part is the training part to train the proposed method. This part is used to modeling to get the forecasting value of time series. The second part is the testing part. This part is used to compering with the forecasting value. After that, the time series are ready to follow this stage:-

1. Firstly, the use of empirical mode decomposition (*EMD*) on the time series. In this stage, the several Intrinsic Mode Functions (*IMFs*) and residue are obtained.
2. Secondly, the KPSS test [5] is applied on each of *IMFs* and residue to select the order of difference (d). While each component has own d . *ACF* is applied on all component to select the order of moving average model (q). If q of time series goes to infinity, the d is exceeded until q be finite. After that, the $\theta_1, \dots, \theta_q$ are selected by using the Corrected Akaike's information criterion (*AICc*) for each component.
3. Thirdly, by using the result p, q and $\theta_1, \dots, \theta_q$ - were found in the last stage - each component is modeling by *MA*(q) to forecasting h days ahead.
4. Finally, in the this stage all the forecasting results for *IMFs* and residue are added up to get the forecasting for the time series.

After that the forecasting results are compared with the forecasting result of random-walk with draft, Holt-winter, ARIMA and moving average model without *EMD*. Figure 2 summarizes all the proposed methodology steps.

3. Result and discussion

In this study, stock market data of fourteen countries are used to present the forecasting accuracy of the proposed forecasting method. Four Models are used in order to validate the forecasting performance of proposed forecasting method. Root Mean Squared Error (RMSE), Mean Absolute Error (MAE), Mean Absolute Percentage Error (MAPE), Mean Absolute Scaled Error (MASE) and Theil's U-statistic (TheilU) will be utilized to evaluate the forecasting accuracy for each method. Equations 3.1, 3.2, 3.3, 3.4 and 3.5 are showed the formula of RMSE,

MAE, MAPE, MASE and TheilU respectively. Where $\hat{y}_i$ is the forecast value of the variable y at time period i from knowledge of the actual series values.

$$(3.1) \quad RMSE = \sqrt{\frac{1}{n} \sum_{i=1}^n (y_i - \hat{y}_i)^2},$$

$$(3.2) \quad MAE = \frac{1}{n} \sum_{i=1}^n |y_i - \hat{y}_i|,$$

$$(3.3) \quad MAPE = \frac{1}{n} \sum_{i=1}^n \frac{|y_i - \hat{y}_i|}{y_i} .100\%,$$

$$(3.4) \quad MASE = \frac{1}{n} \sum_{i=1}^n \left(\frac{|y_i - \hat{y}_i|}{\frac{1}{n-1} \sum_{i=2}^n |y_i - y_{i-1}|} \right),$$

$$(3.5) \quad TheilU = \frac{\sqrt{\sum_{i=1}^{n-1} \left(\frac{\hat{y}_{i+1} - y_{i+1}}{y_i} \right)^2}}{\sqrt{\sum_{i=1}^{n-1} \left(\frac{y_{i+1} - y_i}{y_i} \right)^2}}.$$

Table 2, 3, 4, 5, 6, 7 and Table 8 presented the comparison results of applied proposed forecasting method with four forecasting methods on Australia with Denmark, Estonia with Finland, France with India, Lithuania with Malaysia, Netherlands with Norwegian, Slovenia with Switzerland and Thailand with UK, respectively. When each table presents the result of five measure error for forecasting at $h = 1, 2, 3, 4, 5$ and 6 days ahead. When the four forecasting methods are used to compare with proposed forecasting method. These methods are Holt-winter, moving average, ARIMA and random-walk with draft. The five error measures of RMSE, MAE, MAPE, MASE and TheilU are used to find the forecasting accuracy. The results obtained indicates that the forecast accuracy for proposed forecasting method (denoted by EMD-MA) is generally better than the four selected forecasting models.

4. Conclusion

In this article, the forecasting accuracy has improved in financial time series area since the forecasting accuracy still remains as one of the most difficult area due to the non-stationary and non-linear data. In this study, a new hybrid method has composited which are EMD and MA for modeling and improving forecasting accuracy in content of stock market data for fourteen countries based on several comparison forecast accuracy measurements. Then the findings indicate that proposed is able to outperform the four forecasting models. Thus, this paper has strengthened the idea that proposed forecasting method is suitable for stock market data. These results which will be useful to predict the future event and some structure breaks event.

Table 1: Basic statistics of time series

Country	Mean	Median	SD	Skewness	Kurtosis	N. IMF	N	KPSS p.value	RESET p.value	BP p.value
Australia	4928.42	4939.35	483.66	0.03	-1.05	7	1498	<.01	<.01	<.01
Denmark	586.71	519.34	197.16	0.84	-0.52	7	1466	<.01	<.01	<.01
Estonia	726.91	759.33	112.62	-0.27	-1.29	7	1471	<.01	<.01	<.01
Finland	2554.56	2455.44	474.05	0.39	-0.88	7	1476	<.01	<.01	<.01
France	3968.26	3939.82	557.54	0.21	-0.6	7	1516	<.01	<.01	0.268
India	20964.54	19501.08	4045.57	0.73	-0.94	6	1465	<.01	<.01	<.01
Lithuania	399.79	400.22	61.73	0.02	-1.26	6	1460	<.01	<.01	<.01
Malaysia	1638.2	1643.89	164.52	-0.4	-0.68	7	1459	<.01	<.01	<.01
Netherlands	370.77	355.92	56.19	0.65	-0.32	6	1516	<.01	<.01	<.01
Norwegian	591.02	581.59	97.87	0.05	-0.99	8	1476	<.01	<.01	<.01
Slovenia	710.01	703.86	107.07	0.06	-1	6	1438	<.01	<.01	<.01
Switzerland	7375.98	7226	1191.92	0.08	-1.41	8	1508	<.01	<.01	<.01
Thailand	1254.45	1300.22	247.91	-0.42	-0.94	7	1442	<.01	<.01	<.01
UK	6114.4	6059.3	538.36	-0.15	-1.09	8	1528	<.01	<.01	<.01

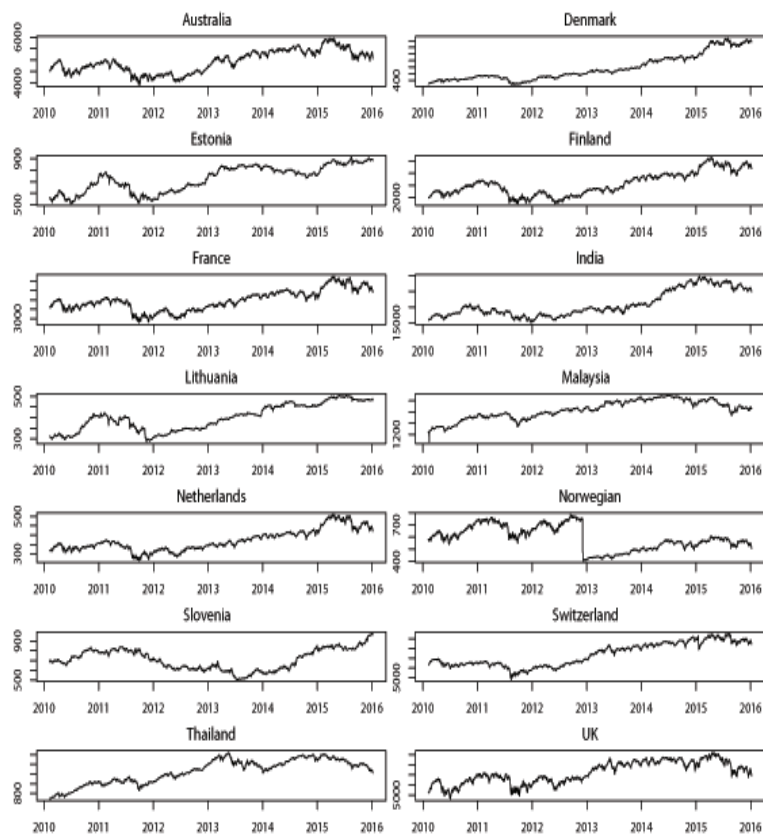


Figure 1: Time series plot

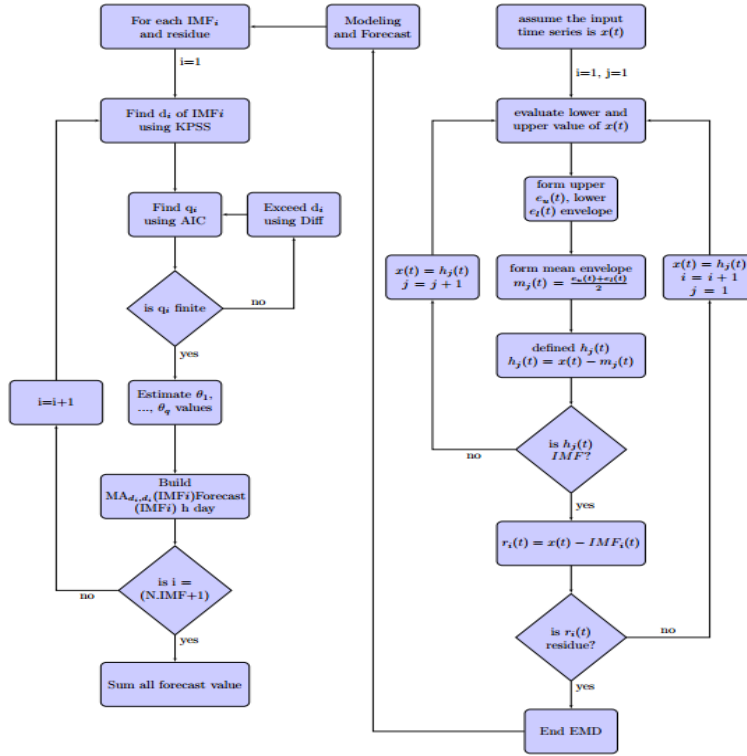


Figure 2: Flowchart of empirical mode decomposition with moving average estimation process

Table 2: Presented the RMSE, MAE, MASE, MAPE and TheilU of proposed method and four forecasting methods for forecasting at $h = 1, 2, 3, 4, 5$ and 6 for stock market data of Australia and Denmark.

Countries	Method	h=1	h=2	h=3	h=4	h=5	h=6
Australia							
MAE	Holt-Winter	107.55	114.143	160.097	142.023	135.866	90.672
	MA	106.836	113.747	159.881	142.801	138.092	91.48
	ARIMA	110.362	99.267	173.24	218.579	277.756	157.907
	R.Walk	109.2	115.8	160.8	142.4	135.72	91.3
	EMD-MA	176.213	153.54	93.04	70.513	59.33	67.869
RMSE	Holt-Winter	107.55	126.292	174.81	169.45	169.046	120.56
	MA	106.836	125.823	174.664	170.114	170.929	123.37
	ARIMA	110.362	117.382	187.391	254	326.04	210.645
	R.Walk	109.2	128.027	175.568	169.857	168.888	121.898
	EMD-MA	176.213	166.799	116.893	82.858	70.487	74.805
MAPE	Holt-Winter	2.078	2.179	3.008	2.658	2.532	1.706
	MA	2.064	2.172	3.004	2.672	2.572	1.72
	ARIMA	2.131	1.899	3.247	4.016	5.004	2.902
	R.Walk	2.109	2.21	3.021	2.664	2.529	1.718
	EMD-MA	3.36	2.907	1.77	1.349	1.136	1.3
TheilU	Holt-Winter	-	1.54	2.325	2.258	2.492	1.905
	MA	-	1.534	2.324	2.266	2.519	1.953
	ARIMA	-	1.483	2.475	3.373	4.786	3.371
	R.Walk	-	1.56	2.335	2.263	2.489	1.927
	EMD-MA	-	2.003	1.616	1.008	0.887	1.017
MASE	Holt-Winter	-	1.045	1.879	1.677	1.971	1.523
	MA	-	1.042	1.877	1.687	2.003	1.537
	ARIMA	-	0.909	2.033	2.582	4.028	2.653
	R.Walk	-	1.06	1.887	1.682	1.968	1.534
	EMD-MA	-	1.406	1.092	0.833	0.86	1.14
Denmark							
MAE	Holt-Winter	18.678	19.455	10.943	16.004	13.465	9.722
	MA	16.882	19.456	11.162	13.539	11.331	9.974
	ARIMA	15.434	29.214	21.378	25.202	14.996	9.684
	R.Walk	18.09	18.415	10.943	14.202	11.314	9.618
	EMD-MA	18.436	10.465	10.23	8.165	6.845	9.553
RMSE	Holt-Winter	18.678	21.603	11.914	19.076	17.569	11.326
	MA	16.882	21.288	11.97	16.788	15.381	11.26
	ARIMA	15.434	31.567	22.076	28.507	18.047	10.715
	R.Walk	18.09	20.516	11.661	17.328	15.44	10.921
	EMD-MA	18.436	12.487	11.333	8.374	7.648	10.437
MAPE	Holt-Winter	1.86	1.919	1.095	1.575	1.324	0.968
	MA	1.684	1.919	1.119	1.336	1.117	0.997
	ARIMA	1.591	2.851	2.18	2.454	1.474	0.967
	R.Walk	1.803	1.818	1.096	1.4	1.116	0.96
	EMD-MA	1.837	1.043	1.024	0.815	0.684	0.956
TheilU	Holt-Winter	-	1.595	0.736	1.36	1.316	0.842
	MA	-	1.553	0.682	1.185	1.152	0.808
	ARIMA	-	2.276	1.373	2.103	1.345	0.796
	R.Walk	-	1.518	0.678	1.219	1.157	0.795
	EMD-MA	-	0.955	0.692	0.523	0.573	0.727
MASE	Holt-Winter	-	1.075	0.797	1.138	0.925	0.833
	MA	-	1.076	0.813	0.963	0.778	0.855
	ARIMA	-	1.615	1.557	1.792	1.03	0.83
	R.Walk	-	1.018	0.797	1.01	0.777	0.824
	EMD-MA	-	0.578	0.745	0.58	0.47	0.819

Table 3: Presented the RMSE, MAE, MASE, MAPE and TheilU of proposed method and four forecasting methods for forecasting at $h = 1, 2, 3, 4, 5$ and 6 for stock market data of Estonia and Finland.

Countries	Method	h=1	h=2	h=3	h=4	h=5	h=6
Estonia							
MAE	Holt-Winter	11.411	7.623	5.487	7.902	6.205	4.618
	MA	11.307	7.095	5.588	6.887	5.13	3.764
	ARIMA	11.381	8.557	5.954	5.239	8.054	4.663
	R.Walk	11.1	7.06	5.487	6.75	5.022	3.768
	EMD-MA	10.613	5.806	5.434	5.958	4.253	3.918
RMSE	Holt-Winter	11.411	9.541	5.778	9.444	8.304	6.623
	MA	11.307	9.091	5.653	8.439	7.109	5.406
	ARIMA	11.381	10.164	7.108	6.858	10.232	6.591
	R.Walk	11.1	8.98	5.66	8.328	7.005	5.473
	EMD-MA	10.613	7.913	7.057	7.831	6.571	6.06
MAPE	Holt-Winter	1.274	0.849	0.615	0.878	0.689	0.514
	MA	1.262	0.791	0.627	0.766	0.571	0.42
	ARIMA	1.27	0.952	0.671	0.584	0.892	0.519
	R.Walk	1.239	0.787	0.615	0.751	0.559	0.42
	EMD-MA	1.186	0.648	0.607	0.663	0.474	0.437
TheilU	Holt-Winter	-	1.204	0.765	1.388	1.291	1.123
	MA	-	1.151	0.713	1.222	1.105	0.911
	ARIMA	-	1.265	0.797	0.951	1.591	1.114
	R.Walk	-	1.136	0.733	1.207	1.088	0.923
	EMD-MA	-	1.007	1.043	1.109	1.022	1.031
MASE	Holt-Winter	-	0.687	0.87	1.319	0.985	0.894
	MA	-	0.639	0.886	1.15	0.814	0.728
	ARIMA	-	0.771	0.944	0.875	1.278	0.902
	R.Walk	-	0.636	0.87	1.127	0.797	0.729
	EMD-MA	-	0.523	0.862	0.995	0.675	0.758
Finland							
MAE	Holt-Winter	73.873	44.396	130.482	116.801	75.751	72.09
	MA	74.028	40.636	127.774	115.829	74.243	71.344
	ARIMA	63.13	36.489	128.425	142.609	733.177	110.78
	R.Walk	73.91	44.345	128.513	113.895	74.474	70.338
	EMD-MA	83.051	50.434	102.561	86.884	70.545	46.876
RMSE	Holt-Winter	73.873	57.775	135.746	133.831	86.803	94.919
	MA	74.028	55.016	132.942	132.376	84.273	93.607
	ARIMA	63.13	39.002	133.648	160.675	870.188	136.865
	R.Walk	73.91	57.725	133.653	130.688	84.876	91.631
	EMD-MA	83.051	66.314	106.423	99.441	74.543	52.285
MAPE	Holt-Winter	2.271	1.362	3.882	3.455	2.274	2.147
	MA	2.276	1.248	3.804	3.428	2.231	2.126
	ARIMA	1.947	1.144	3.823	4.181	34.843	3.246
	R.Walk	2.272	1.36	3.826	3.373	2.237	2.097
	EMD-MA	2.546	1.542	3.079	2.6	2.134	1.419
TheilU	Holt-Winter	-	1.101	2.864	2.188	1.529	1.751
	MA	-	1.052	2.802	2.163	1.477	1.724
	ARIMA	-	0.307	2.813	2.619	15.753	2.508
	R.Walk	-	1.1	2.816	2.137	1.491	1.69
	EMD-MA	-	1.265	2.194	1.623	1.219	0.935
MASE	Holt-Winter	-	0.601	3.21	1.944	1.532	1.466
	MA	-	0.55	3.143	1.928	1.502	1.45
	ARIMA	-	0.494	3.159	2.374	14.83	2.252
	R.Walk	-	0.6	3.161	1.896	1.506	1.43
	EMD-MA	-	0.682	2.523	1.446	1.427	0.953

Table 4: Presented the RMSE, MAE, MASE, MAPE and TheilU of proposed method and four forecasting methods for forecasting at $h = 1, 2, 3, 4, 5$ and 6 for stock market data of France and India.

Countries	Method	h=1	h=2	h=3	h=4	h=5	h=6
France							
MAE	Holt-Winter	77.805	96.562	59.754	153.766	164.095	161.6
	MA	76.768	100.225	62.746	153.597	158.413	156.532
	ARIMA	56.261	71.211	33.074	146.894	224.058	225.527
	R.Walk	76.89	95.605	58.677	151.027	160.902	158.305
	EMD-MA	108.038	111.826	80.201	129.286	93.759	88.662
RMSE	Holt-Winter	77.805	104.061	74.985	162.638	181.513	186.751
	MA	76.768	107.247	78.515	162.253	175.956	181.066
	ARIMA	56.261	77.041	35.562	156.364	251.215	254.839
	R.Walk	76.89	103.045	73.309	159.73	178.052	183.031
	EMD-MA	108.038	116.958	96.167	137.742	108.867	110.694
MAPE	Holt-Winter	1.736	2.128	1.321	3.314	3.506	3.434
	MA	1.713	2.207	1.385	3.31	3.389	3.331
	ARIMA	1.262	1.579	0.741	3.17	4.708	4.716
	R.Walk	1.716	2.107	1.297	3.257	3.44	3.367
	EMD-MA	2.395	2.456	1.764	2.802	2.04	1.922
TheilU	Holt-Winter	-	1.76	1.352	3.133	2.723	2.976
	MA	-	1.8	1.417	3.124	2.641	2.885
	ARIMA	-	1.309	0.353	3.017	3.775	4.053
	R.Walk	-	1.743	1.32	3.074	2.671	2.917
	EMD-MA	-	1.9	1.712	2.655	1.64	1.765
MASE	Holt-Winter	-	1.256	0.892	3.091	2.488	2.659
	MA	-	1.303	0.936	3.088	2.402	2.575
	ARIMA	-	0.926	0.493	2.953	3.397	3.71
	R.Walk	-	1.243	0.875	3.036	2.439	2.604
	EMD-MA	-	1.454	1.197	2.599	1.421	1.459
India							
MAE	Holt-Winter	541.111	433.938	321.473	737.767	487.473	484.4
	MA	560.392	461.068	314.904	763.766	499.961	498.051
	ARIMA	437.048	367.265	198.675	1015.457	635.86	662.853
	R.Walk	554.5	451.26	343.85	752.077	507.155	502.264
	EMD-MA	440.834	555.276	469.23	670.752	402.625	452.674
RMSE	Holt-Winter	541.111	511.861	441.013	797.071	579.256	613.526
	MA	560.392	542.853	448.577	822.565	593.753	630.742
	ARIMA	437.048	455.046	213.497	1097.276	759.687	832.185
	R.Walk	554.5	529.625	463.39	812.503	602.829	636.811
	EMD-MA	440.834	616.216	586.7	731.441	479.589	576.416
MAPE	Holt-Winter	2.131	1.698	1.256	2.827	1.88	1.86
	MA	2.205	1.801	1.23	2.923	1.927	1.91
	ARIMA	1.728	1.441	0.787	3.843	2.432	2.518
	R.Walk	2.183	1.764	1.342	2.88	1.954	1.926
	EMD-MA	1.743	2.162	1.82	2.577	1.56	1.741
TheilU	HW	-	1.272	1.313	2.599	1.695	1.936
	MA	-	1.348	1.337	2.68	1.737	1.99
	ARIMA	-	1.147	0.574	3.613	2.23	2.625
	R.Walk	-	1.314	1.379	2.652	1.767	2.009
	EMD-MA	-	1.483	1.741	2.389	1.401	1.822
MASE	Holt-Winter	-	0.783	0.883	2.869	1.541	1.702
	MA	-	0.832	0.865	2.97	1.58	1.75
	ARIMA	-	0.662	0.545	3.949	2.009	2.329
	R.Walk	-	0.814	0.944	2.924	1.603	1.765
	EMD-MA	-	1.001	1.288	2.608	1.272	1.59

Table 5: Presented the RMSE, MAE, MASE, MAPE and TheilU of proposed method and four forecasting methods for forecasting at $h = 1, 2, 3, 4, 5$ and 6 for stock market data of Lithuania and Malaysia.

Countries	Method	h=1	h=2	h=3	h=4	h=5	h=6
Lithuania							
MAE	Holt-Winter	5.725	2.843	2.507	1.88	3.283	3.138
	MA	5.703	2.823	2.501	1.87	3.153	2.972
	ARIMA	6.312	3.166	2.746	1.986	2.972	3.201
	R.Walk	5.62	2.81	2.507	1.88	3.164	2.967
	EMD-MA	5.058	2.628	2.393	1.817	3.295	2.595
RMSE	Holt-Winter	5.725	3.118	2.604	2.122	3.781	3.782
	MA	5.703	3.088	2.604	2.103	3.672	3.63
	ARIMA	6.312	3.882	2.885	2.261	3.513	3.838
	R.Walk	5.62	3.048	2.61	2.123	3.682	3.626
	EMD-MA	5.058	2.65	2.52	2.15	3.806	3.225
MAPE	Holt-Winter	1.171	0.583	0.516	0.387	0.68	0.65
	MA	1.166	0.579	0.515	0.385	0.653	0.616
	ARIMA	1.289	0.648	0.566	0.407	0.615	0.663
	R.Walk	1.149	0.577	0.516	0.387	0.655	0.615
	EMD-MA	1.036	0.54	0.493	0.374	0.682	0.537
TheilU	Holt-Winter	-	0.734	0.702	0.685	1.288	1.359
	MA	-	0.725	0.702	0.677	1.248	1.305
	ARIMA	-	0.963	0.754	0.646	1.191	1.379
	R.Walk	-	0.71	0.703	0.686	1.251	1.303
	EMD-MA	-	0.528	0.69	0.692	1.303	1.159
MASE	Holt-Winter	-	0.506	0.691	0.616	1.357	1.245
	MA	-	0.502	0.69	0.613	1.303	1.179
	ARIMA	-	0.563	0.758	0.651	1.228	1.27
	R.Walk	-	0.5	0.691	0.616	1.307	1.177
	EMD-MA	-	0.468	0.66	0.596	1.362	1.03
Malaysia							
MAE	Holt-Winter	12.595	6.383	7.365	33.658	27.561	19.803
	MA	12.649	6.452	7.301	32.333	25.885	18.95
	ARIMA	13.391	6.896	7.547	32.068	26.036	18.792
	R.Walk	12.84	6.42	9.563	31.968	26.204	19.033
	EMD-MA	19.113	11.769	7.251	9	12.524	20.504
RMSE	Holt-Winter	12.595	7.182	8.915	34.246	31.188	22.116
	MA	12.649	7.336	8.404	32.951	29.485	20.986
	ARIMA	13.391	7.315	9.121	32.698	29.576	20.846
	R.Walk	12.84	7.644	11.08	32.597	29.71	21.361
	EMD-MA	19.113	12.544	7.534	12.725	13.741	22.445
MAPE	Holt-Winter	0.755	0.383	0.445	1.987	1.626	1.175
	MA	0.758	0.387	0.441	1.91	1.529	1.125
	ARIMA	0.803	0.414	0.456	1.895	1.538	1.116
	R.Walk	0.77	0.385	0.578	1.889	1.548	1.129
	EMD-MA	1.142	0.704	0.434	0.537	0.753	1.241
TheilU	Holt-Winter	-	0.753	0.932	3.099	1.633	1.251
	MA	-	0.774	0.876	2.944	1.543	1.184
	ARIMA	-	0.727	0.989	2.909	1.548	1.175
	R.Walk	-	0.823	1.129	2.901	1.555	1.21
	EMD-MA	-	1.255	0.865	0.479	0.559	1.127
MASE	Holt-Winter	-	0.497	0.975	3.68	1.656	1.473
	MA	-	0.502	0.966	3.535	1.555	1.41
	ARIMA	-	0.537	0.999	3.506	1.564	1.398
	R.Walk	-	0.5	1.266	3.495	1.574	1.416
	EMD-MA	-	0.917	0.96	0.984	0.752	1.525

Table 6: Presented the RMSE, MAE, MASE, MAPE and TheilU of proposed method and four forecasting methods for forecasting at $h = 1, 2, 3, 4, 5$ and 6 for stock market data of Netherlands and Norwegian.

Countries	Method	h=1	h=2	h=3	h=4	h=5	h=6
Netherlands							
MAE	Holt-Winter	8.03	10.315	5.78	12.941	14.83	13.967
	MA	7.648	10.61	5.802	12.738	14.351	13.884
	ARIMA	5.574	12.559	4.829	11.471	18.045	26.872
	RW	8	10.24	5.727	12.765	14.582	13.662
	EMD-MA	10.11	9.374	7.373	7.732	5.342	4.622
RMSE	Holt-Winter	8.03	11.073	6.881	13.999	16.425	16.348
	MA	7.648	11.322	6.837	13.792	15.953	16.18
	ARIMA	5.574	13.299	5.74	12.442	20.288	30.986
	RW	8	10.994	6.812	13.811	16.154	16.014
	EMD-MA	10.11	9.974	8.394	8.662	6.353	5.939
MAPE	Holt-Winter	1.873	2.371	1.339	2.928	3.322	3.117
	MA	1.785	2.437	1.344	2.883	3.218	3.1
	ARIMA	1.307	2.871	1.136	2.605	4.002	5.78
	RW	1.866	2.354	1.327	2.889	3.268	3.052
	EMD-MA	2.346	2.16	1.704	1.773	1.229	1.061
TheilU	Holt-Winter	-	1.793	1.135	2.448	2.499	2.632
	MA	-	1.82	1.117	2.413	2.429	2.603
	ARIMA	-	2.117	0.553	2.168	3.096	4.984
	RW	-	1.78	1.122	2.412	2.458	2.578
	EMD-MA	-	1.597	1.345	1.489	0.965	0.953
MASE	Holt-Winter	-	1.289	0.812	2.213	2.145	2.18
	MA	-	1.326	0.815	2.179	2.075	2.167
	ARIMA	-	1.57	0.678	1.962	2.609	4.195
	RW	-	1.28	0.804	2.183	2.109	2.133
	EMD-MA	-	1.172	1.035	1.323	0.773	0.721
Norwegian							
MAE	Holt-Winter	14.178	13.965	19.711	23.471	19.127	13.468
	MA	14.83	14.768	20.289	23.68	19.539	13.944
	ARIMA	10.697	5.051	24.578	12.095	109.898	192.283
	R.Walk	14.68	14.57	20.223	23.807	19.516	14.023
	EMD-MA	11.692	15.878	20.127	19.108	15.839	13.84
RMSE	Holt-Winter	14.178	15.684	21.631	26.199	23.719	17.192
	MA	14.83	16.539	22.268	26.455	24.138	17.779
	ARIMA	10.697	5.053	28.797	16.574	132.708	233.532
	R.Walk	14.68	16.314	22.183	26.567	24.141	17.928
	EMD-MA	11.692	18.197	22.95	21.467	19.659	17.262
MAPE	Holt-Winter	2.721	2.644	3.658	4.286	3.49	2.488
	MA	2.842	2.791	3.761	4.322	3.562	2.573
	ARIMA	2.066	0.992	5.13	2.466	16.273	116.094
	R.Walk	2.814	2.755	3.749	4.344	3.558	2.586
	EMD-MA	2.254	2.99	3.725	3.523	2.917	2.559
TheilU	Holt-Winter	-	1.438	2.201	2.668	2.513	1.975
	MA	-	1.513	2.268	2.695	2.558	2.043
	ARIMA	-	0.334	3.021	1.717	14.042	27.004
	R.Walk	-	1.493	2.257	2.706	2.558	2.063
	EMD-MA	-	1.687	2.374	2.192	2.084	1.974
MASE	Holt-Winter	-	0.951	1.799	2.172	1.863	1.621
	MA	-	1.006	1.852	2.191	1.903	1.679
	ARIMA	-	0.344	2.244	1.119	10.706	23.15
	R.Walk	-	0.993	1.846	2.203	1.901	1.688
	EMD-MA	-	1.082	1.837	1.768	1.543	1.666

Table 7: Presented the RMSE, MAE, MASE, MAPE and TheilU of proposed method and four forecasting methods for forecasting at $h = 1, 2, 3, 4, 5$ and 6 for stock market data of Slovenia and Switzerland.

Countries	Method	h=1	h=2	h=3	h=4	h=5	h=6
Slovenia							
MAE	Holt-Winter	69.989	62.954	41.532	81.287	68.163	60.134
	MA	73.429	65.575	43.834	81.356	68.328	60.104
	ARIMA	60.641	73.604	26.903	58.056	69.045	116.231
	R.Walk	74.36	67.14	46.5	81.635	68.088	59.45
	EMD-MA	88.208	65.757	50.281	59.943	37.784	38.383
RMSE	Holt-Winter	69.989	72.413	58.256	91.887	84.818	80.085
	MA	73.429	75.381	61.634	92.005	84.916	79.905
	ARIMA	60.641	83.406	27.904	66.642	86.341	141.753
	R.Walk	74.36	76.747	63.918	92.261	84.738	79.364
	EMD-MA	88.208	77.681	68.076	67.297	48.04	49.778
MAPE	Holt-Winter	2.5	2.225	1.469	2.82	2.362	2.081
	MA	2.619	2.315	1.547	2.822	2.368	2.081
	ARIMA	2.173	2.59	0.968	2.035	2.391	3.92
	R.Walk	2.652	2.369	1.64	2.832	2.36	2.058
	EMD-MA	3.13	2.319	1.769	2.1	1.331	1.347
TheilU	Holt-Winter	-	1.328	1.26	2.214	2.05	2.119
	MA	-	1.382	1.333	2.217	2.052	2.114
	ARIMA	-	1.517	0.446	1.589	2.087	3.74
	R.Walk	-	1.403	1.382	2.223	2.048	2.1
	EMD-MA	-	1.44	1.471	1.597	1.16	1.313
MASE	Holt-Winter	-	0.847	0.796	2.299	1.784	1.932
	MA	-	0.882	0.84	2.301	1.788	1.931
	ARIMA	-	0.99	0.516	1.642	1.807	3.735
	R.Walk	-	0.903	0.891	2.309	1.782	1.91
	EMD-MA	-	0.884	0.964	1.696	0.989	1.233
Switzerland							
MAE	Holt-Winter	161.271	167.702	95.689	211.783	235.662	123.634
	MA	154.555	186.691	102.415	197.672	233.418	119.939
	ARIMA	109.378	247.914	151.49	244.356	568.996	100.892
	R.Walk	163.5	169.85	98.166	212.824	235.16	126.467
	EMD-MA	186.58	114.175	85.267	165.944	76.906	100.315
RMSE	Holt-Winter	161.271	186.253	121.217	231.94	264.67	146.838
	MA	154.555	203.947	127.003	219.645	260.841	143.845
	ARIMA	109.378	264.537	185.511	276.961	636.225	123.678
	R.Walk	163.5	188.499	124.478	233.049	264.124	149.913
	EMD-MA	186.58	137.627	108.249	169.317	92.817	115.299
MAPE	Holt-Winter	1.873	1.928	1.106	2.402	2.653	1.415
	MA	1.796	2.141	1.183	2.245	2.629	1.375
	ARIMA	1.278	2.823	1.735	2.755	6.12	1.162
	R.Walk	1.898	1.952	1.134	2.413	2.647	1.447
	EMD-MA	2.16	1.321	0.987	1.893	0.882	1.162
TheilU	Holt-Winter	-	1.521	1.104	2.275	2.361	1.283
	MA	-	1.644	1.143	2.168	2.322	1.236
	ARIMA	-	2.081	1.713	2.8	5.669	0.997
	R.Walk	-	1.539	1.137	2.286	2.356	1.317
	EMD-MA	-	1.168	0.997	1.441	0.793	0.795
MASE	Holt-Winter	-	1.026	0.761	2.141	2.055	1.181
	MA	-	1.142	0.814	1.998	2.036	1.146
	ARIMA	-	1.516	1.204	2.47	4.963	0.964
	R.Walk	-	1.039	0.78	2.151	2.051	1.208
	EMD-MA	-	0.698	0.678	1.677	0.671	0.958

Table 8: Presented the RMSE, MAE, MASE, MAPE and TheilU of proposed method and four forecasting methods for forecasting at $h = 1, 2, 3, 4, 5$ and 6 for stock market data of Thailand and UK.

Countries	Method	h=1	h=2	h=3	h=4	h=5	h=6
Thailand							
MAE	Holt-Winter	34.516	17.239	15.972	36.169	26.028	22.715
	MA	35.23	17.456	15.773	37.428	27.04	23.973
	ARIMA	35.522	17.605	18.143	37.805	27.49	24.438
	R.Walk	35.21	17.605	17.34	37.615	27.548	24.35
	EMD-MA	26.583	16.985	26.014	35.632	12.299	10.186
RMSE	Holt-Winter	34.516	19.832	21.805	39.039	31.022	29.433
	MA	35.23	20.159	21.835	40.347	32.237	31.03
	ARIMA	35.522	20.503	23.874	40.747	32.781	31.629
	R.Walk	35.21	20.709	23.103	40.572	32.858	31.53
	EMD-MA	26.583	21.92	30.12	38.399	16.065	13.672
MAPE	Holt-Winter	2.741	1.377	1.266	2.812	2.031	1.77
	MA	2.796	1.394	1.25	2.906	2.107	1.865
	ARIMA	2.818	1.405	1.435	2.935	2.141	1.9
	R.Walk	2.794	1.405	1.372	2.92	2.146	1.894
	EMD-MA	2.124	1.353	2.045	2.771	0.973	0.806
TheilU	Holt-Winter	-	0.768	1.02	1.996	1.558	1.616
	MA	-	0.782	1.026	2.064	1.62	1.703
	ARIMA	-	0.798	1.116	2.086	1.648	1.736
	R.Walk	-	0.81	1.08	2.077	1.652	1.731
	EMD-MA	-	0.876	1.387	1.951	0.796	0.743
MASE	Holt-Winter	-	0.49	0.762	2.087	1.359	1.405
	MA	-	0.496	0.753	2.16	1.412	1.483
	ARIMA	-	0.5	0.866	2.182	1.436	1.512
	R.Walk	-	0.5	0.827	2.171	1.439	1.506
	EMD-MA	-	0.482	1.241	2.056	0.642	0.63
UK							
MAE	Holt-Winter	116.446	119.834	64.053	173.492	169.506	181.177
	MA	116.798	128.207	70.107	179.265	172.259	184.647
	ARIMA	71.493	99.864	121.873	201.234	170.644	251.493
	R.Walk	119.3	123.45	67.7	177.775	174.02	185.517
	EMD-MA	163.535	127.739	109.383	150.234	114.16	80.502
RMSE	Holt-Winter	116.446	133.327	80.751	185.725	192.645	209.494
	MA	116.798	141.107	87.079	191.694	196.168	213.262
	ARIMA	71.493	109.732	122.871	214.97	197.451	277.314
	R.Walk	119.3	137.106	85.094	190.265	197.528	214.245
	EMD-MA	163.535	147.403	132.608	157.971	127.377	90.184
MAPE	Holt-Winter	1.918	1.954	1.052	2.781	2.704	2.872
	MA	1.924	2.088	1.15	2.871	2.746	2.925
	ARIMA	1.186	1.635	2.052	3.21	2.72	3.939
	R.Walk	1.964	2.012	1.111	2.848	2.774	2.938
	EMD-MA	2.673	2.077	1.78	2.42	1.841	1.298
TheilU	Holt-Winter	-	1.494	0.977	2.398	2.104	2.475
	MA	-	1.569	1.064	2.481	2.144	2.519
	ARIMA	-	1.218	1.258	2.809	2.16	3.261
	R.Walk	-	1.535	1.042	2.462	2.158	2.531
	EMD-MA	-	1.687	1.639	1.966	1.384	0.985
MASE	Holt-Winter	-	1.004	0.7	2.294	1.804	2.222
	MA	-	1.075	0.766	2.37	1.834	2.265
	ARIMA	-	0.837	1.331	2.661	1.816	3.085
	R.Walk	-	1.035	0.739	2.35	1.852	2.276
	EMD-MA	-	1.071	1.195	1.986	1.215	0.988

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COMMON FIXED POINT RESULTS IN S -FUZZY METRIC SPACES

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Abstract. In this paper we prove fixed point results in integral type mapping with rational expression under weak compatible condition.

Keywords: Self mapping, complete metric spaces, t -norm, common fixed point theorem.

1. Introduction

Concept of Fuzzy initiated by Zadeh [8] in 1965 with the publication of his paper fuzzy sets. The notion of fuzzy set is a turning point in the development of mathematics. Consequently the last three decades were very productive of mathematics. In this paper we establish general common fixed point theorems, which generalize the result of Singh and Chouhan [3, 4], Singh and Sharma [5, 6, 7] in S -fuzzy metric space.

In 2002, an analogue of a Banach contraction principle for integral type inequality Branciari [1] obtained a fixed point theorem for a single mapping. A lot of research works have been carried out on generalizing contractive conditions of integral type for different contractive mappings satisfying various known properties.

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Theorem 1.1 ([1]). *Let (X, d) be a complete metric space, $c \in (0, 1)$ and let $f : X \rightarrow X$ be a mapping such that for each $x, y \in X$,*

$$\int_0^{d(fx, fy)} \varphi(t) dt \leq c \int_0^{d(x, y)} \varphi(t) dt.$$

Where $\varphi : [0, +\infty) \rightarrow [0, +\infty)$ is a Lebesgue - integrable mapping which is summable on each compact subset of $[0, +\infty)$, non negative and such that for each $\varepsilon > 0$, $\int_0^\varepsilon \varphi(t) dt > 0$, then f has a unique fixed point $a \in X$ such that for each $x \in X$, $\lim_{n \rightarrow \infty} f^n x = a$.

Theorem 1.2 ([2]). *Let (X, d) be a complete metric space and $f : X \rightarrow X$ such that*

$$\begin{aligned} \int_0^{d(fx, fy)} u(t) dt &\leq \alpha \int_0^{d(x, fx) + d(y, fy)} u(t) dt + \beta \int_0^{d(x, y)} u(t) dt \\ &+ \gamma \int_0^{\max\{d(x, fy), d(y, fx)\}} u(t) dt, \end{aligned}$$

For each $x, y \in X$, with non-negative real α, β, γ such that $2\alpha + \beta + 2\gamma < 1$. Where $u : [0, +\infty) \rightarrow [0, +\infty)$ is a Lebesgue integrable mapping which is summable, non negative and such that for each $\varepsilon > 0$, $\int_0^\varepsilon u(t) dt > 0$. Then f has a unique fixed point in X .

There is a gap in the proof of Theorem 1.2 In fact, the authors [2] used the inequality $\int_0^{a+b} u(t) dt \leq \int_0^a u(t) dt + \int_0^b u(t) dt$ for $0 \leq a < b$, which is not true in general. In this chapter we present in the presence of this inequality an extension of Theorem 1.2 using altering distance functions.

On taking the concept of Branciari [1] we establish common fixed point theorems in S fuzzy space for four integral type mappings.

2. Preliminaries

Definition 2.1. *The 3-tuple $(X, S, *)$ is said to be a S -Fuzzy Metric Space if X is an arbitrary set, $*$ is a continuous t -norm and S is a Fuzzy set on $X^3 \times (0, \infty)$, satisfying the following conditions:*

- (i) $S(x, y, z, t) > 0$;
- (ii) $S(x, y, z, t) = 1$ if and only if $x = y = z$;
- (iii) $S(x, y, z, t) = S(y, z, x, t) = S(z, y, x, t)$ (Symmetry);
- (iv) $S(x, y, z, r + s + t) \geq S(x, y, w, r) * S(x, w, z, s) * S(w, y, z, t)$ (Tetrahedral inequality);
- (v) $S(x, y, z, \cdot) : (0, \infty) \rightarrow [0, 1]$ is continuous for all $x, y, z, w \in X$ and $r, s, t > 0$.

Geometrically $S(x, y, z, t)$ represents the Fuzzy Perimeter of the triangle whose vertices are the points x, y and z with respect to $t > 0$.

Definition 2.2. Two mappings A and S on fuzzy metric space X are weakly compatible iff $S(ATu, TAu, y, t) \geq S(Au, Tu, y, t)$ for $u \in X$.

Lemma 2.3. If for all $x, y \in X$, $t > 0$ and for a number $q \in (0, 1)$,

$$S(x, y, z, qt) \geq S(x, y, z, t),$$

then $x = y = z$.

Lemma 2.4. Let X be a set, f, g be weakly compatible self maps of X . If f and g have a unique point of coincidence, $w = fx = gx$, then w is the unique common fixed point of f and g .

3. Main results

Theorem 3.1. Let A, B, E and F be self mappings of a complete S fuzzy metric space $(X, S, *)$ with continuous t -norm $*$ defined by $a * b * c = \min\{(a, b, c) : a, b, c \in [0, 1]\}$ satisfying the following conditions

$$(3.1) \quad A(X) \subset F(X), B(X) \subset E(X).$$

Mappings E and F are continuous.

$\{A, E\}$ and $\{B, F\}$ are weakly compatible mappings

For all $x, y, z \in X, t > 0, k \in [0, 1]$

$$(3.2) \quad \int_0^{S(Ax, By, z, kt)} \varphi(t) dt \geq \int_0^J \varphi(t) dt.$$

Where

$$J = \min\left\{S(Ax, Ex, z, t), S(By, Fy, z, t), S(Ex, Fy, z, t), \frac{S(Ax, Ex, z, 2t)S(By, Fy, z, 2t)}{S(Ex, Fy, z, t)}, \frac{S(Bx, Ex, z, 2t)}{S(Ex, Fy, z, t)}, \frac{S(Bx, Fy, z, t) + S(By, Ex, z, t)}{2}\right\},$$

for all $x, y, z \in X, \lim S(x, y, z, t) \rightarrow 1$ as $t \rightarrow \infty$.

Then A, B, E and F have a unique common fixed point in X .

Proof. Let $x_0 \in X$ be arbitrary. Since $A(x) \subseteq F(x)$, we can find a point x_1 in X such that $Ax_0 = Fx_1$. Also since $B(x) \subseteq E(x)$, we can choose a point x_2 with $Bx_1 = Ex_2$.

Using this argument repeatedly we can construct sequence $\{y_n\}$ in X such that $y_{2n-1} = Fx_{2n-1} = Bx_{2n-2}$ and $y_{2n} = Ex_{2n} = Bx_{2n-1}$, $n = 1, 2, 3$.

From (3.2) we have

$$\int_0^{S(y_{2n+1}, y_{2n+2}, z, kt)} \varphi(t) dt = \int_0^{S(Ax_{2n}, Bx_{2n+1}, z, kt)} \varphi(t) dt \geq \int_0^{J_1} \varphi(t) dt.$$

Where

$$\begin{aligned} J_1 = & \min \{ S(Ax_{2n}, Ex_{2n}, z, t), S(Bx_{2n+1}, Fx_{2n+1}, z, t) S(Ex_{2n}, Fx_{2n+1}, z, t) \\ & \frac{S(Ax_{2n}, Ex_{2n}, z, t) S(Bx_{2n+1}, Fx_{2n+1}, z, 2t)}{S(Ex_{2n}, Fx_{2n+1}, z, t)}, \\ & \frac{S(Ax_{2n}, Ex_{2n}, z, 2t)}{S(Ex_{2n}, Fx_{2n+1}, z, t)}, \\ & \frac{S(Ax_{2n}, Fx_{2n+1}, z, t) + S(Bx_{2n+1}, Ex_{2n}, z, t)}{2} \} \\ & \int_0^{J_2} \varphi(t) dt. \end{aligned}$$

Where

$$\begin{aligned} J_2 = & \min \{ S(y_{2n}, y_{2n+1}, z, t), S(y_{2n+2}, y_{2n+1}, z, t), S(y_{2n}, y_{2n+1}, z, t), \\ & \frac{S(y_{2n+1}, y_{2n}, z, 2t) S(y_{2n+2}, y_{2n+1}, z, 2t)}{S(y_{2n}, y_{2n+1}, z, t)}, \frac{S(y_{2n+1}, y_{2n}, z, 2t)}{S(y_{2n}, y_{2n+1}, z, t)}, \\ & \frac{S(y_{2n+1}, y_{2n+1}, z, t) + S(y_{2n+2}, y_{2n}, z, t)}{2} \} \\ & \int_0^{S(y_{2n+1}, y_{2n+2}, z, kt)} \varphi(t) dt \geq \int_0^{S(y_{2n}, y_{2n+1}, z, t)} \varphi(t) dt. \end{aligned}$$

Which implies in general

$$(3.3) \quad \int_0^{S(y_n, y_{n+1}, z, kt)} \varphi(t) dt \geq \int_0^{S(y_{n-1}, y_n, z, t)} \varphi(t) dt.$$

To prove that $\{y_n\}$ is a Cauchy sequence we shall prove

$$(3.4) \quad S(y_n, y_{n+m}, z, t) \geq 1 - \lambda$$

is true for all $n \geq n_0$ and every $m \in \mathbb{N}$.

Here we use induction method from (3.3) we have

$$\begin{aligned} & \int_0^{S(y_n, y_{n+1}, z, t)} \varphi(t) dt \geq \int_0^{S(y_{n-1}, y_n, z, \frac{t}{k})} \varphi(t) dt \\ (3.5) \quad & \geq \int_0^{S(y_{n-2}, y_{n-1}, z, t/k^2) \dots S(y_0, y_1, z, t/k^n)} \varphi(t) dt \rightarrow 1 \text{ as } n \rightarrow \infty, \end{aligned}$$

i.e. For $t > 0$, $\lambda \in (0, 1)$ we can choose $n_0 \in N$ such that $S(y_n, y_{n+1}, z, t) \geq 1 - \lambda$. Thus (3.4) is true for $m = 1$ suppose (3.4) is true for m then we shall prove that it is also true for $m + 1$.

Using the definition of fuzzy metric space by (3.3) and (3.4) we have

$$\int_0^{S(y_n, y_{n+m+1}, z, t)} \varphi(t) dt \geq \int_0^{\min\{S(y_n, y_{n+m}, z, t/2), S(y_{n+m}, y_{n+m+1}, z, t/2)\}} \varphi(t) dt \geq 1 - \lambda.$$

Hence (3.4) is true for $m + 1$. Thus $\{y_n\}$ is a Cauchy sequence. By completeness for $(X, S, *)$, $\{y_n\}$ converges to some point z in X . Thus $\{Ax_{2n}\}$, $\{Ex_{2n}\}$, $\{Bx_{2n-1}\}$ and $\{Fx_{2n-1}\}$ also converges to z_1 . Now $Ax_{2n} \rightarrow z$ and E is continuous.

Hence $EAx_{2n} \rightarrow Ez_1$.

Thus for $t > 0$, $\lambda \in (0, 1)$ there exist an $n_0 \in N$ such that $S(EAx_{2n}, Ez_1, z, t/2) \geq 1 - \lambda$, for all $n > n_0$. Using given condition (weak compatible mappings) we have

$$\begin{aligned} \int_0^{S(AEx_{2n}, EAx_{2n}, z, t/2)} \varphi(t) dt &\rightarrow 1 \\ \int_0^{S(AEx_{2n}, Ez, z, t/2)} \varphi(t) dt &\geq \int_0^{\min\{S(AEx_{2n}, EAx_{2n}, z, t/2), S(EAx_{2n}, Ez, z, t/2)\}} \varphi(t) dt \\ &\geq 1 - \lambda \end{aligned}$$

$$(3.6) \quad \text{Hence } AEx_{2n} \rightarrow Ez_1.$$

$$(3.7) \quad \text{Similarly } BFx_{2n-1} \rightarrow Fz_1.$$

Using (3.2) we have

$$\int_0^{S(AEx_{2n}, BFx_{2n-1}, z, kt)} \varphi(t) dt \geq \int_0^{J_3} \varphi(t) dt.$$

Where

$$\begin{aligned} J_3 &= \\ &= \min\{S(AEx_{2n}, E^2x_{2n}, z, t), S(BFx_{2n-1}, F^2x_{2n-1}, z, t), S(E^2x_{2n}, F^2x_{2n-1}, z, t), \\ &= \frac{S(AEx_{2n}, E^2x_{2n}, z, 2t)S(BFx_{2n-1}, F^2x_{2n-1}, z, 2t)}{S(E^2x_{2n}, F^2x_{2n-1}, z, t)}, \frac{S(AEx_{2n}, E^2x_{2n}, z, 2t)}{S(E^2x_{2n}, F^2x_{2n-1}, z, t)}, \\ &\frac{S(AEx_{2n}, F^2x_{2n-1}, z, t) + S(BFx_{2n-1}, E^2x_{2n}, z, t)}{2}\}. \end{aligned}$$

Taking limit as $n \rightarrow \infty$ and using (3.6) and (3.7) we get

$$(3.8) \quad \int_0^{S(Ez_1 Fz_1, z, kt)} \varphi(t) dt \geq \int_0^{S(Ez_1, Fz_1, z, t)} \varphi(t) dt.$$

Which implies that

$$(3.9) \quad Ez_1 = Fz_1.$$

Now

$$(3.10) \quad \int_0^{S(Az_1, BFx_{2n-1}, z, kt)} \varphi(t) dt \geq \int_0^{J_4} \varphi(t) dt.$$

Where

$$J_4 = \min \left\{ S(Az_1, Ez_1, z, t), S(BFx_{2n-1}, F^2x_{2n-1}, z, t), S(Ez_1, F^2x_{2n-1}, t), \right. \\ \left. \frac{S(Az_1, Ez_1, z, 2t)S(BFx_{2n-1}, F^2x_{2n-1}, z, 2t)}{S(Ez_1, F^2x_{2n-1}, z, t)}, \frac{S(AEz_1, Ez_1, z, 2t)}{S(Ez_1, F^2x_{2n-1}, z, t)}, \right. \\ \left. \frac{S(AEz_1, F^2x_{2n-1}, z, t) + S(BFx_{2n-1}, Ez_1, z, t)}{2} \right\}.$$

Taking the limit as $n \rightarrow \infty$ and using (3.7) and (3.9) we get

$$\int_0^{S(Az, Fz_1, z, kt)} \varphi(t) dt \geq \int_0^{S(Az, Fz_1, z, t)} \varphi(t) dt.$$

Which implies that

$$(3.11) \quad Az_1 = Fz_1$$

$$\int_0^{S(Az_1, Bz_1, z, kt)} \varphi(t) dt \geq \int_0^{J_5} \varphi(t) dt.$$

Where

$$J_5 = \min \left\{ S(Az_1, Ez_1, z, t), S(Bz_1, Fz_1, z, t), S(Ez_1, Fz_1, z, t), \right. \\ \left. \frac{S(Az_1, Ez_1, z, 2t)S(Bz_1, Fz_1, z, 2t)}{S(Ez_1, Fz_1, t)}, \frac{S(AEz_1, Ez_1, z, 2t)}{S(Ez_1, Fz_1, z, t)}, \right. \\ \left. \frac{S(AEz_1, Fz_1, z, t) + S(Bz_1, Ez_1, z, t)}{2} \right\}.$$

$$\int_0^{S(Az_1, Bz_1, z, kt)} \varphi(t) dt \geq \int_0^{J_6} \varphi(t) dt.$$

Where

$$J_6 = \min \left\{ S(Fz_1, Ez_1, z, t), S(Bz_1, Az_1, z, t), S(Ez_1, Fz_1, z, t), \right. \\ = \frac{S(Az_1, Ez_1, z, 2t)S(Bz_1, Fz_1, z, 2t)}{S(Ez_1, Fz_1, z, t)}, \frac{S(AEz_1, Ez_1, z, 2t)}{S(Ez_1, Fz_1, z, t)}, \\ \left. \frac{S(AEz_1, Fz_1, z, t) + S(Bz_1, Ez_1, a, t)}{2} \right\}.$$

Which implies that

$$(3.12) \quad Az_1 = Bz_1.$$

Using (3.9) and (3.12) we get

$$(3.13) \quad Az_1 = Bz_1 = Ez_1 = Fz_1$$

Now $\int_0^{S(Ax_{2n}, Bz_1, z, kt)} \varphi(t) dt \geq \int_0^{J_7} \varphi(t) dt$. Where

$$\begin{aligned} J_7 &= \min\{S(Ax_{2n}, Ex_{2n}, z, t), S(Bz_1, Fz_1, z, t), S(Ex_{2n}, Fz_1, z, t), \\ &= \frac{S(Ax_{2n}, Ex_{2n}, z, 2t)S(Bz_1, Fz_1, z, 2t)}{S(Ex_{2n}, Fz_1, z, t)}, \frac{S(Ax_{2n}, Ex_{2n}, z, 2t)}{S(Ex_{2n}, Fz_1, z, t)}, \\ &\frac{S(Ax_{2n}, Fz_1, z, t) + S(Bz_1, Ex_{2n}, z, t)}{2}\} \geq \int_0^{J_8} \varphi(t) dt. \end{aligned}$$

Where

$$\begin{aligned} J_8 &= \min\{S(Ex_{2n}, Fz_1, z, t), S(Ax_{2n}, Ex_{2n}, z, t), S(Bz_1, Fz_1, z, t), \\ &\frac{S(Ax_{2n}, Ex_{2n}, z, 2t)S(Bz_1, Fz_1, z, 2t)}{S(Ex_{2n}, Fz_1, z, t)}, \\ &\frac{S(Ax_{2n}, Ex_{2n}, z, 2t)}{S(Ex_{2n}, Fz_1, z, t)}, \frac{S(Ax_{2n}, Fz_1, z, t) + S(Bz_1, Ex_{2n}, z, t)}{2}\} \end{aligned}$$

Taking limit as $n \rightarrow \infty$ and using (3.13) we get

$$\int_0^{S(z_1, Bz_1, z, kt)} \varphi(t) dt \geq \int_0^{S(z_1, Bz_1, z, t)} \varphi(t) dt.$$

Which implies that

$$(3.14) \quad z_1 = Bz_1.$$

Thus z_1 is a common fixed point of A, B, E and F . For the uniqueness let w be another common fixed point of said mappings. Then from (3.2)

$$\int_0^{S(Az_1, Bw, z, kt)} \varphi(t) dt \geq \int_0^{J_9} \varphi(t) dt.$$

Where

$$\begin{aligned} J_9 &= \min\{S(Az_1, Ez_1, z, t), S(Bw, Fw, z, t), S(Ez, Fw, z, t), \\ &\frac{S(Az_1, Ez_1, z, 2t)S(Bw, Fw, z, 2t)}{S(Ez, Fw, t)}, \frac{S(Az_1, Ez_1, z, 2t)}{S(Ez, Fw, t)}, \\ &\frac{S(Az_1, Fw, z, t) + S(Bw, Ez_1, z, t)}{2}\} \end{aligned}$$

$$\text{i.e. } \int_0^{S(z_1, w, z, kt)} \varphi(t) dt \geq \int_0^{S(z_1, w, z, t)} \varphi(t) dt.$$

Hence $z_1 = w$. This completes the proof. $\square$

Theorem 3.2. *Let p, q, f and g be self mappings of complete S fuzzy metric space $(X, S, *)$ with continuous t -norm $*$ defined by $a * b * c = \min\{(a, b, c) : a, b, c \in [0, 1]\}$ satisfying the following conditions*

(3.15) *Pairs $\{p, f\}$ and $\{q, g\}$ be oscillatory weakly commutative.*

(3.16) *For all $x, y, z \in X, t > 0, h \in [0, 1]$*

$$\int_0^{S(px, qy, z, ht)} \varphi(t) dt \geq \int_0^J \varphi(t) dt.$$

Where $J = \min\{S(fx, gy, z, t), S(fx, px, z, t), S(qy, gy, z, t), [S(px, gy, z, t) + S(fx, px, z, t)]/S(fx, px, z, t), [S(qy, fx, z, t) + S(qy, gy, z, t)]/S(qy, gy, z, t)\}$. For all $x, y, z \in X$, $\lim_{t \rightarrow \infty} S(x, y, z, t) \rightarrow 1$ as $t \rightarrow \infty$. And $\emptyset \in \Phi$, for all $0 < t < 1$, then there exists a unique point $w \in X$ such that $pw = fw = w$ and a unique point $z_1 \in X$ such that $qz_1 = gz_1 = z_1$. Moreover, $z_1 = w$, so that there is a unique common fixed point of p, f, q and g .

Proof. Let the pairs $\{p, f\}$ and $\{q, g\}$ be owc, so there are points $x, y, z \in X$ such that $px = fx$ and $qy = gy$. We claim that $px = qy$. If not, by inequality (3.16)

$$\int_0^{S(px, qy, z, ht)} \varphi(t) dt \geq \int_0^J \varphi(t) dt$$

Where $J = \min\{S(fx, gy, z, t), S(fx, px, z, t), S(qy, gy, z, t), [S(px, gy, z, t) + S(fx, px, z, t)]/S(fx, px, z, t), [S(qy, fx, z, t) + S(qy, gy, z, t)]/S(qy, gy, z, t)\}$

$$\int_0^{S(px, qy, z, ht)} \phi(t) dt \geq \int_0^{J_1} \phi(t) dt,$$

where $J_1 = \min\{S(px, qy, z, t), S(px, px, z, t), S(qy, qy, z, t), [S(px, qy, z, t) + S(px, px, z, t)]/S(px, px, z, t), [S(qy, px, z, t) + S(qy, qy, z, t)]/S(qy, qy, z, t)\}$

$$\int_0^{S(px, qy, z, ht)} \phi(t) dt \geq \int_0^{J_2} \phi(t) dt,$$

where $J_2 = S(px, qy, z, t)$. Therefore $px = qy$, i.e. $px = fx = qy = gy$. Suppose that there is another point z such that $pz = fz$ then by (3.15) we have $pz = fz = qy = gy$, so $px = pz$ and $w = px = fx$ is the unique point of coincidence of p and f . By Lemma 2.4 w is the only common fixed point of p and f . Similarly there is a unique point $z_1 \in X$ such that $z_1 = qz_1 = gz_1$. Assume that $w \neq z z_1$. We have

$$\int_0^{S(w, z_1, z, ht)} \phi(t) dt = \int_0^{S(pw, fz_1, z, ht)} \phi(t) dt \geq \int_0^{S(w, z_1, z, ht)} \phi(t) dt.$$

$$J_3 = \min\{S(fw, gz_1, z, t), S(fw, pw, z, t), S(qz_1, gz_1, z, t), [S(pw, gz_1, z, t) \\ + S(fw, pw, z, t)]/S(fw, pw, z, t), [S(qz_1, fw, z, t) \\ + S(qz_1, gz_1, z, t)]/S(qz_1, gz_1, z, t)\}$$

$$\int_0^{S(w, z_1, z, ht)} \phi(t) dt \geq \int_0^{S(w, z_1, z, ht)} \phi(t) dt.$$

Therefore we have $z_1 = w$. Hence z_1 is a common fixed point of p, f, q and g . The uniqueness of the fixed point holds from (3.15). $\square$

Theorem 3.3. *Let complete S fuzzy metric space $(X, S, *)$ with continuous t -norm $*$ defined by $a * b * c = \min\{(a, b, c) : a, b, c \in [0, 1]\}$ satisfying the following conditions*

- (i) R and T be continuous mappings of X ;
- (ii) if there exists continuous mappings A of X into $R(X) \cap T(X)$ which weakly compatible with R and T and

$$(3.17) \quad \int_0^{S(Ax, Ay, z, ht)} \varphi(t) dt \geq \int_0^J \varphi(t) dt$$

where

$$J = \min\{S(Ty, Ay, z, t), S(Rx, Ax, z, t), S(Rx, Ty, z, t), \frac{S(Rx, Ty, z, t)}{S(Ax, Ty, z, t)}, \\ \frac{S(Ty, Ay, z, t)}{S(Rx, Ax, z, t)}, \frac{S(Rx, Ax, z, t)}{S(Ty, Ay, z, t)}\},$$

for all $x, y, z \in X$, $t > 0$, and $0 < h < 1$. Then R, T and A have a unique common fixed point.

Proof. We define a sequence $\{x_n\}$ such that $Ax_{2n} = Sx_{2n-1}$ and $Ax_{2n-1} = Tx_{2n}$, $n = 1, 2, \dots$. We shall prove that $\{Ax_n\}$ is a Cauchy sequence. For this suppose $x = x_{2n}$ and $y = x_{2n+1}$ in (3.17), we write

$$\int_0^{S(Ax_{2n}, Ax_{2n+1}, z, ht)} \varphi(t) dt \geq \int_0^{J_1} \varphi(t) dt$$

where

$$J_1 = \min\{S(Tx_{2n+1}, Ax_{2n+1}, z, t), S(Rx_{2n}, Ax_{2n}, z, t), S(Rx_{2n}, Tx_{2n+1}, z, t), \\ \frac{S(Rx_{2n}, Tx_{2n+1}, z, t)}{S(Ax_{2n}, Tx_{2n+1}, z, t)}, \\ \frac{S(Tx_{2n+1}, Ax_{2n+1}, z, t)}{S(Rx_{2n}, Ax_{2n}, z, t)}, \frac{S(Rx_{2n}, Ax_{2n}, z, t)}{S(Tx_{2n+1}, Ax_{2n+1}, z, t)} \\ \int_0^{S(Ax_{2n}, Ax_{2n+1}, z, ht)} \varphi(t) dt \geq \int_0^{J_2} \varphi(t) dt$$

where

$$J_2 = \min\{S(Ax_{2n}, Ax_{2n+1}, z, t), S(Ax_{2n+1}, Ax_{2n}, z, t), S(Ax_{2n+1}, Ax_{2n}, z, t), \\ \frac{S(Ax_{2n+1}, Tx_{2n}, z, t)}{S(Ax_{2n}, Ax_{2n}, z, t)}, \frac{S(Ax_{2n}, Ax_{2n+1}, z, t)}{S(Ax_{2n+1}, Ax_{2n}, z, t)}, \frac{S(Ax_{2n+1}, Ax_{2n}, z, t)}{S(Ax_{2n}, Ax_{2n+1}, z, t)}\}.$$

Now

$$J_2 = \min\{S(Ax_{2n}, Ax_{2n+1}, z, t), S(Ax_{2n+1}, Ax_{2n}, z, t), \\ S(Ax_{2n+1}, Ax_{2n}, z, t), 1, 1, 1\} \\ \geq \min\{S(Ax_{2n-1}, Ax_{2n}, z, \frac{t}{h}), S(Ax_{2n}, Ax_{2n-1}, z, \frac{t}{h})\}.$$

Thus

$$\int_0^{S(Ax_{2n}, Ax_{2n+1}, z, ht)} \varphi(t) dt \geq \int_0^{S(Ax_{2n-1}, Ax_{2n}, z, \frac{t}{h})} \varphi(t) dt.$$

By induction

$$\int_0^{S(Ax_{2k}, Ax_{2m+1}, z, ht)} \varphi(t) dt \geq \int_0^{S(Ax_{2m}, Ax_{2k-1}, z, \frac{t}{h})} \varphi(t) dt.$$

For every k and m in N , Further if $2m + 1 > 2k$, then

$$\int_0^{S(Ax_{2k}, Ax_{2m+1}, z, ht)} \varphi(t) dt \geq \int_0^{S(Ax_{2k-1}, Ax_{2m}, z, \frac{t}{h})} \varphi(t) dt \\ \geq \dots \geq \int_0^{S(Ax_0, Ax_{2m+1-2k}, z, \frac{t}{h^{2m+1}})} \varphi(t) dt.$$

If $2k > 2m + 1$, then

$$\int_0^{S(Ax_{2k}, Ax_{2m+1}, z, ht)} \varphi(t) dt \geq \int_0^{S(Ax_{2k-1}, Ax_{2m}, z, \frac{t}{h})} \varphi(t) dt \\ \geq \dots \geq \int_0^{S(Ax_{2k-(2m+1)}, Ax_0, z, \frac{t}{h^{2m+1}})} \varphi(t) dt.$$

By simple induction of above two equation we have

$$\int_0^{S(Ax_n, Ax_{n+p}, z, ht)} \varphi(t) dt \geq \int_0^{S(Ax_0, Ax_p, z, \frac{t}{h^n})} \varphi(t) dt.$$

For $n = 2k, p = 2m + 1$ or $n = 2k + 1, p = 2m + 1$

$$(3.18) \quad S(Ax_n, Ax_{n+p}, z, ht) \geq S(Ax_0, Ax_1, z, \frac{t}{2h^n}) * S(Ax_1, Ax_p, z, \frac{t}{h^n}).$$

If $n = 2k, p = 2m$ or $n = 2k + 1, p = 2m$. For every positive integer p and n in N , by nothing that

$$S(Ax_0, Ax_p, z, \frac{t}{h^n}) \rightarrow 1 \text{ as } n \rightarrow \infty.$$

Thus $\{Ax_n\}$ is a Cauchy sequence. Since the space X is complete there exists $z_1 \in X$, such that $\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Rx_{2n-1} = \lim_{n \rightarrow \infty} Tx_{2n} = z_1$. It follows that $Az_1 = Rz_1 = Tz_1$ and by weakly compatibility therefore

$$\int_0^{S(Az_1, AAz_1, z, ht)} \varphi(t) dt \geq \int_0^{J_3} \varphi(t) dt$$

where

$$(3.19) \quad J_3 = \min\{S(TAz_1, AAz_1, z, t), S(Rz_1, Az_1, z, t), S(Az_1, TAz_1, z, t), \\ \frac{S(Rz_1, TAz_1, z, t)}{S(Az_1, TAz_1, z, t)}, \frac{S(TAz_1, AAz_1, z, t)}{S(Rz_1, Az_1, z, t)}, \frac{S(Rz_1, Az_1, z, t)}{S(TAz_1, AAz_1, z, t)}\}.$$

$$\begin{aligned} \int_0^{S(Az_1, AAz_1, z, ht)} \varphi(t) dt &\geq \int_0^{S(Rz_1, TAz_1, z, t)} \varphi(t) dt \\ &\geq \int_0^{S(Rz_1, ATz_1, z, t)} \varphi(t) dt \geq \int_0^{S(Az_1, A^2z_1, z, t)} \varphi(t) dt \\ &\geq \dots \geq \int_0^{S(Az_1, A^2z_1, z, \frac{t}{h^n})} \varphi(t) dt. \end{aligned}$$

Since $\lim_{n \rightarrow \infty} S(Az_1, A^2z_1, z, \frac{t}{h^n}) = 1 \Rightarrow Az_1 = A^2z_1$.

Thus z_1 is common fixed point of A, R and T . For uniqueness, let $w (w \neq z_1)$ be another common fixed point of R, T and A . By (3.17), we write

$$\int_0^{S(Az_1, Aw, z, ht)} \varphi(t) dt \geq \int_0^{J_4} \varphi(t) dt$$

where

$$J_4 = \min\{S(Tw, Aw, z, t), S(Rz_1, Az_1, z, t), S(Rz_1, Tw, z, t), \\ \frac{S(Rz_1, Tw, z, t)}{S(Az_1, Tw, z, t)}, \frac{S(Tz_1, Aw, z, t)}{S(Rz_1, Az_1, z, t)}, \frac{S(Rz_1, Az_1, z, t)}{S(Tw, Aw, z, t)}\}$$

$$\int_0^{S(Az_1, Aw, z, ht)} \varphi(t) dt \geq \int_0^{S(z_1, w, z, t)} \varphi(t) dt.$$

This implies that

$$\int_0^{S(z_1, w, z, ht)} \varphi(t) dt \geq \int_0^{S(z_1, w, z, t)} \varphi(t) dt.$$

Therefore by Lemma 2.4, we write $z_1 = w$. This complete the proof of Theorem (3.3). $\square$

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RIEMANN-LIOUVILLE FRACTIONAL SIMPSON'S INEQUALITIES THROUGH GENERALIZED (m, h_1, h_2)-PREINVEXITY

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Abstract. The authors first introduce the concept of generalized (m, h_1, h_2) -preinvex function. Second, a new Riemann-Liouville fractional integral identity involving first order differentiable function on an m -invex is found. By using this identity, we present new Riemann-Liouville fractional Simpson's inequalities through generalized (m, h_1, h_2) -preinvexity. These inequalities can be viewed as significant generalization of some previously known results.

Keywords: Simpson's inequality, generalized (m, h_1, h_2) -preinvex functions, Riemann-Liouville fractional integrals.

1. Introduction

The following inequality named Simpson's inequality plays a significant role in analysis.

Theorem 1.1. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a four times continuously differentiable mapping on (a, b) and $\|f^{(4)}\|_\infty := \sup_{x \in (a, b)} |f^{(4)}(x)| < \infty$. Then the following inequality holds:*

$$(1.1) \quad \left| \frac{1}{3} \left[\frac{f(a) + f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{1}{2880} \|f^{(4)}\|_\infty (b-a)^4.$$

The study of Simpson type integral inequalities involving various kinds of convex functions has been carried out by many researchers, including Chun and Qi [7] in the study of Simpson's inequalities for extended s -convex functions, Du et al. [11] in generalization of Simpson type inequality for extended

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(s, m) -convex functions, Hussain and Qaisar [13] in generalizations of Simpson's inequalities through preinvexity and Qaisar et al. [28] in generalizations of Simpson's type inequality via (α, m) -convex functions. For more results and recent development on the Simpson's inequality, see [14, 18, 19, 20, 26, 27, 30, 36, 38, 45, 47] and the references therein.

In 2013, Sarikaya et al. [34] considered the following interesting Hermite-Hadamard-type inequalities containing Riemann-Liouville fractional integrals.

Theorem 1.2 ([34]). *Let $f : [u, v] \rightarrow \mathbb{R}$ be a positive function with $0 \leq u < v$ and let $f \in L^1[u, v]$. Suppose f is a convex function on $[u, v]$, then the following inequalities for fractional integrals hold:*

$$(1.2) \quad f\left(\frac{u+v}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(v-u)^\alpha} [J_{u+}^\alpha f(v) + J_{v-}^\alpha f(u)] \leq \frac{f(u) + f(v)}{2},$$

where the symbol $J_{u+}^\alpha f$ and $J_{v-}^\alpha f$ denote respectively the left-sided and right-sided Riemann-Liouville fractional integrals of the order $\alpha \in \mathbb{R}^+$ defined by

$$J_{u+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_u^x (x-t)^{\alpha-1} f(t) dt, \quad u < x$$

and

$$J_{v-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^v (t-x)^{\alpha-1} f(t) dt, \quad x < v.$$

Here, $\Gamma(\alpha)$ is the Gamma function and its definition is

$$\Gamma(\alpha) = \int_0^\infty e^{-\mu} \mu^{\alpha-1} d\mu.$$

We observe that, for $\alpha = 1$, the inequality (1.2) can be reduced to the following termed Hermite-Hadamard inequality

$$(1.3) \quad f\left(\frac{u+v}{2}\right) \leq \frac{1}{v-u} \int_u^v f(x) dx \leq \frac{f(u) + f(v)}{2},$$

where $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is a convex mapping on the interval I of real numbers and $u, v \in I$ with $u < v$.

For more recent results which generalize, refine, and extend this classic inequality (1.3), One can see contributions [9, 15, 17, 21, 25, 29, 31, 44, 48] and references therein.

Recently, Riemann-Liouville fractional Hermite-Hadamard inequalities and its extensively application have been attached more and more attentions, see [1, 2, 4, 5, 6, 8, 16, 23, 35, 42, 46], however, there are few works on the Simpson type fractional integrals. Thus, it is natural to offer to study Simpson's inequalities involving Riemann-Liouville fractional integral.

Stimulated by above works, in this paper we introduce a class of generalized (m, h_1, h_2) -preinvex functions and derive new Riemann-Liouville fractional Simpson's inequalities involving the class of functions whose first derivatives in absolute values are generalized (m, h_1, h_2) -preinvex functions. Some results can be regarded as generalization of recent results that appeared in Refs. [22, 32, 33, 40].

2. Preliminaries

Definition 2.1 ([43]). A set $S \subseteq \mathbb{R}^n$ is said to be invex set with respect to the mapping $\eta : S \times S \rightarrow \mathbb{R}^n$, if $x + t\eta(y, x) \in S$ for every $x, y \in S$ and $t \in [0, 1]$. The invex set S is also termed an η -connected set.

Definition 2.2 ([10]). A set $K \subseteq \mathbb{R}^n$ is said to be m -invex with respect to the mapping $\eta : K \times K \times (0, 1] \rightarrow \mathbb{R}^n$ for some fixed $m \in (0, 1]$, if $mx + t\eta(y, x, m) \in K$ holds for each $x, y \in K$ and any $t \in [0, 1]$.

We next explore a new concept, to be referred to as the generalized (m, h_1, h_2) -preinvex functions and its variant forms.

Definition 2.3. Let $K \subseteq \mathbb{R}$ be an open m -invex set with respect to $\eta : K \times K \times (0, 1] \rightarrow \mathbb{R}$. A function $f : K \rightarrow \mathbb{R}$, $h_1, h_2 : [0, 1] \rightarrow \mathbb{R}_0$, if

$$(2.1) \quad f(mx + t\eta(y, x, m)) \leq mh_1(t)f(x) + h_2(t)f(y)$$

is valid for all $x, y \in K$ and $t \in [0, 1]$, with $m \in (0, 1]$, then we say that $f(x)$ is a generalized (m, h_1, h_2) -preinvex function with respect to η . If the inequality (2.1) reverses, then f is said to be (m, h_1, h_2) -preincave on K .

Remark 2.1. In Definition 2.3, let $h_1(t) = (1 - t)^{-s}$, $h_2(t) = t^{-s}$ and $s \in [0, 1]$, if

$$(2.2) \quad f(mx + t\eta(y, x, m)) \leq m(1 - t)^{-s}f(x) + t^{-s}f(y)$$

is valid for all $x, y \in K$ and $t \in (0, 1)$, with $m \in (0, 1]$, then we say that $f(x)$ is a generalized (m, s) -Godunova-Levin-preinvex function with respect to η .

We now discuss some special cases of generalized (m, h_1, h_2) -preinvex function, which appears to be new ones.

I) If the mapping $\eta(y, x, m)$ with $m = 1$ degenerates into $\eta(y, x)$, $h_1(t) = h(1 - t)$ and $h_2(t) = h(t)$, then Definition 2.3, reduces to:

Definition 2.4 ([21]). Let $h : [0, 1] \rightarrow \mathbb{R}$ be a non-negative function and $h \neq 0$. The function f on the invex set X is said to be h -preinvex with respect to η , if

$$(2.3) \quad f(x + t\eta(y, x)) \leq h(1 - t)f(x) + h(t)f(y)$$

for each $x, y \in X$ and $t \in [0, 1]$ where $f(\cdot) > 0$.

II) If the mapping $\eta(y, x, m)$ with $m = 1$ degenerates into $\eta(y, x)$, $h_1(t) = (1 - t)^s$ and $h_2(t) = t^s$, then Definition 2.3, reduces to:

Definition 2.5 ([40]). *Let $S \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : S \times S \rightarrow \mathbb{R}^n$. A function $f : S \rightarrow \mathbb{R}_0 = [0, \infty)$ is said to be s -preinvex with respect to η and $s \in (0, 1]$, if for every $x, y \in S$ and $t \in [0, 1]$,*

$$(2.4) \quad f(x + t\eta(y, x)) \leq (1 - t)^s f(x) + t^s f(y).$$

III) If the mapping $\eta(y, x, m)$ with $m = 1$ degenerates into $\eta(y, x)$, $h_1(t) = (1 - t)^{-s}$ and $h_2(t) = t^{-s}$, then Definition 2.3, reduces to:

Definition 2.6 ([24]). *A function $f : K \rightarrow R$ is said to be s -Godunova-Levin preinvex functions of second kind, if*

$$(2.5) \quad f(x + t\eta(y, x)) \leq (1 - t)^{-s} f(x) + t^{-s} f(y)$$

for each $x, y \in K$, $t \in (0, 1)$ and $s \in [0, 1]$.

IV) If the mapping $\eta(y, x, m)$ with $m = 1$ degenerates into $\eta(y, x)$, $h_1(t) = 1 - t^\alpha$ and $h_2(t) = t^\alpha$, then Definition 2.3, reduces to:

Definition 2.7 ([41]). *Let $S \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : S \times S \rightarrow \mathbb{R}^n$. A function $f : S \rightarrow \mathbb{R}$ is said to be α -preinvex with respect to η for $\alpha \in (0, 1]$, if for every $x, y \in S$ and $t \in [0, 1]$,*

$$(2.6) \quad f(x + t\eta(y, x)) \leq (1 - t^\alpha) f(x) + t^\alpha f(y).$$

V) If the mapping $\eta(y, x, m)$ with $m = 1$ degenerates into $\eta(y, x)$, $h_1(t) = 1 - t$ and $h_2(t) = t$, then Definition 2.3, reduces to:

Definition 2.8 ([43]). *A function f defined on the invex set $K \subseteq \mathbb{R}^n$ is said to be preinvex respecting η , if*

$$(2.7) \quad f(x + t\eta(y, x)) \leq (1 - t)f(x) + tf(y), \forall x, y \in K, t \in [0, 1].$$

VI) If the mapping $\eta(y, x, m) = y - mx$ with $m = 1$, $h_1(t) = h(1 - t)$ and $h_2(t) = h(t)$, then Definition 2.3, reduces to:

Definition 2.9 ([39]). *Let $h : J \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a positive function, $h \neq 0$. We say that $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is h -convex, if f is non-negative and for all $x, y \in I$ and $t \in (0, 1)$, one has*

$$(2.8) \quad f((1 - t)x + ty) \leq h(1 - t)f(x) + h(t)f(y).$$

VII) If the mapping $\eta(y, x, m) = y - mx$ with $m = 1$, $h_1(t) = (1 - t)^s$ and $h_2(t) = t^s$, then Definition 2.3, reduces to:

Definition 2.10 ([3]). Let s be a real number, $s \in (0, 1]$. A function $f : [0, \infty) \rightarrow \mathbb{R}$ is said to be s -convex, if

$$(2.9) \quad f((1-t)x + ty) \leq (1-t)^s f(x) + t^s f(y).$$

for all $x, y \in [0, \infty)$ and $t \in [0, 1]$.

VIII) If the mapping $\eta(y, x, m) = y - mx$ with $m = 1$, $h_1(t) = (1-t)^{-s}$ and $h_2(t) = t^{-s}$, then Definition 2.3, reduces to:

Definition 2.11 ([12]). We say that the function $f : K \subseteq X \rightarrow [0, \infty)$ is of s -Godunova-Levin type, with $s \in [0, 1]$, if

$$(2.10) \quad f((1-t)x + ty) \leq (1-t)^{-s} f(x) + t^{-s} f(y)$$

for each $x, y \in X$, $t \in (0, 1)$.

IX) If the mapping $\eta(y, x, m) = y - mx$ with $m = 1$, and $h_1(t) = h_2(t) = t(1-t)$, then Definition 2.3, reduces to:

Definition 2.12 ([37]). Let $f : K \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a nonnegative function, we say that $f : K \rightarrow \mathbb{R}$ is a tgs-convex function on K if the inequality

$$(2.11) \quad f((1-t)x + ty) \leq t(1-t)[f(x) + f(y)]$$

holds for all $x, y \in K$ and $t \in (0, 1)$. We say that f is tgs-concave if $(-f)$ is tgs-convex.

3. Main results

Let $f : K \rightarrow \mathbb{R}$ be a differentiable function, throughout this section we will take

$$(3.1) \quad \begin{aligned} & K_f(\alpha; \eta, m, a, b) \\ &:= \frac{1}{6} \left[f(ma) + 4f\left(ma + \frac{\eta(b, a, m)}{2}\right) + f\left(ma + \eta(b, a, m)\right) \right] \\ &\quad - \frac{2^{\alpha-1} \Gamma(\alpha+1)}{\eta^\alpha(b, a, m)} \left[J_{(ma)^+}^\alpha f\left(ma + \frac{\eta(b, a, m)}{2}\right) \right. \\ &\quad \left. + J_{(ma+\eta(b, a, m))^-}^\alpha f\left(ma + \frac{\eta(b, a, m)}{2}\right) \right], \end{aligned}$$

where $K \subseteq \mathbb{R}$ be an open m -invex subset with respect to $\eta : K \times K \times (0, 1] \rightarrow \mathbb{R}$ for some fixed $m \in (0, 1]$, $a, b \in K$ with $a < b$, $\alpha > 0$ and Γ is the Euler Gamma function.

Before presenting our main results, we claim the following integral identity.

Lemma 3.1. *Let $K \subseteq \mathbb{R}$ be an open m -invex subset with respect to $\eta : K \times K \times (0, 1] \rightarrow \mathbb{R}$ for some fixed $m \in (0, 1]$ and let $a, b \in K$, $a < b$ with $\eta(b, a, m) > 0$. Assume that $f : K \rightarrow \mathbb{R}$ is a first differentiable function, f' is integrable on $[ma, ma + \eta(b, a, m)]$, then the following identity for Riemann-Louville fractional integral with $\alpha > 0$ and $x \in [ma, ma + \eta(b, a, m)]$ holds:*

$$(3.2) \quad K_f(\alpha; \eta, m, a, b) = \frac{\eta(b, a, m)}{2} \int_0^1 \left[\left(\frac{1}{3} - \frac{t^\alpha}{2} \right) f' \left(ma + \frac{1-t}{2} \eta(b, a, m) \right) + \left(\frac{t^\alpha}{2} - \frac{1}{3} \right) f' \left(ma + \frac{1+t}{2} \eta(b, a, m) \right) \right] dt.$$

Proof. By integration by parts, we have

$$(3.3) \quad \begin{aligned} & \int_0^1 \left(\frac{1}{3} - \frac{t^\alpha}{2} \right) f' \left(ma + \frac{1-t}{2} \eta(b, a, m) \right) dt \\ &= \frac{-2}{\eta(b, a, m)} \left[\left(\frac{1}{3} - \frac{t^\alpha}{2} \right) f \left(tma + (1-t) \left(ma + \frac{\eta(b, a, m)}{2} \right) \right) \right] \Big|_0^1 \\ & \quad - \int_0^1 (\alpha t^{\alpha-1}) \frac{1}{\eta(b, a, m)} f \left(tma + (1-t) \left(ma + \frac{\eta(b, a, m)}{2} \right) \right) dt \\ &= \frac{2}{\eta(b, a, m)} \left[\frac{1}{6} f(ma) + \frac{1}{3} f \left(ma + \frac{\eta(b, a, m)}{2} \right) \right] \\ & \quad - \frac{\alpha 2^\alpha}{\eta^{\alpha+1}(b, a, m)} \int_{ma}^{ma + \frac{\eta(b, a, m)}{2}} \left(\left(ma + \frac{\eta(b, a, m)}{2} \right) - u \right)^{\alpha-1} f(u) du \\ &= \frac{2}{\eta(b, a, m)} \left[\frac{1}{6} f(ma) + \frac{1}{3} f \left(ma + \frac{\eta(b, a, m)}{2} \right) \right] \\ & \quad - \frac{2^\alpha \Gamma(\alpha+1)}{\eta^{\alpha+1}(b, a, m)} J_{(ma)+}^\alpha f \left(ma + \frac{\eta(b, a, m)}{2} \right). \end{aligned}$$

Similarly, we get

$$(3.4) \quad \begin{aligned} & \int_0^1 \left(\frac{t^\alpha}{2} - \frac{1}{3} \right) f' \left(ma + \frac{1+t}{2} \eta(b, a, m) \right) dt \\ &= \frac{2}{\eta(b, a, m)} \left[\left(\frac{t^\alpha}{2} - \frac{1}{3} \right) f \left((1-t) \left(ma + \frac{\eta(b, a, m)}{2} \right) + t \left(ma + \eta(b, a, m) \right) \right) \right] \Big|_0^1 \end{aligned}$$

$$\begin{aligned}
& - \int_0^1 (\alpha t^{\alpha-1}) \frac{1}{\eta(b, a, m)} f\left(\left(1-t\right)\left(ma + \frac{\eta(b, a, m)}{2}\right) \right. \\
& \left. + t\left(ma + \eta(b, a, m)\right)\right) dt \\
& = \frac{2}{\eta(b, a, m)} \left[\frac{1}{6} f\left(ma + \eta(b, a, m)\right) + \frac{1}{3} f\left(ma + \frac{\eta(b, a, m)}{2}\right) \right] \\
& \quad - \frac{\alpha 2^\alpha}{\eta^{\alpha+1}(b, a, m)} \int_{ma + \frac{\eta(b, a, m)}{2}}^{ma + \eta(b, a, m)} \left(u - \left(ma + \frac{\eta(b, a, m)}{2}\right)\right)^{\alpha-1} f(u) du \\
& = \frac{2}{\eta(b, a, m)} \left[\frac{1}{6} f\left(ma + \eta(b, a, m)\right) + \frac{1}{3} f\left(ma + \frac{\eta(b, a, m)}{2}\right) \right] \\
& \quad - \frac{2^\alpha \Gamma(\alpha+1)}{\eta^{\alpha+1}(b, a, m)} J_{(ma + \eta(b, a, m))^-}^\alpha f\left(ma + \frac{\eta(b, a, m)}{2}\right).
\end{aligned}$$

From (3.3) and (3.4), we get (3.2). This completes the proof. $\square$

Remark 3.1. If $\eta(b, a, m) = b - ma$ with $m = 1$ in Lemma 3.1, then the identity (3.2) reduces to the following identity

$$\begin{aligned}
(3.5) \quad & \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] \\
& - \frac{2^{\alpha-1} \Gamma(\alpha+1)}{(b-a)^\alpha} \left[J_{a^+}^\alpha f\left(\frac{a+b}{2}\right) + J_{b^-}^\alpha f\left(\frac{a+b}{2}\right) \right] \\
& = \frac{b-a}{2} \int_0^1 \left[\left(\frac{1}{3} - \frac{t^\alpha}{2}\right) f'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) \right. \\
& \quad \left. + \left(\frac{t^\alpha}{2} - \frac{1}{3}\right) f'\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right) \right] dt,
\end{aligned}$$

which is proved by Matłoka in [22]. Based on this identity, he established some interesting inequalities for h -convex functions via fractional integrals.

Remark 3.2. In Lemma 3.1, if the mapping $\eta(b, a, m)$ with $m = 1$ degenerates into $\eta(b, a)$ and we choose $\alpha = 1$, then (3.2) becomes

$$\begin{aligned}
(3.6) \quad & \frac{1}{6} \left[f(a) + 4f\left(a + \frac{\eta(b, a)}{2}\right) + f\left(a + \eta(b, a)\right) \right] \\
& - \frac{1}{\eta(b, a)} \int_a^{a + \eta(b, a)} f(x) dx \\
& = \frac{\eta(b, a)}{4} \int_0^1 \left[\left(\frac{2}{3} - t\right) f'\left(a + \frac{1-t}{2}\eta(b, a)\right) \right. \\
& \quad \left. + \left(t - \frac{2}{3}\right) f'\left(a + \frac{1+t}{2}\eta(b, a)\right) \right] dt,
\end{aligned}$$

which is proved by Wang et al. in [40]. Based on this identity, they established some interesting Simpson type inequalities for s -preinvex functions.

If $\eta(b, a) = b - a$ in (3.6), it follows that

$$\begin{aligned}
 & \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \\
 (3.7) \quad &= \frac{b-a}{2} \int_0^1 \left[\left(\frac{1}{3} - \frac{t}{2} \right) f' \left(\frac{1+t}{2}a + \frac{1-t}{2}b \right) \right. \\
 & \quad \left. + \left(\frac{t}{2} - \frac{1}{3} \right) f' \left(\frac{1+t}{2}b + \frac{1-t}{2}a \right) \right] dt,
 \end{aligned}$$

which is proved by Sarikaya et al. in [33]. Based on this identity, they established some interesting Simpson type inequalities for s -convex functions.

We can achieve the following consequences by the above fractional integral identity.

Theorem 3.1. *Let $K \subseteq \mathbb{R}$ be an open m -invex subset with respect to $\eta : K \times K \times (0, 1] \rightarrow \mathbb{R}$ for some fixed $m \in (0, 1]$ and let $a, b \in K$, $a < b$ with $\eta(b, a, m) > 0$. Suppose that $f : K \rightarrow \mathbb{R}$ is a first differentiable function, $h_1, h_2 : [0, 1] \rightarrow \mathbb{R}_0$, $|f'|$ is generalized (m, h_1, h_2) -preinvex on $[ma, ma + \eta(b, a, m)]$, then the following inequality for Riemann-Louville fractional integral with $\alpha > 0$ and $x \in [ma, ma + \eta(b, a, m)]$ holds:*

$$\begin{aligned}
 & |K_f(\alpha; \eta, m, a, b)| \\
 (3.8) \quad & \leq \frac{\eta(b, a, m)}{3} \left(m|f'(a)| \int_0^1 h_1(t) dt + |f'(b)| \int_0^1 h_2(t) dt \right).
 \end{aligned}$$

Proof. From Lemma 3.1, $|f'|$ is generalized (m, h_1, h_2) -preinvex, we get

$$\begin{aligned}
 & |K_f(\alpha; \eta, m, a, b)| \\
 & \leq \frac{\eta(b, a, m)}{2} \int_0^1 \left| \frac{1}{3} - \frac{t^\alpha}{2} \right| \left[\left| f' \left(ma + \frac{1+t}{2} \eta(b, a, m) \right) \right| \right. \\
 & \quad \left. + \left| f' \left(ma + \frac{1-t}{2} \eta(b, a, m) \right) \right| \right] dt \\
 & \leq \frac{\eta(b, a, m)}{2} \int_0^1 \left| \frac{1}{3} - \frac{t^\alpha}{2} \right| \left[m h_1 \left(\frac{1+t}{2} \right) |f'(a)| + h_2 \left(\frac{1+t}{2} \right) |f'(b)| \right. \\
 (3.9) \quad & \left. + m h_1 \left(\frac{1-t}{2} \right) |f'(a)| + h_2 \left(\frac{1-t}{2} \right) |f'(b)| \right] dt \\
 & = \frac{\eta(b, a, m)}{2} \left\{ \int_0^1 \left| \frac{1}{3} - \frac{t^\alpha}{2} \right| \left[m \left(h_1 \left(\frac{1-t}{2} \right) + h_1 \left(\frac{1+t}{2} \right) \right) |f'(a)| \right] dt \right. \\
 & \quad \left. + \int_0^1 \left| \frac{1}{3} - \frac{t^\alpha}{2} \right| \left[\left(h_2 \left(\frac{1-t}{2} \right) + h_2 \left(\frac{1+t}{2} \right) \right) |f'(b)| \right] dt \right\} \\
 & \leq \frac{\eta(b, a, m)}{3} \left[m|f'(a)| \int_0^1 h_1(t) dt + |f'(b)| \int_0^1 h_2(t) dt \right],
 \end{aligned}$$

where we used the fact that $|\frac{t^\alpha}{2} - \frac{1}{3}| \leq \frac{1}{3}$ for all $t \in [0, 1]$. The proof is completed. $\square$

Corollary 3.1. *Under the assumptions of Theorem 3.1 with $h_1(t) = h(1-t)$, $h_2(t) = h(t)$, if the mapping $\eta(b, a, m)$ with $m = 1$ degenerates into $\eta(b, a)$, then the inequality (3.8) becomes the following fractional inequality for an h -preinvex function*

$$(3.10) \quad \left| \frac{1}{6} \left[f(a) + 4f\left(a + \frac{\eta(b, a)}{2}\right) + f\left(a + \eta(b, a)\right) \right] - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{\eta^\alpha(b, a)} \left[J_{a^+}^\alpha f\left(a + \frac{\eta(b, a)}{2}\right) + J_{(a+\eta(b, a))^-}^\alpha f\left(a + \frac{\eta(b, a)}{2}\right) \right] \right| \leq \frac{\eta(b, a)}{3} \left(|f'(a)| + |f'(b)| \right) \int_0^1 h(t) dt,$$

specially for $\eta(b, a) = b - a$, we achieve a inequality for h -convex function

$$(3.11) \quad \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha} \left[J_{a^+}^\alpha f\left(\frac{a+b}{2}\right) + J_{b^-}^\alpha f\left(\frac{a+b}{2}\right) \right] \right| \leq \frac{b-a}{3} \left(|f'(a)| + |f'(b)| \right) \int_0^1 h(t) dt,$$

which is the same as the inequality established in [22, Theorem 6].

Corollary 3.2. *In Theorem 3.1, when $\eta(b, a, m)$ with $m = 1$ reduces to $\eta(b, a)$, if $s \in (0, 1]$, $h_1(t) = (1-t)^s$, $h_2(t) = t^s$, then the inequality (3.8) becomes the following fractional inequality for s -preinvex function*

$$(3.12) \quad \left| \frac{1}{6} \left[f(a) + 4f\left(a + \frac{\eta(b, a)}{2}\right) + f\left(a + \eta(b, a)\right) \right] - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{\eta^\alpha(b, a)} \left[J_{a^+}^\alpha f\left(a + \frac{\eta(b, a)}{2}\right) + J_{(a+\eta(b, a))^-}^\alpha f\left(a + \frac{\eta(b, a)}{2}\right) \right] \right| \leq \frac{\eta(b, a)}{3(s+1)} \left(|f'(a)| + |f'(b)| \right),$$

specially for $\eta(b, a) = b - a$ and $\alpha = 1$, then from the proof of Theorem 3.1 it follows that the following inequality for s -convex function holds

$$(3.13) \quad \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq (b-a) \frac{(s-4)6^{s+1} + 2 \times 5^{s+2} - 2 \times 3^{s+2} + 2}{6^{s+2}(s+1)(s+2)} \left(|f'(a)| + |f'(b)| \right),$$

which is one of the inequalities given in [33, Theorem 7].

Corollary 3.3. *In Theorem 3.1, if $h_1(t) = (1-t)^{-s}$, $h_2(t) = t^{-s}$ and $s \in [0, 1)$, we get a fractional inequality for generalized (m, s) -Godunova-Levin-preinvex function*

$$(3.14) \quad |K_f(\alpha; \eta, m, a, b)| \leq \frac{\eta(b, a, m)}{3(1-s)} [m|f'(a)| + |f'(b)|],$$

specially for $\alpha = 1$ and $\eta(b, a, m) = b - ma$ with $m = 1$, then from the proof of Theorem 3.1 it follows that the following inequality for s -Godunova-Levin functions holds

$$(3.15) \quad \left| \frac{1}{3} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ \leq (b-a) \frac{(-s-4)6^{1-s} + 2 \times 5^{2-s} - 2 \times 3^{2-s} + 2}{6^{2-s}(1-s)(2-s)} [|f'(a)| + |f'(b)|].$$

Corollary 3.4. *In Theorem 3.1, when $\eta(b, a, m)$ with $m = 1$ degenerates into $\eta(b, a)$, if $h_1(t) = 1 - t^\alpha$, $h_2(t) = t^\alpha$, for $\alpha \in (0, 1]$, then the inequality (3.8) becomes the following fractional inequality for α -preinvex function*

$$(3.16) \quad \left| \frac{1}{6} \left[f(a) + 4f\left(a + \frac{\eta(b, a)}{2}\right) + f\left(a + \eta(b, a)\right) \right] \right. \\ \left. - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{\eta^\alpha(b, a)} \left[J_{a^+}^\alpha f\left(a + \frac{\eta(b, a)}{2}\right) J_{(a+\eta(b, a))}^\alpha f\left(a + \frac{\eta(b, a)}{2}\right) \right] \right| \\ \leq \frac{\eta(b, a)}{3(\alpha+1)} (\alpha|f'(a)| + |f'(b)|),$$

specially for $\eta(b, a) = b - a$ and $\alpha = 1$, then from the proof of Theorem 3.1 it follows that the following inequality holds

$$(3.17) \quad \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ \leq \frac{5(b-a)}{72} (|f'(a)| + |f'(b)|).$$

This is one of the inequalities given in [32, Theorem 5].

Corollary 3.5. *In Theorem 3.1, if $\alpha = 1$, $h_1(t) = h_2(t) = t(1-t)$, and $\eta(b, a, m) = b - ma$ with $m = 1$, then from the proof of Theorem 3.1 it follows that the following inequality for tgs-convex function holds*

$$(3.18) \quad \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ \leq \frac{211(b-a)}{7776} (|f'(a)| + |f'(b)|).$$

Theorem 3.2. Let $K \subseteq \mathbb{R}$ be an open m -invex subset with respect to $\eta : K \times K \times (0, 1] \rightarrow \mathbb{R}$ for some fixed $m \in (0, 1]$ and let $a, b \in K$, $a < b$ with $\eta(b, a, m) > 0$. Assume that $f : K \rightarrow \mathbb{R}$ is a first differentiable function, $h_1, h_2 : [0, 1] \rightarrow \mathbb{R}_0$, $|f'|^q$ is (m, h_1, h_2) -preinvex on $[ma, ma + \eta(b, a, m)]$, $\frac{1}{p} + \frac{1}{q} = 1$ and $q > 1$, then the following inequality for Riemann-Louville fractional integral with $\alpha > 0$ and $x \in [ma, ma + \eta(b, a, m)]$ holds

$$(3.19) \quad \begin{aligned} & |K_f(\alpha; \eta, m, a, b)| \\ & \leq \frac{\eta(b, a, m)}{6} \left\{ \left[(m|f'(a)|^q \int_0^1 h_1\left(\frac{1+t}{2}\right) dt + |f'(b)|^q \int_0^1 h_2\left(\frac{1+t}{2}\right) dt) \right]^{\frac{1}{q}} \right. \\ & \quad \left. + \left[\left(m|f'(a)|^q \int_0^1 h_1\left(\frac{1-t}{2}\right) dt + |f'(b)|^q \int_0^1 h_2\left(\frac{1-t}{2}\right) dt \right) \right]^{\frac{1}{q}} \right\}. \end{aligned}$$

Proof. From Lemma 3.1 and the Hölder inequality, we have

$$(3.20) \quad \begin{aligned} & |K_f(\alpha; \eta, m, a, b)| \\ & \leq \frac{\eta(b, a, m)}{2} \left\{ \left(\int_0^1 \left| \frac{t^\alpha}{2} - \frac{1}{3} \right|^p dt \right)^{\frac{1}{p}} \right. \\ & \quad \cdot \left(\int_0^1 \left| f' \left(ma + \frac{1+t}{2} \eta(b, a, m) \right) \right|^q dt \right)^{\frac{1}{q}} \\ & \quad \left. + \left(\int_0^1 \left| \frac{1}{3} - \frac{t^\alpha}{2} \right|^p dt \right)^{\frac{1}{p}} \left(\int_0^1 \left| f' \left(ma + \frac{1-t}{2} \eta(b, a, m) \right) \right|^q dt \right)^{\frac{1}{q}} \right\}. \end{aligned}$$

Because $|f'|^q$ is generalized (m, h_1, h_2) -preinvex, we get

$$(3.21) \quad \begin{aligned} & \int_0^1 \left| f' \left(ma + \frac{1+t}{2} \eta(b, a, m) \right) \right|^q dt \\ & \leq m |f'(a)|^q \int_0^1 h_1\left(\frac{1+t}{2}\right) dt + |f'(b)|^q \int_0^1 h_2\left(\frac{1+t}{2}\right) dt \end{aligned}$$

and

$$(3.22) \quad \begin{aligned} & \int_0^1 \left| f' \left(ma + \frac{1-t}{2} \eta(b, a, m) \right) \right|^q dt \\ & \leq m |f'(a)|^q \int_0^1 h_1\left(\frac{1-t}{2}\right) dt + |f'(b)|^q \int_0^1 h_2\left(\frac{1-t}{2}\right) dt. \end{aligned}$$

Using the fact that $\left| \frac{t^\alpha}{2} - \frac{1}{3} \right| \leq \frac{1}{3}$ for all $t \in [0, 1]$ and using the last two inequalities in (3.20) we obtain (3.19). This completes the proof of the theorem. $\square$

Corollary 3.6. Under the circumstance of Theorem 3.2 with $h_1(t) = h(1-t)$, $h_2(t) = h(t)$, if the mapping $\eta(b, a, m)$ with $m = 1$ degenerates into $\eta(b, a)$, then

the inequality (3.19) becomes the following inequality for an h -preinvex function

$$\begin{aligned}
 & \left| \frac{1}{6} \left[f(a) + 4f\left(a + \frac{\eta(b,a)}{2}\right) + f\left(a + \eta(b,a)\right) \right] \right. \\
 & \quad \left. - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha} \left[J_{a^+}^\alpha f\left(\frac{a+b}{2}\right) + J_{b^-}^\alpha f\left(\frac{a+b}{2}\right) \right] \right| \\
 (3.23) \quad & \leq \frac{\eta(b,a)}{6} \left\{ \left[|f'(a)|^q \int_0^1 h\left(\frac{1-t}{2}\right) dt + |f'(b)|^q \int_0^1 h\left(\frac{1+t}{2}\right) dt \right]^{\frac{1}{q}} \right. \\
 & \quad \left. + \left[|f'(a)|^q \int_0^1 h\left(\frac{1+t}{2}\right) dt + |f'(b)|^q \int_0^1 h\left(\frac{1-t}{2}\right) dt \right]^{\frac{1}{q}} \right\},
 \end{aligned}$$

specially for $\alpha = 1, \eta(b, a) = b - a$, then from the proof of Theorem 3.2 it follows that the following inequality for h -convex function holds

$$\begin{aligned}
 & \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\
 & \leq \frac{b-a}{12} \left(\frac{1+2^{p+1}}{3^{p+1}} \right)^{\frac{1}{p}} \left\{ \left[|f'(a)|^q \int_0^1 h\left(\frac{1-t}{2}\right) dt \right. \right. \\
 (3.24) \quad & \quad \left. \left. + |f'(b)|^q \int_0^1 h\left(\frac{1+t}{2}\right) dt \right]^{\frac{1}{q}} \right. \\
 & \quad \left. + \left[|f'(a)|^q \int_0^1 h\left(\frac{1+t}{2}\right) dt + |f'(b)|^q \int_0^1 h\left(\frac{1-t}{2}\right) dt \right]^{\frac{1}{q}} \right\}.
 \end{aligned}$$

Corollary 3.7. In Theorem 3.2, when $\eta(b, a, m)$ with $m = 1$ reduces to $\eta(b, a)$, if $s \in (0, 1], h_1(t) = (1-t)^s$ and $h_2(t) = t^s$, then the inequality (3.8) becomes the following inequality for s -preinvex function

$$\begin{aligned}
 & \left| \frac{1}{6} \left[f(a) + 4f\left(a + \frac{\eta(b,a)}{2}\right) + f\left(a + \eta(b,a)\right) \right] \right. \\
 & \quad \left. - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{\eta^\alpha(b,a)} \left[J_{a^+}^\alpha f\left(a + \frac{\eta(b,a)}{2}\right) + J_{(a+\eta(b,a))^-}^\alpha f\left(a + \frac{\eta(b,a)}{2}\right) \right] \right| \\
 (3.25) \quad & \leq \frac{\eta(b,a)}{6} \left(\frac{2}{s+1} \right)^{\frac{1}{q}} \left\{ \left[|f'(a)|^q \frac{1}{2^{s+1}} + |f'(b)|^q \left(1 - \frac{1}{2^{s+1}}\right) \right]^{\frac{1}{q}} \right. \\
 & \quad \left. + \left[|f'(a)|^q \left(1 - \frac{1}{2^{s+1}}\right) + |f'(b)|^q \frac{1}{2^{s+1}} \right]^{\frac{1}{q}} \right\},
 \end{aligned}$$

specially for $\alpha = 1$, then from the proof of Theorem 3.2 it follows that the following inequality holds

$$(3.26) \quad \left| \frac{1}{6} \left[f(a) + 4f\left(a + \frac{\eta(b,a)}{2}\right) + f\left(a + \eta(b,a)\right) \right] - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x) dx \right|$$

$$\leq \frac{\eta(b, a)}{36} \left(\frac{q-1}{2q-1} \right)^{1-\frac{1}{q}} \left[\frac{3}{2^s(s+1)} \right]^{\frac{1}{q}} \left(2^{\frac{2q-1}{q-1}} + 1 \right)^{1-\frac{1}{q}} \\ \times \left\{ \left[|f'(a)|^q + (2^{s+1} - 1) |f'(b)|^q \right]^{\frac{1}{q}} + \left[(2^{s+1} - 1) |f'(a)|^q + |f'(b)|^q \right]^{\frac{1}{q}} \right\},$$

which is the inequalities given in [40, Theorem 3.2].

Specially for $\eta(b, a) = b - a$, then from the proof of Theorem 3.2 it follows that the following inequality for s -convex function holds

$$(3.27) \quad \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ \leq \frac{(b-a)}{12} \left(\frac{1+2^{p+1}}{3(p+1)} \right)^{\frac{1}{p}} \left\{ \left(\frac{(2^{s+1}-1)|f'(a)|^q + |f'(b)|^q}{2^s(s+1)} \right)^{\frac{1}{q}} \right. \\ \left. + \left(\frac{|f'(a)|^q + (2^{s+1}-1)|f'(b)|^q}{2^s(s+1)} \right)^{\frac{1}{q}} \right\}.$$

This is one of the inequalities given in [33, Theorem 9].

Corollary 3.8. In Theorem 3.2, if $h_1(t) = (1-t)^{-s}$, $h_2(t) = t^{-s}$ and $s \in [0, 1)$, we get a fractional inequality for generalized (m, s) -Godunova-Levin-preinvex function

$$(3.28) \quad |K_f(\alpha; \eta, m, a, b)| \\ \leq \frac{\eta(b, a, m)}{6} \left(\frac{2}{1-s} \right)^{\frac{1}{q}} \left\{ \left[m |f'(a)|^q \left(1 - \frac{1}{2^{1-s}} \right) + |f'(b)|^q \frac{1}{2^{1-s}} \right]^{\frac{1}{q}} \right. \\ \left. + \left[m |f'(a)|^q \frac{1}{2^{1-s}} + |f'(b)|^q \left(1 - \frac{1}{2^{1-s}} \right) \right]^{\frac{1}{q}} \right\},$$

specially for $\alpha = 1$, $\eta(b, a, m)$ with $m = 1$ reduces to $\eta(b, a)$, then from the proof of Theorem 3.2 it follows that the following inequality holds

$$(3.29) \quad \left| \frac{1}{6} \left[f(a) + 4f\left(a + \frac{\eta(b, a)}{2}\right) + f\left(a + \eta(b, a)\right) \right] - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(x) dx \right| \\ \leq \frac{\eta(b, a)}{36} \left(\frac{q-1}{2q-1} \right)^{1-\frac{1}{q}} \left[\frac{3}{2^{-s}(1-s)} \right]^{\frac{1}{q}} \left(2^{\frac{2q-1}{q-1}} + 1 \right)^{1-\frac{1}{q}} \\ \times \left\{ \left[(2^{1-s} - 1) |f'(a)|^q + |f'(b)|^q \right]^{\frac{1}{q}} + \left[|f'(a)|^q + (2^{1-s} - 1) |f'(b)|^q \right]^{\frac{1}{q}} \right\},$$

furthermore if $\eta(b, a) = b - a$, then from the proof of Theorem 3.2 it follows that the following inequality for s -Godunova-Levin function holds

$$(3.30) \quad \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right|$$

$$\begin{aligned} &\leq \frac{b-a}{12} \left(\frac{1+2^{p+1}}{3(p+1)} \right)^{\frac{1}{p}} \left\{ \left(\frac{(2^{1-s}-1)|f'(a)|^q + |f'(b)|^q}{2^{-s}(1-s)} \right)^{\frac{1}{q}} \right. \\ &\quad \left. + \left(\frac{(2^{1-s}-1)|f'(b)|^q + |f'(a)|^q}{2^{-s}(1-s)} \right)^{\frac{1}{q}} \right\}. \end{aligned}$$

Corollary 3.9. *In Theorem 3.2, when $\eta(b, a, m)$ with $m = 1$ reduces to $\eta(b, a)$, if $h_1(t) = 1 - t^\alpha$, $h_2(t) = t^\alpha$, for $\alpha \in (0, 1]$, then the inequality (3.8) becomes the following fractional inequality for α -preinvex function*

$$\begin{aligned} &\left| \frac{1}{6} \left[f(a) + 4f\left(a + \frac{\eta(b, a)}{2}\right) + f\left(a + \eta(b, a)\right) \right] \right. \\ &\quad \left. - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{\eta^\alpha(b, a)} \left[J_{a^+}^\alpha f\left(a + \frac{\eta(b, a)}{2}\right) + J_{(a+\eta(b, a))^-}^\alpha f\left(a + \frac{\eta(b, a)}{2}\right) \right] \right| \\ (3.31) \quad &\leq \frac{\eta(b, a)}{6} \left(\frac{1}{2^\alpha(\alpha+1)} \right)^{\frac{1}{q}} \left\{ \left[\left(2^\alpha(\alpha-1)+1 \right) |f'(a)|^q + (2^{\alpha+1}-1) |f'(b)|^q \right]^{\frac{1}{q}} \right. \\ &\quad \left. + \left[\left(2^\alpha(\alpha+1)-1 \right) |f'(a)|^q + |f'(b)|^q \right]^{\frac{1}{q}} \right\}. \end{aligned}$$

Furthermore if $\eta(b, a) = b - a$, with $\alpha = 1$, then from the proof of Theorem 3.2 it follows that the following inequality holds

$$\begin{aligned} &\left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ (3.32) \quad &\leq \frac{b-a}{12} \left(\frac{1+2^{p+1}}{3(p+1)} \right)^{\frac{1}{p}} \left[\left(\frac{1}{4} |f'(a)|^q + \frac{3}{4} |f'(b)|^q \right)^{\frac{1}{q}} \right. \\ &\quad \left. + \left(\frac{3}{4} |f'(a)|^q + \frac{1}{4} |f'(b)|^q \right)^{\frac{1}{q}} \right], \end{aligned}$$

which is given by Sarikaya et al. in [32, Theorem 6].

Corollary 3.10. *In Theorem 3.2, if $h_1(t) = 1 - t$, $h_2(t) = t$, $\eta(b, a, m) = b - ma$ with $m = 1$, then the following fractional inequality holds*

$$\begin{aligned} &\left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] \right. \\ (3.33) \quad &\quad \left. - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha} \left[J_{a^+}^\alpha f\left(\frac{a+b}{2}\right) + J_{b^-}^\alpha f\left(\frac{a+b}{2}\right) \right] \right| \\ &\leq \frac{b-a}{6} \left[\left(\frac{1}{4} |f'(a)|^q + \frac{3}{4} |f'(b)|^q \right)^{\frac{1}{q}} + \left(\frac{3}{4} |f'(a)|^q + \frac{1}{4} |f'(b)|^q \right)^{\frac{1}{q}} \right]. \end{aligned}$$

Corollary 3.11. *In Theorem 3.2, if $h_1(t) = h_2(t) = t(1-t)$, and $\eta(b, a, m) = b - ma$ with $m = 1$, we obtain a fractional inequality for tgs-convex function*

$$(3.34) \quad \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha} \left[J_{a^+}^\alpha f\left(\frac{a+b}{2}\right) + J_{b^-}^\alpha f\left(\frac{a+b}{2}\right) \right] \right| \leq \frac{b-a}{3\sqrt[3]{6}} \left(|f'(a)|^q + |f'(b)|^q \right)^{\frac{1}{q}},$$

furthermore if $\alpha = 1$, then from the proof of Theorem 3.2 it follows that the following inequality holds

$$(3.35) \quad \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{6} \left(\frac{1}{6} \right)^{\frac{1}{q}} \left(\frac{1+2^{p+1}}{3(p+1)} \right)^{\frac{1}{p}} \left(|f'(a)|^q + |f'(b)|^q \right)^{\frac{1}{q}}.$$

Theorem 3.3. *Let $K \subseteq \mathbb{R}$ be an open m -invex subset with respect to $\eta : K \times K \times (0, 1] \rightarrow \mathbb{R}$ for some fixed $m \in (0, 1]$ and let $a, b \in K$, $a < b$ with $\eta(b, a, m) > 0$. Suppose that $f : K \rightarrow \mathbb{R}$ is a first differentiable function, $h_1, h_2 : [0, 1] \rightarrow \mathbb{R}_0$, $|f'|^q$ is (m, h_1, h_2) -preinvex on $[ma, ma + \eta(b, a, m)]$ and $q \geq 1$, then the following inequality for Riemann-Louville fractional integral with $\alpha > 0$ and $x \in [ma, ma + \eta(b, a, m)]$ holds*

$$(3.36) \quad \begin{aligned} & |K_f(\alpha; \eta, m, a, b)| \\ & \leq \frac{\eta(b, a, m)}{6} \left[\frac{1}{\alpha+1} \left(\left(\frac{2}{3} \right)^{\frac{1}{\alpha}} 2\alpha + \frac{1}{2} - \alpha \right) \right]^{1-\frac{1}{q}} \\ & \times \left\{ \left[\left(m|f'(a)|^q \int_0^1 h_1\left(\frac{1+t}{2}\right) dt + |f'(b)|^q \int_0^1 h_2\left(\frac{1+t}{2}\right) dt \right) \right]^{\frac{1}{q}} \right. \\ & \left. + \left[\left(m|f'(a)|^q \int_0^1 h_1\left(\frac{1-t}{2}\right) dt + |f'(b)|^q \int_0^1 h_2\left(\frac{1-t}{2}\right) dt \right) \right]^{\frac{1}{q}} \right\}. \end{aligned}$$

Proof. From Lemma 3.1 and power mean inequality, we have

$$(3.37) \quad \begin{aligned} & |K_f(\alpha; \eta, m, a, b)| \\ & \leq \frac{\eta(b, a, m)}{2} \left(\int_0^1 \left| \frac{t^\alpha}{2} - \frac{1}{3} \right| dt \right)^{1-\frac{1}{q}} \\ & \times \left[\left(\int_0^1 \left| \frac{t^\alpha}{2} - \frac{1}{3} \right| \left| f' \left(ma + \frac{1+t}{2} \eta(b, a, m) \right) \right|^q dt \right)^{\frac{1}{q}} \right. \\ & \left. + \left(\int_0^1 \left| \frac{1}{3} - \frac{t^\alpha}{2} \right| \left| f' \left(ma + \frac{1-t}{2} \eta(b, a, m) \right) \right|^q dt \right)^{\frac{1}{q}} \right]. \end{aligned}$$

By the generalized (m, h_1, h_2) -preinvex of $|f'|^q$ and using the fact that $\left|\frac{t^\alpha}{2} - \frac{1}{3}\right| = \left|\frac{1}{3} - \frac{t^\alpha}{2}\right| \leq \frac{1}{3}$ for all $t \in [0, 1]$, we get

$$(3.38) \quad \begin{aligned} & \int_0^1 \left| \frac{t^\alpha}{2} - \frac{1}{3} \right| \left| f' \left(ma + \frac{1+t}{2} \eta(b, a, m) \right) \right|^q dt \\ & \leq \frac{1}{3} \left(m |f'(a)|^q \int_0^1 h_1 \left(\frac{1+t}{2} \right) dt + |f'(b)|^q \int_0^1 h_2 \left(\frac{1+t}{2} \right) dt \right) \end{aligned}$$

and

$$(3.39) \quad \begin{aligned} & \int_0^1 \left| \frac{1}{3} - \frac{t^\alpha}{2} \right| \left| f' \left(ma + \frac{1-t}{2} \eta(b, a, m) \right) \right|^q dt \\ & \leq \frac{1}{3} \left(m |f'(a)|^q \int_0^1 h_1 \left(\frac{1-t}{2} \right) dt + |f'(b)|^q \int_0^1 h_2 \left(\frac{1-t}{2} \right) dt \right). \end{aligned}$$

By simple computation,

$$(3.40) \quad \left(\int_0^1 \left| \frac{t^\alpha}{2} - \frac{1}{3} \right| dt \right)^{1-\frac{1}{q}} = \left[\frac{1}{3(\alpha+1)} \left(\left(\frac{2}{3} \right)^{\frac{1}{\alpha}} 2\alpha + \frac{1}{2} - \alpha \right) \right]^{1-\frac{1}{q}}.$$

Using the last three inequalities in (3.37) we obtain (3.36). This completes the proof of the theorem. $\square$

Corollary 3.12. *Under the assumptions of Theorem 3.3 with $h_1(t) = h(1-t)$, $h_2(t) = h(t)$, if the mapping $\eta(b, a, m)$ with $m = 1$ degenerates into $\eta(b, a)$, then the inequality (3.36) becomes the following inequality for an h -preinvex function*

$$(3.41) \quad \begin{aligned} & \left| \frac{1}{6} \left[f(a) + 4f \left(a + \frac{\eta(b, a)}{2} \right) + f \left(a + \eta(b, a) \right) \right] \right. \\ & \quad \left. - \frac{2^{\alpha-1} \Gamma(\alpha+1)}{\eta^\alpha(b, a)} \left[J_{a^+}^\alpha f \left(a + \frac{\eta(b, a)}{2} \right) + J_{(a+\eta(b, a))^-}^\alpha f \left(a + \frac{\eta(b, a)}{2} \right) \right] \right| \\ & \leq \frac{\eta(b, a)}{6} \left[\frac{1}{\alpha+1} \left(\left(\frac{2}{3} \right)^{\frac{1}{\alpha}} 2\alpha + \frac{1}{2} - \alpha \right) \right]^{1-\frac{1}{q}} \\ & \quad \times \left\{ \left[|f'(a)|^q \int_0^1 h \left(\frac{1-t}{2} \right) dt + |f'(b)|^q \int_0^1 h \left(\frac{1+t}{2} \right) dt \right]^{\frac{1}{q}} \right. \\ & \quad \left. + \left[|f'(a)|^q \int_0^1 h \left(\frac{1+t}{2} \right) dt + |f'(b)|^q \int_0^1 h \left(\frac{1-t}{2} \right) dt \right]^{\frac{1}{q}} \right\}, \end{aligned}$$

specially for $\eta(b, a) = b - a$ and $\alpha = 1$, we achieve a inequality for h -convex function

$$(3.42) \quad \left| \frac{1}{6} \left[f(a) + 4f \left(\frac{a+b}{2} \right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right|$$

$$\leq \frac{b-a}{6} \left(\frac{5}{12} \right)^{1-\frac{1}{q}} \left\{ \left[|f'(a)|^q \int_0^1 h\left(\frac{1-t}{2}\right) dt + |f'(b)|^q \int_0^1 h\left(\frac{1+t}{2}\right) dt \right]^{\frac{1}{q}} \right. \\ \left. + \left[|f'(a)|^q \int_0^1 h\left(\frac{1+t}{2}\right) dt + |f'(b)|^q \int_0^1 h\left(\frac{1-t}{2}\right) dt \right]^{\frac{1}{q}} \right\}.$$

Corollary 3.13. *In Theorem 3.3, when $\eta(b, a, m)$ with $m = 1$ reduces to $\eta(b, a)$, if $s \in (0, 1]$, $h_1(t) = (1-t)^s$, $h_2(t) = t^s$, then the inequality (3.36) becomes the following fractional inequality for s -preinvex function*

$$\begin{aligned} & \left| \frac{1}{6} \left[f(a) + 4f\left(a + \frac{\eta(b, a)}{2}\right) + f\left(a + \eta(b, a)\right) \right] \right. \\ & \quad \left. - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{\eta^\alpha(b, a)} \left[J_{a^+}^\alpha f\left(a + \frac{\eta(b, a)}{2}\right) + J_{(a+\eta(b, a))^-}^\alpha f\left(a + \frac{\eta(b, a)}{2}\right) \right] \right| \\ (3.43) \quad & \leq \frac{\eta(b, a)}{6} \left[\frac{1}{\alpha+1} \left(\left(\frac{2}{3} \right)^{\frac{1}{\alpha}} 2\alpha + \frac{1}{2} - \alpha \right) \right]^{1-\frac{1}{q}} \left(\frac{2}{s+1} \right)^{\frac{1}{q}} \\ & \quad \times \left\{ \left[|f'(a)|^q \left(1 - \frac{1}{2^{s+1}} \right) + |f'(b)|^q \frac{1}{2^{s+1}} \right]^{\frac{1}{q}} \right. \\ & \quad \left. + \left[|f'(a)|^q \frac{1}{2^{s+1}} + |f'(b)|^q \left(1 - \frac{1}{2^{s+1}} \right) \right]^{\frac{1}{q}} \right\}, \end{aligned}$$

specially for $\eta(b, a) = b - a$, $\alpha = 1$, then from the proof of Theorem 3.3 it follows that the following inequality for s -convex function holds

$$\begin{aligned} & \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)}{2} \left(\frac{5}{36} \right)^{1-\frac{1}{q}} \left\{ \left(\frac{2 \times 5^{s+2} + (s-4) \times 6^{s+1} - (2s+7)3^{s+1}}{3 \times 6^{s+1}(s+1)(s+2)} |f'(a)|^q \right. \right. \\ (3.44) \quad & \left. + \frac{(2s+1)3^{s+1} + 2}{3 \times 6^{s+1}(s+1)(s+2)} |f'(b)|^q \right)^{\frac{1}{q}} + \left(\frac{(2s+1)3^{s+1} + 2}{3 \times 6^{s+1}(s+1)(s+2)} |f'(a)|^q \right. \\ & \left. + \frac{2 \times 5^{s+2} + (s-4) \times 6^{s+1} - (2s+7)3^{s+1}}{3 \times 6^{s+1}(s+1)(s+2)} |f'(b)|^q \right)^{\frac{1}{q}} \Bigg\}, \end{aligned}$$

which is one of the inequalities proved in [33, Theorem 10].

Corollary 3.14. *In Theorem 3.3, if $h_1(t) = (1-t)^{-s}$, $h_2(t) = t^{-s}$ and $s \in [0, 1)$, we get a fractional inequality for generalized (m, s) -Godunova-Levin-preinvex*

function

$$\begin{aligned}
 & |K_f(\alpha; \eta, m, a, b)| \\
 & \leq \frac{\eta(b, a, m)}{6} \left[\frac{1}{\alpha + 1} \left(\left(\frac{2}{3} \right)^{\frac{1}{\alpha}} 2\alpha + \frac{1}{2} - \alpha \right) \right]^{1 - \frac{1}{q}} \left(\frac{2}{1-s} \right)^{\frac{1}{q}} \\
 (3.45) \quad & \times \left\{ \left[m |f'(a)|^q \left(1 - \frac{1}{2^{1-s}} \right) + |f'(b)|^q \frac{1}{2^{1-s}} \right]^{\frac{1}{q}} \right. \\
 & \left. + \left[m |f'(a)|^q \frac{1}{2^{1-s}} + |f'(b)|^q \left(1 - \frac{1}{2^{1-s}} \right) \right]^{\frac{1}{q}} \right\},
 \end{aligned}$$

specially for $\alpha = 1$ and $\eta(b, a, m) = b - ma$ with $m = 1$, then from the proof of Theorem 3.3 it follows that the following inequality for s -Godunova-Levin functions holds

$$\begin{aligned}
 & \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\
 & \leq \frac{(b-a)}{2} \left(\frac{5}{36} \right)^{1 - \frac{1}{q}} \left\{ \left(\frac{2 \times 5^{2-s} + (-s-4) \times 6^{1-s} - (7-2s)3^{1-s}}{3 \times 6^{1-s}(1-s)(2-s)} |f'(b)|^q \right. \right. \\
 (3.46) \quad & \left. + \frac{(1-2s)3^{1-s} + 2}{3 \times 6^{1-s}(1-s)(2-s)} |f'(a)|^q \right)^{\frac{1}{q}} + \left(\frac{(1-2s)3^{1-s} + 2}{3 \times 6^{1-s}(1-s)(2-s)} |f'(b)|^q \right. \\
 & \left. + \frac{(2 \times 5^{2-s} + (-s-4) \times 6^{1-s} - (7-2s)3^{1-s}}{3 \times 6^{1-s}(1-s)(2-s)} |f'(a)|^q \right)^{\frac{1}{q}} \right\}.
 \end{aligned}$$

Corollary 3.15. In Theorem 3.3, when $\eta(b, a, m)$ with $m = 1$ reduces to $\eta(b, a)$, if $h_1(t) = 1 - t^\alpha$, $h_2(t) = t^\alpha$, for $\alpha \in (0, 1]$, then the inequality (3.36) becomes the following fractional inequality for α -preinvex function

$$\begin{aligned}
 & \left| \frac{1}{6} \left[f(a) + 4f\left(a + \frac{\eta(b, a)}{2}\right) + f\left(a + \eta(b, a)\right) \right] \right. \\
 & \left. - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{\eta^\alpha(b, a)} \left[J_{a^+}^\alpha f\left(a + \frac{\eta(b, a)}{2}\right) J_{(a+\eta(b, a))^-}^\alpha f\left(a + \frac{\eta(b, a)}{2}\right) \right] \right| \\
 (3.47) \quad & \leq \frac{\eta(b, a)}{6(\alpha+1)} \left[\left(\left(\frac{2}{3} \right)^{\frac{1}{\alpha}} 2\alpha + \frac{1}{2} - \alpha \right) \right]^{1 - \frac{1}{q}} \left(\frac{1}{2^\alpha} \right)^{\frac{1}{q}} \\
 & \times \left\{ \left[\left(2^\alpha(\alpha-1) + 1 \right) |f'(a)|^q + (2^{\alpha+1} - 1) |f'(b)|^q \right]^{\frac{1}{q}} \right. \\
 & \left. + \left[\left(2^\alpha(\alpha+1) - 1 \right) |f'(a)|^q + |f'(b)|^q \right]^{\frac{1}{q}} \right\},
 \end{aligned}$$

specially for $\eta(b, a) = b - a$, $\alpha = 1$, then from the proof of Theorem 3.3 it follows that the following inequality holds

$$(3.48) \quad \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ \leq \frac{5(b-a)}{72} \left\{ \left(\frac{29|f'(a)|^q + 61|f'(b)|^q}{90} \right)^{\frac{1}{q}} + \left(\frac{61|f'(a)|^q + 29|f'(b)|^q}{90} \right)^{\frac{1}{q}} \right\},$$

which is confirmed by Sarikaya et al. in [32, Theorem 7].

Corollary 3.16. If $\alpha = 1$, $h_1(t) = h_2(t) = t(1-t)$, and $\eta(b, a, m) = b - ma$ with $m = 1$, then from the proof of Theorem 3.3 it follows that the following inequality for tgs-convex function holds

$$(3.49) \quad \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ \leq \frac{5(b-a)}{36} \left(\frac{211}{1080} \right)^{\frac{1}{q}} \left(|f'(a)|^q + |f'(b)|^q \right)^{\frac{1}{q}}.$$

Corollary 3.17. In Theorem 3.3, if $h_1(t) = 1-t$, $h_2(t) = t$, and $\eta(b, a, m) = b - ma$ with $m = 1$, then the following fractional inequality holds

$$(3.50) \quad \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha} \left[J_{a^+}^\alpha f\left(\frac{a+b}{2}\right) + J_{b^-}^\alpha f\left(\frac{a+b}{2}\right) \right] \right| \\ \leq \frac{b-a}{6} \left[\frac{1}{\alpha+1} \left(\left(\frac{2}{3} \right)^{\frac{1}{\alpha}} 2\alpha + \frac{1}{2} - \alpha \right) \right]^{1-\frac{1}{q}} \\ \times \left\{ \left[\left(\frac{1}{4} |f'(a)|^q + \frac{3}{4} |f'(b)|^q \right)^{\frac{1}{q}} + \left(\frac{3}{4} |f'(a)|^q + \frac{1}{4} |f'(b)|^q \right)^{\frac{1}{q}} \right] \right\}.$$

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NOVEL MULTIPLE CRITERIA DECISION MAKING METHODS BASED ON BIPOLAR NEUTROSOPHIC SETS AND BIPOLAR NEUTROSOPHIC GRAPHS

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Abstract. In this research article, we present certain notions of bipolar neutrosophic graphs. We study the dominating and independent sets of bipolar neutrosophic graphs. We describe novel multiple criteria decision making methods based on bipolar neutrosophic sets and bipolar neutrosophic graphs. We develop an algorithm for computing domination in bipolar neutrosophic graphs. We also show that there are some flaws in Broumi et al. [11]’s definition of complement of a bipolar neutrosophic graph.

Keywords: bipolar neutrosophic sets, bipolar neutrosophic graphs, domination number, independent number, decision making, algorithm.

1. Introduction

A fuzzy set [27] is an important mathematical structure to represent a collection of objects whose boundary is vague. Fuzzy models are becoming useful because of their aim in reducing the differences between the traditional numerical models used in engineering and sciences and the symbolic models used in expert systems. In 1994, Zhang [29] introduced the notion of bipolar fuzzy sets and relations. Bipolar fuzzy sets are extension of fuzzy sets whose membership degree ranges $[-1, 1]$. The membership degree $(0, 1]$ indicates that the object satisfies a certain property whereas the membership degree $[-1, 0)$ indicates that the element satisfies the implicit counter property. Positive information represent what is considered to be possible and negative information represent what is granted to be impossible. Actually, a variety of decision making problems are based on two-sided bipolar judgements on a positive side and a negative side. Nowadays bipolar fuzzy sets are playing a substantial role in chemistry, economics, computer science, engineering, medicine and decision making problems. Samarandache [22] introduced the idea of neutrosophic probability, sets and logic. Peng *et al.* [19], in 2014, described some operational properties and

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studied a new approach for multi-criteria decision making problems using neutrosophic sets. Ye [25, 26] discussed trapezoidal neutrosophic sets and simplified neutrosophic sets with applications in multi-criteria decision making problems. The other terminologies and applications of neutrosophic sets can be seen in [23, 25, 26]. In a neutrosophic set, the membership value is associated with truth, false and indeterminacy degrees but there is no restriction on their sum. Deli et al. [12] extended the ideas of bipolar fuzzy sets and neutrosophic sets to bipolar neutrosophic sets and studied its operations and applications in decision making problems.

Graph theory has numerous applications in science and engineering. However, in some cases, some aspects of graph theoretic concepts may be uncertain. In such cases, it is important to deal with uncertainty using the methods of fuzzy sets and logics. Based on Zadeh's fuzzy relations [28], Kaufmann [14] defined a fuzzy graph. The fuzzy relations between fuzzy sets were also considered by Rosenfeld [20] and he developed the structure of fuzzy graphs, obtaining analogs of several graph theoretical concepts. The complement of a fuzzy graph was defined by Mordeson [17]. The concept of strong arcs in fuzzy graphs was discussed in [10]. The theory of fuzzy graphs has extended widely by many researchers as it can be seen in [15, 17]. The idea of domination was first arose in chessboard problem in 1862. Somasundaram and Somasundaram [24] introduced domination and independent domination in fuzzy graphs. Gani and Chandrasekaran [18] studied the notion of fuzzy domination and independent domination using strong arcs. Akram [1, 2] introduced bipolar fuzzy graphs and discuss its various properties. Akram and Dudek [3] studied regular bipolar fuzzy graphs. Several new concepts on bipolar neutrosophic graphs and bipolar neutrosophic hypergraphs have been studied in [4-7]. In this research article, we present certain notions of bipolar neutrosophic graphs. We study the dominating and independent sets of bipolar neutrosophic graphs. We describe novel multiple criteria decision making methods based on bipolar neutrosophic sets and bipolar neutrosophic graphs. We develop an algorithm for computing domination in bipolar neutrosophic graphs. We also show that there are some flaws in Broumi et al. [11]'s definition.

2. Bipolar neutrosophic graphs

In this section, we discuss the concept of a bipolar neutrosophic graph and its various properties.

Definition 2.1 ([23]). A neutrosophic set C on a non-empty set Y is characterized by a truth membership function $t_C : Y \rightarrow [0, 1]$, indeterminacy membership function $I_C : Y \rightarrow [0, 1]$ and a falsity membership function $f_C : Y \rightarrow [0, 1]$. There is no restriction on the sum of $t_C(x)$, $I_C(x)$ and $f_C(x)$, for all $x \in Y$.

Definition 2.2 ([12]). A bipolar neutrosophic set on a empty set Y is an object of the form

$$C = \{(y, t_C^p(y), I_C^p(y), f_C^p(y), t_C^n(y), I_C^n(y), f_C^n(y)) : y \in Y\}$$

where, $t_C^p, I_C^p, f_C^p : Y \rightarrow [0, 1]$ and $t_C^n, I_C^n, f_C^n : Y \rightarrow [-1, 0]$. The positive values $t_C^p(y), I_C^p(y), f_C^p(y)$ denote respectively the truth, indeterminacy and false membership degrees of an element $y \in Y$ whereas $t_C^n(y), I_C^n(y), f_C^n(y)$ denote the implicit counter property of the truth, indeterminacy and false membership degrees of the element $y \in Y$ corresponding to the bipolar neutrosophic set C .

Definition 2.3. A bipolar neutrosophic relation on a non-empty set Y is a bipolar neutrosophic subset of $Y \times Y$ of the form $D = \{(yz, t_D^p(yz), I_D^p(yz), f_D^p(yz), t_D^n(yz), I_D^n(yz), f_D^n(yz)) : yz \in Y \times Y\}$ where, $t_D^p, I_D^p, f_D^p, t_D^n, I_D^n, f_D^n$ are defined by the mappings $t_D^p, I_D^p, f_D^p : Y \times Y \rightarrow [0, 1]$ and $t_D^n, I_D^n, f_D^n : Y \times Y \rightarrow [-1, 0]$.

Definition 2.4. A bipolar neutrosophic graph on a non-empty set Y is a pair $G = (C, D)$, where C is a bipolar neutrosophic set on Y and D is a bipolar neutrosophic relation in Y such that

$$\begin{aligned} t_D^p(yz) &\leq t_C^p(y) \wedge t_C^p(z), & I_D^p(yz) &\leq I_C^p(y) \wedge I_C^p(z), & f_D^p(yz) &\leq f_C^p(y) \vee f_C^p(z), \\ t_D^n(yz) &\geq t_C^n(y) \vee t_C^n(z), & I_D^n(yz) &\geq I_C^n(y) \vee I_C^n(z), & f_D^n(yz) &\geq f_C^n(y) \wedge f_C^n(z), \end{aligned}$$

for all $y, z \in Y$. Note that $D(yz) = (0, 0, 1, 0, 0, -1)$, for all $yz \in Y \times Y \setminus E$.

Example 2.1. Here we discuss an example of a bipolar neutrosophic graph such that $Y = \{x, y, z\}$. Let C be a bipolar neutrosophic set on Y given in Table. 1 and D be a bipolar neutrosophic relation in Y given in Table. 2.

Table 1: Bipolar neutrosophic set C

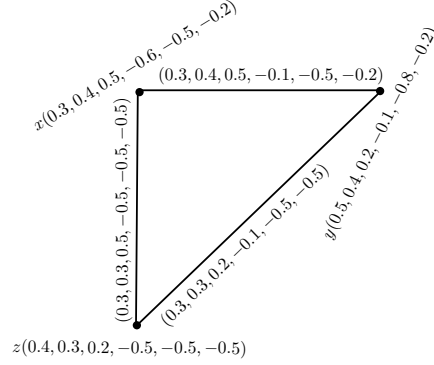
	x	y	z
t_C^p	0.3	0.5	0.4
I_C^p	0.4	0.4	0.3
f_C^p	0.5	0.2	0.2
t_C^n	-0.6	-0.1	-0.5
I_C^n	-0.5	-0.8	-0.5
f_C^n	-0.2	-0.2	-0.5

Table 2: Bipolar neutrosophic relation D

	xy	yz	xz
t_D^p	0.3	0.3	0.3
I_D^p	0.4	0.3	0.3
f_D^p	0.5	0.2	0.5
t_D^n	-0.1	-0.1	-0.5
I_D^n	-0.5	-0.5	-0.5
f_D^n	-0.2	-0.5	-0.5

Routine calculations show that $G = (C, D)$ is a bipolar neutrosophic graph. The bipolar neutrosophic graph G is shown in Fig. 1.

Definition 2.5. Let $G_1 = (C_1, D_1)$ and $G_2 = (C_2, D_2)$ be two bipolar neutrosophic graphs where, C_1 and C_2 are bipolar neutrosophic sets on Y_1 and Y_2 , D_1 and D_2 are bipolar neutrosophic relations in Y_1 and Y_2 , respectively . The

Figure 1: Bipolar neutrosophic graph G

union of G_1 and G_2 is a pair $G_1 \cup G_2 = (C_1 \cup C_2, D_1 \cup D_2)$ such that for all $y, z \in Y$,

1. If $y \in Y_1, y \notin Y_2$ then, $(C_1 \cup C_2)(y) = C_1(y)$.
2. If $y \in Y_2, y \notin Y_1$ then, $(C_1 \cup C_2)(y) = C_2(y)$.
3. If $y \in Y_1 \cap Y_2$ then,

$$(C_1 \cup C_2)(y) = (t_{C_1}^p(y) \vee t_{C_2}^p(y), \frac{I_{C_1}^p(y) + I_{C_2}^p(y)}{2}, f_{C_1}^p(y) \wedge f_{C_2}^p(y), t_{C_1}^n(y) \wedge t_{C_2}^n(y), \frac{I_{C_1}^n(y) + I_{C_2}^n(y)}{2}, f_{C_1}^n(y) \vee f_{C_2}^n(y)),$$

If E_1 and E_2 are the sets of edges in G_1 and G_2 then, $D_1 \cup D_2$ can be defined as:

1. If $yz \in E_1, yz \notin E_2$ then, $(D_1 \cup D_2)(yz) = D_1(yz)$.
2. If $yz \in E_2, yz \notin E_1$ then, $(D_1 \cup D_2)(yz) = D_2(yz)$.
3. If $yz \in E_1 \cap E_2$ then,

$$(D_1 \cup D_2)(yz) = (t_{D_1}^p(yz) \vee t_{D_2}^p(yz), \frac{I_{D_1}^p(yz) + I_{D_2}^p(yz)}{2}, f_{D_1}^p(yz) \wedge f_{D_2}^p(yz), t_{D_1}^n(yz) \wedge t_{D_2}^n(yz), \frac{I_{D_1}^n(yz) + I_{D_2}^n(yz)}{2}, f_{D_1}^n(yz) \vee f_{D_2}^n(yz)).$$

Definition 2.6. The *intersection* of two bipolar neutrosophic graphs $G_1 = (C_1, D_1)$ and $G_2 = (C_2, D_2)$ is a pair $G_1 \cap G_2 = (C_1 \cap C_2, D_1 \cap D_2)$ where, C_1, C_2, D_1 and D_2 are given in Definition 2.5. The membership values of vertices and edges in $G_1 \cap G_2$ can be defined as,

$$(C_1 \cap C_2)(y) = (t_{C_1}^p(y) \wedge t_{C_2}^p(y), \frac{I_{C_1}^p(y) + I_{C_2}^p(y)}{2}, f_{C_1}^p(y) \vee f_{C_2}^p(y), t_{C_1}^n(y) \vee t_{C_2}^n(y), \frac{I_{C_1}^n(y) + I_{C_2}^n(y)}{2}, f_{C_1}^n(y) \wedge f_{C_2}^n(y)), \text{ for all } y \in Y_1 \cap Y_2.$$

$$(D_1 \cap D_2)(yz) = (t_{D_1}^p(yz) \wedge t_{D_2}^p(yz), \frac{I_{D_1}^p(yz) + I_{D_2}^p(yz)}{2}, f_{D_1}^p(yz) \vee f_{D_2}^p(yz), \\ t_{D_1}^n(yz) \vee t_{D_2}^n(yz), \frac{I_{D_1}^n(yz) + I_{D_2}^n(yz)}{2}, f_{D_1}^n(yz) \wedge f_{D_2}^n(yz)), \\ \text{for all } yz \in E_1 \cap E_2.$$

Definition 2.7. The *join* of two bipolar neutrosophic graphs $G_1 = (C_1, D_1)$ and $G_2 = (C_2, D_2)$ is defined by the pair $G_1 + G_2 = (C_1 + C_2, D_1 + D_2)$ such that, $C_1 + C_2 = C_1 \cup C_2$, for all $y \in Y_1 \cup Y_2$, and the membership values of the edges in $G_1 + G_2$ are defined as,

1. $D_1 + D_2 = D_1 \cup D_2$, for all $yz \in E_1 \cup E_2$.
2. Let E' be the set of all edges joining the vertices of G_1 and G_2 then, for all $yz \in E'$, where $y \in Y_1$ and $z \in Y_2$,

$$(D_1 + D_2)(yz) = (t_{D_1}^p(yz) \wedge t_{D_2}^p(yz), I_{D_1}^p(yz) \wedge I_{D_2}^p(yz), f_{D_1}^p(yz) \vee f_{D_2}^p(yz), \\ t_{D_1}^n(yz) \vee t_{D_2}^n(yz), I_{D_1}^n(yz) \vee I_{D_2}^n(yz), f_{D_1}^n(yz) \wedge f_{D_2}^n(yz)).$$

Definition 2.8. The *Cartesian product* of two bipolar neutrosophic graphs G_1 and G_2 is denoted by the pair $G_1 \square G_2 = (C_1 \square C_2, D_1 \square D_2)$ and defined as,

$$t_{C_1 \square C_2}^p(y) = t_{C_1}^p(y) \wedge t_{C_2}^p(y), I_{C_1 \square C_2}^p(y) = I_{C_1}^p(y) \wedge I_{C_2}^p(y), \\ f_{C_1 \square C_2}^p(y) = f_{C_1}^p(y) \vee f_{C_2}^p(y), \\ t_{C_1 \square C_2}^n(y) = t_{C_1}^n(y) \vee t_{C_2}^n(y), I_{C_1 \square C_2}^n(y) = I_{C_1}^n(y) \vee I_{C_2}^n(y), \\ f_{C_1 \square C_2}^n(y) = f_{C_1}^n(y) \wedge f_{C_2}^n(y),'$$

for all $y \in Y_1 \times Y_2$. The membership values of the edges in $G_1 \square G_2$ can be calculated as,

1. $t_{D_1 \square D_2}^p((y_1, y_2)(y_1, z_2)) = t_{C_1}^p(y_1) \wedge t_{D_2}^p(y_2 z_2)$, $t_{D_1 \square D_2}^n((y_1, y_2)(y_1, z_2)) = t_{C_1}^n(y_1) \vee t_{D_2}^n(y_2 z_2)$, for all $y_1 \in Y_1, y_2 z_2 \in E_2$,
2. $t_{D_1 \square D_2}^p((y_1, y_2)(z_1, y_2)) = t_{D_1}^p(y_1 z_1) \wedge t_{C_2}^p(y_2)$, $t_{D_1 \square D_2}^n((y_1, y_2)(z_1, y_2)) = t_{D_1}^n(y_1 z_1) \vee t_{C_2}^n(y_2)$, for all $y_1 z_1 \in E_1, y_2 \in Y_2$,
3. $I_{D_1 \square D_2}^p((y_1, y_2)(y_1, z_2)) = I_{C_1}^p(y_1) \wedge I_{D_2}^p(y_2 z_2)$, $I_{D_1 \square D_2}^n((y_1, y_2)(y_1, z_2)) = I_{C_1}^n(y_1) \vee I_{D_2}^n(y_2 z_2)$, for all $y_1 \in Y_1, y_2 z_2 \in E_2$,
4. $I_{D_1 \square D_2}^p((y_1, y_2)(z_1, y_2)) = I_{D_1}^p(y_1 z_1) \wedge I_{C_2}^p(y_2)$, $I_{D_1 \square D_2}^n((y_1, y_2)(z_1, y_2)) = I_{D_1}^n(y_1 z_1) \vee I_{C_2}^n(y_2)$, for all $y_1 z_1 \in E_1, y_2 \in Y_2$,
5. $f_{D_1 \square D_2}^p((y_1, y_2)(y_1, z_2)) = f_{C_1}^p(y_1) \vee f_{D_2}^p(y_2 z_2)$, $f_{D_1 \square D_2}^n((y_1, y_2)(y_1, z_2)) = f_{C_1}^n(y_1) \wedge f_{D_2}^n(y_2 z_2)$, for all $y_1 \in Y_1, y_2 z_2 \in E_2$,
6. $f_{D_1 \square D_2}^p((y_1, y_2)(z_1, y_2)) = f_{D_1}^p(y_1 z_1) \vee f_{C_2}^p(y_2)$, $f_{D_1 \square D_2}^n((y_1, y_2)(z_1, y_2)) = f_{D_1}^n(y_1 z_1) \wedge f_{C_2}^n(y_2)$, for all $y_1 z_1 \in E_1, y_2 \in Y_2$.

Definition 2.9. The *direct product* of two bipolar neutrosophic graphs $G_1 = (C_1, D_1)$ and $G_2 = (C_2, D_2)$ is denoted by the pair $G_1 \times G_2 = (C_1 \times C_2, D_1 \times D_2)$ such that,

$$\begin{aligned} t_{C_1 \times C_2}^p(y) &= t_{C_1}^p(y) \wedge t_{C_2}^p(y), I_{C_1 \times C_2}^p(y) = I_{C_1}^p(y) \wedge I_{C_2}^p(y), \\ f_{C_1 \times C_2}^p(y) &= f_{C_1}^p(y) \vee f_{C_2}^p(y), \\ t_{C_1 \times C_2}^n(y) &= t_{C_1}^n(y) \vee t_{C_2}^n(y), I_{C_1 \times C_2}^n(y) = I_{C_1}^n(y) \vee I_{C_2}^n(y), \\ f_{C_1 \times C_2}^n(y) &= f_{C_1}^n(y) \wedge f_{C_2}^n(y), \end{aligned}$$

for all $y \in Y_1 \times Y_2$.

1. $t_{D_1 \times D_2}^p((y_1, y_2)(z_1, z_2)) = t_{D_1}^p(y_1 z_1) \wedge t_{D_2}^p(y_2 z_2), t_{D_1 \times D_2}^n((y_1, y_2)(z_1, z_2)) = t_{D_1}^n(y_1 z_1) \vee t_{D_2}^n(y_2 z_2),$ for all $y_1 z_1 \in E_1, y_2 z_2 \in E_2,$
2. $I_{D_1 \times D_2}^p((y_1, y_2)(z_1, z_2)) = I_{D_1}^p(y_1 z_1) \wedge I_{D_2}^p(y_2 z_2), I_{D_1 \times D_2}^n((y_1, y_2)(z_1, z_2)) = I_{D_1}^n(y_1 z_1) \vee I_{D_2}^n(y_2 z_2),$ for all $y_1 z_1 \in E_1, y_2 z_2 \in E_2,$
3. $f_{D_1 \times D_2}^p((y_1, y_2)(z_1, z_2)) = f_{D_1}^p(y_1 z_1) \vee f_{D_2}^p(y_2 z_2), f_{D_1 \times D_2}^n((y_1, y_2)(z_1, z_2)) = f_{D_1}^n(y_1 z_1) \wedge f_{D_2}^n(y_2 z_2),$ for all $y_1 z_1 \in E_1, y_2 z_2 \in E_2.$

Proposition 2.1. Let G_1 and G_2 be any two bipolar neutrosophic graphs then $G_1 \cup G_2, G_1 \cap G_2, G_1 + G_2, G_1 \square G_2$ and $G_1 \times G_2$ are bipolar neutrosophic graphs.

Definition 2.10. A bipolar neutrosophic graph $G = (C, D)$ is called *strong bipolar neutrosophic graph* if $t_D^p(yz) = t_C^p(y) \wedge t_C^p(z), I_D^p(yz) = I_C^p(y) \wedge I_C^p(z), f_D^p(yz) = f_C^p(y) \vee f_C^p(z), t_D^n(yz) = t_C^n(y) \vee t_C^n(z), I_D^n(yz) = I_C^n(y) \vee I_C^n(z), f_D^n(yz) = f_C^n(y) \wedge f_C^n(z),$ for all $yz \in E, E$ is the set of edges.

Definition 2.11. A bipolar neutrosophic graph $G = (C, D)$ is called *complete bipolar neutrosophic graph* if $t_D^p(yz) = t_C^p(y) \wedge t_C^p(z), I_D^p(yz) = I_C^p(y) \wedge I_C^p(z), f_D^p(yz) = f_C^p(y) \vee f_C^p(z), t_D^n(yz) = t_C^n(y) \vee t_C^n(z), I_D^n(yz) = I_C^n(y) \vee I_C^n(z), f_D^n(yz) = f_C^n(y) \wedge f_C^n(z),$ for all $y, z \in Y.$

Definition 2.12. The *complement* of a bipolar neutrosophic graph $G = (C, D)$ is defined as a pair $G^c = (C^c, D^c)$ such that, $C^c(y) = C(y),$ for all $y \in Y,$ and the membership values of the edges in G^c can be calculated as, $t_{D^c}^p(yz) = t_C^p(y) \wedge t_C^p(z) - t_D^p(yz), I_{D^c}^p(yz) = I_C^p(y) \wedge I_C^p(z) - I_D^p(yz), f_{D^c}^p(yz) = f_C^p(y) \vee f_C^p(z) - f_D^p(yz), t_{D^c}^n(yz) = t_C^n(y) \vee t_C^n(z) - t_D^n(yz), I_{D^c}^n(yz) = I_C^n(y) \vee I_C^n(z) - I_D^n(yz), f_{D^c}^n(yz) = f_C^n(y) \wedge f_C^n(z) - f_D^n(yz),$ for all $yz \in \tilde{Y}^2.$

Remark 2.1. A bipolar neutrosophic graph G is called *self complementary* if $G = G^c.$

Theorem 2.1. Let G be a self complementary bipolar neutrosophic graph then,

$$\begin{aligned} \sum_{y \neq z} t_D^p(yz) &= \frac{1}{2} \sum_{y \neq z} t_C^p(y) \wedge t_C^p(z), & \sum_{y \neq z} I_D^p(yz) &= \frac{1}{2} \sum_{y \neq z} I_C^p(y) \wedge I_C^p(z), \\ \sum_{y \neq z} f_D^p(yz) &= \frac{1}{2} \sum_{y \neq z} f_C^p(y) \vee f_C^p(z), & \sum_{y \neq z} t_D^n(yz) &= \frac{1}{2} \sum_{y \neq z} t_C^n(y) \vee t_C^n(z), \\ \sum_{y \neq z} I_D^n(yz) &= \frac{1}{2} \sum_{y \neq z} I_C^n(y) \vee I_C^n(z), & \sum_{y \neq z} f_D^n(yz) &= \frac{1}{2} \sum_{y \neq z} f_C^n(y) \wedge f_C^n(z). \end{aligned}$$

Theorem 2.2. Let $G = (C, D)$ be a bipolar neutrosophic graph such that, for all $y, z \in Y$,

$$\begin{aligned} t_{D^c}^p(yz) &= \frac{1}{2} (t_C^p(y) \wedge t_C^p(z)), & I_{D^c}^p(yz) &= \frac{1}{2} (I_C^p(y) \wedge I_C^p(z)), \\ f_{D^c}^p(yz) &= \frac{1}{2} (f_C^p(y) \vee f_C^p(z)), \\ t_{D^c}^n(yz) &= \frac{1}{2} (t_C^n(y) \vee t_C^n(z)), & I_{D^c}^n(yz) &= \frac{1}{2} (I_C^n(y) \vee I_C^n(z)), \\ f_{D^c}^n(yz) &= \frac{1}{2} (f_C^n(y) \wedge f_C^n(z)). \end{aligned}$$

Then G is self complementary bipolar neutrosophic graph.

Proof. Let $G^c = (C^c, D^c)$ be the complement of a bipolar neutrosophic graph $G = (C, D)$ then, by Definition. 2.12,

$$\begin{aligned} t_{D^c}^p(yz) &= t_C^p(y) \wedge t_C^p(z) - t_D^p(yz) \\ t_{D^c}^p(yz) &= t_C^p(y) \wedge t_C^p(z) - \frac{1}{2} (t_C^p(y) \wedge t_C^p(z)) \\ t_{D^c}^p(yz) &= \frac{1}{2} (t_C^p(y) \wedge t_C^p(z)) \\ t_{D^c}^p(yz) &= t_D^p(yz) \end{aligned}$$

Similarly, it can be proved that $t_{D^c}^n(yz) = t_D^n(yz)$, $I_{D^c}^p(yz) = I_D^p(yz)$, $I_{D^c}^n(yz) = I_D^n(yz)$, $f_{D^c}^p(yz) = f_D^p(yz)$ and $f_{D^c}^n(yz) = f_D^n(yz)$. Hence, G is self complementary. $\square$

Definition 2.13. The *degree* of a vertex y in a bipolar neutrosophic graph $G = (C, D)$ is denoted by $\deg(y)$ and defined by the 6-tuple as,

$$\begin{aligned} \deg(y) &= (\deg_t^p(y), \deg_I^p(y), \deg_f^p(y), \deg_t^n(y), \deg_I^n(y), \deg_f^n(y)), \\ &= (\sum_{yz \in E} t_D^p(yz), \sum_{yz \in E} I_D^p(yz), \sum_{yz \in E} f_D^p(yz), \sum_{yz \in E} t_D^n(yz), \sum_{yz \in E} I_D^n(yz), \\ &\quad \sum_{yz \in E} f_D^n(yz)). \end{aligned}$$

The term degree is also referred as *neighborhood degree*.

Definition 2.14. The closed neighborhood degree of a vertex y in a bipolar neutrosophic graph is denoted by $\deg[y]$ and defined as, $\deg[y] = \deg(y) + C(y)$.

Definition 2.15. A bipolar neutrosophic graph G is known as a *regular* bipolar neutrosophic graph if all vertices of G have same degree. A bipolar neutrosophic graph G is known as a *totally regular* bipolar neutrosophic graph if all vertices of G have same closed neighborhood degree.

Theorem 2.3. A complete bipolar neutrosophic graph is regular.

Theorem 2.4. Let $G = (C, D)$ be a bipolar neutrosophic graph then $C = (t^p, I^p, f^p, t^n, I^n, f^n)$ is a constant function if and only if the following statements are equivalent:

- (1) G is a regular bipolar neutrosophic graph,
- (2) G is totally regular bipolar neutrosophic graph.

Proof. Assume that C is a constant function and for all $y \in Y$,

$$t_C^p(y) = k_t, I_C^p(y) = k_I, f_C^p(y) = k_f, t_C^n(y) = k'_t, I_C^n(y) = k'_I, f_C^n(y) = k'_f$$

where, $k_t, k_I, k_f, k'_t, k'_I, k'_f$ are constants.

(1) $\Rightarrow$ (2) Suppose that G is a regular bipolar neutrosophic graph and $\deg(y) = (p_t, p_I, p_f, n_t, n_I, n_f)$, for all $y \in Y$.

Now consider, $\deg[y] = \deg(y) + C(y) = (p_t + k_t, p_I + k_I, p_f + k_f, n_t + k'_t, n_I + k'_I, n_f + k'_f)$, for all $y \in Y$.

It is proved that G is totally regular bipolar neutrosophic graph.

(2) $\Rightarrow$ (1) Suppose that G is totally regular bipolar neutrosophic graph and for all $y \in Y$ $\deg[y] = (p'_t, p'_I, p'_f, n'_t, n'_I, n'_f)$.

$\Rightarrow \deg(y) + C(y) = (p'_t, p'_I, p'_f, n'_t, n'_I, n'_f) \Rightarrow \deg(y) = (p'_t - k_t, p'_I - k_I, p'_f - k_f, n'_t - k'_t, n'_I - k'_I, n'_f - k'_f)$, for all $y \in Y$. Thus G is a regular bipolar neutrosophic graph.

Conversely, assume that the conditions are equivalent. Let $\deg(y) = (c_t, c_I, c_f, d_t, d_I, d_f)$ and $\deg[y] = (c'_t, c'_I, c'_f, d'_t, d'_I, d'_f)$.

Since by definition of closed neighborhood degree, for all $y \in Y$, $\deg[y] = \deg(y) + C(y) \Rightarrow C(y) = (c'_t - c_t, c'_I - c_I, c'_f - c_f, d'_t - d_t, d'_I - d_I, d'_f - d_f)$, for all $y \in Y$. Hence $C = (c'_t - c_t, c'_I - c_I, c'_f - c_f, d'_t - d_t, d'_I - d_I, d'_f - d_f)$, a constant function which completes the proof. $\square$

Definition 2.16. A bipolar neutrosophic graph G is said to be *irregular* if at least two vertices have distinct degrees. If all vertices do not have same closed neighborhood degrees then G is known as *totally irregular* bipolar neutrosophic graph.

Theorem 2.5. Let $G = (C, D)$ be a bipolar neutrosophic graph and $C = (t_C^p, I_C^p, f_C^p, t_C^n, I_C^n, f_C^n)$ be a constant function then G is an irregular bipolar neutrosophic graph if and only if G is a totally irregular bipolar neutrosophic graph.

Proof. Assume that G is an irregular bipolar neutrosophic graph then at least two vertices of G have distinct degrees. Let y and z be two vertices such that $\deg(y) = (r_1, r_2, r_3, s_1, s_2, s_3)$ and $\deg(z) = (r'_1, r'_2, r'_3, s'_1, s'_2, s'_3)$ where, $r_i \neq r'_i$, for some $i = 1, 2, 3$.

Since, C is a constant function let $C = (k_1, k_2, k_3, l_1, l_2, l_3)$. Therefore,

$$\begin{aligned}\deg[y] &= \deg(y) + (k_1, k_2, k_3, l_1, l_2, l_3) \\ \deg[y] &= (r_1 + k_1, r_2 + k_2, r_3 + k_3, s_1 + l_1, s_2 + l_2, s_3 + l_3) \\ \text{and } \deg[z] &= (r'_1 + k_1, r'_2 + k_2, r'_3 + k_3, s'_1 + l_1, s'_2 + l_2, s'_3 + l_3).\end{aligned}$$

Clearly $r_i + k_i \neq r'_i + k_i$, for some $i = 1, 2, 3$ therefore y and z have distinct closed neighborhood degrees. Hence G is a totally irregular bipolar neutrosophic graph. The converse part is similar. $\square$

Definition 2.17. If $G = (C, D)$ is a bipolar neutrosophic graph and y, z are two vertices in G then we say that y dominates z if

$$\begin{aligned}t_D^p(yz) &= t_C^p(y) \wedge t_C^p(z), & I_D^p(yz) &= I_C^p(y) \wedge I_C^p(z), & f_D^p(yz) &= f_C^p(y) \vee f_C^p(z), \\ t_D^n(yz) &= t_C^n(y) \vee t_C^n(z), & I_D^n(yz) &= I_C^n(y) \vee I_C^n(z), & f_D^n(yz) &= f_C^n(y) \wedge f_C^n(z).\end{aligned}$$

A subset $D' \subseteq Y$ is a *dominating set* if for each $z \in Y \setminus D'$ there exists $y \in D'$ such that y dominates z . A dominating set D' is minimal if for every $y \in D'$, $D' \setminus \{y\}$ is not a dominating set. The *domination number* of G is the minimum cardinality among all minimal dominating sets of G , denoted by $\lambda(G)$.

Example 2.2. Consider a bipolar neutrosophic graph as shown in Fig. 2. The set $\{x, w\}$ is a minimal dominating set and $\lambda(G) = 2$.

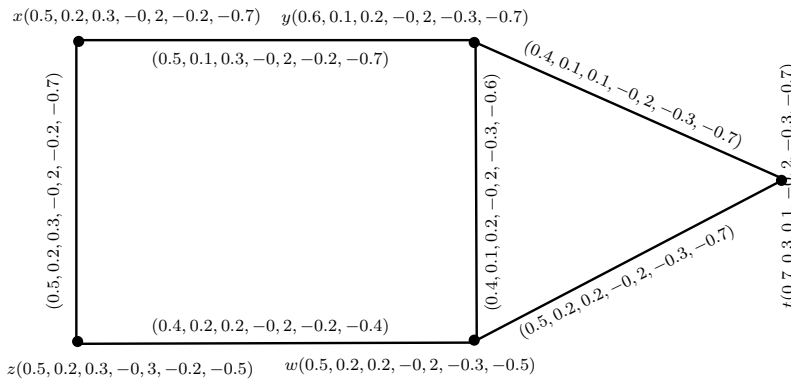


Figure 2: Bipolar neutrosophic graph G .

Theorem 2.6. Let G_1 and G_2 be two bipolar neutrosophic graphs with D'_1 and D'_2 as dominating sets then following conditions hold,

- (1) If $Y_1 \cap Y_2 = \emptyset$ then, $\lambda(G_1 \cup G_2) = \lambda(G_1) + \lambda(G_2)$.

- (2) If for every $y \in Y_1 \cap Y_2 \neq \emptyset$, $C_1(y) = C_2(y)$ and for $yz \in E_1 \cap E_2$, $D_1(yz) = D_2(yz)$ then, $\lambda(G_1 \cup G_2) = \lambda(G_1) + \lambda(G_2) - |D'_1 \cap D'_2|$.

Proof. (1) The proof is obvious.

(2) Since D'_1 and D'_2 are dominating sets of G_1 and G_2 , $D'_1 \cup D'_2$ is a dominating set of $G_1 \cup G_2$. Therefore, $\lambda(G_1 \cup G_2) \leq |D'_1 \cup D'_2|$. It only remains to show that $D'_1 \cup D'_2$ is a minimal dominating set. On contrary, assume that $D' = D'_1 \cup D'_2 \setminus \{y\}$ is a minimal dominating set of $G_1 \cup G_2$. There are two cases, **Case 1.** If $y \in D'_1$ and $y \notin D'_2$, then $D'_1 \setminus \{y\}$ is not a dominating set of G_1 which implies that $D'_1 \cup D'_2 \setminus \{y\} = D'$ is not a dominating set of $G_1 \cup G_2$. A contradiction, hence $D'_1 \cup D'_2$ is a minimal dominating set and

$$\begin{aligned} \lambda(G_1 \cup G_2) &= |D'_1 \cup D'_2|, \\ \Rightarrow \lambda(G_1 \cup G_2) &= \lambda(G_1) + \lambda(G_2) - |D'_1 \cap D'_2|. \end{aligned}$$

Case 2. If $y \in D'_2$ and $y \notin D'_1$, same contradiction can be obtained. $\square$

Theorem 2.7. If G_1 and G_2 are two bipolar neutrosophic graphs then the following conditions are satisfied,

- (1) If $Y_1 \cap Y_2 = \emptyset$ then, $\lambda(G_1 + G_2) = 2$.
- (2) If for every $y \in Y_1 \cap Y_2 \neq \emptyset$, $C_1(y) = C_2(y)$ and for $yz \in E_1 \cap E_2$, $D_1(yz) = D_2(yz)$ then, $\lambda(G_1 + G_2) = \min\{\lambda(G_1), \lambda(G_2), 2\}$.

Proof. (1). Let $y_1 \in Y_1$ and $y_2 \in Y_2$, since $G_1 + G_2$ is a bipolar neutrosophic graph, we have

$$\begin{aligned} t_{D_1+D_2}^p(y_1y_2) &= t_{C_1+C_2}^p(y_1) \wedge t_{C_1+C_2}^p(y_2), t_{D_1+D_2}^n(y_1y_2) = t_{C_1+C_2}^n(y_1) \vee t_{C_1+C_2}^n(y_2) \\ I_{D_1+D_2}^p(y_1y_2) &= I_{C_1+C_2}^p(y_1) \wedge I_{C_1+C_2}^p(y_2), I_{D_1+D_2}^n(y_1y_2) = I_{C_1+C_2}^n(y_1) \vee I_{C_1+C_2}^n(y_2) \\ f_{D_1+D_2}^p(y_1y_2) &= f_{C_1+C_2}^p(y_1) \vee f_{C_1+C_2}^p(y_2), f_{D_1+D_2}^n(y_1y_2) = f_{C_1+C_2}^n(y_1) \wedge f_{C_1+C_2}^n(y_2). \end{aligned}$$

Hence any vertex of G_1 dominates all vertices of G_2 and similarly any vertex of G_2 dominates all vertices of G_1 . So, $\{y_1, y_2\}$ is a dominating set of $G_1 + G_2$.

(2). If D is a minimal dominating set of $G_1 + G_2$ then, D is one of the following forms,

1. $D = D_1$ where, $\lambda(G_1) = |D_1|$,
2. $D = D_2$ where, $\lambda(G_2) = |D_2|$,
3. $D = \{y_1, y_2\}$ where, $y_1 \in Y_1$ and $y_2 \in Y_2$. $\{y_1\}$ and $\{y_2\}$ are not dominating sets of G_1 or G_2 , respectively.

Hence, $\lambda(G_1 + G_2) = \min\{\lambda(G_1), \lambda(G_2), 2\}$. $\square$

Theorem 2.8. *Let $G_1 = (C_1, D_1)$ and $G_2 = (C_2, D_2)$ be two bipolar neutrosophic graphs. If for $y_1 \in Y_1$, $C_1(y_1) \neq (0, 0, 1, 0, 0, -1)$, and y_2 dominates z_2 in G_2 then, (y_1, y_2) dominates (y_1, z_2) in $G_1 \square G_2$.*

Proof. Since y_2 dominates z_2 therefore,

$$\begin{aligned} t_{D_2}^p(y_2 z_2) &= t_{C_2}^p(y_2) \wedge t_{C_2}^p(z_2), \quad I_{D_2}^p(y_2 z_2) = I_{C_2}^p(y_2) \wedge I_{C_2}^p(z_2), \\ f_{D_2}^p(y_2 z_2) &= f_{C_2}^p(y_2) \vee f_{C_2}^p(z_2), \quad t_{D_2}^n(y_2 z_2) = t_{C_2}^n(y_2) \vee t_{C_2}^n(z_2), \\ I_{D_2}^n(y_2 z_2) &= I_{C_2}^n(y_2) \vee I_{C_2}^n(z_2), \quad f_{D_2}^n(y_2 z_2) = f_{C_2}^n(y_2) \wedge f_{C_2}^n(z_2). \end{aligned}$$

For $y_1 \in Y_1$, take $(y_1, z_2) \in Y_1 \times Y_2$. By Definition 2.8,

$$\begin{aligned} t_{D_1 \square D_2}^p((y_1, y_2)(y_1, z_2)) &= t_{C_1}^p(y_1) \wedge t_{D_2}^p(y_2 z_2), \\ &= t_{C_1}^p(y_1) \wedge \{t_{C_2}^p(y_2) \wedge t_{C_2}^p(z_2)\}, \\ &= \{t_{C_1}^p(y_1) \wedge t_{C_2}^p(y_2)\} \wedge \{t_{C_1}^p(y_1) \wedge t_{C_2}^p(z_2)\}, \\ &= t_{C_1 \square C_2}^p(y_1, y_2) \wedge t_{C_1 \square C_2}^p(y_1, z_2). \\ t_{D_1 \square D_2}^n((y_1, y_2)(y_1, z_2)) &= t_{C_1}^n(y_1) \vee t_{D_2}^n(y_2 z_2), \\ &= t_{C_1}^n(y_1) \vee \{t_{C_2}^n(y_2) \vee t_{C_2}^n(z_2)\}, \\ &= \{t_{C_1}^n(y_1) \vee t_{C_2}^n(y_2)\} \vee \{t_{C_1}^n(y_1) \vee t_{C_2}^n(z_2)\}, \\ &= t_{C_1 \square C_2}^n(y_1, y_2) \vee t_{C_1 \square C_2}^n(y_1, z_2). \end{aligned}$$

Similarly, it can be proved that

$$\begin{aligned} I_{D_1 \square D_2}^p((y_1, y_2)(y_1, z_2)) &= I_{C_1 \square C_2}^p(y_1, y_2) \wedge I_{C_1 \square C_2}^p(y_1, z_2), \\ I_{D_1 \square D_2}^n((y_1, y_2)(y_1, z_2)) &= I_{C_1 \square C_2}^n(y_1, y_2) \vee I_{C_1 \square C_2}^n(y_1, z_2), \\ f_{D_1 \square D_2}^p((y_1, y_2)(y_1, z_2)) &= f_{C_1 \square C_2}^p(y_1, y_2) \vee f_{C_1 \square C_2}^p(y_1, z_2), \\ f_{D_1 \square D_2}^n((y_1, y_2)(y_1, z_2)) &= f_{C_1 \square C_2}^n(y_1, y_2) \wedge f_{C_1 \square C_2}^n(y_1, z_2). \end{aligned}$$

Hence (y_1, y_2) dominates (y_1, z_2) and the proof is complete. $\square$

Proposition 2.2. *If G_1 and G_2 are bipolar neutrosophic graphs and for $z_2 \in Y_2$, $C_2(z_2) \neq (0, 0, 1, 0, 0, -1)$, y_1 dominates z_1 in G_1 then (y_1, z_2) dominates (z_1, z_2) in $G_1 \square G_2$.*

Theorem 2.9. *If D'_1 and D'_2 are minimal dominating sets of $G_1 = (C_1, D_1)$ and $G_2 = (C_2, D_2)$, respectively. Then $D'_1 \times Y_2$ and $Y_1 \times D'_2$ are dominating sets of $G_1 \square G_2$ and*

$$(2.1) \quad \lambda(G_1 \square G_2) \leq |D'_1 \times Y_2| \wedge |Y_1 \times D'_2|.$$

Proof. To prove Inequality. (2.1), we need to show that $D'_1 \times Y_2$ and $Y_1 \times D'_2$ are dominating sets of $G_1 \square G_2$. Let $(z_1, z_2) \notin D'_1 \times Y_2$ then, $z_1 \notin D'_1$. Since D'_1 is a dominating set of G_1 , there exists $y_1 \in D'_1$ that dominates z_1 . By Theorem 2.2, (y_1, z_2) dominates (z_1, z_2) in $G_1 \square G_2$. Since (z_1, z_2) was taken to be arbitrary therefore, $D'_1 \times Y_2$ is a dominating set of $G_1 \square G_2$. Similarly, $Y_1 \times D'_2$ is a dominating set if $G_1 \square G_2$. Hence the proof. $\square$

Theorem 2.10. Let D'_1 and D'_2 be the dominating sets of $G_1 = (C_1, D_1)$ and $G_2 = (C_2, D_2)$, respectively then, $D'_1 \times D'_2$ is a dominating set of the direct product $G_1 \times G_2$ and

$$(2.2) \quad \lambda(G_1 \times G_2) = |D'_\times| = |D'_1 \times D'_2 \cup \{(t_1, t_2) : t_1 y_1 \in E_1, y_1 \in D'_1, t_2 \in D'_2\}|.$$

Proof. Let $(z_1, z_2) \in Y_1 \times Y_2 \setminus D'_\times$ then there are two cases.

Case 1: If $z_1 \in Y_1 \setminus D'_1$ and $z_2 \in Y_2 \setminus D'_2$. Since, D'_1 and D'_2 are dominating sets there exist $y_1 \in D'_1$ and $y_2 \in D'_2$ such that y_1 dominates z_1 and y_2 dominates z_2 . Consider,

$$\begin{aligned} t_{D_1 \times D_2}^p((y_1, y_2)(z_1, z_2)) &= t_{D_1}^p(y_1 z_1) \wedge t_{D_2}^p(y_2 z_2), \\ &= \{t_{C_1}^p(y_1) \wedge t_{C_1}^p(z_1)\} \wedge \{t_{C_2}^p(y_2) \wedge t_{C_2}^p(z_2)\}, \\ &= \{t_{C_1}^p(y_1) \wedge t_{C_2}^p(y_2)\} \wedge \{t_{C_1}^p(z_1) \wedge t_{C_2}^p(z_2)\}, \\ &= t_{C_1 \times C_2}^p(y_1, y_2) \wedge t_{C_1 \times C_2}^p(z_1, z_2). \end{aligned}$$

Similarly, it can be proved for other truth, indeterminacy and falsity membership degrees. Hence (y_1, y_2) dominates (z_1, z_2) .

Case 2: If $z_1 \in D'_1$ and $z_2 \in Y_2 \setminus D'_2$ then, there exists $(t_1, t_2) \in \{(t_1, t_2) : t_1 z_1 \in E_1, z_1 \in D'_1, t_2 \in D'_2\}$ such that z_1 dominates t_1 and t_2 dominates z_2 . Consider,

$$\begin{aligned} t_{D_1 \times D_2}^p((t_1, t_2)(z_1, z_2)) &= t_{D_1}^p(t_1 z_1) \wedge t_{D_2}^p(t_2 z_2), \\ &= \{t_{C_1}^p(t_1) \wedge t_{C_1}^p(z_1)\} \wedge \{t_{C_2}^p(t_2) \wedge t_{C_2}^p(z_2)\}, \\ &= \{t_{C_1}^p(t_1) \wedge t_{C_2}^p(t_2)\} \wedge \{t_{C_1}^p(z_1) \wedge t_{C_2}^p(z_2)\}, \\ &= t_{C_1 \times C_2}^p(t_1, t_2) \wedge t_{C_1 \times C_2}^p(z_1, z_2). \end{aligned}$$

On the same lines, the result can be obtained for other truth, indeterminacy and falsity membership degrees. Hence (z_1, z_2) is dominated by (t_1, t_2) .

Since (y_1, y_2) was taken to be arbitrary therefore, every element of $Y_1 \times Y_2 \setminus D'_\times$ is dominated by some element of $D'_\times$.

Clearly, no vertex in $\{(t_1, t_2) : t_1 y_1 \in E_1, y_1 \in D'_1, t_2 \in D'_2\}$ is dominated by any other vertex in $D'_\times$. Therefore, it only remains to show that $D'_1 \times D'_2$ is minimal. On contrary, assume that D' is a minimal such that $|D'| < |D'_1 \times D'_2|$. Let $(t_1, t_2) \in D'_1 \times D'_2$ such that $(t_1, t_2) \notin D'$ i.e., $t_1 \in D'_1$ and $t_2 \in D'_2$ then there exist $t'_1 \in Y_1 \setminus D'_1$ and $t'_2 \in Y_2 \setminus D'_2$ which are only dominated by t_1 and t_2 , respectively. Hence no element other than (t_1, t_2) dominates (t'_1, t'_2) so $(t_1, t_2) \in D'$. A contradiction, thus $D'_1 \times D'_2$ is minimal. $\square$

Corollary 2.1. If G_1 and G_2 are two bipolar neutrosophic graphs, y_1 dominates z_1 in G_1 and y_2 dominates z_2 in G_2 then (y_1, z_1) dominates (y_2, z_2) in $G_1 \times G_2$.

Definition 2.18. In a bipolar neutrosophic graph two vertices y and z are independent if

$$(2.3) \quad \begin{aligned} t_D^p(yz) &< t_C^p(y) \wedge t_C^p(z), I_D^p(yz) < I_C^p(y) \wedge I_C^p(z), f_D^p(yz) < f_C^p(y) \vee f_C^p(z), \\ t_D^n(yz) &> t_C^n(y) \vee t_C^n(z), I_D^n(yz) > I_C^n(y) \vee I_C^n(z), f_D^n(yz) > f_C^n(y) \wedge f_C^n(z). \end{aligned}$$

An *independent set* N of a bipolar neutrosophic graph is a subset N of Y such that, for all $y, z \in N$, Equations (2.3) are satisfied. An independent set is *maximal* if for every $t \in Y \setminus N$, $N \cup \{t\}$ is not an independent set. An *independent number* is the maximal cardinality among all maximal independent sets of a bipolar neutrosophic graph. It is denoted by $\alpha(G)$.

Theorem 2.11. *If G_1 and G_2 are bipolar neutrosophic graphs on Y_1 and Y_2 , respectively such that $Y_1 \cap Y_2 = \emptyset$ then $\alpha(G_1 \cup G_2) = \alpha(G_1) + \alpha(G_2)$.*

Proof. Let N_1 and N_2 be maximal independent sets of G_1 and G_2 . Since $N_1 \cap N_2 = \emptyset$ therefore, $N_1 \cup N_2$ is a maximal independent set of $G_1 \cup G_2$. Hence $\alpha(G_1 \cup G_2) = \alpha(G_1) + \alpha(G_2)$. $\square$

Theorem 2.12. *Let G_1 and G_2 be two bipolar neutrosophic graphs then $\alpha(G_1 + G_2) = \alpha(G_1) \vee \alpha(G_2)$.*

Proof. Let N_1 and N_2 be maximal independent sets. Since every vertex of G_1 dominates every vertex of G_2 in $G_1 + G_2$. Hence, maximal independent set of $G_1 + G_2$ is either N_1 or N_2 . Thus, $\alpha(G_1 + G_2) = \alpha(G_1) \vee \alpha(G_2)$. $\square$

Theorem 2.13. *If N_1 and N_2 are maximal independent sets of G_1 and G_2 , respectively and $Y_1 \cap Y_2 = \emptyset$. Then $\alpha(G_1 \square G_2) = |N_1 \times N_2| + |N|$ where, $N = \{(y_i, z_i) : y_i \in Y_1 \setminus N_1, z_i \in Y_2 \setminus N_2, y_i y_{i+1} \in E_1, z_i z_{i+1} \in E_2, i = 1, 2, 3, \dots\}$.*

Proof. N_1 and N_2 are maximal independent sets of G_1 and G_2 , respectively. Clearly, $N_1 \times N_2$ is an independent set of $G_1 \square G_2$ as no vertex of $N_1 \times N_2$ dominates any other vertex of $N_1 \times N_2$.

Consider the set of vertices $N = \{(y_i, z_i) : y_i \in Y_1 \setminus N_1, z_i \in Y_2 \setminus N_2, y_i y_{i+1} \in E_1, z_i z_{i+1} \in E_2\}$. It can be seen that no vertex $(y_i, z_i) \in N$ for each $i = 1, 2, 3, \dots$ dominates $(y_{i+1}, z_{i+1}) \in N$ for each $i = 1, 2, 3, \dots$. Hence $N' = (N_1 \times N_2) \cup N$ is an independent set of $G_1 \square G_2$.

Assume that $S = N' \cup \{(y_i, z_j)\}$, for some $i \neq j$, $y_i \in Y_1 \setminus N_1$ and $z_j \in Y_2 \setminus N_2$, is a maximal independent set. Without loss of generality, assume that $j = i + 1$ then, (y_i, z_j) is dominated by (y_i, z_i) . A contradiction, hence N' is a maximal independent set and $\alpha(G_1 \square G_2) = |N'| = |N_1 \times N_2| + |N|$. $\square$

Theorem 2.14. *If $D'_\times$ is a minimal dominating sets of $G_1 \times G_2$ then, $Y_1 \times Y_2 \setminus D'_\times$ is a maximal independent set of $G_1 \times G_2$ and $\alpha(G_1 \times G_2) = n_1 n_2 - \lambda(G_1 \times G_2)$ where, n_1 and n_2 are the number of vertices in G_1 and G_2 .*

The proof is obvious.

Theorem 2.15. *An independent set of a bipolar neutrosophic graph $G = (C, D)$ is maximal if and only if it is independent and dominating.*

Proof. If N is a maximal independent set of G , then for every $y \in Y \setminus N$, $N \cup \{y\}$ is not an independent set. For every vertex $y \in Y \setminus N$, there exists some $z \in N$ such that

$$\begin{aligned} t_D^p(yz) &= t_C^p(y) \wedge t_C^p(z), & I_D^p(yz) &= I_C^p(y) \wedge I_C^p(z), & f_D^p(yz) &= f_C^p(y) \vee f_C^p(z), \\ t_D^n(yz) &= t_C^n(y) \vee t_C^n(z), & I_D^n(yz) &= I_C^n(y) \vee I_C^n(z), & f_D^n(yz) &= f_C^n(y) \wedge f_C^n(z). \end{aligned}$$

Thus y dominates z and hence N is both independent and dominating set.

Conversely, assume that D is both independent and dominating set but not maximal independent set. So there exists a vertex $y \in Y \setminus N$ such that $N \cup \{y\}$ is an independent set i.e., no vertex in N dominates y , a contradiction to the fact that N is a dominating set. Hence N is maximal. $\square$

Theorem 2.16. *Any maximal independent set of a bipolar neutrosophic graph is a minimal dominating set.*

Proof. If N is a maximal independent set of a bipolar neutrosophic graph then by Theorem 2.15, N is a dominating set. Assume that N is not a minimal dominating set then, there always exist at least one $z \in N$ for which $N \setminus \{z\}$ is a dominating set. On the other hand if $N \setminus \{z\}$ dominates $Y \setminus \{N \setminus \{z\}\}$, at least one vertex in $N \setminus \{z\}$ dominates z . A contradiction to the fact that N is an independent set of bipolar neutrosophic graph G . Hence N is a minimal dominating set. $\square$

3. Multi-criteria decision making methods

Multiple criteria decision making refers to making decisions in the presence of multiple, usually conflicting criteria. Multi-criteria decision making problems are common in everyday life. In this section, we present multi-criteria decision making methods for the identification of risk in decision support systems. The method is explained by an example for prevention of accidental hazards in chemical industry. The application of domination in bipolar neutrosophic graphs is given for the construction of transmission stations.

(1) An outranking approach for safety analysis using bipolar neutrosophic information

The proposed methodology can be implemented in various fields in different ways e.g., multi-criteria decision making problems with bipolar neutrosophic information. However, our main focus is the identification of risk assessments in industry which is described in the following steps.

The bipolar neutrosophic information consists of a group of risks\alternatives $R = \{r_1, r_2, \dots, r_n\}$ evaluated on the basis of criteria $C = \{c_1, c_2, \dots, c_m\}$. Here r_i , $i = 1, 2, \dots, n$ is the possibility for the criteria c_k , $k = 1, 2, \dots, m$ and r_{ik} are in the form of bipolar neutrosophic values. This method is suitable if we have a small set of data and experts are able to evaluate the data in the form of bipolar neutrosophic information. Take the values of r_{ik} as $r_{ik} = (t_{ik}^p, I_{ik}^p, f_{ik}^p, t_{ik}^n, I_{ik}^n, f_{ik}^n)$.

Step 1. Construct the table of the given data.

Step 2. Determine the average values using the following bipolar neutrosophic average operator,

$$(3.1) \quad A_i = \frac{1}{n} \left(\sum_{j=1}^m t_{ij}^p - \prod_{j=1}^m t_{ij}^p, \prod_{j=1}^m I_{ij}^p, \prod_{j=1}^m f_{ij}^p, \prod_{j=1}^m t_{ij}^n, \sum_{j=1}^m I_{ij}^n - \prod_{j=1}^m I_{ij}^n, \sum_{j=1}^m f_{ij}^n - \prod_{j=1}^m f_{ij}^n \right),$$

for each $i = 1, 2, \dots, n$.

Step 3. Construct the weighted average matrix.

Choose the weight vector $\mathbf{w} = (w_1, w_2, \dots, w_n)$. According to the weights for each alternative, the weighted average table can be calculated by multiplying each average value with the corresponding weight as:

$$\beta_i = A_i w_i, \quad i = 1, 2, \dots, n.$$

Step 4. Calculate the normalized value for each alternative\risk β_i using the formula,

$$(3.2) \quad \alpha_i = \sqrt{(t_i^p)^2 + (I_i^p)^2 + (f_i^p)^2 + (-1 + t_i^n)^2 + (-1 + I_i^n)^2 + (-1 + f_i^n)^2},$$

for each $i = 1, 2, \dots, n$. The resulting table indicate the preference ordering of the alternatives\risks. The alternative\risk with maximum α_i value is most dangerous or more preferable.

Example 3.1. Chemical industry is a very important part of human society. These industries contain large amount of organic and inorganic chemicals and materials. Many chemical products have a high risk of fire due to flammable materials, large explosions and oxygen deficiency etc. These accidents can cause the death of employs, damages to building, destruction of machines and transports, economical losses etc. Therefore, it is very important to prevent these accidental losses by identifying the major risks of fire, explosions and oxygen deficiency.

A manager of a chemical industry Y wants to prevent such types of accidents that caused the major loss to company in the past. He collected data from witness reports, investigation teams and near by chemical industries and found that the major causes could be the chemical reactions, oxidizing materials, formation of toxic substances, electric hazards, oil spill, hydrocarbon gas leakage and energy systems. The witness reports, investigation teams and industries have different opinions. There is a bipolarity in people's thinking and judgement. The data can be considered as bipolar neutrosophic information. The bipolar neutrosophic information about company Y old accidents are given in Table. 3 and Table. 4.

Table 3: Bipolar neutrosophic Data

	Fire	Oxygen Deficiency
Chemical Exposures	(0.5,0.7,0.2,-0.6,-0.3,-0.7)	(0.1,0.5,0.7,-0.5,-0.2,-0.8)
Oxidizing materials	(0.9,0.7,0.2,-0.8,-0.6,-0.1)	(0.3,0.5,0.2,-0.5,-0.5,-0.2)
Toxic vapour cloud	(0.7,0.3,0.1,-0.4,-0.1,-0.3)	(0.6,0.3,0.2,-0.5,-0.3,-0.3)
Electric Hazard	(0.3,0.4,0.2,-0.6,-0.3,-0.7)	(0.9,0.4,0.6,-0.1,-0.7,-0.5)
Oil Spill	(0.7,0.5,0.3,-0.4,-0.2,-0.2)	(0.2,0.2,0.2,-0.7,-0.4,-0.4)
Hydrocarbon gas leakage	(0.5,0.3,0.2,-0.5,-0.2,-0.2)	(0.3,0.2,0.3,-0.7,-0.4,-0.3)
Ammonium Nitrate	(0.3,0.2,0.3,-0.5,-0.6,-0.5)	(0.9,0.2,0.1,0.0,-0.6,-0.5)

Table 4: Bipolar neutrosophic Data

	Large Explosion
Chemical Exposures	(0.6,0.2,0.3,-0.4,0.0,-0.1)
Oxidizing materials	(0.9,0.5,0.5,-0.6,-0.5,-0.2)
Toxic vapour cloud	(0.5,0.1,0.2,-0.6,-0.2,-0.2)
Electric Hazard	(0.7,0.6,0.8,-0.7,-0.5,-0.1)
Oil Spill	(0.9,0.2,0.7,-0.1,-0.6,-0.8)
Hydrocarbon gas leakage	(0.8,0.2,0.1,-0.1,-0.9,-0.2)
Ammonium Nitrate	(0.6,0.2,0.1,-0.2,-0.3,-0.5)

By applying the bipolar neutrosophic average operator 3.1 on Table. 3 and Table. 4, the average values are given in Table. 5.

Table 5: Bipolar neutrosophic average values

	Average Value
Chemical Exposures	(0.39,0.023,0.014,-0.04,-0.167,-0.515)
Oxidizing materials	(0.619,0.032,0.001,-0.08,-0.483,-0.165)
Toxic vapour cloud	(0.53,0.003,0.001,-0.04,-0.198,-0.261)
Electric Hazard	(0.570,0.032,0.032,-0.014,-0.465,-0.422)
Oil Spill	(0.558,0.007,0.014,-0.009,-0.384,-0.445)
Hydrocarbon gas leakage	(0.493,0.004,0.002,-0.011,-0.543,-0.229)
Ammonium Nitrate	(0.546,0.003,0.001,0.0,-0.464,-0.417)

With regard to the weight vector (0.35, 0.80, 0.30, 0.275, 0.65, 0.75, 0.50) associated to each cause of accident, the weighted average values are obtained by multiplying each average value with corresponding weight and are given in Table. 6.

Table 6: Bipolar neutrosophic weighted average table

	Weighted Value
Chemical Exposures	(0.1365,0.0081,0.0049,-0.0140,-0.0585,-0.1803)
Oxidizing materials	(0.4952,0.0256,0.0008,-0.0640,-0.3864,-0.1320)
Toxic vapour cloud	(0.1590,0.0009,0.0003,-0.012,-0.0594,-0.0783)

Continued on next page

Table 6 – *Continued from previous page*

Electric Hazard	(0.2850,0.0160,0.0160,-0.0070,-0.2325,-0.2110)
Oil Spill	(0.1535,0.0019,0.0039,-0.0025,-0.1056,-0.1224)
Hydrocarbon gas leakage	(0.3205,0.0026,0.0013,-0.0072,-0.3530,-0.1489)
Ammonium Nitrate	(0.4095,0.0023,0.0008,0.0,-0.3480,-0.2110)

Using Formula. 3.2, the resulting normalized values are shown in Table. 8.

Table 7: Normalized values

	Normalized value
Chemical Exposures	1.5966
Oxidizing materials	1.5006
Toxic vapour cloud	1.6540
Electric Hazard	1.6090
Oil Spill	1.4938
Hydrocarbon gas leakage	1.6036
Ammonium Nitrate	1.5089

The accident possibilities can be placed in the following order: toxic vapour cloud $\succ$ electric hazard $\succ$ hydrocarbon gas leakage $\succ$ chemical exposures $\succ$ ammonium nitrate $\succ$ oxidizing materials $\succ$ Oil spill where, the symbol $\succ$ represents partial ordering of objects. It can be easily seen that the formation of toxic vapour clouds, electrical and energy systems and hydrocarbon gas leakage are the major dangers to the chemical industry. There is a very little danger due to oil spill. Chemical exposures, oxidizing materials and ammonium nitrate has an average accidental danger. Therefore, industry needs special precautions to prevent the major hazards that could happen due the formation of toxic vapour clouds.

(2) Domination in bipolar neutrosophic graphs

Domination has a wide variety of applications in communication networks, coding theory, fixing surveillance cameras, detecting biological proteins and social networks etc. Consider the example of a TV channel that wants to set up transmission stations in a number of cities such that every city in the country get access to the channel signals from at least one of the stations. To reduce the cost for building large stations it is required to set up minimum number of stations. This problem can be represented by a bipolar neutrosophic graph in which vertices represent the cities and there is an edge between two cities if they can communicate directly with each other. Consider a network of ten cities $\{C_1, C_2, \dots, C_{10}\}$. In the bipolar neutrosophic graph, the positive degree of each vertex represents the level of truth, indeterminacy and falsity of strong signals it can transmit to other cities and the negative degree of each vertex represents

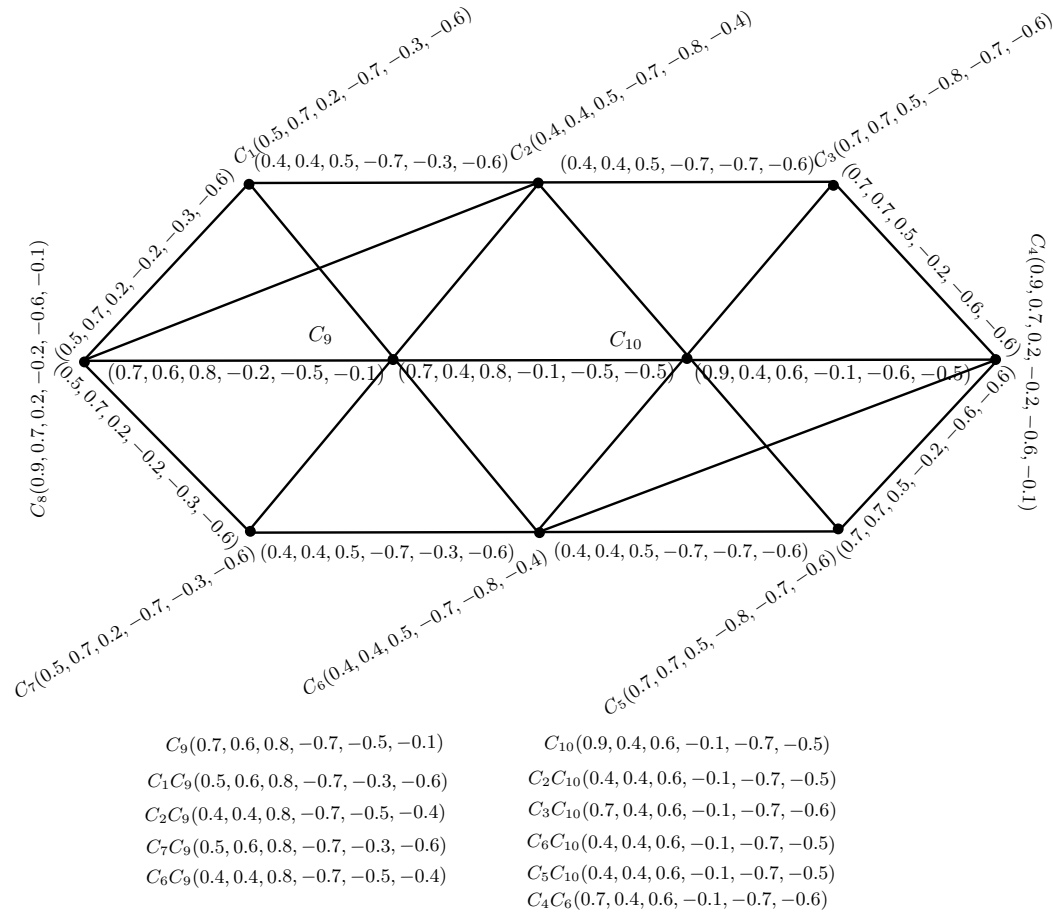


Figure 3: Domination in bipolar neutrosophic graph

the level of truth, indeterminacy and falsity of weaker signals it can transmit to other cities. The bipolar neutrosophic value of each edge represents the degree of truth, indeterminacy and falsity of strong and weak communication between the cities. The graph is shown in Fig. 3. $D = \{C_8, C_{10}\}$ is the minimum dominating set. It is concluded that building only two large transmitting stations in C_8 and C_{10} , a high economical benefit can be achieved. The method of calculating the minimum number of stations is described in the following Algorithm. 1.

Table 8: Algorithm for the selection of minimum locations

Algorithm 1
1. Begin
2. Enter the membership values $B(x_i)$ of n number of possible locations $A = \{C_1, C_2, \dots, C_n\}$.
3. Input the adjacency matrix $[C_{ij}]_{n \times n}$ of transmission stations.

Continued on next page

Table 8 – Continued from previous page

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4.   $k = 0$ 
5.   $D = \emptyset$ 
6.  do  $i$  from  $1 \rightarrow n$ 
7.    do  $j$  from  $i + 1 \rightarrow n$ 
8.      if  $C_{ij} = A(C_i) \cap A(C_j)$  then
9.         $C_i \in D, k = k + 1, x_k = C_i$ 
10.      end if
11.    end do
12.  end do
13.  Arrange  $X \setminus D = \{x_{k+1}, x_{k+2}, \dots, x_n\} = J$ 
14.  do  $i$  from  $1 \rightarrow k$ 
15.     $D' = D \setminus \{x_i\}$ 
16.    if  $D'$  is a dominating set then
17.       $D = D'$ 
18.       $J = J \cup \{x_i\}$ 
19.    end if
20.  end do
21.  if  $D \cup J = Y$  then
22.    print:  $D$  is a minimal dominating set.
23.  else
24.    print: There is no dominating set.
25.  end if

```

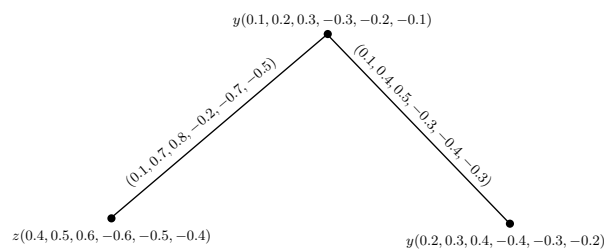
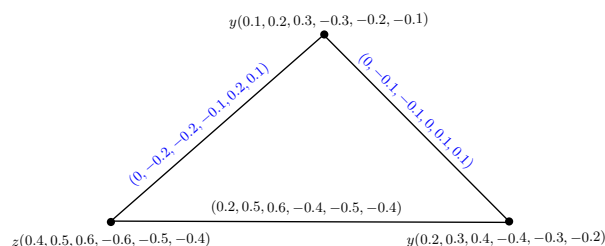
4. Comments on Broumi et al.'s bipolar neutrosophic graphs

Broumi et al. [11] defined bipolar single-valued neutrosophic graphs in the following way:

Definition 4.1 ([11]). Let $A = (T_A^P, I_A^P, F_A^P, T_A^N, I_A^N, F_A^N)$ and $B = (T_B^P, I_B^P, F_B^P, T_B^N, I_B^N, F_B^N)$ be bipolar single valued neutrosophic graph on set Y . If $B = (T_B^P, I_B^P, F_B^P, T_B^N, I_B^N, F_B^N)$ is a bipolar single valued neutrosophic relation on $A = (T_A^P, I_A^P, F_A^P, T_A^N, I_A^N, F_A^N)$ then $T_B^P(xy) \leq T_A^P(x) \wedge T_A^P(y)$, $I_B^P(xy) \geq I_A^P(x) \vee I_A^P(y)$, $F_B^P(xy) \geq F_A^P(x) \vee F_A^P(y)$, $T_B^N(xy) \geq T_A^N(x) \vee T_A^N(y)$, $I_B^N(xy) \leq I_A^N(x) \wedge I_A^N(y)$, $F_B^N(xy) \leq F_A^N(x) \wedge F_A^N(y)$, for all $x, y \in Y$.

Broumi et al. [11] defined complement of a bipolar neutrosophic graph as follows:

Definition 4.2 ([11]). The complement of a bipolar neutrosophic graph $G = (A, B)$ is a bipolar single valued neutrosophic graph $\overline{G} = (\overline{A}, \overline{B})$ where, $\overline{A} = A = (T_A^P, I_A^P, F_A^P, T_A^N, I_A^N, F_A^N)$, $\overline{B} = (\overline{T_B^P}, \overline{I_B^P}, \overline{F_B^P}, \overline{T_B^N}, \overline{I_B^N}, \overline{F_B^N})$ and is defined by $\overline{T_B^P}(xy) = T_A^P(x) \wedge T_A^P(y) - T_B^P(xy)$, $\overline{I_B^P}(xy) = I_A^P(x) \vee I_A^P(y) - I_B^P(xy)$, $\overline{F_B^P}(xy) = F_A^P(x) \vee F_A^P(y) - F_B^P(xy)$, $\overline{T_B^N}(xy) = T_A^N(x) \vee T_A^N(y) - T_B^N(xy)$, $\overline{I_B^N}(xy) = I_A^N(x) \wedge I_A^N(y) - I_B^N(xy)$, $\overline{F_B^N}(xy) = F_A^N(x) \wedge F_A^N(y) - F_B^N(xy)$, for all $xy \in Y \times Y$.

Figure 4: Bipolar single valued neutrosophic graph G according to [11]Figure 5: Complement of G according to [11]

We illustrate Definitions 5.1-5.2 [11] by the following example.

Example 4.1. The complement $\overline{G}$ of G is obtained by using Definitions 5.1-5.2 [11] as shown in Fig. 5. It can be seen from Fig. 5 that $I_B^p(yz)$, $f_B^p(yz)$, $I_B^p(yt)$, $f_B^p(yt) \notin [0, 1]$ and $I_B^n(yz)$, $f_B^n(yz)$, $I_B^n(yt)$, $f_B^n(yt) \notin [-1, 0]$. Hence $\overline{G}$ is not a bipolar single valued neutrosophic graph.

5. Conclusion and future work

Bipolar fuzzy graph theory has many applications in science and technology. A bipolar neutrosophic graph is a generalization of the notion bipolar fuzzy graph. We have introduced the idea of bipolar neutrosophic graph and operations on bipolar neutrosophic graphs. We have investigated the dominating and independent sets of certain graph products. Two applications of bipolar neutrosophic sets and bipolar neutrosophic graphs are studied in chemical industry and construction of radio channels. We are planning to extend our research of fuzzification to (1) Bipolar fuzzy rough graphs; (2) Bipolar fuzzy rough hypergraphs, (3) Bipolar fuzzy rough neutrosophic graphs, and (4) Decision support systems based on bipolar neutrosophic graphs.

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SOME PROPERTIES OF NEAR LEFT ALMOST RINGS BY USING IDEALS

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Abstract. The aim of this paper is to characterize near left almost rings by using ideals. We define a fully idempotent near left almost ring and discuss some of their properties. We then define prime, fully prime and irreducible ideals in a near left almost ring and explore some of their properties. Lastly we define M -system, P -system and I -system in a near left almost ring and discuss some of their properties.

Keywords: Ideal, fully idempotent near left almost ring, M -system, P -system and I -system.

1. Introduction

The notion of a left almost semigroup was introduced by Kazim and Naseeruddin [5]. A groupoid $(G, *)$ is said to be a left almost semigroup, abbreviated as LA-semigroup, if its elements satisfy the left invertive law, that is $(y * x) * z = (z * x) * y \forall x, y, z \in G$. In general a left almost semigroups doesn't satisfy

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associative and commutative laws but, it possesses the properties which are usually valid for associative and commutative algebraic structures. Further in 1993, Kamran [6] extended the concept of an LA-semigroup to a left almost group (abbreviated as LA-group). An LA-semigroup $(G, *)$ is said to be an LA-group if G has a left identity element with respect to “ $*$ ” and every element of G has a left inverse with respect to “ $*$ ”. It is non-associative and non-commutative algebraic structure and is the generalization of a commutative group. In 2006, Yusuf [13] introduced a left almost ring. A non-empty set R together with two binary operations “ $+$ ” and “ $\cdot$ ” is said to be a left almost ring (abbreviated as LA-ring) if $(R, +)$ is a left almost group, $(R, \cdot)$ is a left almost semigroup and distributive laws of “ $\cdot$ ” over “ $+$ ” hold. Further different peoples in [2], [3], [7], [8], [9] and [12] worked on an LA-ring and explored many useful properties of an LA-ring. In 2011, Shah, Rehman and Raees [11] gave the concept of a near left almost ring (abbreviated as nLA-ring). A non-empty set T together with two binary operations “ $+$ ” and “ $\cdot$ ” is said to be a near left almost ring if $(T, +)$ is a left almost group, $(T, \cdot)$ is a left almost semigroup and left distributive property of “ $\cdot$ ” over “ $+$ ” holds. It is the generalization of a left almost ring. The authors of [11] discussed many interesting properties of a near left almost ring. There is another interesting paper [10] in which the authors discussed direct sum of ideals in an nLA-ring. Further in the paper [4], the authors defined direct product of two near left almost rings and explored some useful properties. In this paper, we explore some properties of near left almost rings by using ideals.

2. Preliminaries

In this section, we discuss some of the basic definitions and fundamental results which will be used later. We start with the following definition of a near left almost subring (abbreviated as nLA-subring) which has been taken from [11].

Definition 2.1. *Let $(T, +, \cdot)$ be a near left almost ring and S a non-empty subset of T . If S is itself a near left almost ring under the same binary operations of T , then S is called a near left almost subring of T .*

Let us state some properties of near left almost subrings. The following result gives necessary and sufficient conditions for near left almost subrings. The result has been taken from [11].

Proposition 2.2. *Let S be a non-empty subset of a near left almost ring $(T, +, \cdot)$. Then, S is a near left almost subring of T if and only if $m - n \in S$ and $m \cdot n \in S$ for all $m, n \in S$.*

We are now going to define left (right) ideals. The following definition is taken from the source [11].

Definition 2.3. *A non-empty subset I of a near left almost ring T is called a left ideal of T if $a - b \in I$ and $t \cdot a \in I \forall a, b \in I$ and $\forall t \in T$ and I is called a right ideal of T if $a - b \in I$ and $(i + s) \cdot t - s \cdot t \in I \forall a, b, i \in I$ and $\forall s, t \in T$.*

Let us state some properties of ideals which have been taken from [11].

Proposition 2.4. *Assume that I and J are two left (right) ideals of a near left almost ring T . Then $I \cap J$ is also a left (right) ideal of T .*

We are now going to define sum and product of two ideals of a near left almost ring. The definitions have been taken from the paper [10].

Definition 2.5. *Let $(T, +, \cdot)$ be an nLA -ring. Let I and J be two ideals of T , then we define*

$$I + J = \{x + y : x \in I \text{ and } y \in J\}.$$

Definition 2.6. *If I and J are two ideals of an nLA -ring $(T, +, \cdot)$, then the product of I and J is the set*

$$IJ = \left\{ \sum_{\text{finite}} a_i b_i : a_i \in I, b_i \in J \right\}.$$

It follows from [10] that if I and J are ideals then both $I + J$ and IJ are ideals.

3. Basic properties of ideals

This section is concerned with the fundamental properties of ideals of a near left almost ring. The idea of these properties has come from the source [8] in which the author has done similar calculations for an LA -ring. We start with the result given below which is true for a left almost ring. Here we prove it for a near left almost ring.

Proposition 3.1. *Let $(T, +, \cdot)$ be a near left almost ring and “ e ” the left identity of T with respect to “ $\cdot$ ”. If I is a right ideal of T , then I is a left ideal of T .*

Proof. Suppose I is a right ideal, then I is a near left almost subring of T . Now let $t \in T$ and $c \in I$, then

$$tc = (et)c = (ct)e \in I \quad \because \text{by left invertive law}$$

It follows that I is a left ideal. □

Lemma 3.2. *Let $(T, +, \cdot)$ be a near left almost ring and “ e ” the left identity of T with respect to “ $\cdot$ ”. Let $a, c, x, y \in T$ be such that $ax = cy$, then $xa = yc$.*

Proof. Let $ax = cy$ for all $a, c, x, y \in T$. Now $xa = (ex)a = (ax)e = (cy)e = (ey)c = yc$. □

Proposition 3.3. *Let T be an LA -ring with left identity “ e ” and let N be a right ideal of T . Then the following results are equivalent:*

- (i) $N = T$,

(ii) $e \in N$,

(iii) N contain a unit,

(iv) N contains an element which is left invertible or right invertible.

Proof. (i) $\implies$ (ii) Let us suppose that $N = T$, then $e \in T \implies e \in N$.

(ii) $\implies$ (iii) Let $e \in N$. Then obviously N contains a unit.

(iii) $\implies$ (iv) Suppose N contains a unit. Then N must contain an element that must be left as well as right invertible.

(iv) $\implies$ (i) Let N contain an element r such that r is left invertible or right invertible. But by the above lemma in nLA-rings left invertibility implies right invertibility and right invertibility implies left invertibility. Thus r is a unit. $\square$

Proposition 3.4. Let M be a proper ideal of an nLA-ring T with left identity “ e ”, then $e \notin M$.

Proof. Suppose that $e \in M$. Now let $t \in T$, then $t = et \in MT \subseteq M \implies T \subseteq M$ but $M \subseteq T \implies M = T$. This is a contradiction. Hence $e \notin M$. $\square$

As we know that if T is an nLA-ring and N_1, N_2 are two left (right) ideals of T , then $N_1 \cap N_2$ is a left (right) ideal of T . The following example shows that if T is an nLA-ring and N_1, N_2 are two left (right) ideals of T , then $N_1 \cup N_2$ is not necessary to be a left (right) ideal of T .

Example 3.5. Let $T = \{r, s, t, u, v, w\}$. Define “ $+$ ” and “ $\cdot$ ” in T as follows:

$+$	r	s	t	u	v	w	$\cdot$	r	s	t	u	v	w
r	r	s	t	u	v	w	r	r	r	r	r	r	r
s	w	r	s	t	u	v	s	r	s	t	u	v	w
t	v	w	r	s	t	u	t	r	t	v	r	t	v
u	u	v	w	r	s	t	u	r	u	r	u	r	u
v	t	u	v	w	r	s	v	r	v	t	r	v	t
w	s	t	u	v	w	r	w	r	w	v	u	t	s

Then according to [11], $(T, +, \cdot)$ is an nLA-ring. Now let $N_1 = \{r, t, v\}$ and $N_2 = \{r, u\}$, then it is easy to see that N_1 and N_2 are ideals but $N_1 \cup N_2$ is not an ideal because $t, u \in N_1 \cup N_2$ but $t + u = s$ and $s \notin N_1 \cup N_2$.

The following results shows that when the union of left (right) ideals of an nLA-ring becomes a left (right) ideal.

Theorem 3.6. Let $\{Y_k : k \in \Lambda\}$ be a family of left (right) ideals of an nLA-ring T . If $Y_k \subseteq Y_{k+1}$ for all $k \in \Lambda$. Then $\cup_{k \in \Lambda} Y_k$ is an ideal of T .

Proof. For left ideal, let $x, y \in \cup_{k \in \Lambda} Y_k$, then there exist $i, j \in \Lambda$ such that $x \in Y_i, y \in Y_j$. Let $i > j$, then $x, y \in Y_i$. Now since Y_i is a left ideal, so it follows that $x - y \in Y_i$ for some $i \in \Lambda \implies x - y \in \cup_{k \in \Lambda} Y_k$. Now let $a \in T$, then $a \cdot x \in Y_i$ for some $i \in \Lambda$ because each Y_i is a left ideal. This implies that $a \cdot x \in \cup_{k \in \Lambda} Y_k \implies \cup_{k \in \Lambda} Y_k$ is a left ideal.

For right ideal, let $x, y \in \cup_{k \in \Lambda} Y_k$, then there exist $i, j \in \Lambda$ such that $x \in Y_i, y \in Y_j$. Let $i > j$, then $x, y \in Y_i$. Now since Y_i is a right ideal, so it follows that $x - y \in Y_i$ for some $i \in \Lambda \implies x - y \in \cup_{k \in \Lambda} Y_k$. Now let $a, b \in T$, then $(x + a)b - ab \in Y_i$ for some $j \in \Lambda$ because each Y_i is a right ideal. This implies that $(x + a)b - ab \in \cup_{k \in \Lambda} Y_k \implies \cup_{k \in \Lambda} Y_k$ is a right ideal of T .

This completes the proof. $\square$

4. Fully idempotent near left almost rings

This section is concerned with the fully idempotent near left almost ring. We start with the following definition and then explore some properties of the fully idempotent near left almost ring.

Definition 4.1. An ideal M of an nLA -ring T is said to be idempotent if $M^2 = M$. If every ideal of the nLA -ring T is idempotent, then T is said to be fully idempotent.

Let us describe some properties. The following result gives us equivalent conditions for a fully idempotent nLA -ring. The result is true in case of a ring. Here we prove it for an nLA -ring.

Proposition 4.2. If T is an nLA -ring, then the following conditions are equivalent:

- (i) T is fully idempotent.
- (ii) For each pair of ideals M, N of T , $M \cap N = MN$.

Proof. (i) $\implies$ (ii) As $MN \subseteq MT \subseteq M$ and also $MN \subseteq TN \subseteq N$, so it follows that $MN \subseteq M \cap N$. Conversely $M \cap N = (M \cap N)^2 \subseteq MN$. Hence $M \cap N = MN$.

(ii) $\implies$ (i) By (ii), we have $M \cap M = MM$ for each ideal M of T . Hence $M = MM = M^2$. This shows that T is fully idempotent. $\square$

5. Prime, fully prime and irreducible ideals

In this section, we discuss prime, fully prime and irreducible ideals in an nLA -ring. We define the said terms and then discuss some elegant and intersecting results which give relations among the said terms. Firstly we are going to define prime and semiprime ideals.

Definition 5.1. If T is an nLA-ring and M a two sided ideal of T , then M is known to be a prime ideal if for ideals I and J of T such that $IJ \subseteq M$ implies either $I \subseteq M$ or $J \subseteq M$. If each ideal of an nLA-ring T is prime, then it is called fully prime. M is said to be semiprime if for each ideal I of an nLA-ring T such that $I^2 \subseteq M \implies I \subseteq M$. If each ideal of an nLA-ring T is semiprime, then it is called fully semiprime.

It is clear from the above definition that every prime ideal is semiprime. We are now going to define totally ordered set of an nLA-ring under inclusion.

Definition 5.2. The set of two sided ideals of a near left almost ring T is known as totally ordered under inclusion if $\forall$ ideals M, N of T , either $M \subseteq N$ or $N \subseteq M$. The set of two sided ideals of T is shortly denoted by $\text{ideal}(T)$.

Let us state and prove some properties. The idea of these properties has come from the source [8] in which the author has done similar calculations for an LA-ring. Here we generalize his idea to an nLA-ring.

Theorem 5.3. Suppose " e " is the left identity of an nLA-ring T , then it is fully prime if and only if each ideal of T is idempotent and the set $\text{ideal}(T)$ is totally ordered under inclusion.

Proof. Suppose T is fully prime and let M be an ideal of T . Now clearly $M^2 \subseteq M$. Also $MM \subseteq M^2 \implies M \subseteq M^2$. Therefore $M^2 = M$ and hence M is idempotent. Now let M, N be ideals of T , then $MN \subseteq M$ and $MN \subseteq N \implies MN \subseteq M \cap N$. As M and N are prime ideals, so clearly $M \cap N$ is also a prime ideal of T . Therefore either $M \subseteq M \cap N$ or $N \subseteq M \cap N \implies$ either $M \subseteq N$ or $N \subseteq M$. Therefore the set $\text{ideal}(T)$ is totally ordered under inclusion.

On the other hand suppose each ideal of T is idempotent and $\text{ideal}(T)$ is totally ordered under inclusion. Assume I, J and M are some ideals of T with $IJ \subseteq M$ such that $I \subseteq J$. As I is idempotent, then $I = I^2 = II \subseteq IJ \subseteq M$. This implies that $I \subseteq M$. Hence T is fully prime. $\square$

Let us define strongly irreducible and irreducible ideals.

Definition 5.4. An ideal M of an nLA-ring T is strongly irreducible if for ideals I and J of T such that $I \cap J \subseteq M \implies$ either $I \subseteq M$ or $J \subseteq M$. M is called irreducible if for ideals I and J , $M = I \cap J \implies$ either $M = I$ or $M = J$.

It is clear from the above definition that every strongly irreducible ideal is irreducible. We now state and prove some results. The following results are true in case of a ring. Here we prove them for an nLA-ring.

Proposition 5.5. An ideal M of an nLA-ring T is a prime ideal if and only if M is semiprime and strongly irreducible.

Proof. If M is a prime ideal, then it must be semiprime. Let I and J be ideals of T and $I \cap J \subseteq M$. Now as $IJ \subseteq I \cap J \subseteq M$, so by primeness it follows that either $I \subseteq M$ or $J \subseteq M$. Hence I is strongly irreducible.

Conversely, suppose that M is an ideal of T which is both semiprime and strongly irreducible. If I and J are ideals of T , $IJ \subseteq M$, then $(I \cap J)^2 \subseteq IJ \subseteq M$. By semiprimeness, it follows that $I \cap J \subseteq M$. Thus by strongly irreducible it follows that either $I \subseteq M$ or $J \subseteq M$. Thus M is a prime ideal. $\square$

Before starting the next result firstly we are going to state a result which is called Zorn's Lemma and will be used in the next result. The result has been taken from [1].

Lemma 5.6. *If A is a partially ordered set such that every chain in A has an upper bound in A , then A has at least one maximal element.*

Proposition 5.7. *Let a be an element of a near left almost ring T and I a two sided ideal of T such that $a \notin I$. Then there exist an irreducible ideal M of T such that $I \subseteq M$ and $a \notin M$.*

Proof. If $\{H_i : i \in \Lambda\}$ is a chain of ideals in T containing I and $a \notin I$, then $\cup H_{i \in \Lambda}$ is an ideal of T . Therefore by the above lemma, the set of all ideals of T not containing a has a maximal element M . Assume $M = J \cap K$ where J and K are ideals of T properly containing M . Then by choice of M , $a \in J$ and $a \in K \implies a \in J \cap K = M$, which is a contradiction. Hence M must be irreducible. $\square$

Proposition 5.8. *Let T be an nLA -ring with left identity " e ". Then an ideal M of T such that $M \neq T$ is the intersection of all irreducible ideals containing it.*

Proof. As $e \notin M$, by the above proposition, there exists an irreducible ideal of T containing M . Suppose N is the intersection of all irreducible ideals of T containing M , then $M \subseteq N$. We now show that $N \subseteq M$. For this let $a \notin M$, then again by the above proposition, there exists an irreducible ideal H of T containing M but not containing a . Thus $a \notin N$ and hence $M = N$. $\square$

6. M -system, I -system and P -system in near left almost rings

In this section, we study M -system, I -system and P -system in an nLA -ring containing a left identity element. We define the said terms and relate these concepts through some results. We start with the following definitions.

Definition 6.1. *Let T be an nLA -ring with left identity " e " and $\emptyset \neq S \subseteq T$. Then S is called an M -system if $\forall x, y \in S$ there exist t in T such that $x(ty) \in S$.*

Definition 6.2. Suppose “ e ” is a left identity element of an nLA -ring T , then the principal left ideal generated by an element x of T is defined by

$$\langle x \rangle = Tx = \{tx : t \in T\}.$$

Definition 6.3. Let T be an nLA -ring and N a left ideal of T . Then N is called a quasi-prime ideal if for left ideals I and J of T such that $IJ \subseteq N \implies$ either $I \subseteq N$ or $J \subseteq N$ and N is called quasi-semiprime if $I^2 \subseteq N \implies I \subseteq N$, where I is a left ideal of T . A left ideal N of an nLA -ring T is known to be quasi-strongly irreducible if for left ideals I and J of T such that $I \cap J \subseteq N \implies$ either $I \subseteq N$ or $J \subseteq N$. N is known to be quasi-irreducible if for left ideals I and J , $N = I \cap J \implies$ either $N = I$ or $N = J$.

Let us state and prove some properties. The idea of these properties has come from the source [8] in which the author has done similar calculations for an LA -ring. We generalize his idea to an nLA -ring. The result given below gives us equivalent conditions for a quasi-prime ideal.

Lemma 6.4. Let T be an nLA -ring having left identity “ e ” and N a left ideal of T . Then the under below conditions are equivalent:

- (i) N is quasi-prime.
- (ii) If I and J are any left ideals of T such that $IJ = \langle IJ \rangle \subseteq N \implies$ either $I \subseteq N$ or $J \subseteq N$.
- (iii) Let $I \not\subseteq N$ and $J \not\subseteq N$, then $IJ \not\subseteq N$, where I and J are any left ideals of T .
- (iv) If m_1, m_2 are element of T such that $m_1 \notin N$ and $m_2 \notin N$, then $\langle m_1 \rangle \langle m_2 \rangle \not\subseteq N$.
- (v) If m_1, m_2 are elements of T satisfying $m_1(Tm_2) \subseteq N$, then either $m_1 \in N$ or $m_2 \in N$.

Proof. (i) $\iff$ (ii) Let N be quasi-prime. By definition if $IJ = \langle IJ \rangle \subseteq N$, then it implies that either $I \subseteq N$ or $J \subseteq N$ for all left ideal I and J of T . Conversely suppose that $IJ = \langle IJ \rangle \subseteq N$ implies that $I \subseteq N$ or $J \subseteq N$. Then this implies that N is quasi-prime.

(ii) $\iff$ (iii) is trivial.

(i) $\implies$ (iv) Let $\langle m_1 \rangle \langle m_2 \rangle \subseteq N$ then either $\langle m_1 \rangle \subseteq N$ or $\langle m_2 \rangle \subseteq N \implies$ either $m_1 \in N$ or $m_2 \in N$. So, by contrapositive statement $m_1 \notin N$ and $m_2 \notin N$, then $\langle m_1 \rangle \langle m_2 \rangle \not\subseteq N$.

(i) $\iff$ (v) Let $m_1(Tm_2) \subseteq N$, then $T(m_1(Tm_2)) \subseteq TN \subseteq N$. Now consider

$$\begin{aligned} T(m_1(Tm_2)) &= (TT)(m_1(Tm_2)) \\ &= (Tm_1)(T(Tm_2)) \quad \because \text{by medial law} \end{aligned}$$

$$\begin{aligned}
&= (Tm_1)((TT)(Tm_2)) \\
&= (Tm_1)((m_2T)(TT)) \quad \because \text{by paramedial law} \\
&= (Tm_1)((TT)m_2) \quad \because \text{by left invertive law} \\
&= (Tm_1)(Tm_2) \subseteq N.
\end{aligned}$$

Since Tm_1 and Tm_2 are left ideals, so either $m_1 \in N$ or $m_2 \in N$.

Conversely suppose $IJ \subseteq N$ where I and J are any left ideals of T . Let $I \not\subseteq N$, then there exist $a \in I$ such that $a \notin N$. Now $\forall b \in J$, we have $a(Tb) \subseteq I(TJ) \subseteq IJ \subseteq N \implies J \subseteq N$ and hence N is a quasi-prime ideal of T . $\square$

Further we have the following result which gives equivalent condition for quasi-prime ideal and for M -system.

Theorem 6.5. *Let “e” be a left identity element of an nLA-ring T and N a left ideal of T , then N is quasi-prime if and only if $T \setminus N$ is an M -system.*

Proof. Suppose that N is a quasi-prime ideal. Assume $x, y \in T \setminus N \implies x \notin N$ and $y \notin N$. So by the above lemma, $x(Ty) \not\subseteq N \implies$ there exist $t \in T$ such that $x(ty) \notin N \implies x(ty) \in T \setminus N \implies T \setminus N$ is an M -system.

Conversely suppose $T \setminus N$ is an M -system. Let $x, y \in T$ be such that $x(Ty) \subseteq N$. Suppose that $x \notin N$ and $y \notin N \implies x, y \in T \setminus N$. Since $T \setminus N$ is an M -system, so there exist $t \in T$ such that $x(ty) \in T \setminus N \implies x(Ty) \not\subseteq N$, which is impossible. Hence $x \in N$ or $y \in N \implies N$ is a quasi-prime ideal. $\square$

We are now going to define a P -system.

Definition 6.6. *Let T be an nLA-ring with left identity “e” and $\emptyset \neq S \subseteq T$. Then S is said to be a P -system if $\forall x \in S$, there exist $r \in T$ such that $x(rx) \in S$.*

It is clear from the definition that every M -system is an P -system. Let us state and prove some properties. Again the idea of these properties has come from the source [8]. The following result gives us equivalent conditions for a quasi-semiprime ideal.

Lemma 6.7. *Let T be an nLA-ring having left identity “e” and N a left ideal of T , then the under below conditions are equivalent:*

- (i) N is quasi-semiprime.
- (ii) $I^2 = \langle I^2 \rangle \subseteq N \implies I \subseteq N$, where I is any left ideal of T .
- (iii) For every left ideal I of T such that $I \not\subseteq N$ implies that $I^2 \not\subseteq N$.
- (iv) Let $x \in T$ such that $\langle x \rangle^2 \subseteq N$, then this implies that $x \in N$.
- (v) $\forall y \in T$ such that $y(Ty) \subseteq N$ implies that $y \in N$.

Proof. (i) $\iff$ (ii) Let N be quasi-semiprime. Let $I^2 = \langle I^2 \rangle \subseteq N$, then by definition $I \subseteq N$, where I is any left ideal of T . Conversely, as $I^2 = \langle I^2 \rangle \subseteq N \implies I \subseteq N \implies N$ is quasi-semiprime.

(ii) $\iff$ (iii) is trivial.

(i) $\implies$ (iv) Let $\langle x \rangle^2 \subseteq N$. But by hypothesis N is quasi-semiprime $\implies \langle x \rangle \subseteq N \implies x \in N$.

(iv) $\implies$ (ii) For all left ideal I of T , let $I^2 = \langle I^2 \rangle \subseteq N$. If $x \in I$, then by (iv), $\langle x \rangle^2 \subseteq N \implies x \in N \implies I \subseteq N$.

(i) $\iff$ (v) clear. $\square$

Further the following result gives us equivalent conditions for quasi-semiprime ideal and for P -system.

Proposition 6.8. *Let “ e ” be a left identity element of an nLA -ring T and N a left ideal of T , then N is quasi-semiprime if and only if $T \setminus N$ is a P -system.*

Proof. Suppose N is quasi-semiprime and assume $u \in T \setminus N$. Let us suppose that there does not exist an element $t \in T$ such that $u(tu) \in T \setminus N \implies u(tu) \in N$. As N is quasi-semiprime, so by the above lemma, $u \in N$. This is a contradiction. Thus there exists $t \in T$ such that $u(tu) \in T \setminus N$. Hence $T \setminus N$ is a P -system.

Conversely, suppose that $T \setminus N$ is a P -system. Let $u \in T$ be such that $u(Tu) \subseteq N$. Suppose that $u \notin N \implies u \in T \setminus N$. Since $T \setminus N$ is a P -system, so there exist $t \in T$ such that $u(tu) \in T \setminus N \implies u(Tu) \not\subseteq N$ which is a contradiction. Hence $u \in N$ and so by the above lemma, N is quasi-semiprime $\square$

We are now going to define an I -system.

Definition 6.9. *Let T be an nLA -ring with left identity “ e ” and $\emptyset \neq S \subseteq T$. Then S is said to be an I -system if $(\langle x \rangle \cap \langle y \rangle) \cap S \neq \emptyset \forall x, y \in S$.*

Let us state and prove some properties. Again the idea of these properties has come from the source [8]. The following result gives us equivalent condition for quasi-strongly irreducible ideal and for I -system.

Proposition 6.10. *Let “ e ” be a left identity element of an nLA -ring T . Let N be a left ideal of T , then the following conditions are equivalent:*

(i) N is quasi-strongly irreducible.

(ii) For $x, y \in T$ such that $\langle x \rangle \cap \langle y \rangle \subseteq N \implies x \in N$ or $y \in N$.

(iii) $T \setminus N$ is an I -system.

Proof. (i) $\implies$ (ii) Let $x, y \in T$ such that $\langle x \rangle \cap \langle y \rangle \subseteq N$, then by definition of a quasi-strongly irreducible ideal either $\langle x \rangle \subseteq N$ or $\langle y \rangle \subseteq N$. Thus it follows that either $x \in N$ or $y \in N$.

(ii) $\implies$ (iii) Let $x, y \in T \setminus N$ and let us suppose that $(\langle x \rangle \cap \langle y \rangle) \cap T \setminus N = \emptyset \implies \langle x \rangle \cap \langle y \rangle \subseteq N \implies$ either $x \in N$ or $y \in N$. It follows that our supposition is wrong and so $(\langle x \rangle \cap \langle y \rangle) \cap T \setminus N \neq \emptyset$.

(iii) $\implies$ (i) Let I and J be ideals of T such that $I \cap J \subseteq N$. Suppose $I \not\subseteq N$ and $J \not\subseteq N$, then there exists an element $x \in I$ and $y \in J$ such that $x \in I \setminus N$ and $y \in J \setminus N \implies x, y \in T \setminus N$, so by hypothesis it follows that $(\langle x \rangle \cap \langle y \rangle) \cap T \setminus N \neq \emptyset \implies$ there exist an element $z \in \langle x \rangle \cap \langle y \rangle$ such that $z \in T \setminus N \implies z \in \langle x \rangle \cap \langle y \rangle \subseteq I \cap J$ such that $z \notin N \implies I \cap J \not\subseteq N$, which is a contradiction. Hence either $I \subseteq N$ or $J \subseteq N$ and so N is quasi-strongly irreducible. $\square$

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SOME PROPERTIES OF A NEW KIND OF DOWNWARD SETS IN CERTAIN BANACH SPACES*

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Abstract. Let X be a Banach lattice with strong unit. In this paper, we give some characterizations of certain kind of downward sets in the sequence space $\ell^\infty(X)$. Further some results on best approximation of those sets are presented.

Keywords: Downwards sets, proximinal sets, Banach lattices.

1. Introduction

A vector lattice is an ordered vector space such that $x \vee y = \sup\{x, y\}$ and $x \wedge y = \inf\{x, y\}$ exist for all $x, y \in X$. Vector lattices are also called Riesz spaces or linear lattices, [9]. The most obvious example of a vector lattice is the set of real numbers, $\mathbb{R}$ with all the usual operations. A normed linear lattice X is a real normed vector lattice such that

$$|x| \leq |y| \Rightarrow \|x\| \leq \|y\| \text{ for any } x, y \in X,$$

where, $|x| := \sup\{x, -x\}$ for each $x \in X$. If $(X, \leq)$ is a normed ordered vector space, recall that an element in X , denoted by 1 , is called a strong unit if $\|1\| = 1$ and for each $x \in X$, there exists $0 < \lambda \in \mathbb{R}$ such that $x \leq \lambda 1$. Using the strong unit 1 a norm on X is defined by

$$\|x\| = \inf\{\lambda > 0 : |x| \leq \lambda 1\}$$

for all $x \in X$. It is clear that for all $x \in X$,

$$(1.1) \quad |x| \leq \|x\| 1.$$

Using (1.1), the closed unit ball of X , $B(x, r) = \{y \in X : \|y - x\| \leq r\}$, with center x and radius r can be written as

$$(1.2) \quad B(x, r) = \{y \in X : x - r1 \leq y \leq x + r1\}.$$

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Certain kind of sets in Banach lattices that is called downward sets plays an important role in some part of mathematical economics and game theory. Recall that a subset W of a Banach lattice X is said to be downward, if $(w \in W, x \leq w)$ implies that $x \in W$. The set of the form $\{w \in \mathbb{R}^n : w \leq x\}$, where $x \in \mathbb{R}^n$ is a simple example of a downward set. For more on Banach Lattices we refer the reader to [7, 8, 9].

Convex sets in normed linear spaces and their best approximation properties has many important applications in science. However, since convexity in somehow is a restrictive assumption, so there is a need to study the best approximation by elements of some kind of non convex sets. In [6], Rubinov and Singer developed a theory of best approximation by elements of so-called normal sets in the finite-dimensional coordinate space $\mathbb{R}^n$ endowed with the max-norm. Martinez-Legaz, Rubionv and Singer in [3] have developed a theory of best approximation of downward subsets of the space $\mathbb{R}^n$. While the problem of best approximation by elements of downward sets in a Banach lattice was studied in [4, 5], the problem of best approximation in vector valued functions such as $\ell^p(X)$, $1 \leq p \leq \infty$, where X is a Banach lattice has never been considered.

It is the aim of this paper to give some characterization of some kind of downward sets in the space of bounded sequences $\ell^\infty(X)$ endowed with the max norm in terms of a coupling function. Further we study the problem of best approximation of those kind of sets. Indeed we precisely study proximity of $\ell^\infty(W)$ in $\ell^\infty(X)$, where X is a Banach lattice and W is a downward subset of X .

Throughout of this paper, X is a Banach Lattice with a strong unit and $\mathbb{N}$ is the set of all positive integers. Moreover the interior, the closure and the boundary of the subset W of X will be denoted by $intW$, clW and $bd(W)$ respectively.

2. Characterization of downward sets in $\ell^\infty(X)$

For a Banach space X , let $\ell^\infty(X)$ denotes the space of all sequences $x = (x_i)$, $x_i \in X$, with $\|x\|_\infty = \sup_{i \in \mathbb{N}} \|x_i\| < \infty$. If W is a downward subset of X , by $\ell^\infty(W)$ we denote the subset of all sequences $w = (w_i)$, $w_i \in W$, with $\|w\|_\infty = \sup_{i \in \mathbb{N}} \|w_i\| < \infty$. In this section we characterize some kind of downward sets in the sequence space $\ell^\infty(X)$ in terms of a coupling function. We start by defining a partial order relation " $\leq$ " on $\ell^\infty(X)$, where X is a Banach lattice with strong unit "1" as follows:

Definition 1. For $x = (x_n), y = (y_n) \in \ell^\infty(X)$, we say that $x \leq y$ if and only if $x_n \leq y_n$ for all n .

Proposition 2. A relation $\leq$ is a partial order in $\ell^\infty(X)$.

Proof. Follows from the definition. □

Proposition 3. If X is a Banach lattice with strong unit 1, then $\ell^\infty(X)$ is a Banach lattice with strong unit $(1, 1, \dots, 1, \dots)$.

Proof. Let $(x_n) \in \ell^\infty(X)$. Then, $\sup_{n \in \mathbb{N}} \|x_n\| < \infty$. Since for all $n, x_n \in X$, and 1 is the strong unit of X , there exists $\lambda_n > 0$, such that $x_n \leq \lambda_n 1$. With no loss of generality, we can choose λ_n to be $\|x_n\|$. If $\lambda = \sup_{n \in \mathbb{N}} \lambda_n = \|(x_n)\|_\infty < \infty$, then, for all $n, x_n \leq \|x_n\| 1 \leq (\sup_{n \in \mathbb{N}} \|x_n\|) 1 = \lambda 1$. Hence, $(x_n) \leq (\lambda 1, \lambda 1, \dots, \lambda 1, \dots) = \lambda(1, 1, \dots, 1, \dots) = \lambda 1$. $\square$

Proposition 4. *W is a downward set in X if and only if $\ell^\infty(W)$ is a downward set in $\ell^\infty(X)$.*

Proof. Let $x = (x_n) \in \ell^\infty(W)$ and $w = (w_n) \in \ell^\infty(X)$, such that $w \leq x$. Then for all $n, w_n \leq x_n$. But W is downward set in X and $x_n \in W$ for all n , it follow that $w_n \in W$ for all n , and $w \in \ell^\infty(W)$.

Conversely, let $w \in W, x \in X$, such that $x \leq w$, consider the sequence $u = (x, x, x, \dots) \in \ell^\infty(W)$, $v = (w, w, w, \dots) \in \ell^\infty(X)$. Since $\ell^\infty(W)$ is a downward set and $u \leq v$ it follows that $x \in W$. $\square$

Proposition 5. *If G is a closed subset of X , then $\ell^\infty(G)$ is a closed subset of $\ell^\infty(X)$.*

Proof. Let $(x_n^k), k \geq 1$ be a sequence of $\ell^\infty(G)$, such that $(x_n^k) \rightarrow (x_n)$. Since for all $n, x_n^k \in G$ and G closed, it follows that, $x_n \in G$ for all n . Hence $(x_n) \in \ell^\infty(G)$. $\square$

Theorem 6. *Let W be a closed downward subset of X and $(x_n) \in \ell^\infty(X)$. Then the following are true:*

- (a) *If $(x_n) \in \ell^\infty(W)$, then $(x_n - \lambda_n 1) \in \text{int}(\ell^\infty(W))$, for all $\epsilon > 0$, and all $(\lambda_n) \in \ell^\infty(\mathbb{R})$ with $\inf_{n \in \mathbb{N}} \lambda_n \geq \epsilon$.*
- (b) *$\text{int}(\ell^\infty(W)) = \{(x_n) \in \ell^\infty(X) : (x_n + \epsilon 1) \in \ell^\infty(W) \text{ for some } \epsilon > 0\}$.*

Proof. (a) For $\epsilon > 0$ and $(x_n) \in \ell^\infty(W)$, let, $(\lambda_n) \in \ell^\infty(\mathbb{R})$ with $\inf(\lambda_n) \geq \epsilon$ and

$$V = \{(y_n) \in \ell^\infty(X) : \|(y_n) - (x_n - \lambda_n 1)\|_\infty < \epsilon\},$$

be an open neighborhood for $(x_n - \lambda_n 1)$ in $\ell^\infty(X)$. Then, for all, n

$$\|y_n - (x_n - \lambda_n 1)\| \leq \sup_n \|y_n - (x_n - \lambda_n 1)\| = \|(y_n) - (x_n - \lambda_n 1)\|_\infty < \epsilon.$$

Hence, $|y_n - (x_n - \lambda_n 1)| \leq \|y_n - (x_n - \lambda_n 1)\| < \epsilon$. Using (1.2)

$$\begin{aligned} -\epsilon 1 &< y_n - (x_n - \lambda_n 1) < \epsilon 1 \\ -\epsilon 1 + x_n - \lambda_n 1 &< y_n < \epsilon 1 + (x_n - \lambda_n 1) = x_n + (\epsilon - \lambda_n) 1 < x_n. \end{aligned}$$

Since W is a downward set it follows that $y_n \in W$ for all n . Consequently $(y_n) \in \ell^\infty(W)$ and $V \subset \ell^\infty(W)$. Hence $(x_n - \lambda_n 1) \in \text{int}(\ell^\infty(W))$. Notice that, (λ_n) can be chosen so that $\lambda_n = \epsilon \forall n$.

(b) Let $(x_n) \in \text{int}(\ell^\infty(W))$. Then there exists $\epsilon_o > 0$, such that the closed ball $B((x_n), \epsilon_o) \subseteq \ell^\infty(W)$. That is

$$B((x_n), \epsilon_o) = \left\{ (y_n) \in \ell^\infty(X) : \sup_n \|y_n - x_n\| \leq \epsilon_o \right\} \subseteq \ell^\infty(W).$$

Hence

$$\begin{aligned} B((x_n), \epsilon_o) &= \{(y_n) \in \ell^\infty(X) : \|y_n - x_n\| \leq \sup_n \|y_n - x_n\| \\ &= \|(y_n) - (x_n)\|_\infty \leq \epsilon_o\}. \end{aligned}$$

Consequently using (1.2) we get

$$B((x_n), \epsilon_o) = \{(y_n) \in \ell^\infty(X) : x_n - \epsilon_o 1 \leq y_n \leq x_n + \epsilon_o 1\} \subseteq \ell^\infty(W),$$

and $(\epsilon_o 1 + x_n) \in \ell^\infty(W)$.

Conversely, suppose that there exists $\epsilon > 0$, such that $(x_n + \epsilon 1) \in \ell^\infty(W)$. Then, by part (a), we get $(x_n) = (x_n + \epsilon 1 - \epsilon 1) \in \text{int}(\ell^\infty(W))$, which completes the proof. $\square$

Corollary 7. *Let W be a closed downward subset of X and $(w_n) \in \ell^\infty(W)$. Then, $(w_n) \in \text{bd}(\ell^\infty(W))$ if and only if $(\lambda 1 + w_n) \notin \ell^\infty(W)$ for all $\lambda > 0$.*

Proof. Suppose that $(\lambda 1 + w_n) \in \ell^\infty(W)$ for some $\lambda > 0$. Then

$$(w_n) = (w_n + \lambda 1 - \lambda 1) \in \text{int}(\ell^\infty(W)),$$

which is a contradiction, since $(w_n) \in \text{bd}(\ell^\infty(W))$. Hence, $(\lambda 1 + w_n) \notin \ell^\infty(W)$, for all $\lambda > 0$.

Conversely, suppose that $(w_n) \in \text{int}(\ell^\infty(W))$. Then by Theorem 6, $(\lambda 1 + w_n) \in \ell^\infty(W)$, for some $\lambda > 0$. This is a contradiction, since $(\lambda 1 + w_n) \notin \ell^\infty(W)$. Hence $(w_n) \notin \text{int}(\ell^\infty(W))$. But $(w_n) \in \ell^\infty(W)$, it follows that $(w_n) \in \text{bd}(\ell^\infty(W))$. $\square$

Now, we will define what we call it a coupling ψ function that will be used later to characterize some kind of downward sets as follows:

$$(2.1) \quad \psi : \ell^\infty(X) \times \ell^\infty(X) \rightarrow \ell^\infty(\mathbb{R})$$

$$\psi((x_n), (y_n)) = (\Phi(x_n, y_n)),$$

where, $\Phi(x_n, y_n) = \sup\{\lambda \in \mathbb{R} : \lambda 1 \leq x_n + y_n\}$, for all $(x_n), (y_n) \in \ell^\infty(X)$. Since 1 is a strong unit of X , it follows that the set $\{\lambda \in \mathbb{R} : \lambda 1 \leq x_n + y_n\}$ is non-empty and bounded above (by the number $\|x_n + y_n\|$). Clearly this set is closed.

For each $(y_n) \in \ell^\infty(X)$, define the function $\psi_{(y_n)} : \ell^\infty(X) \rightarrow \ell^\infty(\mathbb{R})$ by

$$(2.2) \quad \psi_{(y_n)}((x_n)) = \psi((x_n), (y_n)) = (\Phi(x_n, y_n)).$$

Proposition 8. *The function ψ satisfies the following properties.*

- (1) *For all $(x_n), (y_n) \in \ell^\infty(X)$, $-\infty \leq \|\psi((x_n), (y_n))\|_\infty \leq \|(x_n) + (y_n)\|_\infty$.*
- (2) *$(\Phi(x_n, y_n)1) \leq (x_n + y_n)$ for all $(x_n), (y_n) \in \ell^\infty(X)$.*
- (3) *$\psi((x_n), (y_n)) = \psi((y_n), (x_n))$ for all $(x_n), (y_n) \in \ell^\infty(X)$.*
- (4) *$\psi((x_n), (-x_n)) = (0, 0, \dots, 0, \dots)$ for all $(x_n) \in \ell^\infty(X)$.*

Proof.

$$(1) \quad -\infty \leq \|\psi((x_n), (y_n))\|_\infty = \sup_n \|\Phi(x_n, y_n)\| \leq \sup_n \|x_n + y_n\| = \|(x_n + y_n)\|_\infty.$$

$$(2) \quad (\Phi(x_n, y_n)1) = ((\sup\{\lambda \in \mathbb{R} : \lambda 1 \leq x_n + y_n\})1) \leq (x_n + y_n).$$

$$(3) \quad \begin{aligned} \psi((x_n), (y_n)) &= (\Phi(x_n, y_n)) = (\sup\{\lambda \in \mathbb{R} : \lambda 1 \leq x_n + y_n\}) \\ &= (\sup\{\lambda \in \mathbb{R} : \lambda 1 \leq y_n + x_n\}) = \psi((y_n), (x_n)). \end{aligned}$$

$$(4) \quad \psi((x_n), (-x_n)) = (\sup\{\lambda \in \mathbb{R} : \lambda 1 \leq x_n - x_n\}) = (0, 0, \dots, 0, \dots).$$

□

A function $f : \ell^\infty(X) \rightarrow \ell^\infty(\mathbb{R})$ is said to be increasing, whenever $(x_n), (y_n) \in \ell^\infty(X)$, $[(x_n) \geq (y_n) \Rightarrow f((x_n)) \geq f((y_n))]$, and plus-homogeneous if

$$(f((x_n) + (\alpha_n)1)) = f((x_n)) + (\alpha_n) \text{ for all } (x_n) \in \ell^\infty(X) \text{ and } (\alpha_n) \in \ell^\infty(\mathbb{R}).$$

A function $f : \ell^\infty(X) \rightarrow \ell^\infty(\mathbb{R})$ is called topical if this function is increasing and plus-homogeneous.

Lemma 9. *The function $\psi_{(y_n)}$ defined by (2.2) is topical.*

Proof. (1) Let $(x_n), (z_n) \in \ell^\infty(X)$ with $(x_n) \leq (z_n)$. Then, since $x_n \leq z_n$ for all n , $\{\lambda \in \mathbb{R} : \lambda 1 \leq x_n + y_n\} \subset \{\lambda \in \mathbb{R} : \lambda 1 \leq z_n + y_n\}$. Hence,

$$\begin{aligned} \psi_{(y_n)}((x_n)) &= \psi((x_n), (y_n)) \\ &= (\sup\{\lambda \in \mathbb{R} : \lambda 1 \leq x_n + y_n\}) \\ &\leq (\sup\{\lambda \in \mathbb{R} : \lambda 1 \leq z_n + y_n\}) \\ &= \psi_{(y_n)}((z_n)). \end{aligned}$$

(2) Let $(x_n) \in \ell^\infty(X)$ and $(\alpha_n) \in \ell^\infty(\mathbb{R})$ be arbitrary. Then

$$\begin{aligned} \psi_{(y_n)}((x_n) + (\alpha_n)1) &= \psi((x_n) + (\alpha_n)1, (y_n)) \\ &= (\sup\{\lambda \in \mathbb{R} : \lambda 1 \leq x_n + \alpha_n 1 + y_n\}) \\ &= (\sup\{\lambda \in \mathbb{R} : (\lambda - \alpha_n)1 \leq x_n + y_n\}). \end{aligned}$$

Let $\lambda - \alpha_n = \beta$. Then $\lambda = \beta + \alpha_n$. Hence

$$\begin{aligned}\psi_{(y_n)}((x_n) + (\alpha_n)1) &= (\sup\{\beta + \alpha_n \in \mathbb{R} : \beta 1 \leq x_n + y_n\}) \\ &= (\sup\{\beta \in \mathbb{R} : \beta 1 \leq x_n + y_n, \}) + (\alpha_n) \\ &= \psi((x_n), (y_n)) + (\alpha_n) \\ &= \psi_{(y_n)}((x_n)) + (\alpha_n).\end{aligned}$$

□

Theorem 10. *The function $\psi_{(y_n)}$ is Lipschitz continuous in the ℓ^∞ norm.*

Proof. Let $(x_n), (z_n) \in \ell^\infty(X)$ be arbitrary. Since $|x_n - z_n| \leq \|(x_n) - (z_n)\|_\infty 1$, it follows that

$$z_n - \|(x_n) - (z_n)\|_\infty \leq x_n \leq z_n + \|(x_n) - (z_n)\|_\infty.$$

In view of (Lemma 9) we have

$$\psi_{(y_n)}((z_n)) - (\|(x_n) - (z_n)\|_\infty 1) \leq \psi_{(y_n)}((x_n)) \leq \psi_{(y_n)}((z_n)) + (\|(x_n) - (z_n)\|_\infty 1),$$

and hence

$$(2.3) \quad \|\psi_{(y_n)}((x_n)) - \psi_{(y_n)}((z_n))\|_\infty \leq \|(x_n) - (z_n)\|_\infty.$$

Therefore, $\psi_{(y_n)}$ is Lipschitz continuous. □

Corollary 11. *The function ψ defined in (2.1) is continuous in the ℓ^∞ norm.*

Proof. It follows directly from (2.3). □

Now we prove one of the main results in this paper

Theorem 12. *Let W be a closed downward subset of X and $(y_k^\circ) \in \ell^\infty(W)$. If $S = \{k \in \mathbb{N}, y_k^\circ \in bd(W)\} \neq \emptyset$, then,*

- (a) $(y_n^\circ) \in bd(\ell^\infty(W))$.
- (b) $\Phi(w_k, -y_k^\circ) \leq 0$, for all $k \in S$ and all $(w_n) \in \ell^\infty(W)$.

Proof. (a) Let $(y_n^\circ) \in \ell^\infty(W)$ and $B(y_n^\circ, \epsilon)$ be any neighborhood of (y_n°) . Then if

$$B(y_n^\circ, \epsilon) = \{(x_n) \in \ell^\infty(X) : \|(x_n) - (y_n^\circ)\|_\infty < \epsilon\},$$

it follows that for all n ,

$$\|x_n - y_n^\circ\| < \|(x_n) - (y_n^\circ)\|_\infty = \sup_n \|x_n - y_n^\circ\| < \epsilon.$$

So, for $k \in S$, $\|x_k - y_k^\circ\| < \epsilon$. Since $y_k^\circ \in bd(W)$, any neighborhood of y_k° contains a point $u_k \in W$ and a point $z_k \notin W$. Now consider the sequence u given by, $u =$

$(y_1^\circ, y_2^\circ, \dots, y_{k-1}^\circ, u_k, y_{k+1}^\circ, \dots) \in \ell^\infty(W)$ and, $z = (y_1^\circ, y_2^\circ, \dots, y_{k-1}^\circ, z_k, y_{k+1}^\circ, \dots) \notin \ell^\infty(W)$. Then,

$$\begin{aligned} \|u_k - y_k^\circ\| &< \epsilon \text{ and } \|z_k - y_k^\circ\| < \epsilon \Rightarrow \|u_k\| \leq \|y_k^\circ\| + \epsilon \text{ and} \\ \|z_k - y_k^\circ\| &< \epsilon \Rightarrow \|z_k\| \leq \|y_k^\circ\| + \epsilon \end{aligned}$$

and so

$$\|u_k\| \leq \|y_k^\circ\| + \epsilon \text{ and } \|z_k\| \leq \|y_k^\circ\| + \epsilon.$$

Therefore, $\|u\|_\infty, \|z\|_\infty \leq \|(y_n^\circ)\|_\infty + \epsilon < \infty$. Hence,

$$\begin{aligned} \phi &\neq B(y_n^\circ, \epsilon) \cap \ell^\infty(W) \supseteq \{u\} \\ \phi &\neq B(y_n^\circ, \epsilon) \cap (\ell^\infty(W))^c \supseteq \{z\}, \end{aligned}$$

and $(y_n^\circ) \in bd(\ell^\infty(W))$.

(b) Let $(w_n) \in \ell^\infty(W)$ such that $\Phi(w_k, -y_k^\circ) = \sup\{\lambda \in \mathbb{R} : \lambda 1 \leq w_k - y_k^\circ\} > 0$ for some $k \in S$. Then there exists $\lambda_0 > 0$ such that $\lambda_0 1 \leq w_k - y_k^\circ$. This means that $\lambda_0 1 + y_k^\circ \leq w_k$. Since W is a downward set and $w_k \in W$, it follows that $\lambda_0 1 + y_k^\circ \in W$. Therefore, by (Proposition 3.1 in [4]) we have, $y_k^\circ \in \text{int}(W)$. This is a contradiction. $\square$

Corollary 13. Let W be a closed downward subset of X , $y_n^\circ \in bd(W)$ for all n . Then $\psi((w_n), (-y_n^\circ)) \leq 0$, for all $(w_n) \in \ell^\infty(W)$.

Proof. Since $y_n^\circ \in bd(W)$, for all n , by Theorem 12, $\Phi(w_n, -y_n^\circ) < 0$. Hence $\psi((w_n), (-y_n^\circ)) \leq 0$. $\square$

In the following two theorems we give some characterizations of the downward set $\ell^\infty(W)$ in terms of the function ψ .

Theorem 14. Let W be a subset of X and ψ be the coupling function of (2.1). Then the following are equivalent:

- (1) $\ell^\infty(W)$ is a downward set.
- (2) For each $(x_n) \in \ell^\infty(X) \setminus \ell^\infty(W)$, there exist $\phi \neq S \subseteq \mathbb{N}$, $\Phi(w_k, -x_k) < 0, \forall k \in S$ and $(w_n) \in \ell^\infty(W)$.
- (3) For each $(x_n) \in \ell^\infty(X) \setminus \ell^\infty(W)$, there exists $(L_n) \in \ell^\infty(X)$ and $\phi \neq S \subseteq \mathbb{N}$,

$$\Phi(w_k, L_k) < 0 \leq \Phi(x_k, L_k), \forall k \in S \text{ and } (w_n) \in \ell^\infty(W).$$

Proof. (1) $\Rightarrow$ (2) Let $\ell^\infty(W)$ be downward set and $(x_n) \in \ell^\infty(X) \setminus \ell^\infty(W)$. Suppose that for all $n \in \mathbb{N}$, $\Phi(w_n, -x_n) \geq 0$. Then by Proposition 8(2), $0 \leq (\Phi(w_n, -x_n) 1) \leq (w_n - x_n)$. Since W is downward set and $w_n \in W$, it follows that for all $n, x_n \in W$, which is a contradiction.

Hence $S = \{k, \Phi(w_k, -y_k^\circ) < 0\} \neq \emptyset$.

(2) $\Rightarrow$ (3). Assume that (2) holds and $(x_n) \in \ell^\infty(X) \setminus \ell^\infty(W)$ is arbitrary. Then, by hypothesis, there exists $\phi \neq S \subseteq \mathbb{N}$, such that $\Phi(w_k, -x_k) < 0, \forall k \in S$.

Now, let $(L_n) = (-x_n) \in \ell^\infty(X)$. Using proposition 8 (2), we have for each $(w_n) \in \ell^\infty(W)$ and $k \in S$.

$$\Phi(w_k, L_k) = \Phi(w_k, -x_k) < 0 = \Phi(x_k, -x_k) = \Phi(x_k, L_k)$$

(3) $\Rightarrow$ (1). Suppose that $\ell^\infty(W)$ is not a downward set. Then there exists $(w_n^\circ) \in \ell^\infty(W)$ and $(x_n^\circ) \in \ell^\infty(X) \setminus \ell^\infty(W)$ with $(x_n^\circ) \leq (w_n^\circ)$. Using (3), there exists $(L_n) \in \ell^\infty(X)$ and $\phi \neq S \subseteq \mathbb{N}$, such that for all $k \in S$.

$$(2.4) \quad \Phi(w_k^\circ, L_k) < 0 \leq \Phi(x_k^\circ, L_k)$$

But ψ is increasing, we have $\psi_{(L_n)}((x_n^\circ)) \leq \psi_{(L_n)}((w_n^\circ))$. This mean

$$\Phi(x_n^\circ, L_n) \leq \Phi(w_n^\circ, L_n),$$

for all $n \in \mathbb{N}$ and this is a contradiction to (2.4). $\square$

Theorem 15. Let ψ be the function defined by (2.1). Then for a subset W of X the following are equivalent:

- (1) $\ell^\infty(W)$ is a closed downward subset of $\ell^\infty(X)$.
- (2) $\ell^\infty(W)$ is downward, and for each $(x_n) \in \ell^\infty(X)$ the set

$$H = \{(\lambda_n) \in \ell^\infty(\mathbb{R}) : (x_n + \lambda_n 1) \in \ell^\infty(W)\}$$

is closed in $\ell^\infty(\mathbb{R})$.

(3) For each $(x_n) \in \ell^\infty(X) \setminus \ell^\infty(W)$, there exists $(L_n) \in \ell^\infty(X)$ and $\phi \neq S \subseteq \mathbb{N}$, such that,

$$\Phi(w_k, L_k) < 0 < \Phi(x_k, L_k),$$

for all $(w_n) \in \ell^\infty(W)$ and for all $k \in S$.

(4) For each $(x_n) \in \ell^\infty(X) \setminus \ell^\infty(W)$, there exists $(L_n) \in \ell^\infty(X)$ and $\phi \neq S \subseteq \mathbb{N}$ such that,

$$\sup_{(w_n) \in \ell^\infty(W)} \Phi(w_k, L_k) < \Phi(x_k, L_k).$$

Proof. (1) $\Rightarrow$ (2). Let $(x_n) \in \ell^\infty(X)$, $(\lambda_n^k) \in \ell^\infty(\mathbb{R})$, $(x_n + \lambda_n^k 1) \in \ell^\infty(W)$ ($k = 1, 2, \dots$) and $(\lambda_n^k) \rightarrow (\lambda_n)$ in ℓ^∞ norm. Then,

$$\begin{aligned} \left\| (x_n + \lambda_n^k 1) - (x_n + \lambda_n 1) \right\|_\infty &= \left\| (\lambda_n^k - \lambda_n) 1 \right\|_\infty \\ &= \sup_n \left| \lambda_n^k - \lambda_n \right| \rightarrow 0 \text{ as } k \rightarrow +\infty. \end{aligned}$$

Since $(x_n + \lambda_n^k 1) \in \ell^\infty(W)$ and $\ell^\infty(W)$ is closed, it follows that $(x_n + \lambda_n 1) \in \ell^\infty(W)$. Hence, $(\lambda_n) \in H$ and H is a closed subset of $\ell^\infty(\mathbb{R})$.

(2) $\Rightarrow$ (3). Let $(x_n) \in \ell^\infty(X) \setminus \ell^\infty(W)$ be arbitrary. We claim that there exists $(\lambda_n^\circ) > (0)$ such that $(-\lambda_n^\circ) \notin H$. Indeed, if $(-\lambda_n) \in H$ for all $(\lambda_n) > (0, 0, \dots, 0, \dots)$. Then due to the closedness of H , we have $(0, 0, \dots, 0, \dots) \in H$. This implies $(x_n) = (x_n + 0 \cdot 1) \in \ell^\infty(W)$. This is a contradiction.

Now, let $(L_n) = (\lambda_n^\circ 1 - x_n) \in \ell^\infty(X)$. We show that, $\exists \phi \neq S \subseteq \mathbb{N}$ such that $\Phi(w_k, L_k) < 0$, for all $k \in S$ and for all $(w_n) \in \ell^\infty(W)$. Assume that there exists $(w_n^\circ) \in \ell^\infty(W)$ such that $\psi((w_n^\circ), (L_n)) \geq (0)$. Then by proposition 8 (2), for all n ,

$$0 \leq \Phi(w_n^\circ, L_n)1 \leq w_n^\circ + L_n$$

and so $w_n^\circ \geq -L_n = x_n - \lambda_n^\circ 1$. Since $\ell^\infty(W)$ is downward and $(w_n^\circ) \in \ell^\infty(W)$, it follows that $(x_n - \lambda_n^\circ 1) \in \ell^\infty(W)$, and consequently $-\lambda_n \in H$. This is a contradiction. Hence, $\exists S \neq \phi$,

$$\Phi(w_k, L_k) < 0 \text{ for all } (w_n) \in \ell^\infty(W), \text{ for all } k \in S.$$

On the other hand, for all $k \in S$

$$\begin{aligned} \Phi(x_k, L_k) &= \sup\{\lambda \in \mathbb{R} : \lambda 1 \leq x_k + L_k\} \\ &= \sup\{\lambda \in \mathbb{R} : \lambda 1 \leq x_k + \lambda_k^\circ 1 - x_k = \lambda_k^\circ 1\} \\ &= \sup\{\lambda \in \mathbb{R} : (\lambda - \lambda_k^\circ)1 \leq 0\}. \end{aligned}$$

Let $\lambda - \lambda_k^\circ = \alpha_k$. Then $\lambda = \lambda_k^\circ + \alpha_k$. Hence

$$\begin{aligned} \Phi(w_k, L_k) &= \sup\{\alpha_k + \lambda_k^\circ \in \mathbb{R} : \alpha_k 1 \leq 0\} \\ &= \sup\{\alpha_k \in \mathbb{R} : \alpha_k 1 \leq 0\} + \lambda_k^\circ \\ &= \lambda_k^\circ > 0. \end{aligned}$$

(3) $\Rightarrow$ (4). By (3) for each $(x_n) \in \ell^\infty(X) \setminus \ell^\infty(W)$, there exists $(L_n) \in \ell^\infty(X)$ and $\phi \neq S \subseteq \mathbb{N}$

$$\Phi(w_k, L_k) < 0 < \Phi(x_k, L_k),$$

for all $(w_n) \in \ell^\infty(W)$. Then

$$\sup_{(w_n) \in \ell^\infty(W)} \Phi(w_k, L_k) < \Phi(x_k, L_k), \text{ for all } k \in S.$$

(4) $\Rightarrow$ (1). Suppose that $\ell^\infty(W)$ is not a downward set. Then there exists $(w_n^\circ) \in \ell^\infty(W)$ and $(x_n^\circ) \in \ell^\infty(X) \setminus \ell^\infty(W)$ with $(x_n^\circ) \leq (w_n^\circ)$. By hypothesis, there exists $(L_n) \in \ell^\infty(X)$ and $\phi \neq S \subseteq \mathbb{N}$,

$$\sup_{(w_n) \in \ell^\infty(W)} \Phi(w_k, L_k) < \Phi(x_k^\circ, L_k),$$

for all $k \in S$. Since $\psi(\cdot, (L_n)) = \psi_{(L_n)}(\cdot)$ is increasing, it follows that

$$\psi((x_n^\circ), (L_n)) \leq \psi((w_n^\circ), (L_n))$$

Hence, for all $k \in S$

$$\Phi(x_k^\circ, L_k) \leq \sup_{(w_n) \in \ell^\infty(W)} \Phi(w_k, L_k) < \Phi(x_k^\circ, L_k).$$

This is a contradiction. Hence, $\ell^\infty(W)$ is a downward set.

Finally, assume that $\ell^\infty(W)$ is not closed. Then there exists a sequence $\{w_n^m\}_{m \geq 1} \subset \ell^\infty(W)$ and $(x_n^\circ) \in \ell^\infty(X) \setminus \ell^\infty(W)$ such that

$$\|w_n^m - x_n^\circ\|_\infty \longrightarrow 0 \text{ as } m \longrightarrow +\infty.$$

Since $(x_n^\circ) \in \ell^\infty(X) \setminus \ell^\infty(W)$, by hypothesis, there exists $(L_n) \in \ell^\infty(X)$ and $\phi \neq S \subseteq \mathbb{N}$, such that

$$\sup_{(w_n) \in \ell^\infty(W)} \Phi(w_k, L_k) < \Phi(x_k^\circ, L_k),$$

for all $k \in S, \forall (w_n) \in \ell^\infty(W)$. Hence

$$\Phi((w_k^m), (L_k)) \leq \sup_{(w_n) \in \ell^\infty(W)} \Phi((w_k), (L_k)),$$

for all $m, \forall k \in S$. By continuity of $\psi_{L_n}(\cdot, (L_n)) = (\Phi_{L_n}(\cdot, L_n))$ it follows that

$$\Phi((x_k^\circ), (L_k)) \leq \sup_{(w_n) \in \ell^\infty(W)} \Phi((w_k), (L_k)),$$

for all $k \in S$. This is a contradiction. $\square$

3. Best approximation of $\ell^\infty(W)$ in $\ell^\infty(X)$

A subset W in a Banach space X is said to be proximal if there corresponds to each $x \in X$ at least one $w \in W$ such that $\|x - w\| = \text{dist}(x, W) = \inf_{z \in W} \|x - z\|$. A necessary condition for proximality of a subset W of a normed linear space X is closeness (see, [2]). The set (possibly empty) of best approximations to x from W is defined by: $P_W(x) = \{w \in W : \|x - w\| = d(x, W)\}$.

In this section we prove that if W is a closed downward set in X , then $\ell^\infty(W)$ is proximal in $\ell^\infty(X)$ and the set $P_{\ell^\infty(W)}((x_n))$ of all of points of best approximation of the point $x = (x_n) \in \ell^\infty(X)$ in $\ell^\infty(W)$ has minimal element.

Theorem 16. *Let W be a closed downward subset of X . Then $\ell^\infty(W)$ is proximal in $\ell^\infty(X)$.*

Proof. Let $(x_n^\circ) \in \ell^\infty(X) \setminus \ell^\infty(W)$ be arbitrary and

$$\begin{aligned} d((x_n^\circ), \ell^\infty(W)) &= \inf_{(w_n) \in \ell^\infty(W)} \|(x_n^\circ) - (w_n)\|_\infty \\ &= \inf_{(w_n) \in \ell^\infty(W)} \sup_n \|x_n^\circ - w_n\| = r > 0. \end{aligned}$$

This implies for all $\epsilon > 0$, there exists $(w_{n\epsilon}) \in \ell^\infty(W)$ such that

$$\|(x_n^\circ) - (w_{n\epsilon})\|_\infty < r + \epsilon.$$

Consequently using (1.2) we get

$$\begin{aligned} B((x_n^\circ), r + \epsilon) &= \left\{ (w_{n\epsilon}) \in \ell^\infty(X) : \begin{aligned} &\|x_n^\circ - w_{n\epsilon}\| \leq \sup_n \|x_n^\circ - w_{n\epsilon}\| \\ &= \|(x_n^\circ) - (w_{n\epsilon})\|_\infty \leq r + \epsilon \end{aligned} \right\} \\ &= \{(w_{n\epsilon}) \in \ell^\infty(X) : x_n^\circ - (r + \epsilon)1 \leq w_{n\epsilon} \leq x_n^\circ + (r + \epsilon)1\}. \end{aligned}$$

If $(w_n^\circ) = (x_n^\circ - r1)$, then

$$\|(x_n^\circ) - (w_n^\circ)\|_\infty = \sup_n \|x_n^\circ - w_n^\circ\| = \sup_n \|r\| = r.$$

Hence $(w_n^\circ - \epsilon 1) = (x_n^\circ - r1 - \epsilon 1) \leq (w_{n\epsilon})$. Since W is closed downward set and $(w_{n\epsilon}) \in \ell^\infty(W)$, it follows that $(w_n^\circ - \epsilon 1) \in \ell^\infty(W)$, for all $\epsilon > 0$ and $w_n^\circ \in W$. So $(w_n^\circ) \in P_{\ell^\infty(W)}((x_n^\circ))$. $\square$

Remark 17. We prove that for each $(x_n^\circ) \in \ell^\infty(X) \setminus \ell^\infty(W)$, the set $P_{\ell^\infty(W)}((x_n^\circ))$ contains $(w_n^\circ) = (x_n^\circ - r1)$ with $r = d((x_n^\circ), \ell^\infty(W))$. If $(x_n^\circ) \in \ell^\infty(W)$, then $(w_n^\circ) = (x_n^\circ)$ and $P_{\ell^\infty(W)}((x_n^\circ)) = \{(w_n^\circ)\}$.

Theorem 18. Let W be a closed downward subset of X and $(x_n^\circ) \in \ell^\infty(X)$.

Then there exists the least element $(w_n^\circ) = \min P_{\ell^\infty(W)}((x_n^\circ))$ of the set $P_{\ell^\infty(W)}((x_n^\circ))$, namely, $(w_n^\circ) = (x_n^\circ - r1)$, where $r = d((x_n^\circ), \ell^\infty(W))$.

Proof. If $(x_n^\circ) \in \ell^\infty(W)$, then the result holds. Assume that $(x_n^\circ) \notin \ell^\infty(W)$ and $(w_n^\circ) = (x_n^\circ - r1)$. Then by (Remark 17), we have

$$(w_n^\circ) = (x_n^\circ - r1) \in P_{\ell^\infty(W)}((x_n^\circ)).$$

Since applying (1.2) and the equality $\|(x_n^\circ) - (w_n)\|_\infty = r$, we get

$$\begin{aligned} B((x_n^\circ), r) &= \{(x_n) \in \ell^\infty(X) : \|(x_n) - (x_n^\circ)\|_\infty \leq r\} \\ &= \left\{ (x_n) \in \ell^\infty(X) : \sup_n \|x_n - x_n^\circ\| \leq r \right\}. \end{aligned}$$

Consequently for all n ,

$$\|x_n - x_n^\circ\| \leq \|(x_n) - (x_n^\circ)\|_\infty = \sup_n \|x_n - x_n^\circ\| \leq r,$$

and using (1.1) we have

$$-r1 \leq x_n - x_n^\circ \leq r1 \Rightarrow x_n^\circ - r1 \leq x_n \leq x_n^\circ + r1.$$

Hence, $w_n^\circ = x_n^\circ - r1 \leq x_n$, and so $(w_n^\circ) \leq (x_n)$ for all $(x_n) \in B((x_n^\circ), r)$, and this implies (w_n°) is the least element of the closed ball $B((x_n^\circ), r)$.

Now, let $(w_n) \in P_{\ell^\infty(W)}(x_n^\circ)$ be arbitrary. Then, $\|(x_n^\circ) - (w_n)\| = r$ and so $(w_n) \in B((x_n^\circ), r)$. Therefore, $(w_n) \geq (w_n^\circ)$. Hence, (w_n°) is the least element of the set $P_{\ell^\infty(W)}(x_n^\circ)$. $\square$

Corollary 19. *Let W be a closed downward subset of X , $(x_n^\circ) \in \ell^\infty(X)$ and $(w_n^\circ) = \min P_{\ell^\infty(W)}(x_n^\circ)$. Then, $(w_n^\circ) \leq (x_n^\circ)$.*

Proof. Since $(w_n^\circ) = \min P_{\ell^\infty(W)}(x_n^\circ)$. Then by Theorem 18, we get $(w_n^\circ) = (x_n^\circ - r1) \leq (x_n^\circ)$. $\square$

Corollary 20. *Let W be a closed downward subset of X and $(x_n) \in \ell^\infty(X)$ be arbitrary. Then $d((x_n), \ell^\infty(W)) = \min\{\lambda \geq 0, (x_n - \lambda 1) \in \ell^\infty(W)\}$.*

Proof. Let $A = \{\lambda \geq 0, (x_n - \lambda 1) \in \ell^\infty(W)\}$. If $(x_n) \in \ell^\infty(W)$, then $(x_n - 0.1) = (x_n) \in \ell^\infty(W)$, and so $\min(A) = 0 = d((x_n), \ell^\infty(W))$. Suppose that $(x_n) \notin \ell^\infty(W)$. Then $r = d((x_n), \ell^\infty(W)) > 0$. Let $\lambda > 0$ be arbitrary such that $(x_n - \lambda 1) \in \ell^\infty(W)$. Thus, we have

$$\lambda = \|(\lambda 1)\|_\infty = \|(x_n - x_n - \lambda 1)\|_\infty = \sup_n \|x_n - (x_n - \lambda 1)\| \geq d((x_n), \ell^\infty(W)) = r.$$

Since by (Theorem 18), $(x_n - r1) \in \ell^\infty(W)$, it follows that $r \in A$. Hence $\min(A) = r$. $\square$

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ON PÁL-TYPE INTERPOLATION II

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Abstract. In this paper, we study the convergence of Pál-type interpolation on two sets of non-uniformly distributed zeros on the unit circle, which are obtained by projecting vertically the nodes of the real line.

Keywords: Pál-type Interpolation, explicit forms, convergence.

1. Introduction

In 1975, L.G. Pál [5] introduced a different type of Hermite interpolation by prescribing the function values at one set of points, whereas its first order derivative values at another set of points. He obtained a unique polynomial of degree at most $2n - 1$ satisfying the interpolating properties.

After that, many mathematicians [1,3] have taken such problem on a different set of nodes *viz.* finite interval, infinite interval or unit circle. Author [2] had also considered some Pál-type Interpolation on the real line and unit circle and established the convergence theorem for them.

In this paper, we consider two pairwise disjoint set nodes $\xi_n = \{t_k\}_{k=0}^{2n-1}$ and $Z_n = \{z_k\}_{k=1}^{2n}$, which are vertically projected zeros of two different polynomials onto the unit circle. On these sets of nodes, we consider the Pál-type Interpolation and obtain a convergence theorem for that interpolatory polynomial. In section 2, we give some preliminaries and in section 3, we describe the problem and in sections 4 and 5, we give explicit representation and estimation of interpolatory polynomials respectively. In section 6, we give the convergence of such polynomials.

2. Preliminaries

In this section, we shall give some well known results, which we shall use in our present paper.

The differential equation satisfied by $\Pi_n(x)$ is

$$(2.1) \quad (1-x^2) \Pi_n''(x) + n(n-1) \Pi_n(x) = 0,$$

$$(2.2) \quad W(z) = K_n \Pi_n(x) z^n,$$

$$(2.3) \quad H(z) = K_n^* \Pi_n'(x) z^{n-1},$$

$$(2.4) \quad R(z) = (z^2 - 1) H(z),$$

we shall require the fundamental polynomials of Lagrange interpolation based on Z_n and ξ_n , respectively

$$(2.5) \quad L_k(z) = \frac{R(z)}{(z-t_k) R'(t_k)}, k = 0(1)2n-1,$$

$$(2.6) \quad l_k(z) = \frac{W(z)}{(z-z_k) W'(z_k)}, k = 1(1)2n,$$

$$(2.7) \quad W'(z_k) = \frac{K_n}{2} \Pi_n'(x_k) (z_k^2 - 1) z_k^{n-2}, k = 1(1)2n,$$

$$(2.8) \quad W'(t_k) = K_n n \Pi_n(u_k) t_k^{n-1} k = 0(1)2n-1,$$

$$(2.9) \quad H'(z_k) = K_n^* (n-1) \Pi_n'(x_k) z_k^{n-2}, k = 1(1)2n,$$

$$(2.10) \quad H'(t_k) = \frac{K_n^*}{2} \Pi_n''(u_k) (t_k^2 - 1) t_k^{n-1}, k = 0(1)2n-1,$$

$$(2.11) \quad I_{1j}(z) = \int_0^z t^{n-j-1} W(t) dt, j = 0, 1$$

such that $I_{1j}(-1) = (-1)^{n-j} I_{1j}(1)$.

We shall also use the following well-known inequalities (see [6])

$$(2.12) \quad |P_n(x)| \leq 1,$$

$$(2.13) \quad |\Pi_n(x)| \leq \left(\frac{2n}{\pi} \right)^{\frac{1}{2}},$$

$$(2.14) \quad (1-x^2)^{\frac{1}{4}} |P_n(x)| \leq \left(\frac{2}{n\pi} \right)^{\frac{1}{2}}.$$

If u_k be the zeros of $P_n'(x)$, then

$$(2.15) \quad P_n(u_k) > \frac{1}{\sqrt{8\pi k}}.$$

Let $x_k = \cos \theta_k$, ($k = 1, 2, \dots, n$) be the zeros of n^{th} Legendre polynomial $P_n(x)$, with $1 > x_1 > x_2 > \dots > -1$, then

$$(2.16) \quad \begin{cases} (1-x_k^2) \geq k^2 n^{-2}, & k = 1, 2, \dots, \left[\frac{n}{2} \right] \\ (1-x_k^2) \geq (n-k+1)^2 n^{-2}, & k = \left[\frac{n}{2} \right] + 1, \dots, n \end{cases}$$

$$(2.17) \quad \begin{cases} |P_n'(x_k)| \geq ck^{-\frac{3}{2}} n^2, & k = 1, 2, \dots, \left[\frac{n}{2} \right] \\ |P_n'(x_k)| \geq c(n-k+1)^{-\frac{3}{2}} n^2, & k = \left[\frac{n}{2} \right] + 1, \dots, n. \end{cases}$$

For more details, see [6].

3. The problem

Let $\{t_k\}_{k=0}^{2n-1}$ and $\{z_k\}_{k=1}^{2n}$ be two disjoint set of nodes obtained by projecting vertically the zeros of $(1-x^2)\Pi'_n(x)$ and $\Pi_n(x)$ onto the unit circle respectively, where

$$(3.1) \quad \Pi_n(x) = (1-x^2)P'_{n-1}(x), n = 2, 3, \dots$$

$P_{n-1}(x)$ stands for $(n-1)^{th}$ Legendre polynomial.

Here we are interested to determine the convergence of interpolatory polynomial satisfying the conditions :

$$(3.2) \quad \begin{cases} R_n(t_k) = \alpha_k, & k = 0(1)2n-1 \\ R'_n(z_k) = 0, & k = 1(1)2n, \end{cases}$$

where α'_k s are arbitrary given complex numbers.

4. Explicit representation of interpolatory polynomial

We shall write $Q_n(z)$ satisfying (3.2) as

$$(4.1) \quad R_n(z) = \sum_{k=0}^{2n-1} \alpha_k A_k(z),$$

where $A_k(z)$ are unique polynomial of degree at most $4n-1$ determined by the following conditions:

For $k = 0(1)2n-1$

$$(4.2) \quad \begin{cases} A_k(t_j) = \delta_{jk}, & j = 0(1)2n-1 \\ A'_k(z_j) = 0, & j = 1(1)2n. \end{cases}$$

Theorem 4.1. For $k = 0(1)2n-1$

$$(4.3) \quad \begin{aligned} A_k(z) &= L_k(z) \frac{W(z)}{W(t_k)} \\ &+ \frac{z^{-n+1}H(z)}{(t_k^2-1)W(t_k)H'(z_k)} \{N_k(z) + b_{10}I_{10}(z) + b_{11}I_{11}(z)\}, \end{aligned}$$

where

$$(4.4) \quad N_k(z) = - \int_0^z z^{n-1}(z^2-1) \frac{W'(z) + c_k W(z)}{(z-t_k)} dz, \quad c_k = - \frac{W'(t_k)}{W(t_k)},$$

$$(4.5) \quad b_{10} = - \frac{N_k(1) + (-1)^{n+1} N_k(-1)}{2I_{10}(1)},$$

$$(4.6) \quad b_{11} = - \frac{N_k(1) + (-1)^n N_k(-1)}{2I_{11}(1)}.$$

Proof. Consider (4.3), we can obtain (4.4)-(4.6) owing to conditions (4.2). The theorem follows. $\square$

5. Estimation of fundamental polynomials

Lemma 5.1. *Let $L_k(z)$ be given by (2.5) Then we have*

$$(5.1) \quad \max_{|z|=1} \sum_{k=0}^{2n-1} |L_k(z)| \leq c \log n,$$

where c is a constant independent of n and z .

Lemma 5.2. *Let $l_k(z)$ be given by (2.6) Then, we have*

$$(5.2) \quad \max_{|z|=1} \sum_{k=1}^{2n} |l_k(z)| \leq c \log n,$$

where c is a constant independent of n and z .

Lemma 5.3. *Let $A_k(z)$ be defined in Theorem 4.1, then for $|z| \leq 1$*

$$(5.3) \quad \sum_{k=0}^{2n-1} |A_k(z)| \leq cn^{\frac{1}{2}} \log n,$$

where c is a constant independent of n and z .

Proof. Using Lemma 5.2 and inequalities (2.11)-(2.16), we get (5.3). $\square$

6. Convergence

In this section, we prove the following theorem:

Theorem 5.1. *Let $f(z)$ be continuous for $|z| \leq 1$ and analytic for $|z| < 1$, then the sequence $\{R_n\}$ defined by*

$$(6.1) \quad R_n(z) = \sum_{k=0}^{2n-1} f(t_k) A_k(z)$$

converges uniformly to $f(z)$.

To prove (6.1), we shall need the following:

Remark. Let $f(z)$ be continuous for $|z| \leq 1$ and $f' \in Lip_{\frac{1}{2}}$, then the sequence $\{R_n\}$ converges uniformly to $f(z)$ provided

$$(6.2) \quad \omega_2(f, n^{-1}) = O(n^{-\frac{3}{2}}).$$

Jackson's Inequality. Let $f(z)$ be continuous for $|z| \leq 1$ and analytic for $|z| < 1$, then there exists a polynomial $F_n(z)$ of degree at most $4n-1$ satisfying

$$(6.3) \quad |F_n(z) - f(z)| \leq c\omega_2(f, n^{-1}), z = e^{i\theta}, (0 \leq \theta < 2\pi).$$

Also an inequality due to O. Kiš [4] viz.

$$(6.4) \quad \left| F_n^{(m)}(z) \right| \leq cn^m \omega_2(f, n^{-1}), \quad \text{for } m \in I^+.$$

Proof of Theorem 5.1. Let $z = e^{i\theta}$ ($0 \leq \theta < 2\pi$), using (6.1)-(6.4) and Lemma 5.3, the theorem follows. $\square$

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COMMON FIXED POINT THEOREMS FOR FUZZY MAPPINGS IN b -METRIC SPACE

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Abstract. In this paper we establish some fixed point results for fuzzy mapping in a complete b -metric space. Our results unify, extend and generalize several results in the existing literature. An example is also given to support our results.

Keywords: Fixed point, complete b -metric space, fuzzy mapping, Hausdorff metric space.

1. Introduction and preliminaries

Fixed point theory plays an important role in the various fields of mathematics. It provides very important tools for finding the existence and uniqueness of the solutions. The Banach contraction theorem has an important role in fixed point theory and became very popular due to iterations which can be easily implemented on the computers. The idea of fuzzy set was first laid down by Zadeh [9]. Later on Weiss [8] and Butnariu [3] obtained many fixed point results for fuzzy mapping in metric spaces. Afterward, Heilpern [4] initiated the idea of fuzzy contraction mappings and proved a fixed point theorem for fuzzy contrac-

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tion mappings which is a fuzzy analogue of Nadler's fixed point theorem for multivalued mappings [6]. Further work on fuzzy mappings can be seen in [7].

In this paper, we obtain a fixed point and a common fixed point for fuzzy mapping in complete b -metric space. An example is also given which supports the obtained results.

Here, the obtained results for fuzzy mapping in b -metric space under certain contractive conditions are helpful for Hausdorff dimensions computing which are helpful in high energy physics to understand e^∞ -spaces. In high energy physics these results are also helpful for solving the arising geometric problems due to the involvement of fuzzy sets.

Definition 1.1 ([2]). Let X be any nonempty set and $b \geq 1$ be any given real number. A function $d : X \times X \rightarrow \mathbb{R}^+$ is called a b -metric, if it satisfies the following conditions for all $x, y, z \in X$:

- i) $d(x, y) = 0$ if and only if $x = y$,
- ii) $d(x, y) = d(y, x)$,
- iii) $d(x, z) \leq b[d(x, y) + d(y, z)]$.

Then, the pair (X, d) is called a b -metric space.

Definition 1.2 ([5]). Let (X, d) be a b -metric space and $\{x_n\}$ be a sequence in X . Then,

- i) $\{x_n\}$ is called a convergent sequence if and only if there exists $x \in X$, such that for all $\epsilon > 0$ there exists $n(\epsilon) \in \mathbb{N}$ such that for all $n \geq n(\epsilon)$, we have $d(x_n, x) < \epsilon$. So, we write $\lim_{n \rightarrow \infty} x_n = x$.
- ii) $\{x_n\}$ is called a Cauchy sequence if and only if for all $\epsilon > 0$ there exists $n(\epsilon) \in \mathbb{N}$ such that for each $m, n \geq n(\epsilon)$, we have $d(x_n, x_m) < \epsilon$.
- iii) (X, d) is called complete if every Cauchy sequence in X converges to a point $x \in X$ such that $d(x, x) = 0$.

Definition 1.3 ([6]). Let (X, d) be a metric space. We define the Hausdorff metric on $CB(X)$ induced by d . Then,

$$H(A, B) = \max\left\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(A, y)\right\},$$

for all $A, B \in CB(X)$, where $CB(X)$ denotes the family of closed and bounded subsets of X and

$$d(x, B) = \inf\{d(x, a) : a \in B\},$$

for all $x \in X$.

A fuzzy set in X is a function with domain X and values in $[0, 1]$, $F(X)$ is the collection of all fuzzy sets in X . If A is a fuzzy set and $x \in X$, then the function value $A(x)$ is called the grade of membership of x in A . The α -level set of fuzzy set A , is denoted by $[A]_\alpha$, and defined as:

$$\begin{aligned} [A]_\alpha &= \{x : A(x) \geq \alpha\}, \quad \text{where } \alpha \in (0, 1], \\ [A]_0 &= \overline{\{x : A(x) > 0\}}. \end{aligned}$$

Let X be any nonempty set and Y be a metric space. A mapping T is called a fuzzy mapping, if T is a mapping from X into $F(Y)$. A fuzzy mapping T is a fuzzy subset on $X \times Y$ with membership function $T(x)(y)$. The function $T(x)(y)$ is the grade of membership of y in $T(x)$. For convenience, we denote the α -level set of $T(x)$ by $[Tx]_\alpha$ instead of $[T(x)]_\alpha$ [1].

Definition 1.4 ([1]). A point $x \in X$ is called a fuzzy fixed point of a fuzzy mapping $T : X \rightarrow F(X)$ if there exists $\alpha \in (0, 1]$ such that $x \in [Tx]_\alpha$.

Lemma 1.5 ([1]). Let A and B be nonempty closed and bounded subsets of a metric space (X, d) . If $a \in A$, then

$$d(a, B) \leq H(A, B).$$

Lemma 1.6 ([1]). Let A and B be nonempty closed and bounded subsets of a metric space (X, d) and $0 < \alpha \in \mathbb{R}$. Then, for $a \in A$, there exists $b \in B$ such that

$$d(a, b) \leq H(A, B) + \alpha.$$

2. Main results

Now, we present our main results.

Theorem 2.1. Let (X, d) be a complete b -metric space with constant $b \geq 1$. Let $T : X \rightarrow F(X)$ be a fuzzy mapping and for $x \in X$, there exist $\alpha(x) \in (0, 1]$ satisfying the following condition:

$$\begin{aligned} H([Tx]_{\alpha(x)}, [Ty]_{\alpha(y)}) &\leq a_1 d(x, [Tx]_{\alpha(x)}) + a_2 d(y, [Ty]_{\alpha(y)}) + a_3 d(x, [Ty]_{\alpha(y)}) \\ (2.1) \quad &+ a_4 d(y, [Tx]_{\alpha(x)}) + a_5 d(x, y) \\ &+ a_6 \frac{d(x, [Tx]_{\alpha(x)})(1 + d(x, [Tx]_{\alpha(x)}))}{1 + d(x, y)}, \end{aligned}$$

for all $x, y \in X$. Also, $a_i \geq 0$, where $i = 1, 2, \dots, 6$ with $ba_1 + a_2 + b(b+1)a_3 + b(a_5 + a_6) < 1$ and $\sum_{i=1}^6 a_i < 1$. Then, T has a fixed point.

Proof. Let x_0 be any arbitrary point in X , such that $x_1 \in [Tx_0]_{\alpha(x_0)}$. Then, by Lemma 1.6 there exists $x_2 \in [Tx_1]_{\alpha(x_1)}$, such that

$$\begin{aligned}
 d(x_1, x_2) &\leq H([Tx_0]_{\alpha(x_0)}, [Tx_1]_{\alpha(x_1)}) + (a_1 + ba_3 + a_5 + a_6) \\
 &\leq a_1 d(x_0, [Tx_0]_{\alpha(x_0)}) + a_2 d(x_1, [Tx_1]_{\alpha(x_1)}) + a_3 d(x_0, [Tx_1]_{\alpha(x_1)}) \\
 &\quad + a_4 d(x_1, [Tx_0]_{\alpha(x_0)}) + a_5 d(x_0, x_1) + \\
 &\quad a_6 \frac{d(x_0, [Tx_0]_{\alpha(x_0)})(1 + d(x_0, [Tx_0]_{\alpha(x_0)}))}{1 + d(x_0, x_1)} + (a_1 + ba_3 + a_5 + a_6) \\
 &\leq a_1 d(x_0, x_1) + a_2 d(x_1, x_2) + ba_3 [d(x_0, x_1) + d(x_1, x_2)] \\
 &\quad + a_5 d(x_0, x_1) + a_6 d(x_0, x_1) + (a_1 + ba_3 + a_5 + a_6) \\
 d(x_1, x_2) &\leq \frac{a_1 + ba_3 + a_5 + a_6}{1 - (a_2 + ba_3)} d(x_0, x_1) \\
 (2.2) \quad &+ \frac{(a_1 + ba_3 + a_5 + a_6)}{1 - (a_2 + ba_3)}.
 \end{aligned}$$

Let

$$\tau = \frac{(a_1 + ba_3 + a_5 + a_6)}{1 - (a_2 + ba_3)} < \frac{1}{b}.$$

Then by (2.2), we have

$$d(x_1, x_2) \leq \tau d(x_0, x_1) + \tau.$$

Again by Lemma 1.6, $x_3 \in [Tx_2]_{\alpha(x_2)}$ such that

$$\begin{aligned}
 d(x_2, x_3) &\leq H([Tx_1]_{\alpha(x_1)}, [Tx_2]_{\alpha(x_2)}) + \frac{(a_1 + ba_3 + a_5 + a_6)^2}{1 - (a_2 + ba_3)} \\
 &\leq a_1 d(x_1, [Tx_1]_{\alpha(x_1)}) + a_2 d(x_2, [Tx_2]_{\alpha(x_2)}) + a_3 d(x_1, [Tx_2]_{\alpha(x_2)}) \\
 &\quad + a_4 d(x_2, [Tx_1]_{\alpha(x_1)}) + a_5 d(x_1, x_2) + \\
 &\quad a_6 \frac{d(x_1, [Tx_1]_{\alpha(x_1)})(1 + d(x_1, [Tx_1]_{\alpha(x_1)}))}{1 + d(x_1, x_2)} + \frac{(a_1 + ba_3 + a_5 + a_6)^2}{1 - (a_2 + ba_3)} \\
 d(x_2, x_3) &\leq a_1 d(x_1, x_2) + a_2 d(x_2, x_3) + ba_3 [d(x_1, x_2) + d(x_2, x_3)] + a_5 d(x_1, x_2) \\
 &\quad + a_6 d(x_1, x_2) + \frac{(a_1 + ba_3 + a_5 + a_6)^2}{1 - (a_2 + ba_3)} \\
 &\leq \frac{(a_1 + ba_3 + a_5 + a_6)}{1 - (a_2 + ba_3)} d(x_1, x_2) + \frac{(a_1 + ba_3 + a_5 + a_6)^2}{(1 - (a_2 + ba_3))^2} \\
 d(x_2, x_3) &\leq \left(\frac{(a_1 + ba_3 + a_5 + a_6)}{1 - (a_2 + ba_3)} \right)^2 d(x_0, x_1) \\
 &\quad + 2 \left(\frac{(a_1 + ba_3 + a_5 + a_6)}{1 - (a_2 + ba_3)} \right)^2 \quad \text{by (2.2)} \\
 d(x_2, x_3) &\leq \tau^2 d(x_0, x_1) + 2\tau^2.
 \end{aligned}$$

Continuing the same way, we obtain a sequence $\{x_n\}$ such that $x_n \in [Tx_{n+1}]_{\alpha(x_{n+1})}$, we have

$$(2.3) \quad d(x_n, x_{n+1}) \leq \tau^n d(x_0, x_1) + n\tau^n.$$

Now, for any positive integers m, n ($n > m$), we have

$$\begin{aligned} d(x_m, x_n) &\leq b[d(x_m, x_{m+1}) + d(x_{m+1}, x_n)] \\ &\leq b(d(x_m, x_{m+1})) + b[b\{d(x_{m+1}, x_{m+2}) + d(x_{m+2}, x_n)\}] \\ &\leq b(d(x_m, x_{m+1})) + b^2(d(x_{m+1}, x_{m+2})) + \cdots + b^{n-m}(d(x_{n-1}, x_n)) \\ &\leq b\tau^m d(x_0, x_1) + mb\tau^m + b^2\tau^{m+1}d(x_0, x_1) + b^2(m+1)\tau^{m+1} + \cdots \\ &\quad + b^{n-m}\tau^{n-1}d(x_0, x_1) + b^{n-m}(n-1)\tau^{n-1} \quad \text{by (2.3)} \\ &\leq b\tau^m(1 + b\tau + \cdots + b^{n-m}\tau^{n-m-1})d(x_0, x_1) + \sum_{i=m}^{n-1} b^{i-m}i\tau^i \\ &\leq \frac{b\tau^m}{1-b\tau}d(x_0, x_1) + \sum_{i=m}^{n-1} b^{n-m}i\tau^i. \end{aligned}$$

Since $b\tau < 1$, it follows from Cauchy root test that $\sum b^{n-m}i\tau^i$ is convergent and hence $\{x_n\}$ is a Cauchy sequence. Since, (X, d) is complete. Then, there exists $z \in X$ such that $x_n \rightarrow z$ as $n \rightarrow \infty$.

Now,

$$\begin{aligned} d(z, [Tz]_{\alpha(z)}) &\leq b[d(z, x_{n+1}) + d(x_{n+1}, [Tz]_{\alpha(z)})] \\ &\leq b[d(z, x_{n+1}) + H([Tx_n]_{\alpha(x_n)}, [Tz]_{\alpha(z)})]. \end{aligned}$$

Using (2.1), with $n \rightarrow \infty$ we get

$$(1 - b(a_2 + a_3))d(z, [Tz]_{\alpha(z)}) \leq 0.$$

So, we get

$$z \in [Tz]_{\alpha(z)}.$$

Hence, $z \in X$ is a fixed point. □

Theorem 2.2. Let (X, d) be a complete b -metric space with constant $b \geq 1$. Let $S, T : X \rightarrow F(X)$ be two fuzzy mappings and for $x \in X$, there exist $\alpha_S(x), \alpha_T(x) \in (0, 1]$ satisfying the following condition:

$$\begin{aligned} H([Tx]_{\alpha_T(x)}, [Sy]_{\alpha_S(y)}) &\leq a_1 d(x, [Tx]_{\alpha_T(x)}) + a_2 d(y, [Sy]_{\alpha_S(y)}) \\ &\quad + a_3 d(x, [Sy]_{\alpha_S(y)}) + a_4 d(y, [Tx]_{\alpha_T(x)}) \\ (2.4) \quad &\quad + a_5 d(x, y). \end{aligned}$$

for all $x, y \in X$. Also $a_i \geq 0$, where $i = 1, 2, \dots, 5$ with $(a_1 + a_2)(b + 1) + b(a_3 + a_4)(b + 1) + 2ba_5 < 2$ and $\sum_{i=1}^5 a_i < 1$. Then, S and T have a common fixed point.

Proof. Let x_0 be any arbitrary point in X , such that $x_1 \in [Tx_0]_{\alpha(x_0)}$. Then, by Lemma 1.6 there exists $x_2 \in [Sx_1]_{\alpha(x_1)}$, such that

$$\begin{aligned}
 d(x_1, x_2) &\leq H([Tx_0]_{\alpha(x_0)}, [Sx_1]_{\alpha(x_1)}) + (a_1 + ba_3 + a_5) \\
 &\leq a_1 d(x_0, [Tx_0]_{\alpha(x_0)}) + a_2 d(x_1, [Sx_1]_{\alpha(x_1)}) + a_3 d(x_0, [Sx_1]_{\alpha(x_1)}) \\
 &\quad + a_4 d(x_1, [Tx_0]_{\alpha(x_0)}) + a_5 d(x_0, x_1) + (a_1 + ba_3 + a_5) \\
 &\leq a_1 d(x_0, x_1) + a_2 d(x_1, x_2) + ba_3 [d(x_0, x_1) + d(x_1, x_2)] \\
 (2.5) \quad &\quad + a_5 d(x_0, x_1) + (a_1 + ba_3 + a_5) \\
 d(x_1, x_2) &\leq \frac{a_1 + ba_3 + a_5}{1 - (a_2 + ba_3)} d(x_0, x_1) + \frac{(a_1 + ba_3 + a_5)}{1 - (a_2 + ba_3)}.
 \end{aligned}$$

Similarly, by symmetry we have

$$\begin{aligned}
 d(x_2, x_1) &\leq H([Sx_1]_{\alpha(x_1)}, [Tx_0]_{\alpha(x_0)}) + (a_2 + ba_4 + a_5) \\
 &\leq a_1 d(x_1, [Sx_1]_{\alpha(x_1)}) + a_2 d(x_0, [Tx_0]_{\alpha(x_0)}) + a_3 d(x_1, [Tx_0]_{\alpha(x_0)}) \\
 &\quad + a_4 d(x_0, [Sx_1]_{\alpha(x_1)}) + a_5 d(x_1, x_0) + (a_2 + ba_4 + a_5) \\
 &\leq a_1 d(x_1, x_2) + a_2 d(x_0, x_1) + ba_4 [d(x_0, x_1) + d(x_1, x_2)] \\
 (2.6) \quad &\quad + a_5 d(x_1, x_0) + (a_2 + ba_4 + a_5) \\
 d(x_2, x_1) &\leq \frac{a_2 + ba_4 + a_5}{1 - (a_1 + ba_4)} d(x_0, x_1) + \frac{(a_2 + ba_4 + a_5)}{1 - (a_1 + ba_4)}.
 \end{aligned}$$

Adding (2.5) and (2.6), we get

$$\begin{aligned}
 d(x_1, x_2) &\leq \frac{a_1 + a_2 + ba_3 + ba_4 + 2a_5}{2 - (a_1 + a_2 + ba_3 + ba_4)} d(x_0, x_1) \\
 (2.7) \quad &\quad + \frac{a_1 + a_2 + ba_3 + ba_4 + 2a_5}{2 - (a_1 + a_2 + ba_3 + ba_4)}.
 \end{aligned}$$

Let

$$\tau = \frac{a_1 + a_2 + ba_3 + ba_4 + 2a_5}{2 - (a_1 + a_2 + ba_3 + ba_4)} < \frac{1}{b}.$$

Then by (2.7), we have

$$d(x_1, x_2) \leq \tau d(x_0, x_1) + \tau$$

Again by Lemma 1.6, $x_3 \in [Tx_2]_{\alpha(x_2)}$ such that

$$\begin{aligned}
 d(x_2, x_3) &\leq H([Sx_1]_{\alpha(x_1)}, [Tx_2]_{\alpha(x_2)}) \\
 &\quad + \frac{(a_1 + a_2 + ba_3 + ba_4 + 2a_5)^2}{2 - (a_1 + a_2 + ba_3 + ba_4)} \\
 &\leq \tau^2 d(x_0, x_1) + 2\tau^2.
 \end{aligned}$$

Continuing the same way, we obtain a sequence $\{x_n\}$ such that $x_{2n+1} \in [Tx_{2n}]_{\alpha(x_{2n})}$ and $x_{2n+2} \in [Sx_{2n+1}]_{\alpha(x_{2n+1})}$, with

$$\begin{aligned}
 d(x_{2n+1}, x_{2n+2}) &\leq H([Tx_{2n}]_{\alpha(x_{2n})}, [Sx_{2n+1}]_{\alpha(x_{2n+1})}) \\
 &\quad + \frac{(a_1 + ba_3 + a_5)^{2n+1}}{(1 - (a_2 + ba_3))^{2n}} \\
 &\leq a_1 d(x_{2n}, [Tx_{2n}]_{\alpha(x_{2n})}) + a_2 d(x_{2n+1}, [Sx_{2n+1}]_{\alpha(x_{2n+1})}) \\
 &\quad + a_3 d(x_{2n}, [Sx_{2n+1}]_{\alpha(x_{2n+1})}) + a_4 d(x_{2n+1}, [Tx_{2n}]_{\alpha(x_{2n})}) \\
 &\quad + a_5 d(x_{2n}, x_{2n+1}) \\
 &\quad + \frac{(a_1 + ba_3 + a_5)^{2n+1}}{(1 - (a_2 + ba_3))^{2n}} \\
 (2.8) \quad d(x_{2n+1}, x_{2n+2}) &\leq \frac{a_1 + ba_3 + a_5}{1 - (a_2 + ba_3)} d(x_{2n}, x_{2n+1}) \\
 &\quad + \frac{(a_1 + ba_3 + a_5)^{2n+1}}{(1 - (a_2 + ba_3))^{2n+1}}.
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 d(x_{2n+2}, x_{2n+1}) &\leq H([Sx_{2n+1}]_{\alpha(x_{2n+1})}, [Tx_{2n}]_{\alpha(x_{2n})}) \\
 &\quad + \frac{(a_2 + ba_4 + a_5)^{2n+1}}{(1 - (a_1 + ba_4))^{2n}} \\
 &\leq a_1 d(x_{2n+1}, [Sx_{2n+1}]_{\alpha(x_{2n+1})}) + a_2 d(x_{2n}, [Tx_{2n}]_{\alpha(x_{2n})}) \\
 &\quad + a_3 d(x_{2n+1}, [Tx_{2n}]_{\alpha(x_{2n})}) + a_4 d(x_{2n}, [Sx_{2n+1}]_{\alpha(x_{2n+1})}) \\
 &\quad + a_5 d(x_{2n+1}, x_{2n}) + \frac{(a_2 + ba_4 + a_5)^{2n+1}}{(1 - (a_1 + ba_4))^{2n}} \\
 (2.9) \quad d(x_{2n+2}, x_{2n+1}) &\leq \frac{(a_2 + ba_4 + a_5)}{(1 - (a_1 + ba_4))} d(x_{2n}, x_{2n+1}) \\
 &\quad + \frac{(a_2 + ba_4 + a_5)^{2n+1}}{(1 - (a_1 + ba_4))^{2n}}.
 \end{aligned}$$

By (2.8) and (2.9),

$$d(x_{2n+1}, x_{2n+2}) \leq \tau d(x_{2n}, x_{2n+1}) + \tau^{2n+1}.$$

Therefore,

$$\begin{aligned}
 d(x_n, x_{n+1}) &\leq \frac{a_1 + a_2 + ba_3 + ba_4 + 2a_5}{2 - (a_1 + a_2 + ba_3 + ba_4)} d(x_{n-1}, x_n) \\
 &\quad + \left(\frac{a_1 + a_2 + ba_3 + ba_4 + 2a_5}{2 - (a_1 + a_2 + ba_3 + ba_4)} \right)^n \\
 d(x_n, x_{n+1}) &\leq \tau d(x_{n-1}, x_n) + \tau^n
 \end{aligned}$$

$$\begin{aligned}
&\leq \tau [\tau d(x_{n-2}, x_{n-1}) + \tau^{n-1}] + \tau^n \\
&= \tau^2 d(x_{n-2}, x_{n-1}) + 2\tau^n \\
&\leq \dots\dots \\
(2.10) \quad d(x_n, x_{n+1}) &\leq \tau^n d(x_0, x_1) + n\tau^n.
\end{aligned}$$

Now, for any positive integers m, n ($n > m$), we have

$$\begin{aligned}
d(x_m, x_n) &\leq b[d(x_m, x_{m+1}) + d(x_{m+1}, x_n)] \\
&\leq b(d(x_m, x_{m+1})) + b[b\{d(x_{m+1}, x_{m+2}) + d(x_{m+2}, x_n)\}] \\
&\leq b(d(x_m, x_{m+1})) + b^2(d(x_{m+1}, x_{m+2})) + \dots + b^{n-m}(d(x_{n-1}, x_n)) \\
&\leq b\tau^m d(x_0, x_1) + mb\tau^m + b^2\tau^{m+1}d(x_0, x_1) + b^2(m+1)\tau^{m+1} + \dots \\
&\quad + b^{n-m}\tau^{n-1}d(x_0, x_1) + b^{n-m}(n-1)\tau^{n-1} \quad \text{by (2.10)} \\
&\leq b\tau^m(1 + b\tau + \dots + b^{n-m}\tau^{n-m-1})d(x_0, x_1) + \sum_{i=m}^{n-1} b^{i-m}i\tau^i \\
&\leq \frac{b\tau^m}{1-b\tau}d(x_0, x_1) + \sum_{i=m}^{n-1} b^{i-m}i\tau^i.
\end{aligned}$$

Since $b\tau < 1$, it follows from Cauchy root test that $\sum b^{n-m}i\tau^i$ is convergent and hence $\{x_n\}$ is a Cauchy sequence in X . Since, (X, d) is complete. Then, there exists $z \in X$ such that $x_n \rightarrow z$ as $n \rightarrow \infty$.

Now, we prove $z \in X$ is a common fixed point of S and T .

$$\begin{aligned}
d(z, [Sz]_{\alpha(z)}) &\leq b[d(z, x_{2n+1}) + d(x_{2n+1}, [Sz]_{\alpha(z)})] \\
&\leq b[d(z, x_{2n+1}) + H([Tx_{2n}]_{\alpha(x_{2n})}, [Tz]_{\alpha(z)})].
\end{aligned}$$

Using (2.4), with $n \rightarrow \infty$ we get

$$(1 - b(a_2 + a_3))d(z, [Sz]_{\alpha(z)}) \leq 0.$$

So, we get

$$z \in [Sz]_{\alpha(z)}.$$

This implies that $z \in X$ is a fixed point for S . Similarly, we can show that $z \in [Tz]_{\alpha(z)}$. Hence, $z \in X$, is a common fixed point. $\square$

Example 2.3. Let $X = [0, 1]$ and $d(x, y) = |x - y|$, whenever $x, y \in X$, then (X, d) is a complete b-metric space. Define a fuzzy mapping $T : X \rightarrow F(X)$ by

$$T(x)(t) = \begin{cases} 1, & 0 \leq t \leq x/4 \\ 1/2, & x/4 < t \leq x/3 \\ 1/4, & x/3 < t \leq x/2 \\ 0, & x/2 < t \leq 1 \end{cases}$$

For all $x \in X$, there exists $\alpha(x) = 1$, such that $[Tx]_{\alpha(x)} = [0, \frac{x}{4}]$. Then,

$$\begin{aligned} H([Tx]_{\alpha(x)}, [Ty]_{\alpha(y)}) &\leq \frac{1}{5} \left| x - \frac{x}{4} \right| + \frac{1}{10} \left| y - \frac{y}{4} \right| + \frac{1}{15} \left| x - \frac{y}{4} \right| \\ &\quad + \frac{1}{20} \left| y - \frac{x}{4} \right| + \frac{1}{25} |x - y| \\ &\quad + \frac{1}{30} \left(\frac{|x - \frac{x}{4}| (1 + |x - \frac{x}{4}|)}{1 + |x - y|} \right) \end{aligned}$$

Since, all the conditions of Theorem 2.1 are satisfied. Therefore, $0 \in X$ is the fixed point of T .

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WIRELESS ACCESS CHANNEL AND BROADBAND DYNAMIC REGULATION BASED ON LAN

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Abstract. As the wireless local area network developed from spare framework to centralized framework, wireless access points tended to interfere with each other, resulting in the fluctuation of network environment and properties. In order to improve such condition in the dense wireless local area network environment, this study calculated interference factors and evaluation factors of information channels, as well as explored channel allocation and interference coordination methods of wireless network based on multiple access control protocol and control and provisioning of wireless access points protocol. Moreover, the study put forward to reduce the interference among wireless access points based on power control. At last, this study put forward a dynamic optimization and regulation scheme based on wireless access points and the scheme was tested. Test results indicated that, dynamic power control could further optimize the wireless local area network environment; the dynamic regulation scheme could effectively reduce the interference of access points as well as improve network performance, which could satisfy users' requirements.

Keywords: Dense, wireless network, wireless access points, information channel, regulation scheme.

1. Introduction

Wireless local area network (WLAN) is the production of wireless communication technique and computer networks [1]. It has constantly changed traditional inter-computer communication methods since the 20th century. With the development of Internet and the wireless communication technique, WLAN, which is featured by convenient usage, appropriate price and easy installation, has been regarded as the optimal broadband access method and widely applied to human life [2]. In addition, as peoples demand for consumption rises, the WLAN has tended to develop towards dense deploy [3]. In WLAN environment that contains dense wireless access points (AP), the network density improves continually and usable efficient resources are developed constantly, thus the user experience becomes worse and worse [4]. The dense AP deployment of WLAN gradually becomes the bottleneck of WLAN development. Moreover, with the development of users intelligent, individualized and diverse demands, the requirements on WLAN become higher [5]. The AP density has an increasing tendency, the coverage rate is high and the mutual interference among AP is severe. Therefore, it is of great significance for optimization of wireless network resources to study interference coordination techniques that can effectively reduce the interference in dense WLAN [6].

Kwon Y. M. et al. [7] put forward the least congested channel search (LCCS), i.e., scanning information channels of every AP one by one to monitor the data transmission of all channels and acquire the load status of each channel, thus to find out the channel with the lightest load (the least occupation of user terminals). De Kerret P et al. [8] analyzed the shared channel interference model under the multiple input multiple output (MIMO) wireless communication system, and thus revealed the changes of system-wide capacity. Besides, they believed that the collaborative communication in densely deployed environment was more beneficial to improvement of system capacity. Ramaiyan V et al. [9] analyzed IEEE 802.11 multiple access control (MAC) using the Fixed-point Theorem. On the basis of the channel access control protocol, this study coordinated interference through power control as well as put forward a regulation scheme based on AP coordination, aiming to provide theoretical basis and technical supports for improvement of internet environment and properties, and thus promote the scientific development of WLAN.

2. MAC protocol and CAPWAP protocol

2.1 MAC protocol

From the physical aspect, MAC ensures wireless nodes sharing media on a fair and orderly basis. Meanwhile, it also plays an important role in network performances. Broadband WLAN has two forms of network architecture: centralized control and distributed control forms. Deployment of MAC to channel resources includes resource utilization and free competition. The deployment of resource

utilization is in the network architecture of centralized control, which is realized through a centrostigma (AP). The free competition method is suitable for distributed environment and the nodes in such method acquire the right to use channels through competition [10, 11].

2.2 CAPWAP protocol

The control and provisioning of wireless access point (CAPWAP) is the standard interface between access controller (AC) and AP, which is a protocol based on centralized WLAN architecture. Such protocol defines the control channel and data channel. AP and AC can make the control frame and data frame interact with each other through corresponding channels, thus to provide power controlled interference coordination with support [12].

3. Channel assignment strategy of dynamic wireless fidelity

The WLAN in current society is distributed densely and characterized by big data flow. Resources of information channels and broadband are limited, thus all AP can not work in different channels by current science and technology. Inefficient distribution of information channels can severely affect the communication performance of all WLAN. Therefore, in IEEE 802.11 20 MHz broadband channels, we distributed channels through channel assessment factors [13].

3.1 Channel interference factors

Channel interference factors (CIF) are mainly used to estimate the degree of interference of different channels [14]. The smaller the CIF, the lower the degree of interference, thus the channel quality is better and the transmission rate is higher. Calculation equations of CIF are:

$$(1) \quad CIF = T_{\partial} - T_{\sigma}$$

$$(2) \quad CIF = \frac{T_{\beta} - T_{\sigma}}{T_{\partial} - T_{\sigma}} \times 2^{10^{N/10} - 10^{N_0/10}}$$

T_{β} refers to the time duration when channels are busy;

T_{σ} refer to the time of data transmission;

T_{∂} refers to the observation time, i.e., scanning and detection time of a single channel; N is the acquired digital noise floor; N_0 is the smallest digital noise floor.

4. Channel assessment factors

Channel assessment factors (CAF) include known inference and potential inference, which are the sum of CIF and potential inference [15]. The calculation

equation is as follow:

$$(3) \quad CAF_n = aCIF_n + \frac{\sum_i (b_i/P_{ni}^\omega)}{1/P^\omega} = aCIF_n + b_i P^\omega \sum_i (1/P_{ni}^\omega).$$

n the equation, n refers to the channel number; P is the covering radius of AP ; P_i is the distance between AP and other interfering AP within the AP covering; i refers to the number of interference AP ; a is the proportion of interference channels; i is the potential interference, the bigger the value, the bigger the potential interference; b is the overlap coefficient of different channels, the bigger the value, the bigger the interference; refers to the index of channel fading.

4.1 Channel allocation

Primary allocation of channels based on CAF

- (1) Collection of AP data
- (2) Calculation of CIF
- (3) Report of received messages
- (4) Sequencing based on CAF
- (5) Work of AP in CAF0 channel

Channel reassignment based on CAF

- (1) Collection of AP data
- (2) Calculation of CIF
- (3) If CIF is lower than the hold value, AP collect data for calculation of CIF
- (4) Report of interference information, including CIF
- (5) Calculation of CAF
- (6) Sequencing is carried out based on CAF and CAFmin is selected for AP
- (7) Channel switch of AP

5. Power controlled interference coordination

6. Power control

In dense WLAN, severe interference can greatly affect the throughput performance of Internet [16]. Therefore, based on known channel interference factors and assessment factors, how to effectively coordinate them affects the development of dense WLAN. Lots of researchers have already carried out studies on interference coordination. However, there are few researches are carried out aiming at AP interference coordination. Therefore, we coordinated interference through power control multiple AP .

As shown in figure 1, the optimal access point of point O is AP1; AP5 and AP6 are out of the covering radius, thus there is no interference. Within the covering radius, channels of AP2, AP3 and AP4 are overlapped. In order to

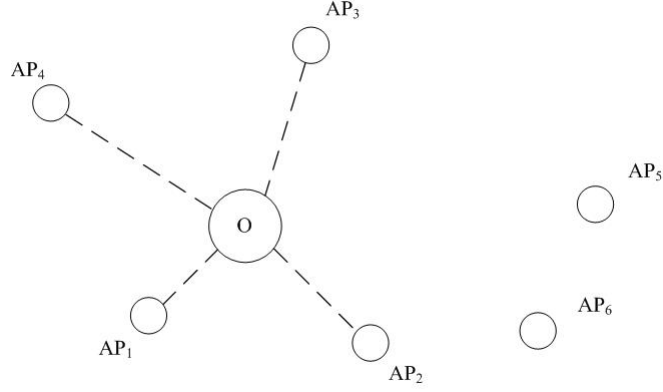


Figure 1: AP covering in dense WLAN

eliminate influence, the power control is necessary. During signal propagation, some factors can weaken the signals which are called path loss. Similarly, in propagation of frequency waves in the air, the filtering effect of air on signals can results in power loss. The path loss equation of TGn channel model (2.4 GHz $\sim$ 5 GHz) is:

$$(4) \quad Z(X) = Z_{\mu}(X_r) + 10n \lg(X/X_r).$$

In the equation, Z_{μ} refers to free-space loss and X_r refers to the distance between the endpoints.

Model	A	B	C	D	E	F
X_r	5	5	5	10	20	30
n before X_r	2	2	2	2	2	2
n after X_r	3.5	3.5	3.5	3.5	3.5	3.5
Shadow fading before X_r	3	3	3	3	3	3
Shadow fading after X_r	4	4	5	5	6	6

Figure 2: Path loss parameters of TGn channel models

In table 1, the received signal power of Model F is

$$(5) \quad P_{\theta} = P_{\rho} - Z(X) - Z_{shadow}.$$

In table 2, the minimum emissivity P_{min} of AP in 802.11 normal sensibility is:

$$(6) \quad P_{\rho \min} = P_{\rho} + Z(X_{mar}) + L_{shadow}.$$

Modulation	Coding rate	Minimum sensitivity
QPSK	0.5	-77
	0.75	-74
BPSK	0.5	-82
	0.75	-79
16-QAM	0.5	-70
	0.75	-66
64-QAM	0.5	-65
	0.75	-64

Figure 3: Receiving sensitivity of 802.11 standard signal

X_{mar} refers to the distance between the most marginal user and AP . In order to enable all users within the covering range to receive signals, the emissivity should be not lower than $P_{\rho \min}$.

Then the power is reported to AC through AP ; the modulating power calculated by AC is the replied to AP ; finally AP adjusts the emissivity.

6.1 Analysis of simulation performance

We carried out a simulation analysis to verify its validity and parameters were set up as follows:

Parameter setting	Value range
Channel	1-13,149,153,157,161,165 (China standard)
Channel width	20MHz
Frequency band	2.4GHz,5GHz
Default emissivity	16dBm (2.4GHz), 20dBm (5GHz)
Maximum emissivity	20dBm(2.4GHz),27dBm (5GHz)
Channel model	TGn
AP density	(5:5:30)*10 ⁻⁵
Radius	160
Channel interference factors	1,0.652,0.325,0.198,0.049,0.007

Figure 4: Values of simulation parameters

Table 3 and figure 2 show that, after AP power control based on channel allocation, the interference on AP is reduced. The intensity of interference signals is in direct proportion to the AP density, thus the lower the AP density, the weaker the intensity of interference signals. In low density, the larger the AP distance, the smaller the interference in the distance. After the emissivity is reduced, the weakening extent of interference intensity is low. In moderate AP density, the gain brought by power control is the maximum. In high AP density, the number of AP increases and the distance reduces, thus the interference intensity increases.

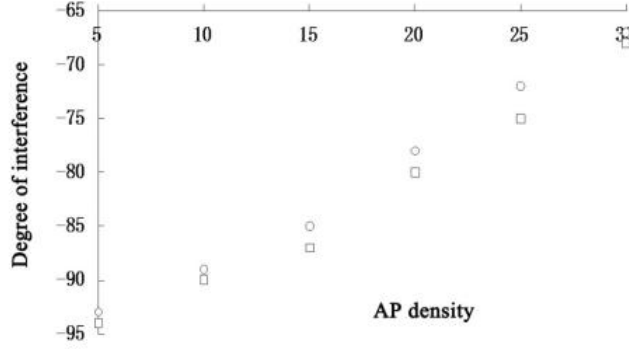


Figure 5: Relationship between interference and AP density

7. Dynamic regulation scheme based on AP coordination

During the AP run phase of this regulation scheme, the minimum broadband channel resources are used to satisfy users demands to an extreme. According to business requirements and environmental information, reasonable broadband extension (compression) schemes are dynamically selected and channel allocation is adjusted through network reconfiguration. The adjustment includes three steps.

7.1 Sensing the environment

The network environment is perceived through following parameters: AP throughput capacity, business volume and demand quantity and interference distribution [17].

7.1.1 Throughput capacity

The throughput capacity refers to the maximum serving rate of AP under the relatively static state.

7.1.2 Business volume and demand quantity

The business volume refers to the current world serving rate of AP. The network demand quantity is to predict users demand by combing the current information with historical information. The calculation equations are:

$$(7) \quad K_{m+1}^{pre} = K_m \times (1 + \Delta K_m^{tre}) \left(1 + \frac{\sum_{i=m-k}^m \nabla Y_i}{k}\right),$$

$$(8) \quad \Delta K_m^{tre} = \frac{K_n - K_{n-1}}{Y},$$

$$(9) \quad \nabla Y_m = \frac{K_m^{pre} - K_{n-1}}{K_n}.$$

In above equations, Y refers to the current business volume, predicted value and predicted time granularity. K_{m+1}^{pre} is the predicted value at the next moment; ΔK_m^{tre} is the actual business growth rate and ∇Y_m refers to the current forecast error.

Firstly, AP learns and processes the surroundings. Then the serving rate is monitored and the maximum throughput capacity of periodic AP is calculated; data are then sent to network resource management (NRM). Then the AP system processes users perception reports and builds environmental information. The environmental information, business volume and throughput capacity are sent to NRM. After that, NRM learns and processes received information, thus to predict the business demand quantity and determine the decision-making time of trigger.

7.2 Decision-making process

The decision includes extension strategy and compression strategy. This study put forward four kinds of strategies: free adjacent channel, free channel selection, adjacent AP channel coordination and broadband compression.

7.2.1 Strategy of free adjacent channel

Suppose there are mutually independent 20 MHz free channels, then the broadband extension can be realized by directly binding free channels.

7.2.2 Strategy of free channel selection

Suppose there is a pair of mutually independent 20 MHz channels and they make up 40 MHz of free channels. Then AP can be applied for transferring to the channel.

7.2.3 Strategy of adjacent AP channel coordination

Suppose a pair of mutually independent 20 MHz channels is occupied by adjacent AP, then adjacent channels that have fewer reconstitution costs are selected as the spreading channel for coordination.

7.2.4 Strategy of broadband compression

Suppose the AP business demand quantity is reduced, then the broadband compression strategy is carried out to save frequency and power resources.

7.3 Network reconfiguration

Network reconfiguration refers to reconstitute AP in order by NRM according to decision results. Different strategies have different reconstitution methods [18]. In strategy of free adjacent channel, the network is reconstituted by changing broadband and mode switch at the same time. In strategy of free channel

selection, channel migration and mode switch of AP and users are carried out at the same time. The method of adjacent AP channel coordination strategy is more complicated than above two: firstly, AP and TRM are given coordinated restructuring; then the method of free adjacent channel is used. In broadband compression strategy, AP is used to directly realize broadband changing and mode switch.

7.4 Simulation analysis of the dynamic regulation scheme

7.4.1 Setting of the simulation environment

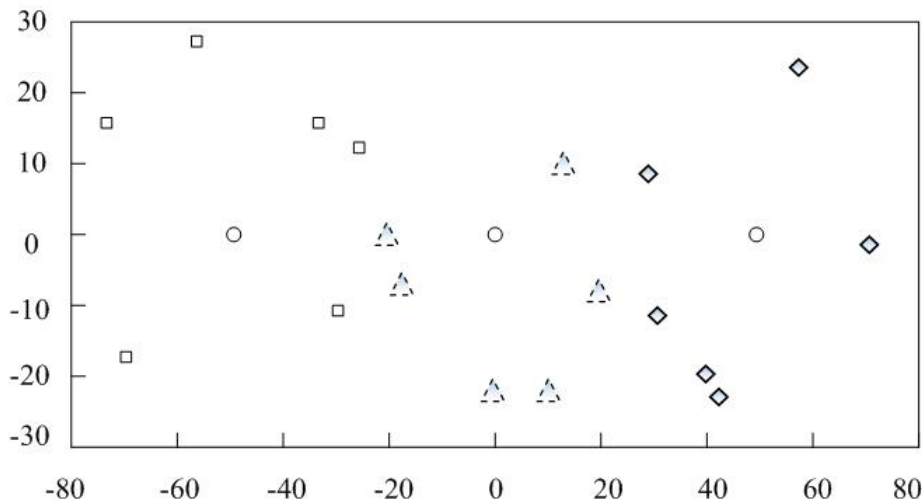


Figure 6: Linear distribution of three AP

As shown in figure 3, the business demand quantity under three AP is big. In traditional mode of distribution, such characteristic can be fully reflected by the method of mixing $20MHz$ and $40MHz$.

7.4.2 Analysis of simulation results

As shown in figure 4, in such network environment, the scheme put forward in this study can improve the actual business throughput capacity of network and the satisfaction of users better compared with that of traditional broadband allocation method.

Figure 5 indicates that resources among each AP flow interactively. The resource allocation method put forward in this study can increase the use ratio of resources and take full advantage of channel resources as well as increase the throughput capacity of network. The good use of channel resources can

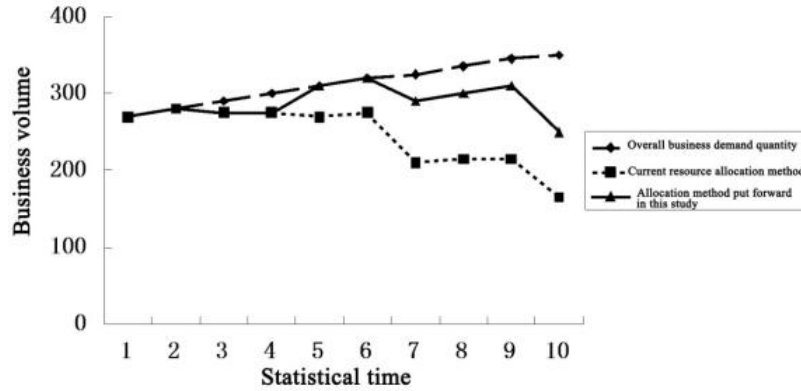


Figure 7: Overall business volume

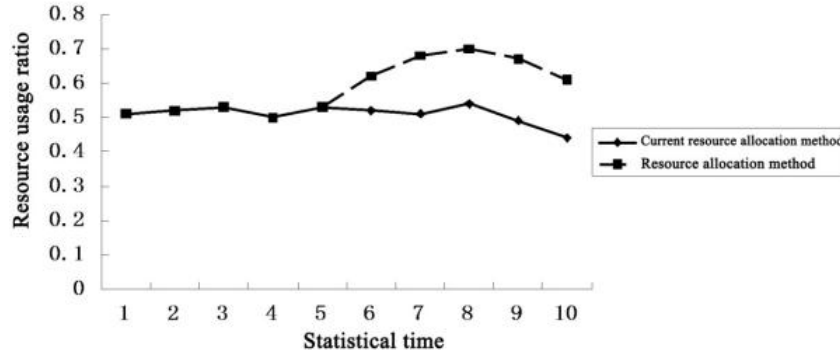


Figure 8: Resource utilization of different channel allocation

reduce the probability of free channel resources, thus the resources can be used effectively.

Figure 6 shows that, the broadband allocation scheme put forward in this study can improve the service ability of network as well as users' degree of satisfaction.

8. Discussion and conclusion

With the rapid development of WLAN, the deployment of AP becomes more and more dense. Thus the interference among AP is severe, which can lead to fluctuation of network environment as well as affect users experience. In recent years, researchers in China and abroad begin to study the reasonable allocation of network resources in large-scale dense WLAN environment [19].

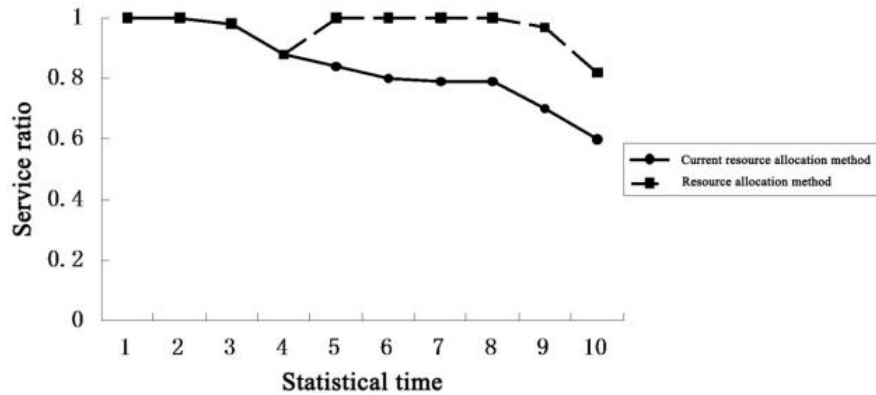


Figure 9: Satisfaction of business requirements

Under the impetus of IEEE802.11n new standard, the 802.11n high throughput mode comes out, which provides new technical supports for resource allocation of large-scale dense WLAN. On the basis of MAC and CAPWAP protocol, channel interference factors are determined and channel assessment factors are calculated to evaluate the quality of channels. The optimal transmitting power is determined by control intervention of power. Relevant frequency resources are allocated according to AP business demand and the resource use ratio is improved under the premise that the business demand is satisfied, which realizes green energy saving and such scheme is feasible. There are also many assumption factors in this study. The interference analyzed in this study belongs to internal interference and the external interference is not analyzed. Therefore, this study should be further improved in the future.

Acknowledgement

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PAIRING-FRIENDLY ELLIPTIC CURVES OF EMBEDDING DEGREE 1 AND APPLICATIONS TO CRYPTOGRAPHY

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Abstract. Recently, Wang et al. [1] proposed a new method for constructing pairing-friendly elliptic curves of embedding degree 1. Authors claim that this method significantly improves the efficiency of generating elliptic curves. In this paper, we give the arithmetic of pairing-friendly elliptic curves of embedding degree 1. We prove that conventional classification of pairings into Type 1, 2, 3 and 4 is applicable for the elliptic curves of embedding degree 1 proposed by Wang et al. We highlight the selection of pairing-friendly elliptic curves of embedding degree 1 for design of efficient cryptosystems. We compare security and efficiency of cryptosystems based on these pairing-friendly elliptic curves with the existing cryptosystems. By using these elliptic curves we propose a new asymmetric group key agreement (ASGKA) scheme from Tate pairing. We discuss the security and efficiency of the proposed ASGKA scheme.

Keywords: Public key cryptography, pairing-friendly elliptic curves, embedding degree, Tate pairing, asymmetric group key agreement scheme.

1. Introduction

Since its applications to cryptography [2, 3], elliptic curves over finite fields have an important role in public key cryptography. Elliptic Curve Cryptography (ECC) [4] is becoming accepted as an alternative to cryptosystems such as RSA and ElGamal over finite field, because it requires less bandwidth and computational efforts when performing key exchange and/or constructing a digital signature. Pairing based cryptography is one of the recent research directions in public key cryptography. To improve the efficiency of cryptosystems based

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on pairings, constructing pairing-friendly elliptic curves with various embedding degrees has been the subject of ongoing research. Pairing-friendly elliptic curves with various embedding degree [5, 6, 7] have been proposed in last few years. If embedding degree k of an elliptic curve is 1, then computation of pairing is related to the base field $\mathbb{F}_p$ rather than the extension field $\mathbb{F}_{p^k}$. This helps to improve the computation efficiency of pairing.

There are many pairing-friendly elliptic curves of embedding degree 1 in cryptographic literature. Recently, Wang et al. [1] proposed a new method for constructing pairing-friendly elliptic curves of embedding degree 1. Authors claim that in their method the parameters are computed under a given expression, which significantly improves the efficiency of generating elliptic curves. Since the discovery of identity-based encryption in the year 2000, pairings have been used in design of hundreds of cryptographic schemes. But computational requirements complicate the practical application of pairing-based cryptography. In this paper, we address the issue of efficiency of pairing-based cryptography. A necessary condition for the security of pairing-based cryptosystems is that the discrete logarithm problem (DLP) in $\mathbb{F}_{p^k}$ is intractable. Experiments conducted by Barbulescu et al. [8] in 2015 illustrated that the DLP is significantly easier in $\mathbb{F}_{p^2}$ than in prime order fields $\mathbb{F}_p$. In this paper, we also highlight the selection of pairing-friendly elliptic curves of embedding degree 1 for design of secure and efficient cryptosystems. By using these curves, we implement some pairing-based protocols and compare the efficiency and security of these protocols with the existing ones.

Many cryptographic protocols rely on the existence of a confidential channel among the users. A major goal of Group Key Agreement (GKA) [9] protocols is to establish a confidential broadcast channel for the users of the group. GKA protocols allow a group of users to establish a common secret key from which a session key can be derived. So these protocols are likely to be used in any group-oriented communication applications to achieve secure broadcasting at the network layer. In GKA protocols, all users should be connected in order to share the secret encryption key. However if users are located in different cities with different time zone, it is very difficult for them to be connected concurrently. To settle this issue, the concept of Asymmetric Group Key Agreement (ASGKA) was introduced in 2009 by Wu et al. [10].

Asymmetric group key agreement is a versatile cryptographic primitive which allows a group of users to negotiate a common encryption key. This encryption key is accessible to any entity and each user only holds her respective decryption key. This primitive not only enables the confidential communication among group users but also permits any outsider to send message to group. Zhang et al. [11] presented asymmetric group key agreement protocol for open networks. The authors of [12] extended this scheme to design a new ASGKA scheme. We further extend the work of Zhang et al. but in a different flavour. In this paper, we propose a new elliptic curve based asymmetric group key agreement (ASGKA) scheme from Tate pairing.

Rest of the paper is organized as follows. In section 2, we briefly describe elliptic curves and pairings. In section 3, we give arithmetic of pairing-friendly elliptic curves of embedding degree 1. Section 4 defines the proposed ASGKA scheme. In section 5, we discuss the security issues and efficiency of the proposed scheme. We finally draw our conclusion in section 6.

2. Preliminaries

In this section we briefly describe the elliptic curves and Tate pairing on elliptic curves. We also define k -Bilinear Diffie-Hellman Exponent (k -BDHE) problem in (G_1, G_2) .

2.1 Elliptic curve

An elliptic curve [13] E over a finite field $\mathbb{F}_p$ is defined by an equation

$$(1) \quad y^2 = x^3 + ax + b; a, b \in \mathbb{F}_p,$$

with the discriminant $\Delta = 4a^3 + 27b^2 \neq 0$. The set of all points on the curve E together with O , called the “point at infinity” forms a group under the operation point addition. This group is denoted by $E(\mathbb{F}_p)$. If $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points from $E(\mathbb{F}_p)$ such that $P, Q \neq O$, then $R(x_3, y_3) = P + Q$ and $2P(x_3, y_3) = P + P$ are defined as:

$$(2) \quad \begin{cases} x_3 = \lambda^2 - x_1 - x_2 \pmod{p} \\ y_3 = \lambda(x_1 - x_3) - y_1 \pmod{p}, \end{cases}$$

where

$$(3) \quad \lambda = \frac{y_2 - y_1}{x_2 - x_1} \pmod{p}, \text{ if } P \neq Q \quad (\text{point addition})$$

$$(4) \quad \text{and } \lambda = \frac{3x_1^2 + a}{2y_1} \pmod{p}, \text{ if } P = Q \quad (\text{point doubling}).$$

Scalar multiplication in elliptic curve group $E(\mathbb{F}_p)$ can be computed as follows:

$$kP = P + P + P + \cdots + P \quad (k \text{ times}).$$

Hasse’s theorem gives a bound on the number of points on an elliptic curve over a finite field. According to this theorem if $E(\mathbb{F}_p)$ has order N , then we have

$$|N - (p + 1)| \leq 2\sqrt{p}.$$

The subgroup $E[r] = \{P \in \bar{E} | [r]P = O\}$ i.e. the subgroup of points of order r on $E(\bar{\mathbb{F}}_p)$, called r -torsion subgroup of $E(\mathbb{F}_p)$.

2.2 Tate pairing

Let G_1, G_2 and G_T be finite groups of prime order. A cryptographic pairing $e : G_1 \times G_2 \longrightarrow G_T$ is a map that satisfies the following properties:

1. Bilinearity: $e(aP, bQ) = e(P, Q)^{ab}$, for all $P \in G_1, Q \in G_2$ and for all $a, b \in \mathbb{Z}_p$.
2. Non-Degeneracy: For all $P \in G_1$, with $P \neq 1_{G_1}$, there is some $Q \in G_2$ such that $e(P, Q) \neq 1_{G_T}$. For all $Q \in G_2$, with $Q \neq 1_{G_2}$, there is some $P \in G_1$ such that $e(P, Q) \neq 1_{G_T}$.
3. Computability: There is an efficient algorithm to compute $e(P, Q)$ for all $P \in G_1, Q \in G_2$.

Cryptographic pairings are constructed from elliptic curves of small embedding degree. More precisely, let $E(\mathbb{F}_p)$ be an elliptic curve defined over the finite field $\mathbb{F}_p$. Let r be a prime divisor of $\#E(\mathbb{F}_p)$ with $\gcd(r, p) = 1$, and let k be the smallest positive integer such that $r | p^k - 1$. This number k is called embedding degree of $E(\mathbb{F}_p)$ with respect to r . Then G_1 is an order- r subgroup of $E(\mathbb{F}_p)$, G_2 is an order- r subgroup of $E(\mathbb{F}_{p^k})$, G_T is the order- r subgroup of $\mathbb{F}_{p^k}^*$, and the map e is derived from the classical Weil and Tate pairings.

Tate pairing [14] is a mapping $e_T : E(\mathbb{F}_{p^k}) \times E(\mathbb{F}_{p^k}) / rE(\mathbb{F}_{p^k}) \longrightarrow \mathbb{F}_{p^k}^* / (\mathbb{F}_{p^k}^*)^r$ defined by $e_T(P, Q) = (f_{r,P}(Q))^{\frac{p^k-1}{r}}$, where $f_{r,P}$ is a normalized Miller function. Tate pairing widely used to design pairing-based protocols because it is twice as efficient as Weil pairing. It is clear from the definition of Tate pairing that if embedding degree $k \neq 1$, then the computation of Tate pairing is related to the extension field $\mathbb{F}_{p^k}$ and hence the computation process will be time consuming. However, if $k = 1$ then the computation of Tate pairing is related to the base field $\mathbb{F}_p$ rather than extension field $\mathbb{F}_{p^k}$. This will significantly improve efficiency of Tate pairing.

2.3 k -Bilinear Diffie-Hellman Exponent (k -BDHE) problem in (G_1, G_2)

Let G_1, G_2 be two elliptic curve groups. Given P in G_1, Q in G_2 and $R_i = \alpha^i P$ for $i = 1, 2, \dots, k, k+2, \dots, 2k$ as input, compute $e_T(P, Q)^{\alpha^{k+1}}$. Since the input lacks the point $\alpha^{k+1}P$, so the bilinear pairing does not seem to help to compute $e_T(P, Q)^{\alpha^{k+1}}$.

k -BDHE assumption is that there is no polynomial-time algorithm that can solve k -BDHE problem in (G_1, G_2) with non-negligible probability.

3. Arithmetic of pairing-friendly elliptic curves of embedding degree 1

In this section we give arithmetic of pairing-friendly elliptic curve of embedding degree 1. We highlight the ρ -value of pairing-friendly elliptic curves of embed-

ding degree 1 proposed by Wang et al. We also highlight security, performance and functionality of pairings on these curves. We discuss some subtleties with using these curves to implement pairing based protocols.

3.1 Pairing-friendly elliptic curves of embedding degree 1

Pairing-friendly elliptic curves of embedding degree 1 were first mentioned in [15, 16, 17] and further studied in [18, 19]. Koblitz and Menezes [18] proposed ordinary elliptic curves of embedding degree 1 with trace 2. In these curves the prime number p is of the form $p = A^2 + 1$, where $A \equiv 2 \pmod{4}$ and the elliptic curve is $E : Y^2 = X^3 - 4X$ over $\mathbb{F}_p$. As it is shown in [18], $\#E(\mathbb{F}_p) = p - 1$, so E is an ordinary elliptic curve of trace 2. Also since $E(\mathbb{F}_p) \cong \mathbb{Z}_A \oplus \mathbb{Z}_A$, so we have $E[n] \subseteq E(\mathbb{F}_p)$, where $n \equiv 3 \pmod{4}$ is a prime. This shows that $E(\mathbb{F}_p)$ has embedding degree $k = 1$ with respect to n . Note that Chatterjee et al. [20] observe that conventional classification of pairings into Type 1, 2, 3 and 4 is not applicable for the elliptic curves of embedding degree 1. Elliptic curves with embedding 1 and trace different 2 can be generated using the Complex Multiplication (CM) method [19, 21]. In these common methods to construct elliptic curves, the equation $u = p + 1 \pm W$ is used to generate parameters of the elliptic curves.

Recently, Wang et al. proposed a new method to construct pairing-friendly elliptic curves of embedding degree 1. In their method, they present a new equation $p = u \pm W + 1$ to generate the parameters of elliptic curves. In the new equation, the order u of elliptic curve is known, and we need to obtain prime p from the order u rather than the order u from p . In other words, this method can generate an elliptic curve under arbitrary order r , while the order u of an elliptic curve E can be trivially obtained using $u = r * r$ (from the security requirement), where r is a large prime and has a low hamming with weight 4. As they claimed, this method improves efficiency of generating elliptic curves significantly.

This method can be summarized in the following three algorithms.

1. The first algorithm is to generate a large prime r of a low hamming with weight 4.
2. The second algorithm is used to generate the characteristic of finite field $\mathbb{F}_p$, the order u of non-supersingular elliptic curve over $\mathbb{F}_p$ and order r of a point on the elliptic curve.
3. The final algorithm is used to construct pairing-friendly elliptic curves of embedding degree 1.

The ρ -value of an elliptic curve E is defined as $\rho(E) = (\log p)/(\log r)$. Also as $\#E \neq p - 1$, so trace of these elliptic curves is different from 2.

3.2 Pairings on elliptic curves of embedding degree 1

In this subsection we will discuss security, performance and functionality characteristics of pairings on pairing-friendly elliptic curves of embedding degree 1 proposed by Wang et al. Chatterjee et al. in their paper proved that conventional classification of pairings into Type 1, 2, 3 and 4 is not applicable for elliptic curves of embedding degree 1. In this subsection we prove that this classification of pairings is applicable for the elliptic curves of embedding 1 proposed by Wang et al.

It is clear from the definition given in section 2 that if embedding of elliptic curve is $k = 1$, then pairings are constructed on the base field $\mathbb{F}_p$ rather than the extension field $\mathbb{F}_{p^k}$. This may improve significantly the computation of pairings. Also since Tate pairing in general is twice as efficient as Weil pairing. So Tate pairing is widely used to design pairing-based cryptosystems.

The function $f_{r,P}(Q)$ in definition of Weil or Tate pairing is a Miller function of length n and we use Miller's algorithm [22] to compute this function. Miller function computation can only fail if $Q \in \langle P \rangle$. In cryptographic protocols it is necessary to have an efficient method for testing whether $Q \in \langle P \rangle$. The properties of Weil pairing imply that $Q \in \langle P \rangle$ if and only if $e_W(P, Q) = 1$. So, subgroup testing can be done by using Weil pairing. But it cannot be done in the same manner by using Tate pairing. Chatterjee et al. highlight in their paper that for elliptic curves of embedding degree 1, there is no natural way of distinguishing any order- r subgroup of $E[r]$ from the other order- r subgroups. So they gave an observation that classification of pairings into Type 1, 2, 3 and 4 is not applicable for elliptic curves of embedding degree 1. They defined three new pairings called Type A, Type B and Type C for elliptic curves of embedding degree 1.

However, third algorithm of the method proposed by Wang et al. for the construction of pairing-friendly elliptic curves of embedding degree 1, ensures that the constructed subgroups G_1 and G_2 satisfy that $G_1 \cap G_2 = \{O\}$, where G_1 is the subgroup generated by P_1 , G_2 is subgroup generated by P_2 and G_1, G_2 are two different subgroups of elliptic curve E with the same order r . Since by construction we get two distinguishing order- r subgroup in $E[r]$, so now we may classify the pairings on these curves into Type 1, 2, 3, and Type 4.

Also in these curves, order r of the subgroups G_1 and G_2 is a prime of low hamming with weight 4 i.e. there are only two '1' bits in addition to the highest bit and lowest bit in the binary representation of the large prime. So it may be more efficient to compute Tate pairing from Miller's algorithms on these elliptic curves of embedding degree 1.

3.3 Implementation of cryptosystems based on elliptic curves proposed by Wang et al.

Information confidentiality is an essential requirement for security in critical infrastructure. Identity-based cryptography [23], which is an increasingly popular

branch of cryptography, widely used to protect the information confidentiality in critical infrastructure due to the ability of directly computing the user's public key from user's identity. However computational requirements complicate the practical application of Identity-based cryptosystems.

Use of pairing-friendly elliptic curves of embedding degree 1 proposed by Wang et al. may improve efficiency of not only identity-based protocols but also of other protocols. In this subsection, we implement some prominent protocols using these pairing-friendly elliptic curves of embedding degree 1.

3.3.1 Boneh-Franklin IBE

Boneh-Franklin IBE [24] is an example of a prominent identity-based encryption scheme. This scheme was designed by using Type 1 Weil pairing. The security of this protocol is based on computational Diffie-Hellman problem in Type 1 setting. We give implementation of this protocol by using Type 3 Tate pairing on pairing-friendly elliptic curves proposed by Wang et al.

Definition 1: Computational Diffie-Hellman Problem (Co-DHP*) in (G_1, G_2)

Let G_1 and G_2 be two elliptic curves groups. For $k \in \mathbb{Z}_q^*$, given P, kP_1 in G_1 and kP_2 in G_2 , compute kP .

This IBE scheme can be described in following four algorithms.

Setup: For security parameter $m \in \mathbb{Z}^+$, the algorithm works as follows:

1. The algorithms of Wang et al. run on m to generate a large prime q , two elliptic curve groups G_1, G_2 of order q , an arbitrary generator P for G_1 , a multiplicative group G_T of order q , and Tate pairing $e_T : G_1 \times G_2 \rightarrow G_T$. Note that since G_1 and G_2 be two distinct subgroups of order q and $G_1 \cap G_2 = \{O\}$, so we can hash onto G_2 .
2. Pick a random $s \in \mathbb{Z}_q^*$ and compute public key $P_{pub} = sP$.
3. Choose cryptographic hash functions $H_1 : \{0, 1\}^* \rightarrow G_2$ and $H_2 : G_T \rightarrow \{0, 1\}^n$ for some n .

The message space is $\mathcal{M} = \{0, 1\}^*$ and the ciphertext space is $\mathcal{C} = G_2 \times \{0, 1\}^n$. The system parameters are $\langle q, G_1, G_2, G_T, e_T, n, P, P_{pub}, H_1, H_2 \rangle$. The master key is s .

Extract: For a given $ID \in \{0, 1\}^*$, the algorithm computes $Q_{ID} = H_1(ID) \in G_2$ and sets the private key d_{ID} as $d_{ID} = sQ_{ID}$, where s is master key.

Encrypt: To encrypt a message $M \in \mathcal{M}$ it first computes $Q_{ID} = H_1(ID) \in G_2$, after that choose a random number $t \in \mathbb{Z}_q^*$ and sets the ciphertext to be $C = \langle tP, M \oplus H_2(g_{ID}^t) \rangle$, where $g_{ID} = e_T(P_{pub}, Q_{ID}) \in G_T$.

Decrypt: To decrypt the ciphertext C using the private key $d_{ID} \in G_2$, compute $V \oplus H_2(e_T(tP, d_{ID})) = M$.

Masks used in encryption and decryption are the same because

$$\begin{aligned}
 e_T(tP, d_{ID}) &= e_T(tP, sQ_{ID}) \\
 &= e_T(P, Q_{ID})^{ts} \\
 &= e_T(sP, Q_{ID})^t \\
 &= e_T(P_{pub}, Q_{ID})^t = g_{ID}^t.
 \end{aligned}$$

The security of Boneh-Franklin IBE from Type 3 Tate pairing depends on (Co-DHP*) in (G_1, G_2) . It is clear from the definition of Co-DHP* that the intractability of DLP in G_1 and DLP in G_2 are both necessary for the hardness of Co-DHP*. So Boneh-Franklin IBE may be more secure in this setting.

3.3.2 BLS signature scheme

Among the short signature schemes from bilinear pairings, the Boneh-Lynn-Shacham (BLS) [25] signature scheme is one of the most important schemes and has been used as a building block for many other protocols. The security of this signature scheme is also based on computational Diffie-Hellman problem. We also give implementation of this protocol by using Tate pairing on pairing-friendly elliptic curves proposed by Wang et al. Sanjeet et al. observe that BLS signature scheme cannot be instantiated in Type 1 and Type 3 setting because there is no efficient method for hashing onto G_2 . Now as for the subgroups G_1 and G_2 proposed by Wang, we can hash onto G_2 efficiently. So we can instantiate this scheme in Type 3 setting without any difficulty.

Let G_1 and G_2 be two order- r subgroups of elliptic curve group E with embedding degree 1 proposed by Wang et al. and let $e_T : G_1 \times G_2 \rightarrow G_T$ be Tate pairing where $G_1 = \langle P \rangle$. Let $H : \{0, 1\}^* \rightarrow G_2$ be a hash function. The public parameters are E, P, e_T and H .

Alice's private key is an arbitrary integer $a \in \mathbb{Z}_r^*$, while her public key is $A = aP$. To sign a message m , Alice computes $M = H(m) \in G_2$ and $\sigma = aM$. Her signature on message m is σ . To verify the signed message (m, σ) , Bob computes $M = H(m)$, verifies that $\sigma \in \langle M \rangle$ and accepts if and only if $e_T(P, \sigma) = e_T(A, M)$.

This verification is correct due to the following properties of pairing $e(P, \sigma) = e_T(P, aM) = e_T(P, M)^a = e_T(aP, M) = e_T(A, M)$. The security of BLS signature scheme in Type 3 setting depends on (Co-DHP*) in (G_1, G_2) , which is a strong assumption than the Co-DHP in Type 1 setting.

Apart from Boneh-Franklin IBE and BLS signature schemes, some other prominent schemes, for examples Boneh-Gentry-Lynn-Shacham aggregate signature scheme [26], Xun Yi identity-based signature scheme [27], Sakai-Oghishi-Kasahara identity-based key agreement scheme [28], can be instantiated by using Tate pairing on elliptic curves of embedding degree 1 proposed by Wang et al.

The computation cost of a pairing-based cryptosystem can be divided mainly into five heads: elliptic curve representation, pairing computation, point multiplication, hashing, and last subgroup membership testing. It is clear from

Wang's method that as the order of the subgroup is a large prime of low hamming with weight 4, so efficiency of generating elliptic curve is improved. More specifically this method requires about 200 ms to generate a suitable elliptic curve. However, the method of Izuta, Nogami and Morikaw [15] takes about 20 hours. Also since the constructed subgroups G_1 and G_2 are distinct subgroups of order r and $G_1 \cap G_2 = \{O\}$, so it is easy to hash onto G_2 and there is no need of subgroup membership test.

It is clear from the paper of Wang et al. that for the order r of the subgroup of 160 bits in PBCs with the parameters of Wang, we need to compute only 2593 multiplications and 1135 inverse operations. However in ordinary PBCs, we need to compute 3429 multiplications and 1515 inverse operations. As inverse operation is estimated to be 5.18 multiplication operations, so may save 24.9% of the time required to compute Tate pairing because

$$\frac{2593 + 1135 \times 5.18}{3429 + 1515 \times 5.18} = 0.751 = 75.1\%.$$

So the computation cost for all the heads mentioned above reduced significantly. Hence implementation of the Boneh-Franklin IBE, BLS signature scheme and other schemes based on elliptic curves of embedding degree 1 is significantly efficient.

4. Proposed Asymmetric Group Key Agreement (ASGKA) scheme

In this section, we propose an identity based asymmetric group key agreement scheme motivated by [11]. We design this scheme by using Tate pairing on pairing-friendly elliptic curve of embedding degree 1 proposed by Wang et al. We considered a group of n members who wanted to receive secure messages from the other participants, who may or may not be group members. Each participant is armed with a private-public key pair for authentication purposes. The algorithm for the scheme works as follows:

System setup: The group controller (U) generates the system parameters $(G_1, G_2, G_T, e_T, H, P, r)$ at this stage. In this tuple, $H : \{0, 1\}^* \rightarrow G_2$ is a cryptographic hash function, G_1 and G_2 are elliptic curve groups of prime order r , $e_T : G_1 \times G_2 \rightarrow G_T$ is a bilinear Tate pairing and P is a generator of G_1 . Also generate and propagate securely the private key (s_i) and public key (p_i) to each user.

Key establishment: At this stage, the participants of the group generate and publish the messages which will be used in generation of group encryption and decryption keys. Let $U_1, U_2, \dots, U_n$ be the participants involved in the group. A participant U_i with identity ID_i for $1 \leq i \leq n$ in group communication will perform the following steps:

1. Randomly choose $a_i \in \mathbb{Z}_r^*$, $Q_i \in G_2$ and compute $R_i = a_i P \in G_1$,
 $E_i = e_T(P, H(ID_i) + Q_i) \in G_T$.

Table 1: Message obtained by the participants

User	U_1	U_2	U_3	$\dots$	U_n	All
U_1	-	$\sigma_{1,2}$	$\sigma_{1,3}$	$\dots$	$\sigma_{1,n}$	(R_1, E_1, ID_1, ρ_1)
U_2	$\sigma_{2,1}$	-	$\sigma_{2,3}$	$\dots$	$\sigma_{2,n}$	(R_2, E_2, ID_2, ρ_2)
U_3	$\sigma_{3,1}$	$\sigma_{3,2}$	-	$\dots$	$\sigma_{3,n}$	(R_3, E_3, ID_3, ρ_3)
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
U_n	$\sigma_{n,1}$	$\sigma_{n,2}$	$\sigma_{n,3}$	$\dots$	-	(R_n, E_n, ID_n, ρ_n)

2. For $1 \leq j \leq n$, compute $\sigma_{i,j} = Q_i + a_i H(ID_j) \in G_2$.
3. Generate a signature ρ_i on R_i by using private key s_i . In order to make the protocol efficient, one may choose an identity based signature scheme on elliptic curve.
4. Publish $\{\sigma_{i,1}, \sigma_{i,2}, \dots, \sigma_{i,i-1}, \sigma_{i,i+1}, \dots, \sigma_{i,n}, (R_i, E_i, ID_i, \rho_i)\}$.

When this stage is completed, each participant can get the messages as shown in table 1, where $\sigma_{i,i} = h_i + H(ID_i)^{r_i}$ is not be published to any other user in the group, but it is kept secret by the user U_i . Since the subgroups G_1 and G_2 are subgroups of an elliptic curve group, so the size of keys will reduce. Hence the key generation becomes more efficient.

Encryption key derivation: A user can compute the group encryption key $(R, E, \mathcal{H})$, where

$$R = \sum_{i=1}^n R_i, \quad E = \prod_{i=1}^n E_i \quad \text{and} \quad \mathcal{H} = \sum_{i=1}^n H(ID_i).$$

The group encryption key $(R, E, \mathcal{H})$ is accepted if all the signature pairs $(R_1, \rho_1), (R_2, \rho_2), \dots, (R_n, \rho_n)$ are valid.

Decryption key derivation: To get the group decryption key, the user U_i computes $d_i = \sum_{l=1}^n \sigma_{l,i}$ and accepts d_i if the signature pairs $(R_1, \rho_1), (R_2, \rho_2), \dots, (R_n, \rho_n)$ are valid.

Encryption: After knowing the public parameters and the group encryption key $(R, E, \mathcal{H})$, anyone can encrypt any message $m \in G_T$ by performing the following steps:

1. Select a random integer $t \in \mathbb{Z}_r$.
2. Compute $C_1 = tP, C_2 = tR$ and $C_3 = mE^t$.
3. Communicate the ciphertext $C = (C_1, C_2, C_3)$ to the receiver.

Decryption: To find the plaintext from the ciphertext, each participant U_i can decrypt

$$m = \frac{C_3}{e_T(C_1, d_i + \mathcal{H})e_T(C_2, H(ID_j)^{-1})}.$$

The correctness of the decryption procedure follows from a direct verification by putting the values of $C_1, C_2, C_3, d_i, \mathcal{H}, H(ID_j)$ and properties of pairing.

5. Security and performance analysis of proposed scheme

In this section, we describe the security and performance analysis of our proposed ASGKA scheme.

5.1 Security analysis

Security of proposed ASGKA scheme depends on k -Bilinear Diffie-Hellman Exponent (k -BDHE) problem in (G_1, G_2) , which is defined in subsection 2.3. Some desirable security attributes of key agreement protocols are identified in [29, 30, 31, 32, 33]. These are known-key security, unknown key-share and key-compromise impersonation etc. Similar to the protocol [11], our ASGKA scheme satisfies these security attributes. Diffie-Hellman problem is believed to be harder for elliptic curves than for finite fields because the subexponential algorithms that apply to finite fields do not translate to the elliptic curve setting, where the best available attacks remain generic, exponential algorithms like Pollard rho are applicable. This means that elliptic curve groups of relatively small size achieve the same security as multiplicative groups in much larger finite fields. Also our proposed scheme is identity based ASGKA scheme in type 3 setting and an identity based protocol is used to protect information confidentiality in critical infrastructure. So our proposed ASGKA scheme is more secure than the existing ASGKA protocols.

5.2 Performance analysis

In this subsection, we give the performance analysis of our proposed ASGKA scheme. Studying the process needed to carryout encryption and decryption in the scheme, we see that we need to do addition and subtraction on the elliptic curve. Although these operations are computationally harder than the corresponding operations on a finite field but the smaller key sizes used for elliptic curve cryptosystems more than make up for this difference. So in our protocol computational overheads and storage overheads are less than that in protocols based on standard pairings.

In our proposed ASGKA scheme the subgroups G_1 and G_2 are elliptic curve subgroups and we have proved in section 3 that the construction of subgroups of prime order- r of a pairing-friendly elliptic curve of embedding degree 1 is efficient. Our proposed ASGKA scheme is based on Type 3 Tate pairing on

subgroups G_1, G_2 and we have also proved that computation of improved Tate pairing for these subgroups is more efficient. We can save 24.9% of the time required to compute Tate pairing on these subgroups. Number of pairing operations in our protocols is less than the number of pairing operations in Zhang et al.'s scheme, which is based on Type 1 pairing. So if we implement proposed ASGKA scheme with improved Tate pairing on pairing-friendly elliptic curves of embedding degree 1 proposed by Wang et al., this scheme may become more efficient than the existing asymmetric group key agreement protocols based on standard pairings.

6. Conclusion

Design and implementation of efficient pairing-based cryptosystems has been the subject of ongoing research. In this paper, we studied pairing-friendly elliptic curves of embedding degree 1 for pairing-based cryptography. We proved that conventional classification of pairings into Type 1, 2, 3 and 4 is applicable for elliptic curves of embedding 1 proposed by Wang et al. We highlighted the selection of pairing-friendly elliptic curves of embedding degree 1 for design of efficient cryptosystems. We developed cryptosystems using these pairing-friendly elliptic curves of embedding degree 1 and compared the security and efficiency aspects of these cryptosystems with the standard cryptosystems. We also proposed a new asymmetric group key agreement scheme from Tate pairing on the selected pairing-friendly elliptic curves of embedding degree 1. The optimal trade off between security and efficiency of the proposed scheme makes it suitable for many of the current practical applications.

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ON GENERALIZED SOME INEQUALITIES FOR CONVEX FUNCTIONS

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Abstract. In this paper, a general integral identity for differentiable mapping is derived. Then, we extend some estimates of the right hand and left hand side of a Hermite-Hadamard-Fejér type inequality for functions whose first derivatives absolute values are convex. Some applications for special means of real numbers are also provided. The results presented here would provide extensions of those given in earlier works.

Keywords: Hermite-Hadamard-Fejér inequality, Trapezoid inequality, convex function, Hölder inequality.

1. Introduction

The following inequality is well known in the literature as the Hermite-Hadamard integral inequality (see, [2], [6]):

$$(1.1) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}$$

where $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$.

The inequalities (1.1) have grown into a significant pillar for mathematical analysis and optimization, besides, by looking into a variety of settings, these inequalities are found to have a number of uses. What is more, for a specific choice of the function f , many inequalities with special means are obtainable. Hermite Hadamard's inequality (1.1), for example, is significant in its rich geometry and hence there are many studies on it to demonstrate its new proofs,

refinements, extensions and generalizations. You can check ([1], [2], [6], [5] and [11]) and the references included there.

In [1], Dragomir and Agarwal proved the following results connected with the right part of (1.1).

Lemma 1. *Let $f : I^\circ \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° , $a, b \in I^\circ$ with $a < b$. If $f' \in L[a, b]$, then the following equality holds:*

$$(1.2) \quad \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx = \frac{b-a}{2} \int_0^1 (1-2t) f'(ta + (1-t)b) dt.$$

Theorem 1. *Let $f : I^\circ \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° , $a, b \in I^\circ$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then the following inequality holds:*

$$(1.3) \quad \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)}{8} (|f'(a)| + |f'(b)|).$$

Theorem 2. *Let $f : I^\circ \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° , $a, b \in I^\circ$ with $a < b$, $f' \in L(a, b)$ and $p > 1$. If the mapping $|f'|^{p/(p-1)}$ is convex on $[a, b]$, then the following inequality holds:*

$$(1.4) \quad \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{2(p+1)^{1/p}} \left(\frac{|f'(a)|^{p/(p-1)} + |f'(b)|^{p/(p-1)}}{2} \right)^{(p-1)/p}.$$

In [5], Kirmacı proved the following results connected with the left part of (1.1).

Lemma 2. *Let $f : I^\circ \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° , $a, b \in I^\circ$ with $a < b$. If $f' \in L(a, b)$, then we have*

$$\begin{aligned} & \frac{1}{b-a} \int_a^b f(s) ds - f\left(\frac{a+b}{2}\right) \\ &= (b-a) \left[\int_0^{\frac{1}{2}} t f'(ta + (1-t)b) dt + \int_{\frac{1}{2}}^1 (t-1) f'(ta + (1-t)b) dt \right]. \end{aligned}$$

Theorem 3. *Let $f : I^\circ \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° , $a, b \in I^\circ$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then we have*

$$(1.5) \quad \left| \frac{1}{b-a} \int_a^b f(s) ds - f\left(\frac{a+b}{2}\right) \right| \leq \frac{b-a}{8} (|f'(a)| + |f'(b)|).$$

The most well-known inequalities related to the integral mean of a convex function are the Hermite Hadamard inequalities or its weighted versions, the so-called Hermite-Hadamard-Fejér inequalities (see, [7, 16]). In [3], Fejér gave a weighted generalization of the inequalities (1.1) as the following:

Theorem 4. $f : [a, b] \rightarrow \mathbb{R}$, be a convex function, then the inequality

$$(1.6) \quad f\left(\frac{a+b}{2}\right) \int_a^b w(x)dx \leq \frac{1}{b-a} \int_a^b f(x)w(x)dx \leq \frac{f(a)+f(b)}{2} \int_a^b w(x)dx$$

holds, where $w : [a, b] \rightarrow \mathbb{R}$ is nonnegative, integrable, and symmetric about $x = \frac{a+b}{2}$.

In [7], some inequalities of Hermite-Hadamard-Fejer type for differentiable convex mappings were proved using the following lemma.

Lemma 3. Let $f : I^\circ \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° , $a, b \in I^\circ$ with $a < b$, and $w : [a, b] \rightarrow [0, \infty)$ be a differentiable mapping. If $f' \in L[a, b]$, then the following equality holds:

$$(1.7) \quad \begin{aligned} & \frac{f(a)+f(b)}{2} \int_a^b w(x)dx - \int_a^b f(x)w(x)dx \\ &= \frac{(b-a)^2}{2} \int_0^1 p(t)f'(ta+(1-t)b)dt \end{aligned}$$

for each $t \in [0, 1]$, where

$$p(t) = \int_t^1 w(as+(1-s)b)ds - \int_0^t w(as+(1-s)b)ds.$$

In this article, using functions whose derivatives absolute values are convex, we obtained new inequalities of Hermite-Hadamard-Fejer type. The results presented here would provide extensions of those given in earlier works.

2. Main results

In order to prove our main results, we need the following lemma:

Lemma 4. Let $f : I^\circ \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° , $a, b \in I^\circ$ with $a < b$ and let $g : [a, b] \rightarrow \mathbb{R}$. If $f', g \in L[a, b]$, then for all $x \in [a, b]$, the following identity holds:

$$(2.1) \quad \begin{aligned} & \int_a^b P_\lambda(x, t) f'(t) dt \\ &= (1-\lambda) f(x) \int_a^b g(s)ds + \lambda \left[f(a) \int_a^x g(s)ds + f(b) \int_x^b g(s)ds \right] \\ & \quad - \int_a^b g(s)f(s)ds \end{aligned}$$

where

$$P_\lambda(x, t) := \begin{cases} (1-\lambda) \int_a^t g(s)ds + \lambda \int_a^x g(s)ds & , a \leq t < x \\ (1-\lambda) \int_b^t g(s)ds + \lambda \int_x^t g(s)ds & , x \leq t \leq b. \end{cases}$$

for $\lambda \in [0, 1]$.

Proof. By integration by parts, we have the following identity:

$$\begin{aligned}
 & \int_a^b P_\lambda(x, t) f'(t) dt \\
 &= \int_a^x \left[(1-\lambda) \int_a^t g(s) ds + \lambda \int_x^t g(s) ds \right] f'(t) dt \\
 &+ \int_x^b \left[(1-\lambda) \int_b^t g(s) ds + \lambda \int_x^t g(s) ds \right] f'(t) dt \\
 &= \left[(1-\lambda) \int_a^t g(s) ds + \lambda \int_x^t g(s) ds \right] f(t) \Big|_{t=a}^x - \int_a^x g(s) f(s) ds \\
 &+ \left[(1-\lambda) \int_b^t g(s) ds + \lambda \int_x^t g(s) ds \right] f(t) \Big|_{t=x}^b - \int_x^b g(s) f(s) ds \\
 &= (1-\lambda) f(x) \int_a^b g(s) ds + \lambda f(a) \int_a^x g(s) ds \\
 &+ \lambda f(b) \int_x^b g(s) ds - \int_a^b g(s) f(s) ds.
 \end{aligned}$$

This completes the proof. $\square$

Remark 1. Under the same assumptions of Lemma 4 with $\lambda = 1$; then the following identity holds:

$$\begin{aligned}
 \int_a^b P_1(x, t) f'(t) dt &= \int_a^b \left[\int_x^t g(s) ds \right] f'(t) dt \\
 &= f(a) \int_a^x g(s) ds + f(b) \int_x^b g(s) ds - \int_a^b g(s) f(s) ds
 \end{aligned}$$

which is proved by Tseng et. al in [11].

Remark 2. Under the same assumptions of Lemma 4 with $\lambda = 0$; then the following identity holds:

$$\begin{aligned}
 \int_a^b P_0(x, t) f'(t) dt &= \int_a^x \left(\int_a^t g(s) ds \right) f'(t) dt - \int_x^b \left(\int_t^b g(s) ds \right) f'(t) dt \\
 &= f(x) \int_a^b g(s) ds - \int_a^b g(s) f(s) ds
 \end{aligned}$$

which is proved by Sarikaya and Erden in [8].

Remark 3. In Lemma 4, let g be symmetric to $\frac{a+b}{2}$ and let $x = \frac{a+b}{2}$. Then (2.1) can be written as

$$\begin{aligned}
 (2.2) \quad & \int_a^b P_\lambda\left(\frac{a+b}{2}, t\right) f'(t) dt = (1-\lambda) f\left(\frac{a+b}{2}\right) \int_a^b g(s) ds \\
 & + \lambda \frac{f(a) + f(b)}{2} \int_a^b g(s) ds - \int_a^b g(s) f(s) ds.
 \end{aligned}$$

Using this Lemma we can obtain the following general integral inequalities:

Theorem 5. Let $f : I^\circ \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° , $a, b \in I^\circ$ with $a < b$ and let $g : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$. If $|f'|$ is convex on $[a, b]$, then, for all $x \in [a, b]$, the following inequalities hold:

$$\begin{aligned}
 (2.3) \quad & \left| (1-\lambda) f(x) \int_a^b g(s) ds - \int_a^b g(s) f(s) ds \right. \\
 & \left. + \lambda \left[f(a) \int_a^x g(s) ds + f(b) \int_x^b g(s) ds \right] \right| \\
 & \leq \frac{\|g\|_{[a,x],\infty}}{6(b-a)} \left[|f'(a)| (x-a)^2 ((1-\lambda)(3b-a-2x) + \lambda(3b-2a-x)) \right. \\
 & \quad \left. + (2-\lambda) |f'(b)| (x-a)^3 \right] \\
 & \quad + \frac{\|g\|_{[x,b],\infty}}{6(b-a)} \left[(2-\lambda) |f'(a)| (b-x)^3 \right. \\
 & \quad \left. + |f'(b)| (b-x)^2 ((1-\lambda)(b-3a+2x) + \lambda(2b-3a+x)) \right] \\
 & \leq \frac{\|g\|_{[a,b],\infty}}{6(b-a)} \left\{ |f'(a)| (x-a)^2 ((1-\lambda)(3b-a-2x) + \lambda(3b-2a-x)) \right. \\
 & \quad + |f'(a)| (2-\lambda) (b-x)^3 + |f'(b)| (2-\lambda) (x-a)^3 \\
 & \quad \left. + |f'(b)| (b-x)^2 ((1-\lambda)(b-3a+2x) + \lambda(2b-3a+x)) \right\}.
 \end{aligned}$$

where $\lambda \in [0, 1]$ and $\|g\|_{[a,b],\infty} = \sup_{s \in [a,b]} |g(s)|$.

Proof. We take absolute of (2.1). Using bounded of the mapping g and the convexity of $|f'|$, we find that

$$\begin{aligned}
 & \left| (1-\lambda) f(x) \int_a^b g(s) ds - \int_a^b g(s) f(s) ds + \lambda \left[f(a) \int_a^x g(s) ds + f(b) \int_x^b g(s) ds \right] \right| \\
 & \leq \int_a^b |P_\lambda(x, t)| |f'(t)| dt \\
 & \leq \int_a^x \left((1-\lambda) \left| \int_a^t g(s) ds \right| + \lambda \left| \int_x^t g(s) ds \right| \right) \left| f' \left(\frac{b-t}{b-a} a + \frac{t-a}{b-a} b \right) \right| dt \\
 & \quad + \int_x^b \left((1-\lambda) \left| \int_b^t g(s) ds \right| + \lambda \left| \int_x^t g(s) ds \right| \right) \left| f' \left(\frac{b-t}{b-a} a + \frac{t-a}{b-a} b \right) \right| dt \\
 & \leq \|g\|_{[a,x],\infty} \int_a^x [(1-\lambda)(t-a) + \lambda(x-t)] \left(\frac{b-t}{b-a} |f'(a)| + \frac{t-a}{b-a} |f'(b)| \right) dt \\
 & \quad + \|g\|_{[x,b],\infty} \int_x^b [(1-\lambda)(b-t) + \lambda(t-x)] \left(\frac{b-t}{b-a} |f'(a)| + \frac{t-a}{b-a} |f'(b)| \right) dt \\
 & = \frac{\|g\|_{[a,x],\infty}}{b-a} \left[(1-\lambda) |f'(a)| \int_a^x (t-a)(b-t) dt + (1-\lambda) |f'(b)| \int_a^x (t-a)^2 dt \right. \\
 & \quad \left. + \lambda |f'(a)| \int_a^x (x-t)^2 dt + \lambda |f'(b)| \int_a^x (x-t)(b-t) dt \right] \\
 & \quad + \frac{\|g\|_{[x,b],\infty}}{b-a} \left[(1-\lambda) |f'(a)| \int_x^b (b-t)^2 dt + \lambda |f'(a)| \int_x^b (b-t)(t-x) dt \right. \\
 & \quad \left. + \lambda |f'(b)| \int_x^b (t-x)^2 dt + (1-\lambda) |f'(b)| \int_x^b (b-t)(t-x) dt \right]
 \end{aligned}$$

$$\begin{aligned}
& +\lambda |f'(a)| \int_a^x (x-t)(b-t) dt + \lambda |f'(b)| \int_a^x (x-t)(t-a) dt \Big] \\
& + \frac{\|g\|_{[x,b],\infty}}{b-a} \left[(1-\lambda) |f'(a)| \int_x^b (b-t)^2 dt + (1-\lambda) |f'(b)| \int_x^b (b-t)(t-a) dt \right. \\
& \left. + \lambda |f'(a)| \int_x^b (t-x)(b-t) dt + \lambda |f'(b)| \int_x^b (t-x)(t-a) dt \right].
\end{aligned}$$

Since

$$\begin{aligned}
\int_a^x (t-a)(b-t) dt &= \frac{(x-a)^2(3b-a-2x)}{6} \\
\int_a^x (t-a)^2 dt &= \frac{(x-a)^3}{3} \\
\int_a^x (x-t)(b-t) dt &= \frac{(x-a)^2(3b-2a-x)}{6} \\
\int_a^x (x-t)(t-a) dt &= \frac{(x-a)^3}{6}
\end{aligned}$$

and

$$\begin{aligned}
\int_x^b (b-t)^2 dt &= \frac{(b-x)^3}{3} \\
\int_x^b (b-t)(t-a) dt &= \frac{(b-x)^2(b-3a+2x)}{6} \\
\int_x^b (t-x)(b-t) dt &= \frac{(b-x)^3}{6} \\
\int_x^b (t-x)(t-a) dt &= \frac{(b-x)^2(2b-3a+x)}{6},
\end{aligned}$$

we obtain (2.3). Hence, this completes the proof. $\square$

Remark 4. Under the same assumptions of Theorem 5 with $\lambda = 1$; then the following inequality holds:

$$\begin{aligned}
& \left| f(a) \int_a^x g(s) ds + f(b) \int_x^b g(s) ds - \int_a^b g(s) f(s) ds \right| \\
& \leq \frac{\|g\|_{[a,x],\infty}}{(b-a)} \left[\frac{|f'(a)|(x-a)^2(3b-2a-x) + |f'(b)|(x-a)^3}{6} \right] \\
& + \frac{\|g\|_{[x,b],\infty}}{(b-a)} \left[\frac{|f'(a)|(b-x)^3 + |f'(b)|(b-x)^2(2b-3a+x)}{6} \right] \\
& \leq \frac{\|g\|_{[a,b],\infty}}{6(b-a)} \left\{ |f'(a)| \left[(x-a)^2(3b-2a-x) + (b-x)^3 \right] \right.
\end{aligned}$$

$$+ |f'(b)| \left[(b-x)^2 (2b-3a+x) + (x-a)^3 \right] \Big\}$$

which is proved by Tseng et. al in [11].

Remark 5. Under the same assumptions of Theorem 5 with $\lambda = 0$; then the following identity holds:

$$\begin{aligned} & \left| f(x) \int_a^b g(s) ds - \int_a^b g(s) f(s) ds \right| \\ & \leq \frac{\|g\|_{[a,x],\infty}}{(b-a)} \left[\frac{|f'(a)| (x-a)^2 (3b-a-2x) + 2|f'(b)| (x-a)^3}{6} \right] \\ & + \frac{\|g\|_{[x,b],\infty}}{(b-a)} \left[\frac{2|f'(a)| (b-x)^3 + |f'(b)| (b-x)^2 (b-3a+2x)}{6} \right] \\ & \leq \frac{\|g\|_{[a,b],\infty}}{6(b-a)} \left\{ |f'(a)| \left[(x-a)^2 (3b-a-2x) + 2(b-x)^3 \right] \right. \\ & \quad \left. + |f'(b)| \left[(b-x)^2 (b-3a+2x) \right] + 2(x-a)^3 \right\} \end{aligned}$$

which is proved by Sarikaya and Erden in [8].

Corollary 1. Let $0 \leq \alpha \leq 1$ and $x = \alpha a + (1-\alpha)b$ in Theorem 5. Then we have

$$\begin{aligned} (2.4) \quad & \left| (1-\lambda) f(\alpha a + (1-\alpha)b) \int_a^b g(s) ds + \lambda f(a) \int_a^{\alpha a + (1-\alpha)b} g(s) ds \right. \\ & \left. + f(b) \int_{\alpha a + (1-\alpha)b}^b g(s) ds - \int_a^b g(s) f(s) ds \right| \\ & \leq \|g\|_{[a, \alpha a + (1-\alpha)b], \infty} (b-a)^2 \left(|f'(b)| \frac{(2-\lambda)(1-\alpha)^3}{6} \right. \\ & \quad \left. + |f'(a)| \frac{(1-\alpha)^2 [(1-\lambda)(2\alpha+1) + \lambda(2+\alpha)]}{6} \right) \\ & + \|g\|_{[\alpha a + (1-\alpha)b, b], \infty} (b-a)^2 \left(|f'(a)| \frac{(2-\lambda)\alpha^3}{6} \right. \\ & \quad \left. + |f'(b)| \frac{\alpha^2 [(1-\lambda)(3-2\alpha) + \lambda(3-\alpha)]}{6} \right) \\ & \leq \|g\|_{[a,b], \infty} (b-a)^2 \\ & \times \left\{ |f'(a)| \frac{(1-\alpha)^2 [(1-\lambda)(2\alpha+1) + \lambda(2+\alpha)] + (2-\lambda)\alpha^3}{6} \right. \\ & \quad \left. + |f'(b)| \frac{\alpha^2 [(1-\lambda)(3-2\alpha) + \lambda(3-\alpha)] + (2-\lambda)(1-\alpha)^3}{6} \right\} \end{aligned}$$

for $\lambda \in [0, 1]$.

Remark 6. If we take $\lambda = 1$ in (2.4), we get

$$\begin{aligned} & \left| f(a) \int_a^{\alpha a + (1-\alpha)b} g(s) ds + f(b) \int_{\alpha a + (1-\alpha)b}^b g(s) ds - \int_a^b g(s) f(s) ds \right| \\ & \leq \|g\|_{[a, \alpha a + (1-\alpha)b], \infty} (b-a)^2 \left[\frac{|f'(b)| (1-\alpha)^3 + |f'(a)| (1-\alpha)^2 (2+\alpha)}{6} \right] \\ & \quad + \|g\|_{[\alpha a + (1-\alpha)b, b], \infty} (b-a)^2 \left[\frac{|f'(a)| \alpha^3 + |f'(b)| \alpha^2 (3-\alpha)}{6} \right] \\ & \leq \|g\|_{[a, b], \infty} (b-a)^2 \\ & \quad \times \left(\frac{|f'(a)| [(1-\alpha)^2 (2+\alpha) + \alpha^3] + |f'(b)| [\alpha^2 (3-\alpha) + (1-\alpha)^3]}{6} \right) \end{aligned}$$

which is proved by Tseng et. al in [11].

Using Theorem 5, we have the following corollary which are connected with the right-hand and left-hand side of Fejér inequality (1.6).

Corollary 2. Let $g : [a, b] \rightarrow \mathbb{R}$ be symmetric to $\frac{a+b}{2}$ and $\alpha = \frac{1}{2}$ in Corollary 1. Then we have the inequalities

$$\begin{aligned} (2.5) \quad & \left| (1-\lambda) f\left(\frac{a+b}{2}\right) \int_a^b g(s) ds \right. \\ & \left. + \lambda \frac{f(a) + f(b)}{2} \int_a^b g(s) ds - \int_a^b g(s) f(s) ds \right| \\ & \leq \|g\|_{[a, \frac{a+b}{2}], \infty} (b-a)^2 \left(\frac{4+\lambda}{48} |f'(a)| + \frac{2-\lambda}{48} |f'(b)| \right) \\ & \quad + \|g\|_{[\frac{a+b}{2}, b], \infty} (b-a)^2 \left(\frac{2-\lambda}{48} |f'(a)| + \frac{4+\lambda}{48} |f'(b)| \right) \\ & \leq \frac{\|g\|_{[a, b], \infty} (b-a)^2 (|f'(a)| + |f'(b)|)}{8} \end{aligned}$$

which is "weighted trapezoid" inequality provided that $|f'|$ is convex on $[a, b]$.

Remark 7. If we take $\lambda = 1$ in (2.5), we obtain

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} \int_a^b g(s) ds - \int_a^b g(s) f(s) ds \right| \\ & \leq \frac{(b-a)^2}{48} (5|f'(a)| + |f'(b)|) \|g\|_{[a, \frac{a+b}{2}], \infty} \\ & \quad + \frac{(b-a)^2}{48} (|f'(a)| + 5|f'(b)|) \|g\|_{[\frac{a+b}{2}, b], \infty} \\ & \leq \frac{(b-a)^2 (|f'(a)| + |f'(b)|)}{8} \|g\|_{[a, b], \infty} \end{aligned}$$

which is proved by Tseng et. al in [11].

Remark 8. If we take $\lambda = 0$ in (2.5), we get

$$\begin{aligned} & \left| f\left(\frac{a+b}{2}\right) \int_a^b g(s) ds - \int_a^b g(s) f(s) ds \right| \\ & \leq \frac{(b-a)^2}{24} (2|f'(a)| + |f'(b)|) \|g\|_{[a, \frac{a+b}{2}], \infty} \\ & \quad + \frac{(b-a)^2}{24} (|f'(a)| + 2|f'(b)|) \|g\|_{[\frac{a+b}{2}, b], \infty} \\ & \leq \frac{(b-a)^2}{8} (|f'(a)| + |f'(b)|) \|g\|_{[a, b], \infty} \end{aligned}$$

which is "weighted midpoint" inequality provided that $|f'|$ is convex on $[a, b]$.

Corollary 3. If we take $g(t) = 1$ in (2.5), we have

$$\begin{aligned} (2.6) \quad & \left| (1-\lambda) f\left(\frac{a+b}{2}\right) (b-a) + \lambda \frac{f(a) + f(b)}{2} (b-a) - \int_a^b f(s) ds \right| \\ & \leq \frac{(b-a)^2}{48} ((4+\lambda)|f'(a)| + (2-\lambda)|f'(b)|) \\ & \quad + \frac{(b-a)^2}{48} ((2-\lambda)|f'(a)| + (4+\lambda)|f'(b)|) \\ & \leq \frac{(b-a)^2}{8} (|f'(a)| + |f'(b)|). \end{aligned}$$

Remark 9. If we choose $\lambda = 1$ in (2.6), then the inequality (2.6) reduces to (1.3).

Remark 10. If we choose $\lambda = 0$ in (2.6), then the inequality (2.6) reduces to (1.5).

Theorem 6. Let $f : I^\circ \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° and let $f' \in L[a, b]$, $a, b \in I^\circ$ with $a < b$, and let $g : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$. If $|f'|^q$ is convex on $[a, b]$, $q > 1$, then for all $x \in [a, b]$, the following inequalities hold: for $\lambda \in [0, 1] \setminus \{\frac{1}{2}\}$

$$\begin{aligned} (2.7) \quad & \left| (1-\lambda) f(x) \int_a^b g(s) ds + \lambda [f(a) \int_a^x g(s) ds + f(b) \int_x^b g(s) ds] \right. \\ & \quad \left. - \int_a^b g(s) f(s) ds \right| \\ & \leq \left(\frac{(1-\lambda)^{p+1} - \lambda^{p+1}}{(p+1)(1-2\lambda)} \right)^{\frac{1}{p}} (b-a)^{\frac{1}{q}} [(x-a)^{p+1} + (b-x)^{p+1}]^{\frac{1}{p}} \\ & \quad \times \left[\frac{|f'(a)|^q + |f'(b)|^q}{2} \right]^{\frac{1}{q}} \|g\|_{[a, b], \infty} \end{aligned}$$

and for $\lambda = \frac{1}{2}$

$$\begin{aligned} & \left| \frac{f(x)}{2} \int_a^b g(s) ds + \frac{1}{2} \left[f(a) \int_a^x g(s) ds + f(b) \int_x^b g(s) ds \right] - \int_a^b g(s) f(s) ds \right| \\ & \leq \frac{\|g\|_{[a,b],\infty} (b-a)^{\frac{1}{q}}}{2} \left[\frac{|f'(a)|^q + |f'(b)|^q}{2} \right]^{\frac{1}{q}} [(x-a)^{p+1} + (b-x)^{p+1}]^{\frac{1}{p}} \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $\|g\|_{\infty} = \sup_{s \in [a,b]} |g(s)|$.

Proof. We take absolute value of (2.1). Using Hölder's inequality and the convexity of $|f'(t)|^q$, we find that

$$\begin{aligned} & \left| (1-\lambda)f(x) \int_a^b g(s) ds + \lambda \left[f(a) \int_a^x g(s) ds + f(b) \int_x^b g(s) ds \right] - \int_a^b g(s) f(s) ds \right| \\ & \leq \int_a^b |P_{\lambda}(x,t)| |f'(t)| dt \leq \left(\int_a^b |P_{\lambda}(x,t)|^p dt \right)^{\frac{1}{p}} \left(\int_a^b |f'(t)|^q dt \right)^{\frac{1}{q}} \\ & \leq \left(\int_a^x \left| (1-\lambda) \int_a^t g(s) ds + \lambda \int_x^t g(s) ds \right|^p dt \right. \\ & \quad \left. + \int_x^b \left| (1-\lambda) \int_b^t g(s) ds + \lambda \int_x^t g(s) ds \right|^p dt \right)^{\frac{1}{p}} \\ & \quad \times \left(\int_a^b \left[\frac{b-t}{b-a} |f'(a)|^q + \frac{t-a}{b-a} |f'(b)|^q \right] dt \right)^{\frac{1}{q}} \\ & = \left(\|g\|_{[a,x],\infty}^p \int_a^x [(1-\lambda)(t-a) + \lambda(x-t)]^p dt \right. \\ & \quad \left. + \|g\|_{[x,b],\infty}^p \int_x^b [(1-\lambda)(b-t) + \lambda(t-x)]^p dt \right)^{\frac{1}{p}} \\ & \quad \times (b-a)^{\frac{1}{q}} \left[\frac{|f'(a)|^q + |f'(b)|^q}{2} \right]^{\frac{1}{q}}. \end{aligned}$$

Now, we make change of variable

$$\begin{aligned} (2.8) \quad & (1-\lambda)(t-a) + \lambda(x-t) = u & dt &= \frac{du}{1-2\lambda} \\ & (1-\lambda)(b-t) + \lambda(t-x) = v & dt &= \frac{dv}{2\lambda-1}. \end{aligned}$$

From (2.8), it follows that

$$\left| (1-\lambda)f(x) \int_a^b g(s) ds + \lambda \left[f(a) \int_a^x g(s) ds + f(b) \int_x^b g(s) ds \right] - \int_a^b g(s) f(s) ds \right|$$

$$\begin{aligned}
&\leq \left(\frac{(1-\lambda)^{p+1} - \lambda^{p+1}}{(p+1)(1-2\lambda)} \right)^{\frac{1}{p}} \left[\|g\|_{[a,x],\infty}^p (x-a)^{p+1} + \|g\|_{[x,b],\infty}^p (b-x)^{p+1} \right]^{\frac{1}{p}} \\
&\times (b-a)^{\frac{1}{q}} \left[\frac{|f'(a)|^q + |f'(b)|^q}{2} \right]^{\frac{1}{q}} \\
&\leq \left(\frac{(1-\lambda)^{p+1} - \lambda^{p+1}}{(p+1)(1-2\lambda)} \right)^{\frac{1}{p}} \|g\|_{[a,b],\infty}^p [(x-a)^{p+1} + (b-x)^{p+1}]^{\frac{1}{p}} \\
&\times (b-a)^{\frac{1}{q}} \left[\frac{|f'(a)|^q + |f'(b)|^q}{2} \right]^{\frac{1}{q}}
\end{aligned}$$

we obtain the inequality (2.7).

For $\lambda = \frac{1}{2}$, because of Lemma 4, and using Hölder's inequality and the convexity of $|f'(t)|^q$, we find that

$$\begin{aligned}
&\left| \frac{f(x)}{2} \int_a^b g(s) ds + \frac{1}{2} \left[f(a) \int_a^x g(s) ds + f(b) \int_x^b g(s) ds \right] - \int_a^b g(s) f(s) ds \right| \\
&\leq \int_a^b \left| P_{\frac{1}{2}}(x, t) \right| |f'(t)| dt \leq \left(\int_a^b \left| P_{\frac{1}{2}}(x, t) \right|^p dt \right)^{\frac{1}{p}} \left(\int_a^b |f'(t)|^q dt \right)^{\frac{1}{q}} \\
&\leq \frac{1}{2} \left(\|g\|_{[a,x],\infty}^p \int_a^x [(t-a) + (x-t)] dt + \|g\|_{[x,b],\infty}^p \int_x^b [(b-t) + (t-x)] dt \right)^{\frac{1}{p}} \\
&\times (b-a)^{\frac{1}{q}} \left[\frac{|f'(a)|^q + |f'(b)|^q}{2} \right]^{\frac{1}{q}}.
\end{aligned}$$

Hence, the proof is completed. $\square$

Remark 11. Under the same assumptions of Theorem 6 with $\lambda = 1$; then the following inequality holds:

$$\begin{aligned}
&\left| f(a) \int_a^x g(s) ds + f(b) \int_x^b g(s) ds - \int_a^b g(s) f(s) ds \right| \\
&\leq \frac{\|g\|_{[a,b],\infty} (b-a)^{\frac{1}{q}}}{(p+1)^{\frac{1}{p}}} [(x-a)^{p+1} + (b-x)^{p+1}]^{\frac{1}{p}} \\
&\times \left[\frac{|f'(a)|^q + |f'(b)|^q}{2} \right]^{\frac{1}{q}}
\end{aligned}$$

which is proved by Tseng et. al in [11].

Corollary 4. *Under the same assumptions of Theorem 6 with $\lambda = 0$; then the following inequality holds:*

$$\begin{aligned} & \left| f(x) \int_a^b g(s) ds - \int_a^b g(s) f(s) ds \right| \\ & \leq \frac{(b-a)^{\frac{1}{q}}}{(p+1)^{\frac{1}{p}}} [(x-a)^{p+1} + (b-x)^{p+1}]^{\frac{1}{p}} \\ & \quad \times \left[\frac{|f'(a)|^q + |f'(b)|^q}{2} \right]^{\frac{1}{q}} \|g\|_{[a,b],\infty} \end{aligned}$$

which is "weighted Ostrowski" inequality provided that $|f'|^q$ is convex on $[a, b]$.

Corollary 5. *Let $0 \leq \alpha \leq 1$ and $x = \alpha a + (1-\alpha)b$ in Theorem 6. Then we have*

$$\begin{aligned} (2.9) \quad & \left| (1-\lambda) f(\alpha a + (1-\alpha)b) \int_a^b g(s) ds + \lambda f(a) \int_a^{\alpha a + (1-\alpha)b} g(s) ds \right. \\ & \left. + f(b) \int_{\alpha a + (1-\alpha)b}^b g(s) ds - \int_a^b g(s) f(s) ds \right| \\ & \leq \left(\frac{(1-\lambda)^{p+1} - \lambda^{p+1}}{(p+1)(1-2\lambda)} \right)^{\frac{1}{p}} (b-a)^2 [(1-\alpha)^{p+1} + \alpha^{p+1}]^{\frac{1}{p}} \\ & \quad \times \left[\frac{|f'(a)|^q + |f'(b)|^q}{2} \right]^{\frac{1}{q}} \|g\|_{[a,b],\infty} \end{aligned}$$

for $\lambda \in [0, 1]$.

Remark 12. If we take $\lambda = 1$ in (2.9), we get

$$\begin{aligned} & \left| f(a) \int_a^{\alpha a + (1-\alpha)b} g(s) ds + f(b) \int_{\alpha a + (1-\alpha)b}^b g(s) ds - \int_a^b g(s) f(s) ds \right| \\ & \leq \frac{(b-a)^2}{(p+1)^{\frac{1}{p}}} [(1-\alpha)^{p+1} + \alpha^{p+1}]^{\frac{1}{p}} \left[\frac{|f'(a)|^q + |f'(b)|^q}{2} \right]^{\frac{1}{q}} \|g\|_{[a,b],\infty} \end{aligned}$$

which is proved by Tseng et. al in [11].

Using Theorem 6, we have the following corollary which are connected with the right-hand and left-hand side of Fejér inequality (1.6).

Corollary 6. *Let $g : [a, b] \rightarrow \mathbb{R}$ be symmetric to $\frac{a+b}{2}$ and $\alpha = \frac{1}{2}$ in Corollary 5. Then we have the inequality*

$$\begin{aligned} (2.10) \quad & \left| (1-\lambda) f\left(\frac{a+b}{2}\right) \int_a^b g(s) ds + \lambda \frac{f(a) + f(b)}{2} \int_a^b g(s) ds - \int_a^b g(s) f(s) ds \right| \\ & \leq \left(\frac{(1-\lambda)^{p+1} - \lambda^{p+1}}{(p+1)(1-2\lambda)} \right)^{\frac{1}{p}} \frac{(b-a)^2}{2} \left[\frac{|f'(a)|^q + |f'(b)|^q}{2} \right]^{\frac{1}{q}} \|g\|_{[a,b],\infty}. \end{aligned}$$

Remark 13 (weighted trapezoid). If we take $\lambda = 1$ in (2.10), we obtain

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} \int_a^b g(s) ds - \int_a^b g(s) f(s) ds \right| \\ & \leq \frac{(b-a)^2}{2(p+1)^{\frac{1}{p}}} \left[\frac{|f'(a)|^q + |f'(b)|^q}{2} \right]^{\frac{1}{q}} \|g\|_{[a,b],\infty} \end{aligned}$$

which is proved by Tseng et. al in [11].

Remark 14. If we take $\lambda = 0$ in (2.10), we get

$$\begin{aligned} & \left| f\left(\frac{a+b}{2}\right) \int_a^b g(s) ds - \int_a^b g(s) f(s) ds \right| \\ & \leq \frac{(b-a)^2}{2(p+1)^{\frac{1}{p}}} \left[\frac{|f'(a)|^q + |f'(b)|^q}{2} \right]^{\frac{1}{q}} \|g\|_{[a,b],\infty} \end{aligned}$$

which is "weighted midpoint" inequality provided that $|f'|^q$ is convex on $[a, b]$ and $f' \in L(a, b)$ where $p > 1$.

Corollary 7. If we take $g(t) = 1$ in (2.10), we have

$$\begin{aligned} (2.11) \quad & \left| (1-\lambda) f\left(\frac{a+b}{2}\right) (b-a) + \lambda \frac{f(a) + f(b)}{2} (b-a) - \int_a^b f(s) ds \right| \\ & \leq \left(\frac{(1-\lambda)^{p+1} - \lambda^{p+1}}{(p+1)(1-2\lambda)} \right)^{\frac{1}{p}} \frac{(b-a)^2}{2} \left[\frac{|f'(a)|^q + |f'(b)|^q}{2} \right]^{\frac{1}{q}} \end{aligned}$$

Remark 15. If we choose $\lambda = 1$ in (2.11), then the inequality (2.11) reduces to (1.4).

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A NEW PROOF FOR THE GLOBAL CONVERGENCE OF THE BFGS METHOD FOR NONCONVEX UNCONSTRAINED MINIMIZATION PROBLEMS

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Abstract. In this paper we give a new proof for the global convergence of the BFGS method for nonconvex unconstrained minimization problems and we prove that the condition of the appropriate method is to satisfy implicitly with inaccurate linear search of Wolfe type. Furthermore, we have checked directly the convergence of the method BFGS with the inaccurate linear search of Wolfe.

Keywords: Wolfe type, global convergence, BFGS method, nonconvex unconstrained minimization problems.

1. Introduction

The BFGS method is a known quasi-Newton method and has been used extensively for solving unconstrained minimization problems in the past two decades [4], [8] and [9]. The global convergence quasi-Newton methods have also established especially for convex unconstrained minimization problems [2], [3], [8], [14], [15], [16] and [17]. However, in [11], the authors studied the nonconvex case. They also proposed a modified BFGS method with global and superlinear convergence. Moreover, the global convergence result for nonlinear equations is due to Griewank [14] for Broyden's rank one method. However, a potential

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trouble with the mentioned method is that the line search may not be executed finitely in a certain special situation ([14], pp. 81-82).

On the other hand, little is known concerning global convergence of the BFGS method for nonconvex optimization problems. In fact, the global convergence of the BFGS method for nonconvex minimization problems has not been proved until now by any one or has given a counter example that shows nonconvergence of the BFGS method. Whether the BFGS method converges globally for a nonconvex function remains unanswered.

In recent work [12], the authors proposed a globally convergent Gauss–Newton-based BFGS method for symmetric nonlinear equations which contain unconstrained optimization problems as a special case. The results obtained in [12] and [11] positively support the unsolved problem. However, their question still remains unanswered. Then in [11], the authors studied the last motioned problem of whether the BFGS method with inexact line search converges globally when applied to nonconvex unconstrained minimization problems. In addition, they proposed a cautious BFGS update and proved that the method with either a Wolfe type [18] or an Armijo-type [13] line search converges globally if the function to be minimized has Lipschitz continuous gradients.

The purpose of this paper is to study this problem further which an extension on the work of Li and Fukushima in [13], as we have mentioned above, the authors proved the convergence of an appropriate method for the BFGS, but they did not completely prove that their method converged to the BFGS. On the other hand, in the current paper we are interested to prove that the condition of the appropriate method is satisfied implicitly with inaccurate linear search of Wolfe type. Furthermore, we have checked directly the convergence of the method BFGS with the inaccurate linear search of Wolfe.

The outline of the paper is as follows: In section 2, we introduce some necessary notations and introduce BFGS method with appropriate update. Then in section 3, we propose an algorithm in order to analyze the convergence of the BFGS method. In addition, in sections 3 and 4, we give a new proposed algorithm and its global convergence with the linear search of Wolfe type is proved. Furthermore, the convergence of the BFGS method for nonconvex unconstrained minimization problem is given as well. We prove that this method is interpreted according to the BFGS method.

2. Global convergence of the BFGS method in the nonconvex case

We introduce some notation: Consider $M \geq 0$ (resp. $M > 0$) is a symmetric positive semi-definite matrix (resp. positive definite) and define the following sets

$$(2.1) \quad S_+^n := \{M \in S^n : M \geq 0, n \in \mathbb{N}\}$$

and

$$(2.2) \quad S_{++}^n := \{M \in S^n : M > 0, n \in \mathbb{N},\}$$

where S^n symmetric matrices of order n such that:

We know that in [1]: to impose on M_{k+1} ($k \in \mathbb{N}$) to be close to M_k ($k \in \mathbb{N}$) and minimize the gap between M_{k+1} and M_k , still requesting that to M_{k+1} is symmetric and satisfies the equation of quasi-Newton. We are thus led to consider the problem in the following variable matrix:

$$(2.3) \quad \begin{cases} \min(\text{gap})(M, M_k), \\ y_k = M s_k, \quad M \in \mathbb{R}^{n \times n}, \\ M = M^\top. \end{cases}$$

So, we say that the matrix is obtained by the variational approach. It is often useful to impose also the positive definition of matrices M_k . Because, for $d_k = -M_k^{-1}g_k$ is a descent direction, M_k is positive definite must be needed, symmetric matrix, for more detail, in fact, we know that any real symmetric and positive definite matrix is invertible, and its inverse is also positive definite. Therefore, it can be written:

$$(2.4) \quad g_k^\top d_k = -g_k^\top M_k^{-1} g_k < 0,$$

so, d_k is a descent direction. This condition defining an open set that cannot directly be used as a constrained in defining the problem M_{k+1} and for this purpose, we first introduce the following function:

$$(2.5) \quad \psi : S^n \rightarrow \mathbb{R},$$

whose domain is S_{++}^n and forms a "barrier" to the edge of the cone S_{++}^n (it devolves to infinity when its argument approaches the edge of S_{++}^n) and infinity:

$$(2.6) \quad \psi(\Upsilon) = \text{tr} \Upsilon + ld \Upsilon,$$

where the function log-determinant $ld : S^n \rightarrow \mathbb{R} \cup \{+\infty\}$ is defined $\Upsilon \in S^n$ by

$$ld(\Upsilon) = \begin{cases} -\log \det \Upsilon, & \text{if } \Upsilon \in S_{++}^n, \\ +\infty, & \text{otherwise.} \end{cases}$$

It can be given the following properties of ψ defined in (2.6):

If we denote $\{\alpha_i\}_{i=1,\dots,n}$ eigenvalues of Υ , it should be got

$$(2.7) \quad \text{tr} \Upsilon = \sum_{i=1}^n \alpha_i \text{ and } \det \Upsilon = \prod_{i=1}^n \alpha_i$$

and so

$$(2.8) \quad \psi(\Upsilon) = \sum_{i=1}^n (\alpha_i - \log \alpha_i), \text{ if } \Upsilon \in S_{++}^n.$$

Being given the shape

$$(2.9) \quad t \in \mathbb{R}_{++} \mapsto t - \log t, \psi(\Upsilon)$$

tends to infinity if one of the eigenvalues of Υ tends to zero or to infinity, i.e.,

$$(2.10) \quad \exists j \in \{1, \dots, n\}; \lim_{\alpha_j \rightarrow 0 \text{ or } \infty} \psi(\Upsilon) = \infty.$$

Formula (2.8) also shows that the only minimizer of ψ is $\Upsilon = I$ the identity matrix.

If M_k is areal symmetric matrix then the matrix M_k is positive definite if and only if there exists a positive definite matrix A_k as: $A_k^2 = M_k$ such that, the positive definite matrix A_k and it can be put that $A_k = M_k^{\frac{1}{2}}$ is a unique.

In fact, if M_k is a real symmetric matrix, then we can write:

$$(2.11) \quad U^\top M_k U = \Lambda$$

where U satisfies

$$(2.12) \quad U^\top U = U U^\top = I$$

and Λ is a diagonal matrix where the diagonal elements are the eigenvalues of M_k which are strictly positive. Because M_k is a positive definite matrix. Thus we can write

$$(2.13) \quad M_k = U \Lambda U^\top = U \Lambda^{\frac{1}{2}} \Lambda^{\frac{1}{2}} U^\top = (U \Lambda^{\frac{1}{2}} U^\top) (U \Lambda^{\frac{1}{2}} U^\top) = M_k^{\frac{1}{2}} M_k^{\frac{1}{2}}.$$

It means that we must find a matrix M that is symmetric and positive definite and be close to M_k . Therefore,

$$(2.14) \quad M \approx M_k^{\frac{1}{2}} M_k^{\frac{1}{2}}$$

implies

$$(2.15) \quad M_k^{-\frac{1}{2}} M M_k^{-\frac{1}{2}} \approx I.$$

In order to minimize the gap between M and M_k , we seek that $M_k^{-\frac{1}{2}} M M_k^{-\frac{1}{2}}$ is close to I ; and this can be get by minimizing the term $\psi(M_k^{-\frac{1}{2}} M M_k^{-\frac{1}{2}})$, so that we shall get M_{k+1} close to M_k by solving :

$$(2.16) \quad \begin{cases} \min \psi(M_k^{-\frac{1}{2}} M M_k^{-\frac{1}{2}}), \\ y_k = M s_k, \\ M \in S_{++}^n \text{ (implicit constraint).} \end{cases}$$

If $s_k = 0$ and $y_k \neq 0$, then (2.16) has no solution or if $s_k = 0$ and $y_k = 0$ so the solution of (2.16) is $M = M_k$, otherwise; the non-trivial case where $s_k \neq 0$ is discussed in the following proposition.

Proposition 1 ([1]). *We assume that M_k is symmetric positive definite and that $s_k \neq 0$. Then, the problem (2.16) has a solution if and only if $y_k^\top s_k > 0$. Under this condition the solution M_{k+1} of (2.16) is unique and is only given by one of the following formulas :*

$$(2.17) \quad M_{k+1} = M_k + \frac{y_k y_k^\top}{y_k^\top s_k} - \frac{M_k s_k s_k^\top M_k}{s_k^\top M_k s_k}$$

and

$$(2.18) \quad B_{k+1} = \left(I - \frac{s_k y_k^\top}{y_k^\top s_k} \right) B_k \left(I - \frac{y_k s_k^\top}{y_k^\top s_k} \right) + \left(I - \frac{s_k s_k^\top}{y_k^\top s_k} \right),$$

where $B_k := M_k^{-1}$ and $B_{k+1} := M_{k+1}^{-1}$.

2.1 Algorithm 1 for BFGS Method

We give the following algorithm

Initial step:

Let $\varepsilon > 0$ be a determined criterion of stopping. Choose $\varkappa_1$ be an initial point and M_1 be any positive definite (e.g. : $M_1 = I$).

Put $k = 1$ and go to the main stages

Main stages.

Step 1:

If $\|\nabla f(\varkappa_k)\| < \varepsilon$ STOP; otherwise, put $d_k = -M_k g_k$ and determine the optimal step λ_k optimal solution of problem

$$(2.19) \quad \min f(\varkappa_k + \lambda d_k), \lambda \geq 0$$

and putting

$$(2.20) \quad \varkappa_{k+1} = \varkappa_k + \lambda_k d_k.$$

Step 2:

Do M_{k+1} as follows:

$$(2.21) \quad M_{k+1} = M_k + \frac{y_k y_k^\top}{y_k^\top s_k} - \frac{M_k s_k s_k^\top M_k}{s_k^\top M_k s_k}$$

with

$$(2.22) \quad \begin{cases} s_k = \varkappa_{k+1} - \varkappa_k, \\ y_k = \nabla f(\varkappa_{k+1}) - \nabla f(\varkappa_k). \end{cases}$$

Replace k by $k + 1$ and go to step 1.

2.2 BFGS method with appropriate update

It has been seen that the properties of the BFGS formula is that the matrix M_{k+1} inherits the positive definiteness of M_k if the condition $y_k^\top s_k > 0$ is checked. It can be noted if one uses an exact linear search or inexact search of Wolf, then the condition $y_k^\top s_k > 0$ is checked. where as, linear search of Armijo [13] does not guarantee this condition, and therefore M_{k+1} is not necessarily positive definite even if M_k is positive definite. To ensure the positive definiteness of M_{k+1} , the condition $y_k^\top s_k > 0$ is sometimes used to decide whether M_{k+1} is an update or not, i.e. we set

$$(2.23) \quad M_{k+1} = \begin{cases} M_k + \frac{y_k y_k^\top}{y_k^\top s_k} - \frac{M_k s_k s_k^\top M_k}{s_k^\top M_k s_k}, & \text{if } y_k^\top s_k > 0, \\ M_k, & \text{otherwise.} \end{cases}$$

The condition $y_k^\top s_k > 0$ is often replaced by $y_k^\top s_k > \eta$ where $\eta > 0$ is a small constant. Li and Fukushima [11] and [12] often appropriate update to the BFGS method similar to what is mentioned before and stating from this they establish a global convergence theorem for nonconvex problems. Before describes the appropriate update, first, we shall need the following important lemma due to Powell [16] which will be useful later.

Lemma 1. (Powell [16]) *If the BFGS method with Wolfe linear research (wolfe1)-(wolfe2) [18] is applied to a function f which is continuously differentiable; and if there exists a constant $c > 0$ such as:*

$$(2.24) \quad \frac{\|y_k\|^2}{y_k^\top s_k} \leq c, \quad \text{for all } k \in \mathbb{N}.$$

Then we have

$$(2.25) \quad \lim_{k \rightarrow \infty} \inf \|g_k\| = 0.$$

Remark 1. If f is two times continuously differentiable and strictly convex, so we shall always get the inequality (2.24), but in the case where f is not convex, it is difficult to guarantee (2.24). May be it is one of the reasons why the global convergence of the BFGS method has not been proven.

Now, we shall present the BFGS method with appropriate update and show later that is globally convergent without the economic function be convex. To be more precise, we determine M_{k+1} depending on M_k function :

$$(2.26) \quad M_{k+1} = \begin{cases} M_k + \frac{y_k y_k^\top}{y_k^\top s_k} - \frac{M_k s_k s_k^\top M_k}{s_k^\top M_k s_k}, & \text{if } \frac{y_k^\top s_k}{\|s_k\|^2} \geq \varepsilon \|g_k\|^\alpha, \\ M_k, & \text{otherwise,} \end{cases}$$

where ε and α are positive constants.

2.3 Algorithm 2 of BFGS method with appropriate update

We give the following algorithm:

Step 0. Choose an initial point $\varkappa_0 \in \mathbb{R}^n$ with an initial matrix $M_0 \in \mathbb{R}^{n \times n}$ which is symmetric and positive definite choose the constants

$$0 < \sigma_1 < \sigma_2 < 1, \alpha > 0 \text{ and } \varepsilon > 0.$$

Let $k = 0$

Step 1: Solve the linear equation $M_k d_k + g_k = 0$ to have d_k .

Step 2: Determine the domain $\lambda_k > 0$ by the inexact linear search of Wolfe or Armijo[18].

Step 3: Calculate $\varkappa_{k+1} := \varkappa_k + \lambda_k d_k$.

Step 4: Determine M_{k+1} by

$$(2.27) \quad M_{k+1} = \begin{cases} M_k + \frac{y_k y_k^\top}{y_k^\top s_k} - \frac{M_k s_k s_k^\top M_k}{s_k^\top M_k s_k}, & \text{if } \frac{y_k^\top s_k}{\|s_k\|^2} \geq \varepsilon \|g_k\|^\alpha \\ M_k, & \text{otherwise} \end{cases}$$

with

$$(2.28) \quad \begin{cases} s_k = \varkappa_{k+1} - \varkappa_k, \\ y_k = \nabla f(\varkappa_{k+1}) - \nabla f(\varkappa_k). \end{cases}$$

Step 5: Replace k by $k + 1$ and go to step1.

Remark 2. It is not difficult to see that the matrix M_k generated by algorithm 2 are symmetric and positive definite for all $k \in \mathbb{N}$. This implies that only with the use of inexact linear search of Wolfe or Armijo, we can obtain that the sequence $\{f(\varkappa_k)\}_{k \in \mathbb{N}}$ is decreasing. Also, we have the considerations of the following: (wolfe1) or (wolfe1)-(wolfe2) and if f is an inferiorly bounded:

$$(2.29) \quad -\sum_{k=0}^{\infty} g_k^\top s_k < \infty.$$

This implies that

$$(2.30) \quad -\lim_{k \rightarrow \infty} (-g_k^\top s_k) = 0$$

and since

$$(2.31) \quad s_k = \varkappa_{k+1} - \varkappa_k = \lambda_k d_k.$$

Thus, we have

$$(2.32) \quad -\lim_{k \rightarrow \infty} (\lambda_k g_k^\top d_k) = 0.$$

3. Global convergence of BFGS method with an appropriate update

In this section, we shall prove the global convergence of algorithm 1 under the following hypothesis:

Hypothesis 1. Consider the following set

$$(3.1) \quad \Omega = \{\varkappa \in IR^n / f(\varkappa) \leq f(\varkappa_0)\}.$$

We assume that Ω is contained in a bounded convex set D and that the economic function f is continuously differentiable on D and there exists a constant $l > 0$ such as:

$$(3.2) \quad \|g(\varkappa) - g(y)\| \leq l \|\varkappa - y\|, \text{ for all } \varkappa, y \in D \text{ i.e., } f \in C^{1,1}(D).$$

Since the sequence $\{f(\varkappa_k)\}_{k \in \mathbb{N}}$ is decreasing, it is clear that the sequence $\{\varkappa_k\}_{k \in \mathbb{N}}$ generated by the algorithm 2 is contained in Ω . Because

$$f(\varkappa_{k+1}) < f(\varkappa_k) < \dots < f(\varkappa_1) < f(\varkappa_0).$$

Define the following sets of indices:

$$(3.3) \quad \bar{K} = \left\{ i \setminus \frac{y_i^\top s_i}{\|s_i\|^2} \geq \varepsilon \|g_i\|^\alpha \right\}$$

and

$$\bar{K}_k = \{i \in \bar{K} \setminus i \leq k\}.$$

We note through i_k , the set of indices $i \in \bar{K}_k$.

We can rewrite (2.26) of the form:

$$(3.4) \quad M_{k+1} = \begin{cases} M_k + \frac{y_k y_k^\top}{y_k^\top s_k} - \frac{M_k s_k s_k^\top M_k}{s_k^\top M_k s_k}, & \text{if } k \in \bar{K}, \\ M_k, & \text{otherwise} \end{cases}$$

and considering the trace of both sides of (3.4), we can write for any $k \in \mathbb{N}$

$$(3.5) \quad Tr(M_{k+1}) = Tr(M_1) + \sum_{i \in \bar{K}_k} \frac{\|y_i\|^2}{y_i^\top s_i} - \sum_{i \in \bar{K}_k} \frac{\|M_i s_i\|^2}{s_i^\top M_i s_i}.$$

Theorem 1. *We assume that the hypothesis 1 is true and be $\{\varkappa_k\}_{k \in \mathbb{N}}$ the sequence generated by algorithm 2. If $\bar{K}$ is a finite set, so*

$$(3.6) \quad \lim_{k \rightarrow \infty} \|g_k\| = 0.$$

Proof. If $\overline{K}$ is a finite set, so, there exists an index k_0 such as

$$M_k = M_{k_0} \triangleq M \text{ for all } k \geq k_0.$$

By the positive definiteness of M , there exists positive constants $c_1 \leq C_1$ such as

$$(3.7) \quad \begin{cases} c_1 \|d\|^2 \leq d^\top M d \leq C_1 \|d\|^2, \\ c_1 \|d\|^2 \leq d^\top M^{-1} d \leq C_1 \|d\|^2, \text{ for all } d \in \mathbb{R}^n. \end{cases}$$

1-If the inexact linear search is of Wolfe, by using (3.2) we have

$$l \|s_k\|^2 \geq \left\| \left(g^\top(\kappa_{k+1}) - g^\top(\kappa_k) \right) s_k \right\| \geq y_k^\top s_k$$

and by using (3.7), we have

$$\begin{aligned} \|s_k\|^2 &\geq \sigma_2 g_k^\top s_k - g_k^\top s_k \\ &\geq (1 - \sigma_2) \lambda_k^\top s_k^\top M^{-1} s_k \\ &\geq (1 - \sigma_2) \lambda_k^\top c_1 \|s_k\|^2, \text{ for all } k \geq k_0, \end{aligned}$$

where

$$s_k = \lambda_k d_k$$

and

$$d_k^\top = \lambda_k^{-1} s_k^\top \text{ and } g_k^\top = -d_k^\top M^{-1}.$$

Therefore, it can be deduced

$$\lambda_k \geq (1 - \sigma_2) c_1 l^{-1}, \text{ for all } k \geq k_0.$$

Thus, we get from (3.2)

$$\lim_{k \rightarrow \infty} \lambda_k g^\top d_k = \lim_{k \rightarrow \infty} \left(-g_k^\top s_k \right) = 0$$

with

$$\lambda_k \geq (1 - \sigma_2) c_1 l^{-1} > 0, \text{ for all } k \geq k_0.$$

We can write

$$g_k^\top M^{-1} g_k = -g_k^\top d_k \rightarrow 0$$

and from (3.7) we have :

$$c_1 \|g_k\|^2 \leq g_k^\top M^{-1} g_k \rightarrow 0,$$

then

$$c_1 \|g_k\|^2 \rightarrow 0$$

with $c_1 \neq 0$, so

$$\lim_{k \rightarrow \infty} \|g_k\| = 0.$$

□

Now we shall prove the global convergence of algorithm 2 in the case where $\bar{K}$ is an infinite set. If by way of contradiction, there exists a constant $\delta > 0$ such as

$$(3.8) \quad \|g_k\| \geq \delta, \text{ for all } k \in \mathbb{N}$$

and we shall see that this produces a contradiction. Before establishing the global convergence theorem of the algorithm 2, we first show some useful lemmas.

Lemma 2. *We assume that the hypothesis 1 be true and that $\{\varkappa_k\}_{k \in \mathbb{N}}$ be a sequence generated by algorithm 2, assume also that the relation (3.8) be true for all $k \in \mathbb{N}$ so, there exists a constant $c_2 > 0$ such as:*

$$(3.9) \quad \text{Tr}(M_{k+1}) \leq c_2 i_k$$

and

$$(3.10) \quad \sum_{i \in \bar{K}_k} \frac{\|M_i s_i\|^2}{s_i^\top M_i s_i} \leq c_2 i_k,$$

for all k sufficiently dig.

Proof. By using (3.3) and (3.8), we have for all $i \in \bar{K}$

$$y_i^\top s_i \geq \varepsilon \|g_i\|^\alpha \|s_i\|^2 \geq \varepsilon \delta^\alpha \|s_i\|^2$$

implies

$$(3.11) \quad y_i^\top s_i \geq \varepsilon \delta^\alpha \|s_i\|^2.$$

Under (3.2) and (3.11), we have for all $i \in \bar{K}$

$$\|g_{i+1} - g_i\| \leq l \|\varkappa_{i+1} - \varkappa_i\|.$$

Therefore,

$$\|y_i\|^2 \leq l^2 \|s_i\|^2.$$

Since

$$\frac{1}{y_i^\top s_i} \leq \frac{1}{\varepsilon \delta^\alpha \|s_i\|^2},$$

we have

$$(3.12) \quad \frac{\|y_i\|^2}{y_i^\top s_i} \leq \frac{l^2}{\varepsilon \delta^\alpha} \triangleq c'_2.$$

Since

$$\begin{aligned} \text{Tr}(M_{k+1}) &= \text{Tr}(M_1) + \sum_{i \in \bar{K}} \frac{\|y_i\|^2}{y_i^\top s_i} - \underbrace{\sum_{i \in \bar{K}} \frac{\|M_i s_i\|^2}{s_i^\top M_i s_i}}_{\text{positive term}} \\ &\leq \text{Tr}(M_1) + \sum_{i \in \bar{K}} \frac{\|y_i\|^2}{y_i^\top s_i} \leq i_k (c_0 + c'_2) = i_k c_2. \end{aligned}$$

Putting

$$c_2 = \max(c_0, c'_2).$$

Thus, we have:

$$\text{Tr}(M_{k+1}) \leq i_k c_2.$$

Since $\text{Tr}(M_{k+1}) > 0$ for any $k \in \mathbb{N}$, we get from (3.5) and (3.12),

$$0 < \text{Tr}(M_1) + \sum_{i \in \bar{K}_k} \frac{\|y_i\|^2}{y_i^\top s_i} - \sum_{i \in \bar{K}_k} \frac{\|M_i s_i\|^2}{s_i^\top M_i s_i}$$

implies

$$\sum_{i \in \bar{K}_k} \frac{\|M_i s_i\|^2}{s_i^\top M_i s_i} < \text{Tr}(M_1) + i_k c'_2 < i_k c_2.$$

Therefore,

$$\sum_{i \in \bar{K}_k} \frac{\|M_i s_i\|^2}{s_i^\top M_i s_i} \leq c_2 i_k.$$

□

4. Global convergence of algorithm 2 with the linear search of Wolfe type

For this purpose, we prove first the following lemma as lemma 2.

Lemma 3. *We assume that the hypothesis 1 be a true. Let $\{\varkappa_k\}_{k \in \mathbb{N}}$ be a sequence generated by algorithm 2 with λ_k determined by the search linear of Wolfe (wolfe1)-(wolfe2) [18]. If we have (3.5) for all $k \in \mathbb{N}$, so there exists a constant $c_3 > 0$ such that for all k big enough we have:*

$$(4.1) \quad \prod_{i \in \tilde{K}_k} \lambda_i \geq c_3^{i_k}.$$

Proof. *The formula (3.4) gives the following recurrence relation*

$$(4.2) \quad \det(M_{i+1}) = \det(M_i) \left(\frac{y_i^\top s_i}{s_i^\top M_i s_i} \right), \text{ for all } i \in \bar{K}$$

and

$$(4.3) \quad \det(M_{i+1}) = \det(M_i), \text{ for all } i \notin \overline{K}.$$

If we note through n_k the largest index in the set $\overline{K}$, so we can write:

$$(4.4) \quad \det(M_{n_k+1}) = \det(M_1) \prod_{i \in \overline{K}_k} \frac{y_i^\top s_i}{s_i^\top M_i s_i}.$$

On the other hand, from (wolfe2) we get:

$$g^\top (\kappa_i + \lambda_i d_i) d_i \geq \sigma_2 g^\top (\kappa_i) d_i$$

implies

$$(4.5) \quad \begin{aligned} y_i^\top s_i &= (g_{i+1} - g_i)^\top s_i \geq \sigma_2 g_i^\top s_i - g_i^\top s_i \\ &\geq -(1 - \sigma_2) g_i^\top s_i = (1 - \sigma_2) \lambda_i^{-1} s_i^\top M_i s_i, \end{aligned}$$

where

$$g^\top = -d_i^\top M_i = -\lambda_i^{-1} s_i^\top M_i.$$

Similarly to the proof of Lemma 2, we obtain (4.1) by using the last inequality (4.5), (3.9) up to (4.4)

Indeed: from the last inequality (4.5), we can write

$$\prod_{i \in \overline{K}_k} \frac{y_i^\top s_i}{s_i^\top M_i s_i} \geq \prod_{i \in \overline{K}_k} \frac{1 - \sigma_2}{\lambda_i}$$

with (4.4), we can deduce

$$(4.6) \quad \det(M_{n_k+1}) \geq \det(M_1) \prod_{i \in \overline{K}_k} \frac{1 - \sigma_2}{\lambda_i}$$

or

$$(4.7) \quad \det(M_{n_k+1}) \leq \left[\frac{\text{Tr}(M_{n_k+1})}{n} \right]^n.$$

Using (4.6), (4.7) and from (3.9)

$$\begin{aligned} \prod_{i \in \overline{K}_k} \frac{1 - \sigma_2}{\lambda_i} &\leq \frac{\det(M_{n_k+1})}{\det(M_1)} \leq \frac{1}{\det(M_1)} \left[\frac{\text{Tr}(M_{n_k+1})}{n} \right]^n \leq \frac{1}{\det(M_1)} \left[\frac{c_2 i_k}{n} \right]^n \\ &\leq \frac{1}{\det(M_1)} \left[\frac{c_2 i_k}{n} \right]^n \\ &\leq \frac{1}{\det(M_1) n^n} [c_2^{i_k}]^n \\ &\leq \frac{1}{\det(M_1) n^n} [c_2^n]^{i_k}, \end{aligned}$$

so, there exists constant c_3 such as

$$\prod_{i \in \tilde{K}_k} \lambda_i \geq c_3^{i_k}.$$

□

Now we are able to prove the global convergence of algorithm 2 with the linear search of Wolfe which given by the following theorem:

Theorem 2. Assume that the hypothesis 1 is true. Be $\{\lambda_k\}_{k \in \mathbb{N}}$ a sequence generated by algorithm 2 with λ_k , $k \in \mathbb{N}$ determined by the linear search of Wolfe (wolfe1)-(wolfe2). So, we have

$$(4.8) \quad \liminf_{k \rightarrow \infty} \|g_k\| = 0.$$

Proof. Taking into consideration the theorem 1, It is sufficient to check (4.8) in the case where $\bar{K}$ is infinite. We note $\bar{K}$ by

$$\bar{K} = \{k_1 < k_2 < \dots < k_n\}.$$

We observe that (2.29) gives that:

$$-\sum_{j=0}^{\infty} g_{k_j}^\top s_{k_j} < \infty$$

and as, $M_{k_j} s_{k_j} = -\lambda_{k_j} g_{k_j}$, then

$$(4.9) \quad \sum_{j=1}^{\infty} \|g_{k_j}\|^2 \lambda_{k_j} \frac{s_{k_j}^\top M_{k_j} s_{k_j}}{\|M_{k_j} s_{k_j}\|^2} = -\sum_{j=0}^{\infty} g_{k_j}^\top s_{k_j} < \infty.$$

Since

$$\begin{aligned} -g_{k_j}^\top s_{k_j} &= -\|g_{k_j}\|^2 \lambda_{k_j}^2 \frac{g_{k_j}^\top s_{k_j}}{\|g_{k_j}\|^2 \lambda_{k_j}^2} = \|g_{k_j}\|^2 \lambda_{k_j} \frac{-\lambda_{k_j} g_{k_j}^\top s_{k_j}}{\|\lambda_{k_j} g_{k_j}\|^2} \\ &= \|g_{k_j}\|^2 \lambda_{k_j} \frac{s_{k_j}^\top M_{k_j} s_{k_j}}{\|M_{k_j} s_{k_j}\|^2}. \end{aligned}$$

If (2.25) is not satisfied, then, there exists a constant $\delta > 0$ such as $\|g_k\| \geq \delta$, for all k . Also (4.9) implies

$$\sum_{j=1}^{\infty} \lambda_{k_j} \frac{s_{k_j}^\top M_{k_j} s_{k_j}}{\|M_{k_j} s_{k_j}\|^2} < \infty.$$

Therefore, for any $\xi > 0$, there exists an integer $j_0 > 0$, such as for all positive integer q , we get

$$\begin{aligned} \left(\prod_{j=j_0+1}^{j_0+q} \lambda_{k_j} \frac{s_{k_j}^\top M_{k_j} s_{k_j}}{\|M_{k_j} s_{k_j}\|^2} \right)^{\frac{1}{q}} &\leq \frac{1}{q} \sum_{j=j_0+1}^{j_0+q} \lambda_{k_j} \frac{s_{k_j}^\top M_{k_j} s_{k_j}}{\|M_{k_j} s_{k_j}\|^2} \\ &\leq \frac{\xi}{q} \end{aligned}$$

implies

$$\begin{aligned} \left(\prod_{j=j_0+1}^{j_0+q} \lambda_{k_j} \right)^{\frac{1}{q}} &\leq \frac{\xi}{q} \left(\prod_{j=j_0+1}^{j_0+q} \frac{\|M_{k_j} s_{k_j}\|^2}{s_{k_j}^\top M_{k_j} s_{k_j}} \right)^{\frac{1}{q}} \\ &\leq \frac{\xi}{q^2} \sum_{j=j_0+1}^{j_0+q} \frac{\|M_{k_j} s_{k_j}\|^2}{s_{k_j}^\top M_{k_j} s_{k_j}} \\ &\leq \frac{\xi}{q^2} \sum_{j=0}^{j_0+q} \frac{\|M_{k_j} s_{k_j}\|^2}{s_{k_j}^\top M_{k_j} s_{k_j}} \end{aligned}$$

Using (3.10), it can be easily deduced

$$\left(\prod_{j=j_0+1}^{j_0+q} \lambda_{k_j} \right)^{\frac{1}{q}} \leq \frac{\xi(j_0 + q + 1)}{q^2} c_2,$$

where

$$i_k = j_0 + q + 1.$$

If $q \rightarrow \infty$, then we obtain a contradiction. Because, Lemma 3 certifies that the left term of the above inequality is larger than a positive constant. $\square$

Remark 3. To show the global convergence of the BFGS method and the inexact linear search of Wolfe, it is sufficient to show implicitly the existence of the condition $\frac{y_k^\top s_k}{\|s_k\|^2} \geq \varepsilon \|g_k\|^\alpha$. That is to say, the BFGSA is devoted to the BFGS method with inexact linear search of Wolfe.

First step: for all $k \geq 1$

$$(4.10) \quad \frac{y_k^\top s_k}{\|s_k\|^2} \geq (1 - \sigma_2) c_1 \lambda_k^{-\frac{1}{k}}$$

indeed, using (Wolfe 2), we have

$$y_k^\top s_k \geq \sigma_2 g_k^\top s_k - g_k^\top s_k \geq (1 - \sigma_2) \lambda_k^{-1} s_k^\top M_k^{-1} s_k,$$

where

$$s_k = \lambda_k d_k$$

and

$$(4.11) \quad d_k^\top = \lambda_k^{-1} s_k^\top \text{ and } g_k^\top = -d_k^\top M_k^{-1}$$

and if the inexact linear search used is Wolfe's, then we have the condition $y_k^\top s_k \geq 0$ i.e. the positive definiteness of M_k is preserved, so, there exists positive constants $c_1 \leq C_1$ such as

$$\begin{aligned} c_1 \|z\|^2 &\leq z^\top M_k z \leq C_1 \|z\|^2, \\ c_1 \|z\|^2 &\leq z^\top M_k^{-1} z \leq C_1 \|z\|^2, \text{ for all } z \in \mathbb{R}^n, \end{aligned}$$

by (4.11), we have

$$y_k^\top s_k \geq (1 - \sigma_2) \lambda_k^{-1} c_1 \|s_k\|^2$$

implies

$$\frac{y_k^\top s_k}{\|s_k\|^2} \geq (1 - \sigma_2) c_1 \lambda_k^{-1} \geq (1 - \sigma_2) c_1 \lambda_k^{-\frac{1}{k}}.$$

Second step: For all $k \geq 1$

$$(4.12) \quad \left(\prod_{i=1}^k \lambda_i \|g_i\|_2^2 \right)^{\frac{1}{k}} \leq \frac{c}{k}, \quad c > 0.$$

Indeed, we have:

$$\sum_{i=1}^k \frac{\|M_i s_i\|_2^2}{s_i^\top M_i s_i} = \sum_{i=1}^k \frac{\lambda_i \|g_i\|_2^2}{-g_i^\top s_i} \leq c_1.$$

Then, we use the inequality of the averages twice, and the first condition of Wolfe so

$$\lambda_i \|g_i\|_2^2 \leq c_1 (-g_i^\top s_i)$$

implies

$$\begin{aligned} \prod_{i=1}^k \lambda_i \|g_i\|^2 &\leq c_1^k \prod_{i=1}^k (-g_i^\top s_i) \\ &\leq \left(\frac{c_1}{k} \sum_{i=1}^k (-g_i^\top s_i) \right)^k \\ &\leq \left(\frac{c_1}{\sigma_1 k} \sum_{i=1}^k (f(\mathbf{x}_i) - f(\mathbf{x}_{i+1})) \right)^k \\ &\leq \left(\frac{c_1}{\sigma_1 k} (f(\mathbf{x}_1) - f(\mathbf{x}_{k+1})) \right)^k \\ &\leq \frac{c}{k}. \end{aligned}$$

We deduce (4.12), with $c = c_1 (f(\mathbf{x}_1) - f_{\min}) / \sigma_1$ where $f_{\min} \in \mathbb{R}$ is a lower bound of $\{f(\mathbf{x}_k)\}_{k \in \mathbb{N}}$.

Third step: To conclude that

$$\frac{y_k^\top s_k}{\|s_k\|^2} \geq \varepsilon \|g_k\|^\alpha.$$

Indeed, from (4.12) we get for all $k \geq 1$

$$\left(\lambda_k \|g_k\|^2\right)^{\frac{1}{k}} \leq \frac{c}{k}$$

implies for all $k \geq 1$

$$(4.13) \quad \frac{k}{c} \|g_k\|^{\frac{2}{k}} \leq \lambda_k^{-\frac{1}{k}}.$$

From (4.13), (4.10), it implies

$$\begin{aligned} \frac{y_k^\top s_k}{\|s_k\|^2} &\geq (1 - \sigma_2) c_1 \lambda_k^{-\frac{1}{k}} \\ &\geq (1 - \sigma_2) c_1 \frac{c}{k} \|g_k\|^{\frac{2}{k}} \end{aligned}$$

and if we put $\alpha = \frac{2}{k}$ and $\varepsilon = (1 - \sigma_2) c_1 \frac{c}{k}$, so we obtain

$$\frac{y_k^\top s_k}{\|s_k\|^2} \geq \varepsilon \|g_k\|^\alpha.$$

5. Conclusion

This paper proposes a new proof for the global convergence of the BFGS method for nonconvex unconstrained minimization problem which an extension of the work of Li and Fukushima in [13]. The authors proved the convergence of an appropriate procedures for the BFGS method, but they did not completely proved that their method converged to the BFGS method. However, in the current paper we are interested in proving that the condition of the appropriate method is to satisfy implicitly with inaccurate linear search of Wolfe type [18]. Furthermore, we have checked directly the convergence of the method BFGS with the inaccurate linear search of Wolfe.

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SOME PROPERTIES OF SOFT β -COMPACT AND RELATED SOFT TOPOLOGICAL SPACES

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Abstract. Benchalli et al. [4] introduced the notion of soft β -compactness by using soft filter basis. In continuation, in this paper we further study some more properties of soft β -compactness in soft topological spaces. Furthermore we introduce and discuss, soft β -first countable, soft β -second countable spaces, soft β -closed spaces and soft generalized β -compact spaces in soft topological spaces.

Keywords: Soft sets, soft β -compact, soft β -first countable spaces, soft generalized β -compact spaces.

1. Introduction

Mathematics is based on exact concepts and there is not vagueness for mathematical theories. In the fields such as medicine, engineering, economics and sociology, the notions are vague, researchers need to define some modern methods for vagueness. To deal with the these problems in real life, researchers anticipated several methods such as fuzzy set theory, rough set theory and soft set theory. Fuzzy set theory [14] proposed by Zadeh in 1965 provides an appropriate framework for representing and processing vague concepts. The basic idea of fuzzy set theory hinges on fuzzy membership function. By fuzzy membership function, we can establish the belonging of an element to set to a degree. Rough set theory [13] which is proposed by Pawlak in 1982 is another mathematical approach to vagueness to catch the granularity induced by vagueness in information systems. It based on equivalence relation. The benefit of rough set method is that it does not need any additional information about data, like membership in fuzzy set theory.

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Theory of fuzzy sets and theory of rough sets can be considered as tools for dealing with vagueness but both of these theories have their own difficulties. The reason for these difficulties is possibly, the inadequacy of the parametrization tool of the theory as mentioned by Molodtsov [10] in 1999. Soft set theory was initiated by Molodtsov [10] as a completely new approach for modeling vagueness and uncertainty. According to Molodtsov [10][11], the soft set theory has been successfully applied to many areas, such as functions smoothness, game theory, Riemann integration, theory of measurement and so on. He also showed how soft set theory is free from the parametrization inadequacy syndrome of, rough set theory, game theory, fuzzy set theory and probability theory.

Recently, weak forms of soft open sets were studied by many researchers like [5] [1] [8] [9] in soft topological spaces respectively. Furthermore, Benchalli et al. [2][3][4] have studied soft β -separation axioms, soft β -compactness, soft β -connectedness in soft topological spaces.

In the present paper, we studied some more properties of soft β -compact spaces in soft topological spaces which are defined over an initial universe with a fixed set of parameters. We have set up a soft topology with the help of soft β -closed spaces. In addition to that the concept of soft β -first countable, soft β -second countable and soft β -Lindelöf spaces are studied. Furthermore we introduced the notion of the soft generalized β -compact spaces in soft topological spaces. Also, we have explored some basic properties of these concepts.

The organization of this paper is as follows: Section 2 briefly reviews some basic concepts about soft sets, definitions of some weaker forms of soft sets and related properties in soft topological spaces; Section 3 defines the concepts of soft β -compact spaces, soft β -spaces and studies some relative properties; Section 4 introduces the concepts of soft β -closed spaces and their properties; Section 5 we give the definition of soft β -first countable and soft β -second countable spaces and their related properties; Section 6 introduces about soft generalized β -compact spaces and section 7 is conclusion of the paper.

2. Preliminary

Through-out this paper (X, τ, E) will be a soft topological space.

Definition 2.1 ([12]). *Let X be an initial universe and let E be a set of parameters. Let $P(X)$ denote the power set of X and let A be a nonempty subset of E . A pair (F, A) is called a soft set over $\tilde{X}$, where F is a mapping given by $F : A \rightarrow P(X)$. In other words, a soft set over $\tilde{X}$ is a parameterized family of subsets of the universe $\tilde{X}$. For $\varepsilon \in A$, $F(\varepsilon)$ may be considered as the set of ε -approximate elements of the soft set (F, A) . Clearly, a soft set is not a set.*

Definition 2.2 ([12]). *Let τ be the collection of soft sets over X ; then τ is said to be a soft topology on X if it satisfies the following axioms:*

- (a) Φ, X belong to τ .
- (b) The union of any number of soft sets in $\tau \in \tau$.

(c) The intersection of any two soft sets in $\tau \in \tau$.

The triplet (X, τ, E) is called a soft topological space over X . The relative complement of a soft set (F, A) is denoted by $(F, A)^c$ and is defined by $(F, A)^c = (F^c, A)$.

Definition 2.3 ([12]). Let (X, τ, E) be a soft topological space over $\tilde{X}$ and let (F, A) be a soft set over X .

(a) *Soft Interior*: The soft interior of (F, A) is the soft set $\text{int}((F, A)) = \cup\{(O, A) : (O, A) \text{ is soft open and } (O, A) \tilde{\subset} (F, A)\}$.

(b) *Soft Closure* : The soft closure of (F, A) is the soft set $\text{cl}((F, A)) = \cap\{(F, E) : (F, E) \text{ is soft closed and } (F, A) \tilde{\subset} (F, E)\}$.

Definition 2.4 ([8]). A soft set (F, A) of a soft topological space (X, τ, E) is said to be

(a) *Soft open* if its complement is soft closed.

(b) *Soft α -open* if $(F, A) \tilde{\subset} \text{int}(\text{cl}(\text{Int}((F, A))))$.

(c) *Soft preopen* if $(F, A) \tilde{\subset} \text{int}(\text{cl}((F, A)))$.

(d) *Soft semiopen* if $(F, A) \tilde{\subset} \text{cl}(\text{int}((F, A)))$.

(e) *Soft β -open* if $(F, A) \tilde{\subset} \text{cl}(\text{int}(\text{cl}((F, A))))$.

The complement of soft open, (resp. soft α -open, soft preopen, soft semiopen, soft β -open) sets are said to be soft closed (resp. soft α -closed, soft preclosed, soft semiclosed, soft β -closed). The intersection of soft closed (resp. soft α -closed, soft preclosed, soft semiclosed, soft β -closed) sets containing (F, A) is called the soft closure (resp. soft α -closure, soft pre-closure, soft semi-closure, soft β -closure) of (F, A) and is denoted by $\text{scl}(F, A)$ (resp. $\text{s}\alpha\text{cl}(F, A)$, $\text{sPcl}(F, A)$, $\text{sScl}(F, A)$, $\text{s}\beta\text{cl}(F, A)$). The soft interior of (F, A) is defined by the union of all soft open (resp. Soft α -open, soft preopen, soft semiopen, soft β -open) sets contained in (F, A) and is denoted by $\text{sint}(F, A)$ (resp. $\text{s}\alpha\text{int}(F, A)$, $\text{sPint}(F, A)$, $\text{sSint}(F, A)$, $\text{s}\beta\text{int}(F, A)$).

Definition 2.5 ([2]). Let (X, τ, E) be a soft topological space over X , (G, E) be soft closed set in X and $x \in X$ such that $x \notin (G, E)$. If there exists soft β -open sets (F_1, E) and (F_2, E) such that $x \in (F_1, E)$, $(G, E) \subseteq (F_2, E)$ and $(F_1, E) \cap (F_2, E) = \phi$, then (X, τ, E) is called a soft β -regular space.

Definition 2.6 ([2]). A space X is said to be a soft β -normal if for any pair of disjoint soft closed sets (F_1, E) and (F_2, E) there exists disjoint soft β -open sets (U, E) and $(V, E) \ni (F_1, E) \subset (U, E)$ and $(F_2, E) \subset (V, E)$.

Definition 2.7 ([7]). Let $SS(X)_E$ and $SS(Y)_E$ be families of soft sets. $u : X \Rightarrow Y$ and $p : E \Rightarrow E'$ be mappings. Then a mapping $f_{pu} : SS(X)_E \Rightarrow SS(Y)_{E'}$ defined as:

(a) Let (F, E) be a soft set in $SS(X)_E$. The image of (F, E) under f_{pu} , written as $f_{pu}(F, E) = (f_{pu}(F), p(E))$, is a soft set in $SS(Y)_{E'}$ such that

$$f_{pu}(F, E) = \begin{cases} \bigcup_{x \in p'(y) \cap A} u(F(x)), & p'(y) \cap A \neq \phi \\ \phi, & \text{otherwise} \end{cases}$$

for all $y \in E'$

(b) Let (G, E') be a soft set in $SS(Y)_{E'}$. Then the inverse image of (G, E') under f_{pu} , written as $f_{pu}^{-1}(G, E') = (f_{pu}^{-1}(G), p^{-1}(E'))$, is a soft set in $SS(X)_E$ such that

$$f_{pu}^{-1}(G, E') = \begin{cases} u^{-1}(G(p(x))), & p(x) \in E' \\ \phi, & \text{otherwise} \end{cases}$$

for all $x \in E$.

Definition 2.8 ([3]). A cover of a soft set is said to be a soft β -open cover if every member of the cover is a soft β -open set.

Definition 2.9 ([3]). A soft topological space (X, τ, E) is said to be soft β -compact space if each soft β -open cover of $\tilde{X}$ has a finite subcover.

3. Some properties of soft β -compact spaces in soft topological spaces

In this section, the concept of soft β -space is introduced and studied the concept of soft β -compactness in terms of soft β -compact spaces. Some more properties of soft β -compactness spaces are studied in detail. Soft β -compactness can be infinite as H. Mahamood[6] introduced and discussed the concept of soft Heine Borel theorem for infinite soft compactness.

Definition 3.1. A soft topological space (X, τ, E) is said to be soft β -space if every soft β -open set of $\tilde{X}$ is soft open in $\tilde{X}$.

Example 3.2. Every soft discrete topology is a soft β -space.

Example 3.3. Every soft indiscrete topology is a soft β -space.

Corollary 3.4. If a soft topological space (X, τ, E) is a soft β -compact space and soft β -space, then (X, τ, E) is soft compact space.

Proof. Let $\{(G_k)_i : i \in I\}$ be a soft open cover of $\tilde{X}$. Since any soft open set is soft β -open set, therefore $\{(G_k)_i : i \in I\}$ is soft β -open cover of $\tilde{X}$. Since $\tilde{X}$ is soft β -compact space and soft β -space, there exists a soft finite subset I_m of I such that $\tilde{X} \subseteq \bigcup \{(G_k)_i : i \in I_m\}$. Hence (X, τ, E) is soft compact space. $\square$

Corollary 3.5. *If $f : X \rightarrow Y$ is a soft β -continuous function and soft β -space, then f is soft continuous function.*

Proof. Let us consider soft β -open set $\{(F_E)_i : i \in I\}$ of $\tilde{Y}$. For f is soft β -continuous function, $\{f^{-1}((F_E)_i) : i \in I\}$ is a soft β -open set of X and for $\tilde{X}$ is soft β -space, $\{f^{-1}((F_E)_i) : i \in I\}$ form a soft open set of $\tilde{X}$. Thus, f is a soft continuous function. $\square$

Corollary 3.6. *Let (X, σ, E) be a soft topological space. If (X, σ_e) is a soft β -compact space, for each $e \in E$, then (X, σ, E) is a soft β -compact space.*

Proof. Let (X, σ_e) is a soft β -compact space and let $E = \{e_1, e_2, \dots, e_n\}$ be a set of parameters, for every $i = 1, 2, 3, \dots, n$. Let $\{(G_k)_i : i \in I\}$ be a soft β -open cover of $\tilde{X}$. For $\bigcup_{i \in I_m} (G_k)_i(e) = \tilde{X}$, for every $e \in E$, and (X, σ_e) is soft β -compact space, there is a finite subset I_m of I such that $\bigcup_{i \in I_m} (G_k)_i(e) = \tilde{X}$. So $\bigcup_{i \in I_m} (G_k)_i(e) = \tilde{X}$, hence $\{(G_k)_i : i \in I_m\}$ is a soft finite subcover of $\{(G_k)_i : i \in I\}$. Therefore, (X, σ, E) is soft β -compact space. $\square$

Definition 3.7. *A soft mapping $f : X \rightarrow Y$ is said to be soft β^* -open if the image of each soft β -open set of $\tilde{X}$ is soft β -open in $\tilde{Y}$.*

Theorem 3.8. *Let (F, A) and (G, B) be the soft subsets of a soft topological space (X, τ, E) , such that (F, A) is soft β -compact in $\tilde{X}$ and (G, B) is soft β -closed set in $\tilde{X}$. Then $(F, A) \cap (G, B)$ is soft β -compact in $\tilde{X}$.*

Proof. Let $\{(H_E)_i : i \in I\}$ be a cover of $(F, A) \cap (G, B)$ consisting of soft β -open subsets of $\tilde{X}$. Since $(G, B)^c$ is a soft β -open set, $\{(H_E)_i : i \in I\} \cup (G, B)^c$ is soft β -open cover of (F, A) . Since (F, A) is soft β -compact in $\tilde{X}$, there exists a soft finite subset $I_m \subset I$ such that $(F, A) \subset \{(H_E)_i : i \in I_m\} \cup (G, B)^c$. Therefore $(F, A) \cap (G, B) \subset \{(H_E)_i : i \in I_m\}$. As a consequence, $(F, A) \cap (G, B)$ is soft β -compact in $\tilde{X}$. $\square$

Theorem 3.9. *Let $f : X \rightarrow Y$ be a soft β -open, soft β -continuous and injective mapping. If a soft subset (H, E) of Y is soft β -compact in Y , then $f^{-1}((H, E))$ is soft β -compact in $\tilde{X}$.*

Proof. Let $\{(F_E)_i : i \in I\}$ be a soft β -open cover of $f^{-1}((H, E))$ in $\tilde{X}$. Then $f^{-1}((H, E)) \subset \bigcup \{(F_E)_i : i \in I\}$ and hence $(H, E) \subset \bigcup \{f(f^{-1}((H, E)))\} \subset f(\bigcup \{(F_E)_i : i \in I\}) = \bigcup \{f(F_E)_i : i \in I\}$. Since (H, E) is soft β -compact in $\tilde{Y}$ there is a soft finite subset $I_m \subset I$ such that $(H, E) \subset \bigcup \{(F_E)_i : i \in I_m\}$. So $f^{-1}((H, E)) \subset f^{-1}(\bigcup \{f(F_E)_i : i \in I_m\}) = \bigcup \{f^{-1}(f(F_E)_i) : i \in I_m\} = \bigcup \{(F_E)_i : i \in I_m\}$. $\square$

Theorem 3.10. *The preimage of soft β -compact space under soft β^* -open bijective mapping is soft β -compact space.*

Theorem 3.11. *If a function $f : X \rightarrow Y$ is soft β^* -open bijective mapping and $\tilde{Y}$ be soft β -compact space, then $\tilde{X}$ is soft β -compact space.*

Proof. Let $\{(F_E)_i : i \in I\}$ be a collection of soft β -open covering of $\tilde{X}$. Then let $\{f(F_E)_i : i \in I\}$ be a soft β -open cover is a collection of soft β -open sets covering $\tilde{Y}$. For $\tilde{Y}$ is soft β -compact space, by definition there exists a finite family $I_m \subset I$ such that $\{f(F_E)_i : i \in I_m\}$ covers $\tilde{Y}$. Since f is soft bijective, we have $\tilde{X} = f^{-1}(Y) = f^{-1}(f(\bigcup_{i \in I_m} (F_E)_i)) = \bigcup_{i \in I_m} (F_E)_i$. Thus $\tilde{X}$ is soft β -compact space. $\square$

4. Soft β -closed spaces

This section deals the concept of soft β -closed spaces and its related concepts in detail.

Definition 4.1. Let (X, τ, E) be a soft topological space and it is said to be soft β -closed space if and only if for every soft family $\{(F_E)_i : i \in I\}$ of soft β -open set such that $\bigcup_{i \in I} (F_E)_i = \tilde{X}$ there is a soft finite subfamily $I_m \subset I$ such that $\bigcup_{i \in I_m} s\beta cl(F_E)_i = \tilde{X}$.

Definition 4.2. A soft set (F, E) in a soft topological space $\tilde{X}$ is said to be soft β -closed relative to $\tilde{X}$ if and only if for every family $\{(H_E)_i : i \in I\}$ of soft β -open set such that $\bigcup_{i \in I_m} (H_E)_i = (F, E)$ there is a soft finite subfamily $I_m \subset I$ such that $\bigcup_{i \in I_m} s\beta cl(H_E)_i = (F, E)$.

Remark 4.3. Every soft β -compact space is soft β -closed but the converse is not true.

Example 4.4. The following example shows that soft β -closed space need not to be soft β -compact space.

Let $\tilde{X} \neq \phi$ be a soft set and $(F, E)_n = 1 - 1/n$ for every $e_x \in \tilde{X}$ and $n \in N^+$. The collection $\{(F, E)_n : n \in N^+\}$ is a soft β -base for a soft topology on $\tilde{X}$. The collection $\{(F, E)_n : n \in N^+\}$ is obviously a soft β -open cover of $\tilde{X}$. On the other hand we have $s\beta cl(F, E)_n = \tilde{X}$ for every $n \geq 3$. Hence (X, τ, E) is soft β -closed but not soft β -compact space.

Theorem 4.5. Soft topological space (X, τ, E) is a soft β -closed if and only if for every soft finite intersection property ψ in $\tilde{X}$ then $\bigcap (H, B) \in \psi s\beta cl(H, B) \neq \phi$.

Proof. Let $\{(G_E)_i : i \in I\}$ be a soft β -open cover of $\tilde{X}$ and let for every finite collection of $\{(G_E)_i : i \in I\}$, $\bigcup_{i \in I_m} (G_A)_i \subset \tilde{X}$ for some $i \in I_m$. Then $\bigcap_{i \in I_m} ((H, B)^c)_i \supset \phi$, for some $i \in I_m$. Thus $\{s\beta cl((G_A)^c)_i : i \in I\} = \psi$ forms a soft β -open finite intersection property in $\tilde{X}$, then $\bigcap_{i \in I} (G_A)_i = \phi$ which implies $\bigcap_{i \in I} s\beta cl(s\beta cl(H, B))^c = \phi$, which is contradiction. Then every soft β -open $\{(G_E)_i : i \in I\}$ of $\tilde{X}$ has a soft finite subfamily I_m such that $\bigcup_{i \in I_m} s\beta cl(G_A)_i = \tilde{X}$ for every $i \in I_m$. Hence $\tilde{X}$ is soft β -closed space.

Conversely, assume there exists a soft β -open finite intersection property ψ in $\tilde{X}$ such that $\bigcap_{(H, B) \in \psi} s\beta cl(H, B) = \phi$. That implies $\bigcup_{(H, B) \in \psi} (s\beta cl(H, B))^c = \tilde{X}$ for every $i \in I$ and hence $\{(G_E)_i : i \in I\} = \{s\beta cl(H, B) : (H, B) \in \psi\}$ is soft

β -open set cover of $\tilde{X}$. For X is soft β -closed, by definition $\{(G_E)_i : i \in I\}$ has a finite subfamily I_m such that $\bigcup_{i \in I_m} s\beta cl(s\beta cl(H, B))^c = \tilde{X}$ for every $I \in I_m$ and hence $\bigcap_{I \in I_0} (s\beta cl(s\beta cl(H, B))^c)^c = \phi$. Thus $\cap(H, B) \in I_m(H, B) \neq \phi$ is a contradiction. Therefore $\cap(H, B) \in \psi s\beta cl(H, B) \neq \phi$. $\square$

Theorem 4.6. *Let $f : X \rightarrow Y$ be a soft β -irresolute function and if $\tilde{X}$ is soft β -closed space, then $\tilde{Y}$ is soft β -closed space.*

Proof. Let $\{(F_E)_i : i \in I\}$ be a soft β -open cover of Y . Since the function f is soft β -irresolute surjection, $\{f^{-1}(F_E)_i : i \in I\}$ is soft β -open cover of $\tilde{X}$. By the hypothesis, there exists a soft finite subset I_m of ψ such that $\bigcup_{i \in I_m} s\beta cl(f^{-1}(F_E)_i) = \tilde{X}$. For f is surjective and by the theorem $\tilde{Y} = f(\tilde{X}) = f(\bigcup_{i \in I_m} s\beta cl(f^{-1}(F_E)_i)) \subset \bigcup_{i \in I_0} s\beta cl(f(f^{-1}(F_E)_i)) = \bigcup_{i \in I_m} s\beta cl(F_E)_i$. Consequently, $\tilde{Y}$ is soft β -closed space. $\square$

5. Soft β -first countable and soft β -second countable spaces

In this section, we proposed the concept of soft β -first countable and soft β -second countable spaces. Their properties are studied with suitable example.

Definition 5.1. *Let (X, τ, E) be a soft topological space and let (U, E) be a family of soft β -neighbourhood of some soft point $e_x \in \tilde{X}$. If for each soft β -neighbourhood (F, E) of e_x , there exists (H, C) in (U, E) such that $e_x \in (H, C) \subseteq (F, E)$ then we say that (U, E) is a soft neighbourhood base at e_x .*

Definition 5.2. *Let (X, τ, E) be a soft topological space, and let e_x be a soft point in $\tilde{X}$. If e_x has a soft β -countable neighbourhood base, then we say that (X, τ, E) is soft β -first countable at the soft point e_x . If (X, τ, E) is soft β -first countable at each of its soft points, then we say that (X, τ, E) is soft β -first countable.*

Definition 5.3. *Let (X, τ, E) be a soft topological space is soft β -second countable if it has a soft β -countable base $\mathcal{U}$ for its soft topology say $\mathcal{U} = \{(F, A_1), (F, A_2), (F, A_3) \dots\}$. That is given any open set (F, A) and point $e_x \in (F, A)$ there is $(F, A_n) \in \mathcal{U}$ such that $(F, A_n) \subseteq (F, A)$ with $e_x \in (F, A_n)$.*

Theorem 5.4. *Every soft β -second countable space is soft β -first countable.*

Proof. Let (X, τ, E) be a soft β -second countable space and suppose

$\mathcal{U} = \{(F, A_1), (F, A_2), (F, A_3) \dots\}$ be the soft β -countable base. We can take for a soft β -basis at e_x the sequence of all (F, A_n) which contain e_x , call this collection as $\mathcal{U}'$. Then $\mathcal{U}'$ is soft β -countable as it is soft subset of the β -countable basis $\mathcal{U}$ and since $\mathcal{U}$ is soft β -basis, for any soft β -neighbourhood (G, B) of e_x , there exists $(F, A_n) \in \mathcal{U}'$ such that $e_x \in (F, A_n) \subseteq (G, B)$. This implies that (X, τ, E) is soft β -first countable. $\square$

Theorem 5.5. *A subspace of soft β -first countable space is soft β -first countable and same holds for soft β -second countability.*

Proof. We prove the results for soft β -second countability and first result follows from second. Let (X, τ, E) is soft β -second countable space and (G, τ', A) be a soft β -subspace of (X, τ, E) . Suppose $\mathcal{U} = \{(G, A_1), (G, A_2), (G, A_3), \dots\}$ is soft β -countable basis for the space (X, τ, E) . Then take the soft β -basis for subspace (G, τ', A) as $\mathcal{U}' = \{(G, A) \cap (K, B) : (K, B) \in \mathcal{U}\}$, which is soft β -countable. Therefore (G, τ', A) is soft β -second countable. $\square$

Remark 5.6. There exist a soft β -first countable space which is not soft β -second countable.

Example 5.7. If we take soft discrete topology τ' on $\tilde{X}$, over the parameter set E , then each soft set in $\tilde{X}$ is soft β -open with respect to soft discrete topology. Take $\mathcal{U}_{e_x} = \{(F, A)(e)\}$ a soft β -neighbourhood base at each soft point e_x . Then $\mathcal{U}_{e_x}$ is soft countable and for each soft β -neighbourhood (G, B) of e_x , there is always $\{e_x\} \in \mathcal{U}_{e_x}$ such that $e_x \in \{e_x\} \subseteq (G, B)$. Therefore (X, τ', E) is soft β -first countable space but it is not soft β -second countable space.

Theorem 5.8. *The image of soft β -first countable space under a soft open continuous map are soft β -first countable.*

Proof. Let (X, τ_1, E) and (Y, τ_2, E) be two soft topological spaces over the parameter set E , and suppose (X, τ_1, E) is soft β -first countable, and let $f : (X, \tau_1, E) \Rightarrow (Y, \tau_2, E)$ be a soft onto and continuous open mapping. Since f is onto, for any soft point e_y in $\tilde{Y}$, there exists a soft point e_x in $\tilde{X}$, such that $f[e_x] = e_y$. Since (X, τ_1, E) is soft β -first countable, there exists a soft β -countable neighbourhood base $\{(F, A_n)\}_{n \in N}$ at e_x . Then it is easy to see that $\{f(F, A_n)\}_{n \in N}$ is soft β -neighbourhood base at e_y . $\square$

Definition 5.9. *A soft topological space (X, τ, E) is said to be soft β -Lindelöf if each soft β -open covering $(G, E_i)_{i \in N}$ of $\tilde{X}$ has a soft β -countable subcover.*

Theorem 5.10. *Each soft β -second countable space is soft β -Lindelöf.*

Proof. Let (X, τ, E) be a soft β -countable space, and let $\mathcal{U}$ be a soft β -countable open base of $\tilde{X}$. Let $(G, E_i)_{i \in N}$ be an arbitrary soft β -open covering of $\tilde{X}$. Put $\mathcal{U}' = \{(F, A) \in \mathcal{U} : \text{there is } (G, B) \in (G, E_i)_{i \in N} \text{ such that } (F, A) \subseteq (G, B)\}$. Then $\mathcal{U}'$ is countable. Denote $\mathcal{U}'$ by $\{(F, A_n) : n \in N\}$. For each $n \in N$, there exists an $(H, A_n) \in (G, E_i)_{i \in N}$ such that $(F, A_n)_{n \in N} \subseteq (H, A_n)_{n \in N}$. Then $\{(H, A_n) : n \in N\}$ is soft countable subfamily of $(G, E_i)_{i \in N}$. Next we shall prove that $\{(H, A_n) : n \in N\}$ is soft β -cover of (X, τ, E) . We take a soft arbitrary point e_x in (X, τ, E) . Since $(G, E_i)_{i \in N}$ is soft β -covering of $\tilde{X}$, there exists a $(K, E) \in (G, E_i)_{i \in N}$ such that $e_x \in (K, E)$. Since (K, E) is soft β -open set, so there is a $(L, E) \in \mathcal{U}$ such that $e_x \in \mathcal{U} \subseteq (K, E)$ because $\mathcal{U}$ is a soft β -base of $\tilde{X}$. Hence $(L, E) \in \mathcal{U}'$ and therefore, there is $n \in N$ such that $(L, E) = (F, A_n)$. Thus $e_x \in (F, A_n)$. As a consequence, $\{(H, A_n) : n \in N\}$ is soft β -cover of (X, τ, E) . $\square$

Theorem 5.11. *Each soft β -regular and soft β -Lindelöf space is soft β -normal.*

Let (X, τ, E) be a soft β -regular and soft β -Lindelöf space. Let (K, E_1) and (K, E_2) be two disjoint soft β -closed sets over $\tilde{X}$. For each soft point $e_x \in (K, E_1) \subseteq (K, E_2)^c$, and since (X, τ, E) is soft β -regular, then there exists a soft β -open neighbourhood (G, B) of e_x such that $e_x \in \overline{(G, B)} \subseteq (F, A_2)^c$, that is $\overline{(G, B)} \cap (F, A_2) = \phi$. Let $\Psi = \{(G, B) : e_x \in (K, E_1)\}$ that is Ψ is the collection of soft β -neighbourhoods of $e_x \in (K, E_1)$, then $\Psi \cup (K, E_1)^c$ is soft β -open cover of (X, τ, E) . Since (X, τ, E) is a soft β -Lindelöf, there exists soft β -countable subcover $\{(G, B_n) : n \in N\} \cup (K, E_1)^c$. Put $(Z, E_n) = (G, B_n)$ for each $n \in N$, then $(K, E_1) \subseteq \bigcup_{n \in N} (Z, E_n)$ and each $(Z, E_n) \cap (K, E_2) = \phi$. Similarly, there exists countably many soft β -open sets $\{(F, E_n) : n \in N\}$ such that $(K, E_2) \subseteq \bigcup_{n \in N} (F, E_n)$ and each $(F, E_n) \cap (K, E_1) = \phi$. For each $n \in N$ put $(Z, E_n)' = (Z, E_n) \cap [\bigcup_{i=1}^n ((F, E_i)^c)]$, $(F, E_n)' = (F, E_n) \cap [\bigcup_{i=1}^n ((Z, E_i)^c)]$.

Then for $m, n \in N$, we have $(Z, E_n)' \cap (F, E_m)' = \phi$, put $(Z, E) = \bigcup_{n \in N} (Z, E_n)'$, $(F, E) = \bigcup_{n \in N} (F, E_n)'$. Then we have $(K, E_1) \subseteq (Z, E)$, $(K, E_2) \subseteq (F, E)$ and $(Z, E) \cap (F, E) = \phi$. Therefore (X, τ, E) is soft β -normal.

6. Conclusion

In this paper, we have studied enriched soft topology. In the sequel, we have introduced soft β -spaces and soft β -compact spaces and obtained some results.

On the other hand, we have given the definition of soft β -closed spaces and studied their basic characteristics in soft topological spaces. We have introduced soft β -first countable and soft β -second countable spaces, and their properties are studied in detail. Finally, we have introduced soft generalized β -compact spaces in soft topological spaces. The results are helpful for the further research on soft topology.

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REPRESENTATION OF UP -ALGEBRAS IN INTERVAL-VALUED INTUITIONISTIC FUZZY ENVIRONMENT

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Abstract. In this paper, the concept of interval-valued intuitionistic fuzzy set to UP -subalgebras and UP -ideals of UP -algebras are introduced. Relations among IVIF UP -subalgebras with IVIF UP -ideals of UP -algebras are investigated. The homomorphic image and inverse image of IVIF UP -subalgebras and IVIF UP -ideals are studied and some related properties are investigated. Equivalence relations on IVIF UP -ideals are discussed. Also, the product of IVIF UP -algebras are investigated.

Keywords: UP -algebra, interval-valued intuitionistic fuzzy set, interval-valued intuitionistic fuzzy UP -subalgebra, interval-valued intuitionistic fuzzy UP -ideal, equivalence relation, upper(lower)-level cuts, product of UP -algebra.

1. Introduction

The theory of the fuzzy set introduced by Zadeh has achieved a great success in various fields. Atanassov [1] introduced the intuitionistic fuzzy set (IFS), which is a generalization of the fuzzy set. The IFS has received more and more attention and has been applied to many fields since its appearance. The theory of the IFS has been found to be more useful to deal with vagueness and uncertainty

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in decision situations than that of the fuzzy set. Atanassov and Gargov further generalized the IFS in the spirit of ordinary interval-valued fuzzy sets (IVFSs) and defined the notion of an interval-valued intuitionistic fuzzy set (IVIFS).

BCK -algebras and BCI -algebras [4] are two important classes of logical algebras introduced by Imai and Iseki. Neggers and Kim [9] introduced a new notion, called a B -algebras which is related to several classes of algebras of interest such as BCK/BCI -algebras. Kim and Kim [8] introduced the notion of BG -algebras, which is a generalization of B -algebras. Senapati together with colleagues [2, 5, 6, 10-23] have done lot of works on BCK/BCI -algebras and related algebras. Iampan [3] introduced a new branch of logical algebra called UP -algebras, which is related to $BCK/BCI/B/BG$ -algebras. Somjanta et al. [24] applied the concept of fuzzy set theory to UP -algebra. Kesorn et al. [7] introduced intuitionistic fuzzy UP -algebras and discussed their properties in details.

The objective of this paper is to introduce the concept of Atanassov's interval-valued intuitionistic fuzzy sets in UP -algebras. The images and preimages of IVIF UP -subalgebras and UP -ideals has been introduced and some important properties of it are also studied. The rest of the paper is organized as follows. Section 2 recalls some definitions, viz., UP -algebra, UP -subalgebra, UP -ideal and refinement of unit interval. In Section 3, UP -subalgebras of IVIFSs are defined with some its properties. In next Section, IVIF UP -ideals are defined and related properties are investigated. In Section 5, homomorphism of IVIF UP -subalgebras and UP -ideals, and some of its properties are studied. In Section 6, equivalence relations on IVIF UP -ideals are introduced. In Section 7, product of IVIF UP -subalgebras and UP -ideals are investigated. Finally, in Section 8, a conclusion of the proposed work is given.

2. Preliminaries

Here we give a brief review of some preliminaries.

Definition 2.1 ([3]). *By a UP -algebra we mean an algebra $(X, *, 0)$ of type $(2, 0)$ with a single binary operation $*$ that satisfies the following axioms: for any $x, y, z \in X$,*

1. $(y * z) * ((x * y) * (x * z)) = 0$,
2. $0 * x = x$,
3. $x * 0 = 0$,
4. $x * y = y * x = 0$ implies $x = y$.

In what follows, let $(X, *, 0)$ denote a UP -algebra unless otherwise specified. For brevity we also call X a UP -algebra. We can define a partial ordering " $\leq$ " by $x \leq y$ if and only if $x * y = 0$.

Proposition 2.2 ([7]). *In a UP -algebra, the following axioms are true: for any $x, y, z \in X$,*

- (i) $x * x = 0$,
- (ii) $x * y = y * z = 0$ implies $x * z = 0$,
- (iii) $x * y = 0$ implies $(z * x) * (z * y) = 0$,
- (iv) $x * y = 0$ implies $(y * z) * (x * z) = 0$,
- (v) $x * (y * x) = 0$,
- (vi) $(y * x) * x = 0$ if and only if $x = y$,
- (vi) $x * (y * y) = 0$.

A non-empty subset S of a UP -algebra X is called a UP -subalgebra [7] of X if $x * y \in S$, for all $x, y \in S$. From this definition it is observed that, if a subset S of a UP -algebra satisfies only the closer property, then S becomes a UP -subalgebra.

A nonempty subset T of X is called an UP -ideal [3] of X if it satisfies the following properties: (I_1) the constant $0 \in T$, (I_2) for ant $x, y, z \in X$, $x * (y * z) \in T$ and $y \in T \Rightarrow x * z \in T$.

Let $(X, *, 0)$ and $(Y, *, 0')$ be UP -algebras. A homomorphism is a mapping $f : X \rightarrow Y$ satisfying $f(x * y) = f(x) *' f(y)$, for all $x, y \in X$.

Let $D[0, 1]$ be the set of all closed subintervals of the interval $[0, 1]$. Consider two elements $D_1, D_2 \in D[0, 1]$. If $D_1 = [a_1, b_1]$ and $D_2 = [a_2, b_2]$, then $rmin(D_1, D_2) = [\min(a_1, a_2), \min(b_1, b_2)]$ which is denoted by $D_1 \wedge^r D_2$ and $rmax(D_1, D_2) = [\max(a_1, a_2), \max(b_1, b_2)]$ which is denoted by $D_1 \vee^r D_2$. Thus, if $D_i = [a_i, b_i] \in D[0, 1]$ for $i = 1, 2, 3, 4, \dots$, then we define $rsup_i(D_i) = [\sup_i(a_i), \sup_i(b_i)]$, i.e, $\vee_i^r D_i = [\vee_i a_i, \vee_i b_i]$. Similarly, we define $rinf_i(D_i) = [\inf_i(a_i), \inf_i(b_i)]$ i.e, $\wedge_i^r D_i = [\wedge_i a_i, \wedge_i b_i]$. Now we call $D_1 \geq D_2$ if and only if $a_1 \geq a_2$ and $b_1 \geq b_2$. Similarly, the relations $D_1 \leq D_2$ and $D_1 = D_2$ are defined.

Our main objective is to investigate the idea of UP -subalgebras and UP -ideals of IVIFS. The IVIFS is a particular type of fuzzy set.

Definition 2.3 ([25]). (*Fuzzy Set*) Let X be the collection of objects denoted generally by x then a fuzzy set A in X is defined as $A = \{ \langle x, \mu_A(x) \rangle : x \in X \}$ where $\mu_A(x)$ is called the membership value of x in A and $0 \leq \mu_A(x) \leq 1$.

Combined the definition of UP -subalgebra and UP -ideal over crisp set and the idea of fuzzy set Somjanta et al. [24] defined fuzzy UP -subalgebra and UP -ideal, which is defined below.

Definition 2.4 ([24]). Let $A = \{ \langle x, \mu_A(x) \rangle : x \in X \}$ be a fuzzy set in a UP -algebra. Then A is called a fuzzy UP -subalgebra of X if $\mu_A(x * y) \geq \min\{\mu_A(x), \mu_A(y)\}$ for all $x, y \in X$.

A is called a fuzzy UP -ideal of X if $\mu_A(0) \geq \mu_A(x)$ and $\mu_A(x * z) \geq \min\{\mu_A(x * (y * z)), \mu_A(y)\}$ for all $x, y, z \in X$.

Definition 2.5 ([1]). (IVIFS) An IVIFS A over X is an object having the form $A = \{\langle x, R_A(x), Q_A(x) \rangle : x \in X\}$, where $R_A(x) : X \rightarrow D[0, 1]$ and $Q_A(x) : X \rightarrow D[0, 1]$. The intervals $R_A(x)$ and $Q_A(x)$ denote the intervals of the degree of belongingness and non-belongingness of the element x to the set A , where $R_A(x) = [R_{AL}(x), R_{AU}(x)]$ and $Q_A(x) = [Q_{AL}(x), Q_{AU}(x)]$, for all $x \in X$, with the condition $0 \leq R_{AU}(x) + Q_{AU}(x) \leq 1$. For the sake of simplicity, we shall use the symbol $A = (R_A, Q_A)$ for the IVIFS $A = \{\langle x, R_A(x), Q_A(x) \rangle : x \in X\}$.

Also note that $\bar{R}_A(x) = [1 - R_{AU}(x), 1 - R_{AL}(x)]$ and $\bar{Q}_A(x) = [1 - Q_{AU}(x), 1 - Q_{AL}(x)]$, where $[\bar{R}_A(x), \bar{Q}_A(x)]$ represents the complement of x in A .

3. IVIF UP -subalgebras of UP -algebras

In this section, we will introduce a new notion called interval-valued intuitionistic fuzzy UP -subalgebra (IVIF UP -subalgebra) of UP -algebras and study several properties of it.

Definition 3.1. Let $A = (R_A, Q_A)$ be an IVIFS in X , where X is a UP -subalgebra, then the set A is IVIF UP -subalgebra over the binary operator $*$ if it satisfies the following conditions:

$$\begin{aligned} (UP1) \quad & R_A(x * y) \geq \min\{R_A(x), R_A(y)\}, \\ (UP2) \quad & Q_A(x * y) \leq \max\{Q_A(x), Q_A(y)\}, \end{aligned}$$

for all $x, y \in X$.

We consider an example of IVIF UP -subalgebra below.

Example 3.2. Let $X = \{0, a, b, c\}$ be a UP -algebra with the following Cayley table:

$*$	0	a	b	c
0	0	a	b	c
a	0	0	0	0
b	0	a	0	c
c	0	a	b	0

Define an IVIFS $A = (R_A, Q_A)$ in X by

$$R_A(x) = \begin{cases} [0.5, 0.6], & \text{if } x \in \{0, a, b\} \\ [0.1, 0.2], & \text{if } x = c \end{cases} \quad \text{and} \quad Q_A(x) = \begin{cases} [0.3, 0.4], & \text{if } x \in \{0, a, b\} \\ [0.4, 0.5], & \text{if } x = c. \end{cases}$$

By routine calculations we get A is an IVIF UP -subalgebra of X .

Proposition 3.3. If $A = (R_A, Q_A)$ is an IVIF UP -subalgebra in X , then for all $x \in X$, $R_A(0) \geq R_A(x)$ and $Q_A(0) \leq Q_A(x)$.

Proof. It is easy and omitted. □

Theorem 3.4. *Let A be an IVIF UP-subalgebra of X . If there exists a sequence $\{x_n\}$ in X such that $\lim_{n \rightarrow \infty} R_A(x_n) = [1, 1]$ and $\lim_{n \rightarrow \infty} Q_A(x_n) = [0, 0]$. Then $R_A(0) = [1, 1]$ and $Q_A(0) = [0, 0]$.*

Proof. By Proposition 3.3, $R_A(0) \geq R_A(x)$ for all $x \in X$, therefore, $R_A(0) \geq R_A(x_n)$ for every positive integer n . Consider, $[1, 1] \geq R_A(0) \geq \lim_{n \rightarrow \infty} R_A(x_n) = [1, 1]$. Hence, $R_A(0) = [1, 1]$.

Again, by Proposition 3.3, $Q_A(0) \leq Q_A(x)$ for all $x \in X$, thus $Q_A(0) \leq Q_A(x_n)$ for every positive integer n . Now, $[0, 0] \leq Q_A(0) \leq \lim_{n \rightarrow \infty} Q_A(x_n) = [0, 0]$. Hence, $Q_A(0) = [0, 0]$. $\square$

Proposition 3.5. *If an IVIFS $A = (R_A, Q_A)$ in X is an IVIF UP-subalgebra, then for all $x \in X$, $R_A(0 * x) \geq R_A(x)$ and $Q_A(0 * x) \leq Q_A(x)$.*

Proof. For all $x \in X$, $R_A(0 * x) \geq \text{rmin}\{R_A(0), R_A(x)\} = \text{rmin}\{R_A(x * x), R_A(x)\} \geq \text{rmin}\{\text{rmin}\{R_A(x), R_A(x)\}, R_A(x)\} = R_A(x)$ and $Q_A(0 * x) \leq \text{rmax}\{Q_A(0), Q_A(x)\} = \text{rmax}\{Q_A(x * x), Q_A(x)\} \leq \text{rmax}\{\text{rmax}\{Q_A(x), Q_A(x)\}, Q_A(x)\} = Q_A(x)$. This completes the proof. $\square$

Theorem 3.6. *An IVIFSs $A = \{[R_{AL}, R_{AU}], [Q_{AL}, Q_{AU}]\}$ in X is an IVIF UP-subalgebra of X if and only if R_{AL} , R_{AU} , Q_{AL} and Q_{AU} are fuzzy UP-subalgebras of X .*

Proof. Let R_{AL} and R_{AU} be fuzzy UP-subalgebra of X and $x, y \in X$. Then $R_{AL}(x * y) \geq \min\{R_{AL}(x), R_{AL}(y)\}$ and $R_{AU}(x * y) \geq \min\{R_{AU}(x), R_{AU}(y)\}$. Now,

$$\begin{aligned} R_A(x * y) &= [R_{AL}(x * y), R_{AU}(x * y)] \\ &\geq [\min\{R_{AL}(x), R_{AL}(y)\}, \min\{R_{AU}(x), R_{AU}(y)\}] \\ &= \text{rmin}\{[R_{AL}(x), R_{AU}(x)], [R_{AL}(y), R_{AU}(y)]\} \\ &= \text{rmin}\{R_A(x), R_A(y)\}. \end{aligned}$$

Again, let Q_{AL} and Q_{AU} be fuzzy UP-subalgebras of X and $x, y \in X$. Then $Q_{AL}(x * y) \leq \max\{Q_{AL}(x), Q_{AL}(y)\}$ and $Q_{AU}(x * y) \leq \max\{Q_{AU}(x), Q_{AU}(y)\}$. Now,

$$\begin{aligned} Q_A(x * y) &= [Q_{AL}(x * y), Q_{AU}(x * y)] \\ &\leq [\max\{Q_{AL}(x), Q_{AL}(y)\}, \max\{Q_{AU}(x), Q_{AU}(y)\}] \\ &= \text{rmax}\{[Q_{AL}(x), Q_{AU}(x)], [Q_{AL}(y), Q_{AU}(y)]\} \\ &= \text{rmax}\{Q_A(x), Q_A(y)\}. \end{aligned}$$

Hence, $A = \{[R_{AL}, R_{AU}], [Q_{AL}, Q_{AU}]\}$ is an IVIF UP-subalgebra of X .

Conversely, assume that, A is an IVIF UP -subalgebra of X . For any $x, y \in X$

$$\begin{aligned} [R_{AL}(x * y), R_{AU}(x * y)] &= R_A(x * y) \geq rmin\{R_A(x), R_A(y)\} \\ &= rmin\{[R_{AL}(x), R_{AU}(x)], [R_{AL}(y), R_{AU}(y)]\} \\ &= [\min\{R_{AL}(x), R_{AL}(y)\}, \min\{R_{AU}(x), R_{AU}(y)\}], \\ [Q_{AL}(x * y), Q_{AU}(x * y)] &= Q_A(x * y) \leq rmax\{Q_A(x), Q_A(y)\} \\ &= rmax\{[Q_{AL}(x), Q_{AU}(x)], [Q_{AL}(y), Q_{AU}(y)]\} \\ &= [\max\{Q_{AL}(x), Q_{AL}(y)\}, \max\{Q_{AU}(x), Q_{AU}(y)\}]. \end{aligned}$$

Thus $R_{AL}(x * y) \geq \min\{R_{AL}(x), R_{AL}(y)\}$, $R_{AU}(x * y) \geq \min\{R_{AU}(x), R_{AU}(y)\}$, $Q_{AL}(x * y) \leq \max\{Q_{AL}(x), Q_{AL}(y)\}$ and $Q_{AU}(x * y) \leq \max\{Q_{AU}(x), Q_{AU}(y)\}$. Therefore, R_{AL} , R_{AU} , Q_{AL} and Q_{AU} are fuzzy UP -subalgebras of X . $\square$

Definition 3.7. Let A and B be two IVIFSs on X , where $A = \{([R_{AL}(x), R_{AU}(x)], [Q_{AL}(x), Q_{AU}(x)]) : x \in X\}$ and $B = \{([R_{BL}(x), R_{BU}(x)], [Q_{BL}(x), Q_{BU}(x)]) : x \in X\}$. Then the intersection of A and B is denoted by $A \cap B$, and is given by $A \cap B = \{([R_{A \cap B}(x), R_{A \cup B}(x)], [Q_{A \cap B}(x), Q_{A \cup B}(x)]) : x \in X\} = \{([min(R_{AL}(x), R_{BL}(x)), min(R_{AU}(x), R_{BU}(x))], [max(Q_{AL}(x), Q_{BL}(x)), max(Q_{AU}(x), Q_{BU}(x))]) : x \in X\}$.

Theorem 3.8. Let A_1 and A_2 be two IVIF UP -subalgebras of X . Then $A_1 \cap A_2$ is an IVIF UP -subalgebra of X .

Proof. Let $x, y \in A_1 \cap A_2$. Then $x, y \in A_1$ and A_2 . Since A_1 and A_2 are IVIF UP -subalgebras of X , by Theorem 3.6,

$$\begin{aligned} R_{A_1 \cap A_2}(x * y) &= [R_{(A_1 \cap A_2)L}(x * y), R_{(A_1 \cap A_2)U}(x * y)] \\ &= [\min(R_{A_1 L}(x * y), R_{A_2 L}(x * y)), \\ &\quad \min(R_{A_1 U}(x * y), R_{A_2 U}(x * y))] \\ &\geq [\min(R_{(A_1 \cap A_2)L}(x), R_{(A_1 \cap A_2)L}(y)), \\ &\quad \min(R_{(A_1 \cap A_2)U}(x), R_{(A_1 \cap A_2)U}(y))] \\ &= rmin\{R_{A_1 \cap A_2}(x), R_{A_1 \cap A_2}(y)\} \\ \text{and } Q_{A_1 \cup A_2}(x * y) &= [Q_{(A_1 \cup A_2)L}(x * y), Q_{(A_1 \cup A_2)U}(x * y)] \\ &= [\max(Q_{A_1 L}(x * y), Q_{A_2 L}(x * y)), \\ &\quad \max(Q_{A_1 U}(x * y), Q_{A_2 U}(x * y))] \\ &\leq [\max(Q_{(A_1 \cup A_2)L}(x), Q_{(A_1 \cup A_2)L}(y)), \\ &\quad \max(Q_{(A_1 \cup A_2)U}(x), Q_{(A_1 \cup A_2)U}(y))] \\ &= rmax\{Q_{A_1 \cup A_2}(x), Q_{A_1 \cup A_2}(y)\}. \end{aligned}$$

This proves the theorem. $\square$

Corollary 3.9. Let $\{A_i | i = 1, 2, 3, 4, \dots\}$ be a family of IVIF UP -subalgebra of X . Then $\bigcap A_i$ is also an IVIF UP -subalgebra of X where, $\bigcap A_i = \{([x, rmin R_{A_i}(x)], [rmax Q_{A_i}(x)]) : x \in X\}$.

Theorem 3.10. *Let $A = (R_A, Q_A)$ be an IVIF UP-subalgebra of X and let $n \in \mathbb{N}$ (the set of natural numbers). Then*

- (i) $R_A(\prod^n x * x) \geq R_A(x)$, for any odd number n ,
- (ii) $Q_A(\prod^n x * x) \leq Q_A(x)$, for any odd number n ,
- (iii) $R_A(\prod^n x * x) = R_A(x)$, for any even number n ,
- (iv) $Q_A(\prod^n x * x) = Q_A(x)$, for any even number n .

Proof. Let $x \in X$ and assume that n is odd. Then $n = 2p - 1$ for some positive integer p . We prove the Theorem by induction.

Now $R_A(x * x) = R_A(0) \geq R_A(x)$ and $Q_A(x * x) = Q_A(0) \leq Q_A(x)$. Suppose that $R_A(\prod^{2p-1} x * x) \geq R_A(x)$ and $Q_A(\prod^{2p-1} x * x) \leq Q_A(x)$. Then by assumption, $R_A(\prod^{2(p+1)-1} x * x) = R_A(\prod^{2p+1} x * x) = R_A(\prod^{2p-1} x * (x * (x * x))) = R_A(\prod^{2p-1} x * x) \geq R_A(x)$ and $Q_A(\prod^{2(p+1)-1} x * x) = Q_A(\prod^{2p+1} x * x) = Q_A(\prod^{2p-1} x * (x * (x * x))) = Q_A(\prod^{2p-1} x * x) \leq Q_A(x)$, which proves (i) and (ii). Proofs are similar for the cases (iii) and (iv). $\square$

We define two operators $\oplus A$ and $\otimes A$ on IVIFS as follows:

Definition 3.11. *Let $A = (R_A, Q_A)$ be an IVIFS defined on X . The operators $\oplus A$ and $\otimes A$ are defined as $\oplus A = \{\langle x, R_A(x), \bar{R}_A(x) \rangle : x \in X\}$ and $\otimes A = \{\langle x, \bar{Q}_A(x), Q_A(x) \rangle : x \in X\}$.*

Theorem 3.12. *If $A = (R_A, Q_A)$ is an IVIF UP-subalgebra of X , then*

- (i) $\oplus A$, and
- (ii) $\otimes A$, both are IVIF UP-subalgebras.

Proof. For (i), it is sufficient to show that $\bar{R}_A$ satisfies the condition (UP2). Let $x, y \in X$. Then $\bar{R}_A(x * y) = [1, 1] - R_A(x * y) \leq [1, 1] - rmin\{R_A(x), R_A(y)\} = rmax\{1 - R_A(x), 1 - R_A(y)\} = rmax\{\bar{R}_A(x), \bar{R}_A(y)\}$. Hence, $\oplus A$ is an IVIF UP-subalgebra of X .

For (ii), it is sufficient to show that $\bar{Q}_A$ satisfies the condition (UP1). Let $x, y \in X$. Then $\bar{Q}_A(x * y) = [1, 1] - Q_A(x * y) \geq [1, 1] - rmax\{Q_A(x), Q_A(y)\} = rmin\{1 - Q_A(x), 1 - Q_A(y)\} = rmin\{\bar{Q}_A(x), \bar{Q}_A(y)\}$. Hence, $\otimes A$ is also an IVIF UP-subalgebra of X . $\square$

The sets $\{x \in X : R_A(x) = R_A(0)\}$ and $\{x \in X : Q_A(x) = Q_A(0)\}$ are denoted by I_{R_A} and I_{Q_A} respectively. These two sets are also UP-subalgebra of X .

Theorem 3.13. *Let $A = (R_A, Q_A)$ be an IVIF UP-subalgebra of X , then the sets I_{R_A} and I_{Q_A} are UP-subalgebras of X .*

Proof. Let $x, y \in I_{R_A}$. Then $R_A(x) = R_A(0) = R_A(y)$ and so, $R_A(x * y) \geq \min\{R_A(x), R_A(y)\} = R_A(0)$. By using Proposition 3.3, we know that $R_A(x * y) = R_A(0)$ or equivalently $x * y \in I_{R_A}$.

Again, let $x, y \in I_{Q_A}$. Then $Q_A(x) = Q_A(0) = Q_A(y)$ and so, $Q_A(x * y) \leq \max\{Q_A(x), Q_A(y)\} = Q_A(0)$. Again, by Proposition 3.3, we know that $Q_A(x * y) = Q_A(0)$ or equivalently $x * y \in I_{Q_A}$.

Hence, the sets I_{R_A} and I_{Q_A} are UP-subalgebras of X . $\square$

Theorem 3.14. Let B be a nonempty subset of X and $A = (R_A, Q_A)$ be an IVIFS in X defined by

$$R_A(x) = \begin{cases} [\alpha_1, \alpha_2], & \text{if } x \in B \\ [\beta_1, \beta_2], & \text{otherwise} \end{cases} \quad \text{and} \quad Q_A(x) = \begin{cases} [\gamma_1, \gamma_2], & \text{if } x \in B \\ [\delta_1, \delta_2], & \text{otherwise} \end{cases}$$

for all $[\alpha_1, \alpha_2], [\beta_1, \beta_2], [\gamma_1, \gamma_2]$ and $[\delta_1, \delta_2] \in D[0, 1]$ with $[\alpha_1, \alpha_2] \geq [\beta_1, \beta_2]$ and $[\gamma_1, \gamma_2] \leq [\delta_1, \delta_2]$ and $\alpha_2 + \gamma_2 \leq 1$; $\beta_2 + \delta_2 \leq 1$. Then A is an IVIF UP-subalgebra of X if and only if B is a UP-subalgebra of X . Moreover, $I_{R_A} = B = I_{Q_A}$.

Proof. Let A be an IVIF UP-subalgebra of X and $x, y \in X$ be such that $x, y \in B$. Then $R_A(x * y) \geq \min\{R_A(x), R_A(y)\} = \min\{[\alpha_1, \alpha_2], [\alpha_1, \alpha_2]\} = [\alpha_1, \alpha_2]$ and $Q_A(x * y) \leq \max\{Q_A(x), Q_A(y)\} = \max\{[\gamma_1, \gamma_2], [\gamma_1, \gamma_2]\} = [\gamma_1, \gamma_2]$. So $x * y \in B$. Hence, B is a UP-subalgebra of X .

Conversely, suppose that B is a UP-subalgebra of X . Let $x, y \in X$. Consider two cases:

Case (i). If $x, y \in B$ then $x * y \in B$, thus $R_A(x * y) = [\alpha_1, \alpha_2] = \min\{R_A(x), R_A(y)\}$ and $Q_A(x * y) = [\gamma_1, \gamma_2] = \max\{Q_A(x), Q_A(y)\}$.

Case (ii). If $x \notin B$ or, $y \notin B$, then $R_A(x * y) \geq [\beta_1, \beta_2] = \min\{R_A(x), R_A(y)\}$ and $Q_A(x * y) \leq [\delta_1, \delta_2] = \max\{Q_A(x), Q_A(y)\}$.

Hence, A is an IVIF UP-subalgebra of X .

Now, $I_{R_A} = \{x \in X, R_A(x) = R_A(0)\} = \{x \in X, R_A(x) = [\alpha_1, \alpha_2]\} = B$ and $I_{Q_A} = \{x \in X, Q_A(x) = Q_A(0)\} = \{x \in X, Q_A(x) = [\gamma_1, \gamma_2]\} = B$. $\square$

Definition 3.15. Let $A = (R_A, Q_A)$ is an IVIF UP-subalgebra of X . For $[s_1, s_2], [t_1, t_2] \in D[0, 1]$, the set $U(R_A : [s_1, s_2]) = \{x \in X : R_A(x) \geq [s_1, s_2]\}$ is called upper $[s_1, s_2]$ -level of A and $L(Q_A : [t_1, t_2]) = \{x \in X : Q_A(x) \leq [t_1, t_2]\}$ is called lower $[t_1, t_2]$ -level of A .

Theorem 3.16. If $A = (R_A, Q_A)$ is an IVIF UP-subalgebra of X , then the upper $[s_1, s_2]$ -level and lower $[t_1, t_2]$ -level of A are subalgebras of X .

Proof. Let $x, y \in U(R_A : [s_1, s_2])$. Then $R_A(x) \geq [s_1, s_2]$ and $R_A(y) \geq [s_1, s_2]$. It follows that $R_A(x * y) \geq \min\{R_A(x), R_A(y)\} \geq [s_1, s_2]$ so that $x * y \in U(R_A : [s_1, s_2])$. Hence, $U(R_A : [s_1, s_2])$ is a subalgebra of X .

Let $x, y \in L(Q_A : [t_1, t_2])$. Then $Q_A(x) \leq [t_1, t_2]$ and $Q_A(y) \leq [t_1, t_2]$. It follows that $Q_A(x * y) \leq \max\{Q_A(x), Q_A(y)\} \leq [t_1, t_2]$ so that $x * y \in L(Q_A : [t_1, t_2])$. Hence, $L(Q_A : [t_1, t_2])$ is a subalgebra of X . $\square$

Theorem 3.17. *Let $A = (R_A, Q_A)$ be an IVIFS in X , such that the sets $U(R_A : [s_1, s_2])$ and $L(Q_A : [t_1, t_2])$ are subalgebras of X for every $[s_1, s_2], [t_1, t_2] \in D[0, 1]$. Then A is an IVIF UP-subalgebra of X .*

Proof. Let for every $[s_1, s_2], [t_1, t_2] \in D[0, 1]$, $U(R_A : [s_1, s_2])$ and $L(Q_A : [t_1, t_2])$ are subalgebras of X . In contrary, let $x_0, y_0 \in X$ be such that $R_A(x_0 * y_0) < rmin\{R_A(x_0), R_A(y_0)\}$. Let $R_A(x_0) = [\vartheta_1, \vartheta_2]$, $R_A(y_0) = [\vartheta_3, \vartheta_4]$ and $R_A(x_0 * y_0) = [s_1, s_2]$. Then $[s_1, s_2] < rmin\{[\vartheta_1, \vartheta_2], [\vartheta_3, \vartheta_4]\} = [\min\{\vartheta_1, \vartheta_3\}, \min\{\vartheta_2, \vartheta_4\}]$. So, $s_1 < \min\{\vartheta_1, \vartheta_3\}$ and $s_2 < \min\{\vartheta_2, \vartheta_4\}$. Consider,

$$\begin{aligned} [\rho_1, \rho_2] &= \frac{1}{2} [R_A(x_0 * y_0) + rmin\{R_A(x_0), R_A(y_0)\}] \\ &= \frac{1}{2} [s_1, s_2] + [\min\{\vartheta_1, \vartheta_3\}, \min\{\vartheta_2, \vartheta_4\}] \\ &= \left[\frac{1}{2}(s_1 + \min\{\vartheta_1, \vartheta_3\}), \frac{1}{2}(s_2 + \min\{\vartheta_2, \vartheta_4\}) \right]. \end{aligned}$$

Therefore, $\min\{\vartheta_1, \vartheta_3\} > \rho_1 = \frac{1}{2}(s_1 + \min\{\vartheta_1, \vartheta_3\}) > s_1$ and $\min\{\vartheta_2, \vartheta_4\} > \rho_2 = \frac{1}{2}(s_2 + \min\{\vartheta_2, \vartheta_4\}) > s_2$. Hence, $[\min\{\vartheta_1, \vartheta_3\}, \min\{\vartheta_2, \vartheta_4\}] > [\rho_1, \rho_2] > [s_1, s_2]$, so that $x_0 * y_0 \notin U(R_A : [s_1, s_2])$ which is a contradiction, since $R_A(x_0) = [\vartheta_1, \vartheta_2] \geq [\min\{\vartheta_1, \vartheta_3\}, \min\{\vartheta_2, \vartheta_4\}] > [\rho_1, \rho_2]$ and $R_A(y_0) = [\vartheta_3, \vartheta_4] \geq [\min\{\vartheta_1, \vartheta_3\}, \min\{\vartheta_2, \vartheta_4\}] > [\rho_1, \rho_2]$. This implies $x_0 * y_0 \in U(R_A : [s_1, s_2])$. Thus $R_A(x * y) \geq rmin\{R_A(x), R_A(y)\}$ for all $x, y \in X$.

Again, in contrary, let $x_0, y_0 \in X$ be such that $Q_A(x_0 * y_0) > rmax\{Q_A(x_0), Q_A(y_0)\}$. Let $Q_A(x_0) = [\psi_1, \psi_2]$, $Q_A(y_0) = [\psi_3, \psi_4]$ and $Q_A(x_0 * y_0) = [t_1, t_2]$. Then $[t_1, t_2] > rmax\{[\psi_1, \psi_2], [\psi_3, \psi_4]\} = [\max\{\psi_1, \psi_3\}, \max\{\psi_2, \psi_4\}]$. So $t_1 > \max\{\psi_1, \psi_3\}$ and $t_2 > \max\{\psi_2, \psi_4\}$. Let us consider,

$$\begin{aligned} [\beta_1, \beta_2] &= \frac{1}{2} [Q_A(x_0 * y_0) + rmax\{Q_A(x_0), Q_A(y_0)\}] \\ &= \frac{1}{2} [t_1, t_2] + [\max\{\psi_1, \psi_3\}, \max\{\psi_2, \psi_4\}] \\ &= \left[\frac{1}{2}(t_1 + \max\{\psi_1, \psi_3\}), \frac{1}{2}(t_2 + \max\{\psi_2, \psi_4\}) \right]. \end{aligned}$$

Therefore, $\max\{\psi_1, \psi_3\} < \beta_1 = \frac{1}{2}(t_1 + \max\{\psi_1, \psi_3\}) < t_1$ and $\max\{\psi_2, \psi_4\} < \beta_2 = \frac{1}{2}(t_2 + \max\{\psi_2, \psi_4\}) < t_2$. Hence, $[\max\{\psi_1, \psi_3\}, \max\{\psi_2, \psi_4\}] < [\beta_1, \beta_2] < [t_1, t_2]$ so that $x_0 * y_0 \notin L(Q_A : [t_1, t_2])$ which is a contradiction, since $Q_A(x_0) = [\psi_1, \psi_2] \leq [\max\{\psi_1, \psi_3\}, \max\{\psi_2, \psi_4\}] < [\beta_1, \beta_2]$ and $Q_A(y_0) = [\psi_3, \psi_4] \leq [\max\{\psi_1, \psi_3\}, \max\{\psi_2, \psi_4\}] < [\beta_1, \beta_2]$. Hence, $x_0 * y_0 \in L(Q_A : [t_1, t_2])$. Thus $Q_A(x * y) \leq rmax\{Q_A(x), Q_A(y)\}$ for all $x, y \in X$. $\square$

Theorem 3.18. *Any subalgebra of X can be realized as both the upper $[s_1, s_2]$ -level and lower $[t_1, t_2]$ -level of some IVIF UP-subalgebra of X .*

Proof. Let P be an IVIF UP-subalgebra of X , and A be an IVIFS on X defined by

$$R_A(x) = \begin{cases} [\xi_1, \xi_2], & \text{if } x \in P \\ [0, 0], & \text{otherwise} \end{cases} \quad \text{and} \quad Q_A(x) = \begin{cases} [\omega_1, \omega_2], & \text{if } x \in P \\ [1, 1], & \text{otherwise,} \end{cases}$$

for all $[\xi_1, \xi_2], [\omega_1, \omega_2] \in D[0, 1]$ and $\xi_2 + \omega_2 \leq 1$. We consider the following cases:

Case (i) If $x, y \in P$, then $R_A(x) = [\xi_1, \xi_2]$, $Q_A(x) = [\omega_1, \omega_2]$ and $R_A(y) = [\xi_1, \xi_2]$, $Q_A(y) = [\omega_1, \omega_2]$. Thus, $R_A(x * y) = [\xi_1, \xi_2] = \text{rmin}\{[\xi_1, \xi_2], [\xi_1, \xi_2]\} = \text{rmin}\{R_A(x), R_A(y)\}$ and $Q_A(x * y) = [\omega_1, \omega_2] = \text{rmax}\{[\omega_1, \omega_2], [\omega_1, \omega_2]\} = \text{rmax}\{Q_A(x), Q_A(y)\}$.

Case (ii) If $x \in P$ and $y \notin P$ then $R_A(x) = [\xi_1, \xi_2]$, $Q_A(x) = [\omega_1, \omega_2]$ and $R_A(y) = [0, 0]$, $Q_A(y) = [1, 1]$. Thus, $R_A(x * y) \geq [0, 0] = \text{rmin}\{[\xi_1, \xi_2], [0, 0]\} = \text{rmin}\{R_A(x), R_A(y)\}$ and $Q_A(x * y) \leq [1, 1] = \text{rmax}\{[\omega_1, \omega_2], [1, 1]\} = \text{rmax}\{Q_A(x), Q_A(y)\}$.

Case (iii) If $x \notin P$ and $y \in P$ then $R_A(x) = [0, 0]$, $Q_A(x) = [1, 1]$, $R_A(y) = [\xi_1, \xi_2]$, $Q_A(y) = [\omega_1, \omega_2]$. Thus, $R_A(x * y) \geq [0, 0] = \text{rmin}\{[0, 0], [\xi_1, \xi_2]\} = \text{rmin}\{R_A(x), R_A(y)\}$ and $Q_A(x * y) \leq [1, 1] = \text{rmax}\{[1, 1], [\omega_1, \omega_2]\} = \text{rmax}\{Q_A(x), Q_A(y)\}$.

Case (iv) If $x \notin P$ and $y \notin P$ then $R_A(x) = [0, 0]$, $Q_A(x) = [1, 1]$ and $R_A(y) = [0, 0]$, $Q_A(y) = [1, 1]$. Now $R_A(x * y) \geq [0, 0] = \text{rmin}\{[0, 0], [0, 0]\} = \text{rmin}\{R_A(x), R_A(y)\}$ and $Q_A(x * y) \leq [1, 1] = \text{rmax}\{[1, 1], [1, 1]\} = \text{rmax}\{Q_A(x), Q_A(y)\}$.

Therefore, A is an IVIF UP -subalgebra of X . $\square$

Theorem 3.19. *Let P be a subset of X and A be an IVIFS on X which is given in the proof of Theorem 3.18. If A be realized as lower level subalgebra and upper level subalgebra of some IVIF UP -subalgebra of X , then P is a IVIF UP -subalgebra of X .*

Proof. Let A be an IVIF UP -subalgebra of X , and $x, y \in P$. Then $R_A(x) = [\xi_1, \xi_2] = R_A(y)$ and $Q_A(x) = [\omega_1, \omega_2] = Q_A(y)$. Thus $R_A(x * y) \geq \text{rmin}\{R_A(x), R_A(y)\} = \text{rmin}\{[\xi_1, \xi_2], [\xi_1, \xi_2]\} = [\xi_1, \xi_2]$ and $Q_A(x * y) \leq \text{rmax}\{Q_A(x), Q_A(y)\} = \text{rmax}\{[\omega_1, \omega_2], [\omega_1, \omega_2]\} = [\omega_1, \omega_2]$, which imply that $x * y \in P$. Hence, the theorem. $\square$

4. IVIF UP -ideals of UP -algebras

In this section we will define IVIF UP -ideal of UP -algebras and prove some propositions and theorems. In what follows, let X denote a UP -algebra unless otherwise specified.

Definition 4.1. *An IVIFS $A = (R_A, Q_A)$ in X is called an IVIF UP -ideal of X if it satisfies:*

- (UP3) $R_A(0) \geq R_A(x)$ and $Q_A(0) \leq Q_A(x)$
- (UP4) $R_A(x * z) \geq \text{rmin}\{R_A(x * (y * z)), R_A(y)\}$
- (UP5) $Q_A(x * z) \leq \text{rmax}\{Q_A(x * (y * z)), Q_A(y)\},$

for all $x, y \in X$.

Example 4.2. Consider a UP -algebra $X = \{0, a, b, c, d\}$ with the following Cayley table

$*$	0	a	b	c	d
0	0	a	b	c	d
a	0	0	b	c	d
b	0	0	0	c	d
c	0	0	b	0	d
d	0	0	0	0	0

Let $A = (R_A, Q_A)$ be an IVIFS in X defined as

$$R_A(x) = \begin{cases} [1, 1], & \text{if } x \in \{0, a, b\} \\ [m_1, m_2], & \text{if } x \in \{c, d\} \end{cases} \quad \text{and} \quad Q_A(x) = \begin{cases} [0, 0], & \text{if } x \in \{0, a, b\} \\ [n_1, n_2], & \text{if } x \in \{c, d\}, \end{cases}$$

where $[m_1, m_2], [n_1, n_2] \in D[0, 1]$ and $m_2 + n_2 \leq 1$. By routine calculations we get A is an IVIF UP -ideal of X .

Lemma 4.3. Let $A = (R_A, Q_A)$ be an IVIF UP -ideal of X . If $x, y \in X$ is such that $y \leq x$, then $R_A(x) \geq R_A(y)$ and $Q_A(x) \leq Q_A(y)$.

Proof. It is immediate and is omitted. $\square$

Lemma 4.4. Let $A = (R_A, Q_A)$ be an IVIF UP -ideal of X and $x, y, z, q \in X$. If $x \leq q * (y * z)$ then $R_A(x * z) \geq \text{rmin}\{R_A(q), R_A(y)\}$ and $Q_A(x * z) \leq \text{rmax}\{Q_A(q), Q_A(y)\}$.

Proof. Let $x, y, z, q \in X$ such that $x \leq q * (y * z)$. Then $x * (q * (y * z)) = 0$ and thus $R_A(x * z) \geq \text{rmin}\{R_A(x * (y * z)), R_A(y)\} \geq \text{rmin}\{R_A\{(x * (q * (y * z)))\}, R_A(q)\}, R_A(y)\} = \text{rmin}\{\text{rmin}\{R_A(0), R_A(q)\}, R_A(y)\} = \text{rmin}\{R_A(q), R_A(y)\}$ and $Q_A(x * z) \leq \text{rmax}\{Q_A(x * (y * z)), Q_A(y)\} \leq \text{rmax}\{\text{rmax}\{Q_A\{(x * (q * (y * z)))\}, Q_A(q)\}, Q_A(y)\} = \text{rmax}\{\text{rmax}\{Q_A(0), Q_A(q)\}, Q_A(y)\} = \text{rmax}\{Q_A(q), Q_A(y)\}$. $\square$

Corollary 4.5. Let $A = (R_A, Q_A)$ be an IVIF UP -ideal of X and $x, y, z \in X$. If $x \leq y * z$ then $R_A(x * z) \geq R_A(y)$ and $Q_A(x * z) \leq Q_A(y)$.

Proof. Let $x, y, z \in X$ be such that $x \leq y * z$. Then by putting $q = 0$ in Lemma 4.4 we have $x * (0 * (y * z)) = 0$ and thus $R_A(x * z) \geq \text{rmin}\{R_A(0), R_A(y)\} = R_A(y)$ and $Q_A(x * z) \leq \text{rmax}\{Q_A(0), Q_A(y)\} = Q_A(y)$. $\square$

Theorem 4.6. Every IVIF UP -ideal of a UP -algebra X is an IVIF UP -subalgebra of X .

Proof. Let $A = (R_A, Q_A)$ is an IVIF UP -ideal of X and $x, y \in X$. By Proposition 2.2, we have $x \leq y * x$. It follows from Lemma 4.3 that $R_A(y * x) \geq R_A(x) \geq \text{rmin}\{R_A(y), R_A(x)\}$ and $Q_A(y * x) \leq Q_A(x) \leq \text{rmax}\{Q_A(y), Q_A(x)\}$. Hence $A = (R_A, Q_A)$ is an IVIF UP -ideal of X . $\square$

The converse of Theorem 4.6 may not be true. For example, the IVIF UP -subalgebra $A = (R_A, Q_A)$ in Example 3.2 is not an IVIF UP -ideal of X since $R_A(b * c) = [0.1, 0.2] < [0.5, 0.6] = \text{rmin}\{R_A(b * (a * c)), R_A(a)\}$.

Theorem 4.7. *An IVIFSs $A = \{[R_{AL}, R_{AU}], [Q_{AL}, Q_{AU}]\}$ in X is an IVIF UP-ideal of X if and only if R_{AL} , R_{AU} , Q_{AL} and Q_{AU} are fuzzy UP-ideals of X .*

Proof. Since $R_{AL}(0) \geq R_{AL}(x)$, $R_{AU}(0) \geq R_{AU}(x)$, $Q_{AL}(0) \leq Q_{AL}(x)$ and $Q_{AU}(0) \leq Q_{AU}(x)$, therefore $R_A(0) \geq R_A(x)$ and $Q_A(0) \leq Q_A(x)$.

Let R_{AL} and R_{AU} are fuzzy UP-ideals of X . Let $x, y, z \in X$. Then

$$\begin{aligned} R_A(x * z) &= [R_{AL}(x * z), R_{AU}(x * z)] \\ &\geq [\min\{R_{AL}(x * (y * z)), R_{AL}(y)\}, \min\{R_{AU}(x * (y * z)), R_{AU}(y)\}] \\ &= rmin\{[R_{AL}(x * (y * z)), R_{AU}(x * (y * z))], [R_{AL}(y), R_{AU}(y)]\} \\ &= rmin\{R_A(x * (y * z)), R_A(y)\}. \end{aligned}$$

Let Q_{AL} and Q_{AU} are fuzzy UP-ideals of X and $x, y \in X$. Then

$$\begin{aligned} Q_A(x * z) &= [Q_{AL}(x * z), Q_{AU}(x * z)] \\ &\leq [\max\{Q_{AL}(x * (y * z)), Q_{AL}(y)\}, \max\{Q_{AU}(x * (y * z)), Q_{AU}(y)\}] \\ &= rmax\{[Q_{AL}(x * (y * z)), Q_{AU}(x * (y * z))], [Q_{AL}(y), Q_{AU}(y)]\} \\ &= rmax\{Q_A(x * (y * z)), Q_A(y)\}. \end{aligned}$$

Hence, $A = \{[R_{AL}, R_{AU}], [Q_{AL}, Q_{AU}]\}$ is an IVIF UP-ideal of X .

Conversely, assume that, A is an IVIF UP-ideal of X . For any $x, y \in X$, we have $[R_{AL}(x * z), R_{AU}(x * z)] = R_A(x * z) \geq rmin\{R_A(x * (y * z)), R_A(y)\} = rmin\{[R_{AL}(x * (y * z)), R_{AU}(x * (y * z))], [R_{AL}(y), R_{AU}(y)]\} = [\min\{R_{AL}(x * (y * z)), R_{AL}(y)\}, \min\{R_{AU}(x * (y * z)), R_{AU}(y)\}]$ and $[Q_{AL}(x * z), Q_{AU}(x * z)] = Q_A(x * z) \leq rmax\{Q_A(x * (y * z)), Q_A(y)\} = rmax\{[Q_{AL}(x * (y * z)), Q_{AU}(x * (y * z))], [Q_{AL}(y), Q_{AU}(y)]\} = [\max\{Q_{AL}(x * (y * z)), Q_{AL}(y)\}, \max\{Q_{AU}(x * (y * z)), Q_{AU}(y)\}]$. Thus, $R_{AL}(x * z) \geq \min\{R_{AL}(x * (y * z)), R_{AL}(y)\}$, $R_{AU}(x * z) \geq \min\{R_{AU}(x * (y * z)), R_{AU}(y)\}$, $Q_{AL}(x * z) \leq \max\{Q_{AL}(x * (y * z)), Q_{AL}(y)\}$, $Q_{AU}(x * z) \leq \max\{Q_{AU}(x * (y * z)), Q_{AU}(y)\}$. Hence, R_{AL} , R_{AU} , Q_{AL} and Q_{AU} are fuzzy UP-ideals of X . $\square$

Theorem 4.8. *Let A_1 and A_2 be two IVIF UP-ideals of a UP-algebras X . Then $A_1 \cap A_2$ is also an IVIF UP-ideal of UP-algebra X .*

Proof. Let $x, y \in A_1 \cap A_2$. Then $x, y \in A_1$ and A_2 . Now, $R_{A_1 \cap A_2}(0) = R_{A_1 \cap A_2}(x * x) \geq rmin\{R_{A_1 \cap A_2}(x), R_{A_1 \cap A_2}(x)\} = R_{A_1 \cap A_2}(x)$ and $Q_{A_1 \cap A_2}(0) = Q_{A_1 \cap A_2}(x * x) \leq rmin\{Q_{A_1 \cap A_2}(x), Q_{A_1 \cap A_2}(x)\} = Q_{A_1 \cap A_2}(x)$. Also,

$$\begin{aligned} R_{A_1 \cap A_2}(x * z) &= [R_{(A_1 \cap A_2)L}(x * z), R_{(A_1 \cap A_2)U}(x * z)] \\ &\geq [\min(R_{(A_1 \cap A_2)L}(x * (y * z)), R_{(A_1 \cap A_2)L}(y)), \end{aligned}$$

$$\begin{aligned}
 & \min(R_{(A_1 \cap A_2)U}(x * (y * z)), R_{(A_1 \cap A_2)U}(y)) \\
 &= rmin\{R_{A_1 \cap A_2}(x * (y * z)), R_{A_1 \cap A_2}(y)\} \\
 \text{and } Q_{A_1 \cup A_2}(x * z) &= [Q_{(A_1 \cup A_2)L}(x * z), Q_{(A_1 \cup A_2)U}(x * z)] \\
 &\leq [\max(Q_{(A_1 \cup A_2)L}(x * (y * z)), Q_{(A_1 \cup A_2)L}(y)), \\
 &\quad \max(Q_{(A_1 \cup A_2)U}(x * (y * z)), Q_{(A_1 \cup A_2)U}(y))] \\
 &= rmax\{Q_{A_1 \cup A_2}(x * (y * z)), Q_{A_1 \cup A_2}(y)\}.
 \end{aligned}$$

Hence, $A_1 \cap A_2$ is also an IVIF UP-ideal of UP-algebra X. $\square$

Corollary 4.9. *Intersection of any family of IVIF UP-ideals of X is again an IVIF UP-ideal of X.*

Corollary 4.10. *If A is an IVIF UP-ideal of X then $\bar{A}$ is also an IVIF UP-ideal of X.*

Theorem 4.11. *If $A = (R_A, Q_A)$ is an IVIF UP-ideal of a UP-algebra X, then*

(i) $\bigoplus A$, and

(ii) $\bigotimes A$, both are IVIF UP-ideals of UP-algebra X.

Proof. For (i), it is sufficient to show that $\bar{R}_A$ satisfies the second part of the conditions (UP3) and (UP5). We have $\bar{R}_A(0) = 1 - R_A(0) \leq 1 - R_A(x) \leq \bar{R}_A(x)$. Let $x, y \in X$. Then $\bar{R}_A(x * z) = 1 - R_A(x * z) \leq 1 - rmin\{R_A(x * (y * z)), R_A(y)\} = rmax\{1 - R_A(x * (y * z)), 1 - R_A(y)\} = rmax\{\bar{R}_A(x * (y * z)), \bar{R}_A(y)\}$. Hence, $\bigoplus A$ is an IVIF UP-ideal of UP-subalgebra X.

For (ii), it is sufficient to show that $\bar{Q}_A$ satisfies the first part of the conditions (UP3) and (UP4). We have $\bar{Q}_A(0) = 1 - Q_A(0) \geq 1 - Q_A(x) \geq \bar{Q}_A(x)$. Let $x, y \in X$. Then $\bar{Q}_A(x * z) = 1 - Q_A(x * z) \geq 1 - rmax\{Q_A(x * (y * z)), Q_A(y)\} = rmin\{1 - Q_A(x * (y * z)), 1 - Q_A(y)\} = rmin\{\bar{Q}_A(x * (y * z)), \bar{Q}_A(y)\}$. Hence, $\bigotimes A$ is an IVIF UP-ideal of UP-algebra X. $\square$

Theorem 4.12. *An IVIFS A is an IVIF UP-ideal of X if and only if the sets $U(R_A : [s_1, s_2])$ and $L(Q_A : [t_1, t_2])$ are either empty or UP-ideal of X for every $[s_1, s_2], [t_1, t_2] \in D[0, 1]$.*

Proof. Suppose that $A = (R_A, Q_A)$ is an IVIF UP-ideal of X. Let $U(R_A : [s_1, s_2])$ and $L(Q_A : [t_1, t_2])$ be non-empty subset of X. Let $[s_1, s_2] \in D[0, 1]$ and $x, y, z \in X$ be such that $x * (y * z) \in U(R_A : [s_1, s_2])$ and $y \in U(R_A : [s_1, s_2])$. Then $R_A(x * z) \geq rmin\{R_A(x * (y * z)), R_A(y)\} \geq [s_1, s_2]$. Thus $x * z \in U(R_A : [s_1, s_2])$. Hence, $U(R_A : [s_1, s_2])$ is a UP-ideal of X.

Let $[t_1, t_2] \in D[0, 1]$ and $x, y, z \in X$ be such that $x * (y * z) \in L(Q_A : [t_1, t_2])$ and $y \in L(Q_A : [t_1, t_2])$. Then $Q_A(x * z) \leq rmax\{Q_A(x * (y * z)), Q_A(y)\} \leq [t_1, t_2]$. Thus $x * z \in L(Q_A : [t_1, t_2])$. Hence, $L(Q_A : [t_1, t_2])$ is a UP-ideal of X.

Conversely, assume that each non-empty level subset $U(R_A : [s_1, s_2])$ and $L(Q_A : [t_1, t_2])$ are UP-ideals of X. If there exist $\alpha, \beta, \gamma \in X$ such that $R_A(\alpha * \beta) \geq [s_1, s_2]$ and $Q_A(\beta * \gamma) \leq [t_1, t_2]$, then $\alpha * (\beta * \gamma) \in U(R_A : [s_1, s_2])$ and $\beta * \gamma \in L(Q_A : [t_1, t_2])$. Thus $\alpha * (\beta * \gamma) \in U(R_A : [s_1, s_2])$ and $\beta * \gamma \in L(Q_A : [t_1, t_2])$. Hence, A is an IVIF UP-ideal of X.

$\gamma) < rmin\{R_A(\alpha * (\beta * \gamma)), R_A(\beta)\}$, then by taking $[s'_1, s'_2] = \frac{1}{2}[R_A(\alpha * \gamma) + rmin\{R_A(\alpha * (\beta * \gamma)), R_A(\beta)\}]$, it follows that $\alpha * (\beta * \gamma) \in U(R_A : [s'_1, s'_2])$ and $\beta \in U(R_A : [s'_1, s'_2])$, but $\alpha * \gamma \notin U(R_A : [s'_1, s'_2])$, which is a contradiction. Hence, $U(R_A : [s'_1, s'_2])$ is not UP -ideal of X .

Again if there exist $\lambda, \delta, \tau \in X$ such that $Q_A(\lambda * \tau) > rmax\{Q_A(\lambda * (\delta * \tau)), Q_A(\delta)\}$, then by taking $[t'_1, t'_2] = \frac{1}{2}[Q_A(\lambda * \tau) + rmax\{Q_A(\lambda * (\delta * \tau)), Q_A(\delta)\}]$, it follows that $\lambda * (\delta * \tau) \in U(Q_A : [t'_1, t'_2])$ and $\delta \in L(Q_A : [t'_1, t'_2])$, but $\lambda * \tau \notin L(Q_A : [t'_1, t'_2])$, which is a contradiction. Hence, $L(Q_A : [t'_1, t'_2])$ is not UP -ideal of X .

Hence, $A = (R_A, Q_A)$ is an IVIF UP -ideal of X since it satisfies (UP3) and (UP4). $\square$

5. Images and preimages of IVIF UP -subalgebras and UP -ideals

In this section we will present some results on images and preimages of IVIF UP -subalgebras and UP -ideals in UP -algebras.

Let f be a mapping from a set X into a set Y . Let $B = (R_B, Q_B)$ be an IVIFS in Y . Then the inverse image of B , is defined as $f^{-1}(B) = (f^{-1}(R_B), f^{-1}(Q_B))$ with the membership function and non-membership function respectively are given by $f^{-1}(R_B)(x) = R_B(f(x))$ and $f^{-1}(Q_B)(x) = Q_B(f(x))$. It can be shown that $f^{-1}(B)$ is an IVIFS.

Theorem 5.1. *Let $f : X \rightarrow Y$ be a homomorphism of UP -algebras. If $B = (R_B, Q_B)$ is an IVIF UP -subalgebra of Y , then the preimage $f^{-1}(B) = (f^{-1}(R_B), f^{-1}(Q_B))$ of B under f is an IVIF UP -subalgebra of X .*

Proof. Assume that B is an IVIF UP -subalgebra of Y and $x, y \in X$. Then $f^{-1}(R_B)(x*y) = R_B(f(x*y)) = R_B(f(x)*f(y)) \geq rmin\{R_B(f(x)), R_B(f(y))\} = rmin\{f^{-1}(R_B)(x), f^{-1}(R_B)(y)\}$ and $f^{-1}(Q_B)(x*y) = Q_B(f(x*y)) = Q_B(f(x)*f(y)) \leq rmax\{Q_B(f(x)), Q_B(f(y))\} = rmax\{f^{-1}(Q_B)(x), f^{-1}(Q_B)(y)\}$. Therefore, $f^{-1}(B)$ is an IVIF UP -subalgebra of X . $\square$

Definition 5.2. *An IVIFS A in the UP -algebra X is said to have the $rsup$ -property and $rinf$ -property if for any subset T of X there exist $t_0 \in T$ such that $R_A(t_0) = rsup_{t_0 \in T} R_A(t)$ and $Q_A(t_0) = rinf_{t_0 \in T} Q_A(t)$ respectively.*

Definition 5.3. *Let f be a mapping from the set X to the set Y . If $A = (R_A, Q_A)$ is an IVIFS in X , then the image of A under f , denoted by $f(A)$, and is defined as*

$$f(A) = \{\langle x, f_{rsup}(R_A), f_{rinf}(Q_A) \rangle : x \in Y\},$$

where

$$f_{rsup}(R_A)(y) = \begin{cases} rsup_{x \in f^{-1}(y)} R_A(x), & \text{if } f^{-1}(y) \neq \phi \\ [0, 0], & \text{otherwise} \end{cases}$$

and

$$f_{\text{rinf}}(Q_A)(y) = \begin{cases} \text{rinf}_{x \in f^{-1}(y)} Q_A(x), & \text{if } f^{-1}(y) \neq \emptyset \\ [1, 1], & \text{otherwise.} \end{cases}$$

Theorem 5.4. Let $f : X \rightarrow Y$ be a homomorphism from a UP-algebra X onto a UP-algebra Y . If $A = (R_A, Q_A)$ is an IVIF UP-subalgebra of X , then the image $f(A) = \{\langle x, f_{\text{rsup}}(R_A), f_{\text{rinf}}(Q_A) \rangle : x \in Y\}$ of A under f is an IVIF UP-subalgebra of Y .

Proof. Let $A = (R_A, Q_A)$ be an IVIF UP-subalgebra of X and let $y_1, y_2 \in Y$. We know that, $\{x_1 * x_2 : x_1 \in f^{-1}(y_1) \text{ and } x_2 \in f^{-1}(y_2)\} \subseteq \{x \in X : x \in f^{-1}(y_1 * y_2)\}$. Now,

$$\begin{aligned} f_{\text{rsup}}(R_A)(y_1 * y_2) &= \text{rsup}\{R_A(x) : x \in f^{-1}(y_1 * y_2)\} \\ &\geq \text{rsup}\{R_A(x_1 * x_2) : x_1 \in f^{-1}(y_1) \text{ and } x_2 \in f^{-1}(y_2)\} \\ &\geq \text{rsup}\{\text{rmin}\{R_A(x_1), R_A(x_2)\} : x_1 \in f^{-1}(y_1) \text{ and } x_2 \in f^{-1}(y_2)\} \\ &= \text{rmin}\{\text{rsup}\{R_A(x_1) : x_1 \in f^{-1}(y_1)\}, \text{rsup}\{R_A(x_2) : x_2 \in f^{-1}(y_2)\}\} \\ &= \text{rmin}\{f_{\text{rsup}}(R_A)(y_1), f_{\text{rsup}}(R_A)(y_2)\} \end{aligned}$$

and

$$\begin{aligned} f_{\text{rinf}}(Q_A)(y_1 * y_2) &= \text{rinf}\{Q_A(x) : x \in f^{-1}(y_1 * y_2)\} \\ &\leq \text{rinf}\{Q_A(x_1 * x_2) : x_1 \in f^{-1}(y_1) \text{ and } x_2 \in f^{-1}(y_2)\} \\ &\leq \text{rinf}\{\text{rmax}\{Q_A(x_1), Q_A(x_2)\} : x_1 \in f^{-1}(y_1) \text{ and } x_2 \in f^{-1}(y_2)\} \\ &= \text{rmax}\{\text{rinf}\{Q_A(x_1) : x_1 \in f^{-1}(y_1)\}, \text{rinf}\{Q_A(x_2) : x_2 \in f^{-1}(y_2)\}\} \\ &= \text{rmax}\{f_{\text{rinf}}(Q_A)(y_1), f_{\text{rinf}}(Q_A)(y_2)\}. \end{aligned}$$

Hence, $f(A) = \{\langle x, f_{\text{rsup}}(R_A), f_{\text{rinf}}(Q_A) \rangle : x \in Y\}$ is an IVIF UP-subalgebra of Y . $\square$

Theorem 5.5. Let $f : X \rightarrow Y$ be a homomorphism of UP-algebras. If $B = (R_B, Q_B)$ is an IVIF UP-ideal of Y , then the pre-image $f^{-1}(B) = (f^{-1}(R_B), f^{-1}(Q_B))$ of B under f in X is an IVIF UP-ideal of X .

Proof. For all $x \in X$ $f^{-1}(R_B)(x) = R_B(f(x)) \leq R_B(0) = R_B(f(0)) = f^{-1}(R_B)(0)$ and $f^{-1}(Q_B)(x) = Q_B(f(x)) \geq Q_B(0) = Q_B(f(0)) = f^{-1}(Q_B)(0)$. Let $x, y \in X$. Then $f^{-1}(R_B)(x * z) = R_B(f(x * z)) = R_B(f(x) * f(z)) \geq \text{rmin}\{R_B(f(x) * (f(y) * f(z))), R_B(f(y))\} = \text{rmin}\{R_B(f(x * (y * z))), R_B(f(y))\} = \text{rmin}\{f^{-1}(R_B)(x * (y * z)), f^{-1}(R_B)(y)\}$ and $f^{-1}(Q_B)(x * z) = Q_B(f(x * z)) = Q_B(f(x) * f(z)) \leq \text{rmax}\{Q_B(f(x) * (f(y) * f(z))), Q_B(f(y))\} = \text{rmax}\{Q_B(f(x * (y * z))), Q_B(f(y))\} = \text{rmax}\{f^{-1}(Q_B)(x * (y * z)), f^{-1}(Q_B)(y)\}$. Hence, $f^{-1}(B)$ is an IVIF UP-ideal of X . $\square$

Theorem 5.6. Let $f : X \rightarrow Y$ be an epimorphism of UP-algebras. Then B is an IVIF UP-ideal of Y , if $f^{-1}(B) = (f^{-1}(R_B), f^{-1}(Q_B))$ of B under f in X is an IVIF UP-ideal of X .

Proof. For any $x \in Y$, $\exists a \in X$ such that $f(a) = x$. Then $R_B(x) = R_B(f(a)) = f^{-1}(R_B)(a) \leq f^{-1}(R_B)(0) = R_B(f(0)) = R_B(0)$ and $Q_B(x) = Q_B(f(a)) = f^{-1}(Q_B)(a) \geq f^{-1}(Q_B)(0) = Q_B(f(0)) = Q_B(0)$. Let $x, y, z \in Y$. Then $f(a) = x$, $f(b) = y$ and $f(c) = z$ for some $a, b, c \in X$. Thus $R_B(x * z) = R_B(f(a) * f(c)) = M_B(f(a * c)) = f^{-1}(R_B)(a * c) \geq \text{rmin}\{f^{-1}(R_B)(a * (b * c)), f^{-1}(R_B)(b)\} = \text{rmin}\{R_B(f(a * (b * c))), R_B(f(b))\} = \text{rmin}\{R_B(f(a) * (f(b) * f(c))), R_B(f(b))\} = \text{rmin}\{R_B(x * (y * z)), R_B(y)\}$ and $Q_B(x * z) = Q_B(f(a) * f(c)) = N_B(f(a * c)) = f^{-1}(Q_B)(a * c) \leq \text{rmax}\{f^{-1}(Q_B)(a * (b * c)), f^{-1}(Q_B)(b)\} = \text{rmax}\{Q_B(f(a * (b * c))), Q_B(f(b))\} = \text{rmax}\{Q_B(f(a) * (f(b) * f(c))), Q_B(f(b))\} = \text{rmax}\{Q_B(x * (y * z)), Q_B(y)\}$. Then B is an IVIF UP -ideal of Y . $\square$

6. Equivalence relations on IVIF UP -ideals

Let $\text{IVIFI}(X)$ denote the family of all interval-valued intuitionistic fuzzy ideals of X and let $\rho = [\rho_1, \rho_2] \in D[0, 1]$. Define binary relations U^ρ and L^ρ on $\text{IVIFI}(X)$ as follows:

$$\begin{aligned} (A, B) \in U^\rho &\Leftrightarrow U(R_A : \rho) = U(R_B : \rho) \text{ and} \\ (A, B) \in L^\rho &\Leftrightarrow L(Q_A : \rho) = L(Q_B : \rho) \end{aligned}$$

respectively, for $A = (R_A, Q_A)$ and $B = (R_B, Q_B)$ in $\text{IVIFI}(X)$. Then clearly U^ρ and L^ρ are equivalence relations on $\text{IVIFI}(X)$. For any $A = (R_A, Q_A) \in \text{IVIFI}(X)$, let $[A]_{U^\rho}$ (respectively, $[A]_{L^\rho}$) denote the equivalence class of A modulo U^ρ (respectively, L^ρ), and denote by $\text{IVIFI}(X)/U^\rho$ (respectively, $\text{IVIFI}(X)/L^\rho$) the collection of all equivalence classes modulo U^ρ (respectively, L^ρ), i.e.,

$$\text{IVIFI}(X)/U^\rho := \{[A]_{U^\rho} | A = (R_A, Q_A) \in \text{IVIFI}(X)\},$$

respectively,

$$\text{IVIFI}(X)/L^\rho := \{[A]_{L^\rho} | A = (R_A, Q_A) \in \text{IVIFI}(X)\}.$$

These two sets are also called the quotient sets.

Now let $T(X)$ denote the family of all ideals of X and let $\rho = [\rho_1, \rho_2] \in D[0, 1]$. Define mappings f_ρ and g_ρ from $\text{IVIFI}(X)$ to $T(X) \cup \{\phi\}$ by $f_\rho(A) = U(R_A : \rho)$ and $g_\rho(A) = L(Q_A : \rho)$, respectively, for all $A = (R_A, Q_A) \in \text{IVIFI}(X)$. Then f_ρ and g_ρ are clearly well-defined.

Theorem 6.1. *For any $\rho = [\rho_1, \rho_2] \in D[0, 1]$, the maps f_ρ and g_ρ are surjective from $\text{IVIFI}(X)$ to $T(X) \cup \{\phi\}$.*

Proof. Let $\rho = [\rho_1, \rho_2] \in D[0, 1]$. Note that $\mathbf{0}_\sim = (\mathbf{0}, \mathbf{1})$ is in $\text{IVIFI}(X)$, where $\mathbf{0}$ and $\mathbf{1}$ are interval-valued fuzzy sets in X defined by $\mathbf{0}(x) = [0, 0]$ and $\mathbf{1}(x) = [1, 1]$ for all $x \in X$. Obviously $f_\rho(\mathbf{0}_\sim) = U(\mathbf{0} : \rho) = U([0, 0] : [\rho_1, \rho_2]) = \phi = L([1, 1] : [\rho_1, \rho_2]) = L(\mathbf{1} : \rho) = g_\rho(\mathbf{0}_\sim)$. Let $P (\neq \phi) \in \text{IVIFI}(X)$. For $P_\sim = (\chi_P, \bar{\chi}_P) \in \text{IVIFI}(X)$, we have $f_\rho(P_\sim) = U(\chi_P : \rho) = P$ and $g_\rho(P_\sim) = L(\bar{\chi}_P : \rho) = P$. Hence f_ρ and g_ρ are surjective. $\square$

Theorem 6.2. *The quotient sets $IVIFI(X)/U^\rho$ and $IVIFI(X)/L^\rho$ are equipotent to $T(X) \cup \{\phi\}$ for every $\rho \in D[0, 1]$.*

Proof. For $\rho \in D[0, 1]$ let f_ρ^* (respectively, g_ρ^*) be a map from $IVIFI(X)/U^\rho$ (respectively, $IVIFI(X)/L^\rho$) to $T(X) \cup \{\phi\}$ defined by $f_\rho^*([A]_{U^\rho}) = f_\rho(A)$ (respectively, $g_\rho^*([A]_{L^\rho}) = g_\rho(A)$) for all $A = (R_A, Q_A) \in IVIFI(X)$. If $U(R_A : \rho) = U(R_B : \rho)$ and $L(Q_A : \rho) = L(Q_B : \rho)$ for $A = (R_A, Q_A)$ and $B = (R_B, Q_B)$ in $IVIFI(X)$, then $(A, B) \in U^\rho$ and $(A, B) \in L^\rho$; hence $[A]_{U^\rho} = [B]_{U^\rho}$ and $[A]_{L^\rho} = [B]_{L^\rho}$. Therefore the maps f_ρ^* and g_ρ^* are injective. Now let $P(\neq \phi) \in IVIFI(X)$. For $P_\sim = (\chi_P, \bar{\chi}_P) \in IVIFI(X)$, we have

$$f_\rho^*([P_\sim]_{U^\rho}) = f_\rho(P_\sim) = U(\chi_P : \rho) = P,$$

and

$$g_\rho^*([P_\sim]_{L^\rho}) = g_\rho(P_\sim) = L(\bar{\chi}_P : \rho) = P.$$

Finally, for $\mathbf{0}_\sim = (\mathbf{0}, \mathbf{1}) \in IVIFI(X)$ we get

$$f_\rho^*([\mathbf{0}_\sim]_{U^\rho}) = f_\rho(\mathbf{0}_\sim) = U(\mathbf{0} : \rho) = \phi$$

and

$$g_\rho^*([\mathbf{0}_\sim]_{L^\rho}) = g_\rho(\mathbf{0}_\sim) = L(\mathbf{1} : \rho) = \phi.$$

This shows that f_ρ^* and g_ρ^* are surjective. This completes the proof. $\square$

For any $\rho \in D[0, 1]$, we define another relation R^ρ on $IVIFI(X)$ as follows:

$$(A, B) \in R^\rho \Leftrightarrow U(R_A : \rho) \cap L(Q_A : \rho) = U(R_B : \rho) \cap L(Q_B : \rho),$$

for any $A = (R_A, Q_A)$ and $B = (R_B, Q_B) \in IVIFI(X)$. Then the relation R^ρ is an equivalence relation on $IVIFI(X)$.

Theorem 6.3. *For any $\rho \in D[0, 1]$, the maps $\psi_\rho : IVIFI(X) \rightarrow T(X) \cup \{\phi\}$ defined by $\psi_\rho(A) = f_\rho(A) \cap g_\rho(A)$ for each $A = (R_A, Q_A) \in X$ is surjective.*

Proof. Let $\rho \in D[0, 1]$. For $\mathbf{0}_\sim = (\mathbf{0}, \mathbf{1}) \in IVIFI(X)$,

$$\psi_\rho(\mathbf{0}_\sim) = f_\rho(\mathbf{0}_\sim) \cap g_\rho(\mathbf{0}_\sim) = U(\mathbf{0} : \rho) \cap L(\mathbf{1} : \rho) = \phi.$$

For any $H \in IVIFI(X)$, there exists $H_\sim = (\chi_H, \bar{\chi}_H) \in IVIFI(X)$ such that

$$\psi_\rho(H_\sim) = f_\rho(H_\sim) \cap g_\rho(H_\sim) = U(\chi_H : \rho) \cap L(\bar{\chi}_H : \rho) = H.$$

This completes the proof. $\square$

Theorem 6.4. *The quotient sets $IVIFI(X)/R^\rho$ are equipotent to $T(X) \cup \{\phi\}$ for every $\rho \in D[0, 1]$.*

Proof. For $\rho \in D[0, 1]$, define a map $\psi_\rho^* : IVIFI(X)/R^\rho \rightarrow T(X) \cup \{\phi\}$ by $\psi_\rho^*([A]_{R^\rho}) = \psi_\rho(A)$ for all $[A]_{R^\rho} \in IVIFI(X)/R^\rho$. Assume that $\psi_\rho^*([A]_{R^\rho}) = \psi_\rho^*([B]_{R^\rho})$ for any $[A]_{R^\rho}$ and $[B]_{R^\rho} \in IVIFI(X)/R^\rho$. Then $f_\rho(A) \cap g_\rho(A) = f_\rho(B) \cap g_\rho(B)$, i.e., $U(R_A : \rho) \cap L(Q_A : \rho) = U(R_B : \rho) \cap L(Q_B : \rho)$. Hence

$(A, B) \in R^\rho$, and so $[A]_{R^\rho} = [B]_{R^\rho}$. Therefore the maps ψ_ρ^* are injective. Now for $\mathbf{0}_\sim = (\mathbf{0}, \mathbf{1}) \in IVIFI(X)$ we have

$$\psi_\rho^*([\mathbf{0}_\sim]_{R^\rho}) = \psi_\rho(\mathbf{0}_\sim) = f_\rho(\mathbf{0}_\sim) \cap g_\rho(\mathbf{0}_\sim) = U(\mathbf{0} : \rho) \cap L(\mathbf{1} : \rho) = \phi.$$

If $H \in IVIFI(X)$, then for $H_\sim = (\chi_H, \bar{\chi}_H) \in IVIFI(X)$, we obtain

$$\psi_\rho^*([H_\sim]_{R^\rho}) = \psi_\rho(H_\sim) = f_\rho(H_\sim) \cap g_\rho(H_\sim) = U(\chi_H : \rho) \cap L(\bar{\chi}_H : \rho) = H.$$

Thus ψ_ρ^* is surjective. This completes the proof. $\square$

7. Product of IVIF UP-subalgebras and UP-ideals

In this section we will provide some new definitions on cartesian product of IVIF UP-subalgebras and UP-ideals in UP-algebras.

Definition 7.1. Let $A = (R_A, Q_A)$ and $B = (R_B, Q_B)$ be two IVIFSs of X and Y respectively. The cartesian product $A \times B = (R_A \times R_B, Q_A \times Q_B)$ of $X \times Y$ is defined by $(R_A \times R_B)(x, y) = \text{rmin}\{R_A(x), R_B(y)\}$ and $(Q_A \times Q_B)(x, y) = \text{rmax}\{Q_A(x), Q_B(y)\}$, where $R_A \times R_B : X \times Y \rightarrow D[0, 1]$ and $Q_A \times Q_B : X \times Y \rightarrow D[0, 1]$ for all $(x, y) \in X \times Y$.

Remark 7.2. Let X and Y be UP-algebras. We define $*$ on $X \times Y$ by $(x, y) * (u, v) = (x * u, y * v)$ for every $(x, y), (u, v)$ belong to $X \times Y$, then clearly $(X \times Y, *, (0, 0))$ is a UP-algebra.

Definition 7.3. An IVIFS $A \times B = (R_A \times R_B, Q_A \times Q_B)$ of $X \times Y$ is called an IVIF UP-subalgebra if it satisfies for all (x_1, y_1) and $(x_2, y_2) \in X \times Y$

- (i) $(R_A \times R_B)((x_1, y_1) * (x_2, y_2)) \geq \text{rmin}\{(R_A \times R_B)(x_1, y_1), (R_A \times R_B)(x_2, y_2)\},$
- (ii) $(Q_A \times Q_B)((x_1, y_1) * (x_2, y_2)) \leq \text{rmax}\{(Q_A \times Q_B)(x_1, y_1), (Q_A \times Q_B)(x_2, y_2)\}.$

Definition 7.4. An IVIFS $A \times B = (R_A \times R_B, Q_A \times Q_B)$ of $X \times Y$ is called an IVIF UP-ideal if it satisfies for all $(x_1, y_1), (x_2, y_2)$ and $(x_3, y_3) \in X \times Y$

- (i) $(R_A \times R_B)(0, 0) \geq (R_A \times R_B)(x, y)$ and $(Q_A \times Q_B)(0, 0) \leq (Q_A \times Q_B)(x, y),$
- (ii) $(R_A \times R_B)((x_1, y_1) * (x_3, y_3)) \geq \text{rmin}\{(R_A \times R_B)((x_1, y_1) * ((x_2, y_2) * (x_3, y_3))), (R_A \times R_B)(x_2, y_2)\}$ and
- (iii) $(Q_A \times Q_B)((x_1, y_1) * (x_3, y_3)) \leq \text{rmax}\{(Q_A \times Q_B)((x_1, y_1) * ((x_2, y_2) * (x_3, y_3))), (Q_A \times Q_B)(x_2, y_2)\}.$

Theorem 7.5. Let $A = (R_A, Q_A)$ and $B = (R_B, Q_B)$ be IVIF UP-subalgebras of X and Y respectively, then $A \times B$ is an IVIF UP-subalgebra of $X \times Y$.

Proof. For any (x_1, y_1) and $(x_2, y_2) \in X \times Y$, we have

$$\begin{aligned} (R_A \times R_B)((x_1, y_1) * (x_2, y_2)) &= (R_A \times R_B)(x_1 * x_2, y_1 * y_2) \\ &= rmin\{R_A(x_1 * x_2), R_B(y_1 * y_2)\} \\ &\geq rmin\{rmin\{R_A(x_1), R_A(x_2)\}, rmin\{R_B(y_1), R_B(y_2)\}\} \\ &= rmin\{rmin\{R_A(x_1), R_B(y_1)\}, rmin\{R_A(x_2), R_B(y_2)\}\} \\ &= rmin\{(R_A \times R_B)(x_1, y_1), (R_A \times R_B)(x_2, y_2)\} \end{aligned}$$

and

$$\begin{aligned} (Q_A \times Q_B)((x_1, y_1) * (x_2, y_2)) &= (Q_A \times Q_B)(x_1 * x_2, y_1 * y_2) \\ &= rmax\{Q_A(x_1 * x_2), Q_B(y_1 * y_2)\} \\ &\leq rmax\{rmax\{Q_A(x_1), Q_A(x_2)\}, rmax\{Q_B(y_1), Q_B(y_2)\}\} \\ &= rmax\{rmax\{Q_A(x_1), Q_B(y_1)\}, rmax\{Q_A(x_2), Q_B(y_2)\}\} \\ &= rmax\{(Q_A \times Q_B)(x_1, y_1), (Q_A \times Q_B)(x_2, y_2)\}. \end{aligned}$$

Hence, $A \times B$ is an IVIF UP -subalgebra of $X \times Y$. $\square$

Definition 7.6. Let $A = (R_A, Q_A)$ and $B = (R_B, Q_B)$ be IVIF UP -subalgebras of X and Y respectively. For $[s_1, s_2], [t_1, t_2] \in D[0, 1]$, the set $U(R_A \times R_B : [s_1, s_2]) = \{(x, y) \in X \times Y \mid (R_A \times R_B)(x, y) \geq [s_1, s_2]\}$ is called upper $[s_1, s_2]$ -level of $A \times B$ and $L(Q_A \times Q_B : [t_1, t_2]) = \{(x, y) \in X \times Y \mid (Q_A \times Q_B)(x, y) \leq [t_1, t_2]\}$ is called lower $[t_1, t_2]$ -level of $A \times B$.

Theorem 7.7. For any IVIFS A and B , if $A \times B$ is an IVIF UP -subalgebra of $X \times Y$ then non-empty upper $[s_1, s_2]$ -level cut $U(R_A \times R_B : [s_1, s_2])$ and non-empty lower $[t_1, t_2]$ -level cut $L(Q_A \times Q_B : [t_1, t_2])$ are UP -subalgebras of $X \times Y$, for all $[s_1, s_2]$ and $[t_1, t_2] \in D[0, 1]$.

Proof. Let A and B be such that $A \times B$ is an IVIF UP -subalgebra of $X \times Y$, therefore, $(R_A \times R_B)((x_1, y_1) * (x_2, y_2)) \geq rmin\{(R_A \times R_B)(x_1, y_1), (R_A \times R_B)(x_2, y_2)\}$ and $(Q_A \times Q_B)((x_1, y_1) * (x_2, y_2)) \leq rmax\{(Q_A \times Q_B)(x_1, y_1), (Q_A \times Q_B)(x_2, y_2)\}$, for all (x_1, y_1) and $(x_2, y_2) \in X \times Y$.

Again, let $(x_1, y_1), (x_2, y_2) \in X \times Y$ be such that (x_1, y_1) and $(x_2, y_2) \in U(R_A \times R_B : [s_1, s_2])$. Then, $(R_A \times R_B)((x_1, y_1) * (x_2, y_2)) \geq rmin\{(R_A \times R_B)(x_1, y_1), (R_A \times R_B)(x_2, y_2)\} \geq rmin([s_1, s_2], [s_1, s_2]) = [s_1, s_2]$. This implies, $((x_1, y_1) * (x_2, y_2)) \in U(R_A \times R_B : [s_1, s_2])$. Thus $U(R_A \times R_B : [s_1, s_2])$ is a UP -subalgebra of $X \times Y$. Similarly, $L(Q_A \times Q_B : [t_1, t_2])$ is a UP -subalgebra of $X \times Y$. $\square$

Proposition 7.8. Let A and B be IVIF UP -ideals of X , then $A \times B$ is an IVIF UP -ideal of $X \times X$.

Proof. For any $(x, y) \in X \times X$, we have $(R_A \times R_B)(0, 0) = rmin\{R_A(0), R_B(0)\} \geq rmin\{R_A(x), R_B(y)\} = (R_A \times R_B)(x, y)$ and $(Q_A \times Q_B)(0, 0) = rmax\{Q_A(0), Q_B(0)\} \leq rmin\{Q_A(x), Q_B(y)\} = (Q_A \times Q_B)(x, y)$.

Let $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in X \times X$. Then,

$$\begin{aligned}
 & (R_A \times R_B)((x_1, y_1) * (x_3, y_3)) \\
 &= (R_A \times R_B)(x_1 * x_3, y_1 * y_3) = rmin\{R_A(x_1 * x_3), R_B(y_1 * y_3)\} \\
 &\geq rmin\{rmin\{R_A(x_1 * (x_2 * x_3)), R_A(x_2)\}, rmin\{R_B(y_1 * (y_2 * y_3)), R_B(y_2)\}\} \\
 &= rmin\{rmin\{R_A(x_1 * (x_2 * x_3)), R_B(y_1 * (y_2 * y_3))\}, rmin\{R_A(x_2), R_B(y_2)\}\} \\
 &= rmin\{(R_A \times R_B)(x_1 * (x_2 * x_3), y_1 * (y_2 * y_3)), (R_A \times R_B)(x_2, y_2)\} \\
 &= rmin\{(R_A \times R_B)((x_1, y_1) * ((x_2, y_2) * (x_3, y_3))), (R_A \times R_B)(x_2, y_2)\}
 \end{aligned}$$

and

$$\begin{aligned}
 & (Q_A \times Q_B)((x_1, y_1) * (x_3, y_3)) \\
 &= (Q_A \times Q_B)(x_1 * x_3, y_1 * y_3) = rmax\{Q_A(x_1 * x_3), Q_B(y_1 * y_3)\} \\
 &\leq rmax\{rmax\{Q_A(x_1 * (x_2 * x_3)), Q_A(x_2)\}, rmax\{Q_B(y_1 * (y_2 * y_3)), Q_B(y_2)\}\} \\
 &= rmax\{rmax\{Q_A(x_1 * (x_2 * x_3)), Q_B(y_1 * (y_2 * y_3))\}, rmax\{Q_A(x_2), Q_B(y_2)\}\} \\
 &= rmax\{(Q_A \times Q_B)(x_1 * (x_2 * x_3), y_1 * (y_2 * y_3)), (Q_A \times Q_B)(x_2, y_2)\} \\
 &= rmax\{(Q_A \times Q_B)((x_1, y_1) * ((x_2, y_2) * (x_3, y_3))), (Q_A \times Q_B)(x_2, y_2)\}.
 \end{aligned}$$

Hence, $A \times B$ is an IVIF UP-ideal of $X \times X$. $\square$

Lemma 7.9. If $A = (R_A, Q_A)$ and $B = (R_B, Q_B)$ are IVIF UP-ideals of X , then $\oplus(A \times B) = (R_A \times R_B, \bar{R}_A \times \bar{R}_B)$ is an IVIF UP-ideals of $X \times X$.

Proof. Let $(R_A \times R_B)(x, y) = rmin\{R_A(x), R_B(y)\}$. Then $1 - (\bar{R}_A \times \bar{R}_B)(x, y) = rmin\{1 - \bar{R}_A(x), 1 - \bar{R}_B(y)\}$. This implies, $1 - rmin\{1 - \bar{R}_A(x), 1 - \bar{R}_B(y)\} = (\bar{R}_A \times \bar{R}_B)(x, y)$. Therefore, $(\bar{R}_A \times \bar{R}_B)(x, y) = rmax\{\bar{R}_A(x), \bar{R}_B(y)\}$. Hence, $\oplus(A \times B) = (R_A \times R_B, \bar{R}_A \times \bar{R}_B)$ is an IVIF UP-ideal of $X \times X$. $\square$

Lemma 7.10. If $A = (R_A, Q_A)$ and $B = (R_B, Q_B)$ are IVIF UP-ideals of X , then $\otimes(A \times B) = (\bar{Q}_A \times \bar{Q}_B, Q_A \times Q_B)$ is an IVIF UP-ideal of $X \times X$.

Proof. Let $(Q_A \times Q_B)(x, y) = rmax\{Q_A(x), Q_B(y)\}$. This implies, $1 - (\bar{Q}_A \times \bar{Q}_B)(x, y) = rmax\{1 - \bar{Q}_A(x), 1 - \bar{Q}_B(y)\}$. This is, $1 - rmax\{1 - \bar{Q}_A(x), 1 - \bar{Q}_B(y)\} = (\bar{Q}_A \times \bar{Q}_B)(x, y)$. Therefore, $(\bar{Q}_A \times \bar{Q}_B)(x, y) = rmin\{\bar{Q}_A(x), \bar{Q}_B(y)\}$. Hence, $\otimes(A \times B) = (\bar{Q}_A \times \bar{Q}_B, Q_A \times Q_B)$ is an IVIF UP-ideal of $X \times X$. $\square$

By the above two lemmas, it is not difficult to verify that the following theorem is valid.

Theorem 7.11. The IVIFSs $A = (R_A, Q_A)$ and $B = (R_B, Q_B)$ are IVIF UP-ideals of X if and only if $\oplus(A \times B) = (R_A \times R_B, \bar{R}_A \times \bar{R}_B)$ and $\otimes(A \times B) = (\bar{Q}_A \times \bar{Q}_B, Q_A \times Q_B)$ are IVIF UP-ideal of $X \times X$.

Theorem 7.12. For any IVIFS A and B , if $A \times B$ is an IVIF UP-ideals of $X \times X$ then the non-empty upper $[s_1, s_2]$ -level cut $U(R_A \times R_B : [s_1, s_2])$ and the non-empty lower $[t_1, t_2]$ -level cut $L(Q_A \times Q_B : [t_1, t_2])$ are UP-ideals of $X \times X$ for any $[s_1, s_2]$ and $[t_1, t_2] \in D[0, 1]$.

Proof. Assume that A and B are IVIF UP -ideals of X . Let $(a, b), (c, d), (e, f) \in X \times X$ be such that $(a, b) * ((e, f) * (c, d)), (e, f) \in U(R_A \times R_B : [s_1, s_2])$. Then $(R_A \times R_B)((a, b) * (c, d)) \geq rmin\{(R_A \times R_B)(a, b) * ((e, f) * (c, d)), (R_A \times R_B)(e, f)\} \geq rmin([s_1, s_2], [s_1, s_2]) = [s_1, s_2]$. This implies, $(a, b) * (c, d) \in U(R_A \times R_B : [s_1, s_2])$. Thus $U(R_A \times R_B : [s_1, s_2])$ is a UP -ideal of $X \times X$. Similarly, $L(Q_A \times Q_B : [t_1, t_2])$ is a UP -ideal of $X \times X$. $\square$

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TRIPLE POSITIVE SOLUTIONS FOR A THIRD-ORDER THREE-POINT BOUNDARY VALUE PROBLEM WITH SIGN-CHANGING GREEN'S FUNCTION

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Abstract. In this paper, we discuss the existence of triple positive solutions for a third-order three-point boundary value problem

$$\begin{cases} u'''(t) = f(t, u(t)), & t \in (0, 1), \\ u'(0) = 0, \quad u(1) = \alpha u(\eta), \quad u''(\eta) = 0, \end{cases}$$

where $0 < \alpha < 1$, $\max\{\frac{1+2\alpha}{1+4\alpha}, \frac{1}{2-\alpha}\} < \eta < 1$. We first study the associated Green's function and obtain some useful properties. Our main tool is the fixed point theorem due to Avery and Peterson. It is to be observed that although the associated Green's function is sign-changing, the solution obtained is still positive. The results of this paper are new. An example demonstrates the main results.

Keywords: Third-order boundary value problem, triple positive solutions, sign-changing Green's function, cone.

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1. Introduction

In recent years, the existence and multiplicity of positive solutions of the boundary value problems with sign-changing Green's function have received much attention from many authors (see [8-13, 15-19]). Specially, in [15, 17], Sun and Zhao considered a class of BVP with an indefinitely signed Green's function of the form

$$\begin{cases} u'''(t) = f(t, u(t)), & t \in [0, 1], \\ u'(0) = u(1) = u''(\eta) = 0. \end{cases}$$

where $\eta \in (\frac{1}{2}, 1)$. Under the premise of $u'(1) \leq 0$, $t \in [0, 1]$ is established, by using the Guo-Krasnoselskii and Leggett-Williams fixed point theorems, the authors established the existence of single or multiple results of positive solutions.

In [11], Li, Sun and Kong considered the following BVP with an indefinitely signed Green's function

$$\begin{cases} u'''(t) = a(t)f(t, u(t)), & t \in (0, 1), \\ u'(0) = u(1) = 0, \quad u''(\eta) + \alpha u(0) = 0. \end{cases}$$

where

$$\alpha \in [0, 2), \eta \in [\frac{\sqrt{121 + 24\alpha}}{3(4 + \alpha)}, 1).$$

By means of the Guo-Krasnosel'skii fixed point theorem, existence results of positive solutions was obtained.

It is to be observed that there are other types of works on sign-changing Green's functions which prove the existence of sign-changing solutions, positive in some cases; see Infante and Webb's papers [5-7], Liu's papers [1, 2], our papers [3, 4].

Inspired by those papers, here we study the problem

$$\begin{cases} u'''(t) = f(t, u(t)), & t \in (0, 1), \\ u'(0) = 0, \quad u(1) = \alpha u(\eta), \quad u''(\eta) = 0, \end{cases}$$

Where $0 < \alpha < 1$,

$$\max \left\{ \frac{1 + 2\alpha}{1 + 4\alpha}, \frac{1}{2 - \alpha} \right\} < \eta < 1, f : [0, 1] \times [0, +\infty) \rightarrow (0, +\infty)$$

is continuous, we give the following assumptions:

(H₀) For $u \in [0, +\infty)$, the mapping $t \mapsto f(t, u)$ is nonincreasing;

(H₁) For $t \in [0, 1]$, the mapping $u \mapsto f(t, u)$ is nondecreasing.

As is known to all, in general, in order to obtain positive solution of the boundary value problems, the Green's functions are positive. It is worth to notice that the Green's function is sign-changing in this paper. Inspired by the work of our

papers [3,4], we try to establish some criteria for the existence of triple positive solutions to the problem (1.1). It is also noted that our method here is different from that of the references [8-13,15-17,19]. This will takes lots of difficulties for us to obtain the existence of positive solutions. To overcome it, we first study the associated Green's function and obtain some useful properties. Meanwhile, by applying the fixed point theorem due to Avery and Peterson, we obtain the existence of triple positive solutions to the problem (1.1) by making full use of the boundary conditions and the concavity and convexity of $u(t)$ skillfully. To the best of our knowledge, no paper has appeared in the literature which discusses the problem. The main purpose of this paper is to fill this gap. An example demonstrates the main results.

2. Preliminaries and lemmas

Let ϑ and θ be nonnegative continuous convex functionals on K , κ be a nonnegative continuous concave functional on K , and ψ be a nonnegative continuous functional on K . Then for positive real numbers c , d , e and q , we define the following convex sets:

$$\begin{aligned} K(\vartheta, q) &= \{u \in K : \vartheta(u) < q\}; \\ K(\vartheta, \kappa, d, q) &= \{u \in K : d \leq \kappa(u), \vartheta(u) \leq q\}; \\ K(\vartheta, \theta, \kappa, d, e, q) &= \{u \in K : d \leq \kappa(u), \theta(u) \leq e, \vartheta(u) \leq q\}; \\ R(\vartheta, \psi, c, q) &= \{u \in K : c \leq \psi(u), \vartheta(u) \leq q\}. \end{aligned}$$

Lemma 2.1 ([14]). *Let K be a cone in a Banach space E . Let ϑ and θ be nonnegative continuous convex functionals on K , κ be a nonnegative continuous concave functional on K , and ψ be a nonnegative continuous functional on K satisfying $\psi(\lambda u) \leq \lambda \psi(u)$ for $0 \leq \lambda \leq 1$, such that for some positive numbers M and q ,*

$$(2.1) \quad \alpha(u) \leq \psi(u) \text{ and } \|u\| \leq M\vartheta(u),$$

for all $u \in \overline{K(\vartheta, q)}$. Suppose $T : \overline{K(\vartheta, q)} \rightarrow \overline{K(\vartheta, q)}$ is completely continuous and there exist positive numbers c , d and e with $c < d$ such that

$$(C_1) \quad \{u \in K(\vartheta, \theta, \kappa, d, e, q) : \kappa(u) > q\} \neq \emptyset \text{ and } \kappa(Tu) > d \text{ for } u \in K(\vartheta, \theta, \kappa, d, e, q);$$

$$(C_2) \quad \kappa(Tu) > d, \text{ for } u \in K(\vartheta, \kappa, d, q) \text{ with } \theta(Tu) > e;$$

$$(C_3) \quad 0 \notin R(\vartheta, \psi, c, q) \text{ and } \psi(Tu) < c \text{ for } u \in R(\vartheta, \psi, c, q), \text{ with } \psi(u) = c.$$

Then T has at least three fixed points u_1 , u_2 and $u_3 \in \overline{K(\vartheta, q)}$ such that

$$\vartheta(u_i) \leq q \text{ for } i = 1, 2, 3, d < \kappa(u_1), c < \psi(u_2) \text{ with } \kappa(u_2) < d \text{ and } \psi(u_3) < c.$$

Lemma 2.2. *We assume that (H_0) holds, then for $y \in C[0, 1]$, the BVP*

$$\begin{aligned} u'''(t) &= y(t), t \in (0, 1), \\ u'(0) &= 0, \quad u(1) = \alpha u(\eta), \quad u''(\eta) = 0, \end{aligned}$$

has the following expression of Green's function:

$$\begin{cases} G_1(t, s), & s \in [0, \eta], \\ G_2(t, s), & s \in [\eta, 1], \end{cases}$$

where

$$\begin{aligned} G_1(t, s) &= \begin{cases} \frac{2s(1-\alpha\eta) - 2ts(1-\alpha)}{2(1-\alpha)}, & s \leq t, \\ \frac{-(1-\alpha)t^2 - (1-\alpha)s^2 + 2(1-\alpha\eta)s}{2(1-\alpha)}, & t \leq s, \end{cases} \\ G_2(t, s) &= \begin{cases} \frac{(1-\alpha)(t-s)^2 - (1-s)^2}{2(1-\alpha)}, & s \leq t, \\ \frac{-(1-s)^2}{2(1-\alpha)}, & t \leq s, \end{cases} \end{aligned}$$

Proof. The proof follows by direct calculations, we omitted it here.

Lemma 2.3. *Suppose (H_0) holds. Then $G(t, s)$ has the following properties:*

- (i) $G_1(t, s) \geq 0$ and $G_2(t, s) \leq 0$, for all $t \in [0, 1]$;
- (ii) $G(t, s_1) \geq -G(t, s_2)$, $t \in [0, 1]$, $s_1 \in [\tau, \eta]$, $s_2 \in [\eta, 1]$, where $\max\{\alpha\eta, \frac{1-\eta}{2\alpha}\} \leq \tau < 2\eta - 1$.

Proof. (i) Since $\frac{\partial}{\partial t} G_1(t, s) \leq 0$, for $t \in [0, 1]$, we have

$$G_1(t, s) \geq G_1(1, s) = \frac{2\alpha(1-\eta)s}{2(1-\alpha)} \geq 0, \quad \forall t \in [0, 1]$$

By $\frac{\partial}{\partial t} G_2(t, s) \geq 0$, for $t \in [0, 1]$, we get

$$G_2(t, s) \leq G_2(1, s) = -\frac{\alpha(1-s)^2}{2(1-\alpha)} \leq 0, \quad \forall t \in [0, 1].$$

(ii) For $s_1 \in [\tau, \eta]$ and $t \in [0, 1]$,

$$G(t, s_1) = G_1(t, s_1) \geq G_1(1, s_1) \geq \frac{2\alpha\tau(1-\eta)}{2(1-\alpha)}$$

and for $s_2 \in [\eta, 1]$ and $t \in [0, 1]$,

$$G(t, s_2) = G_2(t, s_2) \geq G_2(0, s_2) = -\frac{(1-s)^2}{2(1-\alpha)} \geq -\frac{(1-\eta)^2}{2(1-\alpha)}$$

Then,

$$\frac{G(t, s_1)}{-G(t, s_2)} \geq \frac{2\alpha\tau}{1-\eta} \geq 1, \quad \text{for } t \in [0, 1], s_1 \in [\tau, \eta], s_2 \in [\eta, 1].$$

We complete the proof. □

3. The main results and proofs

Let real Banach space $C[0, 1]$ be equipped with the norm $\|u\| = \max_{0 \leq t \leq 1} |u(t)|$. Denote

$$X = \{u \in C[0, 1] : \min_{0 \leq t \leq 1} u(t) \geq 0, u'(0) = 0 \text{ and } u(1) = \alpha u(\eta), u''(\eta) = 0,$$

and define the cone $K \subset X$ and the operator $T : C[0, 1] \rightarrow C[0, 1]$ by

$$K = \{u \in X : u(t) \text{ is concave on } [0, \eta], u(t) \text{ is convex on } [\eta, 1]\},$$

and

$$(Tu)(t) = \int_0^1 G(t, s)f(s, u(s))ds.$$

Lemma 3.1. *If $u \in K$, thus*

$$u(t) \geq h(t)u(\eta), \text{ for } t \in [0, \eta] \text{ and } u(t) \leq h(t)u(\eta), \text{ for } t \in [\eta, 1],$$

where

$$\begin{cases} \frac{t}{\eta}, & t \in [0, \eta], \\ \frac{1 - \alpha\eta - (1 - \alpha)t}{1 - \eta}, & t \in [\eta, 1]. \end{cases}$$

Proof. Since $u \in K$, then $u(t)$ is concave on $[0, \eta]$, convex on $[\eta, 1]$. Thus, we have

$$u(t) \geq u(0) + \frac{u(\eta) - u(0)}{\eta}t \geq \frac{t}{\eta}u(\eta), \text{ for } t \in [0, \eta],$$

and

$$\begin{aligned} u(t) &\leq u(1) + \frac{u(1) - u(\eta)}{1 - \eta}(t - 1) = \frac{t - \eta}{1 - \eta}\alpha u(\eta) + \frac{1 - t}{1 - \eta}u(\eta) \\ &= \frac{1 - \alpha\eta - (1 - \alpha)t}{1 - \eta}u(\eta), \text{ for } t \in [\eta, 1]. \end{aligned}$$

We complete the proof. □

Lemma 3.2. *Suppose (H_0) and (H_1) hold. Thus*

$$(3.1) \quad \int_{\tau}^{\eta} G(t, s)f(s, u(s))ds \geq - \int_{\eta}^1 G(t, s)f(s, u(s))ds.$$

Proof. Since $x \in [0, 1 - \eta]$, thus $\eta - \delta x \in [\tau, \eta]$, $\eta + x \in [\eta, 1]$, where $\delta = \frac{\eta - \tau}{1 - \eta}$, τ as in Lemma 2.3. By Lemma 3.1, we have

$$h(\eta - \delta x) = 1 - \frac{\delta x}{\eta} \text{ and } h(\eta + x) = 1 - \frac{1 - \alpha}{1 - \eta}x, \text{ for } x \in [0, 1 - \eta].$$

By $\max\{\alpha\eta, \frac{1-\eta}{2\alpha}\} < \tau < 2\eta - 1$, we get

$$(3.2) \quad h(\eta - \delta x) \geq h(\eta + x), \quad \text{for all } x \in [0, 1 - \eta].$$

Let $s = \eta - \delta x$, $\forall x \in [0, 1 - \eta]$. From Lemmas 2.3 and 3.1, (3.2) and (H_1) , we have

$$\begin{aligned} \int_{\tau}^{\eta} G(t, s) f(s, u(s)) ds &= \delta \int_0^{1-\eta} G(t, \eta - \delta x) f(\eta - \delta x, u(\eta - \delta x)) dx \\ (3.3) \quad &\geq - \int_0^{1-\eta} G(t, \eta + x) f(\eta - \delta x, h(\eta - \delta x)u(\eta)) dx \\ &\geq - \int_0^{1-\eta} G(t, \eta + x) f(\eta, h(\eta + x)u(\eta)) dx. \end{aligned}$$

On the other hand, let $s = \eta + x$, $x \in [0, 1 - \eta]$, by Lemma 3.1, one has

$$\begin{aligned} - \int_{\eta}^1 G(t, s) f(s, u(s)) ds &= - \int_0^{1-\eta} G_2(t, \eta + x) f(\eta + x, u(\eta + x)) dx \\ (3.4) \quad &\leq - \int_0^{1-\eta} G(t, \eta + x) f(\eta, h(\eta + x)u(\eta)) dx. \end{aligned}$$

Thus, in view of (3.3) and (3.4), we know that (3.1) holds.

Lemma 3.3. *If $u \in K$, then $u(t) \geq \gamma \|u\|$, for $t \in [\alpha\eta, \tau]$, where $\gamma = \frac{\eta - \tau}{\eta}$.*

Proof. Since $u \in K$, thus $u''(t) \leq 0$, for $t \in [0, \eta]$ and $u''(t) \geq 0$, for $t \in [\eta, 1]$, in view of $u'(0) = 0$, $u(1) = \alpha u(\eta) < u(\eta)$, we have

$$(3.5) \quad u'(t) \leq u'(0) = 0, \quad \text{for } t \in [0, \eta]$$

and

$$(3.6) \quad u(t) \leq \max\{u(\eta), u(1)\} = u(\eta), \quad \text{for } t \in [\eta, 1].$$

In view of (3.5), we have

$$(3.7) \quad u(t) \leq u(0), \quad \text{for } t \in [0, \eta].$$

By (3.6) and (3.7), we get

$$\|u\| = \max_{0 \leq t \leq 1} |u(t)| = u(0).$$

If $t \in [\alpha\eta, \tau]$, then by the concavity of $u(t)$, we have

$$u(t) \geq u(0) + \frac{u(\eta) - u(0)}{\eta} t \geq \frac{\eta - \tau}{\eta} \|u\|.$$

Lemma 3.4. *Assumes (H_0) and (H_1) hold, then $T : K \rightarrow K$ is completely continuous.*

Proof. If $u \in K$, we have by Lemma 3.2 that

$$\begin{aligned} (Tu)(t) &= \int_0^1 G(t, s)f(s, u(s))ds \\ &= \left(\int_0^\tau G(t, s)f(s, u(s))ds + \int_\tau^\eta G(t, s)f(s, u(s))ds \right. \\ &\quad \left. + \int_\eta^1 G(t, s)f(s, u(s))ds \right) \geq 0. \end{aligned}$$

Moreover, $(Tu)'(0) = 0$, $(Tu)(1) = \alpha(Tu)(\eta)$, $(Tu)''(\eta) = 0$. Thus, $T : K \rightarrow K$.

On the other hand, by $(Tu)'''(t) \geq 0$, for $t \in [0, 1]$, which together with $(Tu)''(\eta) = 0$ implies that

$$(Tu)''(t) \leq 0, \text{ for } t \in [0, \eta] \text{ and } (Tu)''(t) \geq 0, \text{ for } t \in [\eta, 1],$$

This show that $T : K \rightarrow K$. It can be shown that $T : K \rightarrow K$ is completely continuous by Arzela-Ascoli theorem.

Let the nonnegative continuous concave functional κ on K , the nonnegative continuous convex functionals θ , ϑ and the nonnegative continuous functional ψ be defined on the cone K by

$$\kappa(u) = \min_{\alpha\eta \leq t \leq \tau} |u(t)|, \text{ for } u \in K,$$

and

$$\vartheta(u) = \theta(u) = \psi(u) = \max_{0 \leq t \leq 1} |u(t)|, \text{ for } u \in K.$$

For convenience, we denote

$$N = \int_0^\eta G_1(0, s)ds, \quad M = \int_{\alpha\eta}^\tau G_1(\tau, s)ds.$$

Theorem 3.1. *Suppose that (H_0) and (H_1) hold. If $0 < c < d < \gamma q$ and the following conditions hold:*

$$(H_2) \quad f(t, u) < \frac{q}{N}, \text{ for all } t \in [0, \eta], u \in [0, q],$$

$$(H_3) \quad f(t, u) > \frac{d}{M}, \text{ for all } t \in [\alpha\eta, \tau], u \in [d, \frac{d}{\gamma}],$$

$$(H_4) \quad f(t, u) < \frac{c}{N}, \text{ for all } t \in [0, \eta], u \in [0, c].$$

Thus the BVP (1.1) has at least three nonnegative solutions (positive on $(0, 1)$) u_1 , u_2 and u_3 such that

$$\begin{aligned} \max_{0 \leq t \leq 1} |u_i(t)| &< q \text{ for } i = 1, 2, 3, d < \min_{\alpha\eta \leq t \leq \tau} |u_1(t)|, \\ c &< \max_{0 \leq t \leq 1} |u_2(t)| \text{ with } \min_{\alpha\eta \leq t \leq \tau} |u_2(t)| < d, \text{ and } \max_{0 \leq t \leq 1} |u_3(t)| < c. \end{aligned}$$

Proof. Firstly, if $u \in \overline{K(\vartheta, q)}$, then, we may assert that $T : \overline{K(\vartheta, q)} \rightarrow \overline{K(\vartheta, q)}$ is completely continuous operator.

To see this, suppose $u \in \overline{K(\vartheta, q)}$, then $\vartheta(u) = \max_{0 \leq t \leq 1} |u(t)| \leq q$. It follows from (H_2) that

$$\begin{aligned} \vartheta(Tu) &= \max_{0 \leq t \leq 1} |Tu(t)| = \max_{0 \leq t \leq 1} \int_0^1 G(t, s) f(s, u(s)) ds \\ &= \max_{0 \leq t \leq 1} \left(\int_0^\eta G(t, s) f(s, u(s)) ds + \int_\eta^1 G(t, s) f(s, u(s)) ds \right) \\ &\leq \int_0^\eta \max_{0 \leq t \leq 1} G(t, s) f(s, u(s)) ds = \int_0^\eta G_1(0, s) f(s, u(s)) ds \\ &< \frac{q}{N} \int_0^\eta G_1(0, s) ds = q. \end{aligned}$$

Therefore, $T : \overline{K(\vartheta, q)} \rightarrow \overline{K(\vartheta, q)}$. This together with Lemma 3.4 implies that $T : \overline{K(\vartheta, q)} \rightarrow \overline{K(\vartheta, q)}$ is completely continuous operator.

To check condition (C_1) of Lemma 2.1, we let $u(t) = \frac{d}{\gamma}$, for $t \in [0, 1]$. It is easy to verify that $u(t) = \frac{d}{\gamma} \in K(\vartheta, \theta, \kappa, d, \frac{d}{\gamma}, q)$ and $\kappa(u) = \kappa(\frac{d}{\gamma}) > d$, and so $\{u \in K(\vartheta, \theta, \kappa, d, \frac{d}{\gamma}, q) : \kappa(u) > d\} \neq \emptyset$. Thus, for all $u \in K(\vartheta, \theta, \kappa, d, \frac{d}{\gamma}, q)$, we have that $d \leq u(t) \leq \frac{d}{\gamma}$, for $t \in [\tau_1, \tau]$ and $T(u) \in K$, from Lemma 3.3, (H_3) , one has

$$\begin{aligned} \min_{\alpha\eta \leq t \leq \tau} (Tu)(t) &= (Tu)(\tau) = \int_0^1 G(\tau, s) f(s, u(s)) ds \\ &= \int_0^\tau G(\tau, s) f(s, u(s)) ds + \int_\tau^\eta G(\tau, s) f(s, u(s)) ds \\ &\quad + \int_\eta^1 G(\tau, s) f(s, u(s)) ds \\ &> \frac{d}{M} \int_{\alpha\eta}^\tau G(\tau, s) ds = d. \end{aligned}$$

Consequently,

$$\min_{\alpha\eta \leq t \leq \tau} (Tu)(t) > d, \text{ for } u \in K(\vartheta, \theta, \kappa, d, \frac{d}{\gamma}, q) \text{ with } d \leq u(t) \leq \frac{d}{\gamma}, t \in [\alpha\eta, \tau].$$

This explains condition (C_1) of Lemma 2.1 holds.

Secondly, We verify that (C₂) of Lemma 2.1 is satisfied. By Lemma 3.3 and Lemma 3.4, we have

$$\min_{\alpha\eta \leq t \leq \tau} (Tu)(t) \geq \gamma \|Tu\| = \gamma \theta(Tu) > d, \quad \text{for } u \in K(\vartheta, \kappa, d, q) \text{ with } \theta(Tu) > \frac{d}{\gamma}$$

which shows that $\kappa(Tu) > d$, $u \in K(\vartheta, \kappa, d, q)$ with $\theta(Tu) > \frac{d}{\gamma}$.

Finally, we show condition (C₃) of Lemma 2.1 is also satisfied. Obviously, as $\psi(0) = 0 < c$, then $0 \notin R(\vartheta, \psi, c, q)$. If $u \in R(\vartheta, \psi, c, q)$ with $\psi(u) = c$. Then in view of condition (H₄), we have

$$\begin{aligned} \psi(Tu) &= \max_{0 \leq t \leq 1} |Tu(t)| \\ &= \max_{0 \leq t \leq 1} \int_0^1 G(t, s) f(s, u(s)) ds \\ &= \max_{0 \leq t \leq 1} \left(\int_0^\eta G(t, s) f(s, u(s)) ds + \int_\eta^1 G(t, s) f(u(s)) ds \right) \\ &\leq \int_0^\eta \max_{0 \leq t \leq 1} G(t, s) f(s, u(s)) ds \\ &= \int_0^\eta G_1(0, s) f(s, u(s)) ds \\ &< \frac{c}{N} \int_0^\eta G_1(0, s) ds = c. \end{aligned}$$

To sum up, all the conditions of Lemma 2.1 are satisfied, it follows from Lemma 2.1 that there exist three nonnegative solutions (positive on $(0, 1)$) u_1 , u_2 and u_3 for the BVP (1.1) such that

$$\begin{aligned} \max_{0 \leq t \leq 1} |u_i(t)| &< q \text{ for } i = 1, 2, 3, d < \min_{\alpha\eta \leq t \leq \tau} |u_1(t)|, \\ c &< \max_{0 \leq t \leq 1} |u_2(t)| \text{ with } \min_{\alpha\eta \leq t \leq \tau} |u_2(t)| < d, \text{ and } \max_{0 \leq t \leq 1} |u_3(t)| < c. \end{aligned}$$

4. Example

Consider the BVP

$$\begin{cases} u'''(t) = f(t, u(t)), & t \in (0, 1), \\ u'(0) = 0, \quad u(1) = \frac{1}{2}u(\frac{3}{4}), \quad u''(\frac{3}{4}) = 0, \end{cases}$$

where

$$f(t, u) = \begin{cases} (1-t)u^2, & (t, u) \in [0, 1] \times [0, 1], \\ (1-t)(97u - 96), & (t, u) \in [0, 1] \times [1, 2], \\ 98(1-t)(u-1)^2, & (t, u) \in [0, 1] \times [2, 6], \\ 2450(1-t), & (t, u) \in [0, 1] \times [6, +\infty). \end{cases}$$

It is easy to see that (H_0) and (H_1) are satisfied. Let $\tau = \frac{1}{2}$, an easy computation shows that:

$$\alpha\eta = \frac{3}{8}, \gamma = \frac{1}{3}, N = \frac{9}{32}, M = \frac{21}{512},$$

Thus, if we choose $c = 1$, $d = 2$, $q = 690$, then $f(t, u)$ satisfies the following conditions:

$$(H_2) \quad f(t, u) < \frac{q}{N} \approx 2453.3333, \text{ for all } t \in [0, \frac{3}{4}], u \in [0, 690],$$

$$(H_3) \quad f(t, u) > \frac{d}{M} \approx 48.7619, \text{ for all } t \in [\frac{3}{8}, \frac{1}{2}], u \in [2, 6],$$

$$(H_4) \quad f(t, u) < \frac{c}{N} \approx 3.5556, \text{ for all } t \in [0, \frac{3}{4}], u \in [0, 1].$$

Hence, all the conditions of Theorem 3.1 are satisfied. Thus, the BVP (4.1) has at least three nonnegative solutions (positive on $(0, 1)$) u_1 , u_2 and u_3 satisfying

$$\begin{aligned} \max_{0 \leq t \leq 1} |u_i(t)| &< 690, \quad \text{for } i = 1, 2, 3, 2 < \min_{\frac{3}{8} \leq t \leq \frac{1}{2}} |u_1(t)|, \\ 1 &< \max_{0 \leq t \leq 1} |u_2(t)| \text{ with } \min_{\frac{3}{8} \leq t \leq \frac{1}{2}} |u_2(t)| < 2, \quad \text{and } \max_{0 \leq t \leq 1} |u_3(t)| < 1. \end{aligned}$$

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ON THE ANNIHILATOR INTERSECTION GRAPH OF A COMMUTATIVE RING

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Abstract. Let R be a commutative ring with identity and $A(R)$ be the set of ideals with non-zero annihilator. The annihilator intersection graph of R is defined as the graph $\mathbb{A}\mathbb{I}\mathbb{G}(R)$ with the vertex set $A(R)^* = A(R) \setminus \{0\}$ and two distinct vertices I and J are adjacent if and only if $\text{Ann}(IJ) \neq \text{Ann}(I) \cap \text{Ann}(J)$. It follows that the annihilating-ideal graph $\mathbb{A}\mathbb{G}(R)$ (a well-known graph with the same vertices and two distinct vertices I, J are adjacent if and only if $IJ = 0$) is a subgraph of $\mathbb{A}\mathbb{I}\mathbb{G}(R)$. It is proved that $\mathbb{A}\mathbb{I}\mathbb{G}(R)$ is connected with diameter at most two and with girth at most four, if $\mathbb{A}\mathbb{I}\mathbb{G}(R)$ contains a cycle. Moreover, we characterize all rings whose annihilator intersection graphs are complete or star. Furthermore, we study the affinity between annihilator intersection graph and annihilating-ideal graph associated with a ring.

Keywords: Annihilator intersection graph, annihilating ideal graph, star graph, girth.

1. Introduction

Assigning a combinatorics object to an algebraic structure leads to arising interesting algebraic and combinatorics problems. Therefore, the study of graphs associated with rings has become one of the active area in this field. There are a lot of papers which apply combinatorial methods to obtain algebraic results in ring theory (see for instance [2], [5], [6] and [7]). Moreover, for the most recent study in this field see [1], [4], [10] and [11].

Throughout this paper R is a commutative ring with unity. The sets of all zero-divisors, nilpotent elements, unit elements, minimal prime ideals, maximal

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ideals of R are denoted by $Z(R)$, $\text{Nil}(R)$, $U(R)$, $\text{Min}(R)$ and $\text{Max}(R)$, respectively. For a subset A of a ring R we let $A^* = A \setminus \{0\}$. For every ideal I of R , we denote the *annihilator* of I by $\text{Ann}(I)$. The set of ideals of R with non-zero annihilator is denoted by $\mathbb{A}(R)$ and $I \in \mathbb{A}^*(R)$ is called an *annihilating ideal* of R . The ring R is said to be *reduced* if it has no non-zero nilpotent element. For any undefined notation or terminology in ring theory, we refer the reader to [3, 8].

Let $G = (V, E)$ be a graph, where $V = V(G)$ is the set of vertices and $E = E(G)$ is the set of edges. By $\overline{G}$, $\text{diam}(G)$ and $\text{gr}(G)$, we mean the complement, the diameter and the girth of G , respectively. If $u, v \in V(G)$, then $d(u, v)$ means the distance between u, v . A complete bipartite graph with part sizes m and n is denoted by $K_{m,n}$. If the size of one of the parts is 1, then the graph is said to be a *star graph*. The graph $H = (V_0, E_0)$ is a subgraph of G if $V_0 \subseteq V$ and $E_0 \subseteq E$. Moreover, H is called an *induced subgraph* by V_0 , denoted by $G[V_0]$, if $V_0 \subseteq V$ and $E_0 = \{\{u, v\} \in E \mid u, v \in V_0\}$. Also G is *empty* if it has no edges. For any undefined notation or terminology in graph theory, we refer the reader to [12].

Let R be a commutative ring with $1 \neq 0$. Authors in [6], introduced the *annihilating-ideal graph* of R , denoted by $\mathbb{AG}(R)$, as the graph with the vertex set $\mathbb{A}^*(R)$ and two distinct vertices I and J are adjacent if and only if $IJ = 0$. For a ring R , we define another kind of undirected graph with the vertex set $\mathbb{A}^*(R)$ and two distinct vertices I and J are adjacent if and only if $\text{Ann}(IJ) \neq \text{Ann}(I) \cap \text{Ann}(J)$. This graph is called *annihilator intersection graph* and denoted by $\mathbb{AIG}(R)$. It is not hard to see that $\mathbb{AG}(R)$ is a subgraph of $\mathbb{AIG}(R)$ and so it is interesting to explore the properties of $\mathbb{AIG}(R)$. In this paper, we study some connections between the graph-theoretic properties of $\mathbb{AIG}(R)$ and some algebraic properties of rings. Moreover, we investigate the affinity between annihilator intersection graph and annihilating-ideal graph associated with a ring. Especially, we focus on the conditions under which the annihilator intersection graph is identical to the annihilating-ideal graph.

2. Fundamental properties of $\mathbb{AIG}(R)$

In this section, we study fundamental properties of $\mathbb{AIG}(R)$. It is shown that $\mathbb{AIG}(R)$ is always a connected graph and $\text{diam}(\mathbb{AIG}(R)) \leq 2$. Moreover, we prove that if $\mathbb{AIG}(R)$ contains a cycle, then $\text{gr}(\mathbb{AIG}(R)) \leq 4$. We begin with a lemma containing several useful properties of $\mathbb{AIG}(R)$.

Lemma 2.1 *Let R be a ring and $I, J \in \mathbb{A}^*(R)$. Then the following statements hold:*

- (1) *If $I - J$ is not an edge of $\mathbb{AIG}(R)$, then $\text{Ann}(I) = \text{Ann}(J)$.*
- (2) *If $I - J$ is an edge of $\mathbb{AG}(R)$, then $I - J$ is an edge of $\mathbb{AIG}(R)$.*

- (3) If $I - J$ is not an edge of $\mathbb{A}\mathbb{I}\mathbb{G}(R)$, for some distinct vertices $I, J \in \mathbb{A}^*(R)$, then there exists a vertex $K \in \mathbb{A}^*(R) \setminus \{I, J\}$ such that $I - K - J$ is a path in $\mathbb{A}\mathbb{G}(R)$.

Proof. (1) Suppose that $\text{Ann}(I) \neq \text{Ann}(J)$. Without loss of generality, assume that $x \in \text{Ann}(I) \setminus \text{Ann}(J)$. Thus $x \notin \text{Ann}(I) \cap \text{Ann}(J)$ but $x \in \text{Ann}(IJ)$, a contradiction. Thus $\text{Ann}(I) = \text{Ann}(J)$.

(2) Suppose that $I - J$ is an edge of $\mathbb{A}\mathbb{G}(R)$, for some distinct vertices $I, J \in \mathbb{A}^*(R)$. Then $IJ = (0)$ and hence $\text{Ann}(IJ) = R$. Since $I \neq (0)$ and $J \neq (0)$, $\text{Ann}(I) \neq R$ and $\text{Ann}(J) \neq R$. Thus $I - J$ is an edge of $\mathbb{A}\mathbb{I}\mathbb{G}(R)$.

(3) Suppose that $I - J$ is not an edge of $\mathbb{A}\mathbb{I}\mathbb{G}(R)$, for some distinct vertices $I, J \in \mathbb{A}^*(R)$. By Part 1, $\text{Ann}(I) = \text{Ann}(J)$. Moreover, $\text{diam}(\mathbb{A}\mathbb{G}(R)) \leq 3$, by [6, Theorem 2.1]. Hence we need only to show that $d_{\mathbb{A}\mathbb{G}(R)}(I, J) \neq 3$. If $d_{\mathbb{A}\mathbb{G}(R)}(I, J) = 3$, then $\text{Ann}(I) \not\subseteq \text{Ann}(J)$ and $\text{Ann}(J) \not\subseteq \text{Ann}(I)$, a contradiction. Therefore, $d_{\mathbb{A}\mathbb{G}(R)}(I, J) = 2$. $\square$

Since $\mathbb{A}\mathbb{G}(R)$ is a subgraph of $\mathbb{A}\mathbb{I}\mathbb{G}(R)$, in view of Part (3) in the preceding lemma and [6, Theorem 2.1], we have the following result.

Theorem 2.1 *Let R be a ring. Then $\mathbb{A}\mathbb{I}\mathbb{G}(R)$ is connected and $\text{diam}(\mathbb{A}\mathbb{I}\mathbb{G}(R)) \leq 2$. Moreover, if $\mathbb{A}\mathbb{I}\mathbb{G}(R)$ contains a cycle, then $\text{gr}(\mathbb{A}\mathbb{I}\mathbb{G}(R)) \leq 4$.*

The next theorem shows that $\text{gr}(\mathbb{A}\mathbb{I}\mathbb{G}(R)) = 4$ may occur. First we need the following lemmas.

Lemma 2.2 *Let R be a non-reduced ring. Then every non-zero nilpotent ideal of R is adjacent to all other vertices in $\mathbb{A}\mathbb{I}\mathbb{G}(R)$. In particular, the induced subgraph by nilpotent ideals is a complete subgraph of $\mathbb{A}\mathbb{I}\mathbb{G}(R)$.*

Proof. Assume that I is a non-zero nilpotent ideal of R that is not adjacent to J , for some $J \in \mathbb{A}^*(R) \setminus \{I\}$. Part (1) of Lemma 2.1 implies that $\text{Ann}(I) = \text{Ann}(J)$. If $I^2 = (0)$, then $I \subseteq \text{Ann}(I) = \text{Ann}(J)$ and thus $IJ = (0)$, a contradiction. Let $n \geq 3$ be the least positive integer such that $I^n = (0)$. Since $\text{Ann}(I) = \text{Ann}(J)$, $I^{n-1}J = (0)$ and $I^{n-2}J \neq (0)$. This implies that $I^{n-2}(IJ) = (0)$ but $I^{n-2}I = I^{n-1} \neq (0)$ and $I^{n-2}J \neq (0)$. This means that I is adjacent to J , a contradiction. $\square$

A simple question is posed: If $I \in V(\mathbb{A}\mathbb{I}\mathbb{G}(R))$ is adjacent to every other vertex, then is it nilpotent? The following example shows that the answer is negative.

Example 2.1 Let $D = \mathbb{Z}_2[X, Y, Z]$, $I = (X^2, Y^2, XY, XZ, YZ)D$ be an ideal of D , and let $R = D/I$. Also, let $x = X + I$, $y = Y + I$ and $z = Z + I$ be elements of R . Then $\text{Nil}(R) = R(x, y)$ and $Z(R) = R(x, y, z)$. It is clear that the set $\{Rx, Ry, \text{Nil}(R), Z(R)\}$ is a clique and $\mathbb{A}\mathbb{I}\mathbb{G}(R) = K_4 \vee \overline{K}_\infty$.

To prove Theorem 2.2, the following lemma is needed.

Lemma 2.3 [7, Conjecture 1.11] *Let R be a reduced ring with more than two minimal prime ideals. Then $\text{gr}(\mathbb{A}\mathbb{G}(R)) = 3$.*

Proof. Since R is reduced, by [9, Corollary 2.4], $Z(R) = \cup_{\mathfrak{p} \in \text{Min}(R)} \mathfrak{p}$. Suppose that $\mathfrak{p}_1, \mathfrak{p}_2$ and $\mathfrak{p}_3$ are three distinct minimal prime ideals. If $x \in \mathfrak{p}_1 \setminus \mathfrak{p}_2 \cup \mathfrak{p}_3$, then $\text{Ann}(x) \subseteq \mathfrak{p}_2 \cap \mathfrak{p}_3$. Let $0 \neq y \in \text{Ann}(x)$. Since $Rx\mathfrak{p}_2 \neq (0)$, we may choose $a \in Rx \cap \mathfrak{p}_2$. Also, since R is reduced, $Rx \cap \text{Ann}(x) = (0)$. This implies that $a \notin \text{Ann}(x)$. Since $a, y \in \mathfrak{p}_2$, we have $a + y = z \in \mathfrak{p}_2$ and so $\text{Ann}(z) \neq (0)$. By [9, Corollary 2.2], $hz = 0$, for some $h \notin \mathfrak{p}_2$. This means that $Rh \neq Ra$ and $Rh \neq Ry$. Now, we show that $\text{Ann}(z) = \text{Ann}(y) \cap \text{Ann}(a)$. To see this, it is enough to show that $\text{Ann}(z) \subseteq \text{Ann}(y) \cap \text{Ann}(a)$. Let $fz = 0$. Then $f(a + y) = fa + fy = 0$. If $fa = 0$ (or $fy = 0$), then $fy = 0$ (or $fa = 0$) and thus $f \in \text{Ann}(y) \cap \text{Ann}(a)$. So we let $fa \neq 0$ and $fy \neq 0$. Since $fa + fy = 0$, $0 \neq fa = -fy \in Rx \cap \text{Ann}(x)$. Since R is reduced, $Rx \cap \text{Ann}(x) = (0)$, a contradiction. Hence $\text{Ann}(z) = \text{Ann}(y) \cap \text{Ann}(a)$ and thus $Ra - Ry - Rh - Ra$ is a cycle of length 3. $\square$

Theorem 2.2 *Let R be a ring. Then the following statements are equivalent:*

- (1) $\text{gr}(\mathbb{A}\mathbb{G}(R)) = 4$.
- (2) R is reduced, $|\text{Min}(R)| = 2$ and R contains no minimal ideal.
- (3) $\mathbb{A}\mathbb{G}(R) = K_{\infty, \infty}$.
- (4) $\mathbb{A}\mathbb{G}(R) = K_{\infty, \infty}$.

Proof. (1) $\Rightarrow$ (2) If R is non-reduced, then by Lemma 2.2, $\text{gr}(\mathbb{A}\mathbb{G}(R)) \in \{3, \infty\}$, a contradiction. Hence R is reduced. It follows from Lemma 2.3 that $|\text{Min}(R)| = 2$. Moreover, if R contains a minimal ideal, then $R = R_1 \times R_2$, where R_1 and R_2 are reduced rings. The equality $|\text{Min}(R)| = 2$ implies that one of the rings is a field and the other one is an integral domain. Now, we can easily check that $\text{gr}(\mathbb{A}\mathbb{G}(R)) = \infty$, a contradiction. Thus R contains no minimal ideal.

(2) $\Rightarrow$ (3) Let $\text{Min}(R) = \{\mathfrak{p}_1, \mathfrak{p}_2\}$. Since R is reduced, $Z(R) = \mathfrak{p}_1 \cup \mathfrak{p}_2$ and $\mathfrak{p}_1 \cap \mathfrak{p}_2 = (0)$, by [9, Corollary 2.4]. Let A, B be the sets of all non-zero ideals contained in $\mathfrak{p}_1, \mathfrak{p}_2$, respectively. It is not hard to see that $\mathbb{A}\mathbb{G}(R) = K_{|A|, |B|}$. As R contains no minimal ideal, $|A| = |B| = \infty$.

(3) $\Rightarrow$ (4) Suppose that $\mathbb{A}\mathbb{G}(R) = K_{\infty, \infty}$ with two parts V_1 and V_2 . Let $I, J \in V_i$, for some $1 \leq i \leq 2$. We need only to show that I is not adjacent to J in $\mathbb{A}\mathbb{G}(R)$. With no loss of generality, we can assume that $I, J \in V_1$. If $IJ \in V_1$, then $\text{Ann}(IJ) = \text{Ann}(I) \cap \text{Ann}(J)$ and thus I is not adjacent to J . If $IJ \notin V_1$, then $IJ \in V_2$ and thus $IIJ = I^2J = (0)$. By [7, Theorem 2.3], R is reduced and thus $\text{Ann}(I) = \text{Ann}(I^2)$, for every ideal I of R . This together with $I^2J = (0)$ imply that $IJ = (0)$, a contradiction.

(4) $\Rightarrow$ (1) is clear. $\square$

In Theorem 2.3, all rings R with star $\mathbb{AIG}(R)$ are characterized.

Theorem 2.3 *Let R be a ring. Then $\mathbb{AIG}(R)$ is a star graph if and only if one of the following statements holds:*

- (1) $R = F \times D$, where F is a field and D is an integral domain.
- (2) R has exactly two non-zero proper ideals.
- (3) $\text{Nil}(R)$ is a minimal and prime ideal of R .

Proof. First suppose that $\mathbb{AIG}(R)$ is a star graph. We consider the following cases.

Case 1. R is a reduced ring. Suppose that the vertex $I \in \mathbb{A}^*(R)$ is adjacent to every other vertex. If $I \neq I^2$, then $I - I^2$ is an edge of $\mathbb{AIG}(R)$. Since R is a reduced ring, $\text{Ann}(I) = \text{Ann}(I^2) = \text{Ann}(I^3)$ and thus I is not adjacent to I^2 , a contradiction. Thus $I = I^2$. We claim that I is a minimal ideal. Let $J \subseteq I$, where J is an ideal of R . If $J \neq I$, then $J\text{Ann}(I) = (0)$ implies that J is adjacent to $\text{Ann}(I)$, a contradiction. So the claim is proved and thus $R \cong Ra \times R(1-a)$, by [13, 2.3 and 2.7]. We may assume that $R \cong R_1 \times R_2$ with $R_1 \times (0)$ is adjacent to every other vertex. If R_1 has a non-trivial ideal, say I , then $I \times (0)$ is adjacent to $(0) \times R_2$, a contradiction. Thus R_1 is a field. Similarly, if $xy = 0$, for some $x, y \in R_2$, then $R_2x - R_2y$ is an edge of $\mathbb{AIG}(R)$, a contradiction. Hence $Z(R_2) = (0)$ and so $R = F \times D$, where F is a field and D is an integral domain.

Case 2. R is a non-reduced ring. If $Z(R) = \text{Nil}(R)$, then by Lemma 2.2, $\mathbb{AIG}(R)$ is a complete graph and so it has exactly two non-zero proper ideals. Let $Z(R) \neq \text{Nil}(R)$. By Lemma 2.2, $\text{Nil}(R)$ is a minimal ideal of R . We show that $\text{Nil}(R)$ is a prime ideal of R . Let $IJ \subseteq \text{Nil}(R)$, $I \not\subseteq \text{Nil}(R)$ and $J \not\subseteq \text{Nil}(R)$. Since $\mathbb{AIG}(R)$ is a star graph, $IJ \neq (0)$ and thus $IJ = \text{Nil}(R)$. This implies that $IIJ = I^2J = (0)$ and so $I^2 = \text{Nil}(R)$, a contradiction.

The converse is clear. □

The following example provides an infinite star annihilator intersection graph.

Example 2.2 Let $R = \mathbb{Z}_2[X, Y]/(XY, X^2)$ and let $x = X + (XY, X^2)$, $y = Y + (XY, X^2)$. Then $\text{Nil}(R) = \{0, x\}$ is a minimal and prime ideal of R and $Z(R) = R(x, y)$. It is clear that $\mathbb{AIG}(R) = K_1 \vee \bar{K}_\infty$.

To prove Theorem 2.4, the following lemmas is needed.

Lemma 2.4 *Let R be a non-reduced ring. Suppose that $\mathbb{AIG}(R)$ is a star graph. Then R is indecomposable.*

Proof. Let $R = R_1 \times R_2$ and $\text{Nil}(R_1) \neq (0)$, where R_i is a ring, for $1 \leq i \leq 2$. If $a \in \text{Nil}(R_1)^*$, then $R_1a \times R_2 - R_1a \times (0) - R_1 \times (0) - R_1a \times R_2$ forms a path of length three, a contradiction. So R is indecomposable. □

Next, we give more details on non-reduced ring with star $\mathbb{A}\mathbb{I}\mathbb{G}(R)$.

Lemma 2.5 *Let R be a non-reduced ring with $|\mathbb{A}^*(R)| > 2$. If $\mathbb{A}\mathbb{G}(R)$ is a star graph, then the following statements are equivalent:*

- (1) $\mathbb{A}\mathbb{I}\mathbb{G}(R) \neq \mathbb{A}\mathbb{G}(R)$.
- (2) $\mathbb{A}\mathbb{I}\mathbb{G}(R)$ is a complete graph.
- (3) There are at least two vertices of $\mathbb{A}\mathbb{I}\mathbb{G}(R)$ which are adjacent to every other vertex.

Proof. (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1) are obvious. We have only to prove (1) $\Rightarrow$ (2). Suppose that H is adjacent to every other vertex of $\mathbb{A}\mathbb{G}(R)$. Thus $H = \text{Ann}(Z(R))$ and $\text{Ann}(J) = H$, for every $J \in \mathbb{A}^*(R) \setminus \{H\}$. Let $I, J \in \mathbb{A}^*(R) \setminus \{H\}$ and $I \neq J$. We need only show that I is adjacent to J . Since $\mathbb{A}\mathbb{I}\mathbb{G}(R) \neq \mathbb{A}\mathbb{G}(R)$, $K_1 - K_2$ is an edge of $\mathbb{A}\mathbb{I}\mathbb{G}(R)$ that is not an edge of $\mathbb{A}\mathbb{G}(R)$, for some $K_1, K_2 \in \mathbb{A}^*(R) \setminus \{H\}$. Hence $K_1 K_2 \neq (0)$ and $\text{Ann}(K_1 K_2) \neq \text{Ann}(K_1) \cap \text{Ann}(K_2)$. This means that $K_1 K_2 = H$. Since $IK_1 K_2 = IH = (0)$, $IK_1 = H$. As $JK_1 = JH = (0)$, we deduce that $IJ = H$. This implies that $\text{Ann}(IJ) = Z(R)$ and $\text{Ann}(I) = \text{Ann}(J) = H$. Therefore, I is adjacent to J . $\square$

The following example may explain Lemma 2.5 better.

Example 2.3 Let $R = \mathbb{Z}_2[X, Y]/(X^2, Y^2)$. Then $\mathbb{A}\mathbb{G}(R) = K_{1,4}$ and $\mathbb{A}\mathbb{I}\mathbb{G}(R) = K_5$. Also, $\mathbb{A}\mathbb{G}(\mathbb{Z}_{16}) = K_{1,2}$ and $\mathbb{A}\mathbb{I}\mathbb{G}(\mathbb{Z}_{16}) = K_3$.

Theorem 2.4 *Let R be a non-reduced ring that is not an integral domain. Then the following statements are equivalent:*

- (1) $\mathbb{A}\mathbb{I}\mathbb{G}(R)$ is a star graph.
- (2) $\text{gr}(\mathbb{A}\mathbb{I}\mathbb{G}(R)) = \infty$.
- (3) $\mathbb{A}\mathbb{I}\mathbb{G}(R) = \mathbb{A}\mathbb{G}(R)$ and $\text{gr}(\mathbb{A}\mathbb{G}(R)) = \infty$.
- (4) Either $Z(R) = \text{Nil}(R)$ and $|\mathbb{A}^*(R)| = 2$ or $Z(R) \neq \text{Nil}(R)$ and $\text{Nil}(R)$ is a minimal and prime ideal of R .
- (5) Either $\mathbb{A}\mathbb{I}\mathbb{G}(R) = K_{1,1}$ or $\mathbb{A}\mathbb{I}\mathbb{G}(R) = K_{1,\infty}$.
- (6) Either $\mathbb{A}\mathbb{G}(R) = K_{1,1}$ or $\mathbb{A}\mathbb{G}(R) = K_{1,\infty}$ and $\mathbb{A}\mathbb{I}\mathbb{G}(R)$ is not complete.

Proof. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (3) Since R is a non-reduced ring, by Lemma 2.2, there exists a vertex of $\mathbb{A}\mathbb{I}\mathbb{G}(R)$ which is adjacent to every other vertex. This together with $\text{gr}(\mathbb{A}\mathbb{I}\mathbb{G}(R)) = \infty$ and $\text{diam}(\mathbb{A}\mathbb{I}\mathbb{G}(R)) \leq 2$ imply that $\mathbb{A}\mathbb{I}\mathbb{G}(R)$ is a star graph. Since $\mathbb{A}\mathbb{G}(R)$ is a connected subgraph of $\mathbb{A}\mathbb{I}\mathbb{G}(R)$, $\mathbb{A}\mathbb{I}\mathbb{G}(R) = \mathbb{A}\mathbb{G}(R)$ and so $\text{gr}(\mathbb{A}\mathbb{G}(R)) = \infty$.

(3) $\Rightarrow$ (4) Since R is a non-reduced ring, by Lemma 2.2, it is easily seen that $\mathbb{A}\mathbb{I}\mathbb{G}(R)$ is a star graph and thus by Theorem 2.3, the result holds.

(4) $\Rightarrow$ (5) By Theorem 2.3, we need only to show that if $Z(R) \neq \text{Nil}(R)$, then $|V(\mathbb{A}\mathbb{I}\mathbb{G}(R))| = \infty$. If $|V(\mathbb{A}\mathbb{I}\mathbb{G}(R))| < \infty$, then R is an Artinian ring and thus by Lemma 2.4, R is a local ring. This contradicts $Z(R) \neq \text{Nil}(R)$.

(5) $\Rightarrow$ (6) is clear.

(6) $\Rightarrow$ (1) By Lemma 2.5, $\mathbb{A}\mathbb{I}\mathbb{G}(R) = \mathbb{A}\mathbb{G}(R)$ and thus $\mathbb{A}\mathbb{I}\mathbb{G}(R)$ is a star graph. $\square$

The final results of this section are devoted to study rings R with complete $\mathbb{A}\mathbb{I}\mathbb{G}(R)$.

Theorem 2.5 *Let R be a reduced ring. Then $\mathbb{A}\mathbb{I}\mathbb{G}(R)$ is complete if and only if $\text{Ann}(I) \neq \text{Ann}(J)$, for every distinct pair $I, J \in \mathbb{A}^*(R)$.*

Proof. One implication is clear, by Part (1) of Lemma 2.1. To prove the converse, suppose that $\mathbb{A}\mathbb{I}\mathbb{G}(R)$ is complete. If $\text{Ann}(I) = \text{Ann}(J)$, for some $I, J \in \mathbb{A}^*(R)$, then $\text{Ann}(IJ) \neq \text{Ann}(I) = \text{Ann}(J)$. Thus $rIJ = (0)$, $rI \neq (0)$ and $rJ \neq (0)$, for some $r \in R$. Since $rIJ = (0)$, $rI \subseteq \text{Ann}(J) = \text{Ann}(I)$ and thus $rII = rI^2 = (0)$. Since R is a reduced ring, $rI = (0)$, a contradiction. $\square$

Corollary 2.1 *Let R be a reduced ring and $\mathbb{A}\mathbb{I}\mathbb{G}(R)$ is a complete graph. Then the following statements hold:*

- (1) *For every $I \in \mathbb{A}^*(R)$ and every positive integer $n \geq 2$, $I = I^n$.*
- (2) *$R = Z(R) \cup U(R)$.*

Proof. (1) Suppose that there exists $I \in \mathbb{A}^*(R)$, that $I \neq I^n$, for some positive integer $n \geq 2$. Since $\mathbb{A}\mathbb{I}\mathbb{G}(R)$ is a complete graph, $\text{Ann}(I^{n+1}) \neq \text{Ann}(I^n) \cap \text{Ann}(I)$. As R is a reduced ring, $\text{Ann}(I^{n+1}) = \text{Ann}(I^n) = \text{Ann}(I)$, a contradiction.

(2) Suppose that $a \in Z(R)^*$. By Part (1), $Ra = Ra^2$ and thus there is an $r \in R$ such that $a = ra^2$. Let $e = ra$ and so $e^2 = r^2a^2 = ra = e$. This implies that e is a non-trivial idempotent and hence $R = R_1 \times R_2$, where R_1 and R_2 are two rings. Now, we show that $R_i = Z(R_i) \cup U(R_i)$, for $i = 1, 2$. With no loss of generality, suppose that $x \in R_1 \setminus Z(R_1) \cup U(R_1)$. Then $Rx \neq Rx^2$, a contradiction, by Part (1). $\square$

We close this section with the following example.

Example 2.4 Let $R_1 = \prod_{i \in \Lambda} \mathbb{Z}$ and $R_2 = \prod_{i \in \Lambda} \mathbb{Z}_2$, where $|\Lambda| \geq 2$. By Theorem 2.5, $\mathbb{A}\mathbb{I}\mathbb{G}(R_1)$ is not a complete graph whereas $\mathbb{A}\mathbb{I}\mathbb{G}(R_2)$ is a complete graph.

3. When $\mathbb{AIG}(R) = \mathbb{AG}(R)$? (non-reduced case)

In this section, we study non-reduced rings R whose $\mathbb{AG}(R)$ and $\mathbb{AIG}(R)$ are identical.

Theorem 3.1 *Let R be a non-reduced ring and $\mathbb{AIG}(R) = \mathbb{AG}(R)$. Then one of the following statements holds.*

- (1) $\text{Nil}(R)^2 = (0)$.
- (2) R is a local ring with exactly two non-zero proper ideals $Z(R)$ and $Z(R)^2$.

Proof. Suppose that $\text{Nil}(R)^2 \neq (0)$. By Lemma 2.2, I is adjacent to every other vertex, for some ideal $I \subseteq \text{Nil}(R)$. The equality $\mathbb{AIG}(R) = \mathbb{AG}(R)$ implies that $Z(R)$ is a vertex $\mathbb{AIG}(R)$, by [6, Theorem 2.2]. Since $\text{Nil}(R)^2 \neq (0)$, $I\text{Nil}(R) \neq (0)$, for some ideal $I \subseteq \text{Nil}(R)$ and thus $I Z(R) \neq (0)$, a contradiction unless $I = Z(R) = \text{Nil}(R)$. This means that every vertex of $\mathbb{AIG}(R)$ is a nilpotent ideal of R and thus by Lemma 2.2, $\mathbb{AIG}(R)$ is a complete graph. Since $\mathbb{AIG}(R) = \mathbb{AG}(R)$, by [6, Theorem 2.7], R is a local ring with exactly two non-zero proper ideals $Z(R)$ and $Z(R)^2$. $\square$

Theorem 3.2 *Let R be a ring. If $\mathbb{AIG}(R) = \mathbb{AG}(R)$, then $\text{Ann}(I) = \text{Ann}(I^2)$, for every $I \not\subseteq \text{Nil}(R)$.*

Proof. Assume that $\mathbb{AIG}(R) = \mathbb{AG}(R)$ and $I \not\subseteq \text{Nil}(R)$. If $I = I^2$, then there is nothing to prove. So let $I \neq I^2$. Since $I^3 \neq (0)$, by Part (1) of Lemma 2.1, $\text{Ann}(I) = \text{Ann}(I^2)$. $\square$

Let R be a non-reduced ring and $\mathbb{AIG}(R) = \mathbb{AG}(R)$. We show that $|\text{Min}(R)| = 1$. First we need a series of lemmas.

Lemma 3.1 *Let R be a ring and $I \not\subseteq \text{Nil}(R)$ be an ideal of R . Then $\text{Ann}(I) = \text{Ann}(I^2)$ if and only if $\text{Ann}(I) = \text{Ann}(I^n)$ for every integer $n \geq 2$.*

Proof. One side is clear. To prove the converse first we show that $\text{Ann}(I) = \text{Ann}(I^3)$. If $x \in \text{Ann}(I^3) \setminus \text{Ann}(I)$, then $xI^3 = (0)$ and $xI \neq (0)$. Thus $xI \subseteq \text{Ann}(I^2) = \text{Ann}(I)$ and so $xI^2 = xI = (0)$, a contradiction. Hence $\text{Ann}(I) = \text{Ann}(I^3)$. Similarly, $\text{Ann}(I) = \text{Ann}(I^4)$ and so the result holds by induction on n . $\square$

Lemma 3.2 *Let R be a non-reduced ring and $\mathbb{AIG}(R) = \mathbb{AG}(R)$. Then $|\text{Min}(R)| \leq 2$.*

Proof. Assume that $\mathbb{AIG}(R) = \mathbb{AG}(R)$. Suppose to the contrary, $\mathfrak{p}_1$, $\mathfrak{p}_2$ and $\mathfrak{p}_3$ are three distinct minimal prime ideals. Let $a \in \mathfrak{p}_1 \setminus \mathfrak{p}_2 \cup \mathfrak{p}_3$. Thus $\mathfrak{p}_2 \cup \mathfrak{p}_3 \not\subseteq \text{Ann}(a)$ (as $\text{Ann}(a) \subseteq \mathfrak{p}_2 \cap \mathfrak{p}_3$). So one may assume that $ab \neq 0$, for some $b \in \mathfrak{p}_2 \cup \mathfrak{p}_3 \setminus \mathfrak{p}_1$. With no loss of generality, assume that $b \in \mathfrak{p}_2 \setminus \mathfrak{p}_1$. Obviously,

$\text{Ann}(b) \subseteq \mathfrak{p}_1$. Also, it follows from [9, Theorem 2.1, p. 2], there exists an element $x \in \text{Ann}(a^n)$ (for some positive integer n) such that $x \notin \mathfrak{p}_1$. This, together with Lemma 3.1, imply that $x \in \text{Ann}(a) \setminus \text{Ann}(b)$. Therefore, $\text{Ann}(a) \neq \text{Ann}(b)$. So by Part (1) of Lemma 2.1, $Ra - Rb$ is an edge of $\mathbb{A}\mathbb{I}\mathbb{G}(R)$ that is not an edge of $\mathbb{A}\mathbb{G}(R)$, a contradiction. Hence $|\text{Min}(R)| \leq 2$. $\square$

Lemma 3.3 *Let R be a non-reduced ring. Then $\mathbb{A}\mathbb{I}\mathbb{G}(R) = \mathbb{A}\mathbb{G}(R)$ if and only if one of the following statements holds.*

- (1) *R is a local ring with exactly two non-zero proper ideals $Z(R)$ and $Z(R)^2$.*
- (2) *If $IJ \neq (0)$, for some $I, J \in \mathbb{A}^*(R)$, then $\text{Ann}(I) = \text{Ann}(J)$ and $\text{Ann}(I)$ is a prime ideal of R .*

Proof. Let $\mathbb{A}\mathbb{I}\mathbb{G}(R) = \mathbb{A}\mathbb{G}(R)$ and (1) does not hold, for some ring R . Let $IJ \neq (0)$, for some $I, J \in \mathbb{A}^*(R)$. Then by Part (1) of Lemma 2.1, $\text{Ann}(I) = \text{Ann}(J)$. We show that $\text{Ann}(I)$ is a prime ideal of R . Let $K_1 K_2 \subseteq \text{Ann}(I)$, $K_1 \not\subseteq \text{Ann}(I)$ and $K_2 \not\subseteq \text{Ann}(I)$. By Theorem 3.1, $\text{Nil}(R)^2 = (0)$ and thus $K_1 \neq I$ or $K_2 \neq I$. With no loss of generality, one may assume that $K_2 \neq I$. It is easily seen that $K_2 - I$ is an edge of $\mathbb{A}\mathbb{I}\mathbb{G}(R)$ that is not an edge of $\mathbb{A}\mathbb{G}(R)$, a contradiction. Hence $\text{Ann}(I)$ is a prime ideal of R .

Conversely, assume that one of the conditions is satisfied. If condition (1) holds, there is nothing to prove. Hence assume that condition (2) holds. If $IJ = (0)$, for all $I, J \in \mathbb{A}^*(R)$, then $\mathbb{A}\mathbb{G}(R)$ is complete and so $\mathbb{A}\mathbb{I}\mathbb{G}(R)$ is complete, i.e., $\mathbb{A}\mathbb{I}\mathbb{G}(R) = \mathbb{A}\mathbb{G}(R)$. To complete the proof, we show that if $IJ \neq (0)$ for some $I, J \in \mathbb{A}^*(R)$, then $\text{Ann}(IJ) = \text{Ann}(I) = \text{Ann}(J)$. By (2), $\text{Ann}(I) = \text{Ann}(J)$ and $\text{Ann}(I)$ is a prime ideal of R . Let $x \in \text{Ann}(IJ) \setminus \text{Ann}(I)$. So $RxJI = (0)$, $RxJ \neq (0)$ and $RxI \neq (0)$. Thus $RxJ \subseteq \text{Ann}(I)$, $Rx \not\subseteq \text{Ann}(I)$ and $J \not\subseteq \text{Ann}(I)$, a contradiction. This means that $\text{Ann}(IJ) = \text{Ann}(I) = \text{Ann}(J)$ and hence $\mathbb{A}\mathbb{I}\mathbb{G}(R) = \mathbb{A}\mathbb{G}(R)$. $\square$

Now, we are ready to show that $|\text{Min}(R)| = 1$, if R is a non-reduced ring with $\mathbb{A}\mathbb{I}\mathbb{G}(R) = \mathbb{A}\mathbb{G}(R)$.

Theorem 3.3 *Let R be a non-reduced ring and $\mathbb{A}\mathbb{I}\mathbb{G}(R) = \mathbb{A}\mathbb{G}(R)$. Then $|\text{Min}(R)| = 1$.*

Proof. Suppose that $|\text{Min}(R)| \neq 1$. By Lemma 3.2, $|\text{Min}(R)| = 2$. By Lemma 2.2, I is adjacent to every other vertex, for some ideal $I \subseteq \text{Nil}(R)$. Since $\mathbb{A}\mathbb{I}\mathbb{G}(R) = \mathbb{A}\mathbb{G}(R)$, by [6, Theorem 2.2], we conclude that $Z(R)$ is a vertex $\mathbb{A}\mathbb{I}\mathbb{G}(R)$. Suppose that $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are two distinct minimal prime ideals of R . We show that $Z(R) = \mathfrak{p}_1 \cup \mathfrak{p}_2$. Let $x \in Z(R) \setminus \mathfrak{p}_1 \cup \mathfrak{p}_2$. Since $RxZ(R) \neq (0)$, by Lemma 3.3, $\text{Ann}(x)$ is a prime ideal of R . Obviously, $\text{Ann}(x) \subseteq \mathfrak{p}_1 \cap \mathfrak{p}_2$, a contradiction. Hence $Z(R) = \mathfrak{p}_1 \cup \mathfrak{p}_2$ but in this case since $Z(R)$ is an ideal of R , $\mathfrak{p}_1 \subseteq \mathfrak{p}_2$ or $\mathfrak{p}_2 \subseteq \mathfrak{p}_1$ again a contradiction. Therefore, $|\text{Min}(R)| = 1$. $\square$

Theorem 3.4 *Let R be a non-reduced ring and $\mathfrak{p}$ be the only minimal prime ideal of R .*

(1) *If $\mathfrak{p} \neq Z(R)$, then the following statements are equivalent:*

- (i) $\mathbb{A}\mathbb{G}(R) = K_{|A|} \vee \overline{K}_\infty$, where A is the set of all nilpotent ideals of R .
- (ii) $\mathfrak{p}Z(R) = (0)$.
- (iii) $\mathbb{A}\mathbb{G}(R) = \mathbb{A}\mathbb{I}\mathbb{G}(R)$.

(2) *If $\mathfrak{p} = Z(R)$, then the following statements are equivalent:*

- (i) $\mathbb{A}\mathbb{G}(R)$ is a complete graph.
- (ii) $\mathbb{A}\mathbb{I}\mathbb{G}(R)$ is a complete graph.
- (iii) $\mathbb{A}\mathbb{G}(R) = \mathbb{A}\mathbb{I}\mathbb{G}(R)$.

Proof. (1) (i) $\Rightarrow$ (ii) Since $\mathbb{A}\mathbb{G}(R) = K_{|A|} \vee \overline{K}_\infty$, every vertex of $K_{|A|}$ is adjacent to all other vertices but there is no adjacency between two arbitrary vertices of $\overline{K}_\infty$. This implies that $\text{Ann}(Z(R)) = \text{Nil}(R)$ and $IJ \neq (0)$, for every $I, J \in V(\overline{K}_\infty)$. Now we show that $\text{Ann}(Z(R))$ is a prime ideal of R . For this, let $IJ \subseteq \text{Ann}(Z(R))$, $I \not\subseteq \text{Ann}(Z(R))$ and $J \not\subseteq \text{Ann}(Z(R))$. If $IJ = (0)$, then $I - J$ is an edge of $\overline{K}_\infty$, a contradiction unless $I = J$. But in this end, we have $I^2 = 0$, and so $I \subseteq \text{Ann}(Z(R))$, a contradiction. So $IJ \neq (0)$. Since $IJ \subseteq \text{Ann}(Z(R))$ and $I \not\subseteq \text{Ann}(Z(R))$, $KIJ = (0)$, $KI \neq (0)$ for some $K \in V(\mathbb{A}\mathbb{G}(R))$. This implies that $J \in V(\mathbb{A}\mathbb{G}(R))$, $IJJ = IJ^2 = (0)$, $J^2 \subseteq \text{Ann}(Z(R))$. Hence $J^2J = J^3 = (0)$, a contradiction. Thus $\text{Ann}(Z(R))$ is a prime ideal of R and thus $\mathfrak{p}Z(R) = (0)$.

(ii) $\Rightarrow$ (iii) Since $\text{Ann}(Z(R))$ is a prime ideal of R , $\text{Ann}(Z(R)) = \text{Nil}(R)$ and for every $I \in \mathbb{A}^*(R)$ that $I \not\subseteq \text{Nil}(R)$, $\text{Ann}(I) = \text{Nil}(R)$. Now, it is easy to see that $\mathbb{A}\mathbb{G}(R)[A] = \mathbb{A}\mathbb{I}\mathbb{G}(R)[A]$ and $\mathbb{A}\mathbb{G}(R)[B] = \mathbb{A}\mathbb{I}\mathbb{G}(R)[B]$ are two subgraphs such that $\mathbb{A}\mathbb{I}\mathbb{G}(R)[A]$ is complete, $\mathbb{A}\mathbb{I}\mathbb{G}(R)[B]$ is null and $\mathbb{A}\mathbb{G}(R) = \mathbb{A}\mathbb{I}\mathbb{G}(R) = \mathbb{A}\mathbb{I}\mathbb{G}(R)[A] \vee \mathbb{A}\mathbb{I}\mathbb{G}(R)[B]$, where A is the set of all nilpotent ideals of R and $B = \mathbb{A}^*(R) \setminus A$.

(iii) $\Rightarrow$ (i) Since $\mathbb{A}\mathbb{G}(R) = \mathbb{A}\mathbb{I}\mathbb{G}(R)$, by Lemma 2.2, $\text{Ann}(Z(R)) = \text{Nil}(R)$ and since $\text{Nil}(R)$ is a prime ideal of R , we can easily get $\mathbb{A}\mathbb{G}(R) = K_{|A|} \vee \overline{K}_\infty$.

(2) All cases are clear by Theorem 3.1 and Lemma 2.2. $\square$

We close this paper with the following example.

Example 3.1 Let $D = \mathbb{Z}_2[X, Y, Z]$, $I = (X^2, Y^2, XY, XZ)D$ be an ideal of D , and let $R = D/I$. Also, let $x = X + I$, $y = Y + I$ and $z = Z + I$ be elements of R . Then $\text{Nil}(R) = R(x, y)$ and $Z(R) = R(x, y, z)$. It is clear that $\text{Nil}(R)$ is the only minimal prime ideal of R and $\text{Nil}(R)Z(R) \neq (0)$. Thus by Theorem 3.4, $\mathbb{A}\mathbb{G}(R) \neq \mathbb{A}\mathbb{I}\mathbb{G}(R)$. Also, by construction, $\mathbb{A}\mathbb{I}\mathbb{G}(R) = K_{|A|} \vee \overline{K}_\infty$ but since $\text{Nil}(R)Z(R) \neq (0)$, $\mathbb{A}\mathbb{G}(R) \neq K_{|A|} \vee \overline{K}_\infty$, where A is the set of all nilpotent ideals of R .

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SYMMETRIC METRIC SPACE

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Abstract. We first introduce the concept of symmetric metric space, then we prove that for any point symmetric metric space there is one and only one fixed point, lastly, we list some properties of the finite point symmetric metric spaces.

Keywords: Symmetric metric space, isometric map, fixed point, finite metric space, point symmetric map.

1. Introduction

In mathematics, a metric space is a set where a notion of distance (called a metric) between elements of the set is defined.

The metric space which most closely corresponds to our intuitive understanding of space is the 3-dimensional Euclidean space. In fact, the notion of “metric” is a generalization of the Euclidean metric arising from the four long-known properties of the Euclidean distance. The Euclidean metric defines the distance between two points as the length of the straight line segment connecting them. Other metric spaces occur for example in elliptic geometry and hyperbolic geometry, where distance on a sphere measured by angle is a metric, and the hyperboloid model of hyperbolic geometry is used by special relativity as a metric space of velocities.

A metric space is an ordered pair (M, ρ) where M is a set and ρ is a metric on M , i.e., a function

$$\rho : M \times M \rightarrow \mathbf{R}$$

such that for any $x, y, z \in M$, the following holds:

- 1) $\rho(x, y) \geq 0$ (non-negative),
- 2) $\rho(x, y) = 0$, iff $x = y$, (identity of indiscernibles),
- 3) $\rho(x, y) = \rho(y, x)$ (symmetry)
- 4) $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$ (triangle inequality) .

We suggest the readers to read [2] to learn more about metric space.

In [1] Wang and Bai researched linear structure on translation spaces. They introduced an interesting concept which is called translation space. In this paper, we use the same method of [1] to introduce a point symmetric metric space and research it.

The paper is organized as following. We first introduce a special metric space which is called point symmetric metric space, then we prove that for any point symmetric metric space there is one and only one fixed point, lastly, we list some properties of the finite point symmetric metric spaces.

2. Point symmetric metric space

Let's first give the definition of point symmetric map and point symmetric metric space.

Definition 1. Let (X, ρ) be a metric space, $f : X \rightarrow X$ be an isometric map, i.e., f is bijective and for any $x, y \in X$, we have that $\rho(f(x), f(y)) = \rho(x, y)$. If there exists $x_0 \in X$ such that, for any $x \in X$, we have that

$$\rho(x, x_0) = \rho(f(x), x_0) = \frac{1}{2}\rho(x, f(x)),$$

then f is called a x_0 -symmetric map of X . A metric space (X, ρ) with a x_0 -symmetric map of X , is called a x_0 -symmetric metric space with map f .

Example 1. Let $X = [-1, 1]$ be a real interval, define $f : X \rightarrow X$ by $f(x) = -x$, for any $x \in X$. Then f is a 0-symmetric map of X .

In fact, for any $x \in X$, we have that

$$\rho(x, 0) = |x - 0| = |-x - 0| = \rho(f(x), 0) = \frac{1}{2}|x - (-x)| = \frac{1}{2}\rho(f(x), x).$$

Example 2. Let (X, ρ) be a Banach space over the field of real numbers, for any $a, b \in X$, define $\rho(a, b) = \|a - b\|$.

Choose any $a \in X$, define $f_a : X \rightarrow X$ as $f_a(x) = 2a - x$, $\forall x \in X$.

Then f is bijective and for any $x, y \in X$, we conclude that $\rho(f_a(x), f_a(y)) = \|f_a(x) - f_a(y)\| = \|(2a - x) - (2a - y)\| = \|x - y\| = \rho(x, y)$. and $\rho(x, a) = \|x - a\| = \|(2a - x) - a\| = \rho(f_a(x), a) = \frac{1}{2}\|x - (2a - x)\| = \frac{1}{2}\rho(x, f_a(x))$. Thus f is an a -symmetric map of X .

3. Fixed point of a point symmetric map

In this section we will research the fixed point of a point symmetric map of a metric space.

Proposition 1. Let (X, ρ) be a metric space, f be a x_0 -symmetric map of X and also be a x_1 -symmetric map of X , then $x_0 = x_1$.

Proof. For any $x \in X$, we have that $\rho(x, x_0) = \rho(f(x), x_0) = \frac{1}{2}\rho(x, f(x))$, and $\rho(x, x_1) = \rho(f(x), x_1) = \frac{1}{2}\rho(x, f(x))$. So $\rho(x_0, x_1) \leq \rho(x_0, x) + \rho(x, x_1) = 2\rho(x_0, x)$.

Thus, for any $x \in X$, we have that $\rho(x_0, x_1) \leq 2\rho(x_0, x)$. So $\rho(x_0, x_1) \leq 2\rho(x_0, x_0) = 0$ which conclude that $x_0 = x_1$. $\square$

Proposition 2. Let (X, ρ) be a metric space, f be a x_0 -symmetric map of X , then x_0 is a fixed point of f , i.e., $f(x_0) = x_0$.

Proof. For any $x \in X$, we have that $\rho(x, x_0) = \rho(f(x), x_0) = \frac{1}{2}\rho(x, f(x))$. Thus $\rho(x_0, x_0) = \rho(f(x_0), x_0) = \frac{1}{2}\rho(x_0, f(x_0))$. And so $\rho(f(x_0), x_0) = 0$ which conclude that $f(x_0) = x_0$. $\square$

Proposition 3. Let (X, ρ) be a metric space, f be a x_0 -symmetric map of X , then x_0 is the only fixed point of f .

Proof. Suppose that $x_1 \in X$ is another fixed point of f . Then $f(x_1) = x_1$.

From $\rho(x, x_0) = \rho(f(x), x_0) = \frac{1}{2}\rho(x, f(x))$, $\forall x \in X$, we know that $\rho(x_1, x_0) = \rho(f(x_1), x_0) = \frac{1}{2}\rho(x_1, f(x_1)) = 0$. And we conclude that $x_0 = x_1$. $\square$

4. Further properties of a point symmetric map

Proposition 4. Let (X, ρ) be a metric space and f be a x_0 -symmetric map of X , then for any $x_1 \in X$ such that $x_1 \neq x_0$ and for any positive natural number n , we have that $f^n(x_1) \neq x_0$.

Proof. For any $x_1 \in X$ such that $x_1 \neq x_0$, then from $\rho(x_1, x_0) = \rho(f(x_1), x_0) = \frac{1}{2}\rho(x_1, f(x_1))$, we know that $f(x_1) \neq x_0$.

Suppose that $f^n(x_1) = x_0$, then from the previous proposition, we conclude that $f^{n-1}(x_1) = x_0$, and consequently, $f^{n-2}(x_1) = x_0$, $f^{n-3}(x_1) = x_0$, ..., $f^2(x_1) = x_0$, $f(x_1) = x_0$.

This is a contradiction. $\square$

Proposition 5. Let (X, ρ) be a metric space and f be a x_0 -symmetric map of X , then for any $x_1 \in X$ such that $x_1 \neq x_0$ and for any positive natural number n , we have that $f^n(x_1) \neq f^{n-1}(x_1)$.

Proof. Let's suppose that $f^n(x_1) = f^{n-1}(x_1)$, then $f^{n-1}(x_1)$ is a fixed point of X . So $f^{n-1}(x_1) = x_0$, and this is a contradiction by the previous proposition. $\square$

We will use $|X|$ to denote cardinality of set X .

Proposition 6. Let (X, ρ) be a x_0 -symmetric metric space with map f and X is a finite set, $|X| \geq 2$, for any $x_1 \in X$ such that $x_1 \neq x_0$, then there exists two positive nature numbers m and n which are subjected to $|X| \geq m > n$ and $m - n \geq 2$ such that $f^m(x_1) = f^n(x_1)$.

Proof. Let's suppose $|X| \geq 2$. For any $x_1 \in X$ such that $x_1 \neq x_0$, then from

$$\rho(x_1, x_0) = \rho(f(x_1), x_0) = \frac{1}{2}\rho(x_1, f(x_1)),$$

we know that $f(x_1) \neq x_0$ and $f(x_1) \neq x_1$.

We construct a sequence $\{x_0\}_{i=0}^{|X|} \subseteq X$ as following:

$$x_i = f^i(x_1) (i = 1, 2, \dots, |X|).$$

Because there are only $|X|$ elements in X and $\{x_0\}_{i=0}^{|X|}$ consists of $|X| + 1$ elements, so there must be two elements of $\{x_0\}_{i=0}^{|X|}$, for example, x_m and x_n , such that $x_m = x_n$, i.e., $f^m(x_1) = f^n(x_1)$. We can suppose that $m > n$, and obviously, $|X| > m > n$. From the previous proposition, we know that $m - n \geq 2$.

Example 3. Set $X = C[0, 1]$, $x_0 = x_0(t) = 0$, $x_1 = x_1(t) = 1$, $f(x_1) = -1$, $f^2(x_1) = 2t - 1$. Then $\rho(x_1, x_0) = \rho(f(x_1), x_0) = \frac{1}{2}\rho(x_1, f(x_1)) = 1$, and $\rho(f(x_1), x_0) = \rho(f^2(x_1), x_0) = \frac{1}{2}\rho(f(x_1), f^2(x_1)) = 1$. This shows that $f^2(x_1) \neq x_1$.

From the previous example, we can declare the following remark.

Remark 1. Let (X, ρ) be a metric space, f be a x_0 -symmetric map of X and g also be a x_0 -symmetric map of X , then we can not obtain that $f = g$.

Proof. Let $X = \{x_0, x_1, x_2, x_3\} \subseteq C[0, 1]$, $x_0 = x_0(t) = 0$, $x_1 = x_1(t) = 1$, $x_2 = x_2(t) = -1$, $x_3 = 2t - 1$, define f and g as following:

$$\frac{X}{f(X)} \begin{array}{c|cccc} & x_0 & x_1 & x_2 & x_3 \\ \hline & x_0 & x_2 & x_3 & x_1 \end{array}, \quad \frac{X}{g(X)} \begin{array}{c|cccc} & x_0 & x_1 & x_2 & x_3 \\ \hline & x_0 & x_3 & x_1 & x_2 \end{array},$$

then it is easy to verify that f and g are isometric maps of X . Moreover, they are x_0 -symmetric maps of X . But $f \neq g$. $\square$

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A NOTE ON STRONGLY FULLY STABLE BANACH ALGEBRA MODULES RELATIVE TO AN IDEAL

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Abstract. Let A be a unital algebra, a Banach algebra module M is strongly fully stable Banach A -module relative to ideal K of A , if for every submodule N of M and for each multiplier $\theta : N \rightarrow M$ such that $\theta(N) \subseteq N \cap KM$. In this paper, we adopt the concept of strongly fully stable Banach Algebra modules relative to an ideal which generalizes that of fully stable Banach Algebra modules and we study the properties and characterizations of strongly fully stable Banach A -module relative to ideal K of A .

Keywords: A -module, Banach A -module, fully stable Banach A -module, strongly fully stable Banach A -module relative to ideal.

1. Introduction

In [1], a non-empty set A is an algebra if, $(A, +, \cdot)$ is a vector space over a field F , $(A, +, \circ)$ is a ring and $(\alpha a) \circ b = \alpha(a \circ b) = a \circ (\alpha b)$, for every $a, b \in A$, $\alpha \in F$. A ring R is an algebra $\langle R, +, \cdot, -, 0 \rangle$ where $+$ and $\cdot$ are two binary operations, $-$ is unary and 0 is nullary element satisfying, $\langle R, +, -, 0 \rangle$ is an abelian group, $\langle R, \cdot \rangle$ is a semigroup and $x.(y + z) = (x.y) + (x.z)$ and $(x + y).z = (x.z) + (y.z)$ (see [2]).

Following [1], let A be an algebra, recall that a Banach space E is a Banach left A -module if E is a left A -module, and $\|a.x\| \leq \|a\|\|x\|$ ($a \in A, x \in E$). A map from a left Banach A -module X into a left Banach A -module Y (A is not necessarily commutative) is said a multiplier (homomorphism) if it satisfies $T(a.x) = a.Tx$, for all $a \in A, x \in X$ (see [3]). In [4], a submodule N of an R -module M is said to be stable, if $f(N) \subseteq N$ for each R -homomorphism $f : N \rightarrow M$. M is called a fully stable module, each submodule of M is stable. Following [5], a Banach algebra module M is called fully stable Banach A -module if for every submodule N of M and for each multiplier $\theta : N \rightarrow M$ such that $\theta(N) \subseteq N$. In this paper the concept of strongly full stability relative to ideal for Banach A -modules has been introduced. A Banach algebra module M is called strongly fully stable Banach A -module relative to ideal K of A if

for every submodule N of M and for each multiplier $\theta : N \rightarrow M$ such that $\theta(N) \subseteq N \cap KM$.

Structure of fully stable Banach A -module relative to an ideal in term of their elements is considered see (2.2) Studying Baer criterion gives another characterization of fully stable Banach A -module relative to ideal K of A , corollary (2.8).

2. Main results

In [6], a left Banach A -module X is n -generated for $n \in N$ if there exists $x_1, \dots, x_n \in X$ such that each $x \in X$ can be represented as $x = \sum_{k=1}^n a_k x_k$ for some $a_1, \dots, a_n \in A$. A module which is 1-generated is called a cyclic module.

Definition 2.1. Let X be Banach A -module, X is called fully stable Banach A -module relative to ideal K of A , if for every submodule N of X and for each multiplier $\theta : N \rightarrow X$ such that $\theta(N) \subseteq N \cap KX$. It is clear that every fully stable Banach A -module is fully stable Banach A -module relative to an ideal. Moreover, every fully stable Banach A -modules is strongly fully stable Banach A -modules relative to ideal, therefore X is strongly fully stable Banach A -modules relative to ideal, if and if for every 1-generated submodule L of X and for each multiplier $\theta : L \rightarrow X$ such that $\theta(L) \subseteq L \cap KX$.

Let X be a Banach A -modules and K be a non-zero ideal of A . If M is fully stable Banach A -modules and $X = KX$ then X is strongly fully stable Banach A -modules relative to K , since for each 1-generated submodule N of X and A -homomorphism $f : N \rightarrow X$, $f(N) \subseteq N = N \cap X = N \cap KX$.

Following [7] for a nonempty subset M in a left Banach A -module θ , the annihilator ann_A of M is $\text{ann}_A(M) = \{a \in A; a.x = 0 \text{ for all } x \in M\}$. In [6], Let X be a Banach A -module, $N_x = \{N_x | n \in N, x \in X\}$ and $P_y = \{p_y | p \in P, y \in X\}$, $\text{ann}_A N_x = \{a \in A, a.n_x = 0, \forall n_x \in N_x\}$ and $\text{ann}_A P_y = \{a \in A, a.p_y = 0, \forall p_y \in P_y\}$. The following proposition gives another characterization of strongly fully stable Banach A -modules relative to an ideal.

Proposition 2.2. X is fully stable Banach A -module if and only if for each $x, y \in X$ and N_x, P_y subsets of $X, y \notin N_x \cap KX$ implies $\text{ann}_A(N_x) \subsetneq \text{ann}_A(P_y)$.

Proof. Suppose that X is fully stable Banach A -module relative to ideal K of A , there exists $x, y \in X$ such that $y \notin N_x \cap KX$ and $\text{ann}_A(N_x) \subseteq \text{ann}_A(P_y)$. Define $\theta : \langle N_x \rangle \rightarrow X$ by $\theta(a.n_x) = a.p_y$, for all $a \in A$, if $a.n_x = 0$ then $a \in \text{ann}_A(N_x) \subseteq \text{ann}_A(P_y)$. This implies that $a.p_y = 0$, hence θ is well define. It is clear θ that is a multiplier, because X is strongly fully stable relative to an ideal, there exists an element $t \in A$ such that $\theta(m_x) = tm_x$, for each $m_x \in N_x$. In particular, $p_y = \theta(n_x) = tn_x \in N_x \cap KX$. Which is a contradiction. Thus X is strongly fully stable Banach module relative to an ideal. Conversely, assume that there is a subset N_x of X and a multiplier $\theta : \langle N_x \rangle \rightarrow X$ such that $\theta(N_x) \subsetneq$

$N_x \cap KX$ then there exists an element $m_x \in N_x$ such that $\theta(m_x) \notin N_x \cap KX$. Let $s \in \text{ann}_A(N_x)$ therefore $sm_x = 0$, $s\theta(m_x) = \theta(sm_x) = \theta(0) = 0$. Hence $\text{ann}_A(N_x) \subseteq \text{ann}_A(\theta(m_x))$. Which is a contradiction. $\square$

Corollary 2.3. *Let X be a strongly fully stable Banach A -module relative to an ideal K of A . Then for each $x, y \in X$, $\text{ann}_A(P_y) = \text{ann}_A(N_x)$ implies $N_x \cap KX = P_y \cap KX$.*

Proof. Assume that there are two elements x, y in X such that

$$\text{ann}_A(N_x) = \text{ann}_A(P_y)$$

and $N_x \cap KX \neq P_y \cap KX$. Then without loss of generality there is an element z_x in N_x not in P_y . By proposition (2.2) we have $\text{ann}_A(P_y) \not\subseteq \text{ann}_A(Z_x)$ but $\text{ann}_A(N_x) \subseteq \text{ann}_A(Z_x)$, hence $\text{ann}_A(P_y) \not\subseteq \text{ann}_A(N_x)$ which is a contradiction. $\square$

Definition 2.4. A submodule N of Banach A -module is called pure submodule if $KN = N \cap KX$ for each ideal K of A .

When the submodule of strongly fully stable Banach A -module relative to ideal have been partial answer in the following proposition.

Proposition 2.5. *Let X be a strongly fully stable Banach A -module relative to a non-zero ideal K of A . Then every pure submodule is strongly fully stable Banach A -module relative to an ideal.*

Proof. Let N be pure submodule of X . For each submodule L of N and a multiplier $f : L \rightarrow N$, put $g = i \circ f : L \rightarrow X$ (where i is the inclusion mapping of N to X), then by assumption $f(L) = g(L) \subseteq KX$, and since $f(L) \subseteq N$. Hence $f(L) \subseteq L \cap KX \cap N$. Since N is pure submodule of X then $N \cap KX = KN$, for each ideal K of A , therefore $f(L) \subseteq L \cap KN$. Thus N is strongly fully stable Banach A -module relative to K . $\square$

Definition 2.6. A Banach A -module X is said to satisfy Baer criterion relative to an ideal K of A , if each submodule of X satisfies Baer criterion, that is, for every 1-generated submodule N of X and A -multiplier $\theta : N \rightarrow X$, there exists an element a in A such that $\theta(n) = an \in KX$ for all $n \in N$.

The following proposition and its corollary give another characterization of strongly fully stable Banach A -module relative to ideal.

Proposition 2.7. *Let X be a Banach A -module. Then Baer criterion holds for 1-generated submodules of X if and only if $\text{ann}_X(\text{ann}_A(N_x)) \subseteq N_x \cap KX$, for each $x \in X$.*

Proof. Assume that Baer criterion holds for 1-generated submodule of X . Let $y \in \text{ann}_X(\text{ann}_A(N_x))$ and define $\theta : \langle N_x \rangle \rightarrow X$ by $\theta(a.n_x) = a.py$, for all $a \in A$. Let $a_1.n_x = a_2.n_x$, thus $(a_1 - a_2)n_x = 0$, where $a_1 - a_2 \in \text{ann}_A(N_x)$,

so $(a_1 - a_2) \in \text{ann}_A(P_y)$. Therefore $(a_1 - a_2)p_y = 0$, then $a_1p_y = a_2p_y$, hence θ is well define. It is clear that clear θ is an A -multiplier. By the assumption, there exists an element $t \in A$ such that $\theta(m_x) = tm_x \in KX$ for each $m_x \in N_x$. This implies that, in particular, $p_y = \theta(n_x) = tn_x \in KX$, therefore $\text{ann}_X(\text{ann}_A(N_x)) \subseteq N_x \cap KX$, hence $\text{ann}_X(\text{ann}_A(N_x)) = N_x \cap KX$. Conversely, assume that $\text{ann}_X(\text{ann}_A(N_x)) = N_x \cap KX$. For each $N_x \subseteq X$. Then for each A -multiplier $\theta : N_x \rightarrow X$, and $s \in \text{ann}_A(N_x)$, we have $s\theta(n_x) = \theta(sn_x) = 0$. Thus $\theta(n_x) \in \text{ann}_X(\text{ann}_A(N_x)) = N_x \cap KX$, then $\theta(n_x) = tn_x \in KX$ for some $t \in A$, thus Baer criterion holds. $\square$

Corollary 2.8. *X is strongly fully stable Banach A -module relative to ideal K of A if and only if $\text{ann}_X(\text{ann}_A(N_x)) \subseteq N_x \cap KX$, for each $x \in X$.*

In [9], let A be a unital Banach algebra. A -module X is called quasi α -injective if, $\varphi : N \rightarrow X$ is A -module homomorphism (multiplier) such that $\|\varphi\| \leq 1$, there exists A -module homomorphism (multiplier) $\theta : X \rightarrow X$, such that $\theta \circ i = \varphi$ and $\|\theta\| \leq \alpha$, where i is an isometry (A -module isomorphism is an isometry A -multiplier) from submodule N of X . We shall say that X is quasi injective if it is quasi α -injective for some α .

The concept of strongly quasi α -injective relative to an ideal K of A has been introduce.

Definition 2.9. Let A be a unital Banach algebra. A -module X is called strongly quasi α -injective relative to an ideal K of A if, $\varphi : N \rightarrow X$ is A -module homomorphism (multiplier) such that $\|\varphi\| \leq 1$, there exists A -module homomorphism (multiplier) $\theta : X \rightarrow X$, such that $(\theta \circ i)(n) = \varphi(n) \in KX$ and $\|\theta\| \leq \alpha$ where i is an isometry from submodule N of X to X . We shall say that X is strongly quasi injective relative to ideal if it is strongly quasi α injective relative to ideal for some α .

The following proposition give the relation between strongly quasi α -injective Banach A -module relative to ideal and strongly fully stable Banach A -module relative to an ideal K of A has given

Proposition 2.11. *Let X be Banach A -module and K be a non-zero ideal of algebra A . If X is strongly fully stable Banach A -module relative to ideal then X is strongly quasi injective Banach A -module relative to ideal.*

Proof. Let N be a submodule of X and $f : N \rightarrow X$ be any A -module homomorphism. Since X is a fully stable Banach A -module relative to K , then $f(N) \subseteq N \cap KX$, thus there exist $t \in A$ such that $f(n) = tn$. Define $g : X \rightarrow X$ by $g(x) = tx$, it is clear that g is a well defined A -module homomorphism (multiplier). Now $f(x) = g(x) = tx \in KX$, and for each $y \in N$, $(f \circ i)(y) - g(y) = f(y) - g(y) \in KX$, where i is isometry, and $\|g\| \leq \alpha$ for some α therefore X is strongly quasi injective Banach A -module relative to ideal. $\square$

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FIXED POINT RESULT FOR NEW RATIONAL TYPE CONTRACTION ON CLOSED BALL FOR MULTIVALUED MAPPING

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Abstract. The purpose of this paper is to introduce the idea of new rational type contractive condition on multivalued mapping to find the fixed point results for such mapping on a closed ball in complete metric space. Example has been given to demonstrate the variety of our result. Our results combine, extend and infer several comparable results in the existing literature.

Keywords: Fixed point, complete metric space, closed ball, multivalued mapping, new rational type contractive condition.

1. Introduction and preliminaries

Fixed point theory plays an important role in functional and non linear analysis. Banach proved significant result for contraction mappings. Afterward, a large number of fixed point results have been established by various authors and they showed different generalizations of the Banach's result. In literature, there are many concerning results about the fixed point of mappings which are contractive

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over the whole space. It is very simple to show that $K : G \longrightarrow G$ is not a contraction but $K : D \longrightarrow G$ is a contraction, where D is a subset of G . It is possible to get fixed point for such mappings if they satisfy a certain condition. It has been shown by Hussain et al. [9], the presence of fixed point for such mappings that fulfill a certain condition on a closed ball. For further results on closed ball (see also [2, 3, 4, 5, 8, 12, 13, 14, 15, 16, 17, 18, 19]).

Nadler [11], initiated the study of fixed point theorems for the multivalued mappings (see also [1, 6, 7]). In this paper, the concept of new type of multivalued rational contraction has been introduced. Common fixed point results such contraction on a closed ball in complete metric space have been established. Example has been given. We give the following definitions and results which will be needed in the sequel.

Definition 1.1 ([10]). Let X be a nonempty set. A mapping $d : X \times X \rightarrow [0, \infty)$ is called a metric if the following conditions hold. for any $x, y, z \in X$:

- (i) $d(x, y) \geq 0$,
- (ii) $d(x, y) = 0$, iff $x = y$,
- (iii) $d(x, y) = d(y, x)$,
- (iv) $d(x, y) \leq d(x, z) + d(z, y)$.

Then d is called metric on X , and the pair (X, d) is called metric space.

Definition 1.2 ([10]). Let (X, d) be a metric space.

- (i) A sequence $\{x_n\}$ in (X, d) is called Cauchy sequence if given $\varepsilon > 0$, there corresponds $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$ we have $d(x_m, x_n) < \varepsilon$ or $\lim_{n, m \rightarrow \infty} d(x_n, x_m) = 0$.
- (ii) A sequence $\{x_n\}$ converges (for short d -converges) to x if $\lim_{n \rightarrow \infty} d(x_n, x) = 0$. In this case x is called a d -limit of $\{x_n\}$.
- (iii) (X, d) is called complete if every Cauchy sequence in X converges to a point $x \in X$ such that $d(x, x) = 0$.

Definition 1.3 ([14]). Let K be a nonempty subset of metric space X and let $x \in X$. An element $y_0 \in K$ is called a best approximation in K if

$$d(x, K) = d(x, y_0), \text{ where } d(x, K) = \inf_{y \in K} d(x, y).$$

If each $x \in X$ has at least one best approximation in K , then K is called a proximal set. The set of all proximal subsets of X is denoted by $P(X)$.

Example 1.4. Let $M = \{y \in C[-1, 1] : \int_0^1 y(t)dt = 0\}$. Then M is a closed bounded subset of $C[-1, 1]$ that is not proximal, where $C[-1, 1]$ is the set of all continuous and bounded functions from $[-1, 1]$ to $\mathbb{R}$ and

$$d(f, g) = \sup_{x \in [-1, 1]} |f(x) - g(x)|.$$

Definition 1.5 ([14]). The function $H_d : P(X) \times P(X) \rightarrow R^+$, defined by

$$H_d(A, B) = \max\left\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b)\right\}$$

is called Hausdorff-Pompeiu metric on $P(X)$.

Lemma 1.6 ([14]). Let (X, d) be a metric space. Let $(P(X), H_d)$ is a Hausdorff-Pompeiu metric space on $P(X)$. Then, for all $A, B \in P(X)$ and for each $a \in A$ there exists $b_a \in B$ satisfies $d(a, B) = d(a, b_a)$ then $H_d(A, B) \geq d(a, b_a)$.

2. Main result

Let (X, d) be a complete metric space, $x_0 \in X$ and $S : X \rightarrow P(X)$ be the multivalued mapping on X . Then there exists $x_1 \in Sx_0$ be an element such that $d(x_0, Sx_0) = d(x_0, x_1)$. Let $x_2 \in Sx_1$ be such that $d(x_1, Sx_1) = d(x_1, x_2)$. Let $x_3 \in Sx_2$ be such that $d(x_2, Sx_2) = d(x_2, x_3)$. Continuing this process, we construct a sequence x_n of points in X such that $x_{n+1} \in Sx_n$, $d(x_n, Sx_n) = d(x_n, x_{n+1})$. We denote this iterative sequence by $\{XS(x_n)\}$. We say that $\{XS(x_n)\}$ is a sequence in X generated by x_0 .

Theorem 2.1. Let (X, d) be a complete metric space and x_0 be any arbitrary point in X . Let the mapping $S : X \rightarrow P(X)$ satisfy:

$$(2.1) \quad \begin{aligned} H_d(Sx, Sy) \leq & a_1 d(x, y) + a_2 \frac{(a + d(x, Sx)) \cdot d(y, Sy)}{(a + d(x, y))} \\ & + a_3 [d(x, Sx) + d(y, Sy)], \end{aligned}$$

for all $x, y \in \overline{B_d(x_0, r)} \cap \{XS(x_n)\}$ and $a, r, a_1, a_2, a_3 > 0$, with $a_1 + a_2 + 2a_3 < 1$. Also

$$(2.2) \quad d(x_0, Sx_0) \leq (1 - \lambda)r,$$

where $\lambda = (\frac{a_1 + a_3}{1 - a_2 - a_3})$. Then $\{XS(x_n)\}$ is a sequence in $\overline{B_d(x_0, r)}$ for all $n \in \mathbb{N} \cup \{0\}$ and $\{XS(x_n)\} \rightarrow h \in \overline{B_d(x_0, r)}$. Also, if inequality (2.1) holds for h , then S has fixed point h in $\overline{B_d(x_0, r)}$.

Proof. Let $\{XS(x_n)\}$ is a sequence in X generated by x_0 , then, we have $x_{n+1} \in Sx_n$ where $n = 0, 1, 2, \dots$. By Lemma 1.6, we have

$$\begin{aligned} d(x_1, x_2) &= d(x_1, Sx_1) \leq H_d(Sx_0, Sx_1) \\ d(x_1, x_2) &\leq a_1 d(x_0, x_1) + a_2 \frac{(a + d(x_0, Sx_0)) \cdot d(x_1, Sx_1)}{(a + d(x_0, x_1))} \\ &\quad + a_3 [d(x_0, Sx_0) + d(x_1, Sx_1)] \end{aligned}$$

$$\begin{aligned}
&\leq a_1 d(x_0, x_1) + a_2 \frac{(a + d(x_0, x_1)) \cdot d(x_1, x_2)}{(a + d(x_0, x_1))} \\
&\quad + a_3 [d(x_0, x_1) + d(x_1, x_2)] \\
&\leq a_1 d(x_0, x_1) + a_2 d(x_1, x_2) + a_3 [d(x_0, x_1) + d(x_1, x_2)] \\
&\leq \left(\frac{a_1 + a_3}{1 - a_2 - a_3} \right) d(x_0, x_1) \\
&\leq \lambda(1 - \lambda)r.
\end{aligned}$$

Now,

$$\begin{aligned}
d(x_0, x_2) &\leq d(x_0, x_1) + d(x_1, x_2) \\
&\leq (1 - \lambda)r + \lambda(1 - \lambda)r \\
&\leq (1 - \lambda^2)r \leq r \\
d(x_0, x_2) &\leq r.
\end{aligned}$$

This implies that $x_2 \in \overline{B_d(x_0, r)}$. Similarly, by repeating the same process for

$$\begin{aligned}
d(x_2, x_3) &= d(x_2, Sx_2) \\
d(x_2, x_3) &\leq H_d(Sx_1, Sx_2) \text{ by Lemma 1.6}
\end{aligned}$$

we get

$$d(x_2, x_3) \leq \lambda^2 d(x_0, x_1).$$

Consequently, $x_3, x_4, \dots, x_j \in \overline{B_d(x_0, r)}$, for some $j \in N$. If $j = 2i + 1$, where $i = 0, 1, 2, \dots, \frac{j-1}{2}$ we get

$$(2.3) \quad d(x_{2i+1}, x_{2i+2}) \leq \lambda d(x_{2i}, x_{2i+1}).$$

Similarly, if $j = 2i + 2$, where $i = 0, 1, 2, \dots, \frac{j-2}{2}$, we have

$$(2.4) \quad d(x_{2i+2}, x_{2i+3}) \leq \lambda d(x_{2i+1}, x_{2i+2}).$$

Now, (2.3) implies that

$$(2.5) \quad d(x_{2i+1}, x_{2i+2}) \leq \lambda^{2i+1} d(x_0, x_1).$$

Also, (2.4) implies that

$$(2.6) \quad d(x_{2i+2}, x_{2i+3}) \leq \lambda^{2i+2} d(x_0, x_1).$$

Now, by combining (2.5) and (2.6), we have

$$(2.7) \quad d(x_j, x_{j+1}) \leq \lambda^j d(x_0, x_1) \text{ for all } j \in N.$$

Now,

$$\begin{aligned}
 d(x_0, x_{j+1}) &\leq d(x_0, x_1) + d(x_1, x_2) + \cdots + d(x_j, x_{j+1}) \\
 &\leq d(x_0, x_1) + \lambda d(x_0, x_1) + \cdots + \lambda^j d(x_0, x_1) \text{ by (2.7)} \\
 &\leq (1 + \lambda + \lambda^2 + \cdots + \lambda^j) d(x_0, x_1) \\
 &\leq \frac{1(1 - \lambda^j)}{1 - \lambda} (1 - \lambda) r \text{ as } j \rightarrow \infty \\
 &\leq r.
 \end{aligned}$$

Thus, $x_{j+1} \in \overline{B_d(x_0, r)}$. Hence $x_n \in \overline{B_d(x_0, r)}$ for all $n \in \mathbb{N} \cup \{0\}$, therefore $\{XS(x_n)\}$ is a sequence in $\overline{B_d(x_0, r)}$. Now, the inequality (2.7) can be written as

$$(2.8) \quad d(x_n, x_{n+1}) \leq \lambda^n d(x_0, x_1) \text{ for all } n \in \mathbb{N}.$$

Hence for any $m > n$,

$$\begin{aligned}
 d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\
 &\leq (\lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1}) d(x_0, x_1) \text{ by using (2.8)} \\
 &< \frac{\lambda^n}{1 - \lambda} d(x_0, x_1) \rightarrow 0, \text{ as } m, n \rightarrow \infty.
 \end{aligned}$$

Thus we proved that $\{XS(x_n)\}$ is a Cauchy sequence in $(\overline{B_d(x_0, r)}, d)$. As every closed ball in a complete metric space is complete, so there exists $h \in \overline{B_d(x_0, r)}$ such that $\{XS(x_n)\} \rightarrow h$, it follows that $h \in Sh$, otherwise $d(h, Sh) = z > 0$, that is

$$(2.9) \quad \lim_{n \rightarrow \infty} d(x_n, h) = 0.$$

Now,

$$\begin{aligned}
 d(h, Sh) &\leq d(h, x_{2n+2}) + d(x_{2n+2}, Sh) \\
 &\leq d(h, x_{2n+2}) + H_d(Sx_{2n+1}, Sh), \text{ by Lemma 1.6} \\
 d(h, Sh) &\leq d(h, x_{2n+2}) + H_d(Sh, Sx_{2n+1}) \\
 &\leq d(h, x_{2n+2}) + a_1 d(h, x_{2n+2}) + a_2 \frac{(a + d(h, Sh)).d(x_{2n+1}, Sx_{2n+1})}{(a + d(h, x_{2n+1}))} \\
 &\quad + a_3 [d(h, Sh) + d(x_{2n+1}, Sx_{2n+1})] \\
 &\leq d(h, x_{2n+2}) + a_1 d(h, x_{2n+2}) + a_2 \frac{(a + d(h, Sh)).d(x_{2n+1}, x_{2n+2})}{(a + d(h, x_{2n+1}))} \\
 &\quad + a_3 [d(h, Sh) + d(x_{2n+1}, x_{2n+2})].
 \end{aligned}$$

This implies that

$$\begin{aligned}
 z &\leq d(h, x_{2n+2}) + a_1 d(h, x_{2n+2}) + a_2 \frac{(a + z).d(x_{2n+1}, x_{2n+2})}{(a + d(h, x_{2n+1}))} \\
 &\quad + a_3 [z + d(x_{2n+1}, x_{2n+2})].
 \end{aligned}$$

Letting $n \rightarrow \infty$, it follows that

$$\begin{aligned}(1 - a_3)z &\leq 0 \\ (1 - a_3) &\neq 0 \\ \Rightarrow z &\leq 0.\end{aligned}$$

As $z = d(h, Sh) \leq 0$, rise a contradiction so that $h \in Sh$. $\square$

Example 2.2. Let $X = \mathbb{R}^+ \cup \{0\}$ and let $d : X \times X \rightarrow X$ be the complete metric on X defined by

$$d(x, y) = |x - y| \text{ for all } x, y \in X.$$

Define the multivalued mapping, $S : X \times X \rightarrow P(X)$ by,

$$Sx = \begin{cases} [\frac{x}{3}, \frac{2}{3}x], & \text{if } x \in [0, 9] \\ [x, x + 1], & \text{if } x \in (9, \infty) \end{cases}$$

Considering, $x_0 = 1, r = 8$, then $\overline{B_d(x_0, r)} = [0, 9]$. Now $d(x_0, Sx_0) = d(1, S1) = d(1, \frac{2}{3}) = \frac{1}{3}$. $d(\frac{2}{3}, S\frac{2}{3}) = d(\frac{2}{3}, \frac{4}{9}) = \frac{2}{9}$. So we obtain a sequence $\{XS(x_n)\} = \{1, \frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \dots\}$ in X generated by x_0 . Now, $\overline{B_d(x_0, r)} \cap \{XS(x_n)\} = \{1, \frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \dots\}$. Let $10, 11 \in X$, then $H_d(S10, S11) = 1$. Let $a = 1, x = 10, y = 11, a_1 = \frac{1}{3}, a_2 = \frac{1}{4}$, and $a_3 = \frac{1}{7}$. Then

$$\begin{aligned}H_d(S10, S11) &\geq \frac{1}{3}d(10, 11) + \frac{1}{4} \frac{(1 + d(10, 10)).d(11, 11)}{(1 + d(10, 11))} \\ &\quad + \frac{1}{7}[d(10, 10) + d(11, 11)] = \frac{1}{3}\end{aligned}$$

So, the contractive condition does not hold for whole space. Now, for $1, \frac{2}{3} \in \overline{B_d(x_0, r)} \cap \{XS(x_n)\}$, we have

$$\begin{aligned}H_d(S1, S\frac{2}{3}) &= [\max\{\sup_{a \in S1} d(a, S\frac{2}{3}), \sup_{b \in S\frac{2}{3}} d(S1, b)\}] \\ &= \max \left\{ \sup_{a \in S1} d(a, [\frac{2}{3}, \frac{2}{3}]), \sup_{b \in S\frac{2}{3}} d([\frac{1}{3}, \frac{2}{3}], b) \right\} \\ &= \max \left\{ d(\frac{2}{3}, [\frac{2}{9}, \frac{4}{9}]), d([\frac{1}{3}, \frac{2}{3}], \frac{2}{9}) \right\} \\ &= \max \left\{ d(\frac{2}{3}, \frac{4}{9}), d(\frac{1}{3}, \frac{2}{9}) \right\} \\ &= \max \left\{ \left| \frac{2}{3} - \frac{4}{9} \right|, \left| \frac{1}{3} - \frac{2}{9} \right| \right\} \\ &= \max \left\{ \frac{2}{9}, \frac{1}{9} \right\} = \frac{2}{9}\end{aligned}$$

Now,

$$\begin{aligned} & a_1 d(1, \frac{2}{3}) + a_2 \frac{(a + d(1, S1)) \cdot d(\frac{2}{3}, S\frac{2}{3})}{(a + d(1, \frac{2}{3}))} + a_3 \left[d(1, S1) + d(\frac{2}{3}, S\frac{2}{3}) \right] \\ &= \frac{1}{3} \left| 1 - \frac{2}{3} \right| + \frac{1}{4} \frac{(1 + \frac{1}{3}) \cdot \frac{2}{9}}{1 + \left| 1 - \frac{2}{3} \right|} + \frac{1}{7} \left[\frac{1}{3} + \frac{2}{9} \right] \\ &= \frac{1}{9} + \frac{2}{36} + \frac{5}{63} = \frac{62}{252}. \end{aligned}$$

As $\frac{2}{9} < \frac{62}{252}$. So, the contractive condition holds for $1, \frac{2}{3} \in \overline{B_d(x_0, r)} \cap \{XS(x_n)\}$. Similarly, the contractive condition holds for all $x, y \in \overline{B_d(x_0, r)} \cap \{XS(x_n)\}$. Also,

$$\begin{aligned} \frac{1}{3} &\leq \left(1 - \frac{40}{51}\right) \times 8 \\ d(x_0, Sx_0) &\leq (1 - \lambda)r. \end{aligned}$$

Hence, all the conditions of Theorem 2.1 are satisfied. Now, we have $\{XS(x_n)\}$ is a sequence in $\overline{B_d(x_0, r)}$, for all $n \in \mathbb{N} \cup \{0\}$. Moreover, 0 is a fixed point of S .

Corollary 2.3. Let (X, d) be a complete metric space and let the mapping $S : X \rightarrow P(X)$ satisfy:

$$d(Sx, Sy) \leq a_1 d(x, y) + a_2 \frac{(a + d(x, Sx)) \cdot d(y, Sy)}{(a + d(x, y))},$$

for all $x, y \in \overline{B_d(x_0, r)} \cap \{XS(x_n)\}$ there exist $a, r > 0$,

$$d(x_0, Sx_0) \leq (1 - \lambda)r,$$

where $\lambda = (\frac{a_1}{1-a_2})$ and a_1, a_2 are non negative reals with $a_1 + a_2 < 1$. Then $\{XS(x_n)\}$ is a sequence in $\overline{B_d(x_0, r)}$ for all $n \in \mathbb{N} \cup \{0\}$ and $\{XS(x_n)\} \rightarrow h \in \overline{B_d(x_0, r)}$. Then S has fixed point h in $\overline{B_d(x_0, r)}$.

Proof. By using $a_3 = 0$ in theorem (2.1), we get the required result. $\square$

Corollary 2.4. Let (X, d) be a complete metric space and let the mapping $S : X \rightarrow P(X)$ satisfy:

$$d(Sx, Sy) \leq a_1 d(x, y) + a_3 [d(x, Sx) + d(y, Sy)],$$

for all $x, y \in \overline{B_d(x_0, r)} \cap \{XS(x_n)\}$ there exist $r > 0$,

$$d(x_0, Sx_0) \leq (1 - \lambda)r,$$

where $\lambda = (\frac{a_1+a_3}{1-a_3})$ and a_1, a_3 are non negative reals with $a_1 + 2a_3 < 1$. Then $\{XS(x_n)\}$ is a sequence in $\overline{B_d(x_0, r)}$ for all $n \in \mathbb{N} \cup \{0\}$ and $\{XS(x_n)\} \rightarrow h \in \overline{B_d(x_0, r)}$. Then S has fixed point h in $\overline{B_d(x_0, r)}$.

Proof. By using $a_2 = 0$ in theorem (2.1), we get the result. $\square$

Corollary 2.5. Let (X, d) be a complete metric space and x_0 be any arbitrary point on X let the mapping $S : X \rightarrow P(X)$ satisfy:

$$d(Sx, Sy) \leq a_2 \frac{(a + d(x, Sx)).d(y, Sy)}{(a + d(x, y))} + a_3 [d(x, Sx) + d(y, Sy)],$$

for all $x, y \in \overline{B_d(x_0, r)} \cap \{XS(x_n)\}$ there exist $a, r > 0$,

$$d(x_0, Sx_0) \leq (1 - \lambda)r,$$

where $\lambda = (\frac{a_3}{1-a_2-a_3})$ and a_2, a_3 are non negative reals with $a_2 + 2a_3 < 1$. Then $\{XS(x_n)\}$ is a sequence in $\overline{B_d(x_0, r)}$ for all $n \in \mathbb{N} \cup \{0\}$ and $\{XS(x_n)\} \rightarrow h \in \overline{B_d(x_0, r)}$. Then S has fixed point h in $\overline{B_d(x_0, r)}$.

Proof. By using $a_1 = 0$ in theorem (2.1), we get the result. $\square$

Corollary 2.6. Let (X, d) be a complete metric space and let the mapping $S : X \rightarrow X$ satisfy:

$$d(Sx, Sy) \leq a_1 d(x, y) + a_2 \frac{(a + d(x, Sx)).d(y, Sy)}{(a + d(x, y))} + a_3 [d(x, Sx) + d(y, Sy)],$$

for all $x, y \in X$, there exist $a > 0$, where a_1, a_2, a_4 are non negative reals with $a_1 + 2a_3 < 1$. Then S has a fixed point.

Corollary 2.7. Let (X, d) be a complete metric space and let the mapping $S : X \rightarrow X$ satisfy:

$$d(Sx, Sy) \leq a_2 \frac{(a + d(x, Sx)).d(y, Sy)}{(a + d(x, y))} + a_3 [d(x, Sx) + d(y, Sy)],$$

for all $x, y \in X$, there exist $a > 0$, where a_2, a_3 are non negative reals with $a_2 + 2a_3 < 1$. Then S has a fixed point.

Corollary 2.8. Let (X, d) be a complete metric space and let the mapping $S : X \rightarrow X$ satisfy:

$$d(Sx, Sy) \leq a_1 d(x, y) + a_3 [d(x, Sx) + d(y, Sy)],$$

for all $x, y \in X$, where a_1, a_3 are non negative reals with $a_1 + 2a_3 < 1$. Then S has a fixed point.

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POINTWISE SLANT SUBMERSIONS FROM KENMOTSU MANIFOLDS INTO RIEMANNIAN MANIFOLDS

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Abstract. The purpose of this paper is to study pointwise slant submersions from Kenmotsu manifolds onto Riemannian manifolds admitting vertical and horizontal structure vector fields and find some results related to totally geodesic and harmonic properties.

Keywords: Riemannian submersion, almost contact manifold, pointwise slant submersion

1. Introduction

Immersion and submersions are special tools in Differential Geometry. Both play important role in Riemannian Geometry. O'Neill [19] and Gray [14] introduced Riemannian submersions between Riemannian manifolds. Submersions between Riemannian manifolds equipped with an additional structure of almost contact type on total space, firstly studied by Watson [23] and Chinea [11] independently. We know that Riemannian submersions are related to Mathematical Physics and have their applications in the Kaluza-Klein theory ([7], [16]) and the Yang-Mills theory [6] etc.

On the other hand, submersions have been studied by several authors. Some related research papers are: Geometry of slant submanifolds [10], Slant submersions from almost Hermitian manifolds [20], Slant submanifolds of Lorentzian Sasakian and para Sasakian manifolds [1], Riemannian submersions from almost

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contact metric manifolds [15], On quasi-slant submanifolds of an almost Hermitian manifold [12], Slant submanifolds in Sasakian manifolds [8], Pointwise slant submanifolds in almost Hermitian manifolds [9], Almost contact metric submersions [11], Riemannian Submersions and Related Topics [13], Pointwise slant submersions [17], Slant submanifolds of a Riemannian product manifold [2], Slant submanifolds in contact geometry [18], Point-wise slant submanifolds in almost contact geometry [3], Pointwise slant submersions from Cosymplectic manifolds [21] etc.

In this paper, we study pointwise slant submersions from Kenmotsu manifolds onto Riemannian manifolds. The paper is organized as follows. In section 2, we collect main notions and formulae which are needed for this paper. In section 3, we obtain some results of pointwise slant submersions from Kenmotsu manifolds onto Riemannian manifolds admitting vertical and horizontal structure vector fields.

2. Preliminaries

Let M be an almost contact metric manifold. So there exist on M , a $(1, 1)$ tensor field ϕ , a vector field ξ , a 1-form η and a Riemannian metric g such that

$$(2.1) \quad \phi^2 = -I + \eta \otimes \xi, \quad \phi \circ \xi = 0, \quad \eta \circ \phi = 0,$$

$$(2.2) \quad g(X, \xi) = \eta(X), \eta(\xi) = 1,$$

and

$$(2.3) \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(\phi X, Y) = -g(X, \phi Y),$$

for any vector fields X and Y on M and I is the identity tensor field [5]. An almost contact metric manifold M equipped with an almost contact metric structure (ϕ, ξ, η, g) is denoted by (M, ϕ, ξ, η, g) .

An almost contact metric manifold M is called a Kenmotsu manifold if

$$(2.4) \quad (\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X,$$

for any vector fields X and Y on M , where ∇ is the Riemannian connection of the Riemannian metric g . If (M, ϕ, ξ, η, g) be a Kenmotsu manifold, then the following equation holds:

$$(2.5) \quad \nabla_X \xi = X - \eta(X)\xi.$$

Let M be an m -dimensional Riemannian manifold and N be an n -dimensional Riemannian manifold ($m > n$) with Riemannian metrics g_M and g_N respectively. Let $f : (M, g_M) \rightarrow (N, g_N)$ be a C^∞ map. We denote the kernel space of f_* by $\ker f_*$ and consider the orthogonal complementary space $\mathcal{H} = (\ker f_*)^\perp$ to $\ker f_*$. Then the tangent bundle of M has the following decomposition

$$(2.6) \quad TM = (\ker f_*) \oplus (\ker f_*)^\perp.$$

We also denote the *range* of f_* by $\text{range} f_*$ and consider the orthogonal complementary space $(\text{range} f_*)^\perp$ to $\text{range} f_*$ in the tangent bundle TN of N . Thus the tangent bundle TN of N has the following decomposition

$$(2.7) \quad TN = (\text{range} f_*) \oplus (\text{range} f_*)^\perp.$$

A Riemannian submersion f is a C^∞ map from Riemannian manifold (M, g_M) onto (N, g_N) satisfying the following conditions:

- (i) f has the maximal rank,
- (ii) The differential f_* preserves the lengths of horizontal vectors.

For each $x \in N$, $f^{-1}(x)$ is fiber which is a $(m - n)$ dimensional submanifold of M . If a vector field on M is always tangent (resp. orthogonal) to fibers, then it is called vertical (resp. horizontal). A vector field X on M is said to basic if it is horizontal and f -related to a vector field X_* on N , i.e., $f_*X_p = X_{*f(p)}$ for all $p \in M$. We denote the projection morphisms on the distributions $\ker f_*$ and $(\ker f_*)^\perp$ by $\mathcal{V}$ and $\mathcal{H}$ respectively.

A smooth map $f : (M, g_M) \rightarrow (N, g_N)$ between Riemannian manifolds is a Riemannian submersion if and only if

$$(2.8) \quad g_M(U, V) = g_N(f_*U, f_*V),$$

for every $U, V \in (\ker f_*)^\perp$.

The O'Neill's tensors $\mathcal{T}$ and $\mathcal{A}$ define by

$$(2.9) \quad T_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F,$$

$$(2.10) \quad A_E F = \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F + \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F,$$

for arbitrary vector fields E and F on M , where ∇ is the Riemannian connection on M [19].

Lemma 1. *Let f be a Riemannian submersion between Riemannian manifolds (M, g_M) and (N, g_N) . If X and Y are basic vector fields on M , then:*

- (i) $g_M(X, Y) = g_N(f_*X, f_*Y)$,
- (ii) *the horizontal part $[X, Y]^\mathcal{H}$ of $[X, Y]$ is a basic vector field and corresponds to $[X_*, Y_*]$ i.e., $f_*([X, Y]^\mathcal{H}) = [X_*, Y_*]$,*
- (iii) $[V, X]$ is vertical for any vector field V of $\ker f_*$,
- (iv) $(\nabla_X^M Y)^\mathcal{H}$ is vertical for any vector field corresponding to $\nabla_{X_*}^N Y_*$ where ∇^M and ∇^N are the Riemannian connection on M and N respectively.

Now, from equations (2.9) and (2.10), we get

$$(2.11) \quad \nabla_X Y = \mathcal{T}_X Y + \widehat{\nabla}_X Y,$$

$$(2.12) \quad \nabla_X V = \mathcal{H}\nabla_X V + \mathcal{T}_X V,$$

$$(2.13) \quad \nabla_V X = \mathcal{A}_V X + \mathcal{V}\nabla_V X,$$

$$(2.14) \quad \nabla_V W = \mathcal{H}\nabla_V W + \mathcal{A}_V W,$$

for $X, Y \in \Gamma(\ker f_*)$ and $V, W \in \Gamma(\ker f_*)^\perp$, where $\widehat{\nabla}_X Y = \mathcal{V}\nabla_X Y$. Moreover, if V is basic, then $\mathcal{H}\nabla_X V = \mathcal{A}_V X$.

On the other hand, for any $E \in \Gamma(TM)$, it is seen that $\mathcal{T}$ is vertical, $\mathcal{T}_E = \mathcal{T}_{\mathcal{V}E}$ and $\mathcal{A}$ is horizontal, $\mathcal{A}_E = \mathcal{A}_{\mathcal{H}E}$.

The tensor fields $\mathcal{T}$ and $\mathcal{A}$ satisfy the equations:

$$(2.15) \quad \mathcal{T}_X Y = \mathcal{T}_Y X,$$

$$(2.16) \quad \mathcal{A}_V W = -\mathcal{A}_W V = \frac{1}{2}\mathcal{V}[V, W],$$

for $X, Y \in \Gamma(\ker f_*)$ and $V, W \in \Gamma(\ker f_*)^\perp$.

It can be easily seen that a Riemannian submersion $f : (M, g_M) \rightarrow (N, g_N)$ has totally geodesic fibers if and only if $\mathcal{T}$ identically vanishes.

Now, we consider the notion of harmonic maps between Riemannian manifolds. Let (M, g_M) and (N, g_N) be Riemannian manifolds and suppose that f is a C^∞ mapping between them. Then the differential f_* of f can be considered as a section of the bundle $\text{Hom}(TM, f^{-1}TN) \rightarrow M$, where $f^{-1}TN$ is the pullback bundle that has fibers $(f^{-1}TN)_q = T_{f(q)}N$, $q \in M$. If $\text{Hom}(TM, f^{-1}TN)$ has a connection ∇ induced from the Riemannian connection ∇^M , then the second fundamental form of f is given by

$$(2.17) \quad (\nabla f_*)(X, Y) = \nabla_X^f f_* Y - f_*(\nabla_X^M Y),$$

for any $X, Y \in \Gamma(TM)$, where ∇^f is the pullback connection.

If f is a Riemannian submersion, then we can easily see that

$$(2.18) \quad (\nabla f_*)(V, W) = 0,$$

for any $V, W \in \Gamma(\ker f_*)^\perp$ [4].

3. The pointwise slant submersions from almost contact metric manifolds

Let f be a Riemannian submersion from a Kenmotsu manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N) . If for each $x \in M$, the angle $\theta(X)$ between ϕX and the space $\ker f_*$ is independent of the choice of the non-zero vector field $X \in \Gamma(\ker f_*) - \{\xi\}$, then f is called a pointwise slant submersion and the angle θ is said to be slant function of the pointwise slant submersion.

A pointwise slant submersion is called slant if its slant function θ is independent of the choice of the point on $(M, \phi, \xi, \eta, g_M)$. Then the constant θ is called the slant angle of the slant submersion [21].

3.1 Pointwise slant submersion for $\xi \in \Gamma(\ker f_*)$

Let f be a Riemannian submersion from a Kenmotsu manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N) .

For any $X \in \Gamma(\ker f_*)$, we get

$$(3.1) \quad \phi X = \psi X + \omega X,$$

where ψX and ωX are vertical and horizontal components of ϕX respectively.

For any $V \in \Gamma(\ker f_*)^\perp$, we have

$$(3.2) \quad \phi V = BV + CV,$$

where BV and CV are vertical and horizontal components of ϕV respectively

By using equations (2.3), (3.1) and (3.2), we get

$$(3.3) \quad g_M(\psi X, Y) = -g_M(X, \psi Y),$$

$$(3.4) \quad g_M(\omega X, V) = -g_M(X, BV),$$

for any $X, Y \in \Gamma(\ker f_*)$ and $V \in \Gamma(\ker f_*)^\perp$.

Again using the equations (2.5), (2.11), (2.13), (3.1) and (3.2), we get

$$(3.5) \quad \widehat{\nabla}_X \xi = X - \eta(X)\xi, \mathcal{T}_X \xi = 0,$$

for any $X \in \Gamma(\ker f_*)$ and $V \in \Gamma(\ker f_*)^\perp$.

For any $X, Y \in \Gamma(\ker f_*)$, define

$$(3.6) \quad (\nabla_X \psi)Y = \widehat{\nabla}_X \psi Y - \psi \widehat{\nabla}_X Y,$$

and

$$(3.7) \quad (\nabla_X \omega)Y = \mathcal{H} \nabla_X \omega Y - \omega \widehat{\nabla}_X Y,$$

where ∇ is the Riemannian connection on M . Next, we say that the ω is parallel if

$$(3.8) \quad (\nabla_X \omega)Y = 0.$$

Then we easily have:

Lemma 2. *Let $(M, \phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \rightarrow (N, g_N)$ is a pointwise slant submersion, then*

$$(3.9) \quad (\nabla_X \psi)Y = B\mathcal{T}_X Y - \mathcal{T}_X \omega Y - g(\psi X, Y)\xi + \eta(Y)\psi X,$$

and

$$(3.10) \quad (\nabla_X \omega)Y = C\mathcal{T}_X Y - \mathcal{T}_X \psi Y + \eta(Y)\omega X,$$

for any $X, Y \in \Gamma(\ker f_*)$.

In the same way with the proof of Theorem 1 in [21], we have the following theorem:

Theorem 1. *If $(M, \phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \rightarrow (N, g_N)$ is a pointwise slant submersion, then*

$$\psi^2 = \cos^2 \theta (-I + \eta \otimes \xi).$$

Similarly, as in the proof of Lemma 2 in [21], we have

Corollary 1. Let $(M, \phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \rightarrow (N, g_N)$ is a pointwise slant submersion, then we have

$$\begin{aligned} g_M(\psi X, \psi Y) &= \cos^2 \theta (g_M(X, Y) - \eta(X)\eta(Y)), \\ g_M(\omega X, \omega Y) &= \sin^2 \theta (g_M(X, Y) - \eta(X)\eta(Y)), \end{aligned}$$

for any $X, Y \in \Gamma(\ker f_*)$.

In the same way with the proof of Lemma 3 in [21], we can state the following theorem:

Theorem 2. *Let $(M, \phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. Assume that $f : (M, \phi, \xi, \eta, g_M) \rightarrow (N, g_N)$ is a pointwise slant submersion. If ω is parallel, then we have*

$$T_{\psi X} \psi X = \cos^2 \theta (T_X X - \eta(X)\omega \psi X),$$

for any $X \in \Gamma(\ker f_*)$.

Let f be a C^∞ map from Riemannian manifold (M, g_M) onto (N, g_N) , then the adjoint ${}^*f_*$ map of f_* is characterized by

$$(3.11) \quad g_M(X, {}^*f_*Y) = g_N(f_*X, Y),$$

for any $X \in T_q M, Y \in T_{f(q)} N$ and $q \in M$. For each $q \in M$, f_*^h is a C^∞ map defined by

$$f_*^h : ((\ker f_*)^\perp(q), g_M(\ker f_*)^\perp(q)) \rightarrow (\text{range } f_*(q), g_N(\text{range } f_*)(q)),$$

where denote the adjoint of f_*^h by ${}^*f_*^h$. Let ${}^*f_{*q}$ be the adjoint of f_{*q} that is defined by $f_{*q} : (T_q M, g_M) \rightarrow (T_{f(q)} N, g_N)$.

The linear transformation $({}^*f_*)^h : \text{range } f_*(q) \rightarrow (\ker f_*)^\perp(q)$, defined as $({}^*f_*)^h Y = {}^*f_* Y$, where $Y \in \Gamma(\text{range } f_*)$, is an isomorphism and $(f_{*q}^h)^{-1} = ({}^*f_{*q})^h = {}^*(f_{*q}^h)$.

Theorem 3. *Let $(M, \phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \rightarrow (N, g_N)$ is a pointwise slant*

submersion with non-zero slant function θ , then the fibers are totally geodesic submanifolds of M if and only if

$$\begin{aligned} & g_N(\nabla_{V'}^N f_*(\omega X), f_*(\omega Y)) \\ &= -g_M([X, V], Y) \sin^2 \theta + V(\theta)g_M(\phi X, \phi Y) \sin 2\theta + g_M(\mathcal{A}_V \omega \psi X, Y) \\ & \quad - g_M(\mathcal{A}_V \omega X, \psi Y) - \eta(Y)g_M(BV, \psi X) - \eta(\nabla_V X)\eta(Y) \sin^2 \theta, \end{aligned}$$

for any $X, Y \in \Gamma(\ker f_*)$ and $V \in \Gamma(\ker f_*)^\perp$, where V and V' are f -related vector fields and ∇^N is the Riemannian connection on N .

Proof. For any $X, Y \in \Gamma(\ker f_*)$ and $V \in \Gamma(\ker f_*)^\perp$, using equations (2.3), (2.4), (2.11) and (3.1), we get

$$\begin{aligned} g_M(T_X Y, V) &= -g_M([X, V], Y) + g_M(\nabla_V \psi^2 X, Y) + g_M(\nabla_V \omega \psi X, Y) \\ & \quad - g_M(\nabla_V \omega X, \phi Y) - \eta(\nabla_V X)\eta(Y). \end{aligned}$$

From theorem 1 and using equations (2.11), (2.14) and (3.2), we get

$$\begin{aligned} & g_M(T_X Y, V) \sin^2 \theta \\ &= -g_M([X, V], Y) \sin^2 \theta + V(\theta)g_M(\phi X, \phi Y) \sin 2\theta + g_M(\mathcal{A}_V \omega \psi X, Y) \\ & \quad - g_N(\nabla_{V'}^N f_*(\omega X), f_*(\omega Y)) - g_M(\mathcal{A}_V \omega X, \psi Y) \\ & \quad - \eta(\nabla_V X)\eta(Y) \sin^2 \theta - \eta(Y)g_M(BV, \psi X). \end{aligned}$$

By considering the fibers as totally geodesic, we derive the formula in the above theorem. Conversely, it can be directly verified. $\square$

Theorem 4. Let $(M, \phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \rightarrow (N, g_N)$ be a pointwise slant submersion with non-zero slant function θ , then f is a totally geodesic map if and only if

$$\begin{aligned} & g_N(\nabla_{V'}^N f_*(\omega X), f_*(\omega Y)) \\ &= -g_M([X, V], Y) \sin^2 \theta + V(\theta)g_M(\phi X, \phi Y) \sin 2\theta + g_M(\mathcal{A}_V \omega \psi X, Y) \\ & \quad - g_M(\mathcal{A}_V \omega X, \psi Y) - \eta(Y)g_M(BV, \psi X) - \eta(\nabla_V X)\eta(Y) \sin^2 \theta, \end{aligned}$$

and

$$\begin{aligned} g_M(\mathcal{A}_V \omega X, BW) &= g_N(\nabla_V^f f_*(\omega \psi X), f_*(W)) - g_N(\nabla_{V'}^f f_*(\omega X), f_*(CW)) \\ & \quad - \eta(X)g_M(V, W) \sin^2 \theta, \end{aligned}$$

for $X, Y \in \Gamma(\ker f_*)$ and $V, W \in \Gamma(\ker f_*)^\perp$, where V and V' are f -related vector fields and ∇^f is the pullback connection along f .

Proof. By definition, it follows that f is totally geodesic if and only if $(\nabla f_*)(X, Y) = 0$, for any $X, Y \in \Gamma(TM)$.

From theorem 3, we obtain the first equation. On the other hand, for $X, Y \in \Gamma(\ker f_*)$ and $V, W \in \Gamma(\ker f_*)^\perp$, using equations (2.3), (2.4) and (3.1) we get

$$\begin{aligned} g_M(\nabla_V X, W) &= -g_M(\nabla_V \psi^2 X, W) - g_M(\nabla_V \omega \psi X, W) \\ &\quad + g_M(\nabla_V \omega X, \phi W) + \eta(X)g_M(V, W). \end{aligned}$$

From theorem 1 and using equations (2.8), (2.11), (2.14), (2.17) and (3.2), we get

$$\begin{aligned} &g_N((\nabla f_*)(V, X), f_*(W)) \sin^2 \theta \\ &= -g_N(\nabla_V^f f_*(\omega X), f_*(W)) + g_N(\nabla_V^f f_*(\omega X), f_*(CW)) \\ &\quad + g_M(A_V \omega X, BW) + \eta(X)g_M(V, W) \sin^2 \theta. \end{aligned}$$

Converse is obvious. $\square$

Theorem 5. Let $(M, \phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \rightarrow (N, g_N)$ be a pointwise slant submersion with non-zero slant function θ , then f is harmonic if and only if

$$\text{trace}^* f_*((\nabla f_*)(\omega \psi(\cdot))) - \text{trace} T_{(\cdot)} \omega(\cdot) + \text{trace} C^* f_*(\nabla f_*)(\omega(\cdot)) = 0.$$

Proof. For any $X \in \Gamma(\ker f_*)$ and $V \in \Gamma(\ker f_*)^\perp$, using equations (2.1), (2.3), (2.11), (3.1) and (3.2), we get

$$g_M(\mathcal{T}_X X, V) = -g_M(\phi \nabla_X \psi X, V) + g_M(\nabla_X \omega X, \phi V).$$

From theorem 1 and using equations (2.3), (2.4) and (3.1), we get

$$g_M(\mathcal{T}_X X, V) = g_M(\nabla_X X, V) \cos^2 \theta - g_M(\nabla_X \omega \psi X, V) + g_M(\nabla_X \omega X, \phi V).$$

Using equations (2.12), (2.17), (3.2) and (3.11), we have

$$\begin{aligned} g_M(\mathcal{T}_X X, V) \sin^2 \theta &= g_N(*f_*(\nabla f_*)(X, \omega \psi X), V) - g_M(\omega \mathcal{T}_X \omega X, V) \\ &\quad - g_N(C^* f_*(\nabla f_*)(X, \omega X), V). \end{aligned}$$

Conversely, a direct computation gives the proof. $\square$

3.2 Pointwise slant submersions for $\xi \in \Gamma((\ker f_*)^\perp)$

In this section, we give the basic equations of pointwise slant submersions from Kenmotsu manifolds onto Riemannian manifolds for $\xi \in \Gamma(\ker f_*)^\perp$.

From equations (2.1) and (2.2), we get

$$(3.12) \quad \phi^2 X = -X,$$

and

$$(3.13) \quad g(\phi X, \phi Y) = g(X, Y),$$

for any $X, Y \in \Gamma(\ker f_*)$. Moreover, from equations (2.12), (2.14), (2.5), (3.1) and (3.2), we get

$$(3.14) \quad T_X \xi = X,$$

$$(3.15) \quad A_V \xi = 0,$$

and

$$(3.16) \quad \eta(\nabla_X Y) = -g_M(X, Y),$$

for any $X, Y \in \Gamma(\ker f_*)$ and $V \in \Gamma(\ker f_*)^\perp$.

Theorem 6. *If $(M, \phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \rightarrow (N, g_N)$ is a pointwise slant submersion, then*

$$\psi^2 = -(\cos^2 \theta)I.$$

Corollary 2. *Let $(M, \phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \rightarrow (N, g_N)$ be a pointwise slant submersion, then*

$$g_M(\psi X, \psi Y) = \cos^2 \theta g_M(X, Y),$$

$$g_M(\omega X, \omega Y) = \sin^2 \theta (g_M(X, Y),$$

for any $X, Y \in \Gamma(\ker f_*)$.

Theorem 7. *Let $(M, \phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. Assume that $f : (M, \phi, \xi, \eta, g_M) \rightarrow (N, g_N)$ is a pointwise slant submersion with slant function θ . If ω is parallel, then*

$$T_{\psi X} \psi X = \cos^2 \theta T_X X,$$

for any $X \in \Gamma(\ker f_*)$.

Theorem 8. *Let $(M, \phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \rightarrow (N, g_N)$ is a pointwise slant submersion with non-zero slant function θ , then the fibers are totally geodesic submanifolds of M if and only if*

$$\begin{aligned} & g_N(\nabla_{V'}^N f_*(\omega X), f_*(\omega Y)) \\ &= -g_M([X, V], Y) \sin^2 \theta + V(\theta) g_M(X, Y) \sin 2\theta + g_M(\mathcal{A}_V \omega \psi X, Y) \\ & - g_M(\mathcal{A}_V \omega X, \psi Y), \end{aligned}$$

for any $X, Y \in \Gamma(\ker f_*)$ and $V \in \Gamma(\ker f_*)^\perp$, where V and V' are f -related vector fields and ∇^N is the Riemannian connection on N .

Proof. For any $X, Y \in \Gamma(\ker f_*)$ and $V \in \Gamma(\ker f_*)^\perp$, using equations (2.2), (2.3), (2.4), (2.11), (2.14), (3.1), (3.2), (2.14) and Theorem 6, we get

$$\begin{aligned} & g_M(\mathcal{T}_X Y, V) \sin^2 \theta \\ &= -g_M([X, V], Y) \sin^2 \theta + V(\theta)g_M(X, Y) \sin 2\theta + g_M(\mathcal{A}_V \omega \psi X, Y) \\ & \quad - g_N(\nabla_{V'}^N f_*(\omega X), f_*(\omega Y)) - g_M(\mathcal{A}_V \omega X, \psi Y). \end{aligned}$$

By considering the fibers as totally geodesic, we derive the formula. Conversely, it can be directly verified. $\square$

Theorem 9. Let $(M, \phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. Let $f : (M, \phi, \xi, \eta, g_M) \rightarrow (N, g_N)$ be a pointwise slant submersion with non-zero slant function θ , then f is totally geodesic map if and only if

$$\begin{aligned} & g_N(\nabla_{V'}^N f_*(\omega X), f_*(\omega Y)) \\ &= -g_M([X, V], Y) \sin^2 \theta + V(\theta)g_M(X, Y) \sin 2\theta + g_M(\mathcal{A}_V \omega \psi X, Y) \\ & \quad - g_M(\mathcal{A}_V \omega X, \psi Y), \end{aligned}$$

and

$$\begin{aligned} g_M(\mathcal{A}_V \omega X, BW) &= g_N(\nabla_V^f f_*(\omega \psi X), f_*(W)) - g_N(\nabla_V^f f_*(\omega X), f_*(CW)) \\ & \quad - \eta(W)g_M(BV, \psi X), \end{aligned}$$

for any $X, Y \in \Gamma(\ker f_*)$ and $V, W \in \Gamma(\ker f_*)^\perp$, where V and V' are f -related vector fields and ∇^f is the pullback connection along f .

Proof. By definition, it follows that f is totally geodesic if and only if $(\nabla f_*)(X, Y) = 0$, for any $X, Y \in \Gamma(TM)$.

From Theorem 3, we obtain the first equation. On the other hand, for $X, Y \in \Gamma(\ker f_*)$ and $V, W \in \Gamma(\ker f_*)^\perp$, using equations (2.2), (2.3), (2.4), (2.10), (2.11), (3.1), (3.2), (2.14), and Theorem 6, we obtain

$$\begin{aligned} & g_N((\nabla f_*)(V, X), f_*(W)) \sin^2 \theta \\ &= -g_N(\nabla_V^f f_*(\omega X), f_*(W)) + g_N(\nabla_V^f f_*(\omega X), f_*(CW)) \\ & \quad + g_M(\mathcal{A}_V \omega X, BW) + \eta(W)g_M(BV, \psi X). \end{aligned}$$

Conversely, it can be easily proved. $\square$

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APPLICATION OF MATHEMATICAL MODELING IN MANAGEMENT ACCOUNTING

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Abstract. Mathematics as a basic science plays an important role in the field of economic management field. In processing management accounting business, the combination of mathematical models and the actual condition of enterprises and public institutions can effectively solve various management problems and evade different operational risks, which is of great significance to the prediction, planning and control of enterprise operation activities. This study analyzed five common mathematical models associated to management accounting, made analyses with examples, and pointed out the innovation direction of management accounting based on internet development, aiming to provide a reference for the improvement of management accounting level.

Keywords: Mathematical models, management accounting, economic management.

Management accounting is a subject with strong practicability involving management accounting theory, methodology and practice. Management accounting theory includes contingency theory, management control system theory and agency theory. With the development of social economy and the establishment of national legal systems in recent years, enterprises have paid more and more attentions to internal control management. In the perspective of finance, a feasible management accounting system can provide effective evidences for the inspection of enterprise operation condition and the formulation of management decisions. But according to the modern financial and accounting management theory, scientific management accounting is not post-management, but need the establishment of highly-adaptive mathematical models according to certain mathematical management ideas and principles to measure relevant parameter indexes, solve various difficult problems relating to enterprise management, and achieve effective management.

1. Category of mathematical models in management accounting

Mathematical models involved in management accounting provides evidences for management decisions by making quantitative and qualitative analyses on the correlation between different economic normality factors through mathe-

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mathematical language and thinking logic in the objective perspective and reflecting the objective states of the economic factors based on the scientificity and rigor-ousness of mathematical logical thinking. Considering the practical operation and practices of enterprise management, the mathematical models involved in enterprise management accounting mainly include general mathematical model, mathematical analysis model, input-output model, linear planning model and probability statistic model (Table 1). General mathematical models which are mainly used for calculating enterprise break-even point and carrying out financial and accounting analysis with algebraic formulas are the most common application models. Mathematical analysis models combine multiple functions with management accounting practice and applied elastic analysis and marginal analysis in management accounting. Input-output models are applied to control enterprise system management and comprehensive balance. Linear planning models and probability statistic models are used for performing mathematical analysis on enterprise related data and solving problems based on the characteristics of management accounting. As a result, enterprise management staffs can scientifically and intuitively understand organization operation condition and capital chain risk level to improve organization management level.

Number	Category	Model characteristics
1	Mathematical analysis model	Mathematical calculus theory is applied to calculate the enterprise product marginal effect to guide the correlation matching between cost and products. Moreover, an elastic model can also be used to measure and calculate the correlation between internal and external market price and enterprise product supply quantity.
2	General mathematical model	Financial data are input into a break-even mathematical model to find out the break-even point of some kind of product/service of an enterprise, thus to provide a decision basis for the preservation of the product/service.
3	Linear planning model	Integer programming, linear planning, non-linear planning and analytic hierarchy models all belong to this category, aiming to analyze organizational management data and clarifying development planning.
4	Probability statistic model	Models are established based on mass data. Mathematical analysis is made after the internal and external attributions are clarified. The influence of matter variation rules on organization operation and management is measured and calculated.
5	Input-output model	It aims to control the input-output ratio of different business units. Linear algebra theory is used to establish a mathematical model and reasonably allocate the cost, budget and daily work plan of organizations.

Figure 1: The common mathematical models in management accounting

2. Empirical cases

According to the five common mathematical models in management accounting and considering the practical operation characteristics of management accounting, the application of mathematical models in management accounting was analyzed using examples.

Application of mathematical analysis models

Considering the characteristics of enterprise management accounting practice, functions with regard to cost, profit, supply and demand are extremely common. The management and analysis of those factors require the concept of derivative as well as margin models and elastic models. In modern enterprise management, only when business accounting was performed through fine management and the influence of total cost on yield variation was clarified can enterprises carry out overall planning or fine adjustment in the perspective of cost. The process needs the analysis using enterprise marginal cost models established based on derivatives. Elastic analysis is to measure the sensitivity of product supply quality or demand quality to price based on price variation. Actually, it aims to provide a basis for the decision making of enterprise inventory planning based on derivatives.

Example 1. marginal cost analysis

The changes of enterprise yield could induce the changes of cost. Yield was defined as 1, incremental change as Δq , unit price as p , total cost as C (fixed cost C_0 + variable cost CV), and cost variation as ΔC . The cost function was $C = C(q)$.

With the changes of yield increment, the average increment of the total cost was $\frac{\Delta C}{\Delta q} = \frac{C+(q+\Delta q)-C(q)}{\Delta q}$.

If there was $\lim_{\Delta q \rightarrow 0} \frac{\Delta C}{\Delta q}$, then the marginal cost $C'(q)$ when the enterprise yield was q could be obtained.

In the comparison of marginal cost and unit price, $C'(q) \leq p$ suggested the market feedback on the yield was favorable, demand exceeds supply, and the yield could be increased; otherwise, the yield should be reduced to control cost. Similarly, price elastic analysis reflected the product price variation induced by market demand and supply.

Application of general mathematical models

In the process of development, enterprises usually need to find out the critical points of operation and management, i.e., break even state. The state can guide enterprises to reasonably allocate resources and achieve stable and healthy development though it is not the ultimate purpose of enterprise operation. Management accounting requires the application of break-even models to systematically check the factors such as sales income, cost and profit and find out the critical state when the total cost is equal to the total profit, thereby

scientifically and reasonably allocate enterprise organization cost, price, sales volume and profit and enhance management efficacy.

Example 2: Suppose enterprise operation only include two steps, i.e., production and sales, the extra factors such as tax and nonoperating revenue and expenditure were not taken into account, and enterprise operation profits were only influenced by cost and profits. The following basic variables were defined.

Yield was defined as Q , profit as L , unit price as P , profit as R , unit variable cost as V , and total cost as C (fixed cost C_0 and variable cost CV). Then the premise of enterprise operation was $L = R - C > 0$.

The critical condition was $L = 0$. At that moment,

$$\begin{pmatrix} R = P * Q \\ C = C_v + C_0 \\ C_v = Q * V \\ R = C \end{pmatrix},$$

the sales volume at the critical point was obtained through simultaneous equations.

$Q_0 > 0$ meant enterprises were in a profitable state and its advantage in profit could be extended continuously.

$Q_0 < 0$ meant enterprises were at a loss and the proportion of C_0 , P and V should be adjusted to improve profit.

Certainly, in practical operation, the influence factors for different break-even critical states are different. But general mathematical models can combine mathematical logical thinking with management accounting to provide a reference for enterprise scientific decision-making.

Mathematical planning models

In economic management activities, extreme values such as maximum profit, maximum yield, minimum cost and minimum input are usually obtained under certain limitation conditions for the purpose of best effect of profit maximization. For the requirements of extreme value measurement and calculation in this kind of management, the linear planning model can be used.

Example 3. Linear planning analysis

A chemical engineering enterprise in Shanxi mainly produces product A and C. Input of a certain amount of manpower, electric power and coal resources are required in the process of production. The resources that are needed in the production of unit product are as follows.

The above table demonstrates the input that is needed in the production of product A and B and net output value. Then how to match the yield of product A and B to obtain the maximum net output value when the enterprise owned 600 staffs, 400 degrees of electronic power and 720 tons of coal. It is a common topic in enterprise management. Achieving maximum profit by scientifically and

Category	Manpower/n	Electric power /degree	Coal/ton	Net output value /yuan
A	6	8	18	1400
B	20	10	8	2400

Figure 2: The resources that are needed in the production of unit product in the case enterprise

reasonably allocating internal resources is the target of most of enterprises when resources are limited. Answer was given to solve the case.

Suppose the output of product A and B as X_A and X_B .

The constraint conditions were

$$\left\{ \begin{array}{l} 18X_A + 8X_B \leq 720 \\ 8X_A + 10X_B \leq 400 \\ 6X_A + 20X_B \leq 600 \\ X_A X_B \geq 0 \end{array} \right\}.$$

Objective function was: $Max(X_A, X_B) = 1400X_A + 2400X_B$.

According to the mathematical planning model function, the output of product A and B was 20 and 24 respectively, and the total net output value was 85,600 yuan.

Therefore, enterprises can scientifically and directly determine how to achieve profit maximization through systematic analysis using linear planning function when resources are limited. Moreover, it can also help the decision-making level to figure out how to achieve cost minimization with the consumption of the least resources. Thus enterprises can implement scientific management through applying mathematical models in management accounting during annual strategic planning. Analytic Hierarchy Process is also a common mathematical planning model which is applicable for the situation that multiple influence factors are involved. Using the model, multiple influence factors (indexes) are graded, weight is set, and quantitative data are used for calculation, in order to achieve visual management of numerous influence factors, scientifically determine the correlation between different influence factors and the importance of the factors, and provide management decision making with reference and basis.

Probability statistic model

Throughout various processes involved in organizational management, we find mathematical models not only can solve judgment in decision caused by definite factors, but also can help enterprises scientifically analyze multiple random factors and predict the changes and development of internal and external environment. The application of mathematical models in management accounting can help enterprise managers to predict capital management risks, toler-

ance capacity for internal and external environment changes and future market changes.

Different probability models may be applied when probability was determined in different occasions. The influence factors of enterprises should be considered before selecting models. Among the factors, some are definite factors, some are random factors, and some are between definite factors and random factors as time goes by. Certainly, in the prediction of future, single use of probability models is not enough; relevant data should be statistically analyzed by management accounting to reduce capital risks. Those data include both real data generated by enterprises in the past and referenced data. In this process, budget planning and preparation should be done firstly; the statistical regularity hidden behind random variables, i.e., correlation between variables, should be searched by a large number of tests and observations. Thus, the regression forecasting analysis method can be applied in probability statistics.

Application of input-output models

In the macroscopic view, the operation and management of enterprises need the input of manpower, material resources and financial resources. National economic chain is composed of multiple units, products generated from every unit need raw materials or semi-finished products, and those raw materials or semi-finished products originate from other units. In the supply chain, the raw materials or semi-products which are input for the generation of single department are called input, and the produced final products are called output. In enterprise management accounting practice, the application of input-output models can help scientifically decide issues such as compound interest and annuity in advance. In the perspective of financial input and output, different funding means for the same event may result in different values or significantly different actual input generated by reverse deduction. This study defined the future value of a certain amount of currency (converted according to bank rate) as currency future value. Interest rate was set as r , present value of currency as P , future value of currency as F , and currency cycle as t .

If the number of interest accrual per week was n , then future value of currency $F = Pert$. Thus it could be deduced that, $P = Fe - rt$.

Example 3. Input-output analysis

A mechanical manufacture factory in Shanghai has planned to introduce a set of production line in 2017. Considering the requirements of the seller, two payment schemes have been formulated.

Scheme 1: one-off payment, totally 2.4 million yuan.

Scheme 2: installment payment (six installment), 480 thousand yuan each installment, totally 2.88 million yuan.

The interest rate on borrowings in bank r was supposed as 9%. The input-output effect was analyzed to help the enterprise select the optimal payment scheme.

Scheme 1 was one-off payment; scheme 2 was payment in six installment which spans a long time (six years), which needed to take the input of interest into account. Scheme 2 involved annuity in advance. The selection of the optimal scheme needed the comparison of input of annuity in advance and ordinary annuity increment input. Then we have $P = F \frac{1 - (1+r)^{-t}}{r} (1+r) = 480000 * \frac{1 - (1+0.09)^{-6}}{0.09} (1 + 0.09) \approx 2347032.6$ yuan

$2400000 - 2347032.6 = 52967.4 > 0$, suggesting the total input of scheme 1 was higher than that of scheme 2 when the output was the same. In the perspective of management accounting and to reduce input cost, scheme 2 was better for the enterprise.

The latest research thought

With the rapid development of big data technology, the application of mathematical modeling in management accounting has become increasingly deeper. It is developing in the following three aspects.

Cloud computation of big data: combination of mathematical thinking and management accounting Currently, Made in China 2025 has been the new trend for the optimization and upgrading of industrial structure. Under the background of internet+, enterprise management accounting has been more diversified and intelligent and no longer been limited to mathematical models. Mathematical thinking is integrated with management accounting through cloud computation of big data, which adds new blood to management accounting tools. Integrating internet thinking and mathematical theoretical models with management accounting tools has been a new model which manifests local characteristics and trans-boundary integrative thinking. The butt joint between internet+ and enterprise is closer. Traditional mathematical theories will be constantly deepened and mined. High requirements are proposed to the feasibility and accuracy of management accounting. Moreover, enterprise customized management accounting will be the main trend under the assistance of big data analysis and cloud computation platform. Enterprise customized management accounting is not only beneficial to market resource integration, but also provides a guarantee for the scientificity and accuracy of enterprise management accounting.

Contextual characteristics: optimization of investment decision-making and risk prediction

When Chinese economy develops rapidly, many economic new normal characteristics become obvious and the optimization and upgrading of industrial structure and policy adjustment change quickly, which proposes new development requirements for information acquisition and control inside and outside enterprises. To survive and develop in the market, enterprises should make every investment decision and do risk analysis properly. Those decisions should not be rude behaviors but management decisions which are adapt to the time and area as well as the internal and external development requirements. In the perspective of management accounting, it means applying multi-situation

analysis model and combining qualitative and quantitative to avoid risks and improve decision accuracy.

Transition management: constant innovation of management accounting under mathematical theories

In the new circumstances, the innovation of management accounting is an eternal topic. There is neither changeless mathematical model nor everlasting management mode. Therefore, the thought of transition management is needed to promote system reformation and constant innovation off management accounting. It requires enterprises to focus on investigating management with structural different and optimizing management accounting system. Moreover, in the perspective of system integration, multiple resource channels can be integrated to break the limitation of traditional software and hardware. Accounting management system with high adaptability and practicality and applicability should be established through constant studying.

3. Conclusions

Management accounting plays an important role in enterprise and public institutions and provides important bases for management decision making, risk control and organization development planning. Mathematical models can provide management accounting with important management carriers and scientific basis. Different organizations can apply mathematical models for systematic and scientific management in management accounting or other management processes.

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EFFECT OF ALBEDO ON THE MOTION OF THE INFINITESIMAL BODY IN CIRCULAR RESTRICTED THREE BODY PROBLEM WITH VARIABLE MASSES

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Abstract. This paper presents the investigation of the effect of Albedo on the motion of the infinitesimal body in the circular restricted three body problem with the variation of all masses (primaries as well as infinitesimal body) with time. The radiation pressures fall on the surfaces of the planets, some of radiations are absorbed by them but some radiations are reflected back into the space. These radiations from the planets to the space are known as Albedo (i.e. $\text{Albedo} = (\text{radiation reflected back into the space})/(\text{incident radiation})$).

The equations of motion have been evaluated by using Meshcherskii transformation and found the expression for the variation of the Jacobi integral constant. We have drawn the locations of the equilibrium points, the periodic orbits, Poincaré surface of sections and basins of attraction for four cases (a. Classical Case, b. Variation of mass, c. Solar radiation pressure effect, d. Albedo effect) and also the surfaces of the motion of the infinitesimal body have been drawn with the effect of Albedo only.

Finally, we have examined the stability of the equilibrium points under the effect of Albedo and found that all the equilibrium points are unstable.

Keywords: CR3BP, Variable masses, Albedo, Radiation pressure, Periodic orbits, Poincaré surface of sections, Basins of attraction.

1. Introduction

In our solar system, there are many stars which are source of radiations, one of them is sun. Due to it, the radiation spreads on the planets and after absorption of the radiation by the planets, some radiations reflected back into the space. This reflected radiation is known as Albedo (i.e. $\text{Albedo} = (\text{radiation reflected back into the space})/(\text{incident radiation})$).

The most common problem in the celestial mechanics is the restricted three body problem in which two finite bodies called primaries move around their center of mass in the circular or elliptic orbits under the influence of their mutual gravitational attraction and a third body of the infinitesimal mass is moving in the plane of the primaries, which is influenced by them but not influencing them.

Many scientist studied this model by taking different perturbations as different shapes of the primaries, the solar radiation pressure, the resonance, the variation of masses, the coriolis and centrifugal forces, the Yarkovsky effect, the Poynting-Robertson drags, Albedo effect etc.

The two body problem with variable mass has been discussed by Jeans [19]. Meshcherskii [23] worked on the mechanics of bodies of variable mass. Chernikov [14] studied the stability of equilibrium points by Lyapunov's methods in the restricted three body problem with the effects of solar radiation pressure but the effect of albedo is not considered by him. Perezhogin [28] studied the stability of the sixth and seventh libration points in the photogravitational circular restricted three body problem. Bhatnagar et al. [11, 12] investigated about the equilibrium points in the photogravitational restricted three body problem. They examined the stability of the equilibrium points and observed that the equilibrium points are stable in the linear sense and unstable around the triangular points. Schuerman [31] studied the restricted three body problem including the force of radiation pressure and the Poynting-Robertson effect and found that the triangular equilibrium points are depending on both the solar radiation pressure and the Poynting-Robertson drag. Anselmo et al. [7] studied the effect of Albedo on the orbit of the LAGEOS satellite and proposed that the periodic perturbations is due to Albedo. Mignard [24] studied the restricted three-body problem with the solar radiation pressure. And observed that triangular equilibrium points are no longer exist. Kunitsyn et al. [21] proved the existence of the three-parametric family of the collinear libration points in the photo-gravitational three-body problem. They determined the number and locations of these points and also they examined the stability and found that the points are stable for some domain of parameter space.

Simmons et al.[34] investigated the restricted 3-body problem with radiation pressure and observed that nine equilibrium points exist, five in the plane of motion and four in the out of plane. Sharma [32, 33] investigated the stationary solutions of the planar restricted three-body problem when the more massive primary is a source of radiation and smaller primary is an oblate spheroid with its equatorial plane coincident with the plane of motion. Appel [8] developed a numerical model for altitude estimation from magnetometer and earth-albedo-corrected coarse sun sensor measurement. AbdulRaheem et al. [1] investigated the stability of equilibrium points under the influence of the Coriolis and centrifugal forces together with the effects of oblateness and radiation pressure of the primaries. It is observed that the collinear points are unstable and the triangular points are conditionally stable depending on the radiation factor and oblateness. It is also observed that the Coriolis force has a stabilizing tendency, while the centrifugal force, radiation and oblateness of the primaries have destabilizing effects. Rocco [30] investigated the evaluation of the terrestrial albedo applies to some scientific missions and calculated the irradiance at the satellite with the radiant flux from each cell. Singh et al. [35, 36, 37, 38, 39] studied about the effect of small perturbations in the Coriolis and centrifugal forces on

the locations and stability of the equilibrium points in the restricted three body problem and Robe's restricted three body problem with the variable mass. Ershkov [16] extend the photo-gravitational three-body problem with additional effect of Yarkovsky. Zhang et al. [41] studied about the triangular libration points in photo-gravitational restricted three body problem with variable mass. They have used the space time inverse transformation of Meshcherskii for the linear stability of the triangular points and observed that the motion around the triangular points are unstable with decreasing mass. Abouelmagd et al. [2, 3] studied the effect of oblateness in the perturbed restricted three body problem and also with variable mass. And they found an appropriate approximation for the locations of out-of-plane equilibrium points in the special case of a non-isotropic variation of the mass. They determined the elements of periodic orbits and periodicity around these points under the effects of the perturbations.

Ansari [4] investigated the stability of the equilibrium points in the photo-gravitational circular restricted four body problem with the effect of variable masses, coriolis and centrifugal forces. He used the Jeans law and Meshcherskii law in the space time transformations. It is observed that there exists eight equilibrium points in which three points are asymptotically stable and five points are unstable. Mittal et al. [25] investigated the stability of the Lagrangian solutions for the restricted four-body problem with variable mass. They found at most eight libration points in which two are collinear and rests are non-collinear and observed that all the libration points are unstable. Idrisi [17, 18] investigated the effects of albedo on the libration points and its stability in the restricted three body problem when less massive primary is an ellipsoid. He found five libration points in which three unstable collinear libration points and two stable non-collinear libration points. On the other hand, many scientists have studied about the basin of attraction in both R3BP and R4BP cases. Douskos [15], Assis et al. [10], Kumari et al. [20], Asique et al. [9], Matthies [22], Paricio [27], Ansari [5, 6], Zotos [42, 43, 44, 45].

We have studied the effect of Albedo on the motion of the infinitesimal body in the circular restricted three body problem in which the masses of the primaries as well as the mass of the infinitesimal body vary with time. We have studied our problem in various sections. In the equations of motion section, we have evaluated the equations of motion of the infinitesimal variable mass under the effects of the Albedo and also determined the expression for the variation of Jacobi Integral constant. In the Numerical Analysis section, we have plotted the equilibrium points, the periodic orbits, Poincaré surface of sections and basins of attraction in four cases and also the surfaces of the motion of the infinitesimal body have been drawn under the effect of Albedo only. In the stability section, we have examined the stability of the equilibrium points under the effect of Albedo only. And finally, we have concluded the problem.

Our problem has many applications in scientific research, not only in the fields of celestial mechanics but also in physics and quantum mechanics.

2. Equations of motion

Let there be three variable masses $m_1(t)$, $m_2(t)$ and $m(t)$. The two primary bodies $m_1(t)$ and $m_2(t)$ are moving in the circular orbits around their center of mass O which is considered as origin. The third infinitesimal body $m(t)$ is moving in the plane of motion of $m_1(t)$ and $m_2(t)$ under their gravitational forces F_1 and F_2 respectively but not influencing them. The body $m_1(t)$ is a source of radiation pressure such that $m_1(t) > m_2(t)$. F_p is the solar radiation pressure on $m(t)$ due to $m_1(t)$ and F_a is the Albedo force (solar radiation reflected by $m_2(t)$ in space on $m(t)$). Let the inertial and rotating frames be (XYZ) and (xyz) respectively with the same origin. We also assume that the coordinates of $m_1(t)$, $m_2(t)$ and $m(t)$ in the inertial frame are (X_1, Y_1, Z_1) , (X_2, Y_2, Z_2) and (X, Y, Z) , while in the rotating frame they are $(x_1, 0, 0)$, $(x_2, 0, 0)$ and (x, y, z) respectively (Fig. 1).

The forces acting on $m(t)$ due to $m_1(t)$ and $m_2(t)$ are $F_1(1-\varepsilon_1)$ and $F_2(1-\varepsilon_2)$ respectively, where $\varepsilon_1 = F_p/F_1 \ll 1$ and $\varepsilon_2 = F_a/F_2 \ll 1$. And also ε_1 and ε_2 can be expressed as $\varepsilon_1 = \frac{L_1}{2\pi G m_1(t) c \sigma}$, $\varepsilon_2 = \frac{L_2}{2\pi G m_2(t) c \sigma}$, where L_1 and L_2 are the luminosity of $m_1(t)$ and $m_2(t)$ respectively, G is the gravitational constant, c is the speed of light and σ is the mass per unit area of the infinitesimal mass $m(t)$. $\frac{\varepsilon_2}{\varepsilon_1} = \frac{m_1(t)L_2}{m_2(t)L_1} = \frac{m_1(t)}{m_2(t)}k_0$, $\frac{L_2}{L_1} = k_0$ (constant), $0 \leq \varepsilon_1 < 1$, $0 \leq \varepsilon_2 < \varepsilon_1$ and $0 \leq k_0 < 1$.

The equations of motion of the infinitesimal variable mass with the solar radiation pressure and albedo effect in the inertial frame of reference under the assumptions that the masses that fall on (or ejected from) the bodies have zero momentum, are

$$(1) \quad \begin{cases} \frac{\dot{m}(t)}{m(t)} \dot{X} + \ddot{X} = -\frac{Gm_1(t)(X-X_1)(1-\varepsilon_1)}{r_1^3} - \frac{Gm_2(t)(X-X_2)(1-\varepsilon_2)}{r_2^3}, \\ \frac{\dot{m}(t)}{m(t)} \dot{Y} + \ddot{Y} = -\frac{Gm_1(t)Y(1-\varepsilon_1)}{r_1^3} - \frac{Gm_2(t)Y(1-\varepsilon_2)}{r_2^3}, \\ \frac{\dot{m}(t)}{m(t)} \dot{Z} + \ddot{Z} = -\frac{Gm_1(t)Z(1-\varepsilon_1)}{r_1^3} - \frac{Gm_2(t)Z(1-\varepsilon_2)}{r_2^3}. \end{cases}$$

where, $r_i^2 = (X - X_i)^2 + Y^2 + Z^2$, $(i = 1, 2)$, are the distances from the primaries to the infinitesimal body in the inertial coordinate frame respectively.

The relation between the inertial and the rotating coordinates which has angular velocity $\omega(t)$, is governed by

$$(2) \quad \begin{cases} X = x \cos(\omega(t)t) - y \sin(\omega(t)t), \\ Y = x \sin(\omega(t)t) + y \cos(\omega(t)t), \\ Z = z. \end{cases}$$

Substituting system (2) in system (1), the equations of motion of the infinitesimal variable mass in rotating coordinate system when the variation of masses

is non-isotropic, become

$$(3) \quad \begin{cases} \frac{\dot{m}(t)}{m(t)}(\dot{x} - \omega(t)y) + (\ddot{x} - \dot{\omega}(t)y - 2\omega(t)\dot{y} - \omega^2(t)x) \\ = -\frac{\mu_1(t)(x-x_1)(1-\varepsilon_1)}{r_1^3} - \frac{\mu_2(t)(x-x_2)(1-\varepsilon_2)}{r_2^3}, \\ \frac{\dot{m}(t)}{m(t)}(\dot{y} + \omega(t)x) + (\ddot{y} + \dot{\omega}(t)x + 2\omega(t)\dot{x} - \omega^2(t)y) \\ = -\frac{\mu_1(t)y(1-\varepsilon_1)}{r_1^3} - \frac{\mu_2(t)y(1-\varepsilon_2)}{r_2^3}, \\ \frac{\dot{m}(t)}{m(t)}\dot{z} + \ddot{z} = -\frac{\mu_1(t)z(1-\varepsilon_1)}{r_1^3} - \frac{\mu_2(t)z(1-\varepsilon_2)}{r_2^3}. \end{cases}$$

where, $r_i^2 = (x - x_i)^2 + y^2 + z^2$, ($i = 1, 2$), are the distances from the primaries to the infinitesimal body in the rotating coordinate system respectively, $\mu_i(t) = G m_i(t)$, ($i = 1, 2$).

Using Meshcherskii transformation

$$\begin{aligned} x &= \xi R(t), \quad y = \eta R(t), \quad z = \zeta R(t), \quad \frac{dt}{d\tau} = R^2(t), \quad r_i = \rho_i R(t), \\ \omega(t) &= \frac{\omega_0}{R^2(t)}, \quad x_i = \xi_i R(t), \\ \mu(t) &= \mu_1(t) + \mu_2(t) = \frac{\mu_0}{R(t)}, \quad \mu_i(t) = \frac{\mu_{i0}}{R(t)}, \\ m(t) &= \frac{m_0}{R(t)}, \quad R(t) = \sqrt{a t^2 + 2b t + c}, \end{aligned}$$

where $a, b, c, \mu_0, \mu_{i0}, m_0$ are constants for $i = 1, 2$.

$$(4) \quad \begin{cases} \dot{x} = \frac{\xi' + (a t + b)\xi}{R(t)}, \quad \ddot{x} = \frac{\xi'' + (a c - b^2)\xi}{R^3(t)} \\ \dot{y} = \frac{\eta' + (a t + b)\eta}{R(t)}, \quad \ddot{y} = \frac{\eta'' + (a c - b^2)\eta}{R^3(t)} \\ \dot{z} = \frac{\zeta' + (a t + b)\zeta}{R(t)}, \quad \ddot{z} = \frac{\zeta'' + (a c - b^2)\zeta}{R^3(t)}. \end{cases}$$

The system (3) becomes

$$(5) \quad \begin{cases} \xi'' - 2\omega_0\eta' - (at + b)\xi' = \frac{\partial W}{\partial \xi}, \\ \eta'' + 2\omega_0\xi' - (at + b)\eta' = \frac{\partial W}{\partial \eta}, \\ \zeta'' - (at + b)\zeta' = \frac{\partial W}{\partial \zeta}. \end{cases}$$

where,

$$W = \frac{1}{2}((at+b)^2 + \omega_0^2 - (ac-b^2))(\xi^2 + \eta^2) + \frac{1}{2}((at+b)^2 - (ac-b^2))\zeta^2 \\ - (at+b)\xi\eta + \frac{\mu_{10}(1-\varepsilon_1)}{\rho_1} + \frac{\mu_{20}(1-\varepsilon_2)}{\rho_2}, \\ \rho_i^2 = (\xi - \xi_i)^2 + \eta^2 + \zeta^2, \xi_1 = \frac{\mu_{20}}{\mu_0}R_{12}, \xi_2 = \frac{-\mu_{10}}{\mu_0}R_{12}$$

Prime (') is the differentiation w.r.to τ . R_{12} is the distance between the primaries. Taking unit of mass, distance and time at initial time t_0 such that

$$\mu_0 = 1, G = 1, R_{12} = 1, \omega_0 = 1, at_0 + b = \alpha_1 \text{ (constant).}$$

Introducing the new mass parameter as

$$\frac{\mu_{10}}{\mu_0} = 1 - v, \frac{\mu_{20}}{\mu_0} = v, 0 < v \leq \frac{1}{2}$$

and

$$\varepsilon_2 = \frac{\varepsilon_1(1-v)}{v}k_0,$$

where v is the ratio of the mass of the primaries to the total mass of the primaries.

Finally, the system (5) becomes

$$(6) \quad \begin{cases} \xi'' - 2\eta' - \alpha_1\xi' = \frac{\partial\psi}{\partial\xi}, \\ \eta'' + 2\xi' - \alpha_1\eta' = \frac{\partial\psi}{\partial\eta}, \\ \zeta'' - \alpha_1\zeta' = \frac{\partial\psi}{\partial\zeta}. \end{cases}$$

where,

$$\psi = \frac{1}{2}(\alpha_1^2 + k)(\xi^2 + \eta^2 + \zeta^2) - \frac{1}{2}\zeta^2 - \alpha_1\xi\eta + \frac{(1-v)(1-\varepsilon_1)}{\rho_1} + \frac{v(1-\varepsilon_2)}{\rho_2}, \\ \rho_i^2 = (\xi - \xi_i)^2 + \eta^2 + \zeta^2, ac - b^2 = 1 - k \text{ and } \xi_1 = v, \xi_2 = v - 1.$$

If there are constant masses then there is a constant motion (i.e Moulton [26]), the Jacobi constant defined as

$$(7) \quad C = 2\psi - 2((\xi')^2 + (\eta')^2 + (\zeta')^2),$$

Multiplying in the first equation of (6) by ξ' , in the second equation of (6) by η' and in the third equation of (6) by ζ' and add and using equation (7), we get the variation of the Jacobi constant as

$$(8) \quad \frac{dC}{d\tau} = -2\alpha_1((\xi')^2 + (\eta')^2 + (\zeta')^2),$$

Where C is the Jacobi Integral Constant.

3. Numerical analysis

In this section, we have plotted numerically the locations of the equilibrium points, the periodic orbits, the Poincaré surfaces of section and the basins of attraction by using Mathematica software for four different cases:

- Classical case (i.e. $k = 1, \alpha_1 = 0, \varepsilon_1 = 0, k_0 = 0$),
- Variation of masses (i.e. $k = 0.4, \alpha_1 = 0.2, \varepsilon_1 = 0, k_0 = 0$),
- Solar radiation pressure (i.e. $k = 0.4, \alpha_1 = 0.2, \varepsilon_1 = 0.5, k_0 = 0$),
- Albedo effect (i.e. $k = 0.4, \alpha_1 = 0.2, \varepsilon_1 = 0.5, k_0 = 0.015$).

And also the surfaces of the infinitesimal body have been drawn under the effect of Albedo only. In all the graphical work, the value of v is 0.019.

3.1 Locations of Equilibrium points during in-plane and out of plane motions

In general, the locations of equilibrium points are obtained by solving $\psi_\xi = 0$, $\psi_\eta = 0$, $\psi_\zeta = 0$, but the solutions of these equations when $(\xi \neq 0, \eta \neq 0, \zeta = 0)$, represent the locations during in-plane motions (Fig. 2) and when $(\xi \neq 0, \eta = 0, \zeta \neq 0)$, $(\xi = 0, \eta \neq 0, \zeta \neq 0)$, represent the locations of the out of planes (Fig. 3, Fig. 4). Where

$$(9a) \quad \psi_\xi = (\alpha_1^2 + k)\xi - \alpha_1\eta - \frac{(1-v)(\xi-v)(1-\varepsilon_1)}{\rho_1^3} - \frac{v(\xi-v+1)(1-\varepsilon_2)}{\rho_2^3},$$

$$(9b) \quad \psi_\eta = (\alpha_1^2 + k)\eta - \alpha_1\xi - \frac{(1-\varepsilon_1)(1-v)\eta}{\rho_1^3} - \frac{(1-\varepsilon_2)v\eta}{\rho_2^3},$$

$$(9c) \quad \psi_\zeta = (\alpha_1^2 + k - 1)\zeta - \frac{(1-\varepsilon_1)(1-v)\zeta}{\rho_1^3} - \frac{(1-\varepsilon_2)v\zeta}{\rho_2^3}.$$

3.1.1 Locations of equilibrium points during in-plane motion (i.e. $\xi \neq 0, \eta \neq 0, \zeta = 0$)

During in-plane motion, we have plotted graphs for the locations of the equilibrium points in four cases. In the classical case, we found well known five equilibrium points in which three are collinear and other two are triangular equilibrium points (Fig. 2a). In the variable mass case, we found five equilibrium points but the locations of the equilibrium points are different from the classical case (Fig. 2b). In the solar radiation case, we found seven equilibrium points which have different locations from the classical case (Fig. 2c). On the other hand with the Albedo effects, we found seven equilibrium points which are similar to the case of solar radiation pressure but different from the other two cases (Fig. 2d). In all the figures black stars denote the locations of the primaries.

3.1.2 Locations of equilibrium points during out of plane (i.e.

$\xi \neq 0, \eta = 0, \zeta \neq 0$ and $\xi = 0, \eta \neq 0, \zeta \neq 0$)

During out of plane (i.e. $\xi \neq 0, \eta = 0, \zeta \neq 0$), we found three equilibrium points in all four cases (Fig. 3). In the classical case, L_1 is left side of the primary m_2 , L_2 is in between both the primaries and L_3 is right side of the primary m_1 . But in the other three cases, L_1 is in between both the primaries, L_2 is left side of the primary m_2 and L_3 is right side of the primary m_1 . The black stars denote the locations of the primaries m_1 and m_2 .

On the other hand, during the out of plane (i.e. $\xi = 0, \eta \neq 0, \zeta \neq 0$), we found three equilibrium points (Fig. 4). Where the black orbit is for classical case, green orbit is for the variable mass case, magenta orbit is for the solar radiation case and cyan orbit is for Albedo effect. Magenta orbit is not visible because this and cyan orbits are overlapped on each other.

3.2 Periodic orbits

We have drawn the periodic orbits in four cases. In the classical case, we found that the orbit is periodic (in black color) but in the other three cases orbits are not periodic (Fig. 5).

3.3 Poincaré surface of section

We also have drawn the Poincaré surface of sections for the four cases in both the $(\xi - \xi')$ -plane (Fig. 6) and the $(\eta - \eta')$ -plane (Fig.7-a,b). We found that in the $(\xi - \xi')$ -plane, the surfaces are shrinking for the cases from classical case to Albedo effect and in the $(\eta - \eta')$ -plane, the surface has dumbbell shape in classical case and in the other three cases surfaces have petal shapes which are also shrinking.

3.4 Basins of attraction

In this section, we have drawn the basins of attraction for the circular restricted three body problem by using Newton-Raphson iterative method for the four cases. The iterative algorithm of our problem is given by the system

$$(9) \quad \begin{cases} \xi_{n+1} = \xi_n - \left(\frac{\psi_\xi \psi_{\eta\eta} - \psi_\eta \psi_{\xi\eta}}{\psi_{\xi\xi} \psi_{\eta\eta} - \psi_{\xi\eta} \psi_{\eta\xi}} \right)_{(\xi_n, \eta_n)}, \\ \eta_{n+1} = \eta_n - \left(\frac{\psi_\eta \psi_{\xi\xi} - \psi_\xi \psi_{\eta\xi}}{\psi_{\xi\xi} \psi_{\eta\eta} - \psi_{\xi\eta} \psi_{\eta\xi}} \right)_{(\xi_n, \eta_n)}. \end{cases}$$

Where ξ_n, η_n are the values of ξ and η coordinates of the n^{th} step of the Newton-Raphson iterative process. The initial point (ξ, η) is a member of the basin of attraction of the root if this point converges rapidly to one of the equilibrium points. This process stops when the successive approximation converges to an attractor. For the classification of the equilibrium points on the (ξ, η) -plane, we

used color code. For the classical case (Fig. 8), L_1, L_2 and L_3 represent cyan color region, L_4 represents blue color region and L_5 represents green color region. The basin of attraction corresponding to the equilibrium points L_1, L_2, L_3, L_4 and L_5 extend to infinity. For the variable mass case (Fig. 9), L_1 and L_3 represent red color regions, L_2 represents cyan color region, L_4 represents pink color region and L_5 represents yellow color region. The basin of attraction corresponding to the equilibrium points L_1, L_3, L_4 and L_5 extend to infinity but corresponding to the equilibrium point L_2 , it covers finite area. On the other hand, for the solar radiation pressure and Albedo effect (Fig. 10, Fig. 11), L_1 and L_3 represent red color regions, L_2, L_6 and L_7 represent blue color regions, L_4 represents pink color region and L_5 represents light green color region. The basins of attraction corresponding to the equilibrium points L_1, L_3, L_4 and L_5 extend to infinity but the equilibrium points L_2, L_6 and L_7 cover finite area. In this way a complete view of the basin structures created by the attractors. We can observe in detail from the zoomed part of all the figures in Fig. 8(b), Fig. 9(b), Fig. 10(b), Fig. 11(b). The black points and black stars denote the location of the equilibrium points and the location of the primaries respectively. We also observed that the basins of attraction are shrinking from the classical case to Albedo effect.

3.5 Surfaces

We have drawn the surfaces of the motion of the infinitesimal body under the effect of Albedo in two cases: i- The surfaces with the variation of mass parameter (Fig. 12 (a)). ii. The surfaces with the variation of Albedo (Fig. 12 (b)), by considering equations 9a and 9b. And observed that there are no significant difference in these two surfaces.

4. Stability of the equilibrium points under the effect of Albedo

We can examine the stability of the equilibrium points under the effect of Albedo by taking $\xi = \xi_0 + \alpha$, $\eta = \eta_0 + \beta$, $\zeta = \zeta_0 + \gamma$ in system (6), we get

$$(10) \quad \begin{cases} \alpha'' - 2\beta' - \alpha_1\alpha' = \alpha\psi_{\xi\xi}^0 + \beta\psi_{\xi\eta}^0 + \gamma\psi_{\xi\zeta}^0 \\ \beta'' + 2\alpha' - \alpha_1\beta' = \alpha\psi_{\eta\xi}^0 + \beta\psi_{\eta\eta}^0 + \gamma\psi_{\eta\zeta}^0 \\ \gamma'' - \alpha_1\gamma' = \alpha\psi_{\zeta\xi}^0 + \beta\psi_{\zeta\eta}^0 + \gamma\psi_{\zeta\zeta}^0. \end{cases}$$

Where α , β and γ are the small displacements of the infinitesimal body from the equilibrium point. The superscript zero denotes the value at the equilibrium point.

To solve system (10), let $\alpha = Ae^{\lambda\tau}$, $\beta = Be^{\lambda\tau}$, $\gamma = Ce^{\lambda\tau}$, where A, B and C are parameters. Substituting these values in system (10) and rearranging, we get:

$$(11) \quad \begin{cases} A(\lambda^2 - \alpha_1 \lambda - \psi_{\xi\xi}^0) - B(2\lambda + \psi_{\xi\eta}^0) - C\psi_{\xi\zeta}^0 = 0, \\ A(2\lambda - \psi_{\eta\xi}^0) + B(\lambda^2 - \alpha_1 \lambda - \psi_{\eta\eta}^0) - C\psi_{\eta\zeta}^0 = 0, \\ -A\psi_{\zeta\xi}^0 - B\psi_{\zeta\eta}^0 + C(\lambda^2 - \alpha_1 \lambda - \psi_{\zeta\zeta}^0) = 0. \end{cases}$$

The system (11), will have a non-trivial solution for A, B and C if

$$\begin{vmatrix} \lambda^2 - \alpha_1 \lambda - \psi_{\xi\xi}^0 & -(2\lambda + \psi_{\xi\eta}^0) & -\psi_{\xi\zeta}^0 \\ 2\lambda - \psi_{\eta\xi}^0 & \lambda^2 - \alpha_1 \lambda - \psi_{\eta\eta}^0 & -\psi_{\eta\zeta}^0 \\ -\psi_{\zeta\xi}^0 & -\psi_{\zeta\eta}^0 & \lambda^2 - \alpha_1 \lambda - \psi_{\zeta\zeta}^0 \end{vmatrix} = 0,$$

which is equivalent to

$$(12) \quad \begin{aligned} & \lambda^6 - 3\alpha_1 \lambda^5 + \lambda^4(4 + 3\alpha_1^2 - \psi_{\xi\xi}^0 - \psi_{\eta\eta}^0 - \psi_{\zeta\zeta}^0) \\ & + \alpha_1 \lambda^3(-4 - \alpha_1^2 + 2\psi_{\xi\xi}^0 + 2\psi_{\eta\eta}^0 + 2\psi_{\zeta\zeta}^0) \\ & + \lambda^2(-\psi_{\xi\eta}^0)^2 - (\psi_{\xi\zeta}^0)^2 + \psi_{\xi\xi}^0 \psi_{\eta\eta}^0 - (\psi_{\eta\zeta}^0)^2 - 4\psi_{\zeta\zeta}^0 + \psi_{\xi\xi}^0 \psi_{\zeta\zeta}^0 + \psi_{\zeta\zeta}^0 \psi_{\eta\eta}^0 \\ & - \alpha_1^2 \psi_{\xi\xi}^0 - \alpha_1^2 \psi_{\eta\eta}^0 - \alpha_1^2 \psi_{\zeta\zeta}^0 \\ & + \alpha_1 \lambda((\psi_{\xi\eta}^0)^2 + (\psi_{\xi\zeta}^0)^2 - \psi_{\xi\xi}^0 \psi_{\eta\eta}^0 + (\psi_{\eta\zeta}^0)^2 - \psi_{\xi\xi}^0 \psi_{\zeta\zeta}^0 - \psi_{\eta\eta}^0 \psi_{\zeta\zeta}^0) \\ & + ((\psi_{\xi\zeta}^0)^2 \psi_{\eta\eta}^0 - 2\psi_{\xi\eta}^0 \psi_{\xi\zeta}^0 \psi_{\eta\zeta}^0 + \psi_{\xi\xi}^0 (\psi_{\eta\zeta}^0)^2 \\ & + (\psi_{\xi\eta}^0)^2 \psi_{\zeta\zeta}^0 - \psi_{\xi\xi}^0 \psi_{\eta\eta}^0 \psi_{\zeta\zeta}^0) = 0. \end{aligned}$$

We have solved the equation (12) for each value of the equilibrium point and found that in all the cases, λ has complex values and at least one of them has positive real value. Hence all the equilibrium points are unstable.

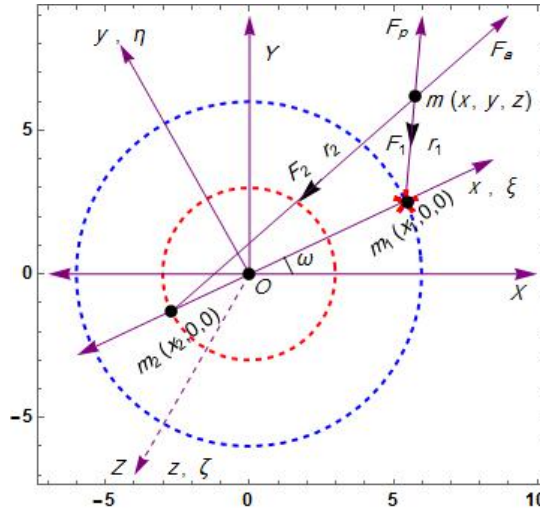


Figure 1: The geometry of the problem with Albedo

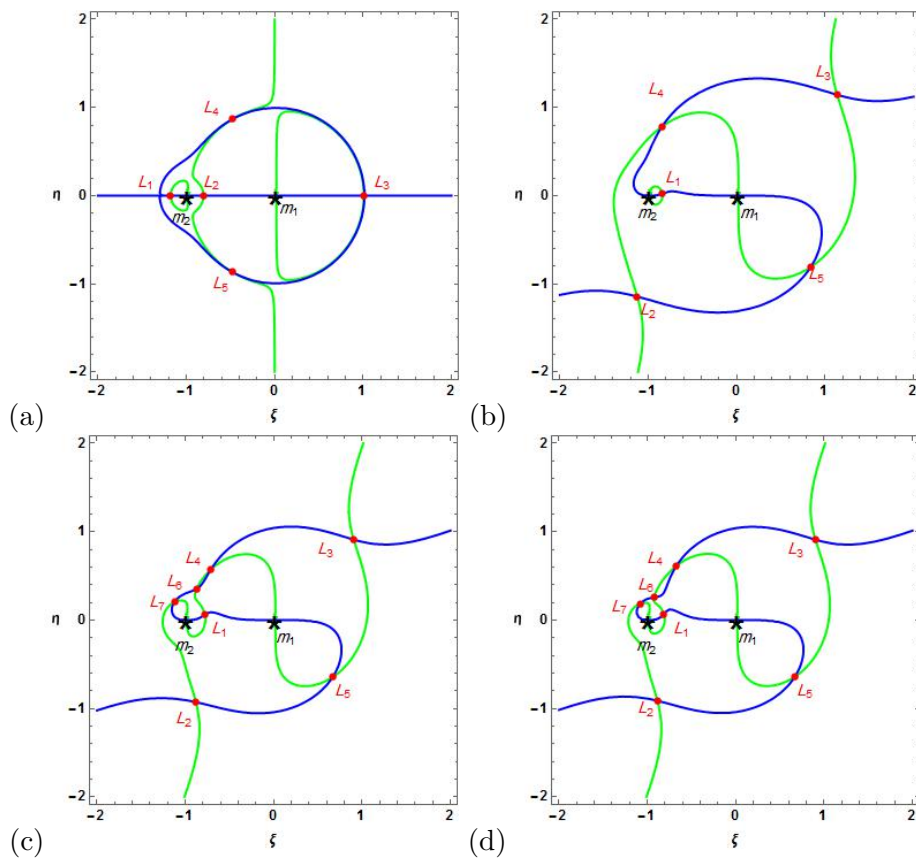


Figure 2: The locations of equilibrium points during in-plane motion (i.e. $\xi \neq 0, \eta \neq 0, \zeta = 0$) in four cases: a. Classical case, b. Variation of masses, c. Solar radiation pressure, d. Albedo effect.

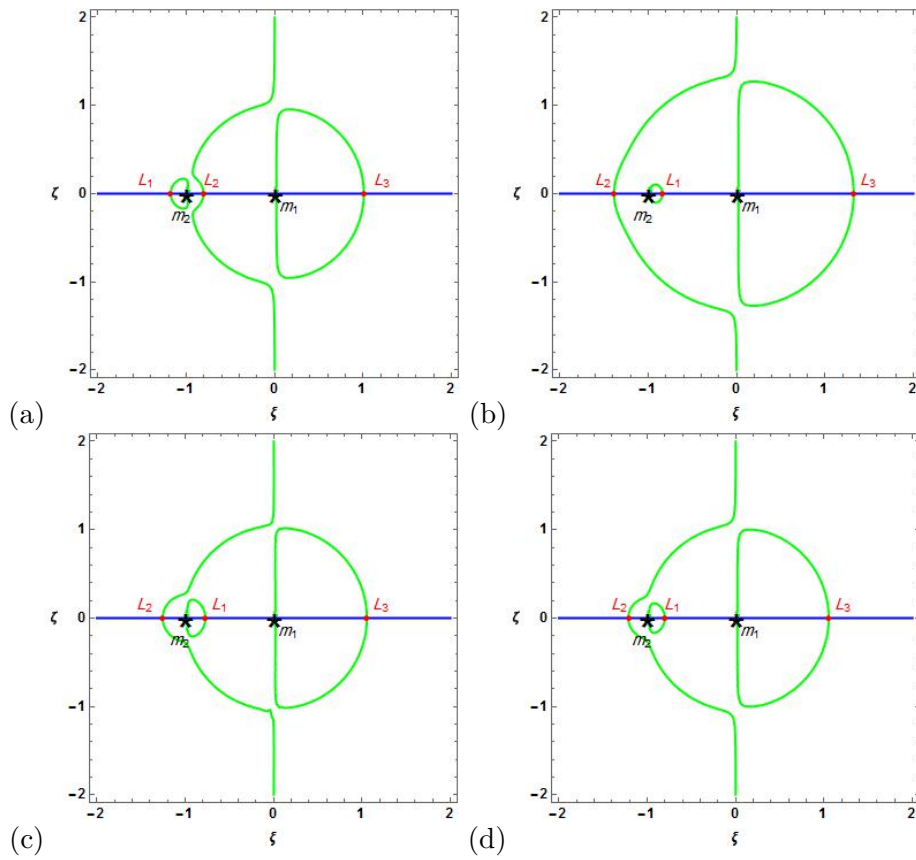


Figure 3: The locations of equilibrium points during out of plane motion (i.e. $\xi \neq 0, \eta = 0, \zeta \neq 0$) in four cases: a. Classical case, b. Variation of masses, c. Solar radiation pressure, d. Albedo effect.

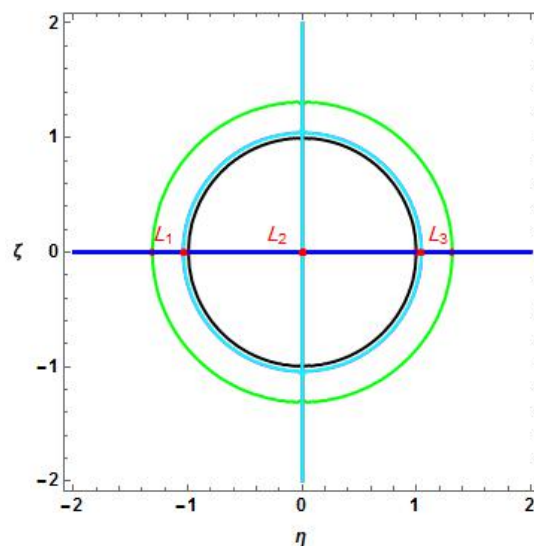


Figure 4: The locations of equilibrium points during out of plane motion (i.e. $\xi = 0, \eta \neq 0, \zeta \neq 0$) in four cases: a. Classical case (Black orbit), b. Variation of masses (Green orbit), c. Solar radiation pressure (Magenta orbit), d. Albedo effect (Cyan orbit). Magenta orbit is covered by Cyan orbit.

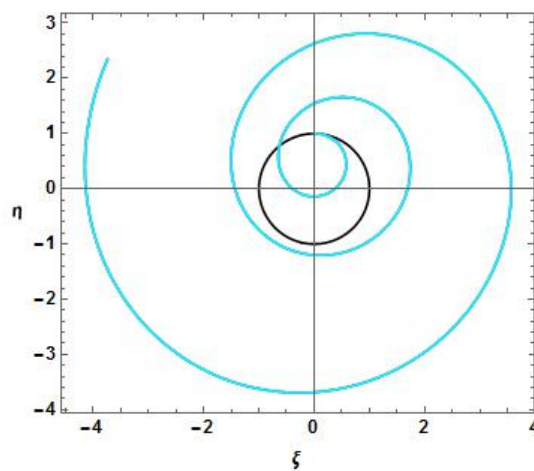


Figure 5: Periodic Orbits in four cases: a- Black, b- Green, c- Magenta, d- Cyan. But Green, Magenta and cyan are overlapped.

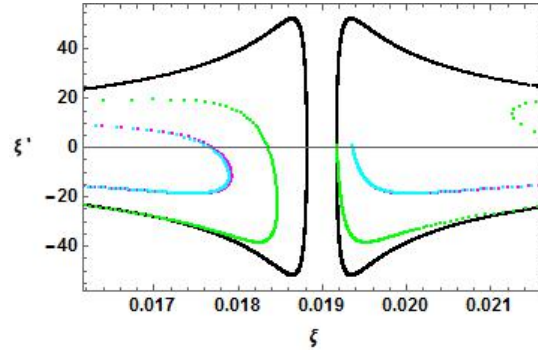


Figure 6: Poincaré surface of sections for four cases: Classical case (Black curve), variable mass case (Green curve), solar radiation pressure (Magenta curve), Albedo effect (Cyan curve).

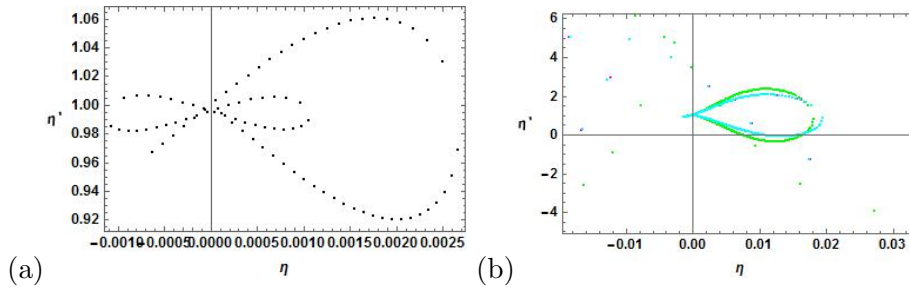


Figure 7: Poincaré surface of sections for four cases: (a) Classical case (Black curve), (b) (variable mass case (Green curve), solar radiation pressure (Magenta curve), Albedo effect (Cyan curve)).

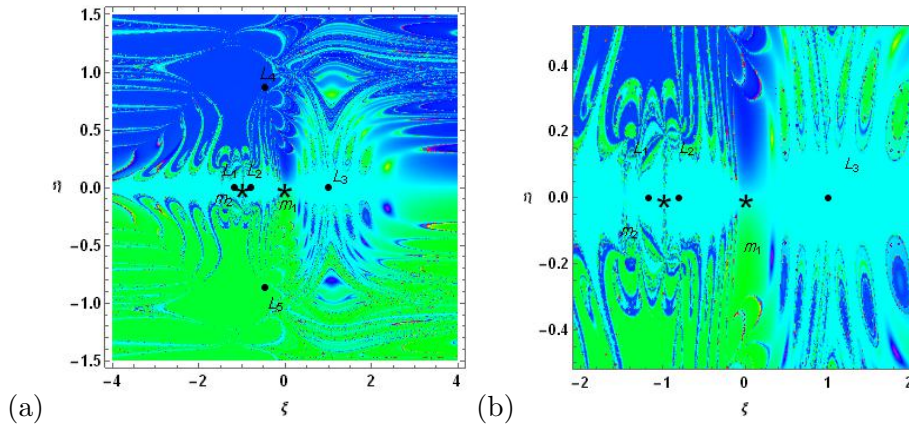


Figure 8: (a): The basin of attraction for the classical case. (b): Zoomed image of (a) near the primaries

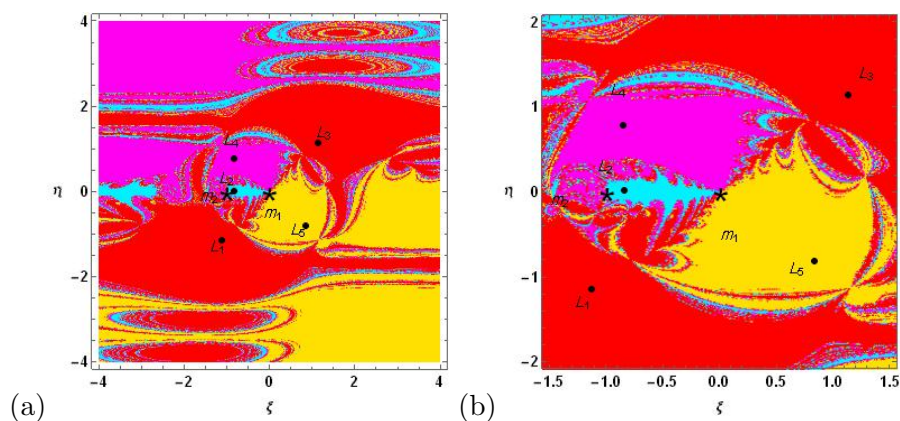


Figure 9: (a): The basin of attraction for the variable mass case. (b): Zoomed image of (a) near the primaries.

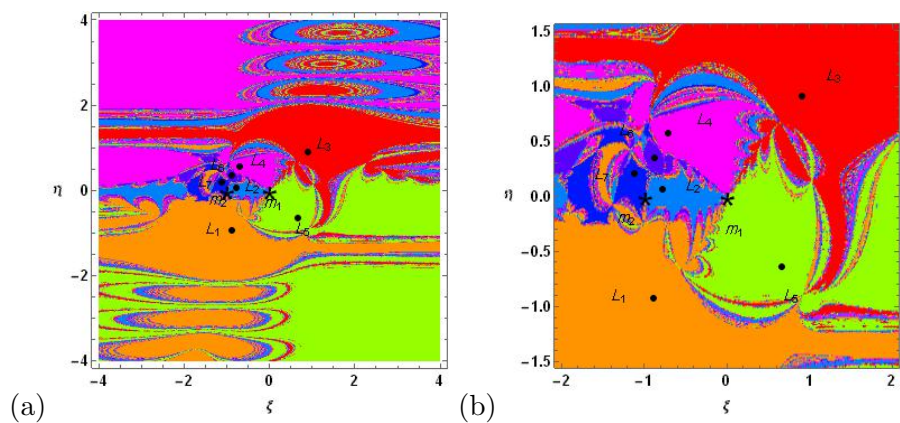


Figure 10: (a): The basin of attraction for the solar radiation pressure case. (b): Zoomed image of (a) near the primaries.

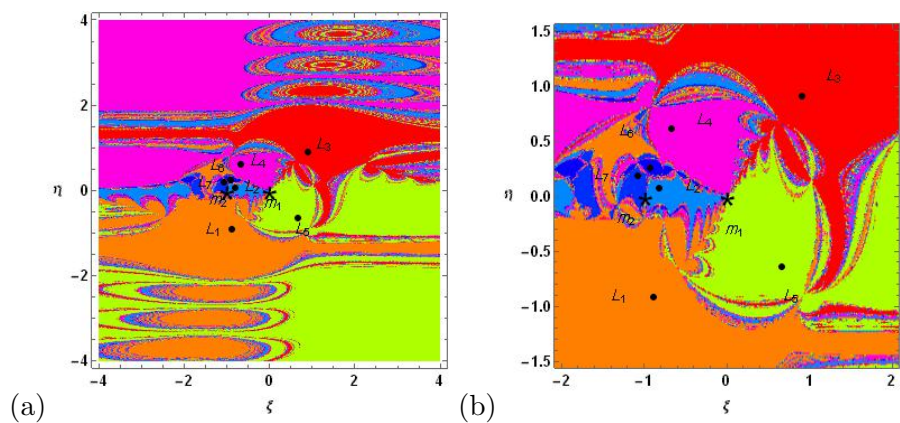


Figure 11: (a): The basin of attraction for the Albedo case. (b): Zoomed image of (a) near the primaries.

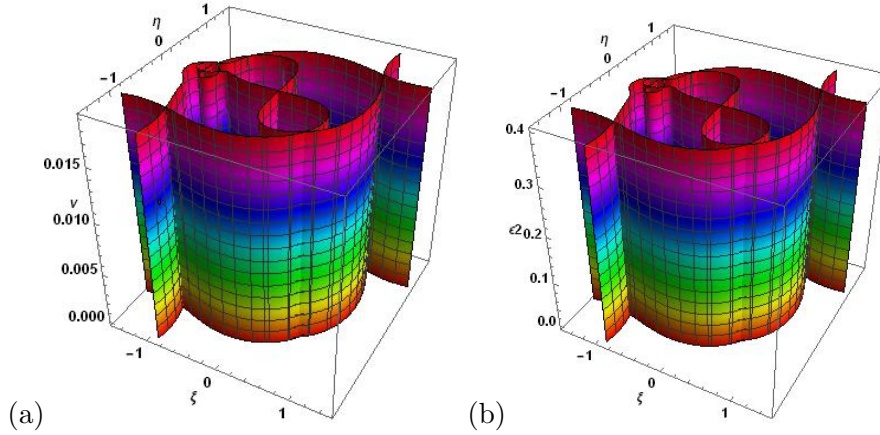


Figure 12: (a) Surfaces with variation of mass parameter. (b): Surfaces with variation of Albedo

5. Conclusion

We have investigated the effect of Albedo in the circular restricted three body problem with variable masses. The equations of motions when the masses of the primaries as well as the infinitesimal body varies, have been evaluated, which are different from the classical case by the variation parameters α_1 and k and the radiations effect ε_1 and ε_2 and also the expression for the variation of Jacobi integral constant have been evaluated. Here all the graphical works have been done by the software package Mathematica in four different cases. We have drawn the equilibrium points during in-plane and out of plane motions. During in-plane motion (Fig. 2), we found five equilibrium points in which three are collinear and two are triangular points in the classical case, in the variable mass case we found five equilibrium points, in the solar radiation and albedo case we found seven equilibrium points. During out-of-plane motions (Fig. 3, Fig. 4), we got three equilibrium points in all four cases. The black stars denote the location of the primaries. The periodic orbits have been drawn in these four cases (Fig. 5). In the classical case orbit is periodic but in the other three cases orbits are not periodic. The poincaré surface of sections have been determined in two phase spaces $(\xi - \xi')$ and $(\eta - \eta')$. In $(\xi - \xi')$ -plane (Fig. 6), the curves are shrinking and in $(\eta - \eta')$ -plane (Fig. 7-a, b), we got dumbbell shape in the classical case and in the other three cases, we got petal shapes. In this case the curves are shrinking.

The Newton-Raphson basins of attraction have been studied for four cases (Fig. 8, Fig. 9, Fig. 10, Fig. 11). For the classification of the equilibrium points on the $(\xi - \eta)$ -plane, we used color code. These points will be clearly visible in the zoomed part of all the figures near the primaries (ie. Fig. 8 b, Fig. 9 b, Fig. 10 b, Fig. 11 b). It is observed that the basins of attraction are shrinking. We also have drawn the surfaces with the variation of mass parameter (Fig. 12

(a)) and with the variation of Albedo (Fig. 12 (b)) and no significant difference have been found in both surfaces. Finally, we have examined the stability of the equilibrium points in the circular restricted three body problem under the effect of Albedo and found that all the equilibrium points are unstable.

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THE MAXIMUM PRINCIPLE OF TSALLIS ENTROPY IN A COMPLEX DOMAIN

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Abstract. In this paper, we extend the concept of Tsallis entropy in a complex domain. Based on this extension, we define some new classes of analytic functions (type Schwarz function). Other geometric properties are illustrated in the sequel. Our technique is delivered by the Maxwell Lemma and Jack Lemma.

Keywords: Fractional calculus, fractional entropy, analytic function, subordination and superordination.

1. Introduction

The theory of operators concerns with the study of different classes of functions on function spaces, establishing with differential operators and integral operators. The operators may be unfilled abstractly by their features, such as compact, bounded or closed operators. This study leads to two types of operators: linear operators and nonlinear operators. The study itself depends on the topological or geometrical properties of the spaces of functions. An amount of motivating physical schemes stands to entropic formulas that are additional commoner than the ordinary entropy. In 1988, Tsallis introduced a new type of fractional entropy. The Tsallis Entropy has been utilized beside with the Principle of maximum entropy to develop the Tsallis distribution. This entropy has been employed in many fields such as thermodynamics, chaos, statistical mechanics and information theory. For continuous probability distributions, the entropy is defined by the form:

$$\mathcal{T}_\mu[\phi] = \frac{1}{\mu - 1} \left(1 - \int (\phi(x))^\mu dx \right), \quad \mu \neq 1,$$

$$\text{or } \mathcal{T}_\mu[\phi](x) = \frac{1}{\mu - 1} \left(1 - \int_0^x (\phi(u))^\mu du \right),$$

where $\phi(x)$ is a probability density function.

Here, we extend the definition into the unit disk $U := \{z : |z| < 1\}$ by applying the analytic function $\varphi(z), z \in U$ (type Schwarz function $|\varphi(z)| < |z|$),

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where it is normalized by $\varphi(0) = 0, \varphi'(0) = 1$. The class of all normalized functions is denoted by $\mathcal{A}_\varphi$. For integrals of the second type defined by [1]-[3]:

$$\mathcal{J}_\mu[\varphi](z) = \int_0^z (\varphi'(t))^\mu dt = \frac{1 - \mathcal{J}_\mu[\varphi](z)}{\mu - 1} \quad \mu \neq 1, z \in U.$$

In this note, we suggest the differential entropy, because this case enables us to study many geometric properties and information regarding the classes of analytic functions. Receiving information about the properties of a function from properties of its derivatives indicates a significant part in many areas of mathematical analysis.

The function φ has enough information in each geometric classes; such as the starlikeness $\Re(z\varphi'(z)/\varphi(z)) > \lambda, \lambda \in [0, 1)$ and convexity $\Re(1 + z\varphi''(z)/\varphi'(z)) > \lambda, \lambda \in [0, 1)$, if the complex entropy satisfies $\Re(\mathcal{T}_\mu[\varphi](z)) > 0$, which is equivalent to $\Re(\rho(z)) > 0, \mu > 1$, where $\rho(z) := 1 - \mathcal{J}_\mu[\varphi](z)$. The set of information for each geometric class can be realized by the conclusion

$$\mathcal{I}_\varphi^\mu = \left\{ \varphi \in \mathcal{A}_\varphi : 0 < \Re\left(\mathcal{T}_\mu[\varphi](z)\right) < \frac{\mu}{\mu - 1}, z \in U, \mu \neq 1 \right\}.$$

Our discussion is based on the Maxwell Lemma as well as Jack Lemma respectively

Lemma 1.1 ([4]). *If ϵ is real and ρ is analytic in the unit disk, then*

$$\Re(\rho(z) + \epsilon z \rho'(z)/\rho(z)) > 0 \implies \Re(\rho(z)) > 0.$$

Lemma 1.2 ([5]). *Let $\psi(z)$ be analytic in U with $\psi(0) = 0$. Then if $|\psi(z)|$ approaches its maximization when $|z| = r$ at a point $z_0 \in U$, then $z_0\psi'(z_0) = \kappa\psi(z_0)$, where $\kappa \geq 1$ is a real number.*

Moreover, we need the subordination concept in the sequel, which is defined as follows: Assume that $\alpha(z)$ and $\beta(z)$ are two analytic functions in U . Then $\alpha(z)$ is said to be subordinate to $\beta(z)$ if there exists an analytic function $\psi(z)$ in U satisfying $\psi(0) = 0, |\psi(z)| < 1 (z \in U)$ and $\alpha(z) = \beta(\psi(z))$. This subordination is noted by $\alpha(z) \prec \beta(z), z \in U$.

2. The main finding

Our aim is to achieve the property of the set $\mathcal{I}_\varphi^\mu$.

Theorem 2.1. *Let $\varphi \in \mathcal{A}_\varphi, \mu \geq 2$, and $1 < \lambda < 2$. If φ satisfies*

$$(1) \quad 0 < \Re\left(1 + \frac{z\varphi''(z)}{\varphi'(z)}\right) < \frac{\lambda(2 - \lambda)}{2(1 + \lambda)},$$

then $\varphi \in \mathcal{I}_\varphi^\mu$. Moreover, φ is starlike in the open unit disk.

Proof. First, we show that $\Re(\mathcal{T}_\mu[\varphi](z)) < \frac{\mu}{\mu-1}$. Define a function ψ as follows:

$$\psi(z) = \mathcal{T}_\mu[\varphi](z) - \frac{1}{\mu-1}, \mu \neq 1$$

such that $\psi'(z) = (\mathcal{T}_\mu[\varphi](z))'$ and $\frac{z\psi'(z)}{\varphi'(z)} = \frac{\lambda(1-\psi(z))}{\lambda-\psi(z)}$, $\psi(z) \neq \lambda$. It is clear that ψ is analytic because φ is analytic in the open disk U . In addition, $\psi(0) = 0$, we need only to show that $|\psi(z)| < 1$. Also, implicitly it is containing λ and μ . A computation implies that

$$\begin{aligned} & \Re\left(1 + \frac{z\varphi''(z)}{\varphi'(z)}\right) \\ &= \Re\left(\frac{\lambda(1-\psi(z))}{\lambda-\psi(z)} - \frac{z\psi'(z)}{1-\psi(z)} + \frac{z\psi'(z)}{\lambda-\psi(z)}\right) \\ &= \Re\left(\frac{\lambda(1 - [\mathcal{T}_\mu[\varphi](z) - \frac{1}{\mu-1}])}{\lambda - [\mathcal{T}_\mu[\varphi](z) - \frac{1}{\mu-1}]} - \frac{z(\mathcal{T}_\mu[\varphi](z))'}{1 - [\mathcal{T}_\mu[\varphi](z) - \frac{1}{\mu-1}]} + \frac{z(\mathcal{T}_\mu[\varphi](z))'}{\lambda - [\mathcal{T}_\mu[\varphi](z) - \frac{1}{\mu-1}]}\right) \\ &= \Re\left(\frac{\lambda(1 + \frac{1}{\mu-1} - \mathcal{T}_\mu[\varphi](z))}{\lambda + \frac{1}{\mu-1} - \mathcal{T}_\mu[\varphi](z)} - \frac{z(\mathcal{T}_\mu[\varphi](z))'}{1 + \frac{1}{\mu-1} - \mathcal{T}_\mu[\varphi](z)} + \frac{z(\mathcal{T}_\mu[\varphi](z))'}{\lambda + \frac{1}{\mu-1} - \mathcal{T}_\mu[\varphi](z)}\right) \\ &< \frac{\lambda(2-\lambda)}{2(1+\lambda)}. \end{aligned}$$

In view of Lemma 1.2, there exists a complex number $z_0 \in U$ such that $\psi(z_0) = e^{i\theta}$ and $z_0\psi'(z_0) = \kappa\psi'(z_0) = \kappa e^{i\theta}$, $\kappa \geq 1$. Therefore, we obtain

$$\begin{aligned} & 1 + \frac{z_0\varphi''(z_0)}{\varphi'(z_0)} \\ &= \frac{\lambda(1 + \frac{1}{\mu-1} - \mathcal{T}_\mu[\varphi](z_0))}{\lambda + \frac{1}{\mu-1} - \mathcal{T}_\mu[\varphi](z_0)} - \frac{z_0(\mathcal{T}_\mu[\varphi](z_0))'}{1 + \frac{1}{\mu-1} - \mathcal{T}_\mu[\varphi](z_0)} + \frac{z_0(\mathcal{T}_\mu[\varphi](z_0))'}{\lambda + \frac{1}{\mu-1} - \mathcal{T}_\mu[\varphi](z_0)} \\ &= \frac{\lambda(1 - e^{i\theta})}{\lambda - e^{i\theta}} - \frac{\kappa e^{i\theta}}{1 - e^{i\theta}} + \frac{\kappa e^{i\theta}}{\lambda - e^{i\theta}} = \frac{\lambda + (\kappa - \lambda)e^{i\theta}}{\lambda - e^{i\theta}} - \frac{\kappa e^{i\theta}}{1 - e^{i\theta}}. \end{aligned}$$

Since

$$\Re\left(\frac{1}{1 - e^{i\theta}}\right) = \frac{1}{2}, \Re\left(\frac{1}{\lambda - e^{i\theta}}\right) = \frac{1}{2\lambda} + \frac{\lambda^2 - 1}{2\lambda(1 + \lambda^2 - 2\lambda \cos(\theta))}$$

then, we attain

$$\Re\left(1 + \frac{z_0\varphi''(z_0)}{\varphi'(z_0)}\right) = \frac{1+\lambda}{2} + \frac{(\lambda^2 - 1)(1 - \lambda + \kappa)}{2(1 + \lambda^2 - 2\lambda \cos \theta)}.$$

For $\lambda \in (1, 2)$, and $\kappa = 1$, we arrive at

$$\Re\left(1 + \frac{z_0\varphi''(z_0)}{\varphi'(z_0)}\right) \geq \frac{2-\lambda}{2(1+\lambda)} + \frac{(\lambda-1)(2-\lambda)}{2(1+\lambda)} = \frac{\lambda(2-\lambda)}{2(1+\lambda)},$$

which is a contradiction. Therefore, $|\psi(z)| < 1$, and $\Re(\mathcal{T}_\mu[\varphi](z)) < \frac{\mu}{\mu-1}$. Obviously,

$$(2) \quad \psi(z) = \mathcal{T}_\mu[\varphi](z) - \frac{1}{\mu-1} = \frac{\lambda(\frac{z\varphi'(z)}{\varphi(z)} - 1)}{\frac{z\varphi'(z)}{\varphi(z)} - \lambda} = \frac{\lambda(1 - \frac{z\varphi'(z)}{\varphi(z)})}{\lambda - \frac{z\varphi'(z)}{\varphi(z)}}.$$

Hence, we obtain $\frac{z\varphi'(z)}{\varphi(z)} \prec \frac{\lambda(1-z)}{\lambda-z}$. This conclude that φ is starlike in U . Now, we proceed to show that $\Re(\mathcal{T}_\mu[\varphi](z)) > 0$. Since φ is starlike, then we define a function $\rho(z)$ such that $\frac{z\varphi'(z)}{\varphi(z)} = \rho(z)$. If, we let $\Lambda(z) := \frac{z\varphi'(z)}{\varphi(z)} = \rho(z)$ then we receive

$$\frac{z\Lambda'(z)}{\Lambda(z)} + \Lambda(z) = 1 + \frac{z\varphi''(z)}{\varphi'(z)} = \rho(z) + \frac{z\rho'(z)}{\rho(z)}.$$

Hence, we obtain

$$\Re\left(1 + \frac{z\varphi''(z)}{\varphi'(z)}\right) = \Re\left(\rho(z) + \frac{z\rho'(z)}{\rho(z)}\right) > 0.$$

By applying Lemma 1.1, we have $\Re(\rho(z)) > 0$ and this implies that $\Re(\mathcal{T}_\mu[\varphi](z)) > 0$. This completes the proof. $\square$

Corollary 2.1. *Let the assumptions of Theorem 2.1 hold. Then $\int_0^z \frac{\varphi(t)}{t} dt$ is convex.*

3. Applications

We illustrate some examples of functions in the set $\mathcal{I}_\varphi^\mu$ by using Theorem 2.1.

Example 3.1. Consider the function

$$\varphi(z) = \left(\frac{1+\lambda}{2}\right)(1-z)^{-\frac{1-\lambda}{1+\lambda}-1} - \left(\frac{1+\lambda}{2}\right), \quad z \in U.$$

Obviously, $\varphi(0) = 0$ and $\varphi'(0) = 1$, where $\varphi'(z) = (1-z)^{-\frac{1-\lambda}{1+\lambda}-2}$, $z \in U$. Now, we have

$$\begin{aligned} 0 < 1 + \Re\left(\frac{z\varphi''(z)}{\varphi'(z)}\right) &= 1 + \left(2 + \frac{1-\lambda}{1+\lambda}\right)\Re\left(\frac{z}{1-z}\right), \quad z \in U \\ &= \frac{\lambda-1}{2(1+\lambda)} < \frac{\lambda(2-\lambda)}{2(1+\lambda)}, \quad z \rightarrow^- 1, \lambda \in (1, 2). \end{aligned}$$

Hence, in view of Theorem 2.1, $\varphi \in \mathcal{I}_\varphi^\mu$ and it is starlike in the open unit disk.

Example 3.2. Define the function

$$\varphi(z) = \left(\frac{2(\lambda+1)}{\lambda^2+2}\right)(1-z)^{-\frac{\lambda(\lambda-2)}{2(1+\lambda)}-1} - \left(\frac{2(\lambda+1)}{\lambda^2+2}\right), \quad z \in U.$$

Obviously, $\varphi(0) = 0$ and $\varphi'(0) = 1$, where $\varphi'(z) = (1-z)^{-\frac{\lambda(\lambda-2)}{2(1+\lambda)}-2}$. A calculation implies

$$\begin{aligned} 0 < 1 + \Re\left(\frac{z\varphi''(z)}{\varphi'(z)}\right) &= 1 + \left(2 + \frac{\lambda(\lambda-2)}{2(1+\lambda)}\right)\Re\left(\frac{z}{1-z}\right), \quad z \in U \\ &= \frac{\lambda(2-\lambda)}{4(1+\lambda)} < \frac{\lambda(2-\lambda)}{2(1+\lambda)}, \quad z \rightarrow^- 1, \lambda \in (1, 2). \end{aligned}$$

Thus, in view of Theorem 2.1, $\varphi \in \mathcal{I}_\varphi^\mu$ and it is starlike in the open unit disk.

Example 3.3. Assume the function

$$\varphi(z) = \left(\frac{\lambda+1}{1-\lambda}\right)(1-z)^{-\frac{\lambda-2}{1+\lambda}-1} - \left(\frac{\lambda+1}{1-\lambda}\right), \quad z \in U.$$

Plainly, $\varphi(0) = 0$ and $\varphi'(0) = 1$, where $\varphi'(z) = (1-z)^{-\frac{\lambda-2}{1+\lambda}-2}$. A computation yields

$$\begin{aligned} 0 < 1 + \Re\left(\frac{z\varphi''(z)}{\varphi'(z)}\right) &= 1 + \left(2 + \frac{\lambda-2}{1+\lambda}\right)\Re\left(\frac{z}{1-z}\right), \quad z \in U \\ &= \frac{2-\lambda}{2(1+\lambda)} < \frac{\lambda(2-\lambda)}{2(1+\lambda)}, \quad z \rightarrow^- 1, \lambda \in (1, 2). \end{aligned}$$

Thus, in view of Theorem 2.1, $\varphi \in \mathcal{I}_\varphi^\mu$ and it is starlike in the open unit disk.

Example 3.4. Consider the function

$$\varphi(z) = \left(\frac{2(1+\lambda)}{1-\lambda}\right)(1-z)^{-\frac{\lambda-2}{2(1+\lambda)}-1} - \left(\frac{2(1+\lambda)}{1-\lambda}\right), \quad z \in U.$$

clearly $\varphi(0) = 0$ and $\varphi'(0) = 1$, where $\varphi'(z) = (1-z)^{-\frac{\lambda-2}{2(1+\lambda)}-2}$. An operation gives

$$\begin{aligned} 0 < 1 + \Re\left(\frac{z\varphi''(z)}{\varphi'(z)}\right) &= 1 + \left(2 + \frac{\lambda-2}{2(1+\lambda)}\right)\Re\left(\frac{z}{1-z}\right), \quad z \in U \\ &= \frac{2-\lambda}{4(1+\lambda)} < \frac{\lambda(2-\lambda)}{2(1+\lambda)}, \quad z \rightarrow^- 1, \lambda \in (1, 2). \end{aligned}$$

Thus, in view of Theorem 2.1, $\varphi \in \mathcal{I}_\varphi^\mu$ and it is starlike in the open unit disk.

4. Conclusion

An extended definition of Tsallis entropy in a complex domain is imposed based on the set of analytic functions. This definition led to define a new class of analytic function, may call the class of geometric information of analytic functions. We introduced a sufficient condition of the existing. Moreover, we studied some geometric properties (see Theorem 2.1 and Corollary 2.1). Recently, the author introduced classes of integral operators in the open unit disk [6]-[9].

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POSITIVE IMPLICATIVE ENERGETIC SUBSETS OF BCK -ALGEBRAS

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Abstract. The notion of PI -energetic subsets in BCK -algebras is introduced, and several properties are investigated. Characterizations of I -energetic subsets and PI -energetic subsets are discussed, and conditions for a subset to be an I -energetic subset and a PI -energetic subset are provided. Relations between I -energetic subsets and PI -energetic subsets are considered, and conditions for an I -energetic subset to be a PI -energetic subset are given. Using PI -energetic subset, a positive implicative ideal is constructed. The condensational property of PI -energetic subset is established.

Keywords: S -energetic subset, I -energetic subset, PI -energetic subset

1. Introduction

Jun et al. [2] introduced the notions of S -energetic subsets and I -energetic subsets in BCK/BCI -algebras, and investigated several properties.

In this paper, we introduce the notion of PI -energetic subsets in BCK -algebras, and investigate several properties. We consider characterizations of I -energetic subsets and PI -energetic subsets. We provide conditions for a subset to be an I -energetic subset and a PI -energetic subset. We discuss relations between I -energetic subsets and PI -energetic subsets. We give conditions for an I -energetic subset to be a PI -energetic subset. Using PI -energetic subset, we make a positive implicative ideal. We establish the condensational property of PI -energetic subset.

2. Preliminaries

A BCK/BCI -algebra is an important class of logical algebras introduced by K. Iséki and was extensively investigated by several researchers.

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An algebra $(X; *, 0)$ of type $(2, 0)$ is called a *BCI-algebra* if it satisfies the following conditions

- (I) $(\forall x, y, z \in X) (((x * y) * (x * z)) * (z * y) = 0),$
- (II) $(\forall x, y \in X) ((x * (x * y)) * y = 0),$
- (III) $(\forall x \in X) (x * x = 0),$
- (IV) $(\forall x, y \in X) (x * y = 0, y * x = 0 \Rightarrow x = y).$

If a BCI-algebra X satisfies the following identity

$$(V) \quad (\forall x \in X) (0 * x = 0),$$

then X is called a *BCK-algebra*. Any *BCK/BCI-algebra* X satisfies the following conditions:

- (2.1) $(\forall x \in X) (x * 0 = x),$
- (2.2) $(\forall x, y, z \in X) (x \leq y \Rightarrow x * z \leq y * z, z * y \leq z * x),$
- (2.3) $(\forall x, y, z \in X) ((x * y) * z = (x * z) * y),$
- (2.4) $(\forall x, y, z \in X) ((x * z) * (y * z) \leq x * y)$

where $x \leq y$ if and only if $x * y = 0$. A nonempty subset S of a *BCK/BCI-algebra* X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$. A subset I of a *BCK/BCI-algebra* X is called an *ideal* of X if it satisfies

$$(2.5) \quad 0 \in I,$$

$$(2.6) \quad (\forall x \in X) (\forall y \in I) (x * y \in I \Rightarrow x \in I).$$

Every ideal I of a *BCK/BCI-algebra* X satisfies the following condition:

$$(2.7) \quad (\forall x, y \in X) (x \in I, y \leq x \Rightarrow y \in I).$$

A subset I of a *BCK-algebra* X is called a *positive implicative ideal* (see [3]) of X if it satisfies (2.5) and

$$(2.8) \quad (\forall x, y, z \in X) ((x * y) * z \in I, y * z \in I \Rightarrow x * z \in I).$$

Observe that every positive implicative ideal is an ideal, but the converse is not true (see [3]).

We refer the reader to the books [1, 3] for further information regarding *BCK/BCI-algebras*.

3. Energetic subsets

In what follows, let X denote a *BCK*-algebra unless otherwise specified.

Definition 3.1 ([2]). A nonempty subset A of X is said to be *S-energetic* if it satisfies

$$(3.1) \quad (\forall a, b \in X) (a * b \in A \Rightarrow \{a, b\} \cap A \neq \emptyset).$$

Definition 3.2 ([2]). A nonempty subset A of X is said to be *I-energetic* if it satisfies

$$(3.2) \quad (\forall x, y \in X) (y \in A \Rightarrow \{x, y * x\} \cap A \neq \emptyset).$$

Proposition 3.3. *Every I-energetic subset A of X which does not contain zero element 0 satisfies the following properties:*

$$(3.3) \quad (\forall x, y \in X) (x \leq y, x \in A \Rightarrow y \in A).$$

Proof. Let $x \leq y$ for $x \in A$ and $y \in X$. Then

$$\{y, 0\} \cap A = \{y, x * y\} \cap A \neq \emptyset.$$

Since $0 \notin A$, it follows that $y \in A$. □

Theorem 3.4. *For a nonempty subset A of X which does not contain zero element 0 , the following are equivalent:*

- (1) A is *I-energetic*.
- (2) $(\forall x, y, z \in X) (z * y \leq x, z \in A \Rightarrow \{x, y\} \cap A \neq \emptyset).$

Proof. Assume that A is *I-energetic* and let $z * y \leq x$ for $z \in A$ and $x, y \in X$. Then $\{y, z * y\} \cap A \neq \emptyset$ by (3.2). Hence $y \in A$ or $z * y \in A$. If $y \in A$, then clearly $\{x, y\} \cap A \neq \emptyset$. If $z * y \in A$, then $\{x, 0\} \cap A = \{x, (z * y) * x\} \cap A \neq \emptyset$ by (3.2). Since $0 \notin A$, it follows that $x \in A$ and so $\{x, y\} \cap A \neq \emptyset$.

Conversely, suppose that (2) is valid and let $x \in A$. Since $x * (x * y) \leq y$, it follows that $\{y, x * y\} \cap A \neq \emptyset$. Therefore A is *I-energetic*. □

Theorem 3.5. *If a nonempty subset A of X satisfies the following condition:*

$$(3.4) \quad (\forall x, y, z \in X) (x * y \in A \Rightarrow \{z, ((x * y) * y) * z\} \cap A \neq \emptyset),$$

then A is I-energetic.

Proof. Assume that $x \in A$. Then $x * 0 = x \in A$ by (2.1), which implies from (2.1) and (3.4) that

$$\{z, x * z\} \cap A = \{z, ((x * 0) * 0) * z\} \cap A \neq \emptyset$$

for all $z \in X$. Hence A is *I-energetic*. □

Lemma 3.6 ([2]). *Let A be a nonempty subset of X with $0 \notin A$. If A is I -energetic, then $X \setminus A$ is an ideal of X .*

Using Theorem 3.5 and Lemma 3.6, we have the following corollary.

Corollary 3.7. *Let A be a nonempty subset of X with $0 \notin A$. If A satisfies the condition (3.4), then $X \setminus A$ is an ideal of X .*

Definition 3.8. A nonempty subset A of X is said to be *positive implicative energetic* (briefly, *PI-energetic*) if it satisfies

$$(3.5) \quad (\forall x, y, z \in X) (x * z \in A \Rightarrow \{(x * y) * z, y * z\} \cap A \neq \emptyset).$$

Example 3.9. (1) Let $X = \{0, 1, 2, 3, 4\}$ be a *BCK*-algebra with the following Cayley table:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	2	0	2	0
3	3	1	1	0	1
4	4	4	4	4	0

It is routine to verify that $A := \{2, 4\}$ is a *PI*-energetic subset of X .

(2) Consider a *BCK*-algebra $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	1	0
2	2	2	0	2	0
3	3	3	3	0	3
4	4	4	4	4	0

Then $A_1 := \{1, 2, 4\}$, $A_2 := \{2, 3, 4\}$ and $A_3 := \{2, 4\}$ are *PI*-energetic subsets of X .

Example 3.10. Let $X = \{0, 1, 2, 3, 4\}$ be a *BCK*-algebra with the following Cayley table:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	2	0
3	3	1	3	0	1
4	4	4	4	4	0

The set $A := \{0, 3\}$ is not a *PI*-energetic subset of X since $3 * 2 = 3 \in A$ but

$$\{(3 * 1) * 2, 1 * 2\} \cap A = \emptyset.$$

Proposition 3.11. *Let A be a *PI*-energetic subset of X . If A does not contain zero element 0 , then the following implication is valid.*

$$(3.6) \quad (\forall x, y \in X) (x * y \in A \Rightarrow (x * y) * y \in A).$$

Proof. Let $x, y \in X$ be such that $x * y \in A$. Then

$$\{0, (x * y) * y\} \cap A = \{y * y, (x * y) * y\} \cap A \neq \emptyset$$

by (III) and (3.5). Since $0 \notin A$, it follows that $(x * y) * y \in A$. $\square$

We provide conditions for a subset to be *PI*-energetic.

Theorem 3.12. *Let A be a nonempty subset of X which does not contain zero element 0 . If A satisfies the condition (3.4), then A is *PI*-energetic.*

Proof. Let $x, z \in X$ be such that $x * z \in A$. If A is not *PI*-energetic, then there exists $y \in X$ such that

$$\{y * z, (x * y) * z\} \cap A = \emptyset.$$

Hence $y * z \in X \setminus A$ and $(x * y) * z \in X \setminus A$. Since

$$((x * z) * z) * (y * z) \leq (x * y) * z$$

and $X \setminus A$ is an ideal of X , it follows from (2.7) that $((x * z) * z) * (y * z) \in X \setminus A$. Thus

$$\{y * z, ((x * z) * z) * (y * z)\} \cap A = \emptyset,$$

which is contradictory to the condition (3.4). Therefore

$$\{(x * y) * z, y * z\} \cap A \neq \emptyset$$

whenever $x * z \in A$ for all $x, y, z \in X$, and so A is *PI*-energetic. $\square$

Theorem 3.13. *For any nonempty subset A of X , if $X \setminus A$ satisfies the condition (2.8), then A is *PI*-energetic.*

Proof. Assume that A is not *PI*-energetic. Then for any $x, z \in X$ with $x * z \in A$, there exists $y \in X$ such that $\{(x * y) * z, y * z\} \cap A = \emptyset$. It follows that

$$(x * y) * z \in X \setminus A \text{ and } y * z \in X \setminus A.$$

Since $X \setminus A$ satisfies the condition (2.8), we have $x * z \in X \setminus A$, that is, $x * z \notin A$. This is a contradiction, and so A is a *PI*-energetic subset of X . $\square$

Corollary 3.14. *For any nonempty subset A of X , if $X \setminus A$ satisfies the condition (2.8), then A is *I*-energetic.*

Corollary 3.15. *For any nonempty subset A of X , if $X \setminus A$ is a positive implicative ideal of X , then A is PI -energetic and so I -energetic.*

The converse of Corollary 3.15 is not true in general as seen in the following example.

Example 3.16. Consider a BCK -algebra $X = \{0, 1, 2, 3\}$ with the following Cayley table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	3	3	0

It is routine to verify that $A := \{0, 1\}$ is a PI -energetic subset of X , but $X \setminus A = \{2, 3\}$ is not a positive implicative ideal of X .

We consider relations between an I -energetic subset and a PI -energetic subset.

Theorem 3.17. *Every PI -energetic subset is I -energetic.*

Proof. Let A be a PI -energetic subset of X . Assume that $x \in A$. Since $x * 0 = x \in A$, it follows from (2.1) and (3.5) that $\{y, x*y\} \cap A = \{y*0, (x*y)*0\} \cap A \neq \emptyset$, for all $y \in X$. Hence A is an I -energetic subset of X . $\square$

The converse of Theorem 3.17 is not true as seen in the following example.

Example 3.18. Let $X = \{0, 1, 2, 3, 4\}$ be a BCK -algebra with the following Cayley table:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	1
2	2	2	0	0	2
3	3	2	1	0	3
4	4	4	4	4	0

The set $A := \{1, 2, 3\}$ is an I -energetic subset of X , but it is not PI -energetic since $3 * 2 = 1 \in A$ but $\{(3 * 2) * 2, 2 * 2\} \cap A = \emptyset$.

We know that the I -energetic subset $A := \{1, 2, 3\}$ in Example 3.18 does not satisfy the condition (3.6). We consider conditions for an I -energetic subset to be PI -energetic.

Theorem 3.19. *Let A be an I -energetic subset of X which does not contain zero element 0. If A satisfies the condition (3.6), then A is PI -energetic.*

Proof. Let $x, z \in X$ be such that $x * z \in A$. We first show that

$$(3.7) \quad (x * z) * (y * z) \in A \Rightarrow (x * y) * z \in A$$

for all $x, y, z \in X$. Let $(x * z) * (y * z) \in A$ and assume that $(x * y) * z \notin A$, that is, $(x * y) * z \in X \setminus A$. Since

$$(3.8) \quad ((x * (y * z)) * z) * z \leq (x * y) * z$$

and $X \setminus A$ is an ideal of X (see Lemma 3.6), we have $((x * (y * z)) * z) * z \in X \setminus A$ by (2.7). It follows from (2.3) and (3.6) that

$$(3.9) \quad (x * z) * (y * z) = (x * (y * z)) * z \in X \setminus A,$$

which is a contradiction. Hence (3.7) is valid. Now suppose that $\{y * z, (x * y) * z\} \cap A = \emptyset$ for some $y \in X$. Then $y * z \in X \setminus A$ and $(x * y) * z \in X \setminus A$. It follows from (3.7) that $(x * z) * (y * z) \in X \setminus A$ and so that $x * z \in X \setminus A$. This is a contradiction, and so $\{y * z, (x * y) * z\} \cap A \neq \emptyset$ whenever $x * z \in A$ for all $x, y, z \in X$. Therefore A is a *PI*-energetic subset of X . $\square$

Corollary 3.20. *Every PI-energetic subset A which does not contain zero element 0 satisfies the condition (3.7).*

Proposition 3.21. *Let A be a nonempty subset of X which does not contain zero element 0. If A satisfies the condition (3.7), then the condition (3.4) is valid.*

Proof. Let $x * y \in A$ for $x, y \in X$. Assume that $\{z, ((x * y) * y) * z\} \cap A = \emptyset$ for some $z \in X$. Then $z \in X \setminus A$ and $((x * z) * y) * y = ((x * y) * y) * z \in X \setminus A$. It follows from (III), (2.1), (2.3) and (3.7) that $(x * y) * z = ((x * z) * y) * (y * y) \in X \setminus A$. Since $X \setminus A$ is an ideal of X , we have $x * y \in X \setminus A$, which is a contradiction. Therefore (3.4) is valid. $\square$

Corollary 3.22. *Let A be an I-energetic subset of X which does not contain zero element 0. If A satisfies the condition (3.6), then A also satisfies the condition (3.4).*

For any nonempty subset A of X , consider a set

$$(3.10) \quad E_a := \{x \in X \mid x * a \in A\}.$$

Theorem 3.23. *Let A be an I-energetic subset of X . Then A is PI-energetic if and only if E_a is an I-energetic subset of X for all $a \in X$.*

Proof. Assume that A is *PI*-energetic and let $y \in E_a$. Then $y * a \in A$, and so $\{x * a, (y * x) * a\} \cap A \neq \emptyset$ by (3.5). Thus $x * a \in A$ or $(y * x) * a \in A$, that is, $x \in E_a$ or $y * x \in E_a$. It follows that $\{x, y * x\} \cap E_a \neq \emptyset$. Therefore E_a is an *I*-energetic subset of X for all $a \in X$.

Conversely, suppose that E_a is an I -energetic subset of X for all $a \in X$. Let $x * z \in A$ for $x, z \in X$. Then $x \in E_z$, and so $\{y, x * y\} \cap E_z \neq \emptyset$. It follows that $y \in E_z$ or $x * y \in E_z$ and so that $y * z \in A$ or $(x * y) * z \in A$. Hence $\{y * z, (x * y) * z\} \cap A \neq \emptyset$, and therefore A is PI -energetic. $\square$

Given a PI -energetic subset A of X , we provide a condition that $X \setminus A$ is a positive implicative ideal of X .

Theorem 3.24. *Let A be a nonempty subset of X with $0 \notin A$. If A is PI -energetic, then $X \setminus A$ is a positive implicative ideal of X .*

Proof. Since $0 \notin A$, we have $0 \in X \setminus A$. Let $x, y, z \in X$ be such that $y * z \in X \setminus A$ and $(x * y) * z \in X \setminus A$. Assume that $x * z \in A$. Then $\{(x * y) * z, y * z\} \cap A \neq \emptyset$ by (3.5), which implies that $y * z \in A$ or $(x * y) * z \in A$. This is a contradiction, and so $x * z \in X \setminus A$. This shows that $X \setminus A$ is a positive implicative ideal of X . $\square$

Corollary 3.25. *Let A be a nonempty subset of X with $0 \notin A$. If A is PI -energetic, then $X \setminus A$ is an ideal and hence a subalgebra of X .*

Let A be an I -energetic subset and B a PI -energetic subset of X such that $A \subseteq B$ and $0 \notin B$. Assume that $(x * z) * (y * z) \in A$ for $x, y, z \in X$. Then $(x * z) * (y * z) \in B$ and so $(x * y) * z \in B$ by Corollary 3.20. If $(x * y) * z \notin A$, then $((x * ((x * y) * z)) * y) * z = ((x * y) * z) * ((x * y) * z) = 0 \in X \setminus B$ by (2.3) and (III). Since $X \setminus B$ is a positive implicative ideal of X , it follows that $((x * z) * (y * z)) * ((x * y) * z) = ((x * ((x * y) * z)) * z) * (y * z) \in X \setminus B \subseteq X \setminus A$. Since $X \setminus A$ is an ideal of X , we have $(x * z) * (y * z) \in X \setminus A$ which is a contradiction. Therefore $(x * y) * z \in A$, and so A satisfies the condition (3.7). Consequently, A is a PI -energetic subset of X .

We summarize this as a theorem, so called the Condensational Property of PI -energetic set.

Theorem 3.26 (Condensational Property of PI -energetic set). *Let A and B be I -energetic subsets of X such that $A \subseteq B$ and $0 \notin B$. If B is PI -energetic, then so is A .*

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(ϵ) -KENMOTSU MANIFOLDS ADMITTING A SEMI-SYMMETRIC METRIC CONNECTION

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Abstract. The object of the present paper is to study some properties of quasi-conformal and concircular curvature tensor on (ϵ) -Kenmotsu manifolds with respect to a semi-symmetric metric connection.

Keywords: Quasi-conformally flat, ϕ -concircularly flat, (ϵ) -Kenmotsu manifold, semi-symmetric metric connection, η -Einstein manifold.

1. Introduction

Takahashi [19] studied Sasakian manifold with associated pseudo-Riemannian metrics and are known as (ϵ) -Sasakian manifolds. Bejancu and Duggal [3] shows the existence of (ϵ) -almost contact metric structures and provide an example of (ϵ) -Sasakian manifolds. Further investigation on this manifold was taken up by Xufeng and Xiaoli [23] and Rakesh kumar et al [13].

The study of manifolds with indefinite metric has a great relevance from the standpoint of geometrization of physics and relativity. Recently De and Sarkar [7] introduced indefinite metrics on Kenmotsu manifold, and are called as (ϵ) -Kenmotsu manifolds. Here they studied conformally flat, Weyl semisymmetric, ϕ -recurrent (ϵ) -Kenmotsu manifolds. Further, Singh et al [18] and Haseeb et al [9] established the relation between Levi-Civita connection and semi-symmetric metric connection and obtained the relation between curvature tensors of Levi-Civita connection and semi-symmetric metric connection in an (ϵ) -Kenmotsu manifold.

After the introduction of an idea of semi-symmetric linear connection in a differentiable manifold [8], Hayden [10] defined a semi-symmetric metric connection on a Riemannian manifold and this was further studied by Yano [24], Barua et al [2], De and Biswas [5].

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Quasi-conformal curvature tensor and concircular curvature tensor are the important curvature tensors from the differential geometry point of view. A quasi-conformal transformation is the one which transforms infinitesimal circles into infinitesimal ellipses, where as a concircular transformation transforms every geodesic circle of an n -dimensional Riemannian manifold M into a geodesic circle.

The quasi-conformal curvature tensor $\bar{C}$ [25] and concircular curvature tensor $\bar{Z}$ [26] with respect to the semi-symmetric metric connection are respectively given by

$$\begin{aligned} \bar{C}(X, Y)Z &= a\bar{R}(X, Y)Z + b[\bar{S}(Y, Z)X - \bar{S}(X, Z)Y \\ &\quad + g(Y, Z)\bar{Q}X - g(X, Z)\bar{Q}Y] \\ &\quad - \frac{\bar{r}}{n} \left\{ \frac{a}{(n-1)} + 2b \right\} [g(Y, Z)X - g(X, Z)Y], \end{aligned} \quad (1.1)$$

$$\bar{Z}(X, Y)Z = \bar{R}(X, Y)Z - \frac{\bar{r}}{n(n-1)} [g(Y, Z)X - g(X, Z)Y]. \quad (1.2)$$

The paper is organized as follows: Section 2 contains the preliminaries of (ϵ) -Kenmotsu manifold and a semi-symmetric metric connection on an (ϵ) -Kenmotsu manifold. Section 3 and 4 are devoted to the study of quasi-conformally flat and quasi-conformally semisymmetric (ϵ) -Kenmotsu manifold admitting a semi-symmetric metric connection. In the next section we study ϕ -concircularly flat (ϵ) -Kenmotsu manifold admitting a semi-symmetric metric connection and shown that ϕ -concircularly flat (ϵ) -Kenmotsu manifold admitting a semi-symmetric metric connection is an η -Einstein manifold. Further in section 6, we prove (ϵ) -Kenmotsu manifold admitting a semi-symmetric metric connection satisfying $\bar{Z}(X, Y) \cdot \bar{S}(U, W) = 0$ is an η -Einstein manifold.

2. Preliminaries

An almost contact structure on a n -dimensional differentiable manifold M is a triple (ϕ, ξ, η) , where ϕ is a tensor field of type $(1, 1)$, η is a 1-form and ξ is a vector field such that

$$\phi^2 = -I + \eta \circ \xi, \quad (2.1)$$

$$\eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0. \quad (2.2)$$

A differential manifold with an almost contact structure is called an almost contact manifold. An almost contact metric manifold is an almost contact manifold endowed with a compatible metric g . An almost contact metric manifold M is said to be an (ϵ) -almost contact metric manifold if

$$g(\xi, \xi) = \pm 1 = \epsilon, \quad (2.3)$$

$$\eta(X) = \epsilon g(X, \xi), \quad \text{rank}(\phi) = n - 1, \quad (2.4)$$

$$g(\phi X, \phi Y) = g(X, Y) - \epsilon \eta(X)\eta(Y), \quad \forall X, Y \in \Gamma(TM), \quad (2.5)$$

holds, where ξ is space-like or time-like but it is never a light like vector field. We say that (ϕ, ξ, η, g) is an (ϵ)-contact metric structure if we have

$$(2.6) \quad d\eta(X, Y) = g(X, \phi Y).$$

In this case, M is an (ϵ)-contact metric manifold. An (ϵ)-contact metric manifold is called an (ϵ)-Kenmotsu manifold [7] if

$$(2.7) \quad (\nabla_X \phi)Y = -g(X, \phi Y)\xi - \epsilon\eta(Y)\phi X,$$

holds, where ∇ is the Riemannian connection of g . An (ϵ)-almost contact metric manifold is a (ϵ)-Kenmotsu manifold if and only if

$$(2.8) \quad \nabla_X \xi = \epsilon(X - \eta(X)\xi).$$

The following conditions holds in an (ϵ)-Kenmotsu manifold [7]:

$$(2.9) \quad (\nabla_X \eta)(Y) = g(X, Y) - \epsilon\eta(X)\eta(Y),$$

$$(2.10) \quad \eta(R(X, Y)Z) = \epsilon\{g(X, Z)Y - g(Y, Z)X\},$$

$$(2.11) \quad R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \quad R(\xi, X)Y = \eta(Y)X - \epsilon g(X, Y)\xi,$$

$$(2.12) \quad S(X, \xi) = -(n-1)\eta(X), \quad Q\xi = -\epsilon(n-1)\xi,$$

$$(2.13) \quad S(\phi X, \phi Y) = S(X, Y) + \epsilon(n-1)\eta(X)\eta(Y).$$

A semi-symmetric metric connection $\bar{\nabla}$ on an n -dimensional (ϵ)-Kenmotsu manifold is given by [18],

$$(2.14) \quad \bar{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X, Y)\xi.$$

A relation between the curvature tensor $\bar{R}$, Ricci curvature $\bar{S}$ and the scalar curvature $\bar{r}$ of M with respect to semi-symmetric metric connection $\bar{\nabla}$ and R, S and r of M with respect to the Riemannian connection ∇ are given by

$$(2.15) \quad \begin{aligned} \bar{R}(X, Y)Z &= R(X, Y)Z + (2 + \epsilon)[g(X, Z)Y - g(Y, Z)X] \\ &+ (1 + \epsilon)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)]\xi \\ &+ (1 + \epsilon)\eta(Z)[\eta(Y)X - \eta(X)Y], \end{aligned}$$

$$(2.16) \quad \begin{aligned} \bar{S}(Y, Z) &= S(Y, Z) + [(\epsilon + 2)(\epsilon - n) + 2]g(Y, Z) \\ &+ (1 + \epsilon)(n - 2\epsilon)\eta(Y)\eta(Z), \end{aligned}$$

$$(2.17) \quad \bar{r} = r + n[(\epsilon + 2)(\epsilon - n) + 2] + \epsilon(1 + \epsilon)(n - 2\epsilon).$$

3. Quasi-conformally flat (ϵ)-Kenmotsu manifold admitting a semi-symmetric metric connection

Definition 3.1. An (ϵ)-Kenmotsu manifold admitting a semi-symmetric metric connection is said to be quasi-conformally flat if $\bar{C}(X, Y)Z = 0$.

Suppose (ϵ) -Kenmotsu manifold admitting a semi-symmetric metric connection is quasi-conformally flat. Then from (1.1) we have

$$(3.1) \quad \begin{aligned} a\bar{R}(X, Y)Z &= b[\bar{S}(X, Z)Y - \bar{S}(Y, Z)X + g(X, Z)\bar{Q}Y - g(Y, Z)\bar{Q}X] \\ &\quad + \frac{\bar{r}}{n} \left\{ \frac{a}{(n-1)} + 2b \right\} [g(Y, Z)X - g(X, Z)Y]. \end{aligned}$$

Taking an inner product of the above equation with ξ , we get

$$(3.2) \quad \begin{aligned} ag(R(X, Y)Z, \xi) &= a\epsilon\{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\} + b[\epsilon S(X, Z)\eta(Y) \\ &\quad - \epsilon S(Y, Z)\eta(X) + \{g(X, Z)Y - g(Y, Z)X\}\{4\epsilon - 3n\epsilon - 2n + 3\}] \\ &\quad + \epsilon\left\{\frac{\bar{r}}{n}\right\}\left\{\frac{a}{(n-1)} + 2b\right\}[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)]. \end{aligned}$$

Setting $X = \xi$ in (3.2) and then using (2.2), (2.3), (2.10) and (2.12), we obtain

$$(3.3) \quad S(Y, Z) = Ag(Y, Z) + B\eta(Y)\eta(Z),$$

where

$$\begin{aligned} A &= \frac{2a}{b} - (4 - 3n) + (2n - 3)\epsilon \\ &\quad + \left\{ \frac{r + n[(\epsilon + 2)(\epsilon - n) + 2] + \epsilon(1 + \epsilon)(n - 2\epsilon)}{n} \right\} \left\{ \frac{a}{(n-1)} + 2b \right\} \end{aligned}$$

and

$$\begin{aligned} B &= -\frac{2a}{b} + (4 - 3n)(1 + \epsilon) \\ &\quad + \left\{ \frac{\epsilon(r + n[(\epsilon + 2)(\epsilon - n) + 2] + \epsilon(1 + \epsilon)(n - 2\epsilon))}{n} \right\} \left\{ \frac{a}{(n-1)} + 2b \right\}. \end{aligned}$$

Thus, we can state the following;

Theorem 3.2. *A quasi conformally flat n -dimensional (ϵ) -Kenmotsu manifold admitting a semi-symmetric metric connection is an η -Einstein manifold.*

4. Quasi-conformally semisymmetric (ϵ) -Kenmotsu manifold admitting a semi-symmetric metric connection

Theorem 4.1. *A quasi-conformally semisymmetric (ϵ) -Kenmotsu manifold admitting a semi-symmetric metric connection is an η -Einstein manifold.*

Proof. Suppose (ϵ) -Kenmotsu manifold admitting a semi-symmetric metric connection satisfies

$$\bar{R}(\xi, Y) \cdot \bar{C}(U, V)W = 0.$$

Which implies

$$(4.1) \quad \begin{aligned} & \bar{R}(\xi, Y)\bar{C}(U, V)W - \bar{C}(\bar{R}(\xi, Y)U, V)W \\ & - \bar{C}(U, \bar{R}(\xi, Y)V)W - \bar{C}(U, V)\bar{R}(\xi, Y)W = 0. \end{aligned}$$

By virtue of (2.15), (4.1) takes the form

$$(4.2) \quad \begin{aligned} & (1 + \epsilon)[\eta(\bar{C}(U, V)W)Y - g(Y, \bar{C}(U, V)W)\xi - \eta(U)\bar{C}(Y, V)W \\ & + g(Y, U)\bar{C}(\xi, V)W - \eta(V)\bar{C}(U, Y)W + g(Y, V)\bar{C}(U, \xi)W \\ & - \eta(W)\bar{C}(U, V)Y + g(Y, W)\bar{C}(U, V)\xi]. \end{aligned}$$

Replacing Y by U in the above equation and then taking inner product with ξ , one can obtain

$$(4.3) \quad \begin{aligned} & (1 + \epsilon)[g(U, \bar{C}(U, V)W) - g(U, U)\eta(\bar{C}(\xi, V)W) \\ & - \epsilon g(U, V)g(\bar{C}(\xi, W)\xi, U) - \epsilon \eta(W)g(\bar{C}(U, V)\xi, U)]. \end{aligned}$$

Now putting $U = e_i$ in (4.3), where $\{e_i\}$, $i = 1, 2, \dots, n$ is an orthonormal basis of the tangent space at each point of the manifold and sum up with respect to i and using (1.1), (2.11), (2.12), (2.15), (2.16) and (2.17), we get

$$(4.4) \quad S(V, W) = \alpha g(V, W) + \beta \eta(V)\eta(W),$$

where $\alpha = -[(\epsilon + 2)(\epsilon - n) + 2] + \frac{1}{(a-b)}[\{a + b(n-2)\}\frac{\bar{r}}{n} - (n-1)\{(a + (n-1)b)(1 + \epsilon) - \frac{\bar{r}}{n(n-1)}(a + 2b(n-1))\}]$
and $\beta = -(\epsilon + 1)(n - 2\epsilon) + \frac{1}{(a-b)}[\{a + 2b(n-1)\}\frac{\bar{r}\epsilon}{n(n-1)} - \epsilon\{(a + (n-2)b)\}\{(n-1)(1 + \epsilon) + \frac{\bar{r}\epsilon}{n}\}]$.

Hence the proof □

5. ϕ -concircularly flat (ϵ)-Kenmotsu manifold admitting a semi-symmetric metric connection

Definition 5.1. An (ϵ)-Kenmotsu manifold admitting a semi-symmetric metric connection is said to be ϕ -concircularly flat if $\bar{Z}(\phi X, \phi Y)\phi Z = 0$.

Assume that (ϵ)-Kenmotsu manifold admitting a semi-symmetric metric connection is ϕ -concircularly flat. Then from (1.2) we have

$$(5.1) \quad \begin{aligned} & g(\bar{R}(\phi X, \phi Y)\phi Z, \phi W) \\ & = \frac{\bar{r}}{n(n-1)}[g(\phi Y, \phi Z)g(\phi X, \phi W) - g(\phi X, \phi Z)g(\phi Y, \phi W)]. \end{aligned}$$

Let $\{e_1, \dots, e_{n-1}, \xi\}$ be a local orthonormal basis of vector fields in M . By using the fact that $\{\phi e_1, \dots, \phi e_{n-1}, \xi\}$ is also a local orthonormal basis, if we

put $X = W = e_i$ in (5.1) and sum up with respect to i , $1 \leq i \leq n-1$, we get

$$(5.2) \quad \begin{aligned} & \sum_{i=1}^{n-1} g(\bar{R}(\phi e_i, \phi Y)\phi Z, \phi e_i) \\ &= \frac{\bar{r}}{n(n-1)} \sum_{i=1}^{n-1} [g(\phi Y, \phi Z)g(\phi e_i, \phi e_i) - g(\phi e_i, \phi Z)g(\phi Y, \phi e_i)]. \end{aligned}$$

It is easy to see that

$$(5.3) \quad \sum_{i=1}^{n-1} g(R(\phi e_i, Y)Z, \phi e_i) = S(Y, Z) + g(Y, Z),$$

$$(5.4) \quad \sum_{i=1}^{n-1} g(\phi e_i, Y)S(\phi e_i, Z) = S(Y, Z),$$

$$(5.5) \quad \sum_{i=1}^{n-1} g(\phi e_i, \phi e_i) = n-1,$$

$$(5.6) \quad \sum_{i=1}^{n-1} g(\phi e_i, Y)g(Z, \phi e_i) = g(Y, Z).$$

And by making use of (5.3)-(5.6), the equation (5.2) turns into

$$(5.7) \quad S(\phi Y, \phi Z) + g(\phi Y, \phi Z) = [2 + \epsilon + \frac{\bar{r}}{n(n-1)}](n-2)g(\phi Y, \phi Z).$$

Thus, by applying (2.5) and (2.13) into (5.7), we get

$$(5.8) \quad \begin{aligned} & S(Y, Z) \\ &= [-1 + \{2 + \epsilon + \frac{r+n[(\epsilon+2)(\epsilon-n)+2] + \epsilon(1+\epsilon)(n-2\epsilon)}{n(n-1)}\}(n-2)]g(Y, Z) \\ &\quad - [3 + \epsilon + \frac{r+n[(\epsilon+2)(\epsilon-n)+2] + \epsilon(1+\epsilon)(n-2\epsilon)}{n(n-1)}](n-2)\epsilon\eta(Y)\eta(Z). \end{aligned}$$

Hence, we have the following;

Theorem 5.2. *A ϕ -concurcularly flat (ϵ) -Kenmotsu manifold admitting a semi-symmetric metric connection is an η -Einstein manifold.*

6. (ϵ) -Kenmotsu manifold admitting a semi-symmetric metric connection satisfying $\bar{Z}(X, Y) \cdot \bar{S}(U, W) = 0$

Theorem 6.1. *An (ϵ) -Kenmotsu manifold admitting a semi-symmetric metric connection satisfying $\bar{Z}(X, Y) \cdot \bar{S}(U, W) = 0$ is an η -Einstein manifold.*

Proof. Let us assume that (ϵ)-Kenmotsu manifold admitting a semi-symmetric metric connection satisfies

$$(6.1) \quad \bar{Z}(X, Y) \cdot \bar{S}(U, W) = 0.$$

Which implies that

$$(6.2) \quad \bar{S}(\bar{Z}(\xi, Y)U, W) + \bar{S}(U, \bar{Z}(\xi, Y)W) = 0.$$

Using (1.2), (2.11) and (2.15) in (6.2), one can get

$$(6.3) \quad \begin{aligned} & -[(1+\epsilon) + \frac{\bar{r}}{n(n-1)}]g(Y, Z)\bar{S}(\xi, U) + [(1+\epsilon) + \frac{\bar{r}\epsilon}{n(n-1)}]\eta(Z)\bar{S}(Y, U) \\ & - [(1+\epsilon) + \frac{\bar{r}}{n(n-1)}]g(Y, U)\bar{S}(Z, \xi) + [(1+\epsilon) + \frac{\bar{r}\epsilon}{n(n-1)}]\eta(U)\bar{S}(Z, Y) = 0. \end{aligned}$$

Plugging $U = \xi$ in (6.3) and then taking into an account of (2.1), (2.3), (2.4), (2.12) and (2.16), we obtain

$$\begin{aligned} S(Y, Z) &= [-\{(2+\epsilon)(\epsilon-n)+2\} + (n-1)(1+\epsilon)]g(Y, Z) \\ &\quad - (1+\epsilon)(n-2\epsilon)\eta(Y)\eta(Z). \end{aligned} \quad \square$$

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A NEW CHARACTERIZATION OF $L_2(p)$ WITH $p \in \{19, 23\}$ BY NSE

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Abstract. Let G be a group. We denote by $nse(G) := \{m_k \mid k \in \pi_e(G)\}$, where $\pi_e(G)$ is the set of element orders of G and m_k is the number of elements of order k in G . In this paper, we characterize simple linear group $L_2(p)$ uniquely by set $nse(L_2(p))$ when $p \in \{19, 23\}$.

Keywords: Finite groups, numbers of elements with the same order, linear groups.

1. Introduction

Throughout this paper, all groups are finite and G denotes a group. We denote by $\pi(G)$ the set of prime divisors of $|G|$, $\pi_e(G)$ the set of element orders of G . We call G a simple K_n -group if G is simple with $|\pi(G)| = n$. If $r \in \pi(G)$, then P_r and $n_r(G)$ denote a Sylow r -subgroup of G and the number of Sylow r -subgroups of G , respectively. Let n be an integer. We denote by $\varphi(n)$ the Euler function of n .

The prime graph $\text{GK}(G)$ of a group G is defined as a graph with vertex set $\pi(G)$ and two distinct primes $p, q \in \pi(G)$ are adjacent if G contains an element of order pq . Moreover, the connected components of $\text{GK}(G)$ are denoted by $\pi_i, 1 \leq i \leq t(G)$, where $t(G)$ is the number of connected components of G . In particular, we define by π_1 the component containing the prime 2 for a group of even order.

Let $k \in \pi_e(G)$ and m_k be the number of elements of order k in G . Set $nse(G) := \{m_k \mid k \in \pi_e(G)\}$. In 1987, Thompson gave an example showing that not all groups can be characterized by $nse(G)$ and $|G|$: Let $G_1 = (C_2 \times C_2 \times$

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$C_2 \times C_2) \rtimes A_7$ and $G_2 = L_3(4) \rtimes C_2$ be two maximal subgroups of M_{23} . Then $nse(G_1) = nse(G_2)$ and $|G_1| = |G_2|$, but $G_1 \not\cong G_2$.

So it is an interesting topic to study class of groups G which can be characterized by the group order $|G|$ and the set $nse(G)$. Then authors of [8] proved that all simple K_4 -groups can be uniquely determined by $nse(G)$ and $|G|$. Also in [9], it is proved that $L_2(3) \cong A_4$, $L_2(4) \cong L_2(5) \cong A_5$ and $L_2(9) \cong A_6$ are uniquely determined by $nse(G)$. M. Khatami, B. Khosravi and Z. Akhlaghi ([4]) deduced that simple groups $L_2(p)$ is characterizable by uniquely the set $nse(L_2(p))$ if $p \in \{7, 8, 11, 13\}$. Recently, the authors of this paper ([6]) proved that simple linear groups $L_2(p)$ with $p \in \{17, 27, 29\}$ can be uniquely determined by its set $nse(L_2(p))$.

By using prime graph of a group as a new skill, in this present paper we prove that simple linear group $L_2(p)$ can be determined by exactly the set $nse(L_2(p))$ if $p \in \{19, 23\}$. Our main theorem is:

Theorem A. *Let G be a group and $p \in \{19, 23\}$. Then $G \cong L_2(p)$ if and only if $nse(G) = nse(L_2(p))$.*

Throughout this paper, we denote $n_r(G)$ by n_r and $m_k(G)$ by m_k if there is no confusion. Further unexplained notation is standard, readers may refer to [2].

2. Preliminaries

In this section we give some lemmas which will be used in the sequel.

Lemma 2.1. *Let G be a group. Let $k \in \pi_e(G)$ and m_k be the number of elements of order k in G . Then $\varphi(k) \mid m_k$. In particular, if there exists some odd integer $n \in nse(G)$, then $2 \in \pi(G)$ and $m_2 = n$. Moreover, $m_k \neq n$ for any $k \geq 3$.*

Proof. Let t be the number of the cyclic subgroups of order k of G , where $1 \neq k \in \pi_e(G)$. Then $m_k = t\varphi(k)$, yielding to $\varphi(k) \mid m_k$. In particular, if there exists some odd integer $n \in nse(G)$, then $m_2 = n$, since otherwise, $\varphi(k)$ is even for any $k \geq 3$. $\square$

Lemma 2.2 ([3]). *Let G be a group and m be a positive integer dividing $|G|$. If $L_m(G) = \{g \in G \mid g^m = 1\}$, then $m \mid |L_m(G)|$.*

Lemma 2.3 ([7, Lemma 2.3]). *Let G be a group and P be a cyclic Sylow p -subgroup of G . Assume further that $|P| = p^a$ and r is an integer such that $p^a r \in \pi_e(G)$. Then $m_{p^a r} = m_r(C_G(P))m_{p^a}$. In particular, $\varphi(r)m_{p^a} \mid m_{p^a r}$.*

Lemma 2.4 ([5]). *Let G be a group and $p \in \pi(G)$ be odd. Suppose that P is a Sylow p -subgroup of G and $n = p^s m$, where $(p, m) = 1$. If P is not cyclic and $s > 1$, then the number of elements of order n is always a multiple of p^s .*

Recall that G is a 2-Frobenius group, if there G has a normal series $1 \triangleleft H \triangleleft K \triangleleft G$ such that G/H and K are Frobenius groups with K/H and H as Frobenius kernels, respectively.

Lemma 2.5 ([11, Theorem]). *Let G be a group such that $t(G) \geq 2$. Then G has one of the following structures:*

- (a) G is a Frobenius or 2-Frobenius group.
- (b) G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $\pi(H) \cup \pi(K/H) \subseteq \pi_1$ and K/H is a nonabelian simple group.

Lemma 2.6 ([1, Theorem 2]). *If G is a 2-Frobenius group of even order, then $t(G) = 2$ and G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $\pi(K/H) = \pi_2$, $\pi(H) \cup \pi(G/K) = \pi_1$, $|G/K| \mid |\text{Aut}(K/H)|$, G/K and K/H are cyclic. In particular, $|G/K| < |K/H|$ and G is solvable.*

Lemma 2.7. *Let G be a simple group. If $\pi(G) = \{2, 3, 5, 19\}$, then $G \cong L_2(19)$; if $\pi(G) = \{2, 3, 11, 23\}$, then $G \cong L_2(23)$.*

Proof. It follows immediately from [10, Corollary 2] and [10, Corollary 4]. $\square$

Lemma 2.8 ([8, Lemma 2.5]). *Let G be a group with a normal series: $K \trianglelefteq L \trianglelefteq G$. Suppose that $P \in \text{Syl}_p(G)$, where $p \in \pi(G)$. If $P \leq L$ and $p \nmid |K|$, then the following statements hold:*

- (1) $|G : N_G(P)| = |L : N_L(P)|$, that is, $n_p(G) = n_p(L)$;
- (2) $|L/K : N_{L/K}(PK/K)|t = |G : N_G(P)| = |L : N_L(P)|$, that is, $n_p(L/K)t = n_p(G) = n_p(L)$ for some positive integer t . Furthermore, $|N_K(P)|t = |K|$.

3. Proof of Theorem A

Proof. The necessity of the theorem is obvious. We only prove the sufficiency and we will discuss it case by case.

Case 1. $nse(G) = \{1, 171, 684, 380, 1140, 360\} = nse(L_2(19))$.

By Lemma 2.1 and Lemma 2.2, we see that $2 \in \pi(G) \subseteq \{2, 3, 5, 7, 19\}$ and $m_2 = 171$. Let $\exp(P_2) = 2^s$. Then $s \leq 4$ since $\varphi(2^s) \mid m_{2^s}$ by Lemma 2.1. Moreover, $|P_2| \leq 2^6$ by Lemma 2.2. Next we show that $\pi(G) = \{2, 3, 5, 19\}$.

If $7 \in \pi(G)$, then Lemma 2.2 gives $m_7 = 1140$. More, $7^2 \notin \pi_e(G)$ since $\varphi(7^2) \mid m_{7^2}$ by Lemma 2.1. Therefore, $|P_7| = 7$ as $|P_7| \mid (1 + m_7)$ by Lemma 2.2. If $14 \in \pi_e(G)$, then $m_7 = m_{14}$ by Lemma 2.3, against $14 \mid (1 + m_2 + m_7 + m_{14})$ by Lemma 2.2. As a consequence, P_7 acts fixed-point-freely on $\Omega_2 := \{\text{all elements of order 2 in } G\}$, leading to $7 \mid m_2$. This contradiction forces $\pi(G) \subseteq \{2, 3, 5, 19\}$.

Similarly, if $3 \in \pi(G)$, then $m_3 = 380$ and $\exp(P_3) \leq 3^2$. Assume that $\exp(P_3) = 9$. Then $m_9 = 1140$ by Lemma 2.1 and Lemma 2.2, which implies that P_3 is cyclic by Lemma 2.4. In this case, $n_3 = m_9/\varphi(9) = 2 \cdot 5 \cdot 19$ and thus

$\pi(G) = \{2, 3, 5, 19\}$, we are done. On the other hand, if $\exp(P_3) = 3$, we also have $\pi(G) = \{2, 3, 5, 19\}$.

Assume now $5 \in \pi(G)$. By a similar argument, we obtain that $m_5 = 684$ and $\exp(P_5) \leq 5^2$. More, if $\exp(P_5) = 5^2$, then $m_{25} = 1140$ and $|P_5| = 5^2$ by Lemma 2.2. In this case, $n_5 = m_{25}/\varphi(25) = 3 \cdot 19$, implying $\pi(G) = \{2, 3, 5, 19\}$; if $\exp(P_5) = 5$, then $|P_5| = 5$, leading to $n_5 = m_5/\varphi(5) = 3^2 \cdot 19$, which also deduces that $\pi(G) = \{2, 3, 5, 19\}$, as we need.

As a consequence, we may consider G as a $\{2, 19\}$ -group. Note that G is not a 2-group since $\exp(P_2) \leq 2^4$ and $|nse(G)| = 6$. Thus $19 \in \pi(G)$. Lemma 2.2 shows that $19^2 \notin \pi_e(G)$. Moreover, $|P_{19}| \mid 19^2$. Assume that $|P_{19}| = 19$. Then $n_{19} = m_{19}/\varphi(19) = 2^2 \cdot 5$, a contradiction. Hence $|P_{19}| = 19^2$. Since $\pi_e(G) \subseteq \{1, 2, \dots, 2^4\} \cup \{19, 19 \cdot 2, \dots, 19 \cdot 2^3\}$, we obtain that

$$(1) \quad |G| = 2736 + 684k_1 + 380k_2 + 1140k_3 + 360k_4$$

with $\sum_{i=1}^4 k_i \leq 3$. That is,

$$(2) \quad 2^a \cdot 19^2 = 2^4 \cdot 3^2 \cdot 19 + 2^2 \cdot 3^2 \cdot 19k_1 + 2^2 \cdot 5 \cdot 19k_2 + 2^2 \cdot 3 \cdot 5 \cdot 19k_3 + 2^3 \cdot 3^2 \cdot 5k_4.$$

Except $2^3 \cdot 3^2 \cdot 5k_4$, 19 divides both sides of the equation (2), we get $k_4 = 0$ as $\sum_{i=1}^4 k_i \leq 3$. Thus,

$$(3) \quad 2^{a-2} \cdot 19 = 2^2 \cdot 3^2 + 3^2k_1 + 5k_2 + 3 \cdot 5k_3.$$

Recall that $a \leq 6$. Then the unique equation of (3) is $k_1 = k_4 = 0, k_2 = k_3 = 2$ with $a = 4$, against $\sum_{i=1}^4 k_i \leq 3$. Consequently, $\pi(G) = \{2, 3, 5, 19\}$, as required.

We claim that $|P_5| = 5$, $|P_{19}| = 19$, where P_5 and P_{19} are Sylow 5-subgroup and Sylow 19-subgroup of G , respectively. Further, $19r \notin \pi_e(G)$ and $5s \notin \pi_e(G)$ for $2 \neq s \in \pi(G)$.

Suppose that $|P_{19}| = 19^2$. If $|P_5| = 25$, then $\exp(P_5) = 5^2$. One has $n_5 = m_{25}/\varphi(25) = 3 \cdot 19$, implying $19 \mid |N_G(P_5)|$. Let $N_{19} \in \text{Syl}_{19}(N_G(P_5))$. By Sylow's Theorem, it follows that $P_5 \times N_{19} \leq G$, leading to $5^2 \cdot 19 \in \pi_e(G)$, against $\varphi(19)m_{5^2} \mid m_{5^2 \cdot 19}$ by Lemma 2.3. Assume that $|P_5| = 5$. In this case, $n_5 = m_5/\varphi(5) = 3^2 \cdot 19$, implying $19 \mid |N_G(P_5)|$. By Sylow's Theorem, $P_5 \times P_{19} \leq G$ with P_{19} a Sylow 19-subgroup of $N_G(P_5)$. Hence $5 \cdot 19 \in \pi_e(G)$, contrary to $\varphi(19)m_5 \mid m_{5 \cdot 19}$ by Lemma 2.3. Consequently, $|P_{19}| = 19$. Furthermore, if there exists some primer r such that $19r \in \pi_e(G)$, then $\varphi(r)m_{19} \mid m_{19r}$. This forces $r = 2$ and $m_{19} = m_{38}$. However, under this situation, $38 \nmid (1 + m_2 + m_{19} + m_{38}) = 892$, against Lemma 2.2.

Analogously, we may prove that $|P_5| = 5$. If not, $|P_5| = 5^2$ and P_5 acts fixed-point-freely on $\Omega_{19} := \{\text{all elements of order 19 of } G\}$. Hence, $25 \mid m_{19}$, a contradiction. As a result, $|P_5| = 5$. Further, and $5s \notin \pi_e(G)$ for any $2 \neq s \in \pi(G)$, as required.

As a result, $t(G) \geq 2$. Assume that $G = K \rtimes H$ is a Frobenius group. Then either $19 \mid |K|$ or $19 \mid |H|$. If the former holds, then $|K| = 19$ as $t(G) = 2$. In

this case, $m_{19} = 18 \in nse(G)$. This contradiction shows that $19 \mid |H|$. Let K_r be a Sylow r -subgroup of K with $r \in \pi(K)$ and H_{19} be a Sylow 19-subgroup of H . Then $K_r \rtimes H_{19}$ is also a Frobenius group, which implies that $19 \mid (|K_r| - 1)$, a contradiction if we consider the order of K .

Suppose then that G is a 2-Frobenius group, then G has the following normal series:

$$1 \triangleleft H \triangleleft K \triangleleft G$$

with $|K/H| = 19$ and $|G/K| \mid |\text{Aut}(K/H)|$ by Lemma 2.6. Hence $5 \mid |H|$ and thus $m_5 = 4$, a contradiction.

By Lemma 2.5, G has the following normal series:

$$1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$$

with K/H a non-solvable simple group and $\pi(H) \cup \pi(G/K) \subseteq \pi_1$. If K/H is a simple K_3 -group, then $19 \nmid |K/H|$ by [2], which leading to $19 \mid |H|$ or $19 \mid |G/K|$. Assume that the former holds. Notice that $5 \mid |K/H|$. Let K_5 be a Sylow 5-subgroup of K and H_{19} be a Sylow 19-subgroup of H . Then $K_5 \times H_{19} \leq G$, yielding to $5 \cdot 19 \in \pi_e(G)$, which is a contradiction to the argument above. Similarly, it will also deduce a contradiction if $19 \mid |G/H|$.

As a consequence, K/H is a non-solvable simple K_4 -group, yielding to $K/H \cong L_2(19)$ by Lemma 2.7. Moreover, it follows by Lemma 2.8 that $n_{19}(K/H) = n_{19}$ and $|N_H(P_{19})|t = |H|$. Since $m_{19} = m_{19}(K/H)$, we obtain $n_{19}(K/H) = n_{19}$, which follows that $t = 1$ and thus $H = N_H(P_{19})$. Note that $H \trianglelefteq G$. Then $HP_{19} = H \times P_{19} \leq G$. As $19r \notin \pi_e(G)$ for any prime $r \in \pi(G)$, we get $H = 1$. Hence $K \cong L_2(19)$ and $|G/K| \mid 2$. If $G = K.2$, then by [2], $m_2 = 361 \notin nse(G)$. This contradiction implies that $G \cong L_2(19)$.

Case 2. $nse(G) = \{1, 253, 506, 2760, 1012, 528\} = nse(L_2(23))$.

By Lemma 2.1 and Lemma 2.2, we see that $2 \in \pi(G) \subseteq \{2, 3, 11, 23, 1013\}$ and $m_2 = 253$. Moreover, $\exp(P_2) \leq 2^5$ and $|P_2| \leq 2^6$.

If $1013 \in \pi(G)$, then $m_{1013} = 1012$, leading to $|P_{1013}| = 1013$ and $P_{1013} \trianglelefteq G$, where P_{1013} is a Sylow 1013-subgroup of G . Let $\Omega_2 := \{\text{all elements of order 2 in } G\}$. We claim that P_{1013} acts fixed-point-freely on Ω . Otherwise, $2026 \in \pi_e(G)$ and $m_{2026} = 1012$ since $\varphi(2026) \mid m_{2026}$, against $m_{2026} \mid (1 + m_2 + m_{1013} + m_{2 \cdot 1013})$ by Lemma 2.2. As a consequence, $|P_{1013}|$ divides $|\Omega_2| = m_2$, again a contradiction. Therefore, $\pi(G) \subseteq \{2, 3, 11, 23\}$.

We assert that $\pi(G) = \{2, 3, 11, 23\}$. Assume first that $11 \in \pi(G)$. Then Lemma 2.2 gives $m_{11} = 2760$, $11^2 \notin \pi_e(G)$ and $|P_{11}| = 11$. By Sylow's Theorem, $|G : N_G(P_{11})| = m_{11}/\varphi(11) = 2^2 \cdot 3 \cdot 23$. This indicates that $\pi(G) = \{2, 3, 11, 23\}$, we are done.

Suppose then that $3 \in \pi(G)$. We see that $m_3 = 506$ by Lemma 2.2. Let $\exp(P_3) = 3^s$. Then $s \leq 3^2$ as $\varphi(3^s) \mid m_{3^s}$. If $\exp(P_3) = 3^2$, then $m_9 = 528$. Lemma 2.4 implies that P_3 is cyclic. Hence $n_3 = m_9/\varphi(9) = 2^3 \cdot 11$; if $\exp(P_3) = 3$, the same arguments gives $n_3 = 11 \cdot 23$. As a result, we always have

$11 \in \pi(G)$ if $3 \in \pi(G)$. By the argument in the previous paragraph, we also have $\pi(G) = \{2, 3, 11, 23\}$.

Consequently, we may consider G as a $\{2, 23\}$ -group. Note that $\exp(P_2) \leq 2^5$. If G is a 2-group, then $|G| = 5060$, which is not a power of 2, a contradiction. Thus $23 \in \pi(G)$. In this case, $m_{23} = 528$ and $\exp(P_{23}) \leq 23^2$. If $23^2 \in \pi_e(G)$, then $m_{23^2} = 506$ or 1012 , against $23^2 \mid (1 + m_{23} + m_{23^2}) = 1035$ or 1541 by Lemma 2.2. Hence $\exp(P_{23}) = 23$. It follows by Lemma 2.2 that $|P_{23}| \mid 23^2$. Assume that $|P_{23}| = 23$, then $n_{23} = m_{23}/\varphi(23) = 2^3 \cdot 3$, a contradiction. Hence $|P_{23}| = 23^2$. Moreover, $\pi_e(G) \subseteq \{1, 2, 2^2, 2^3, 2^4, 2^5\} \cup \{23, 2 \cdot 23, 2^2 \cdot 23, 2^3 \cdot 23, 2^4 \cdot 23\}$. We show that $2^5 \notin \pi_e(G)$. Otherwise, P_{23} act fixed-point-freely on $\Omega_{2^5} := \{\text{all elements of order } 2^5 \text{ in } G\}$, which implies that $|P_{23}| \mid m_{2^5}$, which is a contradiction since $m_{2^5} = 528$. Thus $\pi_e(G) \subseteq \{1, 2, 2^2, 2^3, 2^4\} \cup \{23, 2 \cdot 23, 2^2 \cdot 23, 2^3 \cdot 23, 2^4 \cdot 23\}$. Further,

$$(4) \quad |G| = 2^a \cdot 23^2 = 5060 + 506k_1 + 2760k_2 + 1012k_3 + 528k_4$$

with $\sum_{i=1}^4 k_i \leq 4$. That is,

$$(5) \quad 2^a \cdot 23^2 = 2^2 \cdot 5 \cdot 11 \cdot 23 + 2 \cdot 11 \cdot 23k_1 + 2^3 \cdot 3 \cdot 5 \cdot 23k_2 + 2^2 \cdot 11 \cdot 23k_3 + 2^4 \cdot 3 \cdot 11k_4.$$

Since 23 divides both sides of equation (5) except $2^4 \cdot 3 \cdot 11k_4$, we have $k_4 = 0$. Then the equation becomes

$$(6) \quad 2^{a-1} \cdot 23 = 2 \cdot 5 \cdot 11 + 11k_1 + 2^2 \cdot 3 \cdot 5k_2 + 2 \cdot 11k_3.$$

Recall that $a \leq 6$ and $\sum_{i=1}^4 k_i \leq 4$. It follows that $44 \leq 11k_1 + 2^2 \cdot 3 \cdot 5k_2 + 2 \cdot 11k_3 \leq 240$. Hence the unique possibility is $a = 4$, $k_1 = k_4 = 0$, $k_2 = 1$, $k_3 = 4$, contrary to $\sum_{i=1}^4 k_i \leq 4$. This contradiction shows that $\pi(G) = \{2, 3, 11, 23\}$.

Recall that $|P_{11}| = 11$. If there exists some prime r such that $11r \in \pi_e(G)$, then $(r-1) \cdot m_{11} \mid m_{11r}$ by Lemma 2.3, yielding that $r = 2$ and $m_{22} = 2760$. In this case, $22 \nmid (1 + m_2 + m_{11} + m_{22}) = 5774$, against Lemma 2.2. Therefore, $11 \parallel |G|$ and $11r \notin \pi_e(G)$ for any $r \in \pi(G)$.

Now we prove that $|P_{23}| = 23$. If not, $|P_{23}| = 23^2$. Moreover, P_{23} acts fixed-point-freely on $\Omega_{11} := \{\text{all elements of order 11 in } G\}$. As a consequence, $|P_{23}| \mid m_{11}$, which is a contradiction. Hence $|P_{23}| = 23$. Further, if there exists some prime r such that $23r \in \pi_e(G)$, then Lemma 2.3 shows that $r = 2$. In this case, $2 \cdot 23 \nmid (1 + m_2 + m_{23} + m_{2 \cdot 23})$, against Lemma 2.2. That is to say, for any prime $r \in \pi(G)$, we always have $11r \notin \pi_e(G)$, yielding that $t(G) \geq 3$.

By Lemma 2.5, we see that G is non-solvable and has the following normal series: $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ with K/H a simple K_4 -group and $\pi(H) \cup \pi(G/K) \subseteq \pi_1$.

We see from Lemma 2.7 that $K/H \cong L_2(23)$. Moreover, Lemma 2.8 gives $|N_H(P_{23})|t = |H|$ and $n_{23}(K/H)t = n_{23}$. Since $K/H \cong L_2(23)$, we have $m_{23}(K/H) = m_{23}$, yielding $n_{23}(K/H) = n_{23}(K)$ and thus $t = 1$. Further, $H \leq N_G(P_{23})$, and hence $H \times P_{23} \leq G$. Note that $23r \notin \pi_e(G)$. Then $H = 1$. As a result, $K \cong L_2(23)$ and $|G/K| \mid 2$. Assume that $G = K.2$, then by [2], $m_2 = 529 \notin nse(G)$. This contradiction indicates that $G = K \cong L_2(23)$. $\square$

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n -FOLD (POSITIVE) IMPLICATIVE FILTERS OF HOOPS**Chengfang Luo****Xiaolong Xin****School of Mathematics**Northwest University**Xi'an 710127**P.R. China**luochengfang1992@163.com**xlxin@nwu.edu.cn***Pengfei He***College of Mathematics and Information Science**Shaanxi Normal University**Xi'an, 710062**P.R. China**hepengf1986@126.com*

Abstract. The aim of this paper is to develop the filter theory of hoops. First, the concept of n -fold (positive) implicative (bounded) hoop and n -fold (positive) implicative filter are introduced. Also, the relationship between these filters and other filters on hoops are discussed. Moreover, conditions for a filter becomes a n -fold (positive) implicative filter are given. Finally, we consider some relations between these filters and quotient algebras.

Keywords: Hoop, filter, n -fold (positive) implicative filter, n -fold (positive) implicative (bounded) hoop.

1. Introduction

Hoops are naturally ordered commutative residuated integral monoids, which introduced by B.Bosbach in [6,7], and studied by J.R. Buchi and T.M. Owens in [5]. In the last few years, hoops theory was enriched with deep structure theorems [1,2,6,7]. Many of these results have a strong impact with fuzzy logic. Particularly, from the structure theorem of finite basic hoops [1] one obtains an elegant short proof of the completeness theorem for propositional basic logic, introduced by Hájek in [11]. The filter theory of the logical algebras plays an important role in studying algebras and the completeness of the corresponding non-classical logics. From a logical point of view, various filters correspond to various sets of provable formulas. Also, a filter is called a "deductive systems" [18]. Jun and Ko gave some characterizations of a deductive system and discussed how to generate a deductive system by a set on BL-algebras (cf. [13]).

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In [14] Jun and Ko introduced the notion of n -fold grisly deductive systems, and gave some conditions for a deductive system becomes a n -fold grisly deductive system on BL-algebras. Moreover, they constructed an extension property for n -fold grisly deductive system on BL-algebras. However, n -fold implicative filter and n -fold grisly deductive system are the same on BL-algebras [12]. In [12,17,19] the authors defined the notion of n -fold implicative filters, n -fold positive implicative filters, n -fold boolean filters, n -fold fantastic filters, n -fold normal filters in BL-algebras and studied the relation among many types of n -fold filters in BL-algebra. At the present, the filter theory of hoops has been widely studied and some important results have been obtained. In particular, in [16], some types of filters in hoops were introduced, and some of their characterizations and relations were presented. These filters include Boolean(implicative, positive implicative, prime and ultra, ect.)filters. M.Kondo proved that for any filter of a hoop, it is a positive implicative filter if and only if it is an implicative and fantastic filter.

The aim of this paper is to extend this research to hoops with the connection of the results obtaining in [10,15,16]. First, we give some basic results, which are needed in the rest papers. Next, we define n -fold(positive) implicative filter and n -fold (positive) implicative hoop and derive some of their characterizations. We obtain that if F is a n -fold (positive) implicative filter then it is also a filter. Finally, we discuss the relations between n -fold implicative filter and n -fold positive implicative filter.

2. Preliminaries

In this section, we recollect some definitions and results which will be used in the following:

Definition 2.1 ([3, 4, 6]). *An algebra $(H, \odot, \rightarrow, 1)$ of type $(2, 2, 0)$ is called a hoop if it satisfies the following conditions, for all $x, y, z \in H$:*

- (H1) $(H, \odot, 1)$ is a commutative monoid;
- (H2) $x \rightarrow x = 1$;
- (H3) $x \odot (x \rightarrow y) = y \odot (y \rightarrow x)$;
- (H4) $x \rightarrow (y \rightarrow z) = (x \odot y) \rightarrow z$.

In $(H, \odot, \rightarrow, 1)$, we define a binary relation " $\leq$ ", where $x \leq y$ iff $x \rightarrow y = 1$. One can easily prove the binary relation is partial order.

A hoop H is called bounded if there is a least element 0 in H , such that for all $x \in H$, $0 \leq x$. For all $x \in H$, $x^n = \underbrace{x \odot \dots \odot x}_{n \text{ times}}$ if $n > 0$ and $x^0 = 1$.

Proposition 2.2 ([6, 7]). *Let H be a hoop, then the following properties hold for all $x, y, z \in H$:*

- (1) $(H, \leq)$ is a $\wedge$ -semilattice with $x \wedge y = x \odot (x \rightarrow y)$;
- (2) $x \odot y \leq z$ if and only if $x \leq y \rightarrow z$;

- (3) $x \leq y$ if and only if $x \rightarrow y = 1$;
- (4) $x \odot y \leq x, y$;
- (5) If $x \leq y$, then $x \odot z \leq y \odot z$;
- (6) $x \leq y \rightarrow x$;
- (7) $x \leq (x \rightarrow y) \rightarrow y$;
- (8) $x \rightarrow 1 = 1$;
- (9) $1 \rightarrow x = x$;
- (10) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z) = (x \odot y) \rightarrow z$;
- (11) If $x \leq y$, then $z \rightarrow x \leq z \rightarrow y$, $y \rightarrow z \leq x \rightarrow z$;
- (12) $x \rightarrow y \leq (z \rightarrow x) \rightarrow (z \rightarrow y)$;
- (13) $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$.

Definition 2.3 ([9]). Let H be a hoop and F be a non-empty subset of H . F is called a filter of H if it satisfies the following conditions for every $x, y \in H$:

- (F1) $x, y \in F$ implies $x \odot y \in F$;
- (F2) $x \in F$ and $x \leq y$ imply $y \in F$.

Proposition 2.4 ([9]). Let H be a hoop and F be a subset of H . Then F is a filter if and only if the following conditions hold:

- (1) $1 \in F$;
- (2) $x, y \in H$, if $x, x \rightarrow y \in F$ then $y \in F$.

Theorem 2.5 ([1, 8]). Let F be a filter of H . Define $x\Theta_F y$ if and only if $x \rightarrow y \in F$ and $y \rightarrow x \in F$. Then Θ_F is a congruence relation on H .

The set of all congruence classes is denoted by $H/F \doteq \{x/F | x \in H\}$, where $x/F = \{y \in H | x\Theta_F y\}$. In the H/F we define the binary relation " $\leq$ ", where $x/F \leq y/F$ iff $x \rightarrow y \in F$. Then We can easily prove the binary relation is partial order and $(H/F, \otimes, \rightsquigarrow, 1_{H/F})$ is a hoop, where $1_{H/F} = 1/F$, $x/F \otimes y/F = (x \odot y)/F$, $x/F \rightsquigarrow y/F = (x \rightarrow y)/F$.

3. n -fold implicative filter

From now on, we denote the $(H, \odot, \rightarrow, 1)$ by H .

Definition 3.1. Let F be a subset of H and $n \in \mathbb{N}$. F is called a n -fold implicative filter of H if it satisfies:

- (1) $1 \in F$;
- (2) $x^n \rightarrow (y \rightarrow z) \in F$, $x^n \rightarrow y \in F$, imply $x^n \rightarrow z \in F$, for all $x, y, z \in H$.

Example 3.2. Let $H = \{0, a, b, 1\}$, define $\odot$ and $\rightarrow$ as follows,

$\odot$	0	a	b	1	$\rightarrow$	0	a	b	1
0	0	0	0	0	0	1	1	1	1
a	0	0	0	a	a	b	1	1	1
b	0	0	a	b	b	a	b	1	1
1	0	a	b	1	1	0	a	b	1

Then $(H, \odot, \rightarrow, 1)$ is a hoop, and it is clear that $\{1\}$ is a 3-fold implicative filter.

Theorem 3.3. *Any n -fold implicative filter is a filter of H .*

Proof. Let F be a n -fold implicative filter of H and $x, x \rightarrow y \in F$. Hence $1 \rightarrow x \in F$, $1 \rightarrow (x \rightarrow y) \in F$. But $1 = 1^n$, thus $y = 1 \rightarrow y \in F$, that is, F is a filter of H . $\square$

The following example shows that the converse of Theorem 3.3 is not true.

Example 3.4. Let $H = \{0, a, b, 1\}$, with $0 \leq a \leq b \leq 1$, define $\odot$ and $\rightarrow$ as follows:

$\odot$	0	a	b	1	$\rightarrow$	0	a	b	1
0	0	0	0	0	0	1	1	1	1
a	0	0	a	a	a	a	1	1	1
b	0	a	b	b	b	0	a	1	1
1	0	a	b	1	1	0	a	b	1

Then $(H, \odot, \rightarrow, 1)$ is a hoop, and it is clear that $F = \{b, 1\}$ is a filter of H , while it is not a 1-fold implicative filter. Since $a \rightarrow (a \rightarrow 0) = 1 \in F$ and $a \rightarrow a = 1 \in F$ but $a \rightarrow 0 = a \notin F$.

Theorem 3.5. *For any $a \in H$, $H(a) \doteq \{x \in H \mid a \leq x\}$ the following conditions are equivalent:*

- (1) $H(a)$ is a filter of H ;
- (2) $a \leq y$ whenever $a \leq x \rightarrow y$ and $a \leq x$ for all $x, y \in H$.

Proof. (1) $\Rightarrow$ (2) Let $H(a)$ be a filter of H and $a \leq x \rightarrow y$ and $a \leq x$, then $x \rightarrow y \in H(a)$ and $x \in H(a)$. Since $H(a)$ is a filter, $y \in H(a)$, that is, $a \leq y$.

(2) $\Rightarrow$ (1) Since $a \leq 1$, $1 \in H(a)$. If $a \leq x$, $a \leq (x \rightarrow y)$, then $x \in H(a)$, $x \rightarrow y \in H(a)$. By assumption $a \leq y$ and hence $y \in H(a)$. Hence $H(a)$ is a filter. $\square$

Theorem 3.6. *Let a be an element of H . If $H(a)$ is a n -fold implicative filter of H , then $a^{n+1} \rightarrow (x \rightarrow y) = 1$, $a^{n+1} \rightarrow x = 1$ imply $a^{n+1} \rightarrow y = 1$, for all $x, y \in H$.*

Proof. Let $H(a)$ be a n -fold implicative filter and $a^{n+1} \rightarrow (x \rightarrow y) = 1$, $a^{n+1} \rightarrow x = 1$. Since $a \rightarrow (a^n \rightarrow (x \rightarrow y)) = a^{n+1} \rightarrow (x \rightarrow y) = 1$, $a^n \rightarrow (x \rightarrow y) \in H(a)$. Similarly $a^n \rightarrow x \in H(a)$. Since $H(a)$ is a n -fold implicative filter, $a^n \rightarrow y \in H(a)$, that is, $a \leq a^n \rightarrow y$. Thus $a^{n+1} \rightarrow y = 1$. $\square$

Theorem 3.7. *Let F be a filter of H . Then for all $x, y, z \in H$, the following conditions are equivalent:*

- (1) F is a n -fold implicative filter of H ;
- (2) $x^n \rightarrow x^{2n} \in F$ for all $x \in H$;
- (3) $x^{n+1} \rightarrow y \in F$ implies $x^n \rightarrow y \in F$;
- (4) $x^n \rightarrow (y \rightarrow z) \in F$ implies $(x^n \rightarrow y) \rightarrow (x^n \rightarrow z) \in F$.

Proof. (1) $\Rightarrow$ (2) Let $x \in H$, hence $x^n \rightarrow (x^n \rightarrow x^{2n}) = 1 \in F$ and $x^n \rightarrow x^n = 1 \in F$. Since F is a n -fold implicative filter of H , we get $x^n \rightarrow x^{2n} \in F$.

(2) $\Rightarrow$ (1) Let $x, y, z \in H$ be such that $x^n \rightarrow (y \rightarrow z) \in F$, $x^n \rightarrow y \in F$. Since $(x^n \rightarrow (y \rightarrow z)) \odot (x^n \rightarrow y) \odot x^n \odot x^n = (x^n \wedge (y \rightarrow z)) \odot (x^n \wedge y) \leq (y \rightarrow z) \odot y = y \wedge z \leq z$ then $(x^n \rightarrow (y \rightarrow z)) \odot (x^n \rightarrow y) \leq x^{2n} \rightarrow z$. Since $x^n \rightarrow (y \rightarrow z) \in F$, $x^n \rightarrow y \in F$, we get $(x^n \rightarrow (y \rightarrow z)) \odot (x^n \rightarrow y) \in F$, and so $x^{2n} \rightarrow z \in F$. By Proposition 2.2, $x^n \rightarrow x^{2n} \leq (x^{2n} \rightarrow z) \rightarrow (x^n \rightarrow z)$. On the other hand, $x^{2n} \rightarrow z \in F$ and $x^n \rightarrow x^{2n} \in F$ then $x^n \rightarrow z \in F$. Hence F is a n -fold implicative filter of H .

(2) $\Rightarrow$ (3) Since (2) holds, F is a n -fold implicative filter of H . On the other hand, $x^{n+1} \rightarrow y = x^n \rightarrow (x \rightarrow y) \in F$ and $x^n \rightarrow x = 1 \in F$, hence $x^n \rightarrow y \in F$.

(3) $\Rightarrow$ (2) We have $x^{n+1} \rightarrow (x^{n-1} \rightarrow x^{2n}) = x^{2n} \rightarrow x^{2n} = 1 \in F$, hence by (3) $x^n \rightarrow (x^{n-1} \rightarrow x^{2n}) \in F$. But $x^{n+1} \rightarrow (x^{n-2} \rightarrow x^{2n}) = x^{2n-1} \rightarrow x^{2n} = x^n \rightarrow (x^{n-1} \rightarrow x^{2n}) \in F$, that is, $x^{n+1} \rightarrow (x^{n-2} \rightarrow x^{2n}) \in F$, and so $x^n \rightarrow (x^{n-2} \rightarrow x^{2n}) \in F$. By repeating the process n times we get $x^n \rightarrow (x^0 \rightarrow x^{2n}) = x^n \rightarrow (1 \rightarrow x^{2n}) = x^n \rightarrow x^{2n} \in F$.

(2) $\Rightarrow$ (4) Let $x^n \rightarrow (y \rightarrow z) \in F$, then by Proposition 2.2, $x^n \rightarrow (y \rightarrow z) \leq x^n \rightarrow ((x^n \rightarrow y) \rightarrow (x^n \rightarrow z)) = x^n \rightarrow (x^n \rightarrow ((x^n \rightarrow y) \rightarrow z)) = x^{2n} \rightarrow ((x^n \rightarrow y) \rightarrow z)$. Hence $x^{2n} \rightarrow ((x^n \rightarrow y) \rightarrow z) \in F$. By (2), we have $x^n \rightarrow x^{2n} \in F$, now by Proposition 2.2, $x^{2n} \rightarrow ((x^n \rightarrow y) \rightarrow z) \leq (x^n \rightarrow x^{2n}) \rightarrow (x^n \rightarrow ((x^n \rightarrow y) \rightarrow z))$. Then we get $(x^n \rightarrow y) \rightarrow (x^n \rightarrow z) = x^n \rightarrow ((x^n \rightarrow y) \rightarrow z) \in F$.

(4) $\Rightarrow$ (2) Since $x^n \rightarrow (x^n \rightarrow x^{2n}) = x^{2n} \rightarrow x^{2n} = 1 \in F$, by (4) we have $x^n \rightarrow x^{2n} = (x^n \rightarrow x^n) \rightarrow (x^n \rightarrow x^{2n}) \in F$. $\square$

Theorem 3.8. *If F is a n -fold implicative filter of H , then it is a $(n+1)$ -fold implicative filter.*

Proof. Let $x, y \in H$ be such that $x^{n+2} \rightarrow y \in F$. By Proposition 2.2, $x^{n+1} \rightarrow (x \rightarrow y) = x^{n+2} \rightarrow y$ and since F is a n -fold implicative filter, by Theorem 3.7 $x^n \rightarrow (x \rightarrow y) \in F$. Hence $x^{n+1} \rightarrow y \in F$, that is, F is a $(n+1)$ -fold implicative filter. $\square$

The following example shows that the converse of Theorem 3.8 is not true.

Example 3.9. In Example 3.2, $\{1\}$ is a 3-fold implicative filter, but $\{1\}$ is not a 2-fold implicative filter. Since $b^3 \rightarrow 0 = 1 \in \{1\}$ and $b^2 \rightarrow 0 = b \neq 1$.

Theorem 3.10. *Let F and G be filters of H such that $F \subseteq G$. If F is a n -fold implicative filter, then G is also a n -fold implicative filter.*

Proof. Let F be a n -fold implicative filter of H . Then by Theorem 3.7 $x^n \rightarrow x^{2n} \in F$ for all $x \in H$, and so $x^n \rightarrow x^{2n} \in G$ for all $x \in H$. Hence G is a n -fold implicative filter. $\square$

Definition 3.11. *A hoop H is called a n -fold implicative hoop if it satisfies $x^{n+1} = x^n$, for all $x \in H$.*

Theorem 3.12. *If H is a n -fold implicative hoop, then it is a $(n + k)$ -fold implicative hoop, where $k \in \mathbb{Z}$, $k \geq 0$.*

Proof. Since H is a n -fold implicative hoop, $x^{n+1} = x^n$ for all $x \in H$. But $x^{n+2} = x^{n+1} \odot x = x^n \odot x = x^{n+1}$ and hence H is a $(n + 1)$ -fold implicative hoop. By repeating the process k times, we get H is a $(n + k)$ -fold implicative hoop. $\square$

The following example shows that the converse of Theorem 3.12 is not true.

Example 3.13. In Example 3.2, it is clear that H is a 3-fold implicative hoop, while it is not a 2-fold implicative hoop, since $b^3 \neq b^2$.

Corollary 3.14. *In a n -fold implicative hoop, the concepts of filter and n -fold implicative filter coincide.*

Proof. It follows from Theorem 3.3 and the definition of n -fold implicative hoop. $\square$

Theorem 3.15. *The following conditions are equivalent:*

- (1) H is a n -fold implicative hoop;
- (2) Every filter of H is a n -fold implicative filter;
- (3) $\{1\}$ is a n -fold implicative filter;
- (4) $x^n = x^{2n}$ for all $x \in H$;
- (5) H/F is a n -fold implicative hoop.

Proof. (1) $\Rightarrow$ (2) It is clear by the Corollary 3.14.

(2) $\Rightarrow$ (3) This is clear.

(3) $\Rightarrow$ (1) Let $\{1\}$ be a n -fold implicative filter. Since $x^n \rightarrow (x^n \rightarrow x^{n+1}) = x^{2n} \rightarrow x^{n+1} = 1 \in \{1\}$ and $x^n \rightarrow x^n = 1 \in \{1\}$, we get $x^n \rightarrow x^{n+1} \in \{1\}$, that is, $x^n = x^{n+1}$. Hence H is a n -fold implicative hoop.

(1) $\Rightarrow$ (4) Let H be a n -fold implicative hoop, hence $x^{n+1} = x^n$ for all $x \in H$.

We have $x^{n+2} = x^{n+1} \odot x = x^n \odot x = x^{n+1} = x^n$. By repeating the process n times, we get $x^n = x^{2n}$ for all $x \in H$.

(4) $\Rightarrow$ (1) Since $x^n = x^{2n}$ for all $x \in H$, we have $x = x^2$, $x^3 = x^2 \odot x = x^2 \odot x^2 = (x^2)^2 = x^2 = x$, thus $x^3 = x^2 = x$. By the principle of mathematical induction, we get $x^{n+1} = x^n$. Hence H is a n -fold implicative hoop.

(1) $\Rightarrow$ (5) By Theorem 2.5 and Definition 3.11, it is clear.

(5) $\Rightarrow$ (1) Let H/F be a n -fold implicative hoop. For all $x \in H$, we have $x^{n+1}/F = x^n/F$, then $x^n/F \rightsquigarrow x^{n+1}/F = (x^n \rightarrow x^{n+1})/F = 1/F = 1_{H/F}$ and so $x^n \rightarrow x^{n+1} = 1$, hence H is a n -fold implicative hoop. $\square$

4. n -fold positive implicative filter

In this section, we introduce the notion of n -fold positive implicative filter and n -fold positive implicative bounded hoop. Also, we derive some of their characterizations.

Definition 4.1. Let F be a subset of H and $n \in \mathbb{N}$. F is called a n -fold positive implicative filter of H if it satisfies:

- (1) $1 \in F$;
- (2) $x \rightarrow ((y^n \rightarrow z) \rightarrow y) \in F$ and $x \in F$ imply $y \in F$, for all $x, y, z \in H$.

Example 4.2. In Example 3.2, $\{1\}$ is a 3-fold positive implicative filter. But $\{1\}$ is not a positive implicative filter, since $1 \rightarrow ((b \rightarrow 0) \rightarrow b) = 1 \in F$, while $b \notin F$.

Theorem 4.3. Every n -fold positive implicative filter of H is a filter of H .

Proof. Let F be a n -fold positive implicative filter of H , and $x, x \rightarrow y \in F$. We have that $x \rightarrow ((y^n \rightarrow 1) \rightarrow y) = x \rightarrow y \in F$ and so $x \rightarrow ((y^n \rightarrow 1) \rightarrow y) \in F$. Since $x \in F$ and F is a n -fold positive implicative filter of H , we get $y \in F$. Hence F is a filter. $\square$

The following example shows that the converse of Theorem 4.3 is not true.

Example 4.4. Let $H = \{0, a, b, c, 1\}$, define $\odot$ and $\rightarrow$ as follows,

$\odot$	0	a	b	c	1	$\rightarrow$	0	a	b	c	1
0	0	0	0	0	0	0	1	1	1	1	1
a	0	a	c	c	a	a	0	1	b	b	1
b	0	c	b	c	b	b	0	a	1	a	1
c	0	c	c	c	c	c	0	1	1	1	1
1	0	a	b	c	1	1	0	a	b	c	1

Then $(H, \odot, \rightarrow, 0, 1)$ is a hoop. It is clear that $F = \{b, 1\}$ is a filter of H , but it is not 2-fold positive implicative filter, since $1 \rightarrow ((a^2 \rightarrow 0) \rightarrow a) = 1 \in F$ and $a \notin F$.

Theorem 4.5. Let F be a filter of H , the following conditions are equivalent for all $x, y, z \in H$:

- (1) F is a n -fold positive implicative filter;
- (2) $(x^n \rightarrow y) \rightarrow x \in F$ implies $x \in F$, for all $x, y \in H$;
- (3) $(x^n \rightarrow 0) \rightarrow x \in F$ implies $x \in F$, for all $x \in H$.

Proof. (1) $\Rightarrow$ (2) Let F be a n -fold positive implicative filter of H and $(x^n \rightarrow y) \rightarrow x \in F$. Since $1 \rightarrow ((x^n \rightarrow y) \rightarrow x) = (x^n \rightarrow y) \rightarrow x$, we get $1 \rightarrow ((x^n \rightarrow y) \rightarrow x) \in F$. Now that $1 \in F$, thus $x \in F$.

(2) $\Rightarrow$ (3) This is clear.

(3) $\Rightarrow$ (1) Let $x \rightarrow ((y^n \rightarrow z) \rightarrow y) \in F$ and $x \in F$. Since F is a filter, we have $(y^n \rightarrow z) \rightarrow y \in F$. Since $0 \leq z$, by Proposition 2.2 $y^n \rightarrow 0 \leq y^n \rightarrow z$ and so $(y^n \rightarrow z) \rightarrow y \leq (y^n \rightarrow 0) \rightarrow y$. Hence $(y^n \rightarrow 0) \rightarrow y \in F$ and so by assumption we get $y \in F$. Therefore F is a n -fold positive implicative filter. $\square$

Theorem 4.6. If F is a n -fold positive implicative filter of H , then it is a $(n+1)$ -fold positive implicative filter.

Proof. Let F be a n -fold positive implicative filter and $x \in H$, $(x^{n+1} \rightarrow 0) \rightarrow x \in F$. Since $x^{n+1} \leq x^n$, we have $x^n \rightarrow 0 \leq x^{n+1} \rightarrow 0$ and so $(x^{n+1} \rightarrow 0) \rightarrow x \leq (x^n \rightarrow 0) \rightarrow x$. Since F is a filter, we get $(x^n \rightarrow 0) \rightarrow x \in F$. Note that F is a n -fold positive implicative filter we have $x \in F$ and so F is a $(n+1)$ -fold positive implicative filter. $\square$

The following example shows that the converse of Theorem 4.6 is not true.

Example 4.7. In Example 3.2, $\{1\}$ is a 3-fold positive implicative filter, while $\{1\}$ is not a 2-fold positive implicative filter, since $(b^2 \rightarrow 0) \rightarrow b = 1 \in F$ and $b \neq 1$.

Theorem 4.8. *Every n -fold positive implicative filter of H is a n -fold implicative filter of H .*

Proof. Let F be a n -fold positive implicative filter of H . By Theorem 4.3, F is a filter of H . Let $x, y \in H$ be such that $x^{n+1} \rightarrow y \in F$. Then by Proposition 2.2, we have

$$\begin{aligned}
 (x^{n+1} \rightarrow y)^n \rightarrow (x^n \rightarrow y) &= ((x^{n+1} \rightarrow y)^{n-1} \odot (x^{n+1} \rightarrow y)) \rightarrow (x^n \rightarrow y) \\
 &= (x^{n+1} \rightarrow y)^{n-1} \rightarrow ((x^{n+1} \rightarrow y) \rightarrow (x^n \rightarrow y)) \\
 &= (x^{n+1} \rightarrow y)^{n-1} \rightarrow ((x^{n+1} \rightarrow y) \rightarrow ((x^{n-1} \rightarrow (x \rightarrow y)))) \\
 &= (x^{n+1} \rightarrow y)^{n-1} \rightarrow ((x^{n-1} \rightarrow ((x^{n+1} \rightarrow y) \rightarrow (x \rightarrow y)))) \\
 &= (x^{n+1} \rightarrow y)^{n-1} \rightarrow (x^{n-1} \rightarrow ((x \rightarrow (x^n \rightarrow y)) \rightarrow (x \rightarrow y))) \\
 &\geq (x^{n+1} \rightarrow y)^{n-1} \rightarrow (x^{n-1} \rightarrow (x^n \rightarrow y) \rightarrow y) \\
 &= (x^{n+1} \rightarrow y)^{n-1} \rightarrow ((x^n \rightarrow y) \rightarrow (x^{n-1} \rightarrow y)) \\
 &= (x^n \rightarrow y)^2 \rightarrow ((x^{n+1} \rightarrow y)^{n-2} \rightarrow (x^{n-1} \rightarrow y)).
 \end{aligned}$$

We show that $(x^{n+1} \rightarrow y)^n \rightarrow (x^n \rightarrow y) \geq (x^n \rightarrow y)^2 \rightarrow ((x^{n+1} \rightarrow y)^{n-2} \rightarrow (x^{n-2} \rightarrow y))$.

Note that

$$\begin{aligned}
 (x^n \rightarrow y) \odot (x^{n+1} \rightarrow y)^{n-1} \odot x^{n-1} \\
 &= (x^n \rightarrow y) \odot (x^{n+1} \rightarrow y)^{n-2} \odot (x^{n+1} \rightarrow y) \odot x \odot x^{n-2} \\
 &= (x^n \rightarrow y) \odot (x^{n+1} \rightarrow y)^{n-2} \odot x^{n-2} \odot x \odot (x^{n+1} \rightarrow y).
 \end{aligned}$$

Since $x^{n+1} \rightarrow y \leq x^{n+1} \rightarrow y = x \rightarrow (x^n \rightarrow y)$, then $x \odot (x^{n+1} \rightarrow y) \leq x^n \rightarrow y$, we get $(x^n \rightarrow y) \odot (x^{n+1} \rightarrow y)^{n-1} \odot x^{n-1} \leq (x^n \rightarrow y)^2 \odot (x^{n+1} \rightarrow y)^{n-2} \odot x^{n-2}$.

Hence $((x^n \rightarrow y)^2 \odot (x^{n+1} \rightarrow y)^{n-2} \odot x^{n-2}) \rightarrow y \leq ((x^n \rightarrow y) \odot (x^{n+1} \rightarrow y)^{n-1} \odot x^{n-1}) \rightarrow y$ and so $(x^n \rightarrow y)^2 \rightarrow ((x^{n+1} \rightarrow y)^{n-2} \rightarrow (x^{n-2} \rightarrow y)) \leq (x^n \rightarrow y) \rightarrow ((x^{n+1} \rightarrow y)^{n-1} \rightarrow (x^{n-1} \rightarrow y))$.

Now we have $(x^{n+1} \rightarrow y)^n \rightarrow (x^n \rightarrow y) \geq (x^n \rightarrow y) \rightarrow ((x^{n+1} \rightarrow y)^{n-1} \rightarrow (x^{n-1} \rightarrow y))$ and $(x^{n+1} \rightarrow y)^n \rightarrow (x^n \rightarrow y) \geq (x^n \rightarrow y)^2 \rightarrow ((x^{n+1} \rightarrow y)^{n-2} \rightarrow (x^{n-2} \rightarrow y))$, and hence by repeating the process n times we get $(x^{n+1} \rightarrow$

$$\begin{aligned} y)^n \rightarrow (x^n \rightarrow y) &\geq (x^n \rightarrow y)^2 \rightarrow ((x^{n+1} \rightarrow y)^{n-2} \rightarrow (x^{n-2} \rightarrow y)) \geq \dots \\ &\geq (x^n \rightarrow y)^n \rightarrow ((x^{n+1} \rightarrow y)^0 \rightarrow (x^0 \rightarrow y)) = (x^n \rightarrow y)^n \rightarrow (1 \rightarrow (1 \rightarrow y)) \\ &= (x^n \rightarrow y)^n \rightarrow y. \end{aligned}$$

Hence $((x^n \rightarrow y)^n \rightarrow y) \rightarrow ((x^{n+1} \rightarrow y)^n \rightarrow (x^n \rightarrow y)) = 1$ and so $(x^{n+1} \rightarrow y)^n \rightarrow (((x^n \rightarrow y)^n \rightarrow y) \rightarrow (x^n \rightarrow y)) = 1$. Since F is a filter and $x^{n+1} \rightarrow y \in F$, we get $(x^{n+1} \rightarrow y)^n \in F$ and so $((x^n \rightarrow y)^n \rightarrow y) \rightarrow (x^n \rightarrow y) \in F$. Note that F is a n -fold positive implicative filter, by Theorem 4.5 we have $x^n \rightarrow y \in F$. Hence by Theorem 3.7, F is a n -fold implicative filter. $\square$

The following examples show that the converse of Theorem 4.8 is not true in general.

Example 4.9. In Example 4.4, $F = \{b, 1\}$ is a 2-fold implicative filter but it is not a 2-fold positive implicative filter, since $(a^2 \rightarrow 0) \rightarrow a = 1 \in F$ and $a \notin F$.

Example 4.10. Define for all $x, y \in H$, $x \odot y = \min\{x, y\}$,

$$x \rightarrow y = \begin{cases} 1, & (x \leq y) \\ y, & (x > y) \end{cases}.$$

It is clear that $H = ([0, 1], \odot, \rightarrow, 0, 1)$ is a hoop, $F = [\frac{1}{2}, 1]$ is a 2-fold implicative filter, but it is not 2-fold positive implicative filter, since $\frac{2}{3} \rightarrow ((\frac{1}{3})^2 \rightarrow \frac{1}{10}) \rightarrow \frac{1}{3} = 1 \in F$, $\frac{2}{3} \in F$, but $\frac{1}{3} \notin F$.

Definition 4.11. A bounded hoop $H(H, \odot, \rightarrow, 0, 1)$ is called a n -fold positive implicative bounded hoop if it satisfies $(x^n \rightarrow 0) \rightarrow x = x$ for all $x \in H$.

Example 4.12. In Example 3.2, H is a 3-fold positive implicative bounded hoop.

Theorem 4.13. If H is a n -fold positive implicative bounded hoop, then it is a $(n + 1)$ -fold positive implicative bounded hoop.

Proof. Since H is a n -fold positive implicative bounded hoop, $(x^n \rightarrow 0) \rightarrow x = x$ for all $x \in H$. Since $x^{n+1} \leq x^n$, then $x^n \rightarrow 0 \leq x^{n+1} \rightarrow 0$ and so $(x^{n+1} \rightarrow 0) \rightarrow x \leq (x^n \rightarrow 0) \rightarrow x = x$. Then by Proposition 2.2, $x \leq (x^{n+1} \rightarrow 0) \rightarrow x$. Hence $x = (x^{n+1} \rightarrow 0) \rightarrow x$, that is, H is a $(n + 1)$ -fold positive implicative bounded hoop. $\square$

The following example shows that the converse of Theorem 4.13 is not true.

Example 4.14. In Example 3.2, H is a 3-fold positive implicative bounded hoop, but H is not a 2-fold positive implicative bounded hoop, since $(b^2 \rightarrow 0) \rightarrow b \neq b$.

Corollary 4.15. In a n -fold positive implicative bounded hoop, the concepts of filter and n -fold positive implicative filter coincide.

Proof. It follows from Theorem 4.3 and Theorem 4.5. $\square$

Theorem 4.16. *The following conditions are equivalent:*

- (1) H is a n -fold positive implicative bounded hoop;
- (2) Every filter of H is a n -fold positive implicative filter;
- (3) $\{1\}$ is a n -fold positive implicative filter;
- (4) If F is a filter of H , H/F is a n -fold positive implicative hoop.

Proof. (1) $\Rightarrow$ (2) It is clear by the Corollary 4.15.

(2) $\Rightarrow$ (3) This is clear.

(3) $\Rightarrow$ (1) Let $\{1\}$ be a n -fold positive implicative filter. Consider $x \in H$ and let $t = ((x^n \rightarrow 0) \rightarrow x) \rightarrow x$. Then by Proposition 2.2 we have

$$\begin{aligned} (t^n \rightarrow 0) \rightarrow t &= (t^n \rightarrow 0) \rightarrow (((x^n \rightarrow 0) \rightarrow x) \rightarrow x) \\ &= ((x^n \rightarrow 0) \rightarrow x) \rightarrow ((t^n \rightarrow 0) \rightarrow x) \\ &\geq (t^n \rightarrow 0) \rightarrow (x^n \rightarrow 0) \\ &\geq x^n \rightarrow t^n. \end{aligned}$$

Since $x \leq (x^n \rightarrow 0) \rightarrow x = t$, then $x^n \leq t^n$, that is, $x^n \rightarrow t^n = 1$. Hence $(t^n \rightarrow 0) \rightarrow t = 1 \in \{1\}$, and since $\{1\}$ is a n -fold positive implicative filter, $t = ((x^n \rightarrow 0) \rightarrow x) \rightarrow x = 1$, that is, $(x^n \rightarrow 0) \rightarrow x \leq x$. On the other hand, by Proposition 2.2 we have $(x^n \rightarrow 0) \rightarrow x \geq x$. Hence we get $(x^n \rightarrow 0) \rightarrow x = x$ for all $x \in H$, that is, H is a n -fold positive implicative hoop.

(2) $\Rightarrow$ (4) Let F be a n -fold positive implicative filter of H . By (1) H is a n -fold positive implicative bounded hoop. Then for all $x \in H$, we have $(x^n \rightarrow 0) \rightarrow x = x$. Thus $(x^n/F \rightsquigarrow 0/F) \rightsquigarrow x/F = ((x^n \rightarrow 0) \rightarrow x)/F = x/F$, that is, H/F is a n -fold positive implicative bounded hoop.

(4) $\Rightarrow$ (2) Let F be a filter of H and $x \in H$, $(x^n \rightarrow 0) \rightarrow x \in F$. Since H/F is a n -fold positive implicative bounded hoop, then for all $x \in H$, $(x^n/F \rightsquigarrow 0/F) \rightsquigarrow x/F = x/F$. By Theorem 2.5 we have $((x^n \rightarrow 0) \rightarrow x) \rightarrow x \in F$. Since F is a filter, we get $x \in F$. Hence by Theorem 4.5, F is a n -fold positive implicative filter. $\square$

Theorem 4.17. *Let F and G be filters of H such that $F \subseteq G$. If F is a n -fold positive implicative filter, then G is also a n -fold positive implicative filter.*

Proof. Let $x \in H$ be such that $(x^n \rightarrow 0) \rightarrow x \in G$. Since F is a n -fold positive implicative filter, by Theorem 4.15 H/F is a n -fold positive implicative bounded hoop. Then $((x^n \rightarrow 0) \rightarrow x)/F = (x^n/F \rightsquigarrow 0/F) \rightsquigarrow x/F = x/F$ and so $((x^n \rightarrow 0) \rightarrow x) \rightarrow x \in F \subseteq G$. Since G is a filter and $(x^n \rightarrow 0) \rightarrow x \in G$, we get $x \in G$. Hence by Theorem 4.5, G is a n -fold positive implicative filter. $\square$

5. Conclusion and future research

In this paper, we introduce the concept of n -fold (positive) implicative filter and n -fold (positive) implicative (bounded) hoop. We prove that in any n -fold

(positive)implicative (bounded) hoop, the notion of filter and n -fold (positive) implicative filter are coincide. Also, we prove that F is a n -fold (positive) implicative(bounded) hoop if and only if H/F is a n -fold (positive) implicative (bounded) hoop. Specially, we obtain that every n -fold positive implicative filter is a n -fold implicative filter. For our future research, we will construct n -fold implicative logic and n -fold positive implicative logic respect to many valued logic system correspondence to hoops and investigate under which conditions a n -fold implicative logic is a n -fold positive implicative logic.

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PSEUDO-TOPOLOGICAL HYPERVECTOR SPACES AND THEIR PROPERTIES

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Abstract. In this paper we introduce and study the concepts of a pseudo-topological hypervector space and strongly pseudo-topological hypervector space. Further, we define a regular equivalence relation on a hypervector space and construct a new hypervector space with respect to this regular equivalence relation. Finally, by using a topology on the power set of a hypervector space, we introduce a topological hypervector space and investigate some of its properties.

Keywords: Hypervector space, pseudo-continuous, strongly pseudo-continuous, pseudo-topological hypervector space, strongly pseudo-topological hypervector space, regular equivalence relation.

1. Introduction and preliminaries

Hyperstructures theory was initially introduced in 1934 by Marty [10]. He defined hypergroups that are a generalization of groups and studied some properties and applications of them in non-commutative groups, rational functions and algebraic functions. In the recent years, various aspects of hyperstructures are studied [3, 4]. Especially, many connections between hyperstructures and topological concepts has been established and investigated. For example, the notions of topological hypergroups, topological hypergroupoids, fuzzy pseudo-topological hypergroupoids and intuitionistic fuzzy topological polygroups were introduced and studied in [1, 2, 5, 6, 7, 8, 9].

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In 1998, the notion of hypervector spaces was initially introduced and studied by Tallini [13, 15]. Also some properties of hypervector spaces were studied by Raja and Taghavi in [11, 12].

In this paper we study some new structures. The organization of this paper is as follows:

In section 2; we introduce the concepts of a pseudo-topological hypervector space and strongly pseudo-topological hypervector space and study some of their properties. Further, we introduce a regular equivalence relation on a hypervector space and construct a new hypervector space with respect to this regular equivalence relation.

In section 3, by using a topology on the power set of a hypervector space, we present a new topological hypervector space and investigate some of its properties.

We begin with the following definitions which are used in our paper.

Definition 1.1. A hypervector space over a field K is a quadruplet $(V, +, \circ, K)$, that $(V, +)$ is an abelian group and

$$\circ : K \times V \rightarrow \mathcal{P}^*(V)$$

is a mapping of $K \times V$ into the power set of V (deprived of the empty set), such that the following properties holds:

$$(1.1) \quad (a + b) \circ x \subseteq (a \circ x) + (b \circ x), \quad \forall a, b \in K, \forall x \in V,$$

$$(1.2) \quad a \circ (x + y) \subseteq (a \circ x) + (a \circ y), \quad \forall a \in K, \forall x, y \in V,$$

$$(1.3) \quad a \circ (b \circ x) = (ab) \circ x, \quad \forall a, b \in K, \forall x \in V,$$

$$(1.4) \quad x \in 1 \circ x, \quad \forall x \in V.$$

$$(1.5) \quad a \circ (-x) = -a \circ x, \quad \forall a \in K, \forall x \in V.$$

If in (1.1) the equality holds, the hypervector space is called strongly left distributive. If in (1.2) the equality holds, the hypervector space is called strongly right distributive. The distributive hypervector spaces were studied in [13].

Let V be a hypervector space over the field K ; $H \subseteq V$ is a subspace of V , if

$$H \neq \emptyset, H + H \subseteq H, a \circ H \subseteq H, \quad \forall a \in K.$$

Definition 1.2. Let $(V, +, \circ, K)$ and $(V', +', \circ', K)$ be two hypervector spaces over the same field K . A mapping

$$f : V \rightarrow V'$$

is called a homomorphism between V and V' , if $\forall a \in K, \forall x, y \in V$:

$$(1.6) \quad f(x + y) = f(x) +' f(y),$$

$$(1.7) \quad f(a \circ x) \subseteq a \circ' f(x).$$

If in (1.7) the equality holds, then f is called strong homomorphisms.

Theorem 1.3 ([14]). *Let V and W be hypervector spaces. If H is a subspace of V , then $f(H)$ is a subspace of W . In particular Imf is a subspace of W , conversely if H' is a subspace of W , then $f^{-1}(H')$ is a subspace of V .*

Throught out of the paper, the ground field of a hypervector space V is presented with K , This field is usually considered by $\mathbb{R}$ or $\mathbb{C}$.

2. Pseudo-topological hypervector spaces

In this section, the concepts of a pseudo-topological hypervector space and strongly pseudo-topological hypervector space are introduced and some of their properties are studied.

Definition 2.1. Let $(V, +, \circ, K)$ be a hypervector space and (V, τ) be a topological space. The mapping $\circ$ is called:

- i) pseudo-continuous, or for short p -contiuous, if for every open set U (member of τ), the set

$$U^* = \{(k, x) \in K \times V \mid k \circ x \subseteq U\}$$

is open in $K \times V$, with respect to the product topology on $K \times V$.

- ii) Strongly pseudo-continuous, or for short strongly p -contiuous, if for every open set U (member of τ), the set

$$U_* = \{(k, x) \in K \times V \mid k \circ x \cap U \neq \emptyset\}$$

is open in $K \times V$, with respect to the product topology on $K \times V$.

Definition 2.2. Let $(V, +, \circ)$ be a hypervector space over the field K and also (V, τ) be a topological space endowed with a topology τ . V is called:

- i) Pseudo topological hypervector space (or for short p -topological hypervector space), if the mapping $+$: $V \times V \rightarrow V$ is continuous and the mapping $\circ$: $K \times V \rightarrow \mathcal{P}^*(V)$ is p -contiuous.
- ii) Strongly pseudo topological hypervector space (or for short strongly p -topological hypervector space), if the mapping $+$: $V \times V \rightarrow V$ is continuous and the mapping $\circ$: $K \times V \rightarrow \mathcal{P}^*(V)$ is strongly p -contiuous.

Example 2.3. Let $V = \mathbb{R}$. Define

$$\circ : \mathbb{R} \times \mathbb{R} \rightarrow \mathcal{P}^*(\mathbb{R})$$

$$a \circ x = \{ax\},$$

for all $a, x \in \mathbb{R}$. If $\mathbb{R}$ is endowed with the discrete topology, then for all $U \in \mathbb{R}$,

$$U^* = \{(a, x) : a \circ x \subseteq U\},$$

is an open set in $\mathbb{R} \times \mathbb{R}$ with respect to the product topology. So $(\mathbb{R}, +, \circ, \mathbb{R})$ is a p -topological hypervector space.

Example 2.4. Let $V = \mathbb{R}^2$. Define

$$\circ : \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathcal{P}^*(\mathbb{R}^2)$$

Such that for all $a \in \mathbb{R}$ and $(x, y) \in \mathbb{R}^2$,

$$a \circ (x, y) = \begin{cases} \{(ax, ay)\} & a \neq 0 \\ \{(0, 0)\} & a = 0 \end{cases},$$

Let $\mathbb{R}$ be endowed with the discrete topology, $\mathbb{R}^2$ be endowed with the standard topology and $U = B_r(x', y')$ ($B_r(x', y')$ is an open ball in $\mathbb{R}^2$ centered at a point (x', y') , with radius $r > 0$). So we have

$$U^* = \{(a, (x, y)) : a \circ (x, y) \subseteq U\}.$$

If $(0, 0) \in U$, then

$$U^* = \bigcup_{a \neq 0} \{a\} \times B_{\frac{r}{|a|}}\left(\frac{x'}{a}, \frac{y'}{a}\right) \cup \{0\} \times \mathbb{R}^2,$$

that is open set in $\mathbb{R} \times \mathbb{R}^2$ with respect to the product topology, if $(0, 0) \notin U$, then

$$U^* = \bigcup_{a \neq 0} \{a\} \times B_{\frac{r}{|a|}}\left(\frac{x'}{a}, \frac{y'}{a}\right),$$

that U^* is an open set in $\mathbb{R} \times \mathbb{R}^2$. So $(\mathbb{R}^2, +, \circ, \mathbb{R})$ is a p -topological hypervector space.

Remark 2.5. Let H be a subspace of hypervector space $(V, +, \circ)$. Then $(H, +, \circ|_H)$ is obviously a hypervector space.

Theorem 2.6. Let V be a p -topological hypervector space and H be an open subspace of V . Then H is a p -topological hypervector space with respect to subspace topology.

Proof. According to Remark 2.5, H is a hypervector space. Also $+: H \times H \rightarrow H$ is obviously continuous. Now we show that $\circ : K \times H \rightarrow \mathcal{P}^*(H)$ is p -continuous. Let U_H be an open set in H . So $U_H = U \cap H$ where U is an open set in V . Since V is a p -topological hypervector space, thus

$$U^* = \{(k, x) \in K \times V \mid k \circ x \subset U\}$$

is an open set in $K \times V$. It is enough to show that $U_H^* = U \cap (K \times H)$. Suppose that $(k, h) \in U_H^* \iff$

$$k \circ h \subset U_H = U \cap H \iff (k, h) \in U^* \iff (k, h) \in U^* \cap (K \times H). \quad \square$$

Theorem 2.7. *Let $(V, +, \circ)$ be a p -topological hypervector space and $(V', +', \circ')$ be a hypervector space endowed with topology τ , $f : V \rightarrow V'$ be a strong homomorphism that is open and continuous and H be an open subspace of V . Then the following statements are valid:*

- i) $f(H)$ is a p -topological hypervector space.
- ii) $\text{Im}(f)$ is a p -topological hypervector space.
- iii) If f is onto, then V' is a p -topological hypervector space.

Proof. i) According to Theorem 1.3, $f(H)$ is a subspace of V' . Since f is a continuous strong homomorphism. It can be easily seen that the map $+ : f(H) \times f(H) \rightarrow f(H)$ is continuous.

Suppose that U is an open set in $f(H)$. It is enough to show that

$$U_* = (id_K \times f)((f^{-1}(U))_*).$$

Let $z \in U_*$. Then $z = (k, f(x)) \in K \times f(H)$ and also $k \circ' f(x) \subset U$. Since f is a strong homomorphism, so we have

$$k \circ' f(x) = f(k \circ x) \subset U.$$

It implies that $k \circ x \subset f^{-1}(U)$. So $z = (k, f(x)) \in (id_K \times f)((f^{-1}(U))_*)$. Simply we see that

$$(id_K \times f)((f^{-1}(U))_*) \subset U_*.$$

The proof of (ii) and (iii) is immediate from the part (i). □

Theorem 2.8. *Let $(V, +, \circ)$ be a hyper topological space that endowed with topology τ , $(V', +', \circ')$ is a p -topological hypervector space and $f : V \rightarrow V'$ be a strong homomorphism that is open and continuous. Then the following assertions hold:*

- 1) $f^{-1}(H)$ is a p -topological hypervector space.
- 2) In particular, V is p -topological hypervector space.

Proof. If U is an open set in $f^{-1}(H)$, then $U_* = (id_k \circ f^{-1})(f(U)_*)$. □

Proposition 2.9. *Let $(V, +, \circ)$ and $(V', +', \circ')$ be two hypervector spaces. If $f : V \rightarrow V'$ is an one-to-one open continuous strong homomorphism, then $f^{-1} : V' \rightarrow V$ is an open strong homomorphism.*

Proof. Let $y, z \in V'$. Then there exist $x_1, x_2 \in V$ such that $y = f(x_1), z = f(x_2)$. Now we have

$$\begin{aligned} f^{-1}(y + z) &= f^{-1}(f(x_1) + f(x_2)) \\ &= f^{-1}(f(x_1 + x_2)) \\ &= x_1 + x_2 \\ &= f^{-1}(y) + f^{-1}(z). \end{aligned}$$

and

$$f^{-1}(a \circ' y) = f^{-1}(a \circ' f(x_1)) = f^{-1}(F(a \circ x_1)) = a \circ x_1 = a \circ f^{-1}(y).$$

Since f is continuous, so f^{-1} is an open strong homomorphism. $\square$

Definition 2.10. Let $(V, +, \circ)$ be a hypervector space and ρ be an equivalence relation on V . If A, B are non-empty subset of V , then $A\bar{\rho}B$ means that $\forall a \in A, \exists b \in B$ such that $a\rho b$ and $\forall b' \in B, \exists a' \in A$ such that $a'\rho b'$.

Definition 2.11. Let $(V, +, \circ)$ be a hypervector space and ρ be an equivalence relation on V . The equivalence relation ρ is called regular if for all $r \in K$, from $a\rho b$, it follows that $(r \circ a)\bar{\rho}(r \circ b)$.

Proposition 2.12. Let $(V, +, \circ)$ be a hypervector space, ρ be an equivalence relation on V and $V/\rho = \{[v] : v \in V\}$. Then V/ρ is a hypervector space with respect to actions

$$\oplus : V/\rho \times V/\rho \rightarrow V/\rho,$$

where $([x], [y]) \mapsto [x] \oplus [y] = [x + y], \quad \forall x, y \in V.$ and

$$\odot : K \times V/\rho \rightarrow V/\rho$$

$$k \odot [x] = \{[z] : z \in k \circ x\},$$

if and only if ρ is regular.

Proof. First we check that the mappings $\oplus$ and $\odot$ are well-defined on V/ρ . Simply, we see that $\oplus$ is well-defined. It is enough to show that $\odot$ is well-defined. Let $(r, [x]) = (r', [x'])$. So $r = r'$ and $[x] = [x']$. Now we check that $r \odot [x] = r \odot [x']$. We have $x\rho x'$. Since ρ is regular it follows that $(rox)\bar{\rho}(rox')$. Hence, for all $z \in r \circ x$, there exists $z_1 \in r \circ x'$ such that $z\rho z_1$, which means that $[z] = [z_1]$. It follows that $r \odot [x] \subseteq r \odot [x']$ and similary we obtain the converse inclusion.

Now we show that $(V/\rho, \oplus, \odot)$ is a hypervector space. Simply we see that $(V/\rho, \oplus)$ is an abelian group. It is enough to show that $\odot$ has the properties of Definition 1.1. For all $a, b \in K$ we have

$$\begin{aligned} (a + b) \odot [x] &= \{[z] : z \in (a + b) \circ x\} \\ &\subseteq \{[z] : z \in a \circ x + b \circ x\} \\ &= \{[c] \oplus [d] \mid c \in a \circ x, d \in b \circ x\} \\ &= a \odot [x] \oplus b \odot [x]. \end{aligned}$$

Also the properties (ii), (iii), (iv) in Definition 1.1 are easily obtained. $\square$

Theorem 2.13. Let $(V, +, \circ)$ be a hypervector space and ρ be a regular equivalence relation on V . Then the canonical projection $\pi : V \rightarrow V/\rho$, such that $\pi(x) = [x]$, is a strong epimorphism.

Proof. Suppose that $x, y \in X$. Simply, we see that $\pi(x+y) = \pi(x) \oplus \pi(y)$. Now we show that, for all $r \in K$ and $x \in V$, $\pi(r \circ x) = r \odot \pi(x)$. Let $[z] \in \pi(r \circ x)$, there exists $z' \in r \circ x$ such that $[z] = [z']$. We have $[z] = [z'] \in r \odot [x] = r \odot \pi(x)$. Conversely, if $[z] \in r \odot \pi(x) = r \odot [x]$, then there exists $z_1 \in r \circ x$ such that $[z] = [z_1] \in \pi(r \circ x)$. $\square$

Remark 2.14. Let V be a hypervector space that endowed with the topology τ and ρ be a regular equivalence relation on V . Then V/ρ is a topological space that endowed with the quotient topology. So the canonical projection $\pi : V \rightarrow V/\rho$ is a continuous and open map.

Theorem 2.15. *Let V be a p -topological hypervector space and ρ be a regular equivalence relation on V . Then V/ρ is a p -topological hypervector space.*

Proof. Since the canonical map $\pi : V \rightarrow V/\rho$ is onto, open strong homomorphism and V is a p -topological hypervector space. By Theorem 2.7, V/ρ is a p -topological hypervector space. $\square$

Theorem 2.16. *If $(V, +, \odot)$ and $(V', +', \odot')$ are hypervector spaces over field K and $f : V \rightarrow V'$ is a strong homomorphism, then the equivalence relation R^f associated with the map f , that $xR^f y \Leftrightarrow f(x) = f(y)$, is regular and V/R^f is a hypervector space. And also $\phi : f(V) \rightarrow V/R^f$, defined by $\phi(f(x)) = [x]$, is an isomorphism map.*

Proof. Let $aR^f b$ and r be arbitrary element of K . If

$$u \in r \circ a \implies f(u) \in f(r \circ a) = r \odot' f(a) = r \odot' f(b) = f(r \circ b),$$

then, there exist $v \in r \circ b$ such that $f(u) = f(v)$, which means that $uR^f v$. Hence, R^f is regular. So according to Proposition 2.12, $(V/R^f, \oplus, \odot)$ is a hypervector space. Now we check that ϕ is an isomorphism. Let $f(x), f(y) \in f(V)$. We have $\phi(f(x) + f(y)) = \phi(f(x+y)) = [x+y] = [x] \oplus [y]$. For all $r \in K$ and $f(x) \in f(V)$, we have

$$\phi(r \odot' f(x)) = \phi(f(r \circ x)) = \{[z] : z \in r \circ x\} = r \odot [x] = r \odot \phi(f(x)).$$

Moreover, if $\phi(f(x)) = \phi(f(y))$, then $[x] = [y]$, so $xR^f y$. It implies that $f(x) = f(y)$. Hence ϕ is an injective map. One can easily check that ϕ is a surjective map. $\square$

Remark 2.17. According to Theorem 2.15, we see that if $(V, +, \odot)$ is a p -topological hypervector space, then $(V/R^f, \oplus, \odot)$ is a p -topological hypervector space with respect to quotient topology.

Proposition 2.18. *Let $(V, +, \odot)$ be a hypervector space over the field K and H be a subspace of V . Define an equivalence relation ρ^H on V as follows:*

$$a\rho^H b \iff a - b \in H.$$

If $(V, +, \odot)$ is a strongly right distributive hypervector space, then ρ^H is a regular equivalence relation.

Proof. Let $x\rho^Hy$ and r be an arbitrary element of K . So $x - y \in H$. Since V is strongly right distributive, we have $r \circ (x - y) = r \circ x - r \circ y \subseteq H$. Hence ρ^H is regular. $\square$

Corollary 2.19. *Let V and ρ^H are similar to Proposition 2.18 and $V/\rho^H = V/H = \{[x] = x + H \mid x \in V\}$. Then V/H with respect to actions*

$$\oplus : V/H \times V/H \rightarrow V/H,$$

where $[x] \oplus [y] = [x + y], \forall x, y \in V$, and

$$\odot : K \times V/H \rightarrow V/H,$$

where $k \odot [x] = \{[z] : z \in k \circ x\}$, is a hypervector space.

Proof. According to Propositions 2.12 and 2.18, the proof is clear. $\square$

Proposition 2.20. *Let H be a subspace of a strongly right distributive hypervector space $(V, \circ, +)$, the following statements are valid:*

- 1) $\pi : V \rightarrow V/H$ such that $\pi(x) = [x]$ is a strong homomorphism map.
- 2) If V is a p -topological hypervector space, then $(V/H, \oplus, \odot)$ is a p -topological hypervector space.

3. τ_* -topological hypervector space

In this section, by using a topology on the power set of a hypervector space, we present a new topological hypervector space and investigate some of its properties.

Definition 3.1. Let $(V, +, \circ)$ be a hypervector space over the field K and also (V, τ) and $(\mathcal{P}^*(V), \tau_*)$ be two topological spaces. $\circ : K \times V \rightarrow \mathcal{P}^*(V)$ is called τ_* -continuous, if for every open set U of $\mathcal{P}^*(V)$ (the member of τ_*) the set $\{(k, x) : k \circ x \in U\}$ is an open set in $K \times V$.

Definition 3.2. Let $(V, +, \circ)$ be a hypervector space over the field K and also (V, τ) and $(\mathcal{P}^*(V), \tau_*)$ be two topological spaces. V is called a τ_* -topological hypervector space, if the mapping $+$: $V \times V \rightarrow V$ is continuous and the mapping $\circ : K \times V \rightarrow \mathcal{P}^*(V)$ is τ_* -continuous.

Lemma 3.3. *Let (H, τ) be a topological space and*

$$\mathcal{S}_V = \{U \in \mathcal{P}^*(H) : U \subseteq V\},$$

where $V \in \tau$. Then $\Lambda = \{\mathcal{S}_V : V \in \tau\}$ is a basis for topology τ_Λ on $\mathcal{P}^*(H)$.

Proof. Let $\mathcal{S}_{V_1}, \mathcal{S}_{V_2} \in \Lambda$, where $V_1, V_2 \in \tau$. So we have $U \in \mathcal{S}_{V_1 \cap V_2} = \mathcal{S}_{V_1} \cap \mathcal{S}_{V_2}$, where $V_1 \cap V_2 \in \tau$. Since $H \in \tau$, so we have $\mathcal{S}_H = \mathcal{P}^*(H)$, thus $U \in \mathcal{S}_H$, $\forall U \in \mathcal{P}^*(H)$. $\square$

In above Lemma, τ_Λ that induced by τ is called the upper-topology on $\mathcal{P}^*(H)$.

Theorem 3.4. *Let $(H, +, \circ)$ be a hypervector space over the field K and (H, τ) be a topological space and τ_Λ be the upper-topology on $\mathcal{P}^*(H)$. Then $(H, +, \circ)$ is a p -topological hypervector space if and only if $(H, +, \circ)$ is a τ_Λ -topological hypervector space.*

Proof. Let $\circ : K \times H \rightarrow \mathcal{P}^*(H)$, where $\circ(r, x) = r \circ x$. Then we have

$$\begin{aligned}\circ^{-1}(\mathcal{S}_V) &= \{(r, x) \in K \times H : r \circ x \in \mathcal{S}_V\} \\ &= \{(r, x) \in K \times H : r \circ x \subseteq V\} = V^*.\end{aligned}$$

So the map $\circ$ is τ_Λ -continuous if and only if the map $\circ$ is p -continuous. $\square$

Lemma 3.5. *Let (H, τ) be a topological space and*

$$\mathcal{I}_V = \{U \in \mathcal{P}^*(H) : U \cap V \neq \phi\},$$

where $V \in \tau$. Then $\Gamma = \{\mathcal{I}_V, V \in \tau\}$ is a subbasis for topology τ_Γ on $\mathcal{P}^*(H)$.

Proof. It is enough to show that

$$\bigcup_{V \in \tau} \mathcal{I}_V = \mathcal{P}^*(H).$$

This proof is trivial, because $\mathcal{I}_H = \mathcal{P}^*(H)$. $\square$

In above Lemma, τ_Γ that induced by τ is called the lower-topology on $\mathcal{P}^*(H)$.

Theorem 3.6. *Let $(V, +, \circ)$ be a hypervector space over the field K , (V, τ) be a topological space and τ_Γ be the lower-topology on $\mathcal{P}^*(H)$. Then $(V, +, \circ)$ is a strongly p -topological hypervector space if and only if $(V, +, \circ)$ is a τ_Γ -topological hypervector space.*

Proof. Let $\circ : K \times H \rightarrow \mathcal{P}^*(H)$, where $\circ(r, x) = r \circ x$. Then we have

$$\begin{aligned}\circ^{-1}(\mathcal{I}_V) &= \{(r, x) \in K \times H : r \circ x \in \mathcal{I}_V\} \\ &= \{(r, x) \in K \times H : r \circ x \cap V \neq \phi\} = V_*.\end{aligned}$$

So $\circ$ is a τ_Γ -continuous map if and only if $\circ$ is a strongly p -continuous map. $\square$

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A SEPARATION METHOD FOR MAXIMAL COVERING LOCATION PROBLEMS WITH FUZZY PARAMETERS

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Abstract. Our paper discusses a novel computational approach to the extended Maximal Covering Location Problem (MCLP). We consider a fuzzy-type formulation of the generic MCLP and develop the necessary theoretical and numerical aspects of the proposed Separation Method (SM). A specific structure of the originally given MCLP makes it possible to reduce it to two auxiliary Knapsack-type problems. The equivalent separation we propose reduces essentially the complexity of the resulting computational algorithms. This algorithm also incorporates a conventional relaxation technique and the scalarizing method applied to an auxiliary multiobjective optimization problem. The proposed solution methodology is next applied to Supply Chain optimization in the presence of incomplete information. We study two illustrative examples and give a rigorous analysis of the obtained results.

Keywords: MCLP, integer optimization, numerical optimization

1. Introduction

Optimization of modern technological processes and the corresponding computer oriented methods are nowadays a usual and efficient approach to the practical de-

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velopment of several engineering applications (see e.g., [1,5-7,9,10,11,15,18,23]). In our contribution we study an extended MCLP model with an incomplete information and propose a relative simple approach to the effective numerical treatment of this problem. The obtained theoretic and computational results are next applied to the resilient Supply Chain Management System Optimization. The requested optimal design of an optimal management operation can be formalized as a specific MCLP [10]. In that case the information incompleteness mentioned above can be adequately described by an eligibility matrix with the fuzzy structure and the systems "resilience" is related to this incomplete modelling framework.

Let us recall that the conventional and extended MCLP formulations constitute a family of challenging optimization problems with numerous practical applications. It has a decisive role in the success of a Supply Chain management, with several applications including location of industrial plants, landfills, hubs, cross-docks, etc (see e.g., [1,3,8-10,12-15,18,20,22,24]). A well-known MCLP and the related decision making involve the delivery of a manufactured product to the end customer or/and to a warehouse. In a classical MCLP, one seeks the location of a number of facilities on a network in such a way that the covered "population" is maximized [14,24]. MCLP was first introduced by Church and ReVelle [14] on a network, and since then, several extensions to the original problem have been made. A variety of numerical approaches have been proposed to the practical treatment of distinct MCLPs. Recently several heuristical methods are actively used in the practical treatment of the MCLP based models. We refer to [8-10,12-15,18,20,22] for some effective heuristic and metaheuristic algorithms and for further references. Note that heuristics and metaheuristics have usually been employed in order to solve large size MCLPs (see e.g., [3,13,18,20]). A recent interest to MCLPs has arisen out the uncertainty of model parameters, such as demands or/and locations of demand nodes [9,10,24]. The solution procedure (Separation Method) we propose is generally based on an exact optimization procedure. However it also can incorporate some heuristic procedures for solving the obtained auxiliary problems.

This paper is devoted to a further theoretic and numerical development of a newly elaborated solution method for the MCLPs, namely, to the so called Separation Method (see [7]). The optimization approach we follow includes an equivalent transformation (separation) of the original MCLP and solution of two auxiliary Knapsack-type problems (see e.g., [16] and references therein). The proposed SM reduces the complexity of the original problem. Moreover, one can apply various methods to the resulting auxiliary problems. In this paper we use a usual relaxation scheme for the purpose of a concrete computation [12,16]. We also apply the standard scalarizing of an intermediate multiobjective optimization problem we obtain. And, it should be noted already at this point that the MCLP based optimization approach we propose can be effectively implemented (at the prototype stage) in a concrete optimal design of a decision or management system. Concretely, this SM involved approach is applied in

our paper to the optimal design of a resilient Supply Chain scheme for a typical manufactures - customers delivery. Finally note that SM we propose in fact involves a suitable (equivalent) decomposition of an initially given MCLP. This fact, namely, the consideration of two resulting auxiliary problems makes it also possible to extend this method to some applied large-scale MCLP (see e.g., [3]).

The remainder of our paper is organized as follows: Section 2 contains an abstract problem formulation and some necessary theoretical concepts and facts. In Section 3 we develop a theoretic basis of the SM. This section also includes a necessary characterization of the obtained auxiliary problems. Section 4 discusses the appropriate numerical schemes in the context of the the initially given and auxiliary optimization problems. We use our main theoretic results and finally propose an implementable and well-determined algorithm for an effective numerical treatment of the originally given MCLP. This algorithm also incorporates the conventional relaxation technique. Section 5 contains two computational examples of an optimal resilient Supply Chain design. These practically oriented examples illustrate the implementability of the resulting computational algorithms and usability of the proposed solution procedure. Section 6 summarizes our contribution.

2. Problem formulation and preliminaries

We start by introducing the main optimization problem with a fuzzy structure. The MCLP we study has the following form:

$$(1) \quad \begin{aligned} & \text{maximize } J(z(y)) := \sum_{j=1}^n w_j z_j \\ & \text{subject to } \begin{cases} \sum_{i=1}^l y_i = k \in \mathbb{N}, \quad l > k, \\ z_j \leq \sum_{i=1}^l a_{ij} y_i, \\ z \in \mathbb{B}^n, \quad y \in \mathbb{B}^l \end{cases} \end{aligned}$$

Here $w_j \in \mathbb{R}_+$, $j = 1, \dots, n$ are given nonnegative objective "weights" and variables z_j , $j = 1, \dots, n$ determine the "facilities to be served". By y_i , where $i = 1, \dots, l$, we define the generic decision variables of the problem under consideration and $k \in \mathbb{N}$ in (1) describes the total amount of the facilities to be located. Elements a_{ij} , where

$$1 \geq a_{ij} \geq 0, \quad \sum_{i=1, \dots, l} a_{ij} \geq 1,$$

are components of the so called "eligibility matrix"

$$A := (a_{ij})_{j=1, \dots, n}^{i=1, \dots, l}$$

associated with the eligible sites that provide a covering of the demand points indexed by $j = 1, \dots, n$. The admissible values of the elements of the matrix A are

"distributed" on the interval $[0, 1]$. Note that the second index in (1), namely, $i = 1, \dots, l$ is related to the given "facilities sites". Finally, the admissible sets $\mathbb{B}^n$ and $\mathbb{B}^l$ in the main problem (1) are defined as follows:

$$\mathbb{B}^n := \{0, 1\}^n, \quad \mathbb{B}^l := \{0, 1\}^l.$$

Note that the objective functional $J(\cdot)$ from (1) has a linear structure. We use the following vectorial notation

$$z := (z_1, \dots, z_n)^T, \quad y := (y_1, \dots, y_l)^T.$$

The implicit dependence

$$\begin{aligned} J(z(y)) &= \langle w, z \rangle, \\ w &:= (w_1, \dots, w_n)^T \end{aligned}$$

of the objective functional J on the vector y is given by the corresponding (componentwise) inequalities constraints

$$z \leq A^T y$$

in (1). By $\langle \cdot, \cdot \rangle$ we denote here the scalar product in the corresponding Euclidean space. A vector pair (z, y) that satisfies all the constraints in (1) is next called an admissible pair for the main problem (1). Note that the objective functional does not depend explicitly on the problem variable y .

The abstract optimization framework (1) provides a constructive and modelling approach for various practically oriented problems (see e.g., [1, 9, 11, 13, 18]). Following [14] we next call the main optimization problem (1) a Maximal Covering Location Problem (MCLP). Let us also refer to [24] for a detailed discussion on the applied interpretation of the MCLP (1). The main problem (1) is formulated under the general (non-binary) assumption related to the elements a_{ij} of the eligibility matrix A . This corresponds to a suitable modelling approach under incomplete information (see e.g., [10] and references therein). Roughly speaking every value of an admissible parameter a_{ij} in (1) has a fuzzy nature (similar to [8]). This fuzzy MCLP under consideration provides an adequate formal framework for the resilient Supply Chain Optimization (see Section 5). Let us also observe that the "resilience" concept is understood here as a kind of robustness of the optimization approach we develop. This robustness is considered with respect to a possible incomplete information about the main mathematical model (robustness with respect to uncertainties in the modelling approach). Note that the possible incompleteness of the mathematical model mentioned above and the robustness requirement for a selected optimization approach constitute the common (and adequate) attributes for a realistic Supply Chain optimal design.

The mathematical characterization of (1) can evidently be given in terms of the classic integer programming (see e., g. [11,16,19] for mathematical details). Let us note that (1) possesses an optimal solution (an optimal pair)

$$(z^{opt}, y^{opt}) \in \mathbb{B}^n \otimes \mathbb{B}^l,$$

where

$$\begin{aligned} z^{opt} &:= (z_1^{opt}, \dots, z_n^{opt})^T, \\ y^{opt} &:= (y_1^{opt}, \dots, y_l^{opt})^T. \end{aligned}$$

This fact is a direct consequence of the basic results from [11,16,19]. Let us also note that the conventional problem (1) can also be easily extended to the "multi-valued" version, where the admissible sets $\mathbb{B}^n$ and $\mathbb{B}^l$ are replaced by

$$\begin{aligned} \tilde{\mathbb{B}}^n &:= \{0, 1, \dots, N_n\}^n, \\ \tilde{\mathbb{B}}^l &:= \{0, 1, \dots, N_l\}^l, \end{aligned}$$

where $N_n, N_l \in \mathbb{N}$.

Our aim is to develop a simple and effective numerical approach to the sophisticated MCLP (1). Facility location has a decisive role in success of Supply Chains with applications in many production and service facilities. It has been a focal center of interest in the last century among practitioners and scholars. For a detailed introduction to location models, one may refer to [15,23,24]. In general the literature of covering models is too diverse to be exhaustively studied in this paper. Although some of known publications in the literature of MCLP are included in this paper, one may refer to valuable reviews for further information.

3. Analytical foundations of the separation method

We next separate the originally given MCLP (1) and introduce two auxiliary optimization problems. These formal constructions provide a necessary basis for the future numerical development. The first optimization problem can be formulated as follows

$$\begin{aligned} (2) \quad & \text{maximize} \quad \sum_{j=1}^n \mu_j \sum_{i=1}^l a_{ij} y_i \\ & \text{subject to} \quad \begin{cases} \sum_{i=1}^l y_i = k, & y \in \mathbb{B}^l, \\ \mu_j \in [0, 1] \quad \forall j = 1, \dots, n \end{cases} \end{aligned}$$

The second auxiliary problem has the following specific form:

$$\begin{aligned} (3) \quad & \text{maximize} \quad J(z) := \sum_{j=1}^n w_j z_j \\ & \text{subject to} \quad \begin{cases} z_j \leq \sum_{i=1}^l a_{ij} \hat{y}_i \\ z \in \mathbb{B}^n \end{cases} \end{aligned}$$

where $\hat{y} \in \mathbb{B}^l$ is optimal solution of problem (2). The components of $\hat{y}$ are denoted as $\hat{y}_i$, $i = 1, \dots, l$. The existence of an optimal solution for (2) is a direct consequence of the results from [11,19]. The same is also true with respect to the auxiliary problem (3). Let

$$\hat{z} \in \mathbb{B}^n, \hat{z} := (\hat{z}_1, \dots, \hat{z}_n)^T$$

be an optimal solution to (3). Evidently, problem (3) coincides with the originally given MCLP (1) in a specific case of a fixed variable $y = \hat{y}$. Let us note that in general $\hat{y} \neq y^{opt}$.

The first auxiliary problem, namely, problem (2) can be interpreted as a usual linear scalarization of the following multiobjective optimization problem (vector optimization):

$$(4) \quad \begin{aligned} & \text{maximize} \quad \left\{ \sum_{i=1}^l a_{i1} y_i, \dots, \sum_{i=1}^l a_{in} y_i \right\} \\ & \text{subject to} \quad \begin{cases} \sum_{i=1}^l y_i = k, \\ y \in \mathbb{B}^l \end{cases} \end{aligned}$$

Recall that a scalarizing of a multi-objective optimization problem is an adequate numerical approach, which means formulating a single-objective optimization problem such that optimal solutions to the single-objective optimization problem are Pareto optimal solutions to the multi-objective optimization problem. We next assume that the multipliers

$$\mu_j, \quad j = 1, \dots, n$$

in (2) are chosen by such a way that problems (2) and (4) are equivalent (see e.g., [2,11,19] for necessary details). In this particular case we call (2) an adequate scalarizing of (4). We discuss shortly the adequate scalarizing in Section 4.3.

It is easy to see that problems (2) and (3) have a structure of a so-called Knapsack problem (see [16] and references therein). Various efficient numerical algorithms are recently proposed for a generic Knapsack problem. We refer to [16] for a comprehensive overview about the modern implementable numerical approaches to this basic optimization problem.

The relevance and the main motivation of the auxiliary optimization problems (2) and (3) introduced can be stated by the following abstract result.

Theorem 3.1. *Assume (z^{opt}, y^{opt}) is an optimal solution of (1) and (2) is an adequate scalarizing of (4). Let $\hat{y}$ be an optimal solutions of (2) and $\hat{z}$ be an optimal solution of the auxiliary problem (3). Then (1) and (3) possess the same optimal values, that is*

$$(5) \quad J(z^{opt}(y^{opt})) = J(\hat{z}).$$

Moreover, in the case problems (1), (2), and (3) possess unique solutions we additionally have

$$(z^{opt}, y^{opt}) = (\hat{z}, \hat{y}).$$

Proof. Since

$$\sum_{i=1}^l \hat{y}_i = k,$$

and

$$\hat{z}_j \leq \sum_{i=1}^l a_{ij} \hat{y}_i,$$

we conclude that $(\hat{z}, \hat{y})$ is an admissible pair for the original MCLP (1). Taking into account the definition of an optimal pair for problem (1), we next deduce

$$(6) \quad J(\hat{z}(\hat{y})) \leq J(z^{opt}(y^{opt})).$$

Let

$$\Gamma = \Gamma_z \otimes \Gamma_y \subset \mathbb{B}^n \otimes \mathbb{B}^l$$

be a solutions set (the set of all optimal solutions) for problem (1). We also define the solutions sets

$$\Gamma_{(2)} \subset \mathbb{B}^l, \quad \Gamma_{(3)} \subset \mathbb{B}^n$$

of problems (2) and (3), respectively. From (6) it follows that

$$(7) \quad \Gamma_{(3)} \otimes \Gamma_{(2)} \subset \Gamma.$$

Taking into account the restrictions associated with the variable y in (1) and (2), we next obtain

$$(8) \quad \Gamma_y \equiv \Gamma_{(2)}.$$

Since (2) is an adequate scalarization of the multi-objective maximization problem (4), we deduce

$$z_j \leq \max_{\substack{\sum_{i=1}^l y_i = k, \\ y \in \mathbb{B}^l}} \sum_{i=1}^l a_{ij} y_i.$$

This fact implies

$$(9) \quad \Gamma_z \subset \Gamma_{(3)}.$$

Inclusions (7), (9) and the basic equivalence (8) now imply the following crucial equivalence

$$(10) \quad \Gamma_{(3)} \otimes \Gamma_{(2)} \equiv \Gamma.$$

Taking into account the same form of the objective functionals in (1) and (2.3), we immediately obtain the basic relation (5). In a specific case of the one point sets Γ , $\Gamma_{(3)}$ and $\Gamma_{(2)}$ the expected relation

$$(z^{opt}, y^{opt}) = (\hat{z}, \hat{y})$$

is a direct consequence of (10). The proof is completed. □

Theorem 3.1 makes it possible to separate (decompose equivalently) the original sophisticated problem (1) into two relative simple optimization problems. It provides a theoretical basis for effective numerical approaches to the abstract MCLPs and to possible applications.

4. Numerical analysis of the auxiliary problems

This section is dedicated to the numerical aspects related to the two optimization problems obtained in Section 3. Our aim is to develop a resulting self-closed algorithm for an effective numerical treatment of the original MCLP (1).

4.1 A combinatorial algorithm for the first auxiliary problem

We first observe that the auxiliary optimization problem (2) has a simple combinatorial structure. It can be easily solved using the following natural scheme:

$$(11) \quad \begin{aligned} \hat{y}_i &= 1 \text{ if } i \in \hat{I}; \\ \hat{y}_i &= 0 \text{ if } i \in \{1, \dots, l\} \setminus \hat{I} \end{aligned}$$

where

$$(12) \quad \begin{aligned} \hat{I} &:= \{1 \leq i \leq l \mid S_{\mathcal{A}_i} \in \max_k \{S_{\mathcal{A}_1}, \dots, S_{\mathcal{A}_l}\}\}, \\ S_{\mathcal{A}_i} &:= \sum_{j=1}^n \mu_j a_{ij}, \\ \mathcal{A}_i &:= (a_{i1}, \dots, a_{in})^T. \end{aligned}$$

Here $\mathcal{A}_i$ is a vector of i -row of the eligibility matrix A and operator $\max_k$ determines an array of k -largest numbers from the given array. Evidently, the choice (11)-(12) determines an optimal solution of (2). Roughly speaking the combinatorial algorithm (11)-(12) assigns the maximal value $\hat{y}_i = 1$ for all vectors $\mathcal{A}_i$ which sum of components $S_{\mathcal{A}_i}$ belongs to the array of k -largest sums of components of all vectors

$$\mathcal{A}_i, \quad i = 1, \dots, l.$$

It is easy to see that for the given eligibility matrix A with the specific elements a_{ij} (determined in Section 2) the sum of components $S_{\mathcal{A}_i}$ constitutes a specific norm of the given vector $\mathcal{A}_i$. The total complexity of the combinatorial algorithm (11)-(12) can be easily calculated and is equal to

$$O(l \times \log k) + O(k).$$

We refer to [16] for the necessary details.

Let us denote

$$c := \sum_{j=1}^n \sum_{i=1}^l a_{ij} \hat{y}_i.$$

Then the inequality constraints in (3) imply the generic Knapsack-type constraint with uniform weights

$$\sum_{j=1}^n z_j \leq c.$$

We now present a fundamental solvability result for the second auxiliary optimization problem, namely, the Knapsack problem (3).

Theorem 4.1. *The Knapsack problem (3) can be solved in $O(nc)$ time and*

$$O(n + c)$$

space.

The formal proof of Theorem 4.1 can be found in [16].

4.2 A relaxation based approach and the resulting computational scheme

The theoretic and numerical results obtained above, namely, Theorem 1 and the combinatorial choice algorithm (11)-(12) provide a theoretic basis for a novel exact solution method for the originally given MCLP (1). We now need to establish an implementable solution procedure for the effective numerical treatment of the second auxiliary problem (3) from the obtained decomposition (2)-(2.3). This optimization problem, which is *NP*-hard, has been comprehensively studied in the last few decades and several exact algorithms for its solution can be found in the literature (see [16] and the references therein). Constructive algorithms for this Knapsack problems are mainly based on two basic numerical approaches: branch-and-bound and dynamic programming. Let us also mention here the corresponding combined approach.

In this paper we firstly consider the well-known Lagrange relaxation scheme in the context of the second auxiliary problem (problem (3)). "Relaxing a problem" has various meanings in applied mathematics, depending on the areas where it is defined, depending also on what one relaxes (a functional, the underlying space, etc.). We refer to [2,4-7,12, 21] for various relaxation techniques in the modern optimization. Introducing the Lagrange function

$$\mathcal{L}(z, \lambda) := \sum_{j=1}^n w_j z_j - \sum_{j=1}^n \lambda_j (z_j - \sum_{i=1}^l a_{ij} \hat{y}_i)$$

associated with the Knapsack problem (3), we obtain the following relaxed problem

$$(13) \quad \begin{array}{ll} \text{maximize} & \mathcal{L}(z, \lambda) \\ \text{subject to} & z \in \mathbb{B}^n \end{array}$$

The relaxed problem (13) does not contain the originally given unpleasant inequality constraints. These constraints are now included into the objective function $\mathcal{L}(z, \lambda)$ from (13) as a penalty term

$$\sum_{j=1}^n \lambda_j \left(z_j - \sum_{i=1}^l a_{ij} \hat{y}_i \right).$$

Recall that all feasible solutions to (3) are also feasible solutions to (13). The objective value of feasible solutions to (3) is not larger than the objective value in (13) (see [16] for the necessary proofs). Thus, the optimal solution value to the relaxed problem (13) is an upper bound to the original problem (3) for any vector of nonnegative Lagrange multipliers

$$\begin{aligned} \lambda &:= (\lambda_1, \dots, \lambda_n)^T, \\ \lambda_j &\geq 0, \quad j = 1, \dots, n. \end{aligned}$$

For a concrete numerical solution of the relaxed problem (13) we use here the classic branch-and-bound method (see e.g., [11,16]). In a branch-and-bound algorithm we are interested in achieving the tightest upper bound in (13). Hence, we would like to choose a vector of nonnegative multipliers

$$\begin{aligned} \hat{\lambda}^{\mathcal{L}} &:= (\hat{\lambda}_1^{\mathcal{L}}, \dots, \hat{\lambda}_n^{\mathcal{L}})^T, \\ \hat{\lambda}_j^{\mathcal{L}} &\geq 0, \quad j = 1, \dots, n \end{aligned}$$

such that (13) is minimized. This evidently leads to the generic Lagrangian dual problem

$$(14) \quad \begin{aligned} &\text{minimize } \mathcal{L}(z, \lambda) \\ &\text{subject to } \lambda \geq 0 \end{aligned}$$

It is well-known that the Lagrangian dual problem (14) yields the least upper bound available from all possible Lagrangian relaxations. The problem of finding an optimal vector of multipliers $\hat{\lambda}^{\mathcal{L}} \geq 0$ in (14) is in fact a linear programming problem [11,19]. In a typical branch-and-bound algorithm one will often be satisfied with a sub-optimal choice of multipliers $\lambda \geq 0$ if only the bound can be derived quickly. In this case sub-gradient optimization techniques can be applied [19]. The following analytic result is an immediate consequence of our main Theorem 1 and of the basic properties of the primal-dual system (13)-(14).

Theorem 4.2. *Let $(\hat{z}^{\mathcal{L}}, \hat{\lambda}^{\mathcal{L}})$ be an optimal solution of the primal-dual system (13)-(14) associated with the auxiliary problem (3). Assume that all conditions of Theorem 1 be satisfied. Then*

$$(15) \quad J(z^{\text{opt}}(y^{\text{opt}})) \leq J(\hat{z}^{\mathcal{L}}).$$

The obtained estimation (15) constitutes a tightest upper bound for the optimal value $J(z^{\text{opt}}(y^{\text{opt}}))$.

We are now ready to formulate a complete (conceptual) algorithm for an effective numerical treatment of the basic MCLP (1).

Algorithm 1.

- I. *Given an initial MCLP (1) separate it into two auxiliary problems (2) and (3);*
- II. Apply the combinatorial algorithm (11)-(12) and compute $\hat{y}$;
- III. Using $\hat{y}$, construct the Lagrange function $\mathcal{L}(z, \lambda)$ and solve the primal-dual system (13)-(14).

The numerical consistency of the proposed Algorithm 1 is an immediate consequence of the obtained main theoretic results, namely, of Theorem 3.1 and Theorem 4.2. Recall that the Lagrange relaxation scheme is usually applied to the original MCLP (1) (see e.g., [12,16]). In that case the resulting (relaxed) problem and the corresponding Lagrangian dual problem possess a higher complexity in comparison with the proposed "partial" Lagrange relaxation (13)-(14) associated with the original MCLP (1). This is a simple consequence of the proposed SM that reduces the initial problem (1) to two (more simple) auxiliary optimization problem (2)-(3). This fact makes it possible to apply the proposed separation methodology to the large-scale MCLPs that are important and realistic mathematical models for many practically oriented (optimal) decision making systems (see e.g., [7,9,10,14,15,18,20,22,23,24]).

4.3 A remark on the adequate scalarizing procedure

Let us now make a short remark related to the scalarizing procedure used above (see Section 3, problems (2)-(4)).

It can be shown analytically that the values S_{A_i} in (12) depend on the multipliers vector μ . This is a consequence of the inclusion (9). Recall that (9) constitutes a useful relation of the SM and for the resulting optimization strategy we propose. Since the obtained multiobjective maximization problem (5) has a linear structure, an adequate scalarizing makes it possible to determine every "non-dominant" points (see [11,19] for mathematical details).

On the other hand, a possible "non-adequate" selection of μ geometrically implies a significant "cutting" (restriction) of the feasible region for problem (3). This feasible region restriction can finally eliminate a true optimal solution. Recall that a scalarizing implemented in the objective function from (2) evidently determines the resulting geometry associated with the basic problem (3). On the other side the geometrical properties of a non-adequately scalarized problem can violate the conceptual condition (9).

5. Optimization of the resilient supply chain management system

This section is devoted to applications of the proposed SM to an optimal resilient Supply Chain Management for a system of manufacturing plants - warehouses. Note that the "resilience" of a Supply Chain Management System is modelled here by a fuzzy-type eligibility matrix A (see Section 2). We use here the notation from Section 4 and denote by $\mathcal{A}_i$ a vector of i -row of the eligibility matrix A ($i = 1, \dots, l$) such that

$$A = (\mathcal{A}_1^T \dots \mathcal{A}_l^T)^T.$$

Let us firstly point the common applied meaning of the variables and parameters from the general MCLP (1) in the context of the resilient Supply Chain Management system. The binary variables

$$(z, y) \in \mathbb{B}^n \otimes \mathbb{B}^l$$

constitute the main "decision variables" of the problem under consideration. The vector of weights w can be interpreted as a rentability of the final product. Therefore, the maximization of the cost functional $J(\cdot)$ in (1) expresses the maximization of the total profit (total income) generated by the designed Supply Chain system. The complete "decision resource" associated with the decision variable (vector) y is restricted in (1) by a constant (parameter) $k \in \mathbb{N}$. The eligibility matrix "A" is in fact a useful linear modelling framework that establishes the natural relation between the "producer" decision and "recipient". This relation is formally given by the corresponding elements a_{ij} of the matrix A . Our aim now is to apply the developed SM to two practically oriented examples of the optimal Supply Chain Management design in a classic manufactures - warehouses system.

Example 5.1. The simple Supply Chain system that include $n = 8$ manufacturing plants and $l = 5$ warehouses is indicated on Fig. 1.

We also assume that

$$a_{ij} + a_{i'j} \geq 1, \quad i = 1, \dots, 5 \quad j = 1, \dots, 8.$$

Here i' is an index that corresponds to a resilient cover of a demand point. The last condition means that at least two feasible facilities (warehouse) cover a given demand point (the manufacturing plants). The corresponding eligibility matrix A is given as follows:

$$A^T = \begin{pmatrix} 0.81286 & 0.0 & 0.0 & 0.62968 & 0.0 \\ 0.25123 & 0.58108 & 0.32049 & 0.89444 & 0.79300 \\ 0.0 & 0.0 & 0.64850 & 0.91921 & 0.94740 \\ 0.54893 & 0.90309 & 0.74559 & 0.50869 & 0.99279 \\ 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.77105 & 0.27081 & 0.65883 & 0.60434 & 0.23595 \\ 0.0 & 0.51569 & 0.0 & 0.0 & 0.57810 \\ 0.64741 & 0.91733 & 0.60562 & 0.63874 & 0.71511 \end{pmatrix}$$

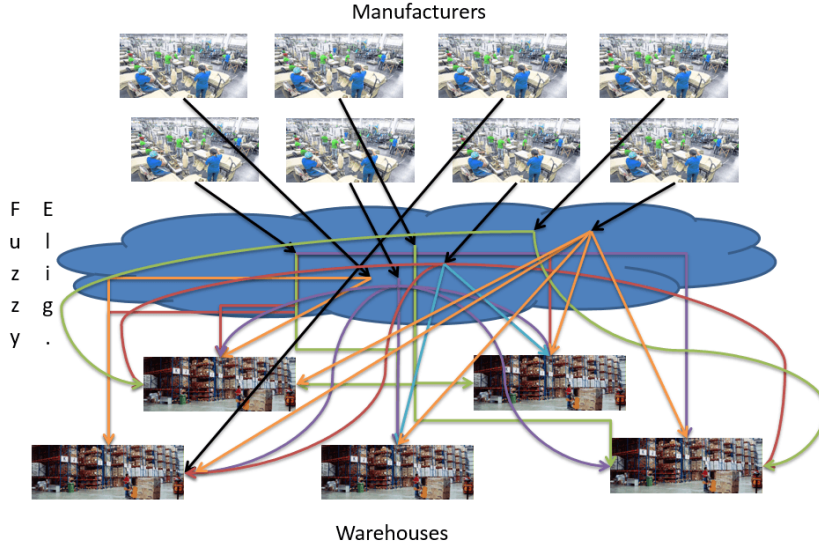


Figure 1: Fuzzy eligibility model

The objective weights

$$w_j \in \mathbb{R}_+, \quad j = 1, \dots, 8$$

indicate the service priority and are selected in this example as follows

$$w = (32.0, 19.0, 41.0, 26.0, 37.0, 49.0, 50.0, 11.0)^T.$$

Note that the fifth demand point in this example has no "resilient" character (only one facility covers this point). We assume that the Supply Chain decision maker is interested opens $k = 2$ facilities. That means

$$\sum_{i=1}^5 y_i = 2.$$

Moreover, we also define the necessary row vectors (see Section 3) for the combinatorial algorithm (11)-(12):

$$\begin{aligned} S_{A_1} &= 8.06295 & S_{A_2} &= 5.86033 \\ S_{A_3} &= 5.30955 & S_{A_4} &= 7.47098 \\ S_{A_5} &= 6.99921 \end{aligned}$$

Application of the basic Algorithm 1 leads to the following computational results:

$$\begin{aligned} (16) \quad z^{opt} &= (1, 1, 0, 1, 1, 1, 0, 1)^T, \\ y^{opt} &= (1, 0, 0, 1, 0)^T, \end{aligned}$$

The corresponding (maximal) value of the objective functional is equal to

$$J(z^{opt}(y^{opt})) = \max_{Problem(1)} J(z(y)) = 174.0$$

Let us also note that the computed scalarizing multiplier μ in the auxiliary problem (2) for the given problem data is equal to

$$\mu = (2.0, 2.0, 1.0, 2.0, 2.0, 2.0, 1.0, 2.0)^T.$$

The practical implementation of Algorithm 1 was carried out by using the standard Python package and an author-written program.

For comparison, the given MCLP problem was also solved by a direct application of the standard CPLEX optimization package. We use the concrete problem parameters given above and obtain the same optimal pair as in (16). The CPLEX integer programming solver proceeds with 6 MIP simplex iterations and 0 branch-and-bound nodes for in total 13 binary variables and 9 linear constraints.

Example 5.2. We now consider a formal extension of the previous example (for a double dimension) and put $n = 16$, $l = 10$, $k = 5$. Let

$$w = (29.0, 37.0, 22.0, 42.0, 26.0, 14.0, 27.0, 30.0, 46.0, 16.0, 10.0, 36.0, 33.0, 39.0, 46.0, 49.0)^T.$$

The eligibility matrix A is given by rows:

$$\mathcal{A}_1 = (0.846109459436, 0.0, 0.0, 0.582693667799, 0.964574511054, 0.798899459366, 0.0, 0.0, 1.0, 0.300320432977, 0.997688107849, 0.3335795069, 0.49602683501, 1.0, 0.0, 0.374671961499)^T,$$

$$\mathcal{A}_2 = (0.0, 1.0, 0.0, 0.0, 0.741552391071, 0.537788748272, 0.883796533814, 0.585368404373, 0.0, 0.860903890172, 0.958028639759, 0.0, 0.186896812387, 0.0, 0.968601622008, 0.579580096602)^T,$$

$$\mathcal{A}_3 = (0.407084305512, 0.0, 0.565187029512, 0.0, 0.420858280659, 0.361836079442, 0.472471488805, 0.0, 0.0, 0.696525107652, 0.436819747759, 0.0, 0.587300759229, 0.0, 0.347864951313, 0.0)^T,$$

$$\mathcal{A}_4 = (0.208102698902, 0.0, 0.0, 0.0, 0.0, 0.346461956794, 0.0, 0.0, 0.0, 0.768124612788, 0.413970925056, 0.0, 0.97348389961, 0.0, 0.0, 0.0)^T,$$

$$\mathcal{A}_5 = (0.0, 0.0, 0.965589029405, 0.0, 0.0, 0.893792904298, 0.0, 0.723969499937, 0.0, 0.562381237935, 0.78216104002, 0.557958082269, 0.671624833192, 0.0, 0.601221801206, 0.0)^T,$$

$$\mathcal{A}_6 = (0.0, 0.0, 0.0, 0.7732353822, 0.0, 0.930557571029, 0.0, 0.427721730484, 0.0, 0.818424694417, 0.795450242494, 0.314453291276, 0.645666417485, 0.0, 0.0, 0.0)^T,$$

$$\mathcal{A}_7 = (0.0, 0.0, 0.0, 0.71613857057, 0.0, 0.573866657173, 0.0, 0.692538237821, 0.0, 0.296797567788, 0.306871729419, 0.334127066948, 0.0, 0.0, 0.0, 0.976783604764)^T,$$

$$\mathcal{A}_8 = (0.448086601628, 0.0, 0.888380378484, 0.576276602931, 0.939065250623, 0.0, 0.0, 0.773234003255, 0.0, 0.414398315721, 0.203669220313, 0.35600682894, 0.523619957827, 0.0, 0.0, 0.527029464076)^T,$$

$$\mathcal{A}_9 = (0.964964029806, 0.0, 0.0, 0.562565185744, 0.0, 0.0, 0.0, 0.0, 0.0, 0.773260049125, 0.468988424786, 0.0, 0.0, 0.0, 0.0, 0.794463270734)^T,$$

$$\mathcal{A}_{10} = (0.0, 0.0, 0.0, 0.545222010668, 0.0, 0.0, 0.536645142919, 0.212898303253, 0.0, 0.197891148706, 0.471120100438, 0.0, 0.0, 0.0, 0.0, 0.0)^T.$$

The basic Algorithm 1 was applied to this example. We obtain the following optimal solution:

$$z^{opt} = (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)^T,$$

$$y^{opt} = (1, 1, 1, 0, 1, 0, 0, 1, 0, 0)^T.$$

The obtained scalarizing multiplier μ in the auxiliary problem (2) for the given problem data is equal to

$$\mu = (2.0, 2.0, 4.0, 4.0, 2.0, 0.0, 8.0, 2.0, 1.0, 2.0, 6.0, 1.0, 0.0, 1.0, 1.0, 1.0)^T.$$

Finally, the calculated optimal value of the objective functional is equal to

$$J(z^{opt}(y^{opt})) = \max_{Problem(1)} J(z(y)) = 502.$$

Let us note that the successful application of the proposed computational algorithm to the above high-dimensional problem indicates a possible usability of this approach in the effective solution procedures of large-scale MCLPs.

Finally let us note that the CPLEX based comparative analysis and the computational results obtained in Example 5.1 and Example 5.2 illustrate the realisability and effectiveness of the Separation Method developed in our paper.

6. Conclusion

In this contribution, we proposed a conceptually new numerical approach to a wide class of Maximal Covering Location Problems with the fuzzy-type eligibility matrices. This computational algorithm is next applied to the optimal design of a practically motivated Resilient Supply Chain Management System. The developed computational scheme is based on a novel separation approach to the initially given maximization problem. The SM we propose makes it possible to reduce the original sophisticated problem to two Knapsack-type optimization

problems. The first one constitutes a generic linear scalarization of a multi-objective optimization problem and the second auxiliary problem is a version of the classic Knapsack formulation. Application of the conventional Lagrange relaxation in combination with a specific combinatorial algorithm leads to an implementable algorithm for the given fuzzy-type Maximal Covering Location Problem.

Theoretical and computational methodologies we present in this contribution can be applied to various generalizations of the basic MCLP. One can combine the elaborated separation scheme with the conventional branch-and-bound method, with the celebrated dynamic programming approach or/and with an alternative exact or heuristic numerical algorithm. Let us finally note that we discussed here only main theoretic aspects of the newly elaborated approach and presented the corresponding conceptual solution procedure. The basic methodology we developed needs further comprehensively numerical examinations that includes solutions of several MCLPs.

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A CHARACTERIZATION OF MATHIEU GROUPS BY THEIR ORDERS AND CHARACTER DEGREE GRAPHS

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Abstract. Let G be a finite group. The character degree graph $\Gamma(G)$ of G is the graph whose vertices are the prime divisors of character degrees of G and two vertices p and q are joined by an edge if pq divides some character degree of G . Let $L_n(q)$ be the projective special linear group of degree n over finite field of order q . Xu et al. proved that the Mathieu groups are characterized by the order and one irreducible character degree. Recently Khosravi et al. have proven that the simple groups $L_2(p^2)$, and $L_2(p)$ where $p \in \{7, 8, 11, 13, 17, 19\}$ are characterizable by the degree graphs and their orders. In this paper, we give a new characterization of Mathieu groups by using the character degree graphs and their orders.

Keywords: Character degree graph, Mathieu group, simple group, character degree.

1. Introduction

All groups in this note are finite. Let G be a finite group and let $\text{Irr}(G)$ be the set of irreducible characters of G . Denote by $\text{cd}(G) = \{\chi(1) : \chi \in \text{Irr}(G)\}$, the set of character degrees of G . Some author have studied the Mathieu groups by considering the properties of element orders [1, 11]. Some authors studied the properties of groups by investigating the character degrees [13]. In this paper, we will study the groups by considering the character degree graph. Recall that the graph $\Gamma(G)$ is called *character degree graph* whose vertices are the prime divisors of character degrees of the group G and two vertices p and q are joined by an edge if pq divides some character degree of G [10]. Xu et al. in [13] have shown that Mathieu groups are determined by some character degree and their orders. Khosravi et. al. in [6, 15, 9] proved that the groups $L_2(p^2)$, where p is a

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prime, and $L_3(q)$ where $q \in \{4, 5, 7, 8, 9\}$, are characterizable by their character degree graphs and orders. Khosravi et. al. in [5] investigated the influence of the character degree graph and order of the simple groups of order less than 6000, on the structure of group. As the development of this topic, we give a new characterization of the Mathieu groups by their character degree graphs and orders. The following theorem is proved.

Main Theorem 1.1. *The following statements hold*

- (1) *Let $L \in \{M_{11}, M_{23}, M_{24}\}$. If G is a finite group such that $\Gamma(G) = \Gamma(L)$ and $|G| = |L|$, then $G \cong L$.*
- (2) *Let $L := M_{12}$. If G is a finite group such that $\Gamma(G) = \Gamma(L)$ and $|G| = |L|$, then $G \cong L$ or $G \cong A_4 \times M_{11}$.*

We introduce some notation here. Let S_n be the symmetric group of degree n . Let $L_n(q)$ be the projective special linear group of degree n over finite field of order q . Let G be a group and let r be a prime, then denote the set of Sylow r -subgroups G_r of G by $\text{Syl}_r(G)$. If H is a characteristic subgroup of G , we write $H \text{ ch } G$. All other symbols are standard (see [2]).

2. Some preliminary results

In this section, we give some lemmas to prove the main theorem.

Lemma 2.1. *Let $A \trianglelefteq G$ be abelian. Then $\chi(1)$ divides $|G : A|$ for all $\chi \in \text{Irr}(G)$.*

Proof. See Theorem 6.5 of [4]. □

Lemma 2.2. *Let $N \triangleleft G$ and let $\chi \in \text{Irr}(G)$. Let θ be an irreducible constituent of χ_N and suppose that $\theta_1, \dots, \theta_t$ are distinct conjugates of θ in G . Then $\chi_N = e \sum_{i=1}^t \theta_i$, where $e = [\chi_N, \theta]$ and $t = |G : I_G(\theta)|$. Also $\theta(1) \mid \chi(1)$ and $\frac{\chi(1)}{\theta(1)} \mid \frac{|G|}{|N|}$.*

Proof. Theorems 6.2, 6.8 and 11.29 of [4]. □

Lemma 2.3. *Let G be a non-solvable group. Then G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that K/H is a direct product of isomorphic non-abelian simple groups and $|G/K| \mid |\text{Out}(K/H)|$.*

Proof. See Lemma 1 of [12]. □

Lemma 2.4. *Let G be a finite solvable group of order $p_1^{a_1} p_2^{a_2} \cdots p_n^{a_n}$, where $p_1, p_2, \dots, p_n$ are distinct primes. If $kp_n + 1 \nmid p_i^{a_i}$ for each $i \leq n-1$ and $k > 0$, then the Sylow p_n -subgroup is normal in G .*

Proof. See Lemma 2 of [13]. □

We also need the structure of non-abelian simple groups whose largest prime divisor is 11 or 23.

Lemma 2.5. *If S is a finite non-abelian simple group such that $\{11\} \subseteq \pi(S) \subseteq \{2, 3, 5, 11\}$, then S is isomorphic to one of the following simple groups listed as in Table 1.*

Proof. See [14]. □

Table 1. Simple groups S with $\{11\} \subseteq \pi(S) \subseteq \{2, 3, 5, 7, 11\}$

S	Order of S	$Out(S)$	S	Order of S	$ Out(S) $
$U_5(2)$	$2^{10} \cdot 3^5 \cdot 5 \cdot 11$	2	$U_6(2)$	$2^{15} \cdot 3^6 \cdot 5 \cdot 7 \cdot 11$	6
$L_2(11)$	$2^2 \cdot 3 \cdot 5 \cdot 11$	2	M_{11}	$2^4 \cdot 3^2 \cdot 5 \cdot 11$	1
M_{12}	$2^6 \cdot 3^3 \cdot 5 \cdot 11$	2	M_{22}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	2
HS	$2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$	2	M^cL	$2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$	2
A_{11}	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11$	2	A_{12}	$2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$	2

Lemma 2.6. *If S is a finite non-abelian simple group except for alternating group such that $\{23\} \subseteq \pi(S) \subseteq \{2, 3, 5, 7, 11, 13, 17, 19, 23\}$, then S is isomorphic to one of the following simple groups listed as in Table 2.*

Proof. See [14]. □

Table 2. Simple group S with $\{23\} \subseteq \pi(S) \subseteq \{2, 3, 5, 7, 11, 13, 17, 19, 23\}$

S	Order of S	$ Out(S) $
$L_2(23)$	$2^3 \cdot 3 \cdot 11 \cdot 23$	2
$U_3(23)$	$2^7 \cdot 3^2 \cdot 11 \cdot 13^2 \cdot 23^2$	4
M_{23}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	1
M_{24}	$2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	1
Co_3	$2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$	1
Co_2	$2^{18} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$	1
Co_1	$2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23$	1
Fi_{23}	$2^{18} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23$	1

3. The proof of Main Theorem

In this section, we will prove the main theorem.

Proof of Main Theorem

Proof. We prove the results by the following cases.

Case 1. $L = M_{11}$.

Then $|L| = 2^4 \cdot 3^2 \cdot 5 \cdot 11$. It is easy to get from [2], that $\text{cd}(L) = \{1, 10, 11, 16, 44, 45, 55\}$. So the graph $\Gamma(L)$ has the vertices $\{2, 3, 5, 11\}$, the prime 5 is adjacent to the primes 2, 3, and 11, but the prime 3 is not adjacent to the primes 2 and 11. By in $\Gamma(G)$, there is a character χ such that $\chi(1)$ is divisible by 55.

We can conclude that $O_{11}(G) = 1 = O_5(G)$. In fact, if $O_{11}(G) \neq 1$, then since $|G_{11}| = 11$, then $O_{11}(G)$ is a Sylow 11-subgroup. Then by Lemma 2.1,

there is a character $\chi \in \text{Irr}(G)$ such that $\chi(1) \mid |G : O_{11}(G)|$, a contradiction. Hence we have $O_{11}(G) = 1$. Similarly, $O_5(G) = 1$.

Assumed that G is a solvable group. Let M be a minimal normal subgroup of G . Then M is an elementary abelian p -group where $p = 2$ or $p = 3$. Since in $\Gamma(G)$ the prime 5 is adjacent to the primes 2 and 3, then we can assume that $|M| \mid 2^3$ or $|M| = 3$.

Case 1.1. Let M be a 3-group. Let H/M be a Hall subgroup of order $2^4 \cdot 5 \cdot 11$. Then $|G/M : H/M| = 3$ and so $(G/M)/(H/M) \hookrightarrow S_3$, where $L/M = \text{Core}_{G/M}(H/M)$. Therefore $11 \mid |L/M|$. By Lemma 2.4, Q/M is normal in L/M , where Q/M is a Sylow 11-subgroup of L/M . Hence $Q \trianglelefteq G$ and $|Q| = 33$. Therefore $O_{11}(G) \neq 1$, a contradiction.

Case 1.2. Let M be a 2-group. Then $|M| = 2^k$ with $1 \leq k \leq 3$. Let H/M be a Hall subgroup of order $3^3 \cdot 5 \cdot 11$. Then $|G/M : H/M| = 2^{4-k}$.

Let $k = 3, 2$ or 1 . Then $G/H_G \hookrightarrow S_2$, $G/H_G \hookrightarrow S_4$ or $G/H_G \hookrightarrow S_8$ respectively. Then in these three cases, $11 \mid |H_G|$. By Lemma 2.4, Q/M , the Sylow 11-subgroup of H_G/M , is also normal in H_G/M . It follows that $Q \trianglelefteq G$. Since $|Q| = 2^k \cdot 11$, then $G_{11} \triangleleft H_G \subset G$ and so G_{11} is normal in G , a contradiction.

Therefore G is non-solvable and so by Lemma 2.3, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that K/H is a direct product of isomorphic non-abelian simple groups and $|G/K| \mid |\text{Out}(K/H)|$.

We will prove that $11 \in \pi(K/H)$. Assume the contrary, then obviously by Lemma 6(d) of [7] and Lemma 2.13 of [8], $|\text{Out}(K/H)|$ is not divisible by 11. If $11 \mid |H|$, then there is a Hall $\{p, 7\}$ -subgroup D of H , where p is a prime and $p \in \{2, 3, 5\}$, then by considering group order and Lemma 2.4, D is cyclic and so D is abelian. By Lemma 2.1, $\chi(1) \mid |G : D|$, a contradiction. Therefore $11 \nmid |K/H|$.

In $\Gamma(G)$, the prime 3 is not adjacent to the primes 2 and 11 and so, by Lemma 2.5 and order consideration, K/H is isomorphic to one of the simple groups: $L_2(11)$ or L .

Let $K/M \cong L_2(11)$. By [2], $\text{cd}(L_2(11)) = \{1, 5, 10, 11, 12\}$ and so in $\Gamma(L_2(11))$, the prime 11 is adjacent to the prime 3. It follows that the prime 2 is adjacent to the prime 3 in $\Gamma(G)$, a contradiction.

Let $K/M \cong L$. Then $M = 1$ and $G \cong L$ by order consideration.

Case 2. $L = M_{12}$.

Then $|L| = 2^6 \cdot 3^3 \cdot 5 \cdot 11$. By [2], $\text{cd}(L) = \{1, 11, 16, 45, 54, 55, 66, 99, 120, 144, 176\}$ and so the graph $\Gamma(G)$ is complete with vertex set $\{2, 3, 5, 11\}$.

Similarly as Case 1, we can prove that $O_{11}(G) = 1$.

Assumed that G is a solvable group. Let M be a minimal normal subgroup of G . Then M is an elementary abelian p -group where $p = 2$ or $p = 3$ (in fact, if $p = 5$, then since $|G_5| = 5 = |M|$, there is a character $\chi \in \text{Irr}(G)$ such that $\chi(1) \mid |G : M|$, contradicting Lemma 2.1).

Let M be a 3-group. Then $|M| = 3^k$ with $1 \leq k \leq 2$ since $\Gamma(G)$ is complete. Let H/M be a Hall subgroup of order $2^6 \cdot 5 \cdot 11$. Then $|G/M : H/M| = 3^{3-k}$ and so $\frac{G}{H_G} \hookrightarrow S_9$ when $k = 1$ or $\frac{G}{H_G} \hookrightarrow S_3$ when $k = 2$. It follows that $11 \mid |H_G|$.

Let Q/M be a Sylow 11-subgroup of H_G/M . Since $|H_G/M||H/M| = 2^6 \cdot 5 \cdot 11$, then Q/M is normal in H_G and so $Q \trianglelefteq G$. Since $|Q| = 3^k \cdot 11$, then $O_{11}(G)$ is normal in G , a contradiction.

Let M be a 2-group. Then $|M| = 2^k$ with $1 \leq k \leq 5$ since $\Gamma(G)$ is complete. Let H/M be a Hall subgroup of order $3^3 \cdot 5 \cdot 11$ of G/M . Then $|G/M : H/M| = 2^{6-k}$.

Let $3 \leq k \leq 5$. Then $G/H_G \cong S_8$ when $k = 3$, $G/H_G \cong S_4$ when $k = 4$, or $G/H_G \cong S_2$ when $k = 5$. In these three cases, $11 \nmid |H_G|$. Let Q/M be a Sylow 11-subgroup of H_G/M . Since $|H_G/M||H/M| = 3^3 \cdot 5 \cdot 11$, then Q/M is normal in H_G/M and so $Q \trianglelefteq G$. Since $|Q| = 2^k \cdot 11$, then $O_{11}(G)$ is normal in G , a contradiction.

Let $1 \leq k \leq 2$. Let Q/M be a Sylow 11-subgroup of H/M . Then by Lemma 2.4, Q/M is normal in H/M , in particular, $Q \trianglelefteq H$. Since $|Q| = 2^k \cdot 11$, then G_{11} is normal in H and so $N/C := \frac{N_G(G_{11})}{C_G(G_{11})} \lesssim Z_{10}$. If $N/C \cong Z_{10}$ or $N/C \cong Z_5$, then $C_G(G_{11})$ is a $\{2, 3, 11\}$ -group. It is easy to see that $G_{11} \text{ ch } C$ and so $G_{11} \trianglelefteq G$. If $N/C \cong Z_2$, then N/C is a $\{2, 3, 5, 11\}$ -group and so $G_{11} \text{ ch } C$ and so $G_{11} \trianglelefteq G$. If $N = C$, then also we have $G_{11} \trianglelefteq G$. So in these cases, we rule out since $O_{11}(G) = 1$.

Therefore, G is non-solvable and so by Lemma 2.3, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that K/H is a direct product of isomorphic non-abelian simple groups and $|G/K||\text{Out}(K/H)|$. Similarly as Case 1, we can show that $11 \nmid |K/H|$. Therefore by Lemma 2.5, K/H is isomorphic to $L_2(11)$, M_{11} or L .

Let $K/H \cong L_2(11)$. Then $L_2(11) \leq G/H \leq \text{Aut}(L_2(11))$ and $|G/K| |\text{Out}(L_2(11))| = 2$. If $G/H \cong L_2(11)$, then $|H| = 2^2 \cdot 3$. By [2], $\text{cd}(L_2(11)) = \{1, 5, 10, 11, 12\}$ and so in $\Gamma(L_2(11))$, the primes 2, 3 are not adjacent to the prime 11. Since $5 \nmid |H|$, then this case can't occur. Similarly we can rule out the two cases $G/H \cong Z_2.L_2(11)$ and $G/H \cong SL_2(11)$.

Let $K/H \cong M_{11}$. Since $|\text{Out}(M_{11})| = 1$ and $\text{cd}(M_{11}) = \{1, 10, 11, 16, 44, 45, 55\}$, then $G/H \cong M_{11}$ and $|H| = 12$. On the other hand, in $\Gamma(M_{11})$, the prime 3 is not adjacent to the primes 2 and 11. Therefore $G = A_4 \times M_{11}$.

Let $K/H \cong L$. Then $H = 1$ and so order consideration implies that $G \cong L$.

Case 3. $L = M_{23}$.

Then $|L| = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$ and $|\text{Out}(L)| = 1$.

By [2], $\text{cd}(L) = \{1, 22, 45, 230, 231, 253, 770, 896, 990, 1035, 2024\}$ and so in $\Gamma(L)$, the prime 7 is not adjacent to the prime 23.

We will prove $O_{23}(G) = 1$. Assume the contrary, then $O_{23}(G)$ is normal in G . But $|G|_{23} = 23$ and so $O_{23}(G)$ is an abelian normal Sylow 23-subgroup of G . It follows that for all $\chi \in \text{Irr}(G)$, $\chi(1) \nmid |G : O_{23}(G)|$, contradicting Lemma 2.1. Similarly, we can conclude that $O_{11}(G) = O_7(G) = O_5(G) = 1$.

Assumed that G is a solvable group. Let M be a minimal normal subgroup of G . Then M is an elementary abelian p -group where $p = 2$ or $p = 3$.

Let M be a 3-group. Then $|M| = 3$ since in $\Gamma(G)$ the prime 3 is adjacent to the prime 23. Let H/M be a Hall subgroup of order $2^7 \cdot 5 \cdot 7 \cdot 11 \cdot 23$. Then

$|G/M : H/M| = 3$ and so $\frac{G}{H_G} \hookrightarrow S_3$. It follows that $23 \mid |H_G|$. Let Q/M be a Sylow 23-subgroup of H_G/M . Since $|H_G/M| \mid |H/M| = 2^7 \cdot 5 \cdot 7 \cdot 11 \cdot 23$, then Q/M is normal in H_G/M and so $Q \trianglelefteq G$. Since $|Q| = 3 \cdot 23$, then $O_{23}(G)$ is normal in G , a contradiction.

Let M be a 2-group. Then $|M| = 2^k$ with $1 \leq k \leq 6$ since in $\Gamma(G)$, the prime 2 is adjacent to the prime 23. Let H/M be a Hall subgroup of order $3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$ of G/M . Then $|G/M : H/M| = 2^{7-k}$.

Let $3 \leq k \leq 6$. Then $G/H_G \cong S_{16}$ when $k = 3$, $G/H_G \cong S_8$ when $k = 4$, $G/H_G \cong S_4$ when $k = 5$, or $G/H_G \cong S_2$ when $k = 6$. In these four cases, $23 \mid |H_G|$. Let Q/M be a Sylow 23-subgroup of H_G/M . Since $|H_G/M| \mid |H/M| = 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$, then Q/M is normal in H_G/M and so $Q \trianglelefteq H_G$. Since $|Q| = 2^k \cdot 23$, then G_{23} is normal in G .

Let $1 \leq k \leq 2$. Let Q/M be a Sylow 23-subgroup of H/M . Then by Lemma 2.4, Q/M is normal in H/M , in particular, $Q \trianglelefteq H$. Since $|Q| = 2^k \cdot 23$, then G_{23} is normal in H and so $N/C := \frac{N_G(G_{23})}{C_G(G_{23})} \lesssim Z_{22}$. If $N/C \cong Z_{11}$ or Z_{22} , then C is a $\{2, 3, 5, 7, 23\}$ -group and by Lemma 2.4 G_{23} is normal in C . If $N/C \cong Z_2$, then C is a $\{2, 3, 5, 7, 11, 23\}$ -group and also G_{23} is normal in C . In these cases, we have $O_{23}(G) \neq 1$, a contradiction.

Therefore, G is non-solvable and so by Lemma 2.3, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that K/H is a direct product of isomorphic non-abelian simple groups and $|G/K| \mid |Out(K/H)|$. Similarly as Case 1, we can show that $7, 11, 23 \mid |K/H|$. Therefore by Lemma 2.6, K/H is isomorphic to L .

It is easy to get that G is isomorphic to L by group order.

Case 4. $L = M_{24}$.

In this case, $|L| = 2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$ and $Mult(L) = 1$.

Similarly as Case 1, we can prove that $O_{23}(G) = O_{11}(G) = O_7(G) = O_5(G) = 1$.

By [2], $cd(L) = \{1, 23, 45, 231, 252, 253, 483, 770, 990, 1035, 1265, 1771, 2024, 2277, 3312, 3520, 5313, 5544, 5796, 10395\}$. It means that the graph $\Gamma(G)$ is complete.

Assumed that G is a solvable group. Let M be a minimal normal subgroup of G . Then M is an elementary abelian p -group with $p = 2$ or 3 .

Let M be a 3-group. Let H/M be a Hall subgroup of G/M of order $2^{10} \cdot 5 \cdot 7 \cdot 11 \cdot 23$. Then similarly as Case 1, we can get that $G/H_G \hookrightarrow S_9$ or $G/H_G \hookrightarrow S_3$ since $\Gamma(G)$ is complete. In both cases, $23 \mid |H_G|$. By Lemma 2.4, we know that $G_{23}M/M$ is normal in H_G/M (note that the Sylow 23-subgroup of H_G is also a Sylow 23-subgroup of G) and so $G_{23}M \trianglelefteq G$. We know that $G_{23} \trianglelefteq G_{23}M$. But $H_G \not\subset G$. Hence $G_{23} \trianglelefteq G$, a contradiction.

Let M be a 2-group. Let H/M be a Hall subgroup of G/M with order $3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$. Let Q/M be a Sylow 23-subgroup of H/M . Then by Lemma 2.4, Q/M is normal in H/M and so $Q \trianglelefteq H$. We have $|Q| = 2^k \cdot 23$ and $G_{23} \subsetneq Q$. So G_{23} is normal in H . Similarly as Case 2, G_{23} is normal in G , a contradiction.

Therefore, G is non-solvable and so by Lemma 2.3, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that K/H is a direct product of isomorphic non-abelian simple groups and $|G/K||\text{Out}(K/H)|$. Similarly as Case 1, we can show that $5, 7, 11, 23 \mid |K/H|$. Therefore by Lemma 2.6, K/H is isomorphic to M_{23} or L .

If $K/H \cong M_{23}$, then $G/H \cong M_{23}$ since $\text{Mult}(M_{23}) = 1$ and $\text{Out}(M_{23}) = 1$. It follows that $|H|$ is a $\{2, 3\}$. Since in $\Gamma(M_{23})$, the prime 7 is not adjacent to the prime 23 and $\Gamma(G)$ is complete, then we can rule out this case.

If $K/H \cong L$, then $H = 1$ and so order consideration forces $G \cong L$.

This completes the proof of Main Theorem. $\square$

4. Some applications

Huppert in [3] gave the following conjecture related to character degrees of finite simple groups.

Conjecture[3] Let H be any simple nonabelian group and G a group such that $cd(G) = cd(H)$. Then $G \cong H \times A$, where A is abelian.

Then we have the following theorem.

Corollary 4.1. *Let $L \in \{M_{11}, M_{12}, M_{23}, M_{24}\}$ and G a group such that $cd(G) = cd(L)$. Then $G \cong L \times A$, where A is abelian.*

We first show the following easy result.

Lemma 4.2. *Let G be a finite group. If $p^a \mid \chi(1)$ for some $\chi \in \text{Irr}(G)$ and $p^{a+1} \nmid \eta(1)$ for all $\eta \in \text{Irr}(G)$. Then $p^a \mid |G|$. In particular, if $a = 1$ and G is simple group, then $|G|_p = p$.*

Proof. It is easy to get from lemma 2.1. If $a = 1$ and G is simple, then by Problem 3.4 of [4], have $|G|_p = p$. $\square$

Proof of Corollary 4.1.

Proof. By Lemma 4.2, and Main Theorem 1.1, $G/H \cong L$. Order consideration implies the result.

This completes the proof. $\square$

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HYPERMATRIX REPRESENTATIONS OF SINGLE POWER CYCLIC HYPERGROUPS

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Abstract. Cyclic hypergroups are special type of hypergroups that have some importance for their applications in different fields. In this paper, we deal with hypermatrix representations of single power cyclic hypergroups. First, we consider single power cyclic hypergroups with infinite period, define a commutative semihyperring and construct non-trivial hypermatrix representations over our defined semihyperring. Then we do the same for single power cyclic hypergroups with finite period. Many properties of these hypermatrix representations are presented.

Keywords: cyclic hypergroup, representation.

1. Introduction

Hypergroup theory was known for the first time in 1934 at the eighth Congress of Scandinavian Mathematicians, when Marty [14] gave the definition of hypergroup as a generalization of the notion of the group, illustrated some applications and showed its utility in the study of groups, algebraic functions and relational fractions. Recently, the hypergroups are studied from the theoretical point of view and for their applications to many subjects of pure and applied mathematics: geometry, topology, cryptography and code theory, graphs and hypergraphs, probability theory, binary relations, theory of fuzzy and rough sets, automata theory, economy, etc. (see [1, 7, 10, 13, 17]). A hypergroup is an algebraic structure similar to a group, but the composition of two elements is a non-empty set. Representation theory is a branch of mathematics, known in 1896 in the work of the German mathematician F. G. Frobenius (see [8]), that has lots of applications in physics, number theory, etc. It was first known to study representations of algebraic structures (groups, rings, topological spaces, etc) by representing their elements as linear transformations of vector spaces. More

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precisely, a representation makes an abstract algebraic object more concrete by transforming it into matrix and its algebraic operation into matrix addition or multiplication. The concept of representation theory was later generalized to study hyperstructures (see [9, 17, 18, 19]) in which the hypergroup's elements are represented as hypermatrices over some semihyperring. The representation of hypergroups depends on the choice of the semihyperring in which the entries of the hypermatrices belong to. This makes representations of hypergroups much harder than that of representation of groups because of the hyperoperations of the semihyperring which are not always the standard operations.

This paper is a connection between hyperstructures and representation theory. It is constructed as follows: after an introduction, Section 2 presents some basic definitions related to hyperstructures and hypermatrix representations. Sections 3 and 4 present non-trivial hypermatrix representations of single power cyclic hypergroups with infinite and finite period respectively and study their properties.

2. Basic definitions

In this section, we present some definitions related to hyperstructures and matrix representations that are used throughout the paper.

Let H be a non-empty set. Then, a mapping $\circ : H \times H \rightarrow \mathcal{P}^*(H)$ is called a *hyperoperation* on H , where $\mathcal{P}^*(H)$ is the family of all non-empty subsets of H . The couple $(H, \circ)$ is called a *hypergroupoid*. In the above definition, if A and B are two non-empty subsets of H and $x \in H$, then we define:

$$A \circ B = \bigcup_{\substack{a \in A \\ b \in B}} a \circ b, \quad x \circ A = \{x\} \circ A \text{ and } A \circ x = A \circ \{x\}.$$

An element $e \in H$ is called an *identity* of $(H, \circ)$ if $x \in x \circ e \cap e \circ x$, for all $x \in H$ and it is called a *scalar identity* of $(H, \circ)$ if $x \circ e = e \circ x = \{x\}$, for all $x \in H$. If e is a scalar identity of $(H, \circ)$, then e is the unique identity of $(H, \circ)$. The hypergroupoid $(H, \circ)$ is said to be *commutative* if $x \circ y = y \circ x$, for all $x, y \in H$. A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if it is associative, i.e., for every $x, y, z \in H$, we have $x \circ (y \circ z) = (x \circ y) \circ z$ and is called a *quasihypergroup* if for every $x \in H$, $x \circ H = H = H \circ x$. This condition is called the reproduction axiom. The couple $(H, \circ)$ is called a *hypergroup* if it is a semihypergroup and a quasihypergroup.

Cyclic semihypergroups have been studied by Corsini [4], De Salvo and Freni [11], Vougiouklis [16], Leoreanu [12]. Cyclic semihypergroups are important not only in the sphere of finitely generated semihypergroups but also for interesting combinatorial implications.

A hypergroup $(H, \circ)$ is cyclic if there exist $h \in H$ and $s \in \mathbb{N}$ such that

$$H = h \cup h^2 \cup \dots \cup h^s \cup \dots.$$

If $H = h \cup h^2 \cup \dots \cup h^s$ then H is a cyclic hypergroup with finite period. Otherwise, H is called cyclic hypergroup with infinite period. Here, $h^s = \underbrace{h \circ h \circ \dots \circ h}_{s \text{ times}}$. It is clear that for all $\alpha \in H$ there exist $r \in \mathbb{N}$ such that $\alpha \in h^r$.

It is a *single-power cyclic hypergroup* if there exist $h \in H$ and $s \in \mathbb{N}$ such that

$$H = h \cup h^2 \cup \dots \cup h^s \cup \dots \text{ and } h \cup h^2 \cup \dots \cup h^{s-1} \subset h^s, \text{ for all } s \in \mathbb{N}.$$

A *semihyperring* $(R, +, \cdot)$ is a hyperstructure with two hyperoperations $+$ and $\cdot$ where $+$ and $\cdot$ are associative hyperoperations and $\cdot$ is distributive with respect to $+$, i.e., $x \cdot (y + z) \subseteq x \cdot y + x \cdot z$ and $(x + y) \cdot z \subseteq x \cdot z + y \cdot z$ for every $x, y, z \in R_1$. An element $0 \in R$ is called the *zero element* in R if $0 + a = a + 0 = a$ and $0 \cdot a = a \cdot 0 = 0$ for all $a \in R$. An element $1 \in R$ is called a *scalar unit* in R if $1 \cdot a = a \cdot 1 = a$ for all $a \in R$.

A *hypermatrix* is a matrix with entries from a semihyperring. The hyperproduct of two $m \times n$ and $n \times r$ hypermatrices (a_{ij}) and (b_{ij}) is the $n \times r$ hypermatrix defined as

$$(a_{ij})(b_{ij}) = \left\{ (c_{ij}) : c_{ij} \in \sum_{k=1}^n a_{ik} b_{kj} \right\}.$$

Let $(R, +, \cdot)$ be a commutative semihyperring and $(H, \circ)$ be a hypergroup then a map $\rho : H \rightarrow M_k(R)$ is said to be an *inclusion representation* if for all $\alpha, \beta \in H$, we have

$$\rho(\alpha \circ \beta) = \{\rho(\lambda) : \lambda \in \alpha \circ \beta\} \subseteq \rho(\alpha)\rho(\beta).$$

It is called a *good representation* if $\rho(\alpha \circ \beta) = \rho(\alpha)\rho(\beta)$. And it is a *faithful representation* if it is an injective good representation.

Throughout this paper, we define $\exp(\alpha) = \min\{i : \alpha \in h^i\} \geq 0$ for all $\alpha \in H$ where H is a cyclic hypergroup with generator h . We call $e \in H$ a trivial element and write $\exp(e) = 0$ if for all $\alpha \in H$, we have $e \circ \alpha = \alpha \circ e = \alpha$.

3. Representation of single power cyclic hypergroups with infinite period

The terms “single-power cyclic hypergroup” and “cyclic hypergroup of infinite period” were introduced by Vougiouklis and appeared in 1981 [16]. Moreover, the theory of “representations” on hyperstructures was introduced by Vougiouklis, as one can see in [17]. Also, see [5].

In this section, we construct a commutative semihyperring on P , the set of non negative integers, introduce some hypermatrix representations of single power cyclic hypergroups with infinite period and present some of their interesting properties. Our work is a generalization of a part of a previous work done by the authors (see [2]) on a special single power cyclic hypergroup with infinite period associated to the braid group.

Throughout this section, we define $M_k(R_1)$ as the set of all $k \times k$ hypermatrices with entries from $R_1 = (P, \oplus, \odot)$, $(H, \circ)$ is a single power cyclic hypergroup with infinite period having a generator $h \in H$ (unless it is mentioned differently). It is clear that $h \in h^2 \subset \dots \subset h^{s-1} \subset h^s \subset \dots$ for all $s > 2$.

Theorem 3.1 ([2]). *Let P be the set of non negative integers and $a, b \in P$. Then $R_1 = (P, \oplus, \odot)$ is a commutative semihyperring with scalar unit and zero element where $\odot$ is the standard multiplication and $a \oplus b = \{0, 1, \dots, a + b\}$ if $a, b \neq 0$, $a \oplus 0 = 0 \oplus a = a$.*

Proposition 3.2. *$(R_1, \oplus)$ is a single power cyclic hypergroup with infinite period and generator 1.*

Proof. We have that $(R_1, \oplus)$ is commutative and associative by Theorem 3.1. We prove now that the reproduction axiom is satisfied. If $m \in R_1$ then we have two cases; $m = 0$ and $m > 0$. If $m = 0$ then $m \oplus R_1 = \{m \oplus n : n \in R_1\} = \{n : n \in R_1\} = R_1$. If $m > 0$ then $m \oplus R_1 = \{m \oplus r : r \in R_1\} = \bigcup_{r \in R_1} \mathbb{Z}_{m+r+1} = R_1$. Thus, $(R_1, \oplus)$ is a hypergroup.

For every $m \neq 0 \in R_1$ we have that

$$m \in 1^m = \underbrace{1 \oplus 1 \oplus \dots \oplus 1}_{m \text{ times}} = \{0, 1, \dots, m\}.$$

It is easy to see that $R_1 = 1 \cup 1^2 \cup \dots$ and $1 \in 1^2 \subset \dots \subset 1^{s-1} \subset 1^s \subset \dots$ for all $s \in \mathbb{N}$. Therefore, $(R_1, \oplus)$ is single power cyclic hypergroup with infinite period and generator 1. $\square$

Definition 3.3. *A representation $\rho : H \rightarrow M_k(R_1)$ is said to be reducible if it has a proper non trivial invariant subspace, i.e., there exist*

$$S \neq \{0\} \subset R_1^k = \underbrace{R_1 \times \dots \times R_1}_{k \text{ times}}$$

such that $\rho(x)u \in S$, for all $x \in H$ and $u \in S$. Otherwise, it is said to be irreducible.

Definition 3.4. *A representation $\rho : H \rightarrow M_k(R_1)$ is said to be unitary relative to a non-zero symmetric hypermatrix M if $\rho(x)M\rho(x)^T = \rho(x)^T M \rho(x) = M$, for all $x \in H$. Here, T is the matrix transpose.*

Proposition 3.5. *Let $(H, \circ)$ be a commutative hypergroup (not necessary a single power cyclic hypergroup) and $\rho : H \rightarrow M_k(R_1)$ be an inclusion (or good) representation of H . Then $\rho^T : H \rightarrow M_k(R_1)$ defined as $\rho^T(\alpha) = (\rho(\alpha))^T$ is a representation of H where T is the matrix transpose.*

Proof. Let $\alpha, \beta \in H$ and ρ be an inclusion representation such that $\rho(\alpha) = A$ and $\rho(\beta) = B$. We have that $\rho(\alpha \circ \beta) = \rho(\beta \circ \alpha)$ as $(H, \circ)$ is commutative. And having ρ an inclusion representation implies that $\rho(\alpha \circ \beta) = \rho(\beta \circ \alpha) \subseteq$

$\rho(\beta)\rho(\alpha) = BA$. We have that $\rho^T(\alpha \circ \beta) = \{\rho^T(\lambda) : \lambda \in \alpha \circ \beta\} = \{(\rho(\lambda))^T : \lambda \in \alpha \circ \beta\}$. And since $\rho(\alpha \circ \beta) = \{\rho(\gamma) : \gamma \in \alpha \circ \beta\} \subseteq BA$, it follows that $\rho^T(\alpha \circ \beta) \subseteq (BA)^T$. Using the definition of hyperproduct and the fact that R_1 is commutative, it is easy to see that $(BA)^T = A^T B^T$. Therefore, $\rho^T(\alpha \circ \beta) \subseteq \rho^T(\alpha)\rho^T(\beta)$.

The case when ρ is a good representation is done in a similar manner. $\square$

Proposition 3.6. *Let $(H, \circ)$ be a commutative hypergroup (not necessary a single power cyclic hypergroup) and $\rho : H \rightarrow M_k(R_1)$ is a unitary representation relative to a hypermatrix M then $\rho^T : H \rightarrow M_k(R)$ is a unitary representation relative to M .*

Proof. Proposition 3.5 asserts that $\rho^T : H \rightarrow M_k(R)$ is a representation of H . Let $\alpha \in H$ such that $\rho(\alpha) = A$. Having ρ unitary relative to M implies that $AM A^T = A^T M A = M$. The latter can be written again as $(A^T)^T M A^T = A^T M (A^T)^T = M$. Thus, ρ^T is unitary relative to M . $\square$

Proposition 3.7. *Let $(H, \circ)$ be a hypergroup (not necessary a single power cyclic hypergroup) and $\rho : H \rightarrow M_k(R_1)$ be the trivial map defined by $\rho(\alpha) = I_k$ where I_k is the $k \times k$ identity matrix. Then ρ is a good representation of H .*

Proof. Let $\alpha, \beta \in H$ with $\rho(\alpha) = \rho(\beta) = I_k$. We have that $\rho(\alpha \circ \beta) = \{\rho(\lambda) : \lambda \in \alpha \circ \beta\} = I_k$ and $\rho(\alpha)\rho(\beta) = I_k I_k = I_k$ using the definition of hyperproduct. Thus, $\rho(\alpha \circ \beta) = \rho(\alpha)\rho(\beta)$. $\square$

Proposition 3.8. *Let $(H, \circ)$ be a cyclic hypergroup with infinite period having h as a generator and $\rho : H \rightarrow M_k(R_1)$ an inclusion (or good) representation of H satisfying $\rho(h) = I_k$. Then ρ is the trivial representation.*

Proof. Let α be a non trivial element in $H = h \cup h^2 \cup \dots$. Then there exist $r \in R_1$ such that $\alpha \in h^r$. The latter implies that $\rho(\alpha) \in \rho(h^r) \subseteq (\rho(h))^r = I_k^r = I_k$. Thus, $\rho(\alpha) = I_k$.

If ρ is a good representation then the proof results from having ρ an inclusion representation. $\square$

Proposition 3.9. *Let $\rho : H \rightarrow GL_k(R_1)$ be an inclusion diagonal representation of H . Then ρ is the trivial representation. Here, $GL_k(R_1)$ is the set of all matrices with entries from R_1 having non-zero determinant.*

Proof. Let $h \in H$ be a generator of $(H, \circ)$ then there exist $a_1, \dots, a_k \in R_1 \setminus \{0\}$ such that

$$\rho(h) = \begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & \dots & 0 & a_k \end{pmatrix}.$$

Since $h \in h^2$ and ρ is an inclusion representation, it follows that $\rho(h) \in \rho(h^2) \subseteq (\rho(h))^2$. We get now that

$$\begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & \dots & 0 & a_k \end{pmatrix} \subseteq \begin{pmatrix} a_1^2 & 0 & \dots & 0 \\ 0 & a_2^2 & \dots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & \dots & 0 & a_k^2 \end{pmatrix}.$$

The latter implies that $a_i^2 = a_i$ for $i = 1, \dots, k$ which is equivalent to $a_i = 1$ or $a_i = 0$. Having $\rho : H \rightarrow GL_k(R_1)$ implies that $a_i = 1$ for $i = 1, \dots, k$. We get now that $\rho(h) = I_k$. Proposition 3.8 completes the proof. $\square$

Proposition 3.10. $\rho : H \rightarrow M_1(R_1)$ is a non-zero inclusion (or good) representation of H of degree 1 if and only if ρ is the trivial representation.

Proof. If ρ is the trivial representation then Proposition 3.7 asserts that ρ is a good representation of H .

Let h be a generator of $(H, \circ)$ and $\rho : H \rightarrow M_1(R)$ is an inclusion representation of H . Then there exist $a \in R_1$ such that $\rho(h) = a$. Having $h \in h^2$ and ρ an inclusion representation imply that $\rho(h) \in \rho(h^2) \subseteq (\rho(h))^2 = a^2$. The latter implies that $a = a^2$. Since $a > 0$, it follows that $\rho(h) = 1$. Proposition 3.8 asserts that ρ is the trivial representation. $\square$

Theorem 3.11. Let $(H, \circ)$ be a cyclic hypergroup with infinite period (not necessary single power), $\rho : H \rightarrow M_k(R_1)$ with $k \geq 2$, $m \in R_1$ and $\alpha \in H$ with $\exp(\alpha) = a$. If $\rho(\alpha) = (a_{ij})$ and

$$a_{ij} = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{if } i < j \text{ or } j < i < k; \\ ma, & \text{if } 1 < i < k \text{ and } j = k \end{cases}$$

i.e.,

$$\rho(\alpha) = \begin{pmatrix} 1 & 0 & \dots & 0 & ma \\ 0 & 1 & \dots & 0 & ma \\ \vdots & 0 & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 1 & ma \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

then ρ and ρ^T are reducible and unitary inclusion representations of H .

Proof. Let $\alpha, \beta \in H$ satisfying $\exp(\alpha) = a$ and $\exp(\beta) = b$. It is easy to see that if $a = 0$ or $b = 0$ ($\alpha = e$ or $\beta = e$ respectively) then

$$\rho(\alpha \circ \beta) = \begin{cases} \rho(\alpha), & \text{if } b = 0; \\ \rho(\beta), & \text{if } a = 0. \end{cases} = \begin{cases} \rho(\alpha)\rho(\beta), & \text{if } b = 0; \\ \rho(\alpha)\rho(\beta), & \text{if } a = 0. \end{cases}$$

If $a, b > 0$ then $\rho(\alpha \circ \beta) = \{\rho(\lambda) : \lambda \in \alpha \circ \beta\}$. Easy computations show that $\rho(\alpha)\rho(\beta) = (c_{ij})$ where

$$(c_{ij}) = \begin{pmatrix} 1 & 0 & \dots & 0 & ma \oplus mb \\ 0 & 1 & \dots & 0 & ma \oplus mb \\ \vdots & 0 & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 1 & ma \oplus mb \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

We have that $\rho(\alpha \circ \beta) = \{\rho(\lambda) : \lambda \in \alpha \circ \beta\}$ and $\exp(\lambda) = i \leq \exp(\alpha) + \exp(\beta)$. We get now that

$$\rho(\lambda) = \begin{pmatrix} 1 & 0 & \dots & 0 & mi \\ 0 & 1 & \dots & 0 & mi \\ \vdots & 0 & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 1 & mi \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

Having $mi \leq ma + mb$ implies that $\rho(\alpha \circ \beta) \subseteq \rho(\alpha)\rho(\beta)$.

Since $\rho(\alpha)(e_1) = e_1$ for all $\alpha \in H$, it follows that ρ is reducible as $\langle e_1 \rangle$ is an invariant subspace of ρ where $e_1 = (1, \underbrace{0, \dots, 0}_{k-1 \text{ times}})^T$ and T is the transpose.

Easy computations show that ρ is unitary relative to

$$M = (m_{ij}) = \begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \dots & 1 & 0 \\ 0 & \dots & \dots & 0 & 0 \end{pmatrix}$$

where

$$m_{ij} = \begin{cases} 1, & \text{if } i = j < k; \\ 0, & \text{otherwise.} \end{cases}$$

The proof that ρ^T is a reducible and unitary inclusion representation of H is done in a similar manner. $\square$

Proposition 3.12. *The matrix found in the proof of Theorem 3.11 is unique if and only if $k = 2$.*

Proof. The matrix found in the proof of Theorem 3.11 is not unique for all $k > 2$ as ρ is unitary also relative to

$$N = \begin{pmatrix} 1 & 2 & 0 & \dots & \dots & 0 \\ 2 & 1 & 0 & \dots & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \dots & \dots & 1 & 0 \\ 0 & \dots & \dots & \dots & 0 & 0 \end{pmatrix}.$$

For the case $k = 2$, we use the same proof as that of Proposition 4.7 in [2]. $\square$

Theorem 3.13. *Let $(H, \circ)$ be a single power cyclic hypergroup with infinite period. The representation $\rho : H \rightarrow M_k(R_1)$ with $k \geq 2$ defined in Theorem 3.11 is a good representation if and only if ρ is the trivial representation or $m = 1$ and for all non trivial elements $\alpha, \beta \in H$ with $r \leq \exp(\alpha) + \exp(\beta)$ there exist $\lambda \in \alpha \circ \beta$ satisfying $\exp(\lambda) = r$.*

Proof. Let ρ be a non trivial good representation and h a generator of H ($m \neq 0$). We have that

$$\rho(h) = \begin{pmatrix} 1 & 0 & \dots & 0 & m \\ 0 & 1 & \dots & 0 & m \\ \vdots & 0 & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 1 & m \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

and $\rho(h^2) = \{\rho(\lambda) : \lambda \in h^2\} \subseteq \{\rho(h), \rho(a_0), \rho(a_2)\}$ where a_0, a_2 are elements in H (if they exist) with $\exp(a_0) = 0, \exp(a_2) = 2$.

Since ρ is a good representation of H , it follows that

$$\rho(h^2) = \rho(h)^2 = \begin{pmatrix} 1 & 0 & \dots & 0 & m \oplus m \\ 0 & 1 & \dots & 0 & m \oplus m \\ \vdots & 0 & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 1 & m \oplus m \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

We get now $|\rho(h)^2| = 2m + 1 = |\rho(h^2)| \leq 3$. The latter implies that $m = 1$.

For all non trivial elements $\alpha, \beta \in H$, we have that $\rho(\alpha \circ \beta) = \{\rho(\lambda) : \lambda \in \alpha \circ \beta\} = \rho(\alpha)\rho(\beta)$. The latter implies that

$$\begin{aligned} & \left\{ \begin{pmatrix} 1 & 0 & \dots & 0 & \exp(\lambda) \\ 0 & 1 & \dots & 0 & \exp(\lambda) \\ \vdots & 0 & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 1 & \exp(\lambda) \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} : \lambda \in \alpha \circ \beta \right\} \\ &= \begin{pmatrix} 1 & 0 & \dots & 0 & \exp(\alpha) \oplus \exp(\beta) \\ 0 & 1 & \dots & 0 & \exp(\alpha) \oplus \exp(\beta) \\ \vdots & 0 & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 1 & \exp(\alpha) \oplus \exp(\beta) \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}. \end{aligned}$$

Since $\exp(\alpha) \oplus \exp(\beta) = \{0, 1, \dots, \exp(\alpha) + \exp(\beta)\}$, it follows that for all $r \leq \exp(\alpha) + \exp(\beta)$ there exist $\lambda \in \alpha \circ \beta$ satisfying $\exp(\lambda) = r$.

Proposition 3.7 asserts that if ρ is the trivial representation then it is a good representation. Also, it is easy to see that ρ is a good representation when the condition: $m = 1$ and for all non trivial elements $\alpha, \beta \in H$ with $r \leq \exp(\alpha) + \exp(\beta)$ there exist $\lambda \in \alpha \circ \beta$ satisfying $\exp(\lambda) = r$, is satisfied. $\square$

Next, we give an example on a faithful representation of a single power cyclic hypergroup with infinite period.

Example 3.14. Let $n \in R_1$ and $\rho : (R_1, \oplus) \rightarrow M_k(R_1)$ be defined as follows:

$$\rho(n) = \begin{pmatrix} 1 & 0 & \dots & 0 & n \\ 0 & 1 & \dots & 0 & n \\ \vdots & 0 & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 1 & n \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

Since $(R_1, \oplus)$ is generated by 1 and $n \in 1^i$ for all $i \geq n$, it follows that $\exp(n) = n$. Theorems 3.11 and 3.13 assert that ρ is a good representation of $(R_1, \oplus)$.

To prove that ρ is faithful let $n, n' \in R_1$ satisfying $\rho(n) = \rho(n')$. It is easy to see that $n = n'$.

Theorem 3.15. *Let $(H, \circ)$ be a commutative single power cyclic hypergroup, $\rho : H \rightarrow M_2(R_1)$, for all non trivial elements $\alpha, \beta \in H$ and $r \leq \exp(\alpha) + \exp(\beta) \in R_1$ there exist $\lambda \in \alpha \circ \beta$ satisfying $\rho(\lambda) = r$. Then ρ is a good homomorphism if and only if ρ is the trivial representation, or ρ is given by*

$$\rho(\alpha) = \begin{pmatrix} 1 & \exp(\alpha) \\ 0 & 1 \end{pmatrix}$$

or by its transpose.

Proof. Propositions (3.5 and 3.7) and Theorem 3.13 assert that ρ is a good homomorphism of H if either ρ is the trivial representation, given by

$$\rho(\alpha) = \begin{pmatrix} 1 & \exp(\alpha) \\ 0 & 1 \end{pmatrix}$$

or by its transpose.. We need to prove the backward direction now.

Since for all non trivial elements $\alpha, \beta \in H$ and $r \leq \exp(\alpha) + \exp(\beta) \in R_1$ there exist $\lambda \in \alpha \circ \beta$ satisfying $\rho(\lambda) = r$, it follows that $(H, \circ)$ admits a trivial element e . Having ρ a good representation of H and $e \circ e = e$ imply that $\rho(e) = \rho(e \circ e) = \rho(e)\rho(e)$. The latter implies that $\rho(e) = I_2$.

Let

$$\rho(h) = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $a, b, c, d \in R_1$. The proof is the same as that of Theorem 4.12 in [2]. $\square$

Next, we give an example on an inclusion representation that is not of the previous forms.

Proposition 3.16. *Let $(H, \circ)$ be a commutative single power cyclic hypergroup and $\rho : H \rightarrow M_2(R_1)$ defined as follows:*

$$\rho(\alpha) = \begin{pmatrix} 1 & \exp(\alpha) \\ \exp(\alpha) & 2 \end{pmatrix}.$$

Then ρ is an inclusion representation of H . Moreover, ρ is not a good representation of H .

Proof. Theorem 3.15 asserts that ρ is not a good representation of H .

To prove that ρ is an inclusion representation of H , let $\alpha, \beta \in H$ with $\exp(\alpha), \exp(\beta) > 0$ (If $\exp(\alpha) = 0$ or $\exp(\beta) = 0$ then $\rho(\alpha \circ \beta) = \rho(\alpha)\rho(\beta)$). We have that

$$\rho(\alpha \circ \beta) = \{\rho(\lambda) : \lambda \in \alpha \circ \beta\} = \left\{ \begin{pmatrix} 1 & \exp(\lambda) \\ \exp(\lambda) & 2 \end{pmatrix} : \lambda \in \alpha \circ \beta \right\}.$$

Also, we have that

$$\rho(\alpha)\rho(\beta) = \begin{pmatrix} 1 \oplus \exp(\alpha)\exp(\beta) & 2\exp(\alpha) \oplus \exp(\beta) \\ \exp(\alpha) \oplus 2\exp(\beta) & \exp(\alpha)\exp(\beta) \oplus 4 \end{pmatrix}.$$

Since $\exp(\lambda) \leq \exp(\alpha) + \exp(\beta)$, it follows that $\rho(\alpha \circ \beta) \subseteq \rho(\alpha)\rho(\beta)$. $\square$

4. Representation of single power cyclic hypergroups with finite period

In this section, we construct a commutative semihyperring on P' , the set of non negative integers less than or equal to r , introduce some hypermatrix representations of single power cyclic hypergroups with finite period r and present some of their interesting properties. Our work is a generalization of a part of a previous work done by the authors on a special single power cyclic hypergroup with finite period associated to the symmetric group.

Throughout this section, we define $M_k(R_2)$ as the set of all $k \times k$ hypermatrices with entries from $R_2 = (P', \oplus, \odot)$ and $(H, \circ)$ is a single power cyclic hypergroup with finite period r and generator h (unless it is mentioned differently). It is clear that $h \in h^2 \subset \dots \subset h^{r-1} \subset h^r$.

Theorem 4.1. *Let $P' = \{0, 1, \dots, r\}$ and $a, b \in P'$. Then $R_2 = (P', \oplus, \odot)$ is a commutative semihyperring with scalar unit and zero element where*

$$a \odot b = \begin{cases} ab, & \text{if } ab \leq r; \\ r, & \text{otherwise} \end{cases}$$

and

$$a \oplus b = \begin{cases} \{0, 1, \dots, a+b\}, & \text{if } a, b \neq 0 \text{ and } a+b \leq r; \\ \{0, 1, \dots, r\}, & \text{if } a+b > r; \\ a, & \text{if } b = 0; \\ b, & \text{if } a = 0. \end{cases}$$

Proof. It is clear that $(P', \oplus)$ and $(P', \odot)$ are commutative. Also, $(P', \oplus)$ is associative as

$$a \oplus (b \oplus c) = (a \oplus b) \oplus c = \begin{cases} \{0, 1, \dots, a+b+c\}, & \text{if } a, b, c \neq 0 \text{ and } a+b+c \leq r; \\ \{0, 1, \dots, r\}, & \text{if } a+b+c > r; \\ b \oplus c, & \text{if } a = 0; \\ a \oplus c, & \text{if } b = 0; \\ a \oplus b, & \text{if } c = 0. \end{cases}$$

In a similar manner, we can show that $(P', \odot)$ is associative with 1 as a scalar unit as $1 \odot a = a$ and

$$a \odot (b \odot c) = (a \odot b) \odot c = \begin{cases} abc, & \text{if } abc \leq r; \\ r, & \text{if } abc > r. \end{cases}$$

Since $(P', \oplus)$ and $(P', \odot)$ are commutative, it suffices to show that $a \odot (b \oplus c) \subseteq (a \odot b) \oplus (a \odot c)$. We have that

$$a \odot (b \oplus c) = \begin{cases} 0, & \text{if } a = 0; \\ \{0, a, \dots, a(b+c)\}, & \text{if } a, b, c \neq 0 \text{ and } a(b+c) \leq r; \\ \{0, a, \dots, r\}, & \text{if } a, b, c \neq 0 \text{ and } a(b+c) > r; \\ a \odot (0 \oplus c) = a \odot c, & \text{if } b = 0; \\ a \odot (b \oplus 0) = a \odot b, & \text{if } c = 0; \end{cases}$$

and

$$(a \odot b) \oplus (a \odot c) = \begin{cases} 0, & \text{if } a = 0; \\ \{0, 1, \dots, ab+ac\}, & \text{if } a, b, c \neq 0 \text{ and } ab+ac \leq r; \\ \{0, 1, \dots, r\}, & \text{if } a, b, c \neq 0 \text{ and } ab+ac > r; \\ a \odot c, & \text{if } b = 0; \\ a \odot b, & \text{if } c = 0. \end{cases}$$

Thus, $a \odot (b \oplus c) \subseteq (a \odot b) \oplus (a \odot c)$. It is easy to see that 1 is the scalar unit and 0 is the zero element of R . $\square$

Proposition 4.2. $(R_2, \oplus)$ is single power cyclic hypergroup with finite period r and generator 1.

Proof. We have that $(R_2, \oplus)$ is commutative and associative by Theorem 4.1. We prove now that the reproduction axiom is satisfied. If $m \in R_2$ then we have two cases: $m = 0$ and $m > 0$. If $m = 0$ then $m \oplus R_2 = \{m \oplus n : n \in R_2\} = \{n : n \in R_2\} = R_2$. If $m > 0$ then $R_2 = m \oplus r \subseteq m \oplus R_2$. Thus, $(R_2, \oplus)$ is a hypergroup.

For every $m \neq 0 \in R_2$ we have that $m \in 1^m = \{0, 1, \dots, m\}$ and $R_2 = 1^r$. It is easy to see that $1 \in 1^2 \subset \dots \subset 1^{r-1} \subset 1^r$. Therefore, $(R_2, \oplus)$ is single power cyclic hypergroup with finite period r and generator 1. $\square$

Proposition 4.3. *Let $(H, \circ)$ be a commutative hypergroup (not necessary single power cyclic hypergroup) and $\rho : H \rightarrow M_k(R_2)$ be an inclusion (or good) representation of H . Then $\rho^T : H \rightarrow M_k(R_2)$ defined as $\rho^T(\alpha) = (\rho(\alpha))^T$ is a representation of H where T is the matrix transpose.*

Proof. The proof is similar to that of Proposition 3.5. $\square$

Proposition 4.4. *Let $(H, \circ)$ be a commutative hypergroup (not necessary single power cyclic hypergroup) and $\rho : H \rightarrow M_k(R_2)$ is a unitary representation relative to a hypermatrix M . Then $\rho^T : H \rightarrow M_k(R_2)$ is a unitary representation relative to M .*

Proof. The proof is similar to that of Proposition 3.6. $\square$

Proposition 4.5. *Let $(H, \circ)$ be a hypergroup (not necessary single power cyclic hypergroup) and $\rho : H \rightarrow M_k(R_2)$ be the trivial map defined by $\rho(\alpha) = I_k$ where I_k is the $k \times k$ identity matrix. Then ρ is a good representation of H .*

Proof. Let $\alpha, \beta \in H$ with $\rho(\alpha) = \rho(\beta) = I_k$. We have that $\rho(\alpha \circ \beta) = \{\rho(\lambda) : \lambda \in \alpha \circ \beta\} = I_k$ and $\rho(\alpha)\rho(\beta) = I_k I_k = I_k$ using the definition of hyperproduct. Thus, $\rho(\alpha \circ \beta) = \rho(\alpha)\rho(\beta)$. $\square$

Proposition 4.6. *Let $(H, \circ)$ be a cyclic hypergroup with finite period r having h as a generator and $\rho : H \rightarrow M_k(R_2)$ an inclusion (or good) representation of H satisfying $\rho(h) = I_k$. Then ρ is the trivial representation.*

Proof. Let $\alpha \in H$. Then there exist $s \in R_2$ such that $\alpha \in h^s$. The latter implies that $\rho(\alpha) \in \rho(h^s) \subseteq (\rho(h))^s = I_k^s = I_k$. Thus, $\rho(\alpha) = I_k$.

If ρ is a good representation then the proof follows from having ρ an inclusion representation. $\square$

Proposition 4.7. $\rho : H \rightarrow M_1(R_2)$ is a non-zero inclusion (or good) representation of H of degree 1 if and only if one of the following is satisfied:

1. ρ is the trivial representation; or
2. $\rho(\alpha) = r$ for all $\alpha \in H$ and H does not have a trivial element.

Proof. If ρ is the trivial representation then Proposition 4.5 asserts that ρ is a representation of H .

Let h be a generator of $(H, \circ)$ and $\rho : H \rightarrow M_1(R_2)$ a representation of H . Then there exist $a \in R_2$ such that $\rho(h) = a \leq r$. Having $h \in h^2$ and ρ an inclusion representation imply that

$$\rho(h) \in \rho(h^2) \subseteq (\rho(h))^2 = a \odot a = \begin{cases} a^2, & \text{if } a^2 \leq r; \\ r, & \text{otherwise.} \end{cases}$$

The latter implies that $a = a \odot a$. Since $a > 0$, it follows that $\rho(h) = 1$ or $\rho(h) = r$. If $\rho(h) = 1$ then Proposition 4.6 asserts that ρ is the trivial representation. If $\rho(h) = r$ then $\rho(h) \in \rho(h^n) \subseteq (\rho(h))^n = \underbrace{r \odot \dots \odot r}_{n \text{ times}} = r$ for

all $n \in \mathbb{N}$. It is easy to see that $\rho(\alpha) = r$ for all $\alpha \in H$.

Let $\alpha, \beta \in H$. we have that $\rho(\alpha \circ \beta) = \{\rho(\lambda) : \lambda \in \alpha \circ \beta\} = r = r \odot r = \rho(\alpha)\rho(\beta)$. Then ρ is a good representation of H . $\square$

Theorem 4.8. Let $(H, \circ)$ be a cyclic hypergroup with finite period r (not necessary single power), $\rho : H \rightarrow M_k(R_2)$ with $k \geq 2$, $m \in R_2$ and $\alpha \in H$ with $\exp(\alpha) = a$. If $\rho(\alpha) = (a_{ij})$ and

$$a_{ij} = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{if } i < j \text{ or } j < i < k; \\ m \odot a, & \text{if } 1 < i < k \text{ and } j = k \end{cases}$$

i.e., $\rho(\alpha) = \begin{pmatrix} 1 & 0 & \dots & 0 & m \odot a \\ 0 & 1 & \dots & 0 & m \odot a \\ \vdots & 0 & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 1 & m \odot a \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$. Then ρ and ρ^T are reducible and unitary inclusion representations of H .

Proof. Let ρ be a non trivial representation of H , i.e., $m \neq 0$ and let $\alpha, \beta \in H$ satisfying $\exp(\alpha) = a \leq r$ and $\exp(\beta) = b \leq r$. It is easy to see that if $a = 0$ or $b = 0$ then $\rho(\alpha \circ \beta) = \rho(\alpha)\rho(\beta)$. If $a, b > 0$ then $\rho(\alpha \circ \beta) = \{\rho(\lambda) : \lambda \in \alpha \circ \beta\}$. Easy computations show that $\rho(\alpha)\rho(\beta) = (c_{ij})$ where

$$(c_{ij}) = \begin{pmatrix} 1 & 0 & \dots & 0 & m \odot a \oplus m \odot b \\ 0 & 1 & \dots & 0 & m \odot a \oplus m \odot b \\ \vdots & 0 & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 1 & m \odot a \oplus m \odot b \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

We have that $\rho(\alpha \circ \beta) = \{\rho(\lambda) : \lambda \in \alpha \circ \beta\}$ and $\exp(\lambda) = i \leq \exp(\alpha) + \exp(\beta)$. We get now that

$$\rho(\lambda) = \begin{pmatrix} 1 & 0 & \dots & 0 & m \odot i \\ 0 & 1 & \dots & 0 & m \odot i \\ \vdots & 0 & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 1 & m \odot i \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

Since

$$m \odot i = \begin{cases} mi \leq ma + mb, & \text{if } mi \leq r; \\ r, & \text{otherwise} \end{cases}$$

and

$$m \odot a \oplus m \odot b = \begin{cases} \mathbb{Z}_{ma+mb+1}, & \text{if } ma + mb \leq r; \\ \mathbb{Z}_{r+1}, & \text{otherwise;} \end{cases}$$

it follows that $\rho(\alpha \circ \beta) \subseteq \rho(\alpha)\rho(\beta)$.

Having $\rho(\alpha)(e_1) = e_1$ for all $\alpha \in H$ implies that ρ is reducible as $\langle e_1 \rangle$ is an invariant subspace of ρ where $e_1 = (1, \underbrace{0, \dots, 0}_{k-1 \text{ times}})^T$ and T is the transpose.

Easy computations show that ρ is unitary relative to

$$M = (m_{ij}) = \begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \dots & 1 & 0 \\ 0 & \dots & \dots & 0 & 0 \end{pmatrix}$$

where

$$m_{ij} = \begin{cases} 1, & \text{if } i = j < k; \\ 0, & \text{otherwise.} \end{cases}$$

The proof that ρ^T is a reducible and unitary inclusion representation of H is done in a similar manner. $\square$

Theorem 4.9. *Let $(H, \odot)$ be a single power cyclic hypergroup with finite period r . The representation $\rho : H \rightarrow M_k(R_2)$ with $k \geq 2$ defined in Theorem 4.8 is a good representation if and only if one of the following is satisfied:*

1. ρ is the trivial representation; or
2. $r = 2$ and H admits a trivial element; or
3. $m = 1$ and for all $\alpha, \beta \in H$ with $s \in \exp(\alpha) \oplus \exp(\beta)$ there exist $\lambda \in \alpha \circ \beta$ satisfying $\exp(\lambda) = s$.

Proof. Let ρ be a good non trivial representation of H and h a generator of H . Then there exist $m \neq 0$ such that

$$\rho(h) = \begin{pmatrix} 1 & 0 & \dots & 0 & m \\ 0 & 1 & \dots & 0 & m \\ \vdots & 0 & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 1 & m \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

And $\rho(h^2) = \{\rho(\lambda) : \lambda \in h^2\} \subseteq \{\rho(h), \rho(a_0), \rho(a_2)\}$ where a_0, a_2 are elements in H (if they exist) with $\exp(a_0) = 0, \exp(a_2) = 2$.

Since ρ is a good representation of H , it follows that

$$\rho(h^2) = \rho(h)^2 = \begin{pmatrix} 1 & 0 & \dots & 0 & m \oplus m \\ 0 & 1 & \dots & 0 & m \oplus m \\ \vdots & 0 & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 1 & m \oplus m \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

Having

$$m \oplus m = \begin{cases} \{0, 1, \dots, 2m\}, & \text{if } 2m < r; \\ \{0, 1, \dots, r\}, & \text{otherwise;} \end{cases}$$

implies that

$$\begin{aligned} |\rho(h)^2| &= \begin{cases} 2m + 1, & \text{if } 2m < r; \\ r + 1, & \text{otherwise} \end{cases} \\ &= |\rho(h^2)| \leq 3. \end{aligned}$$

The latter implies that $m = 1$ or $r = 2$.

We consider first the case $m = 1$. For all non trivial elements $\alpha, \beta \in H$, we have that $\rho(\alpha \circ \beta) = \{\rho(\lambda) : \lambda \in \alpha \circ \beta\} = \rho(\alpha)\rho(\beta)$. The latter implies that

$$\begin{aligned} &\left\{ \begin{pmatrix} 1 & 0 & \dots & 0 & \exp(\lambda) \\ 0 & 1 & \dots & 0 & \exp(\lambda) \\ \vdots & 0 & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 1 & \exp(\lambda) \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} : \lambda \in \alpha \circ \beta \right\} \\ &= \begin{pmatrix} 1 & 0 & \dots & 0 & \exp(\alpha) \oplus \exp(\beta) \\ 0 & 1 & \dots & 0 & \exp(\alpha) \oplus \exp(\beta) \\ \vdots & 0 & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 1 & \exp(\alpha) \oplus \exp(\beta) \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}. \end{aligned}$$

Since

$$\exp(\alpha) \oplus \exp(\beta) = \begin{cases} \{0, 1, \dots, \exp(\alpha) + \exp(\beta)\}, & \text{if } \exp(\alpha) + \exp(\beta) < r; \\ \{0, 1, \dots, r\}, & \text{otherwise;} \end{cases}$$

it follows that for all $s \in \exp(\alpha) \oplus \exp(\beta)$ there exist $\lambda \in \alpha \circ \beta$ satisfying $\exp(\lambda) = s$.

We consider now the case when $r = 2$. If $r = 2$ then $H = h \cup h^2$ and $h \in h^2$. $|(\rho(h))^2| = 3 = |\rho(h^2)| = |\{\rho(\lambda) : \lambda \in h^2\}|$. The latter implies existence of elements of $\exp : 0, 1, 2$ in H .

It is easy to see that ρ is a good representation if $m = 1$ and for all $\alpha, \beta \in H$ with $s \in \exp(\alpha) \oplus \exp(\beta)$ there exist $\lambda \in \alpha \circ \beta$ satisfying $\exp(\lambda) = s$ or if $r = 2$ and H admits a trivial element. $\square$

Next, we give an example on a faithful representation of a single power cyclic hypergroup with finite period.

Example 4.10. Let $n \in R_2$ and $\rho : (R_2, \oplus) \rightarrow M_k(R_2)$ defined as follows:

$$\rho(n) = \begin{pmatrix} 1 & 0 & \dots & 0 & n \\ 0 & 1 & \dots & 0 & n \\ \vdots & 0 & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 1 & n \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

Since $(R_2, \oplus)$ is generated by 1 and $n \in 1^i$ for all $i \geq n$, it follows that $\exp(n) = n$. Theorems 4.8 and 4.9 assert that ρ is a good representation of $(R_2, \oplus)$.

To prove that ρ is faithful let $n, n' \in R_2$ satisfying $\rho(n) = \rho(n')$. It is easy to see that $n = n'$.

5. Conclusion

After the introduction of hyperstructures by Marty, there have been a number of generalizations of this fundamental concept. One of these generalizations was the concept of hypermatrix representations. This paper studied the notion of hypermatrix representations of single power cyclic hypergroups and proved some of their interesting properties. Several results were obtained for both cases; single power cyclic hypergroups with infinite period as well as with finite period.

For a future work, it will be interesting to generalize our work to cyclic hypergroups by studying their non trivial hypermatrix representations.

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EXTENDED d -HOMOLOGY**S.N. Hosseini**

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Abstract. In this article, in more general categories than the abelian categories, we define a homology functor with respect to a kernel transformation d , called the extended d -homology. We then compare the standard homology and the extended d -homology functors.

Keywords: (pre) abelian category, standard homology, extended d -homology, category of R -modules.

1. Introduction and preliminaries

The definition of the standard homology functor has been extended from the category of R -modules to abelian categories in [7]. In [4] we have defined the homology with respect to a kernel transformation d , also called the d -homology, in more general categories. In this section we give the definition of d -homology and some of the results obtained in [4]. In Section 2, we define a second homology functor called the extended homology with respect to a kernel transformation d , or the extended d -homology. We furnish some illustrative examples and also prove the extended d -homology is the cokernel of a certain map. In Section 3, we compare the standard homology as given in [7] and the extended d -homology, by giving a natural transformation from the standard homology functor to the extended d -homology functor. We then consider conditions under which this natural transformation is a natural isomorphism. We also show, in abelian categories, the standard homology is the extended d -homology with respect to a particular kernel transformation d . Some other results are also given at the end of this section.

To this end, for a pointed category $\mathcal{C}$, following the notation of [4], we recall:

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- For $f : A \longrightarrow B$, the maps $K_f \xrightarrow{k_f} A$, $B \xrightarrow{c_f} C_f$ and $P_f \xrightarrow[\pi_2]{\pi_1} A$ are respectively the kernel, the cokernel and the kernel pair of f .
- The image I_f of f is the coequalizer of the kernel pair of f . f can be factorized to $f = m_f e_f$ that e_f is the coequalizer of the kernel pair of f .
- For a pair of maps $A \xrightarrow[g]{f} B$, the maps $Equ(f, g) \xrightarrow{equ(f, g)} A$ and $B \xrightarrow{coe(f, g)} Coe(f, g)$ are respectively the equalizer and the coequalizer of (f, g) .

For a pointed category $\mathcal{C}$ with pullbacks and pushouts, let $\bar{\mathcal{C}}$ be the arrow category and $\hat{\mathcal{C}}$ be the pair-chain category of $\mathcal{C}$. Let $K : \bar{\mathcal{C}} \longrightarrow \mathcal{C}$ be the kernel functor and $I : \bar{\mathcal{C}} \longrightarrow \mathcal{C}$ be the image functor.

- The functor $j : \hat{\mathcal{C}} \longrightarrow \bar{\mathcal{C}}$ takes the object $(f, g) \in \hat{\mathcal{C}}$ to j_{fg} and the morphism (α, β, γ) to $(I(\alpha, \beta), K(\beta, \gamma))$ and we have the following commutative diagram

$$\begin{array}{ccccccc}
 & & & f & & & \\
 & & & \curvearrowright & & & \\
 A & \xrightarrow{e_f} & I_f & \xrightarrow{j_{fg}} & K_g & \xrightarrow{k_g} & B \\
 \alpha \downarrow & & \downarrow I(\alpha, \beta) & & \downarrow K(\beta, \gamma) & & \downarrow \beta \\
 A' & \xrightarrow{e_{f'}} & I_{f'} & \xrightarrow{j_{f'g'}} & K_{g'} & \xrightarrow{k_{g'}} & B' \\
 & & & \curvearrowleft & & & \\
 & & & f' & & &
 \end{array}$$

- The standard homology or s -homology functor H^s , takes $(f, g) \in \hat{\mathcal{C}}$ to $Coker(j_{fg})$, and for a pair chain map $(\alpha, \beta, \gamma) : (f, g) \longrightarrow (f', g')$, we have the following commutative diagram

$$\begin{array}{ccc}
 K_g & \xrightarrow{q} & H_{fg}^s \\
 k(\beta, \gamma) \downarrow & & \downarrow H^s(\alpha, \beta, \gamma) \\
 K_{g'} & \xrightarrow{q'} & H_{f'g'}^s
 \end{array}$$

where $q = coker(j_{fg})$ and $q' = coker(j_{f'g'})$.

2. Extended homology with respect to a kernel transformation

In this section, unless stated otherwise, $\mathcal{C}$ is a pointed category with pullbacks and pushouts.

Definition 2.1. Let $m : A \rightarrow C$ and $j : B \rightarrow C$ be two maps in $\mathcal{C}$. Define $A +_C B$ also denoted by $A + B$ by the pushout:

$$\begin{array}{ccc} P_{jm} & \xrightarrow{\alpha} & A \\ \gamma \downarrow & po & \downarrow h \\ B & \xrightarrow{i} & A +_C B \end{array}$$

where $B \xleftarrow{\gamma} P_{jm} \xrightarrow{\alpha} A$ is the pullback of (j, m) .

With $S : \mathcal{C} \rightarrow \mathcal{C}$ the squaring functor, taking $a \xrightarrow{f} b$ to $a^2 \xrightarrow{f^2} b^2$, a kernel transformation is a natural transformation $d : S \circ K \rightarrow K : \hat{\mathcal{C}} \rightarrow \mathcal{C}$, such that for all $(f, g) \in \hat{\mathcal{C}}$ and diagonal map Δ we have the maps $m_{d_g \Delta_g} : I_{d_g \Delta_g} \rightarrow K_g$ (such that $m_{d_g \Delta_g} e_{d_g \Delta_g} = d_g \Delta_g$) and $j_{fg} : I_f \rightarrow K_g$. The sum $I_{d_g \Delta_g} + I_f$ is therefore obtained by the following diagrams:

$$\begin{array}{ccc} P_{jm} & \xrightarrow{\alpha} & I_{d_g \Delta_g} \\ \gamma \downarrow & pb & \downarrow m_{d_g \Delta_g} \\ I_f & \xrightarrow{j_{fg}} & K_g \end{array} \quad \text{and} \quad \begin{array}{ccc} P_{jm} & \xrightarrow{\alpha} & I_{d_g \Delta_g} \\ \gamma \downarrow & po & \downarrow h \\ I_f & \xrightarrow{i} & I_{d_g \Delta_g} + I_f \end{array}$$

Since $m_{d_g \Delta_g} \alpha = j_{fg} \gamma$, there is a unique map $\beta : I_{d_g \Delta_g} + I_f \rightarrow K_g$ making the following triangles commutative

$$\begin{array}{ccc} P_{jm} & \xrightarrow{\alpha} & I_{d_g \Delta_g} \\ \gamma \downarrow & po & \downarrow h \\ I_f & \xrightarrow{i} & I_{d_g \Delta_g} + I_f \end{array} \quad \begin{array}{c} \searrow m_{d_g \Delta_g} \\ \downarrow \beta \\ K_g \end{array} \quad \begin{array}{c} \nearrow j_{fg} \end{array}$$

Letting $m = m_{d_g \Delta_g}$, $e = e_{d_g \Delta_g}$ and $j = j_{fg}$ for simplicity and setting $\bar{H}_{fg}^d = C_\beta$, the cokernel of β , we have:

Lemma 2.2. For each morphism $(\sigma, \delta, \zeta) : (f, g) \rightarrow (f', g')$ in $\hat{\mathcal{C}}$, there is a unique map $\bar{H}^d(\sigma, \delta, \zeta) : \bar{H}_{fg}^d \rightarrow \bar{H}_{f'g'}^d$, such that the following diagram commutes:

$$\begin{array}{ccc} K_g & \xrightarrow{c_\beta} & \bar{H}_{fg}^d \\ K(\delta, \zeta) \downarrow & & \downarrow \bar{H}^d(\sigma, \delta, \zeta) \\ K_{g'} & \xrightarrow{c_{\beta'}} & \bar{H}_{f'g'}^d \end{array}$$

Proof. Consider the following diagram in which the back squares commute by naturality of d and Δ

$$\begin{array}{ccccc}
 K_g & \xrightarrow{\Delta_g} & K_g^2 & \xrightarrow{d_g} & K_g \\
 \downarrow K(\delta, \zeta) & \searrow e & \downarrow m & \nearrow & \downarrow K(\delta, \zeta) \\
 & & I_{d_g \Delta_g} & & \\
 & & \vdots & & \\
 & & I_{d_{g'} \Delta_{g'}} & & \\
 & \nearrow e' & \downarrow m' & \searrow & \\
 K_{g'} & \xrightarrow{\Delta_{g'}} & K_{g'}^2 & \xrightarrow{d_{g'}} & K_{g'} \\
 & \searrow e' & \downarrow m' & \nearrow & \\
 & & I_{d_{g'} \Delta_{g'}} & &
 \end{array}$$

It follows that $(K(\delta, \zeta), K(\delta, \zeta)) : d_g \Delta_g \rightarrow d_{g'} \Delta_{g'}$ is a map in $\bar{\mathcal{C}}$ and so we get the map $l = I(K(\delta, \zeta), K(\delta, \zeta)) : I_{d_g \Delta_g} \rightarrow I_{d_{g'} \Delta_{g'}}$ such that $le = e'K(\delta, \zeta)$. Since e is epic, $K(\delta, \zeta)m = m'l$. On the other hand $j'I(\sigma, \delta) = K(\delta, \zeta)j$ and we get $m'l\alpha = K(\delta, \zeta)m\alpha = K(\delta, \zeta)j\gamma = j'I(\sigma, \delta)\gamma$. So there is a unique map $y : P_{jm} \rightarrow P_{j'm'}$ such that $\gamma'y = I(\sigma, \delta)\gamma$ and $\alpha'y = l\alpha$. So in the diagram:

$$\begin{array}{ccccc}
 P_{jm} & \xrightarrow{\alpha} & I_{d_g \Delta_g} & & \\
 \downarrow \gamma & \searrow y & \downarrow h & \searrow l & \\
 & & P_{j'm'} & \xrightarrow{\alpha'} & I_{d_{g'} \Delta_{g'}} \\
 & & \downarrow \gamma' & & \downarrow h' \\
 I_f & \xrightarrow{\gamma'} & I_{d_g \Delta_g} + I_f & & \\
 \downarrow I(\sigma, \delta) & \searrow i & \downarrow i' & \searrow z & \\
 I_{f'} & \xrightarrow{i'} & I_{d_{g'} \Delta_{g'}} + I_{f'} & &
 \end{array}$$

the front and back squares are pushouts, and the left and top squares are commutative. It follows that there is a unique map z such that: $zi = i'I(\sigma, \delta)$ and $zh = h'l$. We then get

$$\beta'zi = \beta'iI(\sigma, \delta) = j'I(\sigma, \delta) = K(\delta, \zeta)j = K(\delta, \zeta)\beta i$$

and

$$K(\delta, \zeta)\beta h = K(\delta, \zeta)m = m'l\gamma = \beta'h'l = \beta'zh$$

Pushoutness of $I_{d_g \Delta_g} + I_f$ implies $K(\delta, \zeta)\beta = \beta'z$.

Now $c_{\beta'}K(\delta, \zeta)\beta = c_{\beta'}\beta'z = 0$ and so there is a unique $\bar{H}^d(\sigma, \delta, \zeta) : \bar{H}^d_{fg} \rightarrow \bar{H}^d_{f'g'}$ making the right square in the following diagram commutative:

$$\begin{array}{ccccc} I_{d_g\Delta_g} + I_f & \xrightarrow[\quad 0]{\beta} & K_g & \xrightarrow{c_\beta} & \bar{H}^d_{fg} \\ \downarrow z & & \downarrow K(\delta, \zeta) & & \downarrow \bar{H}^d(\sigma, \delta, \zeta) \\ I_{d_{g'}\Delta_{g'}} + I_{f'} & \xrightarrow[\quad 0]{\beta'} & K_{g'} & \xrightarrow{c_{\beta'}} & \bar{H}^d_{f'g'} \end{array}$$

□

Theorem 2.3. The mapping $\bar{H}^d : \hat{\mathcal{C}} \rightarrow \mathcal{C}$ that takes the object $(f, g) \in \hat{\mathcal{C}}$ to $\bar{H}^d_{fg}$ and the morphism (α, β, γ) to $\bar{H}^d(\alpha, \beta, \gamma)$ is a functor.

The functor $\bar{H}^d : \hat{\mathcal{C}} \rightarrow \mathcal{C}$ is called the extended d -homology or the extended homology functor with respect to the kernel transformation d .

Example 2.4. Let $\mathcal{C} = R\text{mod}$ and $d = +(r \times s) = rpr_1 + spr_2$ with $r, s \in R$. If $A \xrightarrow{f} B \xrightarrow{g} C$ with $gf = 0$, Then:

$$\bar{H}^d_{fg} = \frac{K_g}{(r+s)K_g + I_f}$$

Example 2.5. As a special case of the above example, let $\mathcal{C} = Abgrp$, for $d = +(r \times s)$ with $r, s \in \mathbb{Z}$, $K_g = \mathbb{Z}$, $I_f = n\mathbb{Z}$, we have:

$$\bar{H} = \frac{\mathbb{Z}}{(r+s)\mathbb{Z} + n\mathbb{Z}} = \frac{\mathbb{Z}}{(r+s, n)\mathbb{Z}} = \mathbb{Z}_{(r+s, n)}$$

where $(r+s, n)$ is the greatest common divisor of $r+s$ and n .

Example 2.6 ([7, 4]). Let $\mathcal{C}$ be the category, Sh_R , of short exact sequences of R -modules, $(F, G) \in \hat{\mathcal{C}}$ and $d = +(r \times s)$ $r, s \in R$. Then:

$$\begin{array}{ccccccc} M & 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ F \downarrow & & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\ N & 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \longrightarrow & 0 \\ G \downarrow & & & \downarrow \alpha' & & \downarrow \beta' & & \downarrow \gamma' & & \\ P & 0 & \longrightarrow & A'' & \xrightarrow{f''} & B'' & \xrightarrow{g''} & C'' & \longrightarrow & 0 \end{array}$$

$$\begin{array}{ccccccc} I_F & 0 & \longrightarrow & \frac{I_f + K_\beta}{K_\beta} & \longrightarrow & \frac{B}{K_\beta} & \longrightarrow & \frac{C}{I_{g\beta}} & \longrightarrow & 0 \\ j \downarrow & & & \downarrow \bar{\alpha} & & \downarrow & & \downarrow & & \\ K_G & 0 & \longrightarrow & K_{\alpha'} & \longrightarrow & K_{\beta'} & \longrightarrow & I_{g'\beta'} & \longrightarrow & 0 \end{array}$$

where $I_{g\beta}$ is the image of the restriction of g to K_β and $\bar{\alpha}(f(a) + K_\beta) = \alpha(a)$. We also have:

$$\begin{array}{ccccccc} I_{d_G \Delta_G} & 0 & \longrightarrow & \frac{I_{f'\alpha'} + K_2}{K_2} & \longrightarrow & \frac{K_{\beta'}}{K_2} & \longrightarrow & \frac{I_{g'\beta'}}{K_3} & \longrightarrow & 0 \\ m \downarrow & & & \downarrow & & \downarrow & & \downarrow & & \\ K_G & 0 & \longrightarrow & K_{\alpha'} & \longrightarrow & K_{\beta'} & \longrightarrow & I_{g'\beta'} & \longrightarrow & 0 \end{array}$$

where $K_2 = \{a \in K_{\beta'} \mid (r+s)a = 0\}$ and K_3 is the image of the restriction of g' to K_2 . So the extended homology is:

$$\bar{H}_{FG}^d \quad 0 \longrightarrow K_{\hat{g}} \longrightarrow \frac{K_{\beta'}}{I_{\beta+(r+s)K_{\beta'}}} \xrightarrow{\hat{g}} \frac{I_{g'\beta'}}{I_{\gamma+(r+s)I_{g'\beta'}}} \longrightarrow 0$$

where $\hat{g}$ is the quotient of the restriction of g' to $K_{\beta'}$.

Example 2.7. Let $d = 0$. For any $(f, g) \in \hat{\mathcal{C}}$, $\bar{H}_{fg}^d = H_{fg}^s$.

Lemma 2.8. Let $d = pr_1$ or $d = pr_2$. For any $(f, g) \in \hat{\mathcal{C}}$, $\bar{H}_{fg}^d = 0$.

Proof. Since $pr_1 \Delta = 1$, $I_{d_g \Delta_g} = K_g$, $P_{jm} = I_f$ and $I_f + I_{d_g \Delta_g} = K_g$, $\beta = 1$ and $\bar{H}_{fg}^d = 0$. Similar argument holds for $d = pr_2$. $\square$

Calling the projection transformations, pr_1 and pr_2 , and the zero transformation, the trivial transformations we have:

Proposition 2.9 ([4, 5]). *The only kernel transformations in the categories, Top_* , of pointed topological spaces, Set_* , of pointed sets and the category, $\overrightarrow{Set}$, of partial sets, are the trivial ones.*

Example 2.10. Let $\mathcal{C}$ be the category, Set_* , of pointed sets ($\overrightarrow{Set}$, of partial sets or Top_* , of pointed topological spaces). For $d = pr_1$ and $d = pr_2$, $\beta = 1$ and for $d = 0$, $\beta = j$. Therefore $\bar{H}_{fg}^d = 0$ or $\bar{H}_{fg}^d = H_{fg}^s$.

Given $A \xrightarrow{f} C$ and $B \xrightarrow{g} C$ in $\mathcal{C}$, let $A \xleftarrow{i} P_{fg} \xrightarrow{j} B$ be the pull-back of (f, g) , and $A \xrightarrow{m} A + B \xleftarrow{n} B$ be the pushout of (i, j) . We know there is a unique $\beta : A + B \rightarrow C$ such that $\beta m = f$ and $\beta n = g$. Let $A \coprod B$ be the coproduct of A and B with injections l_1 and l_2 . Then $\beta(m \oplus n) = f \oplus g$.

Lemma 2.11. Let $A \xrightarrow{f} C$ and $B \xrightarrow{g} C$ be in $\mathcal{C}$. Then:

- (i) There is a regular epi $\sigma : A + B \rightarrow I_{f \oplus g}$ such that $\sigma(m \oplus n) = e_{f \oplus g}$.
- (ii) If β is monic, then $A + B \cong I_{f \oplus g}$ and $m_{f \oplus g}$ is monic.
- (iii) $c_{f \oplus g} \cong c_\beta$.

Proof. (i) Since $A + B$ is the pushout of (i, j) , $m \oplus n : A \amalg B \rightarrow A + B$ is the coequalizer of $P_{fg} \xrightarrow[l_2j]{l_1i} A \amalg B$. On the other hand, since $(f \oplus g)l_1i = fi = gj = (f \oplus g)l_2j$, and $P_{f \oplus g}$ is the kernel pair of $f \oplus g$, there is a unique $\xi : P_{fg} \rightarrow P_{f \oplus g}$ such that $\pi_1\xi = l_1i$ and $\pi_2\xi = l_2j$. So we have the following diagram in which the top and bottom rows are coequalizers and the square commutes

$$\begin{array}{ccccc} P_{fg} & \xrightarrow[l_2j]{l_1i} & A \amalg B & \xrightarrow{m \oplus n} & A + B \\ \xi \downarrow & & \downarrow 1_{A \amalg B} & & \downarrow \sigma \\ P_{f \oplus g} & \xrightarrow[\pi_2]{\pi_1} & A \amalg B & \xrightarrow{e_{f \oplus g}} & I_{f \oplus g} \end{array}$$

So there is a unique map $\sigma : A + B \rightarrow I_{f \oplus g}$ such that $\sigma(m \oplus n) = e_{f \oplus g}$ and σ is regular epic.

(ii) Since $\beta(m \oplus n) = f \oplus g = m_{f \oplus g}e_{f \oplus g} = m_{f \oplus g}\sigma(m \oplus n)$ and $m \oplus n$ is epic, $\beta = m_{f \oplus g}\sigma$. If β is monic, then σ will be monic and so is an isomorphism. Then $m_{f \oplus g}$ is monic.

(iii) Since $e_{f \oplus g}$ and σ are epic, we have $c_{f \oplus g} = \text{coker}(f \oplus g) \cong \text{coker}(m_{f \oplus g}e_{f \oplus g}) \cong \text{coker}(m_{f \oplus g}\sigma) \cong \text{coker}(m_{f \oplus g}) \cong \text{coker}(m_{f \oplus g}\sigma) \cong \text{coker}(\beta) = c_\beta$. $\square$

Theorem 2.12. Let $(f, g) \in \hat{\mathcal{C}}$ and d be a kernel transformation. Then $\bar{H}_{fg}^d \cong C_{j \oplus m}$.

Proof. Replace the maps f and g of the previous lemma respectively by the maps $j : I_f \rightarrow K_g$ and $m : I_{d_g \Delta_g} \rightarrow K_g$. $\square$

Lemma 2.13. Let $\mathcal{C}$ be a pre abelian category and $A \xrightarrow{f} C$ and $B \xrightarrow{g} C$ be in $\mathcal{C}$. Then:

(i) There is a regular epi $\delta : I_{f \oplus g} \rightarrow A + B$ such that $\delta e_{f \oplus g} = m \oplus n$.

(ii) If $m_{f \oplus g}$ is monic, then $A + B \cong I_{f \oplus g}$ and β is monic.

Proof. Since $f \oplus g = fpr_1 + gpr_2$, $fpr_1(\pi_1 - \pi_2) = gpr_2(\pi_2 - \pi_1)$. The result then follows similar to the proof given in Lemma 2.11. $\square$

Theorem 2.14. Let $\mathcal{C}$ be a pre abelian category and $A \xrightarrow{f} C$ and $B \xrightarrow{g} C$ be in $\mathcal{C}$. Then β is monic if and only if $m_{f \oplus g}$ is monic. In this case $A + B \cong I_{f \oplus g}$.

Proof. Follows from Lemmas 2.11 and 2.13. $\square$

Example 2.15 ([4]). Let $\mathcal{C}$ be the category, Sh_R , of short exact sequences of R -modules. Then for any pair $A \xrightarrow{f} C$ and $B \xrightarrow{g} C$ be in Sh_R , β is monic, since $m_{f \oplus g}$ is.

Example 2.16. Let $\mathcal{C}$ be any abelian category. Then for any pair $A \xrightarrow{f} C$ and $B \xrightarrow{g} C$ be in $\mathcal{C}$, β is monic, since $m_{f \oplus g}$ is.

3. Standard homology versus extended d -homology

In this section, unless stated otherwise, we assume $\mathcal{C}$ is a category with a zero object, pullbacks and pushouts, and we investigate the relation between the standard homology and the extended d -homology.

Lemma 3.1. *Let d be a kernel transformation in $\mathcal{C}$. There is a natural transformation $p : H^s \rightarrow \bar{H}^d$. Furthermore p is pointwise regular epic.*

Proof. For $(f, g) \in \hat{\mathcal{C}}$, p_{fg} is obtained by applying Lemma 2.2 of [4] to the following diagram

$$\begin{array}{ccccc} I_f & \xrightarrow{j} & K_g & \xrightarrow{c_j} & H_{fg}^s \\ i \downarrow & & \downarrow 1_{K_g} & & \downarrow p_{fg} \\ I_f + I_{d_g \Delta_g} & \xrightarrow{\beta} & K_g & \xrightarrow{c_\beta} & \bar{H}_{fg}^d \end{array}$$

To show naturality, given $(\sigma, \delta, \zeta) : (f, g) \rightarrow (f', g')$ in $\hat{\mathcal{C}}$, we have $p_{f'g'} H^s(\sigma, \delta, \zeta) c_j = p_{f'g'} c_{j'} K(\delta, \zeta) = c_{\beta'} K(\delta, \zeta) = \bar{H}^d(\sigma, \delta, \zeta) c_\beta = \bar{H}^d(\sigma, \delta, \zeta) p_{fg} c_j$. Since c_j is epic, $p_{f'g'} H^s(\sigma, \delta, \zeta) = \bar{H}^d(\sigma, \delta, \zeta) p_{fg}$. $\square$

Lemma 3.2. *If for $(f, g) \in \hat{\mathcal{C}}$, $i : I_f \rightarrow I_f + I_{d_g \Delta_g}$ is epic, then $p_{fg} : H_{fg}^s \cong \bar{H}_{fg}^d$ is an isomorphism.*

Proof. $\bar{H}_{fg}^d = \text{Coker}(\beta) \cong \text{Coker}(\beta i) = \text{Coker}(j) = H_{fg}^s$. $\square$

Corollary 3.3. *If $d\Delta = 0$ (hence if $d = 0$), then $\bar{H}^d \cong_n H^s$, i.e., $\bar{H}^d$ is naturally isomorphic to H^s .*

Proof. Let $(f, g) \in \hat{\mathcal{C}}$. Since $d_g \Delta_g = 0$, $I_{d_g \Delta_g} = I_0 = 0$. It can be easily shown that $I_f + I_{d_g \Delta_g} = I_f + 0 = I_j^c$, where I_j^c is the cokernel of kernel of j , see [4], and $i : I_f \rightarrow I_f + 0 = I_j^c$ is the map c_{k_j} and is therefore epic. The result then follows from Lemma 3.2. $\square$

Corollary 3.4. *Let $\mathcal{C}$ be an abelian category, $d = +(r \times s)$. If $(r + s)K_g = 0$, then $\bar{H}_{fg}^d \cong H_{fg}^s$.*

Proof. Since $I_{d_g \Delta_g} = (r + s)K_g = 0$, the result follows. $\square$

Corollary 3.5. *Let $\mathcal{C}$ be an abelian category and $d = +(r \times -r)$. Then $\bar{H}^d \cong_n H^s$. In particular $\bar{H}^- \cong_n H^s$.*

Proof. Follows from the Corollary 3.3. $\square$

Theorem 3.6. *Let $\mathcal{C}$ be an abelian category. For $(f, g) \in \hat{\mathcal{C}}$, $p_{fg} : H_{fg}^s \rightarrow \bar{H}_{fg}^d$ is an isomorphism if and only if $i : I_f \rightarrow I_f + I_{d_g \Delta_g}$ is an isomorphism.*

Proof. In an abelian category j and β are monic and so $\beta = \ker(\operatorname{coker}(\beta)) = \ker(c_\beta)$ and $j = \ker(\operatorname{coker}(j)) = \ker(c_j)$. So in the following diagram the rows are equalizers and the right square commutes

$$\begin{array}{ccccc} I_f & \xrightarrow{j} & K_g & \xrightarrow{c_j} & H_{fg}^s \\ \downarrow i & & \downarrow 1_{K_g} & & \downarrow p_{fg} \\ I_f + I_{d_g \Delta_g} & \xrightarrow{\beta} & K_g & \xrightarrow{c_\beta} & \bar{H}_{fg}^d \end{array}$$

By dual of Lemma 2.2 in [4], i is a regular mono and if p_{fg} is a mono, i is an isomorphism. $\square$

The converse has been shown previously.

Lemma 3.7. *Let $\mathcal{C}$ be an abelian category. For $(f, g) \in \hat{\mathcal{C}}$, $i : I_f \rightarrow I_f + I_{d_g \Delta_g}$ is an isomorphism if and only if $I_{d_g \Delta_g}$ is a subobject of I_f , i.e., $m : I_{d_g \Delta_g} \rightarrow K_g$ factors through $j : I_f \rightarrow K_g$.*

Proof. Straightforward. $\square$

Corollary 3.8. *Let $\mathcal{C}$ be an abelian category. For $(f, g) \in \hat{\mathcal{C}}$, $p_{fg} : H_{fg}^s \cong \bar{H}_{fg}^d$ is an isomorphism if and only if $I_{d_g \Delta_g}$ is a subobject of I_f .*

Theorem 3.9. *Let $\mathcal{C}$ be the category, $R\operatorname{mod}$. If $H_{fg}^s \cong H_{f'g'}^s$, then $\bar{H}_{fg}^d \cong \bar{H}_{f'g'}^d$ and $H_{fg}^d \cong H_{f'g'}^d$.*

Proof. Let $(f, g), (f', g') \in \hat{\mathcal{C}}$ and suppose $\psi : \frac{K_g}{I_f} \cong \frac{K_{g'}}{I_{f'}}$. Since in $R\operatorname{mod}$, $d = +(r \times s)$ for some $r, s \in R\operatorname{mod}$, $H_{fg}^s = \frac{K_g}{I_f}$ and $\bar{H}_{fg}^d = \frac{K_g}{(r+s)K_g + I_f}$, we have an epi $\phi : K_g \rightarrow \frac{K_{g'}}{(r+s)K_{g'} + I_{f'}}$, which is the composition of the epis $q : K_g \rightarrow \frac{K_g}{I_f}$, $\psi : \frac{K_g}{I_f} \rightarrow \frac{K_{g'}}{I_{f'}}$ and $p_{f'g'} : \frac{K_{g'}}{I_{f'}} \rightarrow \frac{K_{g'}}{(r+s)K_{g'} + I_{f'}}$. Some computations show that $K_\phi = (r+s)K_g + I_f$. The result then follows.

The proof of the second equality follows from the fact that $H_{fg}^d = \{[a] | a \in K_g\}$, where $[a] = \{b \in K_g | r(a-b) \in (r+s)K_g + I_f\} = \{b \in K_g | s(a-b) \in (r+s)K_g + I_f\}$, see [4]. $\square$

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A NOTE ON Z -ALGEBRA

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Abstract. This paper introduces a new notion of algebra called Z -algebra and some of its properties are discussed in detail. It reveals also that Z -algebra is completely different from some of other abstract algebras.

Keywords: Z -algebra, Z -Subalgebra, Z -ideal, Z -filter

1. Introduction

Algebraic structures play an important role in mathematics with wide range of applications in many disciplines such as theoretical physics, computer sciences, control engineering, information sciences, coding theory etc. Several algebraic structure have been introduced in the recent past. K. Iseki and S. Tanaka [5], introduced a class of abstract algebra: BCK-algebra. Also Y. Imai and K. Iseki [6], dealt about BCI-algebras. Then, Hu Q.P. and Li. X [4], have given the

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notion of BCH-algebra which is the generalization of BCI and BCK-algebras. Neggers et.al [13, 14, 15] introduced the notions of B -algebras, Q -algebras and d -algebras. In 2010, K. Megalai and A. Tamilarasi [18], introduced TM-algebra. During 2011, Keawrahan and Leerawat [8] introduced new structured algebra called SU-Algebra. Prabayah and Leerawat [16], discussed ideals and congruences in KU-algebras. With all these ideas, in this paper, a new notion of algebra called Z -algebra is defined and some substructures are also established in this algebra .

2. Preliminares

This section recalls the definitions of various algebras defined by different authors in the algebraic formulation of the propositional calculus and implicational calculus.

Definition 2.1 ([5]). *A BCK-algebra $(X, *, 0)$ is a nonempty set X with constant 0 and a binary operation $*$ by satisfying the following conditions:*

1. $((x * y) * (x * z)) * (z * y) = 0$
2. $0 * x = 0$
3. $x * x = 0$
4. $(x(x * y)) * y = 0$
5. $x * y = 0$ and $y * x = 0$ imply $x = y, \forall x, y \in X$.

Definition 2.2 ([14]). *A Q -algebra $(X, *, 0)$ is a nonempty set X with constant 0 and a binary operation $*$ by satisfying the following conditions:*

1. $x * 0 = x$
2. $x * x = 0$
3. $(x * y) * z = (x * z) * y, \forall x, y \in X$.

Definition 2.3 ([7]). *A BH-algebra $(X, *, 0)$ is a nonempty set X with constant 0 and a binary operation $*$ satisfying the following conditions:*

1. $x * 0 = x$
2. $x * x = 0$
3. $x * y = 0$ and $y * x = 0$ implies $x = y, \forall x, y \in X$.

Definition 2.4 ([18]). *A TM-algebra $(X, *, 0)$ is a nonempty set X with constant 0 and a binary operation $*$ satisfying the following conditions:*

1. $x * 0 = x$

$$2. (x * y) * (x * z) = z * y, \forall x, y \in X.$$

Definition 2.5 ([8]). A *SU-algebra* $(X, *, 0)$ is a nonempty set X with constant 0 and a binary operation $*$ satisfying the following conditions:

$$1. ((x * y) * (x * z)) * (y * z) = 0$$

$$2. x * 0 = x$$

$$3. \text{ if } x * y = 0 \Rightarrow x = y, \forall x, y, z \in X. \forall x, y \in X.$$

Definition 2.6. Let X be any non-empty set. A *filter* of X , is a subset $S (\neq \phi)$ of X , if $0 \notin S$ and $\forall x, y \in S \Rightarrow x \triangle y \in S$, where $x \triangle y = x * (x * y)$.

3. The structure of Z-algebra

In this section, a new structure of algebra, namely *Z-algebra* which is an algebra based on propositional calculus is introduced and the relation between *Z-algebra* and other algebras is investigated. A substructure *Z-Subalgebra* is also defined.

Definition 3.1. A *Z-algebra* $(X, *, 0)$ is a nonempty set X with constant 0 and a binary operation $*$ satisfying the following conditions:

$$(3.1) \quad x * 0 = 0$$

$$(3.2) \quad 0 * x = x$$

$$(3.3) \quad x * x = x$$

$$(3.4) \quad x * y = y * x, \text{ when } x \neq 0 \text{ and } y \neq 0, \forall x, y \in X.$$

Example 3.2. Let $X = \{0, 1, 2, 3\}$ be a set with constant 0 and a binary operation $*$ is defined on X by the Cayley's table as follows.

*	0	1	2	3
0	0	1	2	3
1	0	1	0	1
2	0	0	2	2
3	0	1	2	3

Then, $(X, *, 0)$ is a *Z-algebra*.

Example 3.3. Let $(X = \mathbb{R}, *, 0)$ where $x, y \in \mathbb{R}$. Define a binary operation $*$ on X by,

$$x * y = \begin{cases} y, & \text{if } (x = 0 \text{ and } y \neq 0) \text{ or } (x \neq 0 \text{ and } y = 0) \\ x, & x = y. \\ xy, & x \neq y. \end{cases}$$

Then, $(X, *, 0)$ be *Z-algebra*.

Example 3.4. The following table shows $(X = \{0, a, b, c\}, *, 0)$ is a Z -algebra.

$*$	0	a	b	c
0	0	a	b	c
a	0	a	c	b
b	0	c	b	a
c	0	b	a	c

Definition 3.5. Let S be a non empty subset of a Z -algebra of X . Then, S is called a Z -Subalgebra of X , if $x * y \in S, \forall x, y \in S$.

Example 3.6. Consider the Z -algebra in example 3.2. Then, the subsets $A = \{1, 3\} \subset X$ and $B = \{2, 3\} \subset X$ are Z -Subalgebras of X , but the subset $C = \{1, 2, 3\} \subset X$ is not a Z -Subalgebra of X .

Proposition 3.7. Let $(X, *, 0)$ be a Z -algebra. Then, it is not a BCK-algebra [5], Q -algebra [14], BH-algebra [7], SU -algebra [8], BM-algebra [10], B -algebra [13], BF-algebra [19], BRK-algebra [3], RG-algebra [12] and TM-algebra [18].

Proof. In all the above algebras cited here except a Z -algebra, we have, $x * 0 = x$. But by definition 3.1, for a Z -algebra we have, $x * 0 = 0 \neq x$.

This completes the proof. $\square$

Proposition 3.8. Let $(X, *, 0)$ be a Z -algebra. Then, it is not a d -algebra [15], KU-algebra [16], BE-algebra [9], PS-algebra [17], CI-algebra [2], BCI-algebra [6], BCH-algebra [4], BP-algebra [1], BO-algebra [11] and BCL-algebra [20].

Proof. In all the above algebras cited here except a Z -algebra, we have, $x * x = 0$. But by definition 3.1, for a Z -algebra we have, $x * x = x \neq 0$.

This completes the proof. $\square$

Proposition 3.9. Let X be a Z -algebra. Then, the following results hold for all $x, y, z \in X$.

1. $(x * (x * (y * x))) = x$, if $y = 0$
2. $x * y = (y * 0) * (x * 0)$
3. $(x * y) * [(y * x) * (x * y)] = y * x$
4. $0 * (x * y) = (0 * x) * (0 * y)$
5. $(X, *, 0), x * (0 * y) = y * x, \forall x \neq y$.

Proof. Let $(X, *, 0)$ be Z -algebra, $x, y \in X$.

1. Suppose $y = 0$, then $(x * (x * (y * x))) = (x * (x * (0 * x))) = (x * (x * x))$
by 3.1 $= (x * x)$ by 3.3 $= x$ by 3.3

2. $x * y = (x * y) * 0 = 0 = 0 * 0$ by 3.3 $= (y * 0) * (x * 0)$ by 3.4
3. $(x * y) * [(y * x) * (x * y)] = (x * y) * [(y * x) * (y * x)]$ by 3.4 $= (x * y) * (y * x)$
3.3 $= (y * x) * (y * x)$ by 3.1 $= y * x$ by 3.3
4. $0 * (x * y) = (x * y)$ by 3.2 $= (0 * x) * (0 * y)$ by 3.2
5. $x * (0 * y) = (0 * x) * (0 * y) = x * y = y * x$. □

Definition 3.10. Let $(X, *, 0)$ and $(Y, \Delta, 0')$ be two Z -algebras. A mapping $f : X \rightarrow Y$ of a Z -algebra is called a homomorphism, if $f(x * y) = f(x) \Delta f(y), \forall x, y \in X$.

Definition 3.11. Let $(X, *, 0)$ and $(Y, \Delta, 0')$ be two Z -algebras and $f : X \rightarrow Y$ of is a homomorphism. Then, Kernal of f is the subset of X is defined by $Ker(f) = \{x \in X : f(x) = 0'\}$.

Lemma 3.12. If $f : X \rightarrow Y$ be an homomorphism of Z -algebras, then $f(0) = 0', 0 \in X$.

Proof. Let $f : X \rightarrow Y$ be an homomorphism of Z -algebras. Then, $f(0) = f(0 * 0) = f(0) \Delta f(0) = 0'$ □

Theorem 3.13. Let $(X, *, 0)$ and $(Y, \Delta, 0')$ be two Z -algebras. Let $f : X \rightarrow Y$ be a surjective Z - homomorphism. If A is Z -subalgebra of X , then $f(A)$ is Z -subalgebra of Y .

Proof. Let X and Y be two Z -algebras.

Let $f : X \rightarrow Y$ be a homomorphism and A be a Z -subalgebra of X .

Now, $a, b \in A \Rightarrow a * b \in A, \therefore f(a), f(b) \in f(A)$.

$\Rightarrow f(a) \Delta f(b) = f(a * b) \in f(A)$.

Hence, $f(A)$ is a Z -algebra of Y . □

Theorem 3.14. Let $(X, *, 0)$ and $(Y, \Delta, 0')$ be two Z -algebras and B be Z -subalgebra of Y . Let $f : X \rightarrow Y$ be a homomorphism. Then $f^{-1}(B)$ is a Z -algebra of X .

Proof. It is known that, $f^{-1}(B) = \{x \in X | f(x) = y \text{ for some } y \in B\}$.

Assume that x and $y \in f^{-1}(B)$. Then $f(x)$ and $f(y) \in B$.

Since B is a Z -subalgera of Y , $f(x) \Delta f(y) \in B$. Also, since f is homomorphism, $f(x * y) = f(x) \Delta f(y) \in B$.

$\therefore x * y \in f^{-1}(B)$.

Hence, $f^{-1}(B)$ is a Z -algebra of X . □

4. Z -Ideals

In this section, the notions of Z -ideal and Z -closed ideal of a Z -algebra are discussed.

Definition 4.1. Let X be a Z -algebra and I be a subset of X . Then, I is called a Z -ideal of X , if it satisfies the following conditions:

1. $0 \in I$
2. $x * y \in I$ and $y \in I \Rightarrow x \in I$

Example 4.2. Consider the Z -algebra in example 3.2. Then, $I = \{0, 1, 2\} \subset X$ is a Z -ideal of X .

Definition 4.3. An ideal I of a Z -algebra $(X, *, 0)$ is said to be Z -closed ideal, if $0 * x \in A, \forall x \in I$.

Proposition 4.4. Let I be a Z -ideal of a Z -algebra X . If $x \in I$ and $y * x = 0$, then $y \in I$.

Proof. Let $x \in I$ and $y * x = 0$.

$\Rightarrow x \in I$ and $y * x \in I$.

Since I is a Z -ideal, $y \in I$ is obtained. □

Theorem 4.5. Let $(X, *, 0)$ and $(Y, \Delta, 0')$ be two Z -algebras and $f : X \rightarrow Y$ be a surjective homomorphism. Then, we have,

1. if A is Z -Ideal of X , then $f(A)$ is Z -Ideal of Y .
2. if B is Z -Ideal of Y , then $f^{-1}(B)$ is a Z -Ideal of X .

Theorem 4.6. Let $f : X \rightarrow Y$ be a homomorphism of Z -algebras. Then, $\text{Ker}(f)$ is an ideal of X .

Proof. Obviously, $0 \in \text{Ker}(f)$, $\because f(0) = 0'$.

Let $x * y \in \text{Ker}(f)$ and $y \in \text{Ker}(f)$, so that $f(x * y) = 0', f(y) = 0'$.

That is $f(x) \Delta f(y) = 0' \Rightarrow f(x) \Delta 0' = 0' \Rightarrow f(x) = 0'$.

Hence $x \in \text{Ker}(f)$.

So, $\text{Ker}(f)$ is an ideal of X . □

5. Z -Filters

In this section, the notion of Z -filter on a Z -algebra is studied.

Definition 5.1. Let X be a Z -algebra and $A \subset X$ with $0 \notin A$. Then, A is said to be a Z -filter on X , if $\forall x, y (x \neq y) \in A \Rightarrow x \Delta y = x * (x * y) \in A$.

Example 5.2. Consider the Z -algebra in example 3.4. Then, $A = \{a, b\} \subset X$ and $B = \{a, b, c\} \subset X$ are Z -filters on X .

Remark 5.3. The substructures of Z -filter and Z -subalgebra of X are totally different in general. i.e a Z -subalgebra is not a Z -filter and conversely Z -filter is not a Z -subalgebra.

Example 5.4. Consider the Z -algebra in example 3.4.

Then, $A = \{0, a\} \subset X$ is a Z -subalgebra, but it is not Z -filter, since $0 \in A$.

And, $B = \{a, b\} \subset X$ is a Z -filter, but it is not Z -subalgebra, since $a * b = c \notin B$.

Also, $C = \{0, a, b\} \subset X$ is neither a Z -subalgebra nor a Z -filter.

Remark 5.5. By the definitions 4.1 and 5.1, a Z -filter and Z -ideal are also different substructures in a Z -algebra.

Theorem 5.6. Let X and Y be two Z -algebras. Let $f : X \rightarrow Y$ be a surjective Z -homomorphism. If A is Z -filter of X , then $f(A)$ is Z -filter of Y .

Proof. Let $a, b \in f(A)$. Then, $a = f(x), b = f(y)$ for some $x, y \in X$.

Since A is a Z -filter of X , $x \triangle y = x * (x * y) \in A$.

Also $f(x \triangle y) \in f(A)$.

Now

$$\begin{aligned} a \triangle b &= a * (a * b) \\ &= f(x) * (f(x) * f(y)) \\ &= f(x * (x * y)) \\ &= f(x \triangle y) \end{aligned}$$

Hence, $a \triangle b \in f(A)$, and $f(A)$ is a Z -filter of Y . $\square$

Remark 5.7. Let $f : X \rightarrow Y$ be a homomorphism of Z -algebras. Then, $\text{Ker}(f)$ is not a filter of X .

6. Conclusion and future work

This article introduces the new class of algebra, Z -algebra by taking the theory of sets and propositional calculus as the backend. It has been observed that Z -algebra is not like other algebras. So, it is important that the Z -algebras play an independent role in the axiomatic algebraic system. Interestingly, this concept can further be extended to fuzzification of Z -algebras, Rough set and Soft set application to Z -algebras and derivation on Z -algebras in future.

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ON THE SOLVABILITY OF TWO-POINT IN TIME PROBLEM FOR PDE

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Abstract. We prove that the solution of problem for homogeneous partial differential equation of the second order in time variable in which nonhomogeneous local two-point conditions are given, and infinite order in spatial variables, may not exist in the class of entire functions in the case when the characteristic determinant of the problem equals zero identically. In the case of existence of the solution of the problem, we propose the formula of finding its particular solutions.

Keywords: Multipoint conditions, differential-symbol method, quasipolynomial solutions.

1. Introduction

It is known [1] that multipoint (n -point) problems for PDE are ill-posed problems. The investigations of such problems originate from problems for ODE with multipoint conditions which are called Vallee-Poussin problems [2, 3, 4]. Well-posed solvability of multipoint in time problems for linear PDE, basing on the metric approach, has been investigated in article [5] for the first time. This paper points out the problem of small denominators, which is typical for multipoint problems. Moreover, it was proved that the classes of uniqueness of solution of the multipoint in time problem for PDE are significantly different

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from the classes of uniqueness of the solution of the corresponding Cauchy problem for the same equations. Applying a special technique of below estimation of small denominators to investigation of the n -point problems and the problems with integral conditions for equations and systems of PDEs in the bounded domains, in recent years has been obtained the series of new results (see works [6, 7, 8] and bibliography). For higher order nonlinear hyperbolic equations nonclassical boundary value problems of Vallee-Poussin type are studied in [9].

Papers [10, 11, 12] are devoted to establishing the classes of unique solvability of problems with local multipoint conditions in time for PDEs in unbounded domains $\{(t, x): t \in (0, T), x \in \mathbb{R}^s\}$, where $T > 0$ and $s \in \mathbb{N}$.

In all above-mentioned researches, for correct solvability of multipoint problems for PDEs the authors assumed the characteristic determinant of the problem to be different from identical zero.

This article deals with investigation of solvability of problem with nonhomogeneous local two-point in time conditions for homogeneous differential equations of second order in time and arbitrary order in spatial variables in the class of entire functions in the case when the characteristic determinant of the problem equals zero identically. Note that the set of nontrivial solutions of corresponding homogeneous problem was studied in papers [13, 14].

2. Problem statement

Let's investigate in domain $\mathbb{R}^{s+1}$, $s \in \mathbb{N}$, the solvability of the problem:

$$(2.1) \quad \left[\frac{\partial^2}{\partial t^2} + 2a\left(\frac{\partial}{\partial x}\right)\frac{\partial}{\partial t} + b\left(\frac{\partial}{\partial x}\right) \right] U(t, x) = 0, \quad t \in \mathbb{R}, x \in \mathbb{R}^s,$$

$$(2.2) \quad \begin{aligned} l_{0\partial}U(t, x) &\equiv A_1\left(\frac{\partial}{\partial x}\right)U(0, x) + A_2\left(\frac{\partial}{\partial x}\right)\frac{\partial U}{\partial t}(0, x) = \varphi_0(x), \\ l_{1\partial}U(t, x) &\equiv B_1\left(\frac{\partial}{\partial x}\right)U(h, x) + B_2\left(\frac{\partial}{\partial x}\right)\frac{\partial U}{\partial t}(h, x) = \varphi_1(x), \quad x \in \mathbb{R}^s, \end{aligned}$$

where $\varphi_0(x)$, $\varphi_1(x)$ are given entire functions and at least one of them is nonzero, $h > 0$. In equation (2.1), $a\left(\frac{\partial}{\partial x}\right)$, $b\left(\frac{\partial}{\partial x}\right)$ are differential expressions of the finite or infinite order of the following form:

$$(2.3) \quad a\left(\frac{\partial}{\partial x}\right) = \sum_{|k|=0}^{\infty} a_k \left(\frac{\partial}{\partial x}\right)^k, \quad b\left(\frac{\partial}{\partial x}\right) = \sum_{|k|=0}^{\infty} b_k \left(\frac{\partial}{\partial x}\right)^k,$$

where $k = (k_1, \dots, k_s) \in \mathbb{Z}_+^s$, $|k| = k_1 + \dots + k_s$, $a_k, b_k \in \mathbb{C}$, $\left(\frac{\partial}{\partial x}\right)^k = \frac{\partial^{|k|}}{\partial x_1^{k_1} \dots \partial x_s^{k_s}}$. The symbols $a(\nu)$ and $b(\nu)$ of differential expressions (2.3) are entire functions for $\nu \in \mathbb{C}^s$.

Differential expressions $A_1\left(\frac{\partial}{\partial x}\right)$, $A_2\left(\frac{\partial}{\partial x}\right)$, $B_1\left(\frac{\partial}{\partial x}\right)$, $B_2\left(\frac{\partial}{\partial x}\right)$ in two-point conditions (2.2) are differential polynomials with complex coefficients, such that the

corresponding symbols $A_1(\nu)$, $A_2(\nu)$, $B_1(\nu)$, $B_2(\nu)$ for each $\nu \in \mathbb{C}^s$ satisfy the conditions:

$$|A_1(\nu)|^2 + |A_2(\nu)|^2 \neq 0, \quad |B_1(\nu)|^2 + |B_2(\nu)|^2 \neq 0.$$

The solution of problem (2.1), (2.2) is understood as the following entire function:

$$U(t, x) = \sum_{\widehat{k} \in \mathbb{Z}_+^{s+1}} u_{\widehat{k}} t^{k_0} x^{\widehat{k}}, \quad \widehat{k} = (k_0, k) = (k_0, k_1, \dots, k_s), \quad u_{\widehat{k}} \in \mathbb{C},$$

of variables t and $x = (x_1, \dots, x_s)$, where $x^{\widehat{k}} = x_1^{k_1} \dots x_s^{k_s}$, that satisfies equation (2.1) and two-point conditions (2.2). The action of differential expression $b\left(\frac{\partial}{\partial x}\right)$ onto function U is defined as:

$$b\left(\frac{\partial}{\partial x}\right)U \equiv \sum_{k \in \mathbb{Z}_+^s} b_k \frac{\partial^{|k|} U}{\partial x_1^{k_1} \dots \partial x_s^{k_s}}.$$

The action of differential expression $a\left(\frac{\partial}{\partial t}\right)$ onto $\frac{\partial U}{\partial t}$ is defined in a similar way. Note that papers [15, 16] deal with well-posedness of actions of infinite order differential expressions in the classes of entire functions.

Let's find the conditions under which the solution of problem (2.1), (2.2) exists, and also does not exist in the space of entire functions. By means of the differential-symbol method [17, 18], we shall construct some partial solutions of problem (2.1), (2.2) in the case when the solution of the problem is nonunique.

3. Main results

Consider the ordinary differential equation

$$(3.1) \quad \left[\frac{d^2}{dt^2} + 2a(\nu) \frac{d}{dt} + b(\nu) \right] T(t, \nu) = 0$$

(assume $\nu \in \mathbb{C}^s$) with its normal fundamental system of solutions at the point $t = 0$:

$$T_0(t, \nu) = e^{-a(\nu)t} \left\{ a(\nu) \frac{\sinh \left[t \sqrt{a^2(\nu) - b(\nu)} \right]}{\sqrt{a^2(\nu) - b(\nu)}} + \cosh \left[t \sqrt{a^2(\nu) - b(\nu)} \right] \right\},$$

$$T_1(t, \nu) = e^{-a(\nu)t} \frac{\sinh \left[t \sqrt{a^2(\nu) - b(\nu)} \right]}{\sqrt{a^2(\nu) - b(\nu)}},$$

if $a^2(\nu) \neq b(\nu)$, and $T_0(t, \nu) = e^{-a(\nu)t} \{a(\nu)t + 1\}$, $T_1(t, \nu) = t e^{-a(\nu)t}$, if $a^2(\nu) = b(\nu)$.

We write down the characteristic determinant of problem (2.1), (2.2) of the form

$$\Delta(\nu) = \begin{vmatrix} A_1(\nu) & A_2(\nu) \\ B_1(\nu) T_0(h, \nu) + B_2(\nu) \frac{dT_0}{dt}(h, \nu) & B_1(\nu) T_1(h, \nu) + B_2(\nu) \frac{dT_1}{dt}(h, \nu) \end{vmatrix}.$$

By Poincare Theorem ([19], p. 59), the functions $T_0(t, \nu)$ and $T_1(t, \nu)$, as solutions of Cauchy problem, are entire functions of vector-parameter ν , since the coefficients $a(\nu)$, $b(\nu)$ of equation (3.1) are entire functions by the assumption. Apart from this, the function $\Delta(\nu)$, as a superposition of entire functions, is also entire function.

The order of entire functions $T_0(t, \nu) e^{\nu \cdot x}$, $T_1(t, \nu) e^{\nu \cdot x}$ by the set of variables $\nu_1, \nu_2, \dots, \nu_s$ is denoted by p , where $\nu \cdot x = \nu_1 x_1 + \dots + \nu_s x_s$. Note that $p \in [1; +\infty]$.

Let's denote the class of entire functions $\varphi(x)$, whose the order is less than p' , as $A_{p'}$, where $\frac{1}{p} + \frac{1}{p'} = 1$, when $1 < p < \infty$. Also we assume that if $p = 1$ then $A_{p'}$ is a class of entire functions ($p' = \infty$), and if $p = \infty$ then $A_{p'}$ is a class of entire function of exponential type ($p' = 1$).

Denote by $\mathbb{A}_{p'}$ the class of entire functions $U(t, \cdot)$, which for each fixed $t \in \mathbb{R}$ belong to $A_{p'}$.

Theorem 3.1. *Let $\Delta(\nu) \equiv 0$ in $\mathbb{C}^s$ and for certain $x \in \mathbb{R}^s$ at least one of the following two conditions is satisfied:*

$$(3.2) \quad l_{1\partial} T_1 \left(t, \frac{\partial}{\partial x} \right) \varphi_0(x) \neq A_2 \left(\frac{\partial}{\partial x} \right) \varphi_1(x)$$

or

$$(3.3) \quad l_{1\partial} T_0 \left(t, \frac{\partial}{\partial x} \right) \varphi_0(x) \neq A_1 \left(\frac{\partial}{\partial x} \right) \varphi_1(x)$$

for $\varphi_0, \varphi_1 \in A_{p'}$. Then the solution of problem (2.1), (2.2) does not exist in the class of entire functions $\mathbb{A}_{p'}$.

Proof. Contrary, we assume that in the class $\mathbb{A}_{p'}$ there exists entire solution $U(t, x)$ of equation (2.1), which satisfies conditions (2.2). Let's denote $U(0, x) = \varphi(x)$, $\frac{\partial U}{\partial t}(0, x) = \psi(x)$. Then functions $\varphi(x)$ and $\psi(x)$ are also entire functions and belong to the class $A_{p'}$.

Let's write down the solution of problem (2.1), (2.2) according to the differential-symbol method [17, 18] as the solution of the Cauchy problem for equation (2.1) with initial data φ and ψ in the form

$$(3.4) \quad U(t, x) = \varphi \left(\frac{\partial}{\partial \nu} \right) \left\{ T_0(t, \nu) e^{\nu \cdot x} \right\} \Big|_{\nu=0} + \psi \left(\frac{\partial}{\partial \nu} \right) \left\{ T_1(t, \nu) e^{\nu \cdot x} \right\} \Big|_{\nu=0},$$

where $O = (0, \dots, 0)$.

Since the conditions (2.2) are satisfied for $U(t, x)$, we obtain the matrix equation as follows:

$$\begin{pmatrix} l_{0\partial}T_0\left(t, \frac{\partial}{\partial x}\right) & l_{0\partial}T_1\left(t, \frac{\partial}{\partial x}\right) \\ l_{1\partial}T_0\left(t, \frac{\partial}{\partial x}\right) & l_{1\partial}T_1\left(t, \frac{\partial}{\partial x}\right) \end{pmatrix} \begin{pmatrix} \varphi(x) \\ \psi(x) \end{pmatrix} = \begin{pmatrix} \varphi_0(x) \\ \varphi_1(x) \end{pmatrix}.$$

Let's act onto the last equation by the matrix differential expression

$$\begin{pmatrix} l_{1\partial}T_1\left(t, \frac{\partial}{\partial x}\right) & -l_{0\partial}T_1\left(t, \frac{\partial}{\partial x}\right) \\ -l_{1\partial}T_0\left(t, \frac{\partial}{\partial x}\right) & l_{0\partial}T_0\left(t, \frac{\partial}{\partial x}\right) \end{pmatrix}.$$

From the equalities

$$l_{0\partial}T_1\left(t, \frac{\partial}{\partial x}\right) = A_2\left(\frac{\partial}{\partial x}\right), \quad l_{0\partial}T_0\left(t, \frac{\partial}{\partial x}\right) = A_1\left(\frac{\partial}{\partial x}\right)$$

for arbitrary $x \in \mathbb{R}^s$, it follows that

$$\begin{aligned} \begin{pmatrix} \Delta\left(\frac{\partial}{\partial x}\right)\varphi(x) \\ \Delta\left(\frac{\partial}{\partial x}\right)\psi(x) \end{pmatrix} &= \begin{pmatrix} l_{1\partial}T_1\left(t, \frac{\partial}{\partial x}\right) & -l_{0\partial}T_1\left(t, \frac{\partial}{\partial x}\right) \\ -l_{1\partial}T_0\left(t, \frac{\partial}{\partial x}\right) & l_{0\partial}T_0\left(t, \frac{\partial}{\partial x}\right) \end{pmatrix} \begin{pmatrix} \varphi_0(x) \\ \varphi_1(x) \end{pmatrix} = \\ &= \begin{pmatrix} l_{1\partial}T_1\left(t, \frac{\partial}{\partial x}\right)\varphi_0(x) - A_2\left(\frac{\partial}{\partial x}\right)\varphi_1(x) \\ -l_{1\partial}T_0\left(t, \frac{\partial}{\partial x}\right)\varphi_0(x) + A_1\left(\frac{\partial}{\partial x}\right)\varphi_1(x) \end{pmatrix}. \end{aligned}$$

Since $\Delta(\nu) \equiv 0$ in $\mathbb{C}^s$, then $\Delta\left(\frac{\partial}{\partial x}\right)$ is null operator and $\Delta\left(\frac{\partial}{\partial x}\right)\varphi(x) = \Delta\left(\frac{\partial}{\partial x}\right)\psi(x) = 0$, therefore, we have a contradiction with conditions (3.2) and (3.3). The proof is complete. $\square$

Example 3.1. In the domain $(t, x_1, x_2) \in \mathbb{R}^3$, investigate the solvability of the problem for the equation

$$(3.5) \quad \left[\frac{\partial^2}{\partial t^2} + 2\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right)\frac{\partial}{\partial t} + \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right)^2 + 1 \right] U(t, x) = 0$$

with local two-point conditions

$$(3.6) \quad \begin{aligned} \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right)U(0, x) + \frac{\partial U}{\partial t}(0, x) &= \varphi_0(x), \\ \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right)U(\pi, x) + \frac{\partial U}{\partial t}(\pi, x) &= \varphi_1(x). \end{aligned}$$

For this problem, we have $a(\nu) = \nu_1 + \nu_2$, $b(\nu) = (\nu_1 + \nu_2)^2 + 1$, $A_1(\nu) = B_1(\nu) = \nu_1 + \nu_2$, $A_2(\nu) = B_2(\nu) = 1$, $s = 2$, $h = \pi$.

The normal fundamental system of solutions at the point $t = 0$ of the ODE, corresponding to (3.5)

$$\left[\frac{d^2}{dt^2} + 2(\nu_1 + \nu_2) \frac{d}{dt} + (\nu_1 + \nu_2)^2 + 1 \right] T(t, \nu) = 0$$

has the form

$$T_0(t, \nu) = e^{-(\nu_1 + \nu_2)t} \left\{ (\nu_1 + \nu_2) \sin t + \cos t \right\}, \quad T_1(t, \nu) = e^{-(\nu_1 + \nu_2)t} \sin t.$$

The functions $T_0(t, \nu)$, $T_1(t, \nu)$ are entire functions of the order $p = 1$ by the set of variables ν_1, ν_2 .

Write down the characteristic determinant of problem (3.5), (3.6):

$$\begin{aligned} \Delta(\nu) &= \begin{vmatrix} \nu_1 + \nu_2 & 1 \\ (\nu_1 + \nu_2) T_0(\pi, \nu) + \frac{dT_0}{dt}(\pi, \nu) & (\nu_1 + \nu_2) T_1(\pi, \nu) + \frac{dT_1}{dt}(\pi, \nu) \end{vmatrix} = \\ &= e^{-(\nu_1 + \nu_2)\pi} \sin \pi \equiv 0. \end{aligned}$$

Condition (3.2) for problem (3.5), (3.6) follows from (3.3) and have such a form:

for a certain point $(x_1, x_2) \in \mathbb{R}^2$ inequality

$$(3.7) \quad -\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right) \varphi_0(x_1 - \pi, x_2 - \pi) \neq \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right) \varphi_1(x_1, x_2)$$

is hold for entire ($p' = \infty$) functions φ_0, φ_1 .

So, in the case of fulfillment of condition (3.7), by Theorem 3.1, the solution of nonhomogeneous problem (3.5), (3.6) in the class of entire functions does not exist.

Example 3.2. Investigate the solvability of the two-point problem in the domain of variables $t \in \mathbb{R}$, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ for differential-functional equation

$$(3.8) \quad \frac{\partial^2}{\partial t^2} U(t, x_1, x_2, x_3) + 2 \frac{\partial}{\partial t} U(t, x_1 + 1, x_2 + 1, x_3 - 1) + \\ + 2U(t, x_1 + 1, x_2 + 1, x_3 - 1) - U(t, x_1, x_2, x_3) = 0$$

with nonhomogeneous local conditions

$$(3.9) \quad U(0, x) + \frac{\partial U}{\partial t}(0, x) = \varphi_0(x), \quad U(1, x) + \frac{\partial U}{\partial t}(1, x) = \varphi_1(x).$$

Differential-functional equation (3.8) can be represented as the differential equation of infinite order

$$\left[\frac{\partial^2}{\partial t^2} + 2e^{\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3}} \frac{\partial}{\partial t} + \left(2e^{\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3}} - 1 \right) \right] U(t, x) = 0.$$

For this problem, we have $a(\nu) = e^{\nu_1 + \nu_2 - \nu_3}$, $b(\nu) = 2e^{\nu_1 + \nu_2 - \nu_3} - 1$, $A_1(\nu) = A_2(\nu) = 1$, $B_1(\nu) = B_2(\nu) = 1$, $s = 3$, $h = 1$.

The normal fundamental system of solutions of ODE

$$\left[\frac{d^2}{dt^2} + 2e^{\nu_1 + \nu_2 - \nu_3} \frac{d}{dt} + (2e^{\nu_1 + \nu_2 - \nu_3} - 1) \right] T(t, \nu) = 0$$

has the form

$$(3.10) \quad \begin{aligned} T_0(t, \nu) &= e^{-a(\nu)t} \left\{ a(\nu) \frac{\sinh[t\{a(\nu) - 1\}]}{a(\nu) - 1} + \cosh[t\{a(\nu) - 1\}] \right\}, \\ T_1(t, \nu) &= e^{-a(\nu)t} \frac{\sinh[t\{a(\nu) - 1\}]}{a(\nu) - 1}, \end{aligned}$$

where $a(\nu) = e^{\nu_1 + \nu_2 - \nu_3}$. In the case $a(\nu) = 1$, we have $T_0 = e^{-t}(t+1)$, $T_1 = te^{-t}$.

In our case $p = \infty$. Let's calculate the characteristic determinant $\Delta(\nu)$ of problem (3.8), (3.9):

$$\begin{aligned} \Delta(\nu) &= \\ &= e^{-e^{\nu_1 + \nu_2 - \nu_3}} \frac{\sinh[e^{\nu_1 + \nu_2 - \nu_3} - 1]}{e^{\nu_1 + \nu_2 - \nu_3} - 1} \left[(e^{\nu_1 + \nu_2 - \nu_3} - 1)^2 - (e^{\nu_1 + \nu_2 - \nu_3} - 1)^2 \right] \equiv 0. \end{aligned}$$

Conditions (3.2) and (3.3) for problem (3.8), (3.9) can be written down as one condition:

the following inequality

$$(3.11) \quad \varphi_1(x_1, x_2, x_3) \neq e^{1 - 2e^{\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3}}} \varphi_0(x_1, x_2, x_3)$$

holds for entire functions φ_0, φ_1 of exponential type and for certain point $(x_1, x_2, x_3) \in \mathbb{R}^3$.

Therefore, in the case of fulfillment of condition (3.11), according to Theorem 3.1, the solution of nonhomogeneous problem (3.8), (3.9) in the class of entire functions $\mathbb{A}_1$ does not exist.

4. Constructing partial solutions of the two-point problem in the class of existence of nonunique solution of the problem

Let's consider the case when conditions of Theorem 3.1 are not satisfied.

If for problem (2.1), (2.2) $\Delta(\nu) \equiv 0$ in $\mathbb{C}^s$ and for arbitrary $x \in \mathbb{R}^s$ there hold the equalities

$$l_1 \partial T_1 \left(t, \frac{\partial}{\partial x} \right) \varphi_0(x) = A_2 \left(\frac{\partial}{\partial x} \right) \varphi_1(x),$$

$$l_{1\partial}T_0\left(t, \frac{\partial}{\partial x}\right)\varphi_0(x) = A_1\left(\frac{\partial}{\partial x}\right)\varphi_1(x)$$

for $\varphi_0, \varphi_1 \in A_{p'}$, then the solution of problem (2.1), (2.2) in class of entire functions $A_{p'}$ exists but is not unique.

Let's show in examples constructing partial solutions of the problem.

Example 4.1. Consider the problem

$$(4.1) \quad \left[\frac{\partial^2}{\partial t^2} + 2\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right)\frac{\partial}{\partial t} + \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right)^2 + 1 \right] U(t, x) = 0, (t, x) \in \mathbb{R}^3,$$

$$(4.2) \quad \begin{aligned} &\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right)U(0, x) + \frac{\partial U}{\partial t}(0, x) = \varphi_0(x), \\ &\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right)U(\pi, x) + \frac{\partial U}{\partial t}(\pi, x) = -\varphi_0(x_1 - \pi, x_2 - \pi), \quad x \in \mathbb{R}^2. \end{aligned}$$

Let's consider the problem as problem (3.5), (3.6), in which for entire functions φ_0, φ_1 and for arbitrary $x \in \mathbb{R}^2$ there holds the condition

$$-\varphi_0(x_1 - \pi, x_2 - \pi) = \varphi_1(x_1, x_2).$$

Therefore, the conditions of Theorem 3.1 are not satisfied. We show that the solution of problem (4.1), (4.2) exists in the class of entire functions.

Note that the null space of problem (4.1), (4.2) is infinite-dimensional. It contains not only entire solutions, but also classical solutions of the form

$$U(t, x) = \varphi(x_1 - t, x_2 - t) \cos t,$$

where φ is arbitrary twice continuously differentiable function on $\mathbb{R}^2$.

The entire (partial) solution of problem (4.1), (4.2) can be found, for example, by formula

$$\begin{aligned} U(t, x) &= \varphi_0\left(\frac{\partial}{\partial \nu}\right)\left\{T_1(t, \nu)e^{\nu \cdot x}\right\}\Big|_{\nu=(0,0)} = \\ &= \varphi_0\left(\frac{\partial}{\partial \nu}\right)\left\{e^{-(\nu_1+\nu_2)t+\nu \cdot x} \sin t\right\}\Big|_{\nu=(0,0)} = \varphi_0(x_1 - t, x_2 - t) \sin t. \end{aligned}$$

If in the formula

$$(4.3) \quad U(t, x) = \varphi_0(x_1 - t, x_2 - t) \sin t$$

we take $\varphi_0(x_1, x_2)$ as an arbitrary twice continuously differentiable function on $\mathbb{R}^2$, then function (4.3) is a classical solution of problem (4.1), (4.2).

Example 4.2. Let's consider the problem

$$(4.4) \quad \frac{\partial^2}{\partial t^2} U(t, x_1, x_2, x_3) + 2 \frac{\partial}{\partial t} U(t, x_1 + 1, x_2 + 1, x_3 - 1) + \\ + 2U(t, x_1 + 1, x_2 + 1, x_3 - 1) - U(t, x_1, x_2, x_3) = 0, \quad (t, x) \in \mathbb{R}^4,$$

$$(4.5) \quad \begin{aligned} U(0, x) + \frac{\partial U}{\partial t}(0, x) &= \varphi_0(x), \\ U(1, x) + \frac{\partial U}{\partial t}(1, x) &= e^{1-2e^{\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3}}} \varphi_0(x), \quad x \in \mathbb{R}^3. \end{aligned}$$

Problem (4.4), (4.5) is problem (3.8), (3.9), in which for entire functions $\varphi_0(x)$, $\varphi_1(x)$ of exponential type and for arbitrary $x \in \mathbb{R}^3$ the following condition

$$(4.6) \quad \varphi_1(x_1, x_2, x_3) = e^{1-2e^{\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3}}} \varphi_0(x_1, x_2, x_3).$$

is satisfied.

Hence, the conditions of Theorem 3.1 are not satisfied. The solution of problem (4.4), (4.5) exists in the class of entire functions $\mathbb{A}_1$. Note that the null space of this problem is infinite-dimensional. It contains the functions

$$(4.7) \quad U(t, x) = e^{-t} \varphi(x),$$

where φ is arbitrary continuous on $\mathbb{R}^3$ function.

The entire (partial) solution of problem (4.4), (4.5) for function $\varphi_0(x)$ of the exponential type can be found, for example, using one of the following formulas:

$$U(t, x) = \varphi_0\left(\frac{\partial}{\partial \nu}\right) \left\{ T_0(t, \nu) e^{\nu \cdot x} \right\} \Big|_{\nu=(0,0,0)}$$

or

$$(4.8) \quad U(t, x) = \varphi_0\left(\frac{\partial}{\partial \nu}\right) \left\{ T_1(t, \nu) e^{\nu \cdot x} \right\} \Big|_{\nu=(0,0,0)},$$

where $T_0(t, \nu)$, $T_1(t, \nu)$ are functions (3.10).

Let's consider some cases of different functions $\varphi_0(x)$.

Case 1. Let $\varphi_0(x_1, x_2, x_3) = e^{x_1+x_2+2x_3}$. Then condition (4.6) has the form

$$e^{1-2e^{\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3}}} \varphi_0(x_1, x_2, x_3) = e^{-1+x_1+x_2+2x_3}.$$

By formula (4.8), we find the partial solution of problem (4.4), (4.5):

$$\begin{aligned} U(t, x) &= e^{\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + 2\frac{\partial}{\partial x_3}} \left\{ T_1(t, \nu) e^{\nu \cdot x} \right\} \Big|_{\nu=(0,0,0)} = \\ &= T_1(t, 1, 1, 2) e^{x_1+x_2+2x_3} = t e^{-t+x_1+x_2+2x_3}. \end{aligned}$$

Case 2. Let $\varphi_0(x_1, x_2, x_3) = 1 + 2x_1 - 3x_2 + 2x_3$. Let's calculate

$$\begin{aligned} e^{1-2e^{\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3}}} \varphi_0(x_1, x_2, x_3) &= \left\{ \left(e^{1-2e^{\nu_1 + \nu_2 - \nu_3}} \right) \Big|_{\nu=(0,0,0)} + \right. \\ &+ \frac{\partial}{\partial \nu_1} \left(e^{1-2e^{\nu_1 + \nu_2 - \nu_3}} \right) \Big|_{\nu=(0,0,0)} \frac{\partial}{\partial x_1} + \frac{\partial}{\partial \nu_2} \left(e^{1-2e^{\nu_1 + \nu_2 - \nu_3}} \right) \Big|_{\nu=(0,0,0)} \frac{\partial}{\partial x_2} + \\ &\left. + \frac{\partial}{\partial \nu_3} \left(e^{1-2e^{\nu_1 + \nu_2 - \nu_3}} \right) \Big|_{\nu=(0,0,0)} \frac{\partial}{\partial x_3} + \dots \right\} \varphi_0(x_1, x_2, x_3) = \\ &= e^{-1} \left\{ 1 - 2 \frac{\partial}{\partial x_1} - 2 \frac{\partial}{\partial x_2} + 2 \frac{\partial}{\partial x_3} + \dots \right\} (1 + 2x_1 - 3x_2 + 2x_3) = \\ &= e^{-1} (7 + 2x_1 - 3x_2 + 2x_3). \end{aligned}$$

Therefore, according to equality (4.6), the existence of the solution of problem (4.4), (4.5) is provided by the condition:

$$\varphi_1(x_1, x_2, x_3) = e^{-1} (7 + 2x_1 - 3x_2 + 2x_3).$$

Let's find the partial solution of problem (4.4), (4.5) by formula (4.8):

$$\begin{aligned} U(t, x) &= \left\{ 1 + 2 \frac{\partial}{\partial \nu_1} - 3 \frac{\partial}{\partial \nu_2} + 2 \frac{\partial}{\partial \nu_3} \right\} \left\{ T_1(t, \nu) e^{\nu \cdot x} \right\} \Big|_{\nu=(0,0,0)} = \\ &= T_1(t, 0, 0, 0) + 2 \frac{\partial T_1}{\partial \nu_1}(t, 0, 0, 0) - 3 \frac{\partial T_1}{\partial \nu_2}(t, 0, 0, 0) + \\ &+ 2 \frac{\partial T_1}{\partial \nu_3}(t, 0, 0, 0) + 2x_1 T_1(t, 0, 0, 0) - 3x_2 T_1(t, 0, 0, 0) + 2x_3 T_1(t, 0, 0, 0). \end{aligned}$$

Since

$$\begin{aligned} T_1(t, 0, 0, 0) &= t e^{-t}, \\ \frac{\partial T_1}{\partial \nu_1}(t, 0, 0, 0) &= \frac{\partial T_1}{\partial \nu_2}(t, 0, 0, 0) = -t^2 e^{-t}, \quad \frac{\partial T_1}{\partial \nu_3}(t, 0, 0, 0) = t^2 e^{-t}, \end{aligned}$$

then the solution of problem (4.4), (4.5) has the form

$$U(t, x) = t e^{-t} \{ 1 + 2x_1 - 3x_2 + 2x_3 + 3t \}.$$

Note that the solution above is only partial solution: summed up with elements of null space of problem (4.7) it is also a solution of problem (4.4), (4.5).

5. Conclusions

In this work, we prove that the problem for homogeneous PDE of second order in time variable (in which the nonhomogeneous local two-point conditions are given) and generally infinite order in spatial variables in the class of entire functions, is ill-posed problem in the case when the characteristic determinant identically equals to zero. In the case of existence of nonunique solution of the problem, we propose a method of finding partial solutions. This way is presented on examples.

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MINIMAL MODELS OF SELF-ORGANIZED CRITICALITY

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Abstract. The paper deals with an evaluation of the behavior of non equilibrium systems displaying self-organized criticality, according to the concept of Bak-Tang-Wiesenfeld ([3]). One of the fundamental characteristics of a system in a self-organized state is to exhibit a stationary state with a long-range power law of decay of both spatial and temporal correlations.

Keywords: Self organizing criticalities, Markov Chains, Bak Sneppen processes, econophysics, socio-economic evolution staircase.

Introduction

A rather general frame can be the following:

The spatial structure is discretized in N nodes, that are connected according to some distance rules. Bak-Sneppen original structure (see [1] and [2]) foresaw a

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circular disposition, where each node was connected only with the two adjoining nodes. Many models supplied more sophisticated systems of contact between the nodes, but a relevant structural change seemed to happen only in the case of a bipartite system, as pointed out in Piccinini-Lepellere-Chang ([23]) and Piccinini-Lepellere-Chang-Iseppi ([21]).

Also time is discretized. At each step all nodes have one or more properties. These can be expressed by discrete or continuous variables, numerical or ordinal, or even simply by labels, as in the case of political choices. In numerical cases the function will be called fitness, according to the historical definition of Bak-Sneppen.

The transition from one time step to another involves random changes of the quality of one or more nodes, according to structural distance properties. Hence the process may be represented by a Markov Chain. A typical, but not compulsory, feature is that the changing node is selected among the nodes that have the lowest fitness. Hence it is selected according to a global information. Its fitness will change according to some probability law, so that it may even not increase, but this is inessential. What is essential is that one or more neighbors will also randomly change their fitness even if it is already good. Here lies the clue of the process, in the fact that the behavior of neighbors may even damage their fitness. It is anyhow well proved that in spite of the randomness of evolution, the average level of fitness will be superior to the average of the probability distribution, and that depends on the fact that there exists a moment of rational choice, that is the order of changing given to one of the worst nodes.

Many questions may be asked, but one of the most appealing is the average distribution of the fitness levels. Analytic formulas are very rare because of the extreme complication of the model, but statistical models give good evidences, in particular that in the classical Bak-Sneppen model, for large number of nodes (100 or more), for high numbers of iterations (over 1,000,000), with fitness uniformly distributed in the interval $[0, 1]$, the density of the average distribution follows a sigmoid curve, with values near 0 up to 0.5, and an almost constant asymptotic value in the interval starting from 0.667... ([14]). The sigmoid behavior seems to be very stable and is a characteristic of this type of processes, even if for small systems the left side is less evident.

A recent explicit solving formula for the case of 4 nodes (the smallest non obvious dimension) is to be found in ([24]) and for 5 nodes in ([25]). Small models have been investigated in order to get analytic sharp informations. One of the most celebrated is Barbay-Kanyon [5], with two discrete fitness levels, achieved with a varying probability, so that many models should be approximated. The authors have shown that phenomena may arise that cannot be explained with Barbay-Kanyon model. Anyhow also this case is by no means trivial, as it was shown by Meester and Znamenski in ([16]), see also ([4]).

The aim of this paper is to give a minimal non-trivial model where the standard behavior is still recognizable, and at the same time a Barbay-Kanyon type method allows exact results.

1. The elementary model

In general there exist L^N configurations, where N is the number of nodes and L is the number of fitness levels. The transition matrix of the associated Markov chain is thus a very large matrix, and depends on the structure of the choices that are allowed and the neighborhood table.

The simplest non-trivial case requires three nodes and one neighbor, so that at each step one node is selected for change, another is changed in view of neighborhood and the third one is left unchanged. Anyhow the complexity of computation in general depends on the number of fitness levels. But in the case of three nodes it is possible to reduce every computation to the case of two levels as it is proved in section 3.

Let L be the number of levels. For any given level $L(i)$, $p(i)$ denotes the probability of extracting a fitness equal to $L(i)$, so that $\sum_{i=1}^N p(i) = 1$. The three nodes are disposed on a circle and the (only) neighbor is the left node, therefore the right node remains unchanged.

Since in this problem we are not investigating the return time between configuration we can ignore rotations, so that the L^3 possible configurations can be reduced to $(L^3 - L)/3 + L$, namely L corresponding to a constant sequence and the remaining corresponding each to 3 configurations.

Three simple examples are shown below: the choice of the examples depends on the use made of them in the third section.

1.1 Example 1. Ternary elementary model

The first matrix is referred to three fitness levels, hence it is a 11×11 matrix. The simplest way is to construct an auxiliary column for each of the unchanged items: so there are three auxiliary columns, denoted by $aux1$, $aux2$, $aux3$, and p , q , r denote the probabilities of the three levels.

				aux3	aux2	aux1	
THREE	a	3	3	3	r^2	0	$a = aux3$
	b	3	3	2	$2qr$	r^2	$b = aux3$
TWO	c	3	3	1	$2pr$	0	$c = aux3$
	d	3	2	2	q^2	$2qr$	$d = (aux3 + aux2)/2$
	e	3	2	1	pq	pr	$e = aux3$
	f	3	1	2	pq	pr	$f = aux2$
ONE	g	3	1	1	p^2	0	$g = (aux3 + aux1)/2$
	h	2	2	2	0	q^2	$h = aux2$
	i	2	2	1	0	$2pq$	$i = aux2$
	j	2	1	1	0	p^2	$j = (aux2 + aux1)/2$
ZERO	k	1	1	1	0	0	$k = aux1$

Table 1: *Transition matrix of elementary ternary model*

The choice of the unchanged node depends on the location of the minimum. In case of ties the same probability is assigned to each of the allowed locations, so that the transition column may not be a pure auxiliary column but a weighted

combination of different columns. The final transition matrix, see Table 1, is formed gluing together the auxiliary column according to the rule indicated on the right.

1.2 Example 2. Fundamental elementary transition matrix

An important case is obtained from example 1, stating that one of the probabilities is 0, so that it actually reduces to only two fitness levels. It is obtained from the matrix generated by example 1. Here it is written explicitly, developing the construction up to the end, so that the reader can explicitly check the rule.

The number of states reduces to 4, as they correspond to the cardinality of nodes having the highest level of fitness (probability q), see Table 2.

	# 3	# 2	# 1	# 0
# 3=h	q^2	q^2	$q^2/2$	0
# 2=i	$2pq$	$2pq$	$pq + q^2/2$	q^2
# 1=j	p^2	p^2	$p^2/2 + pq$	$2pq$
# 0=k	0	0	$p^2/2$	p^2

Table 2: *Fundamental elementary transition matrix*

We represent also the minimal problem for the traditional Bak Sneppen model with two neighbors. The minimal non trivial number of nodes is 4, so that also in this case one node is left unchanged. The example is referred to three fitness levels.

1.3 Example 3. Fundamental 3 level Bak Sneppen transition

Traditional Bak Sneppen model with four nodes and three fitness levels has the following basic transition matrix (apart rotations). Since at each step only one node is left unchanged, three auxiliary columns are enough, according to the value of the remaining node. The structure is thus achieved according to the description that follows each line. Ignoring rotations 24 configurations are required, see Table 3.

Aggregating the two lower fitness values one gets the standard 2-level table with 6 configurations.

$$\begin{aligned}
 4 &= a \\
 3 &= b, c \\
 2+ \text{ (two consecutive maxima) } &= d, e, f, g \\
 2- \text{ (intervalled maxima) } &= h, i, j \\
 1 &= k, l, m, n, o, p, q, r \\
 0 &= s, t, u, v, w, x
 \end{aligned}$$

					aux3	aux2	aux1	
a	3	3	3	3	r^3	0	0	$a = aux3$
b	3	3	3	2	$3qr^2$	r^3	0	$b = aux3$
c	3	3	3	1	$3pr^2$	0	r^3	$c = aux3$
d	3	3	2	2	$2q^2r$	$2qr^2$	0	$d = aux3$
e	3	3	2	1	$2pqr$	pr^2	qr^2	$e = aux3$
f	3	3	1	2	$2pqr$	pr^2	qr^2	$f = aux3$
g	3	3	1	1	$2p^2r$	0	$2pr^2$	$g = aux3$
h	3	2	3	2	q^2r	qr^2	0	$h = aux2$
i	3	2	3	1	$2pqr$	pr^2	qr^2	$i = aux2$
j	3	1	3	1	p^2r	0	pr^2	$j = aux1$
k	3	2	2	2	q^3	$3q^2r$	0	$k = (2aux2 + aux1)/3$
l	3	2	2	1	pq^2	$2pqr$	q^2r	$l = aux2$
m	3	2	1	2	pq^2	$2pqr$	q^2r	$m = aux3$
n	3	2	1	1	p^2q	p^2r	$2pqr$	$n = (aux3 + aux2)/2$
o	3	1	1	2	p^2q	p^2r	$2pqr$	$o = (aux3 + aux2)/2$
p	3	1	2	1	p^2q	p^2r	$2pqr$	$p = aux1$
q	3	1	1	1	p^3	0	$3p^2r$	$q = (aux3 + 2aux2)/3$
r	3	1	2	2	pq^2	$2pqr$	q^2r	$r = aux2$
s	2	2	2	2	0	q^3	0	$s = aux2$
t	2	2	2	1	0	$3pq^2$	q^3	$t = aux2$
u	2	2	1	1	0	$2p^2q$	$2pq^2$	$u = aux2$
v	2	1	2	1	0	p^2q	pq^2	$v = aux1$
w	2	1	1	1	0	p^3	$3p^2q$	$w = (aux2 + 2aux1)/3$
x	1	1	1	1	0	0	p^3	$x = aux1$

Table 3: *Fundamental 3 level Bak Sneppen transition*

2. Questions and solutions

Many questions arise when studying even simple models. A main question is the average probability of an assigned fitness level. The average distribution requires that the ergodic state is achieved, what could require very high times, especially in certain partitioned frames where a long transition period must be foreseen as was proved by the authors in ([21]).

If we have the full transition matrix, with all the fitness levels explicitly listed, it would just be required to find the first normalized eigenvector corresponding to the transition matrix, so that the probabilities of the level could be easily derived according the assigned model. In presence of a rectangular distribution, the same probability would be assigned to each level.

Unfortunately this problem, even avoiding duplications as in examples 1, 2 and 3, would fast become very heavy for computation. The only cases that can be easily dominated are binary frames, where all the fitness levels are identified either in a low level, say 0, or in an high level, say 1. Of course in this case the probability of the two events shall be taylored according to the level of the cut.

The division of fitness levels leeds straightforward to the definition of boxed probability.

Definition 1. For each box let l and u denote the minimum level and the maximum level belonging to the box. $B(l, u)$, called boxed probability, thus denotes the probability of lying inside the box.

In case of a set of discrete values usually the lowest minimum is 0, and is never achieved, while the highest level of each box may be an actual value of the fitness.

A binary model would thus be boxed as $B(0, p)$, $B(p, 1)$. A ternary model, as in example 1, would lead to three boxes $B(0, p)$, $B(p, p+q)$, $B(p+q, p+q+r)$, where $p+q+r=1$.

Of course the sum of all the boxes must be equal to 1, since they exhaust all the possibilities.

Unfortunately the solution of the problem requires full analysis and full knowledge of the process. The simple derivation from some transition matrixes with few fitness levels leads, as it was stated before, to the knowledge of the first normalized eigenvector, hence to the knowledge of the frequencies of a certain number of fitness levels, usually two, as in example 2, sometimes three as in example 1 and 3. We call the frequencies respectively $D(p)$ and $D(q)$, with $p+q=1$ and respectively $T(p)$, $T(q)$, $T(r)$ with $p+q+r=1$. Of course it is possible to consider systems with more levels of fitness.

The fundamental question is this: when does it hold the equality?

$$(1) \quad B(p, p+q) = D(p+q) - D(p)$$

That is, when the probability of all the levels between p and $p+q$ is just equal to the difference between the probability of a single level $p+q$ minus the probability of a single level p ?

Should such a fact happen, all the analysis would be reduced to the (anyhow non obvious) problem of estimating the binary frequency $D(x)$.

The reason why the answer need not to be positive lies in the geometry of Bak-Sneppen type process, where the neighborhood relations condition, though slightly, changes the final averages. On the contrary the given heavy geometric conditions substantially change the return times between two states.

2.1 Example 4. Transition matrix of 4 nodes random elementary model

A simpler case arises when no geometry is at hand, namely when the simplified process consists of finding the worst possible value, change at random its owner, then choose at random one other node, and change it at random, independently on its distance from the worst point.

The binary transition matrix then becomes (in the case of 4 nodes) the Table 4.

Remark that for 3 nodes this matrix corresponds to matrix 2. The random transition matrix is obtained in the same way for all numbers of nodes summing up the shifted copies of the first column multiplied respectively for the ratios $i/(N-1)$, $(N-1-i)/(N-1)$.

This process has no geometry, hence additivity holds, since the only parameter is the number of good (1) nodes with respect to bad (0) nodes. Taking the

	# 4	# 3	# 2	# 1	# 0
# 4	q^2	q^2	$q^2/3$	0	0
# 3	$2pq$	$2pq$	$2pq/3 + 2q^2/3$	$2q^2/3$	0
# 2	p^2	p^2	$p^2/3 + 4pq/3$	$4pq/3 + q^2/3$	q^2
# 1	0	0	$2p^2/3$	$2p^2/3 + 2pq/3$	$2pq$
# 0	0	0	0	$p^2/3$	p^2

Table 4: *Transition for Random model*

differences $D(0.2) - D(0.1)$, $D(0.3) - D(0.2)$, we get the following ten boxes, that have a sigmoid behaviour that resembles the experimental Bak-Sneppen behaviour, see Figure 1.

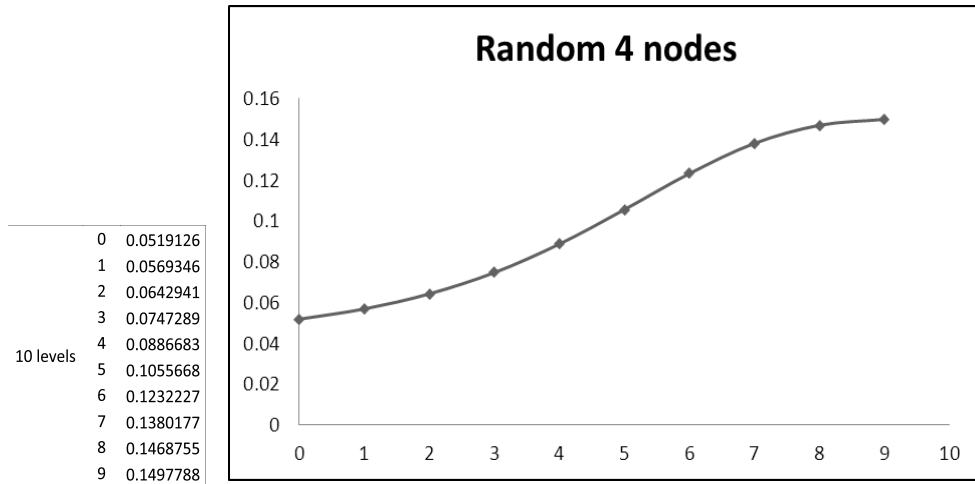


Figure 1: Random model with 4 nodes and 10 levels

3. Some sufficient non trivial conditions for equation 1

The simplest problem is the transition from a ternary model to a dual model in which two values are connected with the ternary values. We call *reducible ternary system* a system where it holds

$$(2) \quad D(p + q) = T(p) + T(q), D(r) = T(r)$$

$$(3) \quad D(q + r) = T(q) + T(r), D(p) = T(p)$$

The main property is given by

Theorem 1. *If the dual and ternary representations of a system lead to a reducible ternary sistem, then the original system allows the boxed representation (1), namely*

$$(4) \quad B(p, p + q) = D(p + q) - D(p)$$

Proof. The reducibility of a ternary system leads to additivity on each subinterval, so that (4) holds for any box in the case of a finite number of fitness levels, provided values in the extreme are chosen consistently. The continuous case is obtained passing to the limit. $\square$

The problem of passing from ternary to dual systems maintaining properties (2) and (3) lies in the fact that in the ternary model the sequences 321 and 312 are different, inasmuch the vanishing minimum once lies in the middle position, once in the right. Passing to the corresponding dual model both sequences are reduced to 3AA, creating a tie. The tie in general cannot be solved at random, since one should a priori know the two separate probabilities of each sequence 321 and 312. So the simplest solution lies in a severe condition as shown in the next theorem.

Theorem 2. *A ternary system is reducible when, for each set of tied configurations, each column of the transition matrix has the same values in the cells corresponding to the elements of the tied set.*

Proof. In this case there is no need to know *a priori* the probability of the tied configurations, since it is enough to know that they have all the same probability. Remark that the condition can be checked directly on the auxiliary columns, without computing explicitly the ternary transition matrix. $\square$

Theorem 3. *The fundamental transition matrix of example 1 leads to a reducible system.*

Proof. In matrix 1 the only tie arises between rows e and f . Looking in the auxiliary columns one finds respectively the equal values of pq , pr , and qr . $\square$

Remark that another way of proof could be showing that symmetry condition is enough for identifying configurations e and f , so that the systems could be reduced to a two level system, as in example 2 (that is a particular case of example 4 where no geometry exist). Numerical results are shown in Figure 2 and 3.

Starting from matrix 2, the 1-eigenvectors are

$$a = \frac{A}{Tot}, \quad b = \frac{B}{Tot}, \quad c = \frac{C}{Tot}, \quad d = \frac{D}{Tot}$$

where

$$\begin{aligned} A &= \frac{p^2}{2(1-p^2)} B \\ C &= \frac{(1-p)(p+4)}{2p(1+p)} B \\ D &= \frac{(1-p)^2}{p^2(1+p)} B \end{aligned}$$

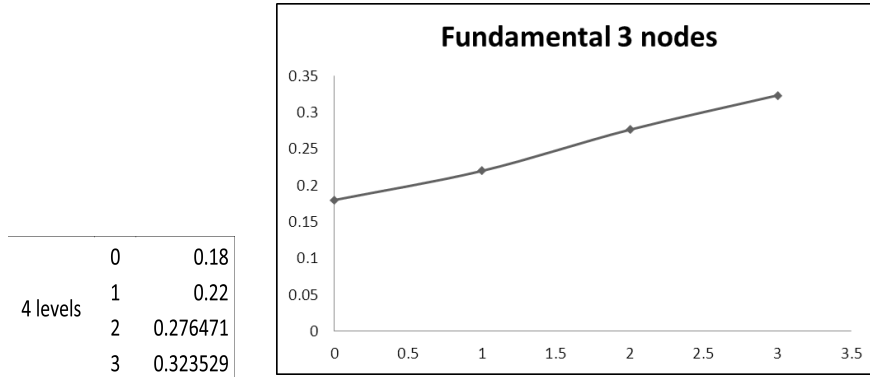


Figure 2: Fundamental model with 3 nodes and 4 levels

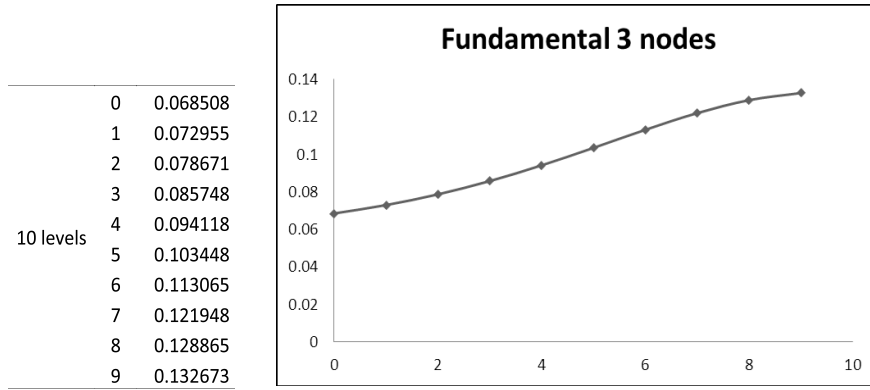


Figure 3: Fundamental model with 3 nodes and 10 levels

and

$$Tot = A + B + C + D$$

so that

$$B(0, p) = 1 - \frac{a \cdot 0 + b \cdot 1 + c \cdot 2 + d \cdot 3}{3}.$$

The numerical result of continuous fundamental model with 3 nodes is shown in Figure 4.

Theorem 4. *The fundamental transition matrix of example 3 leads to a reducible system.*

Proof. A first set of ties is in e, f, i with aux values respectively $2pqr, pr^2, qr^2$. A second set of ties is in l, m, r with aux values respectively $pq^2, 2pqr, q^2r$. The last set of ties is in n, o, p with aux values respectively $p^2q, p^2r, 2pqr$. The use of Theorem 2 leads to the conclusion. The numerical results of discrete fundamental model with 3 nodes and 4 and 10 levels, are shown in Figure 5 and 6 respectively. $\square$

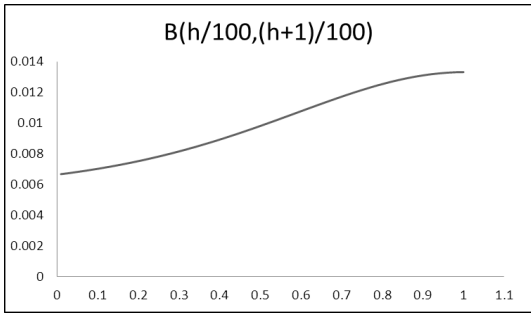


Figure 4: Continuous fundamental model with 3 nodes

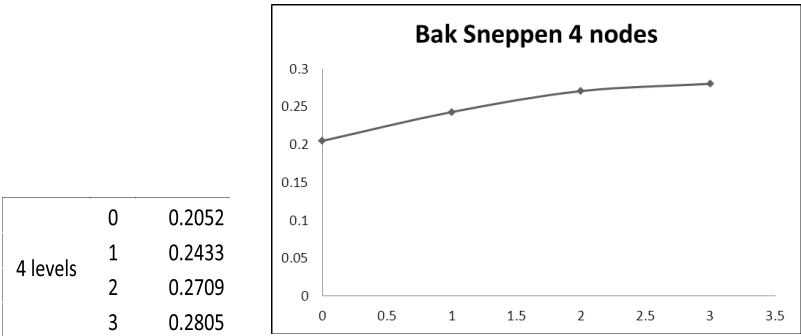


Figure 5: Bak Sneppen model with 4 nodes and 4 levels

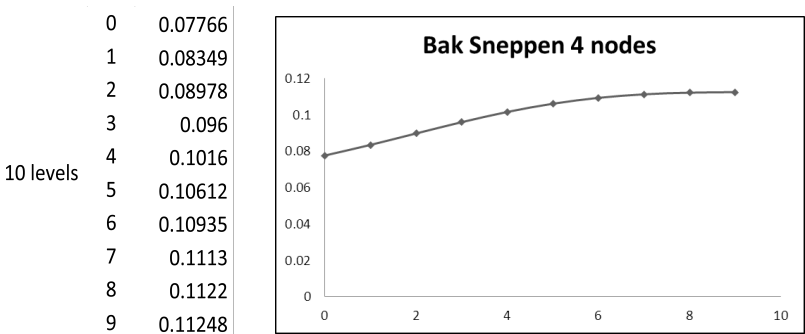


Figure 6: Bak Sneppen model with 4 nodes and 10 levels

4. Conclusions

Bak-Sneppen like models are characterized by some geometry of neighborhood, that overcomes a purely random evolution. Anyhow usually the neighborhood law does not deeply change the distribution of the average of fitness values. What changes in large systems is the speed of propagation of the so called avalanches, and consequently the time elapsed before a section achieves again a stable situation (compare [21]). In that paper, on the contrary, partitioned frames in Bak Sneppen models were introduced and discussed. The situation is as follows: the set of nodes is divided in some distinct sections, and the neighborhood rule allows only evolution in the inside of the section, so that the root change due to minimal fitness can influence only its own section, leaving all the remaining subsystems unchanged, for good and evil. On the contrary, the rule of minimum is global, so that no section should be free from random evolution. But if the local minimum of a section is already equal to the maximum, evolution cannot reach that section unless all other sections reach the maximum, so that maximality is an extremely strong condition of stability.

Of course the condition of maximality of minima can be less strong, inasmuch the minimum in the given section be higher than all minima elsewhere. It can anyhow happen that the minimizing section suddenly jumps to values that are all higher than the previous optimal level of minima, so that overtaking takes place.

The condition for overtaking is the existence of at least two subsections, with no less than two elements, since otherwise the frame becomes trivial. Suppose therefore to consider a simplified system of four nodes, where nodes 1 and 2 belong to the set A, and nodes 3 and 4 belong to the set B. Suppose that fitness levels are 3, otherwise no overtaking can happen, then the transition matrix becomes the Table 5.

Left digit is referred to the set A, right digit to the set B. 0 means that the minimum is 0, 1 that the minimum is 1, 2 that both nodes have the fitness 2. Y means that transition is possible, o that it is impossible.

Start Arrival	00	01	10	11	02	20	12	21	22
00	Y	o	o	o	o	o	o	o	o
01	Y	Y	o	Y	o	o	o	o	o
10	Y	o	Y	Y	o	o	o	o	o
11	o	Y	Y	Y	o	o	o	o	o
02	Y	o	o	o	Y	o	Y	o	Y
20	Y	o	o	o	o	Y	o	Y	Y
12	o	o	Y	Y	Y	o	Y	o	Y
21	o	Y	o	Y	o	Y	o	Y	Y
22	o	o	o	o	Y	Y	Y	Y	Y

Table 5: Transition and overtaking

Remark that the stable set of possible configurations is 02, 20, 12, 21, 22. Overtaking is to be found in the passage from 01 to 21 and from 10 to 12. Of course in the case of more levels a richer choice of cases may arise.

The average value of fitness in the partitioned case is much higher than in non partitioned frames, independently on their detailed geometry.

The confirmed conclusion is that in any Bak-Sneppen model, partition allows higher average levels, but progress may be slower, so that poor subsets only exceptionally make a meaningful jump, an example of weak connection can be found in [15] and [18]. Rich countries have no particular reason to change, somehow following the *happiness paradox* of Easterlin (see [12] and [13]).

In economical history the phenomenon is not frequent but happens and leads to important changes in the world economy. An example of the past is Germany with respect to England, while in the present there is China with respect to USA. Financial evolution, thanks to its volatility and virtuality, is even more subject to sudden changes. Section 5 of [21] gives a useful survey (see also [20]).

A curious phenomenon of overtaking happens with a certain frequency in science, even in hard sciences, where winning ideas arise from *virgin* lands. The case of De Giorgi's theorem is well known and some of its features are explored in detail in [17] in the chapter *Il grande problema* pp. 127-134.

The authors discussed some strange cases of scientific cultural interactions lying between discovery, fashion, teaching (see [22] and [19]).

The problem of economic evolution in restricted frames rather than enlarged ones was analysed in [10] with a long period analysis in Leontief's frame, while in [6] and [7] short period analysis was performed, and efficient strategies were envisaged. Production manufacturing chains are widely subject to Bak-Sneppen phenomena, and are characterized by a continuous attempt to change the structure of neighborhoods, as was well pointed out in [11] and in [8].

Finally it is worth to mention a connection between culture, economy and health management arising in the evolution of food styles, with special reference to the dynamics of Mediterranean Adequacy Index (see [9]).

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HEPTAVALENT SYMMETRIC GRAPHS OF ORDER $6p$

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Abstract. A graph is symmetric if its automorphism group acts transitively on the set of arcs of the graph. In this paper, we classify connected heptavalent symmetric graphs of order $6p$ for each prime p . As a result, there are three sporadic such graphs: one for $p = 5$ and two for $p = 13$.

Keywords: Symmetric graph, s -transitive graph, Cayley graph.

1. Introduction

Throughout this paper graphs are assumed to be finite, simple, connected and undirected. For group-theoretic concepts or graph-theoretic terms not defined here we refer the reader to [22, 25] and [1, 2], respectively. Let G be a permutation group on a set Ω and $v \in \Omega$. Denote by G_v the stabilizer of v in G , that is, the subgroup of G fixing the point v . We say that G is *semiregular* on Ω if $G_v = 1$ for every $v \in \Omega$ and *regular* if G is transitive and semiregular.

For a graph X , denote by $V(X)$, $E(X)$ and $\text{Aut}(X)$ its vertex set, its edge set and its full automorphism group, respectively. A graph X is said to be G -*vertex-transitive* if $G \leq \text{Aut}(X)$ acts transitively on $V(X)$. X is simply called *vertex-transitive* if it is $\text{Aut}(X)$ -vertex-transitive. An s -*arc* in a graph is an ordered $(s+1)$ -tuple $(v_0, v_1, \dots, v_{s-1}, v_s)$ of vertices of the graph X such that v_{i-1} is adjacent to v_i for $1 \leq i \leq s$, and $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s-1$. In particular, a 1-arc is just an arc and a 0-arc is a vertex. For a subgroup $G \leq \text{Aut}(X)$, a graph X is said to be (G, s) -*arc-transitive* and (G, s) -*regular* if G is transitive and regular on the set of s -arcs in X , respectively. A (G, s) -arc-transitive graph is said to be (G, s) -*transitive* if it is not $(G, s+1)$ -arc-transitive. In particular, a $(G, 1)$ -arc-transitive graph is called G -*symmetric*. A graph X is simply called s -*arc-transitive*, s -*regular* and s -*transitive* if it is $(\text{Aut}(X), s)$ -arc-transitive, $(\text{Aut}(X), s)$ -regular and $(\text{Aut}(X), s)$ -transitive, respectively.

As we all known that the structure of the vertex stabilizers of symmetric graphs is very useful to classify such graphs, and this structure of the cubic or tetravalent case was given by Miller [18] and Potočnik [21]. Thus, classifying symmetric graphs with valency 3 or 4 has received considerable attention and a lot of results have been achieved, see [8, 28, 29]. Guo [10] determined the exact structure of pentavalent case. Following this structure, a series of pentavalent

symmetric graphs was classified in [15, 19, 20, 26, 27, 12]. Recently, Guo [11] gave the exact structure of heptavalent case. Thus, as an application, we classify connected heptavalent symmetric graphs of order $6p$ for each prime p in this paper.

2. Preliminary results

Let X be a connected G -symmetric graph with $G \leq \text{Aut}(X)$, and let N be a normal subgroup of G . The *quotient graph* X_N of X relative to N is defined as the graph with vertices the orbits of N on $V(X)$ and with two orbits adjacent if there is an edge in X between those two orbits. In view of [16, Theorem 9], we have the following:

Proposition 2.1. *Let X be a connected heptavalent G -symmetric graph with $G \leq \text{Aut}(X)$, and let N be a normal subgroup of G . Then one of the following holds:*

- (1) N is transitive on $V(X)$;
- (2) X is bipartite and N is transitive on each part of the bipartition;
- (3) N has $r \geq 3$ orbits on $V(X)$, N acts semiregularly on $V(X)$, the quotient graph X_N is a connected heptavalent G/N -symmetric graph.

The following proposition characterizes the vertex stabilizers of connected heptavalent s -transitive graphs (see [11, Theorem 1.1]).

Proposition 2.2. *Let X be a connected heptavalent (G, s) -transitive graph for some $G \leq \text{Aut}(X)$ and $s \geq 1$. Let $v \in V(X)$. Then $s \leq 3$ and one of the following holds:*

- (1) For $s = 1$, $G_v \cong \mathbb{Z}_7, D_{14}, F_{21}, D_{28}, F_{21} \times \mathbb{Z}_3$;
- (2) For $s = 2$, $G_v \cong F_{42}, F_{42} \times \mathbb{Z}_2, F_{42} \times \mathbb{Z}_3, \text{PSL}(3, 2), A_7, S_7, \mathbb{Z}_2^3 \rtimes \text{SL}(3, 2)$ or $\mathbb{Z}_2^4 \rtimes \text{SL}(3, 2)$;
- (3) For $s = 3$, $G_v \cong F_{42} \times \mathbb{Z}_6, \text{PSL}(3, 2) \times S_4, A_7 \times A_6, S_7 \times S_6, (A_7 \times A_6) \rtimes \mathbb{Z}_2, \mathbb{Z}_2^6 \rtimes (\text{SL}(2, 2) \times \text{SL}(3, 2))$ or $[2^{20}] \rtimes (\text{SL}(2, 2) \times \text{SL}(3, 2))$.

To extract a classification of connected heptavalent symmetric graphs of order $2p$ for a prime p from Cheng and Oxley [4], we introduce the graphs $G(2p, r)$. Let V and V' be two disjoint copies of $\mathbb{Z}_p$, say $V = \{0, 1, \dots, p-1\}$ and $V' = \{0', 1', \dots, (p-1)'\}$. Let r be a positive integer dividing $p-1$ and $H(p, r)$ the unique subgroup of $\mathbb{Z}_p^*$ of order r . Define the graph $G(2p, r)$ to have vertex set $V \cup V'$ and edge set $\{xy' \mid x - y \in H(p, r)\}$.

Proposition 2.3. *Let X be a connected heptavalent symmetric graph of order $2p$ with p a prime. Then X is isomorphic to $K_{7,7}$ or $G(2p, 7)$ with $7 \mid (p-1)$. Furthermore, $\text{Aut}(G(2p, 7)) = (\mathbb{Z}_p \rtimes \mathbb{Z}_7) \rtimes \mathbb{Z}_2$.*

From [9, pp.12-14], [24, Theorem 2] and [14, Theorem A], we may obtain the following proposition by checking the orders of non-abelian simple groups:

Proposition 2.4. *Let p be a prime, and let G be a non-abelian simple group of order $|G| \mid (2^{25} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot p)$. Then G has 3-prime factor, 4-prime factor or 5-prime factor, and is one of the following groups:*

Table 1: **Non-abelian simple $\{2, 3, 5, 7, p\}$ -groups**

3-prime factor					
G	Order	G	Order	G	Order
A_5	$2^2 \cdot 3 \cdot 5$	$\text{PSL}(2, 8)$	$2^3 \cdot 3^2 \cdot 7$	$\text{PSU}(4, 2)$	$2^6 \cdot 3^4 \cdot 5$
A_6	$2^3 \cdot 3^2 \cdot 5$	$\text{PSL}(2, 17)$	$2^4 \cdot 3^2 \cdot 17$	$\text{PSU}(3, 3)$	$2^5 \cdot 3^3 \cdot 7$
$\text{PSL}(2, 7)$	$2^3 \cdot 3 \cdot 7$	$\text{PSL}(3, 3)$	$2^4 \cdot 3^3 \cdot 13$		
4-prime factor					
G	Order	G	Order	G	Order
A_7	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	$\text{PSL}(2, 27)$	$2^2 \cdot 3^3 \cdot 7 \cdot 13$	$\text{PSU}(5, 2)$	$2^{10} \cdot 3^5 \cdot 5 \cdot 11$
A_8	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	$\text{PSL}(2, 31)$	$2^5 \cdot 3 \cdot 5 \cdot 31$	$\text{PSp}(4, 4)$	$2^8 \cdot 3^2 \cdot 5^2 \cdot 17$
A_9	$2^6 \cdot 3^4 \cdot 5 \cdot 7$	$\text{PSL}(2, 49)$	$2^4 \cdot 3 \cdot 5^2 \cdot 7^2$	$\text{PSp}(6, 2)$	$2^9 \cdot 3^4 \cdot 5 \cdot 7$
A_{10}	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7$	$\text{PSL}(2, 81)$	$2^4 \cdot 3^4 \cdot 5 \cdot 41$	M_{11}	$2^4 \cdot 3^2 \cdot 5 \cdot 11$
$\text{PSL}(2, 11)$	$2^2 \cdot 3 \cdot 5 \cdot 11$	$\text{PSL}(2, 127)$	$2^7 \cdot 3^2 \cdot 7 \cdot 127$	M_{12}	$2^6 \cdot 3^3 \cdot 5 \cdot 11$
$\text{PSL}(2, 13)$	$2^2 \cdot 3 \cdot 7 \cdot 13$	$\text{PSL}(3, 4)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	J_2	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$
$\text{PSL}(2, 16)$	$2^4 \cdot 3 \cdot 5 \cdot 17$	$\text{PSU}(3, 4)$	$2^6 \cdot 3 \cdot 5^2 \cdot 13$	$\text{P}\Omega^+(8, 2)$	$2^{12} \cdot 3^5 \cdot 5^2 \cdot 7$
$\text{PSL}(2, 19)$	$2^2 \cdot 3^2 \cdot 5 \cdot 19$	$\text{PSU}(3, 5)$	$2^4 \cdot 3^2 \cdot 5^3 \cdot 13$	$\text{Sz}(8)$	$2^6 \cdot 5 \cdot 7 \cdot 13$
$\text{PSL}(2, 25)$	$2^3 \cdot 3 \cdot 5^2 \cdot 13$	$\text{PSU}(3, 8)$	$2^9 \cdot 3^4 \cdot 7 \cdot 19$	${}^2F_4(2)'$	$2^{11} \cdot 3^3 \cdot 5^2 \cdot 13$
5-prime factor					
G	Order	G	Order	G	Order
A_{11}	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11$	$\text{PSL}(2, 449)$	$2^6 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 449$	$\text{PSp}(8, 2)$	$2^{16} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17$
A_{12}	$2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$	$\text{PSL}(2, 2^6)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$	M_{22}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$
$\text{PSL}(2, 29)$	$2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 29$	$\text{PSL}(4, 4)$	$2^{12} \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 17$	$\text{P}\Omega^-(8, 2)$	$2^{12} \cdot 3^4 \cdot 5 \cdot 7 \cdot 17$
$\text{PSL}(2, 41)$	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 41$	$\text{PSL}(5, 2)$	$2^{10} \cdot 3^2 \cdot 5 \cdot 7 \cdot 31$	$G_2(4)$	$2^{12} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13$
$\text{PSL}(2, 71)$	$2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 71$				

Next we construct some heptavalent symmetric graphs of order $6p$ with p a prime. To do this, we need to introduce the so called coset graph (see [18, 23]) constructed from a finite group G relative to a subgroup H of G and a union D of some double cosets of H in G such that $D^{-1} = D$. The *coset graph* $\text{Cos}(G, H, D)$ of G with respect to H and D is defined to have vertex set $[G : H]$, the set of right cosets of H in G , and edge set $\{\{Hg, Hdg\} \mid g \in G, d \in D\}$. The graph $\text{Cos}(G, H, D)$ has valency $|D|/|H|$ and is connected if and only if D generates the group G . The action of G on $V(\text{Cos}(G, H, D))$ by right multiplication induces a vertex-transitive automorphism group, which is arc-transitive if and only if D is a single double coset. Moreover, this action is faithful if and only if $H_G = 1$, where H_G is the largest normal subgroup of G in H . Clearly, $\text{Cos}(G, H, D) \cong \text{Cos}(G, H^\alpha, D^\alpha)$ for every $\alpha \in \text{Aut}(G)$. For more details regarding coset graphs, see, for example, [7, 16, 23].

Construction 2.5. Let $G \cong \text{S}_8$ and $M \cong \text{A}_7$. Then by Atlas [6], G has a maximal subgroup $K \cong \mathbb{Z}_2^4 \rtimes \text{S}_4$ and M has a maximal subgroup $H \cong \mathbb{Z}_2^3 \rtimes \text{PSL}(3, 2)$ such that $K \cap H \cong \mathbb{Z}_2^3 \rtimes \text{S}_4$. Take an element g of order 2 in $K \setminus (K \cap H)$. Then $H \cap H^g \cong \mathbb{Z}_2^3 \rtimes \text{S}_4$, $\langle H, g \rangle = G$ and hence the coset graph:

$$\mathcal{C}_{30} = \text{Cos}(G, H, HgH)$$

is a connected heptavalent symmetric graph of order 30. By Magma [3], $\text{Aut}(\mathcal{C}_{30}) \cong S_8$, and any connected heptavalent symmetric graph of order 30 admitting S_8 as an arc-transitive automorphism group is isomorphic to $\mathcal{C}_{30}$.

The following two graphs are coset graphs of order 78 constructed from the simple group $\text{PSL}(2, 13)$.

Construction 2.6. Let $G \cong \text{PSL}(2, 13)$, and take the following four elements:

$$\begin{aligned} a &= (1, 13, 2, 12, 9, 5, 14)(3, 4, 7, 10, 11, 6, 8), \\ b &= (2, 5)(3, 7)(6, 11)(8, 10)(9, 12)(13, 14), \\ x &= (1, 4)(3, 10)(6, 14)(7, 8)(9, 12)(11, 13), \\ y &= (1, 4)(2, 6)(5, 11)(8, 9)(10, 12)(13, 14). \end{aligned}$$

Then $G = \langle a, b, x \rangle \cong \text{PSL}(2, 13)$ and $H = \langle a, b \rangle \cong D_{14}$. Define the following two coset graphs:

$$\mathcal{C}_{78}^1 = \text{Cos}(G, H, HxH), \quad \mathcal{C}_{78}^2 = \text{Cos}(G, H, HyH).$$

By Magma [3], $\text{Aut}(\mathcal{C}_{78}^1) \cong \text{PSL}(2, 13)$, $\text{Aut}(\mathcal{C}_{78}^2) \cong \text{PGL}(2, 13)$, and any connected heptavalent symmetric graph admitting $\text{PSL}(2, 13)$ as an arc-transitive group is isomorphic to $\mathcal{C}_{78}^1$ or $\mathcal{C}_{78}^2$.

3. Main result

This section is devoted to classify connected heptavalent symmetric graphs of order $6p$ for each prime p .

Theorem 3.1. *Let X be a connected heptavalent symmetric graph of order $6p$ with p a prime. Then X is isomorphic to one of the following graphs:*

Table 2: **Heptavalent symmetric graphs of order $6p$**

X	s -transitivity	$\text{Aut}(X)$	Comments
$\mathcal{C}_{30}$	2-transitive	S_8	Construction 2.5, $p = 5$
$\mathcal{C}_{78}^1$	1-transitive	$\text{PSL}(2, 13)$	Construction 2.6, $p = 13$
$\mathcal{C}_{78}^2$	1-transitive	$\text{PGL}(2, 13)$	Construction 2.6, $p = 13$

Proof. Let $A = \text{Aut}(X)$. If $p = 2$ or 3 , then $|V(X)| = 12$ or 18 . By [17] and [5], there is no connected heptavalent symmetric graph of order 12 or 18. If $p = 5$, then $|V(X)| = 30$. By [5] and Construction 2.5, $X \cong \mathcal{C}_{30}$ and $A \cong S_8$. Thus, we may assume that $p \geq 7$.

Let $v \in V(X)$. Then by Proposition 2.2, $|A_v| \mid 2^{24} \cdot 3^4 \cdot 5^2 \cdot 7$ and hence $|A| \mid 2^{25} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot p$. We separate the proof into two cases: A has a solvable minimal subgroup; A has no solvable minimal normal subgroup.

Case 1. A has a solvable minimal normal subgroup.

Let N be a solvable minimal normal subgroup of A . Then $|N| \mid 2^{25} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot p$, and N is elementary abelian. Thus, $N \cong \mathbb{Z}_q^k$ with $q = 2, 3, 5, 7$ or p and k a positive integer. By Proposition 2.1, N is semiregular and X_N is also a connected heptavalent A/N -symmetric graphs. It follows that $|N| \mid 6p$ and $N \cong \mathbb{Z}_2, \mathbb{Z}_3$ or $\mathbb{Z}_p$. Note that there is no connected heptavalent graph of odd order. Thus, $N \not\cong \mathbb{Z}_2$. Since there is no connected heptavalent regular graph of order 6, we have that $N \not\cong \mathbb{Z}_p$. This forces that $N \cong \mathbb{Z}_3$ and X_N is a heptavalent symmetric graph of order $2p$. By Proposition 2.3, $X_N \cong K_{7,7}$ or $G(2p, 7)$.

Assume that $X_N \cong K_{7,7}$. Then $p = 7$ and $A/N \lesssim S_7 \text{ wr } S_2$. By Magma [3], $(S_7 \times S_7) \rtimes \mathbb{Z}_2$ has minimal arc-transitive subgroups $\mathbb{Z}_7^2 \rtimes \mathbb{Z}_2$ or $\mathbb{Z}_7^2 \rtimes \mathbb{Z}_4$. Thus, A/N has an arc-transitive subgroup $M/N \cong \mathbb{Z}_7^2 \rtimes \mathbb{Z}_2$ or $\mathbb{Z}_7^2 \rtimes \mathbb{Z}_4$. By “N/C-Theorem” (see [13, Chapter I, Theorem 4.5]), $M/C_M(N) \lesssim \text{Aut}(N) \cong \mathbb{Z}_2$. Thus, $7^2 \mid |C_M(N)|$. Let H be a Sylow 7-subgroup of $C_M(N)$. Then $\mathbb{Z}_7^2 \cong H$ is normal in M . Note that $p = 7$. Thus, the quotient graph X_H has order 6. By Propositions 2.1, H is semiregular and hence $H \cong \mathbb{Z}_7$, a contradiction.

Assume that $X_N \cong G(2p, 7)$. Then $A/N \lesssim (\mathbb{Z}_p \rtimes \mathbb{Z}_7) \rtimes \mathbb{Z}_2$. Since A/N is arc-transitive on X_N , we have that $A/N \cong (\mathbb{Z}_p \rtimes \mathbb{Z}_7) \rtimes \mathbb{Z}_2$. Clearly, A/N has a normal subgroup $M/N \cong \mathbb{Z}_p$. By “N/C-theorem”, $M/C_M(N) \lesssim \text{Aut}(N) \cong \mathbb{Z}_2$. It follows that $M = C_M(N)$ and $M = \mathbb{Z}_p \times \mathbb{Z}_3$. This implies that M has a characteristic subgroup $H \cong \mathbb{Z}_p$. Since $M \leq A$, we can deduce that $H \leq A$. Thus, X_H is a connected heptavalent graph of order 6, a contradiction.

Case 2. A has no solvable minimal normal subgroup.

For convenience, we still use N to denote a minimal normal subgroup of A . Then N is non-solvable. Let $N = T^k$ with T a non-abelian simple group and k a positive integer. Then T has at least 3-prime factors. Since $|T| \mid 2^{25} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot p$, we have that T is one of the simple groups listed in Proposition 2.4. By Proposition 2.1, $|N| = 6p|N_v|$ or $3p|N_v|$.

Assume that $k \geq 2$. Since T is a non-abelian simple group, we have that $2^2 \mid |T|$ and $T_v \neq 1$. If $p > 7$, then $p \nmid |T|$ because $p^2 \nmid |N| = |T^k|$. It follows that p divides the order of X_N . By Proposition 2.1, $N = T^k$ is semiregular and hence $T_v = 1$, a contradiction. Thus, $p = 7$ and $N = T^k$ has at most two orbits on $V(X)$. It follows that $3 \cdot 7 \mid |T^k|$. Note that $|T^k| \mid 2^{25} \cdot 3^5 \cdot 5^2 \cdot 7^2$. Thus, T is a simple $\{2, 3, 7\}$ -group or $\{2, 3, 5, 7\}$ -group and $k = 2$. By Proposition 2.4, $T \cong \text{PSL}(2, 7), \text{PSL}(2, 8), A_7, A_8$ or $\text{PSL}(3, 4)$. If $T \cong \text{PSL}(2, 8)$ or A_8 , then $N \cong \text{PSL}(2, 8)^2$ or A_8^2 and $|N_v| = |N|/(3 \cdot 7)$ or $|N|/(2 \cdot 3 \cdot 7)$. However, by Magma [3], $\text{PSL}(2, 8)^2$ or A_8^2 has no subgroups of such orders, a contradiction. If $T \cong \text{PSL}(2, 7)$, then $|N_v| = |N|/(3 \cdot 7) = 2^6 \cdot 3 \cdot 7$ or $|N|/(2 \cdot 3 \cdot 7) = 2^5 \cdot 3 \cdot 7$. By Magma [3], $N_v \cong D_8 \times \text{PSL}(2, 7), \mathbb{Z}_4 \times \text{PSL}(2, 7)$ or $\mathbb{Z}_2^2 \times \text{PSL}(2, 7)$. Note that $N_v \leq A_v$. By Proposition 2.2, $N_v \cong \mathbb{Z}_2^2 \times \text{PSL}(2, 7)$ and $A_v \cong S_4 \times \text{PSL}(2, 7)$.

It follows that N is transitive on $V(X)$ and hence N is arc-transitive. By Magma [3], there is no connected heptavalent symmetric graph of order $6 \cdot 7$ admitting $\text{PSL}(2, 7)^2$ as an arc-transitive group. If $T \cong A_7$, then $|N_v| = 2^6 \cdot 3^3 \cdot 5^2 \cdot 7$ or $2^5 \cdot 3^3 \cdot 5^2 \cdot 7$. By Magma [3], $N_v \cong A_7 \times S_5$ or $A_7 \times A_5$. However, by Proposition 2.2, A_v has no normal subgroups isomorphic to $A_7 \times S_5$ or $A_7 \times A_5$, a contradiction. If $T \cong \text{PSL}(3, 4)$, then by Magma [3], $N_v \cong (\mathbb{Z}_2^4 \rtimes A_5) \times \text{PSL}(3, 4)$. Similarly, A_v has no such normal subgroup by Proposition 2.2, a contradiction.

Thus, $k = 1$ and $N = T$ is a non-abelian simple group. It follows that $2^2 \mid |N|$ and $N_v \neq 1$. By Proposition 2.1, N has at most two orbits on $V(X)$ and $|N| = 6p|N_v|$ or $3p|N_v|$.

Subcase 2.1. Suppose that $p = 7$. Then $|N| \mid 2^{25} \cdot 3^5 \cdot 5^2 \cdot 7^2$.

Note that $3 \cdot 7 \mid |N|$. By Proposition 2.4, N is isomorphic to one of the following simple groups:

$$\begin{aligned} &\text{PSL}(2, 7), \text{PSL}(2, 8), \text{PSU}(3, 3), A_7, A_8, A_9, A_{10}, \\ &\text{PSL}(2, 49), \text{PSL}(3, 4), \text{PSp}(6, 2), J_2, \text{P}\Omega^+(8, 2). \end{aligned}$$

Since N is transitive or has two orbits on $V(X)$, we have that $|N_v| = |N|/42$ or $|N| = |N|/21$. It follows that N has a subgroup of order $|N|/42$ or $|N|/21$. By Magma [3], $\text{PSL}(2, 8)$, $\text{PSU}(3, 3)$, A_8 , A_9 , A_{10} , $\text{PSL}(2, 49)$, $\text{PSL}(3, 4)$, $\text{PSp}(6, 2)$, J_2 and $\text{P}\Omega^+(8, 2)$ have no subgroups of such orders. Thus, $N \cong \text{PSL}(2, 7)$ or A_7 .

Let $N \cong \text{PSL}(2, 7)$. Then $|N_v| = 2^2$ or 2^3 , and by Atlas [6], $N_v \cong D_8$, $\mathbb{Z}_4$ or $\mathbb{Z}_2^2$. The normality of N in A implies that $N_v \leq A_v$. By Proposition 2.2 and Remark of [11, Theorem 1.1], the only possible is $A_v \cong \text{PSL}(3, 2) \times S_4$ and $N_v \cong \mathbb{Z}_2^2$. It follows that N is transitive on $V(X)$ and hence $A = A_v N$. Set $C = C_A(N)$. Then $C \cap N = 1$ because N is a non-abelian simple group. If $C = 1$, then by “N/C-Theorem” (see [13, Chapter I, Theorem 4.5]), $A \cong A/C \lesssim \text{Aut}(N) \cong \text{PGL}(2, 7)$. Since $|A_v| = |A|/42$, we have that $7 \nmid |A_v|$, a contradiction. Note that A has no solvable minimal normal subgroup. Thus, C is non-solvable. Clearly, $C \cong CN/N \leq A/N = A_v N/N \cong A_v/N_v \cong \text{PSL}(3, 2) \times S_3$. It forces that C has a normal subgroup $M \cong \text{PSL}(3, 2)$. Since N is transitive on $V(X)$ and $7 \mid |M|$, we have that $H = M \times N \cong \text{PSL}(2, 7)^2$ is arc-transitive and $H_v \cong \text{PSL}(2, 7) \times \mathbb{Z}_2^2$. However, $\text{PSL}(2, 7) \times \mathbb{Z}_2^2$ can not be as a vertex stabilizer of a heptavalent symmetric graph by Proposition 2.2, a contradiction.

Let $N \cong A_7$. Then $|N_v| = 2^2 \cdot 3 \cdot 5$ or $2^3 \cdot 3 \cdot 5$. By Atlas [6], $N_v \cong A_5$ or S_5 . Since $N \leq A$, we have that $N_v \leq A_v$. Thus, A_v has a normal subgroup isomorphic to A_5 or S_5 . However, by Proposition 2.2, A_v has no such normal subgroup, a contradiction.

Subcase 2.2. Suppose that $p > 7$. Then $|N| \mid 2^{25} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot p$.

Since $3p \mid |N|$, we have that N is simple $\{2, 3, p\}$, $\{2, 3, 5, p\}$, $\{2, 3, 7, p\}$ or $\{2, 3, 5, 7, p\}$ -group, and by Proposition 2.4, we can get the information of the group N , the prime p and $|N_v|$ as the following table:

By Atlas [6] and Magma [3], except for the following groups:

Table 3: The group N , the prime p and the order $|N_v|$

3-prime factor					
N	p	$ N_v $	N	p	$ N_v $
$\text{PSL}(2, 17)$	17	$2^3 \cdot 3$ or $2^4 \cdot 3$	$\text{PSL}(3, 3)$	13	$2^3 \cdot 3^2$ or $2^4 \cdot 3^2$
4-prime factor					
N	p	$ N_v $	N	p	$ N_v $
$\text{PSL}(2, 11)$	11	$2 \cdot 5$ or $2^2 \cdot 5$	$\text{PSL}(2, 127)$	127	$2^6 \cdot 3 \cdot 7$ or $2^7 \cdot 3 \cdot 7$
$\text{PSL}(2, 13)$	13	$2 \cdot 7$ or $2^2 \cdot 7$	$\text{PSU}(3, 4)$	13	$2^5 \cdot 5^2$ or $2^6 \cdot 5^2$
$\text{PSL}(2, 16)$	17	$2^3 \cdot 5$ or $2^4 \cdot 5$	$\text{PSU}(3, 8)$	19	$2^8 \cdot 3^3 \cdot 7$ or $2^9 \cdot 3^3 \cdot 7$
$\text{PSL}(2, 19)$	19	$2 \cdot 3 \cdot 5$ or $2^2 \cdot 3 \cdot 5$	$\text{PSU}(5, 2)$	11	$2^9 \cdot 3^4 \cdot 5$ or $2^{10} \cdot 3^4 \cdot 5$
$\text{PSL}(2, 25)$	13	$2^2 \cdot 5^2$ or $2^3 \cdot 5^2$	$\text{PSp}(4, 4)$	17	$2^7 \cdot 3 \cdot 5^2$ or $2^8 \cdot 3 \cdot 5^2$
$\text{PSL}(2, 27)$	13	$2 \cdot 3^2 \cdot 7$ or $2^2 \cdot 3^2 \cdot 7$	M_{11}	11	$2^3 \cdot 3 \cdot 5$ or $2^4 \cdot 3 \cdot 5$
$\text{PSL}(2, 31)$	31	$2^4 \cdot 5$ or $2^5 \cdot 5$	M_{12}	11	$2^5 \cdot 3^2 \cdot 5$ or $2^6 \cdot 3^2 \cdot 5$
$\text{PSL}(2, 81)$	41	$2^3 \cdot 3^3 \cdot 5$ or $2^4 \cdot 3^3 \cdot 5$	${}^2F_4(2)'$	13	$2^{10} \cdot 3^2 \cdot 5^2$ or $2^{11} \cdot 3^2 \cdot 5^2$
5-prime factor					
N	p	$ N_v $	N	p	$ N_v $
A_{11}	11	$2^6 \cdot 3^3 \cdot 5^2 \cdot 7$ or $2^7 \cdot 3^3 \cdot 5^2 \cdot 7$	$\text{PSL}(4, 4)$	11	$2^{11} \cdot 3^3 \cdot 5^2 \cdot 7$ or $2^{12} \cdot 3^3 \cdot 5^2 \cdot 7$
A_{12}	11	$2^8 \cdot 3^4 \cdot 5^2 \cdot 7$ or $2^9 \cdot 3^4 \cdot 5^2 \cdot 7$	$\text{PSL}(5, 2)$	31	$2^9 \cdot 3 \cdot 5 \cdot 7$ or $2^{10} \cdot 3 \cdot 5 \cdot 7$
$\text{PSL}(2, 29)$	29	$2 \cdot 5 \cdot 7$ or $2^2 \cdot 5 \cdot 7$	$\text{PSp}(8, 2)$	17	$2^{15} \cdot 3^4 \cdot 5^2 \cdot 7$ or $2^{16} \cdot 3^4 \cdot 5^2 \cdot 7$
$\text{PSL}(2, 41)$	41	$2^2 \cdot 5 \cdot 7$ or $2^3 \cdot 5 \cdot 7$	M_{22}	11	$2^6 \cdot 3 \cdot 5 \cdot 7$ or $2^7 \cdot 3 \cdot 5 \cdot 7$
$\text{PSL}(2, 71)$	71	$2^2 \cdot 3 \cdot 5 \cdot 7$ or $2^3 \cdot 3 \cdot 5 \cdot 7$	$\text{P}\Omega^-(8, 2)$	17	$2^{11} \cdot 3^3 \cdot 5 \cdot 7$ or $2^{12} \cdot 3^3 \cdot 5 \cdot 7$
$\text{PSL}(2, 449)$	449	$2^5 \cdot 3 \cdot 5^2 \cdot 7$ or $2^6 \cdot 3 \cdot 5^2 \cdot 7$	$G_2(4)$	13	$2^{11} \cdot 3^2 \cdot 5^2 \cdot 7$ or $2^{12} \cdot 3^2 \cdot 5^2 \cdot 7$
$\text{PSL}(2, 2^6)$	13	$2^5 \cdot 3 \cdot 5 \cdot 7$ or $2^6 \cdot 3 \cdot 5 \cdot 7$			

$\text{PSL}(2, 17)$, $\text{PSL}(3, 3)$, $\text{PSL}(2, 11)$, $\text{PSL}(2, 13)$,
 $\text{PSL}(2, 16)$, $\text{PSL}(2, 19)$, $\text{PSL}(2, 25)$, M_{11} , M_{12} , A_{12} ,

the remaining simple groups listed in Table 3 do not have subgroups of order $|N_v| = |N|/3p$ or $|N|/6p$. Thus, next we deal with these nine groups.

Set $C = C_A(N)$. Then $C \cap N = 1$ and $C \leq A$. If $C \neq 1$, then C is non-solvable because A has no solvable minimal normal subgroup. Note that $p > 7$ and $p \nmid |N|$. Thus, $p \nmid |C|$. It follows that C has at least p orbits on $V(X)$. By Proposition 2.1, C is semiregular and hence $|C| \mid 6p$. This implies that C is solvable, a contradiction. Thus, $C = 1$. By “N/C-Theorem”, $A \cong A/C \lesssim \text{Aut}(N)$, that is, A is almost simple with socle N .

Let $N \cong \text{PSL}(2, 17)$. Then $|N_v| = 2^3 \cdot 3$ or $2^4 \cdot 3$. By Atlas [6], $\text{PSL}(2, 17)$ has no subgroups of order $2^4 \cdot 3$ and $N_v \cong S_4$. Clearly, N is transitive on $V(X)$. Recall that $N_v \trianglelefteq A_v$. By Proposition 2.2, $A_v \cong \text{PSL}(3, 2) \times S_4$. It follows that $\text{PSL}(3, 2) \cong A_v/N_v \cong A/N \lesssim \text{Out}(N) \cong \mathbb{Z}_2$, a contradiction.

Let $N \cong \text{PSL}(3, 3)$. Then $|N_v| = 2^3 \cdot 3^2$ or $2^4 \cdot 3^2$. By Magma [3], N_v has a normal Sylow 3-subgroup $P \cong \mathbb{Z}_3^2$. Thus, P is characteristic in N_v . The normality of N_v in A_v implies that $P \trianglelefteq A_v$. However, by Proposition 2.2, A_v has no normal subgroup isomorphic to $\mathbb{Z}_3^2$, a contradiction.

Let $N \cong \text{PSL}(2, 11)$. Then $|N_v| = 2 \cdot 5$ or $2^2 \cdot 5$. By Atlas [6], $\text{PSL}(2, 11)$ has no subgroups of order $2^2 \cdot 5$ and $N_v \cong D_{10}$. However, by Proposition 2.2, A_v has no normal subgroups isomorphic to D_{10} , a contradiction.

Let $N \cong \text{PSL}(2, 16)$, $\text{PSL}(2, 19)$, $\text{PSL}(2, 25)$, M_{11} , M_{12} or A_{12} . Then by Atlas [6] and Magma [3], $N_v \cong \mathbb{Z}_2^4 \rtimes \mathbb{Z}_5$, A_5 , $\mathbb{Z}_5^2 \rtimes \mathbb{Z}_4$, S_5 , $A_6 \rtimes \mathbb{Z}_2^2$ or S_{10} . Similarly, A_v has no such normal subgroups, a contradiction.

Finally, let $N \cong \text{PSL}(2, 13)$. Then $|N_v| = 2 \cdot 7$ or $2^2 \cdot 7$. By Atlas [6], $\text{PSL}(2, 13)$ has no subgroups of order $2^2 \cdot 7$ and $N_v \cong D_{14}$. Since $|N|/|N_v| = 6 \cdot 13$, we have that N is transitive on $V(X)$, and since $7 \nmid |N_v|$, we have that N is arc-transitive. By Construction 2.6, $X \cong \mathcal{C}_{78}^1$ or $\mathcal{C}_{78}^2$, and $A \cong \text{PSL}(2, 13)$ or $\text{PGL}(2, 13)$. $\square$

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A VIEW ON QUASI λ -OPEN M -SETS IN M -TOPOLOGICAL SPACES

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Abstract. In this article the concepts of Λ -open M -sets, λ -open M -sets, λ -continuous M -set functions, quasi λ -open M -set functions and λ -irresolute M -set functions are introduced and studied. Also, their properties and characterizations are discussed.

Keywords: Λ -open M -sets, λ -open M -sets, λ -continuous M -sets functions, quasi λ -open M -set functions and λ -irresolute M -set functions.

1. Introduction

The concept of multiset have tremendous applications both in Mathematics and Computer Science. Girish and sunil Jacob John [8] introduced the concepts of M -topological spaces and open M -sets. In this article the authors studied many interesting topological properties via M -sets. As an application to M -topological spaces Chakrabarty et al. [6] studied n^k -bags via rough sets. Later on Amudhambigai B and Revathi G.K. et al. [1,2,3] contributed many articles in M -topological spaces. Maki [10] introduced the notion of Λ -sets in topological spaces. Arenas et al. [4] introduced and investigated the notion of λ -sets by involving Λ -sets and closed sets. Caldas et al. [5] introduced the notion of λ -closure of a set by utilizing the notion of λ -open sets defined in [4]. A new class of functions called slightly λ -continuous function has been defined and studied in topological spaces by Duraisamy and Vennila [7].

In this article the concepts of Λ -open M -sets, λ -open M -sets, λ -continuous M -set functions, quasi λ -open M -set functions and λ -irresolute M -set functions are introduced and studied. Also, their properties and characterizations are discussed.

2. Preliminaries

Definition 2.1 ([8]). Let $[X]^w$ be an M -set space and $\{M_1, M_2, \dots\}$ be a collection of M -sets drawn from $[X]^w$. Then the following operations are possible under an arbitrary collection of M -sets.

- The union $\prod_{i \in I} M_i = \{C_{M_i}(x)/x : C_{M_i}(x) = \max\{C_{M_i}(x) : x \in X\}\}$.
- The intersection $I_{i \in I} M_i = \{C_{\cap M_i}(x)/x : C_{\cap M_i}(x) = \min\{C_{M_i}(x) : x \in X\}\}$.
- The M -set addition $\bigoplus_{i \in I} M_i = \{C_{\oplus M_i}(x)/x : C_{\oplus M_i}(x) = \sum_{i \in I} C_{M_i}(x), x \in X\}$.
- M -set complement $M^c = Z \ominus M = \{C_{M^c}(x)/x : C_{M^c}(x) = C_z(x) - C_M(x), x \in X\}$.

Definition 2.2 ([8]). An M -set relation f is called an M -set function if for every element m/x in $\text{Dom } f$, there is exactly one n/y in $\text{Ran } f$ such that $(m/x, n/y)$ is in f with the pair occurring as the product of $C_1(x, y)$ and $C_2(x, y)$.

Definition 2.3 ([8]). Let $M \in [X]^w$ and $\tau \subseteq P^*(M)$. Then τ is called a *Multiset topology* of M if τ satisfies the following properties:

- The M -set M and the empty M -set ϕ are in τ .
- The M -set union of the elements of any sub collection of τ is in τ .
- The M -set intersection of the elements of any finite sub collection of τ is in τ .

Definition 2.4 ([8]). A sub M -set N of an M -topological space M in $[X]^w$ is said to be closed if the M -set $M \ominus N$ is open. i.e., $N^c = M \ominus N$.

Definition 2.5 ([8]). Let M and N be two M -topological spaces. The M -set function $f : M \rightarrow N$ is said to be continuous if for each open sub M -set V of N , the M -set $f^{-1}(V)$ is an open sub M -set of M , where $f^{-1}(V)$ is the M -set of all points m/x in M for which $f(m/x) \in^n V$ for some n .

Definition 2.6 ([5]). Let B be a subset of a space (X, τ) . B is a Λ -set (resp. $\vee$ -set) if $B = B^\Lambda$ (resp. $B = B^\vee$), where $B^\Lambda = \cap\{U | U \supseteq B, U \in \tau\}$ and $B^\vee = \cup\{F | B \supseteq F, F^C \in \tau\}$.

Definition 2.7 ([7]). A subset A of a topological space (X, τ) is said to be λ -closed if $A = B \cap C$, where B is a Λ -set and C is a closed set of X . The complement of λ -closed set is called λ -open.

3. On quasi λ -open M -sets in M -topological spaces

Definition 3.1. Let (M, τ) be an M -topological space. Any sub M -set B of M is said to be a Λ - M -set if $B = B^\Lambda$ with $C_B(x) = C_{B^\Lambda}(x)$, for all $x \in X$, where $B^\Lambda = \cap\{U | U \supseteq B, U \in \tau\}$ with $C_{B^\Lambda}(x) = \min\{C_U(x) : U \supseteq B, U \in \tau\}$ and $B_\vee = \cup\{F | B \supseteq F, F^C \in \tau\}$ with $C_B(x) = \max\{C_F(x) : B \supseteq F, F^C \in \tau\}$, for all $x \in X$.

Example 3.1. Let $X = \{a, b, c\}$, $w = 3$ and $M = \{2/a, 3/b, 1/c\}$. Let $\tau = \{M, \phi, \{2/a\}, \{2/a, 3/b\}\}$. Here, τ is an M -topology and (M, τ) is an M -topological space. Let $B = \{2/a\}$ be a sub M -set of M , then $B^\Lambda = \{2/a\}$ with $C_B(a) = 2$, $C_{B^\Lambda}(b) = 0$ and $C_{B^\Lambda}(c) = 0$. Also $C_B(a) = 2$, $C_B(b) = 0$ and $C_B(c) = 0$. Therefore, $C_B(x) = C_{B^\Lambda}(x)$, for all $x \in X$. Thus, $B = \{2/a\}$ is a Λ - M -set.

Definition 3.2. Let (M, τ) be an M -topological space. Any sub M -set A of an M -topological space (M, τ) is said to be a λ -closed M -set if $A = B \cap C$ with $C_A(x) = C_B \cap C_C(x)$, for all $x \in X$, where B is a Λ - M -set and C is a closed M -set of (M, τ) .

Definition 3.3. Let (M, τ) be an M -topological space. Any sub M -set A of an M -topological space (M, τ) is called λ -open M -set, if $A^C = M \ominus A$ is λ -closed M -set and also $C_{A^c}(x) = W \ominus C_A(x)$, for all $x \in X$.

Example 3.2. From Example 3.1, clearly (M, τ) is an M -topological space. Here, the collection of all Λ - M -sets of (M, τ) is $\{M, \phi, \{2/a\}, \{2/a, 3/b\}\}$. Let $B = M$ be any Λ - M -set in (M, τ) and $C = \{3/b, 1/c\}$ be any closed M -set in (M, τ) . Now, $A = B \cap C = \{3/b, 1/c\}$. Also, $C_A(x) = C_{B \cap C}(x)$, for all $x \in X$. Thus, $A = \{3/b, 1/c\}$ is a λ -closed M -set and $\{2/a\}$ is a λ -open M -sets of (M, τ) .

Definition 3.4. Let (M, τ) be an M -topological space and A be any sub M -set of M . Then the λ -interior of A and λ -closure of A is respectively denoted and defined as follows.

1. $Int_\lambda(A) = \cup\{U | U \subseteq A, \text{ each } U \subseteq M \text{ is a } \lambda\text{-open } M\text{-set}\}$ with $C_{Int_\lambda(A)}(x) = \max\{C_U(x) : U \subseteq A, \text{ each } U \subseteq M \text{ is a } \lambda\text{-open } M\text{-set}\}$ for all $x \in X$.
2. $cl_\lambda(A) = \cap\{G | A \subseteq G, \text{ each } G \subseteq M \text{ is a } \lambda\text{-closed } M\text{-set}\}$ with $C_{cl_\lambda(A)}(x) = \min\{C_G(x) : A \subseteq G, \text{ each } G \subseteq M \text{ is a } \lambda\text{-closed } M\text{-set}\}$, for all $x \in X$.

Example 3.3. From Example 3.1, clearly (M, τ) is an M -topological space. Let $A = \{2/a, 1/b\}$ be a sub M -set of M . Then, $Int(A) = \{2/a\}$ with $C_{Int_\lambda(A)}(x) =$

$\max\{C_U(x) : U \subseteq A, \text{ each } U \subseteq M \text{ is a } \lambda\text{-open } M\text{-set}\}$, for all $x \in X$ and $Cl(A) = \{2/a, 3/b\}$ with $Cl_{cl_\lambda(A)}(x) = \min\{C_G(x) : B \subseteq G, \text{ each } G \subseteq M \text{ is a } \lambda\text{-closed } M\text{-set}\}$, for all $x \in X$.

Definition 3.5. Let (M, τ) and (N, σ) be any two M -topological spaces. An M -set function $f : (M, \tau) \rightarrow (N, \sigma)$ is said to be a λ -continuous M -set function if $f^{-1}(V)$ is a λ -open M -set in (M, τ) for every open M -set V of (N, σ) .

Example 3.4. Let $X = \{a, b, c\}$, $w_1 = 3$ and $Y = \{x, y, z\}$, $w_2 = 3$. Let $M = \{2/a, 3/b, 1/c\}$ and $\tau = \{M, \phi, \{2/a\}, \{2/a, 3/b\}\}$. Here, τ is an M -topology and (M, τ) is an M -topological space. Let $N = \{3/x, 1/y, 2/z\}$ and $\sigma = \{N, \phi, \{1/y\}\}$. Then, σ is an M -topology and (N, σ) is an M -topological space. Let the function $f : (M, \tau) \rightarrow (N, \sigma)$ be defined by $f : \{(2/a, 1/y)/2, (3/b, 2/z)/6, (1/c, 3/x)/3\}$. Clearly $f^{-1}(V)$ is a λ -open M -set in (M, τ) for every open M -set V of (N, σ) . Thus, f is λ - M -set continuous.

Notation 3.1.

- The collection of all open sub M -sets of (M, τ) is denoted by $O(M)$ and $O(M, x) = \{V \in O(M) | x \in {}^m V\}$, for $x \in {}^m M$.
- The collection of all λ -open sub M -sets of (M, τ) is denoted by $\lambda O(M)$ and $O(M, x) = \{V \in \lambda O(M) | x \in {}^m V\}$, for $x \in {}^m M$.

Proposition 3.1. Let (M, τ) and (N, σ) be any two M -topological spaces. For an M -set function $f : (M, \tau) \rightarrow (N, \sigma)$, the following statements are equivalent:

- f is a λ -continuous M -set function;
- For every open sub M -set V of (N, σ) , $f^{-1}(V) \in \lambda O(M)$;
- For each $x \in {}^m M$ and each $V \in O(N, f(x))$, there exists a $U \in \lambda O(M, x)$ such that $f(U) \subseteq V$ with $C_{f(U)}(x) \leq C_V(x)$, for all $x \in X$.

Definition 3.6. Let (M, τ) and (N, σ) be any two M -topological spaces. An M -set function $f : (M, \tau) \rightarrow (N, \sigma)$ is said to be a quasi λ -open (resp., λ -closed) M -set function if for each λ -open (resp., λ -closed) sub M -set V of M , its image $f(V)$ is an open (resp., closed) sub M -set of (N, σ) . That is, $f(V)$ is the M -set of all points n/x in N for $x \in {}^m V$ for some m .

Example 3.5. In Example 3.4, clearly, the image of every λ -open M -set is an open M -set in (N, τ) . Therefore, f is a quasi λ -open M -set function. Consequently, the image of every λ -closed M -set is a closed M -set in (N, σ) . Hence f is a quasi λ -closed M -set function.

Proposition 3.2. *Let (M, τ) and (N, σ) be any two M -topological spaces. An M -set function $f : (M, \tau) \rightarrow (N, \sigma)$ is said to be a quasi λ -open M -set function if and only if for every sub M -set U of M , $f(\text{Int}_\lambda(U)) \subseteq \text{Int}((U))$ with $C_{f(\text{Int}_\lambda(U))}(x) \leq C_{\text{Int}(f(U))}(x)$, for all $x \in X$.*

Proposition 3.3. *Let (M, τ) and (N, σ) be any two M -topological spaces.*

If $f : (M, \tau) \rightarrow (N, \sigma)$ is a quasi λ -open M -set function, then $\text{Int}_\lambda((G)) \subseteq (\text{Int}(G))$ with, for every sub M -set G of N and $x \in X$.

Proposition 3.4. *Let (M, τ) and (N, σ) be any two M -topological spaces. An M -set function $f : (M, \tau) \rightarrow (N, \sigma)$ is said to be a quasi λ -open M -set function if and only if for any sub M -set B of N and for any λ -closed M -set F of M containing $f^{-1}(B)$ there exists a closed M -set G of N containing B such that $f^{-1}(G) \subseteq F$ with $C_{f^{-1}(G)}(x) \leq C_F(x)$, for all $x \in X$.*

Proposition 3.5. *Let (M, τ) and (N, σ) be any two M -topological spaces. An M -set function $f : (M, \tau) \rightarrow (N, \sigma)$ is said to be a quasi λ -open M -set function if and only if $f^{-1}(Cl(B)) \subseteq Cl_\lambda((B))$ with $C_{f^{-1}(Cl(B))}(x) \leq C_{Cl_\lambda(B)}(x)$, for every sub M -set B of N and for all $x \in X$.*

Proposition 3.6. *Let (M, τ) , (N, σ) and (P, η) be any three M -topological spaces and $f : (M, \tau) \rightarrow (N, \sigma)$ and $g : (N, \sigma) \rightarrow (P, \eta)$ be two M -set functions and $g \circ f : (M, \tau) \rightarrow (P, \eta)$ is a quasi λ -open M -set function. If g is M -set continuous injective, then f is quasi λ -open M -set function.*

Definition 3.7. Let (M, τ) and (N, σ) be any two M -topological spaces. An M -set function $f : (M, \tau) \rightarrow (N, \sigma)$ is said to be λ -closed M -set function if for each λ -closed M -set U in (M, τ) , $f(U)$ is a λ -closed M -set in (N, σ) .

Definition 3.8. Let (M, τ) and (N, σ) be any two M -topological spaces. An M -set function $f : (M, \tau) \rightarrow (N, \sigma)$ is said to be a λ -irresolute M -set function if $f^{-1}(V)$ is a λ -closed M -set (resp. λ -open M -set) in (M, τ) for every λ -closed M -set V (resp. λ -open M -set) of (N, σ) .

Proposition 3.7. *Let (M, τ) , (N, σ) and (P, η) be any three M -topological spaces and $f : (M, \tau) \rightarrow (N, \sigma)$ and $g : (N, \sigma) \rightarrow (P, \eta)$ be any two M -set functions. Then:*

- a. *If f is a λ -closed M -set function and g is a quasi λ -closed M -set function, then $g \circ f$ is a quasi λ -closed M -set function;*
- b. *If f is a λ -irresolute M -set function and g is a quasi λ -closed M -set function, then $g \circ f$ is a λ -continuous M -set function;*
- c. *If f is a λ -closed M -set function and g is a λ -continuous M -set function, then $g \circ f$ is a quasi λ -closed M -set function.*

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ALMOST STRONGLY ω -CONTINUOUS FUNCTIONS

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Abstract. The aim of this paper is to introduce and investigate a new class of continuity, called almost strongly ω -continuous function, which contains the class of strongly θ -continuous functions and it is contained in the class of almost ω -continuous functions.

Keywords: ω -open set, ω -continuous function, almost ω -continuous function.

1. Introduction

Throughout this paper, spaces always mean topological spaces with no separation axioms assumed, unless otherwise stated. Let (X, τ) be a space and A be a subset of X . The closure of A and the interior of A are denoted by $cl(A)$ and $Int(A)$, respectively. A point $x \in X$ is called a condensation point of A [4] if for each open set U containing x , the set $U - A$ is uncountable. A is said to be ω -closed [5] if it contains all its condensation points. The complement of an ω -closed set is said to be ω -open. Note that a subset A of a space (X, τ) is ω -open [2] if and only if for each $x \in A$ there exists an open set U containing x such that $U - A$ is countable. The family of all ω -open subsets of a space (X, τ) , forms a topology on X , denoted by τ_ω , finer than τ . The closure of A in (X, τ_ω) and the interior of A in (X, τ_ω) are denoted by $cl_\omega(A)$ and $Int_\omega(A)$. Several characterizations of ω -closed subsets were proved in [5]. A subset A

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is said to be regular open [11] (resp. regular closed) if $Int(cl(A)) = A$ (resp. $cl(Int(A)) = A$).

A point $x \in X$ is called a δ -cluster [12] (resp. θ_{ω} -cluster [3]) point of A if $A \cap Int(cl(U)) \neq \emptyset$ (resp. $A \cap cl(U) \neq \emptyset$) for each open (resp. ω -open) set U containing x . The set of all δ -cluster (resp. θ_{ω} -cluster) points of A is called the δ -closure (resp. the θ_{ω} -closure) of A and is denoted by $[A]_{\delta}$ (resp. $[A]_{\theta_{\omega}}$). If $[A]_{\delta} = A$ (resp. $[A]_{\theta_{\omega}} = A$), then A is said to be δ -closed (resp. θ_{ω} -closed). The complement of a δ -closed (resp. θ_{ω} -closed) set is said to be δ -open (resp. θ_{ω} -open).

A subset A of a space X is said to be an H -set [12] or quasi H -closed relative to X [8] if for every cover $\{U_{\alpha} : \alpha \in \Delta\}$ of A by open sets of X , there exists a finite subset Δ_0 of Δ such that $A \subseteq \cup\{cl(U_{\alpha}) : \alpha \in \Delta_0\}$. A space X is said to be quasi H -closed [8] if the set X is quasi H -closed relative to X . Quasi H -closed Hausdorff spaces are usually said to be H -closed.

For a nonempty set X , τ_{dis} will denote the discrete topology on X . $\mathbb{R}$ and $\mathbb{Q}$ denote the sets of all real numbers and rational numbers. Finally if (X, τ) and (Y, σ) are two space, then $\tau \times \sigma$ will denote the product topology on $X \times Y$.

Definition 1.1. A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be δ -continuous [7] (resp. almost continuous [9], strongly θ -continuous [7]) if for each $x \in X$ and each open set V containing $f(x)$, there exists an open set U of x such that $f(Int(cl(U))) \subseteq Int(cl(V))$ (resp. $f(U) \subseteq Int(cl(V))$, $f(cl(U)) \subseteq V$).

Definition 1.2. A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be ω -continuous [6] (resp. weakly ω -continuous [1], almost ω -continuous [1]) if for each $x \in X$ and each open set V of Y containing $f(x)$ there exists an ω -open set U containing x such that $f(U) \subseteq V$ (resp. $f(U) \subseteq cl(V)$, $f(U) \subseteq Int(cl(V))$).

Definition 1.3. A space (X, τ) is said to be $\omega - T_2[1]$ (resp. ω -Uryshon [1]) if for each pair of distinct points x and y in X , there exist ω -open sets U and V containing x and y , respectively, such that $U \cap V = \emptyset$ (resp. $cl_{\omega}(U) \cap cl_{\omega}(V) = \emptyset$).

Proposition 1.4. [3] *Let A be a subset of a space (X, τ) , A is θ_{ω} -open if and only if for each $x \in A$ there exists an ω -open set U containing x such that $cl(U) \subseteq A$.*

Lemma 1.5. [2] *Let A be a subset of a space (X, τ) . Then:*

- i. $(\tau_{\omega})_{\omega} = \tau_{\omega}$.
- ii. $(\tau_A)_{\omega} = (\tau_{\omega})_A$.

2. Almost strongly ω -continuous functions

Definition 2.1. A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be almost strongly ω -continuous if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists an ω -open set U containing x such that $f(cl(U)) \subseteq Int(cl(V))$.

Clearly, the following diagram follows immediately from the definitions and facts.

$$\begin{array}{ccccccc} \text{Continuous} & \rightarrow & \omega\text{-continuous} & \rightarrow & \text{almost } \omega\text{-continuous} & \rightarrow & \text{weakly } \omega\text{-continuous} \\ \uparrow & & & & & & \uparrow \\ \text{Strongly } \theta\text{-continuous} & \rightarrow & \text{almost strongly } \omega\text{-continuous} & & & & \end{array}$$

Note that almost strong ω -continuity and continuity (resp. ω -continuity) are independent of each other as the following examples show.

Example 2.2. Let $X = \{a, b, c, d\}$ with the topology $\tau = \{\phi, X, \{c\}, \{a, b\}, \{a, b, c\}\}$ and let $Y = \{p, q, r\}$ with the topology $\sigma = \{\phi, Y, \{p\}, \{q\}, \{p, q\}\}$. Define a function $f : (X, \tau) \rightarrow (Y, \sigma)$ as follows:

$$f(x) = \begin{cases} p & : x = a, b \\ r & : x = c, d \end{cases}.$$

Then f is continuous (hence, ω -continuous) but it is not almost strongly ω -continuous at $x = a$.

Example 2.3. Let $X = \mathbb{R}$ with the topologies $\tau = \tau_u$ and $\sigma = \{\phi, \mathbb{R}, \mathbb{R} - \{0\}\}$, where τ_u is the standard topology. Let $f : (X, \tau) \rightarrow (X, \sigma)$ be the function defined by

$$f(x) = \begin{cases} 0, & x \in \mathbb{R} - \mathbb{Q} \\ 1, & x \in \mathbb{Q} \end{cases}.$$

Then f is not ω -continuous since $V = \mathbb{R} - \{0\} \in \sigma$, but $f^{-1}(V) = \mathbb{Q} \notin \tau_\omega$. On the other hand, f is almost strongly ω -continuous.

Next, several characterizations of almost strongly ω -continuous functions are obtained.

Theorem 2.4. For a function $f : (X, \tau) \rightarrow (Y, \sigma)$, the following are equivalent:

- i. f is almost strongly ω -continuous.
- ii. The inverse image of a regular open set in (Y, σ) is θ_{ω} -open in (X, τ) .
- iii. The inverse image of a regular closed set in (Y, σ) is θ_{ω} -closed in (X, τ) .
- iv. For each $x \in X$ and each regular open set V in (Y, σ) containing $f(x)$, there exists an ω -open set U in (X, τ) containing x such that $f(\text{cl}(U)) \subseteq V$.
- v. The inverse image of a δ -open set in (Y, σ) is θ_{ω} -open in (X, τ) .
- vi. The inverse image of a δ -closed set in (Y, σ) is θ_{ω} -closed in (X, τ) .
- vii. $f([A]_{\theta_{\omega}}) \subset [f(A)]_{\delta}$ for each subset A of X .

viii. $[f^{-1}(B)]_{\theta_{\omega}} \subset f^{-1}([B]_{\delta})$ for each subset B of Y .

Proof. (i $\rightarrow$ ii) Let V be any regular open set in (Y, σ) and $x \in f^{-1}(V)$. Then $f(x) \in V$ and there exists an ω -open set U in (X, τ) containing x such that $f(cl(U)) \subset V$. Thus $x \in U \subseteq cl(U) \subseteq f^{-1}(V)$ and hence, by Proposition 1.4, $f^{-1}(V)$ is θ_{ω} -open.

(ii $\rightarrow$ iii) Let F be any regular closed set in (Y, σ) . By (ii), $f^{-1}(F) = X - f^{-1}(Y - F)$ is θ_{ω} -closed in X .

(iii $\rightarrow$ iv) Let $x \in X$ and V be any regular open set in (Y, σ) containing $f(x)$. By (iii), $f^{-1}(Y - V) = X - f^{-1}(V)$ is θ_{ω} -closed in (X, τ) . Since $f^{-1}(V)$ is a θ_{ω} -open set containing x , by Proposition 1.4, there exists an ω -open set U containing x such that $cl(U) \subseteq f^{-1}(V)$; hence $f(cl(U)) \subseteq V$.

(iv $\rightarrow$ v) Let V be a δ -open set in (Y, σ) and $x \in f^{-1}(V)$. There exists a regular open set G in (Y, σ) such that $f(x) \in G \subseteq V$. By (iv), there exists an ω -open set U containing x such that $f(cl(U)) \subseteq G$. Therefore, we obtain $x \in U \subseteq cl(U) \subseteq f^{-1}(V)$. Hence, by Proposition 1.4, $f^{-1}(V)$ is θ_{ω} -open in (X, τ) .

(v $\rightarrow$ vi) Let F be a δ -closed set in (Y, σ) . By (v) we have $f^{-1}(F) = X - f^{-1}(Y - F)$ is θ_{ω} -closed in (X, τ) .

(vi $\rightarrow$ vii) Let A be a subset of X . Since $[f(A)]_{\delta}$ is δ -closed in (Y, σ) , by (vi), $f^{-1}([f(A)]_{\delta})$ is θ_{ω} -closed in (X, τ) . Let $x \notin f^{-1}([f(A)]_{\delta})$. Then for some ω -open set U in (X, τ) containing x , $cl(U) \cap f^{-1}([f(A)]_{\delta}) = \emptyset$ and hence $cl(U) \cap A = \emptyset$. So $x \notin [A]_{\theta_{\omega}}$. Therefore, we have $f([A]_{\theta_{\omega}}) \subseteq [f(A)]_{\delta}$.

(vii $\rightarrow$ viii) Let B be a subset of Y . By (vii) we have $f([f^{-1}(B)]_{\theta_{\omega}}) \subseteq [B]_{\delta}$ and hence $[f^{-1}(B)]_{\theta_{\omega}} \subseteq f^{-1}([B]_{\delta})$.

(viii $\rightarrow$ i) Let $x \in X$ and V be an open set in (Y, σ) containing $f(x)$. Then $G = Y - Int(cl(V))$ is regular closed and hence δ -closed in (Y, σ) . By (viii), $[f^{-1}(G)]_{\theta_{\omega}} \subseteq f^{-1}(G)$ and hence $f^{-1}(G)$ is θ_{ω} -closed in (X, τ) . Therefore, $f^{-1}(Int(cl(V)))$ is a θ_{ω} -open set in (X, τ) containing x . By Proposition 1.4 there exists an ω -open set U containing x such that $x \in U \subseteq cl(U) \subseteq f^{-1}(Int(cl(V)))$. Therefore, we obtain $f(cl(U)) \subseteq Int(cl(V))$. This show that f is almost strongly ω -continuous. $\square$

Note that the family of all θ_{ω} -open [3] (resp. δ -open [12]) sets in a space (X, τ) form a topology for X which is denoted by $\tau_{\theta_{\omega}}$ (resp. τ_{δ}).

Theorem 2.5. For a function $f : (X, \tau) \rightarrow (Y, \sigma)$, the following are equivalent:

- i. $f : (X, \tau) \rightarrow (Y, \sigma)$ is almost strongly ω -continuous.
- ii. $f : (X, \tau_{\theta_{\omega}}) \rightarrow (Y, \sigma)$ is almost continuous.
- iii. $f : (X, \tau_{\theta_{\omega}}) \rightarrow (Y, \sigma_{\delta})$ is continuous.

Proof. (i $\rightarrow$ ii) Let V be any regular open set in (Y, σ) . By Theorem 2.4 $f^{-1}(V)$ is θ_{ω} -open in (X, τ) and hence open in $(X, \tau_{\theta_{\omega}})$, it follows from Theorem 2.2 of [9] that f is almost continuous.

(ii $\rightarrow$ iii) Let V be an open in (Y, σ_δ) . Then V is δ -open in (Y, σ) and it is the union of regular open sets in (Y, σ) . By (ii), $f^{-1}(V)$ is open in $(X, \tau_{\theta_{\omega}})$. Therefore, $f : (X, \tau_{\theta_{\omega}}) \rightarrow (Y, \sigma_\delta)$ is continuous.

(iii $\rightarrow$ i) Let V be a regular open set in (Y, σ) . Since V is open in (Y, σ_δ) , by (iii), $f^{-1}(V)$ is θ_{ω} -open in (X, τ) and hence by Theorem 2.4, f is almost strongly ω -continuous. $\square$

The composition of two almost strongly ω -continuous functions need not be almost strongly ω -continuous as the following examples shows.

Example 2.6. Let $X = \mathbb{R}$, $Y = \{0, 1\}$ and $Z = \{1, 2, 3\}$ with the topologies $\tau = \{\phi, X, \mathbb{Q}\}$, $\sigma = \{\phi, Y, \{0\}\}$, $\rho = \{\phi, Z, \{1\}, \{2\}, \{1, 2\}\}$ defined on X , Y and Z respectively. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be the function defined by

$$f(x) = \begin{cases} 0, & x \in \mathbb{R} - \mathbb{Q} \\ 1, & x \in \mathbb{Q} \end{cases}$$

and let $g : (Y, \sigma) \rightarrow (Z, \rho)$ be the function defined by

$$g(y) = \begin{cases} 1, & y = 1 \\ 3, & y = 0 \end{cases}.$$

Then f and g are almost strongly ω -continuous. However $g \circ f$ is not almost strongly ω -continuous at $x \in \mathbb{Q}$. For more clarify, let $x \in \mathbb{Q} \subseteq X$, $(g \circ f)(x) = g(f(x)) = g(1) = 1 \in V = \{1\} \in \rho$. Now for every ω -open set W containing x , $cl(W) = \mathbb{R}$, therefore $(g \circ f)(cl_\rho(W)) = g(f(\mathbb{R})) = g(\{0, 1\}) = \{1, 3\} \not\subseteq \{1\}$.

Theorem 2.7. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ and $g : (Y, \sigma) \rightarrow (Z, \rho)$ be functions. Then the following hold:

- $g \circ f$ is almost strongly ω -continuous if f is almost strongly ω -continuous and g is δ -continuous.
- $g \circ f$ is almost strongly ω -continuous if f is almost strongly ω -continuous and g is continuous and open.
- Let $p : (X \times Y, \tau \times \sigma) \rightarrow (X, \tau)$ be the projection function. If $(f \circ p)$ is almost strongly ω -continuous, then f is almost strongly ω -continuous.

Proof. The proof of a. follows immediately from Definitions 1.1 and 2.1. Thus we prove only part b and c.

b) Let $x \in X$ and V be any open set in (Z, ρ) such that $(g \circ f)(x) \in V$. Therefore $f(x) \in g^{-1}(V)$ which is open in (Y, σ) . Since f is almost strongly ω -continuous, there exists an ω -open set W in (X, τ) such that $x \in W$ and $f(cl(W)) \subseteq Int(cl(g^{-1}(V)))$. Therefore $(g \circ f)(cl(W)) = g(Int(cl(g^{-1}(V)))) \subseteq Int(cl(V))$.

c) Let $x \in X$ and V be any open set in (Y, σ) such that $f(x) \in V$. Choose $y \in Y$. Then $(f \circ p)(x, y) = f(x) \in V$. Since $(f \circ p)$ is almost strongly ω -continuous, there exists an ω -open set W in $X \times Y$ such that $(x, y) \in W$ and $(f \circ p)(cl_{\tau \times \sigma}(W)) \subseteq Int_{\sigma}(cl_{\sigma}(V))$. Since $(x, y) \in W$, choose $W_1 \in (X, \tau_{\omega})$ and $W_2 \in (Y, \sigma_{\omega})$ such that $x \in W_1$, $y \in W_2$ and $(x, y) \in W_1 \times W_2 \subseteq W$ and so $(f \circ p)(cl_{\tau \times \sigma}(W_1 \times W_2)) = f(cl(W_1)) \subseteq (f \circ p)(cl_{\tau \times \sigma}(W)) \subseteq Int_{\sigma}(cl(V))$. Thus $f(cl_{\tau}(W_1)) \subseteq Int_{\sigma}(cl(V))$ and so f is almost strongly ω -continuous. $\square$

To show that the assumption g is a continuous open function in part (b) of Theorem 2.7 is essential and that the projection function p in the same theorem part (c) can not be replaced by arbitrary open continuous function we consider the following examples.

Example 2.8. Let $X = \mathbb{R}$ with the topologies $\rho = \{\phi, \mathbb{R}, \mathbb{Q}\}$ and $\tau = \{U \subseteq \mathbb{R} : \mathbb{Q} \subseteq U\} \cup \{\phi\}$ and let $Y = \{0, 1, 2\}$ with the topology $\sigma = \{\phi, Y, \{0\}, \{1, 2\}\}$. Let $f : (X, \rho) \rightarrow (Y, \sigma)$ be the function defined by

$$f(x) = \begin{cases} 1, & x \in \mathbb{R} - \mathbb{Q} \\ 0, & x \in \mathbb{Q} \end{cases}$$

and $g : (X, \tau) \rightarrow (X, \rho)$ be the function defined by

$$g(x) = \begin{cases} 1, & x \in \mathbb{R} - \mathbb{Q} \\ x, & x \in \mathbb{Q} \end{cases}.$$

Then g is open and continuous but f is not almost strongly ω -continuous. Note that $g^{-1}(\mathbb{Q}) = \mathbb{Q} \in \tau$ and $g^{-1}(\mathbb{R}) = \mathbb{R}$, so we get that g is continuous and for every open set U in (X, τ) , $g(U) = \mathbb{Q} \in \rho$ therefore g is open. Now $(f \circ g)(x) = 0$ for every $x \in X$ and so $(f \circ g)$ is almost strongly ω -continuous.

Example 2.9. Let $X = \mathbb{R}$ with the topology $\rho = \{\phi, \mathbb{R}, \mathbb{Q}\}$, let $Y = \{0, 1\}$ with the topology $\sigma = \{\phi, Y, \{0\}\}$ and let $Z = \{0, 1, 2\}$ with the topology $\tau = \{\phi, Z, \{0\}, \{1, 2\}\}$. Let $f : (X, \rho) \rightarrow (Y, \sigma)$ be the function defined by

$$f(x) = \begin{cases} 0, & x \in \mathbb{R} - \mathbb{Q} \\ 1, & x \in \mathbb{Q} \end{cases}$$

and let $g : (Y, \sigma) \rightarrow (Z, \tau)$ defined by

$$g(y) = \begin{cases} 2, & y = 0 \\ 1, & y = 1 \end{cases}.$$

Then f is almost strongly ω -continuous, g is continuous function but not open and $(g \circ f)$ is not almost strongly ω -continuous at $x \in \mathbb{Q}$.

Example 2.10. Let $x = \mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{Q}\}$ and let $Y = \{0, 1\}$ with the topology $\sigma = \{\phi, Y, \{0\}\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be the function defined by

$$f(x) = \begin{cases} 0, & x \in \mathbb{R} - \mathbb{Q} \\ 1, & x \in \mathbb{Q} \end{cases}$$

and $g : (Y, \sigma) \rightarrow (Y, \tau_{dis})$ be the identity function. Then f is almost strongly ω -continuous function, g is open but not continuous and $(g \circ f)$ is not almost strongly ω -continuous. For every open set V in (Y, σ) ; $g(V)$ is open in (Y, τ_{dis}) and $g^{-1}(\{0\}) \notin \sigma$ and hence is not continuous. To show that $(g \circ f)$ is not almost strongly ω -continuous. Let $x \in \mathbb{Q} \subseteq X$, $(g \circ f)(x) = g(f(x)) = g(1) = 1 \in V = \{1\} \in \tau_{dis}$. Now for every ω -open set W containing x , $cl(W) = \mathbb{R}$, therefore $(g \circ f)(cl_p(W)) = g(f(\mathbb{R})) = g(Y) = Y \not\subseteq Int(cl(\{1\})) = \{1\}$.

Corollary 2.11. Let Δ be an index set and let $f_\alpha : (X_\alpha, \tau_\alpha) \rightarrow (Y_\alpha, \sigma_\alpha)$ be a function for each $\alpha \in \Delta$. If the product function $f = \prod_{\alpha \in \Delta} f_\alpha : \prod X_\alpha \rightarrow \prod Y_\alpha$ is almost strongly ω -continuous, then f_α is almost strongly ω -continuous for each $\alpha \in \Delta$.

Proof. For each $\beta \in \Delta$, we consider the projections $p_\beta : \prod_{\alpha \in \Delta} X_\alpha \rightarrow X_\beta$ and $q_\beta : \prod_{\alpha \in \Delta} Y_\alpha \rightarrow Y_\beta$. Then we have $q_\beta \circ f = f_\beta \circ p_\beta$ for each $\beta \in \Delta$. Since f is almost strongly ω -continuous and q_β is a continuous open function for each $\beta \in \Delta$, $q_\beta \circ f$ is almost strongly ω -continuous by Theorem 2.7 and hence $f_\beta \circ p_\beta$ is almost strongly ω -continuous. Thus f_β is almost strongly ω -continuous by Theorem 2.7. $\square$

Proposition 2.12. Let $f : (X, \tau) \rightarrow (Y_1 \times Y_2, \sigma_1 \times \sigma_2)$ be a function, where (X, τ) , (Y_1, σ_1) and (Y_2, σ_2) are topological spaces. Let $f_i : (X, \tau) \rightarrow (Y_i, \sigma_i)$ be defined as $f_i = p_i \circ f$ for $i = 1, 2$ where $p_i : (Y_1 \times Y_2, \sigma_1 \times \sigma_2) \rightarrow (Y_i, \sigma_i)$ is the projection function. If f is almost strongly ω -continuous, then f_i is almost strongly ω -continuous for $i = 1, 2$.

Proof. Since p_i is a continuous open function and f is almost strongly ω -continuous, then by Theorem 2.7, $f_i = p_i \circ f$ is almost strongly ω -continuous for $i = 1, 2$. $\square$

Theorem 2.13. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be an almost strongly ω -continuous function. Then the restriction $f|_A : (A, \tau_A) \rightarrow (Y, \sigma)$ is almost strongly ω -continuous for any subset A of X .

Proof. Let $a \in A$ and V be an open set in (Y, σ) containing $f(a)$. Since f is almost strongly ω -continuous, there exists an ω -open set W in (X, τ) such that $x \in W$ and $f(cl(W)) \subseteq Int(cl(V))$. Therefore by Lemma 1.5 $W \cap A \in (\tau_A)_\omega$ and $(f|_A)(cl(W \cap A)) \subseteq Int(cl(V))$. And the result follows. $\square$

The following example shows that the converse of the previous theorem is not true in general.

Example 2.14. Let $X = \mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{Q}\}$ and let $Y = \{0, 1, 2\}$ with the topology $\sigma = \{\phi, Y, \{0\}, \{1, 2\}\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be the function defined by

$$f(x) = \begin{cases} 0, & x \in \mathbb{Q} \\ 1, & x \in \mathbb{R} - \mathbb{Q} \end{cases}.$$

Then f is not almost strongly ω -continuous, since if we take $x \in \mathbb{Q}$, then $f(x) = 0 \in \{0\} \in \sigma$ and for any ω -open set W containing x , $f(cl(W)) = f(\mathbb{R}) = \{0, 1\} \not\subseteq Int(cl(\{0\})) = \{0\}$. Let $A = \mathbb{Q}$. Then $A \in \tau$ and $\tau_A = \{\phi, A\}$. Note that $(f|_A)(x) = 0$ for every $x \in A$ and so $(f|_A)$ is almost strongly ω -continuous.

Note that if A is a clopen subset of a space (X, τ) . Then $cl(U \cap A) = cl(U) \cap A$ for every $U \subseteq X$.

Proposition 2.15. *Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a function and let $x \in X$. If there exists a clopen subset A of X containing x and $(f|_A)$ is almost strongly ω -continuous at x , then f is almost strongly ω -continuous at x .*

Proof. Let V be an open set in (Y, σ) containing $f(x)$. Since $(f|_A)$ is almost strongly ω -continuous at x , there exists an ω -open set W in (A, τ_A) such that $x \in W$ and $(f|_A)(cl_{\tau_A}(W)) = f(cl_{\tau_A}(W)) \subseteq Int(cl(V))$. So by Lemma 1.5 $W \in (\tau_A)_\omega = (\tau_\omega)_A$ and there exists an ω -open set U in (X, τ) such that $W = U \cap A$. Therefore W is an ω -open set in (X, τ) and $f(cl_\tau(W)) = f(cl_\tau(A \cap U)) = f(cl_\tau(U) \cap A) = f(cl_\tau(U \cap A) \cap A) = f(cl_\tau(W) \cap A) = f(cl_{\tau_A}(W)) \subseteq Int(cl(V))$ and the result follows. $\square$

The following example shows that if the set A is ω -clopen then the result in proposition 2.15 need not be true.

Example 2.16. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be the function defined in Example 2.14. Then f is not almost strongly ω -continuous. Let $A = \mathbb{R} - \mathbb{Q}$. Then A is an ω -clopen set in (X, τ) and $(f|_A)$ is almost strongly ω -continuous. Note that $(f|_A)(x) = 1$ for every $x \in A$ so $(f|_A)(x)$ is almost strongly ω -continuous.

3. Basic properties

A space (X, τ) is said to be weakly Hausdorff [10] if each point of X is expressed by the intersection of regular closed sets of (X, τ) and it is said to be ω^* -regular if for every ω -open set U and each point $x \in U$ there exists an open set V such that $x \in U \subseteq cl(U) \subseteq V$.

Theorem 3.1. *Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a function such that X is ω^* -regular and let $g : (X, \tau) \rightarrow (X \times Y, \tau \times \sigma)$ be the graph function of f defined by $g(x) = (x, f(x))$ for each $x \in X$. Then g is almost strongly ω -continuous if and only if f is almost strongly ω -continuous.*

Proof. Necessity. Suppose that g is almost strongly ω -continuous. Let $x \in X$ and V be an open set in (Y, σ) containing $f(x)$. Then $X \times V$ is an open set of $X \times Y$ containing $g(x)$. Since g is almost strongly ω -continuous, there exists an ω -open set U in (X, τ) containing x such that $g(cl(U)) \subseteq Int(cl(X \times V))$. It follows $Int(cl(X \times V)) = X \times Int(cl(V))$. Therefore, we obtain $f(cl(U)) \subseteq Int(cl(V))$.

Sufficiency. Let $x \in X$ and W be any open set of $X \times Y$ containing $g(x)$. There exist open sets $U_1 \subseteq X$ and $V \subseteq Y$ such that $g(x) = (x, f(x)) \in U_1 \times V \subseteq W$. Since f is almost strongly ω -continuous, there exists an ω -open set U_2 in (X, τ) containing x such that $f(cl(U_2)) \subseteq Int(cl(V))$. Let $U = U_1 \cap U_2$, then U is an ω -open in (X, τ) containing x . Since X is ω^* -regular, there exists an open set Z such that $x \in Z \subseteq cl(Z) \subseteq U$. Therefore, we obtain $g(cl(Z)) \subseteq U_1 \times f(U_2) \subseteq Int(cl(W))$. $\square$

Theorem 3.2. *If $f, g : (X, \tau) \rightarrow (Y, \sigma)$ are almost strongly ω -continuous functions and (Y, σ) is a Hausdorff space, then the set $E = \{x \in X : f(x) = g(x)\}$ is θ_{ω} -closed in (X, τ) .*

Proof. By Theorem 2.5 $f, g : (X, \tau_{\theta_{\omega}}) \rightarrow (Y, \sigma_{\delta})$ are continuous functions and hence A is closed in $(X, \tau_{\theta_{\omega}})$. Therefore, A is θ_{ω} -closed in (X, τ) . $\square$

Theorem 3.3. *Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be an almost strongly ω -continuous injection. If Y is a Hausdorff (resp. weakly Hausdorff) space, then X is an ω -Urysohn (resp. ω -Hausdorff) space.*

Proof. Let (Y, σ) be Hausdorff and $x_1 \neq x_2$ for any $x_1, x_2 \in X$ and there exist disjoint open sets V_1 and V_2 containing $f(x_1)$ and $f(x_2)$, respectively. Since V_1 and V_2 are disjoint, we obtain $Int(cl(V_1)) \cap Int(cl(V_2)) = \phi$. Since f is almost strongly ω -continuous, for $i = 1, 2$, there exists an ω -open set U_i containing x_i such that $f(cl(U_i)) \subseteq Int(cl(V_i))$. It follows from $cl(U_1) \cap cl(U_2) = \phi$ that X is an ω -Urysohn space. Next, let Y be weakly Hausdorff and x_1, x_2 distinct points of X . Then $f(x_1) \neq f(x_2)$ and there exists a regular closed set V of Y such that $f(x_1) \notin V$ and $f(x_2) \in V$. Since f is almost strongly ω -continuous, by Theorem 2.4, there exists an ω -open set U containing x_1 such that $f(cl(U)) \subseteq Y - V$. Then we have $x_2 \in f^{-1}(V) \subseteq X - cl(U)$. This show that (X, τ) is ω -Hausdorff. $\square$

For a function $f : (X, \tau) \rightarrow (Y, \sigma)$, the subset $\{x, f(x) : x \in X\} \subseteq X \times Y$ is called the graph of f and is denoted by $G(f)$.

The $G(f)$ is said to be θ_{ω} -closed with respect to $X \times Y$ if for each $(x, y) \notin G(f)$, there exists an ω -open sets U and V containing x and y , respectively, such that $cl(U \times V) \cap G(f) = \phi$. It is easy to see that $G(f)$ is θ_{ω} -closed with respect to $X \times Y$ if and only if for each $(x, y) \notin G(f)$ there exist ω -open subsets $U \subseteq X$ and $V \subseteq Y$ containing x and y , respectively, such that $f(cl(U)) \cap cl(V) = \phi$.

Definition 3.4. A subset S of a space X is said to be quasi H_{ω} -closed (resp. N_{ω} -closed) relative to X if for every cover $\{U_{\alpha} : \alpha \in \Delta\}$ of S by ω -open sets of

X , there exists a finite subset Δ_o of Δ such that $S \subseteq \cup\{cl(U_\alpha) : \alpha \in \Delta_o\}$ (resp. $S \subseteq \cup\{Int(cl(U_\alpha)) : \alpha \in \Delta_o\}$). A space X is said to be quasi H_ω -closed (resp. nearly ω -compact) if the set X is quasi H_ω -closed (resp. N_ω -closed) relative to X .

Theorem 3.5. *Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a function whose graph is $\theta_{o\omega}$ -closed with respect to $X \times Y$. If K is quasi H_ω -closed relative to Y , then $f^{-1}(K)$ is $\theta_{o\omega}$ -closed in X .*

Proof. Let $x \in X - f^{-1}(K)$. For each $y \in K$, $(x, y) \notin G(f)$ and there exist ω -open sets U_y and V_y containing x and y , respectively, such that $f(cl(U_y)) \cap cl(V_y) = \phi$. The family $\{V_y : y \in K\}$ is a cover of K by ω -open sets of Y and $K \subseteq \cup\{cl(V_y) : y \in K_o\}$ for some finite subset K_o of K . Put $U = \cap\{U_y : y \in K_o\}$. Then U is an ω -open set containing x and $f(cl(U)) \cap K = \phi$. Therefore, we have $cl(U) \cap f^{-1}(K) = \phi$ and hence $x \notin [f^{-1}(K)]_{\theta_{o\omega}}$. This shows that $f^{-1}(K)$ is $\theta_{o\omega}$ -closed in X . $\square$

Theorem 3.6. *If $f : (X, \tau) \rightarrow (Y, \sigma)$ almost strongly ω -continuous and K is quasi H_ω -closed relative to X , then $f(K)$ is N_ω -closed relative to Y .*

Proof. Let $\{V_\alpha : \alpha \in \Delta\}$ be a cover of $f(K)$ by ω -open sets of Y . For each $x \in K$, there exists $\alpha_x \in \Delta$ such that $f(x) \in V_{\alpha_x}$. Since f is almost strongly ω -continuous, there exists an ω -open set U_x containing x such that $f(cl(U_x)) \subseteq Int(cl(V_{\alpha_x}))$. The family $\{U_x : x \in K\}$ is a cover of K by ω -open sets of (X, τ) and hence there exists a finite subset K^* of K such that $K \subseteq \cup_{x \in K^*} cl(U_x)$. Therefore, we obtain $f(K) \subseteq f(\cup_{x \in K^*} cl(U_x)) \subseteq \cup_{x \in K^*} Int(cl(V_{\alpha_x}))$. $\square$

Lemma 3.7. *If X is nearly ω -compact and A is regular closed in X , then A is N_ω -closed relative to X (and hence quasi H_ω -closed relative to X).*

Proof. Let $\{U_\alpha : \alpha \in \Delta\}$ be any cover of A by ω -open sets of X . Then $X = \cup\{U_\alpha : \alpha \in \Delta\} \cap (X - A)$. Since $X - A$ is regular open, it is open and hence ω -open. Since X is nearly ω -compact, there exists a finite subset Δ_o of Δ such that $X = [\cup\{Int(cl(U_\alpha)) : \alpha \in \Delta_o\}] \cup Int(cl(X - A)) = (\cup\{Int(cl(U_\alpha)) : \alpha \in \Delta_o\}) \cup (X - A)$. Therefore, $A \subseteq \cup\{Int(cl(U_\alpha)) : \alpha \in \Delta_o\}$ and A is N_ω -closed relative to X . $\square$

Theorem 3.8. *Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a function and (Y, σ) nearly ω -compact Hausdorff. Then, the following are equivalent:*

- i. f is almost strongly ω -continuous.
- ii. $G(f)$ is $\theta_{o\omega}$ -closed with respect to $X \times Y$.
- iii. If K is quasi H_ω -closed relative to Y , then $f^{-1}(K)$ is $\theta_{o\omega}$ -closed in (X, τ) .

Proof. (i $\rightarrow$ ii) Let $(x, y) \in X \times Y - G(f)$. Since (Y, σ) is Hausdorff, there exist two open sets V and W such that $y \in V$, $f(x) \in W$ and $V \cap W = \phi$. This gives $cl(V) \cap Int(cl(W)) = \phi$. By (i), there exists an ω -open set U containing x such that $f(cl(U)) \subseteq Int(cl(W))$. Hence $f(cl(U)) \cap cl(V) = \phi$, that is, $G(f)$ is θ_{ω} -closed with respect to $X \times Y$.

(ii $\rightarrow$ iii) This follows from Theorem 3.5.

(iii $\rightarrow$ i) Let $x \in X$ and V be a regular open subset of Y such that $f(x) \in V$. Then $Y - V$ is a regular closed set and Y is nearly ω -compact, by Lemma 3.7 $Y - V$ is quasi H_{ω} -closed relative to Y . By (iii) $f^{-1}(X - V)$ is θ_{ω} -closed in X and $x \notin f^{-1}(Y - V)$. Hence there exists an ω -open set U containing x such that $cl(U) \cap f^{-1}(Y - V) = \phi$. This implies that $f(cl(U)) \subseteq V$. Therefore, it follows from Theorem 2.4 that f is almost strongly θ -continuous. $\square$

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SCHUR m -POWER CONVEXITY OF GEOMETRIC BONFERRONI MEAN

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Abstract. In this paper the Schur m -power convexity of the geometric Bonferroni mean for n variables is discussed.

Keywords: Schur m -power convexity, geometric Bonferroni means, majorization.

1. Introduction

Throughout the paper $\mathbb{R}$ denotes the set of real numbers, $\mathbf{x} = (x_1, x_2, \dots, x_n)$ denotes n -tuple (n -dimensional real vectors), the set of vectors can be written as

$$\mathbb{R}^n = \{\mathbf{x} = (x_1, x_2, \dots, x_n) : x_i \in \mathbb{R}, i = 1, 2, \dots, n\},$$

$$\mathbb{R}_+^n = \{\mathbf{x} = (x_1, x_2, \dots, x_n) : x_i \geq 0, i = 1, 2, \dots, n\},$$

$$\mathbb{R}_{++}^n = \{\mathbf{x} = (x_1, x_2, \dots, x_n) : x_i > 0, i = 1, 2, \dots, n\}.$$

In particular, the notations $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{R}_{++}$ denote $\mathbb{R}^1$, $\mathbb{R}_+^1$ and $\mathbb{R}_{++}^1$, respectively.

The Bonferroni mean has important application in multi criteria decision-making (see[2–7]), it was initially proposed by Bonferroni [1], as follows

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Definition 1. Let $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}_+^n$ and $p, q \geq 0, p + q \neq 0$. The Bonferroni mean is defined by

$$(1) \quad B^{p,q}(\mathbf{x}) = \left(\frac{1}{n(n-1)} \sum_{i,j=1, i \neq j}^n x_i^p x_j^q \right)^{\frac{1}{p+q}}.$$

Motivated by the Bonferroni mean $B^{p,q}(\mathbf{x})$ and the geometric mean $G(\mathbf{x}) = \prod_{i=1}^n (x_i)^{\frac{1}{n}}$, Xia et al. [4] introduced a new mean which is called the geometric Bonferroni mean, i.e.,

Definition 2. Let $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}_{++}^n$ and $(p, q) \in \mathbb{R}_{++}^2$. The geometric Bonferroni mean is defined by

$$(2) \quad GB^{p,q}(\mathbf{x}) = \frac{1}{p+q} \prod_{i,j=1, i \neq j}^n (px_i + qx_j)^{\frac{1}{n(n-1)}}.$$

Obviously, the geometric Bonferroni mean has the following properties:

- (i) $GB^{p,q}(0, 0, \dots, 0) = 0$.
- (ii) $GB^{p,q}(x, x, \dots, x) = x$, if $x_i = x$ for $i = 1, 2, \dots, n$.
- (iii) $GB^{p,q}(\mathbf{x}) \geq GB^{p,q}(\mathbf{y})$, if $x_i \geq y_i$ for $i = 1, 2, \dots, n$.
- (iv) $\min\{x_1, x_2, \dots, x_n\} \leq GB^{p,q}(\mathbf{x}) \leq \max\{x_1, x_2, \dots, x_n\}$.

Furthermore, if $q = 0$, then the geometric Bonferroni mean reduces to the geometric mean, i.e.,

$$GB^{p,0}(\mathbf{x}) = \frac{1}{p} \prod_{i,j=1, i \neq j}^n (px_i)^{\frac{1}{n(n-1)}} = \prod_{i=1}^n (x_i)^{\frac{1}{n}} = G(\mathbf{x}).$$

In recent years, the theory of majorization has been used as an important tool in studying the properties of the mean. Yang [8],[9],[10] generalized the notion of Schur convexity to Schur f -convexity and discussed the Schur m -power convexity of Stolarsky means [8], Gini means [9] and Daróczy means [10]. Subsequently, the Schur m -power convexity has evoked the interest of many researchers (see [11], [12], [13], [14]).

In this paper, we discuss the Schur m -power convexity of the geometric Bonferroni mean $GB^{p,q}(\mathbf{x})$. Our main results are stated in the following theorem.

Theorem 1. Let $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}_{++}^n$. For fixed $(p, q) \in \mathbb{R}_{++}^2$ and $n \geq 3$,

- (i) if $m < 0$, then $GB^{p,q}(\mathbf{x})$ is Schur m -power convex;
- (ii) if $m = 0$, then $GB^{p,q}(\mathbf{x})$ is Schur m -power convex;
- (iii) if $m = 1$, then $GB^{p,q}(\mathbf{x})$ is Schur m -power concave;
- (iv) if $m \geq 2$, then $GB^{p,q}(\mathbf{x})$ is Schur m -power concave.

2. Preliminaries

We begin with recalling some basic concepts and notations in the theory of majorization. For more details, we refer the reader to [15, 16].

Definition 3. Let $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $\mathbf{y} = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$.

- (i) $\mathbf{x} \geq \mathbf{y}$ means $x_i \geq y_i$ for all $i = 1, 2, \dots, n$.
- (ii) Let $\Omega \subset \mathbb{R}^n$, $\varphi: \Omega \rightarrow \mathbb{R}$ is said to be increasing if $\mathbf{x} \geq \mathbf{y}$ implies $\varphi(\mathbf{x}) \geq \varphi(\mathbf{y})$. φ is said to be decreasing if and only if $-\varphi$ is increasing.

Definition 4. Let $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $\mathbf{y} = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$.

- (i) $\mathbf{x}$ is said to be majorized by $\mathbf{y}$ (in symbols $\mathbf{x} \prec \mathbf{y}$) if $\sum_{i=1}^k x_{[i]} \leq \sum_{i=1}^k y_{[i]}$ for $k = 1, 2, \dots, n-1$ and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$, where $x_{[1]} \geq x_{[2]} \geq \dots \geq x_{[n]}$ and $y_{[1]} \geq y_{[2]} \geq \dots \geq y_{[n]}$ are rearrangements of $\mathbf{x}$ and $\mathbf{y}$ in a descending order.
- (ii) Let $\Omega \subset \mathbb{R}^n$, the function $\varphi: \Omega \rightarrow \mathbb{R}$ is said to be Schur convex on Ω if $\mathbf{x} \prec \mathbf{y}$ on Ω implies $\varphi(\mathbf{x}) \leq \varphi(\mathbf{y})$. φ is said to be Schur concave function on Ω if and only if $-\varphi$ is Schur convex function on Ω .

Definition 5. Let $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $\mathbf{y} = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$.

- (i) $\Omega \subset \mathbb{R}^n$ is said to be a convex set if $\mathbf{x}, \mathbf{y} \in \Omega$, $0 \leq \alpha \leq 1$ implies $\alpha\mathbf{x} + (1-\alpha)\mathbf{y} = (\alpha x_1 + (1-\alpha)y_1, \alpha x_2 + (1-\alpha)y_2, \dots, \alpha x_n + (1-\alpha)y_n) \in \Omega$.
- (ii) Let $\Omega \subset \mathbb{R}^n$ be convex set. A function $\varphi: \Omega \rightarrow \mathbb{R}$ is said to be convex on Ω if

$$\varphi(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) \leq \alpha\varphi(\mathbf{x}) + (1-\alpha)\varphi(\mathbf{y})$$

for all $\mathbf{x}, \mathbf{y} \in \Omega$, and all $\alpha \in [0, 1]$. The function φ is said to be concave on Ω if and only if $-\varphi$ is convex function on Ω .

Definition 6. (i) A set $\Omega \subset \mathbb{R}^n$ is called symmetric, if $\mathbf{x} \in \Omega$ implies $\mathbf{x}P \in \Omega$ for every $n \times n$ permutation matrix P .

(ii) A function $\varphi: \Omega \rightarrow \mathbb{R}$ is called symmetric if for every permutation matrix P , $\varphi(\mathbf{x}P) = \varphi(\mathbf{x})$ for all $\mathbf{x} \in \Omega$.

The first systematical study of the functions preserving the ordering of majorization was made by I. Schur in 1923. In Schur's honor, such functions are said to be Schur convex (see [15]). It can be used extensively in analytic inequalities, combinatorial optimization, quantum physics, information theory, and other related fields.

The following proposition is called Schur's condition (see [15]). It provides an approach for testing whether a vector valued function is Schur convex or not.

Proposition 1. *Let $\Omega \subset \mathbb{R}^n$ be symmetric and have a nonempty interior convex set. Ω^0 is the interior of Ω . $\varphi : \Omega \rightarrow \mathbb{R}$ is continuous on Ω and differentiable in Ω^0 . Then φ is the Schur convex function (Schur concave function) if and only if φ is symmetric on Ω and*

$$(3) \quad (x_1 - x_2) \left[\frac{\partial \varphi(\mathbf{x})}{\partial x_1} - \frac{\partial \varphi(\mathbf{x})}{\partial x_2} \right] \geq 0 \ (\leq 0)$$

holds for any $\mathbf{x} \in \Omega^0$.

A generalization of Schur convex functions was introduced by Yang [8], as follows

Definition 7. *Let $f : \mathbb{R}_{++} \rightarrow \mathbb{R}$ be defined by*

$$(4) \quad f(x) = \begin{cases} \frac{x^m - 1}{m}, & m \neq 0; \\ \ln x, & m = 0. \end{cases}$$

Then a function $\varphi : \Omega \subset \mathbb{R}_{++}^n \rightarrow \mathbb{R}$ is said to be Schur m -power convex on Ω if

$$(f(x_1), f(x_2), \dots, f(x_n)) \prec (f(y_1), f(y_2), \dots, f(y_n))$$

for all $(x_1, x_2, \dots, x_n) \in \Omega$ and $(y_1, y_2, \dots, y_n) \in \Omega$ implies $\phi(x) \leq \phi(y)$.

If $-\varphi$ is Schur m -power convex, then we say that φ is Schur m -power concave.

Similarly to the Schur's condition mentioned above, Yang [8] gave a method of determining the Schur m -power convex functions, i.e.,

Proposition 2. *Let $\Omega \subset \mathbb{R}_{++}^n$ be a symmetric set with nonempty interior Ω° and $\varphi : \Omega \rightarrow \mathbb{R}$ be continuous on Ω and differentiable in Ω° . Then φ is Schur m -power convex (Schur m -power concave) on Ω if and only if φ is symmetric on Ω and*

$$(5) \quad \frac{x_1^m - x_2^m}{m} \left[x_1^{1-m} \frac{\partial \varphi(\mathbf{x})}{\partial x_1} - x_2^{1-m} \frac{\partial \varphi(\mathbf{x})}{\partial x_2} \right] \geq 0 \ (\leq 0), \quad \text{if } m \neq 0$$

and

$$(6) \quad (\log x_1 - \log x_2) \left[x_1 \frac{\partial \varphi(\mathbf{x})}{\partial x_1} - x_2 \frac{\partial \varphi(\mathbf{x})}{\partial x_2} \right] \geq 0 \ (\leq 0), \quad \text{if } m = 0$$

for all $\mathbf{x} \in \Omega^\circ$.

3. Proof of Theorem 1

Proof. Note that the geometric Bonferroni mean is defined by

$$GB^{p,q}(\mathbf{x}) = \frac{1}{p+q} \prod_{i,j=1, i \neq j}^n (px_i + qx_j)^{\frac{1}{n(n-1)}}.$$

Taking the natural logarithm gives

$$\log GB^{p,q}(\mathbf{x}) = \log \frac{1}{p+q} + \frac{1}{n(n-1)} Q$$

where

$$\begin{aligned} Q = & \sum_{j=3}^n [\log(px_1 + qx_j) + \log(px_2 + qx_j)] + \sum_{i=3}^n [\log(px_i + qx_1) + \log(px_i + qx_2)] \\ & + \log(px_1 + qx_2) + \log(px_2 + qx_1) + \sum_{i,j=3, i \neq j}^n \log(px_i + qx_j). \end{aligned}$$

Hence, we have

$$\begin{aligned} \frac{\partial GB^{p,q}(\mathbf{x})}{\partial x_1} &= \frac{GB^{p,q}(\mathbf{x})}{n(n-1)} \left(\sum_{j=3}^n \frac{p}{px_1 + qx_j} + \sum_{i=3}^n \frac{q}{px_i + qx_1} \right. \\ &\quad \left. + \frac{p}{px_1 + qx_2} + \frac{q}{px_2 + qx_1} \right) \\ \frac{\partial GB^{p,q}(\mathbf{x})}{\partial x_2} &= \frac{GB^{p,q}(\mathbf{x})}{n(n-1)} \left(\sum_{j=3}^n \frac{p}{px_2 + qx_j} + \sum_{i=3}^n \frac{q}{px_i + qx_2} \right. \\ &\quad \left. + \frac{q}{px_1 + qx_2} + \frac{p}{px_2 + qx_1} \right) \end{aligned}$$

It is easy to see that $GB^{p,q}(\mathbf{x})$ is symmetric on $\mathbb{R}_+^n$. Without loss of generality, we may assume that $x_1 \geq x_2$.

Direct computation gives

$$\begin{aligned} \Delta &:= \frac{x_1^m - x_2^m}{m} \left(x_1^{1-m} \frac{\partial GB^{p,q}(\mathbf{x})}{\partial x_1} - x_2^{1-m} \frac{\partial GB^{p,q}(\mathbf{x})}{\partial x_2} \right) \\ &= \frac{(x_1^m - x_2^m) GB^{p,q}(\mathbf{x})}{mn(n-1)} \left[p \sum_{j=3}^n \left(\frac{x_1^{1-m}}{px_1 + qx_j} - \frac{x_2^{1-m}}{px_2 + qx_j} \right) \right. \end{aligned}$$

$$\begin{aligned}
 & + q \sum_{i=3}^n \left(\frac{x_1^{1-m}}{px_i + qx_1} - \frac{x_2^{1-m}}{px_i + qx_2} \right) \\
 & + \frac{px_1^{1-m} - qx_2^{1-m}}{px_1 + qx_2} + \frac{qx_1^{1-m} - px_2^{1-m}}{px_2 + qx_1} \Big] \\
 & = \frac{(x_1^m - x_2^m)GB^{p,q}(\mathbf{x})}{mn(n-1)} \left[p \sum_{j=3}^n \frac{px_1x_2(x_1^{-m} - x_2^{-m}) + qx_j(x_1^{1-m} - x_2^{1-m})}{(px_1 + qx_j)(px_2 + qx_j)} \right. \\
 & + q \sum_{i=3}^n \frac{qx_1x_2(x_1^{-m} - x_2^{-m}) + px_i(x_1^{1-m} - x_2^{1-m})}{(px_i + qx_1)(px_i + qx_2)} \\
 & \left. + \frac{x_1x_2(p^2 + q^2)(x_1^{-m} - x_2^{-m}) + 2pq(x_1^{2-m} - x_2^{2-m})}{(px_1 + qx_2)(px_2 + qx_1)} \right].
 \end{aligned}$$

If $m < 0$, then $x_1^m - x_2^m \leq 0$, $x_1^{-m} - x_2^{-m} \geq 0$, $x_1^{1-m} - x_2^{1-m} \geq 0$ and $x_1^{2-m} - x_2^{2-m} \geq 0$. Thus, $\Delta \geq 0$. From Proposition 2, it follows that $GB^{p,q}(\mathbf{x})$ is Schur m -power convex for $\mathbf{x} \in \mathbb{R}_{++}^n$.

If $m \geq 2$, then $x_1^m - x_2^m \geq 0$, $x_1^{-m} - x_2^{-m} \leq 0$, $x_1^{1-m} - x_2^{1-m} \leq 0$ and $x_1^{2-m} - x_2^{2-m} \leq 0$. Thus, $\Delta \leq 0$. By Proposition 2, we conclude that $GB^{p,q}(\mathbf{x})$ is Schur m -power concave for $\mathbf{x} \in \mathbb{R}_{++}^n$.

If $m = 1$, then

$$\begin{aligned}
 \Delta & = -\frac{(x_1 - x_2)^2 GB^{p,q}(\mathbf{x})}{n(n-1)} \left[\sum_{j=3}^n \frac{p^2}{(px_1 + qx_j)(px_2 + qx_j)} \right. \\
 & + \sum_{i=3}^n \frac{q^2}{(px_i + qx_1)(px_i + qx_2)} + \frac{(p-q)^2}{(px_1 + qx_2)(px_2 + qx_1)} \Big] \\
 & \leq 0.
 \end{aligned}$$

By using Proposition 2, we deduce that $GB^{p,q}(\mathbf{x})$ is Schur m -power concave for $\mathbf{x} \in \mathbb{R}_{++}^n$.

If $m = 0$, then

$$\begin{aligned}
 \Delta & = (\log x_1 - \log x_2) \left(x_1 \frac{GB^{p,q}(\mathbf{x})}{\partial x_1} - x_2 \frac{\partial GB^{p,q}(\mathbf{x})}{\partial x_2} \right) \\
 & = \frac{(\log x_1 - \log x_2)GB^{p,q}(\mathbf{x})}{n(n-1)} \left[p \sum_{j=3}^n \left(\frac{x_1}{px_1 + qx_j} - \frac{x_2}{px_2 + qx_j} \right) \right. \\
 & + q \sum_{i=3}^n \left(\frac{x_1}{px_i + qx_1} - \frac{x_2}{px_i + qx_2} \right) + \frac{px_1 - qx_2}{px_1 + qx_2} + \frac{qx_1 - px_2}{px_2 + qx_1} \Big]
 \end{aligned}$$

$$\begin{aligned}
&= \frac{(x_1 - x_2)(\log x_1 - \log x_2)GB^{p,q}(\mathbf{x})}{n(n-1)} \left[p \sum_{j=3}^n \frac{qx_j}{(px_1 + qx_j)(px_2 + qx_j)} \right. \\
&\quad \left. + q \sum_{i=3}^n \frac{px_i}{(px_i + qx_1)(px_i + qx_2)} + \frac{2pq(x_1 + x_2)}{(px_1 + qx_2)(px_2 + qx_1)} \right] \\
&\geq 0.
\end{aligned}$$

From Proposition 2, we conclude that $GB^{p,q}(\mathbf{x})$ is Schur m -power convex for $\mathbf{x} \in \mathbb{R}_{++}^n$.

The proof of Theorem 1 is completed.

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STUDY ON THE SEQUENCE VOLATILITY OF FINANCIAL ASSETS BASED ON MARKOV CHAIN MONTE CARLO SIMULATION

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Abstract. In recent years, a new issue occurs in the financial academy and business circles, i.e., dynamics of the financial asset price and its volatility model. However, lots of problems in the financial asset price and its volatility model at present have made the motor behaviors of emergencies in the fitting financial market become difficult; whats more, the limitation of parameter estimation on the practical application of models can increase with the increase of model complexity. Estimation of financial temporal models is usually based on classical statistical methods, and the measuring standard of the volatility estimation model is calculated using the actual volatility of low-frequency data. Therefore, taking the price fluctuation of Shanghai Stock Exchange A-share index as an example, this study constructed a model and aimed to analyze the sequence volatility of financial assets based on Markov chain Monte Carlo simulation methods.

Keywords: Financial assets, Markov chain Monte Carlo method, volatility, GARCH Jump model.

1. Introduction

In recent years, under the influence of the increasingly innovating science and technology as well as the developing international financial market, the economic society is becoming more and more global. As a consequence, more and more financial products and the derived product financial market develop constantly, leading to the increasingly enhanced volatility of the financial market. This phenomenon can increase the complexity and risk of the financial market (Chaboud et al., 2010; Tadesse et al., 2016). At present, the most widely applied Bayesian statistical method belongs to the Markov chain Monte Carlo (MCMC) simulation method. On one hand, it considers the uncertainty of probability and predictive parameters, which can provide finite samples with precise inference.

In the meantime, it is efficient and flexible in dealing with complex non-standard parameter models, and it can have efficient statistical inference even under the condition that the parameters are restrained (Plotnikov and Shkarupa, 2015). On the other hand, as the Bayesian statistical method is becoming more and more mature, now there is a specific kind of Bayesian statistical software that is used for inference. At present, lots of researchers in China and abroad have carried out relevant studies as well.

For example, in 2012, Oliveira SC and Andrade MG (Oliveira and Andrade, 2012) introduced and explained how to use the MCMC technology to estimate the model parameters of item response theory (IRT). First of all, the basic principle of using MCMC method to estimate model parameters was introduced; secondly, general methods of using MCMC method to estimate model parameters, including Gibbs sampling, alternative sampling and Metropolis-Hastings algorithm, etc., were introduced; at last, the 2 parameters logistic (2PL) model of IRT was taken as an example to specifically introduce the algorithm process of using the M-H algorithm in Gibbs to estimate item parameters (1j2j). In 2012, Ketter W, Collins J and Gini M, et al. (Ketter et al., 2012) took the Standard Poor 500 share index, one of the most mature stock indexes worldwide, as the research object to astringently test the algorithm process of the construction of standard stochastic volatility model and the fat-tail stochastic volatility model based on Gibbs sampling using MCMC. In 2014, Agdas D, Davidson MT and Ellis RD (Agdas et al., 2014) used hybrid MCMC method to predict the permeability more reasonably according to multi-well test information and based on Bayesian method, thus to provide a new method for numerical simulation of fine reservoir. On the basis of MCMC simulation method, this study took the price fluctuation of Shanghai Stock Exchange A-share index as an example to explore the sequence volatility of financial assets.

2. MCMC method

MCMC method is a special kind of Monte Carlo method. In essence, Monte Carlo integrals are used in the Markoff process in Monte Carlo simulation (Nabok et al., 2011; Lynch, 2014). Besides, the traditional Monte Carlo simulation has the deficiency that only the static simulation is feasible; however, the introduction of the MCMC method can make up such deficiency and realize dynamic simulation. A Markov chain that can be used for enough time of simulation of a specific stable transition distribution $P(\theta|X)$ is constructed and the simulated current values are very close to the stable transition distribution $P(\theta|X)$; then based on the Markov chain, samples of $P(\theta|X)$ are obtained for statistical inference, which is the main idea of the MCMC method. Therefore, it is significantly important to construct a Markov chain that is stable to $P(\theta|X)$ using MCMC method. In following content, the MCMC sampling algorithm which is widely applied is mainly introduced (George et al., 2010; Tian et al., 2010; Ruggeri, 2015).

2.1 Gibbs sampling

The Gibbs sampling method is a special kind of MCMC algorithm as well as a kind of iteration sampling method based on conditional distribution. Therefore, the conditional distribution is required, especially full conditional distribution. The required conditional distribution is something like $\pi(x_T|x_{-T})$, in which $X_T = \{X_i, i \in T\}$, $X_{-T} = \{X_i, i \notin T\}$ and $T \subset N = \{1, \dots, n\}$. All variables are included in the above conditional distributions.

2.2 Distribution conditions of MCMC

The MCMC method is mainly constructed on the conditional distribution $\pi(x_T|x_{-T})$, in which $x_T = \{x_i, i \in T\}$, $x_{-T} = \{x_i, i \notin T\}$, $T \subset N = \{1, 2, \dots, N\}$. The definition of full conditional distribution is that, all variables in distribution have been included in all conditions in distribution. Such kind of distribution is defined as the full conditional distribution (Charalambous et al., 2011; Moeini et al., 2011; Trutschnig and Snchez, 2014; Chakraborty et al., 2012). For any given $x \in \psi$ and $T \in N$, there is

$$(2.1) \quad \pi(x_T|x_{-T}) = \frac{\pi(x)}{\int \pi(x) dx_T} \alpha \pi(x).$$

In the equation, α means there may be a scale factor that is not related to x . Thus in the product term of $\pi(x)$, only the items that are related to are remained. If $x, x' \in \psi$, and $x_T = x_{-T}$, the following equation holds.

$$(2.2) \quad \frac{\pi(x'_T|x'_{-T})}{\pi(x_T|x_{-T})} = \frac{\pi(x')}{\pi(x)}.$$

Generally, if y represents the observed data, $x = (\theta, \phi, z)$; θ is parameter, ϕ is hyperparameter, z refers to missed data, $p(y, (z|\theta))$ refers to the density function of observed data, $\pi(\theta|\phi)$ is prior distribution and $\pi(\phi)$ refers to the distribution of hyper-parameter ϕ ; then $\pi(x)$ can be expressed as $\pi(x|y)$ and $\pi(x|y)\alpha p(y, (z|\theta)\pi(\theta|\phi)\pi(\phi))$.

2.3 Calculation of value at risk using MCMC method

WinBUGS is a new software package developed by the Public Health Institute of University of Cambridge, which is exclusively used in the MCMC method and can have Bayesian statistical inference (King et al., 2011; Zhao et al., 2011; Marano et al., 2014). On the basis of such software, the Gibbs sampling can be easily achieved in common models like hierarchical model, cross-over design model and latent variable model, etc. (Bradlow et al., 2012; Sendjaya et al., 2011; Liaw et al., 2010). Moreover, models can be described more directly through digraphs and then Gibbs sampling dynamic graphs with parameters can be presented. Therefore, the software was selected in this study for the calculation of MCMC method.

3. Empirical study on the return rate of Shanghai Stock Exchange A-share index

3.1 Data source

Data samples of the Shanghai Stock Exchange A-share index were share index closing quotation of every R of the China Securities Index Co., Ltd, with totally 1813 trading days from January 4th, 2008 to June 21st, 2015.

3.2 The calculation method of return rates

The calculation equation of the return rate of Shanghai Stock Exchange Ashare index was:

$$(3.1) \quad R1 = 100(InP_t - InP_{t1})$$

P_t is the share index closing quotation at (t) time point; P_{t1} is the former share index closing quotation at ($t - 1$) time.

3.3 Descriptive statistics of the daily return rate of Shanghai Stock Exchange A-share index

Figure 1 shows that, the average value of daily return rates R1 of the Shanghai Stock Exchange A-share index was 0.052163, indicating that the daily return rates of the Shanghai Stock Exchange A-share index from January 4th, 2008 to June 21st, 2015 were generally JH. The skewness was -0.373126 which was smaller than zero and had left skewness. Kurtosis was 5.556129 which was bigger than 3, indicating that the daily return rates R1 of the Shanghai Stock Exchange A-share index was characterized by peak and fat tail. The value of Jarque-Bera statistics was 535.52196 and the significance probability P was smaller than 0.01, indicating that the daily return rates R1 of the Shanghai Stock Exchange A-share index was not normal distribution (Zellner, 2012; Garca et al., 2010; Vazquezleal et al., 2011).

4. Markov switching autoregressive model of the return rates of Shanghai Stock Exchange A-share index

In order to explore the structural changes of the daily return series of Shanghai Stock Exchange A-share index, the Chow Test was adopted to test whether the linear model of series had structural changes or not. The results of Chow test were below the 5% significance level, thus the linear model of series did have structural changes.

Due to the structural changes of the daily return rate R1, the daily return rate of Shanghai Stock Exchange A-share index was considered from the aspect of Markov switching. It could be confirmed from the observation of R1 curve that, the two states of Markov switching autoregressive model were high return rate and low return rate. The values of state variable were 1 and 2, thus the

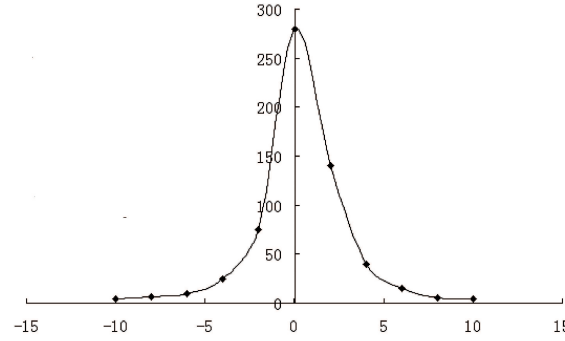


Figure 1: Descriptive statistics of the daily return rates R1 of the Shanghai Stock Exchange A-share index

Parameter	μ	ω	α	β	ν_1	τ	θ	δ^2
Rhat	1.0023	1.0213	1.0012	1.0053	1.0090	1.0090	1.0129	1.0011

Table 1: BGR deviation ratios of each parameter of the model

state variable St transformed between 1 and 2. The maximum likelihood estimation method was adopted in this study to estimate the parameters of Markov switching autoregressive model (Komprej et al., 2013; Zeebari et al., 2012; Murariu et al., 2011). Residual terms were analyzed respectively from the normal distribution and t-distribution aspects. According to the Akaike information criterion (AIC), it was confirmed that the number of lagged ranks was 4.

5. Analysis of GARCH-Jump model using MCMC method

The volatility of Shanghai Stock Exchange A-share index was verified based on the MCMC method. Prior distributions of the parameters of GARCH-Jump model were designed as: $\mu \sim N(0, 1.0E - 6)$, $\omega \sim U(0, 10)$, $\alpha \sim U(0, 0.3)$, $\beta \sim U(0.8, 1)$, $\nu_1 \sim U(0, 1)$, $\tau \sim U(0, 0.5)$, $\theta \sim N(0, 1.0E - 3)$ and $\delta^2 \sim IGamma(4, 0.06)$. Combining with the OpenBUGs software, two-chain setting was carried out to set different initial values to each chain; iterations were increased once and the convergence of model was judged mainly according to the autocorrelation coefficient map and BGR deviation ratio. In this verification, we discovered that it was a convergence model if 12000 times of Gibbs sampling were carried out; and in the former 5000 times of sampling, we adopted the random walk algorithm and took it as the combustion period (Kharol et al., 2012; Sharma et al., 2012); in the latter 7000 times, the Metropolis Hasting (MH) algorithm was used for sampling. Table 1 shows the BGR deviation ratios of each parameter of the model.

After certain times of iterations of the two Markov chains of each parameter, the autocorrelation coefficient approached to zero rapidly. In addition, according to table 1, we could easily find that the deviation ratio of each parameter was very close to 1. Therefore, we could regard that the model had achieved convergence. The posterior distribution images of each parameter were shown in figure 2.

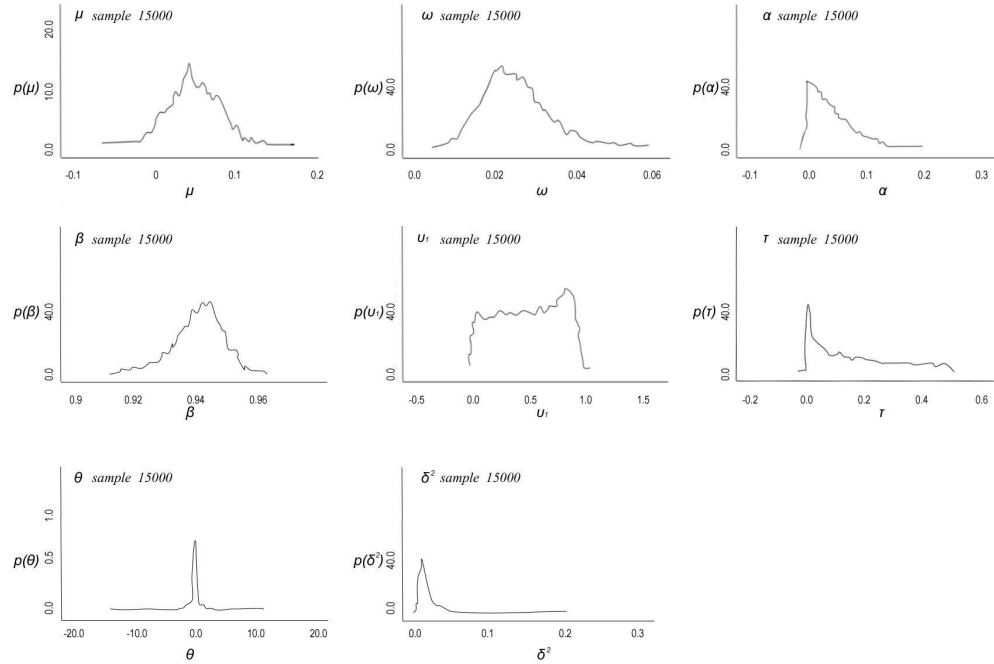


Figure 2: Posterior distribution images of each parameter

The experimental verification showed that, the four kinds of loss function values calculated based on ML algorithm were bigger than function values calculated using the MCMC method. Thus we could infer that, the models constructed using the MCMC method had better results. In order to further verify whether the loss function indexes had significant differences in these two kinds of methods, we used Durbin-Watson (DW) statistics to further verify these loss function indexes. In the meantime, we adopted relevant software to calculate DM values and other relevant values, as shown in table 2. This study showed that, the Shanghai Stock Exchange A-share index was very sensitive to the volatility of return rates, which showed significant jumping characteristic. Because the total volatility was the sum of smooth volatility h_1 and jumping volatility $\sigma_{t,2}^2$, according to estimation results of parameters of the GARCH-Jump model, it was calculated that the jump volatility accounted for 58% of the total volatility. Meanwhile, because $E(\lambda_t) = \tau * \omega / (1 - \alpha - \beta) / (1 - v_1)$, it

Loss function indexes	DW statistics	P value
MSE	-3.1621	0.0015
QLIKE	-27.0921	$< 2.2e - 16$
MAE	-13.9416	$< 2.2e - 16$
R2LN	-34.1524	$< 2.2e - 16$

Table 2: DW values and some relevant values

was calculated that the average unconditional jump intensity was 0.4259, which verified that we could not ignore the jumping characteristic and the jumping volatility of stocks.

6. Conclusion

We discover from the experimental verification that, the GARCH-Jump model estimated based on MCMC method has better results. It can better simulate the volatility characteristics of Shanghai Stock Exchange A-share index. Abnormal fluctuation that occurs to stock market under the effects of events will last for a period and would not disappear in a short term. Stronger persistence of the fluctuation indicates the stronger capability of the predicted GARCH-Jump model in capturing the sequence information of financial events based on MCMC method and higher accuracy in describing the fat tail of the sequence of return on assets. The fluctuation of yield rate of single asset was fitted by a model, and the structural relationship between assets was described by Copula function. The risk value of asset portfolio was obtained based on the above results. Then VaR value was calculated using MCMC method. The risks were estimated, and the optimal decision was made to guide investors in the theoretical level. However, due to the lack of resources and conditions during experimental verification and data acquisition, this study still has some deficiencies to be improved in the future.

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OPTIMAL MATHEMATICAL MODEL OF DELIVERY ROUTING AND PROCESSING TIME OF LOGISTICS

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Abstract. As a newly-developing service industry that has a broad prospect and enormous market potential under the environment of electronic commerce, logistics is characterized by intellectualization, exhibility, informatization, networking and automation, etc. However, with the rapid development of economy, clients requirements on logistics services are increasing gradually. Therefore, how to reasonably control delivery costs while still maintaining the high-standard quality and timeliness has become a new challenge for logistics. On the basis of the earliest and latest service time allowed by clients, the problem of time window constraints should be especially considered for the distribution problems of multiple batches, multiple species and small batch quantity. Due to the practical significance of studying the routing problem of vehicles with time windows, this study explored the key optimization problem of logistics as well as the problem of distribution routing selection and processing time according to characteristics of logistics under the environment of electronic commerce, using ant colony algorithm (ACA) and particle swarm optimization (PSO).

Keywords: Logistics, route optimization, mathematical model, ant colony algorithm, particle swarm optimization.

1. Introduction

With the popularization of Internet in peoples daily life as well as the increasing development of scientific technology, electronic commerce has become a more and more important part in our life. As the key of electronic commerce, commodity delivery is also developing vigorously [1-2]. The key of upgrading the level of the whole logistics industry is to improve the delivery efficiency and thus reduce delivery costs using scientific and reasonable delivery schemes. Functions of logistics include various contents, but the core is article delivery. Delivery refers to products delivery and collection according to requirements of target clients. Delivery flow includes products collection, subpackage, transportation and distribution. In order to provide the best services for clients as well as reduce delivery costs as much as possible at the same time, scheduling and optimization of delivery vehicles has been selected as the research emphasis [3-5].

Ant colony algorithm (ACA) was initially adopted to solve the famous Chinese traveling salesman problem (TSP) [6]. However, with the development of researches carried out by scientific researchers all over the world, ACA was gradually applied to other industries [7-8]; moreover, ACA has transformed from dealing with one-dimensional static optimization problems to multi-dimensional dynamic combination optimization problems, as well as from exploring in discrete type range to exploring in successive type range. Due to great changes of ACA in application spaces, the latest evolutionary optimization algorithm of ACA showed vigorous growth potential [9-10]. At present, the algorithm has been extensively applied to scheduling problems, network route problems, vehicle routing, robot field, image processing, electrical power system, data mining, fault diagnosis and controlling parameter optimization, etc. [11]. After the continuous improvement of worldwide academic researches in several years, particle swarm optimization (PSO) has achieved satisfactory results in dealing with optimization solutions of most continuous problems. During years of improvement, PSO has been successively introduced to a wide range of fields, such as biological medicine, combination optimization, communication network, control engineering, electron and electromagnetic field, finance, image processing, task scheduling and other engineering application problems [12-15].

On the basis of ACA, PSO and ACA combined PSO mixed algorithm, this study analyzed and solved the vehicle scheduling model with time windows, and results showed that the mixed algorithm was superior to single algorithm.

2. Construction of vehicle routing problem with time windows model

Fundamental assumptions

In order to abstract vehicle routing problem with time window (VRPTW) constraints [16] in reality as a mathematical model, following fundamental assumptions were established in this study: products delivery was one-way, i.e., pure products delivery; both the starting point and terminal point of every delivery routine should be the distribution center and there was only one distribution center; position coordinates of distribution center and clients points were known; vehicles types were the same and the maximum capacity of the vehicles was known; product weight should not exceed the rated loading capacity of the vehicle; demanded quantity of each client was known; each client was visited once and only once; each client had a specific service time window, thus the arrival time of vehicles should be within this time range; one vehicle for one routine; all routines were unhampered.

Penalty function

Former researches on delivery problems of vehicles with time window constraints only focused on the delivery costs. But in fact, violation of time window constraints of clients can result in certain economic losses which can be regarded

as the time cost effect caused by violating time window constraints of clients. Therefore, while pursuing the minimization of overall costs, the time-effect costs should also be considered [24].

Costs caused by violation of time window should be expressed in an appropriate way to avoid neglecting intangible costs. Therefore, it can be regarded as the penalty cost function of the distribution center violating time window of clients, i.e., using functions to simulate time windows of clients. Reasonable penalty costs can lower costs of the distribution center as well as guarantee the service quality at the same time; usually the limit of penalty costs is determined by the balance between costs and clients degree of satisfactory. In another word, the more the arrival time window of vehicle deviating the time window constraint, the higher the penalty costs. To simplify the problem, we assume that penalty costs increase linearly.

Therefore, before the determination of penalty cost function, the presumption and hypothesis of penalty cost were as follows: penalty cost is expressed in function; the distribution center does not need to pay any penalty cost if the vehicle serves within the time window; the smaller the width of time window, the higher the marginal effect of penalty costs; the penalty cost increases linearly with the increase of penalty degree whether the vehicle arrives early or late. Function expression of penalty cost is as follow:

$$(2.1) \quad p_i(s_i) = \begin{cases} p(e_i - s_i), & s_i < e_i \\ 0, & e_i \leq s_i \leq l_i \\ q(s_i - l_i), & s_i > l_i. \end{cases}$$

The above function can also be unified as:

$$(2.2) \quad p_i(s_i) = p \max(e_i - s_i, 0) + q \max(s_i - l_i, 0).$$

In above equations, e_i refers to the starting point of service time window allowed by client i ; l_i refers to the end point of client allowed service time window; s_i is the arrival time of the vehicle at client i ; p refers to the waiting opportunity cost of unit time of the vehicle arriving at the location of client in advance; q means the penalty cost of unit time of vehicle arriving late than time window of client. If vehicle k arrives at the client node i before the time window e_i and the vehicle waits there, then the loss of opportunity cost is $p(e_i - s_i)$; if the vehicle arrives late than the time window and the service is delayed, then the delay cost is $q(s_i - l_i)$; if the vehicle arrives within the time window $[e_i, l_i]$, then the time effect cost is 0. Values of p and q are determined by practical situations. For some important clients or clients who have high requirements on time, then values of p and q can be big. Hard time window can be regarded as the soft time window, i.e., when the value of p and q is the infinitely-great positive number M , the soft time window penalty function can be corrected as hard time window and its penalty function expression is:

$$(2.3) \quad p_i(s_i) = M \max(e_i - s_i, 0) + M \max(s_i - l_i, 0).$$

Mathematical model

Suppose a distribution center provides delivery services for n clients and the client set was $C = \{1, 2, \dots, n\}$; the demanded quantity of each client was $q_i (i \in C)$; the distribution center distributed m same type vehicles and the vehicle set is $V = \{1, 2, \dots, n\}$; loading capacity is Q ; d_{ij} is the distance between client i and j ; P refers to the unit delivery cost on the distribution route; t_i is the time of vehicle arriving at client i ; the number of clients served by the k th vehicle is N_k ; the least number of clients served by each vehicle is p , and r_k^q refers to the serial number of the q th client in the client set served by the k th vehicle (r_0^k refers to the serial number of the distribution center); L_i refers to the delayed time of vehicle served for client p and P_1 is the late penalty factor; E_i is the advanced time of vehicle serving client p and P_i is the penalty factor of early wait; thus the mathematical model of VRPTW is:

$$(2.4) \quad F = \min \sum_{k=1}^m [P \cdot d_{r_k^0 r_k^1} + P_1 \cdot L_{r_k^1} + P_2 \cdot E_{R_k^1} + \sum_{i=1}^{N_k} \sum_{j=1}^{N_k} x_{ijk} (P \cdot d_{r_k^i r_k^j} + P_1 L_{r_k^i} + P \cdot d_{r_k^{N_k} r_k^0} + P_1 \cdot L_{r_k^0})].$$

Constraint conditions are:

$$(2.5) \quad \sum_{i=1}^{N_k} q_{r_k^i} \leq Q$$

$$(2.6) \quad \sum_{k=1}^m \sum_{i=0}^{N_k} x_{ijk} = 1$$

$$(2.7) \quad \sum_{k=1}^m \sum_{j=0}^{N_k} x_{ijk} = 1$$

$$(2.8) \quad p \leq N_k \leq n$$

$$(2.9) \quad x_{ijk} = \begin{cases} 1, & \text{Vehicle } k \text{ from } r_k^i \text{ to } r_k^i \\ 0, & \text{others} \end{cases}$$

$$(2.10) \quad L_i = \begin{cases} t_i - b_i, & \text{If } t_i > b_i \\ 0, & \text{or} \end{cases}$$

$$(2.11) \quad E_i = \begin{cases} a_i - t_i, & \text{if } f a_i > t_i \\ 0, & \text{or} \end{cases}$$

$$(2.12) \quad k = 1, 2, \dots, m.$$

Vehicle k from r_i^k to r_j^k Equation (4) is the objective function of minimum delivery cost; equation (5) presents loading capacity constraint of the vehicle; equation (6) and (7) indicate that, every vehicle starts from the distribution

center and each client is served by one vehicle and only for once; all vehicles returns to the distribution center after delivery; equation (8) shows the minimum number of clients served by each vehicle, which should not exceed the overall number of clients; equation (9), (10) and (11) are all decision variables.

3. Swarm intelligence algorithm

In VRPTW mathematical model which takes the minimum total distribution cost as the objective function, the penalty function with regard to the violation of client time window give consideration to the waiting opportunity cost generated because of the early arrival of vehicles and the delay penalty cost generated because of delayed arrival. In VRPTW model, besides the limitation of VRP, time window limits for demand points also need to be satisfied. With the increase of the number of clients and the complexification of constraint conditions, solution becomes increasingly complicated. The traditional accurate algorithm is not quite suitable to the solution of VRPTW as the calculated quantity will be in exponential growth with the increase of issue scale. Swarm intelligence algorithm suggests favorable search effect in discrete and continuous solution; routine selection mechanisms established through swarm intelligence behaviors can make the algorithmic search approximate to the optimal solution in the highest speed; it has become the research hotspot in the solution to VRPTW for its advantages of absence of centralized control, multi-agent mechanism, simple algorithm structure, implied parallelism, easy understanding and easy implementation. Ant colony algorithm and particle swarm optimization among swarm intelligence algorithm have high application values and development potentials.

Vehicle routing problem (VRP) is an important content of logistics system research, which can be transformed to TSP problem according to vehicle routing model. ACA shows significant achievements in dealing with famous problems like TSP; however, when it comes to large-scale problems, disadvantages like slow rate of convergence and long consumed time may occur.

PSO is a global optimizing algorithm which has following advantages [17-18]: ability of global research in a wide range; ability of researching from the colony, which has good stability; evaluation function values are used for inspiration while searching; fast rate of convergence and simple parameter adjustment; it has good extendibility which can be combined with other algorithms. Disadvantages of PSO include insufficient usage of feedback information and poor local search ability at later stage of algorithm.

This study analyzed and solved the constructed vehicle scheduling model with time window using ACA, PSO and ACA combined PSO algorithm.

Ant colony algorithm ACA is a kind of heuristic algorithm of swarm intelligence, which was put forward by Dorigo in his doctoral thesis in 1991 and applied to planning of delivery vehicle routing. ACA is a kind of bionics algorithm and its basic principle is to simulate the foraging of a colony of ants:

during the process of foraging, ants release certain amount of pheromone from their nest to the location where food is discovered as a kind of mark, which can be used by following ants to bypass obstacles to find better foraging routes; the disadvantage is that the algorithm may be caught in local optimum [19-20].

Basic steps of ACA are as follows:

The first step is the initialization of parameters. Initial pheromones are set between client routes.

The second step is to distribute ants randomly to different nodes and the current positions of ants are added to the search tabu table $tabu_k$.

The third step is to calculate the next client according to the equation and thus to determine the next client j .

Step four is to judge whether the loading capacity of the vehicle overpasses the maximum loading capacity; if so, step five is carried out; if not the third step is repeated.

Step five is to judge whether the service arrival time meets the requirements of clients; if so, j is added to $tabu_k$ and costs from position i to position j is calculated; if not, then j is added to $tabu_k$ and costs from position i to position j is calculated, including time penalty costs.

In step six, after all ants have moved once, new pheromones on each route are adjusted according to local pheromone adjusting formula.

Step seven is to check whether the tabu table is completely filled, i.e., whether all points are covered; if so, step 9 is carried out; if not, step three is repeated. Step seven is to check whether the tabu table $tabu_k$ is completely filled, i.e., whether all points are covered; if so, step 9 is carried out; if not, step three is repeated.

In step eight, the optimal path is updated.

In step nine, conditions of circulation termination are judged. If the number of circulation times is equal to or greater than the maximum iterations, the circulation is terminated and the optimal route and costs are output; if not, the tabu table $tabu_k$ is reset and the whole process is repeated from step two.

Particle swarm optimization

Inspired by foraging behaviors of birds, Kennedy and Eberhart in America put forward the PSO theory in 1995. The basic model of PSO is similar to the genetic algorithm and PSO is a kind of optimizing tool based on iteration. Compared with the genetic algorithm, particles can be updated by following two extreme values of particles in PSO and the optimal particle is obtained in iterative process to find the global optimum; however, in genetic algorithm, the optimal values can only be obtained through multi-step operation and multiple iterations such as crossover and variation. Therefore, the advantages of using PSO to solve VRP are that fewer parameters are considered and the algorithm is much simpler, etc. [21-23].

Steps based on PSO algorithm are as follows: In step one, particle position x_i^0 and v_i^0 in n -dimensional space are initialized and the number of iterations is specified.

Every particle is evaluated in step two. Current adaptive values p_i of all particles are obtained according to the fitness function.

In step 3, the obtained adaptive value p_i in step two was compared with the past optimal adaptive value $pBest$; if $p < pBest$, then the history optimal solution value of particle was p_i ; thus the former particle is replaced by the new particle, i.e., $p = pBest$.

In step 4, the history optimal adaptive values $pBest$ of every particle are compared with the optimal value $gBest$ of the whole particle swarm. If $gBest > pBest$, then the former global optimum adaptive value should be replaced by the optimal adaptive value of the new particle; meanwhile, the up-to-date state of the particle is recorded, i.e., $gBest = pBest$.

After the latest speed and position of the particle is calculated according to the adjusting formula, the particle is moved to a new position, thus a new particle comes into being. If the circulation times exceed the maximum iterations in step one or are lower than the expected convergence precision, then the algorithm running is ended and the final solution is output; or the whole process is repeated from step two.

ACA combined PSO

The initial pheromone information of ACA is random, thus there is no reasonable set; moreover, ACA has a slow rate of convergence. In PSO algorithm, when a local optimal solution is obtained after multiple times of circulation calculation of particles, speed change of particles mainly depends on the current flying speed of particles. In order to improve disadvantages of ACA and PSO, ants accomplish traversal once according to ACA, then appropriate adjustment is accomplished according to locally optimal solution and globally optimal solution [25-27]. Specific steps of mixed algorithm are as follows:

Step one: $nc \leftarrow 0$; refers to the number of iterations; after initialization, a large number of routes are generated and some good ones are chosen out of them; pheromones of those routes are left and m ants are placed at n peaks.

Step two: according to the current position, adaptive value $Its_p 0$ is calculated; suppose the current adaptive value is individual extremum $Ptbest$ and the current position is individual extremum position $pcbest$; global extremum $bgtest$ and global extremum position $gcbest$ are discovered according to individual extremum $Ptbest$ of each particle.

Step three: the initial starting point of each ant $k(k = 1, 2, \dots, m)$ is placed in the current solution set; each ant can move to the next peak j according to the probability p_{ij}^k and the peak j is placed in the current solution set.

Step four: following operations are carried out for each ant. The j -th ant route $Co(j)$ intersects with $gcbest$ to obtain $C'_1(j)$; $C'_1(j)$ interacts with $pcbest$ to obtain $C''_1(j)$ and $C''_1(j)$ mutates into $C_1(j)$; route length $Its_p 1$ is calculated

according to current position; if the new target function is good, then the new value is accepted; otherwise, the route $C_1(j)$ of the j -th particle route changes to $Co(j)$. Individual extremum $ptbest$ and extremum position of each ant are discovered again to find the global extremum $gbest$ and global extremum position $gcbest$.

Step five: route length $Lk(k = 1, 2, \dots, m)$ of each ant is calculated and the optimal solution is recorded.

Step six: route length that is smaller than the set value is corrected according to the equation $\Delta\tau_{ij}(t) = \sum_{k=1}^m \Delta\tau_{ij}^k(t)$.

Step seven: $nc \leftarrow nc + 1$.

Step eight: if $nc < \text{predetermined iterations}$ and has no degradation behavior, i.e., the discovered solutions are all the same, then the procedure returns back to step 2.

Step nine: the current optimal solution is output.

Comparison of simulation results

Taking ei151, st70 and wi176 as example, matrix laboratory (MATLAB) is used to verify the effectiveness of algorithms. First, the second-best solution is obtained after 20 iterations using PSO; then based on the route length of the secondbest solution, distribution of initial pheromone in ACA is obtained according to $\tau_S = \tau_\varepsilon + \tau_\rho$; in ACA, $\rho = 0.02$, the number of ants is equal to the number of cities and $\alpha = 1.0$, $\beta = 5.0$ and $T_{min} = 0.0001$. Table 1 shows the comparison of solving capacity and time efficiency of PSO combined ant swarm (PSOAS) algorithm and basic ACA.

	The known optimal solution	ACA		PSO		PSOAS	
		The shortest route	Evolution algebra	The shortest route	Evolution algebra	The shortest route	Evolution algebra
ei151	426	429	823	430	152	429	152
st70	675	678	1034	679	235	678	235
ei176	538	545	1322	546	281	545	281

Table 1. Comparison of simulation experiment results.

4. Conclusion

To solve the optimal distribution route issue in logistic process, this study established VRPTW model and designed the PSOAS algorithm based on ACA and PSO and performed analysis and solution. The algorithm combined the advantages of both ACA and PSO. The simulation experiment suggested that, PSOAS algorithm was advantageous in the solution to VRPTW; it was superior to ant colony algorithm in the aspect of time efficiency and to particle swarm algorithm in the aspect of optimizing capability. But it also has deficiencies. For

example, its universality is weak. More theoretical analyses are required to improve the performance and applicability of the algorithm and further investigate the optimization of distribution route involving multiple distribution centers and vehicle types. In conclusion, PSOAS algorithm is a brand-new heuristic algorithm which is of great significance to solve the route selection and distribution time issues in the field of logistics.

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ON SOFT LA-MODULES AND EXACT SEQUENCES

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Abstract. We introduce the concepts of soft LA-modules, Soft homomorphisms and the exact sequences of soft LA-modules. Some properties concerning the exact sequences of the LA-modules are investigated. A functional approach to study the soft LA-modules is adopted so that a characterization theorem of soft LA-modules is obtained.

Keywords: Logistics, route optimization, mathematical model, ant colony algorithm, particle swarm optimization.

1. Introduction

To deal with uncertainties, though many theories have been developed, yet difficulties are seen to be there. In order to cope with these uncertainties, the theory of soft sets was first proposed by D. Molodtsov [12] in 1999. Nowadays, the theory of soft sets has been successfully applied into a number of mathematical disciplines and scientific fields.

We notice that P. K. Maji, R. Biswas and R. Roy [10], [11] have recently applied the concept soft sets in decision making problems. We also observe that soft set parameterization reduction and a comparison of soft sets with attributes reduction in rough set theory was given by Chen [5]. Later on, A. Sezgin et al., [16], introduced the union soft subnear-rings and union soft ideals of a near-ring.

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H. Aktas and N. Çağman [2], first applied the theory of soft sets in algebraic structures. In addition, X. Liu et al., [8], established some fuzzy isomorphism theorems of soft rings. They also considered the fuzzy ideals of soft rings. In [9], X. Liu et al., have provided the proof of isomorphism theorems of soft rings. In [24], Q. M. Sun et al., have discussed and investigated some of its basic properties of soft modules.

The Left Almost Ring (LA-ring) is actually an off shoot of LA-semigroup and LA-group. In fact, an LA-ring is a non-commutative and non-associative algebraic structure. Due to its atypical characteristics, the class of LA-rings has been emerging as a useful non-associative class which intuitively would have reasonable contribution to enhance non-associative ring theory. By an LA-ring, we mean a non-empty set R together two binary operations “+” and “.”, such that under “+” it is an LA-group and under “.” it is an LA-semigroup and distributive laws hold both from left and right .

In [21], T. Shah et al., generalized a commutative semigroup ring and established several basic results of left almost ring (LA-ring) of finitely nonzero functions. Moreover, T. Shah and I. Rehman [22], have considered ideals in LA-rings. Consequently, the theory of ideals of LA-rings was applied in fuzzy sets. Moreover, T. Shah et al., also considered intuitionistics fuzzy sets and soft sets in LA-rings. For example, T. Shah et al., [19], have applied the concept of intuitionistic fuzzy sets and established some useful results. In [15], some computational research through Mace4, has been obtained and some interesting characteristics of LA-rings are explored. It is noted that in [17], T. Shah et al., have introduced a new approach to apply the Molodtsov’s soft set theory to a class of non-associative rings. And in [18], T. Shah and Asima Razzaque have further discussed some basic properties regarding soft M-systems, soft P-systems and soft I-systems in a non-associative left almost rings. T. Shah et al., [20], have promoted the concept of LA-modules and establish some isomorphism theorems and the direct sum of LA-modules. For more information of LA-rings, the readers are referred to the papers ([14], [23]).

In this paper, we first consider the soft LA-modules. Then we established some results of substructures of soft LA-modules, homomorphisms and finally, we consider exact sequence of soft LA-modules and provide some useful results of exact sequences.

2. Basic definitions and some preliminary results

In this section, we first cite some basic definitions which are closely related with the soft sets and LA-modules.

Definition 1. [12, Def: 2.1] Suppose that U be an initial universe. Let E be a set of parameters. Then we denote the power set of U by $P(U)$ and a non-empty subset α of E . We now call a pair (F, α) a soft set over an initial universe U , where F is a mapping given by $F : \alpha \rightarrow P(U)$.

Now, we call a soft set over the initial universe U , a parametrized family of subsets of the universe U . For $\varepsilon \in \alpha$, $F(\varepsilon)$ may be considered as the set of ε -approximate elements of the soft set (F, α) . Clearly, a soft set is not a set.

Definition 2. [13, Def: 2] Let (F, α) and (G, β) be the two non-empty soft sets over a common universe U . Then we call that (F, α) is a soft subset of (G, β) if

- (i) $\alpha \subseteq \beta$ and
- (ii) for all $\varepsilon \in \alpha$, $F(\varepsilon) \subseteq G(\varepsilon)$.

The above soft sets can be written as $(F, \alpha) \widetilde{\subseteq} (G, \alpha)$. If a soft set (G, β) is a soft subset of a soft set (F, α) . Then (F, α) is said to be a soft super set of (G, β) and it can be defined as $(F, \alpha) \widetilde{\supseteq} (G, \beta)$.

Definition 3. [11, Def: 2.4] Suppose that (F, α) and (G, β) are two non-empty soft sets over a common universe U . Then these soft sets are said to be soft equal if $(F, \alpha) \widetilde{\subseteq} (G, \beta)$ and $(G, \beta) \widetilde{\subseteq} (F, \alpha)$.

Definition 4. [11, Def: 2.7] A soft set (F, A) over an initial universe U is said to be a NULL soft set and is denoted by Φ if for every $\varepsilon \in \alpha$, $F(\varepsilon) = \emptyset$ (null set).

Definition 5. [1, Def: 2.4] (F, α) and (G, β) be two soft sets over a common universe U . Then the bi-intersection of these two soft sets (F, α) and (G, β) is defined as the soft set (K, γ) which satisfies the following conditions:

- (i) $\gamma = \alpha \cap \beta$ and
- (ii) for every $x \in \gamma$, $K(x) = F(x) \cap G(x)$.

Now we denote the bi-intersection of these two soft sets by $(F, \alpha) \widetilde{\cap} (G, \beta) = (K, \gamma)$.

Definition 6. [6, Def: 3.5] Let $(F_i, \alpha_i)_{i \in I}$ be a non-empty family of soft sets over a common universe U . Then the bi-intersection of the family of soft sets is defined to be the soft set (K, γ) , such that $\gamma = \cap_{i \in I} \alpha_i$, and $H(x) = \cap_{i \in I} F_i(x)$ for all $x \in \gamma$. This intersection can be written as $\widetilde{\cap}_{i \in I} (F_i, \alpha_i) = (K, \gamma)$.

Definition 7. [1, Def: 2.7] Consider two non-null soft sets (F, α) and (G, β) over a common universe U , then “AND” of these two soft sets is denoted by $(F, \alpha) \widetilde{\wedge} (G, \beta)$. It is defined as $(K, \gamma) = (F, \alpha) \widetilde{\wedge} (G, \beta)$, where $\gamma = \alpha \times \beta$ and for all $(x, y) \in \gamma$, $K(x, y) = F(x) \cap G(y)$.

Definition 8. [1, Def: 2.3] Let (F, α) and (G, β) be two non-empty soft sets over a common universe U . Then the intersection of (F, α) and (G, β) is defined as the soft set (K, γ) satisfying the following conditions:

- (i) $\gamma = \alpha \cap \beta$ and
- (ii) for all $x \in \gamma$, $H(x) = F(x)$ or $G(x)$.

Hence the intersection of two soft sets (F, α) and (G, β) can be written as $(K, \gamma) = (F, \alpha) \widetilde{\cap} (G, \beta)$.

Definition 9. [1, Def: 2.5] Suppose that (F, α) and (G, β) be two non-empty soft sets over U . Then the union of these two soft sets is defined as (K, γ) which is also a soft set, satisfying the following conditions:

$$\begin{aligned} (i) \quad & \gamma = \alpha \cup \beta \text{ and} \\ (ii) \quad & \text{for all } x \in \gamma, \\ & K(x) = \begin{cases} F(x), & \text{if } x \in \alpha - \beta, \\ G(x), & \text{if } x \in \beta - \alpha, \\ F(x) \cup G(x), & \text{if } x \in \alpha \cap \beta. \end{cases} \end{aligned}$$

The above union of two soft sets can be written as $(F, \alpha) \widetilde{\cup} (G, \beta) = (K, \gamma)$.

Definition 10. [1, Def: 2.9] If (F, α) is a soft set over a universe U , then the set $\text{supp}(F, \alpha) = \{\varepsilon \in \alpha \mid F(\varepsilon) \neq \phi\}$ is called the support of the soft set (F, α) . We now call a soft set to be non null if its support is not equal to the empty set.

Definition 11. [21, Def: 1] Consider an LA-ring $(R, +, \cdot)$ with left identity e . An LA-group $(M, +)$ is said to be an LA-module over R if $R \times M \rightarrow M$ defined as $(a, n) \mapsto an \in M$, where $a \in R$, $n \in M$ satisfies the following conditions:

- (i) $(a + b)n = an + bn$,
- (ii) $a(m + n) = am + an$,
- (iii) $a(bn) = b(an)$,
- (iv) $1.n = n$,

where for all $a, b \in R$, $m, n \in M$.

${}_R M$ is used for left R -LA-module or simply M . Right R -LA-module can be defined in similar manner and is denoted by M_R .

Definition 12. [20, Def: 2] Consider a left R -LA-module M , a sub LA-group N of M over an LA-ring R is called left R -sub LA-module of M , if $RN \subseteq N$, i.e., $rn \in N$ for all $n \in N$ and $r \in R$. We denote this sub LA-module by $N \leq M$.

Definition 13. [20, Def: 3] Let M be an LA-module and $A \subset M$ is a sub LA-module. We define quotient module or factor module M/A by $M/A = \{A + m : m \in M\}$. That is, M/A is the set of equivalence classes of elements of M . An equivalence class is denoted by $A + m$ or by $[m]$. Each element in the class $A + m$ is called a representative of the class.

For the intersection of LA-modules, sums, products of modules, the Jacobson radical of modules and the LA-module homomorphisms, we have the following results and definitions.

Theorem 1. [20, Theorem: 2] If A and B are two sub LA-modules of an LA-module M over an LA-ring R , then the intersection i.e., $A \cap B$ is also a sub LA-module of M .

Corollary 1. *Intersection of any number of sub LA-modules of an LA-module is a sub LA-module.*

Definition 14. [24, Def: 7] *Consider a non-empty family of R-modules i.e., $\{M_i \mid i \in I\}$. Then $P = \prod_{i \in I} M_i = \{(m_i) \mid m_i \in M_i\}$ is a set of direct product, if $(m_i) + (n_i) = (m_i + n_i)$ and $r(m_i) = (rm_i)$ are the operators given on the product, then P induces $\{M_i \mid i \in I\}$, which is denoted by $\prod_{i \in I} M_i$, a left R-module structure called direct product.*

Proposition 1. [24, Prop: 2] *Consider a non-empty family of sub modules $\{M_i \mid i \in I\}$ of M . Then $\bigcap_{i \in I} M_i$ and $\sum_{i \in I} M_i$ are all sub modules of M .*

Definition 15. [24, Def: 8] *In the direct product $\prod_{i \in I} M_i$ all elements (m_i) , where m_i is zero for almost all $i \in I$ except finite one, a sub module of $\prod_{i \in I} M_i$ is said to be direct sum of $\{M_i \mid i \in I\}$, which can be written as $\prod_{i \in I} M_i$ or $\bigoplus_{i \in I} M_i$.*

Definition 16. [20, Def: 4(a)] *Let M, N be the LA-modules over an LA-ring R . Then the map $\varphi : M \longrightarrow N$ is called an LA-module homomorphism(or simply R-homomorphism) if, for all $r \in R$ and $n, m \in M$*

- (i) $\varphi(n + m) = \varphi(n) + \varphi(m)$
- (ii) $\varphi(rn) = r\varphi(n)$

Theorem 2. [20, Theorem: 5] *Let $\varphi : M \longrightarrow N$ be an LA-module homomorphism from an LA-module M to an LA-module N , then*

- (1) *If A is a sub LA-module of M , then $\varphi(A)$ is a sub LA-module of N .*
- (2) *If B is a sub LA-module of N , then $\varphi^{-1}(B)$ is a sub LA-module of M .*

Lemma 1. [20, Lemma: 2] *With the canonical operations, by choosing representatives, $(A+n) + (A+m) = A + (n+m)$, the set M/A is an LA-group. A , the equivalence class of $0 \in M$ is the left identity of M/A . The map $\pi : M \longrightarrow M/A$, $\pi(n) = A + n$ is a surjective LA-group homomorphism.*

Definition 17. [24, Def: 6] *Consider a non-empty family of sub modules $\{M_i \mid i \in I\}$ of M . Then we the following statements:*

- (i) $\bigcap_{i \in I} M_i$ is a sub module of M if $\{M_i \mid i \in I\}$ be a non-empty family of maximal sub modules of M . Then it is called Jacobson radical of the module. It is written as $\text{rad}M$.
- (ii) $\sum_{i \in I} M_i$ is a sub module of M if $\{M_i \mid i \in I\}$ be a non-empty family of minimal sub modules of M . Then it is said to be Socle of module and is written as $\text{soc}M$.

Definition 18. [24, Def: 9] *A sequence of R-homomorphisms and R-modules $\cdots \longrightarrow P_{n-1} \xrightarrow{f_{n-1}} P_n \xrightarrow{f_n} P_{n+1} \longrightarrow \cdots$ is said be an exact sequence if $\text{Im} f_{n-1} = \ker f_n$ for all $n \in \mathbb{N}$. An exact sequence of the form $0 \longrightarrow P' \xrightarrow{f} P \xrightarrow{g} P'' \longrightarrow 0$ is said to be a short exact sequence.*

Proposition 2. [25, 51] Let $f : M \longrightarrow N$ be R -homomorphism for R -modules M and N . Then we have the following properties:

- (1) $0 \longrightarrow M \xrightarrow{f} N$ is exact if and only if f is monomorphism.
- (2) $M \xrightarrow{f} N \longrightarrow 0$ is exact if and only if f is an epimorphism.
- (3) $0 \longrightarrow M \xrightarrow{f} N \longrightarrow 0$ is exact if and only if f is an isomorphism.

3. Soft LA-modules and their related properties

In this section, we study the properties of the Soft LA-modules. We first state the following definitions and results.

Definition 19. Consider a soft set (F, α) over a left R -LA module M . The pair (F, α) is said to be soft LA-module over M if $F(a)$ is a sub LA-module of M for all $a \in \alpha$. It can be written as $F(\varepsilon) \leq M$.

Throughout this paper, M is considered to be a left R -LA module, and α be any non-empty set. The pair (F, α) refers to a soft set over M and $F : \alpha \longrightarrow P(M)$ considered to be a set valued function.

Proposition 3. Consider two soft LA-modules (F, α) and (G, β) over M . Then the following statements hold:

- (1) $(F, \alpha) \widetilde{\cap} (G, \beta)$ is a soft LA-module over M .
- (2) if α and β are disjoint i.e., $A \cap B = \phi$ then $(F, \alpha) \widetilde{\cup} (G, \beta)$ is a soft LA-module over M

Proof. (1) By definition 8, $(F, \alpha) \widetilde{\cap} (G, \beta) = (K, \gamma)$ is a soft set over soft LA-module M , where $\alpha \cap \beta = \gamma$. Since (F, α) and (G, β) are two soft LA-modules over M , so the $K(x) = F(x) \leq M$ and also $K(x) = G(x) \leq M$ for every $x \in \gamma$. Therefore, it follows that $(F, \alpha) \widetilde{\cap} (G, \beta)$ is a soft LA-module over M .

(2) Using the definition 9, we can write $(F, \alpha) \widetilde{\cup} (G, \beta) = (K, \gamma)$ which is a soft set, where $\gamma = \alpha \cup \beta$ and for every $x \in \gamma$,

$$K(x) = \begin{cases} F(x), & \text{if } x \in \alpha - \beta, \\ G(x), & \text{if } x \in \beta - \alpha, \\ F(x) \cup G(x), & \text{if } x \in \alpha \cap \beta. \end{cases}$$

Given that α and β are disjoint, therefore either $x \in \alpha - \beta$ or $x \in \beta - \alpha$. If $x \in \alpha - \beta$, then $K(x) = F(x) \leq M$ as (F, α) is a soft LA-module. Similarly if $x \in \beta - \alpha$, then $K(x) = G(x) \leq M$ as (G, β) is a soft LA-module. Hence $(F, \alpha) \widetilde{\cup} (G, \beta) = (K, \gamma)$ is a soft LA-module over M . $\square$

Theorem 3. Let (F, α) and (G, β) be two soft LA-modules over M . Then the following conditions hold:

- (1) If $(F, \alpha) \widetilde{\cap} (G, \beta)$ is non-null then is a soft LA-module over M .
- (2) If the bi-intersection is non-null then $(F, \alpha) \widetilde{\cap} (G, \beta)$ is a soft LA-module over M .

Proof. (1) By definition 7, let $(F, \alpha) \tilde{\wedge} (G, \beta) = (K, \gamma)$, where $\gamma = \alpha \times \beta$ and $K(x, y) = F(x) \cap G(y)$, for all $(x, y) \in \gamma$. Since by hypothesis (K, γ) is non-null soft set over M and $(x, y) \in \text{supp}(K, \gamma)$, so $K(x, y) = F(x) \cap G(y) \neq \emptyset$. As both $F(x)$ and $G(y)$ are sub LA-modules, hence this result implies that $K(x, y)$ is also a sub LA-modules. Thus we deduce that $(K, \gamma) = (F, \alpha) \tilde{\wedge} (G, \beta)$ is a soft LA-module over M .

(2) By definition 5, we write $(F, \alpha) \tilde{\cap} (G, \beta) = (K, \gamma)$, where $\gamma = \alpha \cap \beta$ and $K(x) = F(x) \cap G(x) \neq \emptyset$ for some $x \in \alpha \cap \beta$. Since the non-empty sets $F(x)$ and $G(x)$ both are sub LA-modules over M and hence, we see that $F(x) \cap G(x)$ is a sub LA-module over M . Consequently $(F, \alpha) \tilde{\cap} (G, \beta) = (K, \gamma)$ is a soft LA-module over M . $\square$

Definition 20. Consider (F, α) and (G, β) be two soft LA-modules over M . Then $(F, \alpha) + (G, \beta) = (K, \gamma)$, where $\gamma = \alpha \times \beta$ and $K(x, y) = F(x) + G(y)$ for every $(x, y) \in \gamma$.

Based on above definition, we have the following theorem:

Theorem 4. Consider (F, A) and (G, B) be two soft LA-modules over M . Then $(F, \alpha) + (G, \beta) = (K, \gamma)$, where $\gamma = \alpha \times \beta$ is a soft LA-module over M .

Proof. By definition 20, we have $(F, \alpha) + (G, \beta) = (K, \gamma)$, where $\gamma = \alpha \times \beta$ and $K(x, y) = F(x) + G(y)$ for every $(x, y) \in \gamma$. Since (F, α) and (G, β) are two soft LA-modules over M , therefore we have $F(x) \leq M$ and $G(y) \leq M$ for every $(x, y) \in \gamma$ respectively. Then $F(x) + G(y) \leq M$ for all $(x, y) \in \gamma$. Hence it follows that $K(x, y) \leq M$ for all $(x, y) \in \gamma$. Thus $(F, \alpha) + (G, \beta) = (K, \gamma)$ is a soft LA-module over M . $\square$

Definition 21. Let (F, α) and (G, β) be two soft LA-modules over M and N respectively. Then $(F, \alpha) \times (G, \beta) = (K, \gamma)$, where $\gamma = \alpha \times \beta$ is defined as $K(x, y) = F(x) \times G(y)$ for every $(x, y) \in \gamma$.

Theorem 5. Let (F, α) and (G, β) be two soft LA-modules over M and N respectively. Then $(F, \alpha) \times (G, \beta) = (K, \gamma)$, where $\gamma = \alpha \times \beta$ is a soft LA-module over $M \times N$.

Proof. From definition 21, we have $(F, \alpha) \times (G, \beta) = (K, \gamma)$, where $\gamma = \alpha \times \beta$ and $K(x, y) = F(x) \times G(y)$ for every $(x, y) \in \gamma$. Since (F, α) and (G, β) are two soft LA-modules over M and N respectively, therefore $F(x) \leq M$ and $G(y) \leq N$ for every $(x, y) \in \gamma$. Hence by definition 15, it can be easily verified that $K(x, y) = F(x) \times G(y) \leq M \times N$ for every $(x, y) \in \gamma$. Thus $(F, \alpha) \times (G, \beta) = (K, \gamma)$ is a soft LA-module over $M \times N$. $\square$

Definition 22. Consider (F, α) and (G, β) be two soft LA-modules over M . Then (G, β) is called a soft sub LA-module of (F, α) , if the following conditions are satisfied:

- (i) $\beta \subseteq \alpha$ and
- (ii) $G(x) \leq F(x)$ for every $x \in \beta$.

The above soft sub LA-module can be written as $(G, \beta) \leq (F, \alpha)$.

If $(G, \beta) \leq (F, \alpha)$ and $(F, \alpha) \leq (G, \beta)$, then these two soft sets over M are called soft equal sub LA-modules and it is written as $(F, \alpha) = (G, \beta)$.

Theorem 6. Consider (F, α) and (G, α) be two soft LA-modules over M . The soft set (G, α) is a soft sub LA-module of (F, α) in case if $G(\varepsilon)$ is a sub LA-module of $F(\varepsilon)$ i.e., $G(\varepsilon) \subseteq F(\varepsilon)$ for all $\varepsilon \in \alpha$.

Proof. By definition 22, the proof is straightforward. $\square$

Definition 23. Every soft LA-module (F, α) over M has at least two soft sub LA-modules (F, α) and (F, E) , which are called the trivial soft sub LA-module over M , where $E = \{e\}$, and e is a unit of α .

Theorem 7. Suppose that (F, α) is a soft LA-module over M , and $\{(G_i, \beta_i) \mid i \in I\}$ be a family of non-empty soft sub LA-modules of (F, α) . Then the following assertions hold:

- (1) The $\sum_{i \in I} (G_i, \beta_i)$ is a soft sub LA-module of (F, α) .
- (2) The $\prod_{i \in I} (G_i, \beta_i)$ is a soft sub LA-module of (F, α) .
- (3) The $\bigcup_{i \in I} (G_i, \beta_i)$ is soft a sub LA-module of (F, α) , if $\beta_i \cap \beta_j = \emptyset$ for every $i, j \in I$.

Proof. (1) By proposition 1, the proof is straightforward.

(2) By definition 6, let $\prod_{i \in I} (G_i, \beta_i) = (K, \gamma)$, where $\gamma = \bigcap_{i \in I} \beta_i$ and $K(x) = \bigcap_{i \in I} G_i(x)$ for all $x \in \gamma$. Since (K, γ) is non-null soft set and if $x \in \text{supp}(K, \gamma)$, then $K(x) = \bigcap_{i \in I} G_i(x) \neq \emptyset$. As the intersection of any number of sub LA-modules over M is a sub LA-module, hence for all $i \in I$, the non-empty set $G_i(x)$ is a sub LA-module of (F, α) over M . Hence (G_i, β_i) is a soft sub LA-module of (F, α) over M and therefore $K(x)$ is a sub LA-module over M for all $x \in \text{supp}(K, \gamma)$. Thus it follows that $\prod_{i \in I} (G_i, \beta_i) = (K, \gamma)$ is a soft sub LA-module of (F, α) over M .

(3) Straightforward by proposition 3, part (2). $\square$

Definition 24. Suppose that (F, α) and (G, β) be two soft LA-modules over M . Let $(G, \beta) \leq (F, \alpha)$. Then (G, β) is called a maximal soft sub LA-module of (F, α) if $G(\eta)$ is maximal sub LA-module of $F(\eta)$ for all $\eta \in \beta$. Similarly (G, β) is called a minimal soft sub LA-module of (F, α) if $G(\eta)$ is minimal sub LA-module of $F(\eta)$ for all $\eta \in \beta$.

Proposition 4. Consider a soft LA-module (F, α) over M . Then the following conditions are true:

- (1) The $\bigcap_{i \in I} (G_i, \beta_i)$ is a maximal soft sub LA-module of (F, α) , if $\{(G_i, \beta_i) \mid i \in I\}$ be a family of non-empty maximal soft sub LA-modules of (F, α) .
- (2) The $\sum_{i \in I} (G_i, \beta_i)$ is a minimal soft sub LA-module of (F, α) , if $\{(G_i, \beta_i) \mid i \in I\}$ be a family of non-empty minimal soft sub LA-modules of (F, α) .

Proof. The proof follows from the definition 17. $\square$

4. Soft LA modules and exact sequences

In this section, we consider the LA-homomorphisms and the quotient soft sub LA-modules. We will establish a theorem concerning the short exact sequence of LA modules. Throughout this section, the homomorphism always means an LA-module homomorphism.

Definition 25. Consider (F, α) and (G, β) be two soft LA-modules over M and N respectively. Then (θ, φ) is a soft homomorphism if the following assertions are true:

- (i) $\theta : M \longrightarrow N$ is a homomorphism
- (ii) $\varphi : \alpha \longrightarrow \beta$ is a mapping
- (iii) $\theta(F(\varepsilon)) = G(\varphi(\varepsilon))$ for all $\varepsilon \in \alpha$.

We now call (F, α) is soft homomorphic to (G, β) if the above definition is satisfied. It can be written as $(F, \alpha) \simeq (G, \beta)$.

If $\theta : M \longrightarrow N$ is a bijective homomorphism (isomorphism) and $\varphi : \alpha \longrightarrow \beta$ is a one-one and onto mapping, then (θ, φ) is called a soft bijective morphism (isomorphism). Moreover, we also call (F, α) is soft isomorphic to (G, β) , and it is defined by $(F, \alpha) \cong (G, \beta)$.

Definition 26. Consider (F, α) be a soft LA-module over M , then the following statements hold:

- (i) If $F(\varepsilon) = \{0\}$ for all $\varepsilon \in \alpha$, where 0 is zero element of M , then (F, α) is said to be trivial (null) soft LA-module over M .
- (ii) If $F(\varepsilon) = M$ for all $\varepsilon \in \alpha$, then (F, α) is said to be whole (absolute) soft LA-module over M .

Proposition 5. Consider two soft LA-modules (F, α) and (G, β) over M , and (G, β) be a soft sub LA-module of (F, α) . If $\theta : M \longrightarrow N$ is a homomorphism then $(\theta(F), \alpha)$ and $(\theta(G), \beta)$ are two soft LA-modules over N . Also $(\theta(G), \beta)$ is a soft sub LA-module of $(\theta(F), \alpha)$.

Proof. Since $\theta : M \longrightarrow N$ is a homomorphism and we know that homomorphic image of sub LA-module is a sub LA-module by proposition 2. Therefore $\theta(F(x))$ and $\theta(G(y))$ are soft LA-modules over N for every $\varepsilon \in \alpha$ and $\eta \in \beta$. Hence $(\theta(F), \alpha)$ and $(\theta(G), \beta)$ are two soft LA-modules over N . If (G, β) is a soft

sub LA-module of (F, α) , then $G(\eta)$ is a sub LA-module of $F(\eta)$ for every $\eta \in \beta$ and θ is a homomorphism so intuitively $\theta(G(\eta))$ is a sub LA-module of $\theta(F(\eta))$ for every $\eta \in \beta$. Therefore, we can say that $(\theta(G), \beta)$ is a soft sub LA-module of $(\theta(F), \alpha)$. $\square$

Proposition 6. (1) Consider a soft LA-module (F, α) over M and a homomorphism $\theta : M \rightarrow N$, if $F(\varepsilon) = \ker \theta$ for all $\varepsilon \in \alpha$, then $(\theta(F), \alpha)$ is a trivial (null) soft LA-module over N .

(2) Consider (F, α) to be an absolute soft LA-module over M and $\theta : M \rightarrow N$ be an onto homomorphism (epimorphism), if $F(\varepsilon) = M$ for all $\varepsilon \in \alpha$, then $(\theta(F), \alpha)$ is a whole (absolute) soft LA-module over N .

Proof. (1) Since $F(\varepsilon) = \ker \theta$ for all $\varepsilon \in \alpha$, and $\theta : M \rightarrow N$ is a homomorphism, so this implies that $\theta(F(\varepsilon)) = \theta(\ker \theta) = \{0_N\}$ for all $\varepsilon \in \alpha$. Therefore, by definition 26, $(\theta(F), \alpha)$ is a trivial (null) soft LA-module over N .

(2) Since (F, α) is an absolute soft LA-module over M , then clearly $F(\varepsilon) = M$ for all $\varepsilon \in \alpha$. Since θ is an epimorphism, therefore it follows that $\theta(F(\varepsilon)) = \theta(M) = N$ for all $\varepsilon \in \alpha$. Hence it is proved that $(\theta(F), \alpha)$ is a whole (absolute) soft LA-module over N . $\square$

Definition 27. A sequence of R -homomorphisms and R -LA-modules $\cdots \rightarrow P_{n-1} \xrightarrow{\theta_{n-1}} P_n \xrightarrow{\theta_n} P_{n+1} \rightarrow \cdots$ is called an exact sequence if $\text{Im} \theta_{n-1} = \ker \theta_n$ for all $n \in \mathbb{N}$. An exact sequence of the form $0 \rightarrow P' \xrightarrow{\theta} P \xrightarrow{\varphi} P'' \rightarrow 0$ is called a short exact sequence.

Example 1. Consider a sequence

$$(*) \quad 0 \rightarrow \langle h(s) \rangle \xrightarrow{i} R[X; S] \xrightarrow{\pi} \frac{R[X; S]}{\langle h(s) \rangle} \rightarrow 0$$

where $i : \langle h(s) \rangle \rightarrow R[X; S]$ is an inclusion map defined as $i[\langle h(s) \rangle] = \langle h(s) \rangle$, where $\langle h(s) \rangle = \{g(X).h(s) : g(X) \in R[X; S]\}$. Let $\pi : R[X; S] \rightarrow \frac{R[X; S]}{\langle h(s) \rangle}$ defined by $\pi[g(X)] = \langle h(s) \rangle + g(X)$ for every $g(X) \in R[X; S]$, where $h(s)$ is an irreducible polynomial in $R[X; S]$. Here $\langle h(s) \rangle$, $R[X; S]$ and $\frac{R[X; S]}{\langle h(s) \rangle}$ being LA-rings are LA-modules over an LA-ring R and S is a commutative semigroup and every commutative semigroup is an LA-semigroup. It is important to note that the additive identity in $\frac{R[X; S]}{\langle h(s) \rangle}$ is $\langle h(s) \rangle$. So if i is monomorphism, π is an epimorphism and $\text{Im} i = \ker \pi$. Then the sequence $(*)$ is an exact sequence of LA-modules over an LA-ring R .

First we show that $i : \langle h(s) \rangle \rightarrow R[X; S]$ defined by $i[\langle h(s) \rangle] = \langle h(s) \rangle$, where $\langle h(s) \rangle = \{g(X).h(s) : g(X) \in R[X; S]\}$, is a homomorphism. Let $h_1(s), h_2(s) \in \langle h(s) \rangle$ where $h_1(s) = g_1(X).h(s)$ and $h_2(s) = g_2(X).h(s)$

$$\begin{aligned}
i\{h_1(s) + h_2(s)\} &= i\{g_1(X).h(s) + g_2(X).h(s)\} \\
&= \{g_1(X).h(s) + g_2(X).h(s)\} \\
&= i\{h_1(s)\} + i\{h_2(s)\} \\
i\{rh_1(s)\} &= i\{rg_1(X).h(s)\} \\
&= rg_1(X).h(s) = r\{g_1(X).h(s)\} = ri\{h_1(s)\}
\end{aligned}$$

so i is an LA-module homomorphism. Now we show that it is monic, as we know that $f_1(s), f_2(s) \in \langle f(s) \rangle$ where $f_1(s) = g_1(X).f(s)$ and $f_2(s) = g_2(X).f(s)$, so

$$\begin{aligned}
i\{f_1(s)\} &\neq i\{f_2(s)\} \\
\implies i\{g_1(X).f(s)\} &\neq i\{g_2(X).f(s)\} \\
\implies g_1(X).f(s) &\neq g_2(X).f(s) \\
\implies f_1(s) &\neq f_2(s)
\end{aligned}$$

Hence we have proved that i is monic and hence it is monomorphism.

Now we have to show that $\pi : R[X; S] \rightarrow \frac{R[X; S]}{\langle h(s) \rangle}$ is an epimorphism defined by $\pi\{g(X)\} = \langle h(s) \rangle + g(X)$ for all $g(X) \in R[X; S]$, where $h(s)$ is an irreducible polynomial in $R[X; S]$.

Let

$$\begin{aligned}
\pi\{g_1(X) + g_2(X)\} &= \langle h(s) \rangle + \{g_1(X) + g_2(X)\} \\
&= \{\langle h(s) \rangle + g_1(X)\} + \{\langle h(s) \rangle + g_2(X)\} \\
&= \pi\{g_1(X)\} + \pi\{g_2(X)\} \\
\pi\{rg(X)\} &= \{\langle h(s) \rangle + rg(X)\} \\
&= r\{\langle h(s) \rangle + g(X)\} = r\pi\{g(X)\}
\end{aligned}$$

Hence π is an LA-module homomorphism. So by definition it is obviously onto and hence π is an epimorphism.

Finally, we proceed to prove that $Im\pi = ker\pi$

As it has been known that $\pi : R[X; S] \rightarrow \frac{R[X; S]}{\langle h(s) \rangle}$ which is defined by $\pi\{g(X)\} = \langle h(s) \rangle + g(X)$ for every $g(X) \in R[X; S]$, where $h(s)$ is an irreducible polynomial in $R[X; S]$. Now $ker\pi = \{g(X) \in R[X; S] : \pi\{g(X)\} = \langle h(s) \rangle\}$. Let $g(X) \in ker\pi$. As $\pi\{g(X)\} = \langle h(s) \rangle \implies \langle h(s) \rangle + g(X) = \langle h(s) \rangle \implies g(X) \in \langle h(s) \rangle \implies ker\pi \subseteq \langle h(s) \rangle = Im\pi \implies ker\pi \subseteq Im\pi$ (**). Let $g(X) \in Im\pi$, this implies that there exist an element $g(X) \in \langle h(s) \rangle$ such that $g(X) = i[g(X)] \implies \pi\{g(X)\} = \pi[i\{g(X)\}] \implies \langle h(s) \rangle + g(X) = \pi\{g(X)\} \implies \langle h(s) \rangle = \pi\{g(X)\} \implies g(X) \in ker\pi \implies Im\pi \subseteq ker\pi$ (* * *). Hence from (**) and (* * *) it is proved that that $ker\pi = Im\pi$. Thus the sequence (*) is an exact sequence of LA-modules over an LA-ring R .

Definition 28. Let (F_n, α) be a soft LA-module over M , where $n \in \mathbb{N}$. Then a sequence of R -homomorphisms and soft LA-modules $\cdots \longrightarrow F_{n-1}(\varepsilon) \xrightarrow{\tilde{\theta}_{n-1}}$

$F_n(\varepsilon) \xrightarrow{\tilde{\theta}_n} F_{n+1}(\varepsilon) \longrightarrow \cdots$ is called an exact sequence if $Im\tilde{\theta}_{n-1} = ker\tilde{\theta}_n$ for every $n \in \mathbb{N}$ and for all $\varepsilon \in \alpha$. An exact sequence of the form $0 \longrightarrow F'(\varepsilon) \xrightarrow{\tilde{\theta}} F(\varepsilon) \xrightarrow{\tilde{\varphi}} F''(\varepsilon) \longrightarrow 0$ is called a short exact sequence.

The followings results are the main results of this paper.

Theorem 8. Let (F, α) to be a trivial (null) soft LA-module over V and let (G, β) be an absolute soft LA-module over W . If $0 \longrightarrow V \xrightarrow{\theta} M \xrightarrow{\varphi} W \longrightarrow 0$ is a short exact sequence, then $0 \longrightarrow F(\varepsilon) \xrightarrow{\tilde{\theta}} M \xrightarrow{\tilde{\varphi}} G(\eta) \longrightarrow 0$ is a short exact sequence for every $\varepsilon \in \alpha$ and $\eta \in \beta$.

Proof. It has been already known that the sequence $0 \xrightarrow{\theta'} V \xrightarrow{\theta} M \xrightarrow{\varphi} W \xrightarrow{\varphi'} 0$ is a short exact sequence, hence by definition 18 and proposition 2, we deduce that $Im\theta' = \{0\} = ker\theta$. Given that (F, α) is a trivial (null) soft LA-module over V , therefore, by definition 26, $F(\varepsilon) = \{0\}$ for all $\varepsilon \in \alpha$. Then from the triplet $0 \xrightarrow{\tilde{\theta}} F(\varepsilon) = \{0\} \xrightarrow{\tilde{\theta}} M$, we have $Im\tilde{\theta} = \{0\} = ker\tilde{\theta}$. This implies that $\tilde{\theta}$ is a one-one and hence it is a monomorphism. Conversely if $\tilde{\theta}$ is a monomorphism, then obviously the triplet $0 \xrightarrow{\tilde{\theta}} F(\varepsilon) = \{0\} \xrightarrow{\tilde{\theta}} M$ becomes an exact sequence. From the hypothesis we have $Im\varphi = ker\varphi'$. Now consider the triplet $M \xrightarrow{\tilde{\varphi}} G(\eta) \xrightarrow{\tilde{\varphi}} 0$, also given that (G, β) is an absolute soft LA-module over W , so $G(\eta) = W$ for all $\eta \in \beta$. From triplet $M \xrightarrow{\tilde{\varphi}} G(\eta) \xrightarrow{\tilde{\varphi}} 0$ it is obvious that $ker\tilde{\varphi} = G(\eta)$ for all $\eta \in \beta$. This implies that $Im\tilde{\varphi} = G(\eta) = ker\tilde{\varphi}$ for all $\eta \in \beta$. It follows that $\tilde{\varphi}$ is an onto and hence it is an epimorphism. Conversely, if $\tilde{\varphi}$ is an epimorphism, then obviously the triplet $M \xrightarrow{\tilde{\varphi}} G(\eta) \xrightarrow{\tilde{\varphi}} 0$ becomes an exact sequence. Now from the proposition 2, it follows that $0 \longrightarrow F(\varepsilon) \xrightarrow{\tilde{\theta}} M \xrightarrow{\tilde{\varphi}} G(\eta) \longrightarrow 0$ is a short exact sequence for all $\varepsilon \in \alpha$ and $\eta \in \beta$. $\square$

Proposition 7. Let (F, α) to be a trivial (null) soft LA-module over V and (G, β) be an absolute soft LA-module over W . If $0 \longrightarrow V \xrightarrow{\theta} M \xrightarrow{\varphi} W \longrightarrow 0$ is a short exact sequence, then $0 \longrightarrow \theta(F(\varepsilon)) \xrightarrow{\tilde{\theta}} M \xrightarrow{\tilde{\varphi}} \varphi(G(\eta)) \longrightarrow 0$ is a short exact sequence for every $\varepsilon \in \alpha$ and $\eta \in \beta$.

Proof. The proof follows from theorem 8 and proposition 2. $\square$

Theorem 9. Let a soft LA-module (F, α) over V and (G, β) is a soft LA-module over W . Then for a short exact sequence $0 \longrightarrow F(\varepsilon) \xrightarrow{\tilde{\theta}} M \xrightarrow{\tilde{\varphi}} G(\eta) \longrightarrow 0$

of soft R -LA-modules and R -homomorphisms, the following assertions are equivalent:

- (1) There exists an R -homomorphism $i : M \longrightarrow F(\varepsilon)$ such that $i\tilde{\theta} = 1_{F(\varepsilon)}$ for all $\varepsilon \in \alpha$.
- (2) There exists an R -homomorphism $\pi : G(\eta) \longrightarrow M$ such that $\tilde{\varphi}\pi = 1_{G(\eta)}$ for all $\eta \in \beta$.
- (3) $Im\tilde{\theta}$ is direct summand of M .

Proof. (1) $\implies$ (3) Suppose that there exists a homomorphism $i : M \longrightarrow F(\varepsilon)$ such that $i\tilde{\theta} = 1_{F(\varepsilon)}$ for all $\varepsilon \in \alpha$. Let $m \in M$. Then $i(m) \in F(\varepsilon)$. We have $i\tilde{\theta} = 1_{F(\varepsilon)}$ for all $\varepsilon \in \alpha$, then $i\tilde{\theta}(i(m)) = i(m) \implies i\tilde{\theta}(i(m) - i(m)) = 0 \implies i(\tilde{\theta}(i(m) - i(m))) = 0$ or $i(m - \tilde{\theta}(i(m))) = 0$. It follows that $m - \tilde{\theta}(i(m)) \in \ker i = K$ (say) $\implies m - \tilde{\theta}(i(m)) \in K \implies m - \tilde{\theta}(i(m)) = k$ for some $k \in K \implies m = k + \tilde{\theta}(i(m))$ for some $k \in K$. Thus $M = K + Im\tilde{\theta}$. Suppose that there is an another element $k' \in K$ and an element $a \in F(\varepsilon)$ such that $m = k + \tilde{\theta}(i(m)) = k' + \tilde{\theta}(a) \dots (*)$. Then $i(k + \tilde{\theta}(i(m))) = i(k' + \tilde{\theta}(a)) \implies i(k) + i(\tilde{\theta}(i(m))) = i(k') + i(\tilde{\theta}(a)) \implies 0 + i(\tilde{\theta}(i(m))) = 0 + i(\tilde{\theta}(a))$, since $k, k' \in K \implies i(\tilde{\theta}(i(m))) = i(\tilde{\theta}(a)) \implies (i\tilde{\theta})(i(m)) = (i\tilde{\theta})(a) \implies 1_{F(\varepsilon)}(i(m)) = 1_{F(\varepsilon)}(a) \implies i(m) = a$. Thus, the relation $(*)$ becomes $k + \tilde{\theta}(a) = k' + \tilde{\theta}(a) \implies k = k'$. Hence every element of M can be uniquely expressed as $k + \tilde{\theta}(a)$ for some $k \in K$ and $a \in F(\varepsilon)$. Therefore $M = K \oplus Im\tilde{\theta}$.

(3) $\implies$ (2) Let K be a soft sub LA-module of M such that $M = K \oplus Im\tilde{\theta}$. Let $a' \in G(\eta)$ and $\tilde{\varphi}$ is an epimorphism so $\tilde{\varphi}(m) = a'$. Let $m = k + \tilde{\theta}(a)$ for some $k \in K$ and $a \in F(\varepsilon)$. Then $\tilde{\varphi}(m) = \tilde{\varphi}(k + \tilde{\theta}(a)) \implies a' = \tilde{\varphi}(k) + \tilde{\varphi}(\tilde{\theta}(a)) \implies a' = \tilde{\varphi}(k) + 0 \implies a' = \tilde{\varphi}(k)$. If k' is another element such that $a' = \tilde{\varphi}(k)$, then $\tilde{\varphi}(k') = \tilde{\varphi}(k) \implies \tilde{\varphi}(k') - \tilde{\varphi}(k) = 0 \implies \tilde{\varphi}(k') + \tilde{\varphi}(-k) = 0 \implies \tilde{\varphi}(k' - k) = 0 \implies k' - k \in \ker \tilde{\varphi} \implies k' - k \in Im\tilde{\theta}$. This implies that there exists an element $a'' \in F(\varepsilon)$ such that $\tilde{\theta}(a'') = k' - k$. The direct summand property of M shows that $k' - k = \tilde{\theta}(a'') = 0 \implies k = k'$. Thus there exists a unique $k \in K$ such that $a' = \tilde{\varphi}(k)$. Define $\pi : G(\eta) \longrightarrow M$ by $\pi(a') = k$, where k is a unique element such that $\tilde{\varphi}(k) = a'$. Let $a'_1, a'_2 \in G(\eta) \implies$ there exist unique elements $k_1, k_2 \in K$ such that $\tilde{\varphi}(k_1) = a'_1$ and $\tilde{\varphi}(k_2) = a'_2 \implies \pi(a'_1) = k_1$ and $\pi(a'_2) = k_2$. Since $\tilde{\varphi}$ is homomorphism so $\tilde{\varphi}(k_1 + k_2) = \tilde{\varphi}(k_1) + \tilde{\varphi}(k_2) = a'_1 + a'_2 \implies \tilde{\varphi}(k_1 + k_2) = a'_1 + a'_2 \implies \pi(a'_1 + a'_2) = k_1 + k_2 = \pi(a'_1) + \pi(a'_2)$. Let $r \in R$ and $a' \in G(\eta) \implies ra' \in G(\eta)$. Consequently, there exists a unique element $k \in K$ such that $\tilde{\varphi}(k) = a' \implies \pi(a') = k$. Also $\tilde{\varphi}(rk) = r\tilde{\varphi}(k) = ra' \implies \pi(ra') = rk = r\pi(a')$. Thus π is an R -homomorphism and since $\tilde{\varphi}\pi(a') = \tilde{\varphi}(k) = a'$ for all $a' \in G(\eta) \implies \tilde{\varphi}\pi = 1_{G(\eta)}$ for all $\eta \in \beta$. (2) $\implies$ (1) Suppose that there exists

an R -homomorphism $\pi : G(\eta) \longrightarrow M$ such that $\tilde{\varphi}\pi = 1_{G(\eta)}$ for all $\eta \in \beta$. Let $m \in M$ and $\tilde{\varphi}(m) = a'$. Then $\tilde{\varphi}(m) = a' = \tilde{\varphi}\pi(a') \implies \tilde{\varphi}(m) - \tilde{\varphi}\pi(a') = 0 \implies \tilde{\varphi}(m - \pi(a')) = 0 \implies m - \pi(a') \in \ker \tilde{\varphi} \implies m - \pi(a') \in \text{Im } \tilde{\theta}$. Intuitively, there exists an element $a \in F(\varepsilon)$ such that $\tilde{\theta}(a) = m - \pi(a') \implies m = \pi(a') + \tilde{\theta}(a)$. To prove the uniqueness, let $\pi(a'') + \tilde{\theta}(a') = \pi(a') + \tilde{\theta}(a)$. Then $\pi(a'') - \pi(a') = \tilde{\theta}(a) - \tilde{\theta}(a') \implies \pi(a'' - a') = \tilde{\theta}(a - a') \implies \tilde{\varphi}\pi(a'' - a') = \tilde{\varphi}\tilde{\theta}(a - a') = 0 \implies a'' - a' = 0 \implies a'' = a'$. Now we have $\pi(a') + \tilde{\theta}(a') = \pi(a') + \tilde{\theta}(a) \implies \tilde{\theta}(a') - \tilde{\theta}(a) = 0 \implies \tilde{\theta}(a' - a) = 0 \implies a' - a \in \ker \tilde{\theta} \implies a' - a \in \{0\}$ since $\tilde{\theta}$ is monic so $\ker \tilde{\theta} = \{0\} \implies a' = a$. Hence every element of M can be written as $\pi(a') + \tilde{\theta}(a)$, $a' \in G(\eta)$ and $a \in F(\varepsilon)$. Now we define a map $i : M \longrightarrow F(\varepsilon)$ by $i(\pi(a') + \tilde{\theta}(a)) = a$. Obviously i is well-defined homomorphism and $i\tilde{\theta} = 1_{F(\varepsilon)}$ for all $\varepsilon \in \alpha$. Let $\pi(a'_1) + \tilde{\theta}(a_1), \pi(a'_2) + \tilde{\theta}(a_2) \in M$, then $i((\pi(a'_1) + \tilde{\theta}(a_1)) + (\pi(a'_2) + \tilde{\theta}(a_2))) = i((\pi(a'_1) + \pi(a'_2)) + (\tilde{\theta}(a_1) + \tilde{\theta}(a_2))) = i(\pi(a'_1 + a'_2) + \tilde{\theta}(a_1 + a_2)) = a_1 + a_2 = i(\pi(a'_1) + \tilde{\theta}(a_1)) + i(\pi(a'_2) + \tilde{\theta}(a_2))$ and $i(r(\pi(a') + \tilde{\theta}(a))) = i(r\pi(a') + r\tilde{\theta}(a)) = i(\pi(ra') + \tilde{\theta}(ra)) = ra = r(i(\pi(a') + \tilde{\theta}(a)))$.

Hence i is a homomorphism. Now the $\tilde{\theta}(a) = m - \pi(a') = i\tilde{\theta}(a) = i(m - \pi(a')) = i(m) - i(\pi(a')) = a - 0 = a \implies i\tilde{\theta}(a) = a \implies i\tilde{\theta} = 1_{F(\varepsilon)}$ for all $\varepsilon \in \alpha$. $\square$

Definition 29. Let (F, α) to be a soft LA-module over M and (G, β) be a soft sub LA-module of (F, α) . Then the soft quotient LA-module is defined as $(F, \alpha)/(G, \beta) = \{(G, \beta) + F(\varepsilon) \mid \text{for every } \varepsilon \in \alpha\}$.

Theorem 10. Suppose (F, α) is a soft LA-module over M . (G, β) be a soft sub LA-module of (F, α) and $\theta : (G, \beta) \longrightarrow (F, \alpha)$ is the inclusion map defined by $\theta(G(\eta)) = G(\eta)$ for all $\eta \in \beta$ and $\varphi : (F, \alpha) \longrightarrow (F, \alpha)/(G, \beta)$ is the natural projection i.e., $\varphi(F(\varepsilon)) = (G, \beta) + F(\varepsilon)$ for all $\varepsilon \in \alpha$. Then θ is a monomorphism, φ is an epimorphism and $\text{Im } \theta = (G, \beta) = \ker \varphi$. Then the sequence $0 \longrightarrow G(\eta) \xrightarrow{\theta} F(\varepsilon) \xrightarrow{\varphi} F(\varepsilon)/G(\eta) \longrightarrow 0$ is a short exact sequence.

Proof. Define $\theta : (G, \beta) \longrightarrow (F, \alpha)$ by $\theta(G(\eta)) = G(\eta)$ for all $\eta \in \beta$. For all $\eta_1, \eta_2 \in \beta$, $G(\eta_1) = G(\eta_2) \implies \theta(G(\eta_1)) = \theta(G(\eta_2))$ by definition, so θ is well defined. Let $\theta(G(\eta_1)) = \theta(G(\eta_2))$ for all $\eta_1, \eta_2 \in \beta \implies G(\eta_1) = G(\eta_2)$ by definition. Hence θ is one-one. Now for all $\eta_1, \eta_2 \in \beta$ $\theta(G(\eta_1) + G(\eta_2)) = G(\eta_1) + G(\eta_2) = \theta(G(\eta_1)) + \theta(G(\eta_2))$, and for all $\eta \in \beta$ and $r \in R$, $\theta(r \cdot G(\eta)) = r \cdot G(\eta) = r \cdot \theta(G(\eta))$. So it follows that θ is a homomorphism and hence monomorphism. Define $\varphi : (F, \alpha) \longrightarrow (F, \alpha)/(G, \beta)$ by $\varphi(F(\varepsilon)) = (G, \beta) + F(\varepsilon)$ for all $\varepsilon \in \alpha$. For all $\varepsilon_1, \varepsilon_2 \in \alpha$, $F(\varepsilon_1) = F(\varepsilon_2) \implies (G, \beta) + F(\varepsilon_1) = (G, \beta) + F(\varepsilon_2) \implies \varphi(F(\varepsilon_1)) = \varphi(F(\varepsilon_2))$ by definition, so φ is well defined. Now to show φ is an onto map, it is obvious that for every $(G, \beta) + F(\varepsilon) \in (F, \alpha)/(G, \beta)$ there

exists an element $F(\varepsilon) \in (F, \alpha)$ such that $\varphi(F(\varepsilon)) = (G, \beta) + F(\varepsilon)$ for all $\varepsilon \in \alpha$, hence φ is an onto. Now $\varphi(F(\varepsilon_1) + F(\varepsilon_2)) = (G, \beta) + (F(\varepsilon_1) + F(\varepsilon_2)) = ((G, \beta) + F(\varepsilon_1)) + ((G, \beta) + F(\varepsilon_2)) = \varphi(F(\varepsilon_1)) + \varphi(F(\varepsilon_2))$. Also for all $r \in R$ and for all $\varepsilon \in \alpha$, $\varphi(r \cdot F(\varepsilon)) = (G, \beta) + r \cdot F(\varepsilon) = r \cdot (G, \beta) + r \cdot F(\varepsilon) = r \cdot [(G, \beta) + F(\varepsilon)] = r \cdot \varphi(F(\varepsilon))$ because (G, β) is zero of $(F, \alpha)/(G, \beta)$ so we can write $(G, \beta) = r \cdot (G, \beta)$. Therefore φ is homomorphism and hence epimorphism. Now we show that $\text{Im}\theta = (G, \beta) = \ker\varphi$. So for this, let $F(\varepsilon) \in \ker\varphi \implies \varphi(F(\varepsilon)) = (G, \beta) \implies (G, \beta) + F(\varepsilon) = (G, \beta) \implies F(\varepsilon) \in (G, \beta)$. Since θ is an inclusion map so $\theta(F(\varepsilon)) = F(\varepsilon) \implies F(\varepsilon) \in \theta(G, \beta) = \text{Im}\theta \implies F(\varepsilon) \in \text{Im}\theta$. Hence it follows that $\ker\varphi \subseteq \text{Im}\theta$(*). Let $F(\varepsilon) \in \text{Im}\theta$. Then there exist an element $F(\varepsilon) \in (G, \beta)$ such that $F(\varepsilon) = \theta(F(\varepsilon)) \implies \varphi(F(\varepsilon)) = \varphi(\theta(F(\varepsilon))) \implies (G, \beta) + F(\varepsilon) = \varphi(F(\varepsilon)) \implies (G, \beta) = \varphi(F(\varepsilon)) \implies F(\varepsilon) \in \ker\varphi \implies \text{Im}\theta \subseteq \ker\varphi$(**). Hence from (*) and (**) we have $\text{Im}\theta = \ker\varphi$. Now we proceed to show $\text{Im}\theta = (G, \beta)$, let $F(\varepsilon) \in \text{Im}\theta$. Then there exists an element $G(\eta) \in (G, \beta)$ such that $\theta(G(\eta)) = F(\varepsilon)$, where $\eta \in \beta$. Now as θ is an inclusion map so $G(\eta) = F(\varepsilon) \implies F(\varepsilon) \in (G, \beta) \implies \text{Im}\theta \subseteq (G, \beta)$(* * *). For all $\eta \in \beta$, let $G(\eta) \in (G, \beta) \implies \theta(G(\eta)) = G(\eta) \implies G(\eta) \in \text{Im}\theta$. Hence $(G, \beta) \subseteq \text{Im}\theta$(* * *). Thus from (* * *) and (* * *) we have $\text{Im}\theta = (G, \beta)$. Finally, let $F(\varepsilon) \in \ker\varphi \iff \varphi(F(\varepsilon)) = (G, \beta) \iff (G, \beta) + F(\varepsilon) = (G, \beta) \iff F(\varepsilon) \in (G, \beta) \iff (G, \beta) = \ker\varphi$. Hence $\text{Im}\theta = (G, \beta) = \ker\varphi$. Consequently, the sequence $0 \longrightarrow G(\eta) \xrightarrow{\theta} F(\varepsilon) \xrightarrow{\varphi} F(\varepsilon)/G(\eta) \longrightarrow 0$ is a short exact sequence. $\square$

In closing this paper, we extend the well known "Five lemma" in exact sequence in terms of the soft LA modules.

Lemma 2. Let (F_i, α_i) and (G_i, β_i) be soft LA-modules over M , where $i = 1, 2, 3, 4, 5$. Consider a commutative diagram

$$\begin{array}{ccccccccc}
 F_1(x) & \xrightarrow{f_1} & F_2(x) & \xrightarrow{f_2} & F_3(x) & \xrightarrow{f_3} & F_4(x) & \xrightarrow{f_4} & F_5(x) \\
 \downarrow t_1 & & \downarrow t_2 & & \downarrow t_3 & & \downarrow t_4 & & \downarrow t_5 \\
 G_1(y) & \xrightarrow{g_1} & G_2(y) & \xrightarrow{g_2} & G_3(y) & \xrightarrow{g_3} & G_4(y) & \xrightarrow{g_4} & G_5(y)
 \end{array}$$

of R - LA-modules and homomorphisms with exact rows

(1) If t_2 and t_4 are epimorphisms and t_5 is a monomorphism, then t_3 is an epimorphism.

(2) If t_2 and t_4 are monomorphisms and t_1 is an epimorphism, then t_3 is a monomorphism.

Proof. (1) Suppose that t_2 and t_4 are epimorphisms and t_5 is a monomorphism. For all $y \in \beta_3$, let $\varepsilon_3 \in G_3(y)$. Then $g_3(\varepsilon_3) \in G_4(y)$ for all $y \in \beta_4$. Since

t_4 is an epimorphism, then there exists an element $\varepsilon'_4 \in F_4(x)$ for all $x \in \alpha_4$ such that $g_3(\varepsilon_3) = t_4(\varepsilon'_4)$ for all $y \in \beta_3$. By commutativity of diagram, $t_5 f_4(\varepsilon'_4) = g_4 t_4(\varepsilon'_4) \implies t_5 f_4(\varepsilon'_4) = g_4(g_3(\varepsilon_3)) = 0$ because $\text{Im} g_3 = \ker g_4$. As t_5 is a monomorphism thus $f_4(\varepsilon'_4) = 0$. The row being exact there exists an element $\varepsilon'_3 \in F_3(x)$ for all $x \in \alpha_3$, such that $f_3(\varepsilon'_3) = \varepsilon'_4$. Then $g_3(\varepsilon_3) = t_4(\varepsilon'_4) = t_4(f_3(\varepsilon'_3)) = g_3 t_3(\varepsilon'_3)$ and so $g_3(\varepsilon_3) - g_3 t_3(\varepsilon'_3) = 0 \implies g_3(\varepsilon_3 - t_3(\varepsilon'_3)) = 0 \implies \varepsilon_3 - t_3(\varepsilon'_3) \in \ker g_3 \implies \varepsilon_3 - t_3(\varepsilon'_3) \in \text{Im} g_2$ because $\text{Im} g_2 = \ker g_3$. Therefore there exists an element $\varepsilon_2 \in G_2(y)$ for all $y \in \beta_2$ such that $g_2(\varepsilon_2) = \varepsilon_3 - t_3(\varepsilon'_3)$. The homomorphism t_2 being an epimorphism, there exists an element $\varepsilon'_2 \in F_2(x)$ for all $x \in \alpha_2$ such that $t_2(\varepsilon'_2) = \varepsilon_2$. But then $g_2(\varepsilon_2) = g_2(t_2(\varepsilon'_2)) \implies \varepsilon_3 - t_3(\varepsilon'_3) = g_2(t_2(\varepsilon'_2)) = t_3 f_2(\varepsilon'_2)$ by commutativity of diagram. It follows that $\varepsilon_3 = t_3 f_2(\varepsilon'_2) + t_3(\varepsilon'_3) = t_3(f_2(\varepsilon'_2) + \varepsilon'_3)$. Hence t_3 is epimorphism.

(2) Suppose t_2 and t_4 are monomorphisms and t_1 is an epimorphism. For all $x \in \alpha_3$, let $\varepsilon'_3 \in F_3(x)$ such that $t_3(\varepsilon'_3) = 0$. Then $0 = g_3(0) = g_3 t_3(\varepsilon'_3) = t_4(f_3(\varepsilon'_3))$ because diagram is commutative, $\implies 0 = t_4(f_3(\varepsilon'_3))$. Now t_4 being a monomorphism we have $f_3(\varepsilon'_3) = 0$. It follows that $\varepsilon'_3 \in \ker f_3$ but due to exact row $\ker f_3 = \text{Im} f_2 \implies \varepsilon'_3 \in \text{Im} f_2$. Then there exists an element $\varepsilon'_2 \in F_2(x)$ for all $x \in \alpha_2$ such that $f_2(\varepsilon'_2) = \varepsilon'_3$. But from commutativity of diagram $g_2 t_2(\varepsilon'_2) = t_3 f_2(\varepsilon'_2)$. Then it follows that $g_2 t_2(\varepsilon'_2) = t_3 f_2(\varepsilon'_2) = t_3(\varepsilon'_3) = 0$. This implies that $g_2 t_2(\varepsilon'_2) = 0 \implies t_2(\varepsilon'_2) \in \ker g_2$ and hence $t_2(\varepsilon'_2) \in \text{Im} g_1$. Then there exists an element $\varepsilon_1 \in G_1(y)$ for all $y \in \beta_1$ such that $g_1(\varepsilon_1) = t_2(\varepsilon'_2)$. Since t_1 is an epimorphism, then there exists an element $\varepsilon'_1 \in F_1(x)$ for all $x \in \alpha_1$ such that $t_1(\varepsilon'_1) = \varepsilon_1$. Then $t_2(\varepsilon'_2) = g_1(\varepsilon_1) = g_1(t_1(\varepsilon'_1)) = t_2 f_1(\varepsilon'_1) \implies t_2(\varepsilon'_2) = t_2 f_1(\varepsilon'_1)$. Now t_2 being a monomorphism, we have $\varepsilon'_2 = f_1(\varepsilon'_1)$ and hence $\varepsilon'_3 = f_2(\varepsilon'_2) = f_2(f_1(\varepsilon'_1)) = 0 \implies \varepsilon'_3 = 0$. Hence proved that t_3 is a monomorphism. $\square$

The following corollary is the consequence of lemma 2.

Corollary 2. *Consider a commutative diagram with exact rows*

$$\begin{array}{ccccccc}
 0 & \longrightarrow & F_1(x) & \xrightarrow{f_1} & F_2(x) & \xrightarrow{f_2} & F_3(x) \longrightarrow 0 \\
 & & \downarrow t_1 & & \downarrow t_2 & & \downarrow t_3 \\
 0 & \longrightarrow & G_1(y) & \xrightarrow{g_1} & G_2(y) & \xrightarrow{g_2} & G_3(y) \longrightarrow 0
 \end{array}$$

If any two t_1, t_2, t_3 are isomorphism, then so is the third.

Proof. The proof follows from lemma 2. $\square$

5. Conclusion

The research done in this paper can be regarded as a systematic study of soft LA-modules. We have presented a detailed theoretical study of the homomorphism

together with the exact sequence of soft LA-modules. A functional approach has been developed to undertake a characterization of soft LA-modules which leads to a determination of some of its interesting theoretic properties. We have established some useful results regarding soft LA-modules, homomorphisms, soft quotient LA-modules and finally the exact sequence. This study can be enhanced by defining projective soft LA-modules, injective soft LA-modules, tensor product of soft LA-modules and their related properties.

Compliance with ethical standards

This study was not funded by any organization, institution or any other person.

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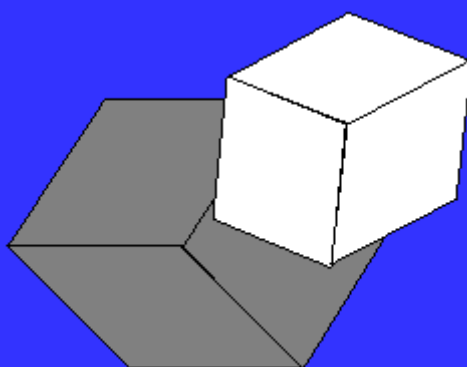
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