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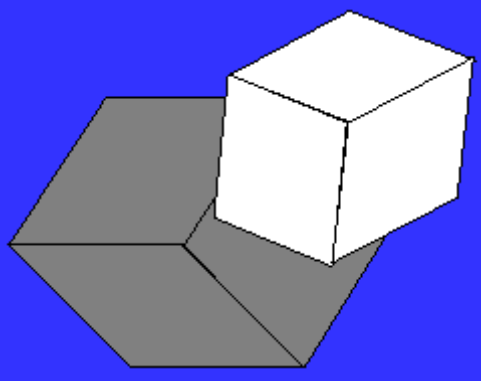

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INTERVAL-VALUED FUZZY HYPERGRAPH AND INTERVAL-VALUED FUZZY HYPEROPERATIONS[1]

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Abstract. We first construct two interval-valued fuzzy hyperoperations by the use of $[\alpha, \beta]$-cut on an interval-valued $[\alpha, \beta]$-fuzzy hypergraph and list some properties of them. Then we construct another two interval-valued fuzzy hyperoperations by the use of $\langle\alpha, \beta\rangle$-cut on an interval-valued $\langle\alpha, \beta\rangle$-fuzzy hypergraph and also list some properties of them. Lastly, we study the interval-valued fuzzy hyperstructure associated to the latter two interval-valued fuzzy hyperoperations and we can see that the hyperstructure is almost an interval-valued superlattice.

Keywords: interval-valued fuzzy hyperoperation; interval-valued fuzzy hypergraph; quasi-hypergroup; superlattice.

1. Introduction and preliminaries

The connections between graphs and hyperoperations had been looked into by several researchers (see for instance [3], [8]). Corsini [4] and Ali [1] studied the connections between hypergraphs and hyperoperations. Feng used fuzzy hypergraphs to construct fuzzy hyperoperations in [6], [7].

In this paper, we first construct two interval-valued fuzzy hyperoperations by the use of $[\alpha, \beta]$-cut on an interval-valued $[\alpha, \beta]$-fuzzy hypergraph and list some properties of them. Then we construct another two interval-valued fuzzy hyperoperations by the use of $\langle\alpha, \beta\rangle$-cut on an interval-valued $\langle\alpha, \beta\rangle$-fuzzy hypergraph

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and also list some properties of them. Lastly, we study the interval-valued fuzzy hyperstructure associated to the latter two interval-valued fuzzy hyperoperations and we can see that the hyperstructure is almost an interval-valued superlattice.

We recall some notations of interval-valued fuzzy hyperstructure theory.

All intervals in $[0,1]$ is denoted by $I([0,1])$.

Definition 1.1. For all $[a,b],[x,y] \in I([0,1])$, we define

$$\begin{aligned} [a,b] \vee [x,y] &\doteq [a \vee x, b \vee y]. \\ [a,b] \wedge [x,y] &\doteq [a \wedge x, b \wedge y]. \\ [a,b]' &= [1-b, 1-a]. \end{aligned}$$

Proposition 1.2. *$\forall A,B,C \in I([0,1])$, we have the following properties*

(1) $A \vee A = A, A \wedge A = A$;

(2) $A \vee B = B \vee A, A \wedge B = B \wedge A$;

(3) $(A \vee B) \vee C = A \vee (B \vee C), (A \wedge B) \wedge C = A \wedge (B \wedge C)$;

(4) $A \wedge (A \vee B) = A, A \vee (A \wedge B) = A$;

(5) $(A \vee B) \wedge C = (A \wedge C) \vee (B \wedge C), (A \wedge B) \vee C = (A \vee C) \wedge (B \vee C)$;

(6) $A \vee [0,0] = A, A \wedge [0,0] = [0,0], A \vee [1,1] = [1,1], A \wedge [1,1] = A$;

(7) $(A')' = A$;

(8) $(A \vee B)' = A' \wedge B', (A \wedge B)' = A' \vee B'$.

Proof. A straightforward verification. ∎

Definition 1.3. $\forall [a,b],[x,y] \in I([0,1])$, we define

$$[a,b] \leq [x,y] \doteq a \leq x, b \leq y.$$

It is easy to see that $\langle I([0,1]); \leq \rangle$ is a partial ordered set.

An interval-valued fuzzy subset of a nonempty set H is a function $M : H \to I([0,1])$. The collection of all interval-valued fuzzy subsets of H is denoted by $IF(H)$.

Given $A, B \in IF(H)$, we give the following definitions

$$\begin{aligned} A \subseteq B &\doteq A(x) \leq B(x), \quad \forall x \in H. \\ A = B &\doteq A(x) = B(x), \quad \forall x \in H. \\ (A \cup B)(x) &\doteq A(x) \vee B(x), \quad \forall x \in H. \\ (A \cap B)(x) &\doteq A(x) \wedge B(x), \quad \forall x \in H. \\ A'(x) &\doteq (A(x))', \quad \forall x \in H. \end{aligned}$$

Note that $\emptyset$ and H can be seen as interval-valued fuzzy subsets of H and we set $\emptyset(x) = [0,0]$, $H(x) = [1,1]$, $\forall x \in H$.

Proposition 1.4. *$\forall A, B, C \in IF(H)$, we have the following properties*

(1) $A \cup A = A,\ A \cap A = A$;

(2) $A \cup B = B \cup A,\ A \cap B = B \cap A$;

(3) $(A \cup B) \cup C = A \cup (B \cup C),\ (A \cap B) \cap C = A \cap (B \cap C)$;

(4) $A \cap (A \cup B) = A,\ A \cup (A \cap B) = A$;

(5) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C),\ (A \cap B) \cup C = (A \cup C) \cap (B \cup C)$;

(6) $A \cup \emptyset = A, A \cap \emptyset = \emptyset,\ A \cup H = H, A \cap H = A$.

(7) $(A')' = A$;

(8) $(A \cup B)' = A' \cap B',\ (A \cap B)' = A' \cup B'$.

Proof. A straightforward verification. ∎

An interval-valued fuzzy hypergroupoid $\langle H; * \rangle$ is a nonempty set H endowed with an interval-valued fuzzy hyperoperation (i.e., a function $*$ from $H \times H$ to $IF(H)$).

Given an interval-valued fuzzy hyperoperation $* : H \times H \to IF(H)$, for all $a \in H$, $B \in IF(H)$, the interval-valued fuzzy subset $a * B$ of H is defined by

$$(a * B)(x) \doteq \vee_{B(b) \subseteq (0,1]} (a * b)(x).$$

The readers can consult [2], [5], [9] to learn more about hyperstructures and fuzzy sets.

2. Interval-valued fuzzy hyperoperations obtained from $[\alpha, \beta]$-cut

The $[\alpha, \beta]$-cut of a fuzzy subset M of H is defined by

$$M_{[\alpha,\beta]} \doteq \{x \in H | M(x) \subseteq [\alpha, \beta]\}.$$

Definition 2.1. H is a nonempty set, $\{A_i\}_i$ is a family of interval-valued fuzzy subsets of H, if there exists $[\alpha, \beta] \in I([0, 1])$ with $\alpha \neq 0$ such that

$$\cup_i (A_i)_{[\alpha,\beta]} = H,$$

then $\langle H; \{A_i\}_i \rangle$ is called an interval-valued $[\alpha, \beta]$-fuzzy hypergraph.

2.1. The interval-valued fuzzy hyperoperation $\overline{*}_{\cup}$

In this section, we use $[\alpha, \beta]$-cut to construct an interval-valued fuzzy hyperoperation $\overline{*}_{\cup}$.

Definition 2.2. Let $\Gamma = \langle H; \{A_i\}_i \rangle$ be an interval-valued $[\alpha, \beta]$-fuzzy hypergraph, set $E_{[\alpha,\beta]}(x) = \cup_{A_i(x) \subseteq [\alpha,\beta]} A_i$. The interval-valued fuzzy hypergroupoid $H_\Gamma = \langle H; \overline{*}_\cup \rangle$ where the interval-valued fuzzy hyperoperation $\overline{*}_\cup$ is defined by

$$x \overline{*}_\cup y = E_{[\alpha,\beta]}(x) \cup E_{[\alpha,\beta]}(y), \quad \forall x, y \in H$$

is called an interval-valued $[\alpha, \beta]$-fuzzy hypergraph hypergroupoid.

Proposition 2.3. *The interval-valued $[\alpha, \beta]$-fuzzy hypergraph hypergroupoid H_Γ has the following properties for any $x, y \in H$:*

(1) $x \overline{*}_\cup y = x \overline{*}_\cup x \cup y \overline{*}_\cup y$;

(2) $x \in (x \overline{*}_\cup x)_{[\alpha,\beta]}$.

Proof. (1) $x \overline{*}_\cup y = E_{[\alpha,\beta]}(x) \cup E_{[\alpha,\beta]}(y) = (E_{[\alpha,\beta]}(x) \cup E_{[\alpha,\beta]}(x)) \cup (E_{[\alpha,\beta]}(y) \cup E_{[\alpha,\beta]}(y)) = x \overline{*}_\cup x \cup y \overline{*}_\cup y$.
(2) $(x \overline{*}_\cup x)(x) = (E_{[\alpha,\beta]}(x))(x) = (\cup_{A_i(x) \subseteq [\alpha,\beta]} A_i)(x) = \vee_{A_i(x) \subseteq [\alpha,\beta]} A_i(x) \subseteq [\alpha, \beta]$. ∎

Remark 2.4. $\forall x, y \in H, \quad y \in (x \overline{*}_\cup x)_{[\alpha,\beta]} \not\Rightarrow x \in (y \overline{*}_\cup y)_{[\alpha,\beta]}$.

For example, $\Gamma = \langle \{a, b\}; \{A_1, A_2, A_3\} \rangle$, where

$$\begin{aligned} A_1 &= \frac{[0.2, 0.25]}{a} + \frac{[0.1, 0.3]}{b}, \\ A_2 &= \frac{[0.2, 0.3]}{a} + \frac{[0.25, 0.3]}{b}, \\ A_3 &= \frac{[0.4, 0.5]}{a} + \frac{[0.2, 0.3]}{b}. \end{aligned}$$

Then $(A_1)_{[0.2,0.3]} = \{a\}, (A_2)_{[0.2,0.3]} = \{a, b\}, (A_3)_{[0.2,0.3]} = \{b\}$, and so, Γ is an interval-valued $[0.2, 0.3]$-fuzzy hypergraph since $\cup_{i=1}^{i=3} (A_i)_{[0.2,0.3]} = \{a, b\}$.

$b \in (a \overline{*}_\cup a)_{[0.2,0.3]}$ since $(a \overline{*}_\cup a)(b) = A_1(b) \vee A_2(b) = [0.1, 0.3] \vee [0.25, 0.3] = [0.25, 0.3] \subseteq [0.2, 0.3]$.

But $a \overline{\in} (b \overline{*}_\cup b)_{[0.2,0.3]}$ since $(b \overline{*}_\cup b)(a) = A_2(a) \vee A_3(a) = [0.2, 0.3] \vee [0.4, 0.5] = [0.4, 0.5] \not\subseteq [0.2, 0.3]$.

Lemma 2.5. *For all $[u, v] \subseteq [\alpha, \beta] \subseteq [0, 1]$ and $[x, y] \subseteq [0, 1]$ we have*

$$[u, v] \vee [x, y] \subseteq [\alpha, 1].$$

Proof. From $[u, v] \vee [x, y] = [u \vee x, v \vee y]$, $\alpha \leq v \vee x$ and $b \vee y \leq 1$ we can easily see that the conclusion is correct. ∎

Proposition 2.6. *If an interval-valued hyperstructure $\langle H; \overline{*}_\cup \rangle$ satisfying* (1) *and* (2) *of the previous proposition, then it also satisfies*

(3) $\{x, y\} \subseteq (x \overline{*}_\cup y)_{[\alpha,1]}$;

(4) $x \overline{*}_\cup y = y \overline{*}_\cup x$;

(5) $(x\overline{*}_{\cup}H)_{[\alpha,1]} = H$;

(6) $\langle H; \{x\overline{*}_{\cup}x\}_{x\in H}\rangle$ *is an interval-valued* $[\alpha,\beta]$*-fuzzy hypergraph;*

(7) $x\overline{*}_{\cup}x\overline{*}_{\cup}x = \cup_{(x\overline{*}_{\cup}x)(z)\subseteq(0,1]} z\overline{*}_{\cup}z$;

(8) $(x\overline{*}_{\cup}x)\overline{*}_{\cup}(x\overline{*}_{\cup}x) = x\overline{*}_{\cup}x\overline{*}_{\cup}x$.

Proof. Let us use (1), (2) of the previous proposition to prove (3)-(8).

(3) $(x\overline{*}_{\cup}y)(x) = (x\overline{*}_{\cup}x \cup y\overline{*}_{\cup}y)(x) = (x\overline{*}_{\cup}x)(x) \vee (y\overline{*}_{\cup}y)(x) \subseteq [\alpha,1]$.
So $x \in (x\overline{*}_{\cup}y)_{[\alpha,1]}$. Similarly we can prove $y \in (x\overline{*}_{\cup}y)_{[\alpha,1]}$.

(4) $x\overline{*}_{\cup}y = x\overline{*}_{\cup}x \cup y\overline{*}_{\cup}y = y\overline{*}_{\cup}y \cup x\overline{*}_{\cup}x = y\overline{*}_{\cup}x$.

(5) for any $y \in H$, $(x\overline{*}_{\cup}H)(y) = (\cup_{t\in H} x\overline{*}_{\cup}t)(y) = (\cup_{t\in H}(x\overline{*}_{\cup}x \cup t\overline{*}_{\cup}t))(y) = ((y\overline{*}_{\cup}y) \cup (\cup_{t\in H}(x\overline{*}_{\cup}x \cup t\overline{*}_{\cup}t)))(y) = (y\overline{*}_{\cup}y)(y) \vee (\cup_{t\in H} x\overline{*}_{\cup}t)(y) \subseteq [\alpha,1]$ and thus $y \in (x\overline{*}_{\cup}H)_{[\alpha,1]}$.
So, $H \subseteq (x\overline{*}_{\cup}H)_{[\alpha,1]}$ and, finally, $H = (x\overline{*}_{\cup}H)_{[\alpha,1]}$.

(6) From $x \in (x\overline{*}_{\cup}x)_{[\alpha,\beta]}$ we know $\cup_{x\in H}(x\overline{*}_{\cup}x)_{[\alpha,\beta]} = H$.
And then $< H; \{x\overline{*}_{\cup}x\}_{x\in H} >$ is an interval-valued $[\alpha,\beta]$-fuzzy hypergraph.

(7) $x\overline{*}_{\cup}x\overline{*}_{\cup}x = \cup_{(x\overline{*}_{\cup}x)(z)\subseteq(0,1]} z\overline{*}_{\cup}x = \cup_{(x\overline{*}_{\cup}x)(z)\subseteq(0,1]}(z\overline{*}_{\cup}z) \cup (x\overline{*}_{\cup}x)$
$= \cup_{(x\overline{*}_{\cup}x)(z)\subseteq(0,1]} z\overline{*}_{\cup}z$.

(8) $(x\overline{*}_{\cup}x)\overline{*}_{\cup}(x\overline{*}_{\cup}x) = \cup_{(x\overline{*}_{\cup}x)(a)\subseteq(0,1],(x\overline{*}_{\cup}x)(b)\subseteq(0,1]} a\overline{*}_{\cup}b$
$= \cup_{(x\overline{*}_{\cup}x)(a)\subseteq(0,1],(x\overline{*}_{\cup}x)(b)\subseteq(0,1]}(a\overline{*}_{\cup}a \cup b\overline{*}_{\cup}b) = \cup_{(x\overline{*}_{\cup}x)(a)\subseteq(0,1]} a\overline{*}_{\cup}a = x\overline{*}_{\cup}x\overline{*}_{\cup}x$. ∎

Remark 2.7. An interval-valued $[\alpha,\beta]$-fuzzy quasi-hypergroup is an interval-valued fuzzy hypergroupoid such that

$$(x * H)_{[\alpha,\beta]} = H = (H * x)_{[\alpha,\beta]}, \quad \forall x \in H.$$

From (4) and (5) of the previous we know that H_Γ is a commutative interval-valued $[\alpha,\beta]$-fuzzy quasi-hypergroup if $\beta = 1$.

2.2. The interval-valued fuzzy hyperoperation $\overline{*}_{\cap}$

In this section, we use $[\alpha,\beta]$-cut to construct an interval-valued fuzzy hyperoperation $\overline{*}_{\cap}$.

Definition 2.8. Let $\Gamma = \langle H; \{A_i\}_i\rangle$ be an interval-valued $[\alpha,\beta]$-fuzzy hypergraph, set $D_{[\alpha,\beta]}(x) = \cap_{A_i(x)\subseteq[\alpha,\beta]} A_i$. The fuzzy hyperoperation $\overline{*}_{\cap}$ is defined by

$$x\overline{*}_{\cap}y \doteq D_{[\alpha,\beta]}(x) \cup D_{[\alpha,\beta]}(y), \quad \forall x,y \in H.$$

Proposition 2.9. *For all* $x,y \in H$*, we have:*

(1) $x\overline{*}_{\cap}y = x\overline{*}_{\cap}x \cup y\overline{*}_{\cap}y$;

(2) $x \in (x\overline{*}_{\cap}x)_{[\alpha,\beta]}$.

Proof. (1) $x\overline{*}_{\cap}y = D_{[\alpha,\beta]}(x) \cup D_{[\alpha,\beta]}(y) = (D_{[\alpha,\beta]}(x) \cup D_{[\alpha,\beta]}(x)) \cup (D_{[\alpha,\beta]}(y) \cup D_{[\alpha,\beta]}(y)) = x\overline{*}_{\cap}x \cup y\overline{*}_{\cap}y$.

(2) $(x\overline{*}_{\cap}x)(x) = (D_{[\alpha,\beta]}(x))(x) = \wedge_{A_i(x)\subseteq[\alpha,\beta]}A_i(x) \subseteq [\alpha,\beta]$. ∎

Proposition 2.10. *$\langle H; \overline{*}_{\cap}\rangle$ also has the following properties.*

(3) $\{x, y\} \subseteq (x\overline{*}_{\cap}y)_{[\alpha,1]}$;

(4) $x\overline{*}_{\cap}y = y\overline{*}_{\cap}x$;

(5) $(x\overline{*}_{\cap}H)_{[\alpha,1]} = H$;

(6) $\langle H; \{x\overline{*}_{\cap}x\}_{x\in H}\rangle$ *is an interval-valued $[\alpha,\beta]$-fuzzy hypergraph;*

(7) $x\overline{*}_{\cap}x\overline{*}_{\cap}x = \cup_{(x\overline{*}_{\cap}x)(z)\subseteq(0,1]}z\overline{*}_{\cap}z$;

(8) $(x\overline{*}_{\cap}x)\overline{*}_{\cap}(x\overline{*}_{\cap}x) = x\overline{*}_{\cap}x\overline{*}_{\cap}x$.

Proof. Applied Proposition 2.6. ∎

Remark 2.11. From (4), (5) of the previous proposition we know that $\langle H; \overline{*}_{\cap}\rangle$ is also a commutative interval-valued $[\alpha,\beta]$-fuzzy quasi-hypergroup if $\beta = 1$.

Remark 2.12. $\forall x, y \in H$,

$$y \in (x\overline{*}_{\cap}x)_{[\alpha,\beta]} \nRightarrow x \in (y\overline{*}_{\cap}y)_{[\alpha,\beta]}.$$

For example,

$$\Gamma = \langle \{a, b\}; \{A_1, A_2, A_3\} \rangle,$$

where

$$\begin{aligned} A_1 &= \frac{[0.2, 0.25]}{a} + \frac{[0.2, 0.4]}{b}, \\ A_2 &= \frac{[0.2, 0.3]}{a} + \frac{[0.2, 0.25]}{b}, \\ A_3 &= \frac{[0.1, 0.2]}{a} + \frac{[0.2, 0.3]}{b}. \end{aligned}$$

Then

$$(A_1)_{[0.2,0.3]} = \{a\},\ (A_2)_{[0.2,0.3]} = \{a, b\},\ (A_3)_{[0.2,0.3]} = \{b\},$$

and so Γ is an interval-valued $[0.2, 0.3]$-fuzzy hypergraph since

$$\cup_{i=1}^{i=3}(A_i)_{[0.2,0.3]} = \{a, b\}.$$

$b \in (a\overline{*}_{\cup}a)_{[0.2,0.3]}$ since

$$(a\overline{*}_{\cap}a)(b) = A_1(b) \wedge A_2(b) = [0.2, 0.4] \wedge [0.2, 0.25] = [0.2, 0.25] \subseteq [0.2, 0.3].$$

But $a\overline{\in}(b\overline{*}_{\cup}b)_{[0.2,0.3]}$ since

$$(b\overline{*}_{\cup}b)(a) = A_2(a) \wedge A_3(a) = [0.2, 0.3] \wedge [0.1, 0.2] = [0.1, 0.2] \not\subseteq [0.2, 0.3].$$

3. Interval-valued fuzzy hyperoperations obtained from $\langle\alpha,\beta\rangle$-cut

Definition 3.1. M is an interval-valued fuzzy subset of H, set

$$M(x) = [M^-(x), M^+(x)],\ \forall x \in H.$$

$M^- : H \to [0,1]$ and $M^+ : H \to [0,1]$ are called lower fuzzy set and upper fuzzy set about M.

The $\langle\alpha,\beta\rangle$-cut of an interval-valued fuzzy subset M of H is defined by

$$M_{\langle\alpha,\beta\rangle} \doteq \{x \in H \mid M^-(x) \geq \alpha, M^+(x) \geq \beta\}.$$

Definition 3.2. H is a nonempty set, $\{A_i\}_i$ is a family of interval-valued fuzzy subsets of H, if there exists $\alpha,\beta \in (0,1]$ such that

$$\cup_i (A_i)_{\langle\alpha,\beta\rangle} = H,$$

then $\langle H; \{A_i\}_i\rangle$ is called an interval-valued $\langle\alpha,\beta\rangle$-fuzzy hypergraph.

3.1. The interval-valued fuzzy hyperoperation $\overline{\circ}_\cup$

In this section, we use $<\alpha,\beta>$-cut to construct an interval-valued fuzzy hyperoperation $\overline{\circ}_\cup$.

Definition 3.3. Let $\Gamma = \langle H; \{A_i\}_i\rangle$ be an interval-valued $\langle\alpha,\beta\rangle$-fuzzy hypergraph, set $E_{\langle\alpha,\beta\rangle}(x) = \cup_{A_i^-(x)\geq\alpha, A_i^+(x)\geq\beta} A_i$. The interval-valued fuzzy hypergroupoid $H_\Gamma = \langle H; \overline{\circ}_\cup\rangle$ where the interval-valued fuzzy hyperoperation $\overline{\circ}_\cup$ is defined by

$$x\overline{\circ}_\cup y = E_{\langle\alpha,\beta\rangle}(x) \cup E_{\langle\alpha,\beta\rangle}(y), \quad \forall x,y \in H$$

is called an interval-valued $<\alpha,\beta>$-fuzzy hypergraph hypergroupoid.

Proposition 3.4. *The interval-valued $\langle\alpha,\beta\rangle$-fuzzy hypergraph hypergroupoid H_Γ has the following properties for any $x,y \in H$:*

(1) $x\overline{\circ}_\cup y = x\overline{\circ}_\cup x \cup y\overline{\circ}_\cup y$;

(2) $x \in (x\overline{\circ}_\cup x)_{\langle\alpha,\beta\rangle}$.

Proof. (1) $x\overline{\circ}_\cup y = E_{\langle\alpha,\beta\rangle}(x) \cup E_{\langle\alpha,\beta\rangle}(y) = (E_{\langle\alpha,\beta\rangle}(x) \cup E_{\langle\alpha,\beta\rangle}(x)) \cup (E_{[\alpha,\beta]}(y) \cup E_{\langle\alpha,\beta\rangle}(y)) = x\overline{\circ}_\cup x \cup y\overline{\circ}_\cup y$.

(2) $(x\overline{\circ}_\cup x)(x) = (E_{\langle\alpha,\beta\rangle}(x))(x) = (\cup_{A_i^-(x)\geq\alpha, A_i^+(x)\geq\beta} A_i)(x) = \vee_{A_i^-(x)\geq\alpha, A_i^+(x)\geq\beta} A_i(x)$.

Since $\cup_i (A_i)_{\langle\alpha,\beta\rangle} = H$, then for all $x \in H$ there exists some A_i such that $x \in (A_j)_{\langle\alpha,\beta\rangle}$ and thus $A_j^-(x) \geq \alpha, A_j^+(x) \geq \beta$.

So $(x\overline{\circ}_\cup x)^-(x) = \vee_{A_i^-(x)\geq\alpha, A_i^+(x)\geq\beta} A_i^-(x) \geq \alpha$.

Similarly, $(x\overline{\circ}_\cup x)^+(x) = \vee_{A_i^-(x)\geq\alpha, A_i^+(x)\geq\beta} A_i^+(x) \geq \beta$.

So $x \in (x\overline{\circ}_\cup x)_{\langle\alpha,\beta\rangle}$. ∎

Proposition 3.5. *If an interval-valued hyperstructure* $< H; \overline{\circ}_{\cup} >$ *satisfying* (1) *and* (2) *of the previous proposition, then it also satisfies*

(3) $\{x, y\} \subseteq (x\overline{\circ}_{\cup}y)_{\langle\alpha,\beta\rangle}$;

(4) $x\overline{\circ}_{\cup}y = y\overline{\circ}_{\cup}x$;

(5) $(x\overline{\circ}_{\cup}H)_{\langle\alpha,\beta\rangle} = H$;

(6) $\langle H; \{x\overline{\circ}_{\cup}x\}_{x\in H}\rangle$ *is an interval-valued* $\langle\alpha, \beta\rangle$*-fuzzy hypergraph;*

(7) $x\overline{\circ}_{\cup}x\overline{\circ}_{\cup}x = \cup_{(x\overline{\circ}_{\cup}x)(z)\subseteq(0,1]}z\overline{\circ}_{\cup}z$;

(8) $(x\overline{\circ}_{\cup}x)\overline{\circ}_{\cup}(x\overline{\circ}_{\cup}x) = x\overline{\circ}_{\cup}x\overline{\circ}_{\cup}x$.

Proof. Let's use the (1) and (2) of the previous proposition to prove (3)-(8).

(3) Since $(x\overline{\circ}_{\cup}y)(x) = (x\overline{\circ}_{\cup}x \cup y\overline{\circ}_{\cup}y)(x) = (\vee_{A_i^-(x)\geq\alpha, A_i^+(x)\geq\beta}A_i(x))$ $\vee\ (\vee_{A_i^-(y)\geq\alpha, A_i^+(y)\geq\beta}A_i(x))$, then

$$(x\overline{\circ}_{\cup}y)^+(x) = (\vee_{A_i^-(x)\geq\alpha, A_i^+(x)\geq\beta}A_i^+(x)) \vee (\vee_{A_i^-(y)\geq\alpha, A_i^+(y)\geq\beta}A_i^+(x)) \geq \beta.$$

Similarly, $(x\overline{\circ}_{\cup}y)^-(x) \geq \alpha$. And so $x \in (x\overline{\circ}_{\cup}y)_{\langle\alpha,\beta\rangle}$. The other part is proved similarly.

(4) $x\overline{\circ}_{\cup}y = x\overline{\circ}_{\cup}x \cup y\overline{\circ}_{\cup}y = y\overline{\circ}_{\cup}y \cup x\overline{\circ}_{\cup}x = y\overline{\circ}_{\cup}x$.

(5) $\forall y \in H$, $(x\overline{\circ}_{\cup}H)(y) = (\cup_{t\in H}(x\overline{\circ}_{\cup}t))(y) = (\cup_{t\in H}(x\overline{\circ}_{\cup}x\cup\overline{\circ}_{\cup}t))(y) = (y\overline{\circ}_{\cup}y\cup$ $(\cup_{t\in H}(x\overline{\circ}_{\cup}t)))(y) = (\vee_{A_i^-(x)\geq\alpha, A_i^+(x)\geq\beta}A_i(y)) \vee (\cup_{t\in H}(x\overline{\circ}_{\cup}t))(y)$.

Thus $(x\overline{\circ}_{\cup}H)^-(y) \geq \alpha$ and $(x\overline{\circ}_{\cup}H)^+(y) \geq \beta$. And so $(x *_{\cup} H)_{\langle\alpha,\beta\rangle} \supseteq H$. Therefore, $(x *_{\cup} H)_{\langle\alpha,\beta\rangle} = H$.

(6) From $x \in (x\overline{\circ}_{\cup}x)_{\langle\alpha,\beta\rangle}$ we know $\cup_{x\in H}(x\overline{\circ}_{\cup}x)_{\alpha,\beta} = H$.

And then $\langle H; \{x\overline{\circ}_{\cup}x\}_{x\in H}\rangle$ is an interval-valued $\langle\alpha, \beta\rangle$-fuzzy hypergraph.

(7) $x\overline{\circ}_{\cup}x\overline{\circ}_{\cup}x = \cup_{(x\overline{\circ}_{\cup}x)(z)\subseteq(0,1]}z\overline{\circ}_{\cup}x = \cup_{(x\overline{\circ}_{\cup}x)(z)\subseteq(0,1]}((z\overline{\circ}_{\cup}z) \cup (x\overline{\circ}_{\cup}x))$ $= \cup_{(x\overline{\circ}_{\cup}x)(z)\subseteq(0,1]}z\overline{\circ}_{\cup}z$.

(8) $(x\overline{\circ}_{\cup}x)\overline{\circ}_{\cup}(x\overline{\circ}_{\cup}x) = \cup_{(x\overline{\circ}_{\cup}x)(u)\subset(0,1],(x\overline{\circ}_{\cup}x)(v)\subset(0,1]}u\overline{\circ}_{\cup}v$ $= \cup_{(x\overline{\circ}_{\cup}x)(u)\subset(0,1],(x\overline{\circ}_{\cup}x)(v)\subset(0,1]}(u\overline{\circ}_{\cup}u \cup v\overline{\circ}_{\cup}v) = \cup_{(x\overline{\circ}_{\cup}x)(u)\subset(0,1]}u\overline{\circ}_{\cup}u = x\overline{\circ}_{\cup}x\overline{\circ}_{\cup}x$. ∎

Remark 3.6. An interval-valued $\langle\alpha, \beta\rangle$-fuzzy quasi-hypergroup is an interval-valued fuzzy hypergroupoid such that

$$(x * H)_{\langle\alpha,\beta\rangle} = H = (H * x)_{\langle\alpha,\beta\rangle}, \quad \forall x \in H.$$

From (4) and (5) of the previous proposition we know that H_Γ is a commutative interval-valued $\langle\alpha, \beta\rangle$-fuzzy quasi-hypergroup.

3.2. The interval-valued fuzzy hyperoperation $\overline{\circ}_{\cap}$

Definition 3.7. *Let* $\Gamma = \langle H; \{A_i\}_i\rangle$ *be an interval-valued* $\langle\alpha, \beta\rangle$*-fuzzy hypergraph, set* $D_{\langle\alpha,\beta\rangle}(x) = \cap_{A_i^-(x)\geq\alpha, A_i^+(x)\geq\beta}A_i$. *The interval-valued fuzzy hyperoperation* $\overline{\circ}_{\cap}$ *is defined by*

$$x\overline{\circ}_{\cap}y = D_{\langle\alpha,\beta\rangle}(x) \cup D_{\langle\alpha,\beta\rangle}(y), \quad \forall x, y \in H.$$

Proposition 3.8. *The interval-valued fuzzy hyperoperation $\overline{\circ}_{\cap}$ has the following properties for any $x, y \in H$:*

(1) $x\overline{\circ}_{\cap}y = x\overline{\circ}_{\cap}x \cup y\overline{\circ}_{\cap}y$;

(2) $x \in (x\overline{\circ}_{\cap}x)_{\langle\alpha,\beta\rangle}$.

Proof. (1) $x\overline{\circ}_{\cap}y = D_{\langle\alpha,\beta\rangle}(x) \cup D_{\langle\alpha,\beta\rangle}(y) = (D_{\langle\alpha,\beta\rangle}(x) \cup D_{\langle\alpha,\beta\rangle}(x)) \cup (D_{[\alpha,\beta]}(y) \cup D_{\langle\alpha,\beta\rangle}(y)) = x\overline{\circ}_{\cap}x \cup y\overline{\circ}_{\cap}y$.

(2) $(x\overline{\circ}_{\cap}x)(x)=(D_{\langle\alpha,\beta\rangle}(x))(x)=(\cap_{A_i^-(x)\geq\alpha,A_i^+(x)\geq\beta}A_i)(x)=\wedge_{A_i^-(x)\geq\alpha,A_i^+(x)\geq\beta}A_i(x)$.

Since $\cup_i(A_i)_{\langle\alpha,\beta\rangle} = H$, then for all $x \in H$ there exists some A_i such that $x \in (A_j)_{\langle\alpha,\beta\rangle}$ and thus $A_j^-(x) \geq \alpha, A_j^+(x) \geq \beta$.

So $(x\overline{\circ}_{\cup}x)^-(x) = \wedge_{A_i^-(x)\geq\alpha,A_i^+(x)\geq\beta}A_i^-(x) \geq \alpha$ and

$(x\overline{\circ}_{\cup}x)^+(x) = \wedge_{A_i^-(x)\geq\alpha,A_i^+(x)\geq\beta}A_i^+(x) \geq \beta$.

So $x \in (x\overline{\circ}_{\cap}x)_{<\alpha,\beta>}$. ∎

Proposition 3.9. *The interval-valued fuzzy hyperoperation $\overline{\circ}_{\cup}$ also satisfies*

(3) $\{x, y\} \subseteq (x\overline{\circ}_{\cap}y)_{\langle\alpha,\beta\rangle}$;

(4) $x\overline{\circ}_{\cap}y = y\overline{\circ}_{\cap}x$;

(5) $(x\overline{\circ}_{\cap}H)_{\langle\alpha,\beta\rangle} = H$;

(6) $\langle H; \{x\overline{\circ}_{\cap}x\}_{x\in H}\rangle$ *is an interval-valued $\langle\alpha,\beta\rangle$-fuzzy hypergraph;*

(7) $x\overline{\circ}_{\cap}x\overline{\circ}_{\cap}x = \cup_{(x\overline{\circ}_{\cap}x)(z)\subseteq(0,1]}z\overline{\circ}_{\cap}z$;

(8) $(x\overline{\circ}_{\cap}x)\overline{\circ}_{\cap}(x\overline{\circ}_{\cap}x) = x\overline{\circ}_{\cap}x\overline{\circ}_{\cap}x$.

Proof. Applied Proposition 3.5. ∎

Remark 3.10. From (4) and (5) of the previous we know that $\langle H; \overline{\circ}_{\cap}\rangle$ is also a commutative interval-valued $\langle\alpha,\beta\rangle$-fuzzy quasi-hypergroup .

Proposition 3.11. $\forall x, y \in H,\ y \in (x\overline{*}_{\cap}x)_{\langle\alpha,\beta\rangle} \Rightarrow (y\overline{*}_{\cap}y)(x) \leq (x\overline{*}_{\cap}x)(x)$.

Proof. Since $(x\overline{*}_{\cap}x)^-(y) = \wedge_{A_i^-(x)\geq\alpha,A_i^+(x)\geq\beta}A_i^-(y) \geq \alpha$, then $A_i^-(y) \geq \alpha$ for all $A_i \in IF(H)$, where $A_i^-(x) \geq \alpha$. Similarly, $A_i^+(y) \geq \beta$ for all $A_i \in IF(H)$, where $A_i^+(x) \geq \beta$. Thus

$$\begin{aligned}(y\overline{*}_{\cap}y)^-(x) &= \wedge_{A_i^-(y)\geq\alpha,A_i^+(y)\geq\beta}A_i^-(x)\\ &\leq (\wedge_{A_i^-(y)\geq\alpha,A_i^+(y)\geq\beta,A_i^-(x)\geq\alpha,A_i^+(x)\geq\beta}A_i^-(x))\\ &= \wedge_{A_i^-(x)\geq\alpha,A_i^+(x)\geq\beta}A_i^-(x)\\ &= (x\overline{*}_{\cap}x)^-(x).\end{aligned}$$

Similarly, $(y\overline{*}_{\cap}y)^+(x) \leq (x\overline{*}_{\cap}x)^+(x)$.

Therefore, $(y\overline{*}_{\cap}y)(x) \leq (x\overline{*}_{\cap}x)(x)$. ∎

3.3. The interval-valued fuzzy hyperstructure $\langle H; \overline{\circ}_{\cap}, \overline{\circ}_{\cup}\rangle$

Definition 3.12. Given the interval-valued fuzzy hyperoperations $\widetilde{\triangle}$, $\widetilde{\bigtriangledown}$ on X, we say that $\left\langle X; \widetilde{\triangle}, \widetilde{\bigtriangledown}\right\rangle$ is an interval-valued $\langle\alpha, \beta\rangle$-fuzzy superlattice iff the following properties hold for all $x, y, z \in X$.

(1) $x \in (x\widetilde{\triangle}x)_{\langle\alpha,\beta\rangle}$, $x \in (x\widetilde{\bigtriangledown}x)_{\langle\alpha,\beta\rangle}$.

(2) $x\widetilde{\triangle}y = y\widetilde{\triangle}x$, $x\widetilde{\bigtriangledown}y = y\widetilde{\bigtriangledown}x$.

(3) $((x\widetilde{\triangle}y)\widetilde{\triangle}z)_{\langle\alpha,\beta\rangle} = (x\widetilde{\triangle}(y\widetilde{\triangle}z))_{\langle\alpha,\beta\rangle}$, $((x\widetilde{\bigtriangledown}y)\widetilde{\bigtriangledown}z)_{\langle\alpha,\beta\rangle} = (x\widetilde{\bigtriangledown}(y\widetilde{\bigtriangledown}z))_{\langle\alpha,\beta\rangle}$.

(4) $x \in ((x\widetilde{\triangle}y)\widetilde{\bigtriangledown}x)_{\langle\alpha,\beta\rangle}$, $x \in ((x\widetilde{\bigtriangledown}y)\widetilde{\triangle}x)_{\langle\alpha,\beta\rangle}$.

(5) $y \in (x\widetilde{\bigtriangledown}y)_{\langle\alpha,\beta\rangle} \Leftrightarrow x \in (x\widetilde{\triangle}y)_{\langle\alpha,\beta\rangle}$.

(6) $x, y \in (x\widetilde{\bigtriangledown}y)_{\langle\alpha,\beta\rangle} \Rightarrow x = y$.

(7) $y \in (x\widetilde{\bigtriangledown}y)_{\langle\alpha,\beta\rangle}$ et $z \in (y\widetilde{\bigtriangledown}z)_{\langle\alpha,\beta\rangle} \Rightarrow z \in (x\widetilde{\bigtriangledown}z)_{\langle\alpha,\beta\rangle}$.

Proposition 3.13. *The fuzzy hyperstructure $\langle H; \overline{*}_{\cap}, \overline{*}_{\cup}\rangle$ has the following properties for all $x, y, z \in H$.*

(1) $x \in (x\overline{\circ}_{\cap}x)_{\langle\alpha,\beta\rangle}$, $x \in (x\overline{\circ}_{\cup}x)_{\langle\alpha,\beta\rangle}$.

(2) $x\overline{\circ}_{\cap}y = y\overline{\circ}_{\cap}x$, $x\overline{\circ}_{\cup}y = y\overline{\circ}_{\cup}x$.

(3) $((x\overline{\circ}_{\cap}y)\overline{\circ}_{\cap}z)_{\langle\alpha,\beta\rangle} \cap (x\overline{\circ}_{\cap}(y\overline{\circ}_{\cap}z))_{\langle\alpha,\beta\rangle} \neq \emptyset$,
$((x\overline{\circ}_{\cup}y)\overline{\circ}_{\cup}z)_{\langle\alpha,\beta\rangle} \cap (x\overline{\circ}_{\cup}(y\overline{\circ}_{\cup}z))_{\langle\alpha,\beta\rangle} \neq \emptyset$.

(4) $x \in ((x\overline{\circ}_{\cap}y)\overline{\circ}_{\cup}x)_{\langle\alpha,\beta\rangle}$, $x \in ((x\overline{\circ}_{\cup}y)\overline{\circ}_{\cap}x)_{\langle\alpha,\beta\rangle}$.

(5) $y \in (x\overline{\circ}_{\cup}y)_{\langle\alpha,\beta\rangle} \Leftrightarrow x \in (x\overline{\circ}_{\cap}y)_{\langle\alpha,\beta\rangle}$.

(6) $x, y \in (x\overline{\circ}_{\cup}y)_{\langle\alpha,\beta\rangle} \nRightarrow x = y$.

(7) $y \in (x\overline{\circ}_{\cup}y)_{\langle\alpha,\beta\rangle}$ *et* $z \in (y\overline{\circ}_{\cup}z)_{\langle\alpha,\beta\rangle} \Rightarrow z \in (x\overline{\circ}_{\cup}z)_{\langle\alpha,\beta\rangle}$.

Proof. It is enough to prove (3), (4) and (6). The others are clear.

(3) Since $((x\overline{\circ}_{\cap}y)\overline{\circ}_{\cap}z)^{-}(y) \geq (y\overline{\circ}_{\cap}z)^{-}(y) \geq \alpha$ and $(x\overline{\circ}_{\cap}(y\overline{\circ}_{\cap}z))^{-}(y) \geq (x\overline{\circ}_{\cap}y)^{-}(y) \geq \alpha$.

Similarly, $((x\overline{\circ}_{\cap}y)\overline{\circ}_{\cap}z)^{+}(y) \geq \beta$ and $(x\overline{\circ}_{\cap}(y\overline{\circ}_{\cap}z))^{+}(y) \geq \beta$.

Then $y \in ((x\overline{\circ}_{\cap}y)\overline{\circ}_{\cap}z)_{\langle\alpha,\beta\rangle} \cap (x\overline{\circ}_{\cap}(y\overline{\circ}_{\cap}z))_{\langle\alpha,\beta\rangle}$.

And so $((x\overline{\circ}_{\cap}y)\overline{\circ}_{\cap}z)_{\langle\alpha,\beta\rangle} \cap (x\overline{\circ}_{\cap}(y\overline{\circ}_{\cap}z))_{\langle\alpha,\beta\rangle} \neq \emptyset$.

The other part is proved similarly.

(4)$((x\overline{\circ}_{\cap}y)\overline{\circ}_{\cup}x)^{-}(x) \geq (x\overline{\circ}_{\cup}x)^{-}(x) \geq \alpha$;

$((x\overline{\circ}_{\cup}y)\overline{\circ}_{\cap}x)^{-}(x) \geq (x\overline{\circ}_{\cap}x)^{-}(x) \geq \alpha$.

Similarly, $((x\overline{\circ}_{\cap}y)\overline{\circ}_{\cup}x)^{+}(x) \geq \beta$; $((x\overline{\circ}_{\cup}y)\overline{\circ}_{\cap}x)^{+}(x) \geq \beta$.

So $x \in ((x\overline{\circ}_{\cap}y)\overline{\circ}_{\cup}x)_{\langle\alpha,\beta\rangle}$.

The other part is proved similarly.

(6)For example, the interval-valued $\langle 0.2, 0.5\rangle$-fuzzy hypergraph is

$$\langle \{x, y\}; \{A_1, A_2\} \rangle ,$$

where

$$A_1 = \frac{[0.2, 0.5]}{x} + \frac{[0.4, 0.8]}{y} \text{ and } A_2 = \frac{[0.2, 0.3]}{x} + \frac{[0.2, 0.5]}{y}.$$

Of course we have $x, y \in x\overline{\circ}_{\cap}y$. But $x \neq y$. ∎

Remark 3.14. From the previous proposition we can see that the hyperstructure $\langle H; \overline{\circ}_{\cap}, \overline{\circ}_{\cup}\rangle$ is almost an interval-valued $\langle \alpha, \beta\rangle$-fuzzy superlattice.

4. Further research

Definition 4.1. for all $[a, b], [x, y] \in I([0, 1])$,$p \in [0, 1]$ we define

$$[a, b] \vee_p [x, y] \doteq [(a \vee x) \wedge p, (b \vee y) \vee p'].$$
$$[a, b] \wedge_p [x, y] \doteq [(a \wedge x) \wedge p, (b \wedge y) \vee p'].$$

Definition 4.2. Given $A, B \in IF(H)$, we give the following definitions

$$(A \cup_p B)(x) \doteq A(x) \vee_p B(x), \forall x \in H.$$
$$(A \cap_p B)(x) \doteq A(x) \wedge_p B(x), \forall x \in H.$$

The present work can be extended in several directions. Let us indicate some possibilities.

1. We can use the above definitions to reconstruct interval-valued fuzzy hyperoperations and study the properties of them and the relations between them.
2. One may explore the interval-valued fuzzy join space by the use of the hyperoperations.
3. One can also investigate the hyperstructure associated to the product of interval-valued hypergraphs (see [1]).

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SYNTACTIC FUZZY MONOIDS

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Abstract. Fuzzy monoids and fuzzy congruences are introduced and the syntactic fuzzy monoid M_L associated to a subset L of a fuzzy monoid M is constructed. It is shown that M_L is minimal among all fuzzy epimorphisms $h : M \to M'$ whose kernel saturates L. The subset L is said to be fuzzy recognizable whenever M_L is finite. The so obtained class is closed under boolean operations and inverse morphisms.

Keywords: fuzzy sets, recognizability, monoid.

1. Introduction

Traditionally, a fuzzy subset F of a semigroup S is called a fuzzy subsemigroup of S if

$$F(ab) \geq F(a) \wedge F(b),$$

for all $a, b \in S$ [6], [7]. On the other hand, a nondeterministic semigroup is a set S endowed with an associative nondeterministic operation $M \times M \to \mathcal{P}(M)$ [1], [4], [5]. In the present paper we extend nondeterminism to the fuzzy setup. Thus a fuzzy semigroup in our sense is a set S equipped with an associative fuzzy operation $S \times S \to Fuz(S)$, which to every pair (s_1, s_2) of elements of S assigns a fuzzy subset $s_1 s_2$ of S.

Hence in the first approach in order to obtain a fuzzy algebraic structure one takes a fuzzy subset of a crisp algebraic structure while in our approach we construct a fuzzy structure by means of a fuzzy operation. Moreover, traditionally a fuzzy congruence on a crisp monoid S is a fuzzy equivalence $R : S \times S \to [0, 1]$ satisfying

$$R(sa, sb) \geq R(ab), \quad R(as, bs) \geq R(ab).$$

for all $a, b \in S$. Whereas, a congruence in our sense, is an equivalence $\sim$ on a fuzzy semigroup S such that $s_1 \sim s_1'$ and $s_2 \sim s_2'$ imply

$$\sup_{n \in C}, (s_1 s_2)(n) = \sup_{n \in C}(s_1' s_2')(n)$$

for all $\sim$-classes C. It is clear that the proposed fuzzy monoid notion significantly diverges from the traditional and that it fuzzifies the nondeterministic case.

Naturally, in the proposed setup, a fuzzy monoid M is a fuzzy semigroup together with a unit element. The quotient $M/\sim$ of a fuzzy monoid M by a fuzzy congruence $\sim$, admits a fuzzy monoid structure rendering the canonical function $m \mapsto [m]$ an epimorphism of fuzzy monoids. The classical Isomorphism Theorem of Algebra still holds in the fuzzy setup.

> For any epimorphism of fuzzy monoids $h : M \to M'$ and every fuzzy congruence $\sim$ on M' its inverse image $h^{-1}(\sim)$ defined by
>
> $$m_1 h^{-1}(\sim) m_2 \quad \text{iff} \quad h(m_1) \sim h(m_2),$$
>
> is again a fuzzy congruence and the quotient fuzzy monoids $M/h^{-1}(\sim)$ and $M'/\sim$ are isomorphic. In particular if $\sim$ is the equality, then $h^{-1}(=)$ is the kernel congruence of h (denoted by $\sim_h$)
>
> $$m_1 \sim_h m_2 \quad \text{iff} \quad h(m_1) = h(m_2),$$
>
> and the fuzzy monoids $M/\sim_h$ and M' are isomorphic.

We show that fuzzy congruences are closed under the join operation. This allows us to construct the greatest fuzzy congruence included in an equivalence $\sim$. It is the join of all fuzzy congruences on M included into $\sim$ and it is denoted by $\sim^{fuz}$. The quotient fuzzy monoid $M/\sim^{fuz}$ is denoted by M^{fuz} and has the following universal property.

> Given an epimorphism of fuzzy monoids $h : M \to M'$ whose kernel $\sim_h$ saturates the equivalence $\sim$ there exists a unique epimorphism of fuzzy monoids $h' : M' \to M^{fuz}$ such that $h' \circ h = h^{fuz}$, where $h^{fuz} : M \to M^{fuz}$ is the canonical epimorphism into the quotient.

This result states that h^{fuz} is minimal among all epimorphisms saturating $\sim$. Let M be a fuzzy monoid and $L \subseteq M$. Denote by $\sim_L$ the greatest congruence of M included in the partition (equivalence) $\{L, M-L\}$, i.e., $\sim_L = \{L, M-L\}^{fuz}$. The quotient fuzzy monoid $M_L = M/\sim_L$ will be called the syntactic fuzzy monoid of L and it is characterized by the following universal property.

> For every fuzzy monoid M and every epimorphism $h : M \to M'$ verifying $h^{-1}(h(L)) = L$, there exists a unique epimorphism $h' : M' \to M_L$ such that $h' \circ h = h_L$ where $h_L : M \to M_L$ is the canonical projection into the quotient.

A subset L of a fuzzy monoid M is fuzzy recognizable if there exist a finite fuzzy monoid M' and a morphism $h : M \to M'$ such that $h^{-1}(h(L)) = L$. By taking into account the previous result we get that L is recognizable if and only if its syntactic fuzzy monoid is finite. Moreover fuzzy recognizable subsets are closed under boolean operations and inverse morphisms.

2. Fuzzy monoids

We first recall that for every $a \in [0,1]$ and every fuzzy set $F : M \to [0,1]$, the fuzzy set $a \wedge F : M \to [0,1]$ is defined by

$$(a \wedge F)(m) = a \wedge F(m),$$

for all $m \in M$. It is easy to verify the rules

$$(a_1 \wedge a_2) \wedge F = a_1 \wedge (a_2 \wedge F), \quad a \wedge (F_1 \wedge F_2) = (a \wedge F_1) \wedge F_2$$

for all $a_1, a_2 \in [0,1]$ and all fuzzy sets F, F_1, F_2 of M.

Every fuzzy set $F : M \to [0,1]$ is written

$$F = \sup_{m \in M} F(m) \wedge \hat{m}$$

where $\hat{m} : M \to [0,1]$ denotes the singleton function: $\hat{m}(\eta) = 1$, if $\eta = m$ and $\hat{m}(\eta) = 0$ if $\eta \neq m$. Indeed, for every $\eta \in M$ we have

$$\left(\sup_{m \in M} F(m) \wedge \hat{m}\right)(\eta) = \sup_{m \in M} F(m) \wedge \hat{m}(\eta) = F(\eta) \wedge \hat{\eta}(\eta) = F(\eta) \wedge 1 = F(m)$$

Convention 1 *From now on, we identify $\hat{m}$ with the element m itself.*

First recall that a non deterministic operation on a set M is a function

$$\triangle : M \times M \to \mathcal{P}(M) = \{0,1\}^M$$

which to every pair (m_1, m_2) of elements of M assigns a subset $m_1 \triangle m_2$ of m [1], [2], [3], [4].

More generally, a fuzzy operation on a set M is a function

$$\triangle : M \times M \to Fuzzy(M) = [0,1]^M$$

which to every pair (m_1, m_2) of elements of M assigns a fuzzy subset

$$m_1 \triangle m_2 : M \to [0,1].$$

The operation $\triangle$ is extended to the fuzzy subsets of M via the formula

$$F_1 \triangle F_2 = \sup_{m_1, m_2 \in M} F_1(m_1) \wedge F_2(m_2) \wedge (m_1 \triangle m_2) \tag{2.1}$$

for all $F_1, F_2 : M \to [0,1]$.

An operation $\triangle$ is associative whenever

$$m_1 \triangle (m_2 \triangle m_3) = (m_1 \triangle m_2) \triangle m_3, \tag{2.2}$$

for all $m_1, m_2, m_3 \in M$. Let us analyze the above equality, by taking into account (2.1) the left hand side member of (2.2) is written

$$\begin{aligned} m_1 \triangle (m_2 \triangle m_3) &= \sup_{\eta_1, \eta_2 \in M} m_1(n_1) \wedge (m_2 \triangle m_3)(\eta_2) \wedge (\eta_1 \triangle \eta_2) \\ &= \sup_{\eta \in M} (m_2 \triangle m_3)(\eta) \wedge (m_1 \triangle \eta) \end{aligned}$$

and, similarly,

$$\begin{aligned}(m_1 \triangle m_2)\triangle m_3 &= \sup_{\eta_1,\eta_2\in M} (m_1\triangle m_2)(n_1) \wedge m_3(\eta_2) \wedge (\eta_1 \triangle \eta_2) \\ &= \sup_{\eta\in M} (m_1\triangle m_2)(\eta) \wedge (\eta \triangle m_3).\end{aligned}$$

Thus associativity is explicitly expressed as

$$\sup_{\eta\in M} (m_2\triangle m_3)(\eta) \wedge (m_1\triangle \eta) = \sup_{\eta\in M} (m_1\triangle m_2)(\eta) \wedge (\eta\triangle m_3).$$

Remark 1 Other variants of associativity can be also formulated but they are unsuitable for the syntactic theory treated in this paper.

A fuzzy monoid is a set M equipped with an associative fuzzy operation $\triangle$ which admits a neutral element $e \in M$

$$m\triangle\eta = m = \eta\triangle m, \quad \text{for all } m \in M.$$

Example 1 Any ordinary monoid can be viewed as a fuzzy monoid.

Example 2 A nondeterministic monoid (ND-monoid) is a set M equipped with a nondeterministic operation which is associative and unitary. Every ND-monoid can be viewed as a fuzzy monoid in the obvious way.

In the present study it is necessary to have a congruence notion. First let us remark that every equivalence relation $\sim$ on a set M induces an equivalence relation $\approx$ on $Fuz(M)$ as follows: for all $F_1, F_2 \in Fuz(M)$

$$F_1 \approx F_2 \quad \text{iff} \quad \sup_{m\in C} F_1(m) = \sup_{m\in C} F_2(m),$$

for all $\sim$-classes C. An equivalence relation $\sim$ on M is termed a left (resp. right) fuzzy congruence if

$$\begin{aligned} m_1 \sim m_2, \quad &\text{implies} \quad m_1 m \approx m_2 m, \quad \text{for all } m \in M \\ (\text{resp. } m_1 \sim m_2, \quad &\text{implies} \quad m m_1 \approx m m_2, \quad \text{for all } m \in M). \end{aligned}$$

A left and right fuzzy congruence is called a fuzzy congruence on M.

Lemma 1 *The equivalence $\sim$ on M is a fuzzy congruence on M if and only if*

$$m_1 \sim m_1', m_2 \sim m_2', \quad \textit{implies} \quad m_1 m_2 \approx m_1' m_2'.$$

Proof. One direction is clear. Conversely, assume that $m_1 \sim m_1'$ and $m_2 \sim m_2'$. Then

$$m_1 m_2 \approx m_1' m_2 \approx m_1' m_2'. \qquad \blacksquare$$

The condition of the above lemma can be explicitly described in the following way: $m_1 \sim m_1'$ and $m_2 \sim m_2'$ imply

$$\sup_{m\in C} (m_1 m_2)(m) = \sup_{m\in C} (m_1' m_2')(m)$$

for all $\sim$-classes C.

Proposition 1 *The quotient set $M/\sim$ is structured into a fuzzy monoid by defining its fuzzy multiplication via the formula*

$$([m_1][m_2])([n]) = \sup_{m\in[n]} (m_1m_2)(m).$$

Proof. First observe that the above multiplication is well defined. Next for every $\sim$-class $[b]$ we have

$$\begin{aligned}(([m_1][m_2])\,[m_3])\,([b]) &= \sup_{[n]\in M/\sim} ([m_1][m_2])([n]) \wedge ([n][m_3])([b]) \\ &= \sup_{[n]\in M/\sim} \sup_{n_1\in[n]} (m_1m_2)(n_1) \wedge \Big(\sup_{b'\in[b]} (nm_3)(b') \Big)\end{aligned}$$

since $n \sim n_1$ we get

$$\begin{aligned}&= \sup_{[n]\in M/\sim} \sup_{n_1\in[n]} (m_1m_2)(n_1) \wedge \Big(\sup_{b'\in[b]} sup(n_1m_3)(b') \Big) \\ &= \sup_{[n]\in M/\sim} \sup_{b'\in[b]} \sup_{n_1\in[n]} (m_1m_2)(n_1) \wedge \big((n_1m_3)(b') \big) \\ &= \sup_{b'\in[b]} \sup_{n_1\in M} (m_1m_2)(n_1) \wedge \big((n_1m_3)(b') \big)\end{aligned}$$

and by taking into account the associativity of M we obtain

$$\begin{aligned}&= \sup_{b'\in[b]} \sup_{n_1\in M} (m_2m_3)(n_1) \wedge \big((m_1n_1)(b') \big) \\ &= ([m_1]([m_2][m_3]))([b]).\end{aligned}$$

Example 3 Congruences on an ordinary monoid M coincide with fuzzy congruences when M is viewed as a fuzzy monoid. On the other hand an equivalence $\sim$ on an ND-monoid M is a congruence whenever $m_1 \sim m_1'$ and $m_2 \sim m_2'$ imply

$$m_1m_2 \cap C \neq \emptyset \Rightarrow m_1'm_2' \cap C \neq \emptyset.$$

for all $\sim$-classes C. The quotient set $M/\sim$ is structured into an ND-monoid by means of the operation

$$[m_1][m_2] = \{C \mid m_1m_2 \cap C \neq \emptyset\}.$$

The first question arising is whether a fuzzy congruence is a good algebraic notion. This is checked by the validity of the known isomorphism theorems in their fuzzy variant.

Given fuzzy monoids M and M', a strict morphism from M to M' is a function $h : M \to M'$ preserving fuzzy multiplication and units, i.e.,

$$\bar{h}(m_1m_2) = h(m_1)h(m_2), \quad h(e) = e',$$

for all $m_1, m_2 \in M$, where e, e' are the units of M, M' respectively, and $\bar{h} : Fuz(M) \to Fuz(M')$ the canonical extension of h defined by

$$\bar{h}(F) = \sup_{m\in M} \left(F(m) \wedge h(m)\right).$$

Theorem 1 *Given an epimorphism of fuzzy monoids $h : M \to M'$ and a fuzzy congruence $\sim$ on M', its inverse image $h^{-1}(\sim)$ defined by*

$$m_1 h^{-1}(\sim) m_2 \quad \textit{if} \quad h(m_1) \sim h(m_2)$$

is also a fuzzy congruence and the fuzzy quotient monoids $M/h^{-1}(\sim)$ and $M'/\sim$ are isomorphic.

Proof. Assume that

$$m_1 h^{-1}(\sim) m_1' \text{ and } m_2 h^{-1}(\sim) m_2'$$

that is

$$h(m_1) \sim h(m_1') \text{ and } h(m_2) \sim h(m_2').$$

Then

$$\bar{h}(m_1 m_2) = h(m_1)h(m_2) \approx h(m_1')h(m_2') = \bar{h}(m_1' m_2').$$

That is for all $C \in M'/\sim$, we have

$$\sup_{c \in C} \bar{h}(m_1 m_2)(c) = \sup_{c \in C} \bar{h}(m_1' m_2')(c)$$

but

$$\begin{aligned} \sup_{c \in C} \bar{h}(m_1 m_2)(c) &= \sup_{c \in C} \sup_{m \in M} (m_1 m_2)(m) \wedge h(m)(c) \\ &= \sup_{m \in M} (m_1 m_2)(m) \wedge \sup_{c \in C} h(m)(c) \\ &= \sup_{m \in h^{-1}(C)} (m_1 m_2)(m). \end{aligned}$$

Recall that all $h^{-1}(\sim)$-classes are of the form $h^{-1}(C)$, $C \in M'/\sim$. Consequently,

$$\sup_{m \in h^{-1}(C)} (m_1 m_2)(m) = \sup_{m \in h^{-1}(C)} (m_1' m_2')(m)$$

which shows that $h^{-1}(\sim)$ is indeed a congruence of the fuzzy monoid M. The desired isomorphism $\hat{h} : M/h^{-1}(\sim) \to M'/\sim$ is given by

$$\hat{h}([m]_{h^{-1}(\sim)}) = [h(m)]_{\sim}.$$ ∎

Corollary 1 *Let $h : M \to M'$ be an epimorphism of fuzzy monoids. Then the kernel equivalence*

$$m_1 \sim_h m_2 \textit{ if } h(m_1) = h(m_2)$$

is a congruence on M and the fuzzy quotient monoid $M/\sim_h$ is isomorphic to M'.

Given fuzzy monoids $M_1, \ldots, M_k$ the fuzzy multiplication

$$[(m_1, \ldots, m_k) \cdot (m_1', \ldots, m_k')](n_1, \cdots, n_k) = (m_1 m_1')(n_1) \wedge \cdots \wedge (m_k m_k')(n_k)$$

structures the set $M_1 \times \cdots \times M_k$ into a fuzzy monoid so that the canonical projection

$$\pi_i : M_1 \times \cdots \times M_k \to M_i, \quad \pi_i(m_1, \ldots, m_k) = m_i$$

becomes a morphism of fuzzy monoids. Notice that the above multiplication is fuzzy because

$$\begin{aligned}\sup_{1\le i\le k}\ \sup_{n_i\in M_i}(m_1m_1')(n_1) \ &\wedge\cdots\wedge(m_km_k')(n_k)\\ &=\sup_{n_1\in M_1}(m_1m_1')(n_1)\wedge\cdots\wedge\sup_{n_k\in M_k}(m_km_k')(n_k)\\ &=1\cdots1=1.\end{aligned}$$

Theorem 2 *Let $\sim_i$ be a fuzzy congruence on the fuzzy monoid M_i $(1\le i\le k)$. Then $\sim_1\times\cdots\times\sim_k$ is a fuzzy congruence on the fuzzy monoid $M_1\times\cdots\times M_k$ and the fuzzy monoids $M_1\times\cdots\times M_k/\sim_1\times\cdots\times\sim_k$ and $M_1/\sim_1\times\cdots\times M_k/\sim_k$ are isomorphic.*

3. The greatest fuzzy congruence saturating an equivalence

First observe that, due to the symmetric property which an equivalence relation satisfies, the sumability condition in the definition of a congruence can be replaced by the weaker condition: $m_1\sim m_1'$ and $m_2\sim m_2'$ implies

$$\sup_{m\in C}(m_1m_2)(m)\le\sup_{m\in C}(m_1'm_2')(m)$$

for all $\sim$-classes C. Equivalently, we may state: $m\sim m'$, implies

$$\sup_{b\in C}(mn)(b)\le\sup_{b\in C}(m'n)(b)\quad\text{and}\quad\sup_{b\in C}(nm)(b)\le\sup_{b\in C}(nm')(b).$$

Next, we demonstrate that fuzzy congruences are closed under the join operation. We recall that the join $\bigvee_{i\in I}\sim_i$ of a family of equivalences $(\sim_i)_{i\in I}$ on a set A is the reflexive and transitive closure of their union:

$$\bigvee_{i\in I}\sim_i=\left(\bigcup_{i\in I}\sim_i\right)^*.$$

Theorem 3 *If $(\sim_i)_{i\in I}$ is a family of fuzzy congruences on M, then their join $\bigvee_{i\in I}\sim_i$ is also a fuzzy congruence.*

Proof. Let $\sim_1,\sim_2$ be two congruences on M and $\sim=\sim_1\vee\sim_2$. First we show that $m\sim_1 m'$ implies

$$\sup_{b\in C}(mn)(b)\le\sup_{b\in C}(m'n)(b),$$

for all $\sim$-classes C. From the inclusion $\sim_1\subseteq\sim$ we get that C is the disjoint union

$$C=\bigcup_{j\in J}C_j^1$$

where C_j^1 denote $\sim_1$-classes and J is an index set. Then

$$\sup_{b\in C}(mn)(b)=\sup_{j\in J}\ \sup_{b\in C_j^1}(mn)(b)\le\sup_{j\in J}\ \sup_{b\in C_j^1}(m'n)(b)=\sup_{b\in C}(m'n)(b).$$

By a similar argument we show that $m \sim_2 m'$ implies

$$\sup_{b\in C}(mn)(b) \leq \sup_{b\in C}(m'n)(b),$$

for all $\sim$-classes C. Now, if $m \sim m'$, without any loss we may assume that

$$m \sim_1 m_1 \sim_2 m_2 \sim_1 \cdots \sim_1 m_{2\lambda-1} \sim_2 m'$$

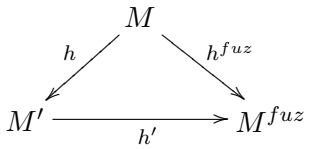
for some elements $m_1, \ldots, m_{2\lambda-1} \in M$. Applying successively the previous facts, we obtain

$$\sup_{b\in C}(mn)(b) \leq \sup_{b\in C}(m_1n)(b) \leq \cdots \leq \sup_{b\in C}(m_{2\lambda-1}n)(b) \leq \sup_{b\in C}(m'n)(b).$$

For an arbitrary set of congruences we proceed in a similar way. ∎

The above fact leads us to introduce the greatest fuzzy congruence included into an equivalence $\sim$ of M. It is the join of all fuzzy congruences on M included into $\sim$ and it is denoted by $\sim^{fuz}$. The quotient fuzzy monoid $M/\sim^{fuz}$ is denoted by M^{fuz} and has the following universal property

Theorem 4 *Given an epimorphism of fuzzy monoids $h : M \to M'$ whose kernel $\sim_h$ saturates k the equivalence $\sim$ there exists a unique epimorphism of fuzzy monoids $h' : M' \to M^{fuz}$ rendering commutative the triangle*



where $h^{fuz} : M \to M^{fuz}$ is the canonical projection $m \mapsto [m]_{fuz}$ sending every element $m \in M$ on its $\sim^{fuz}$-class.

Proof. By virtue of the Isomorphism Theorem the fuzzy monoid M' is isomorphic to the quotient $M/\sim_h$. Since by assumption $\sim_h \subseteq \sim^{fuz}$, h' is the following composition

$$M' \xrightarrow{\sim} M/\sim_h \xrightarrow{f} M/\sim^{fuz} = M^{fuz}$$

with $f([m]_h) = [m]_{fuz}$, $[m]_h$ being the $\sim_h$-class of m. ∎

The previous result states that h^{fuz} is minimal among all epimorphisms saturating $\sim$.

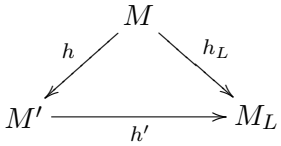
4. Syntactic fuzzy monoids

Let M be a fuzzy monoid and $L \subseteq M$. Denote by $\sim_L$ the greatest congruence of M included in the partition (equivalence) $\{L, M - L\}$, i.e.,

$$\sim_L = \{L, M - L\}^{fuz}.$$

The quotient fuzzy monoid $M_L = M/\sim_L$ will be called the *syntactic fuzzy monoid* of L and it is characterized by the following universal property.

Theorem 5 *For every fuzzy monoid M and every epimorphism $h : M \to M'$ verifying $h^{-1}(h(L)) = L$, there exists a unique epimorphism $h' : M' \to M_L$ rendering commutative the triangle*



where h_L is the canonical morphism sending every element $m \in M$ to its $\sim_L$-class.

Proof. The hypothesis $h^{-1}(h(L)) = L$ means that $\sim_h$ saturates L and so the statement follows immediately by Theorem 4. ∎

Given fuzzy monoids M, M' we write $M' < M$ if there is a fuzzy monoid $\bar{M}$ and a situation

$$M' \xleftarrow{h} \bar{M} \xrightarrow{i} M$$

where i (resp. h) is a monomorphism (resp. epimorphism).

Theorem 6 *Given subsets L_1, L_2, L of a fuzzy monoid M it holds*

(i) $M_{L_1 \cap L_2} < M_{L_1} \times M_{L_2}$,

(ii) $M_L = M_{\bar{L}}$, *where $\bar{L}$ designates the set theoretic complement of L,*

(iii) $M_{L_1 \cup L_2} < M_{L_1} \times M_{L_2}$,

(iv) *If $h : M \to N$ is an epimorphism of ND-monoids and $L \subseteq N$, then $M_{h^{-1}(L)} = M_L$.*

Proof. The proof follows by applying Theorem 5. ∎

A subset L of a fuzzy monoid M is *fuzzy recognizable* if there exist a finite fuzzy monoid M' and a morphism $h : M \to M'$ such that $h^{-1}(h(L)) = L$. The class of fuzzy recognizable subsets of M is denoted by $FRec(M)$. By taking into account Theorem 5 we get

Proposition 2 *$L \subseteq M$ is recognizable if and only if its syntactic fuzzy monoid is finite, i.e., $card(M_L) < \infty$.*

Putting this result together with Theorem 6 we yield

Proposition 3 *The class $FRec(M)$ is closed under boolean operations and inverse morphisms.*

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APPLICATIONS OF THE UNITING ELEMENTS METHOD

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Abstract. We present the Uniting Elements Method introduced by Corsini-Vougiouklis in 1989 [5]. Some applications of the uniting elements method on the classical algebra are presented and finally we connect them with ∂-structures and the other classes of hyperstructures.

Keywords: hyperstructures, H_v-structures, hopes, ∂-structures.

AMS Subject Classification: 20N20, 16Y99.

1. Basic definitions

The largest class of hyperstructures is the one which satisfy the weak properties. These are called H_v-structures introduced in 1990 [10], and they proved to have a lot of applications on several applied science such as linguistic, biology, chemistry, physics, and so on. The H_v-structures satisfy the *weak axioms* where the non-empty intersection replaces the equality. The H_v-structures can be used in models as an organized devise as well.

Recall some basic definitions:

Definitions 1.1. A set H equipped with at least one hyperoperation (we abbreviate by *hope* any *hyperoperation*) $. : H \times H \longrightarrow P(H)$ is called the *hyperstructure*. We abbreviate by *WASS* the *weak associativity*:$(xy)z \cap x(yz) \neq \emptyset,\ \forall x, y, z \in H$ and by *COW* the *weak commutativity*: $xy \cap yx \neq \emptyset,\ \forall x, y \in H$.

The hyperstructure $(H,.)$ is called *H_v-semigroup* if it is WASS and is called H_v-group if it is reproductive H_v-semigroup, i.e., $xH = Hx = H, \forall x \in H$. The hyperstructure $(R,+,.)$ is called H_v-ring if $(+)$ and $(.)$ are WASS, the reproduction axiom is valid for $(+)$ and $(.)$ is *weak distributive* with respect to $(+)$, i.e.,

$$x(y+z) \cap (xy+xz) \neq \emptyset,\ (x+y)z \cap (xz+yz) \neq \emptyset,\ \forall x,y,z \in R.$$

For more definitions and results on H_v-structures one can seen in books and papers as [1], [2], [3], [4], [6], [7], [9], [13]. An extreme class of the H_v-structures is the following: An H_v-structure is called *very thin* iff all hopes are operations except one, which has all hyperproducts singletons except only one, which has cardinality more than one.

The fundamental relations β^*,γ^* and ε^* are defined, in H_v-groups, H_v-rings and H_v-vector spaces, respectively, as the smallest equivalences so that the quotient would be group, ring and vector space, respectively [11],[12],[13]. The way to find the fundamental classes is given by analogous theorems to the following one:

Theorem. *Let $(H,.)$ be an H_v-group and denote by $\mathbf{U}$ the set of all finite products of elements of H. We define the relation β in H as follows: $x\beta y$ iff $\{x,y\} \subset \mathbf{u}$ where $\mathbf{u} \in \mathbf{U}$. Then the fundamental relation β^* is the transitive closure of the relation β.*

Remark that the main point of the proof is that the β guaranties the validity of the following: Take two elements x,y such that $\{x,y\} \subset \mathbf{u} \in \mathbf{U}$ and any hyperproduct where one of the elements x,y, is used. Then, if this element is replaced by the other, the new hyperproduct is inside the same fundamental class where the first hyperproduct is. Therefore, if the 'hyperproducts' of the above β-classes are 'products', then, they are fundamental classes. Analogous remarks for the relations γ and ε, are also applied.

An element is called *single* if its fundamental class is a singleton.

Motivation for H_v-structures. *We know that the quotient of a group with respect to an invariant subgroup is a group. Marty states that, the quotient of a group with respect to any subgroup is a hypergroup. Now, the quotient of a group with respect to any partition is an H_v-group.*

Let $(H,.),(H,\otimes)$ H_v-semigroups defined on the same set H. $(.)$ is called *smaller* than $(\otimes)$, and $(\otimes)$ greater than $(.)$, iff there exists automorphism

$$f \in Aut(H,\otimes) \text{ such that } \quad xy \subset f(x \otimes y), \forall x,y \in H.$$

Then we write $. \leq \otimes$ and we say that $(H,\otimes)$ *contains* the $(H,.)$. If $(H,.)$ is a structure then it is called *basic structure* and $(H,\otimes)$ is called H_b-*structure*.

The Little Theorem. *Greater hopes of hopes which are WASS or COW, are also WASS and COW, respectively.*

The fundamental relations are used for general definitions of hyperstructures. Thus, to define the general H_v-field one uses the fundamental relation γ^*: The H_v-ring $(R,+,.)$ is called *H_v-field* if the quotient R/γ^* is a field [11],[12]. The *H_v-module* is an H_v-group over an H_v-ring if the weak distributivity and a mixed weak associativity on all hopes, is valid. In an analogous way the *H_v-vector spaces* and the *H_v-algebra* can be defined [11], [16].

The definition of the H_v-field introduced a new class of hyperstructures [14]: The H_v-semigroup (H,.) is called *h/v-group* if the quotient H/β^* is a group. The h/v-groups are a generalization of the H_v-groups because in h/v-groups the reproductivity is not necessarily valid. In a similar way the h/v-rings, h/v-fields, h/v-modulus, h/v-vector spaces etc, are defined.

The general definition of an H_v-Lie algebra was given in [13], [16] as follows:

Definitions 1.3. Let $(\mathbf{L},+)$ be H_v-vector space over the field $(\mathbf{F},+,.), \varphi : \mathbf{F} \longrightarrow \mathbf{F}/\gamma^*$, the canonical map and $\omega_F = \{x \in F : \varphi(x) = 0\}$, where 0 is the zero of the fundamental field $\mathbf{F}/\gamma^*$. Similarly, let ω_L be the core of the canonical map $\varphi' : \mathbf{L} \longrightarrow \mathbf{L}/\varepsilon^*$ and denote by the same symbol 0 the zero of $\mathbf{L}/\varepsilon^*$. Consider the *bracket (commutator) hope:*

$$[\ ,\ ] : L \times L \longrightarrow P(L) : (x,y) \longrightarrow [x,y].$$

Then $\mathbf{L}$ is an H_v-Lie algebra over $\mathbf{F}$ if the following axioms are satisfied:

(L1) The bracket hope is bilinear, i.e.,
$[\lambda_1 x_1 + \lambda_2 x_2, y] \cap (\lambda_1[x_1,y] + \lambda_2[x_2,y]) \neq \emptyset$
$[x, \lambda_1 y_1 + \lambda_2 y_2] \cap (\lambda_1[x,y_1] + \lambda_2[x,y_2]) \neq \emptyset,\ \forall x, x_1, x_2, y, y_1, y_2 \in L,\ \lambda_1, \lambda_2 \in F,$

(L2) $[x,x] \cap \omega_L \neq \emptyset,\ \ \forall x \in L,$

(L3) $([x,[y,z] + [y,[z,x]] + [z,[x,y]]) \cap \omega_L \neq \emptyset,\ \ \forall x,y,z \in L.$

A general way to define hopes, from given operations [9], [11] is the following:

Definitions 1.4. Let $(G,.)$ be a groupoid, then for every set $P \subset G, P \neq \emptyset$, we define the following hopes called *P-hopes*:

$$\underline{P} : x\underline{P}y = (xP)y \cup x(Py),$$

$$\underline{P}_r : x\underline{P}_r y = (xy)P \cup x(yP), \underline{P}_l : x\underline{P}_l y = (Px)y \cup P(xy), \forall x,y \in G.$$

The $(G,P),(G,\underline{P}_r)$ and $(G,\underline{P}_l)$ are called *P-hyperstructures.* If $(G,.)$ is semigroup, then (G,P) is a semihypergroup but we do not know for $(G,\underline{P}_r),(G,\underline{P}_l)$. In some cases, mainly depending on the choice of P, the $(G,\underline{P}_r),(G,\underline{P}_l)$ can be associative or WASS. If in G, more operations are defined then for each operation several P-hopes can be defined.

Constructions 1.5. Let $(G,.)$ be abelian group and $P \subset G$, with more than one elements. We define a hope $(\times_P)$ as follows:

$$x \times_P y = \begin{cases} x.P.y = \{x.h.y \mid h \in P\} & \text{if } x \neq e \text{ and } y \neq e \\ x.y & \text{if } x = e \text{ or } y = e \end{cases}$$

and we call this *P_e-hope.* The hyperstructure $(G, \times_P)$ is an abelian H_v-group.

During the last decades, the hyperstructures have a variety of applications not only in other branches of mathematics but also in many other sciences including social ones. These applications are on biomathematics, conchology, inheritance, hadronic physics, to mention but a few. The hyperstructure theory is closely related to fuzzy theory; thus, hyperstructures can now be widely applicable in industry and production, too.

The Lie-Santilli theory on *isotopies* was born in 1970's [8] to solve Hadronic Mechanics problems. Santilli proposed a 'lifting'of the n-dimensional trivial unit matrix of a normal theory into a nowhere singular, symmetric, real-valued, positive defined, n-dimensional new matrix. The original theory is reconstructed such as to admit the new matrix as left and right unit. The *isofields* needed in this theory correspond into the hyperstructures were introduced by Santilli and Vougiouklis in 1999 and they are called *e-hyperfields*. The H_v-fields can give e-hyperfields which can be used in the isotopy theory in applications as in physics or biology, see [8].

A new application, which combines hyperstructure theory and fuzzy, is to replace in questionnaires the scale of Likert by the bar of Vougiouklis & Vougiouklis [17]. The suggestion is the following:

Definition 1.6. In every question substitute the Likert scale with 'the bar' whose poles are defined with $'0'$ on the left end, and $'1'$ on the right end:

The subjects/participants are asked instead of deciding and checking a specific grade on the scale, to cut the bar at any point s/he feels expresses her/his answer to the specific question. The use of the bar of Vougiouklis & Vougiouklis instead of a scale of Likert has several advantages during both the filling-in and the research processing. The final suggested length of the bar, according to the Golden Ratio, is 6.2cm.

Now, we intend to remove elements from hyperstructures or classical structures.

Definitions 1.7. Let $(H, .)$ be hypergroupoid.

We say that we *remove* $h \in H$, if we simply consider the restriction of the hope $(.)$ on the $H - \{h\}$.

We say that an $\underline{h} \in H$ *absorbs* $h \in H$ if we replace h, whenever it appears, by $\underline{h}$.

We say that $\underline{h} \in H$ *merges* with $h \in H$, if we take as the product of any $x \in H$ by $\underline{h}$, the union of the results of x with both h and $\underline{h}$, and we consider h and $\underline{h}$ as one class, with representative $\underline{h}$.

The above definitions can be applied on more complicated structures analogously. When we remove an element, then the hope may become partial. If the removing element does not appeared in any result, then a hyperstructure more strict may obtained.

The representation problem of H_v-structures by H_v-matrices is the following (see [11]):

The *H_v-matrix* is a matrix with entries of an H_v-ring or H_v-field. The hyperproduct of two H_v-matrices $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$, of type $m \times n$ and $n \times r$, respectively, is defined in the usual manner but it is a set of $m \times r$ H_v-matrices:

$$\mathbf{A}.\mathbf{B} = (a_{ij}).(b_{ij}) = \left\{ \mathbf{C} = (c_{ij}) \mid c_{ij} \in \oplus \sum a_{ik} b_{kj} \right\},$$

where $(\oplus)$ denotes the *n-ary circle hope* on the hyperaddition, i.e. the sum of products of elements of the H_v-ring is the union of the sets obtained with all possible parentheses put on them. The hyperproduct is not WASS.

Definitions 1.8. Let $(H, .)$ be H_v-group, consider an H_v-ring or H_v-field, $(R, +, .)$ and a set

$$\mathbf{M}_R = \{(a_{ij}) \mid a_{ij} \in R\},$$

then is called the *H_v-matrix representation*, any map

$$\mathbf{T} : H \longrightarrow M_R, h \longrightarrow \mathbf{T}(h) \text{ such that } \mathbf{T}(h_1 h_2) \cap \mathbf{T}(h_1)\mathbf{T}(h_2) \neq \emptyset, \ \forall h_1, h_2 \in H.$$

If $\mathbf{T}(h_1 h_2) \subset \mathbf{T}(h_1)\mathbf{T}(h_2), \forall h_1, h_2 \in H$, then $\mathbf{T}$ is an *inclusion representation*, if

$$\mathbf{T}(h_1 h_2) = \mathbf{T}(h_1)\mathbf{T}(h_2) = \{\mathbf{T}(h) \mid h \in h_1 h_2\}, \forall h_1, h_2 \in H,$$

then $\mathbf{T}$ is a *good representation*.

The main theorem of the theory of representations is:

Theorem. *A necessary condition in order to have an inclusion representation $\boldsymbol{T}$ of the H_v-group $(H, .)$ by $n \times n$ H_v-matrices over the H_v-ring $(R, +, .)$ is the following:*

For all classes $\beta^*(a), a \in H$ there must exist elements $a_{ij} \in R, i, j \in \{1, ..., n\}$ such that

$$\boldsymbol{T}(\beta^*(a)) \subset \left\{ \boldsymbol{A} = (a'_{ij}) \mid a'_{ij} \in \gamma^*(a_{ij}), i, j \in \{1, ..., n\} \right\}.$$

2. The theta ∂-hopes

We present now a large class of hopes defined in any groupoid with a map f on it, which is denoted by 'theta' ∂, since the motivation is the property which the derivative has on the product of functions, see [15], [16], [17].

Definitions 2.1. Let $(G, .)$ be groupoid (respectively, hypergroupoid) and $f : G \longrightarrow G$ be a map. We define a hope ∂, called *theta-hope*, on G as follows

$$x \partial y = \{f(x).y, x.f(y)\}, \forall x, y \in G. (resp. \ x \partial y = (f(x).y) \cup (x.f(y)), \forall x, y \in G)$$

If (.) is commutative, then (∂) is commutative. If (.) is *COW*, then (∂) is *COW*.

Let $(G,.)$ be groupoid (resp. hypergroupoid) and $f : G \longrightarrow \mathbf{P}(G) - \{\emptyset\}$, multivalued map. We define the *theta-hope* (∂), on G as follows

$$x\partial y = (f(x).y) \cup (x.f(y)), \forall x, y \in G$$

Let $(G,.)$ be a groupoid and $f_i : G \longrightarrow G, i \in I$, be set of maps on G. We consider the map $f_\cup : G \longrightarrow \mathbf{P}(G)$ such that $f_\cup(x) = \{f_i(x) | i \in I\}$, called *the union* of the $f_i(x)$. We define the *union theta-hrope* (∂), on G if we consider the $f_\cup(x)$. A special case for given f, is to take the union with the identity: We consider the map $\underline{f} \equiv f \cup (id)$, so $\underline{f}(x) = \{x, f(x)\}, \forall x \in G$, which we call *b-theta-hope*. Then we have

$$x\partial y = \{xy, f(x).y, x.f(y)\}, \forall x, y \in G.$$

Motivation for the definition of the theta-hope is the map *derivative* where only the multiplication of functions can be used. Therefore, in these terms, for any functions $s(x), t(x)$, we have $s\partial t = \{s't, st'\}$ where $(')$ denotes the derivative.

Properties 2.2. [15] If $(G,.)$ is a semigroup, then:

(a) For every f, the (∂) is *WASS*. If f is homomorphism then (∂) remains *WASS*.

(b) If f is homomorphism and projection, i.e., $f^2 = f$, then (∂) is associative.

(c) If $(G,.)$ is a semigroup then, for every f, the b-theta-operation (∂) is *WASS*.

(d) *Reproductivity.* If (.) is reproductive then $(\partial)$is also reproductive, because

$$x\partial G = \bigcup_{g \in G} \{f(x).g, x.f(g)\} = G \ \textit{and} \ G\partial x = \bigcup_{g \in G} \{f(g).x, g.f(x)\} = G$$

(e) *Commutativity.* If (.) is commutative, then (∂) is commutative. If f is into the center of G, then (∂) is a commutative. If (.) is a COW then, (∂) is a COW.

(f) *Unit elements.* u is a right unit element if $x \in x\partial u = \{f(x).u, x.f(u)\}$. So $f(u) = e$, where e be a unit in $(G,.)$. The elements of the kernel of f, are the units of (G, ∂).

(g) *Inverse elements.* Let $(G,.)$ is a monoid with unit e and u be a unit in (G, ∂), then $f(u) = e$. For given x, the element $x' = (f(x))^{-1}u$ and $x'' = u(f(x))^{-1}$, are the right and left inverses, respectively. We have two-sided inverses iff $f(x)u = uf(x)$.

Definitions 2.3. Let $(R,+,.)$ be a ring and $f : R \longrightarrow R, g : R \longrightarrow R$ be two maps. We define two hopes (∂_+) and $(\partial_.)$, called both *theta-hopes*, on R as follows

$$x\partial_+ y = \{f(x)+y, x+f(y)\} \ \textit{and} \ x\partial_. y = \{g(x)y, xg(y)\}, \forall x, y \in G$$

The hyperstructure $(R, \partial_+, \partial_.)$, called *theta*, is an H_v-near-ring, i.e., satisfy all H_v-ring axioms, except the weak distributivity.

Some results and examples:

Let $(G,.)$ be group and $f(x) = a$, a constant map on G. Then $(G,\partial)/\beta^*$ is singleton. If $f(x) = e$, then $x\partial y = \{x, y\}$ which is the smallest incidence hope.

Consider all polynomials of first degree $g_i(x) = a_i x + b_i$, and as map the derivative, we have

$$g_1 \partial g_2 = \{a_1a_2x + a_1b_2, a_1a_2x + b_1a_2\},$$

so it is a hope inside the set of first degree polynomials. Moreover all polynomials $x + c$, where c be a constant, are units.

Several results can be obtained using ∂-hopes [15]:

Theorems 2.4.

(a) Consider the group of integers $(\mathbb{Z},+)$ and $n \neq 0$ be a natural number. Take the map f such that $f(0) = n$ and $f(x) = x, \forall x \in \mathbb{Z} - \{0\}$. Then

$$(\mathbb{Z},\partial)/\beta^* \cong (\mathbb{Z}_n,+).$$

(b) Take the ring of integers $(\mathbb{Z},+,.)$ and fix $n \neq 0$ a natural number. Consider f such that $f(0) = n$ and $f(x) = x, \forall x \in \mathbb{Z} - \{0\}$. Then $(\mathbb{Z}, \partial_+, \partial_.)$, where ∂_+ and $\partial_.$ are the ∂-hopes refereed to the addition and the multiplication respectively, is an H_v-near-ring, with

$$(\mathbb{Z}, \partial_+, \partial_.)/\gamma^* \cong (\mathbb{Z}_n).$$

(c) Take the $(\mathbb{Z},+,.)$ and $n \neq 0$ a natural. Take f such that $f(n) = 0$ and $f(x) = x, \forall x \in \mathbb{Z} - \{n\}$. Then $(\mathbb{Z}, \partial_+, \partial_.)$ is an H_v-ring, moreover,

$$(\mathbb{Z}, \partial_+, \partial_.)/\gamma^* \cong (\mathbb{Z}_n).$$

Special case of the above is for $n = p$, prime, then $(\mathbb{Z}, \partial_+, \partial_.)$ is an H_v-field.

3. Uniting elements

The *uniting elements* method was introduced by Corsini-Vougiouklis [5] in 1989. With this method one puts in the same class, two or more elements. This leads, through hyperstructures, to structures satisfying additional properties.

Definition 3.1. The *Uniting Elements Method* is described as follows: Let G be an algebraic structure and let d be a property, which is not valid. Suppose that d is described by a set of equations; then, consider the partition in G for which it is put together, in the same partition class, every pair of elements that causes the non-validity of the property d. The quotient by this partition G/d is an H_v-structure. Then, quotient out the H_v-structure G/d by the fundamental relation β^*, a stricter structure $(G/d)\beta^*$ for which the property d is valid, is obtained.

The merging procedure is in fact the first step of the Uniting Elements Method, i.e., we do not take the quotient of the obtaining hyperstructure by the fundamental relation. From the definition of merging elements we obtain the following. If $\underline{h}$ merges with the element h in a groupoid $(H, .)$, then in merged hyperstructure (H, o) we have $\forall x \in H$,

$$\underline{h} o x = \{\underline{h}.x, h.x\}, \;\; x o \underline{h} = \{x.\underline{h}, x.h\}, \;\; \underline{h} o \underline{h} = \{\underline{h}.\underline{h}, \underline{h}.h, h.\underline{h}, h.h\}$$

and if $(H, .)$ is a hypergroupoid, we have $\forall x \in H$,

$$\underline{h} o x = (\underline{h}.x) \cup (h.x), \;\; x o \underline{h} = (x.\underline{h}) \cup (x.h), \;\; \underline{h} o \underline{h} = (\underline{h}.\underline{h}) \cup (\underline{h}.h) \cup (h.\underline{h}) \cup (h.h)$$

Moreover if $(.)$ is *WASS*, *COW*, reproductive then the merge construction remains *WASS*, *COW*, reproductive, respectively.

An interesting application of the Uniting Elements Method is when more, than one, properties are desired. The reason for this is that some of the properties lead straighter to the classes than others. Therefore, it is better to apply the straightforward classes followed by the more complicated ones. The commutativity is one of the easy applicable properties. Moreover it is clear that the reproductivity property is also easily applicable. One can do this because the following is valid.

Theorem 3.2. [11] *Let $(G, .)$ be a groupoid, and*

$$F = \{f_1, ..., f_m, f_{m+1}, ..., f_{m+n}\}$$

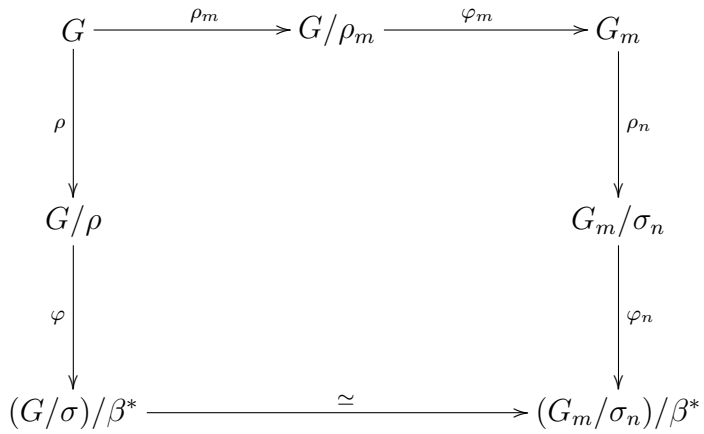
be a system of equations on G consisting of two subsystems

$$F_m = \{f_1, ..., f_m\} \;\; and \;\; F_n = \{f_{m+1}, ..., f_{m+n}\}.$$

Let σ, σ_m be the equivalence relations defined by the Uniting Elements procedure using the systems F and F_m respectively, and let σ_n be the equivalence relation defined using the induced equations of F_n on the groupoid $G_m = (G/\sigma_m)/\beta^*$. Then

$$(G/\sigma)/\beta^* \cong (G_m/\sigma_n)/\beta^*,$$

i.e., the following diagram is commutative



From the above it is clear that the fundamental structure is very important, and even more so if it is known from the beginning. This is the problem to construct hyperstructures with desired fundamental structures [14], [16].

Now, we introduce the following generalization:

Generalization of the Uniting Elements Method. Using this basic Theorem we remark that it is not necessary to apply the Uniting Elements Method to obtain a total property, but to obtain a part of a property and then in next steps to obtain the property. Therefore, one can enlarge some operations or hopes, and even more only some results, by putting together elements which cause the non-validity of the property. Thus, we can enlarge the classes to obtain desired properties.

An opposite problem, to the Uniting Elements Method, is to 'enlarge' given hyperstructures [14]. In representation theory enlargement or reduction, are useful if the H_v-structures have the same fundamental structure.

4. Applications

The applications of the Uniting Elements Method are referred mainly on the classical algebraic structures, however we can apply the method on hyperstructures as well. In the following, we present some special and general applications of the Uniting Elements Method and, moreover, we connect them with ∂-hopes.

Let now present a special example on the generalization of the Uniting Elements Method, where we enlarge only one product:

Example 4.1. Consider the non-commutative quaternion group

$$Q = \{1, -1, i, -i, j, -j, k, -k\}$$

with defining relations:

$$i^2 = j^2 = k^2 = -1, ij = k, jk = i, ki = j.$$

Denote

$$\underline{i} = \{i, -i\}, \underline{j} = \{j, -j\}, \underline{k} = \{k, -k\}, \underline{1} = \{1, -1\}.$$

Suppose that we enlarge only the product $ij = k$ as follows $ij = \{k, -k\} = \underline{k}$. Then we have $i(ij) = i\underline{k} = \underline{j}$, $j(ij) = j\underline{k} = \underline{i}$, $k(ij) = k\underline{k} = \underline{1}$. Therefore, we have four fundamental equivalent classes, the $\underline{1}, \underline{i}, \underline{j}, \underline{k}$. The Cayley table is

*	$\underline{1}$	$\underline{i}$	$\underline{j}$	$\underline{k}$
$\underline{1}$	$\underline{1}$	$\underline{i}$	$\underline{j}$	$\underline{k}$
$\underline{i}$	$\underline{i}$	$\underline{1}$	$\underline{k}$	$\underline{j}$
$\underline{j}$	$\underline{j}$	$\underline{k}$	$\underline{1}$	$\underline{i}$
$\underline{k}$	$\underline{k}$	$\underline{j}$	$\underline{i}$	$\underline{1}$

Therefore, the fundamental group becomes isomorphic to the group $(\mathbb{Z}_2 \times \mathbb{Z}_2, +)$.

Basic general results on the topic are the following (see [5], [11]):

Definition. Denote U the set of all finite products of elements of a hypergroupoid $(H, .)$. Consider the relation L defined as follows:

$$xLy \iff \text{there exists } u \in U \text{ such that } ux \cap uy \neq \emptyset.$$

Then the transitive closure L^* of L is called *left fundamental reproductivity relation.* Similarly, *the right fundamental reproductivity relation* R^* is defined.

Theorem 4.2. *If $(H, .)$ is a commutative semihypergroup, i.e., the strong commutativity and the strong associativity is valid, then the strong expression of the above $L : ux = uy$, has the property: $L^* = L$.*

Theorem 4.3. *Let $(S, .)$ be commutative semigroup with one element $w \in S$ such that the set wS is finite. Consider the transitive closure L^* of the relation L defined as follows:*

$$xLy \iff \textit{there exists } z \in S \textit{ such that } zx = zy.$$

Then $\langle S/L^*, \circ \rangle / \beta^*$ is a finite commutative group, where $(\circ)$ is the induced operation on classes of S/L^*.

Proof. For the proofs one can follow [5] and [11]. ∎

An open problem is to prove that L^*, is the smallest equivalence relation such that H/L^*, is reproductive.

Theorem 4.4. [16] *Let $(H, \otimes)$ be an H_b-semigroup where the basic semigroup $(H, .)$ is commutative and which has at least one element $w \in H$ such that the set $w.H$ is finite. Then $(H/L^*, \otimes)$, where L is the relation: xLy iff there exists $z \in H$ such that $zx = zy$ and $(\otimes)$ is the induced hyperoperation on classes, i.e., L is defined with respect to $(.)$, is finite commutative h/v-group.*

Remarks 4.5. We can see the results of Theorem 2.4, in terms of the generalized Uniting Elements Method, especially in the cases (a) and (c) where strict fundamental structures are obtained. Therefore, in the case (a), since we have, for the given map f,

$$0\partial 0 = n,\ \ 0\partial x = \{n+x, x\}, \forall x \in \mathbb{Z} - \{0\},\ \ x\partial y = x + y, \forall x, y \in \mathbb{Z} - \{0\}$$

we remark that we only enlarge the sums $0\partial x, \forall x \in \mathbb{Z} - \{0\}$, by adding in the results the element $n + x$.

In the case (c), since we have, for the given map f,

$$n\partial_+ n = n,\ \ n\partial_+ x = \{x, n+x\},\ \forall x \in \mathbb{Z} - \{n\},\ \ x\partial_+ y = x + y,\ \forall x \in \mathbb{Z} - \{n\}$$

and

$$n\partial_{\cdot} n = 0,\ \ n\partial_{\cdot} x = \{0, nx\},\ \forall x \in \mathbb{Z} - \{n\},\ \ x\partial_{\cdot} y = xy,\ \forall x \in \mathbb{Z} - \{n\}$$

we remark that we only enlarge the sums $n\partial_+ x, \forall x \in \mathbb{Z} - \{n\}$, by adding in the results the element x, and enlarge the products $n\partial_{\cdot} x, \forall x \in \mathbb{Z} - \{n\}$, by adding the element 0.

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GENERALIZED HESITANT FUZZY SOFT SETS

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Abstract. In this paper, generalized hesitant fuzzy soft sets and some operations on generalized hesitant fuzzy soft sets are defined and some of their properties are studied. Applications of generalized hesitant fuzzy soft sets in decision-making are investigated.

Keywords: soft sets, fuzzy soft sets, generalized hesitant fuzzy soft sets, similarity, decision-making.

1. Introduction

Decision-making problems referring to evaluating, prioritizing or selecting over some available alternatives are very common in practice [1]. Since it was introduced by Zadeh [2], theories of fuzzy sets serve as an excellent resolution of decision-making under uncertainties. But the modeling tools of Zadeh's fuzzy sets (Z-FSs) are limited whereby two or more sources of vagueness appear simultaneously. Thus several generalizations and extensions of Z-FSs are developed, such as type-2 fuzzy sets [3], [4], type-n fuzzy sets [4], intuitionistic fuzzy sets (IFSs) [5], fuzzy multisets [6] and hesitant fuzzy sets (T-HFSs) [7], [8].

T-HFSs are quite suit for the situation where we have a set of possible values, rather than a margin of error (as in IFSs) or some possibility distribution on the possible values (as in type-2 fuzzy sets) [7], [8]. The motivation to propose the T-HFSs is that when people make a decision, they are usually hesitant and irresolute for one thing or another which makes it difficult to reach a final agreement. "For example, three decision makers give the membership of x into A, and they want to assign 0.4, 0.5 and 0.7, which can be considered as a hesitant fuzzy element $\{0.4, 0.5, 0.7\}$ rather than the convex of 0.4 and 0.7, or the interval between 0.4 and 0.7" [13].

There are some developments on T-HFSs. Torra and Narukawa [7] introduced the extension principle to apply it in decision-making. Xu and Xia [9] developed a series of aggregation operators for hesitant fuzzy information and applied to multi-criteria decision-making. Later, some induced aggregation operators in hesitant

fuzzy setting are introduced by Xia et al. [10]. Based on Quasi arithmetic means, Xia et al. [11] discussed some ordered aggregation operators and induced ordered aggregation operators, as well as their application in group decision-making. Some similarity measure and correlation measures are detailed studied in Xu and Xia [12], [13], respectively. Later, Farhadinia study some information measures for hesitant fuzzy sets and interval-valued hesitant fuzzy sets in [14]. Chen et al. [15] study the correlation coefficients of hesitant fuzzy sets and apply to clustering analysis. Bedregal et al. [16] induced some aggregation functions for typical hesitant fuzzy elements.

Qian et al. further generalized the concept of T-HFSs in practice needs and gave the definition of generalized hesitant fuzzy sets [17]. There are mainly three advantages of the extension. First, as the case in T-HFSs, it is very useful to consider all possible memberships with hesitancy rather than considering just an aggregation operator. Second, it can eliminate times of using aggregation operators during the group decision-making process, which can alleviate suffering from less robust decision led by times of aggregations. At last, individual expert can express his/her evaluations by either Z-FSs, IFSs, T-HFSs or the proposed fuzzy sets.

Soft set theory [18] is also a new and important method for dealing with uncertain data. In recent years, research on soft set theory and its generalization has been done by many researchers in mathematics, computer and information science, including the works of fundamental soft set theory [19], soft set theory in abstract algebra and topological space [20]-[23], theory for data analysis, particularly in decision-making [24]-[27]. P.K. Maji et al. [28] defined the fuzzy soft set and studied some properties of this set. And Roy et al. [29] applied fuzzy soft set theory to decision-making problems. Majumdar and Samanta [30] introduced a concept of generalized fuzzy soft sets and their operations and application of generalized fuzzy soft sets in decision-making problem and medical diagnosis problem. Recently, some other generalizations of fuzzy soft sets such as trapezoidal fuzzy soft sets, multi-fuzzy soft sets, Type-2 Fuzzy Soft Sets, generalized interval-valued fuzzy soft sets and fuzzy soft multiset theory are studied in [31]-[35].

From the above analysis, we can see that generalized hesitant fuzzy sets and soft sets are useful tools for dealing with uncertainty and vagueness. Interestingly, it is possible to combine these two sets together and study new operations and properties of these sets, which must make them more important and applicable. In this paper, we will propose new sets based on generalized hesitant fuzzy sets and soft sets and study their properties.

2. Preliminaries

Let U be an initial universe set and E be a collection of all possible parameters with respect to U, where parameters are the characteristics or properties of objects in U. We will call E the universe set of parameters with respect to U.

Definition 2.1. ([18]) A pair (F, A) is called a soft set over U if $A \subset E$ and $F : A \to P(U)$, where $P(U)$ is the set of all subsets of U.

Definition 2.2. ([28]) A pair (F, A) is called a fuzzy soft set over U if $A \subset E$ and $F : A \to I^U$, where I^U denotes the collection of all fuzzy subsets of U.

Definition 2.3. ([5]) Let X be a fixed set, an intuitionistic fuzzy set (IFS) A on X is represented in terms of two functions $\mu : X \to [0, 1]$ and $\nu : X \to [0, 1]$, with the condition $0 \leq \mu(x) + \nu(x) \leq 1$, for all $x \in X$.

Furthermore, $\pi(x) = 1 - \mu(x) - \nu(x)$ is called a hesitancy degree or an intuitionistic index of x in A. In the special case $\pi(x) = 0$, that is, $\mu(x)+\nu(x) = 1$, the IFS A reduces to a FS.

Atanassov [5] gave some basic operations on IFSs, which ensure that the operational results are also IFSs.

Definition 2.4. ([5]) Let a set X be fixed, and let A (represented by the functions μ_A and ν_A), A_1 (represented by the functions μ_{A_1} and ν_{A_1}), A_2 (represented by the functions μ_{A_2} and ν_{A_2}), be three IFSs. Then the following operations are valid:

(1) complement: $A^c = \{\langle x, \nu_A(x), \mu_A(x)\rangle | x \in X\}$.
(2) intersection: $A_1 \cap A_2 = \{\langle x, min\{\mu_{A_1}(x), \mu_{A_2}(x)\}, max\{\nu_{A_1}(x), \nu_{A_2}(x)\}\rangle | x \in X\}$;
(3) union: $A_1 \cup A_2 = \{\langle x, max\{\mu_{A_1}(x), \mu_{A_2}(x)\}, min\{\nu_{A_1}(x), \nu_{A_2}(x)\}\rangle | x \in X\}$.

Sometimes, it is difficult to determine the membership of an element into a fixed set and which may be caused by a doubt among a set of different values. For the sake of a better description of this situation, Torra introduced the concept of T-HFS as a generalization of fuzzy sets. The membership degree of a T-HFS is presented by several possible values in [0, 1]. The definition is cited as follow.

Definition 2.5 ([7],[8]). Let X be a fixed set, then a hesitant fuzzy set (T-HFS) A on X in terms of a function h is that when applied to X returns a subset of $[0, 1]$.

To be easily understood, Xia and Xu [9] expressed the hesitant fuzzy set by the following mathematical symbol: $A = \{\langle x, h_A(x)\rangle \mid x \in X\}$, where $h_A(x)$ is a set of some different values in [0, 1], representing the possible membership degrees of the element $x \in X$ to A. Xia and Xu [9] called $h_A(x)$ a hesitant fuzzy element (HFE), a basic unit of T-HFS.

It is noted that the number of values in different HFE may be different, let $l(h_A(x))$ be the number of values in $h_A(x)$. We arrange the elements in $h_A(x)$ in increasing order, and let $h_A^{\sigma(j)}(x)$ be the jth value in $h_A(x)$.

Example 2.6. Let $X = \{x_1, x_2, x_3\}$ be a reference set. $h_A(x_1) = \{0.1, 0.3, 0.5\}$, $h_A(x_2) = \{0.4, 0.5\}$, $h_A(x_3) = \{0.3, 0.4, 0.5, 0.7\}$. A is a T-HFS, namely

$A = \{\langle x_1, 0.1, 0.3, 0.5\rangle, \langle x_2, 0.4, 0.5\rangle, \langle x_3, 0.3, 0.4, 0.5, 0.7\rangle\}$.

Apparently, this definition encompasses IFSs as a particular case in the form of a nonempty closed interval.

The followings are some basic operations which introduced in [9]-[13] on T-HFS:

Given a hesitant fuzzy set h, we define below its lower and upper bound,

- lower bound: $h^-(x) = minh(x)$ and
- upper bound: $h^+(x) = maxh(x)$.

Qian et.al. [17] defined the generalized hesitant fuzzy set in terms of two functions that return two sets of membership values and nonmembership values, respectively, for each element in the domain as follows.

Definition 2.7. Let X be a fixed set, then a generalized hesitant fuzzy set (GHF set) G on X is described as:

$G = \left\{ \frac{x}{h(x),g(x)} | x \in X \right\}$,

in which $h(x)$ and $g(x)$ are two sets of some values in [0, 1], denoting the possible membership degrees and nonmembership degrees of the element $x \in X$ to the set G, respectively, with the conditions:

$0 \leq \mu_i(x), \nu_i(x) \leq 1, 0 \leq \mu_i(x) + \nu_i(x) \leq 1$, $1 \leq i \leq N_x = |h(x)| = |g(x)|$,

where $\mu_i(x) \in h(x)$, $\nu_i(x) \in g(x)$, for all $x \in X$. And $|h(x)|$ denote the cardinality of the set $h(x)$. For convenience, the pair $G(x) = (h(x), g(x))$ is called a generalized hesitant fuzzy element (GHFE) denoted by $G = (h, g)$.

Definition 2.8. Given a GHFE represented by G, we define its complement as follows:

$$G^c(x) = \bigcup_{\mu_i(x) \in h(x), \nu_i(x) \in g(x)} \{\{\nu_i(x)\}, \{\mu_i(x)\}\}.$$

Definition 2.9. Given two generalized hesitant fuzzy sets $G_1 = (h_1(x), g_1(x))$ and $G_2 = (h_2(x), g_2(x))$, we define their union represented by $G_1 \cup G_2$ as $G_1 \cup G_2(x) = \{(\mu(x), \nu(x)), \mu(x) \in h_1(x) \cup h_2(x)) | \mu(x) \geq max(h_1^-(x), h_2^-(x)), \nu(x) \in (g_1(x) \cup g_2(x)) | \nu(x) \leq min(g_1^+(x), g_2^+(x))\}$.

Definition 2.10. Given two generalized hesitant fuzzy sets $G_1 = (h_1(x), g_1(x))$ and $G_2 = (h_2(x), g_2(x))$, we define their union represented by $G_1 \cap G_2$ as $G_1 \cap G_2(x) = \{(\mu(x), \nu(x)), \mu(x) \in (h_1(x) \cup h_2(x)) | \mu(x) \leq min(h_1^+(x), h_2^+(x)), \nu(x) \in (g_1(x) \cup g_2(x)) | \nu(x) \geq max(g_1^-(x), g_2^-(x))\}$.

Example 2.11. Let $G_1 = \{\{0.1, 0.3, 0.5\}, \{0.8, 0.6, 0.4\}\}$ and $G_2 = \{\{0.2, 0.4\}, \{0.7, 0.5\}\}$ be two GHFEs. Then we have

(1) complement: $G_1^c(x) = \{\{0.8, 0.6, 0.4\}, \{0.1, 0.3, 0.5\}\}$,

(2) union: $G_1 \cup G_2(x) = \{\{0.3, 0.5, 0.2, 0.4\}, \{0.6, 0.4, 0.7, 0.5\}\}$,

(3) intersection: $G_1 \cap G_2(x) = \{\{0.1, 0.3, 0.2, 0.4\}, \{0.8, 0.6, 0.7, 0.5\}\}$.

Assumption 2.12. Notice that the number of values in different GHFEs may be different. Suppose that $|h_M(x)|$ stands for the number of values in $h_M(x)$. Hereafter, the following assumptions are made: (see [12,36-38])

(A1) All the elements in each $h_M(x)$ are arranged in increasing order.

(A2) If, for some $x \in X$, $|h_M(x)| \neq |h_N(x)|$, then $l_x = max\{|h_M(x)|, |h_N(x)|\}$.

To have a correct comparison, the two GHFEs $h_M(x)$ and $h_N(x)$ should have the same length l_x. If there are fewer elements in $h_M(x)$ than $h_N(x)$, an extension of $h_M(x)$ should be considered optimistically by repeating its maximum element until it has the same length with $h_N(x)$.

Remark 2.13. If we arrange the membership sequences in increasing order, then the corresponding non-membership sequence may not be in decreasing or increasing order.

Definition 2.14. Let M, N be two GHF set on X. Then, M is a generalized hesitant fuzzy subset of N, if for each $x \in X$, $1 \leq i \leq l_x$, we have $\mu_i^M(x) \leq \mu_i^N(x)$ and $\nu_i^M(x) \geq \nu_i^N(x)$. And denote by $M \sqsubseteq N$.

Example 2.15. Let $X = \{x_1, x_2, x_3\}$ be the discourse set, and
$M = \{\frac{x_1}{(0.4),(0.5)}, \frac{x_2}{(0.4,0.5),(0.5,0.3)}, \frac{x_3}{(0.2,0.3,0.5,0.6),(0.7,0.6,0.4,0.3)}\}$,
$N = \{\frac{x_1}{(0.5,0.7),(0.4,0.2)}, \frac{x_2}{(0.6),(0.3)}, \frac{x_3}{(0.5,0.6),(0.3,0.2)}\}$,
be two GHFS sets on X. Then, in view of Assumption 2.12, the GHF sets M, N can be respectively represented as
$M = \{\frac{x_1}{(0.4,0.4),(0.5,0.5)}, \frac{x_2}{(0.4,0.5),(0.5,0.3)}, \frac{x_3}{(0.2,0.3,0.5,0.6),(0.7,0.6,0.4,0.3)}\}$,
$N = \{\frac{x_1}{(0.5,0.7),(0.4,0.2)}, \frac{x_2}{(0.6,0.6),(0.3,0.3)}, \frac{x_3}{(0.5,0.6,0.6,0.6),(0.3,0.2,0.2,0.2)}\}$.
We can find that $\mu_j^M(x_i) \leq \mu_j^N(x_i)$ and $\nu_j^M(x_i) \geq \nu_j^N(x_i)$, for each $x_i \in X$ and each $1 \leq j \leq l_{x_i}$. Then, M is a generalized hesitant fuzzy subset of N and denote by $M \sqsubseteq N$.

Definition 2.16. For a GHF set G, $s(h) = \frac{1}{|h|} \cdot \sum_{\gamma \in h} \gamma$ and $s(h) = \frac{1}{|g|} \cdot \sum_{\eta \in g} \eta$ is called the score function of h and g, where $|h|$ and $|g|$ represent the number of the elements in h and g.

3. Generalized hesitant fuzzy soft sets

Obviously, by combining the generalized hesitant fuzzy set and soft set, it is natural to define the generalized hesitant fuzzy soft sets model. We first define the generalized hesitant fuzzy soft sets as follows.

Definition 3.1 Let U be an initial universe and E a set of parameters, $A \subseteq E$. Define a function $\tilde{G} : A \to GHF^U$, where GHF^U denote all of the generalized hesitant fuzzy sets over U. Then, a pair $(\tilde{G}, A)$ is called a generalized hesitant fuzzy soft set (GHFS set).

Example 3.2. Let $U = \{x_1, x_2, x_3, x_4\}$. Let $A = \{e_1, e_2, e_3\} \subseteq E$ a set of parameters. Define $\tilde{G} : A \to HFS^U$, as follows:
$\tilde{G}(e_1) = \{\frac{x_1}{(0.4),(0.5)}, \frac{x_2}{(0.4,0.5),(0.5,0.3)}, \frac{x_3}{(0.2,0.3,0.5,0.6),(0.7,0.6,0.4,0.3)}, \frac{x_4}{(0.5,0.6),(0.3,0.2)}\}$,
$\tilde{G}(e_2) = \{\frac{x_1}{(0.1,0.3),(0.6,0.5)}, \frac{x_2}{(0.7,0.8,0.9),(0.2,0.1,0.1)}, \frac{x_3}{(0.2,0.3),(0.6,0.7)}, \frac{x_4}{(0.2,0.4),(0.6,0.5)}\}$,
$\tilde{G}(e_3) = \{\frac{x_1}{(0.2,0.4,0.6),(0.3,0.5,0.3)}, \frac{x_2}{(0.5,0.7,0.9),(0.4,0.3,0.1)}, \frac{x_3}{(0.1,0.2),(0.8,0.6)}, \frac{x_4}{(0.4,0.7),(0.5,0.2)}\}$.
Then $(\tilde{G}, A) = \{\tilde{G}(e_1), \tilde{G}(e_2), \tilde{G}(e_3)\}$ is a GHFS set.

Definition 3.3. Let $(\tilde{F}, A)$ and $(\tilde{G}, B)$ be two GHFS sets over (U, E). Then $(\tilde{F}, A)$ is called a GHFS subset of $(\tilde{G}, B)$ if
(i) $A \subseteq B$,
(ii) $\tilde{F}(e)$ is a generalized hesitant fuzzy subset of $\tilde{G}(e)$, for each $e \in A$.
In this case, the above relationship is denoted by $(\tilde{F}, A)\tilde{\sqsubseteq}(\tilde{G}, B)$. And $(\tilde{G}, B)$ is said to be a GHFS superset of $(\tilde{F}, A)$.

Definition 3.4. Let $(\tilde{F}, A)$ and $(\tilde{G}, B)$ be two GHFS sets over (U, E). Then $(\tilde{F}, A)$ and $(\tilde{G}, B)$ are said to be GHFS equal if and only if $(\tilde{F}, A)\tilde{\sqsubseteq}(\tilde{G}, B)$ and $(\tilde{G}, B)\tilde{\sqsubseteq}(\tilde{F}, A)$.

Example 3.5. We consider the GHFS set $(\tilde{G}, A)$ given in Example 3.2 and define a GHFS set $(\tilde{M}, B)$ as follows:

$\tilde{M}(e_1) = \{\frac{x_1}{(0.2),(0.7)}, \frac{x_2}{(0.3,0.4),(0.7,0.5)}, \frac{x_3}{(0.1,0.2,0.4,0.5),(0.8,0.7,0.5,0.4)}, \frac{x_4}{(0.2,0.4),(0.6,0.6)}\}$,

$\tilde{M}(e_2) = \{\frac{x_1}{(0.1,0.2),(0.9,0.6)}, \frac{x_2}{(0.3,0.4,0.5),(0.6,0.5,0.4)}, \frac{x_3}{(0.1,0.2),(0.9,0.8)}, \frac{x_4}{(0.2,0.3),(0.7,0.6)}\}$.

Then $(\tilde{M}, B)$ is a GHFS subset of $(\tilde{G}, A)$.

Definition 3.6. The complement of a GHFS set $(\tilde{G}, A)$ is denoted by $(\tilde{G}, A)^c$ and is defined by $(\tilde{G}, A)^c = (\tilde{G}^c, A)$, where $\tilde{G}^c : A \to GHF^U$ is a mapping given by $\tilde{G}^c(e) = \{\frac{x}{g(x),h(x)}, x \in U\}$.

Example 3.7. We consider the GHFS set $(\tilde{G}, A)$ given in Example 3.2. Then,

$\tilde{G}^c(e_1) = \{\frac{x_1}{(0.5),(0.4)}, \frac{x_2}{(0.5,0.3),(0.4,0.5)}, \frac{x_3}{(0.7,0.6,0.4,0.3),(0.2,0.3,0.5,0.6),}, \frac{x_4}{(0.3,0.2),(0.5,0.6)}\}$,

$\tilde{G}^c(e_2) = \{\frac{x_1}{(0.6,0.5),(0.1,0.3)}, \frac{x_2}{(0.2,0.1,0.1),(0.7,0.8,0.9)}, \frac{x_3}{(0.6,0.7),(0.2,0.3)}, \frac{x_4}{(0.6,0.5),(0.2,0.4)}\}$,

$\tilde{G}^c(e_3) = \{\frac{x_1}{(0.3,0.5,0.3),(0.2,0.4,0.6)}, \frac{x_2}{(0.4,0.3,0.1),(0.5,0.7,0.9)}, \frac{x_3}{(0.8,0.6),(0.1,0.2)}, \frac{x_4}{(0.5,0.2),(0.4,0.7)}\}$.

And, in view of Assumption 2.1, the GHFS set $(\tilde{G}, A)^c$ can be respectively represented as

$\tilde{G}^c(e_1) = \{\frac{x_1}{(0.5),(0.4)}, \frac{x_2}{(0.3,0.5),(0.5,0.4)}, \frac{x_3}{(0.3,0.4,0.6,0.7),(0.6,0.5,0.3,0.2),}, \frac{x_4}{(0.2,0.3),(0.6,0.5)}\}$,

$\tilde{G}^c(e_2) = \{\frac{x_1}{(0.5,0.6),(0.3,0.1)}, \frac{x_2}{(0.1,0.1,0.2),(0.8,0.9,0.7)}, \frac{x_3}{(0.6,0.7),(0.2,0.3)}, \frac{x_4}{(0.5,0.6),(0.4,0.2)}\}$,

$\tilde{G}^c(e_3) = \{\frac{x_1}{(0.3,0.3,0.5),(0.2,0.6,0.4)}, \frac{x_2}{(0.1,0.3,0.4),(0.9,0.7,0.5)}, \frac{x_3}{(0.6,0.8),(0.2,0.1)}, \frac{x_4}{(0.2,0.5),(0.7,0.4)}\}$.

Then, $(\tilde{G}, A)^c$ is the complement of the GHFS set $(\tilde{G}, A)$.

Definition 3.8. A GHFS set $(\tilde{G}, A)$ over (U, E) is said to be relative absolute GHFS set denoted by Ω_A, if $h_{\tilde{G}(e)}(x) = 1$ and $g_{\tilde{G}(e)}(x) = 0$ for all $x \in U$ and $e \in A$.

Definition 3.9. A GHFS set $(\tilde{G}, A)$ over (U, E) is said to be relative null GHFS set denoted by Φ_A, if $h_{\tilde{G}(e)}(x) = 0$ and $g_{\tilde{G}(e)}(x) = 1$ for all $x \in U$ and $e \in A$.

Followings are some operations on GHFS set:

Definition 3.10. Union of two GHFS sets $(\tilde{F}, A)$ and $(\tilde{G}, B)$ of over U is the GHFS set $(\tilde{H}, B)$, where $C = A \cup B$, and $\forall e \in C$ and $x \in U$,

$$\tilde{H}(e) = \begin{cases} \tilde{F}(e), & \text{if } e \in A - B; \\ \tilde{G}(e), & \text{if } e \in B - A; \\ \tilde{F}(e) \cup \tilde{G}(e), & \text{if } e \in A \cap B. \end{cases}$$

We write $(\tilde{F}, A)\tilde{\cup}(\tilde{G}, B) = (\tilde{H}, C)$.

Example 3.11. Consider Example 3.2 and define a GHFS set $(\tilde{M}, B)$ as follows:

$\tilde{M}(e_1) = \{\frac{x_1}{(0.2),(0.7)}, \frac{x_2}{(0.3,0.4),(0.7,0.6)}, \frac{x_3}{(0.1,0.2,0.4,0.5),(0.8,0.6,0.5,0.4)}, \frac{x_4}{(0.2,0.4),(0.6,0.6)}\}$,

$\tilde{M}(e_2) = \{\frac{x_1}{(0.1,0.2),(0.9,0.6)}, \frac{x_2}{(0.3,0.4,0.5),(0.6,0.5,0.4)}, \frac{x_3}{(0.1,0.2),(0.9,0.8)}, \frac{x_4}{(0.2,0.3),(0.7,0.6)}\}$.

We have $(\tilde{F}, A)\tilde{\cup}(\tilde{M}, B) = (\tilde{H}, C)$, where

$\tilde{H}(e_1) = \{\frac{x_1}{(0.4),(0.5)}, \frac{x_2}{(0.4,0.4,0.5),(0.6,0.5,0.3)}, \frac{x_3}{(0.2,0.3,0.5,0.6,0.2,0.4,0.5),(0.7,0.6,0.4,0.3,0.6,0.5,0.4)}, \frac{x_4}{(0.4,0.5,0.6),(0.6,0.3,0.2)}\}$,

$\tilde{H}(e_2) = \{\frac{x_1}{(0.1,0.2,0.3),(0.6,0.6,0.5)}, \frac{x_2}{(0.4,0.5,0.7,0.8,0.9),(0.5,0.4,0.2,0.1,0.1)}, \frac{x_3}{(0.2,0.2,0.3),(0.8,0.6,0.7)}, \frac{x_4}{(0.2,0.3,0.4),(0.6,0.6,0.5)}\},$

$\tilde{H}(e_3) = \{\frac{x_1}{(0.2,0.4,0.6),(0.3,0.5,0.3)}, \frac{x_2}{(0.5,0.7,0.9),(0.4,0.3,0.1)}, \frac{x_3}{(0.1,0.2),(0.8,0.6)}, \frac{x_4}{(0.4,0.7),(0.5,0.2)}\}.$

Definition 3.12. Intersection of two GHFS sets $(\tilde{F}, A)$ and $(\tilde{G}, B)$ of over U is the GHFS sets set $(\tilde{H}, B)$, where $C = A \cap B$, and $\forall e \in C$ and $x \in U$, $\tilde{H}(e) = \tilde{F}(e) \cap \tilde{G}(e)$.

We write $(\tilde{F}, A)\widetilde{\cap}(\tilde{G}, B) = (\tilde{H}, C)$.

Example 3.13. Consider Example 3.11. We have $(\tilde{F}, A)\widetilde{\cap}(\tilde{G}, B) = (\tilde{H}, C)$, where

$\tilde{H}(e_1) = \{\frac{x_1}{(0.2),(0.7)}, \frac{x_2}{(0.3,0.4,0.4),(0.7,0.6,0.5)}, \frac{x_3}{(0.2,0.3,0.5,0.1,0.2,0.4,0.5),(0.7,0.6,0.4,0.8,0.6,0.5,0.4)}, \frac{x_4}{(0.2,0.4,0.5),(0.6,0.6,0.3)}\},$

$\tilde{H}(e_2) = \{\frac{x_1}{(0.1,0.2,0.1),(0.9,0.6,0.6)}, \frac{x_2}{(0.3,0.4,0.5,0.7),(0.6,0.5,0.4,0.2)}, \frac{x_3}{(0.1,0.2),(0.9,0.8)}, \frac{x_4}{(0.2,0.3,0.2),(0.7,0.6,0.6)}\}.$

By the suggestions given by Molodtsov in [18], we present the notion of AND and OR operations on two GHFS sets as follows.

Definition 3.14. If $(\tilde{F}, A)$ and $(\tilde{G}, B)$ are two GHFS sets of over U, the "$(\tilde{F}, A)$ AND $(\tilde{G}, B)$", denoted by $(\tilde{F}, A)\wedge(\tilde{G}, B)$ is defined by $(\tilde{F}, A)\wedge(\tilde{G}, B)=(\tilde{H}, A\times B)$, where $\tilde{H}(\alpha, \beta)=\tilde{F}(\alpha) \cap \tilde{G}(\beta)$, for all $(\alpha, \beta) \in A \times B$.

Definition 3.15. If $(\tilde{F}, A)$ and $(\tilde{G}, B)$ are two GHFS sets of over U, the "$(\tilde{F}, A)$ OR $(\tilde{G}, B)$", denoted by $(\tilde{F}, A)\vee(\tilde{G}, B)$ is defined by $(\tilde{F}, A)\vee(\tilde{G}, B) = (\tilde{O}, A\times B)$, where $\tilde{O}(\alpha, \beta) = \tilde{F}(\alpha) \cup \tilde{G}(\beta)$, for all $(\alpha, \beta) \in A \times B$.

Example 3.16. Let $U=\{h_1, h_2, h_3\}$. Let $A=\{e_1, e_2\}\subseteq E$ and $B=\{e_2, e_4, e_5\} \subseteq E$ be sets of parameters. Define $(\tilde{F}, A)$ and $(\tilde{G}, B)$ as follows:

$\tilde{F}(e_1) = \{\frac{x_1}{(0.5,0.7),(0.3,0.2)}, \frac{x_2}{(0.8,0.9,1),(0.2,0.1,0)}, \frac{x_3}{(0.2,0.3),(0.7,0.6)}\},$

$\tilde{F}(e_2) = \{\frac{x_1}{(0.7,1),(0.3,0)}, \frac{x_2}{(0.4,0.5),(0.5,0.4)}, \frac{x_3}{(0.8,0.9),(0.1,0.1)}\},$

$\tilde{G}(e_2) = \{\frac{x_1}{(0.6,0.8),(0.3,0.2)}, \frac{x_2}{(0.5,0.6),(0.3,0.4)}, \frac{x_3}{(0.5,0.7),(0.4,0.3)}\},$

$\tilde{G}(e_4) = \{\frac{x_1}{(0.4,0.6),(0.5,0.3)}, \frac{x_2}{(0.6,0.8,0.9),(0.4,0.1,0.05)}, \frac{x_3}{(0.5),(0.3)}\},$

$\tilde{G}(e_5) = \{\frac{x_1}{(0.8,0.9),(0.2,0.05)}, \frac{x_2}{(0.1,0.3,0.5),(0.8,0.6,0.3)}, \frac{x_3}{(0.4,0.6,0.8),(0.5,0.3,0.2)}\}.$

We have $(\tilde{F}, A) \wedge (\tilde{G}, B) = (\tilde{H}, A \times B)$ and $(\tilde{F}, A) \vee (\tilde{G}, B) = (\tilde{O}, A \times B)$ as follows:

$\tilde{H}(e_1, e_2) = \{\frac{x_1}{(0.5,0.6,0.7),(0.3,0.3,0.2)}, \frac{x_2}{(0.5,0.6,0.8,0.9),(0.3,0.4,0.2,0.1)}, \frac{x_3}{(0.2,0.3,0.5),(0.7,0.6,0.4)},$

$\tilde{H}(e_1, e_4) = \{\frac{x_1}{(0.4,0.5,0.6),(0.5,0.3,0.3)}, \frac{x_2}{(0.6,0.8,0.8,0.9,0.9),(0.4,0.2,0.1,0.1,0.05)}, \frac{x_3}{(0.2,0.3),(0.7,0.6)}\},$

$\tilde{H}(e_1, e_5) = \{\frac{x_1}{(0.5,0.7,0.8),(0.3,0.2,0.2)}, \frac{x_2}{(0.1,0.3,0.5,0.8,0.9),(0.8,0.6,0.4,0.2,0.1)}, \frac{x_3}{(0.2,0.3,0.4,0.6),(0.7,0.6,0.5,0.3)}\},$

$\tilde{H}(e_2, e_2) = \{\frac{x_1}{(0.6,0.7,0.8),(0.3,0.3,0.2)}, \frac{x_2}{(0.4,0.5),(0.5,0.4)}, \frac{x_3}{(0.5,0.7),(0.4,0.3)}\},$

$\tilde{H}(e_2, e_4) = \{\frac{x_1}{(0.4,0.6,0.7),(0.5,0.3,0.3)}, \frac{x_2}{(0.4,0.5,0.6,0.8),(0.5,0.4,0.4,0.1)}, \frac{x_3}{(0.5),(0.3)}\},$

$\tilde{H}(e_2, e_5) = \{\frac{x_1}{(0.7,0.8,0.9),(0.3,0.2,0.05)}, \frac{x_2}{(0.1,0.3,0.4,0.5),(0.8,0.6,0.5,0.4)}, \frac{x_3}{(0.4,0.6,0.8),(0.5,0.3,0.2)}\}.$

and

$\tilde{O}(e_1, e_2) = \{\frac{x_1}{(0.6,0.7,0.8),(0.3,0.2,0.2)}, \frac{x_2}{(0.8,0.9,0.1),(0.2,0.1,0)}, \frac{x_3}{(0.3,0.5,0.7),(0.6,0.4,0.3)},$

$\tilde{O}(e_1, e_4) = \{\frac{x_1}{(0.5,0.6,0.7),(0.3,0.3,0.2)}, \frac{x_2}{(0.8,0.8,0.9,0.9,1),(0.2,0.1,0.1,0.05,0)}, \frac{x_3}{(0.3,0.5),(0.6,0.3)}\}$,
$\tilde{O}(e_1, e_5) = \{\frac{x_1}{(0.7,0.8,0.9),(0.2,0.2,0.05)}, \frac{x_2}{(0.3,0.5,0.8,0.9,1),(0.6,0.3,0.2,0.1,0)}, \frac{x_3}{(0.3,0.4,0.6,0.8),(0.6,0.5,0.3,0.2)}\}$,
$\tilde{O}(e_2, e_2) = \{\frac{x_1}{(0.7,0.8,1),(0.3,0.2,0)}, \frac{x_2}{(0.5,0.5,0.6),(0.4,0.3,0.4)}, \frac{x_3}{(0.7,0.8,0.9),(0.3,0.1,0.1)}\}$,
$\tilde{O}(e_2, e_4) = \{\frac{x_1}{(0.6,0.7,1),(0.3,0.3,0)}, \frac{x_2}{(0.5,0.6,0.8,0.9),(0.4,0.4,0.1,0.05)}, \frac{x_3}{(0.8,0.9),(0.1,0.1)}\}$,
$\tilde{O}(e_2, e_5) = \{\frac{x_1}{(0.8,0.9,1),(0.2,0.05,0)}, \frac{x_2}{(0.3,0.4,0.5,0.5),(0.6,0.5,0.4,0.3)}, \frac{x_3}{(0.6,0.8,0.8,0.9),(0.3,0.2,0.1,0.1)}\}$.

4. An application of GHFS set in decision-making

There are several applications of GHFS set theory in several directions. Here we present an application of GHFS sets in the decision-making problem.

First, we give some definitions which are useful in this section:

Definition 4.1. Let $(\tilde{G}, A)$ be a GHFS set over (U, E). If for each $e \in A$ and $x \in U$, we use $\mu(x) = h^{-}_{\tilde{G}(e)}(x)$ and the corresponding $\nu(x)$ of $\tilde{G}(e)$ to construct an intuitionistic fuzzy soft set (G, A). Then (G, A) is called the pessimistic reduct intuitionistic fuzzy soft set of $(\tilde{G}, A)$.

Example 4.2. Consider Example 3.2, the pessimistic reduct intuitionistic fuzzy soft set (F, A) of $(\tilde{F}, A)$ is defined as follow:

$G(e_1) = \{\frac{x_1}{(0.4,0.5)}, \frac{x_2}{(0.4,0.5)}, \frac{x_3}{(0.2,0.7)}, \frac{x_4}{(0.5,0.3)}\}$,
$G(e_2) = \{\frac{x_1}{(0.1,0.6)}, \frac{x_2}{(0.7,0.2)}, \frac{x_3}{(0.2,0.6)}, \frac{x_4}{(0.2,0.6)}\}$,
$G(e_3) = \{\frac{x_1}{(0.2,0.3)}, \frac{x_2}{(0.5,0.4)}, \frac{x_3}{(0.1,0.8)}, \frac{x_4}{(0.4,0.5)}\}$.

Then $(G, A) = \{G(e_1), G(e_2), G(e_3)\}$ is the pessimistic reduct intuitionistic fuzzy soft set of $(\tilde{G}, A)$.

Definition 4.3. Let $(\tilde{G}, A)$ be a GHFS set over (U, E). If for each $e \in A$ and $x \in U$, we use $\mu(x) = h^{+}_{\tilde{G}(e)}(x)$ and the corresponding $\nu(x)$ of $\tilde{G}(e)$ to construct an intuitionistic fuzzy soft set (G, A). Then (G, A) is called the optimistic reduct intuitionistic fuzzy soft set of $(\tilde{G}, A)$.

Example 4.4. Consider Example 3.2, the optimistic reduct intuitionistic fuzzy soft set (G, A) of $(\tilde{G}, A)$ is defined as follow:

$G(e_1) = \{\frac{x_1}{(0.4,0.5)}, \frac{x_2}{(0.5,0.3)}, \frac{x_3}{(0.6,0.3)}, \frac{x_4}{(0.6,0.2)}\}$,
$G(e_2) = \{\frac{x_1}{(0.3,0.5)}, \frac{x_2}{(0.9,0.1)}, \frac{x_3}{(0.3,0.7)}, \frac{x_4}{(0.4,0.5)}\}$,
$G(e_3) = \{\frac{x_1}{(0.6,0.3)}, \frac{x_2}{(0.9,0.1)}, \frac{x_3}{(0.2,0.6)}, \frac{x_4}{(0.7,0.2)}\}$.

Then $(G, A) = \{G(e_1), G(e_2), G(e_3)\}$ is the optimistic reduct intuitionistic fuzzy soft set of $(\tilde{G}, A)$.

Definition 4.5. Let $(\tilde{G}, A)$ be a GHFS set over (U, E). If for each $e \in A$ and $x \in U$, we use $\mu(x) = s(h_{\tilde{G}(e)}(x))$ and the corresponding $\nu(x) = s(g_{\tilde{G}(e)}(x))$ of $\tilde{G}(e)$ to construct an intuitionistic fuzzy soft set (G, A). Then (G, A) is called the neutral reduct intuitionistic fuzzy soft set of $(\tilde{G}, A)$.

Example 4.6. Consider Example 3.2, the neutral reduct intuitionistic fuzzy soft set (F, A) of $(\tilde{F}, A)$ is defined as follow:

$G(e_1) = \{\frac{x_1}{(0.4,0.5)}, \frac{x_2}{(0.45,0.4)}, \frac{x_3}{(0.4,0.5)}, \frac{x_4}{(0.55,0.25)}\}$,
$G(e_2) = \{\frac{x_1}{(0.2,0.55)}, \frac{x_2}{(0.8,0.13)}, \frac{x_3}{(0.25,0.65)}, \frac{x_4}{(0.3,0.55)}\}$,
$G(e_3) = \{\frac{x_1}{(0.4,0.36)}, \frac{x_2}{(0.7,0.27)}, \frac{x_3}{(0.15,0.7)}, \frac{x_4}{(0.55,0.35)}\}$.
Then $(G, A) = \{G(e_1), G(e_2), G(e_3)\}$ is the neutral reduct intuitionistic fuzzy soft set of $(\tilde{F}, A)$.

In the following, we will use comparison table introduced by A.R. Roy and P.K. Maji [29] to solve a decision-making problem which is based on the operations of generalized hesitant fuzzy soft sets.

Example 4.7. Let us consider a GHFS set which describes evaluations of the investment value of some stocks. Suppose that there are six stocks in the universe $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ under consideration, and $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ is the set of decision parameters, where e_i $(i = 1, 2, 3, 4, 5, 6)$ stands for the parameters "earnings per share(eps)", "net assets per share", "net profit growth rate", "market share growth ratio", "asset-liability ratio" and "price to earning ratio (ps)", respectively. Let A, B and C denote three subsets of the parameters E. Also let A represents the profitability of each stock and B represents the growth of each stock. $A = \{$earnings per share(eps), net assets per share$\}$ and $B = \{$net profit growth rate, market share growth ratio$\}$. The subset C represents the risk of each stock, i.e., $C = \{$asset-liability ratio, price to earning ratio (ps)$\}$.

Assuming that the GHFS set $(\tilde{F}, A)$ describes the "stocks having high profitabi-lity", the GHFS set $(\tilde{G}, B)$ describes the "stocks having high growth" and the GHFS set $(\tilde{H}, C)$ describes the "stocks having low investing risk". Our purpose is to find out the best stock for the investor based on experts' evaluations.

After a serious and careful analysis, a group of stock experts construct the following generalized hesitant fuzzy soft sets:
$(\tilde{F}, A) = \{\tilde{F}(e_1), \tilde{F}(e_2)\}\}$ and
$\tilde{F}(e_1) = \{\frac{x_1}{(0.6,0.7,0.8),(0.3,0.3,0.1)}, \frac{x_2}{(0.2,0.3),(0.6,0.5)}, \frac{x_3}{(0.1,0.3,0.4),(0.8,0.7,0.6)}, \frac{x_4}{(0.5,0.7,0.8),(0.3,0.2,0.2)}, \frac{x_5}{(0.5,0.6,0.7),(0.3,0.4,0.3)}, \frac{x_6}{(0.4,0.6,0.8),(0.4,0.3,0.2)}\}$,
$\tilde{F}(e_2) = \{\frac{x_1}{(0.3,0.5,0.7),(0.6,0.3,0.3)}, \frac{x_2}{(0.2,0.3,0.5),(0.7,0.5,0.2)}, \frac{x_3}{(0.3,0.4,0.5),(0.6,0.4,0.5)}, \frac{x_4}{(0.3,0.5,0.6),(0.6,0.3,0.2)}, \frac{x_5}{(0.7,0.8,0.9),(0.3,0.1,0.05)}, \frac{x_6}{(0.3,0.5,0.7),(0.6,0.5,0.15)}\}$.
$(\tilde{G}, B) = \{\tilde{G}(e_3), \tilde{G}(e_4)\}$ and
$\tilde{G}(e_3) = \{\frac{x_1}{(0.5,0.6),(0.4,0.2)}, \frac{x_2}{(0.4,0.5,0.6),(0.5,0.3,0.3)}, \frac{x_3}{(0.2,0.4,0.6),(0.7,0.5,0.4)}, \frac{x_4}{(0.4,0.5),(0.5,0.4)}, \frac{x_5}{(0.3,0.4,0.5),(0.6,0.5,0.2)}, \frac{x_6}{(0.2,0.4,0.6),(0.7,0.6,0.2)}\}$,
$\tilde{G}(e_4) = \{\frac{x_1}{(0.4,0.6),(0.5,0.2)}, \frac{x_2}{(0.3,0.4,0.5),(0.65,0.5,0.5)}, \frac{x_3}{(0.2,0.3),(0.65,0.6)}, \frac{x_4}{(0.4),(0.3)}, \frac{x_5}{(0.4,0.6,0.9),(0.6,0.3,0.1)}, \frac{x_6}{(0.7,0.8),(0.2,0.1)}\}$.
$(\tilde{H}, C) = \{\tilde{H}(e_5), \tilde{H}(e_6)\}$ and
$\tilde{H}(e_5) = \{\frac{x_1}{(0.75,0.85),(0.15,0.05)}, \frac{x_2}{(0.55,0.65),(0.1,0.35)}, \frac{x_3}{(0.65,0.75),(0.2,0.1)}, \frac{x_4}{(0.75,0.85),(0.15,0.05)}, \frac{x_5}{(0.65,0.85),(0.25,0.15)}, \frac{x_6}{(0.75,0.95),(0.2,0.01)}\}$,
$\tilde{H}(e_6) = \{\frac{x_1}{(0.65),(0.15)}, \frac{x_2}{(0.72,0.83),(0.18,0.12)}, \frac{x_3}{(0.65),(0.22)}, \frac{x_4}{(0.45),(0.15)}, \frac{x_5}{(0.72,0.88),(0.2,0.12)}, \frac{x_6}{(0.66,0.82),(0.22,0.13)}\}$.

Let $(\tilde{F}, A)$ and $(\tilde{G}, B)$ be any two GHFS sets over the common universe U. After performing some operations (like AND,OR etc.) on the GHFS sets for

some particular parameters of A and B, we obtain another GHFS set. The newly obtained GHFS set is termed as resultant GHFS set of $(\tilde{F}, A)$ and $(\tilde{G}, B)$

Considering the above two GHFS sets $(\tilde{F}, A)$ and $(\tilde{G}, B)$ if we perform "$(\tilde{F}, A)$ OR $(\tilde{G}, B)$". Then we can get the following resultant new GHFS set: $(\tilde{K}, A \times B)$ defined as following,

$$\tilde{K}(e_1, e_3) = \{\frac{x_1}{(0.6,0.6,0.7,0.8),(0.2,0.3,0.3,0.1)}, \frac{x_2}{(0.3,0.4,0.5,0.6),(0.5,0.5,0.3,0.3)}, \frac{x_3}{(0.2,0.3,0.4,0.4,0.6),(0.7,0.7,0.6,0.5,0.4)}, \frac{x_4}{(0.5,0.5,0.7,0.8),(0.4,0.3,0.2,0.2)}, \frac{x_5}{(0.4,0.5,0.5,0.6,0.7),(0.5,0.2,0.3,0.4,0.3)}, \frac{x_6}{(0.4,0.4,0.6,0.6,0.8),(0.6,0.4,0.2,0.3,0.2)}\},$$

$$\tilde{K}(e_1, e_4) = \{\frac{x_1}{(0.6,0.6,0.7,0.8),(0.2,0.3,0.3,0.1)}, \frac{x_2}{(0.3,0.4,0.5),(0.5,0.5,0.5)}, \frac{x_3}{(0.2,0.3,0.3,0.4),(0.65,0.7,0.6,0.6)}, \frac{x_4}{(0.5,0.7,0.8),(0.3,0.2,0.2)}, \frac{x_5}{(0.5,0.6,0.6,0.7,0.9),(0.3,0.4,0.3,0.3,0.1)}, \frac{x_6}{(0.6,0.7,0.8,0.8),(0.3,0.2,0.2,0.1)}\},$$

$$\tilde{K}(e_2, e_3) = \{\frac{x_1}{(0.5,0.5,0.6,0.7),(0.4,0.3,0.2,0.3)}, \frac{x_2}{(0.3,0.4,0.5,0.5,0.6),(0.5,0.5,0.2,0.3,0.3)}, \frac{x_3}{(0.3,0.4,0.4,0.5,0.6),(0.6,0.4,0.5,0.5,0.4)}, \frac{x_4}{(0.4,0.5,0.5,0.6),(0.5,0.4,0.3,0.2)}, \frac{x_5}{(0.4,0.5,0.7,0.8,0.9),(0.5,0.2,0.3,0.1,0.05)}, \frac{x_6}{(0.3,0.4,0.5,0.6,0.7),(0.6,0.6,0.5,0.2,0.15)}\},$$

$$\tilde{K}(e_2, e_4) = \{\frac{x_1}{(0.4,0.5,0.6,0.7),(0.5,0.3,0.2,0.3)}, \frac{x_2}{(0.3,0.3,0.4,0.5,0.5),(0.5,0.65,0.5,0.2,0.5)}, \frac{x_3}{(0.3,0.4,0.5),(0.6,0.4,0.5)}, \frac{x_4}{(0.4,0.5,0.6),(0.3,0.3,0.2)}, \frac{x_5}{(0.6,0.7,0.8,0.9,0.9),(0.3,0.3,0.1,0.1,0.05)}, \frac{x_6}{(0.5,0.7,0.7,0.8),(0.5,0.15,0.2,0.1)}\}.$$

Considering the above two GHFS sets $(\tilde{K}, A \times B)$ and $(\tilde{H}, C)$ if we perform "$(\tilde{K}, A \times B)$ AND $(\tilde{H}, C)$". Then we can get the following resultant new GHFS set: $(\tilde{R}, A \times B \times C)$ defined as following,

$$\tilde{R}(e_1, e_3, e_5) = \{\frac{x_1}{(0.6,0.6,0.7,0.75,0.8),(0.2,0.3,0.3,0.15,0.1)}, \frac{x_2}{(0.3,0.4,0.5,0.6),(0.5,0.5,0.3,0.3)}, \frac{x_3}{(0.2,0.3,0.4,0.4,0.6,0.65),(0.7,0.7,0.6,0.5,0.4,0.2)}, \frac{x_4}{(0.5,0.5,0.7,0.75,0.8),(0.4,0.3,0.2,0.15,0.2)}, \frac{x_5}{(0.4,0.5,0.5,0.6,0.65,0.7),(0.5,0.2,0.3,0.4,0.25,0.3)}, \frac{x_6}{(0.4,0.4,0.6,0.6,0.75,0.8),(0.6,0.4,0.2,0.3,0.2,0.2)}\},$$

$$\tilde{R}(e_1, e_3, e_6) = \{\frac{x_1}{(0.6,0.6,0.65,0.7),(0.2,0.3,0.15,0.3)}, \frac{x_2}{(0.3,0.4,0.5,0.6,0.72),(0.5,0.5,0.3,0.3,0.18)}, \frac{x_3}{(0.2,0.3,0.4,0.4,0.6),(0.7,0.7,0.6,0.5,0.4)}, \frac{x_4}{(0.5,0.5,0.7),(0.4,0.3,0.2)}, \frac{x_5}{(0.4,0.5,0.5,0.6,0.7,0.72),(0.5,0.2,0.3,0.4,0.3,0.2)}, \frac{x_6}{(0.4,0.4,0.6,0.6,0.66,0.8),(0.6,0.4,0.2,0.3,0.22,0.2)}\},$$

$$\tilde{R}(e_1, e_4, e_5)\{\frac{x_1}{(0.6,0.6,0.7,0.75,0.8),(0.2,0.3,0.3,0.15,0.1)}, \frac{x_2}{(0.3,0.4,0.5),(0.5,0.5,0.5)}, \frac{x_3}{(0.2,0.3,0.3,0.4,0.65),(0.65,0.7,0.6,0.6,0.2)}, \frac{x_4}{(0.5,0.7,0.75,0.8),(0.3,0.2,0.15,0.2)}, \frac{x_5}{(0.5,0.6,0.6,0.65,0.7,0.85),(0.3,0.4,0.3,0.25,0.3,0.15)}, \frac{x_6}{(0.6,0.7,0.75,0.8,0.8),(0.3,0.2,0.2,0.2,0.1)}\},$$

$$\tilde{R}(e_1, e_4, e_6) = \{\frac{x_1}{(0.6,0.6,0.65,0.7),(0.2,0.3,0.15,0.3)}, \frac{x_2}{(0.3,0.4,0.5,0.72),(0.5,0.5,0.5,0.18)}, \frac{x_3}{(0.2,0.3,0.3,0.4),(0.65,0.7,0.6,0.6)}, \frac{x_4}{(0.5,0.7),(0.3,0.2)}, \frac{x_5}{(0.5,0.6,0.6,0.7,0.72,0.88),(0.3,0.4,0.3,0.3,0.2,0.12)}, \frac{x_6}{(0.6,0.66,0.7,0.8),(0.3,0.22,0.2,0.2)}\},$$

$$\tilde{R}(e_2, e_3, e_5) = \{\frac{x_1}{(0.5,0.5,0.6,0.7,0.75),(0.4,0.3,0.2,0.3,0.15)}, \frac{x_2}{(0.3,0.4,0.5,0.5,0.6),(0.5,0.5,0.2,0.3,0.3)}, \frac{x_3}{(0.3,0.4,0.4,0.5,0.6,0.65),(0.6,0.4,0.5,0.5,0.4,0.2)}, \frac{x_4}{(0.4,0.5,0.5,0.6,0.75),(0.5,0.4,0.3,0.2,0.15)}, \frac{x_5}{(0.4,0.5,0.65,0.7,0.8,0.85),(0.5,0.2,0.25,0.3,0.1,0.15)}, \frac{x_6}{(0.3,0.4,0.5,0.6,0.7,0.75),(0.6,0.6,0.5,0.2,0.15,0.2)}\},$$

$$\tilde{R}(e_2, e_3, e_6) = \{\frac{x_1}{(0.5,0.5,0.6),(0.4,0.3,0.2)}, \frac{x_2}{(0.3,0.4,0.5,0.5,0.6,0.72),(0.5,0.5,0.2,0.3,0.3,0.18)}, \frac{x_3}{(0.3,0.4,0.4,0.5,0.6),(0.6,0.4,0.5,0.5,0.4)}, \frac{x_4}{(0.4,0.5,0.5),(0.5,0.4,0.3)}, \frac{x_5}{(0.4,0.5,0.7,0.72,0.8,0.88),(0.5,0.2,0.3,0.2,0.1,0.12)}, \frac{x_6}{(0.3,0.4,0.5,0.6,0.66,0.7),(0.6,0.6,0.5,0.2,0.22,0.15)}\},$$

$$\tilde{R}(e_2, e_4, e_5) = \{\frac{x_1}{(0.4,0.5,0.6,0.7,0.75),(0.5,0.3,0.2,0.3,0.15)}, \frac{x_2}{(0.3,0.3,0.4,0.5,0.5),(0.5,0.65,0.5,0.2,0.5)}, \frac{x_3}{(0.3,0.4,0.5,0.65),(0.6,0.4,0.5,0.2)}, \frac{x_4}{(0.4,0.5,0.6,0.75),(0.3,0.3,0.2,0.15)}, \frac{x_5}{(0.6,0.65,0.7,0.8,0.85,0.9),(0.3,0.25,0.3,0.1,0.15,0.1)}, \frac{x_6}{(0.5,0.7,0.7,0.75,0.8),(0.5,0.15,0.2,0.2,0.1)}\},$$

$$\tilde{R}(e_2, e_4, e_6) = \{\frac{x_1}{(0.4,0.5,0.6),(0.5,0.3,0.2)}, \frac{x_2}{(0.3,0.3,0.4,0.5,0.5,0.72),(0.5,0.65,0.5,0.2,0.5,0.18)},$$
$$\frac{x_3}{(0.3,0.4,0.5),(0.6,0.4,0.5)}, \frac{x_4}{(0.4,0.5),(0.3,0.3)},$$
$$\frac{x_5}{(0.6,0.7,0.72,0.8,0.88,0.9),(0.3,0.3,0.2,0.1,0.12,0.1)}, \frac{x_6}{(0.5,0.66,0.7,0.7),(0.5,0.22,0.15,0.2)}\}.$$

And the GHFS set $(\tilde{R}, A \times B \times C)$ describes the evaluations of the stocks having high profitability or high growth and low investing risk by the group of experts.

Definition 4.8. [29] (Comparison table) It is a square table in which number of rows and number of column are equal and both are labeled by the object name of the universe such as $x_1, x_2, ..., x_n$ and the entries are c_{ij}, where c_{ij} = the number of parameters for which the value of x_i exceeds or equal to the value of x_j.

Algorithm:

(i) Input the set $A \subseteq E$ of choice of parameters of the investor.

(ii) Consider the reduced intuitionistic fuzzy soft set obtained by the GHFS set in tabular form.

(iii) Compute the comparison table of membership function and non-membership function of the reduced intuitionistic fuzzy soft set.

(iv) Compute the membership score and non-membership score.

(v) Compute the final score by subtracting non-membership score from membership score.

(vi) Find the maximum score, if it occurs in i-th row then the investor will choose x_i.

Note. In step (ii), the reduced intuitionistic fuzzy soft set we can choose the pessimistic reduct intuitionistic fuzzy soft set of $(\tilde{G}, A)$, the optimistic reduct intuitionistic fuzzy soft set of $(\tilde{G}, A)$, or the neutral reduct intuitionistic fuzzy soft set of $(\tilde{G}, A)$, depending on the decision-maker's actual situations. For example, we use the optimistic reduct intuitionistic fuzzy soft set of $(\tilde{G}, A)$ in step (ii). The tabular representation of the optimistic reduct fuzzy soft set of $(\tilde{R}, A \times B \times C)$ will be as.

Table 1. Tabular representation of membership function

·	(e_1, e_3, e_5)	(e_1, e_3, e_6)	(e_1, e_4, e_5)	(e_1, e_4, e_6)
x_1	0.8	0.7	0.8	0.7
x_2	0.6	0.72	0.5	0.72
x_3	0.65	0.6	0.65	0.4
x_4	0.8	0.7	0.8	0.7
x_5	0.7	0.72	0.85	0.88
x_6	0.8	0.8	0.8	0.8

·	(e_2, e_3, e_5)	(e_2, e_3, e_6)	(e_2, e_4, e_5)	(e_2, e_4, e_6)
x_1	0.75	0.6	0.75	0.6
x_2	0.6	0.72	0.5	0.72
x_3	0.65	0.6	0.65	0.5
x_4	0.75	0.5	0.75	0.5
x_5	0.85	0.88	0.9	0.9
x_6	0.75	0.7	0.8	0.7

Table 2. Comparison table of the above table

·	x_1	x_2	x_3	x_4	x_5	x_6
x_1	8	4	8	8	1	3
x_2	4	8	4	4	1	2
x_3	1	4	8	2	0	0
x_4	6	4	7	8	1	3
x_5	7	8	8	7	8	6
x_6	8	6	8	8	2	8

Table 3. Membership score table

·	RowSum(a)	ColumnSum(b)	MembershipScore(a-b)
x_1	32	34	-2
x_2	23	34	-11
x_3	15	43	-28
x_4	29	37	-8
x_5	44	13	31
x_6	40	22	18

Table 4. Tabular representation of non-membership function

·	(e_1, e_3, e_5)	(e_1, e_3, e_6)	(e_1, e_4, e_5)	(e_1, e_4, e_6)
x_1	0.1	0.3	0.1	0.3
x_2	0.3	0.18	0.5	0.18
x_3	0.2	0.4	0.2	0.6
x_4	0.2	0.2	0.2	0.2
x_5	0.3	0.2	0.15	0.12
x_6	0.2	0.2	0.1	0.2

·	(e_2, e_3, e_5)	(e_2, e_3, e_6)	(e_2, e_4, e_5)	(e_2, e_4, e_6)
x_1	0.15	0.2	0.15	0.2
x_2	0.3	0.18	0.5	0.18
x_3	0.2	0.4	0.2	0.5
x_4	0.15	0.3	0.15	0.3
x_5	0.15	0.12	0.1	0.1
x_6	0.2	0.15	0.1	0.2

Table 5. Comparison table of the above table

·	x_1	x_2	x_3	x_4	x_5	x_6
x_1	8	4	0	4	6	6
x_2	4	8	4	4	7	5
x_3	8	4	8	8	7	8
x_4	6	4	2	8	7	7
x_5	3	2	1	3	8	4
x_6	4	3	2	4	6	8

Table 6. Non-membership score table

·	RowSum(a)	ColumnSum(b)	Non-embershipScore(a-b)
x_1	28	33	-5
x_2	32	25	7
x_3	43	17	26
x_4	34	31	3
x_5	21	41	-20
x_6	27	38	-11

Table 7. Final score table

·	MembershipScore(m)	Non-membershipScore(n)	FinalScore(m-n)
x_1	-2	-5	3
x_2	-11	7	-18
x_3	-28	26	-54
x_4	-8	3	-11
x_5	31	-20	51
x_6	18	-11	29

Clearly, the maximum score is 51 scored by the stock x_5. Decision: The investor will choose x_5. In case, if the investor does not want to choose x_5 due to certain reasons, its second choice will be x_6. This is the result obtained by the optimistic reduct fuzzy soft set of $(\tilde{R}, A\times B\times C)$. Other cases, e.g., the pessimistic reduct intuitionistic fuzzy soft set or the neutral reduct intuitionistic fuzzy soft set of $(\tilde{G}, A)$, are similar to the above analysis.

Remark 4.9. After step (ii), we can also adopt Jiang's algorithm by using adjustable approach to intuitionistic fuzzy soft sets based decision-making [39] or Zhang's rough set approach to intuitionistic fuzzy soft set based decision making [25] to get the final optimal decision, especially when there are too many "optimal choices" to be chosen.

5. Similarity between two generalized hesitant fuzzy soft sets

In several problems it is often required to compare two sets. The sets may be fuzzy, may be vague etc. We often interested to know whether two patterns or images are identical or approximately identical or at least to what degree they are identical. Several researchers have studied the problem of similarity measurement between fuzzy sets, fuzzy numbers and vague sets. Recently Majumdar and Samanta [40], [41] have studied the similarity measure of soft sets and fuzzy soft sets. Chang Wang and Anjing Qu studied the entropy, similarity measure and distance on vague soft sets in [42]. Y.C. Jiang et al. studied entropy, similarity measure and distance on intuitionistic fuzzy soft sets interval-valued fuzzy soft sets in [43]. Similarity measures have extensive application in several areas such as pattern recognition [44], image processing [45], cluster analysis [46], approximate reasoning [47], medical diagnosis [48], decision-making [49] etc.

Let $U = \{x_1, x_2, ..., x_n\}$ be the universal set of elements and $E = \{e_1, e_2, ..., e_m\}$ be the universal set of parameters. Let $(\tilde{F}, E)$ and $(\tilde{G}, E)$ be two GHFS sets over the parametrized universe (U, E).

Definition 5.1. We first define the Hamming distance $d((\tilde{F}, E), (\tilde{G}, E))$ by the equation:

$$d((\tilde{F}, E), (\tilde{G}, E)) = \sum_{i=1}^{m}\sum_{j=1}^{n}\frac{1}{2}\cdot\left[\frac{1}{l_{x_j}}\cdot\sum_{k=1}^{l_{x_j}}|\mu_F^{\sigma(k)}(e_i)(x_j) - \mu_G^{\sigma(k)}(e_i)(x_j)| + \frac{1}{l_{x_j}}\cdot\sum_{k=1}^{l_{x_j}}|\nu_F^{\sigma(k)}(e_i)(x_j) - \nu_G^{\sigma(k)}(e_i)(x_j)|\right].$$

Then, define the normalized Hamming distance between $\tilde{F}(e_i)$ and $\tilde{G}(e_i)$ by the following formula:

$D((\tilde{F}, E), (\tilde{G}, E)) = \frac{d((\tilde{F}, E), (\tilde{G}, E))}{nm}$.

And we find the similarity by using the simple formula:

$S((\tilde{F}, E), (\tilde{G}, E)) = 1 - D((\tilde{F}, E), (\tilde{G}, E))$.

Example 5.2. Consider the following two GHFS sets where $U = \{x_1, x_2, x_3\}$ $E = \{e_1, e_2\}$. Define $(\tilde{F}, E)$ and $(\tilde{G}, E)$ as follows:

$\tilde{F}(e_1) = \{\frac{x_1}{(0.3,0.5),(0.5,0.2)}, \frac{x_2}{(0.6,0.9),(0.2,0.1)}, \frac{x_3}{(0.2,0.4),(0.6,0.5)}\}$,

$\tilde{F}(e_2) = \{\frac{x_1}{(0.7,0.8),(0.1,0.2)}, \frac{x_2}{(0.5,0.7),(0.3,0.2)}, \frac{x_3}{(0.5,0.8),(0.3,0.1)}\}$.

$\tilde{G}(e_1) = \{\frac{x_1}{(0.4,0.7),(0.5,0.3)}, \frac{x_2}{(0.6,0.7),(0.3,0.2)}, \frac{x_3}{(0.2,0.5),(0.3,0.4)}\}$,

$\tilde{G}(e_2) = \{\frac{x_1}{(0.6,0.7),(0.3,0.1)}, \frac{x_2}{(0.6,0.8),(0.3,0.1)}, \frac{x_3}{(0.5,0.7),(0.3,0.2)}\}$.

Here, the Hamming distance

$$d((\tilde{F}, E), (\tilde{G}, E)) = \sum_{i=1}^{2}\sum_{j=1}^{3}\frac{1}{2}\cdot\left[\frac{1}{l_{x_j}}\cdot\sum_{k=1}^{l_{x_j}}|\mu_F^{\sigma(k)}(e_i)(x_j) - \mu_G^{\sigma(k)}(e_i)(x_j)| + \frac{1}{l_{x_j}}\cdot\sum_{k=1}^{l_{x_j}}|\nu_F^{\sigma(k)}(e_i)(x_j) - \nu_G^{\sigma(k)}(e_i)(x_j)|\right]$$

$$= \frac{\frac{1}{2}(|0.3-0.4|+|0.5-0.7|+|0.5-0.5|+|0.2-0.3|)+\frac{1}{2}(|0.6-0.6|+|0.9-0.7|+|0.2-0.3|+|0.1-0.2|)+\frac{1}{2}(|0.2-0.2|+|0.4-0.5|+|0.6-0}{2}$$

$$+\frac{\frac{1}{2}(|0.7-0.6|+|0.8-0.7|+|0.1-0.3|+|0.2-0.1|)+\frac{1}{2}(|0.5-0.6|+|0.7-0.8|+|0.3-0.3|+|0.2-0.1|)+\frac{1}{2}(|0.5-0.5|+|0.8-0.7|+|0.3-0.}{2}$$

$= 0.575$.

The normalized Hamming distance

$$D((\tilde{F}, E), (\tilde{G}, E)) = \frac{d((\tilde{F}, E), (\tilde{G}, E))}{nm} = \frac{0.575}{6} \approx 0.096.$$

Hence the similarity between the two GHFS sets $(\tilde{F}, E)$ and $(\tilde{G}, E)$ will be

$$S((\tilde{F}, E), (\tilde{G}, E)) = 1 - 0.096 = 0.904.$$

Proposition 5.3. *Let $(\tilde{F}, E)$, $(\tilde{G}, E)$ and $(\tilde{H}, E)$ be three GHFS sets over the parametrized universe (U, E). Then the following holds:*

(i) $S((\tilde{F},E),(\tilde{G},E)) = S((\tilde{G},E),(\tilde{F},E))$,

(ii) $0 \leq S((\tilde{F},E),(\tilde{G},E)) \leq 1$,

(iii) $(\tilde{F},E) = (\tilde{G},E) \Leftrightarrow S((\tilde{F},E),(\tilde{G},E)) = 1$,

(iv) $(\tilde{F},E)\tilde{\sqsubseteq}(\tilde{G},E)\tilde{\sqsubseteq}(\tilde{H},E) \Rightarrow S((\tilde{F},E),(\tilde{H},E)) \leq S((\tilde{G},E),(\tilde{H},E))$.

Proof. The proofs are straightforward and follow from definition.

6. An application of this similarity measure in decision-making

This technique of similarity measure between two GHFS sets can also be applied to decision-making problems.

In what follows, we give an example adapted from [50] and [12] to illustrate our distance measures for GHFS sets:

Example 6.1. Energy is an indispensable factor for the socio-economic development of societies. Thus the correct energy policy affects economic development and environment, and so, the most appropriate energy policy selection is very important. Suppose that there are four alternatives (energy projects) A_i (i = 1,2,3,4) to be invested, each energy project has two stages x_1, x_2, and four attributes to be considered: e_1: technological; e_2: environmental; e_3: socio-political; e_4: economic (more details about them can be found in [50,12]). Several decision makers are invited to evaluate the performance of the four alternatives. For an alternative under an attribute, although all of the decision makers provide their evaluated values, some of these values may be repeated. However, a value repeated more times does not indicate that it has more importance than other values repeated less times. For example, the value repeated one time may be provided by a decision maker who is an expert at this area, and the value repeated twice may be provided by two decision makers who are not familiar with this area. In such cases, the value repeated one time may be more important than the one repeated twice. To get a more reasonable result, it is better that the decision makers give their evaluations anonymously. We only collect all of the possible values for an alternative under an attribute, and each value provided only means that it is a possible value, but its importance is unknown. Thus the times that the values repeated are unimportant, and it is reasonable to allow these values repeated many times appear only once. The GHFS is just a tool to deal with such cases, and all possible evaluations for an alternative under the attributes can be considered as a GHFS.

Suppose that the ideal alternative is $A = 1$ seen as a special GHFS, we can calculate the distance between each alternative and the ideal alternative using our distance measures.

Our model generalized hesitant fuzzy soft set for ideal alternative A is given in Table 8.

Table 8. Tabular representation of Model GHFS set

$(\tilde{A}, E)$	e_1	e_2	e_3	e_4
x_1	(1, 0)	(1, 0)	(1, 0)	(1, 0)
x_2	(1, 0)	(1, 0)	(1, 0)	(1, 0)

Table 9. Tabular representation of GHFS set for the first energy projects

$(\tilde{A}_1, E)$	e_1	e_2	e_3	e_4
x_1	$(0.6, 0.7, 0.9), (0.2, 0.1, 0.1)$	$(0.7, 0.8), (0.2, 0.1)$	$(0.8, 0.9), (0.2, 0.1)$	$(0.7, 0.9), (0.2, 0.1)$
x_2	$(0.7, 0.8), (0.2, 0.1)$	$(0.6, 0.7), (0.3, 0.2)$	$(0.5, 0.7), (0.3, 0.4)$	$(0.8, 0.9), (0.2, 0.1)$

Table 10. Tabular representation of GHFS set for the second energy projects

$(\tilde{A}_2, E)$	e_1	e_2	e_3	e_4
x_1	(0.3,0.5),(0.5,0.3)	(0.7,0.9),(0.3,0.1)	(0.1,0.5),(0.8,0.5)	(0.3,0.4),(0.6,0.5)
x_2	(0.4,0.6),(0.3,0.3)	(0.6,0.7),(0.2,0.1)	(0.3,0.5),(0.4,0.3)	(0.5,0.6),(0.3,0.3)

Table 11. Tabular representation of GHFS set for the third energy projects

$(\tilde{A}_3, E)$	e_1	e_2	e_3	e_4
x_1	$(0.3, 0.5, 0.7), (0.4, 0.3, 0.2)$	$(0.4, 0.6), (0.3, 0.2)$	$(0.5, 0.7), (0.2, 0.2)$	$(0.3, 0.5), (0.2, 0.4)$
x_2	$(0.1, 0.2), (0.7, 0.8)$	$(0.3, 0.4), (0.1, 0.2)$	$(0.5, 0.6), (0.2, 0.3)$	$(0.7, 0.8), (0.2, 0.1)$

Table 12. Tabular representation of GHFS set for the third energy projects

$(\tilde{A}_4, E)$	e_1	e_2	e_3	e_4
x_1	(0.4,0.7),(0.3,0.2)	(0.8,0.9),(0.1,0.1)	(0.3,0.4),(0.5,0.4)	(0.6,0.7),(0.3,0.2)
x_2	(0.5,0.6),(0.3,0.3)	(0.3,0.5),(0.5,0.3)	(0.6,0.8),(0.3,0.2)	(0.5,0.8),(0.2,0.1)

After simple computing, we have the follow results:

Table 13. Tabular representation of similarity of A_i and A

$S((\tilde{A}, E), (\tilde{A}_1, E))$	$S((\tilde{A}, E), (\tilde{A}_2, E))$	$S((\tilde{A}, E), (\tilde{A}_3, E))$	$S((\tilde{A}, E), (\tilde{A}_4, E))$
0.78125	0.565625	0.590625	0.659375

Thus, the ranking of the four energy projects is $A_1 \succ A_4 \succ A_3 \succ A_2$.

7. Concluding remarks

In this paper, the concept of generalized hesitant fuzzy soft sets and some operations on generalized hesitant fuzzy soft sets are defined and some of their properties

are studied. Applications of generalized hesitant fuzzy soft sets in decision-making are investigated. As we can see, the generalized hesitant fuzzy soft sets are effective for the decision makers and easier to be applied in real-life applications. In the following paper, we will study entropy, similarity and distance measure of generalized hesitant fuzzy soft sets and their applications in decision-making as in [12,42,43]. Also, we will investigate the applications of generalized hesitant fuzzy soft sets with Multiple Criterion Decision-Making (MCDM) problems as in [31] in the following paper. The weighted decision making problem discussed in [32] about generalized hesitant fuzzy soft sets should also be investigated in the future. And the theory of generalized hesitant fuzzy soft sets can also be use in the classification for gene expression engineering as in [51]. We are hopeful that this modified concept will be helpful in dealing with several problems related to uncertainty and will yield more natural results.

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A GENERALIZED COMMON FIXED POINT THEOREM FOR SIX SELF-MAPS

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Abstract. A recent common fixed point theorem of Kikina and Kikina (2011) has been extended to two triads of self-maps through the notions of weak compatibility and the property (EA), under an implicit-type relation and restricted completeness, namely orbital completeness of the space.

Keywords: property (EA), implicit relation, orbital completeness, weak compatibility, common fixed point.

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1. Introduction

Throughout this paper, (X, d) denotes a metric space. Given $x_0 \in X$ and f, g and h self-maps on X, the associated sequence $\langle x_n \rangle_{n=1}^{\infty} \subset X$ with the choice

$$x_{3n-2} = fx_{3n-3},\ x_{3n-1} = gx_{3n-2},\ x_{3n} = hx_{3n-1} \quad \text{for} \quad n = 1, 2, 3, ... \tag{1.1}$$

is an (f, g, h)-orbit at x_0. The metric space X is (f, g, h)-orbitally complete [6] if every Cauchy sequence in the (f, g, h)-orbit at each $x_0 \in X$ converges in X.

Kikina and Kikina [6] proved the following

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Theorem 1.1 *Let f, g and h be self-maps on X satisfying the three conditions:*

$$\begin{aligned}[1+pd(x,y)]d(fx,gy) &\le p[d(x,fx)d(y,gy)+d(x,gy)d(y,fx)]\\ &+q\max\left\{d(x,y),d(x,fx),d(y,gy),\frac{1}{2}[d(x,gy)+d(y,fx)]\right\},\end{aligned}\tag{1.2}$$

$$\begin{aligned}[1+pd(x,y)]d(gx,hy) &\le p[d(x,gx)d(y,hy)+d(x,hy)d(y,gx)]\\ &+q\max\left\{d(x,y),d(x,gx),d(y,hy),\frac{1}{2}[d(x,hy)+d(y,gx)]\right\},\end{aligned}\tag{1.3}$$

$$\begin{aligned}[1+pd(x,y)]d(hx,fy) &\le p[d(x,hx)d(y,fy)+d(x,fy)d(y,hx)]\\ &+q\max\left\{d(x,y),d(x,hx),d(y,fy),\frac{1}{2}[d(x,fy)+d(y,hx)]\right\}\end{aligned}\tag{1.4}$$

for all $x,y\in X$, where

$$p>-\frac{1}{\max\{d(x,y):\ x,y\in X\}}\quad \textit{with}\quad \max\{d(x,y):\ x,y\in X\}>0$$

and $0\le q<1$. If X is (f,g,h)-orbitally complete, then f, g and h will have a unique common fixed point.

In this paper, we first extend the orbital completeness to two triads of self-maps and give an extended version of Theorem 1.1 using weak compatibility and property (EA) under generalized inequalities of (1.2)-(1.4) involving an implicit relation.

2. Preliminaries

Self-maps f and r on (X,d) are known to be commuting if $frx=rfx$ for all $x\in X$, where fr denotes the composition of f and r. As a weaker form of it, Sessa [14] introduced weakly commuting maps f and r on X with the choice:

$$d(frx,rfx)\le d(fx,rx)\quad\text{for all}\quad x\in X.\tag{2.1}$$

As a further weaker form of commuting mappings, Gerald Jungck [3] introduced the notion of compatibility as follows:

Definition 2.1 *Self-maps f and r on X are said to be compatible if there is a sequence $\langle x_n\rangle_{n=1}^{\infty}\subset X$ such that*

$$\lim_{n\to\infty}fx_n=\lim_{n\to\infty}rx_n=p\quad \textit{for some}\quad p\in X\tag{2.2}$$

implies that

$$\lim_{n\to\infty}d(frx_n,rfx_n)=0.\tag{2.3}$$

It is obvious that every commuting pair is weakly commuting and every weakly commuting pair is compatible. However neither reverse implication is true. For examples, one can refer to [3] and [14].

We observe that both compatibility and noncompatibility of (f, r) guarantee the existence of a sequence $\langle x_n \rangle_{n=1}^{\infty} \subset X$ with the choice (2.2). Self-maps f and r on X satify the property (EA) [1] if (2.2) holds good for some $\langle x_n \rangle_{n=1}^{\infty} \subset X$. The class of compatible maps is contained in the class of *weakly compatible* maps [4] which commute at their coincidence points. However, weak compatibility and property (EA) are independent of each other [9]. Various form of compatibility and weak compatibility and their interrelation was presented in [2]. Weak compatibility have nice applications in dynamical programming (See [10]).

The idea of contraction type conditions involving an *implicit relation* was first introduced by Popa [13] which covers several contractive conditions and has the ability to unify several fixed point theorems.

For instance, [5] utilized one such implicit relation $\psi : \mathbb{R}_+^6 \to \mathbb{R}$ which is lower semicontinuous and satisfies the following conditions:

(C_1) ψ is nondecreasing in the fifth and sixth coordinate variables,

(C_2) For every $l \geq 0, m \geq 0$, there is a constant $0 \leq \omega < 1$ such that

$$\min\{\psi(l, m, m, l, l + m, 0), \psi(l, m, l, m, 0, l + m)\} \leq 0 \ \Rightarrow \ l \leq \omega m, \tag{2.4}$$

(C_3) $\psi(l, l, 0, 0, l, l) > 0$ for all $l > 0$.

In this paper, we first extend the property (EA) and orbital completeness to two triads of self-maps and then obtain an extended generalization of Theorem 1.1 involving weak compatibility and an implicit-type relation which does not require the choice (C_1), unlike in [5].

3. Main result

We begin with orbital completeness involving two triads of self-maps as follows:

Given $x_0 \in X$ and two triads (f, g, h) and (r, s, t) of self-maps on X, if there exist points $x_1, x_2, x_3, ...$ in X such that

$$\begin{aligned} y_{3n-2} = fx_{3n-3} = rx_{3n-2}, \ y_{3n-1} = gx_{3n-2} = sx_{3n-1}, \\ y_{3n} = hx_{3n-1} = tx_{3n} \quad \text{for } n = 1, 2, 3, ..., \end{aligned} \tag{3.1}$$

then the associated sequence $\langle y_n \rangle_{n=1}^{\infty}$ is an (f, g, h)-orbit relative (r, s, t) at x_0.

The space X is (f, g, h)-orbitally complete at x_0 relative (r, s, t) if every Cauchy sequence in an (f, g, h)-orbit at x_0 relative (r, s, t) converges in X, and X is (f, g, h)-orbitally complete relative (r, s, t) if it is (f, g, h)-orbitally complete relative to (r, s, t) at each $x_0 \in X$.

It may be noted that if $r = s = t = i$, the identity map, then we get the orbital completeness as discussed in [6].

Similarly, if $g = h = f$ and $t = s = r$ in this notion, we get the f-orbital completeness relative to r as given in [11].

The property (EA) was extended to two pairs of self-maps in [7]. In fact, self-maps (f, r) and (g, s) share the common property (EA) on X if there exist sequences $\langle x_n \rangle_{n=1}^{\infty}$ and $\langle y_n \rangle_{n=1}^{\infty}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} rx_n = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} sy_n = u \quad \text{for some} \quad u \in X. \tag{3.2}$$

We improve the notion of [7] to a pair of triads of maps as follows:

Let f, g, h, r, s and t be self-maps on X. We say that the triads (f, g, h) and (r, s, t) share the common property (EA) if we can find sequences $\langle x_n \rangle_{n=1}^{\infty}, \langle y_n \rangle_{n=1}^{\infty}$ and $\langle z_n \rangle_{n=1}^{\infty}$ in X such that

$$\begin{aligned} \lim_{n\to\infty} fx_n &= \lim_{n\to\infty} gy_n = \lim_{n\to\infty} hz_n = \lim_{n\to\infty} rx_n \\ &= \lim_{n\to\infty} sy_n = \lim_{n\to\infty} tz_n = u \quad \text{for some } u \in X. \end{aligned} \tag{3.3}$$

If $r = s = t = i$ in this notion, we get the property (EA) on the collection $\{f, g, h\}$ as given in [8].

Our main result is

Theorem 3.1 *Let f,g, h, r, s and t be self-maps on X sharing the common property (EA) and satisfying the following inequalities:*

$$\psi(d(fx, gy), d(rx, sy), d(rx, fx), d(sy, gy), d(rx, gy), d(sy, fx)) < 0, \tag{3.4}$$

$$\psi(d(gx, hy), d(sx, ty), d(sx, gx), d(ty, hy), d(sx, hy), d(ty, gx)) < 0, \tag{3.5}$$

$$\psi(d(hx, fy), d(tx, ry), d(tx, hx), d(ry, fy), d(tx, fy), d(ry, hx)) < 0, \tag{3.6}$$

for all $x, y \in X$. Suppose that r, s and t are onto. If (f, r), (g, s) and (h, t) are weakly compatible, then all the six maps f, g, h, r, s and t will have a unique common fixed point.

Proof. Suppose f, g, h, r, s and t share the common property. Then we can find sequences $\langle x_n \rangle_{n=1}^{\infty}, \langle y_n \rangle_{n=1}^{\infty}$ and $\langle z_n \rangle_{n=1}^{\infty}$ in X with the choice (3.3). Since r is onto, we have

$$u = rp \quad \text{for some} \quad p \in X. \tag{3.7}$$

Writing $x = p$ and $y = x_n$ in (3.4), we get

$$\psi(d(fp, gx_n), d(rp, sx_n), d(rp, fp), d(sx_n, gx_n), d(rp, gx_n), d(sx_n, fp)) < 0.$$

Applying the limit as $n \to \infty$ and then using (3.3) and (3.7), this yields

$$\psi(d(fp, rp), 0, d(rp, fp), 0, 0, d(rp, fp)) < 0,$$

which is (2.4) with $l = d(fp, rp)$ and $m = 0$. So, by (C_2) we get $d(fp, rp) \leq 0$, that is $fp = rp = u$. Since (f, r) is weakly compatible, f and r commute at the coincidence point p, that is $frp = rfp$, we get

$$fu = ru. \tag{3.8}$$

Now s is onto implies that

$$u = sq \quad \text{for some} \quad q \in X. \tag{3.9}$$

Then writing $x = q$ and $y = y_n$ in (3.5), we get

$$\psi(d(gq, hy_n), d(sq, ty_n), d(sq, gq), d(ty_n, hy_n), d(sq, hy_n), d(ty_n, gq)) < 0.$$

Applying the limit as $n \to \infty$ and then using (3.3) and (3.9), this yields

$$\psi(d(gq, sq), 0, d(sq, gq), 0, 0, d(sq, gq)) < 0,$$

which is (2.4) with $l = d(gq, sq)$ and $m = 0$. So, by (C_2) we get $d(gq, sq) \leq 0$, that is $gq = sq = u$. By the weak compatibility of, g and s, we get $gsq = sgq$ or

$$gu = su. \tag{3.10}$$

Finally, t is onto implies that

$$u = tw \quad \text{for some} \quad w \in X. \tag{3.11}$$

Then (3.6) with $x = w$ and and $y = w_n$ gives

$$\psi(d(hw, fz_n), d(tw, rz_n), d(tw, hw), d(rz_n, fz_n), d(tw, fz_n), d(rz_n, hw)) < 0.$$

Applying the limit as $n \to \infty$ and then using (3.3) and (3.11), this yields

$$\psi(d(hw, tw), 0, d(tw, hw), 0, 0, d(tw, hw)) < 0.$$

This in view of (C_2) with $l = d(tw, hw)$ and $m = 0$ finally gives $d(tw, hw) \leq 0$, that is $tw = hw = u$. By the weak compatibility of, h and t, we get $thw = htw$ or

$$tu = hu. \tag{3.12}$$

But from (3.4) with $x = y = u$, it follows that

$$\psi(d(fu, gu), d(fu, gu), 0, 0, d(fu, gu), d(gu, fu)) < 0,$$

which is nothing but (2.4) with $l = d(fu, gu)$ and $m = 0$. Hence by (C_2), we get $d(fu, gu) \leq 0$ so that $fu = gu$.

On one hand, from (3.5) with $x = u = y$ it follows that

$$\psi(d(gu, hu), d(ru, ru), d(ru, gu), d(ru, hu), d(ru, hu), d(ru, gu)) < 0,$$

or that $\psi(d(gu,hu),0,0,d(fu,hu),d(ru,hu),0)<0$.

If $d(gu,hu)>0$, from (C_3), we see a contradiction that

$$\psi(d(gu,hu),0,0,d(fu,hu),d(ru,hu),0)>0.$$

Hence we must have $fu=gu$.

Now, writing $x=y=u$ in (3.5) and then using (3.8), (3.10), (3.12) and $fu=gu$, we get

$$\psi(d(gu,hu),d(gu,hu),0,0,d(gu,hu),d(hu,gu))<0$$

which, due to (C_3), implies that $gu=hu$.

In other words, u is a common coincidence point of all the six maps, that is

$$fu=gu=hu=ru=su=tu. \tag{3.13}$$

Further, we see from (3.4) with $x=u$ and $y=x_n$ becomes

$$\psi(d(fu,gx_n),d(ru,rx_n),d(ru,fu),d(rx_n,gx_n),d(ru,gx_n),d(rx_n,fu))<0.$$

Proceeding the limit as $n\to\infty$, in this and using (3.13), we obtain

$$\psi(d(fu,u),d(fu,u),0,0,d(fu,u),d(fu,u))<0, \tag{3.14}$$

which would be a contradiction to $(C3)$ if $d(fu,u)>0$, proving that u is a fixed point of f and hence a common fixed point of f,g,h and r. ∎

Remark 3.1 Writing $s=t=r$ in this main result, recently it has been shown in [15] that any two of the three inequalities (3.4)-(3.6) are sufficient to obtain a common fixed point for the four self-maps f,g,h and r, using the weak compatibility of one of the three pairs $(f,r),(g,r)$ and (h,r).

Now we write

$$\psi(l_1,l_2,l_3,l_4,l_5,l_6)=(1+pl_2)l_1-p[l_3l_4+l_5l_6]-q\max\left\{l_2,l_3,l_4,\frac{1}{2}[l_5+l_6]\right\},$$

where p and q have the same choice as given in Theorem 1.1.

Corollary 3.1 *Let f, g, h, r, s and t be self-maps on X satisfying following conditions:*

$$\begin{aligned}[1+pd(rx,sy)]d(fx,gy)&\le p[d(rx,fx)d(sy,gy)+d(rx,gy)d(sy,fx)]\\&+q\max\{d(rx,sy),d(rx,fx),d(sy,gy),\frac{1}{2}[d(rx,gy)+d(sy,fx)]\},\end{aligned} \tag{3.15}$$

$$\begin{aligned}[1+pd(sx,ty)]d(gx,hy)&\le p[d(sx,gx)d(ty,hy)+d(sx,hy)d(ty,gx)]\\&+q\max\{d(sx,ty),d(sx,gx),d(ty,hy),\frac{1}{2}[d(sx,hy)+d(ty,gx)]\},\end{aligned} \tag{3.16}$$

$$
\begin{aligned}
(3.17)\qquad & [1+pd(tx,ry)]d(hx,fy) \le p[d(tx,hx)d(ry,fy)+d(tx,fy)d(ry,hx)] \\
& +q\max\{d(tx,ry),d(tx,hx),d(ry,fy),\frac{1}{2}[d(tx,fy)+d(ry,hx)]\}
\end{aligned}
$$

for all $x, y \in X$, *where*

$$p > -\frac{1}{\max\{d(x,y) : x, y \in X\}}$$

with $\max\{d(x,y) : x, y \in X\} > 0$ *and* $0 \le q < 1$. *Suppose that*

$$(3.18)\qquad f(X) \subset r(X), g(X) \subset s(X), h(X) \subset t(X).$$

Suppose that X *is* (f,g,h)*-orbitally complete relative to* (r,s,t), *and* r, s *and* t *are onto. If* (f,r), (g,s) *and* (h,t) *are weakly compatible, then all the six maps* f, g, h, r, s *and* t *will have a unique common fixed point.*

Proof. Let $p_0 \in X$. In view of (3.18), there exists an (f,g,h)-orbit relative (r,s,t) at p_0 with the choice (3.1). By a routine iteration procedure just similar to that of Theorem 1.1, one can prove that (f,g,h)-orbit relative (r,s,t) at p_0 is Cauchy, and hence converges to some $u \in X$, since X is (f,g,h)-orbitally complete at p_0. That is

$$
\begin{aligned}
\lim_{n\to\infty} fp_{3n-3} &= \lim_{n\to\infty} gp_{3n-2} = \lim_{n\to\infty} hp_{3n-1} = \lim_{n\to\infty} rp_{3n-2} \\
&= \lim_{n\to\infty} sp_{3n-1} = \lim_{n\to\infty} tp_{3n} = u \quad \text{for some } u \in X.
\end{aligned}
$$

Then, using the inequalities (3.15)-(3.17), it is not difficult to see that

$$
\begin{aligned}
(3.19)\qquad \lim_{n\to\infty} fp_{3n-3} &= \lim_{n\to\infty} rp_{3n-3} = \lim_{n\to\infty} gp_{3n-2} = \lim_{n\to\infty} sp_{3n-2} \\
&= \lim_{n\to\infty} hp_{3n-1} = \lim_{n\to\infty} tp_{3n-1} = u \text{ for some } u \in X.
\end{aligned}
$$

Here we note that (3.19) is a special case of (3.3) with $x_n = p_{3n-3}, y_n = p_{3n-2}$ and $z_n = x_{3n-1}$. Therefore, the common fixed point follows from Theorem 3.1. ∎

When $r = s = t = I$, the identity map on X in Corollary 3.1, we find that (1.2), (1.3) and (1.4)are particular cases of the implicit relations of Theorem 3.1. Also it is well-known that I is onto and commutes with evrey map and hence is weakly compatible with the maps f, g and h, and hence a common fixed point follows directly from Theorem 3.1. Thus Theorem 1.1 is a stronger version of Corollary 3.1.

Finally, writing $h = g = f$ and $r = s = t$ in Theorem 3.1, we get a particular case of each of (3.4)-(3.6) as

$$(3.20)\qquad \psi(d(fx,fy),d(rx,ry),d(rx,fx),d(ry,fy),d(rx,fy),d(ry,fx)) < 0$$

for all $x, y \in X$. Also the space X reduces to *f-orbitly complete* relative to r in the sense that every Cauchy sequence in the (f,r)-orbit $O_{f,r}(x_0)$ at each x_0 converges in X, where $O_{f,r}(x_0)$ has the choice

$$(3.21)\qquad fx_{n-1} = rx_n \quad \text{for} \quad n = 1, 2, 3, \ldots.$$

Then we have

Corollary 3.2 *Let f and r be self-maps on X satisfying the property E. A. and the inequality* (3.20)*. If $r(X)$ is f-orbitally complete relative to r, then f and r will have a coincidence point. Further if (f, r) is weakly compatible, then f and r will have a unique common fixed point.*

The following example shows that f-orbital completeness of $r(X)$ is necessary in Corollary 3.2:

Example 3.1 Let

$$\psi(l_1, l_2, l_3, l_4, l_5, l_6) = l_1^2 - al_2^2 - \frac{bl_5l_6}{l_3^2 + l_4^2 + 1},$$

where $a = 1/2$ and $b = 1/4$. Set $X = \left\{0, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, ...\right\}$ with the usual metric d. Define $f, r : X \to X$ by

$$f0 = \frac{1}{2^2}, f\left(\frac{1}{2^{n-1}}\right) = \frac{1}{2^{n+1}} \quad \text{and} \quad r0 = \frac{1}{2}, r\left(\frac{1}{2^{n-1}}\right) = \frac{1}{2^n},$$

for $n = 1, 2, 3, ...$ Then (f, r) satisfies the property E.A. and

$$r(X) = \left\{\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, ...\right\}.$$

For $x_0 = 0$, choose $x_1 = \frac{1}{2}, x_2 = \frac{1}{2^2}, x_3 = \frac{1}{2^3}, ...$ so that

$$O_{f,r}(x_0) = \left\{\frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, ...\right\}$$

while for $x_0 = \frac{1}{2^{n-1}}$, we have

$$O_{f,r}(x_0) = \left\{\frac{1}{2^{n+1}}, \frac{1}{2^{n+2}}, \frac{1}{2^{n+3}}, ...\right\}$$

for each $n = 1, 2, 3, ...$ In either case, $O_{f,r}(x_0)$ converges to $0 \notin r(X)$.

Thus $r(X)$ is not orbitally complete at each x_0. As such the maps f and r do not have a coincidence point, though X is complete.

Since every complete metric space is orbitally complete at each of its points, we have

Corollary 3.3 (Theorem 3.1, [5]) *Let f and r be self-maps on X satisfying the property E. A. and the inequality* (3.20)*. If $r(X)$ is complete, then f and r will have a coincidence point. Further, f and r will have a unique common fixed point, provided (f, r) is weakly compatible.*

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ON GENERALIZED WEAK I-LIFTING MODULES

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Abstract. In this paper, the concept of $\mathcal{I}$-lifting modules is extended to weak $\mathcal{I}$-lifting and generalized weak $\mathcal{I}$-lifting modules. Some properties of these modules are investigate and some results about $\mathcal{I}$-lifting modules are extended.

Keywords: lifting module; $\mathcal{I}$-lifting module; semiregular ring; weak $\mathcal{I}$-lifting module.

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1. Introduction

Throughout this paper, R will denote an arbitrary associative ring with identity, M a unitary right R-module and $S = End_R(M)$ the ring of all R-endomorphisms of M. We will use the notation $N \ll M$ to indicate that N is small in M (i.e., $\forall L \lneq M, L + N \neq M$); $N \leq^e M$ to indicate that N is an essential submodule of M (i.e., $\forall 0 \neq L \leq M, L \cap N \neq 0$). The notation $N \leq^{\oplus} M$ denotes that N is a direct summand of M. $N \trianglelefteq M$ means that N is a fully invariant submodule of M (i.e., $\forall \phi \in End_R(M)$, $\phi(N) \subseteq N$). For all $I \subseteq S$, the left and right annihilators of I in S are denoted by $\ell_S(I)$ and $r_S(I)$, respectively. We also denote $r_M(I) = \{x \in M \mid Ix = 0\}$, for $I \subseteq S$; $\ell_S(N) = \{\phi \in S \mid \phi(N) = 0\}$, for $N \subseteq M$. $\Delta(M) = \{f \in S \mid Kerf \leq^e M\}$ and $Z(M) = \{x \in M \mid xI = 0$ for some essential right ideal I of $R\}$. A ring R is called a *semiregular* ring if for each $a \in R$, there exists $e^2 = e \in aR$ such that $(1 - e)a \in J(R)$ [7]. A ring R is called *weak semiregular* if, for any $a \in R$, there exists $0 \neq b \in R$ and an idempotent $g \in abR$ such that $(1 - g)ab \in J(R)$. A ring R is called *I-weak semiregular* if, for any $a \in R$, there exist $0 \neq b \in R$ and $e^2 = e \in abR$ such that $(1 - e)ab \in I$ [4].

Lifting modules play important roles in rings and categories of modules, and have been studied extensively by many authors in recent years (see for example, [1], [3], [6]). A module M is called *lifting* if for every $A \leq M$, there exists a direct summand B of M such that $B \subseteq A$ and $A/B \ll M/B$ [6].

In [1], we introduced $\mathcal{I}$-lifting modules as a generalization of lifting modules. Following [1], a module M is called *$\mathcal{I}$-lifting* if for every $\phi \in S$ there exists a

decomposition $M = M_1 \oplus M_2$ such that $M_1 \subseteq \mathrm{Im}\phi$ and $M_2 \cap \mathrm{Im}\phi \ll M_2$. It is obvious that every lifting module is $\mathcal{I}$-lifting while the converse in not true (the $\mathbb{Z}$-module $\mathbb{Q}$ is $\mathcal{I}$-lifting but it is not lifting). It is easily checked that R_R is an $\mathcal{I}$-lifting module if and only if R is a semiregular ring.

In this paper, we call a module M *weak $\mathcal{I}$-lifting* if, for every $\phi \in S$, there exists $0 \neq b \in S$ such that $\phi bM = eM \oplus N$, where $e^2 = e \in S$ and $N \ll M$. We give an example which shows that these modules are non-trivial generalization of $\mathcal{I}$-lifting modules. We call a module M *generalized weak $\mathcal{I}$-lifting* if, for any element $\phi \in S = End_R(M)$, there exist $0 \neq b \in S$ and a decomposition $r_M\ell_S(\phi b) = A \oplus B$ such that $A \subseteq \phi bM$ and $\phi bM \cap B \ll M$. In this note our aim is to investigate and study some properties of these modules.

In Section 2, we introduce weak $\mathcal{I}$-lifting and generalized weak $\mathcal{I}$-lifting modules. We give conditions under which a generalized weak $\mathcal{I}$-lifting module is weak $\mathcal{I}$-lifting (Proposition 2.7 and Corollary 2.2). We also prove the following:

Let $f_1+\cdots+f_n = 1$ in S, where f_i's are orthogonal central idempotents. Then M is a generalized weak $\mathcal{I}$-lifting module if and only if each f_iM is generalized weak $\mathcal{I}$-lifting (see Corollary 2.4).

Let F be a submodule of an R-module M. A module M is called *F-$\mathcal{I}$-lifting* if for every $\phi \in S$ there exist a decomposition $M = A \oplus B$ such that $A \subseteq \mathrm{Im}\ \phi$ and $\mathrm{Im}\ \phi \cap B \leq F$. Let F be a submodule of an R-module M. A module M is called *weak F-$\mathcal{I}$-lifting* if for every $\phi \in S$ there exist $0 \neq b \in S$ and a decomposition $M = A \oplus B$ such that $A \subseteq \phi bM$ and $\phi bM \cap B \leq F$. These modules are non-trivial generalization of F-$\mathcal{I}$-lifting modules. In Section 3, we give a equivalent condition of semi-projective weak FI-$\mathcal{I}$-lifting modules. We also prove the following corollary (see Corollary 3.6):

Let M be a semi-projective retractable module. Then the following are equivalent:

(1) M is weak $Z(M)$-$\mathcal{I}$-lifting.
(2) S is weak $\Delta(M)$-semiregular.
(3) S is weak $Z_r(S)$-semiregular.
(4) M is weak $\Delta(M)M$-$\mathcal{I}$-lifting.

2. Generalized weak $\mathcal{I}$-lifting modules

Definition 2.1 Let M be a right R-module. M is called *weak $\mathcal{I}$-lifting* if, for every $\phi \in S$, there exists $0 \neq b \in S$ such that $\phi bM = eM \oplus N$, where $e^2 = e \in S$ and $N \ll M$.

It is clear that every $\mathcal{I}$-lifting module is weak $\mathcal{I}$-lifting. But the converse is not true as we see in the following example.

Example 2.2 Let $R = M = \{(x_1, x_2, ..., x_n, x, x, ...) \mid x_1, x_2, ..., x_n \in M_2(\mathbb{Z}_2), x \in \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{pmatrix}\}$. Then M_R is a weak $\mathcal{I}$-lifting R-module but not $\mathcal{I}$-lifting.

A module M is called *semi-projective* if for any epimorphism $f : M \to N$, where N is a submodule of M, and for any homomorphism $g : M \to N$, there exists $h : M \to M$ such that $fh = g$.

Lemma 2.3 *Let M be a semi-projective module. Then the following are equivalent for an element $\phi \in S$:*

(1) *There exist $0 \neq b \in S$ and $e^2 = e \in \phi bS$ with $\phi bM \cap (1-e)M \ll M$.*

(2) *M is weak $\mathcal{I}$-lifting.*

Proof. (1) $\Rightarrow$ (2) Note that $\phi bM = eM \oplus [\phi bM \cap (1-e)M]$. The rest is clear.

(2) $\Rightarrow$ (1) By (2), there exists $0 \neq b \in S$ such that $\phi bM = eM \oplus N$ where $e^2 = e \in S$ and $N \ll M$. First we show that $e^2 = e \in \phi bS$. Consider the epimorphisms $\phi b : M \to \phi bM$ and $e : M \to eM$. Since M is semi-projective, there exists a homomorphism $g \in S$ such that $\phi bg = ie = e$, where $i : eM \to \phi bM$ is the inclusion map. Hence $e \in \phi bS$. Since $\phi bM = eM \oplus N$, $\phi bM \cap (1-e)M = N \cap (1-e)M \subseteq N$. As $N \ll M$, $\phi bM \cap (1-e)M \ll M$. ∎

Proposition 2.4 *Let M be a projective weak $\mathcal{I}$-lifting module. Then $Rad(M)$ is small in M.*

Proof. Let $N \subseteq M$ be any submodule with $N + Rad(M) = M$. If $g : M \to M/N$ is the natural map, then there exists $f : M \to Rad(M)$ with $gf = g$. Then $g = gf^2$. Note that if $f = 0$, then $g = 0$ and so $M = N$. Suppose that $f \neq 0$. Since M is weak $\mathcal{I}$-lifting, there exists a decomposition $M = M_1 \oplus M_2$ with $M_1 \subseteq \mathrm{Im}f^2$ and $M_2 \cap \mathrm{Im}f^2 \ll M_2$. Note that $M_1 \subseteq \mathrm{Im}f^2 \subseteq \mathrm{Im}f \subseteq Rad(M)$. By [9, 22.3], $M_1 = 0$ and so $\mathrm{Im}f^2 \ll M$. Hence $f^2 \in \nabla = Jac(S)$, thus $g = 0$ and so $N = M$. This shows that $Rad(M) \ll M$. ∎

Recall that a projective module is *semiperfect* if every homomorphic image has a projective cover [9].

Corollary 2.1 *If R is a semiperfect ring, then the following are equivalent for a projective R-module M:*

(1) *M is semiperfect;*

(2) *$End_R(M)$ is semiregular;*

(3) *$Rad(M) \ll M$;*

(4) *M is $\mathcal{I}$-lifting;*

(5) *M is weak $\mathcal{I}$-lifting.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1) By [7, Corollary 3.7].

(2) $\Rightarrow$ (4) Since M is projective, it is well known that, $J(S) = \nabla(M)$ where $\nabla(M) = \{\alpha \in S \mid \mathrm{Im}\alpha \ll M\}$. Assume that $f \in S$, then there exists an idempotent $e \in S$ such that $eS \subseteq fS$ and $(1-e)fS \subseteq J(S) = \nabla(M)$. Therefore $M = eM \oplus (1-e)M$, $eM \subseteq fM$ and $(1-e)fM \ll M$. Hence M is $\mathcal{I}$-lifting.

(4) $\Rightarrow$ (5) is clear.

(5) $\Rightarrow$ (3) By Proposition 2.4. ∎

Definition 2.5 Let M be a right R-module. M is called *generalized weak $\mathcal{I}$-lifting* if, for every $\phi \in S$, there exist $0 \neq b \in S$ and a decomposition $r_M\ell_S(\phi b) = A \oplus B$ such that $A \subseteq \phi bM$ and $B \cap \phi bM \ll M$.

Proposition 2.6 *Let M be a weak $\mathcal{I}$-lifting module. Then M is generalized weak $\mathcal{I}$-lifting.*

Proof. Let $\phi \in S$. Then there exist $0 \neq b \in S$ and $e^2 = e \in \phi bS$ such t hat $\phi bM \cap (1-e)M \ll M$. Thus $M = eM \oplus (1-e)M$, where $eM \subseteq \phi bM$ and $\phi bM \cap (1-e)M \ll M$. Since $\phi bM \subseteq r_M\ell_S(\phi b)$, it follows by the modular law that $r_M\ell_S(\phi b) = r_M\ell_S(\phi b) \cap (eM \oplus (1-e)M) = eM \oplus (r_M\ell_S(\phi b) \cap (1-e)M)$ and $\phi bM \cap (r_M\ell_S(\phi b) \cap (1-e)M) = \phi bM \cap (1-e)M \ll M$. Hence M is generalized weak $\mathcal{I}$-lifting. ∎

Proposition 2.7 *Let M be a semi-projective generalized weak $\mathcal{I}$-lifting module with $RadM \ll M$. If there exists $e^2 = e \in S$ such that $\ell_S(\phi) = \ell_S(e)$ for any $\phi \in S$, then M is weak $\mathcal{I}$-lifting.*

Proof. Let $\phi \in S$. Then there exist $0 \neq b \in S$ and a decomposition $r_M\ell_S(\phi b) = A \oplus B$ such that $A \subseteq \phi bM$ and $B \cap \phi bM \ll M$. Since $\ell_S(\phi b) = \ell_S(e)$ where $e^2 = e \in S$, we have $r_M\ell_S(\phi b) = r_M\ell_S(e) = eM$ and so $eM = A \oplus B$. As A and B are direct summands of M, we can write $eM = fM \oplus gM$ for some $f^2 = f \in S$, $g^2 = g \in S$. By [9, 18.4], we get $Hom_R(M, eM) = Hom_R(M, fM) + Hom_R(M, gM)$. Since M is semi-projective, $eS = fS + gS$. As $A \cap B = 0$, $fS \cap gS = 0$. Thus $eS = fS \oplus gS$. Since $fM \subseteq \phi bM$ and $gM \cap \phi bM \ll M$, $fS \subseteq \phi bS$ and $gS \cap \phi bS \subseteq Hom_R(M, RadM)$. Since $\ell_S(\phi b) = \ell_S(e)$, $r_S\ell_S(\phi b) = r_S\ell_S(e) = eS$ and so $\phi b = e\phi b$. Let $e = \alpha + \beta$, where $\alpha = \phi bh \in fS$ and $\beta \in gS$. Then $\phi b = e\phi b = \phi bh\phi b + \beta\phi b$ and $\phi bh = \phi bh\phi bh + \beta\phi bh$. As $\phi bh - \phi bh\phi bh = \beta\phi bh \in gS \cap fS = 0$, ϕbh is an idempotent. Moreover, we have $(1-\phi bh)\phi b = \phi b - \phi bh\phi b = \beta\phi b \in gS \cap \phi bS \subseteq Hom_R(M, RadM)$, hence $(1-\phi bh)\phi bM \subseteq RadM \ll M$. Therefore M is weak $\mathcal{I}$-lifting. ∎

Corollary 2.2 *Let $r_M\ell_S(\phi b)$ is a direct summand of a semi-projective module M for any $\phi \in S$. If M is a generalized weak $\mathcal{I}$-lifting module with $RadM \ll M$, then M is weak $\mathcal{I}$-lifting.*

Proof. Let $\phi \in S$. By assumption, $r_M\ell_S(\phi b) = eM$ for some $e^2 = e \in S$. Then $\ell_S(\phi b) = \ell_S(e)$ and so M is weak $\mathcal{I}$-lifting by Proposition 2.7. ∎

A ring R is called *left Rickart* if for every $a \in R$ there exists an idempotent $e \in R$ such that $\ell_R(a) = Re$ [2].

Corollary 2.3 *Let S be a left Rickart ring. If M is a finitely generated semi-projective generalized weak $\mathcal{I}$-lifting module, then M is weak $\mathcal{I}$-lifting.*

Proposition 2.8 *Let e be a central idempotent of S. If M is a generalized weak $\mathcal{I}$-lifting module, then eM is generalized weak $\mathcal{I}$-lifting.*

Proof. Let $\phi \in End_R(eM) = eSe$. Then there exists $0 \neq b \in S$ such that $\phi b \neq 0$ and $r_M\ell_S(\phi b) = P \oplus L$ where $P \subseteq \phi bM$ and $L \cap \phi bM \ll M$. Note that $\phi b = e\phi b = \phi eb$ since $\phi \in eSe$. We claim that $r_{eM}\ell_{eSe}(\phi b) = eP \oplus eL$. Since $\phi \in eSe$, $1 - e \in \ell_S(\phi) \subseteq \ell_S(\phi b)$. Thus for every $t \in L$, we have $(1 - e)t = 0$, which implies that $eL = L$. Similarly, $eP = P$. Take any $y \in eP \subseteq e\phi bM$, where $y = ey_1$, $y_1 \in P \subseteq r_M\ell_S(\phi b)$. Then for every $\psi \in \ell_{eSe}(\phi b) \subseteq \ell_S(\phi b)$, $\psi y_1 = 0$. As $y_1 \in P \subseteq \phi bM$, $y_1 = \phi bm_1$ for some $m_1 \in M$. Thus we have $\psi y = \psi ey_1 = \psi e\phi bm_1 = \psi\phi bm_1 = \psi y_1 = 0$. Hence $y \in r_{eM}\ell_{eSe}(\phi b)$ and $eP \subseteq r_{eM}\ell_{eSe}(\phi b)$. Similarly, $eL \subseteq r_{eM}\ell_{eSe}(\phi b)$. On the other hand, let $x \in r_{eM}\ell_{eSe}(\phi b)$. Then for every $f \in \ell_S(\phi b)$, we have $efe\phi bM = ef\phi bM = 0$. Thus $efe \in \ell_{eSe}(\phi b)$ and so $efex = 0$ which gives $fx = fex = efex = 0$ since $x \in eM$. Hence $r_{eM}\ell_{eSe}(\phi b) \subseteq r_M\ell_S(\phi b)$. Take $x = x_1 + x_2$, where $x_1 \in P$ and $x_2 \in L$. Then $x = ex = ex_1 + ex_2 \in eP + eL$. This shows that $r_{eM}\ell_{eSe}(\phi b) = eP \oplus eL$. Since $eP \subseteq e\phi bM = \phi beM$, it is enough to show that $eL \cap \phi beM \ll eM$. Note that $eL \cap \phi beM = L \cap \phi beM \ll M$. Thus $eL \cap \phi beM \ll eM$ since $eM \leq^{\oplus} M$. Therefore eM is generalized weak $\mathcal{I}$-lifting. ∎

Theorem 2.9 *Let e and f be orthogonal central idempotents of S. If eM and fM are generalized weak $\mathcal{I}$-lifting modules, then $gM = eM \oplus fM$ is generalized weak $\mathcal{I}$-lifting.*

Proof. Let $\phi \in End_R(gM) \cong gSg = gS$. Then $e\phi \in eS$ and $f\phi \in fS$. By assumption, there exists $0 \neq b \in eS$ such that $e\phi b \neq 0$ and $r_{eM}\ell_{eS}(e\phi b) = P_e \oplus L_e$, where $P_e \subseteq e\phi beM = \phi beM$ and $\phi beM \cap L_e \ll eM$. Similarly, there exists $0 \neq t \in fS$ such that $f\phi t \neq 0$ and $r_{fM}\ell_fS(f\phi t) = P_f \oplus L_f$, where $P_f \subseteq \phi tfM$ and $L_f \cap \phi tfM \ll fM$. Note that $g = e + f$ is central idempotent and $\phi be + \phi tf = \phi(be + tf) \neq 0$. Let $h = be + tf \in gS$. Then $\phi h \neq 0$. We claim that $r_{gM}\ell_{gS}(\phi h) = P_e \oplus L_e \oplus P_f \oplus L_f$. Take any $x \in r_{gM}\ell_{gS}(\phi h)$. Then for any $\psi \in \ell_{eS}(e\phi b)$, we have $\psi e\phi b = 0$ and so $\psi\phi h = \psi\phi(be + tf) = \psi\phi be + \psi\phi tf = \psi\phi tf = \psi e\phi tf = 0$. Hence $g\psi\phi h = 0$ and $g\psi \in \ell_{gS}(\phi h)$. Thus $\psi(x) = g\psi(x) = 0$ and so $\psi ex = e\psi(x) = 0$, hence $ex \in r_{eM}\ell_{eS}(e\phi b) = P_e \oplus L_e$. Similarly, $fx \in r_{fM}\ell_{fS}(f\phi t) = P_f \oplus L_f$. Then $x = gx = ex + fx \in P_e \oplus P_f \oplus L_e \oplus L_f$ since e and f are orthogonal. Hence $r_{gM}\ell_{gS}(\phi h) \subseteq P_e \oplus L_e \oplus P_f \oplus L_f$. On the other hand, $P_e \oplus P_f \subseteq \phi beM \oplus \phi tfM \subseteq \phi hgM$. Let $x \in L_e$. For any $\psi \in \ell_{gS}(\phi h)$, we have $\psi\phi h = 0$, and so $e\psi e\phi h = e\psi\phi h = 0$. Thus $e\psi e\phi b = e\psi\phi b = 0$ and $e\psi \in \ell_{eS}(e\phi b)$. As $L_e \subseteq r_{eM}\ell_{eS}(e\phi b)$, $e\psi x = 0$. Note that $L_e \subseteq eM \subseteq gM$ and $\psi x = e\psi x = \psi ex = 0$. Hence $L_e \subseteq r_{gM}\ell_{gS}(\phi h)$. Similarly, $L_f \subseteq r_{gM}\ell_{gS}(\phi h)$. This shows that $r_{gM}\ell_{gS}(\phi h) = P_e \oplus P_f \oplus L_e \oplus L_f$. It is easily checked that $(L_e \oplus L_f) \cap \phi hgM \subseteq (L_e \oplus L_f) \cap (\phi beM + \phi tfM) \subseteq (L_e \cap \phi beM) \oplus (L_f \cap \phi tfM) \ll eM \oplus fM = gM$. Therefore gM is generalized weak $\mathcal{I}$-lifting. ∎

Corollary 2.4 *Let $f_1 + \cdots + f_n = 1$ in S, where f_i's are orthogonal central idempotents. Then M is a generalized weak $\mathcal{I}$-lifting module if and only if each f_iM is generalized weak $\mathcal{I}$-lifting.*

A ring R is called *abelian* if every idempotent is central, that is, $ae = ea$ for any $a, e^2 = e \in R$.

Corollary 2.5 *If S is an abelian ring, then any finite direct sum of generalized weak $\mathcal{I}$-lifting modules is generalized weak $\mathcal{I}$-lifting.*

3. Weak F-$\mathcal{I}$-lifting modules

Definition 3.1 Let F be a submodule of an R-module M. A module M is called *weak F-$\mathcal{I}$-lifting* if for every $\phi \in S$ there exist $0 \neq b \in S$ and a decomposition $M = A \oplus B$ such that $A \subseteq \phi bM$ and $\phi bM \cap B \leq F$.

It is clear that every weak $\mathcal{I}$-lifting module is weak $Rad(M)$-$\mathcal{I}$-lifting.

The following example shows that weak F-$\mathcal{I}$-lifting modules need not be F-$\mathcal{I}$-lifting.

Example 3.2 Let C be a commutative Von Neumann regular ring with no minimal ideal and J a maximal ideal. Let $M_1 = \begin{pmatrix} C & C \\ J & C \end{pmatrix}$, $M_2 = Z_2[[X]]$ be the formal power series ring over Z_2, and $M = R = M_1 \oplus M_2$. Then M_R is a weak $Rad(M)$-$\mathcal{I}$-lifting module but not $Rad(M)$-$\mathcal{I}$-lifting.

Lemma 3.3 *Let F be a fully invariant submodule of a semiprojective module M. Then the following are equivalent for an element $\phi \in S$:*

(1) *There exist $0 \neq b \in S$ and $e^2 = e \in \phi bS$ with $\phi bM \cap (1-e)M \subseteq F$.*

(2) *M is weak F-$\mathcal{I}$-lifting.*

Proof. (1) $\Rightarrow$ (2) Note that $\phi bM = eM \oplus [\phi bM \cap (1-e)M]$. The rest is clear.

(2) $\Rightarrow$ (1) By (2), there exists $0 \neq b \in S$ such that $\phi bM = eM \oplus N$ where $e^2 = e \in S$ and $N \subseteq F$. First we show that $e^2 = e \in \phi bS$. Consider the epimorphisms $\phi b : M \to \phi bM$ and $e : M \to eM$. Since M is semi-projective, there exists a homomorphism $g \in S$ such that $\phi bg = ie = e$, where $i : eM \to \phi bM$ is the inclusion map. Hence $e \in \phi bS$. Since $\phi bM = eM \oplus N$, $\phi bM \cap (1-e)M = N \cap (1-e)M \subseteq N$. As $N \subseteq F$, $\phi bM \cap (1-e)M \subseteq F$. ∎

An R-module M is called *retractable* if $Hom_R(M, N) \neq 0$ for all nonzero submodules N of M.

Lemma 3.4 *Let M be a semi-projective module. Consider the following conditions for $\phi \in S$:*

(1) *There exists $0 \neq b \in S$ such that $\phi bM = eM \oplus N$ where $e^2 = e \in S$ and N is a singular submodule of M.*

(2) *There exists $0 \neq b \in S$ such that $\phi bS = eS \oplus B$ where $e^2 = e \in S$ and $B \subseteq \Delta(M)$ is a right ideal of S.*

Then (1) $\Rightarrow$ (2) holds and, if moreover M is a retractable module, then (2) $\Rightarrow$ (1) holds.

Proof. (1) $\Rightarrow$ (2) Suppose that $\phi bM = eM \oplus N$ as in (1). First we show that $N = \phi bhM$ for some $h \in S$. Consider the homomorphism $\phi b : M \to \phi bM$. Since M is semi-projective, there exists a homomorphism $h : M \to M$ such that $\phi bh = i\pi\phi b$, where $i : N \to \phi bM$ and $\pi : \phi bM \to N$ are injection and projection maps respectively. Hence $\phi bhM = \pi(\phi bM) = N$. Now, by [9, 18.4], we have $Hom_R(M, \phi bM) = Hom(M, eM) + Hom(M, \phi bhM)$. Since M is semi-projective, $\phi bS = eS + \phi bhS$. As $eM \cap N = 0$, $eS \cap \phi bhS = Hom_R(M, eM) \cap Hom_R(M, \phi bhM) = Hom_R(M, eM \cap \phi bhM) = 0$. Thus $\phi bS = eS \oplus \phi bhS$. Finally, since $N = \phi bhM$ is singular and $\phi bhM \cong \frac{M}{\mathrm{Ker}_{\phi bh}}$, $\mathrm{Ker}\phi bh \leq^e M$ by [8, Lemma 2.1]. So $\phi bh \in \Delta(M)$.

(2) $\Rightarrow$ (1) Let $\phi bS = eS \oplus B$ as in (2). Clearly, $\phi bM = eM + BM$. Since $eS \cap B = 0$ and M is semi-projective, we have $Hom_R(M, eM) \cap Hom_R(M, BM) = 0$. Therefore, $Hom_R(M, eM \cap B) = 0$. Hence $eM \cap BM = 0$ by retractability. It follows that $\phi bM = eM \oplus BM$ and $BM \subseteq \Delta(M)M \subseteq Z(M)$. ∎

Corollary 3.6 *Let M be a semi-projective retractable module. Then the following are equivalent:*

(1) *M is weak $Z(M)$-$\mathcal{I}$-lifting.*

(2) *S is weak $\Delta(M)$-semiregular.*

(3) *S is weak $Z_r(S)$-semiregular.*

(4) *M is weak $\Delta(M)M$-$\mathcal{I}$-lifting.*

Proof. (1) $\Leftrightarrow$ (2) By Lemma 3.4. (2) $\Leftrightarrow$ (3) By [5, Proposition 2.4]. (2) $\Leftrightarrow$ (4) Similar to the proof of Lemma 3.4. ∎

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A-NUMERICAL RADIUS OF A-NORMAL OPERATORS IN SEMI-HILBERTIAN SPACES

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Abstract. Let $\mathcal{H}$ be a Hilbert space and A be a positive bounded linear operator on $\mathcal{H}$. The semi-inner product $\langle h,k\rangle_A = \langle Ah,k\rangle$, $h,k \in \mathcal{H}$, induces a seminorm for a bounded linear operator T, which is defined by

$$\|T\|_A = \sup\{\|Th\|_A/\|h\|_A : \|h\| \neq 0\}.$$

The main purpose of this paper is to prove that $\|T\|_A$ equals the A-numerical radius of T, when T is an A-normal operator. This generalizes the similar result for a normal operator on a Hilbert space.

Keywords and phrases: A-normal operator, A-adjoint operator, A-numerical radius.

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1. Introduction

Throughout this paper, $\mathcal{H}$ denotes a complex Hilbert space with the inner product $\langle .,.\rangle$ and $\mathcal{L}(\mathcal{H})$ is the Banach algebra of all bounded linear operators on $\mathcal{H}$, and A always stands for a positive linear operator in $\mathcal{L}(\mathcal{H})$.

We are going to consider an additional semi-inner product $\langle .,.\rangle_A$ on $\mathcal{H}$ given by $\langle h,k\rangle_A = \langle Ah,k\rangle$, $(h,k \in \mathcal{H})$ which defines a seminorm $\|.\|_A$ on $\mathcal{H}$. This makes $\mathcal{H}$ into a semi-Hilbertian space. Then we replace the operator norm with

$$\|T\|_A = \sup\{\|Th\|_A/\|h\|_A : \|h\|_A \neq 0\}.$$

The previously defined concepts about operators can be generalized using this seminorm. For some references, the reader can see [1], [2], [7], [5].

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Our purpose in this paper, is to study the relationship between the new concept of normality of operators, called *A-normality*, and the numerical radius of them, called the *A-numerical radius*, when we consider $\|.\|_A$ instead of the initial norm. A-normal operators are defined in [6], in which some results related to $\|.\|_A$ and A-numerical radius of them are obtained. A question posed in the mentioned paper, states that if $\|T\|_A$ equals the A-numerical radius of T, when T is an A-normal operator. In this direction, Section 2 is devoted to collect some facts about $\|.\|_A$ and the relevant concepts. It is well-known that the numerical radius of a normal operator on a Hilbert space equals its norm [3]. Similar to this fact, the last section is dedicated to proving the same result for operators defined on a semi-Hilbertian space.

2. Preliminaries

The positive operator $A \in \mathcal{L}(\mathcal{H})$ define a positive semi-definite sesquilinear form $\langle .,. \rangle_A : \mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{C}$ given by $\langle h,k \rangle_A = \langle Ah,k \rangle$. Note that $\langle .,. \rangle_A$ defines a semi-inner product on $\mathcal{H}$, and the seminorm induced by it is given by

$$\|h\|_A = \langle h,h \rangle_A^{1/2}.$$

It is easily seen that $\|.\|_A$ is a norm on $\mathcal{H}$ if and only if A is injective.

The above seminorm induces a seminorm on the subspace $\mathcal{L}^A(\mathcal{H})$ of $\mathcal{L}(\mathcal{H})$ consisting of all $T \in \mathcal{L}(\mathcal{H})$ so that for some $c > 0$ and all $h \in \mathcal{H}$, $\|Th\|_A \leq c\|h\|_A$. Indeed, if $T \in \mathcal{L}^A(\mathcal{H})$, then

$$\|T\|_A := \sup \left\{ \frac{\|Th\|_A}{\|h\|_A} : h \in \overline{R(A)}, h \neq 0 \right\} < \infty.$$

Operators in $\mathcal{L}^A(\mathcal{H})$ are called A-bounded operators.

Some equivalent statements for $\|.\|_A$ are summarized in the next two results.

Proposition 1. *If $T \in \mathcal{L}^A(\mathcal{H})$, then*

$$\begin{aligned} \|T\|_A &= \sup\{\|Th\|_A/\|h\|_A : h \notin kerA\} \\ &= \sup\{\|Th\|_A : \|h\|_A = 1\} \\ &= \sup\{|\langle Th,k \rangle_A| : h,k \in \mathcal{H}, \|h\|_A \leq 1,\ \|k\|_A \leq 1\}. \end{aligned}$$

(*The last equality is stated in* [7].)

Proof. Suppose that $h \notin kerA$. Then h can be represented as $h = h_1 + h_2$ in which $h_1 \in \overline{R(A)}$ and $h_2 \in N(A^{1/2})$. Note that $h_1 \neq 0$, thanks to the fact that $A^{1/2}h \neq 0$. Thus,

$$\frac{\|Th\|_A}{\|h\|_A} = \frac{\|A^{1/2}Th_1 + A^{1/2}Th_2\|}{\|A^{1/2}h_1 + A^{1/2}h_2\|}.$$

Since $kerA^{1/2}$ is an invariant subspace for T, $Th_2 \in kerA^{1/2}$, and so the above equality implies that

$$\frac{\|Th\|_A}{\|h\|_A} = \frac{\|Th_1\|_A}{\|h_1\|_A}.$$

Thus,

$$\sup\{\|Th\|_A/\|h\|_A : h \in \mathcal{H}, A^{1/2}h \neq 0\} = \|T\|_A.$$

The second inequality is obvious, and from which it can be easily deduced that

$$\|T\|_A = \sup\{\|Th\|_A : \|h\|_A \leq 1\}. \tag{1}$$

To prove the last equality, suppose that $h, k \in \mathcal{H}$ are so that $\|h\|_A \leq 1$ and $\|k\|_A \leq 1$. Then

$$|\langle Th, k\rangle| \leq \|T\|_A,$$

and so

$$\alpha := \sup\{|\langle Th, k\rangle_A| : h, k \in \mathcal{H}, \|h\|_A \leq 1, \|k\|_A \leq 1\} \leq \|T\|_A.$$

Now suppose that $\|h\| \leq 1$. For $\varepsilon > 0$, let $k = Th/(\|Th\|_A + \varepsilon)$. Then

$$|\langle Th, k\rangle_A| = \|Th\|/(\|Th\|_A + \varepsilon) \leq \alpha.$$

Now, letting $\varepsilon \longrightarrow 0$, we observe that $\|Tx\|_A \leq \alpha$ for each $h \in \mathcal{H}$ with $\|h\|_A \leq 1$. This, coupled with (1), shows that $\|T\|_A \leq \alpha$. Hence $\|T\|_A = \alpha$. ∎

Corollary 1. *If $T \in \mathcal{L}^A(\mathcal{H})$, then*

$$\|T\|_A = \sup\{|\langle Th, k\rangle_A| : h, k \in \mathcal{H}, \|h\|_A = \|k\|_A = 1\}.$$

Proof. Suppose that $\|h\|_A = 1$, and $\|Th\|_A \neq 0$. Then

$$\begin{aligned} \|Th\|_A &= |\langle Th, T(\frac{h}{\|Th\|_A})\rangle_A| \\ &\leq \sup\{|\langle Th, k\rangle| : \|k\|_A = 1\}. \end{aligned}$$

So, in light of Proposition 1, the result holds. ∎

An operator $S \in \mathcal{L}(\mathcal{H})$ is called an A-adjoint of an operator $T \in \mathcal{L}(\mathcal{H})$, if $\langle Th, k\rangle_A = \langle h, Sk\rangle_A$, for every $h, k \in \mathcal{H}$, or equivalently, $AT = S^*A$. If T is an A-adjoint of itself, then T is called an A-selfadjoint operator. It is possible that an operator T does not have an A-adjoint, and if S is an A-adjoint of T we may find many A-adjoints; in fact, if $AV = 0$ for some $V \in \mathcal{L}(\mathcal{H})$, then $S + V$ is an A-adjoint of T. The set of all A-bounded operators which admit an A-adjoint is denoted by $\mathcal{L}_A(\mathcal{H})$. By Douglas' theorem [4], $\mathcal{L}_A(\mathcal{H})$ consists of all operators T such that $R(T^*A) \subseteq R(A)$. If $T \in \mathcal{L}_A(\mathcal{H})$, the reduced solution of the equation $AX = T^*A$ is a distinguished A-adjoint operator of T, which is denoted by $T^\sharp$. Note that $T^\sharp = A^\dagger T^*A$ in which $A^\dagger$ is the Moore-Penrose inverse of A. For more details see [1] and [2].

Definition 1. An operator $T \in \mathcal{L}_A(\mathcal{H})$ is an A-normal operator, if $T^\sharp T = TT^\sharp$.

3. A-numerical radius

The A-numerical radius of an operator $T \in \mathcal{L}(\mathcal{H})$, denoted by $w_A(T)$ is defined as

$$w_A(T) = \sup\{|\langle Th, h\rangle_A| : h \in \mathcal{H}, \|h\|_A = 1\}.$$

It is a generalization of the concept of numerical radius of an operator. Clearly, w_A defines a seminorm on $\mathcal{L}(\mathcal{H})$.

Furthermore, for every $h \in \mathcal{H}$,

$$|\langle Th, h\rangle_A| \leq w_A(T).\|h\|_A{}^2.$$

Taking Corollary 1 into consideration, it is obvious that

$$w_A(T) \leq \|T\|_A. \tag{2}$$

From the proof of part (2) of Corollary 3.2 of [6] we can derive the following lemma.

Lemma 1. *If $T \in \mathcal{L}_A(\mathcal{H})$ is an A-normal operator, then $\|T^n\|_A = \|T\|_A{}^n$ for each positive integer n.*

As it is known [3], for a normal operator T, $w(T) = \|T\|$, where $w(T)$ denotes the numerical radius of T.

In the next theorem, we establish a similar result for A-normal operators.

Theorem 1. *Suppose that $T \in \mathcal{L}_A(\mathcal{H})$ is an A-normal operator. Then*

$$\|T\|_A = w_A(T).$$

Proof. Take $h, k \in \mathcal{H}$ so that $\|h\|_A = \|k\|_A = 1$.

For an arbitrary operator $S \in \mathcal{L}_A(\mathcal{H})$,

$$\begin{aligned} |2\langle Sh, k\rangle_A + 2\langle Sk, h\rangle_A| &= |\langle S(h+k), h+k\rangle_A - \langle S(h-k), h-k\rangle_A| \\ &\leq w_A(S)\|h+k\|_A{}^2 + w_A(S)\|h-k\|_A{}^2 \\ &= 2w_A(S)(\|h\|_A{}^2 + \|k\|_A{}^2) \\ &= 4w_A(S). \end{aligned}$$

If $\|Sh\|_A \neq 0$, substitute k by $\|Sh\|_A{}^{-1}Sh$ in the above computations to get

$$\frac{1}{\|Sh\|_A}|\langle Sh, Sh\rangle_A + \langle S^2h, h\rangle_A| \leq 2w_A(S).$$

Consequently,

$$|\|Sh\|_A{}^2 + \langle S^2h, h\rangle_A| \leq 2w_A(S)\|Sh\|_A. \tag{3}$$

Choose θ so that

$$e^{2i\theta}\langle T^2h, h\rangle_A = |\langle T^2h, h\rangle_A|.$$

Then, applying (3) for $S = e^{i\theta}T$ implies that

$$|\|Th\|_A^2 + \langle e^{2i\theta}T^2h, h\rangle_A| \leq 2w_A(T)\|Th\|_A.$$

Therefore,

$$\|Th\|_A^2 + |\langle T^2h, h\rangle_A| \leq 2w_A(T)\|Th\|_A, \tag{4}$$

which implies that

$$\|Th\|_A \leq 2w_A(T).$$

Note that this is true, even if $\|Th\|_A = 0$. Consequently,

$$\|T\|_A \leq 2w_A(T). \tag{5}$$

Besides, (4) shows that

$$\begin{aligned} 0 &\leq 2w_A(T)\|Th\|_A - \|Th\|_A^2 - |\langle T^2h, h\rangle_A| \\ &\leq w_A(T)^2 - |\langle T^2h, h\rangle_A|, \end{aligned}$$

and so

$$|\langle T^2h, h\rangle_A| \leq w_A(T)^2.$$

Hence

$$w_A(T^2) \leq w_A(T)^2.$$

Now using a mathematical induction, we observe that for every positive integer number k,

$$w_A(T^{2^k}) \leq w_A(T)^{2^k}.$$

This coupled with (5) and Lemma (1) shows that

$$\|T\|_A^{2^k} \leq 2w_A(T)^{2^k},$$

and so

$$\|T\|_A \leq \lim_{k\longrightarrow\infty} 2^{2^{-k}} w_A(T) = w_A(T).$$

Now, taking (2) into account, the equality holds. ∎

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FINITE GROUPS WITH SOME QUASINORMAL AND SELFNORMALIZING SUBGROUPS

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Abstract. A subgroup H of a finite group G is called quasinormal in G if $HK = KH$ holds for every subgroup K of G. In this paper, we mainly give the structure of finite nonabelian simple groups in which every cyclic subgroups of order 2 and order 4 of every second maximal subgroups is either quasinormal or selfnormalizing.

Key words: quasinormal subgroups; selfnormalizing subgroups; second maximal subgroups; minimal subgroups.

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1. Introduction

All groups considered in this paper will be finite. A group G is called a PN-group if all minimal subgroups of G are normal in G. Gaschütz and Itô [6, IV, 5.7 Satz] showed that if G is a PN-group, then G is solvable, and the commutator subgroup G' of G has a normal Sylow 2-subgroup with nilpotent factor. In 1970, Buckley [2] proved that a PN-group of odd order is supersolvable. The PN-groups have been generalized by many authors(see [1], [5] [9], [11] and [10]).

Definition 1.1 Let G be a group. A subgroup H of G is said to be quasinormal in G if $HK = KH$ holds for every subgroup K of G.

Obviously, the quasinormality is a generalization of the normality. Hence the concept of selfnormalizing subgroup is, in a sense, opposite of that of a quasinormal subgroup of G.

In this paper, we will investigate the structure of a group G in which every cyclic subgroups of order 2 and order 4 of every second maximal subgroups is either quasinormal or selfnormalizing. In the light of Feit-Thompson's Theorem,

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all groups of odd order are solvable, and so, the involutions should pose something of special interest. However, if we only give the conditions on the subgroups of order 2 of G, we could not expect a detailed structure. On the other hand, if we assume that every cyclic subgroup of a group G of order 2 and 4 is either quasinormal or selfnormalizing in G, then we are able to get some interesting results.

For convenience, we list the following condition.

($\mathcal{P}$) Every cyclic subgroup of a group G of order 2 and 4 is either quasinormal or selfnormalizing in G.

We first classify the nonsolvable groups all of whose second maximal subgroups satisfy condition ($\mathcal{P}$). In a sense, these groups can be viewed as a generalization of PN-groups. Next, we prove that a group G is solvable if every nonnilpotent subgroup of G is either quasinormal or selfnormalizing in G.

The notation and terminology used in this paper are standard. Particularly, we denote by $A \rtimes B$ the semidirect product of A and B; G_p always denotes a Sylow p-subgroup of G and $\pi(G)$ denotes the set of all prime divisors of $|G|$.

2. Preliminaries

In this section, we give some basic properties and collect some results that are needed in the sequel.

Lemma 2.1 *Let G be a dihedral group of order $2n$, where n is an odd integer. Then, G satisfies condition ($\mathcal{P}$).*

Proof. Let G be a dihedral group of order $2n$, where n is odd. Let a be any involution of G. Then, $N_G(\langle a\rangle) = \langle a\rangle$, and so, G satisfies condition ($\mathcal{P}$). ∎

Lemma 2.2 *Let G be a dihedral group of order $4n$, where n is an odd integer. Then, G does not satisfy condition ($\mathcal{P}$). Moreover, if n is not a prime, then there exist some maximal subgroups of G do not satisfy condition ($\mathcal{P}$).*

Proof. Let G be a dihedral group of order $4n$, where n is odd. We may write

$$G = \langle a, b \mid a^2 = b^{2n} = 1, a^{-1}ba = b^{-1}\rangle.$$

Then $N_G(\langle a\rangle) = \langle a\rangle \times \langle b^n\rangle$. It implies that the involution a is neither quasinormal nor selfnormalizing in G. Hence the dihedral group of order $4n$ does not satisfy condition ($\mathcal{P}$).

If there is an odd prime t such that $t < n$ and $t|n$, then G may have a maximal subgroup of order $\frac{4n}{t}$ which is dihedral. By what has been said above, such a subgroup does not satisfy condition ($\mathcal{P}$). ∎

Lemma 2.3 [7, Theorem B] *Let G be a nonsolvable group. Suppose that solvable subgroups of G are either 2-nilpotent or minimal non-nilpotent. Then G is one of the following groups:*

(1) *$PSL(2, 2^r)$, where $2^r - 1$ is a prime;*

(2) *$PSL(2, q)$, where q is odd and $q \equiv 3, 5 (mod\ 8)$;*

(3) *$SL(2, q)$, where q is odd and $q \equiv 3, 5 (mod\ 8)$.*

It is easy to see that if the exponent of Sylow 2-subgroups of G is not more than 4, then any quasinormal subgroup of order 2 of G is normal. We will use this fact freely in our following proof.

3. The main results and proofs

Theorem 3.1 *If a group G satisfies condition $(\mathcal{P})$, then G is 2-nilpotent.*

Proof. Assume that G is not 2-nilpotent. Then, G has a minimal non-2-nilpotent subgroup K. By a result of Itô [6, III, 5.4 Satz], G is a minimal nonnilpotent group and $K = K_2 \rtimes K_p$ with $\exp K_2 \leq 4$.

Let a be any nontrivially element of K_2. Then $\langle a \rangle$ is quasinormal in G by hypothesis. Hence $\langle a \rangle K_p$ is a subgroup of K. If $\langle a \rangle K_p < K$, then $\langle a \rangle K_p$ is nilpotent. It implies that K is nilpotent, a contradiction. Hence we have $\langle a \rangle K_p = K$ and $K_2 = \langle a \rangle$ is cyclic. Therefore K is 2-nilpotent, which implies that K is nilpotent, a contradiction. Thus, G is 2-nilpotent. ∎

Theorem 3.2 *Let G be a group. If every maximal subgroup of G satisfies condition $(\mathcal{P})$, then one of the following holds:*

(1) *G is 2-nilpotent;*

(2) *$G = PQ$ is a minimal nonnilpotent group, where P is an elementary abelian 2-group and Q is a cyclic q-group $(q > 2)$, or P is a quaternion group of order 8 and Q is a cyclic 3-group.*

Proof. Assume that G is not 2-nilpotent. Then, G has a minimal non-2-nilpotent subgroup K. By our hypothesis and Theorem 3.1, we know that $G = K$. By [6, III, 5.4 Satz], G is a minimal nonnilpotent group. According to a result due to Schmidt [6, III, 5.2 Satz], we have $G = PQ$, where P is a normal Sylow 2-subgroup of G with $\exp P \leq 4$, and Q is a cyclic Sylow q-subgroup $(q > 2)$. Furthermore, if P is abelian, then P is elementary abelian; if P is not abelian, then $Z(P) = \Phi(P) = P'$. In the case P is abelian, it is clear that every maximal subgroup of G satisfies condition $(\mathcal{P})$. Assume that P is not abelian, and let $x \in P$ with $x^4 = 1$. By the hypothesis, we have $\langle x \rangle$ is quasinormal in P. Let $Z = \langle x^2 \rangle$, then $\langle xZ \rangle$ is quasinormal in P/Z. Hence $\langle xZ \rangle$ is normal in P/Z, which implies that $\langle x \rangle$ is normal in P. Therefore, P is a Hamiltonian group. It follows

that $P = Q_8 \times A$, where Q_8 is the quaternion group of order 8, and A is an elementary abelian 2-group. However, $A \leq Z(P) = P' \leq Q_8$. Hence $A = 1$, and thus $P = Q_8$. Since $Aut(Q_8) \cong S_4$, the symmetry group of degree 4, we have $q = 3$. ∎

Theorem 3.3 *Let G be a nonsolvable group. If every second maximal subgroup of G satisfies condition $(\mathcal{P})$, then G is one of the following groups:*

(1) *$PSL(2, 2^r)$, where r is a prime such that $2^r - 1$ is a prime;*

(2) *$PSL(2, p)$, where $p > 3$ is a prime with $p \equiv 3, 5(mod\ 8)$ and $p^2 \not\equiv 1(mod\ 5)$. Moreover, if $p \pm 1 = 4s$, then s is an odd prime;*

(3) *$PSL(2, 3^r)$, where r is an odd prime and $3^r \equiv 3, 5(mod\ 8)$. Moreover, if $3^r \pm 1 = 4s$, then s is an odd prime;*

(4) *$SL(2, p)$, where $p > 3$ is a prime with $p \equiv 3, 5(mod\ 8)$ and $p^2 \not\equiv 1(mod\ 5)$;*

(5) *$SL(2, 3^r)$, where r is an odd prime and $3^r \equiv 3, 5(mod\ 8)$.*

Proof. Let M be any maximal subgroup of G. By the hypothesis and Theorem 3.2, M is 2-nilpotent or a minimal nonnilpotent group. In particular, M is solvable. Then, by Lemma 2.3, G is one of the following groups:

(i) $PSL(2, 2^r)$, where $2^r - 1$ is a prime such that $2^r - 1$ is a prime;

(ii) $PSL(2, q)$, where q is odd and $q \equiv 3, 5(mod\ 8)$;

(iii)) $SL(2, q)$, where q is odd and $q \equiv 3, 5(mod\ 8)$.

Case 1. $G \cong PSL(2, 2^r)$, where r is a prime such that $2^r - 1$ is also a prime.

In this case, any maximal subgroup of G is one of the following groups:

(i) a minimal nonabelian group of order $2^r(2^r - 1)$;

(ii) dihedral groups of order $2(2^r - 1)$ or $2(2^r + 1)$.

By Lemma 2.1, G satisfies condition $(\mathcal{P})$. That is G is of type (1).

Case 2. $G \cong PSL(2, q)$, where $q \equiv 3, 5(mod\ 8)$.

Let $q = p^n$ and let $p > 3$. Since $PSL(2, p^n)$ contains the nonsolvable subgroup $PSL(2, p)$, we have $n = 1$.

If $p^2 \equiv 1(mod\ 5)$, then $PSL(2, p)$ contains the subgroup A_5, and therefore, $p = 5$, which is a contradiction. Hence, $p^2 \not\equiv 1(mod\ 5)$.

By [6, II, Theorem 8.27], G has only three kinds of maximal subgroups:

(i) Frobenius group N and $N = P \rtimes C$ is a minimal nonabelian group, where P is an elementary abelian group and C is a cyclic group of order $\frac{p-1}{2}$;

(ii) dihedral groups of order $p - 1$ or $p + 1$;

(iii) A_4, the alternating group of degree 4.

Clearly, all maximal subgroups of (i) are abelian. All maximal subgroups of A_4 are satisfy condition $(\mathcal{P})$. Now we consider (ii). Since $p = q \equiv 3, 5(mod\ 8)$, we get $p-1 = 2s$ or $4s$ and $p+1 = 2s$ or $4s$, where s is an odd integer. If $p \pm 1 = 2s$, then the groups of type (ii) satisfy condition $(\mathcal{P})$ by Lemma 2.1. If $p \pm 1 = 4s$, then the groups of type (ii) satisfy condition $(\mathcal{P})$ if and only if s is a prime by Lemma 2.2. That is G is of type (2).

When $p = 3$, if n is even, then $PSL(2,9) \leq PSL(2,3^n)$, and therefore, $PSL(2,3^n)$ contains a proper subgroup A_5, which is a contradiction. Hence, n is odd. If n is an odd composite, let $n = st$, where t is a prime with $t < n$. By [6, II, Theorem 8.27], we know $PSL(2,3^n)$ contains a nonsolvable proper subgroup $PSL(2,3^t)$, which is a contradiction. Thus n is an odd prime. Now by the same argument as above, we get G is of type (3) in this case.

Case 3. $G \cong SL(2,q)$, where $q \equiv 3, 5(mod\ 8)$.

Let $q > 5$ and $q = p^n$. It is well known that $SL(2,q)$ has only one element of order 2, say, z, and $SL(2,q)/\langle z\rangle \cong PSL(2,q)$. By the same arguments as in Case 2, we get $n = 1$ and $p^2 \not\equiv 1(mod\ 5)$ if $p > 3$, and n is an odd prime for $p = 3$. By [3, Theorem 6.17], we know that any maximal subgroup of G is one of the following groups:

(i) Frobenius group N and $N = P \rtimes C$ is a minimal nonabelian group, where P is an elementary abelian group and C is a cyclic group of order $p^n - 1$;

(ii) the group M defined by the presentation $\langle x, y | x^m = y^2, y^{-1}xy = x^{-1}\rangle$, where $2m = p^n - 1$ or $p^n + 1$;

(iii) $SL(2,3)$.

We consider these groups in turn.

Since all maximal subgroups of (i) are abelian, the groups of type (i) satisfy condition $(\mathcal{P})$. For the groups of type (ii), we get $m = 2s$ or $2s \pm 1$, where s is odd. In the later case, M is the generalized quaternion group of order $4(2s \pm 1)$. It is well known that the cyclic subgroup of order 2 of M is normal in M, and the cyclic subgroup of order 4 of M is self-normalizing in M. Hence M satisfies condition $(\mathcal{P})$. For the case $m = 2s$, the maximal subgroups of M are

$$M_1 = \langle x^2, y\rangle, M_2 = \langle x^2, xy\rangle, M_3 = \langle x\rangle.$$

Since both M_1 and M_2 are the generalized quaternion group, and M_3 is cyclic, we know that M satisfies condition $(\mathcal{P})$. Since the maximal subgroups of $SL(2,3)$ are the quaternion group of order 8 and the abelian group $Z_2 \times Z_3$. Hence, we get conclusions (4) and (5). The proof of the theorem is now complete. ∎

A group G is called a $\mathcal{QNS}$-group if every minimal subgroup of G is either quasinormal in G or self-normalizing. It is proved that every $\mathcal{QNS}$-group is solvable(see [5]). We now give a slight improvement of this result.

Theorem 3.4 *Let G be a group. If every nonnilpotent proper subgroup of G is a $\mathcal{QNS}$-group, then G is solvable.*

Proof. Assume that the theorem is false, and let G be a minimal counterexample. By [5, Theorem 3.1], we know that the $\mathcal{QNS}$-groups are solvable, and therefore, $G/\Phi(G)$ is a minimal simple group, where $\Phi(G)$ is the Frattini subgroup of G. By Thompson's classification of minimal simple groups (see [12]), we know that $G/\Phi(G)$ is isomorphic to one of the following groups:

(1) $PSL(2,p)$, where p is a prime with $p > 3$ and $5 \nmid p^2 - 1$;

(2) $PSL(2,2^q)$, where q is a prime;

(3) $PSL(2,3^q)$, where q is a prime;

(4) $PSL(3,3)$;

(5) The Suzuki group $Sz(2^q)$, where q is an odd prime.

Let H be the 2-complement of $\Phi(G)$. Then $H \trianglelefteq G$ and H is nilpotent. We have

(1) $H \leq Z(G)$.

Indeed, let $P \in Syl_p(H)$, where p is a prime in $\pi(H)$. Then $P \trianglelefteq G$. Let X be any minimal subgroup of P. Since G is nonsolvable, there exists some nonnilpotent maximal subgroup M of G such that $P \leq M$. By the hypothesis, M is a $\mathcal{QNS}$-group. Then, X is a subnormal subgroup of M. Thus, X is normal in M, and therefore, normal in G. Hence $G/C_G(X) = N_G(X)/C_G(X) \lesssim Aut(X) \cong C_{p-1}$. If $C_G(X)$ is a proper subgroup G, then $C_G(X)$ is solvable and G is hence solvable, a contradiction. Thus $C_G(X) = G$, i.e., $X \leq Z(G)$. Let $G_2 \in Syl_2(G)$ and $K = G_2P$. Applying Ito's Lemma, we see that K is p-nilpotent and so K is nilpotent. Then we have that $G_2 \leq C_G(P) \trianglelefteq G$. Using the simplicity of $G/\Phi(G)$, we conclude that $H \leq Z(G)$. Thus (1) holds.

(2) $H = 1$.

Set $\overline{G} = G/\Phi(G)_2$, where $\Phi(G)_2 \in Syl_2(\Phi(G))$. Then by (1), $\overline{G}/Z(\overline{G}) \cong G/\Phi(G)$ and $\overline{G}$ is a quasisimple group with the center of odd order. Hence in order to prove $H = 1$, i.e., $Z(\overline{G}) = 1$, it will suffice to show that the Schur multiplier of each of the minimal simple groups is a 2-group. Indeed, this is true by checking the table on the Schur multipliers of the known simple groups (see [3, p. 302]).

(3) Every subgroup of order $2^m p$ (p an odd prime) of $\overline{G} = G/\Phi(G)$ is 2-nilpotent.

By (2), $\Phi(G)$ is a 2-group. Assume $L/\Phi(G)$ is a proper subgroup of order $2^m p$. Then L is a proper subgroup of order $2^n p$ for some natural number n. Let $G_p \in Syl_p(L)$. Then $|G_p| = p$ and hence G_p is quasinormal in L. Since G_p is subnormal in L, we have that G_p is normal in L, that is, L is 2-nilpotent. Thus $L/\Phi(G)$ is 2-nilpotent.

(4) Final contradiction.

We know that $\overline{G}$ is isomorphic to one of the above simple groups. Suppose that $\overline{G} \cong PSL(2,p)$, $PSL(2,3^q)$ or $PSL(3,3)$. Indeed, each of $PSL(2,p)$, $PSL(2,3^q)$ and $PSL(3,3)$ contains a subgroup which is isomorphic to A_4, the alternating group of degree 4, by (3) we conclude that $\overline{G}$ cannot be any one of $PSL(2,p), PSL(2,3^q)$ and $PSL(3,3)$. Suppose that $\overline{G} \cong PSL(2,2^q)$ or $Sz(2^q)$. Then $\overline{G}$ is a Zassenhaus group of odd degree and the stabilizer of a point is a Frobenius group with kernel a 2-group. So $\overline{G}$ cannot be any one of $PSL(2,2^q)$ and $Sz(2^q)$ as well. Thus the proof is complete. ∎

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QUOTIENT HYPER HOOP-ALGEBRAS

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Abstract. In this paper, by considering the notion of hyper hoop-algebras, we define the concepts of (strong) regular relations on hyper hoop-algebras and investigate some properties of them. Then we construct a quotient (hyper) hoop-algebra by a (strong)regular relation on hyper hoop-algebras. Finally, we define the notion of maximal filter and we investigate the relation between quotient simple hyper hoop-algebras and maximal filters.

Keywords: Hyper hoop-algebra, (strong) regular relation, quotient hyper hoop-algebra, maximal filter.

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1. Introduction

Hoop-algebras are naturally ordered commutative residuated integral monoids, introduced by B. Bosbach in [5] and then were investigated by Büchi and Owens in an unpublished manuscript [8] of 1975, and they have been studied by Aglianò, Blok and Ferreirim [1], [6], [7]. The study of hoops is motivated by their occurrence

both in universal algebra and algebraic logic. In recent years, hoop theory was enriched with deep structure theorems. Many of these results have a strong impact with fuzzy logic. Particularly, from the structure theorem of finite basic hoops one obtains an elegant short proof of the completeness theorem for propositional basic logic introduced by Hájek [13].

Hyperstructure theory was introduced in 1934 [14], when Marty at the 8th congress of scandinavian mathematicians, gave the definition of hypergroup and illustrated some applications and showed its utility in the study of groups, algebraic functions and rational fraction. Till now, the hyperstructures have been studied from the theoretical point of view for their applications to many subject of pure and applied mathematics. Some fields of applications of the mentioned structures are lattices, graphs, coding, ordered sets, median algebra, automata, and cryptography [9]. Many researchers have worked on this area. The authors applied hyper structure theory on hyper hoop and introduced hyper hoop algebra in [4]. In this paper, we continue the study of hyper hoop-algebras and introduce (strong) regular relations on hyper hoop-algebras and study the quotient (hyper) hoop-algebras that are the results of (strong) regular relations. Then we define the notion of maximal filter and we investigate the relation between quotient simple hyper hoop-algebras and maximal filters.

2. Preliminaries

Let H be a non-empty set. A hypergroupoid is a pair $(H, \odot)$, where $\odot : H \times H \longrightarrow P(H) - \{\emptyset\}$ is a binary hyperoperation on H. If $a \odot (b \odot c) = (a \odot b) \odot c$ holds, for all $a, b, c \in H$ then $(H, \odot)$ is called a semihypergroup, and it is said to be commutative if $\odot$ is commutative. An element $1 \in H$ is called a unit, if $a \in 1 \odot a \cap a \odot 1$, for all $a \in H$ and is called a scaler unit, if $1 \odot a = a \odot 1 = \{a\}$, for all $a \in A$. If $a \odot H = H = H \odot a$, for any element $a \in H$, then the pair $(H, \odot)$ is called a hypergroup. Note that if $A, B \subseteq H$, then we consider $A \odot B$ by $A \odot B = \bigcup_{a \in A, b \in B} (a \odot b)$.

Definition 2.1. [7] A *hoop-algebra* or a *hoop* is an algebra $(A, \odot, \rightarrow, 1)$ of type $(2, 2, 0)$ such that $(A, \odot, 1)$ is a commutative monoid and for all $x, y, z \in A$, $x \rightarrow x = 1$, $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z)$ and $(x \rightarrow y) \odot x = (y \rightarrow x) \odot y$. On hoop A we define "$x \leq y$" if and only if $x \rightarrow y = 1$. It is easy to see that $\leq$ is a partial order relation on A.

Definition 2.2. [4] A *quasi hyper hoop algebra* or briefly, a *quasi hyper hoop* is a non-empty set A endowed with two binary hyperoperations $\odot, \rightarrow$ and a constant 1 such that, for all $x, y, z \in A$ satisfying the following conditions:
(HHA1) $(A, \odot, 1)$ is a commutative semihypergroup with 1 as the unit,
(HHA2) $1 \in x \rightarrow x$,
(HHA3) $(x \rightarrow y) \odot x = (y \rightarrow x) \odot y$,
(HHA4) $x \rightarrow (y \rightarrow z) = (x \odot y) \rightarrow z$,

A quasi hyper hoop $(A, \odot, \rightarrow, 1)$ is called a *hyper hoop* if the following hold;
(HHA5) $1 \in x \rightarrow 1$,
(HHA6) if $1 \in x \rightarrow y$ and $1 \in y \rightarrow x$ then $x = y$,
(HHA7) if $1 \in x \rightarrow y$ and $1 \in y \rightarrow z$ then $1 \in x \rightarrow z$.

In the sequel we will refer to the (quasi) hyper hoop $(A, \odot, \rightarrow, 1)$ by its universe A. On (quasi) hyper hoop A, we define $x \leq y$ if and only if $1 \in x \rightarrow y$. If A is a hyper hoop, it is easy to see that $\leq$ is a partial order relation on A. Moreover, for all $B, C \subseteq A$ we define $B \ll C$ iff there exist $b \in B$ and $c \in C$ such that $b \leq c$ and define $B \leq C$ if and only if for any $b \in B$ there exists $c \in C$ such that $b \leq c$. A (quasi) hyper hoop A is bounded if there is an element $0 \in A$ such that $0 \leq x$, for all $x \in A$.(See [4])

Proposition 2.3. *In (quasi) hyper hoop $(A, \odot, \rightarrow, 1)$, if for any $x, y \in A$, $x \odot y$ and $x \rightarrow y$ are singletons, then $(A, \odot, \rightarrow, 1)$ is a hoop. Hence (quasi) hyper hoops are a generalization of hoops.*

Proposition 2.4. [4] *Let A be a hyper hoop. Then for all $x, y, z \in A$ and $B, C, D \subseteq A$, the following hold;*
(HHA8) $x \odot y \ll z \Leftrightarrow x \leq y \rightarrow z$,
(HHA9) $B \odot C \ll D \Leftrightarrow B \ll C \rightarrow D$,
(HHA10) $z \rightarrow y \leq (y \rightarrow x) \rightarrow (z \rightarrow x)$,
(HHA11) $z \rightarrow y \ll (x \rightarrow z) \rightarrow (x \rightarrow y)$,
(HHA12) $1 \odot 1 = \{1\}$.

Definition 2.5. [4] Let A be a hyper hoop and F be a non-empty subset of A. Then,
(i) F is called an *upset* of A, if $x \in F$ and $x \leq y$ imply $y \in F$, for all $x, y \in A$,
(ii) F is called a *weak filter* of A, if F is an upset and for all $x, y \in F$, $x \odot y \cap F \neq \emptyset$,
(iii) F is called a *filter* of A, if F is an upset and for all $x, y \in F$, $x \odot y \subseteq F$.

Let F be a (weak) filter of A and $x \in F$. Since F is an upset and $x \leq 1$, we get $1 \in F$.

Notation. From now on, in this paper we let A be a hyper hoop, unless otherwise is stated.

3. (Strong) Regular relations on hyper hoops

In this section, we introduce the concepts of regular, strong regular, and fundamental relations on hyper hoops and investigate some properties of them.

Notations. Let $\mathbf{R}$ be an equivalence relation on A and $B, C \subseteq A$. Then $B\mathbf{R}C$, $B\overline{\mathbf{R}}C$ and $B\overline{\overline{\mathbf{R}}}C$ denoted as follows:

(i) $B\mathbf{R}C$ if there exist $b \in B$ and $c \in C$ such that $b\mathbf{R}c$,
(ii) $B\overline{\mathbf{R}}C$ if for all $b \in B$ there exists $c \in C$ such that $b\mathbf{R}c$ and for all $c \in C$ there exists $b \in B$ such that $b\mathbf{R}c$,
(iii) $B\overline{\overline{\mathbf{R}}}C$ if for all $b \in B$ and $c \in C$, we have $b\mathbf{R}c$.

Remark 3.1. It is clear that $B\overline{\mathbf{R}}C$ and $C\overline{\mathbf{R}}D$ imply that $B\overline{\mathbf{R}}D$, for all $B, C, D \subseteq A$.

Definition 3.2. Let $\mathbf{R}$ be an equivalence relation on A. Then $\mathbf{R}$ is called a *regular relation* on A if and only if for all $x, y, z \in A$,

(i) if $x\mathbf{R}y$, then $x \odot z\overline{\mathbf{R}}y \odot z$,

(ii) if $x\mathbf{R}y$, then $x \to z\overline{\mathbf{R}}y \to z$ and $z \to x\overline{\mathbf{R}}z \to y$,

(iii) if $x \to y\mathbf{R}\{1\}$ and $y \to x\mathbf{R}\{1\}$, then $x\mathbf{R}y$.

Example 3.3. Let $A = \{1, a, b, c\}$. Define the hyperoperations $\odot$ and $\to$ on A as follows:

$\odot$	1	a	b	c
1	$\{1\}$	$\{1, a\}$	$\{1, b\}$	$\{1, a, b, c\}$
a	$\{1, a\}$	$\{1, a, b, c\}$	$\{1, a, b, c\}$	$\{1, a, b, c\}$
b	$\{1, b\}$	$\{1, a, b, c\}$	$\{1, b\}$	$\{1, a, b, c\}$
c	$\{1, a, b, c\}$	$\{1, a, b, c\}$	$\{1, a, b, c\}$	$\{1, a, b, c\}$

$\to$	1	a	b	c
1	$\{1, a, b, c\}$	$\{a, c\}$	$\{b, c\}$	$\{c\}$
a	$\{1, a, b, c\}$	$\{1, a, b, c\}$	$\{1, a, b, c\}$	$\{b, c\}$
b	$\{1, a, b, c\}$	$\{a, c\}$	$\{1, a, b, c\}$	$\{a, c\}$
c	$\{1, a, b, c\}$	$\{1, a, b, c\}$	$\{1, a, b, c\}$	$\{1, a, b, c\}$

Then $(A, \odot, \to, 1)$ is a hyper hoop. Let

$$\mathbf{R} = \{(1,1), (a,a), (b,b), (c,c), (1,b), (b,1), (a,c), (c,a)\}$$

Then $\mathbf{R}$ is a regular relation on A.

In the following example, we will show that the condition (iii) in the Definition 3.2, is independent from the other conditions.

Example 3.4. Let $A = \{1, a, b, c\}$. Define hyperoperations $\odot$ and $\to$ on A as follows:

$\odot$	1	a	b	c
1	$\{1\}$	$\{1, a, c\}$	$\{1, b\}$	$\{1, c\}$
a	$\{1, a, c\}$	$\{1, a, c\}$	$\{1, a, b, c\}$	$\{1, a, c\}$
b	$\{1, b\}$	$\{1, a, b, c\}$	$\{1, a, b, c\}$	$\{1, b, c\}$
c	$\{1, c\}$	$\{1, a, c\}$	$\{1, b, c\}$	$\{1, a, c\}$

$\to$	1	a	b	c
1	$\{1, a, c\}$	$\{a\}$	$\{a, b\}$	$\{a, c\}$
a	$\{1, a, c\}$	$\{1, a, c\}$	$\{1, a, b, c\}$	$\{1, a, c\}$
b	$\{1, a, b, c\}$	$\{a, b\}$	$\{1, a, b, c\}$	$\{a, b, c\}$
c	$\{1, a, c\}$	$\{a, c\}$	$\{a, b, c\}$	$\{1, a, c\}$

Then $(A, \odot, \to, 1)$ is a bounded hyper hoop.

Let

$$\mathbf{R} = \{(a,a),(b,b),(c,c),(1,1),(1,c),(c,1),(a,c),(c,a),(a,1),(1,a)\}$$

Then $\mathbf{R}$ is an equivalence relation on A and conditions (i) and (ii) hold. We have $b \to c\mathbf{R}\{1\}$ and $c \to b\mathbf{R}\{1\}$, but b, c are not related.

In the follows, by definition of homomorphism on hyper hoops, we get a class of examples of regular relations on hyper hoops.

Definition 3.5. Let A_1 and A_2 be two hyper hoops. A mapping $f : A_1 \to A_2$ is said to be a *homomorphism* if $f(1) = 1$, $f(x \to y) = f(x) \to f(y)$, and $f(x \odot y) = f(x) \odot f(y)$, for any $x, y \in A$. If f is both one to one and onto, we say that f is an isomorphism.

Example 3.6. Let $f : A_1 \to A_2$ be a homomorphism of hyper hoops. We define relation ρ on A_1, for all $a, b \in A_1$, as following:

$$a\rho b \Leftrightarrow f(a) = f(b)$$

We claim that ρ is a regular relation on A_1. It is clear that ρ is an equivalence relation on A_1. So, we show that ρ is regular. Let $x\rho y$, for $x, y \in A_1$. Then $f(x) = f(y)$ and so $f(x) \to f(z) = f(y) \to f(z)$, for all $z \in A_1$. Then $f(x \to z) = f(y \to z)$. Hence for all $u \in x \to z$ there exists $v \in y \to z$ such that $u\mathbf{R}v$ and for all $v \in y \to z$ there exists $u \in x \to z$ such that $v\rho u$ and so $x \to z\overline{\rho}y \to z$. By the same way, we can show that $x\rho y$ implies $x \odot z\overline{\rho}y \odot z$, for all $z \in A_1$. Now, if $x \to y\rho\{1\}$ and $y \to x\rho\{1\}$, then there exist $u \in (x \to y)$ and $v \in (y \to x)$ such that $u\rho 1$ and $v\rho 1$ and so $f(u) = f(v) = f(1) = 1$. Then $1 \in f(x \to y) = f(x) \to f(y)$ and $1 \in f(y \to x) = f(y) \to f(x)$. Since A_2 is a hyper hoop, we get $f(x) = f(y)$ and so $x\rho y$. Hence ρ is a regular relation on A_1.

Lemma 3.7. *Let F be a non-empty subset of A. Then F is a filter of A if and only if $1 \in F$ and $F \ll x \to y$ and $x \in F$ implies $y \in F$, for any $x, y \in A$.*

Proof. ($\Rightarrow$) Let F be a filter, $F \ll x \to y$ and $x \in F$, for $x, y \in A$. Hence there exist $u \in F$ and $v \in x \to y$ such that $u \leq v$. Since $u \in F$ and F is an upset, we get $v \in F$ and since F is a filter, we get $x \odot v \subseteq F$. By $v \in x \to y$ we have $v \leq x \to y$. Then by (HHA10), $v \odot x \ll y$ and so there exists $t \in v \odot x \subseteq F$ such that $t \leq y$. Since F is an upset, we get $y \in F$.
($\Leftarrow$) Let $x \leq y$ and $x \in F$, for $x, y \in A$. Then $1 \in x \to y$ and since $1 \in F$, we get $F \ll x \to y$. Then, by hypothesis $y \in F$ and so F is an upset. Now, let $x, y \in F$ and $u \in x \odot y$. Then $x \odot y \ll u$ and so by (HHA10), $y \leq x \to u$. Since $y \in F$, we get $F \ll x \to u$ and so by hypothesis, $u \in F$. Hence $x \odot y \subseteq F$ and so F is a filter of A. ∎

Proposition 3.8. *If $\boldsymbol{R}$ is a regular relation on A, then the class* $[1]$ *is a filter of A.*

Proof. Let $F = [1]$. It is clear that $1 \in F$. Let $F \ll x \to y$ and $x \in F$, for $x, y \in A$. Then there exist $u \in F$ and $v \in x \to y$ such that $u \leq v$. Then $1 \in u \to v$ and so $u \to v\mathbf{R}\{1\}$. Since $u \in F$, we have $u\mathbf{R}1$ and by Definition 3.2, $v \to u\overline{\mathbf{R}}v \to 1$. By (HHA5), $1 \in v \to 1$, and so $v \to 1\mathbf{R}\{1\}$. Then by Remark 3.1, $v \to u\mathbf{R}\{1\}$ and then by Definition 3.2(iii), $u\mathbf{R}v$. Since $u\mathbf{R}1$ and $u\mathbf{R}v$, we get $v\mathbf{R}1$ and since $v \in x \to y$, we get $x \to y\mathbf{R}\{1\}$. By $x \in F$, we have $x\mathbf{R}1$ and so by Definition 3.2(ii), $y \to x\overline{\mathbf{R}}y \to 1$. By (HHA5), $1 \in y \to 1$ and so $y \to x\mathbf{R}\{1\}$. By Definition 3.2(iii), since $x \to y\mathbf{R}\{1\}$ and $y \to x\mathbf{R}\{1\}$, we get $x\mathbf{R}y$. Hence $y\mathbf{R}1$ and so $y \in F$. Therefore by Lemma 3.7, F is a filter of A. ∎

Definition 3.9. Let $\mathbf{R}$ be an equivalence relation on A. Then $\mathbf{R}$ is called a *strong regular relation* on A if and only if, for all $x, y, z \in A$,

(i) if $x\mathbf{R}y$, then $x \odot z\overline{\overline{\mathbf{R}}}y \odot z$,

(ii) if $x\mathbf{R}y$, then $x \to z\overline{\overline{\mathbf{R}}}y \to z$ and $z \to x\overline{\overline{\mathbf{R}}}z \to y$.

Example 3.10. Let $A = \{1, a, b, c\}$. Define the hyperoperations $\odot, \to$ on A as follows:

$\odot$	1	a	b	c
1	$\{1\}$	$\{a, b\}$	$\{b\}$	$\{1, c\}$
a	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$
b	$\{b\}$	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$
c	$\{1, c\}$	$\{a, b\}$	$\{a, b\}$	$\{1, c\}$

$\to$	1	a	b	c
1	$\{1, c\}$	$\{a\}$	$\{a, b\}$	$\{c\}$
a	$\{1, c\}$	$\{1, c\}$	$\{1, c\}$	$\{1, c\}$
b	$\{1, c\}$	$\{c\}$	$\{1, c\}$	$\{1, c\}$
c	$\{1, c\}$	$\{a, b\}$	$\{a, b\}$	$\{1, c\}$

Then $(A, \odot, \to, 1)$ is a hyper hoop. Let

$$\mathbf{R} = \{(a, a), (b, b), (c, c), (1, 1), (1, c), (c, 1), (a, b), (b, a)\}.$$

Then $\mathbf{R}$ is a strong regular relation on A.

Proposition 3.11. *If $\boldsymbol{R}$ is a strong regular relation on A, then $\boldsymbol{R}$ is a regular relation on A.*

Proof. Let $\mathbf{R}$ be a strong regular relation on A. Clearly the properties (i) and (ii), of Definition 3.2 are satisfied. If $x \to y\mathbf{R}\{1\}$ and $y \to x\mathbf{R}\{1\}$ then by the condition (ii) of Definition 3.9 we have, $x \odot (x \to y)\overline{\overline{\mathbf{R}}}x \odot 1$ and $y \odot (y \to x)\overline{\overline{\mathbf{R}}}y \odot 1$. By (HHA1)and (HHA3), we get $x\mathbf{R}y$ and so the condition (iii) of Definition 3.2 holds. Hence $\mathbf{R}$ is a regular relation on A. ∎

Example 3.12. In Example 3.3, $\mathbf{R}$ is a regular relation but is not a strong regular relation. Hence every regular relation may not be a strong regular relation.

In the following, by definition of fundamental relation on hyper hoops, we get an example of strong regular relations.

Let $(H, \{*_i\}_{i \in I})$ be an algebraic hyperstructure, where $\{*_i\}_{i \in I}$ are hyperoperations. Consider U be the set of all finite combinations of elements of A with

respect to $*_i$, where $*_i \in \{*_i\}_{i\in I}$. We define the relation β on A by $a\beta b$ if and only if there exists $u \in U$ such that $\{a, b\} \subseteq u$. Let β^* be the transitive closure of β, that is $a\beta^* b$ if and only if there exist $z_1, ..., z_{m+1} \in A$ and $u_i \in U$ such that $z_1 = a$, $z_{m+1} = b$ and $\{z_i, z_{i+1}\} \subseteq u_i$, for $i = 1, ..., m$. Then the relation β^* is an equivalence relation on A and is called the fundamental relation on A (See [10]).

Now, in the following, the well-known idea of β^* relation on hyperstructures that is mentioned above is applied to hyper hoops.

Let $U(A)$ denote the set of all finite combinations of elements A with $\odot$ and $\rightarrow$. Then, for all $a, b \in A$, we have $a\beta b$ if and only if $\{a, b\} \subseteq u$, where $u \in U(A)$, and $a\beta^* b$ if and only if there exist $z_1, ..., z_{m+1} \in A$ with $z_1 = a, z_{m+1} = b$ such that $\{z_i, z_{i+1}\} \subseteq u_i \subseteq U(A)$, for $i = 1, ..., m$. (In fact β^* is the transitive closure of the relation β and so is an equivalence relation on A).

Example 3.13. β^* is a strong regular relation on A. Since, if $a\beta^* b$, for $a, b \in A$, then there exist $x_1, ..., x_{n+1} \in A$ with $x_1 = a, x_{n+1} = b$ and $u_i \in U(A)$ such that $\{x_i, x_{i+1}\} \subseteq u_i$ for $1 \leq i \leq n$. Let $z_i \in x_i \rightarrow c$, for all $1 \leq i \leq n+1, c \in A$. Then we have,

$$\{z_i, z_{i+1}\} \subseteq (x_i \rightarrow c) \cup (x_{i+1} \rightarrow c) \subseteq u_i \rightarrow c \subseteq U(A), \text{ for all } 1 \leq i \leq n.$$

Hence $z_1 \beta^* z_{n+1}$, where $z_1 \in a \rightarrow c$ and $z_{n+1} \in b \rightarrow c$ and so $a \rightarrow c\overline{\overline{\beta^*}}b \rightarrow c$. Similarly, we can show that $c \rightarrow a\overline{\overline{\beta^*}}c \rightarrow b$. Now, by the same way we can prove that $a\beta^* b$ implies $a \odot c\overline{\overline{\beta^*}}b \odot c$, for all $c \in A$. Hence β^* is a strong regular relation on A.

4. Quotient hyper hoops

In this section we construct the quotient hyper hoops by using the concept of regular relations and the quotient hoops by using the concept of strong regular relations and investigate some properties of them.

Lemma 4.1. *Let $\boldsymbol{R}$ be a regular relation on A. If $x \rightarrow y\boldsymbol{R}\{1\}$ and $y \rightarrow z\boldsymbol{R}\{1\}$, then $x \rightarrow z\boldsymbol{R}\{1\}$, for any $x, y, z \in A$.*

Proof. Let $x \rightarrow y\mathbf{R}\{1\}$ and $y \rightarrow z\mathbf{R}\{1\}$. Then it is easy to see that $(x \rightarrow y) \odot (y \rightarrow z)\overline{\mathbf{R}}\{1\} \odot \{1\} = \{1\}$ and so for all $u \in (x \rightarrow y) \odot (y \rightarrow z)$, we have $u\mathbf{R}1$. By (HHA9) and (HHA10), $(x \rightarrow y) \odot (y \rightarrow z) \ll x \rightarrow z$ and so there exists $v \in (x \rightarrow y) \odot (y \rightarrow z)$ and $r \in x \rightarrow z$, such that $v \leq r$. By considering $u = v$, we get $u \leq r$. Since $u \leq r$, we get $1 \in u \rightarrow r$ and so $u \rightarrow r\mathbf{R}\{1\}$. Since $u\mathbf{R}1$, we have $r \rightarrow u\overline{\mathbf{R}}r \rightarrow 1$ and since by (HHA5), $1 \in r \rightarrow 1$, we get $r \rightarrow u\mathbf{R}\{1\}$. Hence $r\mathbf{R}u$. Now, since $u\mathbf{R}1$, we have $r\mathbf{R}1$ and so $x \rightarrow z\mathbf{R}\{1\}$. ∎

Theorem 4.2. *Let $\boldsymbol{R}$ be a regular relation on A and $\frac{A}{\boldsymbol{R}}$ be the set of all equivalence classes respect to $\boldsymbol{R}$, that is $\frac{A}{\boldsymbol{R}} = \{[x] |\ x \in A\}$. Then $(\frac{A}{\boldsymbol{R}}, \otimes, \hookrightarrow, [1])$ is a hyper hoop, which is called the quotient hyper hoop of A respect to $\boldsymbol{R}$, where for all $[x], [y] \in \frac{A}{\boldsymbol{R}}$,*

$$[x] \otimes [y] = \{[t] |\ t \in x \odot y\} \quad \text{and} \quad [x] \hookrightarrow [y] = \{[z] |\ z \in x \rightarrow y\}$$

Proof. First, we show that $\otimes$ and $\hookrightarrow$ are well-defined on $\frac{A}{\mathbf{R}}$. Let $[x_1] = [x_2]$ and $[y_1] = [y_2]$. Then $x_1\mathbf{R}x_2$ and $y_1\mathbf{R}y_2$. Since $\mathbf{R}$ is a regular relation on A, it follows that $x_1 \odot y_1\overline{\mathbf{R}}x_2 \odot y_1$ and $x_2 \odot y_1\overline{\mathbf{R}}x_2 \odot y_2$ and so $x_1 \odot y_1\overline{\mathbf{R}}x_2 \odot y_2$. Let $[t] \in [x_1] \otimes [y_1]$. Then $[t] = [s]$ for some $s \in x_1 \odot y_1$. From $x_1 \odot y_1\overline{\mathbf{R}}x_2 \odot y_2$, we get that $s\mathbf{R}u$ for some $u \in x_2 \odot y_2$. Thus $[s] = [u]$ and then $[t] = [u]$. Hence $[x_1] \otimes [y_1] \subseteq [x_2] \otimes [y_2]$. Similarly, we have $[x_2] \otimes [y_2] \subseteq [x_1] \otimes [y_1]$. Therefore, $\otimes$ is well-defined. By the same way we can prove that $\hookrightarrow$ is well-defined on $\frac{A}{\mathbf{R}}$. Now, we show that $(\frac{A}{\mathbf{R}}, \otimes, \hookrightarrow, [1])$ is a hyper hoop.

(HHA1): First we check the associativity of $\otimes$. Let $[x], [y]$ and $[z]$ be arbitrary elements in $\frac{A}{\mathbf{R}}$ and $[u] \in ([x] \otimes [y]) \otimes [z]$. Then there exists $[v] \in [x] \otimes [y]$ such that$[u] \in [v] \otimes [z]$. In the other words, there exist $v_1 \in x \odot y$ and $u_1 \in v \odot z$, such that $v\mathbf{R}v_1$ and $u\mathbf{R}u_1$. Since $\mathbf{R}$ is regular and A is a hyper hoop, there exists $u_3 \in v_1 \odot z \subseteq (x \odot y) \odot z = x \odot (y \odot z)$ such that $u_1 R u_3$. Then there exists $u_4 \in y \odot z$ such that $u_3 \in x \odot u_4$. Hence $[u] = [u_1] = [u_3] \in [x] \otimes [u_4] \subseteq [x] \otimes ([y] \otimes [z])$ and so $([x] \otimes [y]) \otimes [z] \subseteq [x] \otimes ([y] \otimes [z])$. Similarly, we can prove that $[x] \otimes ([y] \otimes [z]) \subseteq ([x] \otimes [y]) \otimes [z]$. Then $\otimes$ is associative. Now, since $\odot$ is commutative, then $\otimes$ is commutative. Since by (HHA1), $x \in 1 \odot x$, for all $x \in A$, then $[x] \in [x] \otimes [1]$, for all $x \in A$. Hence $[1]$ is a unit of $(\frac{A}{\mathbf{R}}, \otimes)$. Therefore $(\frac{A}{\mathbf{R}}, \otimes)$ is a semihypergroup with $[1]$ as unit.

(HHA2): Since for all $x \in A$, $1 \in x \to x$, we get $[1] \in [x] \hookrightarrow [x]$.

(HHA3): Let $[x], [y], [z]$ be arbitrary elements in $\frac{A}{\mathbf{R}}$ and $[u] \in ([x] \otimes [y]) \hookrightarrow [z]$. This means that there exists $[v] \in [x] \otimes [y]$ such that$[u] \in [v] \hookrightarrow [z]$. Then, there exist $v_1 \in x \odot y$ and $u_1 \in v \to z$, such that $v\mathbf{R}v_1$ and $u\mathbf{R}u_1$. Since $\mathbf{R}$ is regular and A is a hyper hoop, it follows that there exists $u_3 \in v_1 \to z \subseteq (x \odot y) \to z = x \to (y \to z)$ such that $u_1\mathbf{R}u_3$. Then there exists $u_4 \in y \to z$ such that $u_3 \in x \to u_4$. Hence $[u] = [u_1] = [u_3] \in [x] \hookrightarrow [u_4] \subseteq [x] \hookrightarrow ([y] \hookrightarrow [z])$. Then $([x] \otimes [y]) \hookrightarrow [z] \subseteq [x] \hookrightarrow ([y] \hookrightarrow [z])$. Similarly, we can prove that $[x] \hookrightarrow ([y] \hookrightarrow [z]) \subseteq ([x] \otimes [y]) \hookrightarrow [z]$.

(HHA4): Let $[x]$ and $[y]$ be arbitrary elements in $\frac{A}{\mathbf{R}}$ and $[u] \in ([x] \hookrightarrow [y]) \otimes [x]$. This means that there exists $[v] \in [x] \hookrightarrow [y]$ such that $[u] \in [v] \otimes [x]$. Then, there exist $v_1 \in x \to y$ and $u_1 \in v \odot x$, such that $v\mathbf{R}v_1$ and $u\mathbf{R}u_1$. Since $\mathbf{R}$ is regular and A is a hyper hoop, it follows that there exists $u_3 \in v_1 \odot x \subseteq (x \to y) \odot x = (y \to x) \odot y$ such that $u_1\mathbf{R}u_3$. Then there exists $u_4 \in y \to x$ such that $u_3 \in u_4 \odot y$. Hence $[u] = [u_1] = [u_3] \in [u_4] \otimes [y] \subseteq ([y] \hookrightarrow [x]) \otimes [y]$. Then $([x] \hookrightarrow [y]) \otimes [x] \subseteq ([y] \hookrightarrow [x]) \otimes [y]$. Similarly, we can prove the converse and conclude that $([x] \hookrightarrow [y]) \otimes [x] = ([y] \hookrightarrow [x]) \otimes [y]$

(HHA5): Since for all $x \in A$, $1 \in x \to 1$, we get $[1] \in [x] \hookrightarrow [1]$.

(HHA6): Let $[x]$ and $[y]$ be arbitrary elements in $\frac{A}{\mathbf{R}}$, $[1] \in [x] \hookrightarrow [y]$ and $[1] \in [y] \hookrightarrow [x]$. Then there exist $u \in x \to y$ and $v \in y \to x$ such that $[u] = [v] = [1]$. Since $\mathbf{R}$ is a regular relation on A, we get $x\mathbf{R}y$ and so $[x] = [y]$.

(HHA7): Let $[x], [y], [z]$ be arbitrary elements in $\frac{A}{\mathbf{R}}$, $[1] \in [x] \hookrightarrow [y]$ and $[1] \in [y] \hookrightarrow [z]$. Then there exist $u \in x \to y$ and $v \in y \to z$ such that $[u] = [v] = [1]$. Since $\mathbf{R}$ is a regular relation on A, by Lemma 4.1, $x \to z\mathbf{R}\{1\}$ and so there exists $t \in x \to z$ such that $[t] = [1]$. So $[1] \in [x] \hookrightarrow [z]$.

Therefore, $(\frac{A}{\mathbf{R}}, \otimes, \hookrightarrow, [1])$ is a hyper hoop. ∎

Example 4.3. Let $(A, \otimes, \hookrightarrow, 1)$ be hyper hoop as in Example 3.3. Then $[1] = [b] = \{1, b\}$ and $[a] = [c] = \{a, c\}$. Hence $\frac{A}{\mathbf{R}} = \{[1], [a]\}$ and hyperoperations $\otimes, \hookrightarrow$ are as following tables,

$\otimes$	[1]	[a]
[1]	$\{[1]\}$	$\{[a], [1]\}$
$[a]$	$\{[a], [1]\}$	$\{[1]$,[a]$\}$

$\hookrightarrow$	[1]	[a]
[1]	$\{[a], [1]\}$	$\{[a]\}$
$[a]$	$\{[1], [a]\}$	$\{[a], [1]\}$

It is clear that $(\frac{A}{\mathbf{R}}, \otimes, \hookrightarrow, [1])$ is a hyper hoop.

Theorem 4.4. *Let $\boldsymbol{R}$ be an equivalence relation on A. If $(\frac{A}{\boldsymbol{R}}, \otimes, \hookrightarrow, [1])$ is a hyper hoop, then $\boldsymbol{R}$ is a regular relation.*

Proof. Let $\mathbf{R}$ be an equivalence relation on A, $(\frac{A}{\mathbf{R}}, \otimes, \hookrightarrow, [1])$ be a hyper hoop, and $a\mathbf{R}b$, for $a, b \in A$. Then for any $x \in A$ and $u \in a \odot x$, we have $[u] \in [a] \otimes [x] = [b] \otimes [x] = \{[v] | v \in b \odot x\}$ and so, there exists $v \in b \odot x$ such that $u\mathbf{R}v$. Thus $(a \odot x)\bar{\mathbf{R}}(b \odot x)$. Similarly, if $a\mathbf{R}b$, then $(a \to x)\bar{\mathbf{R}}(b \to x)$. Now, let $x \to y\mathbf{R}\{1\}$ and $y \to x\mathbf{R}\{1\}$, for $x, y \in A$. Then $[1] \in [x] \hookrightarrow [y]$ and $[1] \in [y] \hookrightarrow [x]$. Since $(\frac{A}{\mathbf{R}}, \otimes, \hookrightarrow, [1])$ is a hyper hoop by (HHA6), we get $[x] = [y]$ and so $x\mathbf{R}y$. Hence $\mathbf{R}$ is a regular relation. ∎

Corollary 4.5. *Let $\boldsymbol{R}$ be an equivalence relation on A. Then $\boldsymbol{R}$ is a regular relation on A if and only if $(\frac{A}{\boldsymbol{R}}, \otimes, \hookrightarrow, [1])$ is a hyper hoop.*

Proof. By Theorems 4.2 and 4.4, the proof is clear. ∎

Theorem 4.6. *Let $\boldsymbol{R}$ be a strong regular relation on A. Then $(\frac{A}{\boldsymbol{R}}, \otimes, \hookrightarrow, [1])$ is a hoop which is called the quotient hoop of A respect to $\boldsymbol{R}$.*

Proof. Similar to the proof of Theorem 4.2, we can prove that the hyperoperations $\otimes$ and $\hookrightarrow$ are well-defined on $\frac{A}{\mathbf{R}}$ and axioms (HHA1), (HHA2)and (HHA3) hold on $(\frac{A}{\mathbf{R}}, \otimes, \hookrightarrow, [1])$. Hence $(\frac{A}{\mathbf{R}}, \otimes, \hookrightarrow, [1])$ is a quasi hyper hoop. Now, for any $x, y \in A$, since $x\mathbf{R}x$ and $\mathbf{R}$ is a strong regular relation on A, we get $(x \odot y)\overline{\overline{\mathbf{R}}}(x \odot y)$ and $(x \to y)\overline{\overline{\mathbf{R}}}(x \to y)$. Hence for any $z, z' \in x \odot y$ and $w, w' \in x \to y$, $z\mathbf{R}z'$ and $w\mathbf{R}w'$. Then $[x] \otimes [y] = \{[z] | z \in x \odot y\} = \{[z]\}$ and $[x] \hookrightarrow [y] = \{[w] | w \in x \to y\} = \{[w]\}$, which means that $[x] \otimes [y]$ and $[x] \hookrightarrow [y]$ are singletons, for any $[x], [y] \in \frac{A}{\mathbf{R}}$. Therefore, By Proposition 2.3, $(\frac{A}{\mathbf{R}}, \otimes, \hookrightarrow, [1])$ is a hoop. ∎

Example 4.7. Let A be hyper hoop as in Example 3.10. Then $[a] = [b] = \{a, b\}$ and $[c] = [1] = \{1, c\}$. Hence $\frac{A}{\mathbf{R}} = \{[1], [a]\}$ and

$\otimes$	[1]	$[a]$
[1]	$\{[1]\}$	$\{[a]\}$
$[a]$	$\{[a]\}$	$\{[a]\}$

$\hookrightarrow$	[1]	[a]
[1]	$\{[1]\}$	$\{[a]\}$
$[a]$	$\{[1]\}$	$\{[1]\}$

It is clear that $(A, \otimes, \hookrightarrow, [1])$ is a hoop.

Theorem 4.8. *Let $\boldsymbol{R}$ be an equivalence relation on A. If $(\frac{A}{\boldsymbol{R}}, \otimes, \hookrightarrow, [1])$ is a hoop, then $\boldsymbol{R}$ is a strong regular relation on A.*

Proof. Let $\mathbf{R}$ be an equivalence relation on A, $(\frac{A}{\mathbf{R}}, \otimes, \hookrightarrow, [1])$ be a hoop , $a\mathbf{R}b$, for $a, b \in A$, and $x \in A$. Then for any $u \in a \odot x$ and $v \in b \odot x$, we have $[u] = [a] \otimes [x] = [b] \otimes [x] = [v]$, which means that $u\mathbf{R}v$. Hence, $(a \odot x)\overline{\overline{\mathbf{R}}}(b \odot x)$. Similarly, if $a\mathbf{R}b$ then for any $x \in A$, $(a \to x)\overline{\overline{R}}(b \to x)$ and $(x \to a)\overline{\overline{R}}(x \to b)$. Hence $\mathbf{R}$ is a strong regular relation on A. ∎

Corollary 4.9. *Let $\boldsymbol{R}$ be an equivalence relation on A. Then $\boldsymbol{R}$ is a strong regular relation if and only if $(\frac{A}{\boldsymbol{R}}, \otimes, \hookrightarrow, [1])$ is a hoop.*

Proof. By Theorems 4.6 and 4.8, the proof is clear. ∎

Lemma 4.10. *Let $f : A_1 \to A_2$ be a homomorphism of hyper hoops. Then,*

(i) *if $x \leq y$, for all $x, y \in A_1$ then $f(x) \leq f(y)$,*

(ii) *if F is a (weak) filter of A_2, then $f^{-1}(F)$ is a (weak) filter of A_1,*

(iii) *ker(f) $= \{x \in A_1 | f(x) = 1\}$ is a filter of A_1,*

(iv) *f is one to one if and only if $ker(f) = \{1\}$,*

(v) *if f is onto and F is a filter of A_1 which contains ker(f), then f(F) is a filter of A_2.*

Proof. (i). The proof is easy.

(ii) Let $x \leq y$ and $x \in f^{-1}(F)$. Then there exists $u \in F$ such that $f(x) = u$. Since $x \leq y$, by (i), $u = f(x) \leq f(y)$. Since F is a filter and $u \in F$, we get $f(y) \in F$ and so $y \in f^{-1}(F)$. Then $f^{-1}(F)$ is an upset. Let $x, y \in f^{-1}(F)$. Then $f(x) \in F$ and $f(y) \in F$. Hence $f(x \odot y) = f(x) \odot f(y) \subseteq F$ and so $x \odot y \subseteq f^{-1}(F)$. Then $f^{-1}(F)$ is a filter. If F is a weak filter, then $f(x \odot y) = f(x) \odot f(y) \cap F \neq \emptyset$. Hence $x \odot y \cap f^{-1}(F) \neq \emptyset$ and then $f^{-1}(F)$ is a weak filter, too.

(iii) Let $x \leq y$ and $x \in ker(f)$. Then by (i), $f(x) \leq f(y)$. Since $f(x) = 1$, we get $f(y) = 1$ and so $y \in ker(f)$. Hence $ker(f)$ is an upset. Let $x, y \in ker(F)$. Then by (HHA12), $f(x \odot y) = f(x) \odot f(y) = 1 \odot 1 = 1$. Hence $x \odot y \subseteq F$ and so F is a filter of A_1.

(iv) The proof is clear.

(v) Since $1 \in F$, we get $1 = f(1) \in f(F)$. Let $x, y \in A_2$ such that $f(F) \ll x \to y$ and $x \in f(F)$. Since $x \in f(F)$ and f is onto, there are $a \in F$ and $b \in A_1$ such that $y = f(b)$ and $x = f(a)$. Hence $f(a \to b) = f(a) \to f(b) = x \to y \gg f(F)$. Then there are $u \in a \to b$ and $v \in F$ such that $f(v) \leq f(u)$ and so $1 \in f(v) \to f(u) = f(v \to u)$. Then there exists $t \in v \to u$, such that $f(t) = 1$ and so $t \in ker(f) \subseteq F$. Hence $F \ll v \to u$. Since F is a filter of A_1 and $v \in F$, by Theorem 3.7, we get $u \in F$. Then $F \ll a \to b$ and so by Theorem 3.7, $b \in F$. Hence $y = f(b) \in f(F)$ and so, by Theorem 3.7 $f(F)$ is a filter of A_2. ∎

Let $\mathbf{R}$ be a regular relation on A. We define the relation $\ll$ on $\frac{A}{\mathbf{R}}$ by $[x] \ll [y]$ if and only if $[1] \in [x] \hookrightarrow [y]$.

Theorem 4.11. *(First isomorphism theorem) Let A_1 and A_2 be two hyper hoops and $\boldsymbol{R}$ be a regular relation on A_1. If $f : A_1 \to A_2$ is a homomorphism of hyper hoops and $ker(f) = [1]$, then $\frac{A_1}{\boldsymbol{R}} \cong Im(f)$.*

Proof. Since R is a regular relation on A_1, $\frac{A_1}{\mathbf{R}}$ is well-defined. Now, let $\overline{f} : \frac{A1}{\mathbf{R}} \to A_2$ is defined by $\overline{f}[x] = f(x)$, for all $x \in A_1$. If $[x] = [y]$, for $x, y \in A_1$ then $x\mathbf{R}y$ and so $x \to y\overline{\mathbf{R}}y \to y$. Since by (HHA2), $1 \in y \to y$, we get $x \to y\mathbf{R}\{1\}$ and so there exists $a \in x \to y$ such that $a \in [1] = ker(f)$. Then $1 = f(a) \in f(x \to y) = f(x) \to f(y)$ and so $f(x) \leq f(y)$. By the same way, we can show that $f(y) \leq f(x)$. Hence by (HHA6), $f(x) = f(y)$ and so $\overline{f}$ is well-defined. Now, we show that $\overline{f}$ is a homomorphism. Let $[x], [y] \in \frac{A_1}{\mathbf{R}}$. Then,

$$\begin{aligned}
\overline{f}([x] \hookrightarrow [y]) &= \{\overline{f}[z] \mid z \in x \to y\} = \{f(z) \mid z \in x \to y\} = f(x \to y) \\
&= f(x) \to f(y) = \overline{f}([x]) \to \overline{f}([y]) \\
\overline{f}([x] \otimes [y]) &= \{\overline{f}[z] \mid z \in x \odot y\} = \{f(z) \mid z \in x \odot y\} = f(x \odot y) \\
&= f(x) \odot f(y) = \overline{f}([x]) \odot \overline{f}([y])
\end{aligned}$$

Moreover, $\overline{f}([1]) = f(1) = 1$. Then $\overline{f}$ is a homomorphism. Now, we show that $\overline{f}$ is one to one. Suppose that $[x] \in ker(\overline{f})$. Then $f(x) = \overline{f([x])} = 1$ and so $x \in kerf = [1]$ which implies that $x\mathbf{R}1$. i.e. $[x] = [1]$. Hence $ker\overline{f} = \{[1]\}$ and so by Theorem 4.10 (iv), $\overline{f}$ is one to one. Clearly $\frac{A1}{\mathbf{R}} \to Im(f)$ is onto. Therefore, $\frac{A_1}{\mathbf{R}} \cong Im(f)$. ∎

Example 4.12. If $f : A_1 \to A_2$ is a homomorphism of hyper hoops and ρ is regular relation as Example 3.6, then $\frac{A_1}{\rho} \cong Im(f)$.

Notation. In the follows, the set of all filters of A is denoted by $\mathcal{F}(A)$ and the set of all filters of A containing S is denoted by $\mathcal{F}(A, S)$, where S is a non-empty subset of A.

Theorem 4.13. *Let $\boldsymbol{R}$ be a regular relation on A and $F = [1]$. Then there exists a one to one corresponding between $\mathcal{F}(A, F)$ and $\mathcal{F}(\frac{A}{R})$.*

Proof. Consider the canonical homomorphism $\pi : A \to \frac{A}{\mathbf{R}}$. If J is a filter of A containing F, then by Proposition 4.10(ii) and (v), the function $g : \mathcal{F}(A, F) \to \mathcal{F}(\frac{A}{R})$ by $J \to \pi(J)$ is onto. Now, we show that g is one to one. Let $G, H \in \mathcal{F}(A, F)$ and $g(G) = g(H)$. Then for $x \in H$, $[x] \in \pi(H) = \pi(G)$. Hence there exists $y \in G$ such that $[x] = [y]$ and so $x\mathbf{R}y$. Since $\mathbf{R}$ is a regular relation on A, we have $y \to x\overline{\mathbf{R}}y \to y$ and since $1 \in y \to y$ we get $y \to x\mathbf{R}\{1\}$. Then there exists $t \in y \to x$ such that $t\mathbf{R}1$, i.e. $t \in F$. Now, since $F \ll y \to x$ and $F \subseteq G$, we have $G \ll y \to x$ and so by Lemma 3.7 $x \in G$. Hence $H \subseteq G$. Similarly, we can show that $G \subseteq H$. Therefore $G = H$ and so g is one to one. ∎

Definition 4.14. Let F be a proper (weak) filter of A. Then F is called a *(weak) maximal filter* of A, if $F \subseteq J \subseteq A$ for some (weak)filter of A, then $F = J$ or $J = A$.

Proposition 4.15. *Every proper filter of A is contained in a maximal filter of A.*

Proof. Let F be a proper filter of A and S be the collection of all proper filters J of A such that $F \subseteq J$. Then $F \in S$ and $S \neq \emptyset$. S is a partially ordered set by $\subseteq$. Let $C = \{J_\alpha | \alpha \in \Gamma\}$ be a chain in S. We can show that $J = \bigcup_{\alpha\in\Gamma} J_\alpha$ is a filter of A. It is clear that $F \subseteq J$. If $J = A$ then $0 \in J$. Thus there exists $\beta \in \Gamma$ such that $0 \in J_\beta$. Since J_β is an upset, we get $J_\beta = A$ and this is a contradiction. Therefore $J \neq A$ and then $J \in C$. Now, the proof follows by Zorn's Lemma. ∎

Proposition 4.16. *Let $f : A_1 \to A_2$ be an epimorphism of hyper hoops. Then,*

(i) *if F is a maximal filter of A_1 which contains $kerf$, then $f(F)$ is a maximal filter of A_2,*

(ii) *if F is a maximal filter of A_2, then $f^{-1}(F)$ is a maximal filter of A_1 which contains $kerf$,*

(iii) *the map $f : F \to f(F)$ is one to one corresponding between the maximal filters of A_1 containing $kerf$ and maximal filters of A_2.*

Proof. (i) Let $f(F) = A_2$. Since $F \neq A_1$, there exists $x \in A_1$ such that $x \notin F$. So $f(x) \in A_2 = f(F)$. Thus there exists some $a \in F$ such that $f(x) = f(a)$ and then $1 = f(1) \in f(a) \to f(x) = f(a \to x)$. So there exists $t \in a \to x$ such that $f(t) = 1$, i.e., $t \in kerf \subseteq F$. Then $F \ll a \to x$. Thus by Theorem 3.7, $x \in F$ which is a contradiction. Hence $f(F) \neq A_2$. Since F is a filter of A_1 which contains $kerf$, then $f(F)$ is a filter of A_2 by Theorem 4.10 (v). Let J be a filter of A_2 such that $f(F) \subseteq J \subseteq A_2$. By Lemma 4.10 (ii), $f^{-1}(J)$ is a filter of A_1 such that $F \subseteq f^{-1}(J) \subseteq f^{-1}(A_2) = A_1$. Since F is a maximal filter of A_1, we get $f^{-1}(J) = F$ or $f^{-1}(J) = A_1$. Hence $J = f(F)$ or $J = f(A_1) = A_2$, i.e., $f(F)$ is a maximal filter of A_2.
(ii) Let $f^{-1}(F) = A_1$. Since $F \neq A_2$, there exists $y \in A_2$ such that $y \notin F$. So $f^{-1}(y) \in A_1 = f^{-1}(F)$. Thus there exists some $x \in A_1$ such that $y = f(x) \in F$ which is a contradiction. Hence $f^{-1}(F) \neq A_1$. Similarly to the part(i), we can show that $f^{-1}(F)$ is a maximal filter of A_1.

(iii) The proof is straightforward by (i) and (ii). ∎

Definition 4.17. The hyper hoop A is called *simple*, if it has only two filters $\{1\}$ and A.

Example 4.18. Let $A = \{1, a, b, c\}$. Define the hyperoperations $\odot$ and $\to$ on A as follows,

$\odot$	1	a	b	c
1	$\{1\}$	$\{1, a\}$	$\{1, b\}$	$\{1, a, b, c\}$
a	$\{1, a\}$	$\{1, a, b, c\}$	$\{1, a, b\}$	$\{1, a, b, c\}$
b	$\{1, b\}$	$\{1, a, b\}$	$\{1, a, b, c\}$	$\{1, a, b, c\}$
c	$\{1, a, b, c\}$	$\{1, a, b, c\}$	$\{1, a, b, c\}$	$\{1, a, b, c\}$

$\rightarrow$	1	a	b	c
1	$\{1,a,b,c\}$	$\{a,c\}$	$\{b,c\}$	$\{c\}$
a	$\{1,a,b,c\}$	$\{1,a,b,c\}$	$\{a,b,c\}$	$\{a,c\}$
b	$\{1,a,b,c\}$	$\{a,b,c\}$	$\{1,a,b,c\}$	$\{b,c\}$
c	$\{1,a,b,c\}$	$\{1,a,b,c\}$	$\{1,a,b,c\}$	$\{1,a,b,c\}$

Then $(A,\odot,\rightarrow,1)$ is a simple hyper hoop.

Remark 4.19. The hyper hoop A is simple if and only if $\{1\}$ is a maximal filter of A .

Theorem 4.20. *Let $\boldsymbol{R}$ be a regular relation on A and $F=[1]$. Then $\frac{A}{\boldsymbol{R}}$ is simple if and only if F is a maximal filter of A.*

Proof. Let $\frac{A}{\mathbf{R}}$ be simple. Then by Remark 4.19, $\{[1]\}$ is a maximal filter of $\frac{A}{\mathbf{R}}$ and so $\{[1]\}\neq\frac{A}{\mathbf{R}}$. Then there exists $[x]\in\frac{A}{\mathbf{R}}$ such that $[x]\neq[1]$ and so $x\notin F$. Hence $F\neq A$. Suppose that F is not a maximal filter of A. By Proposition 4.15, there exists a maximal filter J of A such that $F\subset J$. Consider canonical epimorphism $\varphi:A\rightarrow\frac{A}{\mathbf{R}}$ in which $ker(\varphi)=F$. Using Proposition 4.16(i), $\frac{J}{\mathbf{R}}$ is a maximal filter of $\frac{A}{\mathbf{R}}$. Also, we have $\{[1]\}=\{[x]|x\in F\}\subseteq\{[x]|x\in J\}=\frac{J}{\mathbf{R}}$. So $\{[1]\}=\frac{J}{\mathbf{R}}$. Then for all $x\in J$, we have $x\mathbf{R}1$, i.e., $x\in F$. Hence $J\subseteq F$ which is a contradiction.

Conversely, let F be a maximal filter of A. By contrary, suppose that $\frac{A}{\mathbf{R}}$ is not simple. Then by Remark 4.19, $\{[1]\}$ is not a maximal filter of $\frac{A}{\mathbf{R}}$. By Proposition 4.15, there exists a maximal filter K of $\frac{A}{\mathbf{R}}$ such that $\{[1]\}\subseteq K$. By Proposition 4.16(ii), $\varphi^{-1}(K)$ is a maximal filter of A such that $F\subseteq\varphi^{-1}(K)$. So $F=\varphi^{-1}(K)$. Hence $\{[1]\}=\varphi(F)=K$ which is a contradiction. ∎

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GENERALIZED CUBIC SOFT SETS AND THEIR APPLICATIONS TO ORDERED ABEL-GRASSMANN'S GROUPOIDS

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Abstract. In this paper, we introduced the concept of generalized cubic soft left (resp., right, bi-) ideals to study the structural properties of ordered $\mathcal{AG}$-groupoids. We characterized intra-regular ordered $\mathcal{AG}$-groupoids using the properties of generalized cubic soft ideals.

Keywords: ordered $\mathcal{AG}$-groupoids, cubic soft sets, left invertive, medial law, para-medial law and generalized cubic soft ideals.

Introduction

In order to deal with many complicated problems in the fields of engineering, economics, social science, medical science involving uncertainties, classical methods are found to be inadequate in recent times. Molodstov [15] pointed out that the important existing theories viz. probability theory, fuzzy set theory, intuitionistic fuzzy set theory, rough set theory etc, which can be considered as mathematical tools for dealing with uncertainties, have their own difficulties. He further pointed out that the reason for these difficulties, is possibly the inadequacy parametrization tool of the theory. Molodtsov in

1999 introduced the fundamental concept of a soft set which provides a natural framework for generalizing several basic notions of algebra. Many related concepts with soft sets, especially soft sets operations, have also undergone tremendous studies. Maji et al. [16] presented some definitions on soft sets and based on the analysis of several operations on soft sets Ali et al. [7] introduced several new algebraic operations on soft sets. Cagman and Enginoglu [3] developed the uni-int decision making method in virtue of soft sets Feng et al. [5] investigated soft semirings by using soft set theory. Aktas and Cagman [1] defined the notion of soft groups and derived some related properties. This initiated an important research direction concerning algebraic properties of soft sets in miscellaneous kinds of algebras such as BCK/BCI-algebras, d-algebras, semirings, rings, Lie algebras and K-algebras.Feng and Li [6] ascertained the relationship among five different types of soft subsets and considered the free soft algebras associated with soft product operations. It has been shown that soft sets have some nonclassical algebraic properties which are distinct from those of crisp sets or fuzzy sets. Zhan et al. [19], worked on soft ideals of BL-algebras. Atagun and Sezgin [2] defined the concept of soft subrings and ideals of a ring, soft subfields of a field and soft sub modules of a module and studied the related properties with respect soft set operations. Jun et al. [10] gave the concept of soft BCK/BCI-algebras. Further, Jun discussed the applications of soft sets in ideal theory of BCK/BCI-algebras and in d-algebras respectively. The theory of soft sets have many applications in different fields of learning such as medical sciences, engineering, economics, computer sciences, artificial intelligence and mathematics etc.

Recently, combining cubic sets and soft sets the first author with Al-roqi [13] introduced the notions of (external, internal) cubic soft sets, P-cubic (resp., R-cubic) soft subsets, R-union (resp., R-intersection, P-union and P-intersection) of cubic soft sets and the complement of a cubic soft set. They investigated several related properties and applied the notion of cubic soft sets to BCK/BCI-algebras. The theory of soft sets introduced in [9], attracted several mathematicians. This theory have been done by many researchers in algebra. In this paper, we applied the theory of cubic soft sets to the ideal theory of $\mathcal{AG}$-groupoid and introduced the concept of generalized cubic soft sets to study the structural properties of non-associative algebraic structure.

The idea of generalization of a commutative semigroup (which we call the left almost semigroup) was introduced by M. A Kazim and M. Naseeruddin in 1972 [11].

A groupoid $(\mathcal{G}, \cdot)$ is called an $\mathcal{AG}$-groupoid (Abel-Grassmann's Groupoid) if it satisfies the left invertive law that is $(ab)c = (cb)a$, $\forall\ a, b, c \in \mathcal{G}$. In an $\mathcal{AG}$-groupoid medial law $(ab)(cd) = (ac)(bd)$, holds $\forall\ a, b, c, d \in \mathcal{G}$ [11]. An $\mathcal{AG}$-groupoid $\mathcal{G}$ with left identity satisfies the paramedial law $(ab)(cd) = (db)(ca)$, $\forall\ a, b, c, d \in \mathcal{G}$. If an $\mathcal{AG}$-groupoid $\mathcal{G}$ which contains left identity. Then, $a(bc) = b(ac), \forall a, b, c \in \mathcal{G}$ holds.

In this paper, first we remind some basic definitions about ordered $\mathcal{AG}$-groupoids, cubic sets, $(\in_{(\widetilde{\gamma}_1,\gamma_2)}, \in_{(\widetilde{\gamma}_1,\gamma_2)} \vee q_{(\widetilde{\delta}_1,\delta_2)})$-cubic sets, $(\in_{(\widetilde{\gamma}_1,\gamma_2)}, \in_{(\widetilde{\gamma}_1,\gamma_2)} \vee q_{(\widetilde{\delta}_1,\delta_2)})$-cubic sets intersection, product and cubic characteristic function. Then, cubic soft sets, $(\in_{(\widetilde{\gamma}_1,\gamma_2)}, \in_{(\widetilde{\gamma}_1,\gamma_2)} \vee q_{(\widetilde{\delta}_1,\delta_2)})$-cubic soft sets $(\in_{(\widetilde{\gamma}_1,\gamma_2)}, \in_{(\widetilde{\gamma}_1,\gamma_2)} \vee q_{(\widetilde{\delta}_1,\delta_2)})$-cubic soft ordered $\mathcal{AG}$-groupoid, $(\in_{(\widetilde{\gamma}_1,\gamma_2)}, \in_{(\widetilde{\gamma}_1,\gamma_2)} \vee q_{(\widetilde{\delta}_1,\delta_2)})$-cubic soft left (resp., right, bi-) ideals in ordered $\mathcal{AG}$-groupoid are defined and studied with respect to cubic soft sets operations and obtained some interesting new results. Moreover, we give some characterizations of intra-regular ordered $\mathcal{AG}$-groupoids using the properties of $(\in_{(\widetilde{\gamma}_1,\gamma_2)}, \in_{(\widetilde{\gamma}_1,\gamma_2)} \vee q_{(\widetilde{\delta}_1,\delta_2)})$-cubic soft ideals.

Preliminaries notes

In this section, we define some basic definitions that are required for next sections.

An ordered $\mathcal{AG}$-groupoid ($\mathfrak{po}$-$\mathcal{AG}$-groupoid) is a structure $(\mathcal{G}, \cdot, \leq)$ in which the following conditions hold [12]:

(i) $(\mathcal{G}, \cdot)$ *is an* $\mathcal{AG}$*-groupoid.*

(ii) $(\mathcal{G}, \leq)$ *is a poset.*

(iii) $\forall\ a, b, x \in \mathcal{G},\ a \leq b \Rightarrow ax \leq bx\ (xa \leq xb)$.

Example 1 Let $\mathcal{G} = \{a, b, c\}$ be an ordered $\mathcal{AG}$-groupoid with the following multiplication table and two different orders below:

$\cdot$	a	b	c
a	a	a	a
b	a	a	c
c	a	a	a

$$\leq := \{(a,a),(b,b),(c,c),(c,a),(c,b)\} \tag{1}$$

$$\leq := \{(a,a),(b,b),(c,c),(a,c),(a,b)\} \tag{2}$$

An ordered $\mathcal{AG}$-groupoid is the generalization of an ordered semigroup. If an ordered $\mathcal{AG}$-groupoid has a right identity, then it becomes an ordered semigroup.

Let A be a non-empty subset of an ordered $\mathcal{AG}$-groupoid $\mathcal{G}$, then

$$(A] = \{t \in \mathcal{G} \mid t \leq a, \text{ for some } a \in A\}.$$

For $A = \{a\}$, we usually written as $(a]$.

Now, we recall the concept of interval valued fuzzy sets.

By an interval number, we mean a closed subinterval $\widetilde{x} = [x^-, x^+]$ of the closed interval $I = [0,1]$,where $0 \leqslant x^- \leqslant x^+ \leqslant 1$. Let $D[0,1]$ denote the family of all closed subintervals of $[0,1]$, i.e.,

$$D[0,1] = \{\widetilde{x} = [x^-, x^+] : x^- \leqslant x^+, \text{ for all } x^-, x^+ \in [0,1]\},$$

where the elements in $D[0,1]$ are called the interval numbers on $[0,1]$, $\widetilde{0} = [0,0]$ and $\widetilde{1} = [1,1]$. For any two elements in $D[0,1]$, the redefined minimum and redefined maximum, respectively, denoted by $r\min$ and $r\max$, and the symbols "$\preceq$","$\succeq$","$=$" are defined. We consider two elements $\widetilde{x} = [x^-, x^+]$ and $\widetilde{y} = [y^-, y^+]$ in $D[0,1]$. Then

$$\begin{aligned}
\widetilde{x} &\succeq \widetilde{y} \text{ if and only if } x^- \geq y^- \text{ and } x^+ \geq y^+ \\
\widetilde{x} &\preceq \widetilde{y} \text{ if and only if } x^- \leqslant y^- \text{ and } x^+ \leqslant y^+ \\
\widetilde{x} &= \widetilde{y} \text{ if and only if } x^- = y^- \text{ and } x^+ = y^+ \\
r\min\{\widetilde{x}, \widetilde{y}\} &= \min\{x^-, y^-\}, \min\{x^+, y^+\} \\
r\max\{\widetilde{x}, \widetilde{y}\} &= \max\{x^-, y^-\}, \max\{x^+, y^+\}.
\end{aligned}$$

Let $\widetilde{x}_i \in D[0,1]$, where $i \in \Lambda$. We define

$$r\inf_{i\in\Lambda}\widetilde{x}_i = \left[\inf x_i^-, \inf x_i^+\right]_{i\in\Lambda} \quad \text{and} \quad r\sup_{i\in\Lambda}\widetilde{x}_i = \left[\sup x_i^-, \sup x_i^+\right]_{i\in\Lambda}.$$

Let A be a non-empty set. A cubic set $\widetilde{\mu}_X$ on A is defined as

$$\widetilde{\mu}_X = \left\{\left\langle a, [\mu_X^-(a), \widetilde{\mu}_X^+(a)]\right\rangle : a \in A\right\},$$

where $\mu_X^-(a) \leq \mu_X^+(a)$, for all $a \in A$. Then the ordinary fuzzy sets $\mu_X^-(a) : A \to [0,1]$ and $\mu_X^+(a) : A \to [0,1]$ are called a lower fuzzy sets and upper fuzzy sets of $\widetilde{\mu}_X$ respectively. Let $\widetilde{\mu}_X(a) = [\mu_X^-(a), \widetilde{\mu}_X^+(a)]$ then

$$X = \{\langle a, \widetilde{\mu}_X(a)\rangle : a \in A\}, \text{ where } \widetilde{\mu}_X : A \to D[0,1].$$

For rest of the study, let us use the symbols $\lambda = (\widetilde{\gamma}_1, \gamma_2)$ and $\mu = (\widetilde{\delta}_1, \delta_2)$ to avoid symbols complications.

$(\in_\lambda, \in_\lambda \vee q_\mu)$-cubic sets

Jun et al. [8] introduced the concept of cubic sets defined on a non-empty set A as objects having the form

$$\Xi = \left\{\left\langle a, \widetilde{\Psi}_\Xi(a), \phi_\Xi(a)\right\rangle : a \in A\right\},$$

which is briefly denoted by $\Xi = \left\langle \widetilde{\Psi}_\Xi, \phi_\Xi\right\rangle$, where the function $\widetilde{\Psi}_\Xi : A \to D[0,1]$ and $\phi_\Xi : A \to [0,1]$.

Let $\Xi = \left\langle \widetilde{\Psi}_\Xi, \phi_\Xi\right\rangle$ and $\digamma = \left\langle \widetilde{\Psi}_\digamma, \phi_\digamma\right\rangle$ be two cubic sets of an ordered $\mathcal{AG}$-groupoid $\mathcal{G}$. Then

$$\begin{aligned}
\Xi \cap \digamma &= \left\{\left\langle a, r\min\left\{\widetilde{\Psi}_\Xi(a), \widetilde{\Psi}_\digamma(a)\right\}, \max\{\phi_\Xi(a), \phi_\digamma(a)\}\right\rangle : a \in \mathcal{G}\right\}, \\
\Xi \odot \digamma &= \left\{\left\langle \left(\widetilde{\Psi}_\Xi \circ \widetilde{\Psi}_\phi\right)(a), \left(\phi_\Xi \circ \phi_\digamma\right)(a)\right\rangle : a \in \mathcal{G}\right\},
\end{aligned}$$

where

$$\begin{aligned}
\left(\widetilde{\Psi}_\Xi \circ \widetilde{\Psi}_\digamma\right)(a) &= \begin{cases} r\sup\limits_{a\leq bc}\{r\min\left\{\widetilde{\Psi}_\Xi(b), \widetilde{\Psi}_\digamma(c)\right\} & \text{if } a \leq bc \\ [0,0] & \text{otherwise} \end{cases} \\
(\phi_\Xi \circ \phi_\digamma)(a) &= \begin{cases} r\min\limits_{a\leq bc}\{\max\{\phi_\Xi(b), \phi_\digamma(c)\}\} & \text{if } a \leq bc \\ 1 & \text{otherwise} \end{cases}
\end{aligned}$$

Let $\mathcal{C}(\mathcal{G})$ denote the family of all cubic sets in $\mathcal{G}$. Then, it becomes an ordered $\mathcal{AG}$-groupoid. Let $\widetilde{\alpha} \in D(0,1]$ and $\beta \in [0,1)$ be such that $\widetilde{0} \prec \widetilde{\alpha}$ and $\beta < 1$. Then by cubic point (CP) we mean $x_{(\widetilde{\alpha},\beta)}(y) = \langle x_{\widetilde{\alpha}}(y), x_\beta(y)\rangle$, where

$$x_{\widetilde{\alpha}}(y) = \begin{cases} \widetilde{\alpha} & \text{if } x \leq y \\ \widetilde{0} & \text{otherwise.} \end{cases}$$

$$x_\beta(y) = \begin{cases} \beta & \text{if } x \leq y \\ 1 & \text{otherwise.} \end{cases}$$

For any cubic set $\Xi = \left\langle \widetilde{\Psi}_\Xi, \phi_\Xi \right\rangle$ and for a cubic point $x_{(\widetilde{\alpha},\beta)}$, with the condition that $[\alpha,\beta]+[\alpha,\beta] = [2\alpha, 2\beta]$ such that $0 \leq 2\alpha < 1$ and $2\beta \leq 1$, we have

(i) $x_{(\widetilde{\alpha},\beta)} \in_\lambda \Xi$ if $\widetilde{\Psi}_\Xi(x) \succeq \widetilde{\alpha} \succ \widetilde{\gamma}_1$ and $\phi_\Xi(x) \leq \beta < \gamma_2$.

(ii) $x_{(\widetilde{\alpha},\beta)} q_\mu \Xi$ if $\widetilde{\Psi}_\Xi(x) + \widetilde{\alpha} \succ 2\widetilde{\delta}_1$ and $\phi_\Xi(x) + \beta < 2\delta_2$.

(iii) $x_{(\widetilde{\alpha},\beta)} \in_\lambda \vee q_\mu \Xi$ if $x_{(\widetilde{\alpha},\beta)} \in_\lambda \Xi$ or $x_{(\widetilde{\alpha},\beta)} q_\mu \Xi$.

(iv) $x_{(\widetilde{\alpha},\beta)} \in_\lambda \wedge q_\mu \Xi$ if $x_{(\widetilde{\alpha},\beta)} \in_\lambda \Xi$ and $x_{(\widetilde{\alpha},\beta)} q_\mu \Xi$.

Definition 1 Let $\mathcal{G}$ be an ordered $\mathcal{AG}$-groupoid. Then, the cubic characteristic function

$$\mathcal{X}^\mu_{\lambda\Sigma} = \left\langle \widetilde{\Psi}_{\mathcal{X}^\mu_{\lambda\Sigma}}, \phi_{\mathcal{X}^\mu_{\lambda\Sigma}} \right\rangle$$

of $\Sigma = \left\langle \widetilde{\Psi}_\Sigma, \phi_\Sigma \right\rangle$ is defined as

$$\widetilde{\Psi}_{\mathcal{X}^\mu_{\lambda\Sigma}}(a) \succeq \begin{cases} \widetilde{\delta}_1 = [1,1] & \text{if } a \in \Sigma \\ \widetilde{\gamma}_1 = [0,0] & \text{if } a \notin \Sigma \end{cases} \qquad \text{and} \qquad \phi_{\mathcal{X}^\mu_{\lambda\Sigma}}(a) \leq \begin{cases} \delta_2 = 0 & \text{if } a \in \Sigma \\ \gamma_2 = 1 & \text{if } a \notin \Sigma \end{cases}$$

where $\widetilde{\delta}_1, \widetilde{\gamma}_1 \in D(0,1]$ such that $\widetilde{\gamma}_1 \prec \widetilde{\delta}_1$ and $\delta_2, \gamma_2 \in [0,1)$ such that $\delta_2 < \gamma_2$.

Now, we define a new relation on $\mathcal{C}(\mathcal{G})$ denoted by $\subseteq \vee q_{(\lambda,\mu)}$follow:

Let $\Sigma = \left\langle \widetilde{\Psi}_\Sigma, \phi_\Sigma \right\rangle, \Omega = \left\langle \widetilde{\Psi}_\Omega, \phi_\Omega \right\rangle \in \mathcal{C}(\mathcal{G})$, by $\Sigma \subseteq \vee q_{(\lambda,\mu)}\Omega$ we mean that $a_{(\widetilde{\alpha},\beta)} \in_\lambda \Sigma$ implies that $a_{(\widetilde{\alpha},\beta)} \in_\lambda \vee q_\mu \Omega$ for all $a \in \mathcal{G}$. Moreover, Σ and Ω are said to be (λ,μ)-equal if $\Sigma \subseteq \vee q_{(\lambda,\mu)}\Omega$ and $\Omega \subseteq \vee q_{(\lambda,\mu)}\Sigma$. The above definitions can be found in [14].

Lemma 1 [14] *Let* $\Sigma = \left\langle \widetilde{\Psi}_\Sigma, \phi_\Sigma \right\rangle, \Omega = \left\langle \widetilde{\Psi}_\Omega, \phi_\Omega \right\rangle$ *be the cubic subsets of* $\mathcal{C}(\mathcal{G})$. *Then,* $\Sigma \subseteq \vee q_{(\lambda,\mu)}\Omega$ *if and only if*

$$r\max\left\{\widetilde{\Psi}_\Omega(a), \widetilde{\gamma}_1\right\} \succeq r\min\left\{\widetilde{\Psi}_\Sigma(a), \widetilde{\delta}_1\right\} \text{ and } \min\{\phi_\Omega(a), \gamma_2\} \leq \max\{\phi_\Sigma(a), \delta_2\}$$

for all $a \in \mathcal{G}$.

Lemma 2 [14] *Let* $\Sigma = \left\langle \widetilde{\Psi}_\Sigma, \phi_\Sigma \right\rangle$, $\Omega = \left\langle \widetilde{\Psi}_\Omega, \phi_\Omega \right\rangle$, $\Pi = \left\langle \widetilde{\Psi}_\Pi, \phi_\Pi \right\rangle \in \mathcal{C}(\mathcal{G})$. *If* $\Sigma \subseteq \vee q_{(\lambda,\mu)}\Omega$ *and* $\Omega \subseteq \vee q_{(\lambda,\mu)}\Pi$, *then,* $\Sigma \subseteq \vee q_{(\lambda,\mu)}\Pi$.

From Lemmas 1 and 2 we say that " $=_{(\lambda,\mu)}$ " is an equivalence relation on $\mathcal{C}(\mathcal{G})$.

Lemma 3 *Let* $\Sigma = \left\langle \widetilde{\Psi}_\Sigma, \phi_\Sigma \right\rangle$ *and* $\Omega = \left\langle \widetilde{\Psi}_\Omega, \phi_\Omega \right\rangle \in \mathcal{C}(\mathcal{G})$. *Then, we have*

(i) $\Sigma \subseteq \Omega$ *if and only if* $\mathcal{X}^\mu_{\lambda\Sigma} \subseteq \vee q_{(\lambda,\mu)} \mathcal{X}^\mu_{\lambda\Omega}$.

(ii) $\mathcal{X}^\mu_{\lambda\Sigma} \cap \mathcal{X}^\mu_{\lambda\Omega} =_{(\lambda,\mu)} \mathcal{X}^\mu_{\lambda(\Sigma\cap\Omega)}$.

(iii) $\mathcal{X}^\mu_{\lambda\Sigma} \odot \mathcal{X}^\mu_{\lambda\Omega} =_{(\lambda,\mu)} \mathcal{X}^\mu_{\lambda(\Sigma\Omega]}$.

Proof. It is simple. ∎

Cubic soft sets

Molodtsov et al. [15], Maji et al. [16] and Muhiuddin et al. [13], introduced soft set theory which provided new definitions and various results on soft set theory.

We introduced the concept of an $(\in_\lambda, \in_\lambda \vee q_\mu)$-cubic soft set which is actually the generalization of soft set.

Definition 2 [13] Let U be an initial universal set and E be set of parameters under consideration. Let $\mathcal{C}^U$ denotes the set all cubic sets of U. Let $A \subseteq E$. A pair $\langle \alpha, A\rangle$ is called cubic soft set over U, where α is a mapping given by $\alpha : A \to \mathcal{C}^U$.

Note that the the pair $\langle \alpha, A\rangle$ can be expressed as the following set:

$$\langle \alpha, A\rangle := \{\alpha(e) : e \in A\}, \text{ where } \alpha(e) = \left\langle \widetilde{\Psi}_{\alpha(e)}, \phi_{\alpha(e)} \right\rangle.$$

In general, for every $e \in A, \alpha(e)$ is a cubic set of U and it is called cubic value set of parameter e. The set of all cubic soft sets over U with parameters from E is called cubic soft class and is denoted by $\mathcal{F}_\rho(U, E)$.

Definition 3 Let $\langle \alpha, A\rangle$ and $\langle \beta, B\rangle$ be two cubic soft sets over U. Then, $\langle \alpha, A\rangle$ is called cubic soft set of $\langle \beta, B\rangle$ and write $\langle \alpha, A\rangle \subset \langle \beta, B\rangle$ if

(i) $A \subseteq B$.

(ii) For any $e \in A$, $\alpha(e) \subseteq \beta(e)$.

$\langle \alpha, A\rangle$ and $\langle \beta, B\rangle$ are said to be cubic soft equal and write $\langle \alpha, A\rangle = \langle \beta, B\rangle$ if $\langle \alpha, A\rangle \subset \langle \beta, B\rangle$ and $\langle \beta, B\rangle \subset \langle \alpha, A\rangle$.

Definition 4 The extended union of two cubic soft sets $\langle \alpha, A\rangle$ and $\langle \beta, B\rangle$ over U is called cubic soft set and is denoted by $\langle \omega, C\rangle$, where $C = A \cup B$ and

$$\omega(e) = \begin{cases} \alpha(e), & \text{if } e \in A - B, \\ \beta(e), & \text{if } e \in B - A, \\ \alpha(e) \cup \beta(e), & \text{if } e \in A \cap B \end{cases}$$

for all $e \in C$. This is denoted by $\langle \omega, C\rangle = \langle \alpha, A\rangle \widetilde{\cup} \langle \beta, B\rangle$.

Definition 5 The extended intersection of two cubic soft sets $\langle \alpha, A\rangle$ and $\langle \beta, B\rangle$ over U is called cubic soft set and is denoted by $\langle \omega, C\rangle$, where $C = A \cup B$ and

$$\omega(e) = \begin{cases} \alpha(e), & \text{if } e \in A - B, \\ \beta(e), & \text{if } e \in B - A, \\ \alpha(e) \cap \beta(e), & \text{if } e \in A \cap B \end{cases}$$

for all $e \in C$. This is denoted by $\langle \omega, C\rangle = \langle \alpha, A\rangle \widetilde{\cap} \langle \beta, B\rangle$.

Definition 6 Let $\langle \alpha, A\rangle$ and $\langle \beta, B\rangle$ be two cubic soft sets over U such that $A \cap B \neq \varphi$. The restricted union of $\langle \alpha, A\rangle$ and $\langle \beta, B\rangle$ is defined to be cubic soft set $\langle \omega, C\rangle$, where $C = A \cap B$ and $\omega(e) = \alpha(e) \cup \beta(e)$ for all $e \in C$. This is denoted by $\langle \omega, C\rangle = \langle \alpha, A\rangle \cup \langle \beta, B\rangle$.

Definition 7 Let $\langle \alpha, A\rangle$ and $\langle \beta, B\rangle$ be two cubic soft sets over U such that $A \cap B \neq \varphi$. The restricted intersection of $\langle \alpha, A\rangle$ and $\langle \beta, B\rangle$ is defined to be cubic soft set $\langle \omega, C\rangle$, where $C = A \cap B$ and $\omega(e) = \alpha(e) \cap \beta(e)$ for all $e \in C$. This is denoted by $\langle \omega, C\rangle = \langle \alpha, A\rangle \cap \langle \beta, B\rangle$.

Definition 8 Let $V \subseteq U$. A cubic soft set $\langle \alpha, A\rangle$ over U is said to be relative whole(λ, μ)-cubic soft set (with respect to universe set V and parameter set A), denoted by $\Sigma(V, A)$, if

$$\alpha(e) = f_{\lambda}^{\mu} V \text{ for all } e \in A.$$

Definition 9 Let $\langle \alpha, A\rangle$ and $\langle \beta, B\rangle$ be two cubic soft sets over U. We say that $\langle \alpha, A\rangle$ is an (λ, μ)-cubic soft subset of $\langle \beta, B\rangle$ and write $\langle \alpha, A\rangle \subset_{(\lambda,\mu)} \langle \beta, B\rangle$ if

(i) $A \subseteq B$.

(ii) For any $e \in A$, $\alpha(e) \subseteq \vee q_{(\lambda,\mu)} \beta(e)$.

$\langle \alpha, A\rangle$ and $\langle \beta, B\rangle$ are said to be (λ, μ)-cubic soft equal and write $\langle \alpha, A\rangle \asymp_{(\lambda,\mu)} \langle \beta, B\rangle$ if $\langle \alpha, A\rangle \subset_{(\lambda,\mu)} \langle \beta, B\rangle$ and $\langle \beta, B\rangle \asymp_{(\lambda,\mu)} \langle \alpha, A\rangle$.

The product of two cubic soft sets $\langle \alpha, A\rangle$ and $\langle \beta, B\rangle$ over an ordered $\mathcal{AG}$-groupoid $\mathcal{G}$, denoted by $\langle \alpha \circ \beta, C\rangle$, where $C = A \cup B$ and

$$(\alpha \circ \beta)(e) = \begin{cases} \alpha(e), & \text{if } e \in A - B, \\ \beta(e), & \text{if } e \in B - A, \\ \alpha(e) \circ \beta(e),, & \text{if } e \in A \cap B, \end{cases}$$

for all $e \in C$. This is denoted by $\langle \alpha \circ \beta, C\rangle = \langle \alpha, A\rangle \odot (\alpha, B)$.

$(\in_{\lambda}, \in_{\lambda} \vee q_{\mu})$-cubic soft ideals over an ordered AG-groupoid

Here, we introduce the concepts of $(\in_{\lambda}, \in_{\lambda} \vee q_{\mu})$-cubic soft left (resp., right) ideals and $(\in_{\lambda}, \in_{\lambda} \vee q_{\mu})$-cubic soft bi-ideals over an ordered $\mathcal{AG}$-groupoid $\mathcal{G}$ and investigate the fundamental properties and relationship.of $(\in_{\lambda}, \in_{\lambda} \vee q_{\mu})$-cubic soft sets and $(\in_{\lambda}, \in_{\lambda} \vee q_{\mu})$-cubic soft ideals.

Definition 10 A cubic soft set $\langle \alpha, A\rangle$ over an ordered $\mathcal{AG}$-groupoid $\mathcal{G}$ is called an $(\in_{\lambda}, \in_{\lambda} \vee q_{\mu})$-cubic soft $\mathcal{AG}$-subgroupoid over $\mathcal{G}$ if it satisfies:

(1) If $y \leq x$, then $x_{(\widetilde{t}_1, t_2)} \in_{\lambda} \alpha(e) \Rightarrow y_{(\widetilde{t}_1, t_2)} \in_{\lambda} \vee q_{\mu} \alpha(e)$ for all $x, y \in \mathcal{G}$, $e \in A$, $\widetilde{t}_1, \widetilde{\delta}_1, \widetilde{\gamma}_1 \in D(0, 1]$ such that $\widetilde{\gamma}_1 \prec \widetilde{\delta}_1$ and $t_2, \delta_2, \gamma_2 \in [0, 1)$ such that $\delta_2 < \gamma_2$.

(2) $\langle \alpha, A\rangle \odot \langle \alpha, A\rangle \subset_{(\lambda,\mu)} \langle \alpha, A\rangle$.

Definition 11 A cubic soft set $\langle \alpha, A\rangle$ over an ordered $\mathcal{AG}$-groupoid $\mathcal{G}$ is called an $(\in_{\lambda}, \in_{\lambda} \vee q_{\mu})$-cubic soft left (resp., right) ideal over $\mathcal{G}$ if it satisfies:

(1) If $y \leq x$, then $x_{(\widetilde{t}_1, t_2)} \in_{\lambda} \alpha(e) \Rightarrow y_{(\widetilde{t}_1, t_2)} \in_{\lambda} \vee q_{\mu} \alpha(e)$ for all $x, y \in \mathcal{G}$, $e \in A$, $\widetilde{t}_1, \widetilde{\delta}_1, \widetilde{\gamma}_1 \in D(0, 1]$ such that $\widetilde{\gamma}_1 \prec \widetilde{\delta}_1$ and $t_2, \delta_2, \gamma_2 \in [0, 1)$ such that $\delta_2 < \gamma_2$.

(2) $\sum(\mathcal{G}, A) \odot \langle \alpha, A\rangle \subset_{(\lambda,\mu)} \langle \alpha, A\rangle \ \left(\langle \alpha, A\rangle \odot \sum(\mathcal{G}, A) \subset_{(\lambda,\mu)} \langle \alpha, A\rangle\right)$.

A cubic soft set $\langle\alpha, A\rangle$ over an ordered $\mathcal{AG}$-groupoid $\mathcal{G}$ is called an $(\in_\lambda, \in_\lambda \vee q_\mu)$-cubic soft ideal over $\mathcal{G}$ if it is both an $(\in_\lambda, \in_\lambda \vee q_\mu)$-cubic soft left ideal and an $(\in_\lambda, \in_\lambda \vee q_\mu)$-cubic soft right ideal over $\mathcal{G}$.

Definition 12 A cubic soft set $\langle\alpha, A\rangle$ over an ordered $\mathcal{AG}$-groupoid $\mathcal{G}$ is called an $(\in_\lambda, \in_\lambda \vee q_\mu)$-cubic soft bi-ideal over $\mathcal{G}$ if it satisfies:

(1) If $y \leq x$, then $x_{(\widetilde{t}_1,t_2)} \in_\lambda \alpha(e) \Rightarrow y_{(\widetilde{t}_1,t_2)} \in_\lambda \vee q_\mu\alpha(e)$ for all $x, y \in \mathcal{G}$, $e \in A$, $\widetilde{t}_1, \widetilde{\delta}_1, \widetilde{\gamma}_1 \in D(0,1]$ such that $\widetilde{\gamma}_1 \prec \widetilde{\delta}_1$ and $t_2, \delta_2, \gamma_2 \in [0,1)$ such that $\delta_2 < \gamma_2$.

(2) $\langle\alpha, A\rangle \odot \langle\alpha, A\rangle \subset_{(\lambda,\mu)} \langle\alpha, A\rangle$.

(3) $\langle\alpha, A\rangle \odot \sum(\mathcal{G}, A) \odot \langle\alpha, A\rangle \subset_{(\lambda,\mu)} \langle\alpha, A\rangle$.

Definition 13 A cubic soft set $\langle\alpha, A\rangle$ over an ordered $\mathcal{AG}$-groupoid $\mathcal{G}$ is called $(\in_\lambda, \in_\lambda \vee q_\mu)$-cubic soft $\mathcal{AG}$-subgroupoid over $\mathcal{G}$ if for all $a, b \in \mathcal{G}$, $e \in A$, $\widetilde{t}_1, \widetilde{t}_2, \widetilde{\delta}_1, \widetilde{\gamma}_1 \in D(0,1]$ such that such that $\widetilde{\gamma}_1 \prec \widetilde{\delta}_1$ and $s_1, s_2, \delta_2, \gamma_2 \in [0,1)$ such that $\delta_2 < \gamma_2$, the following conditions holds:

(i) If $a \leq b$ and $a_{(\widetilde{t}_1,s_1)} \in_\lambda \alpha(e) \Rightarrow a_{(\widetilde{t}_1,s_1)} \in_\lambda \vee q_\mu\alpha(e)$

(ii) $a_{(\widetilde{t}_1,s_1)} \in_\lambda \alpha(e)$ and $b_{(\widetilde{t}_2,s_2)} \in_\lambda \alpha(e) \Rightarrow (ab)_{(r\min\{\widetilde{t}_1,\widetilde{t}_2\},\max\{s_1,s_2\})} \in_\lambda \vee q_\mu\alpha(e)$.

Theorem 1 *Let $\langle\alpha, A\rangle$ be a cubic soft set over an ordered $\mathcal{AG}$-groupoid $\mathcal{G}$ with left identity. Then, $\langle\alpha, A\rangle$ is an $(\in_\lambda, \in_\lambda \vee q_\mu)$-cubic soft $\mathcal{AG}$-subgroupoid over $\mathcal{G}$ if and only if for all $a, b \in \mathcal{G}$, $e \in A, \widetilde{\delta}_1, \widetilde{\gamma}_1 \in D(0,1]$ such that $\widetilde{\gamma}_1 \prec \widetilde{\delta}_1$, and $\delta_2, \gamma_2 \in [0,1)$ such that $\delta_2 < \gamma_2$, the following conditions holds:*

(1) $r\max\{\widetilde{\Psi}_{\alpha(e)}(a), \widetilde{\gamma}_1\} \succeq r\min\left\{\widetilde{\Psi}_{\alpha(e)}(b), \widetilde{\delta}_1\right\}$ *and*
$\min\{\phi_{\alpha(e)}(a), \gamma_2\} \leq \max\{\phi_{\alpha(e)}(b), \delta_2\}$ *with* $a \leq b$.

(2) $r\max\left\{\widetilde{\Psi}_{\alpha(e)}(ab), \widetilde{\gamma}_1\right\} \succeq r\min\left\{\widetilde{\Psi}_{\alpha(e)}(a), \widetilde{\Psi}_{\alpha(e)}(b), \widetilde{\delta}_1\right\}$ *and*
$\min\{\phi_{\alpha(e)}(ab), \gamma_2\} \leq \max\{\phi_{\alpha(e)}(a), \phi_{\alpha(e)}(b), \delta_2\}$.

Proof. (i) $\Rightarrow$ (1) Assume that $a, b \in \mathcal{G}$, $e \in A, \widetilde{t}_1, \widetilde{\delta}_1, \widetilde{\gamma}_1 \in D(0,1], t_2, \delta_2, \gamma_2 \in [0,1)$ such that $r\max\left\{\widetilde{\Psi}_{\alpha(e)}(a), \widetilde{\gamma}_1\right\} \prec \widetilde{t}_1 \preceq r\min\left\{\widetilde{\Psi}_{\alpha(e)}(b), \widetilde{\delta}_1\right\}$ and $\min\{\phi_{\alpha(e)}(a), \gamma_2\} > t_2 \geq \max\{\phi_{\alpha(e)}(b), \delta_2\}$, then $\widetilde{\Psi}_{\alpha(e)}(a) \prec \widetilde{t}_1 \preceq \widetilde{\gamma}_1 \Rightarrow \widetilde{\Psi}_{\alpha(e)}(a) + \widetilde{t}_1 \preceq 2\widetilde{\delta}_1$ and $\phi_{\alpha(e)}(a) > t_2 \geq \gamma_2 \Longrightarrow \phi_{\alpha(e)}(a) + t_2 \geq 2\delta_2$, that is $(a)_{(\widetilde{t}_1,t_2)}\overline{\in_\lambda \vee q_\mu}\alpha(e)$. On the other hand, if $\widetilde{\Psi}_{\alpha(e)}(b) \succeq \widetilde{t}_1 \succ \widetilde{\gamma}_1$ and $\phi_{\alpha(e)}(b) \leq t_2 < \gamma_2$, then $(b)_{(\widetilde{t}_1,t_2)} \in_\lambda \vee q_\mu\alpha(e)$ but $(a)_{(\widetilde{t}_1,t_2)}\overline{\in_\lambda \vee q_\mu}\alpha(e)$. This is a contradiction to the hypothesis. Hence, $r\max\{\widetilde{\Psi}_{\alpha(e)}(a), \widetilde{\gamma}_1\} \succeq r\min\{\widetilde{\Psi}_{\alpha(e)}(b), \widetilde{\delta}_1\}$ and $\min\{\phi_{\alpha(e)}(a), \gamma_2\} \leq \max\{\phi_{\alpha(e)}(b), \delta_2\}$.

(1) $\Rightarrow$ (i) Assume that $a, b \in \mathcal{G}$, $e \in A, \widetilde{t}_1, \widetilde{\delta}_1, \widetilde{\gamma}_1 \in D(0,1], t_2, \delta_2, \gamma_2 \in [0,1)$ and $(b)_{(\widetilde{t}_1,t_2)} \in_\lambda \vee q_\mu\alpha(e)$, then by definition $\widetilde{\Psi}_{\alpha(e)}(b) \succeq \widetilde{t}_1 > \widetilde{\gamma}_1$ and $\phi_{\alpha(e)}(b) \leq t_2 < \gamma_2$.

Now, by (1), $r\max\left\{\widetilde{\Psi}_{\alpha(e)}(a),\widetilde{\gamma}_1\right\}\succeq r\min\left\{\widetilde{\Psi}_{\alpha(e)}(b),\widetilde{\delta}_1\right\}\succeq\min\{\widetilde{t}_1,\widetilde{\delta}_1\}$ and $\min\{\phi_{\alpha(e)}(a),\gamma_2\}\leq\max\{\phi_{\alpha(e)}(b),\delta_2\}\leq\min\{t_2,\delta_2\}$.

We have to consider two cases here.

Case (a). If $\widetilde{t}_1\preceq\widetilde{\delta}_1$ and $t_2\geq\delta_2$, then $\widetilde{\Psi}_{\alpha(e)}(a)\succeq\widetilde{t}_1>\widetilde{\gamma}_1$ and $\phi_{\alpha(e)}(a)\leq t_2<\gamma_2$. This implies that $(a)_{(\widetilde{t}_1,t_2)}\in_\lambda\vee q_\mu\alpha(e)$.

Case (b). $\widetilde{t}_1\succ\widetilde{\delta}_1$ and $t_2<\delta_2$, then $\widetilde{\Psi}_{\alpha(e)}(a)+\widetilde{t}_1\succ 2\widetilde{\delta}_1$ and $\phi_{\alpha(e)}(a)+t_2<2\delta_2$. This implies that $(a)_{(\widetilde{t}_1,t_2)}\in_\lambda\vee q_\mu\alpha(e)$.

Hence, from both cases we can write $(a)_{(\widetilde{t}_1,t_2)}\in_\lambda\vee q_\mu\alpha(e)$.

(ii) $\Rightarrow$ (2) Let $\langle\alpha,A\rangle$ be a cubic soft set over an ordered $\mathcal{AG}$-groupoid $\mathcal{G}$ which is an $(\in_\lambda,\in_\lambda\vee q_\mu)$-cubic soft $\mathcal{AG}$-subgroupoid over $\mathcal{G}$. Let $a,b\in\mathcal{G}$, $e\in A,\widetilde{t}_1,\widetilde{\delta}_1,\widetilde{\gamma}_1\in D(0,1]$, $t_2,\delta_2,\gamma_2\in[0,1)$ such that

$$r\max\left\{\widetilde{\Psi}_{\alpha(e)}(ab),\widetilde{\gamma}_1\right\}\prec\widetilde{t}_1\preceq r\min\left\{\widetilde{\Psi}_{\alpha(e)}(a),\widetilde{\Psi}_{\alpha(e)}(b),\widetilde{\delta}_1\right\}$$

and

$$\min\{\phi_{\alpha(e)}(ab),\gamma_2\}>t_2\geq\max\{\phi_{\alpha(e)}(a),\phi_{\alpha(e)}(b),\delta_2\}.$$

Then

$$r\max\left\{\widetilde{\Psi}_{\alpha(e)}(ab),\widetilde{\gamma}_1\right\}\prec\widetilde{t}_1\Rightarrow\widetilde{\Psi}_{\alpha(e)}(ab)\prec\widetilde{t}_1\prec\widetilde{\gamma}_1$$

and

$$\min\{\phi_{\alpha(e)}(ab),\gamma_2\}>t_2\Rightarrow\phi_{\alpha(e)}(ab)>t_2>\gamma_2.$$

Thus $(ab)_{(\widetilde{t}_1,t_2)}\overline{\in_\lambda\vee q_\mu}\alpha(e)$. On the other hand, if

$$\widetilde{t}_1\preceq r\min\left\{\widetilde{\Psi}_{\alpha(e)}(a),\widetilde{\Psi}_{\alpha(e)}(b),\widetilde{\delta}_1\right\}$$

and

$$t_2\geq\max\{\phi_{\alpha(e)}(a),\phi_{\alpha(e)}(b),\delta_2\}$$

we have

$$\widetilde{\Psi}_{\alpha(e)}(a)\succeq\widetilde{t}_1\succ\widetilde{\gamma}_1,\widetilde{\Psi}_{\alpha(e)}(b)\succeq\widetilde{t}_1\succ\widetilde{\gamma}_1\text{ and }\phi_{\alpha(e)}(a)\leq t_2<\gamma_2,\phi_{\alpha(e)}(b)\leq t_2<\gamma_2.$$

Then $(a)_{(\widetilde{t}_1,t_2)}\in_\lambda\alpha(e)$ and $(b)_{(\widetilde{t}_1,t_2)}\in_\lambda\alpha(e)$ but $(ab)_{(\widetilde{t}_1,t_2)}\overline{\in_\lambda\vee q_\mu}\alpha(e)$.

This is a contradiction to the hypothesis. Hence,

$$r\max\left\{\widetilde{\Psi}_{\alpha(e)}(ab),\widetilde{\gamma}_1\right\}\succeq r\min\left\{\widetilde{\Psi}_{\alpha(e)}(a),\widetilde{\Psi}_{\alpha(e)}(b),\widetilde{\delta}_1\right\}$$

and

$$\min\{\phi_{\alpha(e)}(ab),\gamma_2\}\leq\max\{\phi_{\alpha(e)}(a),\phi_{\alpha(\epsilon)}(b),\delta_2\}.$$

(2) $\Rightarrow$ (ii) Let there exist $a \in \mathcal{G}$, $e \in A, \widetilde{t}, \widetilde{t}_1 \in D(0,1], s, s_1 \in [0,1)$ such that $(a)_{(\widetilde{t},s)} \in_\lambda \alpha(e), b \in \mathcal{G}$ such that $(b)_{(\widetilde{t}_1,s_1)} \in_\lambda \alpha(e)$. This implies that $\widetilde{\Psi}_{\alpha(e)}(a) \succeq \widetilde{t} \succ \widetilde{\gamma}_1$, $\phi_{\alpha(e)}(a) \leq s < \gamma_2$ and $\widetilde{\Psi}_{\alpha(e)}(b) \succeq \widetilde{t}_1 \succ \widetilde{\gamma}_1, \phi_{\alpha(e)}(b) \leq s_1 < \gamma_2$. So

$$r\max\left\{\widetilde{\Psi}_{\alpha(e)}(ab), \widetilde{\gamma}_1\right\} \succeq r\min\left\{\widetilde{\Psi}_{\alpha(e)}(a), \widetilde{\Psi}_{\alpha(e)}(b), \widetilde{\delta}_1\right\} \succeq r\min\{\{\widetilde{t}, \widetilde{t}_1\}, \widetilde{\delta}_1\}$$

and

$$\min\{\phi_{\alpha(e)}(ab), \gamma_2\} \leq \max\{\phi_{\alpha(e)}(a), \phi_{\alpha(e)}(b), \delta_2\} \leq \max\{\{s, s_1\}, \delta_2\}.$$

Now, we have the following cases:

Case (a). If $r\min\{\widetilde{t}, \widetilde{t}_1\} \preceq \widetilde{\delta}_1$ and $\max\{s, s_1\} \geq \delta_2$, then $\widetilde{\Psi}_{\alpha(e)}(ab) \succeq r\min\{\widetilde{t}, \widetilde{t}_1\} \succ \widetilde{\gamma}_1$ and $\phi_{\alpha(e)}(ab) \leq \max\{s, s_1\} < \gamma_2$. This implies that $(ab)_{(r\min\{\widetilde{t},\widetilde{t}_1\}, \max\{\{s,s_1\}\})} \in_\lambda \alpha(e)$.

Case (b). If $r\min\{\widetilde{t}, \widetilde{t}_1\} \succ \widetilde{\delta}_1$ and $\max\{s, s_1\} < \delta_2$, then $\widetilde{\Psi}_{\alpha(e)}(ab) + r\min\{\widetilde{t}, \widetilde{t}_1\} \succ 2\widetilde{\delta}_1$ and $\phi_{\alpha(e)}(ab) + \max\{\{s, s_1\} < 2\delta_2$. This implies that $(ab)_{(r\min\{\widetilde{t},\widetilde{t}_1\}, \max\{s,s_1\})} q_\mu \alpha(e)$.

From both cases we get $(ab)_{(r\min\{\widetilde{t}_1,\widetilde{t}_2\}, \max\{s_1,s_2\})} \in_\lambda \vee q_\mu \alpha(e)$. Hence, $\langle \alpha, A\rangle$ is an $(\in_\lambda, \in_\lambda \vee q_\mu)$-cubic soft $\mathcal{AG}$-subgroupoid over $\mathcal{G}$. ∎

Example 2 Let $\mathcal{G} = \{a, b, c\}$ be an an ordered $\mathcal{AG}$-groupoid with the multiplication table and order given below:

$\cdot$	a	b	c
a	a	a	a
b	a	a	c
c	a	a	a

$$\leq := \{(a,a), (b,b), (c,c), (a,c), (a,b)$$

For $A = \{e_1, e_2, e_3\} \subseteq E$, the cubic soft set $\langle \alpha, A\rangle = \{\alpha(e_1), \alpha(e_2), \alpha(e_3)\}$ over $\mathcal{G}$ is defined as follows.

$$\begin{aligned}
\alpha(e_1) &= \{\langle a, [0.2, 0.3), 0.1\rangle, \langle b, [0.3, 0.4), 0.2\rangle, \langle c, [0.4, 0.5), 0.3\rangle\},\\
\alpha(e_2) &= \{\langle a, [0.15, 0.16), 0.2\rangle, \langle b, [0.16, 0.18), 0.3\rangle, \langle c, [0.18, 0.2), 0.3\rangle\},\\
\alpha(e_3) &= \{\langle a, [0.1, 0.12), 0.1\rangle, \langle b, [0.12, 0.14), 0.2\rangle, \langle c, [0.14, 0.16), 0.2\rangle\},
\end{aligned}$$

such that $\widetilde{\gamma}_1 = [0.1, .18) \prec \widetilde{\delta}_1 = [0.3, 0.4)$ and $\delta_2 = 0.3 < \gamma_2 = 0.4$. Then it is easy to verify that $\langle \alpha, A\rangle$ is an $(\in_{([0.1,.18),0.4)}, \in_{([0.1,.18),0.4)} \vee q_{([0.3,0.4),0.3)})$-cubic soft $\mathcal{AG}$-subgroupoid over $\mathcal{G}$.

Definition 14 A cubic soft set $\langle \alpha, A\rangle$ over an ordered $\mathcal{AG}$-groupoid $\mathcal{G}$ is called $(\in_\lambda, \in_\lambda \vee q_\mu)$-cubic soft left (resp., right) ideal over $\mathcal{G}$ if for all $a, b \in \mathcal{G}$, $e \in A$, $\widetilde{t}_1, \widetilde{\delta}_1, \widetilde{\gamma}_1 \in D(0,1]$ such that $\widetilde{\gamma}_1 \prec \widetilde{\delta}_1$and $s_1, \delta_2, \gamma_2 \in [0,1)$ such that $\delta_2 < \gamma_2$, the following conditions holds:

(i) If $a \leq b$ and $b_{(\widetilde{t}_1,s_1)} \in_\lambda \alpha(e) \Rightarrow (a)_{(\widetilde{t}_1,s_1)} \in_\lambda \vee q_\mu \alpha(e)$.

(ii) If $b_{(\widetilde{t}_1,s_1)} \in_\lambda \alpha(e) \Rightarrow (ab)_{(\widetilde{t}_1,s_1)} \in_\lambda \vee q_\mu \alpha(e)$ $(a_{(\widetilde{t}_1,s_1)} \in_\lambda \alpha(e)$ $\Rightarrow (ab)_{(\widetilde{t}_1,s_1)} \in_\lambda \vee q_\mu \alpha(e))$.

Theorem 2 *A cubic soft set $\langle \alpha, A\rangle$ over an ordered $\mathcal{AG}$-groupoid $\mathcal{G}$ is called $(\in_\lambda, \in_\lambda \vee q_\mu)$-cubic soft left(resp., right) ideal if and only if for all $a, b \in \mathcal{G}, e \in A$, $\widetilde{\delta}_1, \widetilde{\gamma}_1, \in D(0,1]$ such that $\widetilde{\gamma}_1 \prec \widetilde{\delta}_1$, and $\delta_2, \gamma_2 \in [0,1)$ such that $\delta_2 < \gamma_2$, the following conditions holds:*

(1) $r\max\left\{\widetilde{\Psi}_{\alpha(e)}(a), \widetilde{\gamma}_1\right\} \succeq r\min\left\{\widetilde{\Psi}_{\alpha(e)}(b), \widetilde{\delta}_1\right\}$ *and* $\min\{\phi_{\alpha(e)}(a), \gamma_2\} \leq \max\{\phi_{\alpha(e)}(b), \delta_2\}$ *with* $a \leq b$.

(2) $r\max\left\{\widetilde{\Psi}_{\alpha(e)}(ab), \widetilde{\gamma}_1\right\} \succeq r\min\left\{\widetilde{\Psi}_{\alpha(e)}(b), \widetilde{\delta}_1\right\}$ *and* $\min\{\phi_{\alpha(e)}(ab), \gamma_2\} \leq \max\{\phi_{\alpha(e)}(b), \delta_2\}$.
$\Big($*resp.,* $r\max\left\{\widetilde{\Psi}_{\alpha(e)}(ab), \widetilde{\gamma}_1\right\} \succeq r\min\left\{\widetilde{\Psi}_{\alpha(e)}(a), \widetilde{\delta}_1\right\}$ *and* $\min\{\phi_{\alpha(e)}(ab), \gamma_2\} \leq \max\{\phi_{\alpha(e)}(a), \delta_2\}\Big)$.

Proof. (i) $\Leftrightarrow$ (1) It is the same as in Theorem 1.

(ii) $\Rightarrow$ (2) Assume that $a, b \in \mathcal{G}$, $e \in A, \widetilde{t}_1, \widetilde{\delta}_1, \widetilde{\gamma}_1 \in D(0,1], t_2, \delta_2, \gamma_2 \in [0,1)$ such that $r\max\left\{\widetilde{\Psi}_{\alpha(e)}(ab), \widetilde{\gamma}_1\right\} \prec \widetilde{t}_1 \preceq r\min\left\{\widetilde{\Psi}_{\alpha(e)}(b), \widetilde{\delta}_1\right\}$ and $\min\{\phi_{\alpha(e)}(ab), \gamma_2\} > t_2 \geq \max\{\phi_{\alpha(e)}(b), \delta_2\}$, then $r\max\left\{\widetilde{\Psi}_{\alpha(e)}(ab), \widetilde{\gamma}_1\right\} \prec \widetilde{t}_1 \Rightarrow \widetilde{\Psi}_{\alpha(e)}(ab) \prec \widetilde{t}_1 \preceq \widetilde{\gamma}_1$ and $\phi_{\alpha(e)}(ab) > t_2 \Rightarrow \phi_{\alpha(e)}(ab) > t_2 \geq \gamma_2$. This implies that $(ab)_{(\widetilde{t}_1,t_2)} \overline{\in_\lambda \vee q_\mu} \alpha(e)$. On the other hand, if $r\min\left\{\widetilde{\Psi}_{\alpha(e)}(b), \widetilde{\delta}_1\right\} \succeq \widetilde{t}_1 \succ \widetilde{\gamma}_1$ and $\max\{\phi_{\alpha(e)}(b), \delta_2\} \leq t_2 < \gamma_2$, this implies that $(b)_{(\widetilde{t}_1,t_2)} \in_\lambda \vee q_\mu \alpha(e)$, but $(ab)_{(\widetilde{t}_1,t_2)} \overline{\in_\lambda \vee q_\mu} \alpha(e)$ is a contradiction. Hence, $r\max\left\{\widetilde{\Psi}_{\alpha(e)}(ab), \widetilde{\gamma}_1\right\} \succeq r\min\left\{\widetilde{\Psi}_{\alpha(e)}(b), \widetilde{\delta}_1\right\}$ and $\min\{\phi_{\alpha(e)}(ab), \gamma_2\} \leq \max\{\phi_{\alpha(e)}(b), \delta_2\}$.

(2) $\Rightarrow$ (ii) Assume that $a, b \in \mathcal{G}$, $e \in A, \widetilde{t}_1, \widetilde{\delta}_1, \widetilde{\gamma}_1 \in D(0,1], t_2, \delta_2, \gamma_2 \in [0,1)$ such that $b_{(\widetilde{t}_1,s_1)} \in_\lambda \alpha(e)$, then by definition we can write $\widetilde{\Psi}_{\alpha(e)}(a) \succeq \widetilde{t}_1 \succ \widetilde{\gamma}_1$ and $\phi_{\alpha(e)}(a) \leq t_2 < \gamma_2$, therefore $r\max\left\{\widetilde{\Psi}_{\alpha(e)}(ab), \widetilde{\gamma}_1\right\} \succeq r\min\left\{\widetilde{\Psi}_{\alpha(e)}(b), \widetilde{\delta}_1\right\} \succeq \min\{\widetilde{t}_1, \widetilde{\delta}_1\}$ and $\min\{\phi_{\alpha(e)}(ab), \gamma_2\} \leq \max\{\phi_{\alpha(e)}(b), \delta_2\} \leq \{t_2, \delta_2\}$.

We have to consider two cases here.

Case (a). If $\widetilde{t}_1 \preceq \widetilde{\delta}_1$ and $t_2 \geq \delta_2$, then $\widetilde{\Psi}_{\alpha(e)}(ab) \succeq \widetilde{t}_1 > \widetilde{\gamma}_1$ and $\phi_{\alpha(e)}(ab) \leq t_2 < \gamma_2$. This implies that $(ab)_{(\widetilde{t}_1,t_2)} \in_\lambda \vee q_\mu \alpha(e)$.

Case (b). $\widetilde{t}_1 \succ \widetilde{\delta}_1$ and $t_2 < \delta_2$, then $\widetilde{\Psi}_{\alpha(e)}(ab) + \widetilde{t}_1 \succ 2\widetilde{\delta}_1$ and $\phi_{\alpha(e)}(ab) + t_2 < 2\delta_2$. This implies that $(ab)_{(\widetilde{t}_1,t_2)} \in_\lambda \vee q_\mu \alpha(e)$.

Hence, from both cases we can write $(ab)_{(\widetilde{t}_1,t_2)} \in_\lambda \vee q_\mu \alpha(e)$.

The other case can be seen in a similar way. ∎

A cubic soft set $\langle \alpha, A\rangle$ over an ordered $\mathcal{AG}$-groupoid $\mathcal{G}$ is called an $(\in_\lambda, \in_\lambda \vee q_\mu)$-cubic soft ideal over $\mathcal{G}$, if it is both $(\in_\lambda, \in_\lambda \vee q_\mu)$-cubic soft left and an $(\in_\lambda, \in_\lambda \vee q_\mu)$-cubic soft right over $\mathcal{G}$.

Example 3 Let $\mathcal{G} = \{a, b, c\}$ be an ordered $\mathcal{AG}$-groupoid with the multiplication table and order given below:

$\cdot$	a	b	c
a	a	a	a
b	a	a	c
c	a	b	a

$$\leq := \{(a,a),(b,b),(c,c),(a,c),(a,b)$$

For $A = \{e_1, e_2, e_3\} \subseteq E$, the cubic soft set $\langle \alpha, A\rangle = \{\alpha(e_1), \alpha(e_2), \alpha(e_3)\}$ over $\mathcal{G}$ is defined as follows.

$$\begin{aligned}
\alpha(e_1) &= \{\langle a, [0.2, 0.3), 0.1\rangle, \langle b, [0.3, 0.4), 0.2\rangle, \langle c, [0.4, 0.5), 0.3\rangle\},\\
\alpha(e_2) &= \{\langle a, [0.2, 0.15), 0.1\rangle, \langle b, [0.15, 0.18), 0.3\rangle, \langle c, [0.18, 0.2), 0.4\rangle\},\\
\alpha(e_3) &= \{\langle a, [0.1, 0.15), 0.2\rangle, \langle b, [0.15, 0.20), 0.3\rangle, \langle c, [0.20, 0.25), 0.3\rangle\},
\end{aligned}$$

such that $\widetilde{\gamma}_1 = [0.1, .18) \prec \widetilde{\delta}_1 = [0.19, 0.2)$ and $\delta_2 = 0.53 < \gamma_2 = 0.54$. Then it is easy to varify that $\langle \alpha, A\rangle$ is an $(\in_{([0.1,.18),0.54)}, \in_{([0.1,.18),0.54)} \vee q_{([0.19,0.2),0.53)})$-cubic soft ideal over $\mathcal{G}$.

Lemma 4 *Let $\langle \alpha, A\rangle$ be a cubic soft set over an ordered $\mathcal{AG}$-groupoid $\mathcal{G}$, let $e \in A$, $\widetilde{\delta}_1, \widetilde{\gamma}_1, \in D(0,1]$ such that $\widetilde{\gamma}_1 \prec \widetilde{\delta}_1$, and $\delta_2, \gamma_2 \in [0,1)$ such that $\delta_2 < \gamma_2$, then $\langle \alpha, A\rangle$ is an $(\in_\lambda, \in_\lambda \vee q_\mu)$ cubic soft left (resp., right) over $\mathcal{G}$ if and only if $\langle \alpha, A\rangle$ satisfies the following conditions.*

(1) $a \leq b \Rightarrow r\max\left\{\widetilde{\Psi}_{\alpha(e)}(a), \widetilde{\gamma}_1\right\} \succeq r\min\left\{\widetilde{\Psi}_{\alpha(e)}(b), \widetilde{\delta}_1\right\}$ *and* $\min\{\phi_{\alpha(e)}(a), \gamma_2\} \leq \max\{\phi_{\alpha(e)}(b), \delta_2\}, \forall a, b \in \mathcal{G}$.

(2) $\Sigma\langle \mathcal{G}, A\rangle \odot \langle \alpha, A\rangle \subseteq \vee q_{(\lambda,\mu)}\langle \alpha, A\rangle$ $\left(resp.,\ \langle \alpha, A\rangle \odot \Sigma\langle \mathcal{G}, A\rangle \subseteq \vee q_{(\lambda,\mu)}\langle \alpha, A\rangle\right)$.

Proof. The proof is straightforward. ∎

Definition 15 A cubic soft set $\langle \alpha, A\rangle$ over an ordered $\mathcal{AG}$-groupoid $\mathcal{G}$ is called an $(\in_\lambda, \in_\lambda \vee q_\mu)$-cubic soft ideal over $\mathcal{G}$ if for $a, b, c \in \mathcal{G}$, $e \in A, \widetilde{t}_1, \widetilde{t}_2, \widetilde{\delta}_1, \widetilde{\gamma}_1, \in D(0,1]$ such that $\widetilde{\gamma}_1 \prec \widetilde{\delta}_1$ and $s_1, s_2, \delta_2, \gamma_2 \in [0,1)$ such that $\delta_2 < \gamma_2$, the following conditions holds:

(i) If $a \leq b$ and $b_{(\widetilde{t}_1, s_1)} \in_\lambda \alpha(e) \Rightarrow (a)_{(\widetilde{t}_1, s_1)} \in_\lambda \vee q_\mu \alpha(e)$.

(ii) If $a_{(\widetilde{t}_1, s_1)} \in_\lambda \alpha(e)$ and $b_{(\widetilde{t}_2, s_2)} \in_\lambda \alpha(e) \Rightarrow (ab)_{(r\min\{\widetilde{t}_1, \widetilde{t}_2\}, \max\{s_1, s_2\})} \in_\lambda \vee q_\mu \alpha(e)$.

(iii) If $a_{(\widetilde{t}_1, s_1)} \in_\lambda \alpha(e)$ and $c_{(\widetilde{t}_2, s_2)} \in_\lambda \alpha(e) \Rightarrow ((ab)c)_{(r\min\{\widetilde{t}_1, \widetilde{t}_2\}, \max\{s_1, s_2\})} \in_\lambda \vee q_\mu \alpha(e)$.

Theorem 3 *A cubic soft set $\langle\alpha, A\rangle$ over an ordered $\mathcal{AG}$-groupoid $\mathcal{G}$ is called an $(\in_\lambda, \in_\lambda \vee q_\mu)$-cubic soft bi-ideal over $\mathcal{G}$ if for $a, b, c \in \mathcal{G}$, $e \in A, \widetilde{t}_1, \widetilde{t}_2, \widetilde{\delta}_1, \widetilde{\gamma}_1 \in D(0,1]$ such that $\widetilde{\gamma}_1 \prec \widetilde{\delta}_1$ and $s_1, s_2, \delta_2, \gamma_2 \in [0,1)$ such that $\delta_2 < \gamma_2$, the following conditions holds:*

(1) $r\max\left\{\widetilde{\Psi}_{\alpha(e)}(a), \widetilde{\gamma}_1\right\} \succeq r\min\left\{\widetilde{\Psi}_{\alpha(e)}(b), \widetilde{\delta}_1\right\}$ *and* $\min\{\phi_{\alpha(e)}(a), \gamma_2\} \leq \max\{\phi_{\alpha(e)}(b), \delta_2\}$ *with* $a \leq b$.

(2) $r\max\left\{\widetilde{\Psi}_{\alpha(e)}(ab), \widetilde{\gamma}_1\right\} \succeq r\min\left\{\widetilde{\Psi}_{\alpha(e)}(a), \widetilde{\Psi}_{\alpha(e)}(b), \widetilde{\delta}_1\right\}$ *and* $\min\{\phi_{\alpha(e)}(ab), \gamma_2\} \leq \max\{\phi_{\alpha(e)}(a), \phi_{\alpha(e)}(b), \delta_2\}$.

(3) $r\max\left\{\widetilde{\Psi}_{\alpha(e)}((ab)c), \widetilde{\gamma}_1\right\} \succeq r\min\left\{\widetilde{\Psi}_{\alpha(e)}(a), \widetilde{\Psi}_{\alpha(e)}(c), \widetilde{\delta}_1\right\}$ *and* $\min\{\phi_{\alpha(e)}((ab)c), \gamma_2\} \leq \max\{\phi_{\alpha(e)}(a), \phi_{\alpha(e)}(c), \delta_2\}$.

Proof. (i) $\Leftrightarrow$ (1) It is the same as in Theorem 1.
(ii) $\Leftrightarrow$ (2) It is the same as in Theorem 1.
(iii) $\Leftrightarrow$ (3) Assume that $\langle\alpha, A\rangle$ is an $(\in_\lambda, \in_\lambda \vee q_\mu)$-cubic soft bi-ideal over an ordered $\mathcal{AG}$-groupoid $\mathcal{G}$. Let $a, b, c \in \mathcal{G}$, $e \in A$, $\widetilde{t}_1, \widetilde{\delta}_1, \widetilde{\gamma}_1 \in D(0,1], t_2, \delta_2, \gamma_2 \in [0,1)$ such that $r\max\left\{\widetilde{\Psi}_{\alpha(e)}((ab)c), \widetilde{\gamma}_1\right\} \prec \widetilde{t}_1 \preceq r\min\left\{\widetilde{\Psi}_{\alpha(e)}(a), \widetilde{\Psi}_{\alpha(e)}(c), \widetilde{\delta}_1\right\}$ and $\min\{\phi_{\alpha(e)}((ab)c), \gamma_2\} > t_2 \geq \max\{\phi_{\alpha(e)}(a), \phi_{\alpha(e)}(c), \delta_2\}$. Then,

$$r\max\left\{\widetilde{\Psi}_{\alpha(e)}((ab)c), \widetilde{\gamma}_1\right\} \prec \widetilde{t}_1 \;\Rightarrow\; \widetilde{\Psi}_{\alpha(e)}((ab)c) \prec \widetilde{t}_1 \prec \widetilde{\gamma}_1 \quad \text{and}$$
$$\min\{\phi_{\alpha(e)}((ab)c), \gamma_2\} > t_2 \;\Rightarrow\; \phi_{\alpha(e)}((ab)c) > t_2 > \gamma_2.$$

Thus $((ab)c)_{(\widetilde{t}_1, t_2)}\overline{\in_\lambda \vee q_\mu}\alpha(e)$.

On the other hand, if

$$\widetilde{t}_1 \preceq r\min\left\{\widetilde{\Psi}_{\alpha(e)}(a), \widetilde{\Psi}_{\alpha(e)}(c), \widetilde{\delta}_1\right\} \text{ and } t_2 \geq \max\{\phi_{\alpha(e)}(a), \phi_{\alpha(e)}(c), \delta_2\},$$

then we have

$$\widetilde{\Psi}_{\alpha(e)}(a) \succeq \widetilde{t}_1 \succ \widetilde{\gamma}_1, \widetilde{\Psi}_{\alpha(e)}(c) \succeq \widetilde{t}_1 \succ \widetilde{\gamma}_1 \text{ and } \phi_{\alpha(e)}(a) \leq t_2 < \gamma_2, \phi_{\alpha(e)}(c) \leq t_2 < \gamma_2.$$

This implies that $(a)_{(\widetilde{t}_1, t_2)} \in_\lambda \alpha(e)$ and $(c)_{(\widetilde{t}_1, t_2)} \in_\lambda \alpha(e)$, but $((ab)c)_{(\widetilde{t}_1, t_2)}\overline{\in_\lambda \vee q_\mu}\alpha(e)$.

This is a contradiction to the hypothesis. Hence,

$$r\max\left\{\widetilde{\Psi}_{\alpha(e)}((ab)c), \widetilde{\gamma}_1\right\} \succeq r\min\left\{\widetilde{\Psi}_{\alpha(e)}(a), \widetilde{\Psi}_{\alpha(e)}(c), \widetilde{\delta}_1\right\} \text{ and}$$
$$\min\{\phi_{\alpha(e)}((ab)c), \gamma_2\} \leq \max\{\phi_{\alpha(e)}(a), \phi_{\alpha(\epsilon)}(c), \delta_2\}.$$

(2) $\Rightarrow$ (ii) Let there exist $a, b, c \in \mathcal{G}$, $e \in A, \widetilde{t}, \widetilde{t}_1 \in D(0,1], s, s_1 \in [0,1)$ such that $(a)_{(\widetilde{t}, s)} \in_\lambda \alpha(e), b \in \mathcal{G}$ such that $(c)_{(\widetilde{t}_1, s_1)} \in_\lambda \alpha(e)$. This implies that $\widetilde{\Psi}_{\alpha(e)}(a) \succeq \widetilde{t} \succ \widetilde{\gamma}_1$, $\phi_{\alpha(e)}(a) \leq s < \gamma_2$ and $\widetilde{\Psi}_{\alpha(e)}(c) \succeq \widetilde{t}_1 \succ \widetilde{\gamma}_1, \phi_{\alpha(e)}(c) \leq s_1 < \gamma_2$. So

$$r\max\left\{\widetilde{\Psi}_{\alpha(e)}((ab)c), \widetilde{\gamma}_1\right\} \succeq r\min\left\{\widetilde{\Psi}_{\alpha(e)}(a), \widetilde{\Psi}_{\alpha(e)}(b), \widetilde{\delta}_1\right\} \succeq r\min\{\{\widetilde{t}, \widetilde{t}_1\}, \widetilde{\delta}_1\} \text{ and}$$
$$\min\{\phi_{\alpha(e)}((ab)c), \gamma_2\} \leq \max\{\phi_{\alpha(e)}(a), \phi_{\alpha(e)}(b), \delta_2\} \leq \max\{\{s, s_1\}, \delta_2\}.$$

Now, we have the following cases:

Case (a). If $r\min\{\widetilde{t},\widetilde{t}_1\}\preceq\widetilde{\delta}_1$ and $\max\{s,s_1\}\geq\delta_2$, then $\widetilde{\Psi}_{\alpha(e)}((ab)c)\succeq r\min\{\widetilde{t},\widetilde{t}_1\}\succ\widetilde{\gamma}_1$ and $\phi_{\alpha(e)}((ab)c)\leq\max\{s,s_1\}<\gamma_2$. This implies that $((ab)c)_{(r\min\{\widetilde{t},\widetilde{t}_1\},\max\{\{s,s_1\})}\in_\lambda\alpha(e)$.

Case (b). If $r\min\{\widetilde{t},\widetilde{t}_1\}\succ\widetilde{\delta}_1$ and $\max\{s,s_1\}<\delta_2$, then $\widetilde{\Psi}_{\alpha(e)}((ab)c)+r\min\{\widetilde{t},\widetilde{t}_1\}\succ 2\widetilde{\delta}_1$ and $\phi_{\alpha(e)}((ab)c)+\max\{\{s,s_1\}<2\delta_2$. This implies that $(ab)_{(r\min\{\widetilde{t},\widetilde{t}_1\},\max\{s,s_1\})}q_\mu\alpha(e)$.

From both cases we get $((ab)c)_{(r\min\{\widetilde{t}_1,\widetilde{t}_2\},\max\{s_1,s_2\})}\in_\lambda\vee q_\mu\alpha(e)$ for all $a,b,c\in\mathcal{G}$. ∎

Example 4 Let $\mathcal{G}=\{0,1,2,3\}$ be an ordered $\mathcal{AG}$-groupoid with multiplication table and order given below.

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	0	1
3	0	0	1	2

$$\leq:=\{(0,0),(1,1),(2,2),(3,3),(0,1)\}$$

For $A=\{e_1,e_2,e_3,e_4\}\subseteq E$, the cubic soft set $\langle\alpha,A\rangle=\{\alpha(e_1),\alpha(e_2),\alpha(e_3),\alpha(e_4)\}$ over $\mathcal{G}$ is defined as follows.

$\alpha(e_1)=\{\langle 0,[0.2,0.3),0.1\rangle,\langle 1,[0.3,0.4),0.2\rangle,\langle 2,[0.4,0.5),0.3\rangle,\langle 3,[0.6.0.8),0.5\rangle\}$,
$\alpha(e_2)=\{\langle 0,[0.2,0.15),0.1\rangle,\langle 1,[0.15,0.18),0.3\rangle,\langle 2,[0.18,0.2),0.4\rangle,\langle 3,[0.2.0.28),0.4\rangle\}$,
$\alpha(e_3)=\{\langle 0,[0.1,0.15),0.2\rangle,\langle 1,[0.15,0.20),0.3\rangle,\langle 2,[0.20,0.25),0.3\rangle,\langle 3,[0.25.0.28),0.5\rangle\}$,
$\alpha(e_4)=\{\langle 0,[0.1,0.2),0.5\rangle,\langle 1,[0.2,0.25),0.7\rangle,\langle 2,[0.25,0.3),0.9\rangle,\langle 3,[0.3,0.35),0.9\rangle\}$

such that $\widetilde{\gamma}_1=[0.1,.18)\prec\widetilde{\delta}_1=[0.19,0.2)$ and $\delta_2=0.53<\gamma_2=0.54$. Then it is easy to verify that $\langle\alpha,A\rangle$ is an $(\in_{([0.1,.18),0.54)},\in_{([0.1,.18),0.54)}\vee q_{([0.19,0.2),0.53)})$-cubic soft bi-ideal over $\mathcal{G}$.

Theorem 4 *Let A be a non-empty subset of an ordered $\mathcal{AG}$-groupoid $\mathcal{G}$, then for $(\in_\lambda,\in_\lambda\vee q_\mu)$-cubic soft left (resp., right) ideal the following conditions are equivalent.*

(1) (a) $\sum\langle\mathcal{G},A\rangle\odot\langle\alpha,A\rangle\subset_{(\lambda,\mu)}\langle\alpha,A\rangle$ $\left(resp.,\ \langle\alpha,A\rangle\odot\sum\langle\mathcal{G},A\rangle\subset_{(\lambda,\mu)}\langle\alpha,A\rangle\right)$.

(b) *If $y\leq x$, then $x_{(\widetilde{t}_1,t_2)}\in_\lambda\alpha(e)\Rightarrow y_{(\widetilde{t}_1,t_2)}\in_\lambda\vee q_\mu\alpha(e)$ for all $x,y\in\mathcal{G}$, $e\in A$, $\widetilde{t}_1,\widetilde{\delta}_1,\widetilde{\gamma}_1\in D(0,1]$ such that $\widetilde{\gamma}_1\prec\widetilde{\delta}_1$ and $t_2,\delta_2,\gamma_2\in[0,1)$ such that $\delta_2<\gamma_2$.*

(2) $x_{(\widetilde{\alpha},\beta)}\in_\lambda\alpha(e)\Rightarrow(yx)_{(\widetilde{\alpha},\beta)}\in_\lambda\alpha(e)$.

(3) (a) $r\max\left\{\widetilde{\Psi}_{\alpha(e)}(a),\widetilde{\gamma}_1\right\}\succeq r\min\left\{\widetilde{\Psi}_{\alpha(e)}(b),\widetilde{\delta}_1\right\}$ *and* $\min\{\phi_{\alpha(e)}(a),\gamma_2\}\leq\max\{\phi_{\alpha(e)}(b),\delta_2\}$ *with* $a\leq b$.

(b) $r\max\left\{\widetilde{\Psi}_{\alpha(e)}(ab),\widetilde{\gamma}_1\right\}\succeq r\min\left\{\widetilde{\Psi}_{\alpha(e)}(b),\widetilde{\delta}_1\right\}$ *and* $\min\{\phi_{\alpha(e)}(ab),\gamma_2\}\leq\max\{\phi_{\alpha(e)}(b),\delta_2\}$.

Proof. It is straightforward. ∎

Lemma 5 *Let $\phi \neq A \subseteq \mathcal{G}$, then A is a left (resp., right, bi-) ideal of $\mathcal{G}$ if and only if the cubic characteristic function $\mathcal{X}^{\mu}_{\lambda A} = \left\langle \widetilde{\Psi}_{\mathcal{X}^{\mu}_{\lambda A}}, \phi_{\mathcal{X}^{\mu}_{\lambda A}} \right\rangle$ of $A = \left\langle \widetilde{\Psi}_{A}, \phi_{A} \right\rangle$ is an $(\in_{\lambda}, \in_{\lambda} \vee q_{\mu})$-cubic left (resp., right, bi-) ideal over $\mathcal{G}$, where $\widetilde{\delta}_1, \widetilde{\gamma}_1 \in D(0,1]$ such that $\widetilde{\gamma}_1 \prec \widetilde{\delta}_1$, and $\delta_2, \gamma_2 \in [0,1)$ such that $\delta_2 < \gamma_2$.*

Proof. It is the same as in [14]. ∎

Let us define the $\in_{\lambda} \vee q_{\mu}$-cubic level set for the cubic set $\Sigma = \left\langle \widetilde{\Psi}_{\Sigma}, \phi_{\Sigma} \right\rangle$ as

$$[\Sigma]_{(\widetilde{t},\delta)} = \{a \in \mathcal{G} : a_{(\widetilde{t},\delta)} \in_{\lambda} \vee q_{\mu} \Sigma\}.$$

Theorem 5 *A cubic set $\Sigma = \left\langle \widetilde{\Psi}_{\Sigma}, \phi_{\Sigma} \right\rangle$ is said to be an $(\in_{\lambda}, \in_{\lambda} \vee q_{\mu})$-cubic left (resp., right, bi-) ideal of an ordered $\mathcal{AG}$-groupoid $\mathcal{G}$ if and only if $\Phi \neq [\Sigma]_{(\widetilde{t},\delta)}$ is left (resp., right, bi-) ideal of $\mathcal{G}$.*

Proof. It is the same as in [14]. ∎

Theorem 6 *Let $\mathcal{G}$ be an ordered $\mathcal{AG}$-groupoid and $P \subseteq \mathcal{G}$. Then P is left (resp., right, bi-) ideal of $\mathcal{G}$ if and only if $\Sigma(P, A)$ is an $(\in_{\lambda}, \in_{\lambda} \vee q_{\mu})$-cubic soft left (resp., right, bi-) ideal over $\mathcal{G}$ for any $A \subseteq E$.*

Proof. It follows from Theorem 5. ∎

Lemma 6 [12] *In an ordered $\mathcal{AG}$-groupoid $\mathcal{G}$, the following are true.*

(i) $A \subseteq (A], \forall\ A \subseteq \mathcal{G}$.

(ii) $A \subseteq B \subseteq \mathcal{G} \Longrightarrow (A] \subseteq (B], \forall\ A, B \subseteq \mathcal{G}$.

(iii) $(A](B] \subseteq (AB], \forall\ A, B \subseteq \mathcal{G}$.

(iv) $(A] = ((A]], \forall\ A \subseteq \mathcal{G}$.

(vi) $((A](B]] = (AB], \forall\ A, B \subseteq \mathcal{G}$.

Main results

This section contains the main results of the paper.

Definition 16 An element a of an ordered $\mathcal{AG}$-groupoid $\mathcal{G}$ is called **intra-regular** if there exist $x, y \in \mathcal{G}$ such that $a \leq (xa^2)y$ and $\mathcal{G}$ is called **intra-regular**, if every element of $\mathcal{G}$ is intra-regular or equivalently, $A \subseteq ((\mathcal{G}A^2)]$, for all $A \subseteq \mathcal{G}$. In this section, we discuss the characterizations of intra-regular ordered $\mathcal{AG}$-groupoids.

From now on, $\mathcal{G}$ will denote an ordered $\mathcal{AG}$-groupoid unless otherwise specified.

Theorem 7 *The following conditions are equivalent for $\mathcal{G}$ with left identity.*

(i) *$\mathcal{G}$ is intra-regular.*

(ii) *For every left ideal L and bi-ideal B of $\mathcal{G}$, $L \cap B \subseteq ((LB]B]$.*

(iii) *For an $(\in_\lambda, \in_\lambda \vee q_\mu)$-cubic soft left ideal $\langle \alpha, L\rangle$ and for an $(\in_\lambda, \in_\lambda \vee q_\mu)$-cubic soft bi-ideal $\langle \beta, B\rangle$ over $\mathcal{G}$, we have $\langle \alpha, L\rangle \widetilde{\cap} \langle \beta, B\rangle \subseteq \vee\, q_{(\lambda,\mu)}\, [\langle \alpha, L\rangle \odot \langle \beta, B\rangle] \odot \langle \beta, B\rangle$.*

Proof. (i) $\Rightarrow$ (iii) Let $\mathcal{G}$ be intra-regular, $\langle \alpha, L\rangle$ be an $(\in_\lambda, \in_\lambda \vee q_\mu)$-cubic soft left ideal and $\langle \beta, B\rangle$ be an $(\in_\lambda, \in_\lambda \vee q_\mu)$-cubic soft bi-ideal over $\mathcal{G}$, respectively. Now let $e \in L \cup B$ and $\langle \alpha, L\rangle \widetilde{\cap} \langle \beta, B\rangle = \langle \omega, L \cup B\rangle$. We consider the following cases.

Case 1. $e \in L - B$. Then $\omega(e) = \alpha(e) = (\alpha \circ \beta)(e)$.

Case 2. $e \in B - L$. Then $\omega(e) = \beta(e) = (\alpha \circ \beta)(e)$.

Case 3. $e \in L \cap B$. Then $\omega(e) = \alpha(e) \cap \beta(e)$ and $(\alpha \circ \beta)(e) = (\alpha(e) \circ \beta(e)) \circ \beta(e)$. Now we show that $\alpha(e) \cap \beta(e) \subseteq \vee q_{(\lambda,\mu)}(\alpha(e) \circ \beta(e)) \circ \beta(e)$. Since $\mathcal{G}$ is intra-regular, then for any $a \in \mathcal{G}$ there exist $x, y \in \mathcal{G}$ such that

$$\begin{aligned} a &\leq (xa^2)y = (a(xa))y = (y(xa))a \leq (y(x((xa^2)y)))a = (x(y((a(xa)y)))a \\ &= (x(a(xa))y^2)a = ((a(xa))(xy^2))a = (((xy^2)(xa))a)a = (((ay^2)x^2)a)a \\ &= (((x^2y^2)a)a)a = (ba)a, \text{ where } b = (x^2y^2)a. \end{aligned}$$

Thus $(ba, a) \in A_a$. Therefore,

$$\begin{aligned} & r\max\left\{((\widetilde{\Psi}_{\alpha(e)} \circ \widetilde{\Psi}_{\beta(e)}) \circ \widetilde{\Psi}_{\beta(e)})(a), \widetilde{\gamma}_1\right\} \\ &= r\max\left[r \sup_{(ba,a)\in A_a}\left\{r\min\left\{(\widetilde{\Psi}_{\alpha(e)} \circ \widetilde{\Psi}_{\beta(e)})(ba), \widetilde{\Psi}_{\beta(e)}(a)\right\}\right\}, \widetilde{\gamma}_1\right] \\ &\succeq r\max\left[r\min\left\{(\widetilde{\Psi}_{\alpha(e)} \circ \widetilde{\Psi}_{\beta(e)})(ba), \widetilde{\Psi}_{\beta(e)}(a)\right\}, \widetilde{\gamma}_1\right] \\ &= r\max\left[r\min\left\{r \sup_{(ba,a)\in A_a}\left\{r\min\left\{\widetilde{\Psi}_{\alpha(e)}(ba), \widetilde{\Psi}_{\beta(e)}(a)\right\}\right\}, \widetilde{\Psi}_{\beta(e)}(a)\right\}, \widetilde{\gamma}_1\right] \\ &\succeq r\max\left[r\min\left\{r\min\left\{\widetilde{\Psi}_{\alpha(e)}((x^2y^2)a), \widetilde{\Psi}_{\beta(e)}(a)\right\}, \widetilde{\Psi}_{\beta(e)}(a)\right\}, \widetilde{\gamma}_1\right] \\ &= r\min\left[r\max\left\{\widetilde{\Psi}_{\alpha(e)}((x^2y^2)a), \widetilde{\Psi}_{\beta(e)}(a), \widetilde{\Psi}_{\beta(e)}(a)\right\}, \widetilde{\gamma}_1\right] \\ &= r\min\left[r\max\left\{\widetilde{\Psi}_{\alpha(e)}((x^2y^2)a), \widetilde{\gamma}_1\right\}, r\max\left\{\widetilde{\Psi}_{\beta(e)}(a), \widetilde{\gamma}_1\right\}, r\max\left\{\widetilde{\Psi}_{\beta(e)}(a), \widetilde{\gamma}_1\right\}\right] \\ &\succeq r\min\left[r\min\left\{\widetilde{\Psi}_{\alpha(e)}(a), \widetilde{\delta}_1\right\}, r\min\left\{\widetilde{\Psi}_{\beta(e)}(a), \widetilde{\delta}_1\right\}, r\min\left\{\widetilde{\Psi}_{\beta(e)}(a), \widetilde{\delta}_1\right\}\right] \\ &= r\min\left\{(\widetilde{\Psi}_{\alpha(e)} \cap \widetilde{\Psi}_{\beta(e)})(a), \widetilde{\delta}_1\right\} \end{aligned}$$

and

$$
\begin{aligned}
&\min\left\{((\phi_{\alpha(e)}\circ\phi_{\beta(e)})\circ\phi_{\beta(e)})(a),\gamma_2\right\}\\
&=\min\left[r\min_{(ba,a)\in A_a}\left\{\max\left\{(\phi_{\alpha(e)}\circ\phi_{\beta(e)})(ba),\phi_{\beta(e)}(a)\right\}\right\},\gamma_2\right]\\
&\leq\min\left[\max\left\{(\phi_{\alpha(e)}\circ\phi_{\beta(e)})(ba),\phi_{\beta(e)}(a)\right\},\gamma_2\right]\\
&=\min\left[\max\left\{r\min_{(ba,a)\in A_a}\left\{\max\left\{\phi_{\alpha(e)}(ba),\phi_{\beta(e)}(a)\right\}\right\},\phi_{\beta(e)}(a)\right\},\gamma_2\right]\\
&\leq\min\left[\max\left\{\max\left\{\phi_{\alpha(e)}((x^2y^2)a),\phi_{\beta(e)}(a)\right\},\phi_{\beta(e)}(a)\right\},\gamma_2\right]\\
&=\max\left[\min\left\{\phi_{\alpha(e)}((x^2y^2)a),\phi_{\beta(e)}(a),\phi_{\beta(e)}(a)\right\},\gamma_2\right]\\
&=\max\left[\min\left\{\phi_{\alpha(e)}((x^2y^2)a),\gamma_2\right\},\min\left\{\phi_{\beta(e)}(a),\gamma_2\right\},\min\left\{\phi_{\beta(e)}(a),\gamma_2\right\}\right]\\
&\leq\max\left[\max\left\{\phi_{\alpha(e)}(a),\delta_2\right\},\max\left\{\phi_{\beta(e)}(a),\delta_2\right\},\max\left\{\phi_{\beta(e)}(a),\delta_2\right\}\right]\\
&=\max\left\{(\phi_{\alpha(e)}\cap\phi_{\beta(e)})(a),\delta_2\right\}
\end{aligned}
$$

Thus, by Lemma 1, $\langle\alpha,L\rangle\widetilde{\cap}\langle\beta,B\rangle\subseteq\vee q_{(\lambda,\mu)}[\langle\alpha,L\rangle\odot\langle\beta,B\rangle]\odot\langle\beta,B\rangle$.

(iii) $\Rightarrow$ (ii) Let L be left ideal and B be bi-ideal of $\mathcal{G}$ with left identity, then by Theorem 6, $\Sigma(L,E)$ and $\Sigma(B,E)$ is an $(\in_\lambda,\in_\lambda\vee q_\mu)$-cubic soft left ideal and $(\in_\lambda,\in_\lambda\vee q_\mu)$-cubic soft bi-ideal over $\mathcal{G}$, respectively. Now by using assumption we have $\Sigma(L,E)\cap\Sigma(B,E)\subset_{(\lambda,\mu)}[\Sigma(L,E)\odot\Sigma(B,E)]\odot\Sigma(B,E)$. By using Lemma 3, we have $\mathcal{X}^\mu_{\lambda(L\cap B)}=_{(\lambda,\mu)}\mathcal{X}^\mu_{\lambda L}\cap\mathcal{X}^\mu_{\lambda B}\subseteq\vee q_\mu(\mathcal{X}^\mu_{\lambda L}\odot\mathcal{X}^\mu_{\lambda B})\odot\mathcal{X}^\mu_{\lambda B}=\mathcal{X}^\mu_{\lambda(LB]}\odot\mathcal{X}^\mu_{\lambda B}=_{(\lambda,\mu)}\mathcal{X}^\mu_{\lambda((LB]B)}$. By Lemma 3, $(L\cap B)\subseteq((LB]B]$.

(ii) $\Rightarrow$ (i) Sine $(\mathcal{G}a]$ is a both left and bi-ideal of $\mathcal{G}$ with left identity. Therefore, $a\in(\mathcal{G}a]\cap(\mathcal{G}a]=(((\mathcal{G}a](\mathcal{G}a]](\mathcal{G}a]]=(((\mathcal{G}a](\mathcal{G}a])((\mathcal{G}a]=(((\mathcal{G}a)(\mathcal{G}a))(\mathcal{G}a)]=(((\mathcal{G}\mathcal{G})(aa))(\mathcal{G}a]\subseteq(\mathcal{G}a^2)\mathcal{G}$. Hence, $\mathcal{G}$ is intra-regular. ∎

Theorem 8 *The following conditions are equivalent for $\mathcal{G}$ with left identity:*

(i) *$\mathcal{G}$ is intra-regular.*

(ii) *$\langle\alpha,L\rangle\widetilde{\cap}(\beta,B)\widetilde{\cap}\langle\omega,R\rangle\subseteq\vee q_{(\lambda,\mu)}[\langle\alpha,L\rangle\odot\langle\beta,B\rangle]\odot[\langle\alpha,L\rangle\odot\langle\omega,R\rangle]$, where $\langle\alpha,L\rangle$ an $(\in_\lambda,\in_\lambda\vee q_\mu)$-cubic soft left ideal, $\langle\beta,R\rangle$ ia an $(\in_\lambda,\in_\lambda\vee q_\mu)$-cubic soft right ideal and (ω,B) an $(\in_\lambda,\in_\lambda\vee q_\mu)$-cubic soft bi-ideal over $\mathcal{G}$.*

Proof. (i) $\Rightarrow$ (ii) Let $\langle\alpha,L\rangle$ an $(\in_\lambda,\in_\lambda\vee q_\mu)$-cubic soft left ideal, $\langle\beta,R\rangle$ ia an $(\in_\lambda,\in_\lambda\vee q_\mu)$-cubic soft right ideal and (ω,B) an $(\in_\lambda,\in_\lambda\vee q_\mu)$-cubic soft bi-ideal over an intra-regular $\mathcal{G}$ with left identity. Let a be any element of $\mathcal{G}$, $\langle\alpha,L\rangle\widetilde{\cap}\langle\omega,B\rangle\widetilde{\cap}(\beta,R)=\langle\theta,B\cup L\cup R\rangle$. For any $e\in B\cup L\cup R$. We consider the following cases.

Case 1. $e\in L\backslash(B\cap R)$, then $\alpha(e)=[(\alpha\circ\beta)\circ\omega](e)$.

Case 2. $e\in B\backslash(L\cap R)$, then $\beta(e)=[(\alpha\circ\beta)\circ\omega](e)$.

Case 3. $e\in R\backslash(L\cap B)$, then $\omega(e)=[(\alpha\circ\beta)\circ\omega](e)$.

Case 4. $e\in(L\cap B)\cap R$, then $[(\alpha\circ\beta)\circ\omega](e)=[\alpha(e)\circ\beta(e)]\circ\omega(e)=[\alpha(e)\circ\beta(e)]\circ[\alpha(e)\circ\omega(e)]$.We have to show that $\alpha(e)\cap\beta(e)\cap\omega(e)\subseteq\vee q_{(\lambda,\mu)}(\alpha(e)\circ\beta(e))\circ(\alpha(e)\circ\omega(e))$. Since $\mathcal{G}$ is intra-regular, then for any $a\in\mathcal{G}$ there exist $x,y\in\mathcal{G}$ such that $a\leq(xa^2)y=(a(xa))y=y(xa)\leq y(x((xa^2)y)=y(x(a(xa))y)=$

$y((a(xa))(xy)) = (a(xa))(y(xy)) = ((y(xy))(xa))a = ba$, where $b = (y(xy))(xa)$ and $a \leq (xa^2)y = (x(aa))y = (a(xa))y = (y(xa))a = ca$. Thus, $(c, a) \in A_a$ and $(y(xa), a) \in A_a$. Since $A_a \neq \Phi$, therefore

$$\begin{aligned}
&r\max\left\{(\widetilde{\Psi}_{\alpha(e)} \circ \widetilde{\Psi}_{\beta(e)})(a), \widetilde{r}_1\right\} \\
&\quad = r\max\left[r \sup_{(ca,a)\in A_a}\left\{r\min\left\{\widetilde{\Psi}_{\alpha(e)}(c), \widetilde{\Psi}_{\beta(e)}(a)\right\}\right\}, \widetilde{\gamma}_1\right] \\
&\quad \succeq r\max\left[r\min\left\{\widetilde{\Psi}_{\alpha(e)}(y(xa)), \widetilde{\Psi}_{\beta(e)}(a)\right\}, \widetilde{\gamma}_1\right] \\
&\quad = r\min\left[r\max\left\{\widetilde{\Psi}_{\alpha(e)}(y(xa)), \widetilde{\gamma}_1\right\}, r\max\left\{\widetilde{\Psi}_{\beta(e)}(a), \widetilde{\gamma}_1\right\}\right] \\
&\quad \succeq r\min\left[r\min\left\{\widetilde{\Psi}_{\alpha(e)}(a), \widetilde{\delta}_1\right\}, r\min\left\{\widetilde{\Psi}_{\beta(e)}(a), \widetilde{\delta}_1\right\}\right] \\
&\quad = r\min\left\{(\widetilde{\Psi}_{\alpha(e)} \cap \widetilde{\Psi}_{\beta(e)})(a), \widetilde{\delta}_1\right\}
\end{aligned}$$

and

$$\begin{aligned}
\min\left\{(\phi_{\alpha(e)} \circ \phi_{\beta(e)})(a), \gamma_2\right\} &= \min\left[r \min_{(ca,a)\in A_a}\left\{\max\left\{\left(\phi_{\alpha(e)}(c), \phi_{\beta(e)}(a)\right)\right\}\right\}, \gamma_2\right] \\
&\leq \min\left[\max\left\{\phi_{\alpha(e)}(y(xa)), \phi_{\beta(e)}(a)\right\}, \gamma_2\right] \\
&= \max\left[\min\left\{\phi_{\alpha(e)}(y(xa)), \gamma_2\right\}, \min\left\{\phi_{\beta(e)}(a), \gamma_2\right\}\right] \\
&\leq \max\left[\max\left\{\phi_{\alpha(e)}(a), \delta_2\right\}, \max\left\{\phi_{\beta(e)}(a), \delta_2\right\}\right] \\
&= \max\left\{\left(\phi_{\alpha(e)} \cap \phi_{\beta(e)}\right)(a), \delta_2\right\}
\end{aligned}$$

Thus, by Lemma 1 $\langle\alpha, L\rangle\widetilde{\cap}\langle\beta, B\rangle \subseteq \vee q_{(\lambda,\mu)}\langle\alpha, L\rangle \odot \langle\beta, B\rangle$.

Similarly, we can show that $\langle\alpha, L\rangle\widetilde{\cap}\langle\omega, R\rangle \subseteq \vee q_{(\lambda,\mu)}\langle\alpha, L\rangle \odot \langle\omega, R\rangle$. Hence,

$$\langle\alpha, L\rangle\widetilde{\cap}(\beta, B)\widetilde{\cap}\langle\omega, R\rangle \subseteq \vee q_{(\lambda,\mu)}[\langle\alpha, L\rangle \odot \langle\beta, B\rangle] \odot [\langle\alpha, L\rangle \odot \langle\omega, R\rangle].$$

(ii) $\Rightarrow$ (i) Since $\mathcal{G} \subseteq \mathcal{G}$ with left identity therefore by using Theorem 6 $\Sigma\langle\mathcal{G}, E\rangle$ is cubic soft set over $\mathcal{G}$ and $\mathcal{X}^{\mu}_{\lambda\mathcal{G}}$ is an $(\in_\lambda, \in_\lambda \vee q_\mu)$-cubic soft right ideal over $\mathcal{G}$. Therefore,

$$\begin{aligned}
\langle\alpha, L\rangle\widetilde{\cap}(\beta, B) &= \langle\alpha, L\rangle\widetilde{\cap}(\beta, B)\widetilde{\cap}\mathcal{X}^{\mu}_{\lambda\mathcal{G}} \subseteq \vee q_{(\lambda,\mu)}[\langle\alpha, L\rangle \odot (\beta, B)] \odot [(\beta, B) \odot \mathcal{X}^{\mu}_{\lambda\mathcal{G}}] \\
&\subseteq \vee q_{(\lambda,\mu)}[\langle\alpha, L\rangle \odot (\beta, B)] \odot (\beta, B).
\end{aligned}$$

This implies that $\langle\alpha, L\rangle\widetilde{\cap}(\beta, B) \subseteq \vee q_{(\lambda,\mu)}[\langle\alpha, L\rangle \odot (\beta, B)] \odot (\beta, B)$. Hence, by Theorem 7, $\mathcal{G}$ is intra-regular. ∎

Lemma 7 *The following conditions are equivalent for $\mathcal{G}$ with left identity.*

(i) *$\mathcal{G}$ is intra-regular.*

(ii) *$\langle\alpha, R\rangle\widetilde{\cap}\langle\beta, L\rangle = \langle\alpha, R\rangle \odot \langle\beta, L\rangle$,.for an $(\in_\lambda, \in_\lambda \vee q_\mu)$-cubic soft right ideal $\langle\alpha, R\rangle$ and for an $(\in_\lambda, \in_\lambda \vee q_\mu)$-cubic soft left ideal $\langle\beta, L\rangle$ over $\mathcal{G}$, such that $\langle\alpha, R\rangle$ is an $(\in_\lambda, \in_\lambda \vee q_\mu)$-cubic soft semiprime.*

Proof. The proof is straightforward. ∎

Note that every intra-regular $\mathcal{AG}$-groupoid $\mathcal{G}$, with left identity is regular but the converse is not true in general [18].

Theorem 9 *The following conditions are equivalent for $\mathcal{G}$ with left identity:*

(i) *$\mathcal{G}$ is intra-regular.*

(ii) $\langle\alpha, R\rangle\widetilde{\cap}\langle\omega, B\rangle\widetilde{\cap}\langle\beta, L\rangle \subseteq \vee q_{(\lambda,\mu)}[\langle\alpha, R\rangle\odot\langle\omega, B\rangle]\odot\langle\beta, L\rangle$, *for any $(\in_\lambda, \in_\lambda \vee q_\mu)$-cubic soft right ideal $\langle\alpha, R\rangle$, $(\in_\lambda, \in_\lambda \vee q_\mu)$-cubic soft bi-ideal $\langle\omega, B\rangle$ and $(\in_\lambda, \in_\lambda \vee q_\mu)$-cubic soft left ideal $\langle\beta, L\rangle$ over $\mathcal{G}$.*

(iii) $\langle\alpha, R\rangle\widetilde{\cap}\langle\omega, B\rangle\widetilde{\cap}\langle\beta, L\rangle \subseteq \vee q_{(\lambda,\mu)}[\langle\omega, B\rangle\odot\langle\alpha, R\rangle]\odot\langle\beta, L\rangle$, *for any $(\in_\lambda, \in_\lambda \vee q_\mu)$-cubic soft right ideal $\langle\alpha, R\rangle$, $(\in_\lambda, \in_\lambda \vee q_\mu)$-cubic soft generalized bi-ideal $\langle\omega, B\rangle$ and $(\in_\lambda, \in_\lambda \vee q_\mu)$-cubic soft left ideal $\langle\beta, L\rangle$ over $\mathcal{G}$.*

Proof. (i) $\Rightarrow$ (iii) Assume that $\mathcal{G}$ is intra-regular with left identity. Let $\langle\alpha, R\rangle$, $\langle\omega, B\rangle$ and $\langle\beta, L\rangle$ be any $(\in_\lambda, \in_\lambda \vee q_\mu)$-cubic soft right ideal, $(\in_\lambda, \in_\lambda \vee q_\mu)$-cubic soft bi-ideal and $(\in_\lambda, \in_\lambda \vee q_\mu)$-cubic soft left ideal over $\mathcal{G}$, respectively. Let $\langle\alpha, R\rangle\widetilde{\cap}\langle\omega, B\rangle\widetilde{\cap}(\beta, L) = \langle K, R\cup B\cup L\rangle$. For any $e \in R\cup B\cup L$. We consider the following cases.

Case 1. $e \in R\backslash(B\cap L)$, then $\alpha(e) = [(\alpha\circ\omega)\circ\beta](e)$.

Case 2. $e \in B\backslash(R\cap L)$, then $\beta(e) = [(\alpha\circ\omega)\circ\beta](e)$.

Case 3. $e \in L\backslash(R\cap B)$, then $\omega(e) = [(\alpha\circ\omega)\circ\beta](e)$.

Case 4. $e \in (L\cap B)\cap R$, then $[(\alpha\circ\omega)\circ\beta](e) = [\alpha(e)\circ\omega(e)]\circ\beta(e)$. Since $\mathcal{G}$ is intra-regular, then for any $a \in \mathcal{G}$ there exist $x, y, u \in \mathcal{G}$ such that $a \leq (ua)a$ and $a \leq (xa^2)y$. Now, $au \leq ((xa^2)y)u = (uy)(xa^2) = (ux)(ya^2) = (a^2x)(yu) = ((yu)x)(aa) = ((yu)a)(xa) = (ax)(a(yu)) = (aa)(x(yu)) = ((x(yu))a)a = ((x(yu))(ea))a = ((xe)((yu)a))a = ((ae)((yu)x))a$. Thus, $(((ae)((yu)x)), a) = (d, a) \in A_{au}$, where $d = ((ae)((yu)x))$. Therefore

$$
\begin{aligned}
& r\max\left\{((\widetilde{\Psi}_{\alpha(e)}\circ\widetilde{\Psi}_{\beta(e)})\circ\widetilde{\Psi}_{\omega(e)})(a), \widetilde{\gamma}_1\right\} \\
&= r\max\left[r\sup_{(au,a)\in A_a}\left\{r\min\left\{(\widetilde{\Psi}_{\alpha(e)}\circ\widetilde{\Psi}_{\beta(e)})(au), \widetilde{\Psi}_{\omega(e)}(a)\right\}\right\}, \widetilde{\gamma}_1\right] \\
&\succeq r\max\left[r\min\left\{(\widetilde{\Psi}_{\alpha(e)}\circ\widetilde{\Psi}_{\beta(e)})(au), \widetilde{\Psi}_{\omega(e)}(a)\right\}, \widetilde{\gamma}_1\right] \\
&= r\max\left[r\min\left\{r\sup_{(d,a)\in A_{au}}\left\{r\min\left\{\widetilde{\Psi}_{\alpha(e)}(d), \widetilde{\Psi}_{\beta(e)}(a)\right\}\right\}, \widetilde{\Psi}_{\omega(e)}(a)\right\}, \widetilde{\gamma}_1\right] \\
&\succeq r\max\left[r\min\left\{r\min\left\{\widetilde{\Psi}_{\alpha(e)}(d), \widetilde{\Psi}_{\beta(e)}(a)\right\}, \widetilde{\Psi}_{\omega(e)}(a)\right\}, \widetilde{\gamma}_1\right] \\
&= r\min\left[r\max\left\{\widetilde{\Psi}_{\alpha(e)}((ae)((yu)x)), \widetilde{\Psi}_{\beta(e)}(a), \widetilde{\Psi}_{\omega(e)}(a)\right\}, \widetilde{\gamma}_1\right]
\end{aligned}
$$

$$= r\min\left[r\max\left\{\widetilde{\Psi}_{\alpha(e)}((ae)((yu)x)),\widetilde{\gamma}_1\right\},r\max\left\{\widetilde{\Psi}_{\beta(e)}(a),\widetilde{\gamma}_1\right\},r\max\left\{\widetilde{\Psi}_{\omega(e)}(a),\widetilde{\gamma}_1\right\}\right]$$
$$\succeq r\min\left[r\min\left\{\widetilde{\Psi}_{\alpha(e)}(a),\widetilde{\delta}_1\right\},r\min\left\{\widetilde{\Psi}_{\beta(e)}(a),\widetilde{\delta}_1\right\},r\min\left\{\widetilde{\Psi}_{\omega(e)}(a),\widetilde{\delta}_1\right\}\right]$$
$$= r\min\left\{(\widetilde{\Psi}_{\alpha(e)}\cap\widetilde{\Psi}_{\beta(e)}\cap\widetilde{\Psi}_{\omega(e)})(a),\widetilde{\delta}_1\right\}$$

and

$$\min\left\{((\phi_{\alpha(e)}\circ\phi_{\beta(e)})\circ\phi_{\omega(e)})(a),\gamma_2\right\}$$
$$= \min\left[r\min_{(au,a)\in A_a}\left\{\max\left\{(\phi_{\alpha(e)}\circ\phi_{\beta(e)})(au),\phi_{\omega(e)}(a)\right\}\right\},\gamma_2\right]$$
$$\leq \min\left[\max\left\{(\phi_{\alpha(e)}\circ\phi_{\beta(e)})(au),\phi_{\omega(e)}(a)\right\},\gamma_2\right]$$
$$= \min\left[\max\left\{\min_{(d,a)\in A_{au}}\left\{\max\left\{\phi_{\alpha(e)}(d),\phi_{\beta(e)}(a)\right\}\right\},\phi_{\omega(e)}(a)\right\},\gamma_2\right]$$
$$\leq \min\left[\max\left\{\max\left\{\phi_{\alpha(e)}(d),\phi_{\beta(e)}(a)\right\},\phi_{\omega(e)}(a)\right\},\gamma_2\right]$$
$$= \max\left[\min\left\{\phi_{\alpha(e)}((ae)((yu)x)),\phi_{\beta(e)}(a),\phi_{\omega(e)}(a)\right\},\gamma_2\right]$$
$$= \max\left[\min\left\{\phi_{\alpha(e)}((ae)((yu)x)),\gamma_2\right\},\min\left\{\phi_{\beta(e)}(a),\gamma_2\right\},\min\left\{\phi_{\omega(e)}(a),\gamma_2\right\}\right]$$
$$\leq \max\left[\max\left\{\phi_{\alpha(e)}(a),\delta_2\right\},\max\left\{\phi_{\beta(e)}(a),\delta_2\right\},\max\left\{\phi_{\omega(e)}(a),\delta_2\right\}\right]$$
$$= \max\left\{(\phi_{\alpha(e)}\cap\phi_{\beta(e)}\cap\phi_{\omega(e)})(a),\delta_2\right\}.$$

Thus, by Lemma 1, $\langle\alpha,R\rangle\widetilde{\cap}\langle\omega,B\rangle\widetilde{\cap}\langle\beta,L\rangle\subseteq\vee q_{(\lambda,\mu)}[\langle\alpha,R\rangle\odot\langle\omega,B\rangle]\odot\langle\beta,L\rangle$.

(iii) $\Rightarrow$ (ii) It is straightforward.
(ii) $\Rightarrow$ (i) By using the given assumption, it is easy to show that $\langle\alpha,R\rangle\odot\langle\omega,B\rangle\subseteq\vee q_{(\lambda,\mu)}\langle\alpha,R\rangle\widetilde{\cap}\langle\omega,B\rangle$ and therefore by using Lemma 7, $\mathcal{G}$ is intra-regular. ∎

Theorem 10 *The following conditions are equivalent for $\mathcal{G}$ with left identity:*

(i) *$\mathcal{G}$ is intra-regular.*

(ii) $[\langle\alpha,A\rangle\odot\langle\beta,B\rangle]\widetilde{\cap}[\langle\beta,B\rangle\odot\langle\alpha,A\rangle]\supseteq\vee q_{(\lambda,\mu)}\langle\alpha,A\rangle\widetilde{\cap}\langle\beta,B\rangle$, *for any $(\in_\lambda,\in_\lambda\vee q_\mu)$-cubic soft right ideal $\langle\alpha,A\rangle$ and $(\in_\lambda,\in_\lambda\vee q_\mu)$-cubic soft left ideal $\langle\beta,B\rangle$ over $\mathcal{G}$.*

(iii) $[\langle\alpha,A\rangle\odot\langle\beta,B\rangle]\widetilde{\cap}[\langle\beta,B\rangle\odot\langle\alpha,A\rangle]\supseteq\vee q_{(\lambda,\mu)}\langle\alpha,A\rangle\widetilde{\cap}\langle\beta,B\rangle$, *for any $(\in_\lambda,\in_\lambda\vee q_\mu)$-cubic soft right ideal $\langle\alpha,A\rangle$ and $(\in_\lambda,\in_\lambda\vee q_\mu)$-cubic soft bi-ideal $\langle\beta,B\rangle$ over $\mathcal{G}$.*

(iv) $[\langle\alpha,A\rangle\odot\langle\beta,B\rangle]\widetilde{\cap}[\langle\beta,B\rangle\odot\langle\alpha,A\rangle]\supseteq\vee q_{(\lambda,\mu)}\langle\alpha,A\rangle\widetilde{\cap}\langle\beta,B\rangle$, *for any $(\in_\lambda,\in_\lambda\vee q_\mu)$-cubic soft bi-ideal $\langle\alpha,A\rangle$ and for $(\in_\lambda,\in_\lambda\vee q_\mu)$-cubic soft generalized bi-ideal $\langle\beta,B\rangle$ over $\mathcal{G}$.*

(v) $[\langle\alpha,A\rangle\odot\langle\beta,B\rangle]\widetilde{\cap}[\langle\beta,B\rangle\odot\langle\alpha,A\rangle]\supseteq\vee q_{(\lambda,\mu)}\langle\alpha,A\rangle\widetilde{\cap}\langle\beta,B\rangle$, *for $(\in_\lambda,\in_\lambda\vee q_\mu)$-cubic soft bi-ideals $\langle\alpha,A\rangle$ and $\langle\beta,B\rangle$ over $\mathcal{G}$.*

(vi) $[\langle\alpha,A\rangle\odot\langle\beta,B\rangle]\widetilde{\cap}[\langle\beta,B\rangle\odot\langle\alpha,A\rangle]\supseteq\vee q_{(\lambda,\mu)}\langle\alpha,A\rangle\widetilde{\cap}\langle\beta,B\rangle$, *for $(\in_\lambda,\in_\lambda\vee q_\mu)$-cubic soft generalized bi-ideals $\langle\alpha,A\rangle$ and $\langle\beta,B\rangle$ over $\mathcal{G}$.*

Proof. (i) $\Rightarrow$ (vi) Let (α, A) and (β, B) be an $(\in_\lambda, \in_\lambda \vee q_\mu)$-cubic soft generalized bi-ideals over an intra-regular $\mathcal{G}$ with left identity. Let $e \in A \cup B$ and $(\alpha, A)\widetilde{\cap}(\beta, B) = (\omega, A \cup B)$.

Now, we consider the following cases.

Case 1. $e \in A - B$. Then $\omega(e) = \alpha(e) = (\alpha \circ \beta)(e)$.

Case 2. $e \in B - A$. Then $\omega(e) = \beta(e) = (\alpha \circ \beta)(e)$.

Case 3. $e \in A \cap B$. Then $\omega(e) = \alpha(e) \cap \beta(e)$ and $(\alpha \circ \beta)(e) = \alpha(e) \circ \beta(e)$.

We have to show that $\alpha(e) \circ \beta(e) \supseteq \vee q_{(\lambda,\mu)}\alpha(\epsilon) \cap \beta(e)$.

Since $\mathcal{G}$ is intra-regular, then for any $a \in \mathcal{G}$ there exist $x, y, u \in \mathcal{G}$ such that $a \leq (xa^2)y$ and $a \leq (au)a$. Now

$$a \leq (xa^2)y = (a(xa))y = (y(xa))a,$$

and

$$\begin{aligned}
y(xa) &\leq y(x((xa^2)y)) = y((xa^2)(xy)) = (xa^2)(y(xy)) = (xa^2)(xy^2)\\
&= (xx)(a^2y^2) = a^2(x^2y^2) = (aa)(x^2y^2)\\
&= ((x^2y^2)a)a = ((x^2y^2)((xa^2)y))a\\
&= ((xa^2)((x^2y^2)y))a = ((x(x^2y^2))(a^2y))a\\
&= ((y(x^2y^2))(a^2x))a = (a^2((y(x^2y^2))x))a\\
&= ((x(y(x^2y^2)))a^2)a = (a(x(y(x^2y^2)))a)a\\
&= (av)a, \text{ where } v = a(x(y(x^2y^2)))a.
\end{aligned}$$

Hence,

$$a \leq ((av)a)a.$$

Hence, $((av)a, a) \in A_a$. Therefore,

$$\begin{aligned}
&r\max\left\{(\widetilde{\Psi}_{\alpha(e)} \circ \widetilde{\Psi}_{\beta(e)})(a), \widetilde{r}_1\right\}\\
&= r\max\left[r \sup_{((av)a,a)\in A_a}\left\{r\min\left\{\left(\widetilde{\Psi}_{\alpha(e)}((av)a), \widetilde{\Psi}_{\beta(e)}(a)\right)\right\}\right\}, \widetilde{\gamma}_1\right]\\
&\succeq r\max\left[r\min\left\{\widetilde{\Psi}_{\alpha(e)}((av)a), \widetilde{\Psi}_{\beta(e)}(a)\right\}, \widetilde{\gamma}_1\right]\\
&= r\min\left[r\max\left\{\widetilde{\Psi}_{\alpha(e)}((av)a), \widetilde{\gamma}_1\right\}, r\max\left\{\widetilde{\Psi}_{\beta(e)}(a), \widetilde{\gamma}_1\right\}\right]\\
&\succeq r\min\left[r\min\left\{\widetilde{\Psi}_{\alpha(e)}(a), \widetilde{\delta}_1\right\}, r\min\left\{\widetilde{\Psi}_{\beta(e)}(a), \widetilde{\delta}_1\right\}\right]\\
&= r\min\left\{\left(\widetilde{\Psi}_{\alpha(e)} \cap \widetilde{\Psi}_{\beta(e)}\right)(a), \widetilde{\delta}_1\right\}
\end{aligned}$$

and

$$\begin{aligned}
&\min\left\{(\phi_{\alpha(e)}\circ\phi_{\beta(e)})(a),\gamma_2\right\}\\
&\quad=\min\left[r\underset{((av)a,a)\in A_a}{\min}\left\{\max\left\{\left(\phi_{\alpha(e)}((av)a),\phi_{\beta(e)}(a)\right)\right\}\right\},\gamma_2\right]\\
&\quad\leq\min\left[\max\left\{\phi_{\alpha(e)}((av)a),\phi_{\beta(e)}(a)\right\},\gamma_2\right]\\
&\quad=\max\left[\min\left\{\phi_{\alpha(e)}((av)a),\gamma_2\right\},\min\left\{\phi_{\beta(e)}(a),\gamma_2\right\}\right]\\
&\quad\leq\max\left[\max\left\{\phi_{\alpha(e)}(a),\delta_2\right\},\max\left\{\phi_{\beta(e)}(a),\delta_2\right\}\right]\\
&\quad=\max\left\{\left(\phi_{\alpha(e)}\cap\phi_{\beta(e)}\right)(a),\delta_2\right\}
\end{aligned}$$

This shows that

$$\langle\alpha,A\rangle\odot\langle\beta,B\rangle\supseteq\vee q_{(\lambda,\mu)}\langle\alpha,A\rangle\widetilde{\cap}\langle\beta,B\rangle$$

and, similarly, we can show that

$$\langle\beta,B\rangle\odot\langle\alpha,A\rangle\supseteq\vee q_{(\lambda,\mu)}\langle\alpha,A\rangle\widetilde{\cap}\langle\beta,B\rangle.$$

Therefore,

$$[\langle\alpha,A\rangle\odot\langle\beta,B\rangle]\widetilde{\cap}[\langle\beta,B\rangle\odot\langle\alpha,A\rangle]\vee q_{(\lambda,\mu)}\langle\alpha,A\rangle\widetilde{\cap}\langle\beta,B\rangle.$$

(vi) $\Rightarrow$ (v) $\Rightarrow$ (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) are obvious cases.

(ii) $\Rightarrow$ (i) By using (ii), it is easy to show that

$$\langle\alpha,A\rangle\odot\langle\beta,B\rangle\subseteq\vee q_{(\lambda,\mu)}\langle\alpha,A\rangle\widetilde{\cap}\langle\beta,B\rangle$$

and, therefore, by using Lemma 7, $\mathcal{G}$ is intra-regular. ∎

Conclusion

In this paper, we discuss a new approach to soft set through applications of cubic set. By combination of cubic set and soft set, we introduce a new mathematical model which is called cubic soft set. We introduce a new concept of generalized cubic soft sets and apply it to the ideal theory of ordered $\mathcal{AG}$-groupoids. We introduce $(\in_{(\widetilde{\gamma}_1,\gamma_2)},\in_{(\widetilde{\gamma}_1,\gamma_2)}\vee q_{(\widetilde{\delta}_1,\delta_2)})$-cubic soft left (resp., right, bi-) ideals over an ordered $\mathcal{AG}$-groupoids and characterize intra-regular ordered $\mathcal{AG}$-groupoids in terms of $(\in_{(\widetilde{\gamma}_1,\gamma_2)},\in_{(\widetilde{\gamma}_1,\gamma_2)}\vee q_{(\widetilde{\delta}_1,\delta_2)})$-cubic soft sets. In our future research work we will focus on considering other types of $(\in_{(\widetilde{\gamma}_1,\gamma_2)},\in_{(\widetilde{\gamma}_1,\gamma_2)}\vee q_{(\widetilde{\delta}_1,\delta_2)})$-cubic soft ideals over an intra-regular ordered $\mathcal{AG}$-groupoids.

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A STUDY ON FUZZY INTERIOR HYPERIDEALS IN ORDERED SEMIHYPERGROUPS

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Abstract. In this paper, we first introduce the concept of fuzzy interior hyperideals of an ordered semihypergroup S by the ordered fuzzy points of S, and investigate its related properties. In particular, we give the characterization of fuzzy interior hyperideal generated by a fuzzy subset in an ordered semihypergroup. Furthermore, the idea of normal fuzzy interior hyperideals in ordered semihypergroups is given and several related characterization theorems are provided. Finally, some new characterizations of semisimple ordered semihypergroups by the properties of fuzzy interior hyperideals are given.

Keywords: ordered semihypergroup, ordered fuzzy point, interior hyperideal, fuzzy interior hyperideal, normal fuzzy interior hyperideal.

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1. Introduction

As we know, an ordered semigroup is a semigroup together with a partial order that is compatible with the semigroup operation. Ordered semigroups play a prominent role in mathematics with a wider range of applications in many disciplines such as sequential machines, formal languages, computer arithmetics, error-correcting codes and many others. Ordered semigroups have been investigated by several researchers, for example, Alimov [1], Clifford [4] and Conrad [5], also see [38]. The theory of fuzzy sets, which was initially introduced by Zadeh [39], has been applied to many mathematical branches. In particular, Kehayopulu and Tsingelis [22] applied the concept of fuzzy sets to the theory of ordered semigroups. Then they defined "fuzzy" analogue of several notations, which appeared to be useful in the theory of ordered semigroups. The theory of fuzzy sets on ordered semigroups has been recently developed. For more details, the reader is referred to [23], [24], [34], [36]. In [32], Sardar et al. studied the notions of interior ideals and characteristic interior ideals of a Γ-semigroups. In [33], they studied the notion of generalized fuzzy interior ideals in Γ-semigroups. Also, see [13].

The theory of algebraic hyperstructures which is a generalization of the concept of algebraic structures was first introduced by Marty [27]. Later on, hyperstructures are widely studied from the theoretical point of view and for their applications to many subjects of pure and applied mathematics. One of the main reason which attracts researches towards hyperstructures is its unique property that in hyperstructures composition of two elements is a set, while in classical algebraic structures the composition of two elements is an element. Thus hyperstructures are natural extension of classical algebraic structures. Recently, many researchers have worked on algebraic hyperstructures and generalized various classical algebraic structures, for instance, see [6], [7], [11], [17], [28], [37]. Especially, semihypergroups are the simplest algebraic hyperstructures which possess the properties of closure and associativity. Nowadays many authors have studied different aspects of semihypergroups, for example, Davvaz [8], Davvaz and Poursalavati [14], Hila et al. [21] and Leoreanu [25], also see [10], [29]. It is worth pointing out that Heidari and Davvaz [19] applied the theory of hyperstructures to ordered semigroups and introduced the concept of ordered semihypergroups, which is a generalization of ordered semigroups, also see [3], [12].

The study of fuzzy hyperstructures is an interesting research topic of fuzzy set theory. We noticed that the relationships between the fuzzy sets and algebraic hyperstructures have been already considered by Corsini, Davvaz, Leoreanu, Dudek, Jun, Zhan, Hila and others, for instance, the reader can refer to [2], [9], [15], [20], [26], [40]. In [16], Ersoy et al. investigated the properties of fuzzy interior hyperideals of Γ-semihypergroups. It is now natural to investigate similar type of the existing fuzzy subsystems of ordered semihypergroups. In [30], [31], Pibaljommee and Davvaz studied fuzzy hyperideals of ordered semihypergroups. Also, see [35]. As a further study of ordered semihypergroups theory, we attempt in the present paper to study fuzzy interior hyperideals of ordered semihypergroups in detail.

The rest of this paper is organized as follows. After an introduction, in Section 2 we recall some basic definitions and results of ordered semihypergroups

which will be used throughout this paper. In Section 3, we define and study the interior hyperideals of an ordered semihypergroup. Furthermore, we introduce the concept of fuzzy interior hyperideals of ordered semihypergroups by the ordered fuzzy points, and investigate its related properties. In particular, we characterize the fuzzy interior hyperideal generated by a fuzzy subset in an ordered semihypergroup. The idea of normal fuzzy interior hyperideals in ordered semihypergroups is given and several related characterization theorems are provided in Section 4. In Section 5, we give some new characterizations of semisimple ordered semihypergroups in terms of fuzzy interior hyperideals.

2. Preliminaries and some notations

Recall first the basic terms and definitions from the hyperstructure theory.

As we know, a *hypergroupoid* $(S, \circ)$ is a nonempty set S together with a hyperoperation, that is a map $\circ : S \times S \to P^*(S)$, where $P^*(S)$ denotes the set of all nonempty subsets of S (see [6]). The image of the pair (x, y) is denoted by $x \circ y$. If $x \in S$ and A, B are nonempty subsets of S, then $A \circ B$ is defined by $A \circ B = \bigcup_{a \in A, b \in B} a \circ b$. The notations $A \circ x$ and $x \circ A$ are used for $A \circ \{x\}$ and $\{x\} \circ A$, respectively.

We say that a hypergroupoid $(S, \circ)$ is a *semihypergroup* if the hyperoperation " $\circ$ " is associative, that is, $(x \circ y) \circ z = x \circ (y \circ z)$ for all $x, y, z \in S$ (see [7]).

Now, we recall the notion of ordered semihypergroups from [19].

Definition 2.1. An algebraic hyperstructure $(S, \circ, \leq)$ is called an *ordered semihypergroup* (also called *po-semihypergroup* in [19]) if $(S, \circ)$ is a semihypergroup and $(S, \leq)$ is a partially ordered set such that: for any $x, y, a \in S$, $x \leq y$ implies $a \circ x \preceq a \circ y$ and $x \circ a \preceq y \circ a$. Here, if $A, B \in P^*(S)$, then we say that $A \preceq B$ if for every $a \in A$ there exists $b \in B$ such that $a \leq b$. In particular, if $A = \{a\}$, then we write $a \preceq B$ instead of $\{a\} \preceq B$.

Throughout this paper, unless otherwise mentioned, S will denote an ordered semihypergroup.

Let S be an ordered semihypergroup. For $\emptyset \neq H \subseteq S$, we define

$$(H] := \{t \in S \mid t \leq h \text{ for some } h \in H\}.$$

For $H = \{a\}$, we write $(a]$ instead of $(\{a\}]$.

By a *subsemihypergroup* of an ordered semihypergroup S we mean a nonempty subset A of S such that $A \circ A \subseteq A$. A nonempty subset A of an ordered semihypergroup S is called a *left* (resp. *right*) *hyperideal* of S if (1) $S \circ A \subseteq A$ (resp. $A \circ S \subseteq A$) and (2) If $a \in A$ and $S \ni b \leq a$, then $b \in A$. If A is both a left and a right hyperideal of S, then it is called a *(two-sided) hyperideal* of S (see [19]).

Lemma 2.2. *Let S be an ordered semihypergroup. Then the following statements hold:*

(1) $A \subseteq (A]$, $\forall A \subseteq S$.

(2) *If $A \subseteq B \subseteq S$, then $(A] \subseteq (B]$.*

(3) $(A] \circ (B] \subseteq (A \circ B]$ *and* $((A] \circ (B]] = (A \circ B],\ \forall A, B \subseteq S$.

(4) $((A]] = (A],\ \forall A \subseteq S$.

(5) *For any nonempty subset* A, B, C *of* S *such that* $A \preceq B$ *and* $B \preceq C$, *we have* $A \preceq C$.

(6) *For any two nonempty subsets* A, B *of* S *such that* $A \preceq B$, *we have* $C \circ A \preceq C \circ B$ *and* $A \circ C \preceq B \circ C$ *for any nonempty subset* C *of* S.

Proof. Straightforward. ∎

For the sake of simplicity, throughout this paper, we denote $A^n = A \circ A \circ \cdots \circ A$ (n-copies).

An ordered semihypergroup $(S, \circ, \leq)$ is called *regular* if for each $a \in S$, there exists $x \in S$ such that $a \preceq a \circ x \circ a$.

Equivalent Definitions: (1) $A \subseteq (A \circ S \circ A], \forall A \subseteq S$. (2) $a \in (a \circ S \circ a], \forall a \in S$.

An ordered semihypergroup $(S, \circ, \leq)$ is called *semisimple* if $(I^2] = I$ holds for every hyperideal I of S (see [35]).

We next state some fuzzy logic concepts.

Let S be an ordered semihypergroup. By a *fuzzy subset* of S, we mean a function from S into the real closed interval [0,1], that is, $f : S \to [0, 1]$. For an ordered semihypergroup S, the fuzzy subset 1 of S is defined as follows:

$$1 :\ S \to [0, 1],\ x \mapsto 1(x) := 1,\ \forall x \in S.$$

We denote by $F(S)$ the set of all fuzzy subsets of S. Let $f, g \in F(S)$. Then the inclusion relation $f \subseteq g$ is defined by $f(x) \leq g(x)$ for all $x \in S$, and $f \cap g$ is defined by

$$(f \cap g)(x) = \min\{f(x), g(x)\} = f(x) \wedge g(x)$$

for all $x \in S$.

Let $(S, \circ, \leq)$ be an ordered semihypergroup. For $x \in S$, we define $H_x := \{(y, z) \in S \times S |\ x \preceq y \circ z\}$. For any $f, g \in F(S)$, the product $f * g$ of f and g is defined by

$$(\forall x \in S)\ (f * g)(x) = \begin{cases} \bigvee\limits_{(y,z) \in H_x} [f(y) \wedge g(z)], & \text{if } H_x \neq \emptyset\ , \\ 0, & \text{if } H_x = \emptyset. \end{cases}$$

As we know, the multiplication "$*$" on $F(S)$ is associative and $(F(S), *, \subseteq)$ forms an ordered semigroup (see [35]).

Let S be an ordered semihypergroup. A fuzzy subset f of S is called a *fuzzy left* (resp. *right*) *hyperideal* of S if

(1) $x \leq y$ implies $f(x) \geq f(y)$ for all $x, y \in S$, and

(2) $\bigwedge\limits_{z \in x \circ y} f(z) \geq f(y)$ (resp. $\bigwedge\limits_{z \in x \circ y} f(z) \geq f(x)$) for all $x, y \in S$. Equivalently, $1 * f \subseteq f$ (resp. $f * 1 \subseteq f$).

A *fuzzy hyperideal* of S is a fuzzy subset of S which is both a fuzzy left and a fuzzy right hyperideal of S (see [35]).

Definition 2.3. Let S be an ordered semihypergroup and $f \in F(S)$. The set $f_t := \{x \in S \mid f(x) \geq t\}$, where $t \in [0, 1]$ is called a *level subset* of f.

Definition 2.4. ([35]) Let S be an ordered semihypergroup, $a \in S$ and $\lambda \in [0,1]$. An *ordered fuzzy point* a_λ of S is defined by the rule that

$$(\forall x \in S)\ a_\lambda(x) = \begin{cases} \lambda, & \text{if } x \in (a], \\ 0, & \text{if } x \notin (a]. \end{cases}$$

It is evident that every ordered fuzzy point of S is a fuzzy subset of S. For any fuzzy subset f of S, we also denote $a_\lambda \subseteq f$ by $a_\lambda \in f$ in the sequel.

Definition 2.5. ([35]) Let f be a fuzzy subset of an ordered semihypergroup S. We define $(f]$ by the rule that

$$(f](x) = \bigvee_{y \geq x} f(y)$$

for all $x \in S$. A fuzzy subset f of S is called *strongly convex* if $f = (f]$.

Lemma 2.6. ([35]) *Let f be a fuzzy subset of an ordered semihypergroup S. Then f is a strongly convex fuzzy subset of S if and only if $x \leq y$ implies $f(x) \geq f(y)$, for all $x, y \in S$.*

Lemma 2.7. ([35]) *Let a_λ, b_μ $(\lambda > 0, \mu > 0)$ be ordered fuzzy points of S, and $f, g \in F(S)$. Then the following statements are true:*

(1) $a_\lambda * b_\mu = \bigcup_{c \in (a \circ b]} c_{\lambda \wedge \mu}$.

(2) $(a_\lambda * b_\mu) * c_\delta = a_\lambda * (b_\mu * c_\delta) = \bigcup_{d \in (a \circ b \circ c]} d_{\lambda \wedge \mu \wedge \delta}$ *for any ordered fuzzy point a_λ, b_μ and c_δ of S.*

(3) *If $f \subseteq g$ and $h \in F(S)$, then $f * h \subseteq g * h$, $h * f \subseteq h * g$.*

(4) *If f is a strongly convex fuzzy subset of S, then $a_\lambda \in f$ if and only if $f(a) \geq \lambda$.*

Let A be a nonempty subset of an ordered semihypergroup S. Then the *characteristic function* f_A of A is a fuzzy subset of S defined by

$$f_A : S \to [0,1] \mid x \mapsto f_A(x) := \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

Lemma 2.8. ([35]) *Let S be an ordered semihypergroup. A nonempty subset A of S is a hyperideal of S if and only if the characteristic function f_A of A is a fuzzy hyperideal of S.*

The reader is referred to [7], [35], [38] for notation and terminology not defined in this paper.

3. Fuzzy interior hyperideals of ordered semihypergroups

In what follows, let Z^+ denote the set of all positive integers. In this section, we introduce the concepts of interior hyperideals and fuzzy interior hyperideals of an ordered semihypergroup, and investigate their related properties. In particular, we give the characterization of fuzzy interior hyperideal generated by a fuzzy subset in an ordered semihypergroup.

Definition 3.1. Let S be an ordered semihypergroup. A subsemihypergroup A of S is called an *interior hyperideal* of S if

(1) $S \circ A \circ S \subseteq A$, and
(2) If $a \in A$ and $S \ni b \leq a$, then $b \in A$. Equivalently, $(A] \subseteq A$.

Remark 3.2. The concept of interior hyperideals defined in Definition 3.1 is a generalization of the concept of interior hyperideals of semihypergroups (without order) to ordered semihypergroup theory, see [18]. Moreover, it is clear that every hyperideal of an ordered semihypergroup S is an interior hyperideal of S. However, the converse is not true, in general, as shown in the following example.

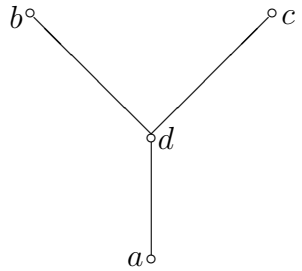
Example 3.3. We consider a set $S := \{a, b, c, d\}$ with the following hyperoperation " $\circ$ " and the order " $\leq$ ":

$\circ$	a	b	c	d
a	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
c	$\{a\}$	$\{a\}$	$\{a, b\}$	$\{a, b\}$
d	$\{a\}$	$\{a\}$	$\{a, b\}$	$\{a\}$

$$\leq := \{(a, a), (a, b), (a, c), (a, d), (b, b), (c, c), (d, b), (d, c), (d, d)\}.$$

We give the covering relation "$\prec$" and the figure of S as follows:

$$\prec = \{(a, d), (d, b), (d, c)\}.$$



Then $(S, \circ, \leq)$ is an ordered semihypergroup. We can easily verify that the sets $\{a\}$, $\{a, d\}$, $\{a, b, d\}$ and S are all interior hyperideals of S. But $\{a, d\}$ is not a hyperideal of S. In fact, since $c \in S, d \in \{a, d\}$, but $c \circ d = \{a, b\} \not\subseteq \{a, d\}$, i.e., $\{a, d\}$ is not a left hyperideal of S.

In the following theorem, we give a condition for an interior hyperideal of an ordered semihypergroup S to be a hyperideal of S.

Theorem 3.4. *Let S be a regular ordered semihypergroup and A an interior hyperideal of S. Then A is a hyperideal of S.*

Proof. Suppose that A is an interior hyperideal of a regular ordered semihypergroup S. Let $x \in A, y \in S$. Then, since S is regular, there exists $x_1 \in S$ such that $x \preceq x \circ x_1 \circ x$, and by Lemma 2.2(6), we have

$$x \circ y \preceq x \circ x_1 \circ x \circ y = (x \circ x_1) \circ x \circ y \subseteq S \circ A \circ S \subseteq A \text{ and}$$
$$y \circ x \preceq y \circ x \circ x_1 \circ x = y \circ x \circ (x_1 \circ x) \subseteq S \circ A \circ S \subseteq A.$$

Then it can be easily shown that $x \circ y \subseteq A, y \circ x \subseteq A$. Thus A is a hyperideal of S. ∎

Theorem 3.5. *Let S be an ordered semihypergroup and $\{A_i \mid i \in I\}$ a family of interior hyperideals of S. Then $\bigcap_{i\in I} A_i$ is an interior hyperideal of S if $\bigcap_{i\in I} A_i \neq \emptyset$.*

Proof. The proof is straightforward verification, and hence we omit the details.

Let now S be an ordered semihypergroup and $\emptyset \neq A \subseteq S$. We denote

$$\Omega = \{I \mid I \text{ is an interior hyperideal of } S \text{ containing } A\}.$$

Clearly, Ω is not empty since $S \in \Omega$. Let $In(A) = \bigcap_{I\in\Omega} I$. It is clear that $In(A) \neq \emptyset$ because $A \subseteq In(A)$. By Theorem 3.5, $In(A)$ is an interior hyperideal of S. Moreover, $In(A)$ is the smallest interior hyperideal of S containing A. The interior hyperideal $In(A)$ is called the *interior hyperideal of S generated by A*. For $A = \{a\}$, let $In(a)$ denote the interior hyperideal of S generated by $\{a\}$.

Theorem 3.6. *Let S be an ordered semihypergroup. Then*

(1) *For every $a \in S$, $In(a) = (a \cup a^2 \cup (S \circ a \circ S)]$.*

(2) *For every $\emptyset \neq A \subseteq S$, $In(A) = (A \cup A^2 \cup (S \circ A \circ S)]$.*

Proof. (1) Let $a \in S$. Clearly, $(a\cup a^2\cup(S\circ a\circ S)]$ is a subsemihypergroup of S. Let $x, y \in S$ and $b \in (a\cup a^2\cup(S\circ a\circ S)]$. Then there exists $w \in a\cup a^2\cup(S\circ a\circ S)$ such that $b \leq w$. Thus, by Lemma 2.2(6), we have $x\circ b\circ y \preceq x\circ w\circ y \subseteq a\cup a^2\cup(S\circ a\circ S)$. It implies that $x\circ b\circ y \subseteq (a\cup a^2\cup(S\circ a\circ S)]$. Consequently, $S\circ(a\cup a^2\cup(S\circ a\circ S)]\circ S \subseteq (a \cup a^2 \cup (S \circ a \circ S)]$. Hence $(a \cup a^2 \cup (S \circ a \circ S)]$ is an interior hyperideal of S containing a. Furthermore, we claim that $In(a) = (a\cup a^2\cup(S\circ a\circ S)]$. To prove our claim, let I be an interior hyperideal of S containing a. Let $c \in (a\cup a^2\cup(S\circ a\circ S)]$. Then, there exists $u \in a \cup a^2 \cup (S \circ a \circ S)$ such that $c \leq u$. Since $a \in I$ and I is an interior hyperideal of S, we have $a^2 \subseteq I, S \circ a \circ S \subseteq S \circ I \circ S \subseteq I$. Thus, $c \leq u \in I$ and then $c \in I$. Hence $(a \cup a^2 \cup (S \circ a \circ S)]$ is the smallest interior hyperideal of S containing a. Therefore, we obtain the request result.

(2) It can be proved similarly as (1). ∎

Example 3.7. Let S be the ordered semihypergroup of Example 3.3 and $A = \{a, b\} \subseteq S$. Then we apply the result of the above theorem, and obtain that $In(A) = \{a, b, d\}$.

Now, we define and study the fuzzy interior hyperideals of ordered semihypergroups.

Definition 3.8. Let S be an ordered semihypergroup. A fuzzy subset f of S is called a *fuzzy subsemihypergroup* of S if $x_t \in f$ and $y_r \in f$ imply $x_t * y_r \in f$ for all $t, r \in (0, 1]$ and $x, y \in S$.

Theorem 3.9. *Let S be an ordered semihypergroup and f a strongly convex fuzzy subset of S. Then f is a fuzzy subsemihypergroup of S if and only if*

$$\bigwedge_{z\in x\circ y} f(z) \geq f(x) \wedge f(y) \text{ for all } x, y \in S.$$

Proof. Let f be a fuzzy subsemihypergroup of S. Assume that $\bigwedge_{z\in x\circ y} f(z) < f(x)\wedge f(y)$ for some $x, y \in S$. Choose $t \in (0, 1]$ such that $\bigwedge_{z\in x\circ y} f(z) < t \leq f(x) \wedge f(y)$. Then, since f is strongly convex, $x_t \in f$ and $y_t \in f$. Thus, by Lemma 2.7(1), $\bigcup_{z\in(x\circ y]} z_t = x_t * y_t \in f$. Then it can be easily shown that $\bigwedge_{z\in x\circ y} f(z) \geq t$, which is a contradiction. Hence $\bigwedge_{z\in x\circ y} f(z) \geq f(x) \wedge f(y)$ for all $x, y \in S$. Conversely, suppose that $\bigwedge_{z\in x\circ y} f(z) \geq f(x) \wedge f(y)$ for all $x, y \in S$. Let $x_t \in f$ and $y_r \in f$ for all $t, r \in (0, 1]$. Then $f(x) \geq t$ and $f(y) \geq r$. By hypothesis, we have $\bigwedge_{z\in x\circ y} f(z) \geq f(x) \wedge f(y) \geq t \wedge r$, which means that $f(z) \geq t \wedge r$ for any $z \in x \circ y$. Let w be any element of $(x \circ y]$. Then there exists $u \in x \circ y$ such that $w \leq u$. Since f is a strongly convex fuzzy subset of S, by Lemma 2.6, $f(w) \geq f(u) \geq t \wedge r$, and we have $w_{t\wedge r} \in f$. Thus, by Lemma 2.7(1), $x_t * y_r = \bigcup_{w\in(x\circ y]} w_{t\wedge r} \in f$. Therefore, f is a fuzzy subsemihypergroup of S. ∎

Definition 3.10. Let S be an ordered semihypergroup. A fuzzy subsemihypergroup f of S is called a *fuzzy interior hyperideal* of S if for all $x, y, z \in S$ and $t \in (0, 1]$, the following conditions hold:

(1) $x \leq y \Rightarrow f(x) \geq f(y)$.

(2) $y_t \in f$ and $x, z \in S \Rightarrow x_t * y_t * z_t \in f$.

Theorem 3.11. *Let S be an ordered semihypergroup and f a fuzzy subsemihypergroup of S. Then f is a fuzzy interior hyperideal of S if and only if f satisfies the following assertions:*

(1) $x \leq y \Rightarrow f(x) \geq f(y)$ *for all* $x, y \in S$.

(2) $\bigwedge_{w\in x\circ y\circ z} f(w) \geq f(y)$.

Proof. The proof is similar to that of Theorem 3.9 with suitable modification by using Lemma 2.7(2). ∎

Lemma 3.12. *Let S be an ordered semihypergroup and f a strongly convex fuzzy subset of S. Then f is a fuzzy subsemihypergroup of S if and only if $f * f \subseteq f$.*

Proof. Suppose that f is a fuzzy subsemihypergroup of S. To prove that $f*f \subseteq f$, it is enough to show that $(f * f)(x) \leq f(x)$ for any $x \in S$. Indeed, let $x \in S$. If $H_x = \emptyset$, then $(f*f)(x) = 0$. Since $f(x) \geq 0$ for all $x \in S$, we have $(f*f)(x) \leq f(x)$. Let $H_x \neq \emptyset$. Then there exist $y, z \in S$ such that $x \preceq y\circ z$, and there exists $w \in y\circ z$ such that $x \leq w$. Since f is a strongly convex fuzzy subsemihypergroup of S, by Lemma 2.6 and Theorem 3.9, we have $f(y) \wedge f(z) \leq \bigwedge_{w\in y\circ z} f(w) \leq \bigwedge_{\substack{w\in y\circ z\\ x\leq w}} f(w) \leq \bigwedge_{\substack{w\in y\circ z\\ x\leq w}} f(x) = f(x)$. Hence we have $f(y) \wedge f(z) \leq f(x)$ for any $x \preceq y \circ z$. Thus

$(f * f)(x) = \bigvee\limits_{x \preceq y \circ z} [f(y) \wedge f(z)] \leq f(x)$. Thus $(f * f)(x) \leq f(x)$ for any $x \in S$.

Conversely, let $x, y \in S$. Then we can prove that $f(z) \geq f(y) \wedge f(z)$ for any $z \in x \circ y$. In fact, since $z \in x \circ y, z \leq z$, we have $z \preceq x \circ y$. Thus, by hypothesis, we have $f(z) \geq (f * f)(z) = \bigvee\limits_{z \preceq u \circ v} [f(u) \wedge f(v)] \geq f(x) \wedge f(y)$ for every $x, y \in S$ such that $z \in x \circ y$, from which we can deduce that $\bigwedge\limits_{z \in x \circ y} f(z) \geq f(x) \wedge f(y)$. By Theorem 3.9, f is a fuzzy subsemihypergroup of S. ∎

Theorem 3.13. *Let S be an ordered semihypergroup and f a strongly convex fuzzy subset of S. Then f is a fuzzy interior hyperideal of S if and only if $f * f \subseteq f$ and $1 * f * 1 \subseteq f$.*

Proof. Let f be a fuzzy interior hyperideal of S. Then $(1 * f * 1)(a) \leq f(a)$ for all $a \in S$. Indeed, if $(1 * f * 1)(a) = 0$, clearly, $(1 * f * 1)(a) \leq f(a)$. Let $(1 * f * 1)(a) \neq 0$. Then we can also prove that $(1 * f * 1)(a) \leq f(a)$. In fact, let $(x, y) \in H_a$ and $(p, q) \in H_x$, i.e., $a \preceq x \circ y, x \preceq p \circ q$. Then, by Lemma 2.2, $a \preceq p \circ q \circ y$, and there exists $u \in p \circ q \circ y$ such that $a \leq u$. Since f is a fuzzy interior hyperideal of S, by Theorem 3.11 we have $f(a) \geq \bigwedge\limits_{\substack{u \in p \circ q \circ y \\ a \leq u}} f(u) \geq \bigwedge\limits_{u \in p \circ q \circ y} f(u) \geq f(q)$. Thus $(1 * f * 1)(a) = \bigvee\limits_{(x,y) \in H_a} [(1 * f)(x) \wedge 1(y)] = \bigvee\limits_{(x,y) \in H_a} (1 * f)(x) = \bigvee\limits_{(x,y) \in H_a} (\bigvee\limits_{(p,q) \in H_x} [1(p) \wedge f(q)]) = \bigvee\limits_{\substack{a \preceq x \circ y \\ x \preceq p \circ q}} f(q) \leq f(a)$, which means that $1 * f * 1 \subseteq f$.

Conversely, for any $x, y, z \in S$, let $w \in x \circ y \circ z$. Then, there exists $u \in x \circ y \subseteq (x \circ y]$ such that $w \in u \circ z \subseteq (u \circ z]$, and we have $(x, y) \in H_u, (u, z) \in H_w$. Since $1 * f * 1 \subseteq f$, we have $f(w) \geq (1 * f * 1)(w) = \bigvee\limits_{(p,q) \in H_w} [(1 * f)(p) \wedge 1(q)] \geq (1 * f)(u) \wedge 1(z) = (1 * f)(u) = \bigvee\limits_{(s,t) \in H_u} [1(s) \wedge f(t)] \geq 1(x) \wedge f(y) = f(y)$. It thus follows that $\bigwedge\limits_{w \in x \circ y \circ z} f(w) \geq f(y)$.

The rest of the proof is a consequence of Theorem 3.11 and Lemma 3.12. ∎

Theorem 3.14. *Let S be an ordered semihypergroup. A nonempty subset A of S is an interior hyperideal of S if and only if the characteristic function f_A of A is a fuzzy interior hyperideal of S.*

Proof. Suppose that A is an interior hyperideal of S. Let $x, y \in S, x \leq y$. Then $f_A(x) \geq f_A(y)$. In fact, if $y \in A$, then $f_A(y) = 1$. Since $S \ni x \leq y \in A$, by hypothesis we have $x \in A$, then $f_A(x) = 1$. Thus $f_A(x) \geq f_A(y)$. If $y \notin A$, then $f_A(y) = 0$. Since $x \in S$, we have $f_A(x) \geq 0$. Thus $f_A(x) \geq 0 = f_A(y)$, and by Lemma 2.6, f_A is a strongly convex fuzzy subset of S. Let now $x, y \in S$. Then we can show that $\bigwedge\limits_{z \in x \circ y} f_A(z) \geq f_A(x) \wedge f_A(y)$. Indeed, if $x \circ y \not\subseteq A$, then there exists $z \in x \circ y$ such that $z \notin A$, and we have $\bigwedge\limits_{z \in x \circ y} f_A(z) = 0$. Besides that, $x \circ y \not\subseteq A$ implies that $x \notin A$ or $y \notin A$. Then $f_A(x) = 0$ or $f_A(y) = 0$, and

$\bigwedge_{z\in x\circ y} f_A(z) = f_A(x) \wedge f_A(y)$. Let $x \circ y \subseteq A$. Then $f_A(z) = 1$ for any $z \in x \circ y$. It implies that $\bigwedge_{z\in x\circ y} f_A(z) = 1$. Since f_A is a fuzzy subset of S, we have $f_A(x) \leq 1$ for any $x \in A$. Thus, in this case, $\bigwedge_{z\in x\circ y} f_A(z) \geq f_A(x) \wedge f_A(y)$. Similar to the above proof, we can prove that $\bigwedge_{w\in x\circ y\circ z} f_A(w) \geq f_A(y)$ for all $x, y, z \in S$. Therefore, f_A is a fuzzy interior hyperideal of S by Theorem 3.11.

Conversely, let $\emptyset \neq A \subseteq S$ such that f_A is a fuzzy interior hyperideal of S. We claim that $A \circ A \subseteq A$. To prove our claim, let $x, y \in A$. By hypothesis, $\bigwedge_{z\in x\circ y} f_A(z) \geq f_A(x) \wedge f_A(y) = 1$, which implies that $f_A(z) \geq 1$ for any $z \in x \circ y$. On the other hand, $f_A(z) \leq 1$ for all $z \in S$. Thus for any $z \in x \circ y$, $f_A(z) = 1$, i.e., $z \in A$. It thus follows that $A \circ A \subseteq A$. In a similar way, we can show that $S \circ A \circ S \subseteq A$. Furthermore, let $x \in A, S \ni y \leq x$. Then $y \in A$. Indeed, it is enough to prove that $f_A(y) = 1$. From $x \in A$, we have $f_A(x) = 1$. Since f_A is a fuzzy interior hyperideal of S and $y \leq x$, we have $f_A(y) \geq f_A(x) = 1$. Since $y \in S$, we have $f_A(y) \leq 1$. Thus A is an interior hyperideal of S. ∎

Now, we shall characterize fuzzy interior hyperideals of ordered semihypergroups by the level subsets.

Theorem 3.15. *Let S be an ordered semihypergroup and f a fuzzy subset of S. Then f is a fuzzy interior hyperideal of S if and only if the level subset f_t ($t \in (0, 1]$) of f is an interior hyperideal of S for $f_t \neq \emptyset$.*

Proof. Assume that f is a fuzzy interior hyperideal of S. Let $t \in (0, 1]$ be such that $f_t \neq \emptyset$. Let now $x, y \in f_t$. Then $f(x) \geq t, f(y) \geq t$. By hypothesis, we have $\bigwedge_{z\in x\circ y} f(z) \geq f(x) \wedge f(y) \geq t \wedge t = t$. Thus for any $z \in x \circ y$, we have $f(z) \geq t$, i.e., $z \in f_t$. It follows that $x \circ y \subseteq f_t$. Hence f_t is a subsemihypergroup of S. Let $y \in f_t, x, z \in S$. Then $f(y) \geq t$. It follows from Theorem 3.11 that $\bigwedge_{w\in x\circ y\circ z} f(w) \geq f(y) \geq t$. Hence $f(w) \geq t$ for any $w \in x \circ y \circ z$. Thus it can be easily shown that $x \circ y \circ z \subseteq f_t$. Furthermore, let $x \in f_t, S \ni y \leq x$. Then $y \in f_t$. Indeed, since $x \in f_t, f(x) \geq t$ and f is a fuzzy interior hyperideal of S, we have $f(y) \geq f(x) \geq t$, so $y \in f_t$. Consequently, f_t is an interior hyperideal of S for $f_t \neq \emptyset$.

Conversely, suppose that f_t ($\neq \emptyset$) is an interior hyperideal of S. If $\bigwedge_{z\in x\circ y} f(z) < f(x) \wedge f(y)$ for some $x, y \in S$, then there exists $t \in (0, 1]$ such that $\bigwedge_{z\in x\circ y} f(z) < t \leq f(x) \wedge f(y)$, which implies that $x, y \in f_t$ and $x \circ y \not\subseteq f_t$. It contradicts the fact that f_t is an interior hyperideal of S. Consequently, $\bigwedge_{z\in x\circ y} f(z) \geq f(x) \wedge f(y)$ for all $x, y \in S$. In a similar way, we can show that $\bigwedge_{w\in x\circ y\circ z} f(w) \geq f(y)$ for all $x, y, z \in S$. Moreover, let $x, y \in S$ and $x \leq y$. Then $f(x) \geq f(y)$. In fact, let

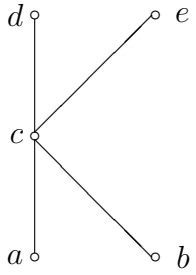
$t = f(y)$. Then $y \in f_t$. Since f_t is an interior hyperideal of S, we have $x \in f_t$. Then $f(x) \geq t = f(y)$. Therefore, f is a fuzzy interior hyperideal of S. ∎

Example 3.16. We consider a set $S := \{a, b, c, d, e\}$ with the following hyperoperation " $\circ$ " and the order " $\leq$ ":

$\circ$	a	b	c	d	e
a	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$
b	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$
c	$\{a, b\}$	$\{a, b\}$	$\{c\}$	$\{c\}$	$\{e\}$
d	$\{a, b\}$	$\{a, b\}$	$\{c\}$	$\{d\}$	$\{e\}$
e	$\{a, b\}$	$\{a, b\}$	$\{c\}$	$\{c\}$	$\{e\}$

$\leq := \{(a, a), (a, c), (a, d), (a, e), (b, b), (b, c), (b, d), (b, e), (c, c), (c, d), (c, e), (d, d), (e, e)\}$.
We give the covering relation "$\prec$" and the figure of S as follows:

$$\prec = \{(a, c), (b, c), (c, d), (c, e)\}.$$



Then $(S, \circ, \leq)$ is an ordered semihypergroup, and the sets $\{a, b\}$, $\{a, b, c, e\}$ and S are all interior hyperideals of S. Now let f be a fuzzy subset of S such that $f(a) = f(b) = 0.8, f(c) = f(d) = f(e) = 0.6$. Then

$$f_t = \begin{cases} S, & \text{if } t \in (0, 0.6], \\ \{a, b\}, & \text{if } t \in (0.6, 0.8], \\ \emptyset, & \text{if } t \in (0.8, 1]. \end{cases}$$

Thus all nonempty level subsets f_t $(t \in (0, 1])$ of f are interior hyperideals of S and by Theorem 3.15, f is a fuzzy interior hyperideal of S.

Theorem 3.17. *Let S be an ordered semihypergroup. If f is a fuzzy hyperideal of S, then f is a fuzzy interior hyperideal of S.*

Proof. Assume that f is a fuzzy hyperideal of S. Then, by Lemma 2.7(3), we have $f * f \subseteq f * 1 \subseteq f$ and $1 * f * 1 = 1 * (f * 1) \subseteq 1 * f \subseteq f$. Thus, by Theorem 3.13, f is a fuzzy interior hyperideal of S. ∎

The converse of Theorem 3.17 is not true in general. We can illustrate it by the following example:

Example 3.18. Suppose that S is the ordered semihypergroup of Example 3.3. Let f be a fuzzy subset of S such that $f(a) = 0.8, f(b) = f(c) = 0.4, f(d) = 0.6$. Then

$$f_t = \begin{cases} S, & \text{if } t \in (0, 0.4], \\ \{a, d\}, & \text{if } t \in (0.4, 0.6], \\ \{a\}, & \text{if } t \in (0.6, 0.8], \\ \emptyset, & \text{if } t \in (0.8, 1]. \end{cases}$$

Similar to Example 3.16, we can verify that f is a fuzzy interior hyperideal of S. But we claim that f is not a fuzzy hyperideal of S. Indeed, since $c \circ d = \{a, b\}$, we have

$$\bigwedge_{z \in c \circ d} f(z) = f(a) \wedge f(b) = 0.4 < 0.6 = f(d),$$

i.e., f is not a fuzzy left hyperideal of S.

The following theorem can be obtained under the assumption of an additional condition.

Theorem 3.19. *Let S be a regular ordered semihypergroup. If f is a fuzzy interior hyperideal of S, then f is a fuzzy hyperideal of S.*

Proof. Assume that f is a fuzzy interior hyperideal of a regular ordered semihypergroup S. Let $x, y \in S$. Then, since S is regular, there exists $x_1 \in S$ such that $x \preceq x \circ x_1 \circ x$. Then, by Lemma 2.2(6), we have $x \circ y \preceq (x \circ x_1 \circ x) \circ y = (x \circ x_1) \circ x \circ y$. Thus, for any $z \in x \circ y$, there exists $w \in (x \circ x_1) \circ x \circ y$ such that $z \leq w$. By $w \in (x \circ x_1) \circ x \circ y$, we have $w \in u \circ x \circ y$ for some $u \in x \circ x_1$. Since f is a fuzzy interior hyperideal of S, by Theorem 3.11 we have

$$f(z) \geq f(w) \geq \bigwedge_{w \in u \circ x \circ y} f(w) \geq f(x).$$

Thus it can be obtained that $\bigwedge_{z \in x \circ y} f(z) \geq f(x)$. Hence f is a fuzzy right hyperideal of S. In the same way we can show that f is also a fuzzy left hyperideal of S. Therefore, f is indeed a fuzzy hyperideal of S. ∎

By Theorems 3.17 and 3.19, we immediately obtain the following corollary:

Corollary 3.20. *In regular ordered semihypergroups the concepts of fuzzy hyperideals and fuzzy interior hyperideals coincide.*

Theorem 3.21. *Let $\{f_i \mid i \in I\}$ be a family of fuzzy interior hyperideals of an ordered semihypergroup S. Then $f := \bigcap_{i \in I} f_i$ is a fuzzy interior hyperideal of S, where $(\bigcap_{i \in I} f_i)(x) = \bigwedge_{i \in I}(f_i(x))$.*

Proof. Let $x, y \in S$. Then, since each f_i $(i \in I)$ is a fuzzy interior hyperideal of S, $\bigwedge_{z \in x \circ y} f_i(z) \geq f_i(x) \wedge f_i(y)$. Thus, for any $z \in x \circ y$, $f_i(z) \geq f_i(x) \wedge f_i(y)$, and we have $f(z) = (\bigcap_{i \in I} f_i)(z) = \bigwedge_{i \in I}(f_i(z)) \geq \bigwedge_{i \in I}(f_i(x) \wedge f_i(y)) = (\bigwedge_{i \in I}(f_i(x))) \wedge (\bigwedge_{i \in I}(f_i(y))) = (\bigcap_{i \in I} f_i)(x) \wedge (\bigcap_{i \in I} f_i)(y) = f(x) \wedge f(y)$, which implies that $\bigwedge_{z \in x \circ y} f(z) \geq f(x) \wedge f(y)$. Similar to the above proof, it can be easily shown that $\bigwedge_{w \in x \circ y \circ z} f(w) \geq f(y)$ for all

$x, y, z \in S$. Furthermore, if $x \leq y$, then $f(x) \geq f(y)$. Indeed, since every f_i $(i \in I)$ is a fuzzy interior hyperideal of S, it can be obtained that $f_i(x) \geq f_i(y)$ for all $i \in I$. Thus $f(x) = (\bigcap_{i\in I} f_i)(x) = \bigwedge_{i\in I}(f_i(x)) \geq \bigwedge_{i\in I}(f_i(y)) = (\bigcap_{i\in I} f_i)(y) = f(y)$. By Theorems 3.9 and 3.11, f is a fuzzy interior hyperideal of S.

Definition 3.22. Let S be an ordered semihypergroup and $f \in F(S)$. The smallest fuzzy interior hyperideal of S containing f is called the *fuzzy interior hyperideal of S generated by f*, denoted by $< f >$.

In order to characterize the fuzzy interior hyperideal generated by a fuzzy subset in an ordered semihypergroup, we need the following lemma.

Lemma 3.23. *Let S be an ordered semihypergroup and $f \in F(S)$. Then $f(x) = \sup\{k \mid x \in f_k\}$ for any $x \in S$, where f_k $(k \in (0, 1])$ is the level subset of f.*

Proof. Let $\alpha = \sup\{k \mid x \in f_k\}$. Then, for any $\varepsilon > 0$, we have $\sup\{k \mid x \in f_k\} > \alpha - \varepsilon$, and thus there exists $t \in \{k \mid x \in f_k\}$ such that $t > \alpha - \varepsilon$. Since $x \in f_t$, we have $f(x) \geq t$. Thus $f(x) > \alpha - \varepsilon$. By the arbitrariness of ε, it follows that $f(x) \geq \alpha$. On the other hand, let $t = f(x)$. Then $x \in f_t$, and $t \in \{k \mid x \in f_k\}$. It thus implies that $f(x) = t \leq \sup\{k \mid x \in f_k\} = \alpha$. Therefore, we obtain the requested result. ∎

Theorem 3.24. *Let S be an ordered semihypergroup and $f \in F(S)$. Then the fuzzy set f^* of S defined by*

$$(\forall x \in S)\ f^*(x) = \sup\{k \mid x \in In(f_k)\}$$

is the fuzzy interior hyperideal $< f >$ generated by f in S, where $In(f_k)$ $(k \in (0, 1])$ is the interior hyperideal of S generated by f_k.

Proof. In order to prove that f^* is the fuzzy interior hyperideal $< f >$ of S generated by f, we now consider the following three steps:

(1) $f \subseteq f^*$. In fact, for any $x \in S$, let $t \in \{k \mid x \in f_k\}$. Then $x \in f_t$, and we have $x \in In(f_t)$. It thus follows that $t \in \{k \mid x \in In(f_k)\}$. Hence we have shown that $\{k \mid x \in f_k\} \subseteq \{k \mid x \in In(f_k)\}$. By Lemma 3.23, we have

$$f(x) = \sup\{k \mid x \in f_k\} \leq \sup\{k \mid x \in In(f_k)\} = f^*(x),$$

from which we can deduce that $f \subseteq f^*$.

(2) f^* is a fuzzy interior hyperideal of S. Indeed, let $t \in (0, 1]$ be such that $f^* \neq \emptyset$. Let $\alpha_n = t - \frac{1}{n}$ for any $n \in Z^+$. We claim that $f_t^* = \bigcap_{n\in Z^+} In(f_{\alpha_n})$. To prove our claim, let $x \in f_t^*$. Then $f^*(x) \geq t$, that is, $\sup\{k \mid x \in In(f_k)\} \geq t > t - \frac{1}{n} = \alpha_n$ for any $n \in Z^+$. It implies that $k_n > \alpha_n$ for some $k_n \in \sup\{k \mid x \in In(f_k)\}$. Thus $f_{k_n} \subseteq f_{\alpha_n}$, and we have $x \in In(f_{k_n}) \subseteq In(f_{\alpha_n})$, which means that $x \in \bigcap_{n\in Z^+} In(f_{\alpha_n})$. Consequently, $f_t^* \subseteq \bigcap_{n\in Z^+} In(f_{\alpha_n})$. To show the inverse inclusion, let $x \in \bigcap_{n\in Z^+} In(f_{\alpha_n})$. Then $\alpha_n \in \{k \mid x \in In(f_k)\}$ for any $n \in Z^+$. Thus we have

$$t - \frac{1}{n} = \alpha_n \leq \sup\{k \mid x \in In(f_k)\} = f^*(x),\ n \in Z^+.$$

Since n is an arbitrary positive integer, we have $t \leq f^*(x)$. Thus $x \in f_t^*$. Therefore, $f_t^* = \bigcap_{n \in Z^+} In(f_{\alpha_n})$, and by Theorem 3.5, f_t^* is an interior hyperideal of S. Thus f^* is a fuzzy interior hyperideal of S by Theorem 3.15.

(3) $f^* =< f >$. In fact, suppose that g is a fuzzy interior hyperideal of S and $f \subseteq g$. Let $x \in S$. If $f^*(x) = 0$, then, obviously, $f^*(x) \leq g(x)$. Let $f^*(x) = t \neq 0$. Then, by (2), we have $x \in f_t^* = \bigcap_{n \in Z^+} In(f_{\alpha_n})$. Since $f \subseteq g$, we have $f_{\alpha_n} \subseteq g_{\alpha_n}$, and $x \in In(f_{\alpha_n}) \subseteq In(g_{\alpha_n}) = (g_{\alpha_n} \cup g_{\alpha_n}^2 \cup (S \circ g_{\alpha_n} \circ S)]$, $n \in Z^+$. Then $x \leq w$ for some $w \in g_{\alpha_n} \cup g_{\alpha_n}^2 \cup (S \circ g_{\alpha_n} \circ S)$. We consider the following three cases:

Case 1. If $w \in g_{\alpha_n}$, then $g(w) \geq \alpha_n$.

Case 2. If $w \in g_{\alpha_n}^2$, then $w \in w_1 \circ w_2$ for some $w_1, w_2 \in g_{\alpha_n}$. Hence $g(w_1) \geq \alpha_n$, $g(w_2) \geq \alpha_n$. Since g is a fuzzy interior hyperideal of S, and it is a fuzzy subsemihypergroup of S, by Theorem 3.9, $g(w) \geq \bigwedge_{w \in w_1 \circ w_2} g(w) \geq g(w_1) \wedge g(w_2) \geq \alpha_n \wedge \alpha_n = \alpha_n$.

Case 3. Let $w \in S \circ g_{\alpha_n} \circ S$. Then there exist $y, z \in S, u \in g_{\alpha_n}$ such that $w \in y \circ u \circ z$, and we have $g(u) \geq \alpha_n$. Since g is a fuzzy interior hyperideal of S, by Theorem 3.11, $g(w) \geq \bigwedge_{w \in y \circ u \circ z} g(w) \geq g(u) \geq \alpha_n$.

Thus, in any case, we have $g(w) \geq \alpha_n$, $n \in Z^+$. Since g is a fuzzy interior hyperideal of S and $x \leq w$, we have $g(x) \geq g(w) \geq \alpha_n = t - \frac{1}{n}$, $n \in Z^+$. By the arbitrariness of n in Z^+, $g(x) \geq t = f^*(x)$. Hence $f^* \subseteq g$, and it is shown that f^* is the fuzzy interior hyperideal $< f >$ of S generated by f. This completes the proof. ∎

4. Normal fuzzy interior hyperideals of ordered semihypergroups

In the current section we define a normal fuzzy interior hyperideal of an ordered semihypergroup, and investigate some related properties.

An element 0 of an ordered semihypergroup S is called *zero element* if $0 \leq x$ and $0 \circ x = x \circ 0 = \{0\}$ for all $x \in S$. In this section, unless stated otherwise S means an ordered semihypergroup containing the zero element 0.

Example 4.1. Suppose that S is the ordered semihypergroup of Example 3.3. Then S is an ordered semihypergroup containing the zero element, and a is the zero element of S.

For a given fuzzy interior hyperideal f of S, we note that $f(0)$ is the largest value of f, that is, $f(0) \geq f(x)$ for all $x \in S$. In general, $f(0) \neq 1$. Particularly, we have the following concept.

Definition 4.2. A fuzzy interior hyperideal f of S is said to be *normal*, if $f(0) = 1$.

Theorem 4.3. *Let f be a fuzzy subset of S. We define the subset S_f of S as follows:*

$$S_f := \{x \in S \mid f(x) = f(0)\}.$$

Then the following statements are true:

(1) *If f is a fuzzy interior hyperideal of S, then S_f is an interior hyperideal of S.*

(2) *If A is an interior hyperideal of S, then the characteristic function f_A of A is a normal fuzzy interior hyperideal of S and $S_{f_A} = A$.*

Proof. (1) Let $x, y \in S_f$. Then $f(x) = f(y) = 0$. Since f is a fuzzy subsemihypergroup of S, by Theorem 3.9, $\bigwedge\limits_{z \in x \circ y} f(z) \geq f(x) \wedge f(y) = f(0)$, and for any $z \in x \circ y$, we have $f(z) \geq f(0)$. Noticing that $f(z) \leq f(0)$ for all $z \in S$, we have $f(z) = f(0)$, that is, $z \in S_f$. It thus follows that $x \circ y \subseteq S_f$. Let now $x, z \in S, y \in S_f$. Then, similar to the above proof, we can deduce that $x \circ y \circ z \subseteq S_f$. Furthermore, let $y \in S_f, S \ni x \leq y$. Then $x \in S_f$. Indeed, it is enough to prove that $f(x) \geq f(0)$. By $y \in S_f$, we have $f(y) = 0$. Since f is a fuzzy interior hyperideal of S, we have $f(x) \geq f(y) = 0$. Hence S_f is an interior hyperideal of S.

(2) Assume that A is an interior hyperideal of S. By Theorem 3.14, f_A is a fuzzy interior hyperideal of S. To prove that f_A is normal, let $x \in A$. Then, since A is an interior hyperideal of S, $0 \in 0 \circ x \circ 0 \subseteq S \circ A \circ S \subseteq A$, and we have $f_A(0) = 1$. Thus f_A is a normal fuzzy interior hyperideal of S. Moreover, it can be easily shown that $S_{f_A} = A$. ∎

Theorem 4.4. *Let f, g be fuzzy interior hyperideals of S. Then the following statements are true:*

(1) $S_f \cap S_g \subseteq S_{f \cap g}$.

(2) *If $f \subseteq g$ and $f(0) = g(0)$, then $S_f \subseteq S_g$.*

Proof. The proof is obvious, and hence we omit the details. ∎

By Theorem 4.4, we immediately obtain the following corollary:

Corollary 4.5. *Let f, g be normal fuzzy interior hyperideals of S. Then the following statements are true:*

(1) $S_f \cap S_g = S_{f \cap g}$.

(2) *If $f \subseteq g$, then $S_f \subseteq S_g$.*

The hypothesis that f and g are normal cannot be removed in the above corollary. Otherwise, Corollary 4.5 does not hold in general. We can illustrate it by the following example.

Example 4.6. Let S be an ordered semihypergroup containing the zero element 0 and $|S| \geq 2$. Define two fuzzy subsets f, g of S as follows:

$$f(x) := 0 \quad \text{and} \quad g(x) := \begin{cases} 1, & \text{if } x = 0, \\ 0, & \text{if } x \neq 0, \end{cases}$$

for any $x \in S$. We can easily verify that f and g are two fuzzy interior hyperideals of S, and $S_f = S, S_g = \{0\}$. Clearly, $f \subseteq g$. But $S_f \cap S_g = \{0\} \subset S = S_f = S_{f \cap g}$ and $S_f \supset S_g$.

Theorem 4.7. *Let f be a fuzzy interior hyperideal of S and let f^+ be a fuzzy subset of S defined by $(\forall x \in S)\ f^+(x) = f(x) + 1 - f(0)$. Then f^+ is a normal fuzzy interior hyperideal of S containing f.*

Proof. Suppose that f is a fuzzy interior hyperideal of S. Let $x, y \in S$. Then $\bigwedge_{z\in x\circ y} f^+(z) = \bigwedge_{z\in x\circ y}(f(z)+1-f(0)) = (\bigwedge_{z\in x\circ y} f(z))+1-f(0) \geq (f(x)\wedge f(y))+1-f(0)) = (f(x)+1-f(0))\wedge(f(y)+1-f(0)) = f^+(x)\wedge f^+(y)$. In a similar way we can show that $\bigwedge_{w\in x\circ y\circ z} f^+(w) \geq f^+(y)$ for all $x, y, z \in S$. Furthermore, let $x, y \in S$ be such that $x \leq y$. Then $f^+(x) \geq f^+(y)$. In fact, since f is a fuzzy interior hyperideal of S, we have $f(x) \geq f(y)$. Thus

$$f^+(x) = f(x)+1-f(0) \geq f(y)+1-f(0) = f^+(y).$$

Consequently, f^+ is a fuzzy interior hyperideal of S. Clearly, $f^+(0) = 1$ and $f \subseteq f^+$. The proof is completed.

Corollary 4.8. *Let f be a fuzzy interior hyperideal of S satisfying $f^+(x) = 0$ for some $x \in S$. Then $f(x) = 0$ and f is normal.*

Theorem 4.9. *Let f be a fuzzy interior hyperideal of S. Then f is normal if and only if $f^+ = f$.*

Proof. The sufficiency is obvious. Assume that f is a normal fuzzy interior hyperideal of S. Let $x \in S$. Then $f^+(x) = f(x)+1-f(0) = f(x)$, and thus $f^+ = f$. ∎

Using Theorems 4.7 and 4.9, we obtain the following corollaries.

Corollary 4.10. *If f is a fuzzy interior hyperideal of S, then $(f^+)^+ = f^+$.*

Corollary 4.11. *If f is a normal fuzzy interior hyperideal of S, then $(f^+)^+ = f$.*

Theorem 4.12. *Let f be a fuzzy interior hyperideal of S. If there exists a fuzzy interior hyperideal g of S satisfying $g^+ \subseteq f$, then f is normal.*

Proof. Assume there exists a fuzzy interior hyperideal g of S such that $g^+ \subseteq f$. Then, by Theorem 4.7, $1 = g^+(0) \leq f(0)$, whence $f(0) = 1$. This completes the proof. ∎

Corollary 4.13. *Let f be a fuzzy interior hyperideal of S. If f is not normal, then f doesn't contain any normal fuzzy interior hyperideal of S.*

Proof. It is a direct consequence of Theorem 4.12. ∎

Theorem 4.14. *Let f be a fuzzy interior hyperideal of S and let $\varphi : [0, f(0)] \to [0, 1]$ be an increasing function. Define a fuzzy subset f_φ of S as follows:*

$$(\forall x \in S)\ f_\varphi(x) = \varphi(f(x)).$$

Then f_φ is a fuzzy interior hyperideal of S. In particular, if $\varphi(f(0)) = 1$, then f_φ is normal, and if $\varphi(t) \geq t$ for all $t \in [0, f(0)]$, then $f \subseteq f_\varphi$.

Proof. Suppose that f is a fuzzy interior hyperideal of S. Let $x, y \in S$. Then, since φ is an increasing function, we have

$$\begin{aligned} \bigwedge_{z\in x\circ y} f_\varphi(z) &= \bigwedge_{z\in x\circ y}(\varphi(f(z))) = \varphi(\bigwedge_{z\in x\circ y} f(z)) \geq \varphi(f(x)\wedge f(y)) \\ &= \varphi(f(x))\wedge\varphi(f(y)) = f_\varphi(x)\wedge f_\varphi(y). \end{aligned}$$

Similar to the above proof, we can prove that $\bigwedge_{w\in x\circ y\circ z} f_\varphi(w) \geq f_\varphi(y)$ for all $x, y, z \in S$. Let now $x, y \in S$ be such that $x \leq y$. Then, by hypothesis, we have

$$f_\varphi(x) = \varphi(f(x)) \geq \varphi(f(y)) = f_\varphi(y).$$

Hence f_φ is a fuzzy interior hyperideal of S. If $\varphi(f(0)) = 1$, then, clearly, f_φ is normal. Assume that $\varphi(t) \geq t$ for all $t \in [0, f(0)]$. Then $f_\varphi(x) = \varphi(f(x)) \geq f(x)$ for any $x \in S$, which implies that $f \subseteq f_\varphi$. ∎

Theorem 4.15. *Let f be a nonconstant normal fuzzy interior hyperideal of S, which is maximal in the poset of normal fuzzy interior hyperideals under set inclusion. Then f takes only the values* 0 *and* 1, *that is, $Im(f) = \{0, 1\}$.*

Proof. Note that $f(0) = 1$. Let $a \in S$ be such that $f(a) \neq 1$. It is sufficient to show that $f(a) = 0$. Assume that $0 < f(a) < 1$. Now define a fuzzy subset g of S as follows:

$$(\forall x \in S)\ g(x) = \frac{1}{2}(f(x) + f(a)).$$

We claim that g is a fuzzy interior hyperideal of S. To prove our claim, let $x, y \in s$. Then, since f is a fuzzy interior hyperideal of S, we have

$$\begin{aligned}
\bigwedge_{z\in x\circ y} g(z) &= \bigwedge_{z\in x\circ y} (\frac{1}{2}(f(z) + f(a))) = \frac{1}{2}(\bigwedge_{z\in x\circ y} f(z) + f(a)) \\
&\geq \frac{1}{2}(f(x) \wedge f(y) + f(a)) = [\frac{1}{2}(f(x) + f(a))] \wedge [\frac{1}{2}(f(y) + f(a))] \\
&= g(x) \wedge g(y).
\end{aligned}$$

In the same way we can show that $\bigwedge_{w\in x\circ y\circ z} g(z) \geq g(y)$ for all $x, y, z \in S$. Moreover, let $x, y \in S$ be such that $x \leq y$. Since f is a fuzzy interior hyperideal of S, by Theorem 3.11, $f(x) \geq f(y)$, and we have

$$g(x) = \frac{1}{2}(f(x) + f(a)) \geq \frac{1}{2}(f(y) + f(a)) = g(y).$$

Thus, g is indeed a fuzzy interior hyperideal of S. By Theorem 4.7, g^+ is a normal fuzzy interior hyperideal of S. In addition, for any $x \in S$, we have

$$g^+(x) = g(x) + 1 - g(0) = \frac{1}{2}(f(x) + f(a)) + 1 - \frac{1}{2}(f(0) + f(a)) = \frac{1}{2}(f(x) + 1),$$

and thus we have

$$g^+(0) = 1 > g^+(a) = \frac{1}{2}(f(a) + 1) > f(a),$$

which means that g^+ is nonconstant. From $g^+(a) > f(a)$ it follows that f is not maximal. This completes the proof.

5. Characterizations of semisimple ordered semihypergroups

In this section, we give some new characterizations of semisimple ordered semihypergroups by fuzzy hyperideals and fuzzy interior hyperideals.

Lemma 5.1 ([35]). *Let S be an ordered semihypergroup. Then S is semisimple if and only if $a \in (S \circ a \circ S \circ a \circ S]$ for all $a \in S$.*

Lemma 5.2. *Let S be a semisimple ordered semihypergroup. If f is a fuzzy interior hyperideal of S, then f is a fuzzy hyperideal of S.*

Proof. The proof is similar to that of Theorem 3.19 with suitable modification by using Lemma 5.1. ∎

In order to characterize the semisimple ordered semihypergroups, we need the following lemma.

Lemma 5.3. *Let A, B be any two nonempty subsets of an ordered semihypergroup S. Then $f_A * f_B = f_{(A\circ B]}$, where f_A, f_B and $f_{(A\circ B]}$ are the characteristic function of A, B and $(A \circ B]$, respectively.*

Proof. Let $x \in S$. If $x \in (A \circ B]$, then $f_{(A\circ B]}(x) = 1$, and $x \preceq a \circ b$ for some $a \in A$ and $b \in B$. Thus $(a, b) \in H_x$, and

$$(f_A * f_B)(x) = \bigvee_{(y,z)\in H_x} [f_A(y) \wedge f_B(z)] \geq f_A(a) \wedge f_B(b) = 1 \wedge 1 = 1.$$

On the other hand, since $f_A(y) \leq 1$ and $f_B(z) \leq 1$ for all $y, z \in S$, we have $(f_A * f_B)(x) \leq 1$. Therefore, $(f_A * f_B)(x) = 1 = f_{(A\circ B]}(x)$. If $x \notin (A \circ B]$, then $f_{(A\circ B]}(x) = 0$. We now prove that $(f_A * f_B)(x) = 0$. Indeed, if $H_x = \emptyset$, then $(f_A * f_B)(x) = 0$, and $(f_A * f_B)(x) = f_{(A\circ B]}(x)$. If $H_x \neq \emptyset$, then $x \leq y \circ z$ for all $(y, z) \in H_x$. If $y \in A$ and $z \in B$, then $y \circ z \subseteq A \circ B$, so $x \in (A \circ B]$, which is impossible. Thus $y \notin A$ or $z \notin B$. If $y \notin A$, then $f_A(y) = 0$. Since $f_B(z) \geq 0$, we have $f_A(y) \wedge f_B(z) = 0$. If $z \notin B$, then, as in the previous case, we have $f_A(y) \wedge f_B(z) = 0$. Thus we have

$$(f_A * f_B)(x) = \bigvee_{(y,z)\in H_x} [f_A(y) \wedge f_B(z)] = 0.$$

This completes the proof. ∎

Now, we give characterizations of an ordered semihypergroup which is semisimple in terms of fuzzy hyperideals and fuzzy interior hyperideals.

Theorem 5.4. *Let S be an ordered semihypergroup. Then the following statements are equivalent:*

(1) *S is semisimple.*

(2) *$f * g = f \cap g$ for all fuzzy interior hyperideals f and g of S.*

(3) *$f * g = f \cap g$ for every fuzzy interior hyperideal f and every fuzzy hyperideal g of S.*

(4) $f * g = f \cap g$ *for every fuzzy hyperideal* f *and every fuzzy interior hyperideal* g *of* S.

(5) $f * g = f \cap g$ *for all fuzzy hyperideals* f *and* g *of* S.

(6) $f * f = f$ *for every fuzzy hyperideal* f *of* S.

(7) $f * f = f$ *for every fuzzy interior hyperideal* f *of* S.

Proof. (1) $\Rightarrow$ (2). Let f and g be fuzzy interior hyperideals of S and $a \in S$. Then, since S is semisimple, by Lemma 5.1 there exist $x, y, z \in S$ such that $a \preceq x \circ a \circ y \circ a \circ z$. Thus, by Lemma 2.2, we have

$$\begin{aligned} a &\preceq x \circ a \circ y \circ a \circ z \preceq x \circ (x \circ a \circ y \circ a \circ z) \circ y \circ a \circ z \\ &= (x^2 \circ a \circ y) \circ ((a \circ z \circ y) \circ a \circ z). \end{aligned}$$

Then there exist $w_1 \in x^2 \circ a \circ y, w_2 \in (a \circ z \circ y) \circ a \circ z$ such that $a \preceq w_1 \circ w_2$, i.e., $(w_1, w_2) \in H_a$. By $w_1 \in x^2 \circ a \circ y$, there exists $u \in x^2$ such that $w_1 \in u \circ a \circ y$. Since f is a fuzzy interior hyperideal of S, by Theorem 3.11 we have

$$f(w_1) \geq \bigwedge_{w \in u \circ a \circ y} f(w) \geq f(a).$$

By $w_2 \in (a \circ z \circ y) \circ a \circ z$, in a similar way we can get $g(w_2) \geq g(a)$. Thus

$$(f * g)(a) = \bigvee_{(s,t) \in H_a} [f(s) \wedge g(t)] \geq f(w_1) \wedge g(w_2) \geq f(a) \wedge g(a) = (f \cap g)(a),$$

which implies that $f \cap g \subseteq f * g$. On the other hand, since S is a semisimple ordered semihypergroup, by Lemmas 2.7(3) and 5.2 we have

$$f * g \subseteq f * 1 \subseteq f \quad \text{and} \quad f * g \subseteq 1 * g \subseteq g,$$

and hence $f * g \subseteq f \cap g$. Thus $f * g = f \cap g$.

Using Theorem 3.17, every fuzzy hyperideal of S is a fuzzy interior hyperideal of S, and so (2) $\Rightarrow$ (3) $\Rightarrow$ (5), (2) $\Rightarrow$ (4) $\Rightarrow$ (5) and (7) $\Rightarrow$ (6) are clear.

(5) $\Rightarrow$ (6). Take $f = g$ in (5), we get $f * f = f \cap f = f$. This proves the (6) holds.

(2) $\Rightarrow$ (7). The proof is similar to that of (5) $\Rightarrow$ (6) with slight modification.

(6) $\Rightarrow$ (1). Let I be a hyperideal of S. By Lemma 2.8, the characteristic function f_I of I is a fuzzy hyperideal of S. Then, by hypothesis and Lemma 5.3, we have

$$f_{(I^2]} = f_I * f_I = f_I,$$

from which we deduce that $(I^2] = I$. Therefore, S is semisimple. ∎

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ON L-FUZZY 2-ABSORBING IDEALS

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Abstract. Let L be a complete lattice. In this paper we introduce various definitions of L-fuzzy 2-absorbing ideals of a commutative ring R and give some basic results concerning these classes of ideals.

Keywords: L-fuzzy 2-absorbing ideal, L-fuzzy strongly 2-absorbing ideal, L-fuzzy weakly completely 2-absorbing ideal, L-fuzzy K-2-absorbing ideal, L-fuzzy prime ideal.
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1. Introduction

Zadeh in [14] introduced the notion of a fuzzy subset μ of a non-empty set X as a function from X to $[0;1]$. Goguen in [2] generalized the notion of a fuzzy subset of X to that of an L-fuzzy subset, namely a function from X to a lattice L.

Later Rosenfeld considered the fuzzification of algebraic structures [12]. Liu [5], introduced and examined the notion of a fuzzy ideal of a ring. Since then several authors have obtained interesting results on L-fuzzy ideals of R and L-fuzzy modules. See [8] for a comprehensive survey of the literature on these developments.

L-fuzzy prime ideals play an important role in fuzzy commutative ring theory. Let R be a commutative ring. Of course for a non-constant fuzzy ideal $\xi : R \to L$, ξ is called an L-fuzzy prime ideal of R if for any L-fuzzy points $x_r, y_s \in F(R)$, $x_r y_s \in \xi$ implies that either $x_r \in \xi$ or $y_s \in \xi$. A particular attention was paid to the fuzzy prime ideals and prime fuzzy ideals (see for example [3], [6], [9], [10], [11], [13], [15], [16] and the papers cited there). In this paper, we introduce some generalizations of L-fuzzy prime ideals, namely *L-fuzzy 2-absorbing ideals*.

2. Preliminaries

Throughout this paper R is a commutative ring with a nonzero identity and L stands for a complete lattice with least element 0 and greatest element 1. Given a nonempty set X, an L-fuzzy subset μ is a function from X to L. We denote by $F(X)$ the set of all L-fuzzy subsets of X. For $\mu, \nu \in F(X)$ we say $\mu \subseteq \nu$ if and only if $\mu(x) \leq \nu(x)$, for all $x \in X$. Also, $\mu \subset \nu$ if and only if $\mu \subseteq \nu$ and $\mu \neq \nu$.

Let $\mu \in F(X)$ and $t \in L$. Then the set $\mu_t = \{x \in X \mid \mu(x) \geq t\}$ is called the t-level subset of X with respect to μ. By an L-fuzzy point x_r of X, $x \in X$; $r \in L\backslash\{0\}$, we mean $x_r \in F(X)$ defined by

$$x_r(y) = \begin{cases} r, & \text{if y=x;} \\ 0, & \text{otherwise.} \end{cases}$$

If x_r is an L-fuzzy point of X and $x_r \subseteq \mu \in F(X)$, we write $x_r \in \mu$. For $A \subseteq X$ the characteristic function of A, $\chi_A \in F(X)$, is defined by

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A; \\ 0, & \text{otherwise.} \end{cases}$$

We recall two following basic definitions given in [8].

Definition 2.1 Let $\xi \in F(R)$. Then ξ is called an L-fuzzy ideal of R if for all $x; y \in R$,

(i) $\xi(x - y) \geq \xi(x) \wedge \xi(y)$,

(ii) $\xi(xy) \geq \xi(x) \vee \xi(y)$.

(iii) $\xi(0) = 1$.

We denote by $I(R)$ the set of all L-fuzzy ideals of R.

The following are two basic operations which will be used to define L-fuzzy prime ideals and their generalizations.

Definition 2.2 Let $\xi, \mu \in F(R)$. Define the composition $\xi o \mu$ and the product $\xi\mu$ (both are L-fuzzy subsets of R) respectively as follows: For all $w \in R$,

$$\begin{aligned} (\xi o\mu)(w) &= \sup\{\xi(r) \wedge \mu(s) \mid r, s \in R,\ w = rs\}, \\ (\xi\mu)(w) &= \sup\left\{\inf_{i=1}^{n}\{\xi(r_i) \wedge \mu(s_i)\} \mid r_i, s_i \in R,\ n \in \mathbb{N},\ w = \sum_{i=1}^{n} r_i s_i\right\}, \end{aligned}$$

where as usual the supremum of an empty set is taken to be 0.

Notice that $\xi o \mu$ is the case $n = 1$ in the definition of $\xi\mu$. Thus $\xi o \mu \subseteq \xi\mu$.

Definition 2.3 ([10]) Let $A \in F(R)$. Then the L-fuzzy ideal of R generated by A, denoted by $\langle A\rangle$, is defined to be the intersection of all L-fuzzy ideals of R containing A.

Lemma 2.4 *Let R be a commutative ring with identity and let x_r and y_s be two L-fuzzy points of R. Then*

(i) $x_r y_s = (xy)_{inf\{r,s\}}$.

(ii) $\langle x_r \rangle \langle y_s \rangle = \langle x_r y_s \rangle$.

Proof. [10]. ∎

Definition 2.5 ([6]) For a non-constant $\xi \in I(R)$, ξ is called an L-fuzzy prime ideal of R if for any L-fuzzy ideals $\mu, \nu \in I(R)$, $\mu\nu \subseteq \xi$ implies that either $\mu \subseteq \xi$ or $\nu \subseteq \xi$.

Definition 2.6 ([11]) For a non-constant $\xi \in I(R)$, ξ is called an L-fuzzy completely prime ideal of R if for any L-fuzzy points $x_r, y_s \in F(R)$, $x_r y_s \in \xi$ implies that either $x_r \in \xi$ or $y_s \in \xi$.

Definition 2.7 ([11]) A fuzzy ideal ξ of R is said to be a fuzzy weakly completely prime ideal if ξ is a non-constant function and for all $x, y \in R$, $\xi(xy) = \max\{\xi(x), \xi(y)\}$.

Definition 2.8 ([4]) Let ξ be a non-constant L-fuzzy ideal of R. ξ is said to be an L-fuzzy K-prime ideal if $\xi(xy) = \xi(0)$ implies either $\xi(x) = \xi(0)$ or $\xi(y) = \xi(0)$ for any $x, y \in R$.

These different definitions of prime are not independent. In fact, in the commutative case, Definition 2.5 is equivalent to Definition 2.6.

Moreover, Definition 2.6 ⇒ Definition 2.7 ⇒ Definition 2.8 ([3]).

Definition 2.9 ([1]) Let R be a commutative ring with identity. A proper ideal I of R is said to be 2-absorbing provided that whenever $a, b, c \in R$ with $abc \in I$ then either $ab \in I$ or $ac \in I$ or $bc \in I$. R is called a 2-absorbing ring if and only if its zero ideal is 2-absorbing.

3. L-fuzzy weakly completely 2-absorbing ideals and L-fuzzy K-2-absorbing ideals

In this section we first give two definitions for fuzzy 2-absorbing ideals of R, and then prove some fundamental properties between these classes of ideals and fuzzy quotient rings.

Definition 3.1 Let η be an L-fuzzy ideal of R.

- η is called an L-fuzzy weakly completely 2-absorbing ideal of R if for all $x, y, z \in R$, $\eta(xyz) = \eta(xy)$ or $\eta(xyz) = \eta(xz)$ or $\eta(xyz) = \eta(yz)$.
- η is called an L-fuzzy K-2-absorbing ideal of R if $\eta(xyz) = \eta(0)$ implies that $\eta(xy) = \eta(0)$ or $\eta(xz) = \eta(0)$ or $\eta(yz) = \eta(0)$.

Let η be a non-constant L-fuzzy ideal of R. It is easy to see that η is an L-fuzzy weakly completely 2-absorbing ideal of R if and only if, for every $x, y, z \in R$, $\eta(xyz) = max\{\eta(xy), \eta(xz), \eta(yz)\}$. Note also that every L-fuzzy weakly completely 2-absorbing ideal is L-fuzzy K-2-absorbing. But the following example shows that the converse is not necessarily true.

Example 3.2 Let $R = Z$, the ring of integers. Define the fuzzy ideal η of $\mathbb{Z}$ by

$$\eta(x) = \begin{cases} 1, & \text{if x=0;} \\ \frac{1}{2}, & \text{if } x \in 8\mathbb{Z} - \{0\}; \\ \frac{1}{3}, & \text{if } x \in \mathbb{Z} - 8\mathbb{Z}. \end{cases} \tag{3.1}$$

Then η is a fuzzy K-2-absorbing ideal of $\mathbb{Z}$. But since

$$\eta(40) = \frac{1}{2} > \frac{1}{3} = \max\{\eta(20), \eta(4), \eta(20)\},$$

η is not a fuzzy weakly completely 2-absorbing ideal.

Proposition 3.3

(1) *Every L-fuzzy weakly completely prime ideal of R is L-fuzzy weakly completely 2-absorbing.*

(2) *Every L-fuzzy K-prime ideal of R is L-fuzzy K-2-absorbing.*

Proof. (1) Let μ be an L-fuzzy weakly completely prime ideal of R. Then, for every $x, y, z \in R$, $\mu(xyz) = \mu(x)$ or $\mu(xyz) = \mu(y)$ or $\mu(xyz) = \mu(z)$. Assume that $\mu(xyz) = \mu(x)$. Then from $\mu(xyz) \geq \mu(xy) \geq \mu(x)$ we get $\mu(xyz) = \mu(xy)$. In a similar way we can show that if $\mu(xyz) = \mu(y)$ or $\mu(xyz) = \mu(z)$, then $\mu(xyz) = \mu(yz)$ or $\mu(xyz) = \mu(xz)$. Hence μ is L-fuzzy weakly completely 2-absorbing.

(2) The proof is similar to that of (1). ∎

Theorem 3.4 *Let η be an L-fuzzy ideal of R. The following statements are equivalent:*

(i) *η is an L-fuzzy weakly completely 2-absorbing ideal of R.*

(ii) *For every $t \in L$, the t-level subset η_t of η is a 2-absorbing ideal of R.*

Proof. (i)⇒(ii) Assume that η is L-fuzzy weakly completely 2-absorbing and let $x, y, z \in R$ be such that $xyz \in \eta_t$ for some $t \in L$. Then $\max\{\eta(xy), \eta(xz), \eta(yz)\} = \eta(xyz) \geq t$. Hence $\eta(xy) \geq t$ or $\eta(xz) \geq t$ or $\eta(yz) \geq t$, that is $xy \in \eta_t$ or $xz \in \eta_t$ or $yz \in \eta_t$. Therefore that is η_t is 2-absorbing in R.

(ii)⇒(i) Assume that η_t is a 2-absorbing ideal of R for every $t \in L$. For $x, y, z \in R$ set $\eta(xyz) = t$. Then $xyz \in \eta_t$ and η_t 2-absorbing gives $xy \in \eta_t$ or $xz \in \eta_t$ or $yz \in \eta_t$. Thus $\eta(xy) \geq t$ or $\eta(xy) \geq t$ or $\eta(yz) \geq t$, that is

$\max\{\eta(xy), \eta(xz), \eta(yz)\} \geq t = \eta(xyz)$. Also since η is a is an L-fuzzy ideal of R, we have $\eta(xyz) \geq \max\{\eta(xy), \eta(xz), \eta(yz)\}$.

Hence $\eta(xyz) = \max\{\eta(xy), \eta(xz), \eta(yz)\}$, that is η is an L-fuzzy weakly completely 2-absorbing ideal. ∎

Here, we recall the definition of an L-fuzzy quotient ring of R induced by an L-fuzzy ideal of R. Let X be a set and μ an L-fuzzy relation on X. Then μ is called a fuzzy equivalence relation if (i) $\mu(x, x) = 1$ for all $x \in X$; (ii) $\mu(x, y) = \mu(y, x)$ for all $x, y \in X$; and (iii) for all $x, y \in X$, $\mu(x, y) \geq \sup_{z \in X} \min\{\mu(x, z), \mu(z, y)\}$. If μ is an L-fuzzy equivalence relation on X, then, for each $a \in X$, we shall denote $\mu[a](x) = \mu(a, x)$ for every $x \in X$. We shall say that $\mu[a]$ is the fuzzy class corresponding to a. In this case the set $X/\mu = \{\mu[a] : a \in X\}$ is called the fuzzy quotient set. Now let ν be an L-fuzzy ideal of R. It is proved in [4] that the relation μ on R defined by $\mu(x, y) = \nu(x - y)$ is a fuzzy equivalence relation. Note that, for $x, y \in R$, $\mu[x] = \mu[y]$ if and only if $\nu(x - y) = 1$. Now define summation and multiplication of L-fuzzy classes as follows:

$$\mu[x] + \mu[y] = \mu[x + y] \quad \text{and} \quad \mu[x]\mu[y] = \mu[xy].$$

Then R/μ is an L-fuzzy ring with this operations. We will write R/ν for R/μ and call it L-fuzzy quotient ring of R induced by the L-fuzzy ideal ν.

Theorem 3.5 *Let η be a non-constant L-fuzzy ideal of R. Then η is an L-fuzzy K-2-absorbing ideal of R if and only if R/η is a 2-absorbing ring.*

Proof. Assume first that η is an L-fuzzy K-2-absorbing ideal of R and let $\mu[x], \mu[y], \mu[z] \in R/\eta$ be such that $\mu[x]\mu[y]\mu[z] = \mu[0]$. Since $\mu[x]\mu[y]\mu[z] = \mu[xyz]$, we have $\eta(xyz) = \eta(xyz - 0) = 1 = \eta(0)$. As η is considered to be L-fuzzy K-2-absorbing, $\eta(xy) = \eta(0) = 1$ or $\eta(xz) = \eta(0) = 1$ or $\eta(yz) = \eta(0) = 1$. This implies that $\mu[x]\mu[y] = \mu[0]$ or $\mu[x]\mu[z] = \mu[0]$ or $\mu[y]\mu[z] = \mu[0]$, that is R/η is a 2-absorbing ring.

Conversely, assume that R/η is a 2-absorbing ring and let $\eta(xyz) = \eta(0) = 1$ for $x, y, z \in R$. Then we have $\mu[x]\mu[y]\mu[z] = \mu[xyz] = \mu[0]$. Since R/η is 2-absorbing, $\mu[x]\mu[y] = \mu[0]$ or $\mu[x]\mu[z] = \mu[0]$ or $\mu[y]\mu[z] = \mu[0]$. Hence $\eta(xy) = \eta(0)$ or $\eta(xz) = \eta(0)$ or $\eta(yz) = \eta(0)$, that is η is L-fuzzy K-2-absorbing. ∎

Corollary 3.6 *If η is an L-fuzzy weakly completely 2-absorbing ideal of R, then R/η is a 2-absorbing ring.*

4. L-fuzzy 2-absorbing ideals and L-fuzzy strongly 2-absorbing ideals

Definition 4.1 Let η be an L-fuzzy ideal of R. η is called an L-fuzzy 2-absorbing ideal of R if for any L-fuzzy points $x_r, y_s, z_t \in F(R)$ ($x, y, z \in R$ and $r, s, t \in L$), $x_r y_s z_t \in \eta$ implies that either $x_r y_s \in \eta$ or $x_r z_t \in \eta$ or $y_s z_t \in \eta$.

Lemma 4.2 *Every L-fuzzy prime ideal of R is L-fuzzy 2-absorbing.*

Proof. The proof is straightforward. ∎

Proposition 4.3 *The intersection of every pair of distinct L-fuzzy prime ideals of R is an L-fuzzy 2-absorbing ideal of R.*

Proof. Let η and μ be two distinct L-fuzzy prime ideals of R. Assume that $x_r, y_s, z_t \in F(R)$ are L-fuzzy points such that $x_r y_s z_t \in \eta \cap \mu$ but $x_r y_s \notin \eta \cap \mu$ and $x_r z_t \notin \eta \cap \mu$. Then we have the following cases:

Case 1. $x_r y_s \notin \eta$ and $x_r z_t \notin \eta$. As η is an L-fuzzy prime ideal of R we will have $z_t \in \eta$. Therefore $x_r z_t \in \eta$ which is a contradiction.

Case 2. $x_r y_s \notin \eta$ and $x_r z_t \notin \mu$. In this case from $x_r y_s z_t \in \eta \cap \mu$ we get $z_t \in \eta$ and $y_s \in \mu$. Hence $y_s z_t \in \eta \cap \mu$.

Case 3. $x_r y_s \notin \mu$ and $x_r z_t \notin \eta$. By a similar argument as in the Case 2 we may show that $y_s z_t \in \eta \cap \mu$.

Case 4. $x_r y_s \notin \mu$ and $x_r z_t \notin \mu$. A similar argument as in the Case 1 leads us to a contradiction.

Therefore $\eta \cap \mu$ is L-fuzzy 2-absorbing. ∎

Theorem 4.4 *Let η be an L-fuzzy 2-absorbing ideal of R. Then, for every $t \in L$ with $\eta_t \neq R$, η_t is a 2-absorbing ideal of R.*

Proof. Assume that $x, y, z \in R$ are such that $xyz \in \eta_t$. Then $\eta(xyz) \geq t$. Then we have $x_t y_t z_t = (xyz)_t \in \eta$. since η is an L-fuzzy 2-absorbing ideal of R, we get $(xy)_t = x_t y_t \in \eta$ or $(xz)_t = x_t z_t \in \eta$ or $(yz)_t = y_t z_t \in \eta$. If $a_t \in \eta$ for some $a \in R$, then $\eta(a) \geq t$. So $a \in \eta_t$. Therefore $xy \in \eta_t$ or $xz \in \eta_t$ or $yz \in \eta_t$. Hence η_t is a 2-absorbing ideal of R. ∎

Corollary 4.5 *If η is an L-fuzzy 2-absorbing ideal of R, then $\eta_* = \{x \in R \mid \eta(x) = \eta(0)\}$ is a 2-absorbing ideal of R.*

Proof. Since η is a non-constant L-fuzzy ideal of R, $\eta_* \neq R$. Now the result follows from Theorem 4.4. ∎

Definition 4.6 Let $\alpha \in L \backslash \{1\}$. Then α is called a 2-absorbing element of L if $r \wedge s \wedge t \leq \alpha$ implies that $r \wedge s \leq \alpha$ or $r \wedge t \leq \alpha$ or $s \wedge t \leq \alpha$ for all $r, s, t \in L$.

Proposition 4.7 *Let η be an L-fuzzy 2-absorbing ideal of R. Then $\alpha = \eta(1)$ is a 2-absorbing element of L.*

Proof. Assume that $x, y, z \in L$ with $x \wedge y \wedge z \leq \alpha$. Consider the three L-fuzzy points $1_x, 1_y, 1_z$ of R. Then we have $1_x 1_y 1_z = 1_{x \wedge y \wedge z} \in \eta$. Since η is 2-absorbing we get $1_{x \wedge y} = 1_x 1_y \in \eta$ or $1_{x \wedge z} = 1_x 1_z \in \eta$ or $1_{y \wedge z} = 1_y 1_z \in \eta$. Hence $x \wedge y \leq \eta(1) = \alpha$ or $x \wedge z \leq \eta(1) = \alpha$ or $y \wedge z \leq \eta(1) = \alpha$, that is α is 2-absorbing in L. ∎

Theorem 4.8 *Let A be an 2-absorbing ideal of R and α a 2-absorbing element of L. If η is the L-fuzzy subset of R defined by*

$$\eta(x) = \begin{cases} 1, & \text{if } x \in A; \\ \alpha, & \text{otherwise.} \end{cases} \tag{4.1}$$

for all $x \in R$, then η is an L-fuzzy 2-absorbing ideal of R.

Proof. (a) Since A is a 2-absorbing ideal of R, $A \neq R$. Therefore η is a non-constant L-fuzzy ideal of R. Suppose that $x_r, y_s, z_t \in F(R)$ are L-fuzzy points such that $x_r y_s z_t \in \eta$ but $x_r y_s \notin \eta$ and $x_r z_t \notin \eta$ and $y_s z_t \notin \eta$. In this case $\eta(xy) = \alpha$ and so $xy \notin A$. Similarly, $xz \notin A$ and $yz \notin A$. As A is assumed to be 2-absorbing, we get $xyz \notin A$. So that $\eta(xyz) = \alpha$. Also from $(xyz)_{r \wedge s \wedge t} = x_r y_s z_t \in \eta$ we have $r \wedge s \wedge t \leq \eta(xyz) = \alpha$. Hence $r \wedge s \leq \alpha$ or $r \wedge t \leq \alpha$ or $s \wedge t \leq \alpha$, since α is a 2-absorbing element, which is a contradiction. Therefore $x_r y_s \in \eta$ or $x_r z_t \in \eta$ or $y_s z_t \in \eta$, that is η is 2-absorbing. ∎

Example 4.9 By Lemma 4.2, every L-fuzzy prime ideal of R is L-fuzzy 2-absorbing, but the converse does not necessarily hold. For example, consider the case where R=$\mathbb{Z}$. Let p and q be a pair of distinct prime numbers, and set A=$pq\mathbb{Z}$. It is not difficult to show that A is a 2-absorbing ideal of $\mathbb{Z}$. Now define $\eta : \mathbb{Z} \rightarrow [0, 1]$ by

$$\eta(x) = \begin{cases} 1, & \text{if } pq|x; \\ 0, & \text{otherwise.} \end{cases}$$

Then η is a fuzzy 2-absorbing ideal of R by Theorem 4.8. Moreover $\eta_0 = A$ is a 2-absorbing ideal of $\mathbb{Z}$ that is not a prime ideal. Hence η is not a fuzzy prime ideal of R.

Definition 4.10 Let η be an L-fuzzy ideal of R. η is said to be an L-fuzzy strongly 2-absorbing ideal of R provided that it is non-constant and whenever $\lambda, \mu, \nu \in I(R)$ with $\lambda\mu\nu \subseteq \eta$, then $\lambda\mu \subseteq \eta$ or $\lambda\nu \subseteq \eta$ or $\mu\nu \subseteq \eta$.

Theorem 4.11

(1) *Every L-fuzzy prime ideal of R is L-fuzzy strongly 2-absorbing.*

(2) *Every L-fuzzy strongly 2-absorbing ideal of R is L-fuzzy 2-absorbing.*

Proof. (1) Straightforward.

(2) Suppose that η is an L-fuzzy strongly 2-absorbing ideal of R. Assume that $x_r, y_s, z_t \in \eta$ for some L-fuzzy points x_r, y_s, z_t of R. Then, by Lemma 2.4, we have $\langle x_r \rangle \langle y_s \rangle \langle z_t \rangle = \langle x_r y_s z_t \rangle \subseteq \eta$. Since η is L-fuzzy strongly 2-absorbing, we have $\langle x_r y_s \rangle = \langle x_r \rangle \langle y_s \rangle \subseteq \eta$ or $\langle x_r z_t \rangle = \langle x_r \rangle \langle z_t \rangle \subseteq \eta$ or $\langle y_s z_t \rangle = \langle y_s \rangle \langle z_t \rangle \subseteq \eta$. Therefore $x_r y_s \in \eta$ or $x_r z_t \in \eta$ or $y_s z_t \in \eta$, that is η is an L-fuzzy 2-absorbing ideal of R. ∎

Let A be an ideal of the commutative ring R. It is proved in [1] that A is a 2-absorbing ideal of R if and only if whenever I, J, K are ideals of R with $IJK \subseteq A$, then $IJ \subseteq A$ or $IK \subseteq A$ or $JK \subseteq A$. It is also well known that a nonconstant ideal $\xi \in I(R)$ is an L-fuzzy prime ideal of R if and only if for any two L-fuzzy ideals μ and ν of R, $\mu\nu \subseteq \xi$ implies that either $\mu \subseteq \eta$ or $\nu \subseteq \eta$. Let I be a nonconstant L-fuzzy ideal of R. We proved in Theorem 4.11 that every L-fuzzy strongly 2-absorbing ideal of R is L-fuzzy 2-absorbing, but I was unable to prove or disprove the converse.

Question 1. Is every L-fuzzy 2-absorbing ideal of R of the form 4.1, where A is a 2-absorbing ideal of R and α is a 2-absorbing element of L?

Question 2. Is every L-fuzzy 2-absorbing ideal of R L-fuzzy strongly 2-absorbing?

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ON GENERALIZED RITT ORDER OF ENTIRE FUNCTIONS REPRESENTED BY VECTOR VALUED DIRICHLET SERIES

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Abstract. In this paper, we introduce the idea of generalized Ritt order of entire function (respectively generalized Ritt lower order) represented by a vector valued Dirichlet series. Hence we study some growth properties of of two of entire functions represented by a vector valued Dirichlet series on the basis of their generalized Ritt orders and generalized Ritt lower orders.

Keywords and phrases: vector valued Dirichlet series (VVDS), generalized Ritt order, generalized Ritt lower order, growth.
AMS Subject Classification (2010): 30B50, 30D15, 30D99.

1. Introduction, definitions and notations

Let $f(s)$ be an entire function of the complex variable $s = \sigma + it$ (σ and t are real variables) defined by everywhere absolutely convergent vector valued Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n} \tag{1}$$

where a_n's belong to a Banach space $(E, \|.\|)$ and λ_n's are non-negative real numbers such that $0 < \lambda_n < \lambda_{n+1}\,(n \geq 1)\,, \lambda_n \to \infty$ as $n \to \infty$ and satisfy the conditions

$$\limsup_{n\to\infty} \frac{\log n}{\lambda_n} = D < \infty$$

and

$$\limsup_{n\to\infty} \frac{\log \|a_n\|}{\lambda_n} = -\infty\ .$$

If σ_a and σ_c are denoted respectively the abscissa of convergence and absolute convergence of (1), then in this case clearly $\sigma_a = \sigma_c = \infty$.

The function $M\,(\sigma, f)$ known as maximum modulus function corresponding to an entire function $f\,(s)$ defined by (1), is written as follows

$$M\,(\sigma, f) = \underset{-\infty<t<\infty}{l.u.b.} |f\,(\sigma + it)|\ .$$

Then, the Ritt order [1] of $f\,(s)\,$, denoted by ρ_f is given by

$$\begin{aligned} \rho_f &= \inf\{\mu > 0 : \log M\,(\sigma, f) < \exp\,(\sigma\mu) \text{ for all } \sigma > R\,(\mu)\} \\ &= \limsup_{\sigma\to\infty} \frac{\log^{[2]} M\,(\sigma, f)}{\sigma} \end{aligned}$$

where

$$\begin{aligned} \log^{[k]} x &= \log\left(\log^{[k-1]} x\right) \text{ for } k = 1, 2, 3, \cdots; \\ \log^{[0]} x &= x\ . \end{aligned}$$

Similarly, one can define the Ritt lower order of $f\,(s)\,$, denoted by λ_f in the following manner:

$$\lambda_f = \liminf_{\sigma\to\infty} \frac{\log^{[2]} M\,(\sigma, f)}{\sigma}\ .$$

During the past decades, several authors (see for example [1],[2],[3],[4],[6]) made close investications on the properties of entire Dirichlet series related to Ritt order. Further, B.L. Srivastava [5] defined different growth parameters such as order and lower order of entire functions represented by vector valued Dirichlet series. He also obtained the results for coefficient characterization of order. Now in the line of Sato [7], it therefore seems reasonable to define suitably the generalized Ritt order denoted as $\rho_f^{[l]}$ of entire function represented by a vector valued Dirichlet series which is as follows:

$$\rho_f^{[l]} = \limsup_{\sigma\to\infty} \frac{\log^{[l]} M\,(\sigma, f)}{\sigma}$$

where l is any positive integer.

Likewise, one can define the generalized Ritt lower order of $f\,(s)\,$, denoted by $\lambda_f^{[l]}$ for any integer $l \geq 1$ in the following manner:

$$\lambda_f^{[l]} = \liminf_{\sigma\to\infty} \frac{\log^{[l]} M\,(\sigma, f)}{\sigma}\ .$$

In this paper we study some growth properties of of two entire functions represented by vector valued Dirichlet series on the basis of generalized Ritt order and generalized Ritt lower order.

2. Theorems

In this section, we present the main results of the paper.

Theorem 1 *If f, g be any two entire vector valued Dirichlet series such that $0 < \lambda_f^{[m]} \leq \rho_f^{[m]} < \infty$ and $0 < \lambda_g^{[n]} \leq \rho_g^{[n]} < \infty$ where m and n are any two positive integers, then*

$$\frac{\lambda_f^{[m]}}{\rho_g^{[n]}} \leq \liminf_{\sigma\to\infty} \frac{\log^{[m]} M(\sigma, f)}{\log^{[n]} M(\sigma, g)} \leq \frac{\lambda_f^{[m]}}{\lambda_g^{[n]}} \leq \limsup_{\sigma\to\infty} \frac{\log^{[m]} M(\sigma, f)}{\log^{[n]} M(\sigma, g)} \leq \frac{\rho_f^{[m]}}{\lambda_g^{[n]}} .$$

Proof. From the definitions of $\rho_g^{[n]}$ and $\lambda_f^{[m]}$, we have for arbitrary positive ε and for all sufficiently large values of σ that

$$\log^{[m]} M(\sigma, f) \geqslant \left(\lambda_f^{[m]} - \varepsilon\right)\sigma \tag{1}$$

and

$$\log^{[n]} M(\sigma, g) \leq \left(\rho_g^{[n]} + \varepsilon\right)\sigma . \tag{2}$$

Now from (1) and (2) it follows for all sufficiently large values of σ that

$$\frac{\log^{[m]} M(\sigma, f)}{\log^{[n]} M(\sigma, g)} \geqslant \frac{\left(\lambda_f^{[m]} - \varepsilon\right)\sigma}{\left(\rho_g^{[n]} + \varepsilon\right)\sigma} .$$

As $\varepsilon\,(> 0)$ is arbitrary, we obtain that

$$\liminf_{\sigma\to\infty} \frac{\log^{[m]} M(\sigma, f)}{\log^{[n]} M(\sigma, g)} \geqslant \frac{\lambda_f^{[m]}}{\rho_g^{[n]}} . \tag{3}$$

Again for a sequence of values of σ tending to infinity,

$$\log^{[m]} M(\sigma, f) \leq \left(\lambda_f^{[m]} + \varepsilon\right)\sigma \tag{4}$$

and for all sufficiently large values of σ,

$$\log^{[n]} M(\sigma, g) \geqslant \left(\lambda_g^{[n]} - \varepsilon\right)\sigma . \tag{5}$$

Combining (4) and (5) we get for a sequence of values of σ tending to infinity that

$$\frac{\log^{[m]} M(\sigma, f)}{\log^{[n]} M(\sigma, g)} \leq \frac{\left(\lambda_f^{[m]} + \varepsilon\right)\sigma}{\left(\lambda_g^{[n]} - \varepsilon\right)\sigma} .$$

Since $\varepsilon\,(>0)$ is arbitrary it follows that

$$\liminf_{\sigma\to\infty}\frac{\log^{[m]} M(\sigma,f)}{\log^{[n]} M(\sigma,g)} \leq \frac{\lambda_f^{[m]}}{\lambda_g^{[n]}} \ . \tag{6}$$

Also for a sequence of values of σ tending to infinity that

$$\log^{[n]} M(\sigma,g) \leq \left(\lambda_g^{[n]}+\varepsilon\right)\sigma \ . \tag{7}$$

Now from (1) and (7) we obtain for a sequence of values of σ tending to infinity that

$$\frac{\log^{[m]} M(\sigma,f)}{\log^{[n]} M(\sigma,g)} \geq \frac{\left(\lambda_f^{[m]}-\varepsilon\right)\sigma}{\left(\lambda_g^{[n]}+\varepsilon\right)\sigma} \ .$$

As $\varepsilon\,(>0)$ is arbitrary, we get from above that

$$\limsup_{\sigma\to\infty}\frac{\log^{[m]} M(\sigma,f)}{\log^{[n]} M(\sigma,g)} \geq \frac{\lambda_f^{[m]}}{\lambda_g^{[n]}} \ . \tag{8}$$

Also for all sufficiently large values of σ,

$$\log^{[m]} M(\sigma,f) \leq \left(\rho_f^{[m]}+\varepsilon\right)\sigma \ . \tag{9}$$

Now, it follows from (5) and (9), for all sufficiently large values of σ that

$$\frac{\log^{[m]} M(\sigma,f)}{\log^{[n]} M(\sigma,g)} \leq \frac{\left(\rho_f^{[m]}+\varepsilon\right)\sigma}{\left(\lambda_g^{[n]}-\varepsilon\right)\sigma} \ .$$

Since $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\limsup_{\sigma\to\infty}\frac{\log^{[m]} M(\sigma,f)}{\log^{[n]} M(\sigma,g)} \leq \frac{\rho_f^{[m]}}{\lambda_g^{[n]}} \ . \tag{10}$$

Thus the theorem follows from (3), (6), (8) and (10). ∎

Theorem 2 *If f, g be any two entire vector valued Dirichlet series such that $0<\rho_f^{[m]}<\infty$ and $0<\rho_g^{[n]}<\infty$ where m and n are any two positive integers, then*

$$\liminf_{\sigma\to\infty}\frac{\log^{[m]} M(\sigma,f)}{\log^{[n]} M(\sigma,g)} \leq \frac{\rho_{f\circ g}^{[m]}}{\rho_g^{[n]}} \leq \limsup_{\sigma\to\infty}\frac{\log^{[m]} M(\sigma,f)}{\log^{[n]} M(\sigma,g)} \ .$$

Proof. From the definition of $\rho_g^{[n]}$, we get for a sequence of values of σ tending to infinity that

$$\log^{[n]} M(\sigma,g) \geqslant \left(\rho_g^{[n]}-\varepsilon\right)\sigma \ . \tag{11}$$

Now from (9) and (11) , it follows for a sequence of values of σ tending to infinity that

$$\frac{\log^{[m]} M(\sigma, f)}{\log^{[n]} M(\sigma, g)} \leq \frac{\left(\rho_f^{[m]} + \varepsilon\right)\sigma}{\left(\rho_g^{[n]} - \varepsilon\right)\sigma} .$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\liminf_{\sigma\to\infty} \frac{\log^{[m]} M(\sigma, f)}{\log^{[n]} M(\sigma, g)} \leq \frac{\rho_f^{[m]}}{\rho_g^{[n]}} . \tag{12}$$

Again for a sequence of values of σ tending to infinity,

$$\log^{[m]} M(\sigma, f) \geqslant \left(\rho_f^{[m]} - \varepsilon\right)\sigma . \tag{13}$$

So combining (2) and (13) , we get for a sequence of values of σ tending to infinity that

$$\frac{\log^{[m]} M(\sigma, f)}{\log^{[n]} M(\sigma, g)} \geqslant \frac{\left(\rho_f^{[m]} - \varepsilon\right)\sigma}{\left(\rho_g^{[n]} + \varepsilon\right)\sigma} .$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows that

$$\limsup_{\sigma\to\infty} \frac{\log^{[m]} M(\sigma, f)}{\log^{[n]} M(\sigma, g)} \geqslant \frac{\rho_f^{[m]}}{\rho_g^{[n]}} . \tag{14}$$

Thus the theorem follows from (12) and (14) . ∎

The following theorem is a natural consequence of Theorem 1 and Theorem 2.

Theorem 3 *If f, g be any two entire vector valued Dirichlet series such that $0 < \lambda_f^{[m]} \leq \rho_f^{[m]} < \infty$ and $0 < \lambda_g^{[n]} \leq \rho_g^{[n]} < \infty$ where m and n are any two positive integers, then*

$$\begin{aligned}
\liminf_{\sigma\to\infty} \frac{\log^{[m]} M(\sigma, f)}{\log^{[n]} M(\sigma, g)} &\leq \min\left\{\frac{\lambda_f^{[m]}}{\lambda_g^{[n]}}, \frac{\rho_f^{[m]}}{\rho_g^{[n]}}\right\} \\
&\leq \max\left\{\frac{\lambda_f^{[m]}}{\lambda_g^{[n]}}, \frac{\rho_f^{[m]}}{\rho_g^{[n]}}\right\} \\
&\leq \leq \limsup_{\sigma\to\infty} \frac{\log^{[m]} M(\sigma, f)}{\log^{[n]} M(\sigma, g)} .
\end{aligned}$$

The proof is omitted.

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COMPARATIVE GROWTH ANALYSIS OF FUNCTIONS ANALYTIC IN THE UNIT DISC DEPENDING UPON THEIR RELATIVE L^*-ORDERS AND RELATIVE L^*-LOWER ORDERS

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Abstract. In the paper the ideas of relative Nevanlinna L^*-order and relative Nevanlinna L^*-lower order of an analytic function with respect to an entire function in the unit disc $U = \{z : |z| < 1\}$ are introduced. Hence, we study some comparative growth properties of composition of two analytic functions in the unit disc U on the basis of relative Nevanlinna L^*-order and relative Nevanlinna L^*-lower order.

Keywords and phrases: growth, analytic function, composition, unit disc, relative Nevanlinna L^*-order, relative Nevanlinna L^*-lower order, slowly changing function in the unit disc.

AMS Subject Classification (2000): 30D35, 30D30, 30D20.

1. Introduction, definitions and notations

A function f, analytic in the unit disc $U = \{z : |z| < 1\}$, is said to be of finite Nevanlinna order [2] if there exist a number μ such that Nevanlinna characteristic function

$$T(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f\left(re^{i\theta}\right)\right| d\theta$$

satisfies $T(r,f) < (1-r)^{-\mu}$ for all r in $0 < r_0(\mu) < r < 1$. The greatest lower bound of all such numbers μ is called the Nevanlinna order of f. Thus the Nevanlinna order ρ_f of f is given by

$$\rho_f = \limsup_{r\to 1} \frac{\log T(r,f)}{-\log(1-r)} .$$

Similarly, the Nevanlinna lower order λ_f of f is given by

$$\lambda_f = \liminf_{r\to 1} \frac{\log T(r,f)}{-\log(1-r)} .$$

Datta et. al. [1] introduced the notion of Nevanlinna L-order for an analytic function f in the unit disc $U = \{z : |z| < 1\}$ where $L = L\left(\frac{1}{1-r}\right)$ is a positive continuous function in the unit disc U increasing slowly i.e., $L\left(\frac{a}{1-r}\right) \sim L\left(\frac{1}{1-r}\right)$ as $r \to 1$, for every positive constant 'a', in the following manner:

Definition 1 If f be analytic in U, then the Nevanlinna L-order ρ_f^L and the Nevanlinna L-lower order λ_f^L of f are defined as

$$\rho_f^L = \frac{\log T(r,f)}{\log\left(\frac{L\left(\frac{1}{1-r}\right)}{(1-r)}\right)} \quad \text{and} \quad \lambda_f = \liminf_{r\to 1} \frac{\log T(r,f)}{\log\left(\frac{L\left(\frac{1}{1-r}\right)}{(1-r)}\right)} .$$

Now we introduce the concepts of relative Nevanlinna L^*-order and relative Nevanlinna L^*-lower order of an analytic function f with respect to another analytic function g in the unit disc U which are as follows:

Definition 2 If f be analytic in U and g be entire, then the relative Nevanlinna L^*-order of f with respect to g, denoted by $\rho_g^{L^*}(f)$ is defined by

$$\rho_f^L = \inf\left\{\mu > 0 : T_f(r) < T_g\left[\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)}\right]^{\mu} \text{ for all } 0 < r_0(\mu) < r < 1\right\} .$$

Similarly, the relative Nevanlinna L^*-order of f with respect to g, denoted by $\lambda_g^{L^*}(f)$ is given by

$$\lambda_g^{L^*}(f) = \liminf_{r\to 1} \frac{\log T_g^{-1} T_f(r)}{\log\left(\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)}\right)} .$$

When $g(z) = \exp z$, the definition coincides with the definition of the Nevanlinna L^*-order and the Nevanlinna L^*-lower order.

In this paper, we study some growth properties of composition of two analytic functions in the unit disc $U = \{z : |z| < 1\}$ on the basis of relative Nevanlinna L^*-order (relative Nevanlinna L^*-lower order). We do not explain the standard

definitions and notations in the theory of entire functions as those are available in [3].

2. Theorems

In this section, we present the main results of the paper.

Theorem 1 *If f, g be any two analytic functions in U and h be an entire function such that $0 < \lambda_h^{L^*}(f \circ g) \leq \rho_h^{L^*}(f \circ g) < \infty$ and $0 < \lambda_h^{L^*}(f) \leq \rho_h^{L^*}(f) < \infty$ then*

$$\frac{\lambda_h^{L^*}(f \circ g)}{\rho_h^{L^*}(f)} \leq \liminf_{r \to 1} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_h^{-1} T_f(r)} \leq \frac{\lambda_h^{L^*}(f \circ g)}{\lambda_h^{L^*}(f)}$$
$$\leq \limsup_{r \to 1} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_h^{-1} T_f(r)} \leq \frac{\rho_h^{L^*}(f \circ g)}{\lambda_h^{L^*}(f)} .$$

Proof. From the definition of $\rho_h^{L^*}(f)$ and $\lambda_h^{L^*}(f \circ g)$, we have for arbitrary positive ε and for all sufficiently large values of $\left(\frac{1}{1-r}\right)$ that

$$\log T_h^{-1} T_{f \circ g}(r) \geqslant \left(\lambda_h^{L^*}(f \circ g) - \varepsilon\right) \log \left(\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)} \right) \tag{1}$$

and

$$\log T_h^{-1} T_f(r) \leq \left(\rho_h^{L^*}(f) + \varepsilon\right) \log \left(\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)} \right) . \tag{2}$$

Now from (1) and (2) it follows for all sufficiently large values of $\left(\frac{1}{1-r}\right)$ that

$$\frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_h^{-1} T_f(r)} \geqslant \frac{\left(\lambda_h^{L^*}(f \circ g) - \varepsilon\right) \log \left(\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)} \right)}{\left(\rho_h^{L^*}(f) + \varepsilon\right) \log \left(\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)} \right)} .$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\liminf_{r \to 1} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_h^{-1} T_f(r)} \geqslant \frac{\lambda_h^{L^*}(f \circ g)}{\rho_h^{L^*}(f)} . \tag{3}$$

Again for a sequence of values of $\left(\frac{1}{1-r}\right)$ tending to infinity,

$$\log T_h^{-1} T_{f \circ g}(r) \leq \left(\lambda_h^{L^*}(f \circ g) + \varepsilon\right) \log \left(\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)} \right) \tag{4}$$

and for all sufficiently large values of $\left(\frac{1}{1-r}\right)$,

$$\log T_h^{-1} T_f(r) \geqslant \left(\lambda_h^{L^*}(f) - \varepsilon\right) \log \left(\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)} \right) . \tag{5}$$

Combining (4) and (5), we get for a sequence of values of $\left(\frac{1}{1-r}\right)$ tending to infinity that

$$\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_f(r)} \leq \frac{\left(\lambda_h^{L^*}(f\circ g)+\varepsilon\right)\log\left(\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)}\right)}{\left(\lambda_h^{L^*}(f)-\varepsilon\right)\log\left(\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)}\right)}.$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows that

$$\liminf_{r\to 1}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_f(r)} \leq \frac{\lambda_h^{L^*}(f\circ g)}{\lambda_h^{L^*}(f)}. \tag{6}$$

Also for a sequence of values of $\left(\frac{1}{1-r}\right)$ tending to infinity that

$$\log T_h^{-1}T_f(r) \leq \left(\lambda_h^{L^*}(f)+\varepsilon\right)\log\left(\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)}\right). \tag{7}$$

Now from (1) and (7), we obtain for a sequence of values of $\left(\frac{1}{1-r}\right)$ tending to infinity that

$$\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_f(r)} \geq \frac{\left(\lambda_h^{L^*}(f\circ g)-\varepsilon\right)\log\left(\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)}\right)}{\left(\lambda_h^{L^*}(f)+\varepsilon\right)\log\left(\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)}\right)}.$$

As $\varepsilon\,(>0)$ is arbitrary, we get from above that

$$\limsup_{r\to 1}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_f(r)} \geq \frac{\lambda_h^{L^*}(f\circ g)}{\lambda_h^{L^*}(f)}. \tag{8}$$

Also for all sufficiently large values of $\left(\frac{1}{1-r}\right)$,

$$\log T_h^{-1}T_{f\circ g}(r) \leq \left(\rho_h^{L^*}(f\circ g)+\varepsilon\right)\log\left(\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)}\right). \tag{9}$$

Now it follows from (5) and (9) for all sufficiently large values of $\left(\frac{1}{1-r}\right)$ that

$$\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_f(r)} \leq \frac{\left(\rho_h^{L^*}(f\circ g)+\varepsilon\right)\log\left(\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)}\right)}{\left(\lambda_h^{L^*}(f)-\varepsilon\right)\log\left(\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)}\right)}.$$

Since $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\limsup_{r\to 1}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_f(r)} \leq \frac{\rho_h^{L^*}(f\circ g)}{\lambda_h^{L^*}(f)}. \tag{10}$$

Thus the theorem follows from (3), (6), (8) and (10). ∎

The following theorem can be proved in the line of Theorem 1 and so its proof is omitted.

Theorem 2 *If f, g be any two analytic functions in U and h be entire function with $0 < \lambda_h^{L^*}(f \circ g) \leq \rho_h^{L^*}(f \circ g) < \infty$ and $0 < \lambda_h^{L^*}(g) \leq \rho_h^{L^*}(g) < \infty$ then*

$$\frac{\lambda_h^{L^*}(f \circ g)}{\rho_h^{L^*}(g)} \leq \liminf_{r \to 1} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_h^{-1} T_g(r)} \leq \frac{\lambda_h^{L^*}(f \circ g)}{\lambda_h^{L^*}(g)}$$
$$\leq \limsup_{r \to 1} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_h^{-1} T_g(r)} \leq \frac{\rho_h^{L^*}(f \circ g)}{\lambda_h^{L^*}(g)} .$$

Theorem 3 *If f, g be any two analytic functions in U and h be entire function such that $0 < \rho_h^{L^*}(f \circ g) < \infty$ and $0 < \rho_h^{L^*}(f) < \infty$ then*

$$\liminf_{r \to 1} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_h^{-1} T_f(r)} \leq \frac{\rho_h^{L^*}(f \circ g)}{\rho_h^{L^*}(f)} \leq \limsup_{r \to 1} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_h^{-1} T_f(r)} .$$

Proof. From the definition of $\rho_h^{L^*}(f)$, we get for a sequence of values of $\left(\frac{1}{1-r}\right)$ tending to infinity that

$$\log T_h^{-1} T_f(r) \geqslant \left(\rho_h^{L^*}(f) - \varepsilon\right) \log \left(\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)} \right) . \tag{11}$$

Now from (9) and (11), it follows for a sequence of values of $\left(\frac{1}{1-r}\right)$ tending to infinity that

$$\frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_h^{-1} T_f(r)} \leq \frac{\left(\rho_h^{L^*}(f \circ g) + \varepsilon\right) \log \left(\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)} \right)}{\left(\rho_h^{L^*}(f) - \varepsilon\right) \log \left(\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)} \right)} .$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\liminf_{r \to 1} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_h^{-1} T_f(r)} \leq \frac{\rho_h^{L^*}(f \circ g)}{\rho_h^{L^*}(f)} . \tag{12}$$

Again for a sequence of values of $\left(\frac{1}{1-r}\right)$ tending to infinity,

$$\log T_h^{-1} T_{f \circ g}(r) \geqslant \left(\rho_h^{L^*}(f \circ g) - \varepsilon\right) \log \left(\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)} \right) . \tag{13}$$

So combining (2) and (13), we get for a sequence of values of $\left(\frac{1}{1-r}\right)$ tending to infinity that

$$\frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_h^{-1} T_f(r)} \geqslant \frac{\left(\rho_h^{L^*}(f \circ g) - \varepsilon\right) \log \left(\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)} \right)}{\left(\rho_h^{L^*}(f) + \varepsilon\right) \log \left(\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)} \right)} .$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows that

$$\limsup_{r\to 1}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_f(r)} \geqslant \frac{\rho_h^{L^*}(f\circ g)}{\rho_h^{L^*}(f)}\ . \tag{14}$$

Thus the theorem follows from (12) and (14). ∎

The following theorem can be carried out in the line of Theorem 3 and therefore we omit its proof.

Theorem 4 *If f, g be any two analytic functions in U and h be an entire function with $0<\rho_h^{L^*}(f\circ g)<\infty$ and $0<\rho_h^{L^*}(g)<\infty$ then*

$$\liminf_{r\to 1}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_g(r)} \leq \frac{\rho_h^{L^*}(f\circ g)}{\rho_h^{L^*}(g)} \leq \limsup_{r\to 1}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_g(r)}\ .$$

The following theorem is a natural consequence of Theorem 1 and Theorem 3.

Theorem 5 *If f, g be any two analytic functions in U and h be an entire function such that $0<\lambda_h^{L^*}(f\circ g)\leq\rho_h^{L^*}(f\circ g)<\infty$ and $0<\lambda_h^{L^*}(f)\leq\rho_h^{L^*}(f)<\infty$ then*

$$\begin{aligned}\liminf_{r\to 1}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_f(r)} &\leq \min\left\{\frac{\lambda_h^{L^*}(f\circ g)}{\lambda_h^{L^*}(f)},\frac{\rho_h^{L^*}(f\circ g)}{\rho_h^{L^*}(f)}\right\}\\ &\leq \max\left\{\frac{\lambda_h^{L^*}(f\circ g)}{\lambda_h^{L^*}(f)},\frac{\rho_h^{L^*}(f\circ g)}{\rho_h^{L^*}(f)}\right\} \leq \limsup_{r\to 1}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_f(r)}\ .\end{aligned}$$

The proof is omitted.

Analogously, one may state the following theorem without its proof:

Theorem 6 *If f, g be any two analytic functions in U and h be an entire function with $0<\lambda_h^{L^*}(f\circ g)\leq\rho_h^{L^*}(f\circ g)<\infty$ and $0<\lambda_h^{L^*}(g)\leq\rho_h^{L^*}(g)<\infty$ then*

$$\begin{aligned}\liminf_{r\to 1}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_g(r)} &\leq \min\left\{\frac{\lambda_h^{L^*}(f\circ g)}{\lambda_h^{L^*}(g)},\frac{\rho_h^{L^*}(f\circ g)}{\rho_h^{L^*}(g)}\right\}\\ &\leq \max\left\{\frac{\lambda_h^{L^*}(f\circ g)}{\lambda_h^{L^*}(g)},\frac{\rho_h^{L^*}(f\circ g)}{\rho_h^{L^*}(g)}\right\} \leq \limsup_{r\to 1}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_g(r)}\ .\end{aligned}$$

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A NEW TYPE OF COMPACT SPACES AND CLOSED SPACES

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Abstract. In this paper, we have introduced and studied the concepts of $sg\alpha$-compact space and $sg\alpha$-closed space by means of nets and filterbases.

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1. Introduction

Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real Analysis concerns the various modified forms of continuity, separation axioms etc. by utilizing generalized open sets. Compactness and properties closely related to compactness play an important role in the applications of General Topology to Real Analysis and Functional Analysis. In this paper, we have introduce and study the concept of $sg\alpha$-compact spaces and $sg\alpha$-closed space by means of nets and filterbases.

2. Preliminaries

Throughout this paper, spaces always means topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X, τ), $\mathrm{cl}(A)$ and $\mathrm{int}(A)$ denote the closure of A and the interior of A in X, respectively.

Definition 2.1 A subset A of a space (X, τ) is called semi-open [2] (resp. α-open [3]) if $A \subset \mathrm{cl}(\mathrm{int}(A))$ (resp. $A \subset \mathrm{int}(\mathrm{cl}(\mathrm{int}(A)))$). The complement of a semiopen (resp. α-open) set is called a semi-closed (resp. α-closed) set.

The semi-closure [1] of a subset A of X, denoted by $s\,\mathrm{cl}(A)$ is defined to be the intersection of all semi-closed sets containing A in X. The α-closure and α-interior of a set A are similarly defined.

Definition 2.2 A subset A of a space X is called semi-generalized closed (briefly $sg\alpha$-closed) [4] if $\alpha\,\mathrm{cl}(A) \subset U$ whenever $A \subset U$ and U is semiopen in X. The complement of a $sg\alpha$-closed set is called a $sg\alpha$-open set.

Definition 2.3 Let (X, τ) be a topological space and $E \subseteq X$. We define the $sg\alpha$-closure [4] of E (briefly $sg\alpha\,\mathrm{cl}(E)$) to be the intersection of all $sg\alpha$-closed sets containing E. In symbols, $sg\alpha\,\mathrm{cl}(E) = \bigcap \{A : E \subset A \text{ and } A \in sg\alpha C(X, \tau)\}$.

The family of all $sg\alpha$-open ($sg\alpha$-closed) sets of (X, τ) is denoted by $sg\alpha O(X)$ ($sg\alpha C(X)$). The family of all $sg\alpha$-open ($sg\alpha$-closed) sets of (X, τ) containing a point $x \in X$ is denoted by $sg\alpha O(X, x)$ ($sg\alpha C(X, x)$).

Proposition 2.4 [4] *Let (X, τ) be a topological space and $E \subseteq X$. The following properties are hold:*

(i) *$sg\alpha\,\mathrm{cl}(E)$ is the smallest $sg\alpha$-closed set containing E;*

(ii) *E is $sg\alpha$-closed if and only if $sg\alpha\,\mathrm{cl}(E) = E$.*

Definition 2.5 [4] Let (X, τ) be a topological space and $E \subset X$. We define the $sg\alpha$-interior of E (briefly $sg\alpha\,\mathrm{int}(E)$) to be the union of all $sg\alpha$-open sets contained in E.

Definition 2.6 The $sg\alpha$-θ-closure [6] of A, denoted by $sg\alpha\,\mathrm{cl}_\theta(A)$, is defined to be the set of all $x \in X$ such that $A \cap sg\alpha\,\mathrm{cl}(U) \neq \varnothing$ for every $sg\alpha$-open set U containing x. A subset A is called $sg\alpha$-θ-closed [6] if and only if $A = sg\alpha\,\mathrm{cl}_\theta(A)$. The complement of $sg\alpha$-θ-closed set is called $sg\alpha$-θ-open. A subset S of a topological space (X, τ) is said to be $sg\alpha$-regular [6] if it is $sg\alpha$-open and $sg\alpha$-closed.

The family of all $sg\alpha$-regular (resp. $sg\alpha$-open, $sg\alpha$-closed) sets of (X, τ) is denoted by $sg\alpha R(X)$ (resp. $sg\alpha O(X)$, $sg\alpha C(X)$). The family of all $sg\alpha$-regular (resp. $sg\alpha$-open, $sg\alpha$-closed) sets of (X, τ) containing a point $x \in X$ is denoted by $sg\alpha R(X, x)$ (resp. $sg\alpha O(X, x)$, $sg\alpha C(X, x)$).

3. $sg\alpha$-compact spaces

Definition 3.1 Let (X, τ) be a topological space. A class $\{G_i\}$ of $sg\alpha$-open subsets of X is said to be $sg\alpha$-open cover of X if each point in X belongs to atleast one G_i that is $\bigcup_i G_i = X$.

Definition 3.2 A subset K of a nonempty set X is said to be $sg\alpha$-compact relative to (X, τ) if every cover of K by sets of $sg\alpha O(X)$ has a finite subcover. We say that (X, τ) is $sg\alpha$-compact if X is $sg\alpha$-compact.

It is clear that every $sg\alpha$-compact space is compact. But the converse is not always true as it can be seen from the following example.

Example 3.3 Let X be the set of real numbers endowed with the cofinite topology. The topological space X is clearly compact. The collection of all $sg\alpha$-open sets in X is precisely the power set of X. Hence the topological space X is not $sg\alpha$-compact.

We will give several characterizations of the $sg\alpha$-compact spaces. The first characterization makes use of the finite intersection condition.

Theorem 3.4 *The following statements are equivalent for any topological space* (X, τ)*:*

(i) *X is $sg\alpha$-compact.*

(ii) *Given any family $\mathcal{F}$ of $sg\alpha$-open sets, if no finite subfamily of $\mathcal{F}$ covers X, then $\mathcal{F}$ does not cover X.*

(iii) *Given any family $\mathcal{F}$ of $sg\alpha$-closed sets, if $\mathcal{F}$ satisfies the finite intersection condition, then $\bigcap\{A : A \in \mathcal{F}\} \neq \emptyset$.*

(iv) *Given any family $\mathcal{F}$ of subsets of X, if $\mathcal{F}$ satisfies the finite intersection condition, then $\bigcap\{sg\alpha\,\mathrm{cl}(A) : A \in \mathcal{F}\} \neq \emptyset$.*

Proof. (i) $\Leftrightarrow$ (ii) and (ii) $\Leftrightarrow$ (iii) are obivious. (iii) $\Rightarrow$ (iv): If $\mathcal{F} \subset P(X)$ satisfies the finite intersection condition, then $\cap\{sg\alpha\,\mathrm{cl}(A) : A \in \mathcal{F}\}$ is a family of $sg\alpha$-closed sets, which obviously satisfies the finite intersection condition. (iv) $\Rightarrow$ (iii) Follows from the fact that $A = b\,\mathrm{cl}(A)$ for every $sg\alpha$-closed set A. ∎

Definition 3.5 A point $x \in X$ is said to be $sg\alpha$-cluster point of a net $\{x_\alpha\}_{\alpha\in\Delta}$ if $\{x_\alpha\}_{\alpha\in\Delta}$ is frequently in every $sg\alpha$-open set containing x. We denote by $sg\alpha$-$\mathrm{cp}\{x_\alpha\}_{\alpha\in\Delta}$ the set of all $sg\alpha$-cluster points of a net $\{x_\alpha\}_{\alpha\in\Delta}$.

Theorem 3.6 *The set of all $sg\alpha$-cluster points of an arbitrary net in X is $sg\alpha$-closed.*

Proof. Let $\{x_\alpha\}_{\alpha\in\Delta}$ be a net in X. Set $A = sg\alpha - cp\{x_\alpha\}_{\alpha\in\Delta}$. Let $x \in X\backslash A$. Then there exists a $sg\alpha$-open set U_x containing x and $\alpha_x \in \Delta$ such that $X_\beta \notin U_x$ whenever $\beta \in \Delta$, $\beta \geq \alpha_x$. It turns out that $U_x \subset X\backslash A$, hence $x \in sg\alpha\,\mathrm{int}(X\backslash A) = X\backslash sg\alpha\,\mathrm{cl}(A)$. This shows that $sg\alpha\,\mathrm{cl}(A) \subset A$; hence A is $sg\alpha$-closed. ∎

Theorem 3.7 *A topological space X is $sg\alpha$-compact if and only if each net $\{x_\alpha\}_{\alpha\in\Delta}$ in X, has atleast one $sg\alpha$-cluster point.*

Proof. Let X be a $sg\alpha$-compact space. Assume that there exist some net $\{x_\alpha\}_{\alpha\in\Delta}$ in X such that $sg\alpha$-cp$\{x_\alpha\}_{\alpha\in\Delta}$ is empty. Then for every $x \in X$, there exist $U(x) \in sg\alpha O(X, x)$ and $\alpha(x) \in \Delta$, such that $x_\beta \notin U(x)$ whenever $\beta \geq \alpha(x)$, $\beta \in \Delta$. Then the family $\{U(x) : x \in X\}$ is a cover of X by $sg\alpha$-open sets and has a finite subcover, say, $\{U_k : k = 1, 2, ..., n\}$ where $U_k = U(x_k)$ for $k =$ 1,2,...,n, $\{x_k : k = 1, 2, ..., n\}$. Let us take $\alpha \in \Delta$ such that $\alpha \geq \alpha(x_k)$ for all $k \in \{1, 2, ..., n\}$. For every $\beta \in \Delta$ such that $\beta \geq \alpha$ we have, $x_\beta \notin U_k$, $k =$ 1, 2,...,n, hence $x_\beta \notin X$, which is a contradiction. Conversely, if X is not $sg\alpha$-compact, there exists $\{U_i : i \in I\}$ a cover of X by $sg\alpha$-open sets, which has no finite subcover. Let $P(I)$ be the family of all finite subsets of I. Clearly, $(P(I), \subseteq)$ is a directed set. For each $J \in \mathcal{I}$. we may choose $x_j \in X\setminus\bigcup\{U_i : i \in J\}$. Let us consider the net $\{x_j\}_{j\in P(I)}$. By hypothesis, the set $sg\alpha$-$cp\{x_j\}_{j\in P(I)}$ is nonempty. Let $x \in sg\alpha$-$cp\{x_j\}_{j\in P(I)}$ and let $i_0 \in I$ such that $x \in U_{i0}$. By the definition of $sg\alpha$-cluster point, for each $J \in P(I)$ there exist $J^* \in P(I)$ such that $J \subset J^*$ and $x_j^* \in U_{i0}$. For $J = \{i_0\}$, there exists $J^* \in P(I)$ such that $i_0 \in J^*$ and $x_j^* \in U_{i0}$. But $x_j^* \in X\setminus\bigcup\{U_i : i \in J^*\} \subset X\setminus U_{i_0}$. The contradiction we obtained shows that X is $sg\alpha$-compact. ∎

In the following, we will give a characterization of $sg\alpha$-compact spaces by means of filterbases.

Let us recall that a nonempty family $\mathcal{F}$ of subsets of X is said to be a filterbase on X if $\emptyset \notin \mathcal{F}$ and each intersection of two members of $\mathcal{F}$ contains a third member of $\mathcal{F}$. Notice that each chain in the family of all filterbase on X (ordered by inclusion) has an upper bound, for example, the union of all members of the chain. Then, by Zorn's Lemma, the family of all filterbases on X has atleast one maximal element. Similarly, the family of all filterbases on X containing a given filterbase $\mathcal{F}$ has atleast one maximal element.

Definition 3.8 A filterbase $\mathcal{F}$ on a topological space X is said to be:

(i) $sg\alpha$-converge to a point $x \in X$ if for each $sg\alpha$-open set U containing x, there exists $B \in \mathcal{F}$ such that $B \subset U$.

(ii) $sg\alpha$-accumulate at $x \in X$ if $U \cap B \neq \emptyset$ for every $sg\alpha$-open set U containing x and every $B \in \mathcal{F}$.

Remark 3.9 A filterbase $\mathcal{F}$ $sg\alpha$-accumulate at x if and only if $x \in \bigcap\{sg\alpha\,\mathrm{cl}(B) : B \in \mathcal{F}\}$. Clearly, if a filterbase $\mathcal{F}$ $sg\alpha$-converges to $x \in X$, then $\mathcal{F}$ $sg\alpha$-accumulates at x.

Lemma 3.10 *If a maximal filterbase $\mathcal{F}$ $sg\alpha$-accumulate at $x \in X$, then $\mathcal{F}$ $sg\alpha$-converges to x.*

Proof. Let $\mathcal{F}$ be a maximal filterbase which $sg\alpha$-accumulate at $x \in X$. If $\mathcal{F}$ does not $sg\alpha$-converges to x, then there exists a $sg\alpha$-open set U_0 containing x such that $U_0 \cap B \neq \emptyset$ and $(X\setminus U_0) \cap B \neq \emptyset$ for every $B \in \mathcal{F}$. Then $\mathcal{F}\bigcup\{U_0 \cap B : B \in \mathcal{F}\}$ is a filterbase which strictly contains $\mathcal{F}$, which is a contradiction. ∎

Theorem 3.11 *For a topological space X, the following statements are equivalent:*

(i) *X is $sg\alpha$-compact;*

(ii) *Every maximal filterbase $sg\alpha$-converges to some point of X;*

(iii) *Every filterbase $sg\alpha$-accumulates at some point of X.*

Proof. (i) $\Rightarrow$ (ii): Let $\mathcal{F}_0$ be a maximal filterbase on X. Suppose that $\mathcal{F}_0$ does not $sg\alpha$-converge to any point of X. Then, by Lemma 3.10, $\mathcal{F}_0$ does not $sg\alpha$-accumulate at any point of X. For each $x \in X$, there exists a $sg\alpha$-open set U_x containing x and $B_x \in \mathcal{F}_0$ such that $U_x \cap B_x = \emptyset$. The family $\{U_x : x \in X\}$ is a cover of X by $sg\alpha$-open sets. By (i), there exists a finite subset $\{x_1, x_2, ..., x_n\}$ of X such that $X = \bigcup\{U_{x_k} : k = 1, 2, ..., n\}$. Since $\mathcal{F}_0$ is a filterbase, there exists $B_0 \in \mathcal{F}_0$ such that $B_0 \subset \bigcap\{B_{x_k} : k = 1, 2, ..., n\} = X \setminus \bigcup\{U_{x_k} : k = 1, 2, ..., n\}$, hence $B_0 = \emptyset$. This is a contradiction. (ii) $\Rightarrow$ (iii): Let $\mathcal{F}$ be a filterbase on X. There exists a maximal filterbase $\mathcal{F}_0$ such that $\mathcal{F} \subset \mathcal{F}_0$. By (ii), $\mathcal{F}_0$ $sg\alpha$-converges to some point $x_0 \in X$. Let $B \in \mathcal{F}$. For every $U \in sg\alpha O(X, x_0)$, there exists $B_U \in \mathcal{F}_0$ such that $B_U \subset U$, hence $U \cap B \neq \emptyset$, since it contains the member $B_U \cap B$ of $\mathcal{F}_0$. This shows that $\mathcal{F}$ $sg\alpha$-accumulates at x_0. (iii) $\Rightarrow$ (i): Let $\{V_i : i \in I\} = \emptyset$ be any family of $sg\alpha$-closed sets such that $\bigcap\{V_i : i \in I\} = \emptyset$. We shall prove that there exists a finite subset I_0 of I such that $\bigcap\{V_i : i \in I\}$. By Theorem 3.4, this implies (i). Let $P(I)$ be the family of finite subsets of I. Assume that

$$\bigcap\{V_i : i \in J\} = \emptyset \text{ for every } J \in P(I). \qquad (*)$$

Then the family $\mathcal{F} = \{\bigcap\{V_i : i \in J\} : J \in P(I)\}$ is a filterbase on X. By (iii), $\mathcal{F}$ $sg\alpha$-accumulates to some point $x_0 \in X$. Since $\{X \setminus V_i : i \in I\}$ is a cover of X, there exists $i_0 \in I$ such that $x_0 \in X \setminus V_{i_0}$. Then $X \setminus V_{i0}$ is a $sg\alpha$-open set containing x_0, $V_{i_0} \in \mathcal{F}$ and $(X \setminus V_{i_0}) \cap V_{i_0} = \emptyset$. This is a contradiction with the fact that $\mathcal{F}$ $sg\alpha$-accumulates at x_0 shows that $(*)$ is false. ∎

Definition 3.12 A point x in a topological space X is said to be a $sg\alpha$-complete accumulation point of a subset A of X if $n(S \cap A) = n(S)$ for each $A \in sg\alpha O(X, x)$, where $n(S)$ denotes the cardinality of S.

Definition 3.13 In a topological space (X, τ), a point x is said to be a $sg\alpha$-adherent point of a filterbase $\mathcal{F}$ on X if it lies in the $sg\alpha$-closure of all sets of $\mathcal{F}$.

Theorem 3.14 *A topological space (X, τ) is $sg\alpha$-compact if and only if each infinite subset of X has a $sg\alpha$-complete accumulation point.*

Proof. Let the topological space (X, τ) be $sg\alpha$-compact and A an infinite subset of X. Let K be the set of all points x in X which are not $sg\alpha$-complete accumulation points of S. Now it is obvious that for each point x in K, we are able to find $U(x) \in sg\alpha O(X, x)$ such that $n(A \cap U(x)) \neq n(S)$. If K is the whole space X, then $\mathcal{F} = \{U(x)x \in X\}$ is a $sg\alpha$-cover of X. By hypothesis, X

is $sg\alpha$-compact. So, there exists a finite subcover $\mathcal{G} = \{U(x_i) : i = 1, 2, ..., n\}$, such that $A \subset \bigcup\{U(x_i) \cap A : i = 1, 2, ..., n\}$. Then $n(S) = \max\{n(U(x_i) \cap A) : i = 1, 2, ..., n\}$ which does not agree with what we assumed. This implies that A has a $sg\alpha$-complete accumulation point. Conversely, assume that X is not $sg\alpha$-compact and that every infinite subset A of X has a $sg\alpha$-complete accumulation point in X. It follows that there exists a $sg\alpha$-cover $\mathcal{S}$ with no finite subcover. Set $\alpha = \min\{n(\Psi)$: $\Psi \subset \mathcal{S}$, where Ψ is a $sg\alpha$-cover of X$\}$. Fix $\Psi = \mathcal{S}$ for which $n(\Psi) = \alpha$ and $\bigcup\{U : U \in \Psi\} = X$. Then, by hypothesis $\alpha \geq n(N)$, where N denotes the set of all natural numbers. By well-ordering of Ψ by some minimal well-ordering "$\sim$", suppose that U is any member of Ψ. By minimal well-ordering "$\sim$" we have $n(\{V : V \in \Psi, V \sim U\}) < n(\{V : V \in \Psi\})$. Since Ψ cannot have any subcover with cardinality less than α, then for each $U \in \Psi$ we have $X \neq \bigcup\{V; V \in \Psi, V \sim U\})$. For each $U \in \Psi$, choose a point $x(U) \in X \backslash \bigcup\{V \cup \{x(V)\}; V \in \Psi, V \sim U\})$. We are always able to do this if not one can choose a cover of smaller cardinality from Ψ. If $H = \{x(U) : U \in \Psi\}$, then to finish the proof we will show that H has no $sg\alpha$-complete accumulation point in X. Suppose $z \in X$. Since Ψ is a $sg\alpha$-cover of X, z is a point of some set, say W in Ψ. By the fact that $U \sim W$, we have $x(U) \in W$. It follows that $T = \{U : U \in \Psi$ and $x(U) \in W\} \subset \{V; V \in \Psi, V \sim W\}$. But $n(T) < \alpha$. Therefore, $n(H \cap W) < \alpha$. But $n(H) = \alpha \geq n(N)$. Since for two distinct points U and W in Ψ, we have $x(U) \neq x(W)$. This means that H has no $sg\alpha$-complete accumulation point in X, which contradicts our assumption. Therefore X is $sg\alpha$-compact. ∎

Theorem 3.15 *For a topological space (X, τ), the following statements are equivalent:*

(i) *X is $sg\alpha$-compact;*

(ii) *Every net in X with a well-ordered directed set as its domain $sg\alpha$-accumulates to some point of X.*

Proof. (i) $\Rightarrow$ (ii): Suppose that X is $sg\alpha$-compact and $A = \{x_\alpha : \alpha \in \Delta\}$ a net with a well-ordered directed set Δ as domain. Assume that A has no $sg\alpha$-adherent point in X. Then for each $x \in X$, there exists $V(x) \in sg\alpha O(X, x)$ and an $\alpha(x) \in \Delta$ such that $V(x) \cap \{x_\alpha : \alpha \geq \alpha(x)\} = \emptyset$. This implies that $\{x_\alpha : \alpha \geq \alpha(x)\}$ is a subset of $X \backslash V(x)$. Then the collection $\mathcal{F} = \{V(x) : x \in X\}$ is a $sg\alpha$-open cover of X. Since X is $sg\alpha$-compact, $\mathcal{F}$ has a finite subfamily $\{V_{xi} : i = 1, 2, ..., n\}$ such that $X = \bigcup_{i=1}^{n}\{V(x_i) : i = 1, 2, ..., n\}$. Suppose that the corresponding elements of Δ are $\{\alpha(x_i)\}$, where $i = 1, 2, ..., n$. Since Δ is well-ordered and $\{\alpha(x_i) : i = 1, 2, ..., n\}$ is finite, the largest element of $\{\alpha(x_i)\}$ exists. Suppose it is $\{\alpha(x_i)\}$. Then for $\beta \geq \{\alpha(x_i)\}$, we have $\{x_\delta : \delta \geq \beta\} \subset \bigcap_{i=1}^{n}\{X \backslash V(x_i)\} = X \backslash \bigcup_{i=1}^{n} V(x_i) = \emptyset$, which is impossible. This shows that A has at least one $sg\alpha$-adherent point in X. (ii) $\Rightarrow$ (i): Now, it is enough to prove that each infinite subset has a $sg\alpha$-complete accumulation point by utilizing Theorem

3.14. Suppose that S is an infinite subset of X. According to Zorn's Lemma, the infinite set S can be well-ordered. This means that we can assume S to be a net with a domain which is a well-ordered index set. It follows that S has a $sg\alpha$-adherent point z. Therefore, z is a $sg\alpha$-complete accumulation point of S. This shows that X is $sg\alpha$-compact. ∎

Theorem 3.16 *A topological space X is $sg\alpha$-compact if and only if each family of $sg\alpha$-closed subsets of X with the finite intersection property has a nonempty intersection.*

Proof. Straightforward. ∎

Theorem 3.17 *A topological space X is $sg\alpha$-compact if and only if each filterbase in X has at least one $sg\alpha$-adherent point.*

Proof. Suppose that X is $sg\alpha$-compact and $\mathcal{F} = \{F_\alpha : \alpha \in \Delta\}$ a filterbase in it. Since all finite intersections of $\mathcal{F}_\alpha$'s are nonempty, it follows that all finite intersection of $sg\alpha\,\mathrm{cl}(F_\alpha)$'s are also nonempty. Now, it follows from Theorem 3.15 that $\bigcap_{\alpha\in\Delta} sg\alpha\,\mathrm{cl}(F_\alpha) \neq \emptyset$. This implies that $\mathcal{F}$ has at least one $sg\alpha$-adherent point. Now suppose $\mathcal{F}$ is a family of $sg\alpha$-closed sets. Let each finite intersection be nonempty. The sets F_α with their finite intersection establish a filterbase $\mathcal{F}$. Therefore, $\mathcal{F}$ $sg\alpha$-accumulates to some point $z \in X$. It follows that $z \in \bigcap_{\alpha\in\Delta} F_\alpha$. Now, we have by Theorem 3.15 X is $sg\alpha$-compact. ∎

Theorem 3.18 *A topological space X is $sg\alpha$-compact if and only if each filterbase on X with at least one $sg\alpha$-adherent point is $sg\alpha$-convergent.*

Proof. Suppose that X is $sg\alpha$-compact, $x \in X$ and $\mathcal{F}$ is a filterbase on X. The $sg\alpha$-adherence of $\mathcal{F}$ is a subset of $\{x\}$. Then the $sg\alpha$-adherence of $\mathcal{F}$ is equal to $\{x\}$ by Theorem 3.15. Assume that there exists $V \in sg\alpha O(X, x)$ such that for all $F \in \mathcal{F}$, $F \cap (X\backslash V) \neq \emptyset$. Then $\Psi = \{F\backslash V : F \in \mathcal{F}\}$ is a filterbase on X. It follows that the $sg\alpha$-adherence of Ψ is nonempty. However, $\bigcap_{F\in\mathcal{F}} sg\alpha\,\mathrm{cl}(F\backslash V) = (\bigcap_{F\in\mathcal{F}} sg\alpha\,\mathrm{cl}(F)) \cap (X\backslash V) = \{x\} \cap (X\backslash V) = \emptyset$, a contradiction. Hence for each $V \in sg\alpha O(X, x)$, there exists an $F \in \mathcal{F}$ with $F \subset V$. This shows that $\mathcal{F}$ $sg\alpha$-converges to x. To prove the converse, it suffices to show that each filterbase in X has at least one $sg\alpha$-accumulation point. Assume that $\mathcal{F}$ is a filterbase on X with no $sg\alpha$-adherent point. By hypothesis, $\mathcal{F}$ $sg\alpha$-converges to some point $z \in X$. Suppose F_α is an arbitrary element of $\mathcal{F}$. Then for each $V \in sg\alpha O(X, x)$, there exists $F_\beta \in \mathcal{F}$ such that $F_\beta \subset V$. Since $\mathcal{F}$ is a filterbase, there exists a γ such that $F_\gamma \subset F_\alpha \cap F_\beta \subset F_\alpha \cap V$, where $F_\alpha \neq \emptyset$. This means that $F\alpha \cap V \neq \emptyset$ for every $V \in sg\alpha O(X, x)$ and corresponding for each α, z is a point of $sg\alpha\,\mathrm{cl}(F_\alpha)$. It follows that $z \in \bigcap_{\alpha} sg\alpha\,\mathrm{cl}(F_\alpha)$. Therefore, z is a $sg\alpha$-adherent point of $\mathcal{F}$, a contradiction. This shows that X is $sg\alpha$-compact. ∎

Definition 3.19 A subset K of a topological space X is said to be $sg\alpha$-closed relative to X if for any cover $\{U_i : i \in I\}$ of K by $sg\alpha$-open sets, there exists a finite subset I_0 of I such that $K \subset \bigcup\{sg\alpha\,\mathrm{cl}(U_i) : i \in I_0\}$. We say that X is $sg\alpha$-closed if X is $sg\alpha$-closed relative to X. Obviously every set which is $sg\alpha$-compact relative to X is also $sg\alpha$-closed relative to X.

Definition 3.20 Let X be a topological space. A point $x \in X$ is said to be $sg\alpha$-θ-cluster point of a net $\{x_\alpha\}_{\alpha\in\Delta}$ if $\{x_\alpha\}_{\alpha\in\Delta}$ is frequently in the $sg\alpha$-closure of every $sg\alpha$-open set containing x.

Theorem 3.21 *A topological space X is $sg\alpha$-compact if and only if each net $\{x_\alpha\}_{\alpha\in\Delta}$ in X, has atleast one $sg\alpha$-cluster point.*

Proof. Similar to the proof of Theorem 3.7. ∎

Definition 3.22 A filterbase $\mathcal{F}$ on a topological space X is said to be:

(i) $sg\alpha$-θ-converge to $x \in X$ if for each $sg\alpha$-open set U containing x, there exists $B \in \mathcal{F}$ such that $B \subset sg\alpha\,\mathrm{cl}(U)$.

(ii) $sg\alpha$-θ-accumulate at $x \in X$ if $sg\alpha\,\mathrm{cl}(U) \cap B \neq \emptyset$ for each $sg\alpha$-open set U containing x and every $B \in \mathcal{F}$.

Remark 3.23 A filterbase $\mathcal{F}$ is $sg\alpha$-θ-convergent to a point $x \in X$ if and only if $\mathcal{F}$ contains the collection $\{sg\alpha\,\mathrm{cl}(U) : U \in sg\alpha O(X, x)\}$.

Theorem 3.24 *For a topological space X, the following statements are equivalent:*

(i) *X is $sg\alpha$-closed;*

(ii) *Every maximal filterbase $sg\alpha$-θ-converges to some point of X;*

(iii) *Every filterbase $sg\alpha$-θ-accumulates to some point of X;*

(iv) *For every family $\{V_i : i \in I\}$ of $sg\alpha$-closed sets that $\cap\{V_i : i \in I\} = \emptyset$, there exists a finite subset I_0 of I such that $\bigcap\{sg\alpha\,\mathrm{int}(V_i) : i \in I_0\} = \emptyset$.*

Proof. (i) ⇒ (ii): Let $\mathcal{F}$ be a maximal filterbase on X. Suppose that $\mathcal{F}$ does not $sg\alpha$-θ-converge to any point of X. Then, by Theorem 3.21, $\mathcal{F}$ does not $sg\alpha$-θ-accumulates at any point of X. For each $x \in X$, there exist a $sg\alpha$-open set U_x containing x and $B_x \in \mathcal{F}$ such that $sg\alpha\,\mathrm{cl}(U_x) \cap B_x = \emptyset$. The family $\{U_x : x \in X\}$ is a cover of X by $sg\alpha$-open sets. By (i), there exists a finite subset $\{x_1, x_2, ..., x_n\}$ of X such that $X = \bigcup\{sg\alpha\,\mathrm{cl}(U_{xk}) : k = 1, 2, ..., n\}$. Since $\mathcal{F}$ is a filterbase, there exists $F \in \mathcal{F}$ such that $\mathcal{F}_0 \subset \bigcap\{B_{x_k} : k = 1, 2, ..., n\}$. It follows that $\mathcal{F}_0 \subset \bigcap\{X \backslash sg\alpha\,\mathrm{cl}(U_{x_k} : k = 1, 2, ..., n\} = X \backslash \bigcup\{sg\alpha\,\mathrm{cl}(U_)x_k : k = 1, 2, ..., n\} = \emptyset$, hence $F = \emptyset$. This is a contradiction with the definition of filterbase. (ii) ⇒ (iii): Let $\mathcal{F}_0$ be a filterbase on X. There exists a maximal filterbase $\mathcal{F}$ such that

$\mathcal{F}_0 \subset \mathcal{F}$. By (ii), $\mathcal{F}$ $sg\alpha$-θ-converges to some point $x_0 \in X$. Let $F \in \mathcal{F}_0$. For every $U \in sg\alpha O(X, x_0)$, there exists $B_U \in \mathcal{F}$ such that $B_U \subset sg\alpha\,\mathrm{cl}(U)$, hence $sg\alpha\,\mathrm{cl}(U) \cap B \supset B_U \cap B$ is nonempty, since it contains a member of $\mathcal{F}$. This shows that $\mathcal{F}_0$ $sg\alpha$-θ-accumulates at x_0. *(iii)* $\Rightarrow$ *(iv)*: Let $\{V_i : i \in I\}$ be any family of $sg\alpha$-closed set such that $\bigcap\{V_i : i \in I\} = \emptyset$. Let $P(I)$ be the family of all finite subsets of I. Assume that

$$\bigcap\{sg\alpha\,\mathrm{int}(V_i) : i \in J\} = \emptyset \text{ for every } J \in P(I). \qquad (**)$$

Then, the family $\mathcal{F} = \{\bigcap\{sg\alpha\,\mathrm{int}(V_i) : i \in I\} : J \in \mathcal{I}\}$ is a filterbase on X. By (iii), $\mathcal{F}$ $sg\alpha$-accumulates at some point $x \in X$. Since $\{X\backslash V_i : i \in I\}$ is a cover of X, there exists $i_0 \in I$ such that $x \in X\backslash V_{i_0}$. Then $X\backslash V_{i_0}$ is a $sg\alpha$-open set containing x, $sg\alpha\,\mathrm{int}(V_{i_0}) \in \mathcal{F}$ and $sg\alpha\,\mathrm{cl}(X\backslash V_{i_0}) \cap sg\alpha\,\mathrm{int}(V_{i_0}) = \emptyset$. This contradicts the fact that that $\mathcal{F}$ $sg\alpha$-accumulates at x. It follows that $(**)$ is false. (iv) $\Rightarrow$ (i): Let $\{U_i : i \in I\}$ be a cover of X by $sg\alpha$-open sets. Denote $V_i = X\backslash U_i$. Then $\{V_i : i \in I\}$ is a family of $sg\alpha$-closed sets such that $\bigcap\{V_i : i \in I\} = \emptyset$. There exists a finite subset $\mathcal{I}_0$ of $\mathcal{I}$ such that $\bigcap\{sg\alpha\,\mathrm{int}(V_i) : i \in I\} = \emptyset$. Then $X = X\backslash\bigcap\{sg\alpha\,\mathrm{int}(V_i) : i \in I_0\} = \bigcup\{X\backslash sg\alpha\,\mathrm{int}(V_i) : i \in I_0\} = \bigcup\{sg\alpha\,\mathrm{cl}(U_i) : i \in I_0\}$. ∎

Theorem 3.25 *A topological space X is $sg\alpha$-closed if and only if each net $\{x_\alpha\}_{\alpha\in\Delta}$ in X has atleast one $sg\alpha$-θ-cluster point.*

Proof. Similar to the proof of Theorem 3.7. ∎

Definition 3.26 Let Δ be a directed set.

(i) A set $\mathcal{F} = \{x_\alpha : \alpha \in \Delta\}$ $sg\alpha$-accumulates at a point $x \in X$ if the net is frequently in every $U \in sg\alpha O(X, x)$, that is, for each $U \in sg\alpha O(X, x)$ and for each $\alpha_0 \in A$, there is some $\alpha \geq \alpha_0$ such that $x_\alpha \in U$.

(ii) The net $\mathcal{F}$ $sg\alpha$-converges to a point $x \in X$ if it is eventually in every $U \in sg\alpha O(X, x)$.

(iii) A filterbase $\mathcal{F} = \{F_\alpha : \alpha \in \Gamma\}$ $sg\alpha$-converges at a point $x \in X$ if $sg\alpha$-open subsets U_α of X for each $\alpha \in \Gamma$ such that $S \subset \bigcup_{\alpha\in\mathcal{F}} U_\alpha$.

(iv) A filterbase $\mathcal{F} = \{F_\alpha : \alpha \in \Gamma\}$ $sg\alpha$-converges to a point x in X if for each $U \in sg\alpha O(X, x)$, there exists an F_α in $\mathcal{F}$ such that $F_\alpha \subset U$.

Definition 3.27 If $\mathcal{F}$ is a filterbase on a topological space X, then the section of $\mathcal{F}$, denoted by $sec\mathcal{F}$, is given by $sec\mathcal{F} = \{A \subset X : A \cap G \neq \emptyset, \text{ for all } G \in \mathcal{F}\}$.

Theorem 3.28 *If a filterbase $\mathcal{F}$ on a topological space X, $sg\alpha$-θ-adheres to some point $x \in X$, then $\mathcal{F}$ is $sg\alpha$-θ-convergent to x.*

Proof. Let a filterbase $\mathcal{F}$ on X, $sg\alpha$-θ-adhere at $x \in X$. Then for each $U \in sg\alpha O(X, x)$ and each $G \in \mathcal{F}$, $sg\alpha\,\mathrm{cl}(U) \cap G \neq \emptyset$ so that $sg\alpha\,\mathrm{cl}(U) \in sec\mathcal{F}$, for each $U \in sg\alpha O(X, x)$, and hence $X\backslash sg\alpha\,\mathrm{cl}(U) \notin \mathcal{F}$. Then $sg\alpha\,\mathrm{cl}(U) \in \mathcal{F}$ (as $\mathcal{F}$ is a filterbase and $X \in \mathcal{F}$), for each $U \in sg\alpha O(X, x)$. Hence $\mathcal{F}$ must $sg\alpha$-θconverge to x. ∎

Remark 3.29 Let X be a topological space. Then, for any $x \in X$, we adopt the following notions:

(i) $\mathcal{F}(sg\alpha_\theta, x) = \{A \subset X : x \in sg\alpha\,\mathrm{cl}_\theta(A)\}$.

(ii) $sec\mathcal{F}(sg\alpha_\theta, x) = \{A \subset X : A \cap G \neq \emptyset$, for all $G \in \mathcal{F}(sg\alpha_\theta, x)\}$

In the next two theorems, we characterize the $sg\alpha$-θ-adherence and $sg\alpha$-θ-convergence of filterbase in terms of the above notions.

Theorem 3.30 *The filterbase $\mathcal{F}$ on a topological space X, $sg\alpha$-θ-adheres to a point $x \in X$ if and only if $\mathcal{F} \subset \mathcal{F}(sg\alpha_\theta, x)$.*

Proof. A filterbase $\mathcal{F}$ on a topological space X $sg\alpha$-θ-converges at $x \in X$.
$\Rightarrow sg\alpha\,\mathrm{cl}(U) \cap G \neq \emptyset$ for all $U \in sg\alpha O(X, x)$ and all $G \in \mathcal{F}$.
$\Rightarrow x \in sg\alpha\,\mathrm{cl}_\theta$ for all $G \in \mathcal{F}$.
$\Rightarrow G \in \mathcal{F}(sg\alpha_\theta, x)$ for all $G \in \mathcal{F}$.
$\Rightarrow \mathcal{F} \subset \mathcal{F}(sg\alpha_\theta, x)$.
Conversely, let $\mathcal{F} \subset \mathcal{F}(sg\alpha_\theta, x)$. Then, for all $G \in \mathcal{F}$, $x \in sg\alpha\,\mathrm{cl}_\theta$, so that for all $U \in sg\alpha O(X)$ and for all $G \in \mathcal{F}$, $sg\alpha\,\mathrm{cl}(U) \cap G \neq \emptyset$. Hence, $\mathcal{F}$ $sg\alpha$-θ-adheres to a point x. ∎

Theorem 3.31 *A filterbase $\mathcal{F}$ on a topological space X is $sg\alpha$-θ-convergent to a point x of X if and only if $sec\mathcal{F}(sg\alpha_\theta, x) \subset \mathcal{F}$.*

Proof. Let $\mathcal{F}$ be a filterbase on X, $sg\alpha$-θ-converging to $x \in X$. Then, for each $U \in sg\alpha O(X)$, there exist $G \in \mathcal{F}$ such that $G \subset b\,\mathrm{cl}(U)$; hence

$$sg\alpha\,\mathrm{cl}(U) \in \mathcal{F} \text{ for each } U \in sg\alpha O(X). \qquad (*)$$

Now, $B \in sec\mathcal{F}(sg\alpha_\theta, x) \Rightarrow X\backslash B \notin sec\mathcal{F}(sg\alpha_\theta, x) \Rightarrow x \notin sg\alpha\,\mathrm{cl}_\theta(X\backslash B) \Rightarrow$ there exists $U \in sg\alpha O(X)$ such that $sg\alpha\,\mathrm{cl}(U) \cap (X\backslash B) = \emptyset \Rightarrow sg\alpha\,\mathrm{cl}(U) \subset B$, where $U \in sg\alpha O(X) \Rightarrow B \in \mathcal{F}$ (by $(*)$). Conversely, let if possible, $\mathcal{F}$ not belong to $sg\alpha$-θ-converge to x. Then for some $U \in sg\alpha O(X)$, $sg\alpha\,\mathrm{cl}(U) \notin \mathcal{F}$ and hence $sg\alpha\,\mathrm{cl}(U) \notin sec\mathcal{F}(sg\alpha_\theta, x)$. Thus, for some $A \in \mathcal{F}(sg\alpha_\theta, x)$,

$$A \cap sg\alpha\,\mathrm{cl}(U) = \emptyset \qquad (**).$$

But $A \in \mathcal{F}(sg\alpha_\theta, x) \Rightarrow x \in sg\alpha\,\mathrm{cl}_\theta(A) \Rightarrow sg\alpha\,\mathrm{cl}(U) \cap A \neq \emptyset$, contradiction $(**)$. ∎

We shall now derive some new characterizations of $sg\alpha$-closedness in terms of filterbase and the associated concepts.

Theorem 3.32 *A subset A of a topological space X is $sg\alpha$-closed relative to X if and only if every filterbase $\mathcal{F}$ on X with $A \in \mathcal{F}$, $sg\alpha$-θ-converges to a point in A.*

Proof. Let A be $sg\alpha$-closed relative to X and $\mathcal{F}$ a filterbase on X satisfying $A \in \mathcal{F}$ such that $\mathcal{F}$ does not $sg\alpha$-θ-converge to any $a \in A$. Then to each $a \in A$, there corresponds some $U_\alpha \in sg\alpha O(X, a)$ such that $sg\alpha\,\mathrm{cl}(U_\alpha) \notin \mathcal{F}$. Now $\{U_a : a \in A\}$

is a cover of A by $sg\alpha$-open sets of X. Then $A \subset \bigcup_{i=1}^{n} sg\alpha\,\mathrm{cl}(U_{ai}) = U$ (say) for some positive integer n. Since $\mathcal{F}$ is a filterbase, $U \in \mathcal{F}$ and hence $A \in \mathcal{F}$, which is a contradiction. Conversely, let A be not $sg\alpha$-closed relative to X. Then for some cover $U = \{U_\alpha : \alpha \in \Lambda\}$ of A by $sg\alpha$-open sets of X, $\mathcal{F} = \{A \setminus \bigcup_{\alpha\in\Lambda_0} : \Lambda_0 \subset \Lambda\}$ is a filterbase on X. Then the family $\mathcal{F}$ can be extended to an ultra-filter $\mathcal{F}^*$ on X. Then $\mathcal{F}^*$ is a filterbase on X with $A \in \mathcal{F}^*$ (as each F of $\mathcal{F}$ is a subset of A). Now for each $x \in A$, there exist $\beta \in \Lambda$ such that $x \in U_\beta$, as U is a cover of A. Then for any $G \in \mathcal{F}^*$, $G \cap (A \setminus sg\alpha\,\mathrm{cl}(U_\beta) \neq \emptyset$, so that $G \nsubseteq sg\alpha\,\mathrm{cl}(U_\beta)$ for all $G \in \mathcal{G}$. Hence $\mathcal{F}^*$ cannot $sg\alpha$-θ-converge to any point of A. This contradiction proves the desired result. ∎

Theorem 3.33 *Let X be any topological space such that every filterbase $\mathcal{F}$ on X, with the property that $\bigcap_{i=1}^{n} sg\alpha\,\mathrm{cl}_\theta(G_i) \neq \emptyset$ for every finite subfamily $\{G_1, G_2, ...G_n\}$ of $\mathcal{F}$ $sg\alpha$-θ-adheres in X, then X is a $sg\alpha$-closed space.*

Proof. Let U be any ultrafilter on X. Then $\mathcal{U}$ is a filterbase on X and also for each finite subcollection $\{U_1, U_2, ...U_n\}$ of $\mathcal{U}$, $\bigcap_{i=1}^{n} sg\alpha\,\mathrm{cl}_\theta(U_i) \supset \bigcap_{i=1}^{n} U_i \neq \emptyset$ so that $\mathcal{U}$ is a filterbase on X with the given condition. Hence by hypothesis, $\mathcal{U}$ $sg\alpha$-θ-adheres. Consequently, the space X is $sg\alpha$-closed. ∎

Definition 3.34 A filterbase $\mathcal{F}$ on a topological space X is said to be:

(i) $sg\alpha$-θ-linked if for any two members $A, B \in \mathcal{F}$ $sg\alpha\,\mathrm{cl}_\theta(A) \cap sg\alpha\,\mathrm{cl}_\theta(B) = \emptyset$;

(ii) $sg\alpha$-θ-conjoint if for every finite subfamily $A_1, A_2, ..., A_n$ of $\mathcal{F}$ $sg\alpha\,\mathrm{int}(\bigcap_{i=1}^{n} sg\alpha\,\mathrm{cl}_\theta(A_i) \neq \emptyset$.

Theorem 3.35 *In a $sg\alpha$-closed topological space X, every $sg\alpha$-θ-conjoint filterbase $sg\alpha$-θ-adheres in X.*

Proof. Consider any $sg\alpha$-θ-conjoint filterbase $\mathcal{F}$ on a $sg\alpha$ -closed space X. We first note from Theorem 3.31 that, for $A \subset X$, $sg\alpha\,\mathrm{cl}_\theta(A)$ is $sg\alpha$-closed. Thus, $\{sg\alpha\,\mathrm{cl}_\theta(A) : A \in \mathcal{F}\}$ is a collection of $sg\alpha$-closed sets in X, such that $sg\alpha\,\mathrm{int}(\bigcap_{i=1}^{n} sg\alpha\,\mathrm{cl}_\theta(A_i)) \neq \emptyset$ for any finite subcollection $A_1, A_2....A_n$ of $\mathcal{F}$. Thus, by Theorem 3.32, $\bigcap_{i=1}^{n}\{sg\alpha\,\mathrm{cl}_\theta(A) : A \in \mathcal{F}\} \neq \emptyset$, that is, there exists $x \in X$ such that $x \in sg\alpha\,\mathrm{cl}_\theta(A)$ for all $A \in \mathcal{F}$. Hence $\mathcal{F} \subset \mathcal{F}(sg\alpha_\theta, x)$ so that by Theorem 3.30, $\mathcal{F}$ $sg\alpha$-θ-adheres at $x \in X$. ∎

Theorem 3.36 *Every filterbase $\mathcal{F}$ on a topological space X with the property that $\{sg\alpha\,\mathrm{cl}_\theta(G) : G \in \mathcal{F}_0\} \neq \emptyset$ for every finite subset $\mathcal{F}_0$ of $\mathcal{F}$, $sg\alpha$-θ-adheres in X if and only if for every family $\mathcal{F}$ of subsets of X for which the family $\{sg\alpha\,\mathrm{cl}_\theta(F) : F \in \mathcal{F}\}$ has the finite intersection property, we have $\bigcap\{sg\alpha\,\mathrm{cl}_\theta(F) : F \in \mathcal{F}\} \neq \emptyset$.*

Proof. Let every filterbase on a topological space X satisfying the given condition, $sg\alpha$-θ-adheres in X, and suppose that $\mathcal{F}$ is a family of subsets of X such that the family $\mathcal{F}^* = \{sg\alpha\,\mathrm{cl}_\theta(F) : F \in \mathcal{F}\}$ has the finite intersection property. Let U be the collection of all those families of subsets of X for which $\mathcal{F}^* = \{sg\alpha\,\mathrm{cl}_\theta(G) : G \in \mathcal{F}\}$ has the finite intersection property and $\mathcal{F} \subset \mathcal{G}$. Then $\mathcal{F} \in \mathcal{U}$ and $\mathcal{U}$ is a partially ordered set under inclusion in which every chain clearly has an upper bound. By Zorn's lemma, $\mathcal{F}$ is then contained in a maximal family $\mathcal{U}^* \in \mathcal{U}$. It is easy to verify that $\mathcal{U}^*$ is a filterbase with the stipulated property. Hence $\bigcap\{sg\alpha\,\mathrm{cl}_\theta(F) : F \in \mathcal{F}\} \supset \cap\{sg\alpha\,\mathrm{cl}_\theta(U) : U \in \mathcal{U} \neq \emptyset$. Conversely, if $\mathcal{F}$ is a filterbase on X with the given property, then for every finite subfamily $\mathcal{F}_0$ of $\mathcal{F}$, $\bigcap\{sg\alpha\,\mathrm{cl}_\theta(F) : \mathcal{F} \in \mathcal{F}_0\} \neq \emptyset$. So, by hypothesis, $\bigcap\{sg\alpha\,\mathrm{cl}_\theta(F) : F \in \mathcal{F}\} \neq \emptyset$. Hence $\mathcal{F}$ $sg\alpha$-θ-adheres in X. ∎

Theorem 3.37 [5] *For a topological space X, the following statements are equivalent:*

(i) *X is $sg\alpha$-regular;*

(ii) *For each and each open set U of X containing x, there exists a $sg\alpha$-open set V such that $\alpha\,\mathrm{cl}(V) \subset U$.*

Theorem 3.38 *Let X be a $sg\alpha$-regular space. Then a subset K of X is $sg\alpha$-compact if and only if K is $sg\alpha$-closed relative to X.*

Proof. Let $\{U_i : i \in I\}$ be a cover of K by $sg\alpha$-open sets. For each $x \in K$ there exists $i(x) \in I$ such that $x \in U_{i(x)}$ and by the assumption that X is $sg\alpha$-regular, there exists, according to Theorem 3.37, a $sg\alpha$-open set V_x such that $x \in V_x \subset sg\alpha\,\mathrm{cl}(V_x) \subset U_{i(x)}$. The family $\{V_x : x \in X\}$ is a cover of K by $sg\alpha$-open sets. Since K is $sg\alpha$-closed relative to X, there exists a finite subset $\{x_1, x_2, ..., x_n\}$ of K such that $K \subset \cup\{sg\alpha\,\mathrm{cl}(V_{xk}) : k = 1, 2, ..., n\}$. Which shows that K is $sg\alpha$-compact relative to X. The necessity is obvious. ∎

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ANALYSIS OF STEADY THREE–DIMENSIONAL HYDROMAGNETIC STAGNATION POINT FLOW TOWARDS A STRETCHING SHEET WITH HEAT GENERATION

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Abstract. This paper is concerned with the three-dimensional hydromagnetic stagnation point flow towards a stretching sheet with heat generation. It is assumed that a uniform magnetic field is applied normal to the plate which is maintained at a constant temperature. The coupled partial differential equations are reduced into ordinary differential equations by using similarity transformations. The series solutions of the coupled non-linear system is obtained using an analytical technique namely the homotopy analysis method (HAM). We use a two-stage method, where both the convergence

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control parameter and the initial guess are optimally selected to minimize the residual error due to the approximation. To do the latter, we consider a family of initial guesses parameterized by a constant which gives the decay rate of the solutions. In several cases considered, we are able to obtain solutions with extremely small residual errors after relatively few iterations are computed. The convergence, salient features of the flow and heat transfer characteristics are analyzed and discussed in detail through graphs. We also discuss the effect of the strength of the uniform magnetic field, the surface stretching velocity, and the heat generation/absorption coefficient on both the flow and heat transfer.

Keywords: boundary layer; heat generater; homotopy analysis method; homotopy solution; laminar flow; stretching sheet.

1. Introduction

Stagnation-point flow is a topic of significance in fluid mechanics, in the sense that stagnation points appear in virtually all flow fields of science and engineering. In some situations flow is stagnated by a solid wall, while in others a free stagnation point or a line exists in the interior of the fluid domain. After Weidman and Putkaradze [1], these flows may be characterized as inviscid or viscous, steady or unsteady, two-dimensional or three-dimensional, symmetric or asymmetric, normal or oblique, homogeneous or two fluid, and forward or reverse. The flow near a stagnation point has an important bearing on several technological processes. Such processes are cooling of nuclear reactors during emergency shutdown, heat exchangers placed in a law-velocity environment, cooling of electronic devices by fans, solar central receivers exposed to wind currents and many hydrodynamic processes [2].

The two-dimensional flow of a fluid near a stagnation point is a classical problem in fluid dynamics and its solution was first given by Hiemenz [3]. Later, the problem of stagnation point flow was extended numerically by Schlichting and Bussmann [4] and analytically by Ariel [5] to include the effect of suction. The problem of stagnation point flow was extended in numerous ways to include various physical effects. The axisymmetric three dimensional stagnation point flow was first studied by Homann [6]. Howarth [7] and Davey [8] considered the three-dimensional orthogonal stagnation-point flow of which the axisymmetric case was studied as a special case. Yeckel et al. [9] showed that the classical Homann flow can be used to supply boundary conditions for Reynolds' lubrication equations for flow within a thin viscous film, thereby leading to a model for the thinning of the film in the stagnation region of a turbulent water jet. Wang [10] studied the stagnation-point flow with surface slip that occurs in rarefied flow or under high pressure, while Blyth and Pozrikidis [11] investigated the stagnation-point flow of a Newtonian fluid against a Newtonian liquid film resting on a plane wall. They individually considered several cases for orthogonal two-dimensional, axisymmetric, three-dimensional and oblique two-dimensional flow.

In the present study the steady and hydromagnetic laminar axisymmetric

three dimensional stagnation-point flow of an incompressible viscous fluid impinging on a permeable stretching surface will be considered. Such flows of gases or liquids impinging on a permeable surface have several industrial applications which encompass the need to enhance heat and/or mass transfer. Typical examples are associated with the annealing of metals, cooling of gas turbine blades, cooling in grinding processes and cooling of photovoltaic cells [12]. The shear stress induced by the impinging fluid stream is utilized in surface cleaning, paper and photographic film manufacturing, wire coating and finishing of metal strips [13].

The steady hydromagnetic laminar axisymmetric three dimensional stagnation point flow of an incompressible viscous fluid impinging on a permeable stretching surface with heat generation/absorption was studied by Attia [14]. He applied a uniform magnetic field directed normal to the plate where the induced magnetic field was neglected. The wall and stream temperatures were assumed to be constants. This model come as a coupled system of nonlinear differential equations and was numerically solved.

Liao [15] used the concept of homotopy to propose an analytic method for highly nonlinear problems, namely the homotopy analysis method (HAM) (see [16], [17]). Different from perturbation techniques, HAM does not require small parameters, so that it is valid for highly nonlinear problems, especially for those with small or large physical parameters (see [18]). Different from all previous analytic methods, HAM always gives us a family of series solutions whose convergence region can be adjusted and controlled by an auxiliary parameter $\hbar$. This parameter can be obtained in different ways such as an $\hbar$–curve plot or by minimizing the residual error [19], [20], [21], [22], [23], [24]. HAM has been successfully applied to many types of nonlinear problems in science and engineering [25], [26], [27], [28], [29], [30], [31], [32], [33], [34, 35].

The aim of the present work is to study analytically the governing momentum and energy equations of the steady three-dimensional hydromagnetic stagnation point flow towards a stretching sheet with heat generation derived in [14]. Homotopy analysis method (HAM) is investigated in obtaining the analytical solution. In order to obtain accurate approximate analytical solutions in the whole spatial region, we consider multiple initial guesses, in order to find the best initial guess which permits convergence and accuracy after relatively few terms are calculated. We also select the convergence control parameter optimally, through the construction of an optimal control problem for the minimization of the accumulated residual errors. To the best of our knowledge, this is the first work that provides an analytic solution for this problem.

2. Formulation of the problem

The steady three-dimensional stagnation point flow of a viscous incompressible fluid near a stagnation point at a surface coinciding with the plane $z = 0$ is considered where the flow is in the region $z > 0$. Two equal and opposing forces

are applied along the radial direction so that the surface is stretched, keeping the origin fixed. Cylindrical coordinates r, ϕ, z are used with assumptions that the wall is at $z = 0$, the stagnation point is at the origin and that the flow is in the direction of the negative z-axis. We denote the radial and axial velocity components in frictionless flow by U and W, respectively, whereas those in viscous flow will be denoted by $u = u(r; z)$ and $w = w(r; z)$ where the component in the ϕ direction vanishes. A uniform magnetic field B_0 is applied normal to the plate where the induced magnetic field is neglected by assuming very small magnetic Reynolds number. For three-dimensional flow let the fluid far from the plate, as z tends infinity, be given by the potential flow

$$U = ar, \qquad W = -2az, \tag{2.1}$$

where $a > 0$ is a constant characterizing the velocity of the mainstream flow. Then, from Euler equation the pressure distribution will be

$$p = p_0 - \frac{\rho a^2}{2}(r^2 + 4z^2),$$

where ρ is the density of the fluid and p_0 is the pressure at the stagnation point. Thus the continuity and momentum equations for the three dimensional steady state flows, using the usual boundary layer approximations reduce to

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0, \tag{2.2}$$

$$\rho(u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z}) = -\frac{\partial \rho}{\partial r} + \mu(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2}) + \sigma B_0^2(U(r) - u), \tag{2.3}$$

$$\rho(u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z}) = -\frac{\partial \rho}{\partial r} + \mu(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} - \frac{u}{r^2} + \frac{\partial^2 w}{\partial z^2}), \tag{2.4}$$

where μ is the coefficient of viscosity of the fluid and σ is the electrical conductivity of the fluid. With boundary conditions

$$\begin{aligned} z = 0 : u &= cr, w = 0, \\ z \to \infty : \quad & u \to ar, \end{aligned} \tag{2.5}$$

where c is a positive constant related to the stretching velocity.

The governing boundary layer equation of energy, neglecting the dissipation, with temperature dependent heat generation or absorption is

$$\rho c_\rho(u\frac{\partial \theta}{\partial r} + w\frac{\partial \theta}{\partial z}) = k\frac{\partial^2 T}{\partial z^2} + Q(T - T_\infty), \tag{2.6}$$

subject to the boundary conditions

$$\begin{aligned} z = 0 : T &= T_w, \\ z \to \infty : T &\to T_\infty. \end{aligned} \tag{2.7}$$

Using the similarity solution

$$u = crf'(\eta), \quad w = -2\sqrt{cv}f(\eta), \quad \eta = \sqrt{c/vz}, \tag{2.8}$$

where $v = \mu/\rho$ is the kinematic viscosity of the fluid and prime denotes differentiation with respect to η, equations (2.2)-(2.4) subject to the boundary conditions (2.5) can be

$$f''' = -2ff'' + f'^2 - C^2 - Ha^2(C - f') \tag{2.9}$$

$$f(0) = 0, \quad f'(0) = 1, \quad f'(\infty) = C, \tag{2.10}$$

where $Ha^2 = \sigma B_0^2/c\rho$, Ha is the modified Hartmann number and $C = a/c$ is the stretching parameter. Using equation (2.8) with the help of the non-dimensional variable

$$\theta = \frac{T - T_\infty}{T_w - T_\infty},$$

we can reduce equations (2.6) and (2.7) to

$$\theta'' + 2Prf\theta' + PrB\theta = 0 \tag{2.11}$$

$$\theta(0) = 1, \quad \theta(\infty) = 0, \tag{2.12}$$

where $Pr = \mu c_\rho/k$ is the prandtl number and $B = Q/c\rho c_\rho$ is the dimensionless heat generation/absorption coefficient.

3. Mathematical analysis for HAM solution

Based on the boundary conditions (2.10) and (2.12), the solution can be expressed by a set of base functions

$$\{\eta^k e^{-s\eta} | k, s \geq 0\}$$

in the form

$$\begin{aligned} f(\eta) &= a_{1,0} + \sum_{k=0}^{+\infty}\sum_{s=0}^{+\infty} a_{1,k}\eta^k e^{-s\eta}, \\ \theta(\eta) &= a_{2,0} + \sum_{k=0}^{+\infty}\sum_{s=0}^{+\infty} a_{2,k}\eta^k e^{-s\eta}, \end{aligned} \tag{3.1}$$

where $a_{1,k}$ and $a_{2,k}$ $(k = 1, 2, \ldots)$ are coefficients to be determined. Moreover, according to the *Rule of Solution Expression* denoted by (3.1) and the boundary conditions (2.10) and (2.12), it is natural to choose initial approximations of $f(\eta)$ and $\theta(\eta)$ in the form

$$f_0(\eta) = (1 - C)/\gamma + C\eta + \frac{(C - 1)e^{-\gamma\eta}}{\gamma}, \quad \theta_0(\eta) = e^{-\beta\eta},$$

where γ and β are constants to be determined later. Based on the governing equations (2.9) and (2.11) and the *Rule of Solution Expression* (3.1), it is reasonable to choose the linear operators

$$L_f[\hat{f}(\eta;q)] = \hat{f}'''(\eta;q) + \gamma\hat{f}''(\eta;q), \tag{3.2}$$

$$L_\theta[\hat{\theta}(\eta;q)] = \hat{\theta}''(\eta;q) + \beta\hat{\theta}'(\eta;q), \tag{3.3}$$

with the properties

$$L_f[C_1 + C_2\eta + \frac{C_3e^{-\gamma\eta}}{\gamma^2}] = 0, \quad L_\theta[C_4 - \frac{C_5e^{-\beta\eta}}{\beta}] = 0, \tag{3.4}$$

where C_i ($i = 1, 2, \ldots, 5$) are the integration constants that will be determined by the boundary conditions. From the equations (2.9) and (2.11), we define the nonlinear operators N_f and N_θ as

$$N_f[\hat{f}(\eta;q)] = \hat{f}'''(\eta;q) - \hat{f}'^2(\eta;q) + 2\hat{f}(\eta;q)\hat{f}''(\eta;q) + C^2 + Ha^2(C - \hat{f}'(\eta;q)),$$
$$N_\theta[\hat{f}(\eta;q), \hat{\theta}(\eta;q)] = \hat{\theta}''(\eta;q) + 2Pr\hat{f}(\eta;q)\hat{\theta}'(\eta;q) + PrB\hat{\theta}(\eta;q)).$$

According to the underlying principle of the HAM [36, 37], the *zeroth-order deformation* problems can be constructed as follows:

$$(1-q)L_f[\hat{f}(\eta;q) - f_0(\eta)] = q\hbar N_f[\hat{f}(\eta;q)], \tag{3.5}$$

$$(1-q)L_\theta[\hat{\theta}(\eta;q) - \theta_0(\eta)] = q\hbar N_\theta[\hat{\theta}(\eta;q), \hat{f}(\eta;q)], \tag{3.6}$$

subject to the boundary conditions

$$\begin{aligned} &\hat{f}(0;q) = 0, \quad \hat{f}'(0;q) = 1, \quad \hat{\theta}(0;q) = 1, \\ &\hat{f}'(\eta;q) = C, \quad \hat{\theta}(\eta;q) = 0 \quad \text{as} \quad \eta \to \infty. \end{aligned} \tag{3.7}$$

In the *zeroth-order deformation equations* (3.5)-(3.6), $q \in [0, 1]$ is an embedding parameter and $\hbar$ is the auxiliary parameter. For $q = 0$ and $q = 1$, the above *zeroth-order deformation* equations (3.5)-(3.6) have the solutions

$$\hat{f}(\eta;0) = f_0(\eta), \quad \hat{\theta}(\eta;0) = \theta_0(\eta), \tag{3.8}$$

$$\hat{f}(\eta;1) = f(\eta), \quad \hat{\theta}(\eta;1) = \theta(\eta). \tag{3.9}$$

When q increases from 0 to 1, then $\hat{f}(\eta;q)$ and $\hat{\theta}(\eta;q)$ vary from the initial guesses $f_0(\eta)$ and $\theta_0(\eta)$ to the exact solutions $f(\eta)$ and $\theta(\eta)$, respectively.

Setting

$$f_m(\eta) = \frac{1}{m!}\frac{\partial^m\hat{f}(\eta;q)}{\partial q^m}\bigg|_{q=0},$$

$$\theta_m(\eta) = \frac{1}{m!}\frac{\partial^m\hat{\theta}(\eta;q)}{\partial q^m}\bigg|_{q=0},$$

and expanding $\hat{f}$ and $\hat{\theta}$ into the Taylor series expansion with respect to the embedding parameter q, we have

$$\hat{f}(\eta;q) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta) q^m, \tag{3.10}$$

$$\hat{\theta}(\eta;q) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta) q^m. \tag{3.11}$$

If the auxiliary parameter $\hbar$ is chosen in such a way that these series are convergent at $q = 1$, we arrive at

$$f(\eta) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta), \tag{3.12}$$

$$\theta(\eta) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta). \tag{3.13}$$

In this line we define the mth-order derivative of ϕ as

$$D_m(\phi) = \frac{1}{m!} \frac{\partial^m(\phi)}{\partial q^m}\bigg|_{q=0}.$$

Using the properties of the mth-order derivative [36], we obtain

$$\begin{aligned} D_m\{(1-q)L_f[\hat{f}(\eta;q) - f_0(\eta)]\} &= D_m\{L_f[\hat{f}(\eta;q) - q\hat{f}(\eta;q) + f_0(\eta)q - f_0(\eta)]\} \\ &= L_f[D_m(\hat{f}(\eta;q)) - D_m(q\hat{f}(\eta;q)) + f_0(\eta)D_m(q)] \\ &= L_f[f_m(\eta) - f_{m-1}(\eta) + f_0(\eta)D_m(q)] \\ &= \begin{cases} L_f[f_m], & m \le 1 \\ L_f[f_m - f_{m-1}], & m > 1 \end{cases} \\ &= L_f[f_m - \chi_m f_{m-1}], \end{aligned} \tag{3.14}$$

where

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1. \end{cases}$$

Apply the operator D_m to the *zeroth-order deformation equation* (3.5), to have

$$D_m\{(1-q)L_f[\hat{f}(\eta) - f_0(\eta)]\} = \hbar D_{m-1}\{N_f[\hat{f}(t;q)]\},$$

where

$$\begin{aligned} D_{m-1}\{N_f[\hat{f}(t;q)]\} &= f'''_{m-1}(\eta) + 2\sum_{i=0}^{m-1} f_i(\eta) f''_{m-1-i}(\eta) \\ &\quad - \sum_{i=0}^{m-1} f'_i(\eta) f'_{m-1-i}(\eta) - Ha^2 C f'_{m-1}(\eta) + (C^2 + Ha^2 C)(1 - \chi_m). \end{aligned}$$

Then the *mth-order deformation equation* related to f_m becomes

$$\begin{aligned} L_f[f_m(\eta) - \chi_m f_{m-1}(\eta)] &= \hbar \Bigg(f'''_{m-1}(\eta) + 2\sum_{i=0}^{m-1} f_i(\eta) f''_{m-1-i}(\eta) - \sum_{i=0}^{m-1} f'_i(\eta) f'_{m-1-i}(\eta) \\ &\quad - Ha^2 C f'_{m-1}(\eta) + (C^2 + Ha^2 C)(1-\chi_m) \Bigg), \end{aligned}$$

or what amounts to the same as

$$\begin{aligned} f_m &= \hbar L_f^{-1} \Bigg[f'''_{m-1}(\eta) + 2\sum_{i=0}^{m-1} f_i(\eta) f''_{m-1-i}(\eta) - \sum_{i=0}^{m-1} f'_i(\eta) f'_{m-1-i}(\eta) \\ &\quad - Ha^2 C f'_{m-1}(\eta) + (C^2 + Ha^2 C)(1-\chi_m) \Bigg] + \chi_m f_{m-1}. \end{aligned} \tag{3.15}$$

Applying similar procedure for equation (3.6), we obtain the *mth-order deformation equation* related to θ_m as

$$\theta_m = \hbar L_\theta^{-1} \left[\theta''_{m-1} + 2Pr \sum_{i=0}^{m-1} f_i \theta'_{m-1-i} + PrB f_{m-1} \right] + \chi_m \theta_{m-1}, \tag{3.16}$$

with boundary conditions

$$\begin{aligned} &f_m(0) = 0, \quad f'_m(0) = 0, \quad \theta_m(0) = 0 \\ &f'_m(\eta) = 0, \quad \theta_m(\eta) = 0 \qquad \text{as } \eta \to \infty. \end{aligned} \tag{3.17}$$

The general solutions of equations (3.15)-(3.16) are

$$\begin{aligned} f_m(\eta) &= f_m^*(\eta) + C_1 + C_2\eta + \frac{C_3 e^{-\gamma\eta}}{\gamma^2}, \\ \theta_m(\eta) &= \theta_m^*(\eta) + C_4 - \frac{C_5 e^{-\beta\eta}}{\beta}, \end{aligned}$$

where $f_m^*(\eta)$ and $\theta_m^*(\eta)$ are the particular solutions and the constants C_i $(i = 1, 2, 3, 4)$ are determined by the boundary conditions (3.17).

This way, it is easy to solve the linear non-homogeneous equations (3.15) and (3.16) with boundary conditions (3.17) by using symbolic computational software such as Mathematica. Thus, the mth-order approximate analytic solutions for the system of equations (2.9)-(2.10) and (2.11)-(2.12) are

$$f(\eta) \simeq \sum_{i=0}^{m} f_i(\eta), \tag{3.18}$$

$$\theta(\eta) \simeq \sum_{i=0}^{m} \theta_i(\eta). \tag{3.19}$$

4. Results and discussion

The series given by equations (3.18) and (3.19) are the analytical solutions of the present problem of laminar fluid flow on a permeable stretching sheet. The convergence region and rate of approximation of these series solutions depend upon the auxiliary parameter $\hbar$. Firstly, we consider a family of initial guesses parameterized by a constant which gives the decay rate of the solutions. In several cases considered, we are able to obtain solutions with extremely small residual errors after relatively few iterations are computed. In order to find the admissible values of this parameter, the $\hbar$-curves of $f''(0)$ and $\theta'(0)$ are displayed in Fig.1 when $\gamma = 7$ and $\beta = 5$. These $\hbar$-curves have been drawn for the physical quantities such as the reduced skin friction coefficient $f''(0)$ and the wall heat transfer rate $-\theta'(0)$.

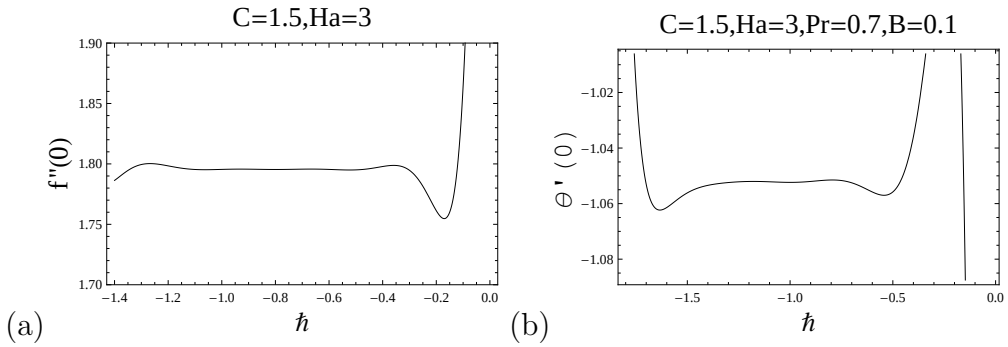

Figure 1: The $\hbar$-curve for (a) Eq. (3.18), (b) Eq. (3.19)

In our approach $\hbar$ is an optimal convergence-control parameter that can be used to accelerate the convergence of the homotopy-series solution. At the m-th order of approximation, one can define the square residual error

$$\Delta_m = \int_0^{+\infty} \left(N \left[\sum_{i=0}^{m} f_i(\eta) \right] \right)^2 d\eta. \tag{4.1}$$

Note that Δ_m contains $\hbar$ as an unknown parameter. At a given order of approximation m, the optimal value of $\hbar$ is given by the minimum of Δ_m, corresponding to the nonlinear algebraic equation

$$\frac{d\Delta_m}{d\hbar} = 0.$$

The optimal value of $\hbar$ when $m = 10$ is -0.97858.

The residual error of the solution is given via Fig. 2. The velocity profile of f and f' are respectively presented in Figs. 3 and 4.

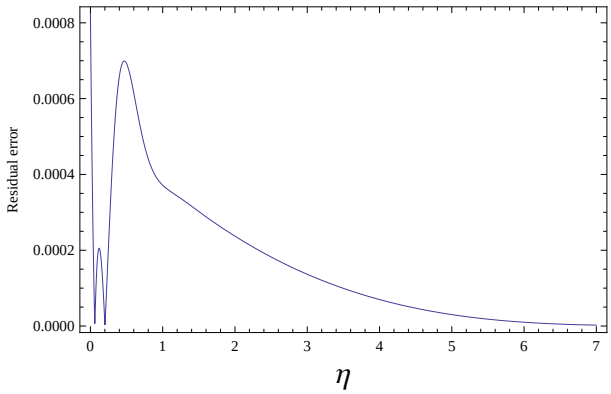

Figure 2: The residual error of the solution using 10-term

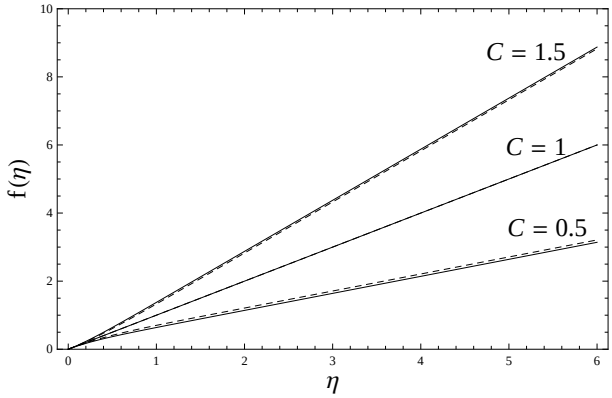

Figure 3: The profile of $f(\eta)$ with different values of Ha and C. Dash line for $Ha = 0$, solid line for $Ha = 3$

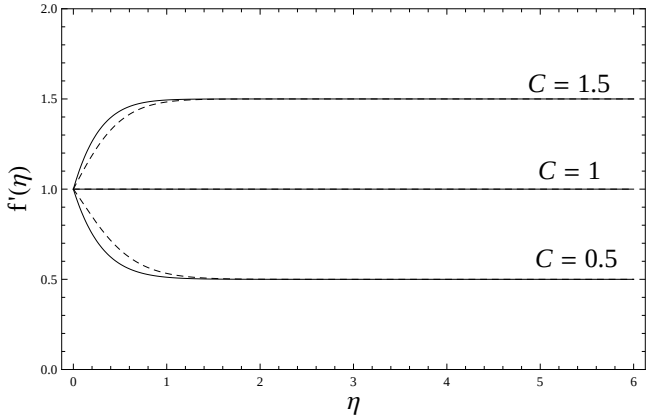

Figure 4: The velocity profile for $f'(\eta)$ with different values of Ha and C. Dash line for $Ha = 0$, solid line for $Ha = 3$

The figures show that as the modified Hartmann number (Ha) or the stretching parameter (C) increase then the velocity profile of f and f' are increased. The effect of Ha on both f and f' depends on C. For $C < 1$, increasing Ha decreases f and f' while for $C > 1$, increasing Ha increases them. The figures indicate also that the effect of C on f and f' is more pronounced for smaller values of Ha. Also, increasing C decreases the velocity boundary layer thickness. The profile of temperature θ for several values of Pr and B is presented in Fig. 5 with $Pr = 0.7$ and $B = 0.1$.

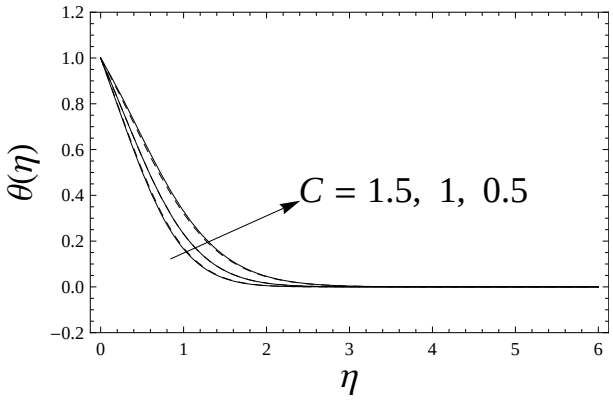

Figure 5: Temperature profile for several values of Ha and C with $Pr = 0.7$, $B = 0.1$. Dash line for $Ha = 0$, solid line for $Ha = 3$

It is clear that while C increasing θ decreasing, and for small value of Ha the effect of C becomes more apparent. The thermal boundary layer thickness decreases as C increases. Moreover, small C give more effect for the decreasing of θ as Ha increasing.

The efficacy of the Prandtl number (Pr) and C are studied in Fig. 6.

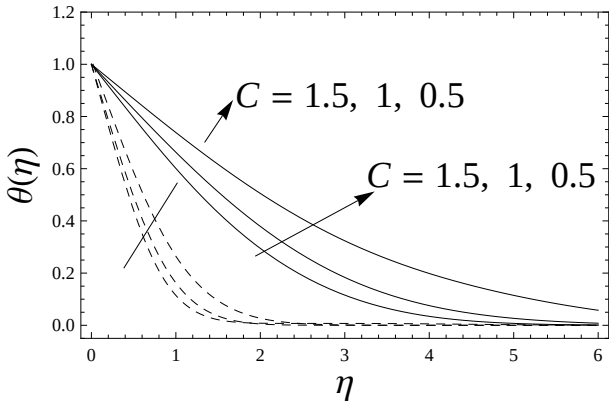

Figure 6: Temperature profile for several values of Pr and C with $Ha = 1$, $B = 0.1$. Dash line for $Pr = 1$, solid line for $Pr = 0.1$

It is clear that the θ increases as C decreases, also increasing Pr makes θ and thermal boundary layer thickness decrease.

Finally, Fig. 7 shows the effect of C and dimensionless heat generation coefficient (B) on the temperature profile for $Ha = 0.5$ and $Pr = 0.7$. When B increases, then the temperature θ and the boundary layer thickness increase. The effect of B on θ is more pronounced for smaller C. However, the effect of C on θ is more apparent for higher B.

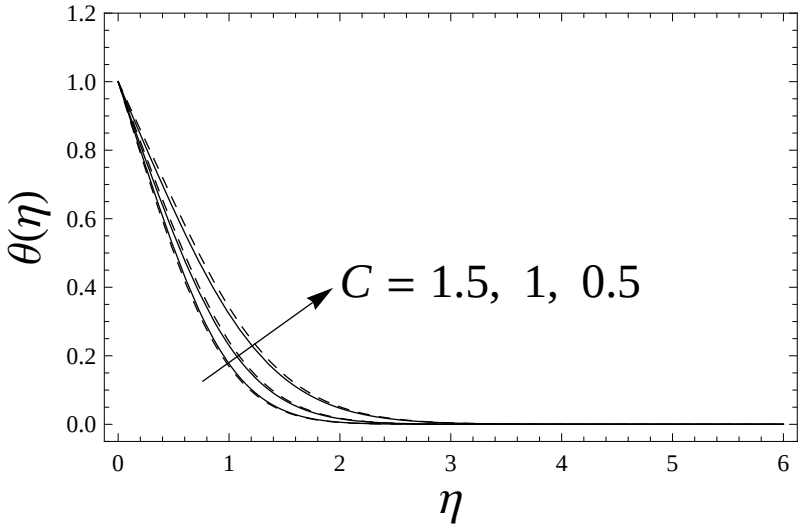

Figure 7: Temperature profile for several values of B and C with $Ha = 0.5$, $Pr = 0.7$. Dash line for $B = 0.1$, solid line for $B = -0.1$

5. Conclusions

The three dimensional hydromagnetic stagnation point flow of a viscous incompressible fluid impinging on a permeable stretching surface is studied in the presence of uniform magnetic field with heat generation/absorption. By a powerful and newly developed technique, based on the homotopy analysis method, the convergence series solutions are obtained. The proper values for $\hbar$, the proper initial guess and the proper linear operator were chosen to give more accuracy of the homotopy solution. Graphs are plotted to analyze the variation of the pertinent flow parameters including the modified Hartmann number Ha, the stretching velocity, the heat generation/absorption parameter and the Prandtl number Pr. From the present analysis, we note that the effect of the stretching parameter on the velocity and temperature is more apparent for smaller values of the magnetic field. The variation of velocity components as well as the rate of heat transfer at the wall with the magnetic field depends on the magnitude of the stretching velocity. On the other hand, increasing the stretching velocity decreases the temperature as well as the thermal boundary layer thickness. We have found that an

increase in the stretching velocity increases the velocity components but decreases the velocity boundary layer thickness. The sign of the wall shear stress was shown to depend on the stretching velocity. Finally, the effect of the heat generation/ absorption parameter B on the rate of heat transfer at the wall becomes more apparent for smaller C. The results demonstrate that selecting a parameterized initial guess can be extremely useful for minimizing residual errors when used concurrently with the optimal homotopy analysis method. This approach can prove useful for a number of nonlinear differential equations arising in physics and nonlinear mechanics.

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THE INFLUENCE ON THE ASYMPTOTICS OF THE RANDOM WALKS CAUSED BY THE VARIATION OF THE INCREMENTS

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Abstract. Let $\{X_n : n \geq 1\}$ be a sequence of independent and identically distributed random variables with a common mean $\mu \in (-\infty, 0)$ and $\{S_n : n \geq 0\}$ be the random walk generated by $\{X_n : n \geq 1\}$. For any $\varepsilon \in (0, -\mu)$, let $\{S_n^{(\pm\varepsilon)} : n \geq 0\}$ be the random walks generated by $\{X_n \pm \varepsilon : n \geq 1\}$. This paper considers the limits of $P\left(\sup_{n\geq 0} S_n^{(\pm\varepsilon)} > x\right)/P\left(\sup_{n\geq 0} S_n > x\right)$ as $x \to \infty$.

Keywords: random walks; supremum; asymptotics.

1. Introduction

Let $\{X_n : n \geq 1\}$ be a sequence of independent and identically distributed random variables with a common (non-degenerate) distribution F on $\mathbf{R} = (-\infty, \infty)$. Let $\{S_n : n \geq 0\}$ be the random walk generated by $\{X_n : n \geq 1\}$ (or by F), where $S_0 = 0$ and $S_n = \sum_{i=1}^{n} X_i$ for all $n \geq 1$. Let $\tau_+^s = \inf\{n \geq 1 : S_n > 0\}$ and $\tau_-^w = \inf\{n \geq 1 : S_n \leq 0\}$, then we call $S_{\tau_+^s}$ the first strictly ascending ladder height and $S_{\tau_-^w}$ the first weakly descending ladder height. Denote the distributions of $S_{\tau_+^s}$, $S_{\tau_-^w}$ by F_+^s, F_-^w, respectively. Put

$$A = \sum_{n=1}^{\infty} n^{-1} P(S_n \leq 0) \text{ and } B = \sum_{n=1}^{\infty} n^{-1} P(S_n > 0).$$

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Since $A + B = \infty$, only three cases may occur:

(i) $A < \infty$ (and $B = \infty$); i.e., $F_+^s(\infty) = 1$ and $F_-^w(0^+) = 1 - e^{-A}$;

(ii) $B < \infty$ (and $A = \infty$); i.e., $F_+^s(\infty) = 1 - e^{-B}$ and $F_-^w(0^+) = 1$;

(iii) $A = B = \infty$; i.e., $F_+^s(\infty) = F_-^w(0^+) = 1$.

If $\int_{-\infty}^{\infty} |x| \mathrm{d}F(x) < \infty$, then we define $\mu = \int_{-\infty}^{\infty} x \mathrm{d}F(x)$. Then it is well-known that the cases (i)-(iii) are implied by $\mu > 0$, $\mu < 0$ and $\mu = 0$, respectively. The converse implications are not true. See, for instance, Chapter VIII of Asmussen (2003).

We now state our motivation concretely. In the case $\mu \in (-\infty, 0)$, the random walk $\{S_n : n \geq 0\}$ drifts to $-\infty$, and its supremum M is finite almost surely. And for any $\varepsilon \in (0, -\mu)$, let $\{S_n^{(\pm\varepsilon)} : n \geq 0\}$ be the random walks generated by $X_n \pm \varepsilon, n \geq 1$. Then $\{S_n^{(\pm\varepsilon)} : n \geq 0\}$ also drift to $-\infty$ and their suprema $M^{(\pm\varepsilon)}$ are finite almost surely. Denote by W and $W^{(\pm\varepsilon)}$ the distributions of M and $M^{(\pm\varepsilon)}$, respectively. What we are concerned about is, whether the limits $\lim\limits_{x\to\infty} \frac{\overline{W^{(\pm\varepsilon)}}(x)}{\overline{W}(x)}$ exist? If the limits exist and are denoted by $h(\pm\varepsilon)$, then as ε tends to 0, whether $h(\pm\varepsilon)$ tend to 1? We discuss these two problems in three cases in which the asymptotics of the suprema can be derived, namely, the Cramér case, the intermediate case and the subexponential case, see, for instance, Veraverbeke (1977).

We now introduce some basic notations and theories on random walks. Let $\tau_+^w = \inf\{n \geq 1 : S_n \geq 0\}$ and $\tau_-^s = \inf\{n \geq 1 : S_n < 0\}$, then we call $S_{\tau_+^w}$ the first weakly ascending ladder height and $S_{\tau_-^s}$ the first strictly descending ladder height. Denote the distributions of $S_{\tau_+^w}$, $S_{\tau_-^s}$ by F_+^w, F_-^s, respectively. Then it is well-known that

$$F_+^w = \zeta\delta_0 + (1-\zeta)F_+^s$$

and

$$F_-^w = \zeta\delta_0 + (1-\zeta)F_-^s,$$

where $\zeta = P(S_{\tau_-^w} = 0) = P(\tau_-^w \neq \tau_-^s) < 1$ and δ_0 is the distribution degenerated at zero, see, for instance, Proposition 1.1 of Section VIII of Asmussen (2003). By Theorem 2 of XII.7 of Feller (1971), we know that $F_+^w(\infty) < 1$ if and only if $F_+^s(\infty) < 1$, and when this happens, $F_+^w(\infty) = 1 - \exp\{-\sum\limits_{n=1}^{\infty} n^{-1}P(S_n \geq 0)\}$. By simple calculation, one has $\zeta = 1 - \exp\{-\sum\limits_{n=1}^{\infty} n^{-1}P(S_n = 0)\}$.

Let f, f_+^s, f_-^w be the Fourier-Stieltjes transforms of X_1, $S_{\tau_+^s}$, $S_{\tau_-^w}$, that is,

$$\begin{aligned} f(t) &= \int_{-\infty}^{\infty} e^{itx} \mathrm{d}F(x), t \in \mathbf{C}, \\ f_+^s(t) &= \int_{0^+}^{\infty} e^{itx} \mathrm{d}F_+^s(x), t \in \mathbf{C}, \end{aligned}$$

and

$$f_-^w(t) = \int_{-\infty}^{0^+} e^{itx} \mathrm{d}F_-^w(x), t \in \mathbf{C},$$

where $\mathbf{C}$ denotes the complex plane. The well-known Wiener-Hopf factorization says

$$1 - f(t) = (1 - f_+^s(t))(1 - f_-^w(t)), t \in \mathbf{C}.$$

Further, we have the following representations:

$$1 - f_+^s(t) = \exp\left\{ -\sum_{n=1}^{\infty} n^{-1} \int_{0^+}^{\infty} e^{itx} \mathrm{d}F^{*n}(x) \right\}, \quad Im(t) \geq 0, \tag{1.1}$$

$$1 - f_-^w(t) = \exp\left\{ -\sum_{n=1}^{\infty} n^{-1} \int_{-\infty}^{0^+} e^{itx} \mathrm{d}F^{*n}(x) \right\}, \quad Im(t) \leq 0, \tag{1.2}$$

where $Im(t)$ is the imaginary part of t and F^{*n} is the n-fold convolution of F, with the convention that $F^{*1} = F$ and F^{*0} is the distribution degenerated at 0. See, for instance, Veraverbeke (1977, Pages 27-28). Moreover, denote by f_+^w, f_-^s the Fourier-Stieltjes transforms of $S_{\tau_+^w}$, $S_{\tau_-^s}$, similar to (1.1) and (1.2), we have

$$1 - f_+^w(t) = \exp\left\{ -\sum_{n=1}^{\infty} n^{-1} \int_{0^-}^{\infty} e^{itx} \mathrm{d}F^{*n}(x) \right\}, \quad Im(t) \geq 0,$$

$$1 - f_-^s(t) = \exp\left\{ -\sum_{n=1}^{\infty} n^{-1} \int_{-\infty}^{0^-} e^{itx} \mathrm{d}F^{*n}(x) \right\}, \quad Im(t) \leq 0, \tag{1.3}$$

see, for instance, (3.9) and (3.17) of Section XVIII.3 of Feller (1971).

In particular, for any $t \in \mathbf{R}$, we denote $f(-it)$, $f_+^s(-it)$, $f_-^w(-it)$ by $\widehat{F}(t)$, $\widehat{F_+^s}(t)$, $\widehat{F_-^w}(t)$, respectively. Denote $\gamma_F = \sup\{t \in \mathbf{R} : \widehat{F}(t) < \infty\}$ and $\beta_F = \inf\{t \in \mathbf{R} : \widehat{F}(t) < \infty\}$, then by the Wiener-Hopf factorization, it is obvious that $\gamma_F = \sup\{t \in \mathbf{R} : \widehat{F_+^s}(t) < \infty\}$ and $\beta_F = \inf\{t \in \mathbf{R} : \widehat{F_-^w}(t) < \infty\}$. We may find that γ_F has an important influence on the asymptotics of the supremum of a random walk. For convenience, we write $\gamma = \gamma_F$.

For any $\varepsilon \in (0, -\mu)$, we denote the first ladder epochs of the random walk $\{S_n^{(+\varepsilon)} : n \geq 0\}$ by

$$\begin{aligned}
\tau_+^{s,(+\varepsilon)} &= \inf\{n \geq 1 : S_n^{(+\varepsilon)} > 0\}, \\
\tau_-^{w,(+\varepsilon)} &= \inf\{n \geq 1 : S_n^{(+\varepsilon)} \leq 0\}, \\
\tau_+^{w,(+\varepsilon)} &= \inf\{n \geq 1 : S_n^{(+\varepsilon)} \geq 0\}, \\
\tau_-^{s,(+\varepsilon)} &= \inf\{n \geq 1 : S_n^{(+\varepsilon)} < 0\},
\end{aligned}$$

denote the corresponding ladder heights by $S_{\tau_+^{s,(+\varepsilon)}}$, $S_{\tau_-^{w,(+\varepsilon)}}$, $S_{\tau_+^{w,(+\varepsilon)}}$, $S_{\tau_-^{s,(+\varepsilon)}}$, and denote the distributions of the ladder heights by $F_+^{s,(+\varepsilon)}$, $F_-^{w,(+\varepsilon)}$, $F_+^{w,(+\varepsilon)}$, $F_-^{s,(+\varepsilon)}$,

respectively. Similarly, let $\tau_+^{s,(-\varepsilon)}$, $\tau_-^{w,(-\varepsilon)}$, $\tau_+^{w,(-\varepsilon)}$, $\tau_-^{s,(-\varepsilon)}$ be the ladder epochs of the random walk $\{S_n^{(-\varepsilon)} : n \geq 0\}$, $S_{\tau_+^{s,(-\varepsilon)}}$, $S_{\tau_-^{w,(-\varepsilon)}}$, $S_{\tau_+^{w,(-\varepsilon)}}$, $S_{\tau_-^{s,(-\varepsilon)}}$ be corresponding ladder heights, and $F_+^{s,(-\varepsilon)}$, $F_-^{w,(-\varepsilon)}$ $F_+^{w,(-\varepsilon)}$, $F_-^{s,(-\varepsilon)}$ be the distributions of corresponding ladder heights.

Hereafter, unless otherwise stated, a limit is taken as $x \to \infty$. For two positive functions $a(\cdot)$ and $b(\cdot)$, we write $a(x) = O(b(x))$ if $\limsup a(x)/b(x) < \infty$, $a(x) = o(b(x))$ if $\lim a(x)/b(x) = 0$ and $a(x) \sim b(x)$ if $\lim a(x)/b(x) = 1$.

We say that a distribution V belongs to the exponential distribution class with index $\alpha \geq 0$, denoted by $V \in \mathcal{L}(\alpha)$, if $\overline{V}(x) > 0$ for all x and $\overline{V}(x-t) \sim e^{\alpha t}\overline{V}(x)$ for any $t \in \boldsymbol{R}$. It is obvious that if $V \in \mathcal{L}(\alpha)$ for some $\alpha \geq 0$, then $\alpha = \gamma_V$.

We say that a proper distribution V belongs to the convolution equivalent distribution class with some index $\alpha \geq 0$, denoted by $V \in \mathcal{S}(\alpha)$, if $V \in \mathcal{L}(\alpha)$, $\widehat{V}(\alpha) < \infty$, and $\overline{V^{*2}}(x) \sim 2\widehat{V}(\alpha)\overline{V}(x)$.

In particular, we call $\mathcal{L}(0)$ the long-tailed distribution class and $\mathcal{S}(0)$ the subexponential distribution class, denoted by $\mathcal{L}$ and $\mathcal{S}$ respectively. In addition, if V is supported on $[0,\infty)$ and $\alpha = 0$, the premise condition $V \in \mathcal{L}$ can be omitted in the definition of the subexponential distribution class.

The rest of this paper is organized as follows. The main results of this paper, as well as some existing results on the asymptotics of the suprema of random walks, are presented in Section 2. And in Section 3, the proofs of our main results are presented.

2. Some related theories and the main results

For a random variable X or its distribution F with $\int_0^\infty \overline{F}(y)\mathrm{d}y < \infty$, we define its integrated tail distribution by

$$F^I(x) = \begin{cases} \int_0^x \overline{F}(t)dt, & \text{if } x > 0; \\ 0, & \text{if } x \leq 0. \end{cases}$$

Theorem 2 of Veraverbeke (1977) presented the following results.

Theorem 2.A. *Suppose $B < \infty$.* (A) *Suppose $\gamma > 0$.*

(i) *If $\widehat{F}(\gamma) > 1$, then*

$$\overline{W}(x) \sim \frac{1 - F_+^s(\infty)}{\kappa} \cdot \frac{1 - \widehat{F_-^w}(\kappa)}{\widehat{F}'(\kappa)} e^{-\kappa x}, \tag{2.1}$$

where $\kappa \in (0,\gamma)$ such that $\widehat{F}(\kappa) = 1$.

(ii) *If $\widehat{F}(\gamma) = 1$ and $\widehat{F}'(\gamma) < \infty$, then*

$$\overline{W}(x) \sim \frac{1 - F_+^s(\infty)}{\gamma} \cdot \frac{1 - \widehat{F_-^w}(\gamma)}{\widehat{F}'(\gamma)} e^{-\gamma x}. \tag{2.2}$$

If $\widehat{F}(\gamma) = 1$ *and* $\widehat{F}'(\gamma) = \infty$, *then*

$$\overline{W}(x) = o(e^{-\gamma x}). \tag{2.3}$$

(iii) *If* $\widehat{F}(\gamma) < 1$, *then* (2.3) *holds. Further, if* $F \in \mathcal{S}(\gamma)$, *then* $W \in \mathcal{S}(\gamma)$ *and*

$$\overline{W}(x) \sim \frac{1 - F_+^s(\infty)}{(1 - \widehat{F}(\gamma))(1 - \widehat{F_+^s}(\gamma))} \overline{F}(x); \tag{2.4}$$

(B) *Suppose* $\gamma = 0$. *If* $\mu < 0$ *and* $F^I \in \mathcal{S}$, *then* $W \in \mathcal{S}$ *and*

$$\overline{W}(x) \sim -\frac{1}{\mu} \int_x^\infty \overline{F}(t)\mathrm{d}t. \tag{2.5}$$

We may find that in the above theorem, the variations of the increments affect the asymptotics of the supremum. Below are our main results.

Theorem 2.1. *Suppose* $\mu < 0$.

(A) *Suppose* $\gamma > 0$.

(i) *If* $\widehat{F}(\gamma) \geq 1$, *then for any* $\varepsilon \in (0, -\mu)$,

$$\overline{W}(x) = o\left(\overline{W^{(+\varepsilon)}}(x)\right). \tag{2.6}$$

In particular, if $\widehat{F}(\gamma) > 1$, *then*

$$\overline{W^{(-\varepsilon)}}(x) = o(\overline{W}(x)).$$

(ii) *Suppose that* $\widehat{F}(\gamma) < 1$. *If* $F \in \mathcal{S}(\gamma)$, *then for any* $\varepsilon \in (0, -\frac{\ln \widehat{F}(\gamma)}{\gamma})$, *the limits* $\lim \frac{\overline{W^{(\pm\varepsilon)}}(x)}{\overline{W}(x)}$ *exist and*

$$\lim_{\varepsilon \downarrow 0} \lim \frac{\overline{W^{(\pm\varepsilon)}}(x)}{\overline{W}(x)} = 1. \tag{2.7}$$

(B) *Suppose* $\gamma = 0$. *If* $F^I \in \mathcal{S}$, *then for any* $\varepsilon \in (0, -\mu)$,

$$\overline{W^{(\pm\varepsilon)}}(x) \sim \frac{\mu}{\mu \pm \varepsilon} \overline{W}(x). \tag{2.8}$$

3. Proof of Theorem 2.1

The following two lemmas play important roles in the proof of Theorem 2.1. By the dominanted convergence theorem, the first one is obvious.

Lemma 3.1. *If $\mu < 0$, then as $\varepsilon \downarrow 0$,*

$$\begin{aligned} F_+^{s,(+\varepsilon)}(\infty) &\longrightarrow F_+^w(\infty), \\ F_+^{s,(-\varepsilon)}(\infty) &\longrightarrow F_+^s(\infty), \\ F_+^{w,(+\varepsilon)}(\infty) &\longrightarrow F_+^w(\infty), \\ F_+^{w,(-\varepsilon)}(\infty) &\longrightarrow F_+^s(\infty). \end{aligned}$$

Lemma 3.2. *Suppose that $\mu < 0$ and $\gamma > 0$. For any $\alpha \in [0, \gamma]$, if $\widehat{F}(\alpha) < 1$, then as $\varepsilon \downarrow 0$,*

$$\begin{aligned} \widehat{F_-^{w,(+\varepsilon)}}(\alpha) &\longrightarrow \widehat{F_-^s}(\alpha), \\ \widehat{F_-^{w,(-\varepsilon)}}(\alpha) &\longrightarrow \widehat{F_-^w}(\alpha), \\ \widehat{F_-^{s,(+\varepsilon)}}(\alpha) &\longrightarrow \widehat{F_-^s}(\alpha), \\ \widehat{F_-^{s,(-\varepsilon)}}(\alpha) &\longrightarrow \widehat{F_-^w}(\alpha). \end{aligned}$$

Proof. We only prove the first conclusion, and the others can be proved similarly.

If $\widehat{F}(\alpha) < 1$, then there exist two positive constants ε_0 and δ such that for all $0 < \varepsilon < \varepsilon_0$, $Ee^{\alpha(X_1+\varepsilon)} < e^{-\delta}$, then for all $n \geq 1$,

$$\int_{-\infty}^{\infty} e^{\alpha x} \mathrm{d}P(S_n \leq x) e^{n\alpha\varepsilon} < e^{-\delta n}.$$

So by the dominated convergence theorem, when $\varepsilon \downarrow 0$, we have

$$\sum_{n=1}^{\infty} n^{-1} \int_{-\infty}^{0^+} e^{\alpha(x+n\varepsilon)} \mathrm{d}P(S_n \leq x) \to -\log\{1 - \widehat{F_-^w}(\alpha)\}, \tag{3.1}$$

and

$$\sum_{n=1}^{\infty} n^{-1} \int_{-n\varepsilon}^{0^+} e^{\alpha(x+n\varepsilon)} \mathrm{d}P(S_n \leq x) \to \sum_{n=1}^{\infty} n^{-1} [F^{*n}(0^+) - F^{*n}(0^-)]. \tag{3.2}$$

By (3.1) and (3.2), when $\varepsilon \downarrow 0$, we have

$$\begin{aligned} 1 - \widehat{F_-^{w,(+\varepsilon)}}(\alpha) &= \exp\left\{ -\sum_{n=1}^{\infty} n^{-1} \int_{-\infty}^{0^+} e^{\alpha x} \mathrm{d}P(S_n \leq x - n\varepsilon) \right\} \\ &= \exp\left\{ -\sum_{n=1}^{\infty} n^{-1} \left(\int_{-\infty}^{0^+} - \int_{-n\varepsilon}^{0^+} \right) e^{\alpha(x+n\varepsilon)} \mathrm{d}P(S_n \leq x) \right\} \\ &\to \left(1 - \widehat{F_-^w}(\alpha)\right) \exp\left\{ \sum_{n=1}^{\infty} n^{-1} [F^{*n}(0^+) - F^{*n}(0^-)] \right\} \\ &= 1 - \widehat{F_-^s}(\alpha), \end{aligned}$$

where (1.2) and (1.3) were used in the last step. This completes the proof. ∎

Proof of Theorem 2.1. We only prove the results for $\overline{W^{(+\varepsilon)}}$, the results for $\overline{W^{(-\varepsilon)}}$ may be proved similarly.

(A) Suppose that $\gamma > 0$.

(i) If $\widehat{F}(\gamma) > 1$, then by Theorem 2.A (A)(i), there exists $\kappa \in (0, \gamma)$ satisfying $Ee^{\kappa X_1} = 1$ such that (2.1) holds. Moreover, for any $\varepsilon \in (0, -\mu)$, the random walk $\{S_n^{(+\varepsilon)} : n \geq 0\}$ also drifts to $-\infty$ and $E^{\kappa(X_1+\varepsilon)} > 1$. Again by Theorem 2.A (A)(i), there exists $\kappa' \in (0, \kappa)$ satisfying $Ee^{\kappa'(X_1+\varepsilon)} = 1$ such that

$$\overline{W^{(+\varepsilon)}}(x) \sim K(\varepsilon)e^{-\kappa' x}. \tag{3.3}$$

where $K(\varepsilon) = \frac{1-F_+^{s,(+\varepsilon)}(\infty)}{\kappa'} \cdot \frac{1-\widehat{F_-^{w,(+\varepsilon)}}(\kappa')}{e^{\kappa'\varepsilon}(\widehat{F}'(\kappa')+\varepsilon\widehat{F}(\kappa'))}$. Thus (2.6) follows from (2.1) and (3.3).

If $\widehat{F}(\gamma) = 1$, then for any $\varepsilon \in (0, -\mu)$, $Ee^{\gamma(X_1+\varepsilon)} > 1$. Then according to Theorem 2.A (A)(i), there exists $\kappa' \in (0, \gamma)$ satisfying $Ee^{\kappa'(X_1+\varepsilon)} = 1$ such that (3.3) holds. And by (2.2) and (2.3),

$$\overline{W}(x) = O(e^{-\gamma x}). \tag{3.4}$$

Thus (2.6) follows by (3.3) and(3.4).

(ii) If $\widehat{F}(\gamma) < 1$ and $F \in \mathcal{S}(\gamma)$, then, for any $\varepsilon \in (0, -\frac{\ln \widehat{F}(\gamma)}{\gamma})$, $Ee^{\gamma(X_1+\varepsilon)} < 1$ and $E(X_1+\varepsilon) < 0$ (see Remark 3.1 below). In addition, the distribution of $X_1+\varepsilon$ belongs to the class $\mathcal{S}(\gamma)$. By Theorem 2.A (A)(iii),

$$\overline{W^{(+\varepsilon)}}(x) \sim \frac{(1-F_+^{s,(+\varepsilon)}(\infty))(1-\widehat{F_-^{w,(+\varepsilon)}}(\gamma))}{(1-e^{\gamma\varepsilon}\widehat{F}(\gamma))^2} e^{\gamma\varepsilon}\overline{F}(x). \tag{3.5}$$

By (2.4) and (3.5)

$$\lim \frac{\overline{W^{(+\varepsilon)}}(x)}{\overline{W}(x)} = \frac{(1-F_+^{s,(+\varepsilon)}(\infty))(1-\widehat{F_-^{w,(+\varepsilon)}}(\gamma))(1-\widehat{F}(\gamma))^2}{(1-F_+^{s}(\infty))(1-\widehat{F_-^{w}}(\gamma))(1-e^{\gamma\varepsilon}\widehat{F}(\gamma))^2} e^{\gamma\varepsilon},$$

which tends to 1 as $\varepsilon \downarrow 0$ by Lemmas 3.1 and 3.2. Thus we get (2.7) immediately.

(B) If $F^I \in \mathcal{S}$, then the integrated tail distribution of $X_1 + \varepsilon$ belongs to the subexponential distribution class. Then by Theorem 2.A (B),

$$\overline{W^{(+\varepsilon)}}(x) \sim -\frac{1}{\mu+\varepsilon}\int_x^\infty \overline{F}(t)dt. \tag{3.6}$$

Thus (2.8) follows immediately by (2.5) and (3.6). □

Remark 3.1. For any given $\alpha \in (0, \infty)$, the function $f_\alpha(x) = e^{\alpha x}, x \in \mathbf{R}$ is a convex function. So for any $\alpha \in (0, \gamma]$, one has $Ef_\alpha(X) > f_\alpha(EX)$, that is, $\widehat{F}(\alpha) > e^{\alpha\mu}$, which implies

$$-\frac{\ln \widehat{F}(\alpha)}{\alpha} < -\mu.$$

So in case (ii), the condition $\varepsilon \in \left(0, -\frac{\ln \widehat{F}(\gamma)}{\gamma}\right)$ implies $E(X_1 + \varepsilon) < 0$.

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HYPERCYCLIC AND SUPERCYCLIC OPERATORS SATISFYING GENERALIZED A-WEYL'S THEOREM

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Abstract. A Banach space operator T satisfies generalized a-Weyl's theorem if the complement of its upper semi B-Weyl spectrum in its approximate point spectrum is the set of eigenvalues of T which are isolated in the approximate spectrum of T. In this note we characterize hypecyclic and supercyclic operators satisfying generalized a-Weyl's theorem.

Keywords: Weyl's and a-Weyl's theorems;generalized Weyl's and generalized a-Weyl's theorems; Browder's and a-Browder's theorems; generalized Browder's and generalized a-Browder's; hypercyclicity; supercyclicity.

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1. Introduction

The open problem in operator theory known as the invariant subspace problem assert that: any Hilbert space operator acting on an infinitely-dimensional, separable, complex Hilbert space operator has proper invariant subspaces. In [23] the author gave a negative answer to this problem in the case of a Banach space operators. As for Hilbert space operators, this is also an open problem. In [22], it is proved that this problem has a negative answer if and only if there is some hypercyclic operator T on the Hilbert space H such that any nonzero x in H is a hypercyclic vector of T. Hence interest in hypercyclic operators arises from the invariant subspace problem. Recall that an operator T on the Hilbert space H is hyponormal if $T^*T \geq TT^*$. In [13], it is proved that hyponormal operators satisfy

generalized Weyl's theorem. However, in [19, Corollary 4.5], it is proved that hyponormal operator are not hypercyclic. Hence it is naturel to ask about the relation between hypercyclic/supercyclic operators and Weyl type theorems. In this direction and using Herrero's results [16] about spectral properties satisfied by hypercyclic/supercyclic operators, Duggal in [14] gave necessary and sufficient conditions for hypercyclic/supercyclic Banach spaces operator to satisfy a-Weyl's theorem. Hypercyclic/supercyclic operators which satisfy Weyl type theorems are also recently studied in [12], [25].

In this note, we prove that if T is a hypercyclic or supecyclic Banach space operator, then T and T^* satisfy generalized a-Browder's theorem. More, we give necessary and/ or sufficient conditions for hypercyclic and supercyclic Banach space operators to satisfies generalized a-Weyl's theorem.

2. Notation and terminology

Let X an infinite complex separable Banach space and $T \in \mathcal{L}(X)$ be the algebra of all bounded linear operator on X. The T-orbit of a vector space $x \in X$ is the set

$$O(x,T) = \{T^n x \ ; \ n \in \mathbb{N}\}.$$

The operator T is said to be *hypercyclic* if there is some vector $x \in X$ such that $O(x,T)$ is dense in X. Similarly, T is said to be *supercyclic* if there exists a vector $x \in X$ such that the set

$$\mathbb{C}O(x,T) = \{\lambda T^n x \ ; \ n \in \mathbb{N}, \ \lambda \in \mathbb{C}\}$$

is dense in X. The sets of all hypercyclic operators and supercyclic operators will be denoted by $HC(X)$ and $SC(X)$ respectively.

Recall that $T \in \mathcal{L}(X)$ is said to be bounded below, if T is injective and the range $R(T)$ of T is closed. Denote the approximate point spectrum of T by

$$\sigma_a(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not bounded below}\}.$$

Let

$$\sigma_s(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not surjective}\}$$

denote the surjectivity spectrum of T. Clearly, $\sigma(T) = \sigma_a(T) \cup \sigma_s(T)$, the spectrum of T. Denote by $\alpha(T)$ the dimension of the kernel $\ker T$, and by $\beta(T)$ the codimension of the range $R(T)$. An operator $T \in \mathcal{L}(X)$ is said to be upper semi-Fredholm (respectively, lower semi-Fredholm) operators if $R(T)$ is closed and $\alpha(T)$ is finite (respectively, $\beta(T)$ is finite). In this case the index of T is defined by $ind(T) = \alpha(T) - \beta(T)$. An operator T is said to be a Fredholm operator if it is both upper semi-Fredholm and lower semi-Fredholm. $SF_+(X)$ and $SF_-(X)$ will stand for the set of all upper semi-Fredholm operators and by the set of all lower semi-Fredholm operators, respectively. This two classes of operators generate the upper semi-Fredholm spectrum and lower semi-Fredholm spectrum defined by

$$\sigma_{SF_+}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin SF_+(X)\}$$

and

$$\sigma_{SF_-}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin SF_-(X)\}.$$

In addition the set

$$SF_+^-(X) = \{T \in SF_+(X) : ind(T) \leq 0\}$$

generates the following Weyl essential approximate point spectrum

$$\sigma_{SF_+^-}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin SF_+^-(X)\}.$$

Recall that the ascent $a(T)$ of an operator T, is defined by $a(T) = \inf\{n \in \mathbb{N} : \ker T^n = \ker T^{n+1}\}$ and the descent $d(T) = \inf\{n \in \mathbb{N} : T^n X = T^{n+1}X\}$, with $\inf \emptyset = \infty$. It is well known that if $a(T)$ and $d(T)$ are both finite, then $a(T) = d(T)$ (see [17] for more details).

The Weyl essential approximate point spectrum and the Browder essential approximate point spectrum of $T \in \mathcal{L}(X)$ are the sets

$$\sigma_{aw}(T) = \{\lambda \in \sigma_a(T) : T - \lambda I \quad \text{is not upper semi-Fredholm or} \quad 0 < ind(T - \lambda I)\}$$

and

$$\sigma_{ab}(T) = \{\lambda \in \sigma_a(T) : \lambda \in \sigma_{aw}(T) \quad \text{or} \quad a(T - \lambda I) = \infty\}.$$

It is clear that

$$\sigma_{SF_+}(T) \subseteq \sigma_{aw}(T) \subseteq \sigma_{ab}(T) \subseteq \sigma_a(T).$$

An operator $T \in \mathcal{L}(X)$ is said to be Weyl operator if it is Fredholm operator of index zero. The Weyl spectrum $\sigma_w(T)$ is defined by

$$\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Weyl operator}\}.$$

For $T \in \mathcal{L}(X)$ and a nonnegative integer n define T_n to be the restriction of T to $R(T^n)$ viewed as a map from $R(T^n)$ into $R(T^n)$. If for some integer n the range space $R(T^n)$ is closed and T_n is semi-Fredholm operator, upper or lower, in the usual sense, then T is called semi B-Fredholm operator. In this case the index of T is defined as the index of the semi-Fredholm operator T_n.

Similarly, we define the upper semi B-Fredholm, lower semi B-Fredholm and B-Fredholm operator of T by:

$$\sigma_{SBF_+}(T) = \{\lambda \in \sigma(T) : T - \lambda \text{ is not upper semi B-Fredholm }\},$$

$$\sigma_{SBF_-}(T) = \{\lambda \in \sigma(T) : T - \lambda \text{ is not lower semi B-Fredholm }\}$$

and

$$\sigma_{BF}(T) = \{\lambda \in \sigma(T) : T - \lambda \text{ is not B-Fredholm }\},$$

respectively. If T is B-Fredholm operator of index zero, then T is said to be a B-Weyl operator and the B-Weyl spectrum is defined by:

$$\sigma_{bw}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not B-Weyl operator}\}.$$

A complex number $\lambda \in \sigma(T)$ is a pole of resolvent of $T \in \mathcal{L}(X)$ if $T - \lambda I$ has finite ascent and descent. Apparently, if $\lambda \in \sigma(T)$ is a pole of the resolvent of $T \in \mathcal{L}(X)$, then $\lambda \in iso\sigma(T)$, the set of isolated points of $\sigma(T)$. To introduce the definitions of the main notions studied in this work, some notations are needed. Let $T \in \mathcal{L}(X)$ and denote by $\Pi(T) = \{\lambda \in \mathbb{C} : a(T - \lambda I) = d(T - \lambda I) < \infty\}$ (respectively, $\Pi_0(T) = \{\lambda \in \Pi(T) : \alpha(T - \lambda I) < \infty\}$) the set of poles of T (respectively, the set of poles of finite rank of T). Similarly denote by $\Pi^a(T) = \{\lambda \in \sigma_a(T) : a(T - \lambda I) = d < \infty$ and $(T - \lambda I)^{d+1}E$ is closed$\}$ (respectively, $\Pi_0^a(T) = \{\lambda \in \Pi^a(T) : \alpha(T - \lambda I) < \infty\}$) the set of left poles of T (respectively, the set of left poles of finite rank of T). In the sequel let $I(T) = iso\sigma(T) \setminus \Pi(T)$ and $I^a(T) = iso\sigma_a(T) \setminus \Pi^a(T)$. In addition, given $T \in \mathcal{L}(X)$, let $E(T) = \{\lambda \in iso\sigma(T) : 0 < \alpha(T - \lambda I)\}$ (respectively, $E_0(T) = \{\lambda \in E(T) : \alpha(T - \lambda I) < \infty\}$) the set of eigenvalues of T which are isolated in the spectrum of T (respectively, the set of eigenvalues of finite multiplicity isolated in $\sigma(T)$). Now we introduce the main notions that has been studied in [5], [9], [10].

Definition 2.1 Let $T \in \mathcal{L}(X)$. Then it will be said that

(i) Browder's theorem holds for T, if $\sigma_w(T) = \sigma(T) \setminus \Pi_0(T)$,

(ii) Generalized Browder's theorem holds for T, if $\sigma_{bw}(T) = \sigma(T) \setminus \Pi(T)$,

(iii) a-Browder's theorem holds for T, if $\sigma_{aw}(T) = \sigma_a(T) \setminus \Pi_0^a(T)$,

(iv) Generalized a-Browder's theorem holds for T, if $\sigma_{SBF_+^-}(T) = \sigma_a(T) \setminus \Pi^a(T)$.

Definition 2.2 Let $T \in \mathcal{L}(X)$. Then it will be said that

(i) Weyl's theorem holds for T, if $\sigma_w(T) = \sigma(T) \setminus E_0(T)$,

(ii) Generalized Weyl's theorem holds for T, if $\sigma_{bw}(T) = \sigma(T) \setminus E(T)$,

(iii) Property (w) holds for T, if $\sigma_{aw}(T) = \sigma_a(T) \setminus E_0(T)$,

(iv) Property (gw) holds for T, if $\sigma_{SBF_+^-}(T) = \sigma_a(T) \setminus E(T)$.

Recall that an operator $T \in \mathcal{L}(X)$ is said to have the single valued extension property at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at λ_0), if for every open disc $\mathbb{D}$ centered at λ_0, the only analytic function $f : \mathbb{D} \to \mathcal{X}$ which satisfies the equation $(T - \lambda I)f(\lambda) = 0$ for all $\lambda \in \mathbb{D}$ is the function $f \equiv 0$. An operator $T \in L(\mathcal{X})$ is said to have SVEP if T has SVEP at every $\lambda \in \mathbb{C}$, see [1], [21] for more details about the SVEP.

3. Hypercyclic/Supercyclic operators satisfying generalized a-Weyls's theorem

In the sequel, we will use the following lemma.

Lemma 3.1 *Let $T \in \mathcal{L}(X)$. If $T \in HC(\mathrm{X}) \cup SC(\mathrm{X})$, then the SVEP holds for T^*.*

Proof. if $T \in SC(\mathrm{X})$, then from [7]

$$\sigma_p(\mathrm{T}^*) = \emptyset \text{ or } \sigma_p(\mathrm{T}^*) = \{\alpha\}, \text{ where } \alpha \in \mathbb{C}\backslash\{0\}.$$

Hence T^* satisfies the SVEP. ∎

Recall that an operator $T \in \mathcal{L}(X)$ is Drazin invertible if and only if it has a finite ascent and descent, which is also equivalent to the fact $T = T_0 \oplus T_1$, where T_0 is a nilpotent operator and T_1 is an invertible operator, see [20, Proposition A]. The Drazin spectrum is given by

$$\sigma_D(T) = \{\lambda \in \mathbb{C} \text{ such that } T - \lambda \text{ is not Drazin invertible }\}.$$

We observe that

$$\sigma_D(T) = \sigma(T) \setminus \Pi(T).$$

An operator $T \in \mathcal{L}(X)$ is called left Drazin invertible if $a(T) < \infty$ and $R(T^{a(T)+1})$ is closed [10, Definition 2.4]. The left Drazin spectrum is given by

$$\sigma_{ld}(T) = \{\lambda \in \mathbb{C} \text{ such that } T - \lambda \text{ is not lefty Drazin invertible }\}.$$

Similarly, we have that $\sigma_{ld}(T) = \sigma_a(T) \setminus \Pi^a(T)$. In [2] the authors proved that:

T satisfies generalized Browder's theorem if and only if $\sigma_{bw}(T) = \sigma_D(T)$

and

T satisfy generalized a-Browder's theorem if and only if $\sigma_{SBF_+^-}(T) = \sigma_{ld}(T)$.

Using this results, we will prove the following theorem.

Theorem 3.2 *Let $T \in \mathcal{L}(X)$. If $T \in HC(\mathrm{X}) \cup SC(\mathrm{X})$, then T and T^* satisfy generalized a-Browder's theorem.*

Proof. By Lemma 3.1, the SVEP holds for T^*. Hence from [4, Theorem 3.2], T satisfies generalized a-Browder's theorem. Now, we shows that generalized a-Browder's theorem holds for T^*. We have that

$$\sigma_{SBF_+^-}(T^*) \subset \sigma_{ld}(T^*).$$

To prove that T^* also satisfies a-Browder's theorem it suffice to prove that

$$\sigma_{SBF_+^-}(T^*) \supset \sigma_{ld}(T^*).$$

For this let $\lambda \notin \sigma_{SBF_+^-}(T^*)$, then $T^* - \lambda$ is upper semi B-Fredholm and $ind(T^* - \lambda) > 0$. Since T^* has SVEP, then

$$d(T^* - \lambda) < \infty.$$

By [1, Theorem 3.17], we have that $ind(T^* - \lambda) \leq 0$. Hence $ind(T^* - \lambda) = 0$ and $T^* - \lambda$ is B-Fredholm. Since $d(T^* - \lambda) < \infty$, then by [1, Theorem 3.3] $a(T^* - \lambda) = d(T^* - \lambda)$ and hence$\lambda \notin \sigma_{ld}(T^*)$. This complete the result since $HC(\mathrm{X}) \subset SC(\mathrm{X})$. ∎

For $T \in \mathcal{L}(X)$ denote by $H(\sigma(T))$ the set of all analytic function on a neighbourhood of $\sigma(T)$. In the following corollary, we prove that the conclusion of Theorem 3.2 holds for $f(T)$ whenever $T \in HC(\mathrm{X}) \cup SC(\mathrm{X})$.

Corollary 3.3 *Let $T \in \mathcal{L}(X)$. If $T \in HC(\mathrm{X}) \cup SC(\mathrm{X})$, then $f(T)$ satisfies generalized a-Browder's theorem for every $f \in H(\sigma(T))$.*

Proof. Since from Lemma 3.1 T^* satisfies the SVEP, then by [21] $f(T^*)$ also satisfies the SVEP for every $f \in H(\sigma(T))$. Hence from [4, Theorem 3.2], $f(T)$ also satisfies generalized a-Browder's theorem. ∎

In what follow, we will prove that the adjoint of hypercyclic or supercyclic Banach space operator satisfy generalized Weyl's theorem.

Theorem 3.4 *Let $T \in \mathcal{L}(X)$.*

(i) *If $T \in HC(\mathrm{X}) \cup SC(\mathrm{X})$, then T^* satisfies generalized Weyl's theorem.*

(ii) *If $T \in HC(\mathrm{X}) \cup SC(\mathrm{X})$ and $E(T) \subseteq E(T^*)$, then T satisfies generalized a-Weyl's theorem.*

Proof. (i) If $T \in SC(\mathrm{X})$, then we can conclude that $E(T^*) = \Pi(T^*)$. Indeed, if $T \in SC(\mathrm{X})$ then either $E(T^*) = \emptyset$ or $E(T^*) = \{\alpha\}$ for non zero α such that $\alpha \notin \sigma_b(T^*)$. If $E(T^*) = \emptyset$, then $E(T^*) = \Pi(T^*) = \emptyset$. If $E(T^*) = \{\alpha\}$ with $\alpha \notin \sigma_b(T^*)$, then $\alpha \in \Pi(T^*)$ and hence $E(T^*) = \Pi(T^*)$. Since, by Theorem 3.2, T^* satisfies generalized a-Browder's theorem, then T^* satisfies Browder's theorem. It then follows from [6, Corollaray 2.1] that T^* satisfies generalized Weyl's theorem.

(ii) Assume that $E(T) \subseteq E(T^*)$. From Theorem 3.2, we conclude that T satisfies generalized a-Browder's theorem, and hence generalized Browder's theorem that is $\sigma(T)/\sigma_{bw}(T) = \Pi(T)$, hence

$$\sigma(T)/\sigma_{bw}(T) = \Pi(T) \subseteq E(T).$$

From the proof of (i), T^* satisfies the generalized Weyl's theorem, hence by [6, Corollary 2.1] we have that $E(T^*) = \Pi(T^*)$. Finally,

$$\begin{aligned} \sigma(T)/\sigma_{bw}(T) &= \Pi(T) \\ &\subseteq E(T) \\ &\subseteq E(T^*) \\ &= \Pi(T^*) \\ &= \Pi(T). \end{aligned}$$

Hence $\sigma(T)/\sigma_{bw}(T) = E(T)$ that is T satisfies generalized Weyl's theorem. Now, since T^* satisfies the SVEP, then $\sigma(T) = \sigma_a(T)$, $\Pi(T) = \Pi^a(T)$ and then $\sigma_{bw}(T) = \sigma_{SBF_{+}^{-}}(T)$ and from the fact that T satisfies generalized Weyl's theorem, we conclude from [6, Corollary 2.1] that $E(T) = \Pi(T)$, and hence $E(T) = \Pi^a(T)$.

Finally,

$$\sigma(T)/\sigma_{bw}(T) = E(T) = \Pi(T) = \sigma_a(T)/\sigma_{SBF_{+}^{-}}(T) = \Pi^a(T).$$

That is, T satisfies generalized a-Weyl's theorem. ∎

In the following, we will characterize generalized a-Weyl's theorem for hypercyclic and supercyclic Banach space operator in term of isolated eigenvalues in its spectrum.

Theorem 3.5 *For $T \in \mathcal{L}(X)$. We have the following assertions:*

(i) *Let $T \in HC(X)$. Then T satisfies generalized a-Weyl's theorem if and only if $E(T) = \emptyset$.*

(ii) *Let $T \in SC(X)$. Then T satisfies generalized a-Weyl's theorem if and only if there is an $\alpha \in \mathbb{C}\backslash\sigma_b(T)$ such that $E(T) \subset \{\alpha\}$.*

Proof. (i) If $T \in HC(X)$, then $E(T^*) = \emptyset$. Since by Theorem 3.4 T^* satisfies generalized Weyl's theorem, then from [6, Corollary 2.1], $E(T^*) = \Pi(T^*)$. If also T satisfies generalized a-Weyl's theorem, then it satisfies generalized Weyl's theorem and again we have $\Pi(T) = E(T) = \emptyset$. So $E(T^*) = \Pi(T^*) = \Pi(T) = E(T) = \emptyset$. For the converse, Assume that $E(T) = \emptyset$. From Theorem 3.2 T satisfies generalized a-Browder's theorem, hence we have $\sigma_a(T)/\sigma_{SBF_+^-}(T) = \Pi^a(T) = \emptyset = E(T) = E^a(T)$. That is T satisfies generalized a-Weyl's theorem.

(ii) If $T \in SC(X)$, then $E(T^*) = \emptyset$ or $E(T^*) = \{\alpha\}$ for a non zero $\alpha \in \mathbb{C}$ such that $\alpha \notin \sigma_b(T)$, that is $E(T^*) \subset \{\alpha\}$. If $E(T^*) = \emptyset$, we conclude as in the proof of (i). If $E(T^*) = \{\alpha\}$, then again $E(T^*) = \Pi(T^*) = \Pi(T) = E(T) = \{\alpha\}$, this since T and T^* satisfy generalized Weyl's theorem. For the converse, assume that there is an $\alpha \in \mathbb{C}\backslash\sigma_b(T)$ such that $E(T) \subset \{\alpha\}$. If $E(T^*) = \emptyset$, then as in the proof of i) T satisfies generalized a-Weyl' theorem. If $E(T^*) = \{\alpha\}$, then $E(T) \subset E(T^*) = \{\alpha\}$. By Theorem 3.4, T satisfies generalized a-Weyl's theorem. ∎

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THE QUOTIENT ULTRA-GROUPS

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Abstract. The main purpose of this paper is to construct the quotient ultra-groups, which are based on congruences. By these means we can present fundamental theorem and Lagrange theorem for the ultra-groups. Also, we show not necessary that the order of each element of the ultra-group divided by the order of the ultra-group.

Keywords: homomorphism, congruence, ultra-group.

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1. Introduction

The notion of congruence was first introduced by Karl Fredrich Gauss in the beginning of the nineteenth century. Congruence is a special type of equivalence relation which plays a vital role in the study of quotient structures of different algebraic structures (see [6]). In this paper we study the quotient structure of ultra-group by using the notion of congruence in ultra-group and study some interesting properties ultra-groups homomorphism. The notion of ultra-group over a group was introduced in [4]. In fact an ultra-group is an algebraic structure whose underlying set is depend on a group and its subgroup. Moreover, an abelian ultra-group is equivalence to quasigroup and finite loop (see [5]). We present now some basic definitions and results about ultra-groups. Let H be a subgroup in group G. A subset M of group G is called a (right) transversal to H in G if $G = \bigcup_{m\in M} Hm$. Therefore, the pair (H, M) is a (right) transversal if and only if

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the subset M of the group G obtained by selecting one and only one member from each right coset of G modulo subgroup H of G. Throughout the paper, a right (left) transversal will be assumed to contain the identity of the group (see [1]). For the group G, which satisfies the above conditions, we have $MH \subseteq G = HM$, and subset M of G is called (right unitary) complementary set with respect to subgroup H. This notion was introduced by Kurosh in [3] which is the base of the concept of an ultra-group. Therefore, for every element $mh \in MH$ there exists unique $h' \in H$ and $m' \in M$ such that $mh = h'm'$. We denote h' and m' by ${}^m h$ and m^h, respectively. Similarly, for any elements $m_1, m_2 \in M$, there exists unique elements $[m_1, m_2] \in M$ and $(m_1, m_2) \in H$ such that $m_1 m_2 = (m_1, m_2)[m_1, m_2]$. Similarly for every element $a \in M$, there exists a^{-1} belonging to G. As $G = HM$, there is $a^{(-1)} \in H$ and $a^{[-1]} \in M$ such that $a^{-1} = a^{(-1)} a^{[-1]}$. Now, we are ready to define an ultra-group.

Definition 1.1 A (*right*) *ultra-group* ${}_H M$ is a complementary set of H over group G with a binary operation $\alpha :_H M \times_H M \to_H M$ and unary operation $\beta_h :_H M \to_H M$ defined by $\alpha((m_1, m_2)) := [m_1, m_2]$ and $\beta_h(m) := m^h$ for all $h \in H$.

It is easy to see that every group is an ultra-group. The following example is an ultra-group which is not a group.

Example 1.2 Let $G = S_3$. Then $H = \{(1), (13)\}$ is a subgroup of G. By $H \setminus G = \{\{(1), (13)\}, \{(23), (123)\}, \{(12), (132)\}\}$ we have four possibility for set M. For example let $M = \{(1), (12), (23)\}$, then M is an ultra-group with the following maps :

α	(1)	(23)	(12)
(1)	(1)	(23)	(12)
(23)	(23)	(1)	(23)
(12)	(12)	(12)	(1)

β	(1)	(13)
(1)	(1)	(1)
(23)	(23)	(12)
(12)	(12)	(23)

Example 1.3 Let $G = C^*$ and $H = U$, where C^* be the set of complex numbers without zero and $U = \{z \in C^* : |z| = 1\}$. The set C^* under Multiplication is an infinite abelian group and $U \leq C^*$. The set $U \backslash C^*$ consists of concentric circles. So, an obvious (left or right) ultra-group for U in C^* is positive real numbers R^+, which is even a subgroup of $U \backslash C^*$. Another (left or right) ultra-group for U in C^* is $R^- = \{x : x < 0\}$, which is not a subgroup of C^*.

Notice that there are many other choices of ultra-groups are available.

Example 1.4 Let G be the symmetric group of order 4. This group contains three subgroups H_i of order 8 where $i = 1, 2, 3$:

$$H_1 = \{(1), (12)(34), (13)(24), (14)(23), (12), (34), (1324), (1423)\};$$
$$H_2 = \{(1), (12)(34), (13)(24), (14)(23), (13), (24), (1234), (1432)\};$$
$$H_3 = \{(1), (12)(34), (13)(24), (14)(23), (14), (23), (1243), (1342)\}.$$

Every H_i/S_4 has three elements for $i = 1, 2, 3$. It is not hard to show that any subgroup H_i has 64 (right) complementary sets M. Therefore ultra-group of order 3 in S_4 are 192.

A (*left*) *ultra-group* M_H is defined similarly via (left unitary) complementary set. From now on, unless specified otherwise, ultra-group means right ultra group. Although every element in right ultra-groups do not have right inverse, but they have left inverse with respect to the first binary operation α. The first binary operation of an ultra-group M has the right cancellation. We observe that we have not left cancellation for right ultra-groups. A subset $S \subseteq {}_HM$ which contains e, is called a *subultra-group* of H over G, if S is closed under the operations α and β_h in the Definition 1.1. It is obvious that $\{e\}$ is a trivial subultra-group for all ultra-groups ${}_HM$ where e is the identity element of H. A subultra-group N of ${}_HM$ is called *normal* if $[a,[N,b]] = [N,[a,b]]$, for all $a, b \in {}_HM$. In addition $[N,S]$ is a subultra-group of ${}_HM$, where S is a subultra-group of ${}_HM$. Moreover, $[N,S]$ is a normal subultra-group of ${}_HM$ if S is also normal subultra-group of ${}_HM$. Suppose A, B are two subsets of the ultra-group ${}_HM$. Moreover we use the notation $[A,B]$ for the set of all $[a,b]$, where $a \in A$ and $b \in B$. If B is a singleton $\{b\}$, then we denote $[A,B]$ by $[A,b]$.

Definition 1.5 Suppose ${}_{H_i}M_i$ is ultra-group of H_i over group G_i, $i = 1, 2$. A function $f : {}_{H_1}M_1 \longrightarrow {}_{H_2}M_2$ is an ultra-group homomorphism provided that for all $m, m_1, m_2 \in {}_{H_1}M_1$ and $h \in H_1$.

(i) $f([m_1, m_2]) = [f(m_1), f(m_2)]$,

(ii) $(f(m))^{\varphi(h)} = f(m^h)$,
where φ is a group homomorphism between two subgroups H_1 and H_2.

If f is a surjective and injective ultra-group homomorphism, we call it isomorphism and denote it by ${}_{H_1}M_1 \cong {}_{H_2}M_2$.

Now, some properties of the ultra-group homomorphism as many algebras reviews.

Theorem 1.6 *Let $f : {}_{H_1}M_1 \longrightarrow {}_{H_2}M_2$ be an ultra-group homomorphism. Then*

(i) $f(e_{{}_{H_1}M_1}) = e_{H_2M_2}$,

(ii) $f(a^{[-1]}) = (f(a))^{[-1]}$ *for all* $a \in {}_{H_1}M_1$.

Proof. The assertion follows immediately from the first binary operation α of an ultra-group has the right cancellation and Definition 1.5. ∎

Definition 1.7 Let $f :_{H_1} M_1 \to {}_{H_2}M_2$ be ultra-groups homomorphism. Then $\mathrm{Ker}(f)$ is defined by

$$\{(m_1, m_2) \in_{H_1} M_1 \times {}_{H_1}M_1 : f(m_1) = f(m_2)\}.$$

In other words, if $(m_1, m_2) \in \mathrm{Ker}(f)$ then $f([m_1^{(m_2)^{(-1)}}, m_2^{[-1]}]) = f(e)$. Therefore $([m_1^{(m_2)^{(-1)}}, m_2^{[-1]}], e) \in \mathrm{Ker}(f)$. We can consider $\mathrm{Ker}(f)$ as a subset of the ${}_HM$ defined the following

$$\mathrm{Ker}(f) = \{m \in {}_{H_1}M_1 : f(m) = e\}.$$

Congruence is a special type of equivalence relation which plays a vital role in the study of quotient structures of different algebraic structures (see [2]).

2. Main result

The structure of an ultra-group ${}_HM$ over the group G is revealed by its subultra-groups and homomorphisms into other ultra-groups.

Lemma 2.1 *Let S be a subultra-group of ultra-group ${}_HM$ over the group G and $a, b \in {}_HM$. Then the following conditions are equivalent.*

(i) $a \in [S, b]$,

(ii) $[S, a] = [S, b]$,

(iii) $[a^{(b^{(-1)})}, b^{[-1]}] \in S$.

Proof. It follows that from the Definition of subultra-group. ∎

By Lemma 2.1 we have $[S, a] = [S, b]$ or $[S, a] \cap [S, b] = \emptyset$, which implies ${}_HM = \cup_{a \in {}_HM}[S, a]$.

Lemma 2.2 *For the ultra-group homomorphism f between two ultra groups ${}_{H_1}M_1$ and ${}_{H_2}M_2$ with kernel K, we deduce that $f(m_1) = f(m_2)$ is equivalent to the fact $m_1 = [k, m_2]$ for some $k \in K$.*

Proof. Suppose that $f(m_1) = f(m_2)$, so we have $[(f(m_1))^{(f(m_2))^{(-1)}}, (f(m_2))^{[-1]}]$ $= f([m_1^{m_2^{(-1)}}, m_2^{[-1]}]) = e$. This means that $[m_1^{m_2^{(-1)}}, m_2^{[-1]}] \in K$, thus $m_1 = [k, m_2]$ for some $k \in K$. Conversely $f(m_1) = f([k, m_2]) = [f(k), f(m_2)] = f(m_2)$. ∎

Theorem 2.3 (see [4]) *Suppose the same notations in the Definition* 1.7. *Then* $\mathrm{Ker}(f)$ *is a congruence on ${}_{H_1}M_1$ and it is a subultra-group of ${}_{H_1}M_1$.*

In the following establish a connection between congruence on an ultra-group and a normal subultra-group.

Lemma 2.4 *Let S be a subultra-group of ${}_HM$ and θ be a relation on ${}_HM$ defined by $a\theta b$ if and only if there is $s \in S$ such that $a = [s, b]$. Then θ is an equivalence relation.*

Proof. The assertion follows immediately from Lemma 2.1. ∎

From now on, we use the notation θ for the equivalence relation which is satisfied in the Lemma 2.4. For instance, we denote the equivalence class of e with respect to the equivalence relation of θ in the Lemma 2.4 by $[e]_\theta = S$ is a subultra-group of ${}_HM$.

Theorem 2.5 *If S is a subultra-group of ultra-group ${}_HM$ over the group G, then the equivalence relation θ is a congruence if and only if S is a normal subultra-group ${}_HM$.*

Proof. Let θ be a congruence. For every $b \in {}_HM$ by definition of θ we have $[s,b]\ \theta\ b$ for every $s \in S$. By θ is a congruence we have $[a,[s,b]]\theta[a,b]$. Therefore $[a,[s,b]] = [s^{'},[a,b]]$ for some $s^{'} \in S$. Thus S is a normal subultra-group of ${}_HM$. For converse, let $a\theta b$ and $a'\theta b'$, where $a,b,a',b' \in {}_HM$. Then by the definition of normal subultra-group we have $[a,a'] = [[n,b],[n',b]] = [n'',[b,b']]$. This means $[a,a']\theta[b,b']$, for $n,n',n'' \in N$. For compatibility of the second operation of the ultra-group, assume $a\theta b$. Thus $a^h = [n',b^h]$ and the definition of θ implies $a^h\theta b^h$, for all $h \in H$. ∎

A right coset of subultra-group S in ultra-group ${}_HM$ is a subset of the form $[S,a] = \{[s,a] : s \in S\}$. By two right cosets are either equal or disjoint. The set of right cosets of subultra-group S in ultra-group ${}_HM$ is denoted M/θ.

Theorem 2.6 *All right cosets of subultra-group S of the ultra-group ${}_HM$ over group G have the same cardinality.*

Proof. Define a mapping $f : [S,a] \to [S,b]$ by $f(x) = [[x^{a^{(-1)}},a^{[-1]}],b]$. f is a mapping one-to-one and onto. ∎

This observation is the key to the very important theorem Lagrange theorem for ultra-groups.

Theorem 2.7 (Lagrange's theorem) *If S is a subultra-group of ${}_HM$, then the cardinality of S divides the cardinality of ${}_HM$.*

Proof. The distinct right cosets of S are mutually disjoint by Theorem 2.1 and each has the same size by Theorem 2.6. Since the union of the right cosets is ${}_HM$, the cardinality of ${}_HM$ is the cardinality of S times the number of distinct right cosets of S. ∎

Definition 2.8 An element a of an ultra-group ${}_HM$ over group G is an idempotent element, if $\alpha(a,a) = a$ where α is its first binary operation over it.

Corollary 2.9 *The only idempotent element in ultra-group ${}_HM$ is $e_{{}_HM}$.*

Proof. Suppose that a is idempotent element in ultra-group ${}_HM$. By $a.a = (a,a)[a,a]$ and $a = (a,a)$ we have $a \in {}_HM \cap H$. ∎

Corollary 2.10 *Every subultra-group of order 2 of ultra-group ${}_HM$ is a group.*

Proof. This follows immediately from $[a,a] = e$, where a of ultra-group ${}_HM$ over group G. ∎

Example 2.11 Suppose $M = \{e,a,a^5b,a^3,a^4,a^2b\}$ is an ultra-group over subgroup $H = \{e,ab\}$ arise from group

$$G = \{a^ib^j \mid a^6 = b^2 = e,\ a^ib = ba^{6-i},\ i = 0,\dots,5,\ j = 0,1\}.$$

Then, the only subultra-groups of ultra-group M are

$$N_1 = \{e\},\ N_2 = \{e,a^3\},\ N_3 = \{e,a^5b\},\ N_4 = \{e,a^2b\},\ N_5 = \{e,a^4,a^5b\}\ N_6 = G.$$

Although we have an associativity property for the groups, but this property is not valid for the binary operation α of the ultra-groups. Therefore, we convent $\alpha(a,a,a) = \alpha(\alpha(a,a),a) = \alpha^3(a)$ where a is the element of the ultra-group M and α is its first binary operation over it.

Definition 2.12 The order of $a \in_H M$ is the smallest positive power k such that $\alpha^k(a) = e$.

In group theory, the order of any element of a finite group divides the order of the group. In this case, with following example we show that this is not established in ultra-groups.

Example 2.13 Let $M = \{e, ab, a^2b\}$ be an ultra-group of order 3 arise from subgroup $H = \langle a^3, b\rangle = \{e, a^3, a^6, b, a^3b, a^6b\}$ over group D_9. Then the order of element ab is 2 with following table :

α	e	a^2b	ab
e	e	a^2b	ab
a^2b	a^2b	e	a^2b
ab	ab	ab	e.

3. Quotient ultra-groups and homomorphism theorems

As respects, in some algebraic structure can be defined the quotient algebra with congruence, we also define the quotient ultra-group.

Definition 3.1 Let M be an ultra-group over subgroup H_1 of group G and θ be a congruence over M. The set $M/\theta = \{[m]_\theta; m \in M\}$ with the operations α_θ and β_{θ_h},

(i) $\alpha_\theta([m]_\theta, [m']_\theta) = [\alpha(m,m')]_\theta$,

(ii) $\beta_{\theta_h}([m]_\theta) = [\beta_h(m)]_\theta$,

is an ultra-group of H_2 over the group G, where $H_1 \leq H_2 \leq G$. This ultra-group is called a *quotient ultra-group.*

Any ultra-group homomorphism $f :_{H_1} M_1 \to_{H_2} M_2$ induces an isomorphism from a quotient of ${}_{H_1}M_1$ to a subgroup of ${}_{H_2}M_2$. More precisely, we have the following.

Theorem 3.2 *(First isomorphism theorem for ultra-groups) Let f be a surjective ultra-group homomorphism between two ultra-groups ${}_{H_1}M_1$ and ${}_{H_2}M_2$ and θ a congruence over ${}_{H_1}M_1$ such that $\theta \subseteq \mathrm{Ker}(f)$. If $\pi :_{H_1} M_1 \longrightarrow_{H_1} M_1/\theta$ is the natural ultra-group homomorphism, then there exists a homomorphism g so that $g\pi = f$.*

Proof. It is enough to define the map $g :_{H_1} M_1/\theta \longrightarrow {}_{H_2}M_2$ by $g([m]_\theta) = f(m)$. Since $[m_1]_\theta = [m_2]_\theta \Leftrightarrow m_1 \ \theta \ m_2$, by the hypothesis $\theta \subseteq \mathrm{K}er(f)$ we have $f(m_1) = f(m_2)$ which implies g is well-defined. The map g preserves the first operation α_θ and the unary operation β_{θ_h}. Thus we can conclude that g is an ultra homomorphism. The map g preserve the first operation α_θ as follows, $g(\alpha_\theta([m_1]_\theta, [m_2]_\theta)) = g([[m_1, m_2]]_\theta) = [f(m_1), f(m_2)] = [g([m_1]_\theta), g([m_2]_\theta)]$. For the second operation β_{θ_h} we have, $g(\beta_{\theta_h}([m]_\theta)) = g([m^h]_\theta) = f(m^h) = (f(m))^{\varphi(h)} = \beta_{\varphi(h)}(g([m]_\theta))$. ∎

Theorem 3.3 *Let γ be a congruence on ${}_HM$ and θ be a relation on ${}_HM$ defined by $a\theta b$ if and only if there is $s \in S$ such that $a = [s, b]$ where $S = [e]_\gamma$. Then γ and θ are equivalent.*

Proof. By congruence definition and right cancellation property we conclude,

$$\begin{array}{lllll} a \ \gamma \ b & \Longleftrightarrow & a^{b^{(-1)}} \ \gamma \ b^{b^{(-1)}} & \Longleftrightarrow & [a^{b^{(-1)}}, b^{[-1]}] \ \gamma \ [b^{b^{(-1)}}, b^{[-1]}] \\ & \Longleftrightarrow & [a^{b^{(-1)}}, b^{[-1]}] \ \gamma \ e & \Longleftrightarrow & [a^{b^{(-1)}}, b^{[-1]}] \in S \\ & \Longleftrightarrow & a \ \theta \ b. & & \end{array}$$

∎

Theorem 3.4 (see [4]) *If γ is a congruence on the ultra-group ${}_HM$ and $[e]_\gamma = N$, then ${}_HM/\theta \cong_H M/N$.*

If γ, θ are congruences on ${}_HM$ and $\theta \subseteq \gamma$, then clearly $\gamma/\theta = \{(a/\theta, b/\theta) \in ({}_HM/\theta)^2 : (a, b) \in \gamma\}$ is a congruence on ${}_HM/\theta$. Recall that two sets S and T have the same cardinality if and only if there is a one-to-one mapping from one set onto the other.

Theorem 3.5 (Second isomorphism theorem of ultra-groups) *If N', N are normal subultra-groups of ultra-group ${}_HM$ such that $N \subseteq N'$, then $\frac{\frac{HM}{N}}{\frac{N'}{N}} \cong \frac{HM}{N'}$.*

Proof. By the argument before the theorem N'/N is a normal subultra-group of ${}_HM/N$. The map $\psi :_H M/N/N'/N \longrightarrow M/N'$ is a homomorphism with $N'/N \subseteq \mathrm{K}er(\psi)$ so the result follows by the first isomorphism theorem. ∎

The third isomorphism theorem [2, Theorem 2.6.18] which is valid for any algebra can be translate for ultra-groups. Although, we can prove the third isomorphism theorem for ultra-groups by the same method of the proof of the first isomorphism theorem. We need Lemma 3.6 in order to mimic the proof of Theorem 2.6.18 in [2].

Lemma 3.6 *Let B be a subultra-group of ${}_HM$ and θ a congruence on ${}_HM$. Then*

(i) $B^2 \cap \theta = (B \cap N)^2$,

(ii) $B^\theta = [N, B]$,

where N is $[e]_\theta$.

Proof. (i) With some basic properties of the congruence in hand, we have: $B^2 \cap \theta = \{(b_1, b_2) : b_1, b_2 \in B \text{ and } b_1 \ \theta \ b_2\} = \{(b_1, b_2) : b_1, b_2 \in N\} = B^2 \cap N^2$.

(ii) By [2, Definition 2.6.16] we deduce $B^\theta = \{a \in_H M : \exists\ b \in B, a \ \theta \ b\} = \{a \in {}_H M : \exists\ b \in B, a = [n, b] \text{ for some } n \in N\} \subseteq [N, B]$. The rest is clear. ∎

Theorem 3.7 (Third isomorphism theorem) *If B is a subultra-group of ${}_H M$ and N is a normal subultra-group of ${}_H M$, then $\frac{B}{B \cap N} \cong \frac{[N,B]}{N}$.*

Proof. Since $[e]_\theta = B \cap N$ the proof is straightforward. ∎

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ON SEMI p-COVER-AVOIDING SUBGROUPS AND $\mathscr{H}$-SUBGROUPS OF FINITE GROUPS

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Abstract. In recent years, some authors gave many valuable results on the structure of a finite group, provided its some subgroups have semi p-cover-avoiding property. At the same time, some interesting conclusions about finite group were also obtained under the assumption that some subgroups of G belong to $\mathscr{H}(G)$. All these results are important in the research of finite groups. Here the authors discussed the connection between the semi p-cover-avoiding subgroups and the $\mathscr{H}$-subgroups on the structure of a finite group, and again obtained some more interesting results about finite groups.

Keywords: semi p-cover-avoiding subgroups, $\mathscr{H}$-subgroups, p-nilpotent, p-supersolvable.

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1. Introduction

It is well known that some properties of some subgroups of finite groups play important roles in the study of the structure of finite groups, especially various generalizations of normality were defined and used to make restrictions on various kinds of subgroups of a finite group, many useful results of finite group have been obtained under the assumption that some certain subgroups of G of prime power orders are well situated in G.

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A cover-avoiding subgroup was introduced in [1] in 1962, and recently, semi p-cover-avoiding subgroup was defined in [2]. It is worth to mention that a semi p-cover-avoiding subgroup is also known as a partial cover-avoiding subgroup, for example in [18]. Such kinds of subgroups have many interesting properties. For example, Gillam [3] and Tomkinson [4] have found some kind of subgroups of a finite sovable group G having the cover-avoiding properties. Fan, Guo and Shum [2] proved that the C-normal subgroup and the near normal subgroup are all semi cover-avoiding subgroups, and gave many meaningful results using the semi p-cover-avoiding subgroups, especially about the supersolvability and p-nilpotency of finite groups.

The concept of $\mathscr{H}$-subgroup was defined by Bianchi et al. in [5]. It is easy to prove by the definition that, the Sylow p-subgroups, the normal subgroups and the self-normalizing subgroups of an arbitrary group are all $\mathscr{H}$-subgroups. Many facts have proved that the $\mathscr{H}$-property of some subgroups can give a good insight into the structure of a group. For example, M. Asaad [6] studied the p-nilpotency and supersolvability of finite group under the assumption that some subgroups of G of prime power orders having $\mathscr{H}$-property.

Although " semi p-cover-avoiding subgroups " and " $\mathscr{H}$-subgroups " both can characterize the supersolvability and nilpotency of groups, there is no implications between them. In this paper, we will give two examples to explain. We study the p-supersolvability and p-nilpotency of a finite groups by assuming that a special kinds of subgroups of G are semi cover-avoiding subgroups or $\mathscr{H}$-subgroups. We got some more interesting results about the p-supersolvability and p-nilpotency of finite groups. Some well-known results in [6], [7] and [8] would be the direct corollaries of our results.

Throughout this paper, all groups considered are finite groups. The terminology and notations employed are standard and referred to Robinson [9]. For example, we denote by $F_p(G)$ the $p-$nilpotent radical, also known as the p-Fitting subgroup of a group G, which is equal to the product of all p-nilpotent normal subgroups of G, i.e. , $F_p(G) = O_{pp'}(G)$. $F(G)$ the Fitting subgroup of a group G (the product of all nilpotent normal subgroups of G).

2. Basic definitions and preliminaries

In this section, we first state some preliminaries in the literature which will be used in the sequel.

Let M and N be normal subgroups of a group G with $N \leq M$. Then the quotient group M/N is called a normal factor of G. Furthermore, if $p \mid |M/N|$, then M/N is called a p-singular. A subgroup H of G is said to cover M/N if $HM = HN$. And in the case of $H \cap M = H \cap N$, H is said to avoid M/N.

Definition 2.1 Let H be a subgroup of a group G. H is said to be semi p-cover-avoiding in G if there exists a chief series such that either H covers or avoids its every p-singular chief factor.

Definition 2.2 Let H be a subgroup of a group G. H is called an $\mathscr{H}$-subgroup of G if the following condition is satisfied:

$$N_G(H) \cap H^g \leq H, for\ all\ g \in G.$$

The set of all $\mathscr{H}$-subgroups of a group G will be denoted by $\mathscr{H}(G)$.

Note that there is no implications between the two concepts, our studies are meaningful:

Example 2.1 (a semi p-cover-avoiding subgroup but not a $\mathscr{H}$-subgroup) Let $G = S_4$ and $H = \langle(12)\rangle$. Easy to know $1 \trianglelefteq K_4 \trianglelefteq A_4 \trianglelefteq G$ is the unique chief series of G and H is semi 2-cover-avoiding in G. But there exists an element $g = (13)(24)$ in G such that $N_G(H) \cap H^g = \{(1), (34)\} \not\leq H$, hence H is not in $\mathscr{H}(G)$.

Example 2.2 (a $\mathscr{H}$-subgroup but not a semi p-cover-avoiding subgroup) Let $G = A_n (n \geq 5)$. Then G is a simple group and has only one chief series $1 \trianglelefteq G$. Take S be any Sylow p-subgroup of G. Then clearly S belongs to $\mathscr{H}(G)$, but not semi p-cover-avoiding in G.

Lemma 2.3 [2, Lemma 2] *Let H be a semi p-cover-avoiding subgroup of G, and N be a normal subgroup of G. Then HN/N is semi p-cover-avoiding in G/N if one of the following holds:*

(1) *$N \subseteq H$;*

(2) *$(|H|, |N|) = 1$, where $(-,-)$ denotes the greatest common divisor.*

Similar to [8, Lemma 2.5], we have:

Lemma 2.4 *Let H be a subgroup of a group G. If H is semi p-cover-avoiding in G, then H is semi p-cover-avoiding in K for every subgroup K of G with containing H.*

Lemma 2.5 [5, Lemma 2(1)] *If $L \leq H \leq G$ and $L \trianglelefteq G$, then $H \in \mathscr{H}(G)$ if and only if $H/L \in \mathscr{H}(G/L)$.*

Lemma 2.6 [5, Theorem 6(2)] *Let G be a group and $H \in \mathscr{H}(G)$. If H is subnormal in G, then $H \trianglelefteq G$.*

Lemma 2.7 [5, Lemma 7(2)] *Let G be a group and H, K be subgroups of G such that $H \leq K$. If $H \in \mathscr{H}(G)$, then $H \in \mathscr{H}(K)$.*

Lemma 2.8 [10, Lemma 6] *If $L \trianglelefteq G$, P is a p-subgroup of G which belongs to $\mathscr{H}(G)$, and $(|L|, |P|) = 1$, then $PL/L \in \mathscr{H}(G/L)$.*

Lemma 2.9 [11, Lemma 2.9] *Let P be an elementary abelian p-subgroup of G of order p^n, where $n \geq 2$. Then the following two statements are equivalent:*

(i) *The subgroups of order p in P are normal in G;*

(ii) *The maximal subgroups of P are normal in G.*

Lemma 2.10 [12, Theorem 3.1] *Let p be an odd prime divisor of the order of G with $(|G|, p-1) = 1$, $P \in Syl_p(G)$. If $N_G(P)$ is p-nilpotent, then G is p-nilpotent.*

Lemma 2.11 [13, Lemma 2.6] *Let G be a group and p a prime divisor of $|G|$ with $(|G|, p-1) = 1$. Suppose M is a subgroup of G with $|G : M| = p$, then M is normal in G.*

Lemma 2.12 [14, Lemma 2.6] *Let N be a solvable normal subgroup of a group G with $N \neq 1$. If every minimal normal subgroup of G which is contained in N is not contained in $\Phi(G)$, then the Fitting subgroup $F(N)$ of N is the direct product of minimal normal subgroups of G which are contained in N.*

Lemma 2.13 [7, Lemma 2.6] *Let G be a p-solvable group. Suppose that G has a chief series:*

$$1 \leq \cdots \leq \Phi(G) = H_0 \leq H_1 \cdots \leq H_s = F_p(G) \leq \cdots \leq G$$

such that H_i/H_{i-1} are cyclic groups of order p or p'-groups for all $1 \leq i \leq s$, then G is p-supersolvable, where $F_p(G)$ is the p-Fitting subgroup of G.

Lemma 2.14 *Let G be a p-solvable group and p a prime divisor of order G. If $F_p(G) = N_1 \times \cdots \times N_s$, where $N_i (i = 1, \cdots, s)$ are minimal normal subgroups of G of order p, then G is p-supersolvable.*

Proof. Let $K_0 = 1$ and $K_j = N_1 \times \cdots \times N_j (j = 1, \cdots, s)$. Then the following series is a chief series of G:

$$1 = \Phi(G) = K_0 \leq K_1 \leq \cdots \leq K_s = F_p(G) \leq \cdots \leq G$$

Since $|K_j/K_{j-1}| = p (j = 1, \cdots, s)$. By Lemma 2.13, G is p-supersolvable. ∎

Lemma 2.15 [9, Theorem 9.3.1] *If G is a $\pi-$separable group, then $C_G(O_{\pi'\pi}(G)/O_{\pi'}(G)) \leqslant O_{\pi'\pi}(G)$.*

3. On p-nilpotency of finite groups

Theorem 3.1 *Let G be a p-solvable group and p a prime divisor of $|G|$ such that $(|G|, p-1) = 1$. If every maximal subgroup of Sylow p-subgroup P of G is semi p-cover-avoiding in G or belongs to $\mathscr{H}(G)$, then G is p-nilpotent.*

Proof. Assume the theorem is false and let G be a counterexample of minimal order. Choose a minimal normal subgroup N of G, where N is a p-group or p'-group. We have the following conditions:

(1) N is a p-group and the unique minimal normal subgroup of G, $\Phi(G) = 1$, and G/N is p-nilpotent.

Consider the quotient group G/N. If N is a p'-group, then the maximal subgroup of the Sylow p-subgroup of G/N is the form P_0N/N, where P_0 is a maximal subgroup of P. By Lemma 2.3 and Lemma 2.8, the hypotheses are inherited by G/N. Thus G/N is p-nilpotent by the choice of G and it follows that G is p-nilpotent, a contradiction. Hence N is a p-group.

If $N = P$, then G/N is a p'-group and of course G/N is p-nilpotent. If $N < P$, the maximal subgroup of the Sylow p-subgroup of G/N is the form P_0/N, where P_0 is a maximal subgroup of P. Then by Lemma 2.3 and Lemma 2.5, G/N satisfies the hypotheses of the theorem. By the minimality of G, G/N is p-nilpotent.

Therefore, N is the unique minimal normal subgroup of G and $\Phi(G) = 1$, since the class of p-nilpotent groups forms a saturated formation.

(2) $|N| = p$.

Let P be a Sylow p-subgroup of G. If $N_G(P) < G$, then $N_G(P)$ is p-nilpotent by the minimality of G. Then it follows from Lemma 2.10 that G is p-nilpotent, a contradiction. This forces that $N_G(P) = G$ and then $P \trianglelefteq G$. By (1), $\Phi(P) = 1$ and hence P is elementary abelian.

By (1), $N \leq P$. Let $|P| = p^n$. If $n = 1$, we have done. Suppose $n \geq 2$. Let $\mathbb{F}_p$ be the field with p-elements and view P as a vector space of dimension n over $\mathbb{F}_p$. Obviously, the subgroups of order p in P are 1-dimensional subspaces and the maximal subgroups of P are $(n-1)$-dimensional subspaces. Given a subgroup $X = \langle x \rangle$ in N of order p. Consider the basis $B = \{x = x_1, x_2, ..., x_n\}$ of P as vector space. Denote M_i the $(n-1)$-dimensional subspace generated by $B \backslash \{x_i\} (i = 2, ..., n)$. Then it is easy to see that $X = \langle x \rangle = \bigcap_{i=2}^{n} M_i$.

By the hypotheses, $M_i (i = 2, ..., n)$ either is semi p-cover-avoiding in G or belongs to $\mathscr{H}(G)$. Without loss of generality, we assume that $M_2, ..., M_t$ are semi p-cover-avoiding in G; $M_{t+1}, \cdots, M_n$ belong to $\mathscr{H}(G)$.

Since N is the unique minimal normal subgroup of G, $M_iN{=}M_i$ or $M_i \cap N{=}1$ for $i = 2, ..., t$. But $x \in M_i \cap N$, hence $M_iN = M_i$ $(i = 2, ..., t)$. Then $N \leq M_i$ $(i = 2, ..., t)$. On the other hand, $M_j \trianglelefteq P \trianglelefteq G$ for $j = t+1, ..., n$. Then by Lemma 2.6, $M_j \trianglelefteq G$, where $j = t+1, ..., n$. Hence $N \cap M_j \trianglelefteq G$ for $j = t+1, ..., n$. By the minimality of N, $N \cap M_j = 1$ *or* $N (j = t+1, ..., n)$. But $x \in M_j \cap N$, hence $N \cap M_j = N$ $(j = t+1, ..., n)$. Then $N \leq M_j$ $(j = t+1, ..., n)$.

Now the above paragraph implies that $N \leq \bigcap_{i=2}^{n} M_i = \langle x \rangle$, hence $N = \langle x \rangle$ and it follows then $|N| = p$.

(3) The final contradiction.

By (1), G/N is p-nilpotent. Let H/N be a normal p-complement of G/N. Then $H = KN$, where K is a $p'-$group and $N \in Syl_p(H)$. Since $|N| = p$, $|H : K| = p$. It follows then $K \trianglelefteq H$ by Lemma 2.15. So K is a normal Hall p'-subgroup of H and hence $K \trianglelefteq G$. This implies that K is a normal p-complement of G and then G is p-nilpotent, the final contradiction. ∎

If we consider the smallest prime divisor of the order of G, we have the following results, actually, it is covered by Theorem 3.1.

Corollary 3.2 *Let G be a $p-$solvable group and p the smallest prime divisor of the order of G. If every maximal subgroup of Sylow p-subgroups of G is semi p-cover-avoiding in G or belongs to $\mathscr{H}(G)$, then G is p-nilpotent.*

Remark 3.3 By Theorem 3.1 and Corollary 3.2, the results [8, Theorem 3.2] and [6, Corollary 1.2] are immediate.

Actually, we can generalizes various results, for example, replace the condition "p is the smallest prime divisor of $|G|$" with "p is a prime divisor of $|G|$ such that $(|G|, p-1) = 1$" in [6, Corollary 1.2] and [8, Theorem 3.2].

Remark 3.4 The assumption that " $(|G|, p-1) = 1$ " or " p is the smallest prime " in the above Theorems and Corollaries could not be removed.

Example: let $G = S_3$ be a the symmetric group of degree 3 and choose $p = 3((|G|, 3-1) = 2 \neq 1)$. It is clear that the order of Sylow 3-subgroup P_3 of G is 3 and the maximal subgroup of P_3 is 1. Clearly, 1 is not only semi p-cover-avoiding in G but also belongs to $\mathscr{H}(G)$. However, G is not 3-nilpotent. ∎

Remark 3.5 The authors do not know if the hypothesis " G is p-solvable" can be removed or not. But It looks ok to be removed. If it is really removed then the minimal counter example may be a simple group of even order by assumption $(|G|, p-1) =$ and Theorem of Solvability of Finite Groups of Odd order, so we can use the Classification Theorem of Finite simple Groups to check if such kinds of simple counter examples go to contradictions. Surely it may be a long paper and we don't know if it can be realized.

4. On p-supersolvablity of finite groups

Theorem 4.1 *Let G be a p-solvable group and $P \in Syl_p(F_p(G))$, where p is a prime divisor of $|G|$. If every maximal subgroup of P is semi p-cover-avoiding in G or belongs to $\mathscr{H}(G)$, then G is p-supersolvable.*

Proof. Let G be a counterexample of minimal order.

Let $\mathscr{M}(P) = \{P_1, ..., P_m\}$ be the set of all the maximal subgroups of P. By the hypotheses, $P_i (i = 1, ..., m)$ either is semi p-cover-avoiding in G or belongs to $\mathscr{H}(G)$. Without loss of generality, let $1 \leq k \leq m$ such that $P_1, ..., P_k$ are semi p-cover-avoiding in G and $P_{k+1}, ..., P_m$ belong to $\mathscr{H}(G)$. Then

(1) $O_{p'}(G) = 1$

Consider the quotient group $G/O_{p'}(G)$. The fact $F_p(G/O_{p'}(G))$=$F_p(G)/O_{p'}(G)$ implies that $F_p(G/O_{p'}(G)) = PO_{p'}(G)/O_{p'}(G)$, therefore $F_p(G/O_{p'}(G))$ is a p-group. For any maximal subgroup $H/O_{p'}(G)$ of $F_p(G/O_{p'}(G)) = PO_{p'}(G)\ /O_{p'}(G)$, there exists a maximal subgroup P_0 of P such that $H/O_{p'}(G) = P_0O_{p'}(G)/\ O_{p'}(G)$. Hence $P_iO_{p'}(G)/O_{p'}(G)(i = 1, ..., m)$ are all maximal subgroups of $F_p(G/O_{p'}(G))$. By Lemma 2.3 and Lemma 2.8, $P_iO_{p'}(G)/O_{p'}(G)(i = 1, ..., k)$ are semi p-cover-avoiding in G and $P_iO_{p'}(G)/O_{p'}(G)(i = k + 1, ..., m)$ belong to $\mathscr{H}(G/O_{p'}(G))$. Hence $G/O_{p'}(G)$ satisfies the hypotheses of the theorem. The minimality of G implies that $O_{p'}(G) = 1$.

(2) $P = N_1 \times ... \times N_s$, where $N_1, ..., N_s$ are minimal normal subgroups of G.

Consider $G/\Phi(G)$. Since $G/\Phi(G)$ is a p-solvable group, $F_p(G/\Phi(G)) > 1$. Since $O_{p'}(G) = 1$, we have that $F_p(G) = P$. Clearly, $F_p(G/\Phi(G)) = F_p(G)/\Phi(G)$. By Lemma 2.3 and Lemma 2.5, $P_1/\Phi(G), ..., P_k/\Phi(G)$ are semi p-cover-avoiding in $G/\Phi(G)$ and $P_{k+1}/\Phi(G), ..., P_m/\Phi(G)$ belong to $\mathscr{H}(G/\Phi(G))$. It follows that $G/\Phi(G)$ satisfies the hypotheses of the theorem. The minimality of G implies that $\Phi(G) = 1$ and then $\Phi(P) = 1$. It follows then P is an elementary abelian p-group.

Let N be a minimal normal subgroup of G. Since G is p-solvable and $O_{p'}(G)$=1, $N \leq F_p(G) = O_p(G) = P$. By Lemma 2.14, the minimality of G implies that $|P| > p$. If $k = m$, then G is already p-supersolvable by [7, *Theorem* 3.1]. So now we assume $k < m$. Then $P_{k+1}, ..., P_m$ are normal in G by Lemma 2.6. Hence $N < F_p(G) = O_p(G) = P$.

By Lemma 2.12, there exist minimal normal subgroups $N_1, ..., N_s$ of G which are contained in $F(F_p(G)) = F_p(G) = O_p(G) = P$ such that$P = N_1 \times ... \times N_s$, $s > 1$.

(3) $|N_i| = p$, $i = 1, ..., s$.

We just need to prove $|N_1| = p$. Since $\Phi(G) = 1$, there exists a maximal subgroup M of G such that $G = N_1M$ and $M \cap N_1 = 1$. Let $R = N_2 \times ... \times N_s$. It is easy to see that $R \leq M$. Let P_0 be a maximal subgroup of P such that $R \leq P_0$. Then $|N_1/N_1 \cap P_0| = p$. By the hypotheses, P_0 is semi p-cover-avoiding in G or belongs to $\mathscr{H}(G)$.

If P_0 belongs to $\mathscr{H}(G)$, then by Lemma 2.6, $P_0 \trianglelefteq G$. Then $P_0 \cap N_1 = 1$ or $P_0 \cap N_1 = N_1$. If $P_0 \cap N_1 = N_1$, then $N_1 \leq P_0 = G_p^* \cap P \leq G_p^*$, a contradiction. Hence $P_0 \cap N_1 = 1$. It follows from $P = P_0N_1$ that $|N_1| = p$.

If P_0 is semi p-cover-avoiding in G, then it is clear that $|N_1| = p$ when P is cyclic. Now assume that P is not cyclic. Then P_0/R is semi p-cover-avoiding in G/R by Lemma 2.3. So there is a chief series of G/R:

$$1 = G_0/R < G_1/R < G_2/R < ... < G_s/R = G/R$$

such that P_0/R covers or avoids every p-singular chief factor of it. It is easy to see

that P_0 covers or avoids every p-singular chief factor of the following chief series of G:

$$1 < N_2 < N_2N_3 < ... < N_2N_3...N_s = R = G_0 < G_1 < ... < G_s = G$$

Since $P \nleqslant G_0 \cdot P_0$ but $P \leqslant G_s \cdot P_0 = G$, there is a chief factor G_i/G_{i-1} such that $P \leqslant G_i \cdot P_0$ but $P \nleqslant G_{i-1} \cdot P_0$. Also since P is a p-group, it is clear that G_i/G_{i-1} is a p-singular chief factor. Noticing that P_0 covers or avoids G_i/G_{i-1} and $G_i \cdot P_0 \neq G_{i-1} \cap P_0$, we have $G_i \cap P_0 = G_{i-1} \cap P_0$. Since $P = P \cap G_iP_0 = P_0(G_i \cap P)$, $G_i \cap P > G_0 = R$. It follows from $G_i \cap P \trianglelefteq G$ that $G_i \cap P = P$ and therefore $P_0 = G_i \cap P_0 = G_{i-1} \cap P_0$. On the other hand, $P > P \cap G_{i-1}P_0 = P_0(G_{i-1} \cap P)$. Thus $P \cap G_{i-1} = G_0 = M$ since $P \cap G_{i-1} \trianglelefteq G$, so $G_{i-1} \cap P_0 = R$. Hence $P_0 = R$ and therefore $|N_1| = p$.

Similarly, we can get $|N_i| = p$ for $i > 1$.

(4) The final contradiction.

By (3), we have got that $F_p(G) = N_1 \times ... \times N_s$, where $N_i (i = 1, ..., s)$ are minimal normal subgroups of G of order p. Then G is p-supersolvable by Lemma 2.14, the final contradiction. ∎

From Theorem 4.1, we immediate get the following corollaries:

Corollary 4.2 [7, Theorem 3.1] *Let p be a prime divisor of the order of a p-solvable group G and P a Sylow p-subgroups of the p-Fitting subgroup of G. If P is cyclic or every maximal subgroup of P is semi p- cover-avoiding in G, then G is p-supersolvable.*

Corollary 4.3 [15, Theorem 3.1] *Let G be a p-solvable group and P a Sylow p-subgroup of the p-Fitting subgroup $F_p(G)$ of G, where p is a prime dividing the order of G. If every maximal subgroup of P belongs to $\mathscr{H}(G)$, then G is p-supersolvable.*

Corollary 4.4 *Let H be a p-solvable normal subgroup of a group G such that G/H is p-supersolvable. If every maximal subgroup of Sylow p-subgroups of $F_p(H)$ is semi p-cover-avoiding in G or belongs to $\mathscr{H}(G)$, then G is p-supersolvable.*

Proof. Assume that the corollary is incorrect and let G be a counterexample with the smallest order.

Consider the quotient group $G/O_{p'}(H)$. By using the same arguments as in the proof of Theorem 4.1, we can see that $G/O_{p'}(H)$ satisfies the hypotheses of the corollary. The minimality of G implies that $O_{p'}(H) = 1$.

Similar to the proof of Theorem 4.1, we may assume that $\Phi(H) = 1$. Thus, $F(F_p(H)) = F_p(H) = O_p(H)$ is the direct product of minimal normal subgroups of G contained in H by Lemma 2.12. Let $F_p(H) = N_1 \times ... \times N_s$, where $N_i (i = 1, ..., s)$ are minimal normal subgroups of G contained in H. By the same arguments as in Theorem 4.1, we have that $N_i (i = 1, ..., s)$ are all cyclic groups of order p. Then we can assume that $F_p(H) = \langle a_1 \rangle \times \langle a_2 \rangle \times \cdots \times \langle a_s \rangle$, where $\langle a_i \rangle (i = 1, ..., s)$ are normal in G with prime order p.

Since $G/C_G(\langle a_i\rangle) \lesssim Aut(\langle a_i\rangle)$ and $\langle a_i\rangle$ is cyclic of order p, $G/C_G(\langle a_i\rangle)$ is cyclic and of course p-supersolvable for each i. It follows then $G/(\cap_{i=1}^s C_G(\langle a_i\rangle))$ is also p-supersolvable. Then $G/C_G(F_p(H))$ is p-supersolvable since $C_G(F_p(H)) = \cap_{i=1}^s C_G(\langle a_i\rangle)$. Now we see that $G/C_H(F_p(H)) = G/(H \cap C_G(F_p(H)))$ is p-supersolvable. Clearly $F_p(H) \leq C_H(F_p(H))$. On the other hand, $C_H(F_p(H)) = C_H(O_p(H)) \leq O_p(H) = F_p(H)$ by Lemma 2.15 since H is p-solvable and $O_{p'}(H)$=1. Hence $F_p(H) = C_H(F_p(H))$ and so $G/F_p(H)$ is p-supersolvable. Since $F_p(H) = \langle a_1\rangle \times \langle a_2\rangle \times \cdots \times \langle a_s\rangle$ is a direct product of cyclic normal subgroups of G, G is p-supersolvable, the final contradiction. ∎

Theorem 4.5 *Let $\mathcal{F}$ be a saturated formation containing the class of all supersolvable groups $\mathcal{U}$ and let H be a solvable normal subgroup of a group G such that $G/H \in \mathcal{F}$. If, for every prime p dividing the order of $F(H)$, every maximal subgroup of Sylow p-subgroups of $F(H)$ is semi p-cover-avoiding in G or belongs to $\mathscr{H}(G)$, then $G \in \mathcal{F}$.*

Proof. Suppose that the theorem is false and let G be a counterexample of minimal order. Then we have:

(1) $\Phi(H) = 1$.

Otherwise, let r be a prime divisor of $|\Phi(H)|$ and take $R \in Syl_r(\Phi(H))$. Easy to see $R \trianglelefteq G$ and $(G/R)/(H/R) \cong G/H \in \mathcal{F}$. Clearly, $R \leq F(H)$ and $F(H/R) = F(H)/R$.

It is easy to see that G/R always satisfies the hypotheses of the theorem respect to the normal subgroup H/R. The minimality of G implies that $G/R \in \mathcal{F}$. Since $G/\Phi(G) \cong (G/R)/(\Phi(G)/R) \in \mathcal{F}$ and $\mathcal{F}$ is a saturated formation, we see that $G \in \mathcal{F}$, a contradiction.

(2) $F(H) = \langle a_1\rangle \times \langle a_2\rangle \times \cdots \times \langle a_n\rangle$, where $\langle a_i\rangle (i = 1, ..., n)$ are normal in G with prime order. Furthermore, $G/F(H) \in \mathcal{F}$.

Let P be a Sylow p-subgroup of $F(H)$. Then $P \trianglelefteq G$. By $\Phi(H) = 1$ and Lemma 2.12, $P = N_1 \times N_2 \times \cdots \times N_s$, where N_i are all minimal normal in G. As in Theorem 4.1, we can see that every N_i is a cyclic group of order p. Then $F(H) = \langle a_1\rangle \times \langle a_2\rangle \times \cdots \times \langle a_n\rangle$, where $\langle a_i\rangle (i = 1, ..., n)$ are normal subgroups of prime orders in G.

Since $G/C_G(\langle a_i\rangle)$ is isomorphic to a subgroup of $Aut(\langle a_i\rangle)$, $G/C_G(\langle a_i\rangle)$ is cyclic and therefore it lies in $\mathcal{U}$ for each i. Thus $G/(\cap_{i=1}^n C_G(\langle a_i\rangle)) \in \mathcal{U}$. Since $C_G(F(H)) = \cap_{i=1}^n C_G(\langle a_i\rangle)$, $G/C_G(F(H)) \in \mathcal{U} \subseteq \mathcal{F}$. Then $G/C_H(F(H)) = G/(H \cap C_G(F(H))) \in \mathcal{F}$. Noticing that $F(H)$ is abelian, we have $F(H) \leq C_H(F(H))$. On the other hand, $C_H(F(H)) \leq F(H)$ since H is solvable. Hence $F(H) = C_H(F(H))$ and so $G/F(H) \in \mathcal{F}$.

(3) If N is a minimal normal subgroup of G which is contained in H, then $G/N \in \mathcal{F}$.

Suppose that N is a minimal normal subgroup of G which is contained in H. Since H is solvable, N is an elementary abelian p-group for some prime p and so $N \leq F(H)$. Now consider the solvable normal subgroup $F(H)/N$ in G/N. It is clear that $(G/N)/(F(H)/N) \simeq G/F(H) \in \mathcal{F}$ and $F(F(H)/N) = F(H)/N$.

So it is easy to see that G/N satisfies the hypotheses of our theorem respect to the solvable normal subgroup $F(H)/N$. By the minimality of G, we have that $G/N \in \mathcal{F}$.

(4) The final contradiction.

Since G is a counterexample of minimal order and take into account (2), we see that $F(H) = N$ is a unique minimal normal subgroup of G which is contained in H and $|F(H)| = |N| = p$. Suppose that K is a minimal normal subgroup of G such that $K \neq N$. Then $NK/K \simeq N$ is a solvable normal subgroup of G/K and $F(NK/K) = NK/K$ is a cyclic group. Thus G/K satisfies the hypotheses of our theorem for the normal subgroup KN/K. So by the minimality of G, we have $G/K \in \mathcal{F}$. It follows then $G \in \mathcal{F}$ since $K \neq N$, a contradiction. Hence $F(H) = N$ is the unique minimal normal subgroup of G.

If $N \leqslant \Phi(G)$, then $G/\Phi(G) \in \mathcal{F}$ by $G/N \in \mathcal{F}$, so that $G \in \mathcal{F}$, a contradiction. Otherwise, there exists a maximal subgroup M of G such that $N \nleq M$. Then $G = MN$ and $M \cap N = 1$. If $N < C_G(N)$, then $1 < C_G(N) \cap M$. It is clear that $C_G(N) \cap M \trianglelefteq G$. The unique minimal normality of N implies that $N \leq C_G(N) \cap M \leq M$, a contradiction. Hence $C_G(N) = N$, which yields that $G/N = G/C_G(N)$ is a cyclic group of order dividing $p-1$ and therefore $G \in \mathcal{U} \subseteq \mathcal{F}$ by noticing that N is of prime order, the final contradiction. ∎

Corollary 4.6 *Let H be a solvable normal subgroup of a group G such that G/H is supersolvable. If, for every prime p dividing the order of $F(H)$, every maximal subgroup of Sylow p-subgroups of $F(H)$ is semi p-cover-avoiding in G or belongs to $\mathscr{H}(G)$, then G is supersolvable, where $F(H)$ is the Fitting subgroup of H.* ∎

Finally, we get a sufficient condition of p-nilpotency by using Theorem 4.1 and limiting p be the smallest prime dividing the order of a group G.

Theorem 4.7 *Let p be the smallest prime divisor of the order of a p-solvable group G. If every maximal subgroup of Sylow p-subgroups of $F_p(G)$ is semi p-cover-avoiding in G or belongs to $\mathscr{H}(G)$, then G is p-nilpotent.*

Proof. Assume that the theorem is false and let G be a counterexample with the minimal order. Then, by the arguments used in the proof of Theorem 4.1, it is easy to see that $O_{p'}(G) = 1$, $\Phi(G) = 1$ and $F_p(G) = N_1 \times ... \times N_s$, where $N_i (i = 1, ..., s)$ are minimal normal subgroups of G of order p.

For any N_i, $G/C_G(N_i)$ is a subgroup of a cyclic group of order $p - 1$. The minimality of p implies that $C_G(N_i) = G$, hence $G = C_G(O_p(G))$. On the other hand, by Lemma 2.15, $C_G(O_p(G)) \leq O_p(G)$. It follows that $G = O_p(G)$ is p-nilpotent, a contradiction. The proof of the theorem is complete. ∎

From Theorem 4.7, we immediately get the following results:

Corollary 4.8 [7, Theorem 3.3] *Let p be the smallest prime dividing the order of a p-solvable group G. If Sylow p-subgroup of $F_p(G)$ are cyclic or every maximal subgroup of Sylow p-subgroups of $F_p(G)$ is semi p- cover-avoiding in G, then G is p-nilpotent.*

Corollary 4.9 [15, Theorem 3.3] *Let p be the smallest prime divisor of the order of a p-solvable group G. If every maximal subgroup of Sylow p-subgroups of $F_p(G)$ belongs to $\mathscr{H}(G)$, then G is p-nilpotent.*

Corollary 4.10 *Let p be the smallest prime divisor of the order of a group G and H be a p-solvable normal subgroup of G such that G/H is p-nilpotent. If every maximal subgroup of Sylow p-subgroups of $F_p(H)$ is semi p-cover-avoiding in G or belongs to $\mathscr{H}(G)$, then G is p-nilpotent, where $F_p(H)$ is the p-Fitting subgroup of H.*

Proof. Since the proof is similar to the ones of Corollary 4.4 and Theorem 4.7, we omit the proof. ∎

Remark 4.11 The assumption that p is the smallest prime divisor of the order of G in Theorem 4.7 and Corollary 4.8, 4.9, 4.10 could not be removed.

Example: let $G = S_4$, the symmetric group of degree 4 and $p = 3$. It is clear that $F_p(G) = A_4$ and the maximal subgroup 1 of Sylow 3-subgroups of $F_p(G)$ is not only semi p-cover-avoiding in G but also belongs to $\mathscr{H}(G)$. However, G is not 3-nilpotent. ∎

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DYNAMICS IN A COMPETITIVE LOTKA-VOLTERRA PREDATOR-PREY MODEL WITH MULTIPLE DELAYS

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Abstract. In this paper, we investigate a discrete competitive Lotka-Volterra predator-prey model with multiple delays. For general non-autonomous case, sufficient conditions which ensure the permanence and the global stability of the system are derived by applying the differential inequality theory; For periodic case, sufficient conditions which guarantee the existence of an unique globally stable positive periodic solution are established. Some numerical simulations which illustrate our theoretical findings are carried out.

Keywords: competitive Lotka-Volterra predator-prey model; permanence; global attractivity; multiple delays.

Mathematics Subject Classification (2000): 34K20; 34C25; 92D25.

1. Introduction

Since the work of Berryman [1], the research on dynamical behavior of predator-prey models has become one of the dominant themes in both ecology and mathematical ecology. In particular, the stability of the equilibrium, the permanence and extinction of species, the existence of periodic solutions and positive almost periodic solutions, bifurcation and chaos of predator-prey models have been investigated in a number of notable studies [2]-[26]. In many applications, the nature of permanence is of great interest. For example, Fan and Li [27] made a theoretical discussion on the permanence of a cooperative scalar population models with

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delays, Wang and Huang [28] addressed the permanence of a stage-structured predator-prey system with impulsive stocking prey and harvesting predator, Liu et al. [29] analyzed the permanence and global attractivity of an impulsive ratio-dependent predator-prey system in a patchy environment. Zhao and Teng et al. [30] established the permanence criteria for a delayed discrete nonautonomous-species Kolmogorov systems. For more details, we refer to [31]-[40].

In 2008, Qiu and Cao [41] investigated the exponential stability of the following competitive Lotka-Volterra predator-prey system with delays

$$\begin{cases} \dot{x}(t) = x(t)[b_1 - a_{11}x(t-\tau_{11}) - a_{12}y(t-\tau_{12}], \\ \dot{y}(t) = y(t)[b_2 - a_{21}x(t-\tau_{21}) - a_{22}y(t-\tau_{22})], \end{cases} \tag{1.1}$$

where $x(t)$ and $y(t)$ denotes the density of population at time t, respectively, $b_i, a_{ij},\ i,j = 1,2$, and $\tau_{ij}, i,j = 1,2$ are positive constants. Using the linear matrix inequality(LMI) optimization approach and constructing suitable Lyapunov function, Qiu and Cao [41] obtained a set of easily verifiable sufficient conditions which guarantee the exponential stability of the positive equilibrium of system (1.1). Considering that the environment fluctuates randomly, we think that it is more reasonable to consider varying parameters of predator-prey systems. Inspired by the viewpoint, we modify system (1.1) as follows.

$$\begin{cases} \dot{x}(t) = x(t)[b_1(t) - a_{11}(t)x(t-\tau_{11}(t)) - a_{12}(t)y(t-\tau_{12}(t)], \\ \dot{y}(t) = y(t)[b_2(t) - a_{21}(t)x(t-\tau_{21}(t)) - a_{22}(t)y(t-\tau_{22}(t))]. \end{cases} \tag{1.2}$$

Here we shall point out that discrete time models governed by difference equations are more appropriate to describe the dynamics relationship among populations than continuous ones when the populations have non-overlapping generations. Moreover, discrete time models can also provide efficient models of continuous ones for numerical simulations [4], [16], [42]. Thus it is reasonable and interesting to investigate discrete time systems governed by difference equations. The principle object of this article is to propose a discrete analogue system (1.2) and explore its dynamics.

Following the ideas of [4], [11], we will discretize the system (1.2). Assume that the average growth rates in system (1.2) change at regular intervals of time, then we can obtain the following modified system:

$$\begin{cases} \dfrac{\dot{x}(t)}{x(t)} = b_1([t]) - a_{11}([t])x([t]-\tau_{11}([t])) - a_{12}([t])y([t]-\tau_{12}([t])), \\ \dfrac{\dot{y}(t)}{y(t)} = b_2([t]) - a_{21}([t])x([t]-\tau_{21}([t])) - a_{22}([t])y([t]-\tau_{22}([t])), \end{cases} \tag{1.3}$$

where $[t]$ denotes the integer part of $t, t \in (0,+\infty)$ and $t \neq 0,1,2,....$ We integrate (1.3) on any interval of the form $[k,k+1), k = 0,1,2,...,$ and obtain

$$\begin{cases} x(t) = x(k)\exp\{b_1(k) - a_{11}(k)x(k-\tau_{11}(k)) - a_{12}(k)y(k-\tau_{12}(k)(t-k)\}, \\ y(t) = y(k)\exp\{b_2(k) - a_{21}(k)x(k-\tau_{21}(k)) - a_{22}(k)y(k-\tau_{22}(k))(t-k)\}, \end{cases} \tag{1.4}$$

where for $k \leq t < k+1, k = 0, 1, 2, ...$ Let $t \to k+1$, then (1.4) takes the following form:

$$(1.5)\quad \begin{cases} x(k+1) = x(k)\exp\{b_1(k) - a_{11}(k)x(k-\tau_{11}(k)) - a_{12}(k)y(k-\tau_{12}(k)\}, \\ y(k+1) = y(k)\exp\{b_2(k) - a_{21}(k)x(k-\tau_{21}(k)) - a_{22}(k)y(k-\tau_{22}(k))\}, \end{cases}$$

which is a discrete time analogue of system (1.2), where $k = 0, 1, 2, ...$ For the point of view of biology, we shall consider (1.5) together with the initial conditions $x(0) \geq 0, y(0) \geq 0$. The principle object of this article is to explore the dynamics of system (1.5). Applying the differential inequality theory and the method of Lyapunov function, we investigate the permanence and the globally asymptotically stability of system (1.5).

Throughout this paper, we assume that

(H1) b_i, a_{ij}, τ_{ij} with $i, j = 1, 2$ are non-negative sequences bounded above and below by positive constants.

Let $\tau = \sup_{1\leq i,j\leq 2, k\in Z}\{\tau_{ij}(k)\}$. We consider (1.5) together with the following initial conditions

$$(1.6)\quad x(\theta) = \varphi(\theta) \geq 0, y(\theta) = \psi(\theta) \geq 0, \theta \in N[-\tau, 0] = \{-\tau, -\tau+1, ..., 0\}, \quad \varphi(0) > 0, \psi(0) > 0.$$

Obviously, the solutions of (1.5) and (1.6) are well defined for all $k \geq 0$ and satisfy

$$x(k) > 0, y(k) > 0, \text{for } k \in Z.$$

The remainder of the paper is organized as follows: in Section 2, we introduce some basic definitions and lemmas, some sufficient conditions for the permanence of system (1.5) are obtained. In Section 3, some sufficient conditions for the global stability of system (1.5) are established. The existence and stability of system (1.5) are analyzed in Section 4. In Section 5, an example which shows the feasibility of the main results is given. A brief conclusion is drawn in Section 6.

2. Permanence

For convenience, we always use the notations:

$$f^l = \inf_{k\in Z} f(k), \ f^u = \sup_{k\in Z} f(k),$$

where $f(k)$ is a non-negative sequence bounded above and below by positive constants. In order to obtain the main result of this paper, we shall first state the definition of permanence and several lemmas which will be useful in the proof of the main result.

Definition 2.1 [43] We say that system (1.5) is permanent if there are positive constants M_i, m_i $(i = 1, 2)$ such that for each positive solution $(x(k), y(k))$ of system (1.5) satisfies

$$m_1 \leq \lim_{k\to+\infty} \inf x(k) \leq \lim_{k\to+\infty} \sup x(k) \leq M_1,$$

$$m_2 \leq \lim_{k\to+\infty} \inf y(k) \leq \lim_{k\to+\infty} \sup y(k) \leq M_2.$$

Lemma 2.1 [43] *Assume that* $\{x(k)\}$ *satisfies* $x(k) > 0$ *and*

$$x(k+1) \leq x(k) \exp\{a(k) - b(k)x(k)\}$$

for $k \in N$, *where* $a(k)$ *and* $b(k)$ *are non-negative sequences bounded above and below by positive constants. Then*

$$\lim_{k\to+\infty} \sup x(k) \leq \frac{1}{b^l} \exp(a^u - 1).$$

Lemma 2.2 [43] *Assume that* $\{x(k)\}$ *satisfies*

$$x(k+1) \geq x(k) \exp\{a(k) - b(k)x(k)\}, \ k \geq N_0,$$

$\lim_{k\to+\infty} \sup x(k) \leq x^*$ *and* $x(N_0) > 0$, *where* $a(k)$ *and* $b(k)$ *are non-negative sequences bounded above and below by positive constants and* $N_0 \in N$. *Then*

$$\lim_{k\to+\infty} \inf x(k) \geq \min\left\{\frac{a^l}{b^u} \exp\{a^l - b^u x^*\}, \frac{a^l}{b^u}\right\}.$$

Now we state our permanence result for system (1.5).

Theorem 2.1 *Let* M_1, M_2, m_1 *and* m_2 *be defined by* (2.4), (2.9), (2.15) *and* (2.20), *respectively. In addition to condition* (H1), *assume that the following condition*

$$\text{(H2)} \ b_1^l > a_{12}^u M_2, b_2^l > a_{21}^u M_1$$

holds, then system (1.5) *is permanent, that is, there exist positive constants* m_i, M_i $(i = 1, 2)$ *which are independent of the solution of system* (1.5), *such that for any positive solution* $(x(k), y(k))$ *of system* (1.5) *with the initial condition* $x(0) \geq 0$, $y(0) \geq 0$, *one has*

$$m_1 \leq \lim_{k\to+\infty} \inf x(k) \leq \lim_{k\to+\infty} \sup x(k) \leq M_1,$$

$$m_2 \leq \lim_{k\to+\infty} \inf y(k) \leq \lim_{k\to+\infty} \sup y(k) \leq M_2.$$

Proof. Let $(x_1(k), x_2(k))$ be any positive solution of system (1.5) with the initial condition $(x(0), y(0))$. It follows from the first equation of system (1.5) that

$$\begin{aligned} x(k+1) &= x(k) \exp\{b_1(k) - a_{11}(k)x(k - \tau_{11}(k)) - a_{12}(k)y(k - \tau_{12}(k)\} \\ &\leq x(k) \exp\{b_1(k)\} \leq x(k) \exp\{b_1^u\}. \end{aligned} \tag{2.1}$$

It follows from (2.1) that

$$x(k-\tau_{11}(k)) \geq x(k)\exp\{-b_1^u\tau^u\}. \tag{2.2}$$

Substituting (2.2) into the first equation of system (1.5), it follows that

$$x(k+1) \leq x(k)[b_1^u - a_{11}^l \exp\{-b_1^u\tau^u\}x(k)]. \tag{2.3}$$

It follows from (2.3) and Lemma 2.1 that

$$\lim_{k\to+\infty}\sup x(k) \leq \frac{1}{a_{11}^l}\exp\{b_1^u\tau^u + b_1^u - 1\} := M_1. \tag{2.4}$$

For any positive constant $\varepsilon > 0$, it follows from (2.4) that there exists a $N_1 > 0$ such that for all $k > N_1$

$$x(k) \leq M_1 + \varepsilon. \tag{2.5}$$

By (1.5), we have

$$\begin{aligned} y(k+1) &= y(k)\exp\{b_2(k) - a_{21}(k)x(k-\tau_{21}(k)) - a_{22}(k)y(k-\tau_{22}(k))\} \\ &\leq y(k)\exp\{b_2(k)\} \leq y(k)\exp\{b_2^u\}, \end{aligned} \tag{2.6}$$

which leads to

$$y(k-\tau_{22}(k)) \geq y(k)\exp\{-b_2^u\tau^u\}. \tag{2.7}$$

Substituting (2.7) into the second equation of system (1.5), we have

$$\begin{aligned} y(k+1) &= y(k)\exp\{b_2(k) - a_{21}(k)x(k-\tau_{21}(k)) - a_{22}(k)y(k-\tau_{22}(k))\} \\ &\leq y(k)[b_2^u - a_{22}^l\exp\{-b_2^u\tau^u\}y(k)]. \end{aligned} \tag{2.8}$$

Thus it follows from Lemma 2.1 and (2.8) that

$$\lim_{k\to+\infty}\sup y(k) \leq \frac{1}{a_{22}^l}\exp\{b_2^u\tau^u + b_2^u - 1\} := M_2. \tag{2.9}$$

For any positive constant $\varepsilon > 0$, it follows from (2.9) that there exists a $N_2 > 0$ such that for all $k > N_2$

$$y(k) \leq M_2 + \varepsilon. \tag{2.10}$$

For $k \geq \max\{N_1, N_2\} + \tau^u$, it follows from the first equation of system (1.5) that

$$\begin{aligned} x(k+1) &= x(k)[b_1(k) - a_{11}(k)x(k-\tau_{11}(k)) - a_{12}(k)y(k-\tau_{12}(k))] \\ &\geq x(k)[b_1^l - a_{11}^u(M_1+\varepsilon) - a_{12}^u(M_2+\varepsilon)], \end{aligned} \tag{2.11}$$

which leads to

$$x(k-\tau_{11}(k)) \leq x(k)\exp\{-[b_1^l - a_{11}^u(M_1+\varepsilon) - a_{12}^u(M_2+\varepsilon)]\tau^u\}. \tag{2.12}$$

Substituting (2.12) into the first equation of system (1.5), it follows that

$$\begin{aligned} x(k+1) &\geq x(k)\{b_1^l - a_{12}^u(M_2+\varepsilon) \\ &\quad -a_{11}^u \exp\{-[b_1^l - a_{11}^u(M_1+\varepsilon) - a_{12}^u(M_2+\varepsilon)]\tau^u\}x(k)\}. \end{aligned} \tag{2.13}$$

According to Lemma 2.2, it follows from (2.13) that

$$\lim_{k\to+\infty} \inf x(k) \geq \min\{A_{1\varepsilon}, A_{2\varepsilon}\}, \tag{2.14}$$

where

$$\begin{aligned} A_{1\varepsilon} &= \frac{b_1^l - a_{12}^u(M_2+\varepsilon)}{a_{11}^u \exp\{-[b_1^l - a_{11}^u(M_1+\varepsilon) - a_{12}^u(M_2+\varepsilon)]\tau^u\}} \\ &\quad \times \exp\Big\{b_1^l - a_{12}^u(M_2+\varepsilon) - a_{11}^u \exp\{-[b_1^l - a_{11}^u(M_1+\varepsilon) \\ &\quad -a_{12}^u(M_2+\varepsilon)]\tau^u\}\Big\}M_1, \\ A_{2\varepsilon} &= \frac{b_1^l - a_{12}^u(M_2+\varepsilon)}{a_{11}^u \exp\{-[b_1^l - a_{11}^u(M_1+\varepsilon) - a_{12}^u(M_2+\varepsilon)]\tau^u\}}. \end{aligned}$$

Setting $\varepsilon \to 0$ in (2.14), we can get

$$\lim_{k\to+\infty} \inf x(k) \geq \frac{1}{2}\min\{A_1, A_2\} := m_1, \tag{2.15}$$

where

$$\begin{aligned} A_1 &= \frac{b_1^l - a_{12}^u M_2}{a_{11}^u \exp\{-[b_1^l - a_{11}^u M_1 - a_{12}^u M_2]\tau^u\}} \\ &\quad \times \exp\Big\{b_1^l - a_{12}^u M_2 - a_{11}^u \exp\{-[b_1^l - a_{11}^u M_1 - a_{12}^u M_2]\tau^u\}\Big\}M_1, \\ A_2 &= \frac{b_1^l - a_{12}^u M_2}{a_{11}^u \exp\{-[b_1^l - a_{11}^u M_1 - a_{12}^u M_2]\tau^u\}}. \end{aligned}$$

For $k \geq \max\{N_1, N_2\} + \tau^u$, from the second equation of system (1.5), we have

$$\begin{aligned} y(k+1) &= y(k)[b_2(k) - a_{21}(k)x(k-\tau_{21}(k)) - a_{22}y(k-\tau_{22}(k)] \\ &\geq y(t)[b_2^l - a_{21}^u(M_1+\varepsilon) - a_{22}^u(M_2+\varepsilon)], \end{aligned} \tag{2.16}$$

which leads to

$$y(k-\tau_{22}(k)) \leq y(k)\exp\{b_2^l - a_{21}^u(M_1+\varepsilon) - a_{22}^u(M_2+\varepsilon)]\tau^u\}. \tag{2.17}$$

Substituting (2.17) into the second equation of system (1.5), it follows that

$$\begin{aligned} y(k+1) &\geq y(k)\{b_2^l - a_{22}^u \exp\{[b_2^l - a_{21}^u(M_1+\varepsilon) - a_{22}^u(M_2+\varepsilon)]\tau^u\}y(k) \\ &\quad -a_{21}^u(M_1+\varepsilon)\}. \end{aligned} \tag{2.18}$$

By Lemma 2.2 and (2.18), we can get

$$\lim_{k\to+\infty}\inf y(k) \geq \min\{B_{1\varepsilon}, B_{2\varepsilon}\}, \tag{2.19}$$

where

$$\begin{aligned} B_{1\varepsilon} &= \frac{b_2^l - a_{21}^u(M_1+\varepsilon)}{a_{22}^u \exp\{-[b_2^l - a_{21}^u(M_1+\varepsilon) - a_{22}^u(M_2+\varepsilon)]\tau^u\}} \exp\Big\{b_2^l - a_{21}^u(M_1+\varepsilon) \\ &\quad -a_{22}^u \exp\{-[b_2^l - a_{21}^u(M_1+\varepsilon) - a_{22}^u(M_2+\varepsilon)]\tau^u\}\Big\}M_2, \\ B_{2\varepsilon} &= \frac{b_2^u - a_{21}^u(M_1+\varepsilon)}{a_{22}^u \exp\{[b_2^l - a_{21}^l(M_1+\varepsilon) - a_{22}^u(M_2+\varepsilon)]\tau^u\}}. \end{aligned}$$

Setting $\varepsilon \to 0$ in above inequality, it follows that

$$\lim_{k\to+\infty}\inf y(k) \geq \frac{1}{2}\min\{B_1, B_2\} := m_2, \tag{2.20}$$

where

$$\begin{aligned} B_1 &= \frac{b_2^l - a_{21}^u M_1}{a_{22}^u \exp\{-[b_2^l - a_{21}^u M_1 - a_{22}^u M_2]\tau^u\}} \\ &\quad \times \exp\Big\{b_2^l - a_{21}^u M_1 - a_{22}^u \exp\{-[b_2^l - a_{21}^u M_1 - a_{22}^u M_2]\tau^u\}\Big\}M_2, \\ B_2 &= \frac{b_2^u - a_{21}^u M_1}{a_{22}^u \exp\{[b_2^l - a_{21}^l M_1 - a_{22}^u M_2]\tau^u\}}. \end{aligned}$$

By (2.4), (2.9), (2.15) and (2.20), we can conclude that system (1.5) is permanent. The proof of Theorem 2.1 is complete. ∎

Remark 2.1. Under the assumption of Theorem 2.1, the set $[m_1, M_1]\times[m_2, M_2]$ is an invariant set of system (1.5).

3. Global stability

In this section, we formulate the stability property of positive solutions of system (1.5) when all the time delays are zero.

Theorem 3.1 *Let $\tau_{ij} = 0 (i, j = 1, 2)$. In addition to* (H1)–(H2)*, assume further that* (H3)

$$\begin{aligned} \Theta_1 &= \max\{|1 - a_{11}^u M_1|, |1 - a_{11}^l m_1|\} + a_{12}^u M_2 < 1, \\ \Theta_2 &= \max\{|1 - a_{22}^u M_2|, |1 - a_{22}^l m_3|\} + a_{21}^u M_1 < 1. \end{aligned}$$

Then for any positive solutions $(x(k), y(k))$ and $(x^*(k), y^*(k))$ of system (1.5)*, we have*

$$\lim_{k\to\infty}[x^*(k) - x(k)] = 0, \lim_{k\to\infty}[y^*(k) - y(k)] = 0.$$

Proof. Let

$$x(k) = x^*(k)\exp(u(k)), y(k) = y^*(k)\exp(v(k)). \tag{3.1}$$

Then system (1.5) is equivalent to

$$\begin{cases} u(k+1) = u(k) - a_{11}(k)x^*(k)(\exp(u(k)) - 1) \\ \qquad\qquad -a_{12}(k)y^*(k)(\exp(v(k)) - 1) \\ v(k+1) = v(k) - a_{21}(k)x^*(k)(\exp(u(k)) - 1), \\ \qquad\qquad -a_{22}(k)y^*(k)(\exp(v(k)) - 1). \end{cases} \tag{3.2}$$

Then

$$\begin{cases} u(k+1) = u(k) - a_{11}(k)x^*(k)(\exp(\theta_{11}(k)u(k))u(k) \\ \qquad\qquad -a_{12}(k)y^*(k)(\exp(\theta_{12}(k)y(k))y(k) \\ v(k+1) = v(k) - a_{21}(k)x^*(k)(\exp(\theta_{21}(k)u(k))u(k), \\ \qquad\qquad -a_{22}(k)y^*(k)(\exp(\theta_{22}(k)v(k))v(k), \end{cases} \tag{3.3}$$

where $\theta_{ij}(k) \in [0,1](i,j = 1,2)$. To complete the proof, it suffices to show that

$$\lim_{k\to+\infty} u(k) = 0, \lim_{k\to+\infty} v(k) = 0. \tag{3.4}$$

In view of (H3), we can choose $\varepsilon > 0$ small enough such that

$$\Theta_1^\varepsilon = \max\{|1 - a_{11}^u(M_1+\varepsilon)|, |1 - a_{11}^l(m_1-\varepsilon)|\} + a_{12}^u(M_2+\varepsilon) < 1, \tag{3.5}$$

$$\Theta_2^\varepsilon = \max\{|1 - a_{22}^u(M_2+\varepsilon)|, |1 - a_{22}^l(m_2-\varepsilon)|\} + a_{21}^u(M_1+\varepsilon) < 1. \tag{3.6}$$

For above $\varepsilon > 0$, in view of Theorem 2.1 in Section 2, there exists a $k^* \in N$ such that

$$m_1 - \varepsilon \le x^*(k) \le M_1 + \varepsilon, m_2 - \varepsilon \le y^*(k) \le M_2 + \varepsilon \text{ for all } k \ge k^*.$$

Noticing that $\theta_{ij}(k) \in [0,1]$ $(i,j = 1,2)$ implies that $x^*(k)\exp(\theta_{ij}(k)u(k))$ lies between $x^*(k)$ and $x(k)$ and $y^*(k)\exp(\theta_{ij}(k)v(k))$ lies between $y^*(k)$ and $y(k)$. From (3.3), we have

$$\begin{aligned} u(k+1) \le\ & \max\{|1 - a_{11}^u(M_1+\varepsilon)|, |1 - a_{11}^l(m_1-\varepsilon)|\}|u(k)| \\ & + a_{12}^u(M_2+\varepsilon)|v(k)|, \end{aligned} \tag{3.7}$$

$$\begin{aligned} v(k+1) \le\ & \max\{|1 - a_{22}^u(M_2+\varepsilon)|, |1 - a_{22}^l(m_3-\varepsilon)|\}|v(k)| \\ & + a_{21}^u(M_1+\varepsilon)|u(k)|. \end{aligned} \tag{3.8}$$

Let $\rho = \max\{\Theta_1^\varepsilon, \Theta_2^\varepsilon\}$, then $\rho < 1$. By (3.7) and (3.8), for $k \ge k^*$, we have

$$\max\{|u(k+1)|, |v(k+1)|\} \le \rho \max\{|u(k)|, |v(k)|\},$$

which implies

$$\max\{|u(k)|, |v(k)|\} \leq \rho^{k-k^*} \max\{|u(k^*)|, |v(k^*)|\}.$$

Thus (3.4) holds true and the proof is completed. ∎

4. Existence and stability of periodic solution

In this section, we further assume that $\tau_{ij} = 0 (i, j = 1, 2)$ and the coefficients of system (1.5) satisfy the following condition

(H4) There exists a positive integer ω such that for $k \in N$, $0 < b_i(k+\omega) = b_i(k)$, $0 < a_{ij}(k+\omega) = a_{ij}(k)$ $(i, j = 1, 2)$.

Theorem 4.1 *Assume that* (H1)–(H4) *are satisfied, then system* (1.5) *with all the delays* $\tau_{ij} = 0$ $(i, j = 1, 2)$ *admits at least one positive* ω*-periodic solution which we denote by* $(x^*(k), y^*(k))$.

Proof. As pointed out in Remark 2.1 of Section 2,

$$D^2 \stackrel{\text{def}}{=} [m_1, M_1] \times [m_2, M_2]$$

is an invariant set of system (1.5). Then we define a mapping F on D^2 by

$$F(x_1(0), x_2(0)) = (x_1(\omega), x_2(\omega)), \text{for } (x_1(0), x_2(0)) \in D^2.$$

Clearly, F depends continuously on $(x_1(0), x_2(0))$. Thus F is continuous and maps the compact set D^2 into itself. Therefore , F has a fixed point. It is not difficult to see that the solution $(x_1^*(k), x_2^*(k))$ passing through this fixed point is an ω-periodic solution of the system (1.5). The proof of Theorem 4.1 is complete. ∎

Theorem 4.2 *Assume that* (H1)–(H4) *are satisfied, then system (1.5) with all the delays* $\tau_{ij} = 0 (i, j = 1, 2)$ *has a global stable positive* ω*-periodic solution.*

Proof. Under the assumptions (H1)–(H4), it follows from Theorem 4.1 that system (1.5) with all the delays $\tau_{ij} = 0 (i, j = 1, 2)$ admits at least one positive ω-periodic solution. In addition, Theorem 3.1 ensures the positive solution to be globally stable. Hence the proof. ∎

5. Numerical example

In this section, we will give an example which shows the feasibility of the main results (Theorem 2.1) of this paper. Let us consider the following discrete system:

$$(5.1)\quad \begin{cases} x(k+1) = x(k) \exp\left\{ \left(2 + 0.03 \sin \dfrac{k\pi}{2}\right) - \left(0.4 + 0.01 \sin \dfrac{k\pi}{2}\right) x(k-1) \right. \\ \qquad\qquad \left. - \left(0.05 + 0.02 \cos \dfrac{k\pi}{2}\right) y(k-1) \right\}, \\ y(k+1) = y(k) \exp\left\{ \left(3 + 0.05 \cos \dfrac{k\pi}{2}\right) - \left(0.5 + 0.2 \sin \dfrac{k\pi}{2}\right) x(k-1) \right. \\ \qquad\qquad \left. - \left(0.4 + 0.02 \sin \dfrac{k\pi}{2}\right) y(k-1) \right\}. \end{cases}$$

Here $b_1(k) = 2 + 0.03\sin\frac{k\pi}{2}$, $b_2(t) = 3 + 0.05\cos\frac{k\pi}{2}$, $a_{11}(k) = 0.4 + 0.01\sin\frac{k\pi}{2}$, $a_{12}(k) = 0.05 + 0.02\cos\frac{k\pi}{2}$, $a_{21}(k) = 0.5 + 0.2\sin\frac{k\pi}{2}$, $a_{22}(k) = 0.4 + 0.02\sin\frac{k\pi}{2}$, $\tau_{ij}(k) = 1$ $(i, j = 1, 2)$. All the coefficients $b_i(k)$ $(i = 1, 2)$, $a_{ij}(k)$ $(i, j = 1, 2)$, $\tau_{ij}(k)$ $(i, j = 1, 2)$ are functions with respect to k, and it is not difficult to obtain that $b_1^u = 2.03$, $b_1^l = 1.97$, $b_2^u = 3.05$, $b_2^l = 2.95$, $a_{11}^u = 0.401$, $a_{11}^l = 0.39$, $a_{12}^u = 0.07$, $a_{12}^l = 0.03$, $a_{21}^u = 0.7$, $a_{21}^l = 0.3$, $a_{22}^u = 0.42$, $a_{22}^l = 0.38$, $\tau_{ij}^u = \tau_{ij}^l = 1$ $(i, j = 1, 2)$. It is easy to check that all the conditions of Theorem 2.1 are fulfilled. Thus system (5.1) is permanence which is shown in Figure 1.

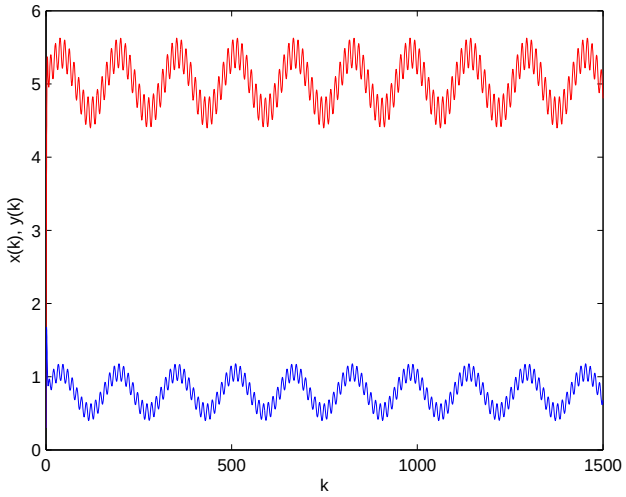

Figure 1: The dynamical behavior of the first the second components of the solution $(x(k), y(k))$ system (5.1), where the red line stands for $x(k)$ and the blue line stands for $y(k)$.

6. Conclusions

In this paper, we have considered the dynamical behavior of a discrete competitive Lotka-Volterra predator-prey model with multiple delays. A set of sufficient conditions which ensure the permanence of the model are obtained. Moreover, we also discuss the global stability of the model with all the delays $\tau_{ij}(k) = 0$ $(i, j = 1, 2)$ and deal with the existence and stability of the system. It is showed that delay has important influence on the permanence of system. Therefore, delay plays an key role in the permanence of the model. When all the delays are zero, we establish some sufficient conditions which guarantee the global stability of the model. Some simulations are presented to illustrate our main theoretical findings.

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SEPARATION AXIOMS BETWEEN T_0 AND T_1 ON LATTICES AND LATTICE MODULES

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Abstract. In this study, we characterize the lattices and lattice modules whose prime spectrum satisfy some of the separation axioms between T_0 and T_1. This characterizations are the notion of $pm-$lattice, $m-$lattice, $\varepsilon-$lattice, e.t.c. Similarly, this characterizations are the notion of $PM-$lattice module, $\mu-$lattice module, $\varepsilon-$lattice module, etc.

Keywords: separation axioms, prime element, prime spectrum.

1. Introduction

A multiplicative lattice L is a complete lattice in which there is defined a commutative, associative multiplication which distributes over arbitrary joins and has compact greatest element 1_L (least element 0_L) as a multiplicative identity (zero). Multiplicative lattices have been studied extensively by E.W.Johnson, C.Jayaram, the current authors, and others, see, for example, [8]-[17]. An element a of a multiplicative lattice L is called compact if $a \leq \vee b_\alpha$ implies $a \leq b_{\alpha_1} \vee b_{\alpha_2} \vee ... \vee b_{\alpha_n}$ for some subset $\{\alpha_1, \alpha_2, ..., \alpha_n\}$. Throughout this paper, L denotes multiplicative

lattice and L_* denotes the set of all compact elements of L. An element $a \in L$ is said to be proper if $a < 1_L$. An element $p < 1_L$ in L is said to be prime if $ab \leq p$ implies $a \leq p$ or $b \leq p$. An element $m < 1_L$ in L is said to be maximal if $m < x \leq 1_L$ implies $x = 1_L$. By a minimal prime element over an element n of L (or a prime element minimal over n), we mean a prime element which is minimal in the collection of all prime elements p with $n \leq p$. Minimal prime elements over 0_L are simply called the minimal prime elements. We define the dimension of L, denoted with $\dim(L)$, to be the maximal positive integer k, if such exists, such that there exists a chain of prime different elements of L as $p_0 < p_1 <\cdots< p_k$.

By a $C-$lattice we mean a (not necessarily modular) complete multiplicative lattice, with least element 0_L and compact greatest element 1_L (a multiplicative identity), which is generated under joins by multiplicatively closed subset C of compact elements. We note that in $C-$lattice, a finite product of compact elements is again compact. Since 1_L is compact element, then maximal elements exist in L. Further, if L is a $C-$lattice, then there is a minimal prime element of L (See, [15]). If $\dim(L) = 0$, then L is said to be zero-dimensional $C-$lattice. Note that L is a zero-dimensional lattice if and only if every prime element of L is maximal.

We denote by $\sigma(L)$ the set of prime elements of L. We put $V(a) = \{p \in \sigma(L) \mid a \leq p\}$ for any $a \in L$. Then we introduce a topology on $\sigma(L)$ such that $\{V(a) \mid a \in L\}$ is the family of all closed sets. With the above definition the following axioms are hold:

(i) $V(0_L) = \sigma(L)$ and $V(1_L) = \emptyset$.

(ii) $\bigcap_\alpha V(a_\alpha) = V(\bigvee_\alpha a_\alpha)$ for any index set $\triangle$.

(iii) $V(a) \cup V(b) = V(a \wedge b) = V(ab)$ (See, [16]).

We define $\zeta(L) = \{V(a) \mid a \text{ is an element of } L\}$ as it is family of all closed sets that satisfies (i), (ii) and (iii) conditions for L. Then there is a topology on $\sigma(L)$, say τ, and τ is called the Zariski topology. Any open set in $\sigma(L)$ is denoted $\sigma(L)-V(a)$ such that $a \in L$ since $V(a)$ is closed set in $\sigma(L)$. Let $D_a = \sigma(L)-V(a)$ for $a \in L$ (See, [7]).

Let M be a complete lattice. Recall that M is a lattice module over the multiplicative lattice L, or simply an L-module in case there is a multiplication between elements of L and M, denoted by lB for $l \in L$ and $B \in M$, which satisfies the following properties:

(i) $(lb)\, B = l\,(bB)$;

(ii) $\left(\bigvee_\alpha l_\alpha\right)\left(\bigvee_\beta B_\beta\right) = \bigvee_{\alpha,\beta} l_\alpha B_\beta$;

(iii) $1_L B = B$;

(iv) $0_L B = 0_M$;

for all l, l_α, b in L and for all B, B_β in M.

Let M be an L-module. The greatest element of M will be denoted by 1_M. An element $N \in M$ is said to be proper if $N < 1_M$. An element $N < 1_M$ in M is said to be prime if $aX \leq N$ implies $X \leq N$ or $a1_M \leq N$ for $a \in L$, $X \in M$. If N is a prime element of M, then $(N : 1_M)$ is a prime element of L (See [10]). An element $N < 1_M$ in M is said to be maximal if $N < X \leq 1_M$ implies $X = 1_M$. If 1_M is compact element of M, then M has a maximal element by (See, [6]). If N and K belong to M, $(N : K)$ is the join of all $a \in L$ such that $aK \leq N$. By a minimal prime element over an element N of M (or a prime element minimal over N), we mean a prime element which is minimal in the collection of all prime elements P with $N \leq P$. Minimal prime elements over 0_M are simply called the minimal prime elements. We define the dimension of M, denoted with $\dim(M)$, to be the maximal positive integer k, if such exists, such that there exists a chain of prime distinguished elements of L as $P_0 < P_1 < \cdots < P_k$.

Let $\sigma(M)$ be the set of all prime elements of M. Next we define $V(N) = \{P \in \sigma(M) \mid (N : 1_M) \leq (P : 1_M)\}$. Then the following axioms are hold:

(i) $V(0_M) = \sigma(M)$ and $V(1_M) = \emptyset$.

(ii) $\bigcap_{\alpha \in \triangle} V(N_\alpha) = V(\bigvee_{\alpha \in \triangle} (N_\alpha : 1_M)1_M)$ for any index set $\triangle$.

(iii) $V(N) \cup V(K) = V(N \wedge K)$ (See, [7]).

We define $\zeta(M) = \{V(N) \mid N \text{ is an element of } M\}$ as it is family of all closed sets that satisfies (i), (ii) and (iii) conditions for M. Then there is a topology on $\sigma(M)$, say τ, and τ is called the Zariski topology. Any open set in $\sigma(M)$ is denoted by $\sigma(L) - V(N)$ such that $N \in M$ since $V(N)$ is closed set in $\sigma(M)$. We define $X_a = \sigma(M) - V(a1_M)$ for each $a \in L$ and it is clear that every X_a is an open set in $\sigma(M)$ (See, [7]).

For a topological space (X, τ) and $x \in X$, define the kernel of x, $\widehat{\{x\}}$, as the set $\{y \in X \mid x \in \overline{\{y\}}\}$. The set $\widehat{\{x\}}\backslash\{x\}$ is called shell of x and it will be denoted in this work by $\widetilde{\{x\}}$. The closure of a subset A of a topological space (X, τ) is the intersection of the members of the family of all closed sets containing A. The set A is called a point closure if $A = \overline{\{p\}}$ for some $p \in X$. A point x is an accumulation point of a subset A of a topological space (X, τ) if every neighborhood of x contains points of A other than x. Then it is true that each neighborhood of a point x intersects A if and only if x is either a point of A or an accumulation point of A. The set of all accumulation points of a set A is called the derived set of A and it will be denoted by A'. The closure of any set is the union of the set and the derived set of it (For more detail, see [5]).

With the Zariski topology on $\sigma(L)$, since $\overline{\{p\}} = V(p)$ for $p \in \sigma(L)$, it is clear that $\widetilde{\{p\}} = \widehat{\{p\}}\backslash\{p\} = \{q \in \sigma(L) \mid p \in V(q) \text{ and } p \neq q\}$ and $\{p\}' = V(p)\backslash\{p\} = \{q \in \sigma(L) \mid p < q\}$. For this topology, $\{p\}$ is a closed set if and only if p is a maximal element of L (See, [7]).

With the Zariski topology on $\sigma(M)$, the above definitions and $\overline{\{P\}} = V(P) = \{Q \in \sigma(M) \mid (P : 1_M) \leq (Q : 1_M)\}$ for $P \in \sigma(M)$, it is clear that $\widetilde{\{P\}} = \widetilde{\{P\}} \backslash \{P\} = \{Q \in \sigma(M) \mid P \in V(Q) \text{ and } P \neq Q\}$ and $\{P\}' = V(P) \backslash \{P\} = \{Q \in \sigma(M) | \ (P : 1_M) \leq (Q : 1_M) \text{ and } P \neq Q\}$ (see,[7]), Proposition 25).

An $L-$module M is called a multiplication lattice module if for every element $N \in M$ there exists an element $a \in L$ such that $N = a1_M$. In this case, $N = (N : 1_M)1_M$ (see [10] for more detail). If the set $\{P\}$ is a closed in $\sigma(M)$ then P is a maximal element of M. The converse is also true, when M is a multiplication lattice $L-$module(See [7]).

Throughout this paper, we assume that 1_L and 1_M are compact element in L and M, respectively.

2. Separation axioms between T_0 and T_1 on $\sigma(L)$

Definition 1 A topological space (X, τ) is:

1. $T(\beta)$ if for any $x \in X$, $\{x\}'$ is empty or singleton.
2. $T(\beta')$ if for any $x \in X$, $\widetilde{\{x\}}$ is empty or singleton.
3. $T(\varepsilon)$ if for any $x, y \in X$, $x \neq y$, $\{x\}' \cap \{y\}'$ is empty or singleton.
4. T_{UD} if for any $x \in X$, $\{x\}'$ is the union of disjoint closed sets.
5. T_D if for any $x \in X$, $\{x\}'$ is a closed set.
6. T_{DD} if it is T_D and for any $x, y \in X,\ x \neq y,\ \{x\}' \cap \{y\}' = \emptyset$.
7. T_Y if for any $x, y \in X$, $x \neq y$, $\overline{\{x\}} \cap \overline{\{y\}}$ is empty or singleton.
8. T_{YS} for any $x, y \in X$, with $x \neq y$, $\overline{\{x\}} \cap \overline{\{y\}}$ is either empty or $\{x\}$ or $\{y\}$.

Definition 2 L is said to be $pm-$lattice if every prime element is contained a unique maximal element.

Proposition 1 *$\sigma(L)$ is a $T(\beta)-$space if and only if L is a $pm-$lattice and* $\dim(L) \leq 1$.

Proof. $(\Rightarrow)$: Let $\sigma(L)$ be a $T(\beta)-$space. Assume that L is not a $pm-$lattice. Then there is a prime element p such that $p < k$ and $p < t$ with k and t two different maximal elements. So $k, t \in V(p)$ and since $p \neq k$ and $p \neq t$, then $k, t \in \{p\}'$, a contradiction.

Assume that $\dim(L) \geq 2$. Then there exist prime elements p, q, k such that $p < q < k$. So $q, k \in \{p\}'$, a contradiction.

$(\Leftarrow)$: Let L be a $pm-$lattice and $\dim(L) \leq 1$. Suppose that $\sigma(L)$ is not a $T(\beta)-$space. Then $\{p\}' \neq \emptyset$ and $\{p\}'$ is not singleton for some $p \in \sigma(L)$. Thus there are two different prime elements k, t with $k, t \in \{p\}'$. So $p < k$ and $p < t$. Since $\dim(L) \leq 1$, then k and t are maximal elements, a contradiction. ∎

Definition 3 A lattice L is said to be $m-$lattice if every prime element contains a unique minimal prime element of L.

Proposition 2 *Let L be a $C-$lattice. Then $\sigma(L)$ is a $T(\beta')-$space if and only if L is an $m-$lattice and* $\dim(L) \leq 1$.

Proof. ($\Rightarrow$) : Let $\sigma(L)$ be a $T(\beta')-$space. Assume that L is not an $m-$lattice. Then there is a prime element p such that $u < p$ and $v < p$ with u and v two different minimal prime elements. Hence $p \in V(u)$ and $p \in V(v)$, that is, $p \in \overline{\{u\}}$ and $p \in \overline{\{u\}}$. Since $p \neq u$ and $p \neq v$, then $u, v \in \widetilde{\{p\}} = \widehat{\{p\}} \backslash \{p\}$, a contradiction.

Assume that $\dim(L) \geq 2$. Then there exist prime elements p, q, k such that $p < q < k$. Thus $k \in V(p)$ and $k \in V(q)$, that is, $k \in \overline{\{p\}}$ and $k \in \overline{\{q\}}$. Therefore, $p, q \in \widetilde{\{k\}}$, a contradiction.

($\Leftarrow$) : Let L be a $m-$lattice and $\dim(L) \leq 1$. Suppose that $\sigma(L)$ is not a $T(\beta')-$space. Then $\widetilde{\{p\}} \neq \emptyset$ and $\widetilde{\{p\}}$ is not singleton for some $p \in \sigma(L)$. Thus there are two different prime elements k, t with $k, t \in \widetilde{\{p\}}$. So $k < p$ and $t < p$. Since $\dim(L) \leq 1$, then k and t are minimal prime elements, a contradiction. ∎

Definition 4 Let m, n be maximal elements of L and k, t be minimal prime elements of L. Then $\rho_{\subset}(m, n) = \{p \in \sigma(L) \mid p < m \wedge n\}$ and $\rho_{\supset}(k, t) = \{p \in \sigma(L) \mid k < p \text{ and } t < p\}$.

Definition 5 A lattice L is called $\varepsilon-$lattice if $\rho_{\subset}(m, n)$ and $\rho_{\supset}(k, t)$ are empty or singleton for any different maximal elements m, n of L and different minimal prime elements k, t of L.

Proposition 3 *Let L be a $C-$lattice. Then $\sigma(L)$ is a $T(\varepsilon)-$space if and only if L is a $\varepsilon-$lattice and* $\dim(L) \leq 2$.

Proof. ($\Rightarrow$) : Let $\sigma(L)$ be a $T(\varepsilon)-$space. If L is not a $\varepsilon-$lattice, then $\rho_{\subset}(m, n)$ or $\rho_{\supset}(k, t)$ is neither empty nor singleton for any different maximal elements m, n of L and different minimal prime elements k, t of L. Assume that $\rho_{\subset}(m, n)$ is neither empty nor singleton. Then there are different prime elements p, q such that $p, q \in \rho_{\subset}(m, n)$. Thus $p < m \wedge n$ and $q < m \wedge n$, that is, $p < m$ and $p < n$, $q < m$ and $q < n$. Then $m, n \in \{p\}'$ and $m, n \in \{q\}'$. Hence $m, n \in \{p\}' \cap \{q\}'$, a contradiction. Suppose that $\rho_{\supset}(k, t)$ is neither empty nor singleton. Then there exist different prime elements u, v such that $u, v \in \rho_{\supset}(k, t)$. Hence $k < u$, $t < u$ and $k < v$, $t < v$. Therefore, $u, v \in \{k\}'$ and $u, v \in \{t\}'$, a contradiction. If $\dim(L) \geq 3$, then there are $p, q, k, t \in \sigma(L)$ with $p < q < k < t$. So $k, t \in \{p\}' \cap \{q\}'$, which is a contradiction.

($\Leftarrow$) : Let L be a $\varepsilon-$lattice and $\dim(L) \leq 2$. Assume that $\sigma(L)$ is not a $T(\varepsilon)-$space. Then $\{p\}' \cap \{q\}'$ is neither empty nor singleton for different prime elements p, q of L. Thus there are different prime elements k, t of L such that $k, t \in \{p\}' \cap \{q\}'$. Thus we get $p < k$, $p < t$ and $q < k$, $q < t$. If p, q are minimal prime elements of L, then $k, t \in \rho_{\supset}(p, q)$, which is a contradiction. If p, q are not

minimal prime elements of L, then there exist minimal prime elements u, v of L such that $u < p$ and $v < q$. Then $u < p < k,\ u < p < t$ and $v < p < k,\ v < q < t$. Hence k, t are maximal elements of L since $\dim(L) \leq 2$. It is clear that $p < k \wedge t$ and $q < k \wedge t$. Indeed, if $p = k \wedge t$ or $q = k \wedge t$, then $k \leq p$ or $t \leq p$ or $k \leq q$ or $t \leq q$ as $p,\ q$ are prime elements of L. This is contradict with k, t where are maximal elements of L. So we get $p < k \wedge t$ and $q < k \wedge t$. Hence $p, q \in \rho_{\subset}(k, t)$, a contradiction. ∎

Definition 6 Let $p \in \sigma(L)$ be not maximal. Then L is called $u-$lattice if $a \vee b = 1_L$ for every elements a, b in $\{p\}'$.

Proposition 4 *Let* $\dim(L)$ *be finite and* L *be a* $u-$*lattice. Then* $\sigma(L)$ *is a* $T_{UD}-$*space.*

Proof. Let $p \in \sigma(L)$ and β be the set of minimal prime elements of $\{p\}'$. Assume that $x \in \{p\}'$. Then $p < x$. We have that there is a $q \in \beta$ such that $p < q \leq x$. So $x \in V(q)$. Thus $x \in \bigcup_{q \in \beta} V(q)$. Conversely, let $x \in \bigcup_{q \in \beta} V(q)$. Then $x \in V(q)$ for some $q \in \beta$. So $q \leq x$ and since $p < q$, then $p < x$. Hence $x \in \{p\}'$. Therefore $\{p\}' = \bigcup_{q \in \beta} V(q)$. Also $V(q_1) \cap V(q_2) = \emptyset$ for minimal prime elements q_1, q_2 with $p < q_1$ and $p < q_2$. Indeed, if $V(q_1) \cap V(q_2) \neq \emptyset$, then there is a $t \in \sigma(L)$ such that $t \in V(q_1) \cap V(q_2)$. Thus $q_1 \leq t$ and $q_2 \leq t$. Therefore, $q_1 \vee q_2 \leq t$. Hence $t = 1_L$, a contradiction. ∎

Definition 7 A lattice L is called $d-$*lattice* if every prime non maximal element is different to the meet of the prime elements properly containing it.

Proposition 5 $\sigma(L)$ *is a* T_D-*space if and only if* L *is a* $d-$*lattice.*

Proof. $(\Rightarrow)$: Let $\sigma(L)$ be a T_D-space. If L is not a $d-$lattice, then there exists a prime non maximal element p such that p is the meet of the prime elements properly containing it, that is, $p = \bigwedge_{p<q} q$. Since $\sigma(L)$ is a T_D-space, then $\{p\}'$ is closed set. Then $\{p\}' = V(a)$ for $a \in L$. Since $a \leq q$ for every $q \in \{p\}'$, $a \leq \bigwedge_{q \in \{p\}'} q = p$. Thus $p \in V(a)$, a contradiction.

$(\Leftarrow)$: Let L be a $d-$lattice. If $p \in \sigma(L)$ is a maximal, then it is clear. Let $p \in \sigma(L)$ be a non maximal element. Since L is a $d-$lattice, then $p \neq \bigwedge_{p<q} q$. But $V(\bigwedge_{p<q} q) \subseteq V(p)$ since $p \leq \bigwedge_{p<q} q$. Since $p \notin V(\bigwedge_{p<q} q)$, then $V(\bigwedge_{p<q} q) \subseteq \{p\}'$. Now, let $k \in \{p\}'$. Then $p < k$ and so $\bigwedge_{p<q} q \leq k$, that is, $k \in V(\bigwedge_{p<q} q)$. ∎

Definition 8 A lattice L is called $y-$lattice if any two different minimal prime element are contained in at most one maximal element.

Proposition 6 *Let L be a $C-$lattice. $\sigma(L)$ is a T_Y-space if and only if L is a $y-$lattice and* $\dim(L) \leq 1$.

Proof. ($\Rightarrow$) : Let $\sigma(L)$ be a T_Y-space. Assume that L is not a $y-$lattice. Let p, q be minimal prime elements of L. Then there are two different maximal elements m, n of L containing p, q, that is, $p < m$, $p < n$ and $q < m$, $q < n$. Then $m, n \in V(p) = \overline{\{p\}}$ and $m, n \in V(q) = \overline{\{q\}}$. Thus $m, n \in \overline{\{p\}} \cap \overline{\{q\}}$, a contradiction. Suppose that $\dim(L) \geq 2$. Then there exist prime elements p, q, k such that $p < q < k$. So $q, k \in V(p) \cap V(q)$, that is, $q, k \in \overline{\{p\}} \cap \overline{\{q\}}$, a contradiction.

($\Leftarrow$) : Let L be a $y-$lattice and $\dim(L) \leq 1$. Assume that $\sigma(L)$ is not a T_Y-space. Then $\overline{\{p\}} \cap \overline{\{q\}}$ is neither empty nor singleton for two different minimal prime elements p, q. Thus we get $k, t \in \overline{\{p\}} \cap \overline{\{q\}}$. Then $p \leq k$, $p \leq t$ and $q \leq k$, $q \leq t$. Also $p \neq k$ and $p \neq t$. Indeed, if $p = k$ or $p = t$, then $p \in V(q) = \overline{\{q\}}$. Thus $q < p$. We get that p is maximal since $\dim(L) \leq 1$.Thus $V(p) = \{p\}$ and so $\overline{\{p\}} \cap \overline{\{q\}} = \{p\}$, a contradiction. In a similar way, $q \neq k$ and $q \neq t$. Hence $p < k$, $p < t$ and $q < k$, $q < t$. Since $\dim(L) \leq 1$, then k and t are maximal elements, a contradiction. ∎

Proposition 7 *Let L be a $C-$lattice. $\sigma(L)$ is a $T_{YS}-$space if and only if L is an $m-$lattice and* $\dim(L) \leq 1$.

Proof. ($\Rightarrow$) : Let $\sigma(L)$ be a $T_{YS}-$space. Suppose that L is not an $m-$lattice. Let $p \in \sigma(L)$. Then there are two different minimal prime elements u, v such that $u < p$ and $v < p$. Thus $p \in V(u) = \overline{\{u\}}$ and $p \in V(v) = \overline{\{v\}}$ and so $p \in \overline{\{u\}} \cap \overline{\{v\}}$, a contradiction. Assume that $\dim(L) \geq 2$. Then there exist prime elements p, q, k such that $p < q < k$. Thus $k \in V(p) = \overline{\{p\}}$ and $k \in V(q) = \overline{\{q\}}$. Therefore, $k \in \overline{\{p\}} \cap \overline{\{q\}}$, a contradiction.

($\Leftarrow$) : Let L be an $m-$lattice and $\dim(L) \leq 1$. Let $p, q \in \sigma(L)$ with $p \neq q$. Assume that $\overline{\{p\}} \cap \overline{\{q\}} \neq \emptyset$ and $\overline{\{p\}} \cap \overline{\{q\}} \neq \{p\}$. Let $k \in \overline{\{p\}} \cap \overline{\{q\}}$. Then $k \neq p$ and $p < k$, $q \leq k$. Since $\dim(L) \leq 1$, then p is minimal prime element. Then $q = k$ because L is an $m-$lattice. Hence $\overline{\{p\}} \cap \overline{\{q\}} = \{q\}$. ∎

3. Separation axioms between T_0 and T_1 on $\sigma(M)$

In this section, we will investigate the like of our studying on lattices for lattice modules.

Definition 9 An $L-$module M is said to be $PM-$lattice module if every prime element is contained a unique maximal element.

Proposition 8 *Let M be an $L-$module. If $\sigma(M)$ is a $T(\beta)-$space, then the followings are hold:*

a) *M is a $PM-$lattice module.*

b) $\dim(M) \leq 1$.

Proof. Let $\sigma(M)$ be a $T(\beta)-$space.

a) Assume that M is not a $PM-$lattice module. Then there exists a prime element P such that $P < N$ and $P < K$ with N and K two different maximal elements. Since $(P : 1_M) \leq (N : 1_M)$, $(P : 1_M) \leq (K : 1_M)$ and $P \neq N$, $P \neq K$, then $N, K \in \{P\}'$, a contradiction.

b) If $\dim(M) \geq 2$, then there are prime elements P, N, K such that $P < N < K$. So $(P : 1_M) \leq (N : 1_M) \leq (K : 1_M)$. Therefore $N, K \in \{P\}'$, which is a false. ∎

Definition 10 An $L-$module M is called $\mu-$lattice module if every prime element contains a unique minimal prime element.

Proposition 9 *Let M be an $L-$module. Let $\sigma(M)$ be a $T(\beta')-$space. Then the following statements are hold:*

a) *M is a $\mu-$lattice module.*

b) $\dim(M) \leq 1$.

Proof. Let $\sigma(M)$ be a $T(\beta')-$space.

a) If M is not a $\mu-$lattice module, then there is a prime element P such that $K < P$ and $T < P$ with K and T two different minimal prime elements. Then $(K : 1_M) \leq (P : 1_M)$ and $(T : 1_M) \leq (P : 1_M)$. Hence, $P \in V(K)$ and $P \in V(T)$. Then $K, T \in \widetilde{\{P\}} = \widehat{\{P\}} \backslash \{P\}$, which is a contradiction.

b) If $\dim(M) \geq 2$, then there are prime elements P, N, K such that $P < N < K$. Hence $(P : 1_M) \leq (N : 1_M) \leq (K : 1_M)$. Then $K \in V(N)$ and $K \in V(P)$. Thus, $P, N \in \widetilde{\{K\}} = \widehat{\{K\}} \backslash \{K\}$, a contradiction. ∎

Definition 11 Let M be an $L-$module. Let N, K be maximal elements of M and U, V be minimal prime elements of M. Then

$$\rho(N, K) = \{P \in \sigma(M) \mid (P : 1_M) \leq (N \wedge K : 1_M) \text{ and } P \neq N \wedge K\} \text{ and}$$

$$\delta(U, V) = \{P \in \sigma(M) \mid P \in \{U\}' \text{ and } P \in \{V\}'\}.$$

Definition 12 An $L-$module M is a $\varepsilon-$lattice module if the set $\rho(N, K)$ and $\delta(U, V)$ are empty or singleton for any different maximal elements N, K and different minimal prime elements U, V.

Let M be an $L-$module. We call M is a multiplication lattice $L-$module if for every element N of M there exists an element a of L such that $N = a1_M$. In this case, we know that $N = (N : 1_M)1_M$ (For more detail, see (10)).

Proposition 10 *Let M be a multiplication lattice $L-$module. Then M is a $\varepsilon-$module and* $\dim(M) \leq 2$ *if and only if $\sigma(M)$ is a $T(\varepsilon)-$space.*

Proof. Let M be an $\varepsilon-$module and $\dim(M) \leq 2$. Suppose that $\sigma(M)$ is not a $T(\varepsilon)-$space. Then $\{P\}' \cap \{Q\}'$ is neither empty nor singleton for different prime elements P, Q of M. Thus there are different prime elements N, K of M such that $N, K \in \{P\}' \cap \{Q\}'$. Thus we get $(P : 1_M) \leq (N : 1_M)$, $(P : 1_M) \leq (K : 1_M)$, $P \neq N, K$ and $(Q : 1_M) \leq (N : 1_M)$, $(Q : 1_M) \leq (K : 1_M)$, $Q \neq N, K$. Since M is a multiplication lattice $L-$module, we obtain $P < N, P < K$ and $Q < N, Q < K$. If P, Q are minimal prime elements, then $N, K \in \delta(P, Q)$, which is a contradiction. If P, Q are not minimal prime elements, then there are different minimal prime elements U, V such that $U < P$ and $V < Q$. Then we have $U < P < N, U < P < K$ and $V < Q < N, V < Q < K$. Thus N, K are maximal elements since $\dim(M) \leq 2$. It is clear that $P < N \wedge K$ and $Q < N \wedge K$. Indeed, if $P = N \wedge K$ or $Q = N \wedge K$, $N \leq P$ or $K \leq P$ or $N \leq Q$ or $K \leq Q$ as P, Q are prime elements of M. This is contradict with N, K where are maximal elements of M. So we get $P < N \wedge K$ and $Q < N \wedge K$. Hence $P, Q \in \rho(N, K)$, a contradiction.

Conversely, let $\sigma(M)$ be a $T(\varepsilon)-$space. If M is not a $\varepsilon-$lattice module, then the set $\rho(N, K)$ or $\delta(U, V)$ is neither empty nor singleton for any different maximal elements N, K and different minimal prime elements U, V. If $\rho(N, K)$ is neither empty nor singleton, then there are $P, Q \in \sigma(M)$ such that $P, Q \in \rho(N, K)$, that is, $(P : 1_M) \leq (N \wedge K : 1_M)$, $P \neq N \wedge K$ and $(Q : 1_M) \leq (N \wedge K : 1_M)$, $Q \neq N \wedge K$. Since M is a multiplication lattice $L-$module, then $P < N \wedge K \leq N$, $P < N \wedge K \leq K$ and $Q < N \wedge K \leq N$, $Q < N \wedge K \leq K$. Thus $N \in \{P\}'$, $K \in \{P\}'$and $N \in \{Q\}'$, $K \in \{Q\}'$. Thus, $N, K \in \{P\}' \cap \{Q\}'$, a contradiction. If $\delta(U, V)$ is neither empty nor singleton, then there are $S, T \in \sigma(M)$ such that $S, T \in \delta(U, V)$. Thus $S, T \in \{U\}'$ and $S, T \in \{V\}'$. Consequently, $S, T \in \{U\}' \cap \{V\}'$, a contradiction. If $\dim(M) \geq 3$, then there are prime elements P, Q, S, T such that $P < Q < S < T$. So $(P : 1_M) \leq (Q : 1_M) \leq (S : 1_M) \leq (T : 1_M)$. Thus, $S, T \in \{P\}' \cap \{Q\}'$, which is a contradiction. ∎

Definition 13 An $L-$module M is a $D-$lattice module if every prime non-maximal element is not equal to the meet of the prime elements properly containing it.

Proposition 11 *Let M be an $L-$module. If $\sigma(M)$ is a T_D-space, then M is a $D-$lattice module. The converse is also true when M is a multiplication lattice $L-$module.*

Proof. Assume that $\sigma(M)$ is a T_D-space. Let $P \in \sigma(M)$ be a non-maximal element. Assume that $P = \bigwedge_{\substack{Q \in \sigma(M) \\ P < Q}} Q$. Since $\{P\}'$ is a closed set, there is an element K of M such that $\{P\}' = V(K)$. Then $(K : 1_M) \leq (Q : 1_M)$ for every $Q \in \{P\}' \Rightarrow (K : 1_M) \leq \bigwedge_{Q \in \{P\}'} (Q : 1_M) \Rightarrow (K : 1_M) \leq (\bigwedge_{Q \in \{P\}'} Q : 1_M) \leq (\bigwedge_{\substack{Q \in \sigma(M) \\ P < Q}} Q : 1_M) = (P : 1_M) \Rightarrow (K : 1_M) \leq (P : 1_M)$. Thus $P \in V(K) \Rightarrow P \in \{P\}'$,

which is a contradiction. Hence we get $P \neq \bigwedge_{\substack{Q\in\sigma(M)\\P<Q}} Q$.

Conversely, let M be a multiplication lattice $L-$module. If $P \in \sigma(M)$ is maximal element, then $V(P) = \{P\}$. Then $\{P\}' = \emptyset$, that is, $\{P\}'$ is a closed set. If P is prime non-maximal element, then P is not equal to the meet of the prime elements properly containing it, that is, $P \neq \bigwedge_{\substack{Q\in\sigma(M)\\P<Q}} Q$. Since $P < \bigwedge_{\substack{Q\in\sigma(M)\\P<Q}} Q$, then $V(\bigwedge_{\substack{Q\in\sigma(M)\\P<Q}} Q) \subseteq V(P)$. Then $P \in V(P)$ but $P \notin V(\bigwedge_{\substack{Q\in\sigma(M)\\P<Q}} Q)$. Indeed, if $P \in V(\bigwedge_{\substack{Q\in\sigma(M)\\P<Q}} Q)$, then $(\bigwedge_{\substack{Q\in\sigma(M)\\P<Q}} Q : 1_M) \leq (P : 1_M)$ and since M is a multiplication lattice $L-$module, then $\bigwedge_{\substack{Q\in\sigma(M)\\P<Q}} Q \leq P$, it is a contradiction. Hence $V(\bigwedge_{\substack{Q\in\sigma(M)\\P<Q}} Q) \subseteq \{P\}'$. Conversely, let $K \in \{P\}'$. Then $(P : 1_M) \leq (K : 1_M)$ and $P \neq K$. Since M is a multiplication lattice $L-$module, then $P < K$. Thus $\bigwedge_{\substack{Q\in\sigma(M)\\P<Q}} Q \leq K$ and so $(\bigwedge_{\substack{Q\in\sigma(M)\\P<Q}} Q : 1_M) \leq (K : 1_M)$. Then $K \in V(\bigwedge_{\substack{Q\in\sigma(M)\\P<Q}} Q)$ and so $\{P\}' \subseteq V(\bigwedge_{\substack{Q\in\sigma(M)\\P<Q}} Q)$. Consequently, $\{P\}' = V(\bigwedge_{\substack{Q\in\sigma(M)\\P<Q}} Q)$, that is, $\{P\}'$ is a closed set. ∎

Proposition 12 *Let $\sigma(M)$ be a $T_{DD}-$space. Then the following statements are hold:*

a) *M is a $\mu-$lattice module.*

b) $\dim(M) \leq 1$.

Proof. Let $\sigma(M)$ be a $T_{DD}-$space.

a) Assume that M is not a $\mu-$lattice module. Then there is a prime element P of M such that $U < P$ and $T < P$ with U and T two different minimal prime elements. So $(U : 1_M) \leq (P : 1_M)$ and $(T : 1_M) \leq (P : 1_M)$. Hence $P \in \{U\}' \cap \{T\}'$, which is a contradiction.

b) Suppose that $\dim(M) \geq 2$. Then there are prime elements P, N, K such that $P < N < K$. Hence $(P : 1_M) \leq (N : 1_M) \leq (K : 1_M)$. Thus $K \in \{P\}' \cap \{N\}'$, a contradiction. ∎

Definition 14 An $L-$module M is a $Y-$lattice module if any two distinct minimal prime elements are contained in at most one maximal element.

Proposition 13 *Let M be an $L-$module. If $\sigma(M)$ is a T_Y-space, then M is a $Y-$lattice module and* $\dim(M) \leq 1$. *The converse is also true when M is a multiplication lattice $L-$module.*

Proof. Let $\sigma(M)$ be a T_Y-space. Suppose that M is not a $Y-$lattice module. Let P, Q be two different minimal prime elements. Then there are two different maximal elements N, K such that $P < N, P < K$ and $Q < N, Q < K$. Then $(P : 1_M) \leq (N : 1_M)$, $(P : 1_M) \leq (K : 1_M)$ and $(Q : 1_M) \leq (N : 1_M)$, $(Q : 1_M) \leq (K : 1_M)$. Thus $N, K \in V(P) = \overline{\{P\}}$ and $N, K \in V(Q) = \overline{\{Q\}}$. Hence $N, K \in \overline{\{P\}} \cap \overline{\{Q\}}$, this is a contradiction. Let $\dim(M) \geq 2$. Then there are prime elements P, Q, T such that $P < Q < T$. Thus $(P : 1_M) \leq (Q : 1_M) \leq (T : 1_M)$. So $Q, T \in V(Q)$ and $Q, T \in V(P)$. Then $Q, T \in \overline{\{P\}} \cap \overline{\{Q\}}$, which is a false.

Conversely, let M be a multiplication lattice $L-$module. Let M be a $Y-$lattice module and $\dim(M) \leq 1$. Assume that $\sigma(M)$ is not a T_Y-space. Then $\overline{\{P\}} \cap \overline{\{Q\}} \neq \emptyset$ and $\overline{\{P\}} \cap \overline{\{Q\}}$ is not singleton for two different prime elements P, Q. So there are two prime elements N, K such that $N, K \in \overline{\{P\}} \cap \overline{\{Q\}}$ and so $(P : 1_M) \leq (N : 1_M), (P : 1_M) \leq (K : 1_M)$ and $(Q : 1_M) \leq (N : 1_M), (Q : 1_M) \leq (K : 1_M)$. Since M is a multiplication lattice $L-$module, $P \leq N, P \leq K$ and $Q \leq N, Q \leq K$. Also, $P \neq N$ and $P \neq K$. Indeed, if $P = N$ or $P = K$, then P is maximal as $\dim(M) \leq 1$. Hence $V(P) = \{P\} = \overline{\{P\}}$ and so $\overline{\{P\}} \cap \overline{\{Q\}} = \{P\}$, a contradiction. In a similar way, $Q \neq N$ and $Q \neq K$. Then we get $P < N$, $P < K$ and $Q < N$, $Q < K$, so N and K are maximal elements, since $\dim(M) \leq 1$. But this contradicts with M which is a $Y-$lattice module. ∎

Proposition 14 *Let M be an $L-$module. If $\sigma(M)$ is a $T_{YS}-$space, then M is a $\mu-$lattice module and* $\dim(M) \leq 1$. *The converse is also true when M is a multiplication lattice $L-$module.*

Proof. Let $\sigma(M)$ be a $T_{YS}-$space. Suppose that M is not a $\mu-$lattice module. Then there exists a prime element P containing two different minimal prime elements Q, T. So $P \in V(Q) \cap V(T)$, that is, $P \in \overline{\{Q\}} \cap \overline{\{T\}}$. This is a contradiction. Suppose that $\dim(M) \geq 2$. Then there are prime elements P, Q, T such that $P < Q < T$. Hence $(P : 1_M) \leq (Q : 1_M) \leq (T : 1_M)$. Thus $T \in V(P) \cap V(Q) \Rightarrow T \in \overline{\{P\}} \cap \overline{\{Q\}}$, which is a false.

Conversely, assume that M is a multiplication lattice $L-$module. Let M be a $\mu-$lattice module and $\dim(M) \leq 1$. Let P, Q be different prime elements of M. Assume that $\overline{\{P\}} \cap \overline{\{Q\}} \neq \emptyset$ and $\overline{\{P\}} \cap \overline{\{Q\}} \neq \{P\}$. Let $K \in \overline{\{P\}} \cap \overline{\{Q\}}$. Then $K \neq P$, also $(P : 1_M) \leq (K : 1_M)$ and $(Q : 1_M) \leq (K : 1_M)$. Since M is a multiplication lattice $L-$module, then $P < K$ and $Q \leq K$. As $\dim(M) \leq 1$, then P is a minimal prime element and since M is a $\mu-$lattice module, $Q = K$. Thus $\overline{\{P\}} \cap \overline{\{Q\}} = \{Q\}$. ∎

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A NEW CHARACTERIZATION OF THE ALTERNATING GROUP A_8 BY ITS ORDER AND LARGE DEGREES OF ITS IRREDUCIBLE CHARACTER

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Abstract. It is well-known that characters of a finite group can give information about its structure. Also it is known that a finite simple group can uniquely determined by its character table. Here the authors attempt to investigate how to characterize a finite group by using less information of its character table, and successfully characterize the alternating group A_8 by its order and at most two irreducible character degrees of the character table.

Keywords: finite group, irreducible character, simple group, character degree.
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1. Introduction and preliminary results

Let G be a finite group, $Irr(G)$ be the set of all irreducible complex characters of G, and denote the set of character degrees of G by $cd(G) = \{\chi(1)|\chi \in Irr(G)\}$. In stances, where the context is clear, we will refer to character degrees as degrees. On the other hand, we use $cd^*(G)$ to denote the multi-set of degrees of irreducible characters, i.e., each element of the set $cd^*(G)$ can occur many times upon on the number of characters of the same degree. In particular, $|cd^*(G)| = |Irr(G)|$. Let $\pi(G)$ denote the set of all primes dividing the order of G. For each prime $p \in \pi(G)$, Let $Syl_p(G)$ denote the set of all Sylow p-subgroups of G and G_p denote a Sylow p-subgroup of G. $L_1(G)$ and $L_2(G)$ are the largest and the second largest irreducible character degree of character table of G, respectively. The other notations and terminologies in this paper are standard and can be found in [1], [2], for instance.

It is a well-known fact that characters of a finite group can give some information of its structure. It was proved in [3] that a finite simple group can be uniquely determined by its character table. In 2000, Huppert in [4] conjectured that each finite non-abelian simple group G can be characterized by the set $cd(G)$ of degrees of its complex irreducible characters; in [4], [5], [6], Huppert confirmed that the conjecture holds for the simple groups such as $L_2(q)$ and $S_z(q)$; moreover, he also proved that the conjecture follows for 19 out of 26 sporadic simple groups, and a few others (cf.[4], [5], [6]). In [7], [8], Daneshkhah, et al. showed that the conjecture holds for another three sporadic simple groups Co_1, Co_2 and Co_3. In [9], Xu, Chen and Yan attempted to characterize the finite simple groups by less information of its characters, and for the first time successfully characterized the simple K_3-groups by their orders and one or both of its largest and second largest irreducible character degrees. Simple K_3-groups are eight simple groups with their orders having exactly three distinct prime divisors, which are A_5, A_6, $L_2(7)$, $L_2(8)$, $L_2(17)$, $L_3(3)$, $U_3(3)$, and $U_4(2)$. In [9], the authors proved that the simple groups A_5, $L_2(7)$, $L_2(17)$, $L_3(3)$ and $U_4(2)$ can be uniquely determined by their orders and the largest degrees of its irreducible characters, the alternating group A_6 can be uniquely determined by its order and the second largest degree of its irreducible characters and the remaining simple groups $L_2(8)$ and $U_3(3)$ can be uniquely determined by their orders and the largest and the second largest degrees of their irreducible characters.

In this paper, we continue this investigation, and show that the alternating group A_8 can be characterized by its order and at most two irreducible character degrees of its character table.

We come to the following main theorem:

Main Theorem. *Let G be a finite group and $|G| = |A_8|$. Then $G \cong A_8$ if and only if*

(i) $L_1(G)$=$L_1(A_8)$;

(ii) $L_2(G)$=$L_2(A_8)$.

In order to prove the above main theorem, we need the following lemmas:

Lemma 1. (c.f [9]) *Let G be a non-solvable group. Then G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic non-abelian simple groups and $|G/K| \mid |Out(K/H)|$.*

Lemma 2. (c.f [9]) *Let G be a finite solvable group of order $q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_s^{\alpha_s}$, where $q_1, q_2, \cdots, q_s$ are distinct primes. If $(kq_s + 1) \nmid q_i^{\alpha_i}$ for each $i \leq s-1$ and $k > 0$, then the Sylow q_s-subgroup is normal in G.*

Lemma 3. (cf.[1]) *Let S be a finite non-abelian simple group with $\pi(S) \subseteq \{2, 3, 5, 7\}$, then S is isomorphic to one of the following simple groups listed in Table 1. In particular, if $|\pi(Out(S))| \neq 1$, then $\pi(Out(S)) \subseteq \{2, 3\}$.*

Table 1. **Finite nonabelian simple groups with** $\pi(S) \subseteq \{2, 3, 5, 7\}$

G	$\lvert G\rvert$	$Out(G)$	G	$\lvert G\rvert$	$Out(G)$
A_5	$2^2 \cdot 3 \cdot 5$	2	$U_3(5)$	$2^4 \cdot 3^2 \cdot 5^3 \cdot 7$	6
$L_2(7)$	$2^3 \cdot 3 \cdot 7$	2	$S_4(7)$	$2^8 \cdot 3^2 \cdot 5^2 \cdot 7^4$	2
A_6	$2^3 \cdot 3^2 \cdot 5$	4	A_9	$2^6 \cdot 3^4 \cdot 5 \cdot 7$	2
$L_2(8)$	$2^3 \cdot 3^2 \cdot 7$	3	$O_8^+(2)$	$2^{12} \cdot 3^5 \cdot 5^2 \cdot 7$	6
A_8	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	2	$U_4(3)$	$2^7 \cdot 3^6 \cdot 5 \cdot 7$	8
$L_3(4)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	12	$S_6(2)$	$2^9 \cdot 3^4 \cdot 5 \cdot 7$	1
A_7	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	2	$L_2(49)$	$2^4 \cdot 3 \cdot 5^2 \cdot 7^2$	4
$U_3(3)$	$2^5 \cdot 3^3 \cdot 7$	2	J_2	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$	2
$U_4(2)$	$2^6 \cdot 3^4 \cdot 5$	2	A_{10}	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7$	2

Proof of Main Theorem

In this section, we give the proof of the main theorem.

Proof of Main Theorem. It is enough to prove the sufficiency.

By hypotheses, we have that $|G| = 2^6 \cdot 3^2 \cdot 5 \cdot 7$. Let $\beta, \chi \in Irr(G)$ such that $\beta(1) = L_1(G) = 70$, $\chi(1) = L_2(G) = 64$. In the following, we will prove that $G \cong A_8$.

We first assert that G is nonsolvable. If G is solvable, let T be a Hall-subgroup of G with order $|T| = 2^6 \cdot 3^2 \cdot 7$. Let $T_G = \bigcap_{g \in G} T^g \leq T$, then we have $G/T_G \lesssim S_5$. Since orders of solvable subgroups of S_5 divided by 5 are 5, 10 and 20, it follows that $|T_G|$ may be one of $2^6 \cdot 3^2 \cdot 7$, $2^5 \cdot 3^2 \cdot 7$ or $2^4 \cdot 3^2 \cdot 7$.

If $|T_G| = 2^6 \cdot 3^2 \cdot 7$, let $\theta \in Irr(T_G)$ such that $[\chi_{T_G}, \theta] \neq 0$, then $\chi(1)/\theta(1) \big| |G : T_G| = 5$. Thus $\theta(1) = 64$. But $\theta(1)^2 > |T_G|$, a contradiction.

If $|T_G| = 2^5 \cdot 3^2 \cdot 7$, since T_G is solvable, then T_G has a subgroup D such that $|T_G : D| = 7$. Considering the permutation representation of T_G on the right cosets of D with kernel D_{T_G}, one has that $T_G/D_{T_G} \lesssim S_7$. Using Magma soft, we know that orders of solvable subgroups of S_7 divided by 7 are 7, 14 and 21. Hence, $|D_{T_G}|$ can be equal to one of $2^5 \cdot 3^2$, $2^4 \cdot 3^2$ or $2^5 \cdot 3$.

If $|D_{T_G}|=2^5 \cdot 3^2$, let $\varsigma \in Irr(D_{T_G})$ such that $[\chi_{D_{T_G}}, \varsigma] \neq 0$, then $\chi(1)/\varsigma(1) \big| |G : D_{T_G}|=5$, and so $\varsigma(1) = 32$. But $\varsigma(1)^2 > |D_{T_G}|$, a contradiction.

Similarly, we can prove that $|D_{T_G}| \neq 2^4 \cdot 3^2$ or $2^5 \cdot 3$.

If $|T_G| = 2^4 \cdot 3^2 \cdot 7$, since T_G is solvable, let I be a Hall-subgroup of T_G with order $|I| = 2^4 \cdot 3^2$ and $I_{T_G} = \bigcap_{g \in T_G} I^g \leq I$, then $T_G/I_{T_G} \lesssim S_7$. Since orders of solvable subgroups of S_7 divided by 7 are 7, 14 and 21, it follows that the order of I_{T_G} is one of $2^4 \cdot 3^2$, $2^3 \cdot 3^2$ and $2^4 \cdot 3$.

If $|I_{T_G}|=2^4 \cdot 3^2$, let $\Lambda \in Irr(I_{T_G})$ such that $[\chi_{I_{T_G}}, \Lambda] \neq 0$, then $\chi(1)/\Lambda(1) \big| |G : I_{T_G}|=70$. Hence $\Lambda(1) = 16$. But $\Lambda(1)^2 > |I_{T_G}|$, a contradiction.

By the same arguments as before, we can prove that $|I_{T_G}| \neq 2^4 \cdot 3$.

If $|I_{T_G}| = 2^3 \cdot 3^2$, let $\mu \in Irr(I_{T_G})$ such that $[\chi_{I_{T_G}}, \mu] \neq 0$, then we have $\chi(1)/\mu(1) \big| |G : I_{T_G}| = 2^3 \cdot 5 \cdot 7$, and so $\mu(1) = 8$. In this case, we see that there exists an irreducible character of degree 8 in I_{T_G}. By using Magma soft (cf. [10]), it is easy to check that the set $cd^*(I_{T_G})$ satisfying $8 \in Irr(I_{T_G})$ can only be one of

$$cd^*(I_{T_G}) = \{1, 1, 1, 1, 1, 1, 1, 1, 8\} \tag{1}$$

$$cd^*(I_{T_G}) = \{1, 1, 1, 1, 2, 8\} \tag{2}$$

Since $|T_G/I_{T_G}| = 2 \cdot 7$, $U/I_{T_G} \in Syl_7(T_G/I_{T_G})$, by Sylow Theorem, one has that $U/I_{T_G} \trianglelefteq T_G/I_{T_G}$. Hence $U \trianglelefteq T_G$ and $|U| = 2^3 \cdot 3^2 \cdot 7$. Let $\vartheta \in Irr(U)$ such that $[\beta_U, \vartheta] \neq 0$. Then $\beta(1)/\vartheta(1) \big| |G : U| = 2^3 \cdot 5$. Furthermore, we get that $\vartheta(1) = 7$ or 14. Let $\pi \in Irr(I_{T_G})$ such that $[\vartheta_{I_{T_G}}, \pi] \neq 0$. Set $e = [\vartheta_{I_{T_G}}, \pi]$, $t = |U : I_U(\pi)|$, one has that $\vartheta(1) = et\pi(1)$. By the structure of $cd^*(I_{T_G})$ in (1) and (2), we deduce that $e = 1$ and $t = 7$ since otherwise $[\vartheta_{I_{T_G}}, \vartheta_{I_{T_G}}] = e^2 t > |U : I_{T_G}|$, a contradiction. By the structure of I_{T_G}, it is easy to see that there exists no such irreducible character with an orbit of length 7 in I_{T_G}, a contradiction.

Therefore, G is nonsolvable. By Lemma 1, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic non-abelian simple groups and $|G/K| \big| |Out(K/H)|$. Since $|G| = 2^6 \cdot 3^2 \cdot 5 \cdot 7$, then by Table 1 we get that K/H can only be isomorphic to one of the groups A_5, A_6, $L_2(7)$, $L_2(8)$, A_7, $L_3(4)$ or A_8. And we write the proof up to what K/H is case by case.

Case 1. $K/H \ncong A_5$.

If $K/H \cong A_5$, then by Lemma 1 one has that $|G:K| = 1$ or 2.

If $|G:K| = 1$, then $|H| = 2^4 \cdot 3 \cdot 7$. If H is solvable, then H has a subgroup T with index 3. Considering the permutation of H on T with kernel T_H, we have $H/T_H \lesssim S_3$. Hence $|T_H| = 2^4 \cdot 7$ or $2^3 \cdot 7$.

If $|T_H|=2^4 \cdot 7$, let $\Pi \in Irr(T_H)$ such that $[\chi_{T_H}, \Pi] \neq 0$, then $\chi(1)/\Pi(1)\big||G:T_H|=2^2 \cdot 3^2 \cdot 5$. Thus $\Pi(1) = 16$. But $\Pi(1)^2 > |T_H|$, a contradiction. By the similar reason we have $|T_H| \neq 2^3 \cdot 7$.

We may assume that H is nonsolvable. By Lemma 1, H has a normal series: $1 \trianglelefteq N \trianglelefteq M \trianglelefteq H$ such that M/N is a direct product of isomorphic non-abelian simple groups and $|H/M|\big||Out(M/N)|$. Since $|H| = 2^4 \cdot 3 \cdot 7$, we obtain that $M/N \cong L_2(7)$. Hence, we have $|H/M|\big||Out(L_2(7))| = 2$, and so $|H:M| = 1$ or 2.

If $|H:M| = 1$, then $|N| = 2$. By [2, Clifford Theorem 6.2], N has an irreducible character of degree 2, which is impossible. If $|H:M| = 2$, then $|N| = 1$. In this case, we have $M \cong L_2(7) \cdot 2$. Let $\nu \in Irr(M)$ such that $[\chi_M, \nu] \neq 0$, then we have $\chi(1)/\nu(1)\big||G:M| = 2^2 \cdot 3 \cdot 5$, and hence $\nu(1) = 16$. By [1], we see that the largest irreducible character degree of $L_2(7) \cdot 2$ is 8. Therefore, M has no such irreducible character of degree 16, a contradiction.

If $|G:K| = 2$, then $|H| = 2^3 \cdot 3 \cdot 7$. If H is solvable, then H has a Hall-subgroup P with index 3. Considering the permutation of H on the right cosets of P with kernel P_H, one has that $H/P_H \lesssim S_3$. Therefore, $|P_H| = 2^3 \cdot 7$ or $2^2 \cdot 7$.

If $|P_H| = 2^3 \cdot 7$, let $\varpi \in Irr(P_H)$ such that $[\chi_{P_H}, \varpi] \neq 0$, $\chi(1)/\varpi(1)\big||G:P_H| = 2^3 \cdot 3^2 \cdot 5$, so $\varpi(1) = 8$. But $\varpi(1)^2 > |P_H|$, a contradiction. If $|P_H| = 2^2 \cdot 7$, let $P_7 \in Syl_7(P_H)$, by Sylow Theorem, we have $P_7 \trianglelefteq P_H$. On the other hand, since $P_H \lhd \lhd G$, one has that $P_7 \trianglelefteq G$. Thus $\beta(1)\big||G:P_7|$, a contradiction.

If H is nonsolvable, by Table 1, we can get that $H \cong L_2(7)$. Therefore, $K \cong A_5 \times L_2(7)$. Let $\xi \in Irr(K)$ such that $[\beta_K, \xi] \neq 0$, then we have $\beta(1)/\xi(1)\big||G:K| = 2$, and thus $35\big|\xi(1)$. Let $e = [\beta_K, \xi]$, $t = |G : I_G(\xi)|$, one has that $\beta(1) = et\xi(1)$. By the structure of K, we have $\xi(1) = 35$, $e = 2$ and $t = 1$. It is easy to check that $[\beta_K, \beta_K] = e^2t = 4 > |G:K| = 2$, a contradiction to [2, Lemma 2.29].

Case 2. $K/H \ncong A_6$.

If not, since $|Out(A_6)| = 4$, then by Lemma 1 we have $|G:K| = 1$, 2 or 4.

If $|G:K| = 1$, then $|H| = 2^3 \cdot 7$. Let $\mu \in Irr(H)$ such that $[\chi_H, \mu] \neq 0$, then we get that $\chi(1)/\mu(1)\big||G:H| = 2^3 \cdot 3^2 \cdot 5$, and hence $\mu(1) = 8$. But $\mu(1)^2 > |H|$, a contradiction.

If $|G:K|=2$, then $|H|=2^2\cdot 7$. Let $\kappa\in Irr(H)$ such that $[\chi_H,\kappa]\neq 0$, one has that $\chi(1)/\kappa(1)\big||G:H|=2^4\cdot 3^2\cdot 5$, and so $\kappa(1)=7$. But $\kappa(1)^2>|H|$, a contradiction. Similarly, we can prove that $|G:K|\neq 4$.

Case 3. $K/H\ncong L_2(7)$.

Otherwise, we assume that $K/H\cong L_2(7)$. In this case, we have $|G:K|=1$ or 2.

If $|G:K|=1$, then $|H|=2^3\cdot 3\cdot 5$. If H is solvable, by Lemma 2, the Sylow 5-subgroup H_5 is normal in H. Since $H\trianglelefteq G$, we have $H_5\trianglelefteq G$. Therefore, $\beta(1)\big||G:H_5|=2^6\cdot 3^2\cdot 7$, a contradiction. If H is nonsolvable, then by Lemma 1 one has that H has a normal series: $1\trianglelefteq N\trianglelefteq M\trianglelefteq H$ such that M/N is a direct product of isomorphic non-abelian simple groups and $|H/M|\big||Out(M/N)|$. Since $|H|=2^3\cdot 3\cdot 5$, then by Table 1 we have $M/N\cong A_5$. Thus $|H/M|\big||Out(A_5)|=2$. And so $|H:M|=1$ or 2.

If $|H:M|=1$, then $|N|=2$. By [2, Clifford Theorem 6.2], N has an irreducible character of degree 2, a contradiction. If $|H:M|=2$, then $|N|=1$. In the case that $M\cong A_5\cdot 2$. Let $\nu\in Irr(M)$ such that $[\chi_M,\nu]\neq 0$, then we have $\chi(1)/\nu(1)\big||G:M|=2^3\cdot 3\cdot 7$, and hence $\nu(1)=8$. By [1], we see that the largest irreducible character degree of $A_5\cdot 2$ is 6. Therefore, M has no such irreducible character of degree 8, a contradiction.

If $|G:K|=2$, then $|H|=2^2\cdot 3\cdot 5$. If H is solvable, then by Lemma 2 we have that $H_5\trianglelefteq H$, where $H_5\in Syl_5(H)$. Since $H\trianglelefteq G$, we have $H_5\trianglelefteq G$. Hence $\beta(1)\big||G:H_5|=2^6\cdot 3^2\cdot 7$, a contradiction. If H is nonsolvable, then by Lemma 1 and Table 1 one has that $H\cong A_5$. Therefore $K\cong A_5\times L_2(7)$. Let $\xi\in Irr(K)$ such that $[\beta_K,\xi]\neq 0$, then we have $\beta(1)/\xi(1)\big||G:K|=2$, and thus $35\big|\xi(1)$. Let $e=[\beta_K,\xi]$, $t=|G:I_G(\xi)|$, one has that $\beta(1)=et\xi(1)$. By the structure of K, we obtain that $\xi(1)=35$, and so $e=2$, $t=1$. But $[\beta_K,\beta_K]=e^2t=4>|G:K|=2$, a contradiction to [2, Lemma 2.29].

Case 4. $K/H\ncong L_2(8)$.

Assume the contrary, suppose that $K/H\cong L_2(8)$. By Lemma 1, one has that $|G:K|=1$ or 3. Since $3^2\parallel|L_2(8)|$, we have $|G:K|=1$. In the case that $|H|=2^3\cdot 5$. By Sylow Theorem, the 5-Sylow subgroup H_5 is normal in H. As $H\trianglelefteq G$, we have $H_5\trianglelefteq G$. Therefore, $\beta(1)\big||G:H_5|=2^6\cdot 3^2\cdot 7$, a contradiction.

Case 5. $K/H\ncong A_7$.

If $K/H\cong A_7$, then by Lemma 1 we have $|G:K|=1$ or 2, and so $|H|=4$ or 8. By [2, Clifford Theorem 6.2], H has an irreducible character of degree 4 or 8, a contradiction to the structure of H.

Case 6. $K/H \ncong L_3(4)$.

We may suppose that $K/H \cong L_3(4)$. By comparing the orders of G and $L_3(4)$, we have $H = 1$, and so $G \cong L_3(4)$. But we know that the largest irreducible character degree of $L_3(4)$ equals to 64, a contradiction to $L_1(G) = 70$.

Case 7. $K/H \cong A_8$.

In this case, we have $H = 1$, and hence $G \cong A_8$.

Thus, we complete the proof of Main Theorem.

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ON SOME ALGEBRAIC PROPERTIES OF SOFT SETS

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Abstract. This paper studies some algebraic and lattice properties of soft sets. A soft binary operation is introduced and a few interesting results are investigated in this context.

Keywords: soft set, soft binary operation, soft relation, soft group.

AMS Subject Classification: 11E57, 20-XX, 20K27, 03Exx.

Introduction

The soft set theory was introduced by Molodtsov [6] in 1999 as a general mathematical tool to deal with problems involving uncertainty or vagueness. Molodtsov recognized the importance of the role of parameters in modeling various problems involving uncertainty and the soft set models worked very well in dealing with such problems. He has shown several applications of this theory in many fields like economics, engineering, medical sciences, etc. Among many theories like Fuzzy set theory, Rough set theory etc., the soft set theory became very significant because of its wide range of applicability. The development in the fields of soft set theory and its application has been taking place in a rapid pace.

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In 2010, Babitha and Sunil [2] introduced the concepts of Soft set relations and Soft equivalence relation. In this present work, we define a soft binary operation and investigate some interesting results. We also introduce a soft relation in a different approach unlike the work of Babitha and Sunil. Some algebraic and lattice properties of soft sets are established.

In what follows, U, E and $P(U)$ stand for the universe set, the set of parameters and the collection of all subsets of U respectively.

1. Preliminaries

Definition 1.1. A pair (F, A) is called a soft set over U if $A \subset E$ and $F : A \to P(U)$. We write F_A for the pair (F, A). The soft set $F : E \to P(U)$ is denoted by F_E .

Definition 1.2. Let F_A and G_B be soft sets over a common universe set U and $A, B \subset E$. Then we say that

(a) F_A is a *soft subset* of G_B, denoted by $F_A \widetilde{\subset} G_B$, if

(i) $A \subset B$, and

(ii) $F(e) \subset G(e) \forall e \in A$.

(b) F_A equals G_B , denoted by $F_A \cong G_B$, if $F_A \widetilde{\subset} G_B$ and $G_B \widetilde{\subset} F_A$.

Definition 1.3. A soft set F_A over U is called a *null soft set*, denoted by Φ, if $e \in A$, $F(e) = \phi$.

Definition 1.4. A soft set F_A over U is called an *absolute soft set*, denoted by $\widetilde{A}$, if $e \in A$, $F(e) = U$

Definition 1.5. The *union* of two soft sets F_A and G_B over a common universe U is the soft set H_C, where $C = A \cup B$, and for all $e \in C$,

$$H(e) = \begin{cases} F(e) & \text{if } e \in A - B, \\ G(e) & \text{if } e \in B - A, \\ F(e) \cup G(e) & \text{if } e \in A \cap B. \end{cases}$$

We write $F_A \cup G_B \cong H_C$.

Definition 1.6. The intersection of two soft sets F_A and G_B over a common universe U is the soft set H_C, where $C = A \cap B$, and for all $e \in C$, $H(e) = F(e) \cap G(e)$. We write $F_A \cap G_B \cong H_C$.

Definition 1.7. For a soft set F_A over U, the relative *complement* of F_A is denoted by F_A^C and is defined by $F_A^C \cong F_A^1$, where $F^1 : A \to P(U)$ is a mapping given by $F^1(e) = U - F(e)$ for all $e \in A$.

2. Algebraic properties

This section is devoted to some algebraic notions of Soft set theory which are different from others. We introduce and define some soft operations and also we establish a few interesting results in this context.

Definition 2.1. Let $*$ be a binary operation on U. If F_E and G_E are two soft sets over U, then we define $F_E \circ G_E$ as a soft set $H : E \to P(U)$ such that

$$H(e) = \{x * y : x \in F(e) \text{ and } y \in G(e)\}.$$

We denote the collection of all soft sets over U with domain E by the symbol $S_E(U)$.

Proposition 2.2. *$\circ$ is a binary operation on $S_E(U)$.*

Proof. Let $F_E \in S_E(U)$ and $G_E \in S_E(U)$. Then $F_E \circ G_E$ is a soft set $H : E \to P(U)$, defined by

$$H(e) = \{x * y : x \in F(e) \text{ and } y \in G(e)\}.$$

Since $*$ is a binary operation on U, $x * y \in U$ for every $x, y \in U$, it results

$$H(e) = \{x * y : x \in F(e) \text{ and } y \in G(e)\} \subseteq U.$$

Hence $H_E \in S_E(U)$. This proves that $\circ$ is a binary operation on $S_E(U)$. ∎

Proposition 2.3. *If $(U, *)$ is a group then so is $((S_E(U), \circ)$.*

Proof. Suppose that $(U, *)$ is a group. Then, by Proposition 2.2, $\circ$ is a binary operation on $S_E(U)$.

Let F_E, G_E and H_E be any three soft sets in $S_E(U)$.

$$\begin{aligned}[F_E \circ (G_E \circ H_E)](e) &= \{x * (y * z) : x \in F(e),\ y \in G(e) \text{ and } z \in H(e)\} \\ &= \{x * y) * z) : x \in F(e),\ y \in G(e) \text{ and } z \in H(e)\} \\ &= [(F_E \circ G_E) \circ H_E](e) \ \ \forall\ e \in E.\end{aligned}$$

Hence

$$F_E \circ (G_E \circ H_E) = (F_E \circ G_E) \circ H_E.$$

Thus $\circ$ is associative on $S_E(U)$.

Let $\theta \in U$ be the identity element in $(U, *)$. Define a soft set $I : E \to P(U)$ by

$$I(e) = \{\theta\} \ \ \forall\ e \in E.$$

Clearly, $I_E \in S_E(U)$. Then,

$$\begin{aligned}(F_E \circ I_E)(e) &= \{x * \theta : x \in F(e)\} \\ &= F_E(e) \ \ \forall\ e \in E \\ &\Rightarrow \ F_E \circ I_E \cong F_E.\end{aligned}$$

Similarly, $I_E \circ F_E \cong F_E$. Thus I_E is the identity element in $S_E(U)$.

Let $F_E \in S_E(U)$. Define a soft set $F^{-1} : E \to P(U)$ by

$$F^{-1}(e) = \{x^{-1} : x \in F(e)\} \ \ \forall\ e \in E.$$

Then $F_E^{-1} \in S_E(U)$ and $F_E \circ F_E^{-1}$ $F_E^{-1} \circ F_E \cong I_E$. Hence $(S_E(U), \circ)$ is a group. ∎

Proposition 2.4. *$*$ is Commutative on $U \Leftrightarrow \circ$ is Commutative on $S_E(U)$.*

Proof. Suppose $*$ is commutative on U. Let $F_E, G_E \in (S_E(U)$. Then

$$\begin{aligned}(F_E \circ G_E)(e) &= \{x * y : x \in F(e) \text{ and } y \in G(e)\} \\ &= \{y * x : y \in G(e) \text{ and } x \in F(e)\} \\ &= (G_E \circ F_E)(e) \ \ \forall\ e \in E.\end{aligned}$$

Hence

$$F_E \circ G_E \cong G_E \circ F_E.$$

Thus $\circ$ is commutative on $S_E(U)$.

Conversely, suppose that $\circ$ is commutative on $S_E(U)$. Let $x, y \in U$. Define $F : E \to P(U)$ and $G : E \to P(U)$ by

$$F(e) = \{x\} \text{ and } G(e) = \{y\} \ \forall\ e \in E.$$

Since $\circ$ is commutative on $S_E(U)$, we have

$$\begin{aligned}F_E \circ G_E \cong G_E \circ F_E &\Rightarrow \ (F_E \circ G_E)(e) = (G_E \circ F_E)(e) \ \ \forall\ e \in E \\ &\Rightarrow \ \{x * y\} = \{y * x\} \\ &\Rightarrow \ x * y = y * x.\end{aligned}$$

Hence $*$ is commutative on U. ∎

Definition 2.5. Let $(U, +, \cdot)$ be a ring. Let $F_E, G_E \in S_E(U)$. Then we define

$$\begin{aligned}(F_E \oplus G_E)(e) &= \{x + y : x \in F(e) \text{ and } y \in G(e)\} \\ (F_E \odot G_E)(e) &= \{x \cdot y : x \in F(e) \text{ and } y \in G(e)\}.\end{aligned}$$

Proposition 2.6. *If $(U, +, \cdot)$ is a commutative ring then $(S_E(U), \oplus, \odot)$ is a soft commutative ring.*

3. Lattice properties

Proposition 3.1. *The collection $S_E(U)$ is a partially ordered set with respect to the relation $\leq$ defined by*

$$F_E \leq G_E \Leftrightarrow F(x) \subseteq G(x) \text{ for all } x \in E.$$

Proof. Since $F(x) \subseteq F(x)$ for all $x \in E$, we have $F_E \leq F_E$. Hence $\leq$ is reflexive on $S_E(U)$.

Let F_E and G_E be two soft sets over U such that $F_E \leq G_E$ and $G_E \leq F_E$.

$\Rightarrow \quad F(x) \subseteq G(x)$ and $G(x) \subseteq F(x)$ for all $x \in E$.

$\Rightarrow \quad F(x) = G(x)$ for all $x \in E$.

$\Rightarrow \quad F_E \cong G_E$.

Hence $\leq$ is anti-symmetric on $S_E(U)$.

Let F_E, G_E and H_E be three soft sets over U such that $F_E \leq G_E$ and $G_E \leq H_E$.

$\Rightarrow \quad F(x) \subseteq G(x)$ and $G(x) \subseteq H(x)$ for all $x \in E$.

$\Rightarrow \quad F(x) \subseteq H(x)$ for all $x \in E$.

$\Rightarrow \quad F_E \leq H_E$.

$\Rightarrow \quad \leq$ is transitive on $S_E(U)$.

Thus $\leq$ is a partially ordered relation on $S_E(U)$. ∎

Proposition 3.2. *$(S_E(U), \leq)$ is a lattice.*

Proof. If $F_E \in S_E(U)$ and $G_E \in S_E(U)$ then the lub and glb of F_E and G_E are defined by the soft sets $F_E \vee G_E$ and $F_E \wedge G_E$ respectively as follows.

$(F_E \vee G_E)(e) = F(e) \cup (G(e)$ for all $e \in E$

$(F_E \wedge G_E)(e) = F(e) \cap (G(e)$ for all $e \in E$

Hence $F_E \vee G_E$ and $F_E \wedge G_E$ are lub and glb of F_E and G_E respectively in $S_E(U)$. ∎

Definition 3.3. Let $\{(F_\alpha)_E : \alpha \in \Delta\}$ be any collection of soft sets in $S_E(U)$. Then we define $\bigcup_{\alpha \in \Delta} (F_\alpha)_E$ to be a soft set $H : E \to P(U)$ such that

$$H(x) = \bigcup_{\alpha \in \Lambda} (F_\alpha(x))$$

for all $x \in E$.

Definition 3.4. Let $\{(F_\alpha)_E : \alpha \in \Delta\}$ be any collection of soft sets in $S_E(U)$. Then we define $\bigcap_{\alpha \in \Delta} (F_\alpha)_E$ to be a soft set $H : E \to P(U)$ such that

$$H(x) = \bigcap_{\alpha \in \Lambda} (F_\alpha(x))$$

for all $x \in E$.

Proposition 3.5. *$(S_E(U), \leq)$ is a complete lattice.*

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FUZZY PARAMETERIZED FUZZY SOFT NORMAL SUBGROUPS OF GROUPS

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Abstract. In the present paper, we redefine the concept of $FPFS$-sets in algebra systems, which is a novel way that is different from the definition of $FPFS$-sets in hemirings by Liu [10]. Based on the definition, we put forth $FPFS$-groups, $FPFS$-normal subgroups and FP-equivalent fuzzy soft normal subgroups of groups. Further, some properties and characterizations are investigated. Finally, aggregate fuzzy normal subgroups of groups are given.

Keyword: $FPFS$-sets; $FPFS$-groups; $FPFS$-normal subgroups; FP-equivalent fuzzy soft normal subgroups; Aggregate fuzzy normal subgroups.
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1. Introduction

In the real world, there are many problems that we met are full of indeterminacy and vagueness. Facing so many uncertain data, classical methods are not always successful, the reason is that these methods are designed for certain situations and the actual situations that under consideration are often more complex. In 1965, Zadeh proposed the fuzzy set theory, which could as a mathematical approach to deal with inexact and uncertain knowledge. After that, fuzzy set theory is progressing rapidly. In 1971, Rosenfeld [17] applied this concept to the theory of groupoids and groups. Especially Liu [11] investigated the fuzzy isomorphism theorems of groups. However, fuzzy set theory have its inherent difficulties which was pointed out in [15]. To overcome the difficulties, in 1999, Molodtsov [15] first put forward the concept of soft set theory as a new mathematical tool for dealing with uncertainty and vagueness. It has been proven useful in many other fields. Moreover, it has opened a new direction, new exploration, new path of thinking to

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mathematicians, engineers, computer scientists and many other researchers and so on. Up to the present, the research on soft sets is progressing rapidly. Several researchers have point out several directions for the applications of soft sets, more details (see e.g. [1], [5], [8], [9], [13], [14], [16]), and some of them also investigated the operations on the theory of soft sets. In particular, an adjustable approach to fuzzy soft set based decision making was applied by Feng [7]. Also, Maji [12] applied the soft set theory to the fuzzy set theory and investigated some related properties of them.

Following the discovery of the soft set theory, some researchers applied the theory to the algebraic structure. In 2007, Aktas et al. [1] applied the notion of soft sets to the theory of groups and investigated some properties of them. Further, Yang [18] given the notions of fuzzy soft semigroups and fuzzy soft ideals and discussed fuzzy soft images and fuzzy soft inverse images of fuzzy soft semigroups (ideals). In particular, Zhan investigated the ideal theory on hemirings, most of relevant conclusions have been already demonstrated in Zhan's book, which is referred to [19].

Recently, Çağman et al. [3], [4] put forward the concepts of fuzzy parameterized fuzzy soft set ($FPFS$-sets) and fuzzy parameterized soft set (FP-soft sets), respectively. Moreover, Liu [10] applied this theory to hemirings and some basic properties are investigated. The main purpose of this paper is to give a novel definition of $FPFS$-sets by a new way and study some related properties. This paper is organized as follows: we recall some concepts and results on groups and $FPFS$-sets in Section 2. In Section 3, we introduce the concepts of $FPFS$-normal subgroups and FP-equivalent fuzzy soft normal subgroups by making the set of parameters to be an algebraic structure. In addition, some properties and characterizations are investigated. Finally, we give the aggregate fuzzy normal subgroups of groups in Section 4.

2. Preliminaries

Definition 2.1 [17] A fuzzy subset μ of a group G is said to be a fuzzy subgroup of G if

(1) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$,

(2) $\mu(x^{-1}) \geq \mu(x)$,

holds for all $x, y \in G$.

Equivalently, $\mu(xy^{-1}) \geq \mu(x) \wedge \mu(y)$ for all $x, y \in G$.

Definition 2.2 [6] A fuzzy subset μ of a group G is said to be a fuzzy normal subgroup of G if the following axioms hold:

$$\mu(xyx^{-1}) \geq \mu(y) \text{ for all } x, y \in G.$$

Equivalently, $\mu(xyx^{-1}) = \mu(y)$ for all $x, y \in G$ or $\mu(xy) = \mu(yx)$ for all $x, y \in G$.

The following concepts are referred to [3].

Definition 2.3 Let U be an initial universe, E be the set of all parameters and X be a fuzzy set over E with the membership function $\mu_X : E \to [0,1]$ and $\gamma_X(x)$ be a fuzzy set over E for all $x \in E$, $F(U)$ be the set of all fuzzy set of U. Then a fuzzy parameterized fuzzy soft set Γ_X on U is defined by a function $\gamma_X(x)$ representing a mapping

$$\gamma_X : E \to F(U) \text{ such that } \gamma_X(x) = \emptyset \text{ if } \mu_X(x) = 0.$$

Here γ_X is called fuzzy approximate function of the fuzzy parameterized fuzzy soft set Γ_X, and the value $\gamma_X(x)$ is a fuzzy set called x-element of the fuzzy parameterized fuzzy soft set for all $x \in E$. Thus a fuzzy parameterized fuzzy soft set Γ_X over U can be represented by the set of ordered pairs

$$\Gamma_X = \{(\mu_X(x)/x, \gamma_X(x))) : x \in E, \gamma_X(x) \in F(U), \mu_X(x) \in [0,1]\}.$$

A fuzzy parameterized fuzzy soft set is briefly said to be an $FPFS$-set. The set of all $FPFS$-sets over U is denoted by $FPFS(U)$.

Definition 2.4 Let $\Gamma_X \in FPFS(U)$.

(1) If $\gamma_X(x) = \emptyset$ for all $x \in E$, then Γ_X is called an X-empty $FPFS$-set, denoted by $\Gamma_{\emptyset_X}$.

(2) If $X = \emptyset$, then the Γ_X is called an empty $FPFS$-set, denoted by $\Gamma_\emptyset$.

(3) If $\mu_X(x) = 1$ and $\gamma_X(x) = U$ for all $x \in E$, then Γ_X is called an X-universal $FPFS$-set, denoted by Γ_X.

(4) If $X = E$, then the X-universal $FPFS$-set is called an universal $FPFS$-set, denoted by Γ_E.

Definition 2.5 Let $\Gamma_X, \Gamma_Y \in FPFS(U)$.

(1) Γ_X is an $FPFS$-subset of Γ_Y, denoted by $\Gamma_X \widetilde{\subseteq} \Gamma_Y$, if $\mu_X(x) \leq \mu_Y(x)$ and $\gamma_X(x) \subseteq \gamma_Y(x)$ for all $x \in E$.

(2) Γ_X and Γ_Y are FP-equal, denoted by $\Gamma_X = \Gamma_Y$, if $\mu_X(x) = \mu_Y(x)$ and $\gamma_X(x) = \gamma_Y(x)$ for all $x \in E$.

Definition 2.6 Let $\Gamma_X \in FPFS(U)$. Then the complement of Γ_X, denoted by $\Gamma_X^{\widetilde{c}}$, is an $FPFS$-set defined by

$$\mu_X^{\widetilde{c}}(x) = 1 - \mu(x) \text{ and } \gamma_Y^{\widetilde{c}}(x) = U \setminus \gamma_X(x).$$

Definition 2.7 Let $\Gamma_X, \Gamma_Y \in FPFS(U)$.

(1) The intersection of Γ_X and Γ_Y, denoted by $\Gamma_X \widetilde{\cap} \Gamma_Y$, is defined by $\mu_{X\widetilde{\cap}Y}(x) = \min\{\mu_X(x), \mu_Y(x)\}$ and $\gamma_{X\widetilde{\cap}Y}(x) = \gamma_X(x) \cap \gamma_Y(x)$ for all $x \in E$.

(2) The union of Γ_X and Γ_Y, denoted by $\Gamma_X \widetilde{\cup} \Gamma_Y$, is defined by $\mu_{X\widetilde{\cup}Y}(x) = \max\{\mu_X(x), \mu_Y(x)\}$ and $\gamma_{X\widetilde{\cap}Y}(x) = \gamma_X(x) \cup \gamma_Y(x)$ for all $x \in E$.

3. Fuzzy parameterized fuzzy soft normal subgroups

In this section, we give the definitions of $FPFS$-groups ($FPFS$-normal subgroups) and FP-equivalent fuzzy normal subgroups of groups, then some related properties and characterizations of them are investigated.

Definition 3.1 Let G_1 and G_2 be two groups as an initial universe and a set of all parameters, respectively. Let X be a fuzzy set over G_2 with $\mu_X : G_2 \to [0,1]$ and γ_X be a fuzzy set over G_1 for all $x \in G_2$. Assume that $\Gamma_X = \{(\mu_X(x)/x, \gamma_X(x)) \mid x \in G_2, \gamma_X(x) \in F(G_1), \mu_X(x) \in [0,1]\} \in FPFS(G_1)$. Then Γ_X is called a fuzzy parameterized fuzzy soft group (fuzzy parameterized fuzzy soft normal subgroup) (briefly, $FPFS$-group or $FPFS$-normal subgroup) over G_1 if for all $x \in G_2$, X and $\gamma_X(x)$ are two fuzzy subgroups (fuzzy normal subgroups) of G_2 and G_1, respectively.

Example 3.2

(1) Let $G_1 = S_3 = \{e, (12), (23), (13), (123), (132)\}$ be a group as an initial universe and $G_2 = \{e, (12)\}$ be a group as a set of all parameters, respectively. If $X = \{0.5/e, 0.4/(12)\}$,

$$\begin{aligned} \gamma_X(e) &= \{0.9/e, 0.8/(12), 0.4/(13), 0.3/(23), 0.5/(123), 0.5/(132)\}, \\ \gamma_X((12)) &= \{0.8/e, 0.7/(12), 0.4/(13), 0.4/(23), 0.5/(123), 0.5/(132)\}, \end{aligned}$$

for all $x \in G_2$, we can verify that X and $\gamma_X(x)$ are two fuzzy subgroups of G_2 and G_1, respectively. So Γ_X is an $FPFS$-group over G_1.

(2) Let $G_1 = G_2 = \{e, (12)\}$ be two groups as an initial universe and a set of all parameters, respectively. If X=$\{0.5/e, 0.4/(12)\}$, $\gamma_X(e)$=$\{0.9/e, 0.8/(12)\}$, $\gamma_X((12)) = \{0.8/e, 0.7/(12)\}$, for all $x \in G_2$, we can verify that X and $\gamma_X(x)$ are two fuzzy normal subgroups of G_2 and G_1, respectively. So Γ_X is an $FPFS$-normal subgroup over G_1.

Proposition 3.3 *Let Γ_X and Γ_Y be two $FPFS$-normal subgroups over G_1. Then their intersection $\Gamma_X \widetilde{\cap} \Gamma_Y$ is also an $FPFS$-normal subgroup over G_1.*

Proof. We can write $\Gamma_X \widetilde{\cap} \Gamma_Y = \Gamma_{X \widetilde{\cap} Y}$. Since Γ_X and Γ_Y are two $FPFS$-normal subgroups over G_1, it follows that X and Y are two fuzzy normal subgroups of G_2, then for all $x, y \in G_2$,

$$\begin{aligned} \mu_{X \widetilde{\cap} Y}(xyx^{-1}) &= \min\{\mu_X(xyx^{-1}), \mu_Y(xyx^{-1})\} \\ &\geq \min\{\mu_X(y), \mu_Y(y)\} \\ &= \mu_{X \widetilde{\cap} Y}(y). \end{aligned}$$

So $X \cap Y$ is a fuzzy normal subgroup of G_2. Now we shall prove $\gamma_X(x) \cap \gamma_Y(x)$ is a fuzzy normal subgroup of G_1.

For all $x \in G_2$, $s, t \in G_1$,

$$\begin{aligned}(\mu_{\gamma_X(x)} \cap \mu_{\gamma_Y(x)})(sts^{-1}) &= \min\{\mu_{\gamma_X(x)}(sts^{-1}), \mu_{\gamma_Y(x)}(sts^{-1})\} \\ &\geq \min\{\mu_{\gamma_X(x)}(t), \mu_{\gamma_Y(x)}(t)\} \\ &= (\mu_{\gamma_X(x)} \cap \mu_{\gamma_Y(x)})(t).\end{aligned}$$

Hence $\mu_{\gamma_X(x)} \cap \mu_{\gamma_Y(x)}$ is a fuzzy normal subgroup of G_1, that is to say $\gamma_{X\widetilde{\cap}Y} = \gamma_X(x) \cap \gamma_Y(x)$ is a fuzzy normal subgroup of G_1. Therefore, $\Gamma_X \widetilde{\cap} \Gamma_Y$ is an $FPFS$-normal subgroup over G_1. ∎

We know that the intersection of all $FPFS$-normal subgroups over a group G_1 is also an $FPFS$-normal subgroup over G_1. Then we would consider whether the union of $FPFS$-normal subgroups over G_1 is also an $FPFS$-normal subgroup over G_1.

Definition 3.4 Let Γ_X and Γ_Y be two $FPFS$-normal subgroups over G_1. Then we said the sequence of values are ordered, if for any $x, y \in G_2$, $s, t \in G_1$, $\mu_X(x) \geq \mu_X(y)$, $\gamma_X(x)(s) \geq \gamma_X(x)(t)$ implies $\mu_Y(x) \geq \mu_Y(y)$, $\gamma_Y(x)(s) \geq \gamma_Y(x)(t)$.

Proposition 3.5 *Let Γ_X and Γ_Y be two $FPFS$-normal subgroups over G_1 with ordered sequence of values. Then their union $\Gamma_X \widetilde{\cup} \Gamma_Y$ is still an $FPFS$-normal subgroup over G_1.*

Proof. We can write $\Gamma_X \widetilde{\cup} \Gamma_Y = \Gamma_{X\widetilde{\cup}Y}$. For all $x, y \in G_2$, $s, t \in G_1$. Let

$$\begin{aligned}P_1 &= \max\{\min\{\mu_X(x), \mu_X(y)\}, \min\{\mu_Y(x), \mu_Y(y)\}\}, \\ P_2 &= \min\{\max\{\mu_X(x), \mu_Y(x)\}, \max\{\mu_X(y), \mu_Y(y)\}\}, \\ S_1 &= \max\{\min\{\mu_{\gamma_X(x)}(s), \mu_{\gamma_X(x)}(t)\}, \min\{\mu_{\gamma_Y(x)}(s), \mu_{\gamma_Y(x)}(t)\}\}, \\ S_2 &= \min\{\max\{\mu_{\gamma_X(x)}(s), \mu_{\gamma_Y(x)}(s)\}, \max\{\mu_{\gamma_X(x)}(t), \mu_{\gamma_Y(x)}(t)\}\}.\end{aligned}$$

We know that $P_1 \geq P_2$. In fact, since the sequence of values are ordered. Let $\mu_X(x) \geq \mu_X(y)$, $\gamma_X(x)(s) \geq \gamma_X(x)(t)$. We have $\mu_Y(x) \geq \mu_Y(y)$, $\gamma_Y(x)(s) \geq \gamma_Y(x)(t)$. It follows that $P_1 = \max\{\mu_X(y), \mu_Y(y)\}$.
(1) if $\max\{\mu_X(x), \mu_Y(x)\} > \max\{\mu_X(y), \mu_Y(y)\}$, then $P_2 = \max\{\mu_X(y), \mu_Y(y)\}$, in this case, we get $P_1 = P_2$;
(2) if $\max\{\mu_X(x), \mu_Y(x)\} \leq \max\{\mu_X(y), \mu_Y(y)\}$, then $P_2 = \max\{\mu_X(x), \mu_Y(x)\}$, we get $P_1 \geq P_2$. Thus, in any case, we have $P_1 \geq P_2$. Similarly, we have $S_1 \geq S_2$.

Now, we have

$$\begin{aligned}\max\{\mu_X(x), \mu_Y(x)\} &\geq \max\{\mu_X(y), \mu_Y(y)\}, \\ \max\{\mu_{\gamma_X(x)}(s), \mu_{\gamma_Y(x)}(s)\} &\geq \max\{\mu_{\gamma_X(x)}(t), \mu_{\gamma_Y(x)}(t)\}.\end{aligned}$$

$$\begin{aligned}\mu_{X\widetilde{\cup}Y}(xyx^{-1}) &= \max\{\mu_X(xyx^{-1}), \mu_Y(xyx^{-1})\} \\ &\geq \max\{\min\{\mu_X(x), \mu_X(y)\}, \min\{\mu_Y(x), \mu_Y(y)\}\} \\ &\geq \min\{\max\{\mu_X(x), \mu_Y(x)\}, \max\{\mu_X(y), \mu_Y(y)\}\} \\ &= \max\{\mu_X(y), \mu_Y(y)\} \\ &= \mu_{X\widetilde{\cup}Y}(y).\end{aligned}$$

So $X \cup Y$ is a fuzzy normal subgroup of G_2. In a similar way, we have $(\mu_{\gamma_X(x)} \cup \mu_{\gamma_Y(x)})(st^{-1}s) \geq (\mu_{\gamma_X(x)} \cup \mu_{\gamma_Y(x)})(t)$. That is $(\gamma_{X\widetilde{\cup}Y})(x) = \gamma_X(x) \cup \gamma_Y(x)$ is a fuzzy normal subgroup of G_1.
Therefore, $\Gamma_X \widetilde{\cup} \Gamma_Y$ is an $FPFS$-normal subgroup over G_1. ∎

Definition 3.6 [2] Let μ and ν be fuzzy sets in a set G. The Cartesian product of μ and ν is defined by

$$(\mu \times \nu)(x, y) = \min\{\mu(x), \nu(y)\}, \forall x, y \in G.$$

Definition 3.7 The multiplication of Γ_X and Γ_Y, denoted by $\Gamma_X \widetilde{\times} \Gamma_Y$, is defined by $\mu_{X\tilde{\times}Y}(x, y) = \min\{\mu_X(x), \mu_Y(y)\}$ and $\gamma_{X\tilde{\times}Y}(x) = \gamma_X(x) \times \gamma_Y(x)$ for all $x \in G_2$.

Proposition 3.8 *Let Γ_X and Γ_Y be two $FPFS$-normal subgroups over G_1. Then $\Gamma_X \widetilde{\times} \Gamma_Y$ is an $FPFS$-normal subgroup over $G_1 \times G_1$.*

Proof. We can write $\Gamma_X \widetilde{\times} \Gamma_Y = \Gamma_{X\tilde{\times}Y}$. Since Γ_X and Γ_Y are two $FPFS$-normal subgroups over G_1, it follows that X and Y are two fuzzy normal subgroups of G_2, then for all $x, y, z, w \in G_2$,

$$\begin{aligned} \mu_{X\tilde{\times}Y}(xyx^{-1}, zwz^{-1}) &= \min\{\mu_X(xyx^{-1}), \mu_Y(zwz^{-1})\} \\ &\geq \min\{\mu_X(y), \mu_Y(w)\} \\ &= \mu_{X\tilde{\times}Y}(y, w). \end{aligned}$$

So $X \times Y$ is a fuzzy normal subgroup of $G_2 \times G_2$. Now we shall prove $\gamma_X(x) \times \gamma_Y(x)$ is a fuzzy normal subgroup of $G_1 \times G_1$.

For all $x \in G_2$, $s, t, a, b \in G_1$,

$$\begin{aligned} (\mu_{\gamma_X(x)} \times \mu_{\gamma_Y(x)})(sts^{-1}, aba^{-1}) &= \min\{\mu_{\gamma_X(x)}(sts^{-1}), \mu_{\gamma_Y(x)}(aba^{-1})\} \\ &\geq \min\{\mu_{\gamma_X(x)}(t), \mu_{\gamma_Y(x)}(b)\} \\ &= (\mu_{\gamma_X(x)} \times \mu_{\gamma_Y(x)})(t, b). \end{aligned}$$

Hence $\mu_{\gamma_X(x)} \times \mu_{\gamma_Y(x)}$ is a fuzzy normal subgroup of $G_1 \times G_1$, that is to say $\gamma_{X\tilde{\times}Y} = \gamma_X(x) \times \gamma_Y(x)$ is a fuzzy normal subgroup of $G_1 \times G_1$. Therefore, $\Gamma_X \widetilde{\times} \Gamma_Y$ is an $FPFS$-normal subgroup over $G_1 \times G_1$. ∎

Definition 3.9 Let Γ_X be an $FPFS$-normal subgroup over G_1, Γ_Y be an $FPFS$-group over G_1. Then Γ_X is said to be an $FPFS$-normal subgroup of Γ_Y, if for all $x \in G_2$, $\mu_X(x) \leq \mu_Y(x)$ and γ_X is an $FPFS$-subset of γ_Y.

Example 3.10 Assume that $G_1 = \{1, -1, i, -i\}$ is a group, where $i^2 = -1$ and $G_2 = \{1, -1\}$ is a set of parameters. If $X = \{0.3/1, 0.1/-1\}$, $\gamma_X(1) = \{0.5/1, 0.5/-1, 0.3/i, 0.3/-i\}$, $\gamma_X(-1) = \{0.5/1\}$, and $Y = \{0.4/1, 0.2/-1\}$, $\gamma_Y(1) = \{0.6/1, 0.6/-1, 0.4/i, 0.4/-i\}$, $\gamma_Y(-1) = \{0.5/1, 0.5/-1, 0.3/i, 0.3/-i\}$, then Γ_X is an $FPFS$-normal subgroup over G_1, Γ_Y is an $FPFS$-group over G_1 and Γ_X is an $FPFS$-normal subgroup of Γ_Y.

Theorem 3.11 *Let $\Gamma_X, \Gamma_Y, \Gamma_Z$ be FPFS-groups over G_1. If Γ_X is an FPFS-normal subgroup of Γ_Y and Γ_Y is an FPFS-normal subgroup of Γ_Z, then Γ_X is an FPFS-normal subgroup of Γ_Z.*

Proof. The proof is obvious and is omitted. ∎

Definition 3.12 Let $\Gamma_X = \{(\mu_X(x)/x, \gamma_X(x))|x \in G_2, \gamma_X(x) \in F(G_1), \mu_X(x) \in [0,1]\}$ and $\Gamma_Y = \{(\mu_Y(y)/y, \gamma_Y(y))|y \in K_2, \gamma_Y(y) \in F(K_1), \mu_Y(y) \in [0,1]\}$ be $FPFS$-sets over groups G_1 and K_1, respectively. If $f : G_1 \to K_1$ and $g : G_2 \to K_2$ are two functions, then (f, g) is called an FP-fuzzy soft homomorphism from Γ_X to Γ_Y, denoted by $\Gamma_X \sim \Gamma_Y$, if the following conditions are satisfied:

(l) f is an epimorphism from G_1 to K_1,

(2) g is a surjective mapping,

(3) $f(\gamma_X(x)) = \gamma_Y(g(x))$ and $\mu_X(x) = \mu_Y(g(x))$ for all $x \in G_2$.

In the above definition, if f is an isomorphism from G_1 to K_1 and g is a bijective mapping, then (f, g) is called an FP-fuzzy soft isomorphism from Γ_X to Γ_Y, denoted by $\Gamma_X \simeq \Gamma_Y$.

Example 3.13 Consider $G_1 = G_2 = \{1, -1\}$ and $K_1 = K_2 = \{e, (12)\}$. Define a homomorphism $f : G_1 \to K_1$ by $f(1) = e$, $f(-1) = (12)$ for all $s \in G_1$, and a mapping $g : G_2 \to K_2$ by $g(1) = e$, $g(-1) = (12)$ for all $x \in G_2$.

Let X be a fuzzy set over G_2 defined by $X = \{0.5/1, 0.5/-1\}$, Y be a fuzzy set over K_2 defined by $Y = \{0.5/e, 0.5/(12)\}$.

Let $\gamma_X : G_2 \to F(G_1)$ defined by

$$(\gamma_X(1))(s) = \begin{cases} 0.4, & s = 1, s \in G_1, \\ 0.1, & s = -1, s \in G_1. \end{cases}$$

$$(\gamma_X(-1))(s) = \begin{cases} 0.6, & s = 1, s \in G_1, \\ 0.2, & s = -1, s \in G_1. \end{cases}$$

$\gamma_Y : K_2 \to F(K_1)$ defined by

$$(\gamma_Y(e))(k) = \begin{cases} 0.4, & k = e, k \in K_1, \\ 0.1, & k = (12), k \in K_1. \end{cases}$$

$$(\gamma_Y(12))(k) = \begin{cases} 0.6, & k = e, k \in K_1 \\ 0.2, & k = (12), k \in K_1. \end{cases}$$

It is clear that Γ_X and Γ_Y are $FPFS$-sets over G_1 and K_1, respectively. We can immediately see that $\mu_X(x) = \mu_Y(g(x))$ and we can deduce that $f(\gamma_X(x)) = \gamma_Y(g(x))$ for all $x \in G_2$. Hence (f, g) is an FP- fuzzy soft homomorphism from Γ_X to Γ_Y.

Lemma 3.14 [10] *Let $f : G \to K$ be an epimorphism of groups and μ be a fuzzy normal subgroup of G, then $f(\mu)$ is a fuzzy normal subgroup of K.*

Theorem 3.15 *Let* $\Gamma_X = \{(\mu_X(x)/x, \gamma_X(x)) \mid x \in G_2,\ \gamma_X(x) \in F(G_1),\ \mu_X(x) \in [0,1]\}$ *be an FPFS-normal subgroup over group* G_1 *and* $\Gamma_Y = \{(\mu_Y(y)/y, \gamma_Y(y)) \mid y \in K_2,\ \gamma_Y(y) \in F(K_1), \mu_Y(y) \in [0,1]\}$ *be an FPFS-set over group* K_1*. If* Γ_X *is FP-fuzzy soft homomorphic to* Γ_Y*, then* Γ_Y *is an FPFS-normal subgroup over* K_1*.*

Proof. Let (f, g) be an FP-fuzzy soft homomorphism from Γ_X to Γ_Y. Since Γ_X is an $FPFS$-normal subgroup over group G_1, $f(G_1) = K_1$, $g(G_2) = K_2$ and for all $x \in G_2$, X, $\gamma_X(x)$ are two fuzzy normal subgroups of G_2 and G_1, respectively. Now, for all $y \in K_2$, there exists $x \in G_2$ such that $g(x) = y$. Since (f, g) is an FP-fuzzy soft homomorphism from Γ_X to Γ_Y, so $\gamma_Y(y) = \gamma_Y(g(x)) = f(\gamma_X(x))$ and $\mu_Y(y) = \mu_Y(g(x)) = \mu_X(x)$. By Lemma 3.14, we have that Y and $\gamma_Y(y)$ are two fuzzy normal subgroups of K_2 and K_1, respectively. Hence Γ_Y is an $FPFS$-normal subgroup over K_1. ∎

Definition 3.16 Let $\Gamma_X = \{(\mu_X(x)/x, \gamma_X(x)) \mid x \in G_2,\ \gamma_X(x) \in F(G_1), \mu_X(x) \in [0,1]\}$ be an $FPFS$-normal subgroup over group G_1. Then Γ_X is said to be FP-equivalent fuzzy soft normal subgroup over G_1 if, for any $x, y \in G_2$, $\mu_X(x) = \mu_X(y)$, we have $\gamma_X(x) = \gamma_X(y)$.

Example 3.17 Assume that $G_1 = \{1, -1, i, -i\}$ is a group, where $i^2 = -1$, and $G_2 = \{e, (123), (132)\}$ is a group as a set of parameters and X is a fuzzy set over G_2 defined by $X = \{0.5/e, 04/(123), 0.4/(132)\}$. Let γ_X be defined by $\gamma_X(e) = \{0.8/1, 0.8/-1, 0.7/i, 0.7/-i\}$, $\gamma_X(123) = \{0.6/1, 0.6/-1, 0.4/i, 0.4/-i\}$, $\gamma_X(132) = \{0.6/1, 0.6/-1, 0.4/i, 0.4/-i\}$. It is clearly that Γ_X is an FP-equivalent fuzzy soft normal subgroup over G_1.

Theorem 3.18 *Let* $\Gamma_X = \{(\mu_X(x)/x, \gamma_X(x)) | x \in G_2, \gamma_X(x) \in F(G_1), \mu_X(x) \in [0,1]\}$ *be an FP-equivalent fuzzy soft normal subgroup over group* G_1 *and* $\Gamma_Y = \{(\mu_Y(y)/y, \gamma_Y(y)) | y \in K_2, \gamma_Y(y) \in F(K_1), \mu_Y(y) \in [0,1]\}$ *be an FPFS-set over group* K_1*. If* Γ_X *is FP-fuzzy soft homomorphic to* Γ_Y*, then* Γ_Y *is an FP-equivalent fuzzy soft normal subgroup over* K_1*.*

Proof. Let (f, g) be an FP-fuzzy soft homomorphism from Γ_X to Γ_Y. Since Γ_X is an FP-equivalent fuzzy soft normal subgroup over group G_1, we have $\gamma_X(x_1) = \gamma_X(x_2)$, if $\mu_X(x_1) = \mu_X(x_2)$ for any $x_1, x_2 \in G_2$. By Theorem 3.15, we have Γ_Y is an $FPFS$-normal subgroup over K_1. Now, for all $y_1, y_2 \in K_2$, then there exist $x_1, x_2 \in G_2$ such that $g(x_1) = y_1$, $g(x_2) = y_2$. Since $\mu_Y(y_1) = \mu_Y(g(x_1)) = \mu_X(x_1)$, $\mu_Y(y_2) = \mu_Y(g(x_2)) = \mu_X(x_2)$, then $\mu_Y(y_1) = \mu_Y(y_2)$. And we have $\gamma_Y(y_1) = \gamma_Y(g(x_1)) = f(\gamma_X(x_1)) = f(\gamma_X(x_2)) = \gamma_Y(g(x_2)) = \gamma_Y(y_2)$, hence Γ_Y is an FP-equivalent fuzzy soft normal subgroup over K_1. ∎

Definition 3.19 Let $\Gamma_X = \{(\mu_X(x)/x, \gamma_X(x)) | x \in G_2, \gamma_X(x) \in F(G_1), \mu_X(x) \in [0,1]\}$ be an $FPFS$-normal subgroup over group G_1. Then Γ_X is said to be FP-increasing fuzzy soft normal subgroup over G_1 if, for any $x, y \in G_2$, $\mu_X(x) \leq \mu_X(y)$, we have $\gamma_X(x) \subseteq \gamma_X(y)$, and Γ_X is said to be FP-decreasing fuzzy soft normal subgroup over G_1 if, for any $x, y \in G_2$, $\mu_X(x) \leq \mu_X(y)$, we have $\gamma_X(x) \supseteq \gamma_X(y)$.

Example 3.20 Let $G_1 = \{e, (13)\}$, $G_2 = \{e, (123), (132)\}$ and X be a fuzzy set over G_2 defined by $X = \{0.6/e, 03/(123), 0.3/(132)\}$, γ_X be defined by $\gamma_X(e) = \{0.7/e, 0.5/(13)\}$, $\gamma_X(123) = \{0.2/e, 0.1/(13)\}$, $\gamma_X(132) = \{0.2/e, 0.1/(13)\}$. It is clear that Γ_X is an FP-increasing fuzzy soft normal subgroup over G_1.

Theorem 3.21 *Let* $\Gamma_X = \{(\mu_X(x)/x, \gamma_X(x))|x \in G_2,\ \gamma_X(x) \in F(G_1), \mu_X(x) \in [0,1]\}$ *be an* FP*-increasing (decreasing) fuzzy soft normal subgroup over group* G_1 *and* $\Gamma_Y = \{(\mu_Y(y)/y, \gamma_Y(y))|y \in K_2, \gamma_Y(y) \in F(K_1), \mu_Y(y) \in [0,1]\}$ *be an* $FPFS$*-set over group* K_1*. If* Γ_X *is* FP*-fuzzy soft homomorphic to* Γ_Y*, then* Γ_Y *is an* FP*-increasing (decreasing) fuzzy soft normal subgroup over* K_1*.*

Proof. Let (f, g) be an FP-fuzzy soft homomorphism from Γ_X to Γ_Y. Since Γ_X is an FP-increasing fuzzy soft normal subgroup over group G_1, we have $\gamma_X(x_1) \subseteq \gamma_X(x_2)$, if $\mu_X(x_1) \leq \mu_X(x_2)$ for any $x_1, x_2 \in G_2$. By Theorem 3.15, we have Γ_Y is an $FPFS$-normal subgroup over K_1. Now, for all $y_1, y_2 \in K_2$, then there exist $x_1, x_2 \in G_2$ such that $g(x_1) = y_1$, $g(x_2) = y_2$. Since $\mu_Y(y_1) = \mu_Y(g(x_1)) = \mu_X(x_1)$, $\mu_Y(y_2) = \mu_Y(g(x_2)) = \mu_X(x_2)$, then $\mu_Y(y_1) \leq \mu_Y(y_2)$. And we have $\gamma_Y(y_1) = \gamma_Y(g(x_1)) = f(\gamma_X(x_1)) \subseteq f(\gamma_X(x_2)) = \gamma_Y(g(x_2)) = \gamma_Y(y_2)$, hence Γ_Y is an FP-increasing fuzzy soft normal subgroup over K_1. ∎

4. Aggregate fuzzy normal subgroups

An aggregate fuzzy set of an $FPFS$-set has been defined by Çağman et al. (see [3]). They also defined $FPFS$-aggregation operator that produced an aggregate fuzzy set from an $FPFS$-set and its fuzzy parameter set.

Definition 4.1 Let $\Gamma_X \in FPFS(G_1)$. Then $FPFS$-aggregation operator, denoted by $FPFS_{agg}$, is defined by

$$FPFS_{agg} : F(G_2) \times FPFS(G_1) \to F(G_1),$$

$$FPFS_{agg}(X, \Gamma_X) = \Gamma_X^*.$$

where

$$\Gamma_X^* = \{\mu_{\Gamma_X^*}(u)/u|u \in G_1\},$$

which is a fuzzy set over G_1. The value Γ_X^* is called an aggregate fuzzy set of the Γ_X. Here the membership degree $\mu_{\Gamma_X^*}(u)$ of u is defined as follows

$$\mu_{\Gamma_X^*}(u) = \frac{1}{|G_2|} \sum_{x \in G_2} \mu_X(x) \mu_{\gamma_X(x)}(u),$$

where $|G_2|$ is the cardinality of G_2.

Theorem 4.2 *Let* $\Gamma_X = \{(\mu_X(x)/x, \gamma_X(x))|x \in G_2, \gamma_X(x) \in F(G_1), \mu_X(x) \in [0,1]\}$ *be an* $FPFS$*-normal subgroup over group* G_1*. Then the aggregate fuzzy set* Γ_X^* *of* Γ_X *is a fuzzy normal subgroup of* G_1*.*

Proof. Since Γ_X is an $FPFS$-normal subgroup over group G_1, so $\gamma_X(x)$ is a fuzzy normal subgroup of G_1 for all $x \in G_2$. Equivalently, $\mu_{\gamma_X(x)}(sts^{-1}) \geq \mu_{\gamma_X(x)}(t)$ for all $s, t \in G_1$. Then

$$\begin{aligned} \mu_{\Gamma_X^*}(sts^{-1}) &= \frac{1}{|G_2|}\sum_{x\in G_2}\mu_X(x)\mu_{\gamma_X(x)}(sts^{-1}) \\ &\geq \frac{1}{|G_2|}\sum_{x\in G_2}\mu_X(x)\mu_{\gamma_X(x)}(t) \\ &= \mu_{\Gamma_X^*}(t). \end{aligned} \quad .$$

Then Γ_X^* is a fuzzy normal subgroup of G_1. ∎

Remark 4.3 (1) The above Γ_X^* is called an aggregate fuzzy normal subgroup of $FPFS$-normal subgroup Γ_X; (2) Γ_X^* is a fuzzy normal subgroup of G_1, but Γ_X is not always $FPFS$-normal subgroup of G_1.

Example 4.4

(1) Let $G_1 = M_n$ be a matrix group, A, B be a lower triangular matrix and a diagonal matrix. And let $G_2 = \{e, (12)\}$, the parameters e, (12) stand for "lower triangular" and "diagonal", respectively. And X is a fuzzy set over G_2 defined by

$$\mu_X(x) = \begin{cases} 0.5, & x = e, \\ 0.3, & x = (12). \end{cases}$$

Let γ_X be defined by

$$\mu_{\gamma_X(e)}(r) = \begin{cases} 0, & r \text{ is not a lower triangular matrix}, \\ 1, & r \text{ is a lower triangular matrix}. \end{cases}$$

$$\mu_{\gamma_X(12)}(r) = \begin{cases} 0, & r \text{ is not a diagonal matrix}, \\ 1, & r \text{ is a diagonal matrix}. \end{cases}$$

It is clear that Γ_X is an $FPFS$-normal subgroup over G_1. The aggregate fuzzy set can be found as

$$\mu_{\Gamma_X^*}(u) = \begin{cases} 0.2, & u \in B, \\ 0.1, & u \in A \setminus B, \\ 0, & \text{otherwise}. \end{cases}$$

We can verify that Γ_X^* is a fuzzy normal subgroup of G_1. Then Γ_X^* is called an aggregate fuzzy normal subgroup of $FPFS$-normal subgroup Γ_X.

(2) Let $G_1 = \{1, -1, i, -i\}$ be a group, where $i^2 = -1$, $G_2 = \{e, (12)\}$, X be a fuzzy set over G_2 defined by $X = \{0.3/e, 0.4/(12)\}$, γ_X be defined by $\gamma_X(e) = \{0.8/1, 0.8/-1, 0.6/i, 0.6/-i\}$, $\gamma_X(12) = \{0.7/1, 0.7/-1, 0.5/i, 0.5/-i\}$. So $\Gamma^*_X = \{0.26/1, 0.26/-1, 0.19/i, 0.19/-i\}$. It is clear that Γ^*_X is a fuzzy normal subgroup of G_1, but Γ_X is not $FPFS$-normal subgroup of G_1 because of X is not a fuzzy normal subgroup of G_2.

Proposition 4.5 *Γ_X is an $FPFS$-normal subgroup of G_1 if and only if Γ^*_X is a fuzzy normal subgroup of G_1 and X is a fuzzy normal subgroup of G_2.*

Proof. Let Γ_X be an $FPFS$-normal subgroup of G_1, it is obvious that Γ^*_X is a fuzzy normal subgroup of G_1 and X is a fuzzy normal subgroup of G_2.

Conversely, let Γ^*_X be a fuzzy normal subgroup of G_1 and X be a fuzzy normal subgroup of G_2. Assume that Γ_X is not an $FPFS$-normal subgroup of G_1, so $\gamma_X(x)$ is not a fuzzy normal subgroup of G_1 for all $x \in G_2$ or X is not a fuzzy normal subgroup of G_2, then Γ^*_X is not a fuzzy normal subgroup of G_1 or X is not a fuzzy normal subgroup of G_2. Which is a contradiction. So Γ_X is an $FPFS$-normal subgroup of G_1. The proof is complete. ∎

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PASSAGE OF PROPERTY (gw) FROM TWO OPERATORS TO THEIR TENSOR PRODUCT

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Abstract. A Banach space operator satisfies property (gw) if the complement of its B-Weyl essential approximate point spectrum in its approximate point spectrum is the set of isolated eigenvalues of the operator. We give necessary and/or sufficient conditions ensuring the passage of property (gw) from two Banach space operators A and B to their tensor product. In particular, we present a revised version of Theorem 2.3 in [20].

Keywords: property (gw), generalized a-Weyl's theorem, tensor product.

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1. Introduction and preliminary

Let $T \in \mathcal{L}(X)$ be a bounded linear operator acting on a Banach space X. We denote by T^*, $N(T)$, $R(T)$, $\sigma(T)$, $\sigma_p(T)$ and $\sigma_{ap}(T)$ denote the adjoint, the null space, the range, the spectrum, the point spectrum and the approximate point spectrum of T respectively. If the range $R(T)$ of T is closed and $\alpha(T) := dimN(T) < \infty$ (resp. $\beta(T) := codimR(T) < \infty$) then T is called an upper (resp. a lower) semi-Fredholm operator. If T is either an upper or a lower semi-Fredholm operator, then T is called a semi-Fredholm operator, and the index of T is defined by

$$ind(T) = \alpha(T) - \beta(T).$$

If both $\alpha(T)$ and $\beta(T)$ are finite, then T is called a Fredholm operator. If T is Fredholm of index zero, then T is said to be a Weyl operator. The Weyl spectrum of T is defined by

$$\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\}$$

and the *Weyl essential approximate point spectrum* is defined by

$$\sigma_{SF_-^+}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not upper semi-Fredholm with } ind(T - \lambda) \leq 0\}.$$

For each nonnegative integer n let $T_{[n]}$ to be the restriction of T to $R(T^n)$ viewed as a map from $R(T^n)$ into $R(T^n)$ (in particular $T_{[0]} = T$). If for some n, $R(T^n)$ is closed and $T_{[n]}$ is an upper (resp. lower) semi-Fredholm operator then T is called an *upper* (resp. *lower*) *semi-B-Fredholm* operator. A *semi-B-Fredholm* operator is an upper or lower semi-B-Fredholm operator. If, moreover, $T_{[n]}$ is a Fredholm operator then T is called a *B-Fredholm* operator. From [13, Proposition 2.1] if $T_{[n]}$ is a semi-Fredholm operator then $T_{[m]}$ is also a semi-Fredholm operator for each $m \geq n$, and $ind(T_{[m]}) = ind(T_{[n]})$. Then the *index* of a semi-B-Fredholm operator is defined as the index of the semi-Fredholm operator $T_{[n]}$ (see [12, 13]). $T \in \mathcal{L}(X)$ is said to be a *B-Weyl* operator if it is a B-Fredholm operator of index zero. The *B-Weyl spectrum* $\sigma_{BW}(T)$ of T is defined by

$$\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not a B-Weyl operator}\}.$$

We shall denote by $SBF_+^-(X)$ (or $SBF_-^+(X)$) the class of all T upper semi-B-Fredholm operators such that $\text{ind}(T) \leq 0$ (respectively, T lower semi-B-Fredholm operators such that $\text{ind}(T) \geq 0$). The spectrum associated with $SBF_+^-(X)$ is called the *B-Weyl essential approximate point spectrum* and is denoted by

$$\sigma_{SBF_+^-}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin SBF_+^-(X)\},$$

while the spectrum associated with $SBF_-^+(X)$ is denoted by

$$\sigma_{SBF_-^+}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin SBF_-^+(X)\}.$$

The *ascent* $a(T)$ and the *descent* $d(T)$ of T are given by

$$a(T) = \inf\{n : N(T^n) = N(T^{n+1})\} \text{ and } d(T) = \inf\{n : R(T^n) = R(T^{n+1})\},$$

with $\inf \emptyset = \infty$. It is well-known that if $a(T)$ and $d(T)$ are both finite then they are equal, see [17, Proposition 38.3]. An operator $T \in \mathcal{L}(X)$ is said to be Browder if it is Weyl of finite ascent and descent. Let $\sigma_b(T)$ be the Browder spectrum defined by

$$\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not a Browder operator}\}.$$

Following Coburn [15], Weyl's theorem holds for T if $\sigma(T) \setminus \sigma_w(T) = E^0(T)$ where $E^0(T) := \{\lambda \in iso\sigma(T) : 0 < \alpha(T - \lambda) < \infty\}$. Here and elsewhere in this paper, for $K \subset \mathbb{C}$, $isoK$ is the set of isolated points of K.

We say that a-Weyl's theorem holds for T if $\sigma_{ap}(T) \setminus \sigma_{SF_-^+}(T) = E_a^0(T)$ where $E_a^0(T) := \{\lambda \in iso\sigma_{ap}(T) : 0 < \alpha(T-\lambda) < \infty\}$.

T is said to have the property (w) if $\sigma_a(T) \setminus \sigma_{SF_-^+}(T) = E^0(T)$. Originally introduced by Rakočević in [19], property (w) has been intensively studied in the recent past, see [2], [3], [4], [5], [6]. Following Berkani and Koliha [10], [12], generalized Weyl's theorem holds of T if $\sigma(T) \setminus \sigma_{BW}(T) = E(T)$ where $E(T) := \{\lambda \in iso\sigma(T) : \alpha(T-\lambda > 0)\}$; and generalized a-Weyl's theorem holds for T if $\sigma_{ap}(T) \setminus \sigma_{SBF_+^-}(T) = E_a(T)$ where $E_a(T) := \{\lambda \in iso\sigma_{ap}(T) : \alpha(T-\lambda) > 0\}$.

Operators satisfying property (gw), an extension of property (w) have been introduced and studied in [9], [11], [7]. T is said to have the property (gw), if $\sigma_{ap}(T) \setminus \sigma_{SBF_+^-}(T) = E(T)$. Note that the property (gw) implies generalized Weyl's theorem but (gw) is not intermediate between Weyl's theorems and generalized a-Weyl's theorem, see [9].

The problem of transferring Weyl's theorem, a-Weyl's theorem and Property (w) from operators A and B to their tensor product $A \otimes B$ was considered in [16], [20], [18]. The main objective of this paper is to study the transfer of property (gw) from a bounded linear operator A acting on a Banach space X and a bounded linear operator B acting on a Banach space Y to their tensor product $A \otimes B$. In Section 2, after having recalled some preliminary definitions and facts, we give necessary and sufficient condition for transferring property (gw) from isoloid operators A and B to their tensor product $A \otimes B$ in term of the B-Weyl essential approximate point spectrum equality. Section 3 is devoted to characterize the transference of generalized a-Weyl's theorem from a-isoloid operators A and B to their tensor product $A \otimes B$.

2. Property (gw) and tensor product

Let $T \in \mathcal{L}(X)$, then T is said to be *isoloid* , if $iso\sigma(T) = E(T)$. In [20, Theorem 2.3], it was stated that if A and B are isoloid operators that possess property (gw), then $A \otimes B$ satisfies property (gw) if and only if

$$\sigma_{SBF_+^-}(A \otimes B) = \sigma_{SBF_+^-}(A)\sigma_{ap}(B) \cup \sigma_{ap}(A)\sigma_{SBF_+^-}(B).$$

We point out that Theorem 2.3 in [20] is accurate as shown by the following example.

Example 2.1 Let A be a nonzero nilpotent operator and let B be a quasi-nilpotent which is not nilpotent. Then it is easy to see that

$$\sigma_{ap}(A) = \{0\},\ \sigma_{SBF_+^-}(A) = \emptyset \text{ and } \sigma_{ap}(B) = \sigma_{SBF_+^-}(B) = \{0\}.$$

Hence A and B satisfy property (gw). Since $A \otimes B$ is nilpotent then 0 is a pole and then $\sigma_{SBF_+^-}(A \otimes B) = \emptyset$. Hence $A \otimes B$ satifies property (gw). However

$$\sigma_{SBF_+^-}(A)\sigma(B) \cup \sigma(A)\sigma_{SBF_+^-}(B) = \{0\} \neq \sigma_{SBF_+^-}(A \otimes B).$$

Here $0 \in iso\sigma(A \otimes B)$.

However, under the assumption $0 \notin iso\sigma(A \otimes B)$, we have the following result.

Theorem 2.2 *Let X and Y two Banach spaces and consider $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(Y)$ such that A and B are isoloid and $0 \notin iso\sigma(A \otimes B)$. If property (gw) holds for A and B, then the following statements are equivalent.*

(i) *$A \otimes B$ satisfies property (gw).*

(ii) $\sigma_{SBF_+^-}(A \otimes B) = \sigma_{SBF_+^-}(A)\sigma_{ap}(B) \cup \sigma_{ap}(A)\sigma_{SBF_+^-}(B)$.

Proof. (ii) $\Rightarrow$ (i): Since A and B obey property (gw), then

$$\sigma_{ap}(A) \setminus_{SBF_+^-}(A) = E(A) \text{ and } \sigma_{ap}(B) \setminus_{SBF_+^-}(B) = E(B).$$

Assume that

$$\sigma_{SBF_+^-}(A \otimes B) = \sigma_{SBF_+^-}(A)\sigma_{ap}(B) \cup \sigma_{ap}(A)\sigma_{SBF_+^-}(B).$$

Let $\lambda \in E(A \otimes B)$. Then there exists $\mu \in iso\sigma(A)$ and $\nu \in iso\sigma(B)$ such that $\lambda = \mu\nu$. Since A and B are isoloid, then $\mu \in E(A)$ and $\nu \in E(A)$. Hence $\mu \notin \sigma_{SBF_+^-}(A)$ and $\nu \notin \sigma_{SBF_+^-}(B)$. Then $\lambda \notin \sigma_{SBF_+^-}(A \otimes B)$. Thus

$$E(A \otimes B) \subseteq \sigma_{ap}(A \otimes B) \setminus \sigma_{SBF_+^-}(A \otimes B).$$

Conversely, suppose that $\lambda \notin \sigma_{ap}(A \otimes B) \setminus \sigma_{SF_+^-}(A \otimes B)$, then there exists $\mu \in \sigma_{ap}(A) \setminus \sigma_{SBF_+^-}(A)$ and $\nu \in \sigma_{ap}(B) \setminus \sigma_{SBF_+^-}(B)$ such that $\lambda = \mu\nu$. Since

$$A \otimes B = (A - \mu) \otimes B + \mu I \otimes (B - \nu),$$

then we can see that $\lambda \in E(A \otimes B)$. Therefore $A \otimes B$ satisfies property (gw).

(i) $\Rightarrow$ ii): Assume that $A \otimes B$ satisfies property (gw). Let

$$\lambda \in E(A \otimes B) = \sigma_{ap}(A \otimes B) \setminus_{SBF_+^-}(A \otimes B).$$

Since $0 \notin iso\sigma(A \otimes B)$, then $\lambda \neq 0$. Hence $\lambda \in iso\sigma(A \otimes B) = iso\sigma(A)iso\sigma(B)$. That is $\lambda = \mu\nu$ with $\mu \in iso\sigma(A)$ and $\nu \in iso\sigma(B)$. Since A and B are isoloid, then $\mu \in E(A) = \sigma_{ap}(A) \setminus_{SBF_+^-}(A)$ and $\nu \in E(B) = \sigma_{ap}(B) \setminus_{SBF_+^-}(B)$, and hence $\lambda = \mu\nu \notin \sigma_{SBF_+^-}(A)\sigma_{ap}(B) \cup \sigma_{ap}(A)\sigma_{SBF_+^-}(B)$. Thus

$$\sigma_{SBF_+^-}(A)\sigma_{ap}(B) \cup \sigma_{ap}(A)\sigma_{SBF_+^-}(B) \subseteq \sigma_{SBF_+^-}(A \otimes B).$$

Conversely, let $\lambda \in \sigma_{ap}(A \otimes B) \setminus (\sigma_{SBF_+^-}(A)\sigma_{ap}(B) \cup \sigma_{ap}(A)\sigma_{SBF_+^-}(B))$, then for $\lambda = \mu\nu$ we have that $\mu \in \sigma_{ap}(A)$ and $\nu \in \sigma_{ap}(B)$, hence $\mu \in E(A)$ and $\nu \in E(B)$. Thus $\lambda = \mu\nu \in E(A \otimes B) = \sigma_{ap}(A \otimes B) \setminus_{SBF_+^-}(A \otimes B)$. Finally,

$$\sigma_{SBF_+^-}(A \otimes B) = \sigma_{SBF_+^-}(A)\sigma_{ap}(B) \cup \sigma_{ap}(A)\sigma_{SBF_+^-}(B).$$

■

Remark 2.3 Let $T \in \mathcal{L}(X)$ without eigenvalue, then $\sigma_{SBF_+^-}(T) = \sigma_{ap}(T)$. Indeed, Let $\lambda \in \sigma_{ap}(T) \setminus \sigma_{SBF_+^-}(T)$. Without loss of generality we may assume that $\lambda = 0$. Then there exists some positive integer n such that $T_{[n]}$ is upper semi-Fredhom. Hence $R(T_{[n]}) = R(T^{n+1})$ is closed. Since T has no eigenvalue, then $N(T^{n+1}) = \{0\}$. Thus $0 \notin \sigma_{ap}(T^{n+1})$ which is a contradiction. Therefore, $\sigma_{SBF_+^-}(T) = \sigma_{ap}(T)$. In particular T satisfies generalized a-Weyl's theorem.

Also, the assumption "A and B are isoloid " is crucial in Theorem 2.2, as shown by the following example.

Example 2.4 Let I_1 and I_2 be the identities acting on $\mathbb{C}$ and l_2, respectively. Let S_1 and S_2 defined on l_2 by

$$S_1(x_1, x_2, ...) = \left(0, \frac{1}{3}x_1, \frac{1}{3}x_2, ...\right) \text{ and } S_2(x_1, x_2, ...) = \left(0, \frac{1}{2}x_1, \frac{1}{3}x_2, ...\right).$$

Let $A = I_1 \oplus S_1$ and $B = S_2 - I_2$. Clearly,

$$\sigma(A) = \left\{\lambda \in \mathbb{C} : |\lambda| \leq \frac{1}{3}\right\} \cup \{1\} \text{ and } \sigma(B) = \{-1\}.$$

We have $\sigma(B) = \sigma_{ap}(B) = \sigma_{BW}(B) = \{-1\}$. Since B and B^* have the SVEP, then by [1, Corollary 3.53] and [1, Theorem 3.66]

$$\sigma_{SBF_+^-}(B) = \sigma_{BW}(B) = \sigma(B) = \{-1\}.$$

Now, $\sigma_{ap}(A) = \{\lambda \in \mathbb{C} : |\lambda| = \frac{1}{3}\} \cup \{1\}$. We claim that $\sigma_{SBF_+^-}(A) = \{\lambda \in \mathbb{C} : |\lambda| = \frac{1}{3}\}$. Indeed, we have that $\sigma_{SBF_+^-}(A) \subseteq \sigma_{ap}(A) = \{\lambda \in \mathbb{C} : |\lambda| = \frac{1}{3}\} \cup \{1\}$ and $1 \notin \sigma_{SBF_+^-}(A)$. If there exists $\lambda \in \mathbb{C}$, $|\lambda| = \frac{1}{3}$ such that $A - \lambda(I_1 \oplus I_2)$ is upper semi-B-Fredholm of negative index, then $S_1 - \lambda I_2$ is upper semi-B-Fredholm of negative index. Since S_1 has no eigenvalue, then it follows from Remark 2.3 that $\lambda \notin \sigma_{ap}(S_1) = \{\lambda \in \mathbb{C} : |\lambda| = \frac{1}{3}\}$. This is a contradiction. Since A and B have the SVEP, then from [1, Corollary 3.72], we conclude that

$$\sigma_{SBF_+^-}(A^2) = \sigma_{SBF_+^-}(A)^2 = \{\lambda \in \mathbb{C} : |\lambda| = \frac{1}{9}\} \text{ and } \sigma_{SBF_+^-}(B^2) = \sigma_{SBF_+^-}(B)^2 = \{1\}.$$

Now,

$$\sigma_{ap}(A) \setminus \sigma_{SBF_+^-}(A) = \{\lambda \in \mathbb{C} : |\lambda| = \frac{1}{3}\} \cup \{1\}) \setminus \{\lambda \in \mathbb{C} : |\lambda| = \frac{1}{3}\} = \{1\} = E(A)$$

and $\sigma_{ap}(B) \setminus \sigma_{SBF_+^-}(B) = \emptyset = E(B)$. Also,

$$\sigma_{ap}(A^2) \setminus \sigma_{SBF_+^-}(A^2) = \{\lambda \in \mathbb{C} : |\lambda| = \frac{1}{9}\} \cup \{1\}) \setminus \{\lambda \in \mathbb{C} : |\lambda| = \frac{1}{9}\} = \{1\} = E(A^2)$$

and $\sigma_{ap}(B^2) \setminus \sigma_{SBF_+^-}(B^2) = \emptyset = E(B^2)$. Thus the property (gw) holds for A, B, A^2 and B^2.

In the other hand, since $A \otimes B$ has no eigenvalue then it follows from Remark 2.3 that

$$\sigma_{SBF_+^-}(A \otimes B) = \{\lambda \in \mathbb{C} : |\lambda| = \sigma_{ap}(A \otimes B) = \{\lambda \in \mathbb{C} : |\lambda| = \frac{1}{3}\} \cup \{-1\}$$

and

$$\sigma_{SBF_+^-}(A^2 \otimes B^2) = \{\lambda \in \mathbb{C} : |\lambda| = \sigma_{ap}(A^2 \otimes B^2) = \{\lambda \in \mathbb{C} : |\lambda| = \frac{1}{9}\} \cup \{1\}.$$

Hence

$$\sigma_{SBF_+^-}(A \otimes B) = \sigma_{SBF_+^-}(A)\sigma_{ap}(B) \cup \sigma_{ap}(A)\sigma_{SBF_+^-}(B),$$

and

$$\sigma_{SBF_+^-}(A^2 \otimes B^2) = \sigma_{SBF_+^-}(A^2)\sigma_{ap}(B^2) \cup \sigma_{ap}(A^2)\sigma_{SBF_+^-}(B^2).$$

Also

$$\sigma_{ap}(A \otimes B) \setminus \sigma_{SBF_+^-}(A \otimes B) = \emptyset = E(A \otimes B)$$

and

$$\sigma_{ap}(A^2 \otimes B^2) \setminus \sigma_{SBF_+^-}(A^2 \otimes B^2) = \emptyset \neq 1 = E(A^2 \otimes B^2).$$

Thus the property (gw) holds for $A \otimes B$ but not for $A^2 \otimes B^2$. Note here that B and B^2 are not isoloid.

The following example show that there exists two operators $A, B \in \mathcal{L}(X)$ such that $A \otimes B$ satisfies the property (gw) but A and B do not satisfy the property (gw).

Example 2.5 Let $B = U + U^*$ where U is the unilateral shift on l_2. Since B is self-adjoint, then

$$\sigma(B) = \sigma_{ap}(B) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$$

and from [1],

$$\sigma_{BW}(B) = \sigma_{SBF_+^-}(B) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}.$$

Hence

$$\sigma_{ap}(B) \setminus \sigma_{SBF_+^-}(B) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\} \setminus \{\lambda \in \mathbb{C} : |\lambda| = 1\}.$$

Since $E(B) = \emptyset$, then property (gw) fails for B. In the other hand, if I is the identity acting on l_2, then $I \otimes B$ is self-adjoint, hence

$$\sigma(I \otimes B) = \sigma_{ap}(I \otimes B) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}.$$

$\sigma_{BW}(I \otimes B) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$, and $\sigma_{SBF_+^-}(I \otimes B) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$. Hence,

$$\sigma_{ap}(I \otimes B) \setminus \sigma_{SBF_+^-}(I \otimes B) = \emptyset = E(I \otimes B).$$

Thus $I \otimes B$ satisfies property (gw).

3. Generalized a-Weyl's theorem and tensor product

Recall that given $T \in \mathcal{L}(X)$, T is said to be *a-isoloid*, if $iso\sigma_{ap}(T) = E_a(T)$.

Theorem 3.1 *Let X and Y two Banach spaces. Suppose that $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(Y)$ are a-isoloid and satisfy generalized a-Weyl's theorem. If $0 \notin iso\sigma_{ap}(A \otimes B)$ and $\sigma_{SBF_+^-}(A \otimes B) = \sigma_{SBF_+^-}(A)\sigma_{ap}(B) \cup \sigma_{ap}(A)\sigma_{SBF_+^-}(B)$, then $A \otimes B$ satisfies generalized a-Weyl's theorem.*

Proof. Let $\lambda \in \sigma_{ap}(A \otimes B) \setminus \sigma_{SBF_+^-}(A \otimes B)$. Assume for the sake of contradiction that $\lambda \in acc\,\sigma_{ap}(A \otimes B)$. Since $\lambda \in acc\,(\sigma_{ap}(A)\sigma_{ap}(B))$, it follows that

$$\lambda \in acc\,\sigma_{ap}(A)\sigma_{ap}(B) \cup \sigma_{ap}(A)acc\,\sigma_{ap}(B).$$

Since generalized a-Weyl's theorem holds for both A and B then

$$\lambda \in \sigma_{SBF_+^-}(A)\sigma_{ap}(B) \cup \sigma_{ap}(A)\sigma_{SBF_+^-}(B).$$

By hypothesis, we get that $\lambda \in \sigma_{SBF_+^-}(A \otimes B)$. This is a contradiction. Hence $\lambda \in iso\,\sigma(A \otimes B)$. Since $A \otimes B - \lambda$ is upper semi B-Fredholm, then $\lambda \in E^a(A \otimes B)$. For the reverse inclusion, let $\lambda \in E^a(A \otimes B)$. By hypothesis, $\lambda \neq 0$. With the same argument as in the proof of Theorem 2.2 we get $\lambda = \mu\nu$ with $\mu \in iso\,\sigma_{ap}(A)$ and $\nu \in iso\,\sigma_{ap}(B)$. Since A and B are a-isoloid, then $\mu \in E^a(A)$ and $\nu \in E^a(B)$. Hence from the fact that generalized a-Weyl's theorem holds for A and B, we deduce that

$$\mu \in \sigma_{ap}(A) \setminus \sigma_{SBF_+^-}(A) \text{ and } \nu \in \sigma_{ap}(B) \setminus \sigma_{SBF_+^-}(B).$$

Thus

$$\lambda \notin \sigma_{ap}(A)\sigma_{SBF_+^-}(B) \cup \sigma_{SBF_+^-}(A)\sigma_{ap}(B).$$

By hypothesis, we conclude that $\lambda \notin \sigma_{SBF_+^-}(A \otimes B)$, and hence

$$\lambda \in \sigma_{ap}(A \otimes B) \setminus \sigma_{SBF_+^-}(A \otimes B). \qquad \blacksquare$$

4. Concluding remarks

A bounded linear operator T is said to satisfy a-Browder's theorem if $\sigma_{ap}(T) \setminus \sigma_{SF_+^-}(T) = \pi_0^a(T)$; where $\pi_0^a(T)$ is the set of all left pole of finite rank defined by $\pi_0^a(T) = \{\lambda \in \mathbb{C} : a(T-\lambda) < \infty; R(T^{a(T-\lambda)+1}) \text{ is closed and } \alpha(T-\lambda) < \infty\}$. It is well known that a-weyl's theorem implies a-Browder's theorem and the reverse is not true.

T is said to satisfy the generalized a-Browder's theorem if $\sigma_{ap}(T) \setminus \sigma_{SBF_+^-}(T) = \pi^a(T)$; where $\pi^a(T)$ is the set of all left pole defined by $\pi^a(T) = \{\lambda \in \mathbb{C} : a(T-\lambda) < \infty \text{ and } R(T^{a(T-\lambda)+1}) \text{ is closed}\}$. Generalized a-Weyl's theorem implies

generalized a-Browder's theorem and the reverse is not true. Recently we proved that a-Browder's theorem is equivalent to generalized a-Browder's theorem [8].

In [16, Theorem 1] it was shown that if $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(Y)$ are isoloid operators which satisfy property (w), then equality

$$\sigma_{SF_+^-}(A \otimes B) = \sigma_{SF_+^-}(A)\sigma_{ap}(B) \cup \sigma_{ap}(A)\sigma_{SF_+^-}(B)$$

is equivalent to $A \otimes B$ satisfies a-Browder's theorem.

In [14, Remark 4.6] Boasso, Duggal and Jeon asked:

If A and B satisfy generalized a-Browder's theorem, does the equality

$$\sigma_{SBF_+^-}(A \otimes B) = \sigma_{SBF_+^-}(A)\sigma_{ap}(B) \cup \sigma_{ap}(A)\sigma_{SBF_+^-}(B)$$

equivalent to $A \otimes B$ satisfies generalized a-Browder's theorem?

In [20, Theorem 2.1] it is apparent that the author answered this question positively. However, it is not true as shown by the following example.

Example 4.1 Let A be a nonzero nilpotent operator and let B be a quasi-nilpotent which is not nilpotent. Then A and B satisfies generalizerd a-Browder's theorem. Hence

$$\sigma(A) = \{0\},\ \sigma_{SBF_+^-}(A) = \emptyset \text{ and } \sigma(B) = \sigma_{SBF_+^-}(B) = \{0\}.$$

Then

$$\sigma_{SBF_+^-}(A)\sigma(B) \cup \sigma(A)\sigma_{SBF_+^-}(B) = \{0\}.$$

However, since $A \otimes B$ is nilpotent then 0 is a pole and then $\sigma_{SBF_+^-}(A \otimes B) = \emptyset$. Here $A \otimes B$ satisfies generalized a-Browder's theorems.

Remarks 4.2

(1) In [20, Theorem 2.2], it was stated that if A and B are isoloid and satisfy generalized a-Weyl's theorem, then $\sigma_{SBF_+^-}(A \otimes B) = \sigma_{SBF_+^-}(A)\sigma_{ap}(B) \cup \sigma_{ap}(A)\sigma_{SBF_+^-}(B)$ implies that $A \otimes B$ satisfies generalized a-Weyl's theorem. In the proof, the author used Theorem 2.1 which is false as mentioned above.

(2) In the proof of [20, Lemma 2.1], the author used the following equivalence "$A \otimes B - \frac{1}{n}I$ is injective if and only if A and B are injective". We would like to point out that this equivalence is not true. For instance, if A and B are nonzero nilpotent operators then A and B are not injective however $A \otimes B - \frac{1}{n}I$ is injective.

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ADAPTIVE CONTROLLER FOR GENERAL NETWORKED MANIPULATORS WITH UNCERTAIN DYNAMICS

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Abstract. In this paper, two adaptive tracking control algorithms for redundant and non-redundant networked nonidentical robot manipulators are proposed, where the networked topology graph is direct and has a spanning tree. It is shown that, the distributed feedback controllers can be achieve end-effectors time-varying positions, velocities and sub-task tracking for manipulators with parametric uncertainty in the Lagrange dynamics. Moreover, the designed distributed tracking control algorithms in this paper is even a small portion of followers can get the leader's position information. Simulation example is presented to illustrate the performance of the proposed controller.
Keywords: redundant manipulators, Lagrange dynamics, dynamic friction, adaptive tracking, uncertain dynamics.

1. Introduction

Lagrange dynamics represent a class of mechanical systems including formation flying spacecraft, autonomous underwater vehicle, robotic manipulators etc. [5], [7], [9], [19], [20], [23]. Recently, coordinated motion and coordinated control of networked Lagrange systems have become important topics and numerous applications including coverage control, consensus, spacecraft formation control, adaptive tracking control, flocking control etc. [5], [17], [20], but it is also very challenging due to the highly inherent nonlinearity of the Lagrange systems [5]. Nair and Leonard [14] addressed stable synchronization of a network of rotating and translating rigid bodies and designed control laws so that the exponential stability of Lagrangian dynamics is proved. Ren [16] presented some distributed, leaderless, model-independent consensus algorithms for Lagrange systems relied on undirected communication topology. Wu and Zhou [25], [26] studied impulsive synchronization motion in networked open-loop multibody systems formulated by

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Lagrange dynamics with undirected graph. Chung et al. [6] studied the cooperative robot control and concurrent synchronization of nonidentical Lagrange systems.

In most applications of robot, the assumption of having exact models of robot systems also means that the robot is not able to adapt to any changes and uncertainties in its models environment. Meanwhile, the exact dynamics of robot manipulators is hard to be obtained due to the imprecision and measurement of manipulator parameters and interactions between manipulator and environments, such as load variation, friction and external disturbances. In order to compensate the nonlinearity and parametric uncertainty, many robot adaptive controllers have been proposed [4], [11], [15], [24]. By using linear parametrization method of Lagrange dynamics, Chopra et al. [4] proposed adaptive coordination control passivity-based architecture and guarantee state synchronization of master-slave robots in free motion. Nuño, et al. [15] investigated the adaptive synchronization of networked nonidentical Lagrange systems with unknown parameters. It is challenging problem in control field to find an effective control scheme to achieve accurate tracking of the desired motion or leader, recently, tracking and cooperative control of multiple robot manipulators have become an attractive area of research owing to its important role in the assembly automation and flexible manufacturing system. The tracking control of the Lagrange systems is relatively challenging due to the highly coupled dynamics, nonlinearities and uncertainties in the system models [3], [13]. Mei et al. [13] investigated the distributed coordinated tracking problem for networked Lagrange systems on undirected graphs. Chen and Lewis [3] developed the cooperative tracking control for networked Lagrange vehicle systems with directed graph topology.

The assigned task to robot manipulators are specified in task space, by using the inverse of kinematic of the non-redundant robot manipulator, the end-effector adaptive tracking controller with uncertainty in the kinematic and dynamic models is developed in [1], [2], [8], [18]. Nevertheless, the redundant manipulator having more degree of freedom than the number of independent task coordinates is being widely used in various applications in industry. The control of redundant robot manipulators is particulary interesting, and has become a very active active research because of the extra degrees of freedom of redundant manipulator allow the robot manipulator to perform more dextrous manipulation and increase flexibility for the task execution. This fact is commonly referred to as self-motion since it is not observed in the operational space. To achieve accurate end-effector tracking while allowing the self-motion of the system to be available for performance augmentation [12], [21], Zergerolu et al. [21] considered the nonlinear tracking control of kinematically redundant robot manipulators, specifically, the designed controller realize the subtasks tracking effectively. In [12], based on the neural network learning of the parametric uncertainties in the dynamical model, nonlinear tracking control of redundant robot manipulator is considered, the end-effector tracking as well as subtask tracking are achieved effectively.

However, in some special engineering case, more than one manipulator is needed to cooperate with each other to perform the desired operation in task-

space by utilizing synchronous track motions. The above mentioned tracking control strategies can only deal with the single robot manipulator [1], [2], [8], [12], [18], [21], the robotic network and its topological structure is not considered. Recently, based on the input-output passivation framework, Wang [25] investigated the adaptive synchronization of networked non-redundant and identical robotic systems with uncertainties in kinematics and dynamics. As to a networked robot manipulators systems which may include the non-identical structures (such as different physical parameters or different degree of freedom), furthermore, these non-identical robots are used to achieve the same designed task. By using linear parametrization approach, passivity-based synchronization control algorithms of heterogeneous redundant robotic manipulators with uncertainties in dynamics are proposed [10]. However, the end-effectors adaptive synchronization control algorithms of networked robot manipulators in [10], [22] are relied on passivity feature and strongly connected graph theory, and all the needed information of leader or desired is accessed to manipulator.

With the aforementioned background, in this paper based on combination of linear parametrization approach and directed graph theory, our main interest is in in end-effector adaptive tracking control of nonidentical networked manipulators systems under the assumptions that the graph contains a spanning tree and only a small part of the followers can get access to the leader's position vector in the task space. Two adaptive tracking algorithms for redundant and non-redundant networked robot manipulators with unknown parametric uncertainties dynamics are proposed. Moreover, the proposed algorithms work to track time-varying positions and velocities vector of the leader, do not require the computation of inverse kinematics and do not place restriction on the self-motion of the manipulator, hence, the extra degrees of freedom are available for sub-tasks.

The outline of the paper is organized as follows: in Section 2, notations, properties, assumptions, dynamical model and its decomposition of networked manipulator systems are shown; in Section 3, under direct graph theory, adaptive tracking control algorithms for redundant and non-redundant manipulators with uncertainty dynamics are designed; in Section 4, an example of application and simulation for three link networked redundant manipulators are presented and validated the results; final conclusion is given in Section 5.

2. Preliminary and problem statement

Let $\mathbb{R} = (-\infty, +\infty)$ be the set of real numbers, $\mathbb{N} = \{1, 2, ...\}$ be the set of nonnegative integer numbers. For the vector $u \in \mathbb{R}^n$, u^T denotes its transpose. The norm of the vector u is defined as $\|u\| = \sqrt{u^T u}$. $\mathbb{R}^{n\times n}$ stands for $n \times n$ the set of real matrices. I_n denotes identity matrix of n order. Let $A \otimes B$ be the Kronecker product of two matrices A and B. Assume matrix A is symmetric, the negative (positive) definite matrix is denoted by $A < 0, (> 0)$. Let $\mathbb{G} = (V, E, A)$ be a directed graph of order N ($N \geq 2$) with the set of nodes $V = \{v_1, v_2, ..., v_N\}$, an edge $e_{ij} \in E \subset V \times V$ if and only if manipulators i and j exchange information, and a adjacency matrix $A = [a_{ij}] \in \mathbb{R}^{N\times N}$ with nonnegative adjacency elements a_{ij}. The edge (i, j) in the edge set of a directed graph denotes that agent j can

obtain information from agent i, but not necessarily vice versa. A is defined as $a_{ij} > 0$ if $(j,i) \in E$, otherwise $a_{ij} = 0$ for all $i \neq j$, and $a_{ii} = 0$ for all $i \in V$. Let Laplacian matrix $L = [l_{ij}] \in \mathbb{R}^{N\times N}$ associated with A be defined as $l_{ii} = \sum_{k=1,k\neq i}^{N} a_{ik}$ and $l_{ij} = -a_{ij}$, where $i \neq j$.

Assume there are N followers, marking agents 1 to N, the $(N+1)$th manipulator indexed by 0 is treated as leader, the general graph $\overline{\mathbb{G}}$ associated with the considered $N+1$ manipulators, the Laplacian matrix of directed $\overline{\mathbb{G}}$ is denoted as $\overline{L}$. Let $B = \mathrm{diag}(b_{1d}, b_{2d}, ..., b_{Nd})$, where $b_{id} > 0$ if $0 \in N_i$, otherwise $b_{id} = 0$, where $N_i = \{j : (j,i) \in E\}$ denote the set of manipulators transmitting their information to the ith manipulators, suppose that leader can't obtain the followers information. define $H = L + B$, thus, there exists positive definite matrix P such that $Q \triangleq HP + PH^T > 0$.

The dynamics equation of n-degree N manipulators can be described as

$$M_i(q_i)\ddot{q}_i + C_i(q_i,\dot{q}_i)\dot{q}_i + G_i(q_i) + F_i\dot{q}_i = \tau_i, \quad i = 1,2,...,N, \tag{1}$$

where $q_i \in \mathbb{R}^n$ is the vector of generalized coordinates, $M_i(q_i) \in \mathbb{R}^{n\times n}$ is inertia matrix of the ith manipulator, $C_i(q_i,\dot{q}_i) \in \mathbb{R}^{n\times n}$ is Coriolis and centripetal matrix, $G_i(q_i) \in \mathbb{R}^n$ is gravitational vector of the ith manipulator, F_i is a diagonal matrix of viscous and dynamic friction coefficients, $\tau_i \in \mathbb{R}^n$ denotes ith manipulator's control input generalized vector.

Three properties and assumptions [17] of dynamics system (1) are given as follows:

Property 2.1. *The inertia matrix M_i is symmetric and positive definite, and has the following relationship $k_{\overline{m}i}I_n \leq M_i(q_i) \leq k_{\underline{m}i}I_n$, where $k_{\overline{m}i}$, $k_{\underline{m}i}$ are positive constants.*

Property 2.2. *The matrix $\dot{M}_i(q_i) - 2C_i(q_i,\dot{q}_i)$ is skew-symmetric, i.e., $\zeta^T[\dot{M}_i(q_i) - 2C_i(q_i,\dot{q}_i)]\zeta = 0$, for $\forall \zeta, q_i, \dot{q}_i \in \mathbb{R}^n$.*

Property 2.3. *The general form of dynamics system (1) can be linearly parameterized in constant unknown parameters $\Theta_i = (\Theta_{i1}, \Theta_{i2}, ..., \Theta_{id})^T \in \mathbb{R}^d$ as*

$$M_i(q_i)\dot{\xi}_i + C_i(q_i,\dot{q}_i)\xi_i + G_i(q_i) + F_i\xi_i = Y_i(q_i,\dot{q}_i,\xi_i,\dot{\xi}_i)\Theta_i,$$

where $Y_i(q_i,\dot{q}_i,\xi_i,\dot{\xi}_i)$ is a known regressor matrix depending on $q_i,\dot{q}_i,\xi_i,\dot{\xi}_i$, suppose if the parameters of Y_i are limited, Y_i is bounded $i = 1,2,...,N$.

Assumption 2.1. *The end-effector position vector $x_d(t)$, velocity $\dot{x}_d(t)$ and acceleration $\ddot{x}_d(t)$ of the leader are bounded.*

The relation between joint space and task space of the ith manipulator is defined as the following kinematics

$$x_i = f_i(q_i), \tag{2}$$

where $f_i(q_i) \in \mathbb{R}^n \to \mathbb{R}^m$ denotes the nonlinear forward kinematics transformation. Suppose $x_d(t) \in \mathbb{R}^m$ and $\dot{x}_d(t) \in \mathbb{R}^m$ be the end-effector position and velocity of the leader.

The differential relation between joint space and task space can be calculated as:

$$\begin{aligned} \dot{x}_i &= J_i(q_i)\dot{q}_i, \\ \ddot{x}_i &= \dot{J}_i(q_i)\dot{q}_i + J_i(q_i)\ddot{q}_i,\ i = 1, 2, ..., N, \end{aligned} \tag{3}$$

where $J_i(q_i) \in \mathbb{R}^{m\times n}$ Jacobian matrix, $J_i(q_i)$ and $\dot{J}_i(q_i)$ are described as follows:

$$\begin{aligned} J_i(q_i) &= \frac{\partial f_i(q_i)}{\partial q_i}, \\ \dot{J}_i(q_i) &= \frac{dJ_i(q_i)}{dt} = \sum_{j=1}^{n} \frac{\partial J_i(q_i)}{\partial q_{ij}} \dot{q}_{ij}. \end{aligned} \tag{4}$$

Suppose that (1) is redundant dynamic system, i.e., $n > m$. For convenience, the pseudo-inverse of $J_i(q_i)$, denoted by $J_i^+(q_i) \in \mathbb{R}^{n\times m}$ define as follows:

$$J_i^+ = J_i^T (J_i J_i^T)^{-1}, \tag{5}$$

where the pseudo-inverse J_i^+ satisfies the following properties:

$$\begin{aligned} &J_i J_i^+ = I_m,\ J_i J_i^+ J_i = J_i,\ J_i^+ J_i J_i^+ = J_i^+,\ (J_i J_i^+)^T = J_i J_i^+, \\ &(J_i^+ J_i)^T = J_i^+ J_i,\ (I_n - J_i^+ J_i) J_i^+ = \mathbf{0},\ J_i(I_n - J_i^+ J_i) = \mathbf{0}. \end{aligned}$$

3. Main results

In this section, we present two distribute coordinated adaptive tracking control algorithms that account for networked redundant and non-redundant robot manipulators in the presence of Lagrange dynamics uncertain parameters being updated online by the proposed parameter updated laws. Additionally, the designed tracking control algorithms allow the redundancy of Lagrange system to perform the sub-task defined by at least one motion optimization index such as maintaining and avoidance of mechanical joint limits.

Before moving on, introducing assistant variables variables:

$$\begin{aligned} p_i &= \underbrace{J_i^+ \Big[\dot{x}_d - \Big(\sum_{j\in N_i} a_{ij}(x_i - x_j) + b_{id}(x_i - x_d) \Big) \Big]}_{task-space\ tracking\ control} + \underbrace{(I_n - J_i^+ J_i)\phi_i}_{sub-task\ tracking\ control}, \\ \dot{p}_i &= J_i^+ \Big[\ddot{x}_d - \Big(\sum_{j\in N_i} a_{ij}(\dot{x}_i - \dot{x}_j) + b_{id}(\dot{x}_i - \dot{x}_d) \Big) \Big] \\ &\quad + \dot{J}_i^+ \Big[\dot{x}_d - \Big(\sum_{j\in N_i} a_{ij}(x_i - x_j) + b_{id}(x_i - x_d) \Big) \Big] \\ &\quad + \dot{\phi}_i - \dot{J}_i^+ J_i \phi_i - J_i^+ \dot{J}_i \phi_i - J_i^+ J_i \dot{\phi}_i, \\ s_i &= J_i^+ \Big[-\dot{x}_d + \sum_{j\in N_i} a_{ij}(x_i - x_j) + b_{id}(x_i - x_d) \Big] - (I_n - J_i^+ J_i)\phi_i + \dot{q}_i, \end{aligned} \tag{6}$$

where $p_i = \dot{q}_i - s_i$, $\ddot{q}_i = \dot{p}_i + \dot{s}_i$, $\phi_i \in \mathbb{R}^n$ is an auxiliary signal that is constructed in view of the sub-task of control targets, for instance mechanical limit avoidance or eluding obstacles. $b_{id} > 0$ if follower i can obtain leader's information, or $b_{id} = 0$.

Let $r_i = J_i s_i$, $e_i = x_i - x_d$, thus

$$\begin{aligned} r_i &= -\dot{x}_d + \sum_{j\in N_i} a_{ij}(x_i - x_j) + b_{id}(x_i - x_d) + J_i\dot{q}_i \\ &= \dot{e}_i + \sum_{j\in N_i} a_{ij}(e_i - e_j) + b_{id}e_i, \end{aligned} \tag{7}$$

The subtask tracking error denoted by $e_{si} \in \mathbb{R}^n$ which is defined as follows:

$$\begin{aligned} e_{si} &= (I_n - J_i^+ J_i)(\dot{q}_i - \phi_i) \\ &= (I_n - J_i^+ J_i)J_i^+ \Big[-\dot{x}_d + \sum_{j\in N_i} a_{ij}(x_i - x_j) + b_{id}(x_i - x_d)\Big] \\ &- (I_n - J_i^+ J_i)(I_n - J_i^+ J_i)\phi_i + (I_n - J_i^+ J_i)\dot{q}_i \\ &= (I_n - J_i^+ J_i)s_i. \end{aligned} \tag{8}$$

The properties of J_i indicate the regulation of s_i implies the regulation of tracking error e_{si}, so the subtask tracking control can be realized.

The inverse kinematics do not computer by J_i , we first consider the coordinated regulation algorithm as

$$\begin{aligned} \tau_i &= \widehat{M}_i(q_i)\dot{p}_i + \widehat{C}_i(q_i,\dot{q}_i)p_i + \widehat{G}_i(q_i) + \widehat{F}_i p_i - J_i^T \sum_{j=1}^{N} P_{ij}e_i \\ &= \underbrace{Y_i(q_i,\dot{q}_i,p_i,\dot{p}_i)\widehat{\Theta}_i}_{adaptive\ feedback\ control} - \underbrace{J_i^T \sum_{j=1}^{N} P_{ij}e_i}_{tracking\ control}, \end{aligned} \tag{9}$$

where $P = (P_{ij})_{N\times N}$ is a candidate positive definite matrix, since the value of the dynamic parameter Θ_i is hard to be known exactly in practice, one defines $\widehat{\Theta}_i$ as the estimate of Θ_i. $\widehat{M}_i$, $\widehat{C}_i, \widehat{G}_i$ and $\widehat{F}_i$ represent estimation of $M_i(q_i, \widehat{\Theta}_i)$, $C_i(q_i, \dot{q}_i, \widehat{\Theta}_i)$, $G_i(q_i, \widehat{\Theta}_i)$ and $F_i(\widehat{\Theta}_i)$, respectively.

The parameter estimation is selected as the following adaptive law:

$$\dot{\widehat{\Theta}}_i = -\Gamma_i^{-1} Y_i^T s_i, \tag{10}$$

where Γ_i is a constant diagonal positive definite matrix.

Theorem 3.1. *If control input torque τ_i and adaptive compensator are designed as* (9) *and* (10)*, assume that the topology $\overline{\mathbb{G}}$ is directed, and at least one follower can get leader's information, then $x_i(t) \to x_d(t)$ and $\dot{x}_i(t) \to \dot{x}_d(t)$ as $t \to \infty$, $i = 1, 2, ..., N$.*

Proof. The matrix or vector of $q, M, C, G, F, \tau, J, x, J^+, s, r, p, e, e_s, \Theta$ and Γ are the same definition in [10], [22], [25], [26].

From the distributed algorithm (9), system (1) can be written in the compact form as follows:

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) + F\dot{q} = Y(q,\dot{q},p,\dot{p})\widehat{\Theta} - J^T(P \otimes I_m)e. \tag{11}$$

By $\ddot{q} = \dot{p} + \dot{s}$, (11) can be reformulated

$$M\dot{s} = Y\widetilde{\Theta} - J^T(P \otimes I_m)e - Cs - Fs. \tag{12}$$

where $\widetilde{\Theta} = \widehat{\Theta} - \Theta$ denote the estimation error, clearly $\dot{\widetilde{\Theta}} = \dot{\widehat{\Theta}}$.

Consider the following candidate Lyapunov function

$$V = \frac{1}{2}\big(s^T M s + e^T(P \otimes I_m)e + \widetilde{\Theta}^T\Gamma\widetilde{\Theta}\big). \tag{13}$$

The derivative of V along the trajectory of (12) is

$$\begin{aligned}
\dot{V} &= \frac{1}{2}s^T\dot{M}s + s^T M\dot{s} + e^T(P \otimes I_m)\dot{e} + \widetilde{\Theta}^T\Gamma\dot{\widetilde{\Theta}} \\
&= \frac{1}{2}s^T\dot{M}s + s^T\big[Y\widetilde{\Theta} - J^T(P \otimes I_m)e - Cs - Fs\big] \\
&\quad + e^T(P \otimes I_m)\dot{e} + \widetilde{\Theta}^T\Gamma\dot{\widetilde{\Theta}} \\
&= \frac{1}{2}s^T\dot{M}s - s^TCs + s^TY\widetilde{\Theta} - s^TJ^T(P \otimes I_m)e - s^TFs \\
&\quad + e^T(P \otimes I_m)[r - (H \otimes I_m)e] + \widetilde{\Theta}^T\Gamma(-\Gamma^{-1}Y^Ts).
\end{aligned} \tag{14}$$

By Property 2, we have

$$\begin{aligned}
\dot{V} &= s^TY\widetilde{\Theta} - r^T(P \otimes I_m)e - s^TFs + e^T(P \otimes I_m)r \\
&\quad - e^T\big[(PH) \otimes I_m\big]e - \widetilde{\Theta}^TY^Ts = -s^TFs - \tfrac{1}{2}e^T(Q \otimes I)e.
\end{aligned} \tag{15}$$

It is important to note that F is a positive definite diagonal matrix and there exist matrix $P > 0$ such that $Q > 0$, hence, $V(t) \geq 0$, $\dot{V}(t) \leq 0$.

Therefore, $V(t) \in L_\infty$, then $s \in L_\infty$, $e \in L_\infty$, $\widetilde{\Theta} \in L_\infty$, from (3) and (6) $r \in L_\infty, x \in L_\infty,$, and then by $J \in L_\infty$ and (7), $\dot{e} \in L_\infty\, \dot{q} \in L_\infty$. By(12), $\dot{s} \in L_\infty$, and the bounded of $\ddot{x}_d \in L_\infty$ and $J^+ \in L_\infty$, so $\dot{J}^+ \in L_\infty$, hence $\ddot{V}(t) \in L_\infty$. By the Barbalat's Lemma, $\dot{V}(t) \to \infty$ as $t \to \infty$, then $x_i(t) \to x_d(t)$, $\dot{x}_i(t) \to \dot{x}_d(t)$ as $t \to \infty$.

In the meantime, from $e_s = (I_{nN} - J^+J)s$, since $s \to 0$, then $e_s \to 0$, i.e., the sub-task tracking can be realize. ∎

Second, consider the system (1) with non-redundant degree case, i.e., $m = n$ and $J^+ = J^{-1}$. In this case, the auxiliary variables (6) can be rewritten as follows:

$$
\begin{aligned}
y_i &= J_i^{-1}\Big[\dot{x}_d - \Big(\sum_{j\in N_i} a_{ij}(x_i - x_j) + b_{id}(x_i - x_d)\Big)\Big],\\
\dot{y}_i &= J_i^{-1}\Big[\ddot{x}_d - \Big(\sum_{j\in N_i} a_{ij}(\dot{x}_i - \dot{x}_j) + b_{id}(\dot{x}_i - \dot{x}_d)\Big)\Big]\\
&\quad + \dot{J}_i^{-1}\Big[\dot{x}_d - \Big(\sum_{j\in N_i} a_{ij}(x_i - x_j) + b_{id}(x_i - x_d)\Big)\Big],\\
s_i &= J_i^{-1}\Big[-\dot{x}_d + \sum_{j\in N_i} a_{ij}(x_i - x_j) + b_{id}(x_i - x_d)\Big] + \dot{q}_i, \qquad (16)
\end{aligned}
$$

We propose an adaptive control as

$$
\tau_i \;=\; Y_i(q_i, \dot{q}_i, y_i, \dot{y}_i)\widehat{\Theta}_i - J_i^T \sum_{j=1}^{N} P_{ij} e_i, \qquad (17)
$$

where Θ_i, e_i, P_{ij} and the updated adaptive law are the same as that of (9) and (10).

Similar to the proof of Theorem 3.1, it is obviously easy to get following results for the case of non-redundant manipulators systems.

Corollary 3.1. *Assume the communication graph $\overline{\mathbb{G}}$ is directed and and at least one follower can obtain the leader's information. For networked manipulator system* (1)*, under adaptive controller* (17)*, then $x_i(t) \to x_d(t)$ and $\dot{x}_i(t) \to \dot{x}_d(t)$ as $t \to \infty$.*

Remark 3.1. Theorem 3.1 is suitable for the systems with nonidentical manipulators, where the network can contain redundant and non-redundant manipulators simultaneously, i.e., all the manipulators can have different degree of freedom. So our result is general and practical.

Remark 3.2. Noted that from the distributed algorithms (9) and (6), (17) and (16), only small number of followers' end-effectors need to get information of x_d, however all information of the desired trajectory needs to be accessed to all manipulators in [25], [26].

4. Illustrative example

For a better illustration of the proposed adaptive control algorithm we put forward an example in numerical simulation. For simplicity, planar motion of nonidentical redundant manipulators with the same degree of freedom is consider in the task space. The simulations are displayed with networked planar manipulators with three-link (see Fig. 1(a)). The joint q_{i1}, q_{i2} and q_{i3}, spiale O_1, O_2, O_3 and moments of inertias I_{i1}, I_{i2} and I_{i3} are shown in Fig. 1(a). The general communication graph is shown in Fig. 1(b).

Example 4.1. *Consider networked three-link manipulators with communication topology Fig.* 1(b)*, each manipulator nonlinear dynamics follows the Lagrange systems:*

$$M_i(q_i)\ddot{q}_i + C_i(q_i,\dot{q}_i)\dot{q}_i + G_i(q_i) + F_i\dot{q}_i = \tau_i,\ i = 1,...,4, \tag{18}$$

where $q_i = (q_{i1}, q_{i2}, q_{i3})^T \in \mathbb{R}^3$*, the matrices* $M_i(q_i)$*,* $C_i(q_i,\dot{q}_i)$*,* Y_i*, vector* $G_i(q_i)$ *and constant parameter* Θ_i *are represented as shown below.*

Let

$$M_i(q_i) = \begin{bmatrix} M_{i11} & M_{i12} & M_{i13} \\ M_{i21} & M_{i22} & M_{i23} \\ M_{i31} & M_{i32} & M_{i33} \end{bmatrix},\ C_i(q_i,\dot{q}_i) = \begin{bmatrix} C_{i11} & C_{i12} & C_{i13} \\ C_{i21} & C_{i22} & C_{i23} \\ C_{i31} & C_{i32} & C_{i33} \end{bmatrix},$$

$$G_i(q_i) = \begin{bmatrix} G_{i1} \\ G_{i2} \\ G_{i3} \end{bmatrix},$$

where the entries of $M_i(q_i)$, $C_i(q_i,\dot{q}_i)$ and $G_i(q_i)$ are given as $M_{i11} = \Theta_{i1} + 2(\Theta_{i2}\cos_{i2} + \Theta_{i3}\cos_{i3} + \Theta_{i4}\cos_{i23})$, $M_{i12} = M_{i21} = \Theta_{i5} + \Theta_{i2}\cos_{i2} + 2\Theta_{i3}\cos_{i3} + \Theta_{i4}\cos_{i23}$, $M_{i13} = M_{i31} = \Theta_{i6} + \Theta_{i3}\cos_{i3} + \Theta_{i4}\cos_{i23}$, $M_{i22} = \Theta_{i5} + 2\Theta_{i3}\cos_{i3}$, $M_{i23} = M_{i32} = \Theta_{i6} + \Theta_{i3}\cos_{i3}$, $M_{i33} = \Theta_{i6}$. $C_{i11} = -\Theta_{i2}\sin_{i2}\dot{q}_{i2} - \Theta_{i4}\sin_{i23}(\dot{q}_{i2} + \dot{q}_{i3}) - \Theta_{i3}\sin_{i3}\dot{q}_{i3}$, $C_{i12} = -\Theta_{i2}\sin_{i2}(\dot{q}_{i1} + \dot{q}_{i2}) - \Theta_{i4}\sin_{i23}(\dot{q}_{i1} + \dot{q}_{i2} + \dot{q}_{i3}) - \Theta_{i3}\sin_{i3}\dot{q}_{i3}$, $C_{i13} = -\Theta_{i3}\sin_{i3}(\dot{q}_{i1} + \dot{q}_{i2} + \dot{q}_{i3}) - \Theta_{i4}\sin_{i23}(\dot{q}_{i1} + \dot{q}_{i2} + \dot{q}_{i3})$, $C_{i21} = \Theta_{i2}\sin_{i2}\dot{q}_{i1} + \Theta_{i4}\sin_{i23}\dot{q}_{i1} - \Theta_{i3}\sin_{i3}\dot{q}_{i3}$, $C_{i22} = -\Theta_{i3}\sin_{i3}\dot{q}_{i3}$, $C_{i23} = -\Theta_{i3}\sin_{i3}(\dot{q}_{i1} + \dot{q}_{i2} + \dot{q}_{i3})$, $C_{i31} = \Theta_{i3}\sin_{i3}(\dot{q}_{i1} + \dot{q}_{i2}) + \Theta_{i4}\sin_{i23}\dot{q}_{i1}$, $C_{i32} = \Theta_{i3}\sin_{i3}(\dot{q}_{i1} + \dot{q}_{i2})$, $C_{i33} = 0$, $G_{i1} = \Theta_{i7}\sin_{i1} + \Theta_{i8}\sin_{i12} + \Theta_{i9}\sin_{i123}$, $G_{i2} = \Theta_{i8}\sin_{i12} + \Theta_{i9}\sin_{i123}$, $G_{i3} = \Theta_{i9}\sin_{i123}$.

Let

$$Y_i(q_i,\dot{q}_i,p_i,\dot{p}_i) = \begin{bmatrix} Y_{i1}^T \\ Y_{i2}^T \\ Y_{i3}^T \end{bmatrix},\ \Theta_i = \begin{bmatrix} \Theta_{i1} \\ \vdots \\ \Theta_{i12} \end{bmatrix} \text{ and } F_i = \begin{bmatrix} \Theta_{i10} & 0 & 0 \\ 0 & \Theta_{i11} & 0 \\ 0 & 0 & \Theta_{i12} \end{bmatrix},$$

where the vector of unknown estimate parameters $\Theta_i \in \mathbb{R}^{12}$ and regressor matrix $Y_i(q_i,\dot{q}_i,p_i,\dot{p}_i) \in \mathbb{R}^{3\times 12}$ are given as $\Theta_{i1} = l_{ci1}^2 m_{i1} + (l_{i1}^2 + l_{ci2}^2)m_{i2} + (l_{i1}^2 + l_{i2}^2 + l_{ci3}^2)m_{i3} + I_{i1} + I_{i2} + I_{i3}$, $\Theta_{i2} = l_{i1}l_{ci2}m_{i2} + l_{i1}l_{i2}m_{i3}$, $\Theta_{i3} = l_{i2}l_{ci3}m_{i3}$, $\Theta_{i4} = l_{i1}l_{ci3}m_{i3}$, $\Theta_{i5} = l_{ci2}^2 m_{i2} + (l_{i2}^2 + l_{ci3}^2)m_{i3} + I_{i2} + I_{i3}$, $\Theta_{i6} = l_{ci3}^2 m_{i3} + I_{i3}$, $\Theta_{i7} = (l_{ci1}m_{i1} + l_{i1}m_{i2} + l_{i1}m_{i3})g$, $\Theta_{i8} = (l_{ci2}m_{i2} + l_{i2}m_{i3})g$, $\Theta_{i9} = l_{ci3}m_{i3}g$, $\Theta_{i10} = F_{i1}$, $\Theta_{i11} = F_{i2}$, $\Theta_{i12} = F_{i3}$, $Y_{i1} = [\dot{p}_{i1}, \alpha_1, \alpha_2, \alpha_3, \dot{p}_{i2}, \dot{p}_{i3}, \sin_{i1}, \sin_{i12}, \sin_{i123}, p_{i1}, 0, 0]^T$, $Y_{i2} = [0, \beta_1, \beta_2, \beta_3, \dot{p}_{i1} + \dot{p}_{i2}, \dot{p}_{i3}, 0, \sin_{i12}, \sin_{i123}, 0, p_{i2}, 0]^T$, $Y_{i3} = [0, 0, \gamma_1, \gamma_2, 0, \dot{p}_{i1} + \dot{p}_{i2} + \dot{p}_{i3}, 0, 0, \sin_{i123}, 0, 0, p_{i3}]^T$, where $l_{c_{i1}} = \frac{1}{2}l_{i1}$, $l_{c_{i2}} = \frac{1}{2}l_{i2}$, $l_{c_{i3}} = \frac{1}{2}l_{i3}$. $I_{i1} = \frac{1}{3}m_{i1}l_{c_{i1}}^2$, $I_{i2} = \frac{1}{3}m_{i2}l_{c_{i2}}^2$ and $I_{i3} = \frac{1}{3}m_{i3}l_{c_{i3}}^2$ are the moments of inertia. $\alpha_1 = 2\cos_{i2}\dot{p}_{i1} + \cos_{i2}\dot{p}_{i2} - \sin_{i2}\dot{q}_{i2}p_{i1} - \sin_{i2}(\dot{q}_{i1} + \dot{q}_{i2})p_{i2}$, $\alpha_2 = 2\cos_{i3}\dot{p}_{i1} + 2\cos_{i3}\dot{p}_{i2} + \cos_{i3}\dot{p}_{i3} - \sin_{i3}[\dot{q}_{i3}p_{i1} + \dot{q}_{i3}p_{i2} + (\dot{q}_{i1} + \dot{q}_{i2} + \dot{q}_{i3})p_{i3}]$, $\alpha_3 = \cos_{i23}(2\dot{p}_{i1} + \dot{p}_{i2} + \dot{p}_{i3}) - \sin_{i23}[(\dot{q}_{i2} + \dot{q}_{i3})p_{i1} + (\dot{q}_{i1} + \dot{q}_{i2} + \dot{q}_{i3})(p_{i2} + p_{i3})]$, $\beta_1 = \cos_{i2}\dot{p}_{i1} + \sin_{i2}\dot{q}_{i1}p_{i1}$, $\beta_2 = \cos_{i3}(2\dot{p}_{i1} + 2\dot{p}_{i2} + \dot{p}_{i3}) - \sin_{i3}[\dot{q}_{i3}p_{i1} + \dot{p}_{i3}p_{i2} + (\dot{q}_{i1} + \dot{q}_{i2} + \dot{q}_{i3})p_{i3}]$, $\beta_3 = \cos_{i23}\dot{p}_{i1} + \sin_{i23}\dot{q}_{i1}p_{i1}$, $\gamma_1 = \cos_{i3}(\dot{p}_{i1} + \dot{p}_{i2}) + \sin_{i3}(\dot{p}_{i1} + \dot{p}_{i2})(p_{i1} + p_{i2})$, $\gamma_2 = \cos_{i23}\dot{p}_{i1} + \sin_{i23}\dot{q}_{i1}p_{i1}$, where

$\cos_{i1} = \cos q_{i1}$, $\cos_{i12} = \cos(q_{i1}+q_{i2})$, $\cos_{i123} = \cos(q_{i1}+q_{i2}+q_{i3})$, $\cos_{i23} = \cos(q_{i2}+q_{i3})$, $\sin_{i1} = \sin q_{i1}$, $\sin_{i12} = \sin(q_{i1}+q_{i2})$, $\sin_{i123} = \sin(q_{i1}+q_{i2}+q_{i3})$, $\sin_{i23} = \sin(q_{i2}+q_{i3})$. Obviously, $M_i(q_i)$ is symmetric, positive definite, differentiable, and satisfies Properties 2.1 and 2.2. Assume that system (18) is using distributed the control input (9) and the updated adaptive law (10).

The agents are assumed to be communicated using directed topology graph $\overline{\mathbb{G}}$ is illustrated in Fig. 1(b). Matrix H is represented as shown below

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & -1 & 2 \end{bmatrix}.$$

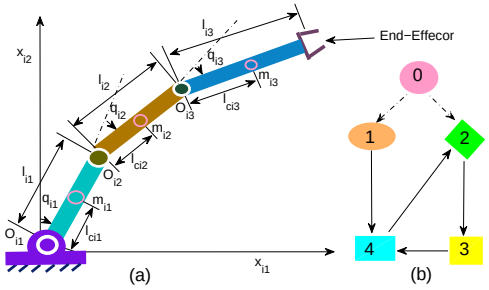

Fig. 1. (a) The link parameters of ith three-link manipulators. (b) The direct communication graph of four nonidentical three-link manipulators followers and one dynamic leader.

In task-space, the ith manipulator position is expressed as:

$$x_i = \begin{bmatrix} -l_{i3}\sin_{i123} - l_{i2}\sin_{i12} - l_{i1}\sin_{i1} \\ l_{i3}\cos_{i123} + l_{i2}\cos_{i12} + l_{i1}\cos_{i1} \end{bmatrix}, \tag{19}$$

where $\sin_{i123} = \sin(q_{i1}+q_{i2}+q_{i3})$, $\sin_{i12} = \sin(q_{i1}+q_{i2})$, $\sin_{i1} = \sin(q_{i1})$, $\cos_{i123} = \cos(q_{i1}+q_{i2}+q_{i3})$, $\cos_{i12} = \cos(q_{i1}+q_{i2})$, $\cos_{i1} = \cos(q_{i1})$.

The Jacobian matrix is shown as:

$$J_i = \begin{bmatrix} -l_{i3}\cos_{i123} - l_{i2}\cos_{i12} - l_{i1}\cos_{i1} & -l_{i3}\cos_{i123} - l_{i2}\cos_{i12} & -l_{i3}\cos_{i123} \\ -l_{i3}\sin_{i123} - l_{i2}\sin_{i12} - l_{i1}\sin_{i1} & -l_{i3}\sin_{i123} - l_{i2}\sin_{i12} & -l_{i3}\sin_{i123} \end{bmatrix} \in \mathbb{R}^{2\times 3}.$$

From (4), the derivative of the manipulator Jacobian matrix with respect to time is as follows:

$$\begin{aligned} \dot{J}_i = & \begin{bmatrix} l_{i3}\sin_{i123} + l_{i2}\sin_{i12} + l_{i1}\sin_{i1} & l_{i3}\sin_{i123} + l_{i2}\sin_{i12} & l_{i3}\sin_{i123} \\ -l_{i3}\cos_{i123} - l_{i2}\cos_{i12} - l_{i1}\cos_{i1} & -l_{i3}\cos_{i123} - l_{i2}\cos_{i12} & -l_{i3}\cos_{i123} \end{bmatrix} \dot{q}_{i1} \\ & + \begin{bmatrix} l_{i3}\sin_{i123} + l_{i2}\sin_{i12} & l_{i3}\sin_{i123} + l_{i2}\sin_{i12} & l_{i3}\sin_{i123} \\ -l_{i3}\cos_{i123} - l_{i2}\cos_{i12} & -l_{i3}\cos_{i123} - l_{i2}\cos_{i12} & -l_{i3}\cos_{i123} \end{bmatrix} \dot{q}_{i2} \\ & + \begin{bmatrix} l_{i3}\sin_{i123} & l_{i3}\sin_{i123} & l_{i3}\sin_{i123} \\ -l_{i3}\cos_{i123} & -l_{i3}\cos_{i123} & -l_{i3}\cos_{i123} \end{bmatrix} \dot{q}_{i3}. \end{aligned} \tag{20}$$

From (5), we can copulate J_i^+ and $\dot{J}_i^+ = \left[\dot{J}_i^T - J_i^T\left(J_i J_i^T\right)^{-1}\left(\dot{J}_i J_i^T + J_i \dot{J}_i^T\right)\right]\left(J_i J_i^T\right)^{-1}$, clearly, the kinematics of each manipulator are also highly inherent nonlinear.

The leader's position and velocity vectors are chosen as follows

$$x_d(t) = \begin{bmatrix} x_{d1}(t) \\ x_{d2}(t) \end{bmatrix} = \begin{bmatrix} 4.2 + 0.76\sin(0.5t) \\ 3.6 + 0.3\cos(t) \end{bmatrix}, \ \dot{x}_d(t) = \begin{bmatrix} 0.38\cos(0.5t) \\ -0.3\sin(t) \end{bmatrix} \tag{21}$$

The $\phi_i(t)$ for ith manipulator is selected as follows:

$$\phi_i(t) = \begin{bmatrix} \phi_{i1}(t) \\ \phi_{i2}(t) \\ \phi_{i3}(t) \end{bmatrix} = -2 \begin{bmatrix} q_{i3}(t) - q_{i2}(t) + 0.5q_{i1}(t) \\ -q_{i3}(t) + q_{i2}(t) - 0.5q_{i1}(t) \\ q_{i3}(t) - q_{i2}(t) + 0.5q_{i1}(t) \end{bmatrix}. \tag{22}$$

Let m_{i1}=0.52 + 0.1i, m_{i2}=0.61 + 0.06i, m_{i3}=0.54 + 0.08i, l_{i1}=3.2 + 0.08i, $l_{i2} = 2.3 + 0.04i$, $l_{i3} = 3.38 + 0.06i$. Let $l_{ci1} = \frac{1}{2}l_{i1}$, $l_{ci2} = \frac{1}{2}l_{i2}$, $l_{ci3} = \frac{1}{2}l_{i3}$, $I_{i1} = \frac{1}{3}m_{i1}l_{ci1}^2$, $I_{i2} = \frac{1}{3}m_{i2}l_{ci2}^2 I_{i3} = \frac{1}{3}m_{i3}l_{ci3}^2$, $F_i = \mathrm{diag}(F_{i1}, F_{i2}, F_{i3})$, $F_{i1} = 5.81 + 0.02i$, $F_{i2} = 5.81 + 0.021i$, $F_{i3} = 5.81 + 0.023i$, $\Gamma_i = 8.5I_3$, $i = 1, ..., 4$, the gravity acceleration be $g = 9.8m/s^2$.

The initial value $q_{ij}, \dot{q}_{ij}, \widehat{\Theta}_{ik}$, $i = 1, ..., 4$, $j = 1, ..., 3$, $k = 1, ..., 12$ can be randomly chosen in $(-\pi, \pi)$.

Let $P = 9.5I_4$, clearly, $Q > 0$, based on Theorem 3.1, $x_i(t) \to x_d(t)$ and $\dot{x}_i(t) \to \dot{x}_d(t)$ as $t \to \infty$, (see Figs. 2, 3). Meanwhile, Figs. 4 and 5 show tracking effect of positions and velocities in task space.

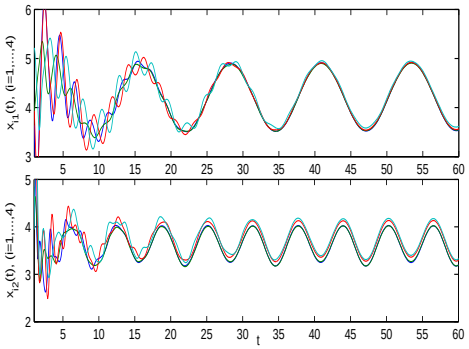

Fig. 2. Desired and actual position task-space trajectories.

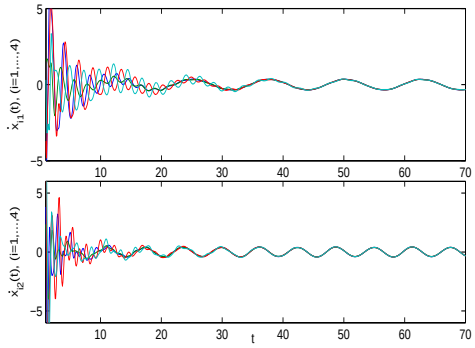

Fig. 3. Desired and actual velocity task-space trajectories.

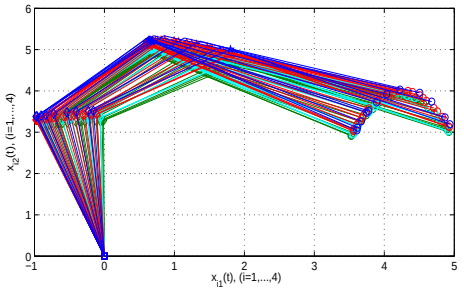

Fig. 4. The position trajectories of four manipulators end-effectors.

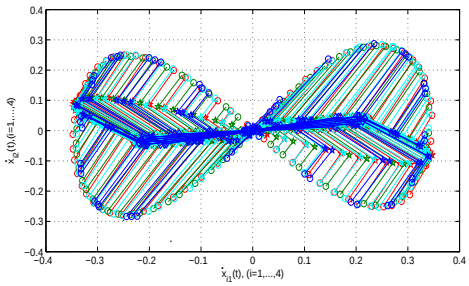

Fig. 5. The velocity trajectories of four manipulators end-effectors.

5. Conclusions

This paper presents two tracking control algorithms of nonidentical networked robot manipulators systems using directed graph theory. While parameters of the considered Lagrange dynamics are uncertain, the proposed adaptive algorithms work to track leader's time-varying positions and velocities vector. Moreover, only a small number of the followers' end-effectors need to obtain position signal of leader. Furthermore, the proposed adaptive algorithms can apply to the tracking control of the nonidentical manipulators with different degree of freedom. The effectiveness of the proposed control algorithms is examined by a simulation example for networked nonidentical manipulator systems with three-links.

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REPRODUCING KERNEL HILBERT SPACE METHOD FOR SOLVING FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS OF FRACTIONAL ORDER

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Abstract. This paper presents a computational technique for solving linear and nonlinear Fredholm integro-differential equations of fractional order. In addition, examples that illustrate the pertinent features of this method are presented, and the results of the study are discussed. Results have revealed that the RKHSM yields efficiently a good approximation to the exact solution.

Keywords: Reproducing Kernel Hilbert Space Method (RKHSM), fractional integro-differential equations and numerical solution.

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1. Introduction

One of the most important topics in applied mathematics is the integro-differential equations. This topic has many applications in engineering, economics, physics, chemistry, astronomy, biology, potential theory and electronics [24].

Solving integro-differential equations becomes one of the most interesting problems to many mathematicians. Numerous methods have been used to solve such problems. For instance, Adomian decomposition method [14], [15], variational iteration method [16], [17], generalized differential transform method [18], [19], nonstandard difference method [8], Adams-Bashforth-Moulton method [7], [22] and homotopy analysis method [9], [20], [23] have been used for this purpose.

The Fredholm integro-differential equations pop up when the differential equations are converted into integro-differential equations. The Reproducing Kernel Hilbert Space Method (RKHSM), which has been first introduced by Aronszajn [1], has played a major role in solving differential and integro-differential equations [2], [3], [4], [5], [6], [11], [12], [25].

This paper is organized as follows. In Section 2, we introduce some basic definitions about fractional calculus [13], [21] and reproducing kernel Hilbert spaces. The Reproducing Kernel Hilbert Space Method (RKHSM) is presented in Section 3. In Section 4, we examine the method by four numerical examples along with a comparison to the exact solutions. Finally, in Section 5, we give conclusions about the examples.

2. Mathematical preliminaries

Definition 2.1 The fractional derivative D^α of $f(t)$ in the Caputo's sense is defined as:

$$D^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{(m-\alpha-1)} f^{(m)}(\tau)\, d\tau,$$
$$m-1 < \alpha \leq m,\ m \in,\ t > 0,\ f(t) \in C^m_{-1}.$$

Definition 2.2 Let H be a Hilbert space of function $f : X \to F$ on a set X. A function $K : X \times X \to C$ is a reproducing kernel of H if the following conditions are satisfied:

1. $K(.,t) \in H$ for all $t \in X$.
2. $\langle f, K(.,t)\rangle = f(t)$ for all $f \in H$ for all $t \in X$.

Note that the reproducing kernel is unique, symmetrical and positively definite.

Definition 2.3 The space $W_2^m[a,b]$ is defined by:

$$W_2^m[a,b] = \{\ u : u^{(j)} \text{ are absolutely continuous functions,}$$
$$j = 1,2,...,m-1 \text{ and} u^{(m)} \in L^2[a,b]\}$$

with the inner product:

$$\langle u, v\rangle_{W_2^m} = \sum_{i=0}^{m-1} u^{(i)}(a)v^{(i)}(a) + \int_a^b u^{(m)}(t)v^{(m)}(t)\ dt$$

and the norm:

$$\|u\|_{W_2^m} = \sqrt{\langle u, u\rangle_{W_2^m}}.$$

3. Reproducing Kernel Hilbert Space Method (RKHSM)

In this article, we try to utilize the Reproducing Kernel Hilbert Space Method (RKHSM) to find a numerical solution of the following Fredholm integro-differential equation of fractional order:

$$D^\alpha u(x) = F(x, u(x), Tu(x)), a \le x \le b,\ m-1 < \alpha \le m, \tag{3.1}$$

with the initial condition

$$u^{(i)}(a) = \beta_i,\ i = 0, 1, ..., m-1, \tag{3.2}$$

where D^α denotes the Caputo fractional derivative of order α, $Tu(x)=\int_a^b h(x,t)u(t)dt$, β_i, $i = 0, 1, 2, ..., m-1$, are real finite constants, $h(x,t)$ is a known continuous function, $u(x)$ is an unknown function and $F(x, u(x), Tu(x))$ is a linear or nonlinear function. In particular, we present a numerical solution for Fredholm integro-differential equation of first and second fractional order.

The Reproducing Kernel Hilbert Space Method has the following steps:

Step(1): Homogenize the initial conditions by some kind of transformation.

Step(2): Construct a reproducing kernel space $W_2^{m+1}[a, b]$, in which every function satisfies the initial condition (3.2).

- [5] The inner product space $W_2^1[a, b] = \{u | u$ is an absolutely continuous real valued function, $u^{'} \in L^2[a, b]\}$ with the inner product $\langle u, v\rangle_{W_2^1} = \int_a^b (u(t)v(t) + u^{'}(t)v^{'}(t))\ dt$ is a complete reproducing kernel and its reproducing kernel is given by:

$$R(x, y) = \frac{1}{\sinh(b-a)}[\cosh(x + y - b - a) + \cosh(|x - y| - b - a)]$$

- [6] The inner product space $W_2^2[a, b] = \{u | u, u^{'}$ are absolutely continuous real valued functions, $u^{''} \in L^2[a, b], u(a) = 0\}$ with the inner product $\langle u, v\rangle_{W_2^2} = u(a)v(a)+u^{'}(a)v^{'}(a)+\int_a^b u^{''}(t)v^{''}(t)\ dt$ is a complete reproducing kernel and its reproducing kernel is given by:

$$K(x, y) = \frac{-1}{6}\begin{cases} (a-y)(2a^2 - y^2 + 3x(2+y) - a(6+3x+y)) & \text{if } y \le x \\ (a-x)(2a^2 - x^2 + 3y(2+x) - a(6+3y+x)) & \text{if } y > x \end{cases}$$

- The inner product space $W_2^3[a,b] = \{u | u, u', u''$ are absolutely continuous real valued functions, $u''' \in L^2[a,b], u(a) = u'(a) = 0\}$ with the inner product $\langle u, v\rangle_{W_2^3} = u(a)v(a) + u'(a)v'(a) + u''(a)v''(a) + \int_a^b u'''(t)v'''(t)\,dt$ is a complete reproducing kernel and its reproducing kernel is given by:

$$S(x,y) = \frac{-1}{120}\begin{cases} h(x,y) & \text{if } y \le x \\ h(y,x) & \text{if } y > x \end{cases}$$

where; $h(x,y) = (a-y)^2(6a^3 + 5xy^2 - y^3 - 10x^2(3+y) - 3a^2(10+5x+y) + 2a(5x^2 - y^2 + 5x(6+y)))$

Step(3): Define the linear operator $L : W_2^{m+1}[a,b] \to W_2^1[a,b]$ such that $Lu(x) = D^\alpha u(x) = F(x, u(x), Tu(x))$.

It is clear that L is a bounded linear operator from $W_2^{m+1}[a,b]$ to $W_2^1[a,b]$.

Step(4): Construct an orthogonal function system of $W_2^{m+1}[a,b]$.

To do this, let $\{x_i\}_{i=1}^\infty$ be a countable dense set in $[a,b]$. Let $\varphi_i(x) = R(x_i, x)$ and $\psi_i(x) = L^*\varphi_i(x)$; where L^* is the self-adjoint operator of L.

The orthonormal system $\{\overline{\psi}_i(x)\}_{i=1}^\infty$ of the space $W_2^{m+1}[a,b]$ can be determined by the Gram–Schmidt orthogonalization process of $\{\psi_i(x)\}_{i=1}^\infty$, $\overline{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x)$, $i = 1, 2, ...$, where β_{ik} are the orthogonalization coefficients.

Theorem 3.1 *Assume that the inverse operator L^{-1} exists. Then if $\{x_i\}_{i=1}^\infty$ is dense on $[a,b]$, then $\{\psi_i(x)\}_{i=1}^\infty$ is a complete system of $W_2^{m+1}[a,b]$.*

Proof. Let $u(x) \in W_2^{m+1}[a,b]$ such that $\langle u(x), \psi_i(x)\rangle_{W_2^{m+1}} = 0$, $\forall i = 1, 2, ...$ Then

$$\begin{aligned} 0 &= \langle u(x), \psi_i(x)\rangle_{W_2^{m+1}} = \langle u(x), L^*\varphi_i(x)\rangle_{W_2^{m+1}} \\ &= \langle Lu(x), \varphi_i(x)\rangle_{W_2^{m+1}} = \langle Lu(x), R(x_i, x)\rangle_{W_2^{m+1}} = Lu(x_i), \end{aligned}$$

and $\{x_i\}_{i=1}^\infty$ is dense on $[a,b]$, then $Lu(x) = 0 \Rightarrow u \equiv 0$ since L^{-1} exists.

Theorem 3.2 *If $\{x_i\}_{i=1}^\infty$ is dense on $[a,b]$ and the solution of equation (3.1) is unique, then the solution is given by:*

$$u(x) = \sum_{i=1}^{\infty} \beta_{ik} A_i \overline{\psi}_i(x), \text{ where } A_i = \sum_{k=1}^{i} \beta_{ik} F(x_k, u(x_k), Tu(x_k)).$$

Proof. $\{\overline{\psi}_i(x)\}_{i=1}^\infty$ is a complete orthonormal basis of $W_2^{m+1}[a,b]$.

$$\begin{aligned} u(x) &= \sum_{i=1}^{\infty} \langle u(x), \overline{\psi}_i(x)\rangle_{W_2^{m+1}} \overline{\psi}_i(x) = \sum_{i=1}^{\infty} \langle u(x), \sum_{k=1}^{i} \beta_{ik}\psi_k(x)\rangle_{W_2^{m+1}} \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} \langle u(x), \psi_k(x)\rangle_{W_2^{m+1}} \overline{\psi}_i(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} \langle u(x), L^*\varphi_k(x)\rangle_{W_2^{m+1}} \overline{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} \langle Lu(x), \varphi_k(x)\rangle_{W_2^{1}} \overline{\psi}_i(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} \langle Lu(x), R(x_k, x)\rangle_{W_2^{1}} \overline{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} Lu(x_k) \overline{\psi}_i(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} F(x_k, u(x_k), Tu(x_k)) = \sum_{i=1}^{\infty} A_i \overline{\psi}_i(x) \end{aligned}$$

The approximate solution $u_n(x)$ can be obtained by taking finitely many terms in the series representation of $u(x)$ and so $u_n(x) = \sum_{i=1}^{n} A_i \overline{\psi}_i(x)$. Note that, since $W_2^{m+1}[a,b]$ is a Hilbert space, then $\sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} F(x_k, u(x_k), Tu(x_k))\overline{\psi}_i(x) < \infty$. Also, note that, if $u(x) \in W_2^{m+1}[a,b]$, then $\exists c_i > 0$ such that $|u_n^{(i)}(x)| \leq c_i ||u^{(i)}(x)||$, $i = 0, 1, 2, ...$

Theorem 3.3 *The approximate solution $u_n(x)$ and its derivatives $u_n^{(i)}(x)$ are uniformly convergent to $u(x)$ and $u^{(i)}(x)$, respectively.*

Proof.

$$\begin{aligned} |u_n(x) - u(x)| &= |\langle u_n(x) - u(x), K(x,y)\rangle_{W_2^{m+1}}| \\ &\leq ||K(x,y)||_{W_2^{m+1}} |||| u_n(x) - u(x)||_{W_2^{m+1}}|| \\ &\leq c_0 ||u_n(x) - u(x)||_{W_2^{m+1}} \end{aligned}$$

and

$$\begin{aligned} |u_n^{(i)}(x) - u^{(i)}(x)| &= |\langle u_n^{(i)}(x) - u^{(i)}(x), K^{(i)}(x,y)\rangle_{W_2^{m+1}}| \\ &\leq ||K^{(i)}(x,y)||_{W_2^{m+1}} |||| u_n^{(i)}(x) - u^{(i)}(x)||_{W_2^{m+1}}|| \\ &\leq c_i ||u_n^{(i)}(x) - u^{(i)}(x)||_{W_2^{m+1}}, \ i = 1, 2, ... \end{aligned}$$

Thus $u_n(x)$ and $u_n^{(i)}(x)$ converge uniformly to $u(x)$ and $u^{(i)}(x)$ respectively.

4. Numerical examples

In this section, we propose four numerical examples to illustrate the accuracy and efficiency of the Reproducing Kernel Hilbert Space Method. The computations are performed by Mathematica 9.0.We compare the results by this method with the exact solution of each example.

Example 4.1 [10] Consider the following Fredholm integro-differential equation:

$$\begin{aligned} D^{\alpha} u(x) &= \frac{159}{160} - \frac{x^2}{64} + \int_0^{0.5} (x^2 + t)(u(t))^2 \, dt, \ 0 \leq x \leq 0.5, \ 0 < \alpha \leq 1, \\ u(0) &= 0. \end{aligned}$$

The exact solution for $\alpha = 1$ is $u(x) = x$.

Solution: Using the RKHSM, taking $x_i = \frac{i}{n}$, $i = 1, 2, ..., 100$, the numerical results are given in Table 1 and the graphs of the approximate solutions for different values of α are given in Figure 1.

Table 1: Numerical solution for Example 4.1

x	Exact solution	Approximate solution	$\alpha = 0.95$	$\alpha = 0.9$	$\alpha = 0.85$	Absolute Error
0.1	0.1	0.099939991363	0.113016834790	0.128634865162	0.146699446320	6.008636902×10^{-5}
0.2	0.2	0.199939610015	0.220010263397	0.242736648403	0.267905445031	6.038998433×10^{-5}
0.3	0.3	0.299939275627	0.323254327854	0.349509106541	0.377709699677	6.072437286×10^{-5}
0.4	0.4	0.399938895226	0.425587656134	0.455226839973	0.485948442440	6.11047734×10^{-5}
0.5	0.5	0.499938453679	0.526740419380	0.559172708891	0.591464486819	6.154632005×10^{-5}
0.6	0.6	0.599937976527	0.627358967434	0.662044915706	0.695387823420	6.202347289×10^{-5}
0.7	0.7	0.699937499374	0.727977515489	0.764917122520	0.799311178222	6.250062573×10^{-5}
0.8	0.8	0.799937022221	0.828596063543	0.867789329335	0.903234533023	6.297777857×10^{-5}
0.9	0.9	0.899936545068	0.929214611597	0.970661536149	1.007157887824	6.345493142×10^{-5}
1.0	1.0	0.999936067915	1.029833159652	1.073533742964	1.111081242626	6.393208426×10^{-5}

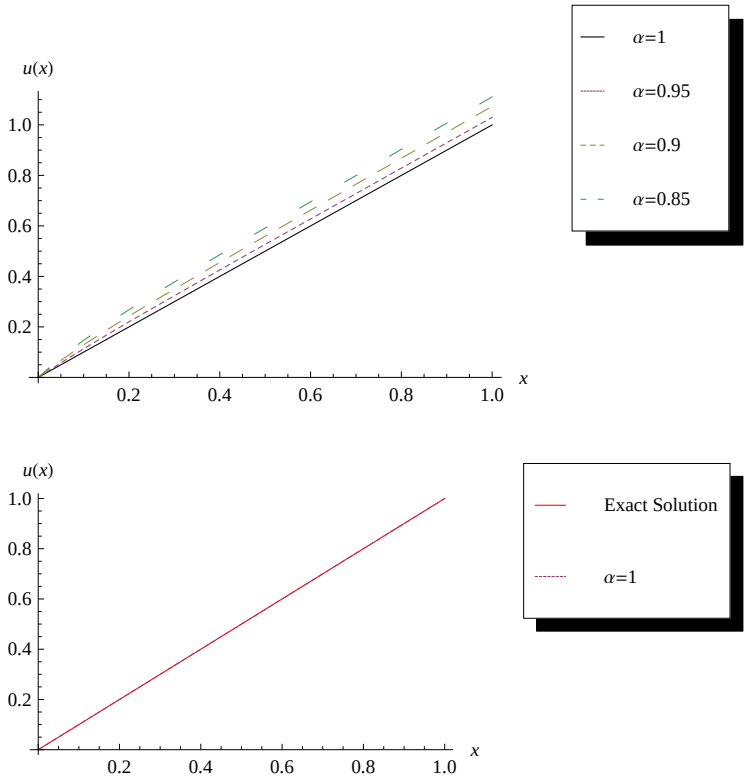

Figure 1: The comparison of approximate solution for $\alpha = 1,\ 0.95,\ 0.9,\ 0.85$ and the exact solution of Example 4.1

From the numerical results in Table 1 and Figure 1, it is clear that the approximate solutions are in good agreement with the exact solutions when $\alpha = 1$ and the solution continuously depends on the fractional derivative.

Example 4.2 Consider the following Fredholm integro-differential equation:

$$\begin{aligned} D^{\alpha}u(x) &= 1 - e^{-1} + \int_0^1 e^{-u'(t)}\, dt,\ 0 \le x \le 1,\ 0 < \alpha \le 1 \\ u(0) &= 0. \end{aligned}$$

The exact solution for $\alpha = 1$ is $u(x) = x$.

Solution: Using the RKHSM, taking $x_i = \dfrac{i}{n},\ i = 1, 2, ..., 100$, the numerical results are given in Table 2 and the graphs of the approximate solutions for different values of α are given in Figure 2.

Table 2: Numerical solution for Example 4.2

x	Exact solution	Approximate solution	$\alpha = 0.95$	$\alpha = 0.9$	$\alpha = 0.85$	Absolute Error
0.1	0.1	0.108054998483	0.156451786148	0.173281483420	0.191698426480	8.054998483×10^{-3}
0.2	0.2	0.207890498221	0.257439610084	0.278328548533	0.302058264862	7.890498222×10^{-3}
0.3	0.3	0.307726094251	0.358085456916	0.380495866370	0.405858362179	7.726094251×10^{-3}
0.4	0.4	0.407561690281	0.457995881326	0.480896758596	0.506474316490	7.561690282×10^{-3}
0.5	0.5	0.507397288668	0.557094440490	0.579691700902	0.604675994093	7.397288668×10^{-3}
0.6	0.6	0.607232887414	0.655462544381	0.677052169259	0.700787627295	7.232887415×10^{-3}
0.7	0.7	0.707068486161	0.753185454380	0.773140771156	0.795040013697	7.068486161×10^{-3}
0.8	0.8	0.806904084907	0.850329848517	0.868089303354	0.887619221137	6.904084908×10^{-3}
0.9	0.9	0.906739683654	0.946946541978	0.961999911229	0.978674307554	6.739683654×10^{-3}
1.0	1.0	1.006575282400	1.043073761178	1.054948132982	1.068315777047	6.575282401×10^{-3}

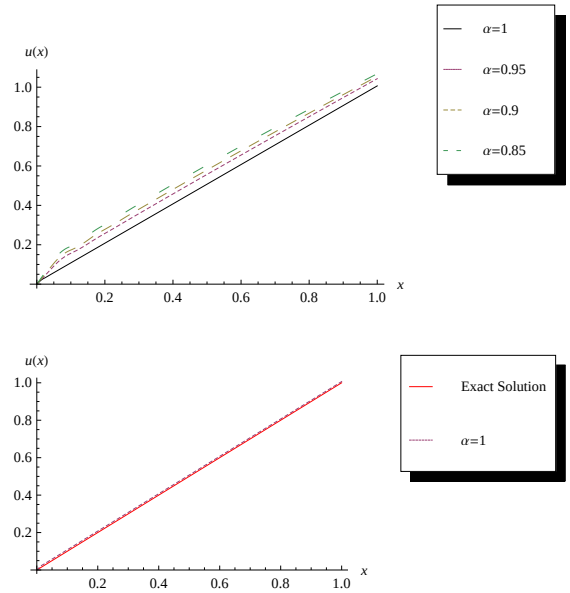

Figure 2: The comparison of approximate solution for $\alpha = 1,\ 0.95,\ 0.9,\ 0.85$ and the exact solution of Example 4.2

Example 4.3 *Consider the following Fredholm integro-differential equation:*

$$\begin{aligned} D^{\alpha}u(x) &= -\sin x + \cos 1 - \sin 1 + \int_0^1 tu(t)\, dt,\ 0 \le x \le 1,\ 1 < \alpha \le 2 \\ u(0) &= 0,\ u'(0) = 1. \end{aligned}$$

The exact solution for $\alpha = 2$ *is* $u(x) = \sin x$.

Solution: After homogenizing the initial condition, using the RKHSM, taking $x_i = \frac{i}{n},\ i = 1, 2, ..., 100$, the numerical results are given in Table 3 and the graphs of the approximate solutions for different values of α are given in Figure 3.

Table 3: Numerical solution for Example 4.3

x	Exact solution	Approximate solution	$\alpha = 1.9$	$\alpha = 1.8$	$\alpha = 1.7$	Absolute Error
0.1	0.099833	0.099989718827	0.099870015390	0.099833619706	0.099776985988	1.563021804×10^{-4}
0.2	0.198669	0.199240887840	0.198769728794	0.198769728794	0.197772917339	5.715570458×10^{-4}
0.3	0.29552	0.296675726695	0.295414566297	0.295414566297	0.292143843212	$1.155520034\times1-^{-3}$
0.4	0.389418	0.391264329285	0.388662210215	0.388662210215	0.381667812369	1.845986977×10^{-3}
0.5	0.479426	0.482028377768	0.477490789153	0.472134288222	0.465372844980	2.602839165×10^{-3}
0.6	0.564642	0.568044521139	0.560976049482	0.552662840607	0.542447997876	3.402047744×10^{-3}
0.7	0.644518	0.648448284409	0.638283208666	0.626439778742	0.612213179934	4.230597172×10^{-3}
0.8	0.717356	0.722438475921	0.708664341922	0.692797709199	0.674107303611	5.082385022×10^{-3}
0.9	0.783327	0.789282034337	0.771458588879	0.751171033040	0.727681590487	5.95512471×10^{-3}
1.0	0.841471	0.848319221670	0.826093568986	0.801094676834	0.772596495739	6.848236863×10^{-3}

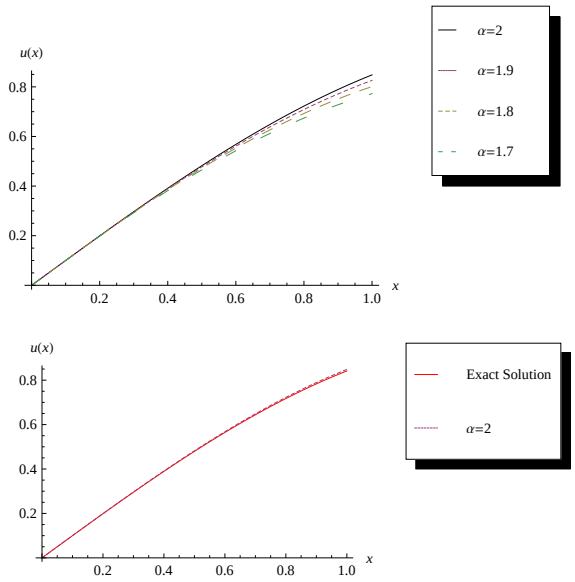

Figure 3: The comparison of approximate solution for $\alpha = 2,\ 1.9,\ 1.8,\ 1.7$ and the exact solution of Example 4.3

Example 4.4 Consider the following Fredholm integro-differential equation:

$$\begin{aligned} D^{\alpha}u(x) &= \frac{7}{3} - \int_0^1 u(t)\,dt,\ 0 \le x \le 1,\ 1 < \alpha \le 2 \\ u(0) &= 0,\ u'(0) = 0. \end{aligned}$$

The exact solution for $\alpha = 2$ is $u(x) = x^2$.

Solution: After homogenizing the initial condition, using the RKHSM, taking $x_i = \frac{i}{n},\ i = 1, 2, ..., 100$, the numerical results are given in Table 4 and the graphs of the approximate solutions for different values of α are given in Figure 4.

Table 4: Numerical solution for Example 4.4

x	Exact solution	Approximate Solution	$\alpha = 1.9$	$\alpha = 1.8$	$\alpha = 1.7$	Absolute Error
0.1	0.01	0.010405401644	0.014490416891	0.018809205338	0.024515140267	4.054016446×10^{-4}
0.2	0.04	0.040822066202	0.052658527691	0.064101825247	0.077782662525	8.220662025×10^{-4}
0.3	0.09	0.091219921199	0.112365258336	0.132306142532	0.155163543905	1.219921199×10^{-3}
0.4	0.16	0.161598966635	0.196030155195	0.224104427911	0.257037814014	1.598966635×10^{-3}
0.5	0.25	0.251959202510	0.300380543172	0.335078830520	0.376862841928	1.95920251×10^{-3}
0.6	0.36	0.362300628824	0.423965990357	0.464756071741	0.514218183642	2.300628825×10^{-3}
0.7	0.49	0.492623245669	0.566517177780	0.612556734097	0.668320944474	2.623245669×10^{-3}
0.8	0.64	0.642927053202	0.727761383896	0.777968145385	0.838464560659	2.927053202×10^{-3}
0.9	0.81	0.813212051427	0.907459460749	0.960544497758	1.024040283810	3.212051427×10^{-3}
1.0	1.0	1.003478240343	1.105400019562	1.159895873357	1.224521962802	3.478240344×10^{-3}

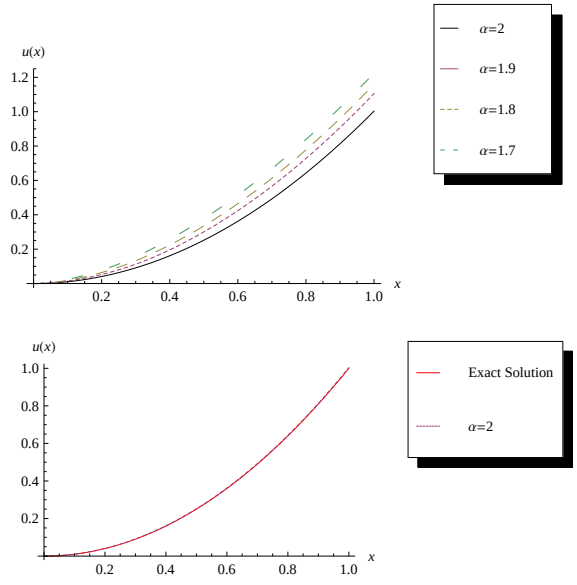

Figure 4: The comparison of approximate solution for $\alpha = 2,\ 1.9,\ 1.8,\ 1.7$ and the exact solution of Example 4.4

5. Conclusion

In this paper, the Reproducing Kernel Hilbert Space Method (RKHSM) has been successfully applied to find an approximate solution for Fredholm integro-differential equations of fractional order. The above examples show that the RKHSM yields a good approximation to the exact solution in an efficient way.

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PRIMAL-DUAL INTERIOR-POINT ALGORITHM FOR LO BASED ON A NEW KERNEL FUNCTION

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Abstract. Based on a new kernel function, a large-update primal-dual interior-point algorithm for solving linear optimization is proposed. The kernel function is used both for determining the search directions and for measuring the distance between the given iterate and the μ-center for the algorithm. By using several new technical lemmas, the iteration complexity bound as $O(\sqrt{n}\log n \log \frac{n}{\varepsilon})$ is obtained, which coincides with the currently best iteration complexity bounds for large-update methods. In addition, we present some preliminary numerical results.

Keywords: linear optimization; kernel function; primal-dual interior-point algorithm; large-update methods; iteration complexity bound.

1. Introduction

In this paper, we consider the standard form of the linear optimization (LO) problem

$$(P) \qquad min\ \{c^T x : Ax = b,\ \ x \geq 0\},$$

where $A \in \mathbb{R}^{m\times n}$ with $rank(A) = m \leq n,\ x, c \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$. The dual problem of (P) is given by

$$(D) \qquad \max\ \{b^T y : A^T y + s = c,\ \ s \geq 0\},$$

where $y \in \mathbb{R}^m$ and $s \in \mathbb{R}^n$.

After the landmark paper of Karmarkar [9], interior point methods (IPMs) for solving LO attract much attention, since IPMs not only are the most effective methods in practice but also have polynomial complexity. The primal-dual IPMs for LO were first introduced by Kojima et al. in [11]. Lately, a large amount of

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beautiful results have been reported. For a survey, we refer to recent books on the subject [18], [22], [23].

In the IPMs, the choice of the barrier function is crucial not only for the quality and elegance of the analysis but also for the performance of the algorithm. Most IPMs for LO were based on the classical logarithmic barrier function. However, today there is still a gap between the practical behavior of these algorithms and the theoretical performance results. Especially, the so-called large-update methods for which the theoretical iteration complexity bound is $O(n \log \frac{n}{\varepsilon})$. In practice, the so-called large-update methods are much more efficient than the so-called small-update methods for which the theoretical iteration complexity bound is only $O(\sqrt{n} \log \frac{n}{\varepsilon})$. This significant gap between theory and practice has been referred to as the irony of IPMs [3].

Recently, a new variant of primal-dual IPM for LO and semi-definite optimization (SDO) based on the so-called self-regular kernel functions was proposed by Peng et al. in [15] and aforementioned gap was narrowed. They obtained the currently best iteration complexity bound, namely, $O(\sqrt{n} \log n \log \frac{n}{\varepsilon})$, for large-update IPMs for LO and SDO. Subsequently, Bai et al. [3] proposed a class of primal-dual IPMs for LO based on an eligible kernel function (neither logarithmic nor self-regular function). The algorithm derived the same favourable iteration complexity bound with large-update strategy as [15]. Kim et al. [10] proposed a new adaptive single-step primal-dual IPM with wide neighborhood for LO based on a simple kernel function, and showed that the algorithm has $O(q\sqrt{n\tau} \log \frac{n}{\varepsilon})$ iteration complexity bound for large-update methods. Later, Liu et al. [13] introduced a new kind of kernel function, and concluded that in some situations the iteration complexity bound as $O(m^{\frac{3m+1}{2m}} n^{\frac{m+1}{2m}} \log \frac{n}{\varepsilon})$. Very recently, Peyghami et al. [17] proposed a primal-dual IPM based on a trigonometric kernel function for LO, and obtained the iteration complexity bound as $O(n^{\frac{2}{3}} \log \frac{n}{\varepsilon})$ for large-update methods. Cai et. al [5] introduced a new parametric kernel function, which is a combination of the classic kernel function and a trigonometric barrier term, and obtained the iteration bounds for large-update methods the same as [17]. Li et al. [12] presented a new primal-dual interior-point algorithm for linear optimization based on a trigonometric kernel function, and derived the worst case complexity for a large-update primal-dual interior-point method based on the kernel function. For more studies with primal-dual IPMs based on kernel functions please refer to [1], [2], [4], [6], [7], [8], [16], [19], [21], [24].

Motivated by their works, we present a large-update primal-dual IPM for LO based on a new kernel function and the corresponding barrier function. Since our kernel function is exponentially convex and is not self-regular and logarithmic barrier function, which is different from the analysis provided to self-regular and logarithmic barrier kernel function. By using several technical lemmas, we derive that the algorithm has favorable iteration complexity iteration bound, namely, $O(\sqrt{n} \log n \log \frac{n}{\varepsilon})$, which is currently the best known iteration complexity bound for large-update IPMs. Furthermore, some preliminary numerical results are presented to show that the algorithm is efficient and reliable.

The paper is organized as follows. In Section 2, after recalling some basic

concepts, we study the central path and the new search directions for LO. The generic primal-dual interior-point algorithm for LO is also presented. In Section 3, we define a new kernel function and the corresponding barrier function and present some properties of kernel function and the barrier function. The amount of decrease of the barrier function during an inner iteration and the default step size are given in Section 4. The complexity analysis of primal-dual IPMs based on our kernel function is given in Section 5. In Section 6, we give some preliminary numerical results. Finally, some concluding remarks are given in Section 7.

We use the following notational conventions.

Throughout the paper, $\| x \| = \sqrt{x^T x}$ denotes the 2-norm of the vector x. The nonnegative orthant and positive orthant are denoted as $\mathbb{R}^n_+$ and $\mathbb{R}^n_{++}$, respectively. If $x, s \in \mathbb{R}^n$, then xs denotes the coordinatewise (or Hadamard) product of the vectors x and s and e denotes the all-one vector of length n, i.e., $e = (1, 1, ..., 1)^T$. The diagonal matrix with the vector x is denoted $X = diag(x)$. Furthermore, $x^T y = \sum_{i=1}^{n} x_i y_i$ for $x, y \in \mathbb{R}^n$. For any $x = (x_1, x_2, ..., x_n)^T \in \mathbb{R}^n$, $x_{\min}$ denotes the smallest and $x_{\max}$ denotes the largest value of the components of x. Finally, if $f(x) \geq 0$ is a real valued function of a real nonnegative variable, the notation $f(x) = O(x)$ means that $f(x) \leq cx$ for some positive constant c and $g(x) = \Theta(x)$ that $c_1 x \leq g(x) \leq c_2 x$ for two positive constants c_1 and c_2.

2. Preliminaries

Without loss of generality, we may assume that both (P) and (D) satisfy the interior-point condition (IPC) [11], i.e., there exist x_0 and (y_0, s_0) such that

$$\begin{aligned} Ax_0 &= b, \quad x_0 > 0, \\ A^T y_0 + s_0 &= c, \quad s_0 > 0. \end{aligned}$$

The optimality condition for (P) and (D) is equivalent to solving the following nonlinear system

$$\begin{aligned} Ax &= b, \quad x \geq 0, \\ A^T y + s &= c, \quad s \geq 0, \\ xs &= 0. \end{aligned} \tag{1}$$

The basic idea of primal-dual IPMs is to replace the third equation in system (1) by a parametric equation $xs = \mu e$, where $\mu > 0$. This leads the system (1) to the following parametric system

$$\begin{aligned} Ax &= b, \quad x \geq 0, \\ A^T y + s &= c, \quad s \geq 0, \\ xs &= \mu e. \end{aligned} \tag{2}$$

Surprisingly enough, if the IPC is satisfied, the parameterized system (2) has a unique solution $(x(\mu), y(\mu), s(\mu))$ for each $\mu > 0$, which is called a μ-center of (P)

and (D). The set of μ-centers is said to be the central path of (P) and (D). The relevance of the central path for LO was recognized first by Sonnevend in [20] and Megiddo in [14]. If $\mu \to 0$, then the limit of the central path exists. Since the limit point satisfies the complementarity condition, the limit point yields optimal solutions for (P) and (D).

For fixed $\mu > 0$, by applying Newton's method to the parameterized system (2), we obtain the search direction $(\triangle x, \triangle y, \triangle s)$ from the following Newton system

$$\begin{aligned} A\Delta x &= 0, \\ A^T\Delta y + \Delta s &= 0, \\ s\Delta x + x\Delta s &= \mu e - xs. \end{aligned} \tag{3}$$

Since A has full row rank, the system (3) uniquely defines $(\triangle x, \triangle y, \triangle s)$ for any $x > 0$ and $s > 0$. By taking a step along the search direction $(\triangle x, \triangle y, \triangle s)$, we can construct a new tripe (x_+, y_+, s_+) according to

$$x_+ := x + \alpha\triangle x, \qquad y_+ := y + \alpha\triangle y, \qquad s_+ := s + \alpha\triangle s, \tag{4}$$

where $\alpha \in (0, 1]$ is obtained by using some rules so that the new iterate satisfies $(x_+, y_+, s_+) > 0$.

Introducing the notations

$$v := \sqrt{\frac{xs}{\mu}}, \quad d_x := \frac{v\triangle x}{x}, \quad d_s := \frac{v\triangle s}{s}, \tag{5}$$

then the Newton system (3) can be rewritten as

$$\begin{aligned} \overline{A}d_x &= 0, \\ \overline{A}^T\triangle y + d_s &= 0, \\ d_x + d_s &= v^{-1} - v, \end{aligned} \tag{6}$$

where $\overline{A} := \frac{1}{\mu}AV^{-1}X$, $V := diag(v)$, $X := diag(x)$, $S := diag(s)$.

Note that d_x and d_s are orthogonal vectors, since d_x belongs to the null space and d_s to the row space of the matrix $\bar{A}$. Hence, we will have

$$d_x = d_s = 0 \Leftrightarrow v - v^{-1} = 0 \Leftrightarrow x = x(\mu), s = s(\mu).$$

It is surprising that the right-side of the third equation in (6) is called the scaled centering equation, which equals the minus gradient of the barrier function

$$\Psi_c(v) := \sum_{i=1}^{n} \left(\frac{v_i^2 - 1}{2} - \log v_i \right).$$

In this paper, we replace $\Psi_c(v)$ by a strictly convex function $\Psi(v)$, $v \in \mathbb{R}^n_{++}$ such that $\Psi(v)$ is minimal at $v = e$ and $\Psi(e) = 0$. Thus the new scaled centering equation becomes

$$d_x + d_s = -\nabla\Psi(v).$$

As before, we will have

$$d_x = d_s = 0 \Leftrightarrow \nabla\Psi(v) = 0 \Leftrightarrow v = e \Leftrightarrow x = x(\mu), s = s(\mu).$$

To simplify matters, we restrict ourselves to the case where $\Psi(v)$ is separable with identical coordinate functions. Thus, letting $\Psi(v)$ denote the function on the coordinates, we write

$$\Psi(v) := \sum_{i=1}^{n} \psi(v_i),$$

where $\psi(t) : D \to R_+^n$ with $R_{++}^n \subseteq D$, is called the kernel function of barrier function $\Psi(v)$ if $\psi(t)$ is twice differentiable and satisfies the following condition

$$\psi(1) = \psi'(1) = 0, \quad \psi''(t) > 0, \quad \lim_{t\to 0^+} \psi(t) = \lim_{t\to\infty} \psi(t) = \infty. \tag{7}$$

Now the new search directions d_x, $\triangle y$, d_s are obtained by solving the following system so that $(\triangle x, \triangle y, \triangle s)$ is computed via (5)

$$\begin{aligned} \overline{A} d_x &= 0, \\ \overline{A}^T \triangle y + d_s &= 0, \\ d_x + d_s &= -\nabla\Psi(v). \end{aligned} \tag{8}$$

Thus, the generic primal-dual interior-point algorithm works as follows.

Algorithm 1: Generic Primal-Dual Algorithm for LO

Input:
A threshold parameter $\tau \geq 1$;
a barrier function $\Psi(v)$;
a fixed barrier update parameter θ, $0 < \theta < 1$;
an accuracy parameter $\varepsilon > 0$;
begin
$x := x^0$; $s := s^0$; $\mu := \mu^0$;
while $n\mu > \varepsilon$ do
begin
$\mu := (1-\theta)\mu$;
while $\Psi(v) > \tau$ do
begin
$x := x + \alpha\triangle x$; $y := y + \alpha\triangle y$; $s := s + \alpha\triangle s$;
$v := \sqrt{\frac{xs}{\mu}}$;
end
end
end

Remark 1 The choice of the barrier update parameter θ plays an important role in both theory and practice of IPMs. Usually, if θ is a constant which is independent of the dimension n of the problem, such as $\theta = \frac{1}{2}$, then the algorithm

is called a large-update (or long-step) method. If θ depends on the dimension of the problem, for instance $\theta = \frac{1}{\sqrt{n}}$, then the algorithm is called a small-update (or short-step) method.

Remark 2 The choice of the step size α $(0 < \alpha \leq 1)$ is another crucial issue in the analysis of the algorithm. It has to be taken such that the closeness of the iterates to the current μ-center improves by a sufficient amount. In the theoretical analysis the step size α is usually given a value that depends on the closeness of the current iterates to the μ-center.

3. The new kernel (barrier) function and its properties

This section is devoted to present a new kernel (barrier) function and give its properties, which are used in the complexity analysis of the algorithm. Let's define a univariate function $\psi(t) : D \to R^n_+$ with $R^n_{++} \subseteq D$ as follows

$$\psi(t) = \frac{t^2 - 1}{2} - \int_1^t \frac{1}{x^{2p}} e^{\frac{p}{x} - p} dx, \qquad p \geq 1. \tag{9}$$

The first three derivatives of the function $\psi(t)$ are given by

$$\psi'(t) = t - \frac{1}{t^{2p}} e^{\frac{p}{t} - p}, \tag{10}$$

$$\psi''(t) = 1 + \frac{2p}{t^{2p+1}} e^{\frac{p}{t} - p} + \frac{p}{t^{2p+2}} e^{\frac{p}{t} - p}, \tag{11}$$

$$\psi'''(t) = -\frac{2p(2p+1)}{t^{2p+2}} e^{\frac{p}{t} - p} - \frac{4p^2 + 2p}{t^{2p+3}} e^{\frac{p}{t} - p} - \frac{p^2}{t^{2p+4}} e^{\frac{p}{t} - p}. \tag{12}$$

Using (9)-(11), we can easily get

$$\psi(1) = \psi'(1) = 0, \psi''(t) > 1, \lim_{t \to 0^+} \psi(t) = \lim_{t \to \infty} \psi(t) = \infty,$$

which show that $\psi(t)$ is indeed a kernel function. According to the definition of self-regular function in [15], we can easily verify that the kernel function $\psi(t)$ does not belong to the family of self-regular functions. Due to conditions $\psi(1) = \psi'(1) = 0$, one can completely describe $\psi(t)$ by its second derivative

$$\psi(t) = \int_1^t \int_1^\xi \psi''(\zeta) d\zeta d\xi. \tag{13}$$

The next lemma serves to prove that the new kernel function $\psi(t)$ is an eligible kernel function according to [3].

Lemma 1 *For the function $\psi(t)$, defined by* (9)*, we have*

$$\text{(a)} \quad \psi''(t) > 1, \quad \forall t > 0, \tag{14}$$

$$\text{(b)} \quad t\psi''(t) + \psi'(t) > 0, \quad \forall t > 0, \tag{15}$$

$$\text{(c)} \quad t\psi''(t) - \psi'(t) > 0, \quad \forall t > 0, \tag{16}$$

$$\text{(d)} \quad \psi'''(t) < 0, \quad \forall t > 0. \tag{17}$$

Proof. Inequalities (a) and (d) immediately follows from (11) and (12), respectively. Next, we prove that (b) holds. Using (10), (11) and $p \geq 1$, we get

$$\begin{aligned} t\psi''(t) + \psi'(t) &= t + \frac{2p}{t^{2p}} e^{\frac{p}{t}-p} + \frac{p}{t^{2p+1}} e^{\frac{p}{t}-p} + t - \frac{1}{t^{2p}} e^{\frac{p}{t}-p} \\ &= 2t + \frac{2p-1}{t^{2p}} e^{\frac{p}{t}-p} + \frac{p}{t^{2p+1}} e^{\frac{p}{t}-p} > 0. \end{aligned}$$

Furthermore, for proving (c), we have

$$\begin{aligned} t\psi''(t) - \psi'(t) &= t + \frac{2p}{t^{2p}} e^{\frac{p}{t}-p} + \frac{p}{t^{2p+1}} e^{\frac{p}{t}-p} - t + \frac{1}{t^{2p}} e^{\frac{p}{t}-p} \\ &= \frac{2p+1}{t^{2p}} e^{\frac{p}{t}-p} + \frac{p}{t^{2p+1}} e^{\frac{p}{t}-p} > 0. \end{aligned}$$

This completes the proof. ∎

Lemma 2 (Lemma 2.1.2 in [15]) *$\psi(t)$ is exponentially convex (e-convex), namely,*

$$\psi(\sqrt{t_1 t_2}) \leq \frac{1}{2}(\psi(t_1) + \psi(t_2)).$$

Proof. This result follows easily by using Lemma 2.1 in [3], which states that the above inequality holds if and only if $t\psi''(t) + \psi'(t) \geq 0$ for all $t > 0$. From (15) the result holds. ∎

In the analysis of the algorithm, we introduce the norm-based proximity measure $\delta(v)$ defined by

$$\delta(v) := \frac{1}{2} \parallel \nabla\Psi(v) \parallel = \frac{1}{2}\sqrt{\sum_{i=1}^{n} (\psi'(v_i))^2}, \qquad v \in \mathbb{R}^n_{++}. \tag{18}$$

Since $\psi(t)$ is an eligible kernel function, we can get the following lemma which will be used in next section. Its proof is similar to the proof of Lemma 2.1, Corollary 2.2 and 2.3 in [3], and is therefore stated without proof.

Lemma 3 *Let $\psi(t)$ be as defined in (9), we have*

$$\begin{aligned} &\text{(a)} \quad \frac{1}{2}(t-1)^2 \leq \psi(t) \leq \frac{1}{2}\psi'(t)^2, \quad \text{if} \quad t > 0, \\ &\text{(b)} \quad \Psi(v) \leq 2\delta(v)^2, \\ &\text{(c)} \quad \parallel v \parallel \leq \sqrt{n} + \sqrt{2\Psi(v)} \leq \sqrt{n} + 2\delta(v). \end{aligned}$$

Corollary 1 *If $\Psi(v) \geq 1$, then we have*

$$\delta(v) \geq \frac{1}{\sqrt{2}}. \tag{19}$$

Proof. It is easily followed from the (b) of Lemma 3. ∎

4. Analysis of the algorithm

In this section, we deal with the growth behavior of the barrier function. A default value for the step size is also presented in this section.

4.1. Growth behavior for the barrier function

For given threshold parameter τ, we have $\Psi(v) \leq \tau$ before the update of μ with the factor $1-\theta$. According to the update of μ, the vector v is divided by a factor $\sqrt{1-\theta}$, which leads to an increase of the value $\Psi(v)$ in general. The subsequent inner iterations are performed in order to bring the value of $\Psi(v)$ decrease until it passes the threshold parameter τ again. Therefore, the largest values of $\Psi(v)$ occur just after the update of μ. In order to investigate the growth behavior of $\Psi(v)$, we need the following lemmas which are important in deriving the iteration complexity bound.

Lemma 4 *Let $\beta \geq 1$. Then*

$$\psi(\beta t) \leq \psi(t) + \frac{t^2}{2}(\beta^2 - 1). \tag{20}$$

Proof. Defining

$$\psi_b(t) := -\int_1^t \frac{1}{x^{2p}} e^{\frac{p}{x}-p} dx,$$

we have

$$\psi_b'(t) = -\frac{1}{t^{2p}} e^{\frac{p}{t}-p} < 0,$$

i.e., $\psi_b(t)$ is a decreasing function with $t > 0$. Thus $\psi_b(\beta t) \leq \psi_b(t)$ for $\beta \geq 1$. So

$$\psi(\beta t) - \psi(t) = \frac{1}{2}(\beta^2 - 1)t^2 + \psi_b(\beta t) - \psi_b(t) \leq \frac{1}{2}(\beta^2 - 1)t^2.$$

That implies the lemma. ∎

Lemma 5 *Assume that $0 < \theta < 1$ and $v_+ := \frac{v}{\sqrt{1-\theta}}$, then*

$$\Psi(v_+) \leq \Psi(v) + \frac{\theta}{2(1-\theta)}(2\Psi(v) + 2\sqrt{2n\Psi(v)} + n).$$

Proof. We can conclude that the above relations by using Lemma 4, with $\beta = \frac{1}{\sqrt{1-\theta}}$, and Lemma 3. ∎

By the assumption $\Psi(v) \leq \tau$ just before the update of μ, we have

$$\Psi\left(\frac{v}{\sqrt{1-\theta}}\right) \leq \tau + \frac{\theta}{2(1-\theta)}(2\tau + 2\sqrt{2n\tau} + n).$$

We define

$$L(n, \theta, \tau) := \tau + \frac{\theta}{2(1-\theta)}(2\tau + 2\sqrt{2n\tau} + n).$$

Since $\tau = O(n)$ and $\theta = \Theta(1)$, we have

$$L := L(n, \theta, \tau) = O(n).$$

4.2. A default value for the step size

After a damped step, using (5), the new point is computed as

$$x_+ := x + \alpha\triangle x = \frac{x}{v}(v + \alpha d_x), \qquad y_+ := y + \alpha\triangle y,$$
$$s_+ := s + \alpha\triangle s = \frac{s}{v}(v + \alpha d_s),$$

where α is a step size which is obtained by using a line search strategy. Thus, we get

$$v_+^2 = \frac{x_+ s_+}{\mu} = (v + \alpha d_x)(v + \alpha d_s).$$

From Lemma 2, we have

$$\Psi(v_+) = \Psi(\sqrt{(v + \alpha d_x)(v + \alpha d_s)}) \leq \frac{1}{2}(\Psi(v + \alpha d_x) + \Psi(v + \alpha d_s)).$$

By defining

$$f(\alpha) := \Psi(v_+) - \Psi(v) = \Psi(\sqrt{(v + \alpha d_x)(v + \alpha d_s)}) - \Psi(v),$$

and

$$f_1(\alpha) := \frac{1}{2}(\Psi(v + \alpha d_x) + \Psi(v + \alpha d_s)) - \Psi(v), \tag{21}$$

we have $f(\alpha) \leq f_1(\alpha)$ and $f_1(\alpha)$ is a convex function with $f(0) = f_1(0) = 0$. Taking the derivative with respect to α, we further get

$$f_1'(\alpha) := \frac{1}{2}\sum_{i=1}^{n}(\psi'(v_i + \alpha d_{xi})d_{xi} + \psi'(v_i + \alpha d_{si})d_{si}),$$
$$f_1''(\alpha) = \frac{1}{2}\sum_{i=1}^{n}(\psi''(v_i + \alpha d_{xi})d_{xi}^2 + \psi''(v_i + \alpha d_{si})d_{si}^2).$$

By using the third equation of the system (8), we obtain

$$f_1'(0) = \frac{1}{2}\nabla\Psi^T(v)(d_x + d_s) = -\frac{1}{2}\nabla\Psi^T(v)\nabla\Psi(v) = -2\delta^2(v).$$

For simplicity, in the sequel we use the following notations

$$v_1 := \min(v), \qquad \delta := \delta(v).$$

Since our kernel function is an eligible function, so we only state the following important results without proof.

Lemma 6 (Lemma 4.1 in [3]) *Let $f_1(\alpha)$ be defined as* (21). *Then*

$$f_1''(\alpha) \leq 2\delta^2\psi''(v_1 - 2\alpha\delta).$$

Lemma 7 (Lemma 3.4 in [2]) *One has $f_1(\alpha) \leq 0$, if α satisfies the inequality*

$$-\psi'(v_1 - 2\alpha\delta) + \psi'(v_1) \leq 2\delta. \tag{22}$$

Lemma 8 (Lemma 4.3 in [16]) *Let $\rho : [0, \infty) \to (0, 1]$ denote the inverse function of $-\frac{1}{2}\psi'(t)$ restricted to the interval $(0, 1]$. Then, the step size*

$$\bar{\alpha} := \frac{1}{2\delta}(\rho(\delta) - \rho(2\delta)), \tag{23}$$

is the largest possible solution of inequality (22).

Lemma 9 (Lemma 4.4 in [6]) *Let ρ and $\bar{\alpha}$ be as define in Lemma 8. Then*

$$\bar{\alpha} \geq \frac{1}{\psi''(\rho(2\delta))}. \tag{24}$$

We are using the following step size as our default step size in Algorithm 1

$$\tilde{\alpha} := \frac{1}{\psi''(\rho(2\delta))}. \tag{25}$$

By (24) and (25), we have $\tilde{\alpha} \leq \bar{\alpha}$.

Lemma 10 (Lemma 4.6 in [16]) *If the step size α is such that $\alpha \leq \bar{\alpha}$, then*

$$f(\alpha) \leq -\alpha\delta^2. \tag{26}$$

Theorem 1 *Let ρ be as defined in Lemma* 8, *$\tilde{\alpha}$ as define in* (25) *and $\Psi(v) \geq 1$. Then*

$$f(\tilde{\alpha}) \leq -\frac{\delta^2}{\psi''(\rho(2\delta))} \leq -\frac{\delta}{(18p+2)(1+\frac{1}{p}\log(4\delta+1))^2}. \tag{27}$$

Proof. Lemma 10 and the fact $\tilde{\alpha} \leq \bar{\alpha}$ imply that $f(\tilde{\alpha}) \leq -\tilde{\alpha}\delta^2$, which follows the first inequality.

To obtain the inverse function of the restriction of $-\frac{1}{2}\psi'(t)$ in the interval $t \in (0, 1]$, we need to solve the equation $-\frac{1}{2}\psi'(t) = s$ for t. To do so, we have

$$-(t - \frac{1}{t^{2p}}e^{\frac{p}{t}-p}) = 2s.$$

This implies that

$$\frac{1}{t^{2p}}e^{\frac{p}{t}-p} = 2s + t \leq 2s + 1, \tag{28}$$

where the last inequality is obtained from the fact that $t \leq 1$. Hence, putting $t = \rho(2\delta)$, which is equivalent to $4\delta = -\psi'(t)$, we get

$$\begin{aligned} \frac{1}{t^{2p}} e^{\frac{p}{t}-p} = 4\delta + t \leq 4\delta + 1, &\Rightarrow e^{\frac{p}{t}-p} \leq t^{2p}(4\delta+1) \leq 4\delta + 1, \\ &\Rightarrow \frac{p}{t} - p \leq \log(4\delta+1), \\ &\Rightarrow \frac{1}{t} \leq 1 + \frac{1}{p}\log(4\delta+1). \end{aligned} \tag{29}$$

Furthermore, using (11) and (29), we conclude that

$$\begin{aligned} \tilde{\alpha} = \frac{1}{\psi''(t)} &= \frac{1}{1 + \frac{2p}{t^{2p+1}} e^{\frac{p}{t}-p} + \frac{p}{t^{2p+2}} e^{\frac{p}{t}-p}} \\ &\geq \frac{1}{1 + 2p(4\delta+1)(1+\frac{1}{p}\log(4\delta+1)) + p(4\delta+1)(1+\frac{1}{p}\log(4\delta+1))^2} \\ &\geq \frac{1}{1 + 2p(4\delta+1)(1+\frac{1}{p}\log(4\delta+1))^2 + p(4\delta+1)(1+\frac{1}{p}\log(4\delta+1))^2} \\ &\geq \frac{1}{1 + 12p\delta(1+\frac{1}{p}\log(4\delta+1))^2 + 3p(1+\frac{1}{p}\log(4\delta+1))^2} \\ &= \frac{1}{1 + (12p\delta+3p)(1+\frac{1}{p}\log(4\delta+1))^2}. \end{aligned}$$

Using Corollary 1, one has

$$\begin{aligned} \tilde{\alpha} &\geq \frac{1}{\sqrt{2}\delta + (12p\delta + 3\sqrt{2}p\delta)(1+\frac{1}{p}\log(4\delta+1))^2} \\ &\geq \frac{1}{(2\delta + 12p\delta + 6p\delta)(1+\frac{1}{p}\log(4\delta+1))^2} \\ &\geq \frac{1}{(18p+2)\delta(1+\frac{1}{p}\log(4\delta+1))^2}. \end{aligned}$$

This implies that

$$f(\tilde{\alpha}) \leq -\frac{\delta^2}{\psi''(\rho(2\delta))} \leq -\frac{\delta}{(18p+2)(1+\frac{1}{p}\log(4\delta+1))^2}.$$

Thus the theorem follows. ∎

A direction consequence of applying the Lemma 3 to (27) is

$$\begin{aligned} f(\tilde{\alpha}) &\leq -\frac{\delta}{(18p+2)(1+\frac{1}{p}\log(4\delta+1))^2} \\ &\leq -\frac{\Psi(v)^{\frac{1}{2}}}{\sqrt{2}(18p+2)(1+\frac{1}{p}\log(2\sqrt{2\Psi(v)}+1))^2} \\ &\leq -\frac{\Psi(v)^{\frac{1}{2}}}{\sqrt{2}(18p+2)(1+\frac{1}{p}\log(2\sqrt{2\Psi_0(v)}+1))^2}, \end{aligned} \tag{30}$$

where the last inequality follows from $\Psi_0(v) \geq \Psi(v) \geq \tau \geq 1$.

5. Iteration complexity

We need to count how many inner iterations are required to return to the situation where $\Psi(v) \leq \tau$ after a μ-update. We define the value of $\Psi(v)$ after μ-update as $\Psi_0(v)$ and the subsequent values as Ψ_k, for $k = 1, 2, \ldots, K$. Let K stands for the total number of inner iterations in an outer iteration. Then we have

$$\Psi_0(v) \leq L = O(n), \quad \Psi_{K-1} > \tau, \quad 0 \leq \Psi_K \leq \tau. \tag{31}$$

To get the iteration complexity bound of algorithm, we need the following technical result.

Lemma 11 (Lemma 4.7 in [21]) *Let $t_0, t_1, \ldots, t_K$ be a sequence of positive numbers such that*

$$t_{k+1} \leq t_k - \beta t_k^{1-\gamma}, \quad k = 0, \ldots, K-1, \tag{32}$$

where $\beta > 0$ and $0 < \gamma \leq 1$. Then $K \leq \lfloor t_0^{\gamma}/\beta\gamma \rfloor$.

Lemma 12 *If K denote the total number of inner iterations in the outer iteration. Then we have*

$$K \leq 2\sqrt{2}(18p+2)\left(1 + \frac{1}{p}\log(2\sqrt{2\Psi_0(v)}+1)\right)^2 \Psi_0(v)^{\frac{1}{2}}.$$

Proof. From (30), we have

$$\Psi_{k+1}(v) \leq \Psi_k(v) - \frac{\Psi_k(v)^{\frac{1}{2}}}{\sqrt{2}(18p+2)(1+\frac{1}{p}\log(2\sqrt{2\Psi_0(v)}+1))^2}.$$

Letting $t_k = \Psi_k(v)$, $\beta = \frac{1}{\sqrt{2}(18p+2)(1+\frac{1}{p}\log(2\sqrt{2\Psi_0(v)}+1))^2}$ and $\gamma = \frac{1}{2}$, we can get the result of the lemma from Lemma 11. ∎

The following theorem gives the iteration bound for the algorithm with large-update methods.

Theorem 2 *For large-update methods, one takes $\theta = \Theta(1)$ and $\tau = O(n)$, then Algorithm 1 requires at most*

$$O\big(p\sqrt{n}(1+\frac{1}{p}\log n)^2 \log\frac{n}{\varepsilon}\big)$$

iterations. The output gives an ε-approximate solution of (P) and (D).

Proof. It is well known that the number of outer iterations is bounded above by

$$\frac{1}{\theta}\log\frac{n}{\varepsilon}$$

(see, e.g.,[11]). By multiplying this result and the upper bound for the number of inner iterations per outer iteration, we can get an upper bound for the total number of iterations, namely,

$$\frac{2\sqrt{2}(18p+2)(1+\frac{1}{p}\log(2\sqrt{2\Psi_0(v)}+1))^2\Psi_0(v)^{\frac{1}{2}}}{\theta}\log\frac{n}{\varepsilon}.$$

Considering that $\theta=\Theta(1)$, $\tau=O(n)$ and $\Psi_0^{\frac{1}{2}}=O(\sqrt{n})$, some elementary transformation reduce this iteration complexity bound in the theorem. ∎

Corollary 2 *If $p=\log n$, the iteration bound for large-update methods reduces to $O(\sqrt{n}\log n\log\frac{n}{\varepsilon})$, which matches the currently best known iteration bound for large-update IPMs.*

6. Numerical results

In this section, we present some numerical results. Number results are obtained by using Matlab 2012b. We consider the following simple LO problem

$$A=\begin{bmatrix}-3 & 2 & 1 & 0 & 0\\ -1 & 2 & 0 & 1 & 0\\ 1 & 1 & 0 & 0 & 1\end{bmatrix},\qquad b=\begin{bmatrix}0\\2\\3\end{bmatrix},\qquad C=\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}.$$

The main goal of this section is to compare iteration numbers of the algorithm for the following kernel functions

$$\psi_0(t)=\frac{t^2-1}{2}-\int_1^t\frac{1}{x^{2p}}e^{\frac{p}{x}-p}dx,\ p\geq 1,\quad \psi_1(t)=\frac{t^2-1}{2}-(t-1)e^{\frac{1}{t}-1},$$
$$\psi_2(t)=\frac{t^2-1}{2}-\log t,\qquad \psi_3(t)=t+t^{-1}-2,$$

$\psi_1(t)$ is the new kernel function proposed in [24], $\psi_2(t)$ is the classical logarithmic kernel function in [18], and $\psi_3(t)$ is the non-self-regular kernel function in [21]. We take $p\in\{1,2,2.5\}$ and the barrier update parameter $\theta\in\{0.1,0.3,0.7\}$, the step size $\alpha\in\{0.1,0.3,0.5\}$, the threshold parameter $\tau=3$, and the accuracy parameter $\varepsilon=10^{-8}$ in all experiments. Table 1 shows that the iteration numbers of the algorithm based on the above kernel functions. From Table 1, it is found that all cases our new kernel function $\psi_0(t)$ can produce better iteration numbers than those by the kernel functions $\psi_1(t)$, $\psi_2(t)$ and $\psi_3(t)$. Obviously, $\psi_3(t)$ gives iteration numbers that are definitely worst compared to the other kernel functions, especially compared to our kernel function. The results in table 1 also show that the algorithm based on our new kernel function $\psi_0(t)$ is effective and the iteration numbers of the algorithm depend on the values of the parameter θ and step size α. In fact, for each θ that considered, larger values of α gives better iteration numbers. However, the step size α should have an upper bound in practical

Table 1: The number of iterations for kernel function $\psi_i(t)$, i=0,1,2,3.

α	θ	$\psi_0(t)$ $p=1$	$\psi_0(t)$ $p=2$	$\psi_0(t)$ $p=2.5$	$\psi_1(t)$	$\psi_2(t)$	$\psi_3(t)$
$\alpha=0.1$	0.1	258	218	209	308	334	617
	0.3	243	210	204	292	279	718
	0.7	217	199	196	259	263	987
$\alpha=0.3$	0.1	80	66	63	96	105	294
	0.3	74	63	60	91	84	332
	0.5	66	59	59	80	82	432
$\alpha=0.5$	0.1	44	35	34	53	62	201
	0.3	40	34	32	56	47	223
	0.5	33	32	31	43	44	289

computation. In most cases, for each α, larger θ gives better iteration numbers for $\psi_0(t)$, $\psi_1(t)$ and $\psi_2(t)$, while for $\psi_3(t)$, smaller θ gives better results.

7. Concluding remarks

In this paper, we have presented a primal-dual large-update IPM for LO based on a new kernel function and derived favorable complexity bound for the algorithm, namely, $O(\sqrt{n}\log n\log\frac{n}{\varepsilon})$. This bound is the currently best known bound for primal-dual IPMs. Some preliminary numerical results are presented, which show that by using new kernel function, the best iteration numbers were achieved in most cases.

Some interesting topics remain for the further research. Firstly, finding a suitable kernel function for which the complexity of large-update methods is equal to (or even better than) $O(\sqrt{n}\log\frac{n}{\varepsilon})$ or showing that such a kernel function does not exist. Secondly, the extensions to cone optimization seems an interesting topic. Furthermore, the numerical test is an interesting topic for investigating the behavior of the algorithm so as to be compared with other approaches.

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SIMPLE ITERATIVE TECHNIQUE FOR SOLVING SOME MODELS OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS USING HAAR WAVELET

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Abstract. We present here, simple iterative technique for solving some models of nonlinear partial differential equations using Haar wavelet. Numerical examples are given to establish the efficiency and accuracy of the present method.

Keywords: nonlinear partial differential equation; Haar wavelet method; function approximation.
AMS Subject Classifications: 65M99.

1. Introduction

Nonlinear partial differential equations has a wide variety in scientific and engineering applications. Mostly, mathematical models can be expressed in terms of nonlinear partial differential equations. For solving such mathematical models different approaches and techniques were established. Wavelet technique is one of the easiest and accurate technique for solving a variety of such models. Wavelet, being a powerful mathematical tool, has been widely used in image digital processing, quantum field theory and numerical analysis. In this paper, we study the physical behaviour of numerical solution of nonlinear partial differential equations. Consider the most general nonlinear partial differential equation of the form:

$$f\left(t, x, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}\right) = 0, \tag{1}$$

subject to the initial and boundary conditions:

$$u(x,0) = f(x), \quad 0 \le x \le L, \tag{2}$$

and

$$u(0,t) = g_1(t), \quad u(L,t) = g_2(t), \quad t \ge 0. \tag{3}$$

In recent years, being a powerful mathematical tool, Haar wavelets have been used extensively in signal processing and in research of different branches of science. The main disadvantage of Haar wavelets is their discontinuity and at the points of discontinuity, derivatives do not exist. Due to this shortcoming, Haar wavelets are not applied directly for solving differential equations. There are two possibilities for overcoming these shortcomings. First, to regularize the piecewise constant Haar functions with interpolation splines: this technique has been applied by Cattani in [1], [2]. But, by this technique, it is difficult to find the solution easily and simplicity of Haar wavelets gets lost. Another possibility, which is proposed by Chen and Hsiao in [3], [4] is that they recommended to expand the highest order derivative appearing in the differential equation into the Haar series, instead of the function itself. The other derivatives (and the function) are obtained through integrations. Chen and Hsiao established the possibilities of their method by solving linear systems of ordinary and partial differential equations. Using the application of Haar analysis into the dynamical systems, Chen and Hsiao [3], had first derived an operational matrix of integration based on Haar wavelets. Numerical solutions of Telegraph equation, Fisher's equation and generalized Burger-Huxley equation using Haar wavelets have been presented in [5], [6] and [7] respectively. A survey of numerical solutions of differential equation is presented in [8]. Lepik [9], [10], [11], [12] had solved differential and integral equations using Haar wavelet. The numerical solutions of wave equation and convection-diffusion equation using Haar wavelets have been presented in [13], [14] respectively.

In Section 2, we briefly describe Haar wavelet. In Section 3, we have described Haar wavelet method for solving nonlinear partial differential equations. Function approximation has been presented in Section 4. In Section 5, numerical examples have been solved using the present method to illustrate the efficiency and accuracy of the method.

2. Haar wavelet

The Haar functions are an orthogonal family of switched rectangular waveforms where amplitudes can differ from one function to another. They are defined in the interval $[0,1]$ in general. Haar wavelet is a sequence of rescaled square shaped functions which together forms a wavelet family or basis. The Haar wavelet function $h_i(x)$ is defined in the interval $[\alpha,\gamma]$ as:

$$h_i(x) = \begin{cases} 1, & \alpha \le x < \beta, \\ -1, & \beta \le x < \gamma, \\ 0, & \text{elsewhere}, \end{cases} \tag{4}$$

where

$$\alpha = \frac{k}{m},\ \beta = \frac{k+0.5}{m},\ \gamma = \frac{k+1}{m},\ \ m = 2^j \text{ and } j = 0, 1, 2, ..., J.$$

J denotes the level of resolution. The integer $k = 0, 1, 2, ..., m-1$ is the translation parameter. The index i is calculated as: $i = m + k + 1$. The minimal value of $i = 2$. The maximal value of i is 2^{j+1}.

The collocation points are calculated as:

$$x_l = \frac{l - 0.5}{2M}, \qquad l = 1, 2, 3, ..., 2M. \tag{5}$$

The Haar coefficient matrix is defined as:

$$H(i, l) = (h_i(x_l)). \tag{6}$$

This is a $2M \times 2M$ square matrix. The Haar coefficient matrix of order 8, is:

$$H_8(x) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}. \tag{7}$$

Integrating (4), we obtain:

$$q_i(x) = \int_0^x h_i(t)dt, \tag{8}$$

using collocation points, it is defined as:

$$Q(i, l) = (q_i(x_l)), \tag{9}$$

which is again $2M \times 2M$ square matrix. Chen and Hsiao [3], presented this matrix in the form $Q_n = P_n H_n$, where $P_n H_n$ is interpreted as the product of the matrices, P_n and H_n, called Haar integration and Haar coefficient matrix, respectively. There are two possibilities for calculating the integrals of the Haar wavelets. In the first case, Chen and Hsiao [3], introduced the operational matrices of integration. But, in the second case, these integrals are calculated directly as discussed by Lepik in [12]. We follow the work as discussed in [12] for calculating integrals. The operational matrix P, which is $2M \times 2M$, is calculated as below:

$$P_{1,i}(x) = \int_0^x h_i(x)dx, \tag{10}$$

and

$$P_{n+1,i}(x) = \int_0^x P_{n,i}(x)dx, \quad n = 1, 2, 3, ... \tag{11}$$

The integrals of Haar wavelets considered here are:

$$P_{1,i}(x) = \begin{cases} x - \alpha, & \alpha \le x < \beta, \\ \gamma - x, & \beta \le x < \gamma, \\ 0, & \text{elsewhere}, \end{cases} \tag{12}$$

$$P_{2,i}(x) = \begin{cases} \frac{1}{2}(x-\alpha)^2, & \alpha \le x < \beta, \\ \frac{1}{2}[(x-\alpha)^2 - 2(x-\beta)^2], & \beta \le x < \gamma, \\ \frac{1}{2}[(x-\alpha)^2 - 2(x-\beta)^2 + (x-\gamma)^2], & \gamma \le x < 1, \\ 0, & \text{elsewhere}, \end{cases} \tag{13}$$

In general,

$$P_{n,i}(x) = \begin{cases} \frac{1}{n!}(x-\alpha)^n, & \alpha \le x < \beta, \\ \frac{1}{n!}[(x-\alpha)^n - 2(x-\beta)^n], & \beta \le x < \gamma, \\ \frac{1}{n!}[(x-\alpha)^n - 2(x-\beta)^n + (x-\gamma)^n], & \gamma \le x < 1, \\ 0, & \text{elsewhere}, \end{cases} \tag{14}$$

Further, Haar wavelets are orthogonal as shown below:

$$\int_0^1 h_m(x)h_n(x)dx = \begin{cases} 1, & m = n, \\ 0, & m \ne n. \end{cases} \tag{15}$$

3. Haar wavelet method [12]

Consider the nonlinear partial differential equation (1) with initial and boundary conditions. Assume that $\dot{u}'(x,t)$ can be expanded in terms of Haar wavelets as below:

$$\dot{u}'(x,t) = \sum_{i=1}^{2M} a_i h_i(x), \qquad t\epsilon(t_s, t_{s+1}]. \tag{16}$$

Here ($\dot{}$) means differentiation with respect to t and ($'$) means differentiation with respect to x. Integrating the above equation with respect to t from t_s to t and then, with respect to x; from 0 to x, we obtain successively:

$$u'(x,t) = u'(x,t_s) + (t-t_s)\sum_{i=1}^{2M} a_i h_i(x), \tag{17}$$

$$u(x,t) = u(0,t) + u(x,t_s) - u(0,t_s) + (t-t_s)\sum_{i=1}^{2M} a_i P_{1,i}(x). \tag{18}$$

Differentiating above equation with respect to t, we obtain:

$$\dot{u}(x,t) = \dot{u}(0,t) + \sum_{i=1}^{2M} a_i P_{1,i}(x). \tag{19}$$

Substituting $x \to x_l$ and $t \to t_{s+1}$ from Equation (16) to Equation (19), we successively obtain:

$$\dot{u}'(x_l,t_{s+1}) = \sum_{i=1}^{2M} a_i h_i(x_l), \tag{20}$$

$$u'(x_l,t_{s+1}) = u'(x_l,t_s) + (t_{s+1}-t_s)\sum_{i=1}^{2M} a_i h_i(x_l), \tag{21}$$

$$u(x_l,t_{s+1}) = u(0,t_{s+1}) + u(x_l,t_s) - u(0,t_s) + (t_{s+1}-t_s)\sum_{i=1}^{2M} a_i P_{1,i}(x_l), \tag{22}$$

$$\dot{u}(x_l,t_{s+1}) = \dot{u}(0,t_{s+1}) + \sum_{i=1}^{2M} a_i P_{1,i}(x_l). \tag{23}$$

For iteration, substitute $\dot{u}(x,t) = \dot{u}(x,t_{s+1})$, $u(x,t) = u(x,t_s)$, $u'(x,t) = u'(x,t_s)$ and $u''(x,t) = u''(x,t_s)$ in Equation (1). After substitution the Equation (1) takes the form:

$$f(t,x,u(x,t_s),\frac{\partial u}{\partial x}(x,t_s),\frac{\partial u}{\partial t}(x,t_{s+1})) = 0. \tag{24}$$

Wavelet coefficients are obtained by substituting the values from Equation (20) to Equation (23) in (24). The numerical solution is obtained from (22) by substituting the wavelet coefficients.

4. Function approximation

The function $u(x,t) \in L^2(0,1)$ can be approximate as:

$$u(x,t) = \sum_{i=0}^{\infty} a_i h_i(x), \tag{25}$$

where the coefficient a_i are determined as:

$$a_i = 2^j \int_0^1 u(x,t)h_i(x)dx, \tag{26}$$

where $i = 2^j + k$, $j \geq 0$ and $0 \leq k < 2^j$. The series expansion of $u(x,t)$ contains infinite terms. If $u(x,t)$ is piecewise constant by itself, or may be approximated as piecewise constant during each subinterval, then $u(x,t)$ will be terminated at finite number of terms, that is:

$$u(x,t) = \sum_{i=0}^{m-1} a_i h_i(x) = a_m^T h_m(x), \tag{27}$$

where $a_m^T = [a_0, a_1, ..., a_{m-1}]$, $h_m = [h_0, h_1, ..., h_{m-1}]^T$ and T is transpose.

5. Numerical examples

Example 1: Consider the hyperbolic nonlinear partial differential equation of the form:

$$\frac{\partial u}{\partial t} = u\left(\frac{\partial u}{\partial x}\right), \qquad 0 < x < 1, \qquad t \geq 0, \tag{28}$$

subject to the initial conditions $u(x,0) = \dfrac{x}{10}$. The exact solution of the problem is given by:

$$u(x,t) = -\frac{x}{t-10}. \tag{29}$$

The process is started with $u(x_l, 0) = \dfrac{x_l}{10}$. Absolute errors so obtained are presented in Table 1, for $t = 0.01$, $t = 0.1$, $t = 0.2$, $t = 0.3$ and $J = 4$. The value of $\Delta t = 0.001$.

Table 1: Absolute errors of Example 1 for different values of t.

xL/64	$t = 0.01$	$t = 0.1$	$t = 0.2$	$t = 0.3$
1	1.5662E-010	1.6020E-009	3.2863E-009	5.0574E-009
3	4.6985E-010	4.8060E-009	9.8590E-009	1.5172E-008
5	7.8309E-010	8.0100E-009	1.6432E-008	2.5287E-008
7	1.0963E-009	1.1214E-008	2.3004E-008	3.5402E-008
57	8.9272E-009	9.1314E-008	1.8732E-007	2.8827E-007
59	9.2404E-009	9.4518E-008	1.9389E-007	2.9839E-007
61	9.5537E-009	9.7722E-008	2.0047E-007	3.0850E-007
63	9.8669E-009	1.0093E-007	2.0704E-007	3.1862E-007

Example 2: Consider the hyperbolic nonlinear partial differential equation of the form:

$$\frac{\partial u}{\partial t} = x^2 - \frac{1}{4}\left(\frac{\partial u}{\partial x}\right)^2, \qquad 0 < x < 1, \qquad t \geq 0, \tag{30}$$

subject to the initial condition $u(x, 0) = 0$. The exact solution of the problem is given below:

$$u(x, t) = x^2 \tanh(t). \tag{31}$$

The process is started with $u(x_l, 0) = 0$. Absolute errors so obtained are presented in Table 2, for $t = 0.01$, $t = 0.1$, $t = 0.2$, $t = 0.3$ and $J = 4$. The value of $\Delta t = 0.001$.

Table 2: Absolute errors of Example 2 for different values of t.

xL/64	$t = 0.01$	$t = 0.1$	$t = 0.2$	$t = 0.3$
1	6.3982E-011	6.1035E-008	4.7988E-007	1.5792E-006
3	1.1992E-010	2.4614E-007	1.9445E-006	6.1841E-006
5	6.2541E-010	4.0055E-007	2.8610E-006	8.2946E-006
7	7.3893E-011	4.9182E-007	3.5299E-006	9.0496E-006
57	4.2191E-008	3.4936E-006	1.4759E-005	3.2748E-005
59	3.7054E-008	4.6152E-006	1.6860E-005	3.4737E-005
61	4.8018E-008	4.0442E-006	1.6479E-005	3.5813E-005
63	4.2555E-008	5.1879E-006	1.9363E-005	4.0680E-005

Conclusion

From above discussions, it is concluded here that the Haar wavelet method is more accurate, simple, fast and computationally efficient for solving nonlinear partial differential equations. The above examples demonstrates the simplicity of the Haar wavelet method. For getting the necessary accuracy the number of calculation points may be increased.

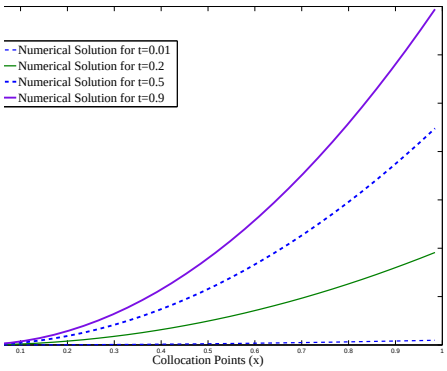

Figure 1: Numerical solutions of Example 2 at different t with $J = 4$ and $\Delta t = 0.001$.

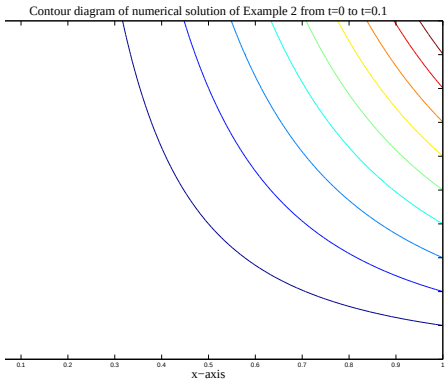

Figure 2: Contour diagram of numerical solutions of Example 2 from $t = 0 - 0.1$.

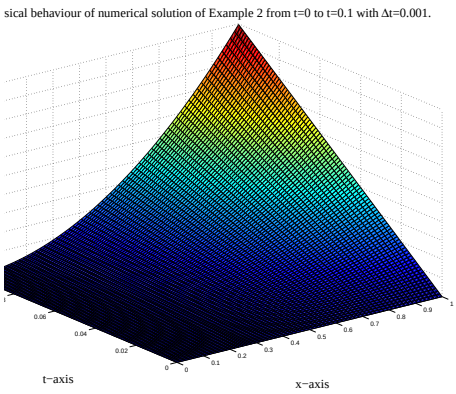

Figure 3: Physical behavior of numerical solutions of Example 2 from $t = 0 - 0.1$ with $\Delta t = 0.001$ and $J = 4$.

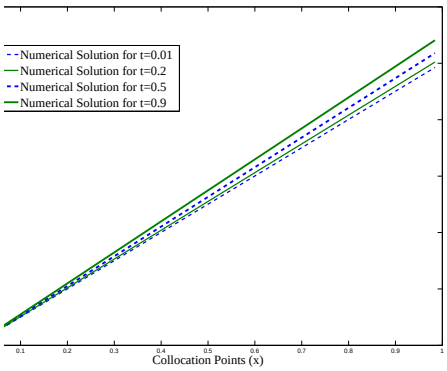

Figure 4: Numerical solutions of Example 1 at different t with $J = 4$ and $\Delta t = 0.001$.

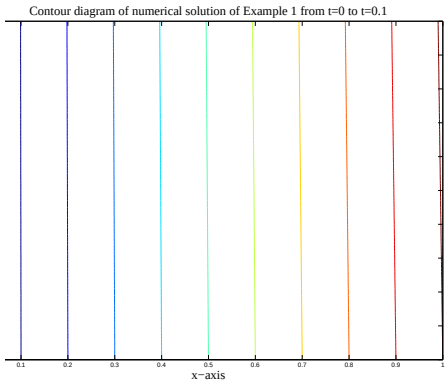

Figure 5: Contour diagram of numerical solutions of Example 1 from $t = 0 - 0.1$.

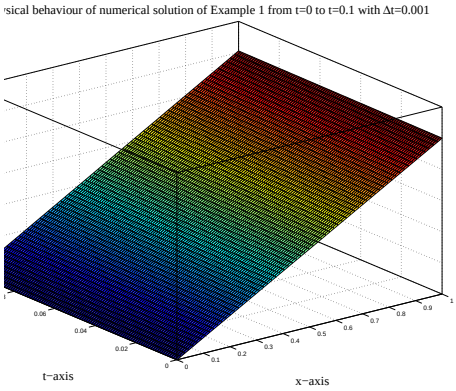

Figure 6: Physical behavior of numerical solutions of Example 1 from $t = 0 - 0.1$ with $\Delta t = 0.001$ and $J = 4$.

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SOLVING THE FRACTIONAL FORNBERG-WHITHAM EQUATION BY MEANS OF THE OPTIMAL q-HOMOTOPY ANALYSIS METHOD (Oq-HAM)

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Abstract. The main aim of this paper is to propose a new and simple algorithm namely optimal q-homotopy analysis method (Oq-HAM), to obtain the approximate analytical solutions to solve the nonlinear Fornberg–Whitham equation with fractional time derivative. The fractional derivatives are taken in the Caputo sense. Comparison of Oq- HAM with the homotopy analysis method (HAM) and the homotopy perturbation method (HPM) are made. The results reveal that the Oq-HAM has more accuracy than the others. Finally, numerical example is given to illustrate the accuracy and stability of this method. Comparison of the approximate solution with the exact solutions also we show that the proposed method is very efficient and computationally attractive. A new efficient approach is proposed to obtain the optimal value of convergence controller parameter $\hbar$ to guarantee the convergence of the obtained series solution.

Keywords: optimal q-homotopy analysis method; convergence-control parameter; fractional Fornberg–Whitham equation; Caputo derivative.

1. Introduction

Fractional differential equation is a fascinating area of study in mathematics. Over the past decades, many methods have been developed to handle these differential equations. Fractional order ordinary and partial differential equations, asgeneralization of classical integer order differential equations, are increasingly used to model problems in fluid mechanics, viscoelasticity, biology, physics and

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engineering, and others applications [3]. Several methods have been suggested to solve nonlinear equations. These methods include the homotopy perturbation method (HPM) [11], Luapanov's artificial small parameter method [24], Adomian decomposition method [1], [31], variation iterative method [33] and so on. Homotopy analysis method (HAM), first proposed by Liao in his Ph.D dissertation [23], is an elegant method which has proved its effectiveness and efficiency in solving many types of nonlinear equations [19]-[8]. The HAM contains a certain auxiliary parameter h, which provides us with a simple way to adjust and control the convergence region and rate of convergence of the series solution [22]. In 2005, Liao [21] has pointed out that the HPM is only a special case of the HAM (The case of $h = -1$). El-Tawil and Huseen [5] proposed a method namely q-homotopy analysis method (q-HAM) which is more general method of homotopy analysis method (HAM), the q-HAM contains an auxiliary parameter n as well as h such that the cases of (q-HAM; $n = 1$) the standard homotopy analysis method (HAM) can be reached. The q-HAM has been successfully applied to solve many types of nonlinear problems [4]-[17].

In this paper, we have to solve the nonlinear time-fractional Fornberg-Whitham equation by the Oq-HAM. This equation can be written in operator form as

$$u_t^{\alpha} - u_{xxt} + u_x = uu_{xxx} - uu_x + 3u_x u_{xx}, \quad \mathbf{0} < \alpha \leq \mathbf{1}, \quad t > 0, \tag{1}$$

subject to the initial condition

$$u(x, 0) = e^{\frac{x}{2}}, \tag{2}$$

where $u(x, t)$ is the fluid velocity, x and t represent the spatial coordinate and the time respectively. To obtain the approximate or numerical solution of fractional Fornberg–Whitham equation, many effective methods have been developed, such as homotopy perturbation method (HPM) [9], variational iteration method (VIM) [29], combination Laplace transform and HPM [30], homotopy analysis method (HAM) [28], differential transform method (DTM) [25] and fractional homotopy analysis transform method (FHATM) [18]. Y.S. Hamed et al. [27], they applied a complex transformation using a modified chain rule to convert fractional generalized coupled MKDV and KDV of an ordinary differential equation and then we obtain to a new exact solution. Y.S. Hamed et al. [10], they applied a fractional complex transform with optimal homotopy analysis method (OHAM) to obtain numerical and analytical solutions for the nonlinear time-space fractional Fornberg-Whitham. The results reveal that the method is very effective, simple and control the convergence region of convergence of the series solution. Mohamed S. Mohamed and Shaheed N. Huseen [12], illustrated the application of a newly developed efficient method namely, optimal q-homotopy analysis method (Oq-HAM) for solving second order initial and boundary value problems. The results reveal that the Oq-HAM has high accuracy to determine the convergence-control parameter; hence the results match well with the exact solutions and this proves the effectiveness of the method.

The paper has been organized as follows. Notations and basic definitions are given in Section 2. In Section 3 the homotopy analysis method is described. In Section 4 applying Oq-HAM for nonlinear time-fractional Fornberg-Whitham equation. Discussion and conclusions are presented in Section 5.

2. Preliminaries and notations

In this section, we give some basic definitions and properties of fractional calculus theory which are further used in this paper.

Definition 2.1. A real function $h(t)$, $t > 0$, is said to be in the space C_μ, $\mu \in \mathrm{R}$ if there exists a real number $p > \mu$, such that $h(t) = t^p h_1(t)$, where $h_1(t) \in C[0, \infty)$, and it is said to be in the space C_μ^n if and only if $h^{(n)} \in C_\mu$, $n \in \mathrm{N}$.

Definition 2.2. The Riemann-Liouville fractional integral operator (J^α) of order $\alpha \geq 0$, of a function $h \in C_\mu$, $\mu \geq -1$, is defined as

$$J^\alpha h(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} h(\tau) d\tau \quad (\alpha > 0),$$
$$J^0 h(t) = -h(t),$$

$\Gamma(\alpha)$ is the well-known Gamma function. Some of the properties of the operator J^α, which we will need here, are as follows:

(1) $J^\alpha J^\beta h(t) = J^{\alpha+\beta} h(t)$,

(2) $J^\alpha J^\beta h(t) = J^\beta J^\alpha h(t)$,

(3) $J^\alpha t^\gamma = \dfrac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}$, where $\beta \geq 0$, and $\gamma \geq -1$.

Definition 2.3. The fractional derivative (D^α) of $h(t)$ in the Caputo's sense is defined as

$$D^\alpha h(t) = -\frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} h^{(n)}(\tau) d\tau,$$
$$\text{for } n-1 < -\alpha \leq n,\ n \in \mathrm{N},\ t > 0,\ h \in C_{-1}^n.$$

The following are two basic properties of Caputo's fractional Derivative [8]:

(1) Let $h \in C_{-1}^n$, $n \in \mathrm{N}$. Then $D^\alpha h$, $0 \leq \alpha \leq n$ is well defined and $D^\alpha h \in C_{-1}$.

(2) Let $n-1 < \alpha \leq n$, $n \in \mathrm{N}$ and $h \in C_\mu^n$, $\mu \geq -1$. Then

$$(J^\alpha D^\alpha) h(t) = h(t) - \sum_{k=0}^{n-1} h^{(k)}(0^+) \frac{t^k}{k!}.$$

Definition 2.4. The fractional derivative of $f(t)$ in the Caputo sense is defined as

$$D^{\alpha}f(t) = J^{m-\alpha}D^{m}f(t),$$

For $m-1 < \alpha \leq m$, $m \in N$, $t > 0$ and $f \in C^{n}_{-1}$.

3. Basic idea of the optimal q-homotopy analysis method (Oq-HAM)

To describe the basic ideas of the optimal Oq-HAM for nonlinear partial differential equations. Let us consider the following nonlinear partial differential equation:

$$N[D^{\alpha}_{t}u(x,\, t)] - f(x,t) = 0, \tag{3}$$

where N is linear and nonlinear operator for this problem, x and t denote the independent variables, $D^{\alpha}_{t}u(x,t)$ denotes the Caputo fractional derivative, $u(x,t)$ is an unknown function and $f$is a known function. We first construct the zero-order deformation equation as follows:

$$(1-nq)L[\phi(x,t;q) - u_0(x,t)] = qhH(x,t)N[\mathrm{D}^{\alpha}_{\mathrm{t}}\phi(x,t;q) - f(x,t)], \tag{4}$$

where $n > 1$, $q \in [0,\, \frac{1}{n}]$ is the embedding parameter, $h \neq 0$ is an auxiliary parameter, $H(x,t) \neq 0$ is an auxiliary function, L is an auxiliary linear operator and $u_0(x,t)$ is an initial guess. Clearly, when $q = 0$ and $q = \frac{1}{n}$, equation (4) becomes:

$$\phi(x,t;0) = u_0(x,t),\ \phi\left(x,t;\frac{1}{n}\right) = u(x,t), \tag{5}$$

respectively. So, as q increases from 0 to $\frac{1}{n}$, the solution $\phi(x,t,q)$ varies from the initial guess $u_0(x,t)$ to the solution $u(x,t)$. If $u_0(x,t)$, $L, h, H(x,t)$ are chosen appropriately, the solution of equation (5) exists for $q \in [0, \frac{1}{n}]$.

Taylor series expression of $\phi(x,t,q)$ with respect to q in the form

$$\phi(x,t;q) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)q^m, \tag{6}$$

where

$$\phi_m(x,t) = \frac{1}{m!}\ \frac{\partial^m \phi(x,t;q)}{\partial q^m}\Big|_{q=0}. \tag{7}$$

We assume that the auxiliary linear operator, the initial guess, the auxiliary parameter h and the auxiliary function $H(x,t)$ is selected such that the series (7) is convergent when $q \to \dfrac{1}{n}$, then the approximate solution (6) takes the form:

$$u(x,t) = \phi(x,t;\frac{1}{n}) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)\left(\frac{1}{n}\right)^m. \tag{8}$$

Let us define the vector

$$u_n^{\rightarrow}(t) = \{u_0(x,t), u_1(x,t), u_2(x,t), ..., u_n(x,t)\}.$$

Differentiating (4) m times with respect to q , then setting $q = 0$ and dividing then by $m!$, we have the mth-order deformation equation (Lioa [23]-[19]) as

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = hH(x,t)\mathbf{R}_m(u_{m-1}^{\rightarrow}(x,t)), \tag{9}$$

with the initial conditions

$$u_m^{(k)}(x,t) = 0, \quad k = 0, 1, 2, 3, ..., m-1,$$

where

$$\mathbf{R}_m(u_{m-1}^{\rightarrow}(x,t)) = \frac{1}{(m-1)!} \frac{\partial^{m-1}\mathbf{N}[\mathrm{D}_t^{\alpha}\phi(x,t;q) - f(x,t)]}{\partial q^{m-1}}\Big|_{q=0}, \tag{10}$$

and

$$\chi_m = \begin{cases} 0 & m \leq 1, \\ n & m > 1. \end{cases} \tag{11}$$

It should be emphasized that $u_m(x,t)$ for $m \geq 1$ is governed by the linear equation (9) with linear boundary conditions that come from the original problem. Due to the existence of the factor$\left(\frac{1}{n}\right)^m$, more chances for convergence may occur or even much faster convergence can be obtained better than the standard HAM. It should be noted that the cases of $n = 1$ in equation (4), standard HAM can be reached. The h-curves cannot tell us the best convergence-control parameter, which corresponds to the fastest convergent series. In 2007, Yabushita et al. [32] applied the HAM to solve two coupled nonlinear ODEs. They suggested the so-called optimization method to find out the two optimal convergence-control parameters by means of the minimum of the squared residual error of governing equations. In 2008, Akyildiz and Vajravelu [2] gained optimal convergence-control parameter by the minimum of squared residual of governing equation, and found that the corresponding homotopy-series solution converges very quickly.

S.J. Liao [20] and M.S. Mohamed et al. [7], [8] have discussed the optimization method to find out the optimal convergence control parameters by minimum of the square residual error integrated in the whole region having physical meaning. Their approach is based on the square residual error.

Let $\Delta(h)$ denote the square residual error of the governing equation (3) and express as:

$$\Delta(h) = \int_{\Omega} (N[u_n(t)])^2 d\Omega, \tag{12}$$

where

$$u_m(t) = u_0(t) + \sum_{k=1}^{m} u_k(t). \tag{13}$$

The optimal value of the auxiliary parameter h is given by solving the following nonlinear algebraic equation

$$\frac{d\Delta(h)}{dh} = 0. \tag{14}$$

4. Numerical results

To demonstrate the effectiveness of the Oq-HAM algorithm discussed above, example of variation problems will be studied in this section. In this section, example is solved according to the mentioned algorithm in previous section. The results have been provided by Mathematica. We first consider the following time-fractional Fornberg-Whitham equation:

$$u_t^{\alpha} - u_{xxt} + u_x = uu_{xxx} - uu_x + 3u_xu_{xx}, \quad 0 < \alpha \leq l, \ t > 0, \tag{15}$$

with the initial condition

$$u(x,0) = Ae^{\frac{x}{2}},$$

with the exact solution at $\alpha = 1$

$$\mathbf{u(x,t)} = \mathrm{Ae}^{-\left(\frac{\mathrm{x}}{2} - \frac{2}{3}\mathrm{t}\right)}, \tag{16}$$

where A is an arbitrary constant.

For Oq-HAM solution we choose the linear operator

$$L[\phi(x,t;q)] = \frac{\partial^{\alpha}\phi(x,t;q)}{\partial t^{\alpha}},$$

with the property that $L[c] = 0$, where c is constant. We define a nonlinear operator as

$$\mathbf{N}[\phi(x,t;q)] = \frac{\partial^{\alpha}\phi(x,t;q)}{\partial t^{\alpha}} - \phi_{xxt} + \phi_x - \phi\phi_{xxx} + \phi\phi_x - 3\phi_x\phi_{xx}. \tag{17}$$

We construct the zero order deformation equation

$$(1-nq)L[\phi(x,t;q) - u_0(x,t)] = qhH(x,t)N[\mathrm{D}_{\mathrm{t}}^{\alpha}\phi(x,t;q)].$$

For $q = 0$ and $q = 1$, we can write

$$\phi(x,t;0) = u_0(x,t) = u(x,0),$$
$$\phi(x,t;1) = u(x,t).$$

We can take $H(x,t) = 1$, and the mth-order deformation equation is

$$L(u_m(x,t) - \chi_m u_{m-1}(x,t)) = h\mathbf{R}_m(\vec{u_{m-1}}(x,t)), \tag{18}$$

with the initial conditions for $m \geq 1$

$$\mathbf{u_m(x,0)} = 0, \tag{19}$$

where χ_m as defined by (11) and

$$\begin{aligned}\mathbf{R}_m(\vec{u_{m-1}}) = {} & D_t^{\alpha} u_{m-1} - (u_{m-1})_{xxt} + (u_{m-1})_x \\ & - \sum_{\mathrm{i}=0}^{m-1} u_{\mathrm{i}}(u_{m-1-\mathrm{i}})_{xxx} + \sum_{\mathrm{i}=0}^{m-1} u_{\mathrm{i}}(u_{m-1-\mathrm{i}})_x - 3\sum_{\mathrm{i}=0}^{m-1}(u_{\mathrm{i}})_x(u_{m-1-\mathrm{i}})_{xx}.\end{aligned} \tag{20}$$

Now, the solution of the mth-order deformation equations for $m \geq 1$ becomes

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + h\, j_t^{\alpha}(\mathbf{R}_m(\vec{u_{m-1}})) + c_1 \tag{21}$$

where the constant of integration c_1 is determined by the initial conditions (19). Then, the components of the solution using Oq-HAM are

$$\begin{aligned} & u(x,\,0) = \mathrm{e}^{\frac{\mathrm{x}}{2}}, \\ & u_1\,(x,t) = \frac{\mathrm{e}^{\frac{\mathrm{x}}{2}}}{2\alpha\Gamma(\alpha)} h\, t^{\alpha}, \\ & u_2\,(x,t) = \frac{\mathrm{e}^{\frac{\mathrm{x}}{2}}}{2\alpha\Gamma(\alpha)} h\, \mathrm{n}t^{\alpha} + h^2 \mathrm{e}^{\frac{\mathrm{x}}{2}} \left(\frac{t^{2\alpha-1}(2t-\alpha)}{8\alpha^2\Gamma^2(\alpha)} + \frac{t^{\alpha}}{2\alpha\Gamma(\alpha)} \right), \\ & \vdots \end{aligned} \tag{22}$$

According to the optimal q-homotopy analysis, we can conclude that

$$u(x,t;n;h) \simeq U_m(x,t;n;h) = \sum_{i=0}^{M} u_i(x,t;n;h) \left(\frac{1}{n}\right)^i. \tag{23}$$

Equation (23) is an approximate solution to the problem (15) in terms of convergence parameter h and n. Then we have at $\alpha = 1$,

$$\begin{aligned} u_{app} = {} & -u_0\,(x,t) + \left(\frac{1}{n}\right) u_1\,(x,t) + \left(\frac{1}{n}\right)^2 u_2\,(x,t) + \left(\frac{1}{n}\right)^3 u_3\,(x,t) \\ & + \left(\frac{1}{n}\right)^4 u_4\,(x,t) + \left(\frac{1}{n}\right)^5 u_5\,(x,t) + ... \end{aligned} \tag{24}$$

As special case, if $n = 1$ and $h = 1$, then we obtain the same result in [9]-[18]. Equation (24) is an approximate solution to the problem (15) in terms of the convergence parameters h and n. To find the valid region of h, the h-curves given by the 5th order q-HAM approximation at different values of $x, t, \alpha = 1$ and n are drawn in Figures 1-9. These figures show the interval of h at which the value of $u_5(x,t;n)$ is constant at certain values of x, t and n. We choose the horizontal line parallel to $x-$axis(h) as a valid region which provides us with a simple way to adjust and control the convergence region of the series solution. From these figures, the valid intersection region of h for the values of x, t and n in the curves becomes larger as n increase.

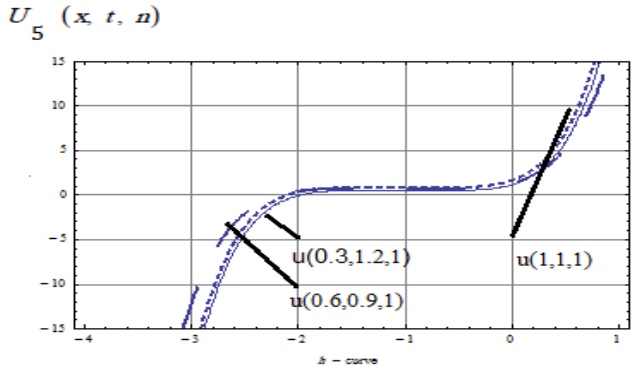

Figure 1. h-curve for the HAM (q-HAM; $n = 1$) approximation solution $u_5(x, t; 1)$ of problem (15) at different values of x, t and $h_{optimal} = -0.95$.

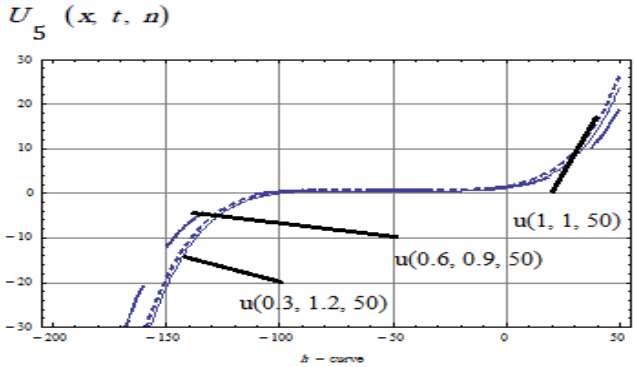

Figure 2. h-curve for the HAM (q-HAM; $n = 50$) approximation solution $u_5(x, t; 50)$ of problem (15) at different values of x, t and $h_{optimal} = -15.03$.

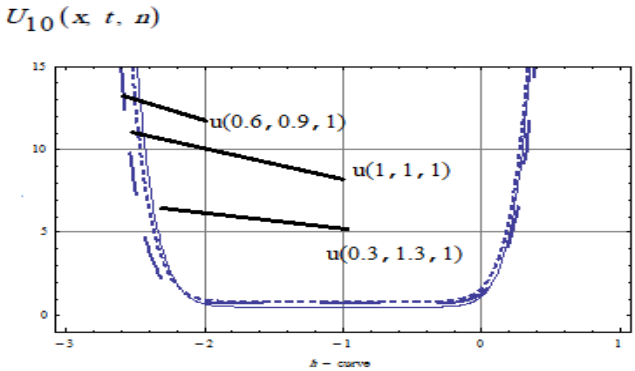

Figure 3. h-curve for the HAM (q-HAM; $n = 1$) approximation solution $u_{10}(x, t; 1)$ of problem (15) at different values of x, t and $h_{optimal} = -0.97$.

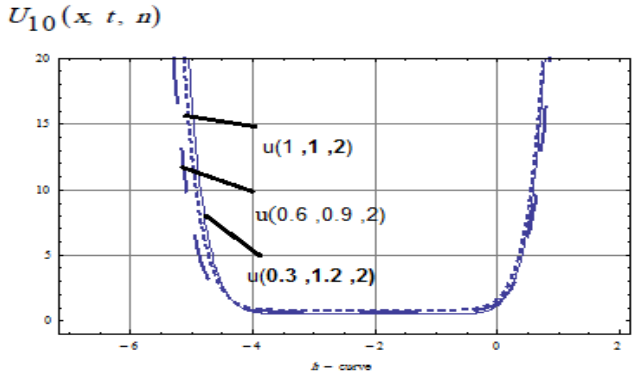

Figure 4. h-curve for the HAM (q-HAM; $n = 2$) approximation solution $u_{10}(x, t; 2)$ of problem (15) at different values of x, t and $h_{optimal} = -1.75$.

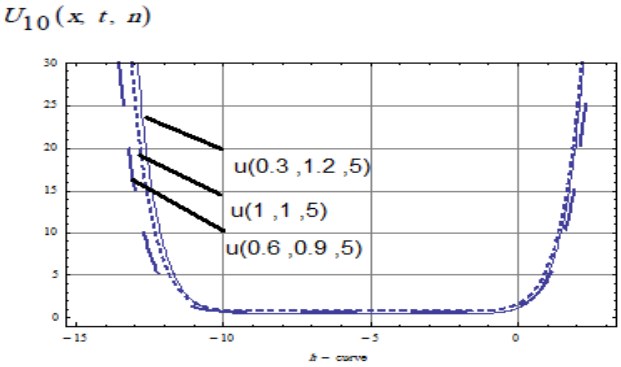

Figure 5. h-curve for the HAM (q-HAM; $n = 5$) approximation solution $u_{10}(x, t; 5)$ of problem (15) at different values of x, t and $h_{optimal} = -2.15$.

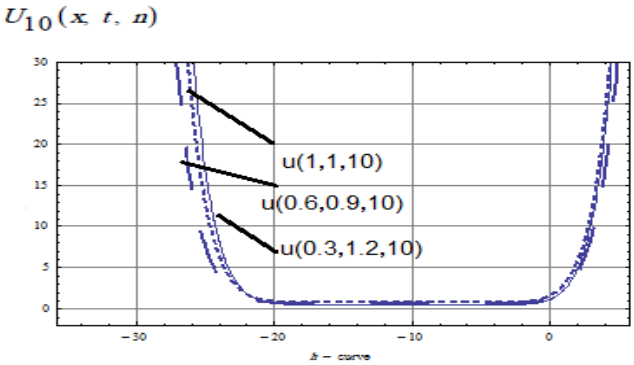

Figure 6. h-curve for the HAM (q-HAM; $n = 10$) approximation solution $u_{10}(x, t; 10)$ of problem (15) at different values of x, t and $h_{optimal} = -5.32$.

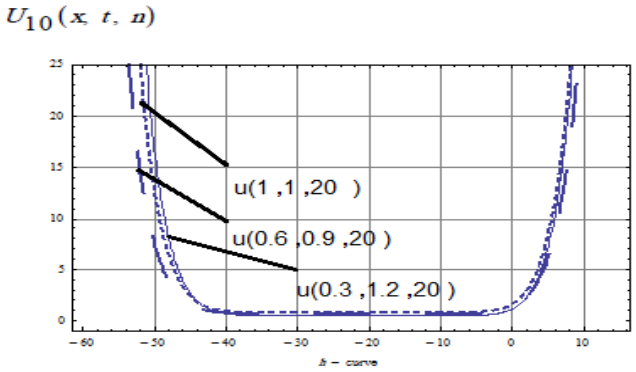

Figure 7. h-curve for the HAM (q-HAM; $n = 20$) approximation solution $u_{10}(x, t; 20)$ of problem (15) at different values of x, t and $h_{optimal} = -8.73$.

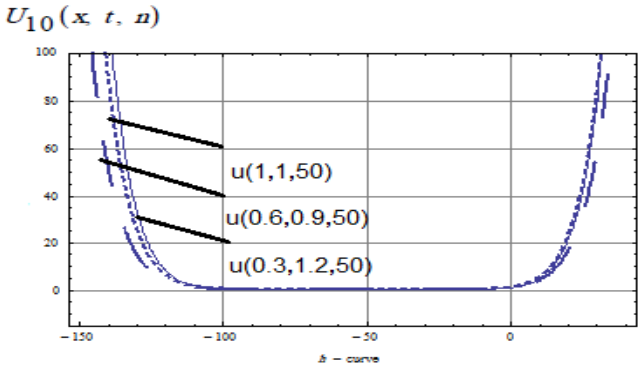

Figure 8. h-curve for the HAM (q-HAM; $n = 50$) approximation solution $u_{10}(x, t; 50)$ of problem (15) at different values of x, t and $h_{optimal} = -12.05$.

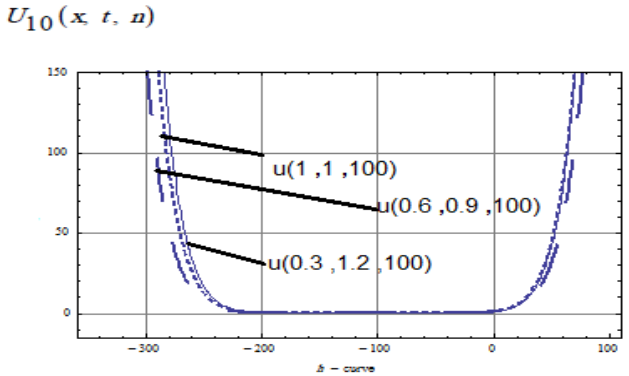

Figure 9. h-curve for the HAM (q-HAM; $n = 100$) approximation solution $u_{10}(x, t; 100)$ of problem (15) at different values of x, t and $h_{optimal} = -19.14$.

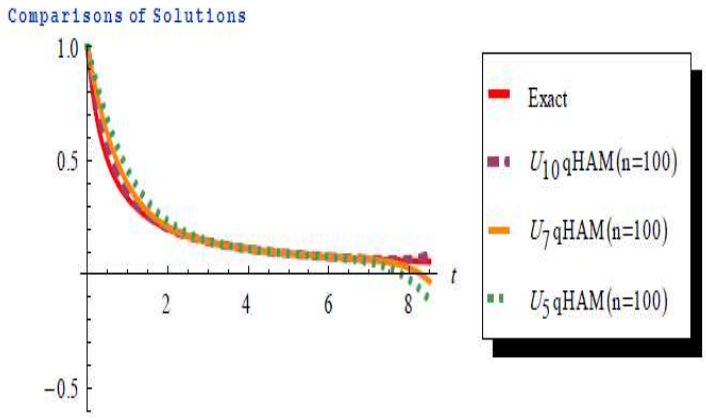

Figure 10. Comparison between u_5, u_7 and u_{10} of (q-HAM; $n = 100$ and $\alpha = 1$) and exact solution (16) at $x = 1$ with $h_{optimal} = -12.45, 0 < t \leq 8$.

Figure 10 shows the comparison between u_5, u_7 and u_{10} using different values of n with the exact solution (16). Therefore, based on these present results, we can say that q-HAM is more effective than HAM and HPM.

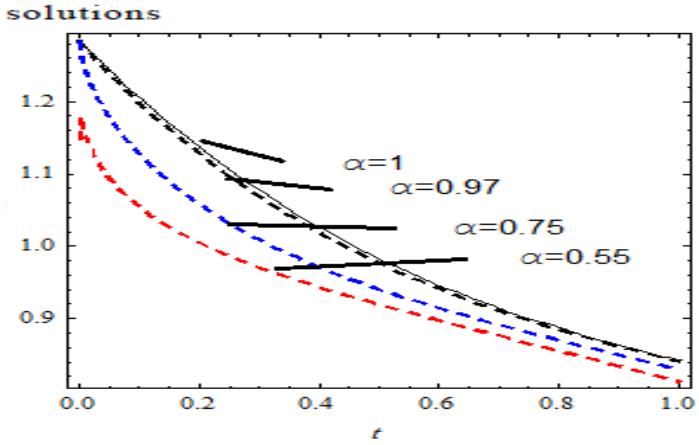

Figure 11. Comparison between $u_5(\alpha = 0.97, 0.75, 0.55)$ of (q-HAM; $n = 1$) and exact solution (16) at $x = 0.5$ with $h_{optimal} = -0.97,\ 0 < t \leq 1$.

We present the numerical results of the series solution of fractional Fornberg–Whithem equation obtained by oq-HAM. The graphs give clear pictures of the simple but elegant nature of the method used to solve strongly nonlinear problems of the fractional type.

5. Conclusions

An approximate solution of nonlinear time-fractional Fornberg-Whitham equation was found by using the Oq-homotopy analysis method (q-HAM).The results show that the convergence region of series solutions obtained by Oq-HAM is increasing as q is decreased. The comparison of Oq-HAM with the HAM and HPM was made. It was shown that the convergence of Oq-HAM is faster than the convergence of HAM and HPM. The results show that the method is powerful and efficient techniques in finding exact and approximate solutions for equations and also, this method uses simple computation with acceptable solution.

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QUASI-FREDHOLM, SAPHAR SPECTRA FOR C_0 SEMIGROUPS GENERATORS

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Abstract. In this work, we show that the spectral inclusion of semigroups hold for Saphar, essentially Saphar and quasi-Fredholm spectra. Some stabilities results are also established.

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1. Introduction and preliminaries

Throughout, X denotes a complex Banach space, let A be a closed linear operator on X with domain $D(A)$, we denote by A^*, $R(A)$, $N(A)$, $R^\infty(A) = \bigcap_{n\geq 0} R(A^n)$, $\sigma(A)$, respectively the adjoint, the range, the null space, the hyper-range, the spectrum of A.

Recall that a closed operator A is said to be Kato operator or semi-regular if $R(A)$ is closed and $N(A) \subseteq R^\infty(A)$. Denote by $\rho_K(A)$:

$\rho_K(A) = \{\lambda \in \mathbb{C} : A-\lambda I$ is Kato $\}$ the Kato resolvent and $\sigma_K(A) = \mathbb{C}\backslash\rho_K(A)$ the Kato spectrum of A. It is well known that $\rho_K(A)$ is an open subset of $\mathbb{C}$.

For subspaces M, N of X we write $M \subseteq^e N$ (M is essentially contained in N) if there exists a finite-dimensional subspace $F \subset X$ such that $M \subseteq N + F$.

A closed operator S is called a generalized inverse of A if $R(A) \subseteq D(S)$, $R(S) \subseteq D(A)$, $ASA = A$ on $D(A)$ and $SAS = S$ on $D(S)$, wich equivalent to the fact that $R(A) \subseteq D(S)$, $R(S) \subseteq D(A)$, $ASA = A$ on $D(A)$.

A closed operator A is called a Saphar operator if A has a generalized inverse and $N(A) \subseteq R^\infty(A)$, which equivalent to the fact that A is Kato operator and has a generalized inverse, see [6].

If we assume in the definition above that $N(A) \subseteq^e R^\infty(A)$, A is said to be a essentially Saphar operator. The Saphar and essentially Saphar spectra are defined by

$$\begin{aligned}\sigma_{Sap}(A) &= \{\lambda \in \mathbb{C} : A - \lambda \text{ is not Saphar}\}.\\ \sigma^e_{Sap}(A) &= \{\lambda \in \mathbb{C} : A - \lambda \text{ is not essentially Saphar}\}.\end{aligned}$$

$\sigma_{Sap}(A)$ is a compact non empty set of $\mathbb{C}$ and we have

$$\partial\sigma(A) \subseteq \sigma_K(A) \subseteq \sigma_{Sap}(A) \subseteq \sigma(A).$$

A one-parameter family $\{T(t)\}_{t\geq 0}$ of operators on X is called a C_0-semigroup of operators if:

1. $T(0) = I$.
2. $T(t+s) = T(t)T(s)$, $\forall t, s \geq 0$.
3. $\lim\limits_{t\to 0} T(t)x = x$, $\forall x \in X$.

$\{T(t)\}_{t\geq 0}$ has a unique infinitesimal generator A defined in the domain $D(A)$ by:

$$\begin{aligned}D(A) &= \left\{x \in X : \lim_{t\to 0}\frac{T(t)x - x}{t} \text{ exists}\right\} \text{ and}\\ Ax &= \lim_{t\to 0}\frac{T(t)x - x}{t}, \forall x \in D(A).\end{aligned}$$

Recall that for all $t \geq 0$, $T(t)$ is a bounded linear operator in X and A is a closed operator. Details for all this may be found in [7], [4].

Next, we introduce the following operator acting on X and depending on the parameters $\lambda \in \mathbb{C}$ and $t \geq 0$:

$$B_\lambda(t)x = \int_0^t e^{\lambda(t-s)}T(s)xds, x \in X.$$

It is well known that $B_\lambda(t)$ is a bounded linear operator on X [7, 4] and we have:

$$\begin{aligned}(e^{\lambda t} - T(t))^n x &= (\lambda - A)^n B^n_\lambda(t)x, \quad \forall x \in X,\ n \in \mathbb{N}\\ (e^{\lambda t} - T(t))^n x &= B^n_\lambda(t)(\lambda - A)^n x, \quad \forall x \in D(A^n),\ n \in \mathbb{N};\\ R^\infty(e^{\lambda t} - T(t)) &\subseteq R^\infty(\lambda - A);\\ N((\lambda - A)^n) &\subseteq N(e^{\lambda t} - T(t))^n.\end{aligned}$$

2. Spectral inclusions and stability for Saphar spectrum

Lemma 2.1 [8] *Let $\{T(t)\}_{t\geq 0}$ a C_0-semigroup on X with infinitesimal generator A. For $\lambda \in \mathbb{C}$ and $t \geq 0$, let $F_\lambda(t)x = \int_0^t e^{-\lambda s}B_\lambda(s)xds$, then:*

1. *There exist a* $M \geq 1$ *and* $\omega > Re(\lambda)$ *such that*

$$F_\lambda(t) \leq \frac{M}{(\omega - Re(\lambda))^2} e^{(\omega - Re(\lambda))t}.$$

2. $\forall x \in X$, $F_\lambda(t)x \in D(A)$ *and* $(\lambda - A)F_\lambda(t) + G_\lambda(t)B_\lambda(t) = tI$ *with*

$$G_\lambda(t) = e^{-\lambda t} I.$$

3. *The operators* $F_\lambda(t)$, $G_\lambda(t)$ *and* $B_\lambda(t)$ *are pairwise commute and for all* $x \in D(A)$*:*

$$\begin{aligned} (\lambda - A)F_\lambda(t)x &= F_\lambda(t)(\lambda - A)x \\ (\lambda - A)G_\lambda(t)x &= G_\lambda(t)(\lambda - A)x \\ (\lambda - A)B_\lambda(t)x &= B_\lambda(t)(\lambda - A)x \end{aligned}$$

Lemma 2.2 *Let* $\{T(t)\}_{t\geq 0}$ *a* C_0*-semigroup on* X *with infinitesimal generator* A*. Then for all* $t > 0$ *we have:*

$$e^{\lambda t} - T(t) \text{ has a generalized inverse} \Longrightarrow \lambda - A \text{ has a generalized inverse.}$$

Proof. Suppose that $e^{\lambda t} - T(t)$ has a generalized inverse then there exists a $S \in \mathcal{B}(X)$ such that:

$$(e^{\lambda t} - T(t))S(e^{\lambda t} - T(t)) = e^{\lambda t} - T(t)$$

According to Lemma 2.1, we have $(\lambda - A)F_\lambda(t) + G_\lambda(t)B_\lambda(t) = tI$. Then:

$$\begin{aligned} t(\lambda - A) &= (\lambda - A)F_\lambda(t)(\lambda - A) + B_\lambda(t)G_\lambda(t)(\lambda - A) \\ &= (\lambda - A)F_\lambda(t)(\lambda - A) + (\lambda - A)B_\lambda(t)G_\lambda(t) \\ &= (\lambda - A)F_\lambda(t)(\lambda - A) + (e^{\lambda t} - T(t))G_\lambda(t) \\ &= (\lambda - A)F_\lambda(t)(\lambda - A) + (e^{\lambda t} - T(t))S(e^{\lambda t} - T(t))G_\lambda(t) \\ &= (\lambda - A)F_\lambda(t)(\lambda - A) + (\lambda - A)B_\lambda(t)S(\lambda - A)B_\lambda(t)G_\lambda(t) \\ &= (\lambda - A)F_\lambda(t)(\lambda - A) + (\lambda - A)B_\lambda(t)SB_\lambda(t)G_\lambda(t)(\lambda - A) \\ &= (\lambda - A)[F_\lambda(t) + B_\lambda(t)SB_\lambda(t)G_\lambda(t)](\lambda - A). \end{aligned}$$

Hence $\lambda - A$ has a generalized inverse. ∎

Theorem 2.1 *Let* $\{T(t)\}_{t\geq 0}$ *be a* C_0*-semigroup on* X *with infinitesimal generator* A*. Then for all* $t > 0$*:*

$$e^{t\sigma_{Sap}(A)} \subseteq \sigma_{Sap}(T(t)),\ e^{t\sigma^e_{Sap}(A)} \subseteq \sigma^e_{Sap}(T(t)).$$

Proof. Assume that $e^{\lambda t} - T(t)$ is a Saphar operator, then $e^{\lambda t} - T(t)$ has a generalized inverse and $N(e^{\lambda t} - T(t)) \subseteq R(e^{\lambda t} - T(t))$.

By Lemma 2.2, $\lambda - A$ has a generalized inverse, and we have:

$$N(\lambda - A) \subseteq N(e^{\lambda t} - T(t)) \subseteq R(e^{\lambda t} - T(t))^{\infty} \subseteq R(\lambda - A)^{\infty}.$$

Therefore, $\lambda - A$ is a Saphar operator.

Let M a finite dimensional subspace of X. We have

$$N(\lambda - A) \subseteq N(e^{\lambda t} - T(t)) \subseteq R(e^{\lambda t} - T(t))^{\infty} + M \subseteq R(\lambda - A)^{\infty} + M.$$

Hence $e^{\lambda t} - T(t)$ is an essentially Saphar operator implying that $\lambda - A$. ∎

Let $\{T(t)\}_{t\geq 0}$ a C_0-semigroup on X with infinitesimal generator A. $\{T(t)\}_{t\geq 0}$ is said to be strongly stable if $\lim_{t\to\infty} ||T(t)x|| = 0$ for all $x \in X$. We say that $\{T(t)\}_{t\geq 0}$ is uniformly stable if $\lim_{t\to\infty} ||T(t)|| = 0$.

In [2], A. Elkoutri and M.A. Taoudi showed that $\{T(t)\}_{t\geq 0}$ is strongly stable if $\sigma_K(A) \cap i\mathbb{R} = \emptyset$.

In the following, we give a stability result for strongly continuous semigroups using the Saphar spectrum:

Proposition 2.1 *Let $\{T(t)\}_{t\geq 0}$ a C_0-semigroup on X with infinitesimal generator A. The following assertions are equivalent:*

1. $\sigma(A) \cap i\mathbb{R} = \emptyset$,
2. $\sigma_K(A) \cap i\mathbb{R} = \emptyset$,
3. $\sigma_{Sap}(A) \cap i\mathbb{R} = \emptyset$.

If the infinitesimal generator A verified one of this properties, $\{T(t)\}_{t\geq 0}$ is strongly stable.

Proof. Since $\sigma_K(A) \subset \sigma_{Sap}(A) \subset \sigma(A)$, the result is immediately comes from [2, Corollary 2.1]. ∎

Recall that a C_0-semigroup $\{T(t)\}_{t\geq 0}$ is said uniformly exponentially stable if there exists $\epsilon > 0$ such that

$$\lim_{t\to\infty} e^{\epsilon t}||T(t)|| = 0.$$

Proposition 2.2 *Let $\{T(t)\}_{t\geq 0}$ a C_0-semigroup on X with infinitesimal generator A, Γ is the unit circle in $\mathbb{C}$. The following assertions are equivalent:*

1. *$\{T(t)\}_{t\geq 0}$ is uniformly exponentially stable,*
2. *there exists $t_0 > 0$ such that $\sigma_K(T(t_0)) \cap \Gamma = \emptyset$,*
3. *there exists $t_0 > 0$ such that $\sigma_{Sap}(T(t_0)) \cap \Gamma = \emptyset$.*

Proof. 1) $\Longleftrightarrow$ 2): see [2, Corollary 2.2].

1) $\Longrightarrow$ 3) is obvious, see [4].

3) $\Longrightarrow$ 1) Suppose that there exists $t_0 > 0$ such that $\sigma_{Sap}(T(t_0)) \cap \Gamma = \emptyset$. Since $\sigma_K(T(t_0)) \subseteq \sigma_{Sap}(T(t_0))$, hence $\sigma_K(T(t_0)) \cap \Gamma = \emptyset$. By [2, Corollary 2.2] $\{T(t)\}_{t\geq 0}$ is uniformly exponentially stable. ∎

3. Spectral inclusion for quasi-Fredholm spectrum

Recall from [5] some definitions:

Definition 3.1 Let T a closed linear operator on X and let

$$\Delta(T) = \{n \in \mathbb{N}, \forall m \geq n, R(T^n) \cap N(T) = R(T^m) \cap N(T)\}$$

The degree of stable iteration $dis(T)$ of T is defined as $dis(T) = \inf \Delta(T)$, where $dis(T) = \infty$ if $\Delta(T) = \emptyset$.

Definition 3.2 Let T a closed linear operator on X. T is called a quasi-Fredholm operator of degree d if there exists an integer $d \in \mathbb{N}$ such that:

1. $dis(T) = d$;
2. $R(T^n)$ is closed in X for all $n \geq d$;
3. $R(T) + N(T^n)$ is closed in X for all $n \geq d$.

The quasi-Fredholm spectrum is defined by:

$$\sigma_{qF}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a quasi-Fredholm}\}.$$

Proposition 3.3 *Let $\{T(t)\}_{t\geq 0}$ a C_0-semigroup on X with infinitesimal generator A. Then:*

$$dis(\lambda - A) \leq dis(e^{\lambda t} - T(t)).$$

Proof. If $dis(e^{\lambda t} - T(t)) = +\infty$ obvious.

If $dis(e^{\lambda t} - T(t)) = d \in \mathbb{N}^*$ then for all $n \geq d$, we have:

$$R((e^{\lambda t} - T(t))^n) \cap N((e^{\lambda t} - T(t))) = R((e^{\lambda t} - T(t))^d) \cap N(e^{\lambda t} - T(t)).$$

Then, for all $n \geq d$, we have:

$$R((\lambda - A)^n) \cap N(\lambda - A) = R((\lambda - A)^d) \cap N(\lambda - A).$$

Indeed, let $y \in R((\lambda - A)^d) \cap N(\lambda - A)$. Then there exists $x \in X$ such that $y = (\lambda - A)^d x$ and, by Lemma 2.1, $(\lambda - A)^d F_d(t) + B_\lambda^d(t) G_d(t) = I$ implies that $(\lambda - A)^d x = (\lambda - A)^{2d} F_d(t)x + (e^{\lambda t} - T(t))^d G_d(t)x$, since $y \in N(\lambda - A)$,

$y = (\lambda - A)^d x = (e^{\lambda t} - T(t))^n z = (\lambda - A)^n B_\lambda^n(t) z \in R((\lambda - A)^n) \cap N(\lambda - A)$ therefore, $dis(\lambda - A) \leq d$.

If $d = 0$, for all $n \geq d$, we have:

$$R((e^{\lambda t} - T(t))^n) \cap N((e^{\lambda t} - T(t))) = N(e^{\lambda t} - T(t)).$$

Then, $\forall n \in \mathbb{N}$,

$$N((e^{\lambda t} - T(t))) \subset R((e^{\lambda t} - T(t))^n).$$

Then

$$N(\lambda - A) \subset N((e^{\lambda t} - T(t))) \subset R((e^{\lambda t} - T(t))^n) \subset R((\lambda - A)^n).$$

Hence

$$N(\lambda - A) \cap R((\lambda - A)^n) = N(\lambda - A) \cap R((\lambda - A)^0).$$

Therefore, $dis(\lambda - A) = 0$. ∎

Proposition 3.4 *Let $\{T(t)\}_{t\geq 0}$ a C_0-semigroup on X with infinitesimal generator A. If $R((e^{\lambda t} - T(t))^n)$ is closed for all $n \geq d$, then $R((\lambda - A^n))$ is closed for all $n \geq d$.*

Proof. Let $y_n = (\lambda - A)^n x_n \to y$, as $n \to \infty$, we show that $y \in R((\lambda - A)^n)$. According to Lemma 2.1 there exists $F_d(t)$ and $G_d(t)$ to bounded linear operators such that

$$(\lambda - A)^d F_d(t) + B_\lambda^d(t) G_d(t) = I. \tag{1}$$

$B_\lambda^n(t) y_n = B_\lambda^n(t)(\lambda - A)^n x_n = (e^{\lambda t} - T(t))^n x_n \in R((e^{\lambda t} - T(t))^n)$.
Since $B_\lambda^n(t) y_n \to B_\lambda^n(t) y$, then there exists $z \in X$ such that

$$B_\lambda^n(t) y = (e^{\lambda t} - T(t))^n z.$$

By (1):

$$(\lambda - A)^n x_n = (\lambda - A)^{2n} F_n(t) x_n + (e^{\lambda t} - T(t))^n G_n(t) x_n \text{ as } n \to \infty.$$

We have:

$$\begin{aligned} y &= (\lambda - A)^n F_n(t) y + (e^{\lambda t} - T(t))^n G_n(t) z \\ &= (\lambda - A)^n [F_n(t) y + B_\lambda^n(t) G_n(t) z] \in R((\lambda - A)^n). \end{aligned}$$ ∎

Proposition 3.5 *Let $\{T(t)\}_{t\geq 0}$ a C_0-semigroup on X with infinitesimal generator A and $d \in \mathbb{N}$. If $R(e^{\lambda t} - T(t)) + N((e^{\lambda t} - T(t))^d)$ is closed in X, then $R(\lambda - A) + N((\lambda - A)^d)$ is closed.*

Proof. Suppose that $R(e^{\lambda t} - T(t)) + N((e^{\lambda t} - T(t))^d)$ is closed in X.
Let $y_n = (\lambda - A)x_n + z_n \to y$, as $n \to \infty$, $z_n \in N((\lambda - A)^d)$.
$B^d_\lambda(t)y_n = B^d_\lambda(t)(\lambda - A)x_n + B^d_\lambda(t)z_n \in R(e^{\lambda t} - T(t)) + N((e^{\lambda t} - T(t))^d)$, then

$$B^d_\lambda(t)y \in R(e^{\lambda t} - T(t)) + N((e^{\lambda t} - T(t))^d)$$
$$B^d_\lambda(t)y = ((e^{\lambda t} - T(t))x + z \text{ with } z \in N((e^{\lambda t} - T(t)^d))$$
$$B^{2d}_\lambda(t)y = B^d_\lambda(t)(e^{\lambda t} - T(t))x + B^d_\lambda(t)z \text{ and } B^d_\lambda(t)z \in N((\lambda - A)^d).$$

Then

$$\begin{aligned} y &= (\lambda - A)^{2d}F_d(t)y + B^{2d}_\lambda(t)G_d(t)y \\ &= (\lambda - A)^{2d}F_d(t)y + G_d(t)[B^d_\lambda(t)(e^{\lambda t} - T(t))x + B^d_\lambda(t)z] \\ &= (\lambda - A)[(\lambda - A)^{2d-1}F_d(t)y + G_d(t)B^{d-1}_\lambda(t)] \\ &\quad + G_d(t)B^d_\lambda(t)z \in R(\lambda - A) + N((\lambda - A)^d). \end{aligned}$$

∎

Corollary 3.1 *Let $\{T(t)\}_{t\geq 0}$ a C_0-semigroup on X with infinitesimal generator A. If $R(e^{\lambda t} - T(t))$ is closed in X, then $R(\lambda - A)$ is closed.*

Theorem 3.2 *Let $\{T(t)\}_{t\geq 0}$ be a C_0-semigroup on X with infinitesimal generator A. Then, for all $t > 0$,*

$$e^{t\sigma_{qF}(A)} \subseteq \sigma_{qF}(T(t)).$$

Proof. Direct consequence of the three last proposition. ∎

If $dis(T) = 0$, T to be a Kato operator and by using the Corollary 3.1 and Proposition 3.3 we have the result of A. Elkoutri and M.A. Taoudi [2]:

Corollary 3.2 *Let $\{T(t)\}_{t\geq 0}$ be a C_0-semigroup on X with infinitesimal generator A. Then for all $t > 0$:*

$$e^{t\sigma_K(A)} \subseteq \sigma_K(T(t)).$$

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NEW SOLITARY WAVE AND MULTIPLE SOLITON SOLUTIONS FOR THE TIME-SPACE FRACTIONAL BOUSSINESQ EQUATION

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Abstract. The aim of this paper are in two fold: First, introduce a new simplified bilinear method based on a transformation method combined with the Hirota bilinear sense. Second, apply this new technique to study the time-space fractional Boussinesq equation. To the best of my knowledge, the proposed work present new N-soliton solutions of the fractional Boussinesq equation. These new exact solutions extend previous results and help us to explain the properties of multidimensional nonlinear solitary waves in shallow water. Parametric analysis is carried out in order to illustrate that the soliton amplitude, width and velocity are affected by the coefficient parameters in the equation.

Keywords: Hirota bilinear method, N-soliton solutions, modified Riemann-Liouville derivative, fractional Boussinesq equation.
2010 Mathematics Subject Classification: 34A08,35A25, 35B25, 35C08.

1. Introduction

It has been a great deal and efforts of interest in fractional differential equations. But, there were almost no practical applications of fractional calculus, and this area has been started as an abstract containing only mathematical manipulations of little or no use. The last three decades, the paradigm began to shift from pure mathematical formulations to applications in various fields. During the last decade fractional calculus has been applied to almost every field of science, engineering, and mathematics. Several fields of application of fractional differentiation and fractional integration are already well established, some others have just started. Fractional differential equations (FDEs) have found applications in many problems in physics and engineering [14], [16], [17], [18]. Phenomena in electromagnetic, acoustics, viscoelasticity, electrochemistry, and material science are also well described by differential equations of fractional order [19], [16], [20], [21]. The solution of differential equations of fractional order is much involved

too. In general, there exists no method that yields an exact solution for nonlinear fractional differential equations. Only approximate solutions can be derived using linearization or perturbation method. Since most of the nonlinear FDEs cannot be solved exactly, approximate and numerical methods must be used. Not much work has been done on nonlinear problems and only a few numerical schemes have been proposed for solving nonlinear fractional differential equations. More recently, applications have included classes of nonlinear equation with multi-order fractional derivatives and this motivates researchers to develop a numerical scheme for their solution [22]. Numerical and analytical methods have included the Adomian decomposition method (ADM) [23], [24], the variational iteration method (VIM) [25], the homotopy perturbation method [26], [27], [28] and more recently the residual power series method [29]. In this work, a new simplified bilinear method based on a transformation method combined with the Hirota's bilinear sense will be introduced to study the fractional Boussinesq equation. Our goal from applying this method is to construct multiple regular soliton solutions and multiple singular soliton solutions. The introduced simplified algorithm derives the auxiliary functions, obtained in Hirota's method, without using the bilinear forms. Although there are studies for the classical Boussinesq equation [30] and some profound results have been established, it seems that detailed studies of the fractional Boussinesq equation are only beginning. To the best of my knowledge the proposed work will represent the first available $N-$ soliton solution of the time-space fractional Boussinesq equation.

2. Preliminaries

There are different definitions for fractional derivatives; for more details see [14]. In our paper we use the modified Riemann–Liouville derivative which was defined by Jumarie [15]

$$\begin{cases} D_x^\alpha f(x) &= \dfrac{1}{\Gamma(1-\alpha)}\displaystyle\int_0^x (x-y)^{-1-\alpha}[f(y)-f(0)]dy, \alpha<0 \\ &= \dfrac{1}{\Gamma(1-\alpha)}\dfrac{d}{dx}\displaystyle\int_0^x (x-y)^{-1-\alpha}[f(y)-f(0)]dy, 0<\alpha<1 \\ &= [f^{(\alpha-n)}(x)]^{(n)}, n\le\alpha<n+1 \end{cases}$$

which has merits over the original one, for example, the α-order derivative of a constant is zero. The main properties of the modified Riemann–Liouville derivative were summarized in [15] and three useful formulas of them are given as follows:

$$\begin{cases} D_x^\alpha x^\beta &= \dfrac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)}x^{\beta-\alpha}, \beta>0 \\ D_x^\alpha(u(x)v(x)) &= u(x)D_x^\alpha v(x)+v(x)D_x^\alpha u(x) \\ D_x^\alpha[f(v(x))] &= \dfrac{df}{du}D_x^\alpha u(x)=\left(\dfrac{du}{dx}\right)^\alpha D_u^\alpha f(u). \end{cases}$$

3. Multiple soliton solutions for the fractional Boussinesq equation

To derive N-soliton solutions of any completely integrable equation, we will mainly use the Hirota's direct method. The Hirota method relies on a transformation for considered equation to a bilinear form. The bilinear forms are usually used to enable us deriving the auxiliary function. It is remarkable to mention that it is not easy to find the bilinear form for many equations and sometimes it requires the introduction of new dependent and sometimes even independent variables. However, Hereman et al. [7], [8], [9], formally introduced the simplified algorithm to derive the auxiliary functions without using the bilinear forms. The Cole–Hopf transformation method combined with the simplified Hirota's sense is a powerful method to determine multiple soliton solutions and multiple singular soliton solutions for integrable systems[2], [3], [4], [5], [6], [10], [11], [12], [13].

In this section, we apply simplified bilinear method, to construct soliton solutions of fractional equation:

$$u_t^{(2\alpha)} + au_{xx} + 2b(uu_x^{(2\beta)} + (u_x^{(\beta)})^2) + cu_x^{(4\beta)} = 0, \tag{3.1}$$

where the coefficients a, b, c are real-valued parameters that are physically meaningful.

The following fractional transforms are introduced

$$T = \frac{p_1 t^\alpha}{\Gamma(1+\alpha)}, \quad X = \frac{p_2 x^\beta}{\Gamma(1+\beta)},$$

where p_1 and p_2 are constants. Using the above transforms, we can convert fractional derivatives into the following classical derivatives:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = p_1 \frac{\partial u}{\partial T}, \quad \frac{\partial^\beta u}{\partial x^\beta} = p_2 \frac{\partial u}{\partial X},$$

see [1]. Therefore, equation (3.1) becomes

$$u_{TT} + au_{XX} + b(u^2)_{XX} + cu_{XXXX} = 0. \tag{3.2}$$

Now, we substitute

$$u(X,T) = e^{\theta_i(X,T)} \tag{3.3}$$

$$\theta_i(X,T) = k_i X - \omega_i T \tag{3.4}$$

into the linear terms of equation (3.2) to determine the dispersion relation as follows

$$\omega_i = ik_i\sqrt{a + ck_i^2}. \tag{3.5}$$

As a result, we obtain

$$\theta_i = k_i x - ik_i\sqrt{a + ck_i^2}T. \tag{3.6}$$

The single soliton solution of equation (3.2) is assumed to be

$$u(X,T) = R\,(\ln f)_{XX} \tag{3.7}$$

where the auxiliary function $f(X,T)$ is given by

$$f(X,T) = 1 + e^{\theta_1(X,T)} = 1 + e^{k_1 X - ik_1\sqrt{a+ck_1^2}T.} \tag{3.8}$$

Substituting (3.7) into equation (3.2) and solving for R we find $R = 6c/b$. By substituting $f(X,T)$ into (3.7), we obtain the single soliton solution

$$u(X,T) = \frac{6c}{b}k_1^2\frac{e^{\theta_1(X,T)}}{\left(1+e^{\theta_1(X,T)}\right)^2} = \frac{6c}{b}k_1^2\frac{e^{k_1X-ik_1\sqrt{a+ck_1^2}T}}{\left(1+e^{k_1X-ik_1\sqrt{a+ck_1^2}T}\right)^2}. \tag{3.9}$$

The solution in (3.9) can be written as

$$u(X,T) = \frac{3c}{2b}k_1^2\sec h^2\left(\frac{\theta_1(X,T)}{2}\right), \tag{3.10}$$

then we obtain the single-soliton solution $u(x,t)$

$$u(x,t) = \frac{3c}{2b}k_1^2\sec h^2\left(\frac{\theta_1(x,t)}{2}\right), \tag{3.11}$$

where

$$\theta_1(x,t) = k_1\frac{x^\beta}{\Gamma(1+\beta)} - \frac{ik_1\sqrt{a+ck_1^2}t^\alpha}{\Gamma(1+\alpha)}, \tag{3.12}$$

which is a bell-shaped solitary wave solution of the Boussinesq equation (3.1).

For two-soliton solutions, we set

$$f(X,T) = 1 + e^{\theta_1(X,T)} + e^{\theta_2(X,T)} + a_{12}e^{\theta_1(X,T)+\theta_2(X,T)}, \tag{3.13}$$

where θ_1 and θ_2 are defined in (3.6). Using (3.13) in

$$u(X,T) = \frac{6c}{b}\,(\ln f)_{XX}\,. \tag{3.14}$$

and substituting the result in equation (3.2), we obtain the following phase shift :

$$a_{12} = \frac{2ck_1^2 - 3ck_1k_2 - \sqrt{a+ck_1^2}\sqrt{a+ck_2^2} + 2ck_2^2 + a}{2ck_1^2 + 3ck_1k_2 - \sqrt{a+ck_1^2}\sqrt{a+ck_2^2} + 2ck_2^2 + a}. \tag{3.15}$$

The two-soliton solutions of the Boussinesq equation (3.2) are obtained in an explicit form as

$$\begin{aligned} &u(X,T)\\ &= \frac{6c}{b}\left[\begin{array}{c} k_1^2e^{\theta_1(X,T)} + k_2^2e^{\theta_2(X,T)} + a_{12}\left(k_2^2e^{\theta_1(X,T)} + k_1^2e^{\theta_2(X,T)}\right)e^{\theta_1(X,T)+\theta_2(X,T)} \\ + \left[a_{12}\left(k_1+k_2\right)^2 + \left(k_1-k_2\right)^2\right]e^{\theta_1(X,T)+\theta_2(X,T)} \end{array}\right] \Bigg/ \\ &\quad \left[\left(1+e^{\theta_1(X,T)}+e^{\theta_2(X,T)}+a_{12}e^{\theta_1(X,T)+\theta_2(X,T)}\right)^2\right]. \end{aligned}$$

As a result, we obtain the two-soliton solutions of the fractional Boussinesq equation (3.1)

$$
\begin{aligned}
&u(x,t)\\
&= \frac{6c}{b}\left[\begin{array}{c} k_1^2 e^{\theta_1(x,t)} + k_2^2 e^{\theta_2(x,t)} + a_{12}\left(k_2^2 e^{\theta_1(x,t)} + k_1^2 e^{\theta_2(x,t)}\right) e^{\theta_1(x,t)+\theta_2(x,t)} \\ + \left[a_{12}\left(k_1+k_2\right)^2 + \left(k_1-k_2\right)^2\right] e^{\theta_1(x,t)+\theta_2(x,t)} \end{array}\right] \Bigg/ \\
&\qquad \left[\left(1+e^{\theta_1(x,t)}+e^{\theta_2(x,t)}+a_{12}e^{\theta_1(x,t)+\theta_2(x,t)}\right)^2\right]
\end{aligned} \tag{3.16}
$$

where θ_1 and θ_2 are defined as

$$
\theta_i(x,t) = k_i \frac{x^\beta}{\Gamma(1+\beta)} - \frac{i k_i \sqrt{a+ck_i^2}\, t^\alpha}{\Gamma(1+\alpha)},\ i = 1, 2. \tag{3.17}
$$

The three-soliton solutions can be obtained by using

$$
\begin{aligned}
f(X,T) = 1 &+ e^{\theta_1(X,T)} + e^{\theta_2(X,T)} + e^{\theta_3(X,T)} + a_{12}e^{\theta_1(X,T)+\theta_2(X,T)} \\
&+ a_{13}e^{\theta_1(X,T)+\theta_3(X,T)} + a_{23}e^{\theta_2(X,T)+\theta_3(X,T)} \\
&+ a_{123}e^{\theta_1(X,T)+\theta_2(X,T)+\theta_3(X,T)},
\end{aligned} \tag{3.18}
$$

where

$$
a_{ij} = \frac{2ck_i^2 - 3ck_ik_j - \sqrt{a+ck_i^2}\sqrt{a+ck_j^2} + 2ck_j^2 + a}{2ck_i^2 + 3ck_ik_j - \sqrt{a+ck_i^2}\sqrt{a+ck_j^2} + 2ck_j^2 + a}. \tag{3.19}
$$

Substituting (3.18) and (3.14) into equation (3.2), we find that

$$
a_{123} = a_{12}a_{13}a_{23}. \tag{3.20}
$$

Since (3.20) holds, we can use the results indicated in [31, 32] to conclude that the N-soliton solutions for the fractional Boussinesq equation (3.1) can be obtained for finite N, where $N \geq 1$.

4. Multiple singular soliton solutions for the fractional Boussinesq equation

To obtain a single singular soliton solution, we substitute

$$
u(X,T) = e^{k_iX - \omega_iT}, \tag{4.1}
$$

into the linear terms of equation (3.2). This gives the dispersion relation (3.5) and as a result we obtain

$$
\theta_i(X,T) = k_iX - ik_i\sqrt{a+ck_i^2}T. \tag{4.2}
$$

The singular single soliton solution of equation (3.2) is assumed to be

$$
u(X,T) = R\left(\ln f\right)_{XX} \tag{4.3}
$$

where the auxiliary function $f(X,T)$ is given by

$$f(X,T) = 1 - e^{\theta_1(X,T)} = 1 - e^{k_1 X - ik_1\sqrt{a+ck_1^2}T}. \tag{4.4}$$

Substituting (4.3) into equation (3.2) and solving for R, we find $R = \frac{6c}{b}$. Now, (4.3) gives the single singular soliton solution for the Boussinesq equation (3.2)

$$u(X,T) = -\frac{6c}{b}k_1^2 \frac{e^{k_1 X - ik_1\sqrt{a+ck_1^2}T}}{(e^{k_1 X - ik_1\sqrt{a+ck_1^2}T} - 1)^2}, \tag{4.5}$$

and so

$$u(x,t) = -\frac{6c}{b}k_1^2 \frac{e^{k_1 \frac{x^\beta}{\Gamma(1+\beta)} - \frac{ik_1\sqrt{a+ck_1^2}t^\alpha}{\Gamma(1+\alpha)}}}{(e^{k_1 \frac{x^\beta}{\Gamma(1+\beta)} - \frac{ik_1\sqrt{a+ck_1^2}t^\alpha}{\Gamma(1+\alpha)}} - 1)^2} = -\frac{3c}{2b}k_1^2 \csc h^2\left(\frac{\theta_1}{2}\right),$$

where θ_1 is defined by (3.12). Multiple singular soliton solutions for equation (3.2) can be expressed in the following form:

$$u(X,T) = \frac{6c}{b}\left(\ln f\right)_{XX}. \tag{4.6}$$

To determine the two singular soliton solutions explicitly, we substitute

$$f(X,T) = 1 - e^{\theta_1(X,T)} - e^{\theta_2(X,T)} + a_{12}e^{\theta_1(X,T)+\theta_2(X,T)}, \tag{4.7}$$

where $\theta_1(X,T)$, $\theta_2(X,T)$ are defined in (4.2) into (4.6) and substituting the result in equation (3.2), to obtain the phase shift a_{12} as in (3.15). As a result, we obtain the two singular soliton solutions of the fractional Boussinesq equation (3.1)

$$\begin{aligned} &u(x,t) \\ &= -\frac{6c}{b}\left[\begin{array}{c} k_1^2 e^{\theta_1(x,t)} + k_2^2 e^{\theta_2(x,t)} + a_{12}\left(k_2^2 e^{\theta_1(x,t)} + k_1^2 e^{\theta_2(x,t)}\right) e^{\theta_1(x,t)+\theta_2(x,t)} \\ -\left[a_{12}\left(k_1+k_2\right)^2 + \left(k_1-k_2\right)^2\right] e^{\theta_1(x,t)+\theta_2(x,t)} \end{array}\right] \Big/ \\ &\quad \left[\left(1 - e^{\theta_1(x,t)} - e^{\theta_2(x,t)} + a_{12}e^{\theta_1(x,t)+\theta_2(x,t)}\right)^2\right] \end{aligned}$$

where $\theta_1(x,t)$ and $\theta_2(x,t)$ are defined in (3.17).

For the singular three-soliton solutions we use

$$\begin{aligned} f(X,T) = 1 &- e^{\theta_1(X,T)} - e^{\theta_2(X,T)} - e^{\theta_3(X,T)} + a_{12}e^{\theta_1(X,T)+\theta_2(X,T)} \\ &+ a_{13}e^{\theta_1(X,T)+\theta_3(X,T)} + a_{23}e^{\theta_2(X,T)+\theta_3(X,T)} \\ &- a_{123}e^{\theta_1(X,T)+\theta_2(X,T)+\theta_3(X,T)}, \end{aligned} \tag{4.8}$$

where $\theta_i(X,T)$ $(i = 1,2,3)$ are defined in (4.2) and a_{ij} are defined in (3.19). Proceeding as before, we obtain$a_{123} = a_{12}a_{13}a_{23}$. The singular three-soliton solution can be obtained explicitly using (4.6) for the function f in (4.8).

5. Analysis of the parameters

The solution (3.11) gives a profile of bell-shaped solitary wave with soliton amplitude amp and width Δ can be expressed as

$$amp = \left|\frac{3c}{2b}k_1^2\right|, \qquad \Delta = \frac{2}{|k_1|}. \tag{5.1}$$

With the characteristic-line method [2], [3], [10], the characteristic line for each solitary wave can be defined by

$$x^{\beta} - \frac{i\Gamma(1+\beta)\sqrt{a+ck_i^2}t^{\alpha}}{\Gamma(1+\alpha)} = 0, \qquad i = 1, 2, 3, \ldots, \tag{5.2}$$

which can be derived from relations (3.5) and (3.6). Correspondingly, the velocity v of each solitary wave can be expressed by

$$v_i = \frac{1}{\beta}\frac{i\alpha\Gamma(1+\beta)\sqrt{a+ck_i^2}t^{\alpha-1}}{\Gamma(1+\alpha)}\left(\frac{i\Gamma(1+\beta)\sqrt{a+ck_i^2}t^{\alpha}}{\Gamma(1+\alpha)}\right)^{\frac{1}{\beta}-1}. \tag{5.3}$$

Further, the absolute value of velocity v determines the speed, namely, velocity in magnitude, and propagation direction of soliton is decided by the sign of v.

The analysis is illustrated by studying the time fractional Boussinesq equation

$$u_t^{(2\alpha)} + au_{xx} + b(u)_{xx}^2 + cu_{xxxx} = 0, \tag{5.4}$$

From (3.11) we obtain the single-soliton solution $u(x,t)$

$$u(x,t) = \frac{3c}{2b}k_1^2 \sec h^2\left(\frac{\theta_1(x,t)}{2}\right), \tag{5.5}$$

where

$$\theta_i = k_i x - \frac{ik_i\sqrt{a+ck_i^2}t^{\alpha}}{\Gamma(1+\alpha)}. \tag{5.6}$$

For the single-soliton solution (5.5), it is obvious that the amplitude of the soliton is $\left|\frac{3c}{2b}k_1^2\right|$, which keeps invariant during the propagation. Following the characteristic-line method, the characteristic face for each solitary wave can be defined by

$$x = \frac{i\sqrt{a+ck_i^2}t^{\alpha}}{\Gamma(1+\alpha)}, \qquad i = 1, 2, \ldots, \tag{5.7}$$

which can be derived from relation (5.6). Correspondingly, the velocity of the wave at time t can be expressed as

$$v_i = \frac{i\alpha\sqrt{a+ck_i^2}t^{\alpha-1}}{\Gamma(1+\alpha)}. \tag{5.8}$$

Further, the absolute value of velocity v determines the speed, namely, velocity in magnitude, and propagation direction of soliton is decided by the sign of v.

We now analytically examine the effects of the parameters a, b, c on the behavior of the solitary waves. The soliton amplitude *amp* is dependent on the ratio of c to b. The solitonic amplitude increases with an increase in the ratio c/b. However, Expression (5.8) indicates that the propagation velocity of the solitary wave is influenced by the coefficient parameters a and c, while it is independent of the coefficient parameter b. It can be observed that the initial superposed solitons travel different distances over a period of time for the different choices of a and c, and it can be concluded that the larger the absolute value of v is the greater the speed. Moreover, if the sign of v reverses, the solitonic direction will be changed accordingly.

6. Conclusions

In this paper we succeeded in obtaining single-soliton, two-soliton, and three-soliton solutions of the time-space fractional Boussinesq model. Also, multiple-singular soliton solutions are obtained for the same model. The adopted method being used in this work is a simplified form of the bilinear method combining with a transformation method, where the needed auxiliary functions obtained in Hirota method can be derived without using bilinear forms. The obtained results reveal that the proposed method is effective and powerful tool in studying N-soliton and N-singular-soliton solutions for both classical and fractional nonlinear equations. Furthermore, based on the one-soliton solution in expression (3.11), we have carried out the parametric analysis in order to investigate the effects of the parameters a, b, c on the soliton amplitude, width and velocity. To our knowledge, the solutions we have constructed are new and different from those in the existing papers.

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THE COMPOSITION, CONVERGENCE AND TRANSITIVITY OF POWERS AND ADJOINT OF GENERALIZED FUZZY MATRICES

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Abstract. Path algebras are additively idempotent semirings and generalize Boolean algebras, fuzzy algebras, distributive lattices and inclines. Thus the Boolean matrices, the fuzzy matrices, the lattice matrices and the incline matrices are prototypical examples of matrices over path algebras. In this paper, generalized fuzzy matrices are considered as matrices over path algebras. Compositions of generalized fuzzy matrices are discussed, and a new transitive matrix is constructed from given matrices. Furthermore, the transitivity and the convergent index for powers of generalized fuzzy matrices are studied, some properties of powers are also established through adjoint matrix, and finally the invertibility of a matrix is investigated. Some results obtained here generalize and develop the corresponding ones on fuzzy matrices, lattice matrices and incline matrices shown in the references.

Keywords: generalized fuzzy matrix, compositions, transitivity, convergence, path algebra.

1. Introduction

Generalized fuzzy matrices [24] over a special type of semiring are considered. The semiring is called path algebra (see [19]) (or additively idempotent semring (see [12]). Path algebras are useful tools in diverse domains such as the design of switching circuits, automata theory, information systems, dynamic programming, decision theory and so on [2], [13]. Path algebras generalize Boolean algebra, fuzzy algebra, distributive lattice, incline, max-plus algebra and min-plus algebra. Then, the Boolean matrices, the fuzzy matrices, the lattice matrices and the incline matrices are prototypical examples of generalized fuzzy matrices over path algebras.

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The techniques of matrices over semirings play a very important role in optimization theory, models of discrete event networks, graph theory and so on [1]. The study of matrices over general semirings has a long history. In 1964, Rutherford [30] gave a proof of the Cayley-Hamilton theorem for a commutative semiring avoiding the use of determinants. Since then, a number of works on the theory of matrices over semirings were published (see e.g. [10], [27], [29]). At the same time, these issues about compositions, transitivity, convergence and invertibility of matrices over some special path algebras were discussed by many authors.

The study of compositions of matrices over some special types of semirings also has a long history. In 1985, Hashimoto [18] discussed the problem of decomposition of fuzzy rectangular matrices. In 1995, some properties of compositions of fuzzy matrices were given by Ragab, Emam and Hashimaton (see [8]). In 2002, Tan generalized some previous results related to compositions of fuzzy matrices to the matrices over distributive lattices (see [32]). In 2011, Jiang et al. [22] obtained some properties of compositions of transitive matrices over inclines. In this paper, we continue to concentrate on the compositions of generalized fuzzy matrices over path algebras.

Many results about properties of powers of matrices over some special semirings have been obtained (see e.g. [19], [17]-[33], [35], [16], [14], [34], [9]-[5]). Transitive matrix is an important type of generalized fuzzy matrices and it represents transitive relation (see e.g. [7],[11]). Transitive relation is widely used in clustering and information retrieval preference (see e.g. [28], [31], [20]). The transitivity of matrices over some special semirings has been studied in [19], [17]-[35], [14], [9], [22], [36]. In 1982, The concept of transitive binary Boolean matrices was introduced by Kim (see [23]). Hashimoto [17] presented the concept of transitive fuzzy matrices and considered the convergence of powers of transitive fuzzy matrices. Tan [37], [35] discussed the convergence of powers of transitive lattice matrices. In 2011, he [33] also described the transitive closure of a generalized fuzzy matrix with periods and obtained an expression for the transitive closure. Jiang et al. [22] discussed the transitive closure, the convergence and the composition of incline transitive matrix. In 2013, Tan [36] gave some elementary characteristics of transitive matrices and obtained some properties of the transitive closure of matrix over path algebra.

The convergence of powers of matrices over some special semirings has been considered by many authors, see e.g. [17], [6], [33], [35], [14], [38]-[22]. In 1977, Thomason [38] proved that the power sequence of a fuzzy matrix either converges in finite steps or oscillates with finite period. Since then, many authors have studied the convergence of powers of fuzzy matrices (see e.g. [26]-[21]). In [35], [4], some previous results related to the index and period of a fuzzy matrix were generalized to the matrices over distributive lattices and some results about the index and period were developed. In [22], Han and Li generalized some previous results about the index and period to the matrices over inclines. Recently, Tan [33] studied the generalized fuzzy matrices over commutative path algebras with periodic elements.

The property of adjoint matrix seems to have appeared firstly in the work of Ragab et al. [9], in [9], some properties of a square fuzzy matrix, such as reflexivity, transitivity, circularity and construction, have been proposed through the adjoint matrix. In 2004, Han et al. [15] studied the inverse of incline matrix from the adjoint matrix. Duan [6] proved that A^{n-1} was equal to the adjoint matrix of A when $A \geq I_n$. In 2013, Tan [36] discussed the transitivity of generalized fuzzy matrix with the adjoint matrix over path algebra.

In this paper, we continue to study the matrices over more general path algebras, namely over a class of commutative path algebras. In Section 3, some properties of compositions of generalized fuzzy matrices are obtained, and a new transitive matrix such as the idempotent matrix is constructed from given matrices. In Section 4, the power sequence of generalized fuzzy matrices is studied in detail. Firstly, the transitivity of generalized fuzzy matrices is discussed. As a result, we get a result $t(A) = A^s$, where A^s is a transitive matrix and the result simplifies the expression of the transitive closure $t(A)$. Secondly, the convergent index for powers of generalized fuzzy matrices is considered. Finally, some properties of powers are established with the adjoint matrix. In section 5, the invertibility of a matrix is also investigated, and some necessary and sufficient conditions for a matrix to be invertible are studied. Some results obtained in this paper generalize the corresponding results in [32], [6], [18], [16], [34], [9], [22].

2. Definitions and preliminaries

At this section, we shall give some definitions and lemmas.

Definition 2.1. [12] An algebraic system $(L, +, \cdot)$ is called a semiring if $(L, +)$ is an Abelian monoid with identity element 0 and $(L, \cdot)$ is another monoid with identity element 1 and $0 \neq 1$, connected by ring-like distributivity, and $0r = r0 = 0$ for all $r \in L$.

A semiring L is called commutative if $ab = ba$ for all $a, b \in L$; L is called an antiring if $a + b = 0$ implies that $a = b = 0$ for all $a, b \in L$. Antirings have been studied in [12] under the name of zerosumfree semirings.

Definition 2.2. [19], [2] A semiring L is called a path algebra if $a + a = a$ for all $a \in L$.

Path algebras have been studied in [12] under the name of additively idempotent semiring. It is easy to see that every path algebra is an antiring. Path algebras are abundant, for example, every Boolean algebra, the fuzzy algebra $F = ([0, 1], \vee, \top)$, where $\vee = max$ and $\top$ is a $t-$norm (for $t-$norm, refer to [25]), every bounded distributive lattice, any incline [12], the max-plus algebra $(R \cup \{-\infty\}, \vee, +)$ and the min-plus algebra $(R \cup \{+\infty\}, \wedge, +)$ [39].

In this paper, the path algebra $(L, +, \cdot)$ is always assumed to be a commutative path algebra with the least element 0. A path algebra L is commutative if $ab = ba$ for all $a, b \in L$. Boolean algebra, bounded distributive lattice, incline,

max-plus algebra $(R \cup \{-\infty\}, \vee, +)$ and min-plus algebra $(R \cup \{+\infty\}, \wedge, +)$ are commutative path algebras.

In a path algebra L, the relation $\leq$ is defined by $x \leq y$ if $x + y = y$. It is easy to see that $\leq$ is a partial order relation over L and satisfies the following properties:

Proposition 2.1. *Let L be a path algebra and $a, b, c, d \in L$, then*

(1) $a \geq 0$;

(2) *if $a \leq b$, then $a + c \leq b + c$, $ac \leq bc$ and $ca \leq cb$;*

(3) *if $a \leq b$ and $c \leq d$, then $a + c \leq b + d$ and $ac \leq bd$;*

(4) *$a \leq a + b$, and $a + b$ is the least upper bound of a and b, in other words, if there is an element c satisfying $a \leq c$ and $b \leq c$, then $a + b \leq c$;*

(5) *$a + b = 0$ if and only if $a = b = 0$.*

Let L be a path algebra. An element a in L is said to be idempotent if $a^2 = a$. The set of all idempotent elements in L is denoted by $I(L)$, i.e., $I(L) = \{a \in L \mid a^2 = a\}$. An element a in L is said to be nilpotent if $a^k = 0$ for some positive integer k. The least positive integer k satisfying $a^k = 0$ is called the nilpotent index of a and denoted by $h(a)$. An element a in L is said to be almost periodic if there exist positive integers k and d such that $a^k = a^{k+d}$. The least such positive integers k and d are called the index and the period of a, and denoted by $k(a)$ and $d(a)$, respectively.

A matrix is called a generalized fuzzy matrix if its entries belong to a path algebra. Let $M_{m\times n}(L)$ be the set of all $m \times n$ matrices over L. Especially, put $M_n(L) = M_{n\times n}(L)$ and $V_n(L) = M_{n\times 1}(L)$. The elements in $V_n(L)$ are called the column vectors of order n over L. For any A in $M_{m\times n}(L)$, we denote by a_{ij} or A_{ij} the element of L which stands in the (i, j)-entry of A, and denote by A^T the transpose of A. We shall use Z_+ to denote the set of all positive integers. For convenience, we define:

$\underline{n} = \{1, 2, ..., n\}$;
$\underline{n}_i = \{1, 2, ..., n\}/\{i\} = \{1, 2, ..., i-1, i+1, ..., n\}$;
$a^k = \underbrace{a \cdot a \cdots a}_{k \ \ times}$.
For any $A,\ B \in M_{m\times n}(L)$ and $C \in M_{n\times l}(L)$, we define:
$A + B = (a_{ij} + b_{ij})_{m\times n}$;
$AC = \left(\sum_{k=1}^{n} a_{ik}c_{kj}\right)_{m\times l}$;
$A^T = D$ if and only if $d_{ij} = a_{ji}$ for all $i \in \underline{n}$ and $j \in \underline{m}$;
$kA = B$ $(k \in L)$ if and only if $b_{ij} = ka_{ij}$ for all $i \in \underline{m}$ and $j \in \underline{n}$;
$A \leq B$ if and only if $a_{ij} \leq b_{ij}$ for all $i \in \underline{m}$ and $j \in \underline{n}$;
$I_n = (t_{ij})$, where $t_{ij} = \begin{cases} 1, i = j \\ 0, i \neq j \end{cases}$ for $i, j \in \underline{n}$.

For any A in $M_n(L)$, the powers of A are defined as follows:
$A^0 = I_n$, $A^l = A^{l-1}A$, $l \in Z_+$.
The (i,j)-entry of A^l is denoted by $a_{ij}^{(l)}$ $(l \in Z_+)$, and obviously

$$a_{ij}^{(l)} = \sum_{1 \le i_1, i_2, \ldots, i_{l-1} \le n} a_{ii_1} a_{i_1 i_2} \cdots a_{i_{l-1} j}.$$

It is easy to see that $(M_n(L), +, \cdot)$ is a path algebra, and we have that $A + C \le B + D$ and $AC \le BD$ for all $A, B, C, D \in M_n(L)$ with $A \le B$ and $C \le D$.

Let $A \in M_n(L)$, A is said to be almost periodic if there exist positive integers k and d such that $A^k = A^{k+d}$. The least positive integers k and d are called the index and the period of A, and denoted by $k(A)$ and $d(A)$, respectively. In particular, if $d(A) = 1$ then A is said to converge in a finite number of steps. If A converges to the zero matrix in a finite number of steps, then A is said to be nilpotent. The least positive integer k satisfying $A^k = 0$ is called the nilpotent index of A, and denoted by $h(A)$.

Let $A = (a_{ij}) \in M_n(L)$. If $A^2 \le A$, then A is called transitive; if $A^2 = A$, then A is called idempotent (any idempotent matrix is transitive); if $A^T = A$, then A is called symmetric; if $A \ge I_n$, then A is called reflexive; if $a_{ii} = 0$ for all $i \in \underline{n}$, then A is called irreflexive; A is called a permutation matrix if exact one of the elements of its every row and every column is 1 and the others are 0.

Let $B \in M_n(L)$. The matrix B is called the transitive closure of A if B is transitive and $A \le B$ and if for any transitive matrix C in $M_n(L)$ satisfying $A \le C$ we have $B \le C$. The transitive closure of A is denoted by $t(A)$. It is clear that if A has a transitive closure then it is unique. However, for general path algebra L, there may be some matrices which have no transitive closure over L.

Definition 2.3. Let $A = (a_{ij}) \in M_{m \times n}(L)$ and $m \le n$. The determinant $\det(A)$ of A is defined by

$$det(A) = \sum_{\sigma \in s} \prod_{i \in \underline{m}} a_{i\sigma(i)},$$

where s is the set of all injective mappings from $\underline{m}$ to $\underline{n}$.

If $A \in M_n(L)$, then the determinant $\det(A)$ is the sum of $n!$ terms which are all possible products of n entries of A taken exactly one in each row and each column.

Definition 2.4. *Let $A = (a_{ij}) \in M_n(L)$. The adjoint matrix $adj(A) \in M_n(L)$ of A is the matrix whose (i,j)-entry is $det(A(j|i))$ for all $i, j \in \underline{n}$, where $A(j|i)$ is the matrix of order $n-1$ formed by deleting row j and column i from A.*

3. Compositions of generalized fuzzy matrices

In this section, some properties of compositions of generalized fuzzy matrices are given.

Definition 3.1. A matrix $A = (a_{ij}) \in M_n(L)$ is said to be

(1) row diagonally dominant if $a_{ij} = a_{ii}a_{ij}$ for all $i, j \in \underline{n}$;

(2) column diagonally dominant if $a_{ij} = a_{ij}a_{jj}$ for all $i, j \in \underline{n}$;

(3) weakly diagonally dominant if for any $i \in \underline{n}$, either $a_{ii}a_{ij} = a_{ij}$ for all $j \in \underline{n}$ or $a_{ii}a_{ji} = a_{ji}$ for all $j \in \underline{n}$;

(4) strongly diagonally dominant if $a_{ij} = a_{ii}a_{ij} = a_{ij}a_{jj}$ for all $i, j \in \underline{n}$;

(5) nearly irreflexive if $a_{ii}a_{kj} = a_{ii}$ for all $i, j, k \in \underline{n}$.

Theorem 3.1. *Let $A = (a_{ij}) \in M_n(L)$. If A is transitive and row diagonally dominant, then $C = (c_{ij})$ is idempotent, where $c_{ij} = a_{ij}a_{ji}$.*

Proof. For any $i, j \in \underline{n}$,

$$c_{ij}^{(2)} = \sum_{k=1}^{n} c_{ik}c_{kj} = \sum_{k=1}^{n} a_{ik}a_{ki}a_{kj}a_{jk} = \sum_{k=1}^{n} (a_{ik}a_{kj})(a_{jk}a_{ki}) \leq \sum_{k=1}^{n} (a_{ij}a_{ji})$$

(because A is transitive) $= \sum_{k=1}^{n} c_{ij} = c_{ij}$. Thus, $C^2 \leq C$. On the other hand, since $c_{ij}^{(2)} = \sum_{k=1}^{n} c_{ik}c_{kj} \geq c_{ii}c_{ij} = a_{ii}a_{ii}a_{ij}a_{ji} = a_{ij}a_{ji}$ (because A is row diagonally dominant) $= c_{ij}$, we have $C^2 \geq C$. Therefore, $C^2 = C$. This completes the proof. ∎

Definition 3.2. Let $A = (a_{ik}) \in M_{m\times n}(L)$, $D = (d_{kj}) \in M_{n\times l}(L)$. We define $A \circ D = C$ if and only if $c_{ij} = \prod_{k=1}^{n} (a_{ik} + d_{kj})$ for any $i \in \underline{m}, k \in \underline{n}$ and $j \in \underline{l}$.

Proposition 3.1. *Let $A = (a_{ij}) \in M_n(L)$ be a symmetric and nearly irreflexive matrix. Then $A \circ A$ is symmetric and nearly irreflexive.*

Proof. Let $S = A \circ A$. Then

$$s_{ji} = \prod_{l=1}^{n} (a_{jl} + a_{li}) = \prod_{l=1}^{n} (a_{il} + a_{lj}) = s_{ij},$$

so that S is symmetric. Also,

$$\begin{aligned} s_{ii}s_{kj} &= \left[\prod_{l=1}^{n} (a_{il} + a_{li})\right] \left[\prod_{l=1}^{n} (a_{kl} + a_{lj})\right] = \prod_{l=1}^{n} a_{il} \left[\prod_{l=1}^{n} (a_{kl} + a_{lj})\right] \\ &= a_{ii} \left[\prod_{l=1}^{n} (a_{kl} + a_{lj})\right] = a_{ii} = \prod_{l=1}^{n} a_{il} = \prod_{l=1}^{n} (a_{il} + a_{li}) = s_{ii}, \end{aligned}$$

and so S is nearly irreflexive. This completes the proof. ∎

Proposition 3.1 generalizes Proposition 3.1 of Tan [32].

Proposition 3.2. *Let $A, B \in M_{m\times n}(L), C \in M_{n\times l}(L)$ and $D \in M_{p\times m}(L)$. Then*

(1) $(B \circ C)^T = C^T \circ B^T$;

(2) *If $A \leq B$, then $D \circ A \leq D \circ B$ and $A \circ C \leq B \circ C$.*

The proof is trivial.

Proposition 3.3. *Let $A = (a_{ij}) \in M_{m\times n}(L)$ be a symmetric and nearly irreflexive matrix. Then $A \circ A^T$ is symmetric and nearly irreflexive.*

Proof. Let $R = A \circ A^T$. Then

$$r_{ji} = \prod_{l=1}^{n}(a_{jl} + a_{il}) = \prod_{l=1}^{n}(a_{il} + a_{jl}) = r_{ij},$$

so that R is symmetric. Furthermore, since

$$\begin{aligned} r_{ii}r_{kj} &= \left[\prod_{l=1}^{n}(a_{il} + a_{il})\right]\left[\prod_{l=1}^{n}(a_{kl} + a_{jl})\right] = \prod_{l=1}^{n} a_{il}\left[\prod_{l=1}^{n}(a_{kl} + a_{lj})\right] \\ &= a_{ii}\left[\prod_{l=1}^{n}(a_{kl} + a_{lj})\right] \text{ (because } A \text{ is nearly irreflexive)} \\ &= a_{ii} = \prod_{l=1}^{n} a_{il} \text{ (because } A \text{ is nearly irreflexive)} \\ &= \prod_{l=1}^{n}(a_{il} + a_{il}) = r_{ii}, \end{aligned}$$

we draw the conclusion that $R = A \circ A^T$ is nearly irreflexive. This completes the proof. ∎

Proposition 3.3 generalizes Theorem 3.3 of Tan [32] and Theorem 3.3 of Ragab and Emam [8].

Theorem 3.2. *Let $A = (a_{ij}) \in M_n(L)$ be a symmetric and nearly irreflexive matrix. Then the matrix $R = B \circ A$ is idempotent and constant, where*

$$B = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}, \ a_{ii} \in L, i \in \underline{n}.$$

Proof. Because the hypothesis $a_{ii}a_{kj} = a_{ii}$ for all $i, j, k \in \underline{n}$, we have $a_{11} = a_{22} = \cdots = a_{nn}$. Furthermore, based on the definitions of the matrices A, B and R, we have

$$\begin{aligned} r_{ij} &= \left(\prod_{k\neq i} a_{kj}\right)(a_{ii} + a_{ij}) = \left(\prod_{k\neq i} a_{jk}\right)(a_{ii} + a_{ij}) \\ &= a_{ii} + \prod_{k\neq i} a_{jk}a_{ij} \text{(because } A \text{ is nearly irreflexive)} \\ &= a_{ii} = a_{jj}. \end{aligned}$$

For any $i,j\in \underline{n}$, $r_{ij}^{(2)} = \sum_{k=1}^{n} r_{ik}r_{kj} = \sum_{k=1}^{n} a_{ii}a_{jj} = a_{ii} = a_{jj}$. Thus, $R^2 = R = (a_{ii})_{n\times n}$. Therefore, R is idempotent and constant. This completes the proof of the theorem. ∎

Theorem 3.2 generalizes Theorem 3.5 of Tan [32] and Corollary 3.6 of Ragab and Emam [8].

Theorem 3.3. *Let $A = (a_{ij}) \in M_{m\times n}(L)$ be a symmetric and nearly irreflexive matrix. Then $I_m \circ (A \circ A^T)$ is idempotent, where $A = (a_{ij})$ with $(a_{it}+a_{kj})(a_{it}+a_{kj}) = (a_{it}+a_{kj})(i,k \in \underline{m}; t,j \in \underline{n})$.*

Proof. Let $B = A \circ A^T$ and $S = I_m \circ (A \circ A^T)$. Then

$$s_{ij} = \prod_{l\neq i} b_{lj}(1+b_{ij}) = \prod_{l\neq i} b_{lj} + b_{ij}\prod_{l\neq i} b_{lj}$$
$$= \prod_{l\neq i} b_{lj} + \prod_{l=1}^{n} b_{lj} = \prod_{l\neq i} b_{lj} + b_{jj} = \begin{cases} b_{jj} + \prod_{l\neq i} b_{lj} = b_{ii} + \prod_{l\neq i} b_{li}, i = j, \\ b_{jj}, i \neq j. \end{cases}.$$

The $(i,j)th$ entry of S^2 is $s_{ij}^{(2)} = \sum_{k=1}^{n} s_{ik}s_{kj}$.

Case 1: $i \neq j$. In this case, we have

$$s_{ij}^{(2)} = \sum_{k\neq i,j} s_{ik}s_{kj} + s_{ii}s_{ij} + s_{ij}s_{jj}$$
$$= \sum_{k\neq i,j} b_{kk}b_{jj} + \left(\prod_{l\neq i} b_{li} + b_{ii}\right) b_{jj} + b_{jj}\left(\prod_{l\neq j} b_{lj} + b_{jj}\right)$$
$$= b_{jj} + b_{jj} + b_{jj} \text{ (by Proposition 3.3, } B \text{ is nearly irreflexive)}$$
$$= b_{jj} = s_{ij},$$

Case 2: $i = j$. In this case, we have

$$s_{ii}^{(2)} = \sum_{l\neq i} s_{il}s_{li} + s_{ii}s_{ii} = \sum_{l\neq i} b_{ll}b_{ii} + \left(\prod_{l\neq i} b_{li} + b_{ii}\right)\left(\prod_{l\neq i} b_{li} + b_{ii}\right)$$
$$= b_{ii} + \prod_{l\neq i} b_{li}\prod_{l\neq i} b_{li} = b_{ii} + \prod_{l\neq i} b_{li} \text{ (since } A = (a_{ij})$$
$$\text{with } (a_{it}+a_{kj})(a_{it}+a_{kj}) = (a_{it}+a_{kj})(i,k \in \underline{m}; t,j \in \underline{n}))$$
$$= s_{ii}.$$

Thus, $S^2 = S$. Therefore, $S = I_m \circ (A \circ A^T)$ is idempotent. This completes the proof of the theorem. ∎

Theorem 3.3 generalizes Corollary 3.7(1) of Tan [32].

Corollary 3.1. *Let $A = (a_{ij}) \in M_{m\times n}(L)$ be a symmetric and nearly irreflexive matrix. Then $(A\circ A^T)\circ I_m$ is idempotent, where $A = (a_{ij})$ with $(a_{it}+a_{kj})(a_{it}+a_{kj}) = (a_{it}+a_{kj})(i,k \in \underline{m}; t,j \in \underline{n})$.*

Proof. The proof of Corollary 3.1 is similar to that of Theorem 3.3. This completes the proof. ∎

Corollary 3.1 generalizes Corollary 3.8 of Tan [32].

Theorem 3.4. *Let $A = (a_{ij}) \in M_n(L)$ be a symmetric and nearly irreflexive matrix. Then $I_n\circ(A\circ A)$ is idempotent, where $A = (a_{ij})$ with $(a_{im}+a_{kj})(a_{im}+a_{kj}) = (a_{im}+a_{kj})(i,m,k,j \in \underline{n})$.*

Proof. The proof of Theorem 3.4 is similar to that of Theorem 3.3. This completes the proof. ∎

Corollary 3.2. *Let $A = (a_{ij}) \in M_n(L)$ be a symmetric and nearly irreflexive matrix. Then $(A\circ A)\circ I_n$ is idempotent, where $A = (a_{ij})$ with $(a_{im}+a_{kj})(a_{im}+a_{kj}) = (a_{im}+a_{kj})(i,m,k,j \in \underline{n})$.*

Proof. The proof of Corollary 3.2 is similar to that of Theorem 3.3. This completes the proof. ∎

Theorem 3.5. *Let $A = (a_{ij}) \in M_{m\times n}(L)$ be a symmetric and nearly irreflexive matrix. Then $I_m \circ A$ is transitive, where $A \geq A^2$.*

Proof. Let $T = I_m \circ A$. Then

$$t_{ij} = \prod_{l\neq i} a_{lj}(1+a_{ij}) = \prod_{l\neq i} a_{lj} + a_{ij}\prod_{l\neq i} a_{lj} = \prod_{l\neq i} a_{lj} + a_{jj}$$
$$= \begin{cases} a_{jj} + \prod_{l\neq i} a_{lj} = a_{ii} + \prod_{l\neq i} a_{li}, i = j, \\ a_{jj}, i \neq j. \end{cases}.$$

The $(i,j)th$ entry of T^2 is $t_{ij}^{(2)} = \sum_{k=1}^{n} t_{ik}t_{kj}$.

Case 1: $i \neq j$. In this case, we have

$$t_{ij}^{(2)} = \sum_{k\neq i,j} t_{ik}t_{kj} + t_{ii}t_{ij} + t_{ij}t_{jj}$$
$$= \sum_{k\neq i,j} a_{kk}a_{jj} + \left(\prod_{l\neq i} a_{li} + a_{ii}\right) a_{jj} + a_{jj}\left(\prod_{l\neq j} a_{lj} + a_{jj}\right)$$
$$= a_{jj} = t_{ij},$$

Case 2: $i = j$. In this case, we have

$$\begin{aligned} t_{ii}^{(2)} &= \sum_{l\neq i} t_{il}t_{li} + t_{ii}t_{ii} = \sum_{l\neq i} a_{ll}a_{ii} + \left(\prod_{l\neq i} a_{li} + a_{ii}\right)\left(\prod_{l\neq i} a_{li} + a_{ii}\right) \\ &= a_{ii} + \prod_{l\neq i} a_{li} \prod_{l\neq i} a_{li} \text{ (because } A \text{ is nearly irreflexive)} \\ &= a_{ii} + \prod_{l\neq i} a_{il} \prod_{l\neq i} a_{li} \text{ (since } A \text{ is symmetric)} \\ &\leq a_{ii} + a_{ii} \text{ (since } A \text{ is transitive)} \\ &= a_{ii} \leq a_{ii} + \prod_{l\neq i} a_{li} = t_{ij}, \end{aligned}$$

Thus, $T^2 \leq T$. Therefore, $T = I_n \circ A$ is transitive. This completes the proof of the theorem. ∎

Theorem 3.6. *Let $A = (a_{ij}) \in M_n(L)$ be an irreflexive and transitive. Then*

(1) $A \circ A^T = 0$;

(2) $A^T \circ A = 0$.

Proof. (1) Let $R = A \circ A^T$. Then

$$\begin{aligned} r_{ij} &= \prod_{k=1}^{n}(a_{ik} + a_{jk}) = (a_{i1} + a_{j1})(a_{i2} + a_{j2}) \cdots (a_{in} + a_{jn}) \\ &\leq \sum_{k\in N} a_{ij}a_{ji}a_{ik} + \sum_{k,l\in N} a_{ij}a_{ji}a_{ik}a_{jl} + \sum_{l\in N} a_{ij}a_{ji}a_{jl} \\ &\leq \sum_{k\in N} a_{ii}a_{ik} + \sum_{k,l\in N} a_{ii}a_{ik}a_{jl} + \sum_{l\in N} a_{ii}a_{jl} \text{ (since } A \text{ is transitive)} \\ &= 0. \end{aligned}$$

Thus, $R = 0$.

(2) The proof of (2) is similar to that of (1). This completes the proof of the theorem. ∎

Theorem 3.6 generalizes Proposition 3.9 of Tan [32].

Definition 3.3. A path algebra $\widetilde{L}$ is said to be a Brouwerian path algebra if for any $a, b \in L$, there exists an element $b \longrightarrow a \in \widetilde{L}$ such that $bx \leq a \Leftrightarrow x \leq b \longrightarrow a$.

Definition 3.4. $A \leftarrow D = C$ if and only if $c_{ij} = \prod_{k=1}^{n}(d_{kj} \rightarrow a_{ik})$ for any $i, j \in \underline{n}$.

Lemma 3.1. *Let $\widetilde{L}$ be a Brouwerian path algebra. Then, for any $a \in \widetilde{L}$, $1 \rightarrow a = a$.*

Proof. By $1x = x$ for any $x \in \widetilde{L}$, the proof is trivial. ∎

Theorem 3.7. *Let $A \in M_{m\times n}(\widetilde{L})$. Then $A \leftarrow A^T$ is transitive.*

Proof. Let $R = A \leftarrow A^T$. Then $r_{ij} = \prod_{k=1}^{n}(a_{jk} \to a_{ik})$ for any $i, j \in \underline{n}$. Since $a_{jk}(a_{lk} \to a_{ik})(a_{jk} \to a_{lk}) = (a_{lk} \to a_{ik})(a_{jk}(a_{jk} \to a_{lk}) \leq (a_{lk} \to a_{ik})a_{lk} \leq a_{ik}$, we have $(a_{lk} \to a_{ik})(a_{jk} \to a_{lk}) \leq a_{jk} \to a_{ik}$. Thus, $r_{il}r_{lj} = \prod_{k=1}^{n}(a_{lk} \to a_{ik})(a_{jk} \to a_{lk}) \leq \prod_{k=1}^{n}(a_{jk} \to a_{ik}) = r_{ij}$, and so $r_{ij}^{(2)} = \sum_{l=1}^{n} r_{il}r_{lj} \leq \sum_{l=1}^{n} r_{ij} = r_{ij}$. Therefore, $R^2 \leq R$. This proves the theorem. ∎

Theorem 3.7 generalizes Lemma 4.1 of Tan [32].

Theorem 3.8. *Let $A \in M_n(I(\widetilde{L}))$is transitive. Then $A \leq A \leftarrow A^T$.*

Proof. Let $T = A \leftarrow A^T$. Then $t_{ij} = \prod_{k=1}^{n}(a_{jk} \to a_{ik})$ for all $i, j \in \underline{n}$. Since A is transitive, we have $a_{ik} \geq a_{ij}a_{jk}$ for all $i, j, k \in \underline{n}$. Thus, $a_{ij} \leq (a_{jk} \to a_{ik})$, and so $a_{ij} \leq \prod_{k=1}^{n}(a_{jk} \to a_{ik}) = t_{ij}$ (because $A \in M_n(I(\widetilde{L}))$). Therefore, $A \leq A \leftarrow A^T$. This proves the theorem. ∎

4. Basic properties of powers of generalized fuzzy matrices

This section studies the power sequence of generalized fuzzy matrices in detail. First, the reduction of the transitive closure of a matrix with a period over L is posed. Second, the convergent index for powers of generalized fuzzy matrices is discussed. Finally, some properties of powers are established by adjoint matrix.

Lemma 4.1. [33] *Let $A \in M_n(L)$ be almost periodic, then A has transitive closure and $t(A) = \sum_{k=1}^{m} A^k$ for all $m \geq k(A) + d(A) - 1$.*

Lemma 4.2. [33] *Let $A \in M_n(L)$ be almost periodic, then there exist a positive integers s such that A^s is idempotent.*

Proof. Since A is almost periodic, we have $A^k = A^{k+d}$ for some positive integers k and d. Then $A^{(k+t)d} = A^{(k+t-1)d}$ for all $t = 1, 2, \cdots, k$. Therefore, $(A^{kd})^2 = A^{(k+k)d} = A^{(2k-1)d} = A^{(2k-2)d} = \cdots = A^{kd}$. Taking $s = kd$, we can get that A^s is idempotent. ∎

Proposition 4.1. *Let $A = (a_{ij}) \in M_n(L)$ be almost periodic. If A is row (or column) diagonally dominant, then $t(A) = A^s$, where A^s is idempotent.*

Proof. We first consider the case A is row diagonally dominant. By Lemma 4.2, A^s is idempotent, then $A^{ks} = A^s$ for any integer $k > 0$, and so $a_{ij}^{(ks)} = a_{ij}^{(s)}$,

$\forall i, j \in \underline{n}$. If A is row diagonally dominant, then $a_{ij}^{(k)} = a_{ii} \cdots a_{ii} a_{ij}^{(k)} = a_{ii}^{ks-k} a_{ij}^{(k)} \leq a_{ij}^{(ks)}$ (since $a_{ij}^{ks-k} a_{ij}^{(k)}$ is the sum of some term in $a_{ij}^{(ks)}$). Thus, $a_{ij}^{(k)} \leq a_{ij}^{(s)}$. Hence, $\sum_{k=1}^{m} a_{ij}^{(k)} \leq a_{ij}^{(s)} (\forall m \geq k(A) + d(A) - 1)$, then $t(A) \leq A^s$. On the other hand, since $t(A) = \sum_{k=1}^{m} A^k$for all $m \geq k(A) + d(A) - 1$, we have $t(A) \geq A^s$. Therefore, $t(A) = A^s$.

The proof of the case that A is column diagonally dominant is similar. This completes the proof. ∎

Corollary 4.1. *Let $A = (a_{ij}) \in M_n(L)$ be almost periodic. If A is strongly diagonally dominant, then $t(A) = A^s$, where A^s is transitive.*

Proof. Since A is strongly diagonally dominant, we have A is column (or row) diagonally dominant. Therefore, the conclusion is from Proposition 4.1. ∎

Proposition 4.2. *Let $A = (a_{ij}) \in M_n(L)$ be almost periodic. If A is weakly diagonally dominant, then $t(A) = A^s$, where A^s is idempotent.*

Proof. By Lemma 4.2, A^s is idempotent, then $A^{ks} = A^s$ for any integer $k > 0$, and so $a_{ij}^{(ks)} = a_{ij}^{(s)}$ for any $i, j \in \underline{n}$. If A is weakly diagonally dominant, then for any $i \in \underline{n}$, either $a_{ii} a_{ij} = a_{ij}$ for all $j \in \underline{n}$ or $a_{ii} a_{ji} = a_{ji}$ for all $j \in \underline{n}$.

Let $a_{ij}^{(k)} = \sum_{1 \leq i_1, \cdots, i_{k-1} \leq n} a_{ii_1} a_{i_1 i_2} \cdots a_{i_{k-1} j}$, $(\forall i, j \in \underline{n})$. If $a_{i_1 i_1} a_{i_1 j} = a_{i_1 j}$ for all $j \in \underline{n}$, we have $a_{i_1 i_1} a_{i_1 i_2} = a_{i_1 i_2}$. Then

$$\begin{aligned} a_{ij}^{(k)} &= \sum_{1 \leq i_1, \cdots, i_{k-1} \leq n} a_{ii_1} a_{i_1 i_2} \cdots a_{i_{k-1} j} = \sum_{1 \leq i_1, \cdots, i_{k-1} \leq n} a_{ii_1} a_{i_1 i_1} \cdots a_{i_1 i_1} a_{i_1 i_2} \cdots a_{i_{k-1} j} \\ &= \sum_{1 \leq i_1, \cdots, i_{k-1} \leq n} a_{ii_1} a_{i_1 i_1}^{ks-k} a_{i_1 i_2} \cdots a_{i_{k-1} j} \\ &\leq a_{ij}^{(ks)} (\text{because} \sum_{1 \leq i_1, \cdots, i_{k-1} \leq n} a_{ii_1} a_{i_1 i_1}^{ks-k} a_{i_1 i_2} \cdots a_{i_{k-1} j} \\ &\qquad \text{is the sum of some term in } a_{ij}^{(ks)}) \\ &= a_{ij}^{(s)}. \end{aligned}$$

If $a_{i_1 i_1} a_{j i_1} = a_{j i_1}$, for all $j \in \underline{n}$, we have $a_{i_1 i_1} a_{i i_1} = a_{i i_1}$. Then

$$\begin{aligned} a_{ij}^{(k)} &= \sum_{1 \leq i_1, \cdots, i_{k-1} \leq n} a_{ii_1} a_{i_1 i_2} \cdots a_{i_{k-1} j} = \sum_{1 \leq i_1, \cdots, i_{k-1} \leq n} a_{i_1 i_1} \cdots a_{i_1 i_1} a_{i i_1} a_{i_1 i_2} \cdots a_{i_{k-1} j} \\ &= \sum_{1 \leq i_1, \cdots, i_{k-1} \leq n} a_{i_1 i_1}^{ks-k} a_{i i_1} a_{i_1 i_2} \cdots a_{i_{k-1} j} = \sum_{1 \leq i_1, \cdots, i_{k-1} \leq n} a_{ii_1} a_{i_1 i_1}^{ks-k} a_{i_1 i_2} \cdots a_{i_{k-1} j} \\ &\leq a_{ij}^{(ks)} \text{ (because} \sum_{1 \leq i_1, \cdots, i_{k-1} \leq n} a_{ii_1} a_{i_1 i_1}^{ks-k} a_{i_1 i_2} \cdots a_{i_{k-1} j} \\ &\qquad \text{is the sum of some term in } a_{ij}^{(ks)}) \\ &= a_{ij}^{(s)}. \end{aligned}$$

From above, we have $a_{ij}^{(k)} \leq a_{ij}^{(s)}$ for any $i, j \in \underline{n}$. Hence, $\sum_{k=1}^{m} a_{ij}^{(k)} \leq a_{ij}^{(s)}$ for all $m \geq k(A)+d(A)-1$, and then $t(A) \leq A^s$. On the other hand, since $t(A) = \sum_{k=1}^{m} A^k$ for all $m \geq k(A) + d(A) - 1$, we have $t(A) \geq A^s$. Therefore, $t(A) = A^s$. ∎

Proposition 4.3. *If $t(A)$ is reflexive, then $t(t(A)) = t(A)$.*

Proof. By the definition of $t(A)$, $t(A) \geq (t(A))^2$. Since $t(A)$ is reflexive, we have $t(A) \geq I_n$, and so $(t(A))^2 \geq t(A)$. Hence, $(t(A))^2 = t(A)$. Therefore, $t(t(A)) = t(A)$. This completes the proof. ∎

In [37], Tan studied the convergence for powers of a lattice matrix. In the following, the convergent index for powers of transitive generalized fuzzy matrices over a path algebra is discussed.

Proposition 4.4. *Let $A = (a_{ij}) \in M_n(L)$ be almost periodic, then*

(1) $A^m \leq \sum_{k=1}^{k(A)+d(A)-1} A^k$ *for any integer $m \geq 1$;*

(2) $\sum_{l=1}^{+\infty} A^l = \sum_{l=1}^{k(A)+d(A)-1} A^l$.

Proof. Since A is almost periodic, the proof is obvious.

Theorem 4.1. *Let $A = (a_{ij}) \in M_n(L)$. If $A \geq A^2$ and $a_{ii} \in I(L)(\forall i \in \underline{n})$, then*

(1) $a_{ii} = a_{ii}^{(s)}$ *for any integer $s \geq 1$;*

(2) *A converges to $A^{k(A)}$ with $k(A) \leq n$.*

Proof. (1) By the hypothesis $A \geq A^2$, we have $A^k \geq A^{k+1}(\forall k \in Z_+)$. Hence $A^n \geq A^{n+1}$ and $a_{ii} \geq a_{ii}^{(s)}(\forall s \in Z_+)$. Since $a_{ii} \in I(L)$, we can get $a_{ii} = a_{ii} \cdots a_{ii} = a_{ii}^s \leq a_{ii}^{(s)}(\forall s \in Z_+)$ (since a_{ii}^s is a term of $a_{ii}^{(s)}$). Therefore, $a_{ii} = a_{ii}^{(s)}$ and $a_{ii}^{(s)} \in I(L)$.

(2) Next, we shall verify $A^n \leq A^{n+1}$.

Let $a_{ij}^{(n)} = \sum_{1 \leq i_1, i_2, \cdots, i_{n-1} \leq n} a_{ii_1} a_{i_1 i_2} \cdots a_{i_{n-1} j} (\forall i, j \in \underline{n})$. Since the number of indices in any term $a_{ii_1} a_{i_1 i_2} \cdots a_{i_{n-1} j}$ is greater than n, there must be two indices i_u and i_v such that $i_u = i_v$ for some u, v $(0 \leq u < v \leq n, i_0 = i, i_n = j)$. Then

$$\begin{aligned} a_{ii_1} a_{i_1 i_2} \cdots a_{i_{n-1} j} &= a_{ii_1} a_{i_1 i_2} \cdots a_{i_{u-1} i_u} a_{i_u i_{u+1}} \cdots a_{i_{v-1} i_v} a_{i_v i_{v+1}} \cdots a_{i_{n-1} j} \\ &\leq a_{ii_1} a_{i_1 i_2} \cdots a_{i_{u-1} i_u} a_{i_u i_u}^{(v-u)} a_{i_v i_{v+1}} \cdots a_{i_{n-1} j} \\ &\quad \text{(since } a_{i_u i_{u+1}} \cdots a_{i_{v-1} i_v} \text{ is a term of } a_{i_u i_u}^{(v-u)}\text{)} \\ &= a_{ii_1} a_{i_1 i_2} \cdots a_{i_{u-1} i_u} a_{i_u i_u}^{(v-u)} a_{i_u i_u}^{(v-u)} a_{i_v i_{v+1}} \cdots a_{i_{n-1} j} \\ &\quad \text{(since } a_{ii} = a_{ii}^{(s)}(\forall i, s \in \underline{n}) \text{ and } a_{ii} \in I(L)\text{)} \end{aligned}$$

$$\leq a_{ij}^{(n+(v-u))}(v-u \geq 1)$$
(since $a_{ii_1}a_{i_1i_2}\cdots a_{i_{u-1}i_u}a_{i_ui_u}^{(v-u)}a_{i_ui_u}^{(v-u)}a_{i_vi_{v+1}}\cdots a_{i_{n-1}j}$ is the sum of some term in $a_{ij}^{(n+(v-u))}$)
$$\leq a_{ij}^{(n+1)} \text{ (since } A^k \geq A^{k+1} \text{ for any } k \in Z_+).$$

Thus, $a_{ij}^{(n)} \leq a_{ij}^{(n+1)}$. Therefore, $A^n \leq A^{n+1}$. From above, we can get $A^n = A^{n+1}$. This completes the proof. ∎

Proposition 4.5. *If $A \in M_n(L)$ is almost periodic and $A \geq I_n$, then*

(1) *A converges to $A^{m(A)}$ with $m(A) \leq k(A)$;*

(2) $t(A) = \sum_{l=1}^{k(A)} A^l$;

(3) *$t(A)$ converges to $A^{k(A)}$.*

The proof is trivial.

Theorem 4.2. *Let $A = (a_{ij}) \in M_n(L)$. If the entries of A satisfy $a_{ii} \geq a_{jk}$ and $a_{ii}a_{jk} = a_{jk}$ $(\forall i, j, k \in \underline{n})$, then*

(1) *$a_{ii} = a_{ii}^s = a_{ii}^{(s)}$ for any integer $s \geq 1$;*

(2) *A converges to $A^{k(A)}$ with $k(A) \leq n-1$.*

Proof. (1) By the hypothesis $a_{ii}a_{jk} = a_{jk}$ for all $i, j, k \in \underline{n}$, we have $a_{11} = a_{22} = \cdots = a_{nn}$ and $a_{ii} = a_{ii}^s$ $(\forall i \in \underline{n}$ and $\forall s \in Z_+)$. Since a_{ii}^s is a term of $a_{ii}^{(s)}$, we can get $a_{ii} = a_{ii}^s \leq a_{ii}^{(s)}$. Let $a_{ii}^{(s)} = \sum_{1\leq i_1,\cdots,i_{s-1}\leq n} a_{ii_1}a_{i_1i_2}\cdots a_{i_{s-1}i} \leq \sum_{1\leq i_1,\cdots,i_{s-1}\leq n} a_{ii}a_{i_1i_1}\cdots a_{i_{s-1}i_{s-1}}$ (by the hypothesis $a_{ii} \geq a_{jk}$ for any $i, j, k \in \underline{n}$) $= \sum_{1\leq i_1,\cdots,i_{s-1}\leq n} a_{ii}^s$ (since $a_{11} = a_{22} = \cdots = a_{nn}$) $= \sum_{1\leq i_1,\cdots,i_{s-1}\leq n} a_{ii}$ (since $a_{ii} = a_{ii}^s$) $= a_{ii}$. Therefore, $a_{ii} = a_{ii}^s = a_{ii}^{(s)}$ for any integer $s \in Z_+$.

(2) Since $a_{ij}^{(2)} = \sum_{k=1}^n a_{ik}a_{kj}$ for any $i, j \in \underline{n}$ and the hypothesis $a_{ii}a_{jk} = a_{jk}$ for any $i, j, k \in \underline{n}$, we can get $a_{ij}^{(2)} = \sum_{k=1}^n a_{ik}a_{kj} = \sum_{k=1}^n a_{ik}a_{kk}a_{kj} \leq a_{ij}^{(3)}$ (because $\sum_{k=1}^n a_{ik}a_{kk}a_{kj}$ is the sum of some term in $a_{ij}^{(3)}$). Thus, $A^2 \leq A^3$, and so $A^{n-1} \leq A^n$.

In the following, we will show that $A^{n-1} \geq A^n$. Since (1)$a_{ii} = a_{ii}^{(s)}$ for any $i \in \underline{n}$ and $s \in Z_+$, we only need to prove that $a_{ij}^{(n-1)} \geq a_{ij}^{(n)}$ for $i \neq j$. Let $a_{ij}^{(n)} = \sum_{1\leq i_1,\cdots,i_{n-1}\leq n} a_{ii_1}a_{i_1i_2}\cdots a_{i_{n-1}j}(\forall i, j \in \underline{n}$ and $i \neq j)$. Since the number of

indices in $a_{ii_1}a_{i_1i_2}\cdots a_{i_{n-1}j}(i \neq j)$ is $n+1$, there must be two indices i_u and i_v such that $i_u = i_v$ for some $u, v(u < v)$.

If $v - u = 1$, then we can get

$$\begin{aligned}
&a_{ii_1}a_{i_1i_2}\cdots a_{i_{n-1}j}\\
&\quad = a_{ii_1}a_{i_1i_2}\cdots a_{i_{u-1}i_u}a_{i_ui_{u+1}}a_{i_{u+1}i_{u+2}}\cdots a_{i_{n-1}j}\\
&\quad = a_{ii_1}a_{i_1i_2}\cdots a_{i_{u-1}i_u}a_{i_ui_v}a_{i_vi_{v+1}}\cdots a_{i_{n-1}j} \text{ (since } v = u+1)\\
&\quad = a_{ii_1}a_{i_1i_2}\cdots a_{i_{u-1}i_u}a_{i_ui_u}a_{i_vi_{v+1}}\cdots a_{i_{n-1}j} \text{ (since } i_u = i_v)\\
&\quad = a_{ii_1}a_{i_1i_2}\cdots a_{i_{u-1}i_u}a_{i_vi_{v+1}}\cdots a_{i_{n-1}j} \text{ (since } a_{ii}a_{jk} = a_{jk}(\forall i,j,k \in \underline{n}))\\
&\quad \leq a_{ij}^{(n-1)} \text{ (since } a_{ii_1}a_{i_1i_2}\cdots a_{i_{u-1}i_u}a_{i_vi_{v+1}}\cdots a_{i_{n-1}j} \text{ is a term of } a_{ij}^{(n-1)}),
\end{aligned}$$

and so $a_{ij}^{(n)} \leq a_{ij}^{(n-1)}$. Thus, $A^n \leq A^{n-1}$.

If $v - u > 1$, then we can get

$$\begin{aligned}
&a_{ii_1}a_{i_1i_2}\cdots a_{i_{n-1}j}\\
&\quad = a_{ii_1}a_{i_1i_2}\cdots a_{i_{u-1}i_u}a_{i_ui_{u+1}}\cdots a_{i_{v-1}i_v}a_{i_vi_{v+1}}\cdots a_{i_{n-1}j}\\
&\quad \leq a_{ii_1}a_{i_1i_2}\cdots a_{i_{u-1}i_u}a_{i_ui_v}^{(v-u)}a_{i_vi_{v+1}}\cdots a_{i_{n-1}j} \text{ (since } a_{i_ui_{u+1}}\cdots a_{i_{v-1}i_v}\\
&\quad \text{ is a term of } a_{i_ui_u}^{(v-u)})\\
&\quad = a_{ii_1}a_{i_1i_2}\cdots a_{i_{u-1}i_u}a_{i_ui_u}a_{i_vi_{v+1}}\cdots a_{i_{n-1}j} \text{ (since } a_{ii} = a_{ii}^{(s)}(\forall i \in \underline{n}, \forall s \in Z_+))\\
&\quad \leq a_{ij}^{(n+1-(v-u))} \text{ (since } a_{ii_1}a_{i_1i_2}\cdots a_{i_{u-1}i_u}a_{i_ui_u}a_{i_vi_{v+1}}\cdots a_{i_{n-1}j} \text{ is a term of}\\
&\quad a_{ij}^{(n+1-(v-u))})\\
&\quad \leq a_{ij}^{(n-1)} \text{ (because } A^{k-1} \leq A^k \text{ for any integer } k \geq 3),
\end{aligned}$$

and so $a_{ij}^{(n)} \leq a_{ij}^{(n-1)}$. Thus, $A^n \leq A^{n-1}$.

From above, we have $A^{n-1} = A^n$. This proves the conclusion. ∎

Proposition 4.6. *If $A = (a_{ij}) \in M_n(L)$ is irreflexive and transitive, then A converges to $A^{k(A)} = 0$ with $k(A) \leq n$.*

Proof. We only need to show $A^n = 0$. Let $a_{ij}^{(n)} = \sum_{1\leq i_1,\ldots,i_{n-1}\leq n} a_{ii_1}a_{i_1i_2}\cdots a_{i_{n-1}j}$ for any $i, j \in \underline{n}$. We consider any term $a_{ii_1}a_{i_1i_2}\cdots a_{i_{n-1}j}$ of $a_{ij}^{(n)}$. Since $i, i_1, ..., i_{n-1}$, $j \in \underline{n}$, there are u, v such that $i_u = i_v$ $(0 \leq u < v \leq n, i_0 = i,\ i_n = j)$, and so any term $a_{ii_1}a_{i_1i_2}\cdots a_{i_{n-1}j}$ has the factor $a_{i_ui_{u+1}}\cdots a_{i_{v-1}i_v}$. Since $a_{i_ui_{u+1}}\cdots a_{i_{v-1}i_v} = a_{i_ui_{u+1}}\cdots a_{i_{v-1}i_u}$ (since $i_u = i_v$)$\leq$ $a_{i_ui_u}^{(v-u)}$ (since $a_{i_ui_{u+1}}\cdots a_{i_{v-1}i_u}$ is a term of $a_{i_ui_u}^{(v-u)}$) $\leq$ $a_{i_ui_u}$ (because A is transitive)$=$ 0 (since A is irreflexive), we have $a_{i_ui_{u+1}}\cdots a_{i_{v-1}i_v} = 0$, and so $a_{ii_1}a_{i_1i_2}\cdots a_{i_{n-1}j} = 0$ (by Lemma 4.2). Hence, $a_{ij}^{(n)} = 0$. Therefore, $A^n = 0$. This proves the conclusion. ∎

Proposition 4.6 generalizes Proposition 3.3 of Tan [34] and Corollary 3.1 of Han [16].

At the end of this section, some properties and characterizations for powers of generalized fuzzy matrices are showed from adjoint matrix.

Lemma 4.3. *Let $A = (a_{ij}) \in M_n(L)$. If the entries of A satisfy $a_{ii} \geq a_{jk}$ $(\forall i, j, k \in \underline{n})$, then $A^{n-1} \geq adj(A)$.*

Proof. In order to get $A^{n-1} \geq adj(A)$, we need to prove $a_{ij}^{(n-1)} \geq det(A_{ji})$.

By the hypothesis $a_{ii} \geq a_{jk}(\forall i, j, k \in \underline{n})$, then $a_{11} = a_{22} = \cdots = a_{nn}$.

First, we consider the case $i = j$. We see that $det(A_{ii})$ can be obtained from $det(A)$ by replacing a_{ii} by 1 and all other row-i entries $a_{ik}(k \neq i)$ by 0. Thus,

$$det(A_{ii}) = \sum_{\sigma \in s_n, \sigma(i)=i} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{i-1,\sigma(i-1)} a_{i+1,\sigma(i+1)} \cdots a_{n\sigma(n)},$$

where s_n denotes the symmetric group of all permutations of the indices $\{1, 2, ..., n\}$. We consider any term $a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{i-1,\sigma(i-1)} a_{i+1,\sigma(i+1)} \cdots a_{n\sigma(n)}$ of $det(A_{ii})$, since

$$\begin{aligned}
&a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{i-1,\sigma(i-1)} a_{i+1,\sigma(i+1)} \cdots a_{n\sigma(n)} \\
&\quad \leq a_{11} a_{22} \cdots a_{i-1,i-1} a_{i+1,i+1} \cdots a_{nn} (\text{by the hypothesis} a_{ii} \geq a_{jk} \ (\forall i, j, k \in \underline{n})) \\
&\quad = a_{ii}^{n-1} \ (\text{since } a_{11} = a_{22} = \cdots = a_{nn}) \\
&\quad \leq a_{ii}^{(n-1)} \ (\text{since } a_{ii}^{n-1} \text{ is a term of } a_{ii}^{(n-1)}),
\end{aligned}$$

we have $det(A_{ii}) \leq a_{ii}^{(n-1)}$. On the other hand, since $a_{11} a_{22} \cdots a_{i-1,i-1} a_{i+1,i+1} \cdots a_{nn}$ is a term of the expansion of $det(A_{ii})$, we have $a_{11} a_{22} \cdots a_{i-1,i-1} a_{i+1,i+1} \cdots a_{nn} = a_{ii}^{n-1} \leq det(A_{ii})$. Thus, $a_{ii}^{n-1} \leq det(A_{ii}) \leq a_{ii}^{(n-1)}$. Since

$$\begin{aligned}
a_{ii}^{(n-1)} &= \sum_{1 \leq i_1, \cdots, i_{n-2} \leq n} a_{ii_1} a_{i_1 i_2} \cdots a_{i_{n-2} i} \\
&\leq \sum_{1 \leq i_1, \cdots, i_{n-2} \leq n} a_{ii} a_{i_1 i_1} \cdots a_{i_{n-2} i_{n-2}} (\text{by the hypothesis } a_{ii} \geq a_{jk} (\forall i, j, k \in \underline{n})) \\
&= a_{ii}^{n-1} \ (\text{since } a_{11} = a_{22} = \cdots = a_{nn}),
\end{aligned}$$

we can get $a_{ii}^{(n-1)} \leq a_{ii}^{n-1}$.

From above, we can get $a_{ii}^{n-1} \leq det(A_{ii}) \leq a_{ii}^{(n-1)} \leq a_{ii}^{n-1}$, and so $det(A_{ii}) = a_{ii}^{(n-1)} = a_{ii}^{n-1}$.

In the following, we consider the case $i \neq j$. Similarly, we note that $det(A_{ji})$ can be obtained from $det(A)$ by replacing a_{ji} by 1 and all other row-j entries $a_{jk}(k \neq i)$ by 0. Thus, we have

$$det(A_{ji}) = \sum_{\sigma \in s_n, \sigma(j)=i} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{j-1,\sigma(j-1)} a_{j+1,\sigma(j+1)} \cdots a_{n\sigma(n)},$$

where s_n denotes the symmetric group of all permutations of the indices $\{1, 2, ..., n\}$. Since $\sigma \in s_n$ and $\sigma(j) = i$, there must be an integer $t \geq 0$ such that $\sigma^{t+1}(i) = j$ and $i, \sigma(i), \cdots, \sigma^t(i), j$ are different each other.

We consider any term $a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{j-1,\sigma(j-1)} a_{j+1,\sigma(j+1)} \cdots a_{n\sigma(n)}$ of $det(A_{ji})$, since

$$\begin{aligned}
&a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{j-1,\sigma(j-1)} a_{j+1,\sigma(j+1)} \cdots a_{n\sigma(n)} \\
&\quad \leq a_{i\sigma(i)} a_{\sigma(i)\sigma^2(i)} \cdots a_{\sigma^t(i)j} a_{jj}^{n-t-2} \ (\text{by the hypothesis } a_{ii} \geq a_{jk} (\forall i, j, k \in \underline{n})) \\
&\quad \leq a_{ij}^{(n-1)} \ (\text{since } a_{i\sigma(i)} a_{\sigma(i)\sigma^2(i)} \cdots a_{\sigma^t(i)j} a_{jj}^{n-t-2} \text{ is a term of } a_{ij}^{(n-1)}),
\end{aligned}$$

we have $det(A_{ji}) \leq a_{ij}^{(n-1)}$.

From above, we can get $det(A_{ji}) \leq a_{ij}^{(n-1)} (\forall i, j \in \underline{n})$, i.e., $adj(A) \leq A^{n-1}$. This proves the conclusion. ∎

Lemma 4.3 generalizes Theorem 4 of Duan [6].

Theorem 4.3. *Let $A = (a_{ij}) \in M_n(L)$. If the entries of A satisfy $a_{ii} \geq a_{jk}$ $(\forall i, j, k \in \underline{n})$, then $A^{n-1} = adj(A)$.*

Proof. By Lemma 4.3, we have $det(A_{ji}) \leq a_{ij}^{(n-1)} (\forall i, j \in \underline{n})$. Thus, we need only to show that $a_{ij}^{(n-1)} \leq det(A_{ji})$.

If $i = j$. By the proof of Lemma 4.3, we have $det(A_{ii}) = a_{ii}^{(n-1)}$. Then $a_{ii}^{(n-1)} \leq det(A_{ii})$.

If $i \neq j$. Let $a_{ij}^{(n-1)} = \sum_{1 \leq i_1, \cdots, i_{n-2} \leq n} a_{ii_1} a_{i_1 i_2} \cdots a_{i_{n-2} j}$. We consider any term $a_{ii_1} a_{i_1 i_2} \cdots a_{i_{n-2} j}$ of $a_{ij}^{(n-1)}$. If the subscripts $i, i_1, ..., i_{n-2}, j$ are pairwise different, we can see that $a_{ii_1} a_{i_1 i_2} \cdots a_{i_{n-2} j}$ serves as a term of $det(A_{ji})$, and so

$$a_{ii_1} a_{i_1 i_2} \cdots a_{i_{n-2} j} \leq det(A_{ji}).$$

If $\exists i_u = i_v$ $(u < v$ and $i_u, i_v \in \{i, i_1, ..., i_{n-2}, j\})$, deleting $a_{i_u i_{u+1}} \cdots a_{i_{v-1} i_v}$ from $a_{ii_1} a_{i_1 i_2} \cdots a_{i_{n-2} j}$, we obtain a new term $a_{ii_1} \cdots a_{i_{u-1} i_u} a_{i_v i_{v+1}} \cdots a_{i_{n-2} j}$. If there are still two identical numbers in the subscripts $i, i_1, ..., i_{u-1}, i_u, i_{v+1}, ..., i_{n-2}, j$, we apply the deleting method used in the above. This method can be applied repeatedly until the subscripts left are pairwise different. Finally, we can get a term $a_{ij_1} a_{j_1 j_2} \cdots a_{j_m j}$ such that $i, j_1, j_2, ..., j_m, j$ are pairwise different. Thus, $a_{ii_1} a_{i_1 i_2} \cdots a_{i_{n-2} j} \leq a_{ij_1} a_{j_1 j_2} \cdots a_{j_m j} a_{jj}^{n-m-2}$ (by $a_{ii} \geq a_{jk}$ $(\forall i, j, k \in \underline{n})$).

Let $\{k_1, k_2, ..., k_{n-m-2}\} = \underline{n} \,/\{i, j_1, j_2, ..., j_m, j\}$, then

$$\begin{aligned} a_{ii_1} a_{i_1 i_2} \cdots a_{i_{n-2} j} &\leq a_{ij_1} a_{j_1 j_2} \cdots a_{j_m j} a_{jj}^{n-m-2} \\ &= a_{ij_1} a_{j_1 j_2} \cdots a_{j_m j} a_{k_1 k_1} a_{k_2 k_2} \cdots a_{k_{n-m-2} k_{n-m-2}} \\ &\text{(since } a_{11} = a_{22} = \cdots = a_{nn}\text{)} \\ &\leq det(A_{ji}) \text{ (since } a_{ij_1} a_{j_1 j_2} \cdots a_{j_m j} a_{k_1 k_1} a_{k_2 k_2} \cdots a_{k_{n-m-2} k_{n-m-2}} \\ &\text{is a term of } det(A_{ji})\text{).} \end{aligned}$$

Hence, $a_{ii_1} a_{i_1 i_2} \cdots a_{i_{n-2} j} \leq det(A_{ji})$. Therefore, $a_{ij}^{(n)} \leq det(A_{ji})$.

From above, we have $det(A_{ji}) = a_{ij}^{(n-1)}$, i.e., $adj(A) = A^{n-1}$. This completes the proof. ∎

Theorem 4.3 generalizes Theorem 6 of Duan [6].

Theorem 4.4. *Let $A = (a_{ij}) \in M_n(L)$. If the entries of A satisfy $a_{ii} \geq a_{jk}$ and $a_{ii} a_{jk} = a_{jk}$ $(\forall i, j, k \in \underline{n})$, then*

(1) *$adj(A)$ is idempotent;*

(2) *A is idempotent $\Leftrightarrow adj(A) = A$;*

(3) *$adj(adj(A)) = adj(A)$;*

(4) *if $A \leq A^2$, then $Aadj(A) = adj(A)A = adj(A)$.*

Proof. (1) By Theorems 4.2 and 4.3, we can get $(adj(A))^2 = (A^{n-1})^2 = A^{2n-2} = A^{n-1} = adj(A)$.

(2) By (1), the proof is trivial.

(3) By (2), the proof is obvious.

(4) Let $C = Aadj(A)$. Then $c_{ij} = \sum_{k=1}^{n} a_{ik}det(A_{jk}) \geq a_{ii}det(A_{ji}) = det(A_{ji})$ (by $a_{ii}a_{jk} = a_{jk}(\forall i, j, k \in \underline{n})$), i.e., $Aadj(A) \geq adj(A)$. On the other hand, by (1), Theorem 4.3 and the hypothesis $A \leq A^2$, we have $Aadj(A) \leq A^{n-1}adj(A) = (adj(A))^2 = adj(A)$. Therefore, $Aadj(A) = adj(A)$. The proof of the equality $adj(A)A = adj(A)$ is similar. The proof is completed. ∎

Theorem 4.4 generalizes Corollary 2 of Duan [6].

Theorem 4.5. *Let $A = (a_{ij}) \in M_n(L)$. If the entries of A satisfy $a_{ik} = a_{jk}$ for any $i, j, k \in \underline{n}$, then*

(1) *the entries of $(adjA)^T = B = (b_{ij})$ satisfy $b_{ik} = b_{jk}$ for any $i, j, k \in \underline{n}$;*

(2) *the entries of $A(adjA) = C = (c_{ij})$ satisfy $c_{ik} = c_{jk}$ for any $i, j, k \in \underline{n}$ and $A(adjA) = (c_{ij}) = (det(A))$.*

Proof. (1) By $B = (b_{ij}) = (adjA)^T$, we have

$$b_{ij} = det(A_{ij}) = \sum_{\sigma \in s_n, \sigma(i)=j} a_{1\sigma(1)}a_{2\sigma(2)} \cdots a_{i-1,\sigma(i-1)}a_{i+1,\sigma(i+1)} \cdots a_{n\sigma(n)} \text{ and}$$

$$b_{kj} = det(A_{kj}) = \sum_{\sigma \in s_n, \sigma(k)=j} a_{1\sigma(1)}a_{2\sigma(2)} \cdots a_{k-1,\sigma(k-1)}a_{k+1,\sigma(k+1)} \cdots a_{n\sigma(n)}, \ (k \neq i).$$

Since $a_{ik} = a_{jk}$ for any $i, j, k \in \underline{n}$ and the numbers $\sigma(t)$ of columns cannot be changed in the two expansions of b_{ij} and b_{kj}, we can get $b_{ij} = b_{kj}$.

(2) By $a_{ik} = a_{jk}$, we have $A_{jk} = A_{ik}$, and so $det(A_{jk}) = det(A_{ik})$ for any $i, j, k \in \underline{n}$. Thus $c_{ij} = \sum_{k=1}^{n} a_{ik}det(A_{jk}) = \sum_{k=1}^{n} a_{ik}det(A_{ik}) = det(A)$. Therefore, $A(adjA) = (det(A))$. This completes the proof. ∎

Theorem 4.5 generalizes Proposition 4.6 [9].

Finally, we characterize the nilpotency with determinant zero from their cut matrices.

Let $A = (a_{ij}) \in M_n(L)$ and $k \in L$. The $k-cut$ of A means the $\{0, 1\}-$matrix and $A(k) = (a_{ij}(k)) \in M_n(L)$ defined as

$$a_{ij}(k) = \begin{cases} 1, & a_{ij} \geq k \\ 0, & \text{otherwise} \end{cases} \quad \text{for } i, j \in \underline{n}.$$

Proposition 4.7. *If $A \in M_n(L)$ and $det(A(k)) = 0$ for all $k \in L/\{0\}$, then $det(A) = 0$.*

Proof. Assume $det(A) > 0$. There exists a $\tau \in s_n$ such that $k = \prod_{i \in \underline{n}} a_{i\tau(i)} > 0$. Hence $det(A(k)) \geq \prod_{i \in \underline{n}} a_{i\tau(i)}(k) = 1$, i.e., $det(A(k)) > 0$. This is a contradiction. Therefore, $det(A) = 0$. This proves the conclusion. ∎

Proposition 4.7 generalizes Theorem 2.1 of Han [16].

Proposition 4.8. *If L has no nilpotent elements and $A \in M_n(L)$, then the following statements are equivalent:*

(1) $det(A) = 0$.

(2) $det(A(k)) = 0$ *for all* $k \in L/\{0\}$.

Proof. (1) $\Rightarrow$ (2) Assume that $det(A(k)) > 0$ for some $k \in L/\{0\}$. Thus, $1 = det(A(k)) = \sum_{\sigma \in s_n} \prod_{i \in \underline{n}} a_{i\sigma(i)}(k)$ (since $A(k)$ is a $\{0,1\}$-matrix). Hence, there exists a $\tau \in s_n$ such that $\prod_{i \in \underline{n}} a_{i\tau(i)}(k) = 1$, and so $det(A) \geq \prod_{i \in \underline{n}} a_{i\tau(i)} \geq k^n$. Since L has no nilpotent elements, we have $k^n > 0$. Therefore, $det(A) \geq k^n > 0$, which is a contradiction.

(2) $\Rightarrow$ (1) By Proposition 4.7, the proof is obvious. ∎

5. The invertibility of generalized fuzzy matrices

In this section, the invertibility of a generalized fuzzy matrix is investigated.

Definition 5.1. A matrix $A = (a_{ij}) \in M_n(L)$ is said to be

(1) right invertible if there exists a matrix $B \in M_n(L)$ such that $AB = I_n$;

(2) left invertible if there exists a matrix $B \in M_n(L)$ such that $BA = I_n$;

(3) invertible if A is both right and left invertible.

Definition 5.2. Let $A \in M_n(L)$. A mapping $f_A : L^n \longrightarrow L^n$ is defined by $f_A(x) = Ax$ for $x \in L^n$.

Theorem 5.1. *Let $A = (a_{ij}) \in M_n(L)$ be almost periodic and right invertible. Then $A^{d(A)} = I_n$.*

Proof. Since A is right invertible, we have $AX = I_n$ for some $X \in M_n(L)$. Since A is almost periodic, $A^{k(A)} = A^{k(A)+d(A)}$. Then $A^{d(A)} = A^{d(A)} \cdot I_n = A^{d(A)}\underbrace{(A \cdots (A(A}_{k \text{ times}}X)X) \cdots X) = A^{d(A)} \cdot A^{k(A)} \cdot X^{k(A)} = A^{k(A)+d(A)} \cdot X^{k(A)} = A^{k(A)} \cdot X^{k(A)} = I_n$, and so $A^{d(A)} = I_n$. ∎

Theorem 5.2. *Let $A = (a_{ij}) \in M_n(L)$ be almost periodic and left invertible, then $A^{d(A)} = I_n$.*

Proof. Similar to that of Theorem 5.1. ∎

6. Conclusions

A path algebra is a special type of semiring but it generalizes Boolean algebra, fuzzy algebra, distributive lattice and incline. In this paper, we study matrices over path algebras. Firstly, we discuss compositions of matrices over path algebras and obtain a new matrix from given matrices. Secondly, we give some properties and characteristics of transitivity and convergence of powers of a matrix, and the adjoint matrix is also studied. Finally, the invertibility of a matrix is considered. The main results in the present paper are the generalizations of the corresponding results in the literatures which have studied the Boolean matrices, the fuzzy matrices, the lattice matrices and the incline matrices.

In this paper, L is always assumed to be a commutative path algebra. But, as for the non-commutative algebra, are this paper's conclusions still justified? It still deserves further study.

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NUMERICAL SOLUTION OF FRACTIONAL RELAXATION–OSCILLATION EQUATION USING CUBIC B-SPLINE WAVELET COLLOCATION METHOD

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Abstract. A relaxation oscillator is a kind of oscillator based on a behavior of physical system's return to equilibrium after being disturbed. The relaxation-oscillation equation is the primary equation of relaxation and oscillation processes. The relaxation-oscillation equation is a fractional differential equation with initial conditions. In this paper, the approximate solutions of relaxation-oscillation equation are obtained by developing the wavelet collocation method to fractional differential equations using cubic B-spline wavelet. Analytical expressions of fractional derivatives in caputo sense for cubic B-spline functions are presented. The main advantage of the proposed method is that it transforms such problems into a system of algebraic equations which is suitable for computer programming. The reliability and efficiency of the proposed method are demonstrated in the numerical examples.

Keywords: fractional relaxation-oscillation equation; fractional differential equation; cubic B-spline function; wavelet collocation method.

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1. Introduction

The concept of fractional or non-integer order derivative and integration can be traced back to the genesis of integer order calculus itself [13]. The fractional differential equations have received considerable interest in recent years. Fractional differential equations have shown to be adequate models for various physical phenomena in areas like damping laws, diffusion processes etc. Most fractional differential equations do not have analytical solutions so we need approximate approach. Solution techniques for fractional differential equations have been studied extensively by many researchers such as Collocation method [8], [17], Adomian decomposition method [7], [18], Operational matrix method [9], [15], Variational iteration method [4], Tau method [16].

A relaxation-oscillator is a kind of oscillator based on a behavior of physical system's return back to equilibrium after being disturbed. There are many relaxation-oscillation models such as positive fractional derivative, fractal derivative and fractional derivative [11], [12], [21]. The relaxation-oscillation equation is the primary equation of relaxation and oscillation processes. The standard relaxation equation is

$$\frac{dy}{dx} + By = f(x)$$

where B denotes the Elc, E is the elastic modulus, $f(x)$ denotes E multiplying the strain rate. When $f(x) = 0$, we have the analytic solution $y(x) = Ce^{-Bx}$, where C is a constant determined by the initial condition.

The standard oscillation equation is

$$\frac{d^2y}{dx^2} + By = f(x)$$

where B equals $\frac{k}{m} = \omega^2$, k is the stiffness coefficient, m is the mass, ω the angular frequency. When $f(x) = 0$, we have the analytic solution $y(x) = C\cos\sqrt{B}x + D\sin\sqrt{B}x$, where C and D are constants determined by the initial conditions.

The fractional derivatives are employed in the relaxation and oscillation models to represent slow relaxation and damped oscillation [11], [12]. Fractional relaxation-oscillation model can be depicted as

$$D_x^\beta y(x) + Ay(x) = f(x), \quad x > 0$$
$$y(0) = a \text{ if } 0 < \beta \leq 1$$

or

$$y(0) = \lambda, \ \ y^{'}(0) = \mu \text{ if } 1 < \beta \leq 2$$

where A is a positive constant. For $0 < \beta \leq 2$, this equation is called the fractional relaxation-oscillation equation. When $0 < \beta < 1$, the model describes the relaxation with the power law attenuation. When $1 < \beta < 2$, the model depicts the damped oscillation with viscoelastic intrinsic damping of oscillator [3], [19].

This model has been applied in electrical model of the heart [20], signal processing [3], modeling cardiac pacemakers [5], Predator–Prey system [1], Spruce-budworm interactions [14] etc.

In the present paper, we intend to extend the cubic B-spline wavelet collocation method to solve fractional relaxation-oscillation equation. Expanding the unknown function as a linear combination of wavelet basis functions with unknown coefficients, the method transforms the differential equation into a system of algebraic equations. Whether the method can be extended to fractional differential equation depends on the calculation of fractional derivatives for all wavelet basis functions. In this paper, analytical expressions of fractional derivatives in caputo sense for wavelet basis functions are presented, which can save memory space and reduce computational complexity.

2. Cubic spline basis functions on $H^2(I)$

Let $I = [0, L]$ be an interval with $4 < L$ and $H^2(I)$ be a Sobolev space which contains functions with square integrable second derivatives and the homogeneous Sobolev space $H_0^2(I)$ can be defined by

$$H_0^2(I) = \big\{ f(t) \in H^2(I), \;\; f(0) = f^{'}(0) = f(L) = f^{'}(L) = 0 \big\}$$

which is a Hilbert space equipped with inner product

$$\langle f, g \rangle = \int_I f^{''}(t) g^{''}(t) dt$$

Cai and Wang [2] gave a multi-resolution analysis (MRA) and a wavelet decomposition for $H_0^2(I)$ by constructing scaling spline functions

$$\varphi(t) = \frac{1}{6} \sum_{l=0}^{4} \binom{4}{l} (-1)^l (t-l)_+^3$$

$$\varphi_b(t) = \frac{3}{2} t_+^2 - \frac{11}{12} t_+^3 + \frac{3}{2}(t-1)_+^3 - \frac{3}{4}(t-2)_+^3 + \frac{1}{6}(t-3)_+^3$$

and wavelet functions

$$\psi(t) = -\frac{3}{7}\varphi(2t) + \frac{12}{7}\varphi(2t-1) - \frac{3}{7}\varphi(2t-2)$$

$$\psi_b(t) = \frac{24}{13}\varphi_b(2t) - \frac{6}{13}\varphi(2t)$$

which have compact supports, where

$$t_+^n = \begin{cases} t^n, & t > 0 \\ 0, & t \leq 0 \end{cases}$$

Then the scaling spaces $V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_\infty = H_0^2(I)$ of MRA and a wavelet decomposition $H_0^2(I) = V_0 \oplus W_0 \oplus W_1 \oplus \cdots \oplus W_{j-1} \oplus \cdots$ corresponding to MRA and obtained by defining

$$V_j = span\big\{ \varphi_{j,k}(t), \varphi_{bj}(t), \varphi_{bj}(L-t), 0 \leq k \leq n_j - 4 \big\}$$

$$W_j = span\big\{ \psi_{j,k}(t), k = -1, 0, ..., n_j - 2 \big\}$$

in which $n_j = 2^j L$

$$\varphi_{j,k}(t) = \varphi(2^j t - k), \;\; \varphi_{bj}(t) = \varphi_b(2^j t), \;\; 0 \le k \le n_j - 4$$

$$\psi_{j,k}(t) = \psi(2^j t - k), \;\; 0 \le j, \;\; k = 0, 1, ..., n_j - 3$$

$$\psi_{j,-1}(t) = \psi_b(2^j t), \;\; \psi_{j,n_j-2}(t) = \psi_b(2^j(L-t))$$

For convenience, we set $\psi_{-1,k}(t) = \varphi_{0,k}(t)$, $0 \le k \le L-4$, $\psi_{-1,-1}(t) = \varphi_b(t)$, $\psi_{-1,L-3}(t) = \varphi_b(L-t)$, $n_{-1} = L-1$.

Let

$$B_j = \left\{ 2^{-3j/2} \psi_{j,k(t)} \right\}_{k=-1}^{n_j-2}, \;\; -1 \le j \le \infty.$$

Wang [22] proved $B = \bigcup_{j=-1}^{\infty} B_j$ is an unconditional basis of $H_0^2(I)$, which turns out to be a basis of continuous space $C_0(I)$. For non-homogeneity Sobolev space $H^2(I)$, Cai and Wang [2] introduced boundary spline functions,

$$\eta_1(t) = (1-t)_+^3$$

$$\eta_2(t) = 2t_+ - 3t_+^2 + \frac{7}{6}t_+^3 - \frac{4}{3}(t-3)_+^3 + \frac{1}{6}(t-2)_+^3$$

to deal with the values of functions at boundary points.

2.1 The function approximation and convergence

Any function $f(t) \in H_0^2(I)$ can be uniquely expanded into cubic spline wavelet series by

$$f(t) = \sum_{j=-1}^{\infty} \sum_{k=1}^{n_j} d_{j,k} \psi_{j,k-2}(t) \tag{1}$$

with

$$d_{j,k} = \int_I f''(t) \left(\psi_{j,k-2}^* \right)''(t) dt$$

where $\psi_{j,k}^*(t)$ are dual functions of $\psi_{j,k}$. Truncating the infinite series (1) at $J-1$, we get

$$f_J(t) = \sum_{j=-1}^{J-1} \sum_{k=1}^{n_j} d_{j,k} \psi_{j,k-2}(t) \tag{2}$$

From Wang [22], we have

$$\| f(t) - f_J(t) \|_{H_0^2}^2 \to 0 \;\; \text{as} \;\; J \to \infty$$

Hence any function $f(t) \in H_0^2(I)$ can be approximated by $f_J(t)$ defined in (2) and any function $f(t) \in H^2(I)$ can be approximated by,

$$f_J(t) = I_{b,J} f(t) + \sum_{j=-1}^{J-1} \sum_{k=1}^{n_j} d_{j,k} \psi_{j,k-2}(t) \tag{3}$$

and the approximation order is $O(2^{-4J})$ if $f(t)$ is sufficiently smooth [6], [10], [23], where

$$I_{b,J}f(t) = a_1\eta_1(2^J t) + a_2\eta_2(2^J t) + a_3\eta_2(2^J(L-t)) + a_4\eta_1(2^J(L-t))$$

Suppose $N = 2^J L + 3$ and $\Omega_J(t)$ is a $1 \times N$ vector as

$$\begin{aligned}\Omega_J(t) = \Big[&\eta_1(2^J t), \eta_2(2^J t), \eta_2(2^J(L-t)), \eta_1(2^J(Lt)), \psi_{-1,-1}(t), \psi_{-1,0}(t), ..., \psi_{-1,n_{-1}-2}(t),\\ &\psi_{0,-1}(t), \psi_{0,0}(t), \psi_{0,1}(t), ..., \psi_{0,L-3}(t), \psi_{0,n_0-2}(t)\\ &\psi_{1,-1}(t), \psi_{1,0}(t), \psi_{1,1}(t), ..., \psi_{1,2L-3}(t), \psi_{1,n_1-2}(t)\\ &...\\ &\psi_{J-1,-1}(t), \psi_{J-1,0}(t), \psi_{J-1,1}(t), ..., \psi_{J-1,n_J-3}(t), \psi_{J-1,n_J-2}(t)\Big]\\ &\triangleq \Big[\omega_1(t), \omega_2(t), ..., \omega_N(t)\Big]\end{aligned}$$

f_J (t) defined in (3) can be rewritten as

$$f_J(t) = \sum_{k=1}^{N} \widehat{f}_k \omega_k(t) = \Omega_J(t)\hat{f} \tag{4}$$

where $\hat{f} = [\hat{f}_1, \hat{f}_2, ..., \hat{f}_N]^T$ are the wavelet expansion coefficients, which can be determined by interpolating at collocation points,

$$t_{-1}^{(-1)} = \frac{1}{2^{J+1}}, \quad t_{L+1}^{(-1)} = L - \tfrac{1}{2^{J+1}},\ t_k^{(-1)} = k,\ k = 0, 1, ..., L \tag{5}$$

$$t_k^{(j)} = \frac{k+1.5}{2^j}, \quad -1 \le k \le 2^j L - 2,\ 0 \le j \le J-1. \tag{6}$$

The point value vanishing property of the wavelet function $\psi_{j,k}(t)$ and the compact supports of scaling spline functions $\varphi(t),\ \varphi_b(t)$ can be used to reduce computational complexity [2].

Since all the wavelet basis functions are composed by one-sided power functions $(at-b)_+^k$ and $(b-at)_+^k,\ a > 0,\ b \ge 0,\ k = 1, 2, 3$, if the analytical expressions of $D^\alpha(at-b)_+^k$ and $D^\alpha(b-at)_+^k,\ a > 0,\ b \ge 0,\ k = 1, 2, 3$ are obtained, those of $D^\alpha\Omega_J(t)$ can be naturally achieved.

For one-sided power function $(at-b)_+^k$ and $(b-at)_+^k,\ a > 0,\ b \ge 0,\ k = 1, 2, 3$, analytical expressions of their fractional derivative can be obtained by the properties of Laplace transform and fractional derivatives.

For $(at-b)_+^k,\ a > 0,\ b > 0$ the results are as follows:

Theorem 2.1. *For $m < \alpha \le m+1,\ m \in \mathbb{N}, k > 0, t > 0, a > 0, b \ge 0$, if $\alpha \le k$ or $\alpha \notin \mathbb{N}$, we have*

$$D^\alpha(at-b)_+^k = a^\alpha \frac{\Gamma(1+k)}{\Gamma(1+k-\alpha)}(at-b)_+^{k-\alpha}. \tag{7}$$

Proof. Let

$$f(t) = t_+^k, \;\; g(t) = f(at-b) = (at-b)_+^k.$$

Then

$$F(s) = \int_0^\infty e^{-st} f(t)dt = \Gamma(1+k)s^{-1-k}. \tag{8}$$

According to the property of the Laplace transform, we have

$$G(s) = \mathcal{L}\big[f(at-b)\big](s) = \frac{1}{a}e^{-\left(\frac{b}{a}\right)s}F\Big(\frac{s}{a}\Big). \tag{9}$$

Now, from the property of fractional derivative, we can obtain

$$\begin{aligned}
\mathcal{L}\big[D^\alpha (at-b)_+^k\big](s) &= \mathcal{L}\big[D^\alpha g(t)\big](s) \\
&= s^\alpha G(s) - \sum_{n=0}^{m} s^{\alpha-n-1} g^{(n)}(0) \\
&= a^k \frac{\Gamma(1+k)}{\Gamma(1+k-a)} \mathcal{L}\left[\Big(t-\frac{b}{a}\Big)_+^{k-\alpha}\right] \\
&= a^\alpha \frac{\Gamma(1+k)}{\Gamma(1+k-\alpha)} \mathcal{L}\big[(at-b)_+^{k-\alpha}\big].
\end{aligned}$$

From the uniqueness of Laplace transform, we get

$$D^\alpha (at-b)_+^k = a^\alpha \frac{\Gamma(1+k)}{\Gamma(1+k-\alpha)} (at-b)_+^{k-\alpha}.$$

Similarly we can derive the analytical expressions of $D^\alpha (b-at)_+^k, \;\; 0<a, 0<b.$

Theorem 2.2. *For $m<\alpha\le m+1,\; m\in\mathbb{N},\; k\in\mathbb{N},\; 0<t,\; 0<a,\; 0\le b,$ if $0<\alpha\le k$, we have*

$$D^\alpha (b-at)_+^k = \frac{(-1)^{k-1}a^\alpha k!}{\gamma_{k+1-\alpha}}(at-b)_+^{k-\alpha} + \sum_{l=m+1}^{k} \binom{k}{l} (-1)^l a^k b^{k-l} \frac{l!}{\gamma_{l+1-\alpha}} t_+^{l-\alpha}.$$

If $k<\alpha$ and $\alpha\notin\mathbb{N}$, we have

$$D^\alpha (b-at)_+^k = \frac{(-1)^{k-1}a^\alpha k!}{\gamma_{k+1-\alpha}}(at-b)_+^{k-\alpha},$$

where $\gamma_n = \Gamma(n-\alpha),\; n\in\mathbb{N}.$

Using the expressions of fractional derivative for one-sided power functions, we can get analytical expressions of fractional derivative for the wavelet basis functions as follows, if $\alpha\notin\mathbb{N}$

$$D^{\alpha}\varphi(t) = \frac{1}{\gamma_4}\sum_{l=0}^{4}\binom{4}{l}(-1)^l(t-l)_+^{3-\alpha} \triangleq \Phi(t)$$

$$D^{\alpha}\varphi(L-t) = \frac{1}{\gamma_4}\sum_{l=0}^{4}\binom{4}{l}(-1)^l(t-L+l)_+^{3-\alpha} \triangleq \Phi_r(t,L)$$

$$\begin{aligned} D^{\alpha}\varphi_b(t) = {} & \frac{3}{\gamma_3}t_+^{2-\alpha} - \frac{11}{2\gamma_4}t_+^{3-\alpha} + \frac{9}{\gamma_4}(t-1)_+^{3-\alpha} - \frac{9}{2\gamma_4}(t-2)_+^{3-\alpha} \\ & + \frac{1}{\gamma_4}(t-3)_+^{3-\alpha} \triangleq \Phi_{bl}(t) \end{aligned}$$

$$\begin{aligned} D^{\alpha}\varphi_b(L-t) = {} & -\frac{3}{\gamma_3}(t-L)_+^{2-\alpha} - \frac{11}{2\gamma_4}(t-L)_+^{3-\alpha} + \frac{9}{\gamma_4}(t-L+1)_+^{3-\alpha} \\ & - \frac{9}{2\gamma_4}(t-L+2)_+^{3-\alpha} + \frac{1}{\gamma_4}(t-L+3)_+^{3-\alpha} \triangleq \Phi_{br}(t,L) \end{aligned}$$

$$D^{\alpha}\varphi_{j,k}(t) = 2^{j\alpha}\Phi(2^jt-k),\ 0 \le j, k = 0,1,...,2^jL-4,$$

$$D^{\alpha}\psi(t) = 2^{\alpha}\left(-\frac{3}{7}\Phi(2t) + \frac{12}{7}\Phi(2t-1) - \frac{3}{7}\Phi(2t-2)\right) \triangleq \Psi(t)$$

$$D^{\alpha}\psi_b(t) = 2^{\alpha}\left(\frac{24}{13}\Phi_{bl}(2t) - \frac{6}{13}\Phi(2t)\right) \triangleq \Psi_{bl}(t)$$

$$D^{\alpha}\psi_b(L-t) = 2^{\alpha}\left(\frac{24}{13}\Phi_{br}(2t,2L) - \frac{6}{13}\Phi_r(2t,2L)\right) \triangleq \Psi_{br}(t,L)$$

$$D^{\alpha}\psi_{j,k}(t) = 2^{j\alpha}\Psi(2^jt-k),\ 0 \le j,\ k = 0,1,...,2^jL-3$$

$$D^{\alpha}\psi_b(2^jt) = 2^{j\alpha}\Psi_{bl}(2^jt),\quad D^{\alpha}\Psi_b(2^j(L-t)) = 2^{j\alpha}\Psi_{br}(2^jt,2^jL),$$

$$D^{\alpha}\eta_1(x) = \frac{6}{\gamma_4}\begin{cases} (x-1)_+^{3-\alpha} - x_+^{3-\alpha} + \frac{\gamma_4}{\gamma_3}x_+^{2-\alpha} - \frac{\gamma_4}{2\gamma_2}x_+^{1-\alpha}, & 0 < \alpha \le 1 \\ (x-1)_+^{3-\alpha} - x_+^{3-\alpha} + \frac{\gamma_4}{\gamma_3}x_+^{2-\alpha}, & 1 < \alpha \le 2 \\ (x-1)_+^{3-\alpha} - x_+^{3-\alpha}, & 2 < \alpha \le 3 \\ (x-1)_+^{3-\alpha}, & 3 < \alpha \text{ and } \alpha \notin \mathbb{N} \end{cases}$$

$$D^{\alpha}\eta_2(t) = \frac{2}{\gamma_2}t_+^{1-\alpha} - \frac{6}{\gamma_3}t_+^{2-\alpha} + \frac{7}{\gamma_4}t_+^{3-\alpha} - \frac{8}{\gamma_4}(t-1)_+^{3-\alpha} + \frac{1}{\gamma_4}(t-2)_+^{3-\alpha}$$

$$\begin{aligned} D^{\alpha}\eta_2(L-t) = {} & \frac{2}{\gamma_2}(t-L)_+^{1-\alpha} + \frac{6}{\gamma_3}(t-L)_+^{2-\alpha} + \frac{7}{\gamma_4}(t-L)_+^{3-\alpha} - \frac{8}{\gamma_4}(t-L+1)_+^{3-\alpha} \\ & + \frac{1}{\gamma_4}(t-L+2)_+^{3-\alpha} \end{aligned}$$

$$D^{\alpha}\eta_1(L-t) = \frac{6}{\gamma_4}(t-L+1)_+^{3-\alpha}$$

where $\gamma_n = \Gamma(n-\alpha),\ n \in \mathbb{N}$.

3. Solving fractional relaxation-oscillation equation by cubic B-spline wavelet collocation method

We consider the linear fractional differential equation

$$\left\{\begin{array}{rcll} D^{\alpha}y(t)+y(t) & = & f(t), & 0<t\leq L \\ y^{(n)}(0) & = & y_0^{(n)}, & n=0,1,2,...,m \end{array}\right. \tag{10}$$

where $m<\alpha\leq m+1,\ m\in\mathbb{N}$ and D^{α} denotes the Caputo fractional derivative of order α. To solve problem (10), we approximate $y(t)$ by

$$y(t)\approx\sum_{k=1}^{N}\hat{y}_k\omega_k(t)=\Omega_J(t)\hat{y}\triangleq y_J(t) \tag{11}$$

where vector $\hat{y}=[\hat{y}_1,...,\hat{y}_N]^T$ is unknown. The α order derivative of $y(t)$ is approximated by

$$D^{\alpha}y(t)\approx\sum_{k=1}^{N}\hat{y}_kD^{\alpha}\omega_k(t)=D^{\alpha}\Omega_J(t)\hat{y}\triangleq D^{\alpha}y_J(t) \tag{12}$$

Denote all collocation points defined in (5) and (6) in an order set $\{t_i\}_{i=1}^{N}$, where $t_1=0$. The expansion coefficients $\hat{y}$ can be determined by interpolating conditions at all collocation points, namely

$$\begin{array}{rcll} y_J(t_k) & = & y(t_k), & k=1,2,...,N \\ D^{\alpha}y_J(t_k) & = & D^{\alpha}y(t_k), & k=2,3,...,N \end{array}$$

Consequently, interpolating the fractional differential equation (10) by $y_J(t)$ at all collocation points, we obtain

$$\left.\begin{array}{rcll} D^{\alpha}y_J(t_k) & = & -y_J(t_k)+f(t_k), & 2\leq k\leq N \\ y_J^{(n)}(0) & = & y^{(n)}(0), & n=0,1,2,...,m \end{array}\right\} \tag{13}$$

where $m<\alpha\leq m+1,\ m\in\mathbb{N}$.

Denote

$$B_1 = \begin{pmatrix} \omega_1(t_2) & \omega_2(t_2) & \cdots & \omega_N(t_2) \\ \cdots & \cdots & \ddots & \cdots \\ \omega_1(t_N) & \omega_2(t_N) & \cdots & \omega_N(t_N) \end{pmatrix}$$

$$B_2 = \begin{pmatrix} D^{\alpha}\omega_1(t_2) & D^{\alpha}\omega_2(t_2) & \cdots & D^{\alpha}\omega_N(t_2) \\ \cdots & \cdots & \ddots & \cdots \\ D^{\alpha}\omega_1(t_N) & D^{\alpha}\omega_2(t_N) & \cdots & D^{\alpha}\omega_N(t_N) \end{pmatrix}$$

where B_1 and B_2 are obtained by analytical method. Then, according to equations (11), (12), equation (13) can be represented as,

$$\begin{cases} A\hat{y} = b \\ c_n\hat{y} = y_0^{(n)}, \quad n = 0, ..., m, \end{cases} \tag{14}$$

where $A = B_1 + B_2$ and

$$b = \Big(f(t_2), ..., f(t_N)\Big)^T,$$
$$c_n = \Big(\omega_1^{(n)}(0), \omega_2^{(n)}(0), ..., \omega_N^{(n)}(0)\Big),\ n = 0, 1, ..., m.$$

Let

$$A_m^T = \Big(A^T, c_0^T, c_1^T, ..., c_m^T\Big), \quad b_m^T = \Big(b^T, y_0^{(0)}, y_0^{(1)}, ..., y_0^{(m)}\Big).$$

Then (14) can be written as,

$$A_m\hat{y} = b_m. \tag{15}$$

Consequently, the wavelet expansion coefficient $\hat{y}$ can be obtained by solving linear equations (15), the approximated solution $y_J(t)$ in (10) can be effectively constructed by discrete wavelet transform technique.

If $0 < \alpha < 1$, the coefficient matrix A_0 is nonsingular, so equation (15) have and only have one solution. If $1 < \alpha$, equations (15) are over-determined, coefficient matrix A_m is column full rank and least squares solution can be regarded as the approximate solution.

4. Numerical Examples

In order to illustrate the effectiveness of the proposed method, some numerical examples are given in this section. The examples presented have exact solutions and also have been solved by other numerical methods. This allows us to compare the numerical results obtained by proposed method with the analytical solutions or those obtained by the other methods. Absolute errors between approximate solutions y_N and the corresponding exact solutions y, i.e., $N_e = |y_N - y|$ are considered.

Example 1. Let us consider the fractional relaxation-oscillation equation [8], [13]

$$D_a^{3/2}y(x) = -y(x)$$

with initial conditions $y(0) = 1, \ \ y'(0) = 0.$

For $0 < \alpha \leq 2$, the exact solution of this problem is $y(x) = E_\alpha(-x^\alpha)$. Here $E_\alpha(x)$ is called the Mittag-Leffler function,

$$E_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + 1)}.$$

The numerical results are obtained by proposed cubic B-spline wavelet collocation method for $L = 10$ and $J = 3, 4, 5$. Comparisons of numerical and exact solutions are shown in Table 1. Additionally, we tabulated the comparison results between proposed method for $J = 5$ and Taylor collocation method [8] in Table 2.

Table 1. Numerical results for Example 1.

x	Exact Solution	Present Method					
		$J = 3$	Ne for $J = 3$	$J = 4$	Ne for $J = 4$	$J = 5$	Ne for $J = 5$
0.0	1.0000000	1.0000000	0.0000E-0	1.0000000	0.0000E-0	1.0000000	0.0000E-0
0.1	0.9763777	0.9763799	0.2200E-5	0.9763777	0.1000E-10	0.9763777	0.2000E-14
0.2	0.9340362	0.9340468	0.1064E-4	0.9340362	0.5810E-10	0.9340362	0.7000E-13
0.3	0.8808084	0.8808726	0.6425E-4	0.8808084	0.2528E-9	0.8808084	0.9025E-12
0.4	0.8200563	0.8202624	0.2061E-3	0.8200563	0.6709E-8	0.8200563	0.2500E-12
0.5	0.7540488	0.7547919	0.7431E-3	0.7540488	0.7232E-7	0.7540488	0.3251E-11
0.6	0.6845298	0.6863748	0.1845E-2	0.6845298	0.4312E-7	0.6845298	0.6219E-11
0.7	0.6129215	0.6165955	0.3674E-2	0.6129217	0.2279E-6	0.6129215	0.8174E-11
0.8	0.5404169	0.5473109	0.6894E-2	0.5404165	0.4852E-6	0.5404169	0.4549E-10
0.9	0.4680306	0.4797706	0.1174E-1	0.4680322	0.1658E-5	0.4680306	0.6731E-10
1.0	0.3966293	0.4146093	0.1798E-1	0.3966272	0.2127E-5	0.3966293	0.9272E-10

Table 2. Comparisons of errors between present method and Taylor Collocation method [8]

x	Exact Solution	Present met. for $J = 5$	Taylor Col. Met. [8]
0.0	1.00000000	0.00000E-0	0.00000E-0
0.2	0.93403621	0.52954E-15	0.44805E-7
0.4	0.82232699	0.12194E-14	0.60316E-7
0.6	0.68452989	0.37252E-11	0.67627E-7
0.8	0.54041695	0.21659E-10	0.69751E-7
1.0	0.39662936	0.75281E-10	0.66570E-7

Figure 1 shows the comparison of approximate solutions and exact solutions. Figure 2 shows the comparison of error functions.

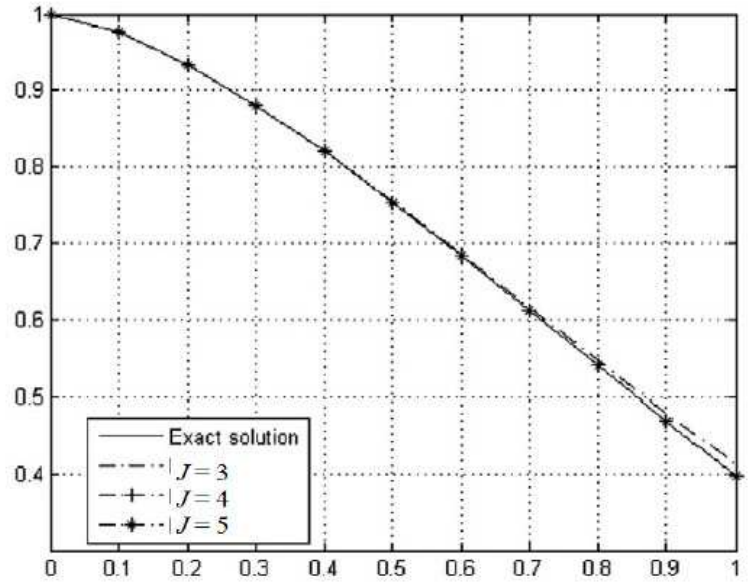

Figure 1. Comparison of approximate solutions and exact solutions.

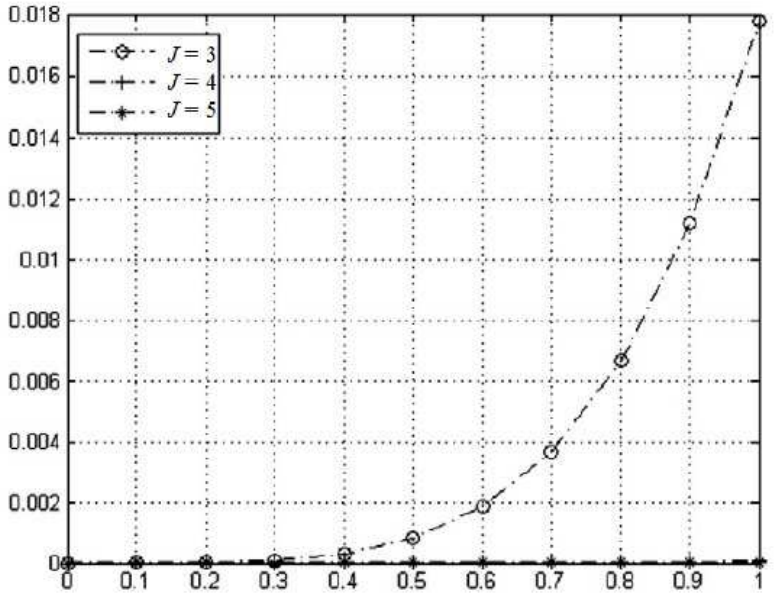

Figure 2. Comparison of errors at different levels.

Example 2. Consider the following fractional relaxation-oscillation equation,

$$D_a^{1/2} y(x) = -y(x)$$

with initial condition $y(0) = 1$.

For $0 \leq x \leq 1$, the approximate solutions for Example 2 are obtained by using the proposed cubic B-spline wavelet collocation method for $L = 10$ and $J = 3, 4, 5$. Comparison of numerical results with the exact solution is shown in table 3 and plotted the numerical results in Figure 3 for various J.

Table 3. Numerical results for Example 2.

x	Exact Solution	Present Method					
		$J = 3$	Ne for $J = 3$	$J = 4$	Ne for $J = 4$	$J = 5$	Ne for $J = 5$
0.0	1.0000000	1.0000000	0.0000$E-0$	1.0000000	0.0000$E-0$	1.0000000	0.0000$E-0$
0.1	0.7235784	0.7235782	0.1618$E-6$	0.7235784	0.3174$E-10$	0.7235784	0.7416$E-13$
0.2	0.6437882	0.6437925	0.4382$E-5$	0.6437882	0.3472$E-10$	0.6437882	0.3095$E-12$
0.3	0.5920184	0.5920375	0.1916$E-4$	0.5920184	0.9437$E-9$	0.5920184	0.6723$E-12$
0.4	0.5536062	0.5539199	0.3137$E-3$	0.5536062	0.6128$E-8$	0.5536062	0.1846$E-11$
0.5	0.5231565	0.5303615	0.7205$E-2$	0.5231565	0.1215$E-7$	0.5231565	0.4251$E-11$
0.6	0.4980245	0.5088645	0.1084$E-1$	0.4980245	0.2189$E-7$	0.4980245	0.8736$E-11$
0.7	0.4767027	0.4950143	0.1834$E-1$	0.4767025	0.2891$E-6$	0.4767027	0.2519$E-10$
0.8	0.4582460	0.4929617	0.3471$E-1$	0.4582469	0.9954$E-6$	0.4582460	0.5291$E-10$
0.9	0.4420214	0.4879841	0.4596$E-1$	0.4420289	0.7534$E-5$	0.4420214	0.9254$E-10$
1.0	0.4275835	0.4801973	0.5261$E-1$	0.4275934	0.9915$E-5$	0.4275835	0.4518$E-9$

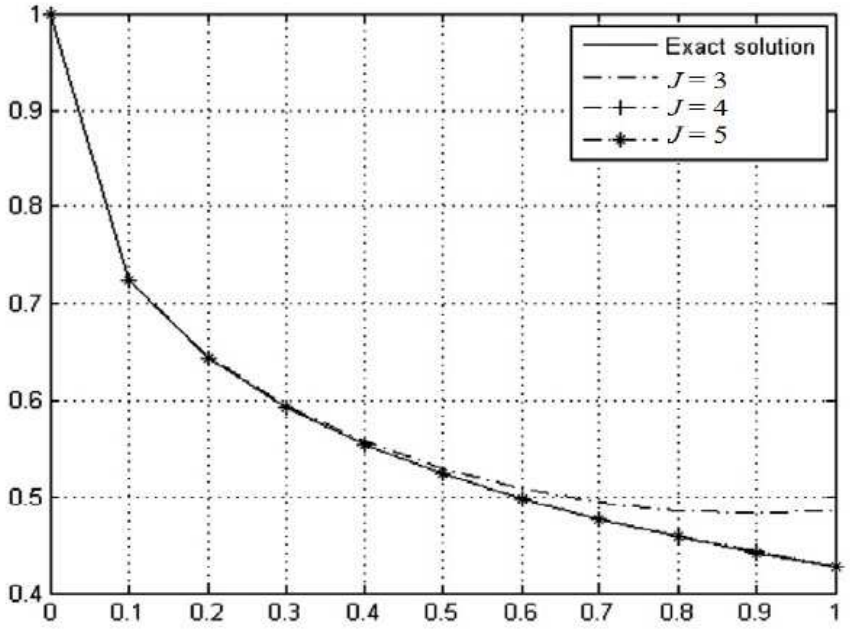

Figure 3. Comparison of approximate solutions and exact solutions.

Example 3. Consider the following fractional relaxation-oscillation model,

$$D_x^{\alpha} y(x) + Ay(x) = f(x), \quad x > 0,$$

with initial condition

$$y(0) = a, \quad \text{if } 0 < \alpha \leq 1$$

or

$$y(0) = \lambda \text{ and } y'(0) = \mu, \quad \text{if } 1 < \alpha \leq 2$$

where A is a positive constant.

First, consider $A = 1$ and $f(x) = 0$.

For $L = 10, J = 5$ on applying the proposed method for $\alpha = 0.25, 0.5, 0.75$ and 1 with initial condition $y(0) = 1$, Figure 4 shows that the numerical results are consistent with the exact ones and as α approaches to 1, the corresponding solutions of Example 3, approach that of integer-order differential equation.

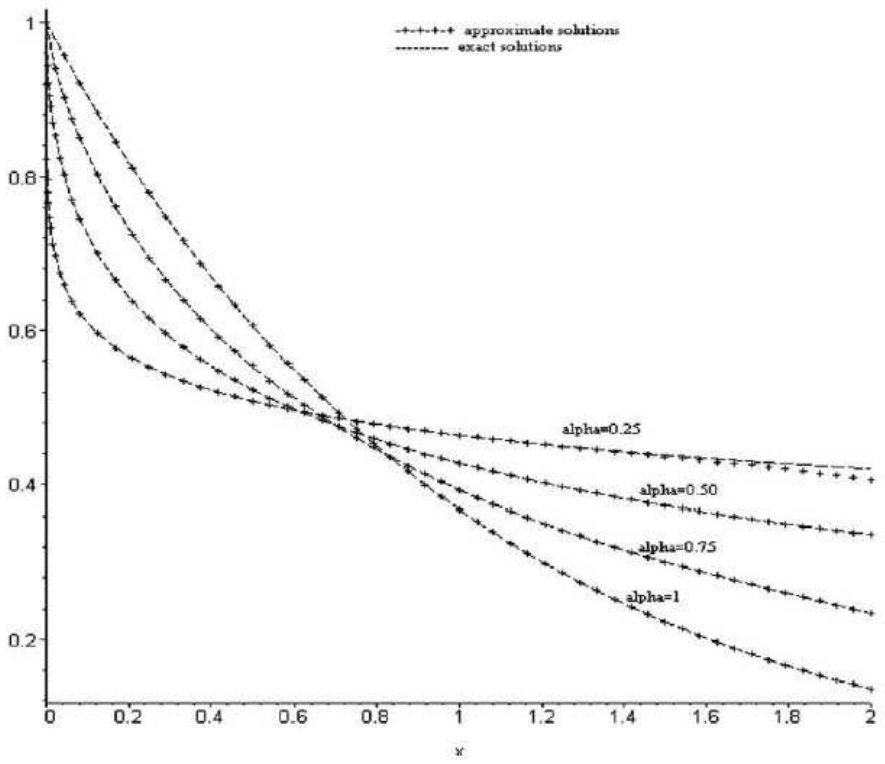

Figure 4. Comparison of approximate results with exact solutions for different values of α.

For $\lambda = 1$ and $\mu = 0$, Figure 5 illustrates the numerical solutions by proposed method and exact solution for $\alpha = 1.25, 1.5, 2$. Obviously, the numerical results are in good agreement with the exact ones. For $\alpha = 2$, the above fractional relaxation-oscillation equation given in Example 3, is the oscillation equation and the exact solution is $y(x) = cosx$.

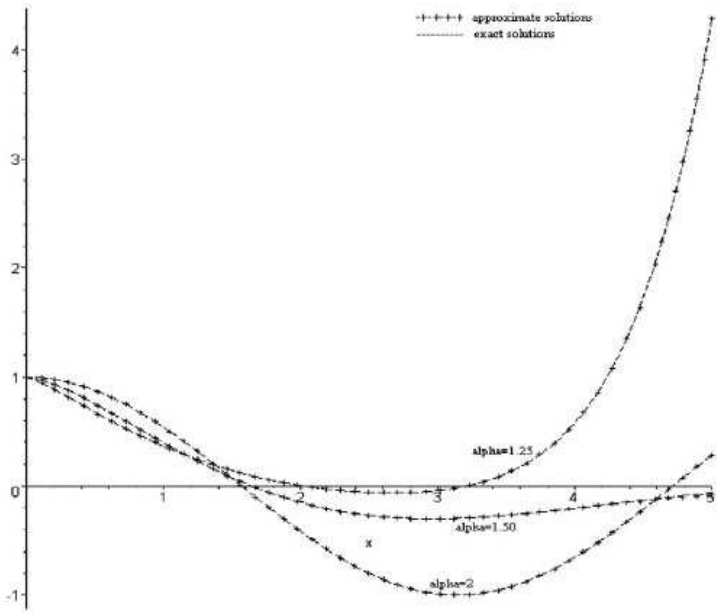

Figure 5. Comparison of approximate solutions with exact solutions for different values of α.

5. Conclusion

In this study, the cubic B-spline wavelet collocation method has been applied to obtain approximate solutions of fractional relaxation-oscillation equation. We have demonstrated the accuracy and efficiency of the proposed technique. The convergence of the method can be seen from the given figures. The better approximations may be obtained by increasing the Value of J. Table 2 shows that the present method is more accurate than Taylor collocation method. Numerical results obtained by the proposed method fairly match with exact solutions. The error of wavelet collocation method does not accumulate over time, so the proposed cubic B-spline wavelet collocation method is very simple, accurate and valid to solve fractional relaxation-oscillation equation.

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APPLICATION OF NEW GENERALIZED (G'/G)-EXPANSION METHOD TO THE $(3+1)$-DIMENSIONAL KADOMTSEV-PETVIASHVILI EQUATION

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Abstract. In this research article, seeking parameters dependent exact solutions, we implement the new generalized (G'/G)-expansion to the (3+1)-dimensional Kadomtsev-Petviashvili equation. The traveling wave solutions are expressed in terms of the hyperbolic functions, trigonometric functions, as well as rational functions. Herein, established is therefore the fact that the new generalized (G'/G)-expansion method offers an efficient and influential mathematical tool for constructing exact solutions of nonlinear evolution equations (NLEEs). In mathematical physics, finding the exact solutions of NLEEs reveals the salient features of the inner mechanism of possibly hidden complex physical phenomena, modeled by the given equations. In consequence to our current work and setup, not only does the new method appear to be straightforward and user-friendly, but also, it turns out easily implementable by computer programmed and symbolic algebra packages, yielding fast, albeit accurate results.

Keywords: homogeneous balance; new generalized (G'/G)-expansion method; nonlinear evolution equation; (3+1)-dimensional Kadomtsev-Petviashvili equation; solitary wave solutions; traveling wave solutions.
Mathematics Subject Classification: 35C07, 35C08, 35P99.

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1. Introduction

In recent years, scholars and researchers became highly interested in obtaining exact solutions for nonlinear partial differential equations (NLEEs). NLEEs are mathematical models of complex physical phenomena that may arise in engineering, applied mathematics, chemistry, biology, mechanics, physics, the exact solutions of which, when found, reveal the salient features of the hidden nonlinear dynamics. Therefore, developing means to crack such models and extract the exact solutions of NLEEs has become of utmost importance.

For the past three decades, searching for methods to solve NLEEs explicitly has been a central targets of numerical mathematics. Many reputed such methods have been developed. The list of methods include homogeneous balance [1], [2], hyperbolic tangent expansion [3], [4], trial function [5], nonlinear transform [6], theta function [7]–[9], inverse scattering transform [10], $\exp(-\varphi(\xi))$-expansion [11]–[14], Exp-function [15], [16], Hirota bilinear [17], Painleve expansion [18], (G'/G)-expansion [19]–[25, 36], improved (G'/G)-expansion [26], [27], new generalized (G'/G)-expansion [28]–[35], and Sumudu transform method [37]–[46]. For purpose, this paper aims to innovatively solve the (3+1)-dimensional Kadomtsev-Petviashvili equation by using the new generalized (G'/G)-expansion to show its suitability. The remainder of the paper is organized as follows: Section 2 is set for the new expansion method description while Section 3 is reserved for the method application to the (3+1)-dimensional Kadomtsev–Petviashvili equation. Results comparisons discussions are provided in Section 4, followed by the conclusion in Section 5. After acknowledgments, the paper culminates into a rich, albeit not exhaustive, set of scholarly references for current and future interest.

2. The new generalized (G'/G)-expansion method

Let us consider a general nonlinear PDE in the form,

$$P(u,\, u_t,\, u_x,\, u_{tt},\, u_{t\,x},\, u_{xx}, \ldots) = 0, \tag{1}$$

where $u = u(x,t)$is an unknown, P is a polynomial in $u(x,\, t)$and its derivatives including nonlinear terms are involved and the subscripts stand for the partial derivatives.

Step 1: We combine the real variables x and t by a complex variable, ξ,

$$u(x,\, t) = u(\xi), \xi = x \pm V\, t, \tag{2}$$

where, V is the speed of the traveling wave. The traveling wave transformation (2) converts Eq.(1)into an ODE, for $u = u(\xi)$, and with and superscripts indicating differentiation with respect to ξ, we obtain the following polynomial Q of u and it derivatives,

$$Q(u,\, u',\, u'',\, u''', \ldots) = 0. \tag{3}$$

Step 2: Accordingly, Eq. (3) can be integrated term by term once or more times, to yield constant(s) of integration. The integral constant (s) may be set to zero for simplicity.

Step 3: Suppose the traveling wave solution of Eq. (3) can be expressed as follows:

$$u(\xi) = \sum_{i=0}^{N} a_i (d+H)^i + \sum_{i=1}^{N} b_i (d+H)^{-i}, \tag{4}$$

where, either a_N or b_N may be zero, but both a_N and $b_N$could be zero at a time, a_i and b_i $(i = 1, 2, \ldots, N)$ and d, are arbitrary constants to be determined, and $H(\xi)$ is given by

$$H(\xi) = (G'/G) \tag{5}$$

where $G = G(\xi)$ satisfies the following auxiliary nonlinear ordinary differential equation:

$$AG\,G'' - BG\,G' - E\,G^2 - C\,(G')^2 = 0 \tag{6}$$

where the prime stands for derivative with respect to ξ; A, B, C and E are real parameters.

Step 4: To determine the positive integer N, taking the homogeneous balance between the highest order nonlinear terms and the derivatives of the highest order appearing in Eq. (3).

Step 5: Substituting Eq. (4) and Eq. (6) and Eq. (5) into Eq. (3) with the value of N obtained in Step 4, we get polynomials in $(d+H)^N$and $(d+H)^{-N}$, $(N = 0, 1, 2, ...)$. We then collect coefficient a set of algebraic equations for d and V, a_i and b_i for $(i = 0, 1, 2, ..., N)$.

Step 6: Suppose that the value of the constants a_i and b_i $(i = 1, 2, ..., N)$, d and V can be found. Since, the general solution of Eq. (6) is known to us, inserting the values of a_i and b_i $(i = 1, 2, ..., N)$, d and V into Eq. (4), we obtain more general types and new exact traveling wave solutions of the NLEE (1). Using Eq.(6) solution, we get the solutions of Eq. (5):

When $B \neq 0$, $\psi = A - C$ and $\Omega = B^2 + 4E(A - C) > 0$,

$$H(\xi) = \left(\frac{G'}{G}\right) = \frac{B}{2\psi} + \frac{\sqrt{\Omega}}{2\psi} \frac{C_1 \sinh\left(\frac{\sqrt{\Omega}}{2A}\xi\right) + C_2 \cosh\left(\frac{\sqrt{\Omega}}{2A}\xi\right)}{C_1 \cosh\left(\frac{\sqrt{\Omega}}{2A}\xi\right) + C_2 \sinh\left(\frac{\sqrt{\Omega}}{2A}\xi\right)} \tag{7}$$

when $B \neq 0$, $\psi = A - C$ and $\Omega = B^2 + 4E(A - C) < 0$,

$$H(\xi) = \left(\frac{G'}{G}\right) = \frac{B}{2\psi} + \frac{\sqrt{-\Omega}}{2\psi} \frac{-C_1 \sin\left(\frac{\sqrt{-\Omega}}{2A}\xi\right) + C_2 \cos\left(\frac{\sqrt{-\Omega}}{2A}\xi\right)}{C_1 \cos\left(\frac{\sqrt{-\Omega}}{2A}\xi\right) + C_2 \sin\left(\frac{\sqrt{-\Omega}}{2A}\xi\right)} \tag{8}$$

when $B \neq 0$, $\psi = A - C$ and $\Omega = B^2 + 4E(A - C) = 0$,

$$H(\xi) = \left(\frac{G'}{G}\right) = \frac{B}{2\psi} + \frac{C_2}{C_1 + C_2\xi} \tag{9}$$

when $B = 0$, $\psi = A - C$ and $\Delta = \psi E > 0$,

$$H(\xi) = \left(\frac{G'}{G}\right) = \frac{\sqrt{\Delta}}{\psi} \frac{C_1 \sinh(\frac{\sqrt{\Delta}}{A}\xi) + C_2 \cosh(\frac{\sqrt{\Delta}}{A}\xi)}{C_1 \cosh(\frac{\sqrt{\Delta}}{A}\xi) + C_2 \sinh(\frac{\sqrt{\Delta}}{A}\xi)} \tag{10}$$

when $B = 0$, $\psi = A - C$ and $\Delta = \psi E < 0$,

$$H(\xi) = \left(\frac{G'}{G}\right) = \frac{\sqrt{-\Delta}}{\psi} \frac{-C_1 \sin(\frac{\sqrt{-\Delta}}{A}\xi) + C_2 \cos(\frac{\sqrt{-\Delta}}{A}\xi)}{C_1 \cos(\frac{\sqrt{-\Delta}}{A}\xi) + C_2 \sin(\frac{\sqrt{-\Delta}}{A}\xi)} \tag{11}$$

3. Application of the method

In this section, we use the new generalized (G'/G)-expansion method to look for the solitary wave solutions to the (3+1)-dimensional Kadomtsev–Petviashvili equation,

$$(u_t + 6\,u\,u_x + u\,u_{x\,x\,x})_x - 3\,u_{y\,y} - 3\,u_{z\,z} = 0. \tag{12}$$

Using the variable, $u(\xi) = u(x, y, z, t)$, $\xi = x+y+z-Vt$, we consider the resulting ODE,

$$(-V\,u' + 6\,u\,u' + u''')' - 6\,u'' = 0. \tag{13}$$

Then, integrating twice, and for K, the constant of integration to be determined, we obtain,

$$K - Vu - 6u + 3\,u^2 + u'' = 0. \tag{14}$$

Taking the homogeneous balance between u^2 and u'' in Eq. (14), we get$N = 2$. Therefore, with, a_0, a_1, a_2, b_1, b_2 and d constants to be determined, the solution of Eq. (14) has the form,

$$u(\xi) = a_0 + a_1(d + H) + a_2(d + H)^2 + b_1(d + H)^{-1} + b_2(d + H)^{-2}, \tag{15}$$

Substituting Eqs. (15), (5) and (6) into Eq. (14), the left-hand side is converted into polynomials in $(d + H)^N$ and $(d + H)^{-N}(N = 1, 2, \ldots)$. We collect each coefficient of these resulting polynomials to zero, yields a set of simultaneous algebraic equations, (for simplicity which are not presented here) to be Maple solved for a_0, a_1, a_2, b_1, b_2, d, K and V.

Case 1: $a_0 = a_0$, $a_1 = 0$, $a_2 = 0$, $d = d$, $b_1 = \frac{2}{A^2}(-2Ed\psi - EB + B^2d + 2d^3\psi^2 - 3Bd^2\psi)$, $b_2 = -\frac{2}{A^2}(d^4\psi^2 - 2Ed^2\psi + E^2 + 2Bd^3\psi + B^2d^2 - 2BdE)$,

$$V = \frac{1}{A^2}(6a_0A^2 - 6A^2 + 12d^2\psi^2 + 12Bd\psi - 8E\psi + B^2), \tag{16}$$

$K = \frac{1}{A^4}(48Cd^2EA^2 - 72Bd^3CA^2 - 16BdEA^2 + 2B^3Ad - 48C^3Ad^4 + 16C^3Ed^2$
$+14C^2B^2d^2 - 24C^3Bd^3 - 2B^3Cd - 24a_0A^3Cd^2 + 12a_0A^3Bd^2 + 12a_0A^2C^2d^2$
$+8a_0A^2EC + 2CB^2E - 2EB^2A + 4E^2C^2 + 4A^2E^2 - 12a_0A^2BdC + 12a_0A^4d^2$
$-8a_0A^3E + a_0A^2B^2 + 32BAECd - 28B^2Ad^2C + 72BAC^2d^3 - 48C^2Ad^2E$
$-16C^2BdE + 3a_0A^4 + 12d^4A^4 + 12C^4d^4 - 8AE^2C + 72C^2d^4A^2 + 24Bd^3A^3$
$-16Ed^2A^3 - 48Cd^4A^3 + 14B^2d^2A^2),$

where $\psi = A - C$, a_0, A, B, C and E are free parameters.

Case 2: $a_1 = \frac{2}{A^2}(B\psi + 2d\psi^2)$, $a_2 = -\frac{2\psi^2}{A^2}$, $b_1 = 0$, $b_2 = 0$, $d = d$, $a_0 = a_0$,

$$K = \tfrac{1}{A^4}(48Cd^2EA^2-72Bd^3CA^2-16BdEA^2+2B^3Ad-48C^3Ad^4+16C^3Ed^2 +14C^2B^2d^2-24C^3Bd^3-2B^3Cd-24a_0A^3Cd^2+12a_0A^3Bd^2+12a_0A^2C^2d^2 +8a_0A^2EC+2CB^2E-2EB^2A+4E^2C^2+4A^2E^2-12a_0A^2BdC+12a_0A^4d^2 -8a_0A^3E+a_0A^2B^2+32BAECd-28B^2Ad^2C+72BAC^2d^3-48C^2Ad^2E -16C^2BdE+3a_0A^4+12d^4A^4+12C^4d^4-8AE^2C+72C^2d^4A^2+24Bd^3A^3 -16Ed^2A^3-48Cd^4A^3+14B^2d^2A^2),$$

$$V = \frac{1}{A^2}(6a_0A^2 - 6A^2 + 12d^2\psi^2 + 12Bd\psi - 8E\psi + B^2). \tag{17}$$

where $\psi = A - C$, a_0, A, B, C and E are free parameters.

Case 3: $a_1 = 0$, $b_1 = 0$, $a_2 = -\frac{2\psi^2}{A^2}$, $d = -\frac{B}{2\psi}$, $b_2 = -\frac{1}{8A^2\psi^2}(16E^2\psi^2+8EB^2\psi+B^4)$, $V = \frac{2}{A^2}(3a_0A^2 - 3A^2 - B^2 - 4E\psi)$,

$$\begin{aligned} K = \frac{1}{A^4}(&3a_0^2A^4 - 8a_0A^3E + 8a_0A^2EC - 2a_0A^2B^2 - 16A^2E^2 \\ &- 8EB^2A + 32AE^2C - B^4 - 16E^2C^2 + 8CB^2E). \end{aligned} \tag{18}$$

where $\psi = A - C$, a_0, A, B, C and E are free parameters.

For Case 1, substituting Eq. (16) into Eq. (15), along with Eq. (7) and simplifying yields following travelling wave solutions (if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$) respectively:

$$\begin{aligned} u_{1_1}(\xi) = a_0 &+ b_1\left(d + \frac{B}{2\psi} + \frac{\sqrt{\Omega}}{2\psi}\coth\left(\frac{\sqrt{\Omega}}{2A}\xi\right)\right)^{-1} \\ &+ b_2\left(d + \frac{B}{2\psi} + \frac{\sqrt{\Omega}}{2\psi}\coth\left(\frac{\sqrt{\Omega}}{2A}\xi\right)\right)^{-2}. \end{aligned}$$

$$\begin{aligned} u_{1_2}(\xi) = a_0 &+ b_1\left(d + \frac{B}{2\psi} + \frac{\sqrt{\Omega}}{2\psi}\tanh\left(\frac{\sqrt{\Omega}}{2A}\xi\right)\right)^{-1} \\ &+ b_2\left(d + \frac{B}{2\psi} + \frac{\sqrt{\Omega}}{2\psi}\tanh\left(\frac{\sqrt{\Omega}}{2A}\xi\right)\right)^{-2}, \end{aligned}$$

where $\xi = x - \left\{\frac{1}{A^2}(6a_0A^2 - 6A^2 + 12d^2\psi^2 + 12Bd\psi - 8E\psi + B^2)\right\}t$.

Substituting Eq. (16) into Eq. (15), along with Eq. (8) and simplifying, our exact solutions become (if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$) respectively:

$$u_{1_3}(\xi) = a_0 + b_1 \left(d + \frac{B}{2\psi} + \frac{\sqrt{-\Omega}}{2\psi} \cot\left(\frac{\sqrt{-\Omega}}{2A}\xi\right)\right)^{-1} + b_2 \left(d + \frac{B}{2\psi} + \frac{\sqrt{-\Omega}}{2\psi} \cot\left(\frac{\sqrt{-\Omega}}{2A}\xi\right)\right)^{-2}.$$

$$u_{1_4}(\xi) = a_0 + b_1 \left(d + \frac{B}{2\psi} - \frac{\sqrt{-\Omega}}{2\psi} \tan\left(\frac{\sqrt{-\Omega}}{2A}\xi\right)\right)^{-1} + b_2 \left(d + \frac{B}{2\psi} - \frac{\sqrt{-\Omega}}{2\psi} \tan\left(\frac{\sqrt{-\Omega}}{2A}\xi\right)\right)^{-2}.$$

Substituting Eq. (16) into Eq. (15) together with Eq. (9) and simplifying, we obtain

$$u_{1_5}(\xi) = a_0 + b_1 \left(d + \frac{B}{2\psi} + \frac{C_2}{C_1 + C_2\xi}\right)^{-1} + b_2 \left(d + \frac{B}{2\psi} + \frac{C_2}{C_1 + C_2\xi}\right)^{-2}.$$

Substituting Eq. (16) into Eq. (15), along with Eq. (10) and simplifying, we obtain following traveling wave but solutions (if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$) respectively:

$$u_{1_6}(\xi) = a_0 + b_1 \left(d + \frac{\sqrt{\Delta}}{\psi} \coth\left(\frac{\sqrt{\Delta}}{A}\xi\right)\right)^{-1} + b_2 \left(d + \frac{\sqrt{\Delta}}{\psi} \coth\left(\frac{\sqrt{\Delta}}{A}\xi\right)\right)^{-2}.$$

$$u_{1_7}(\xi) = a_0 + b_1 \left(d + \frac{\sqrt{\Delta}}{\psi} \tanh\left(\frac{\sqrt{\Delta}}{A}\xi\right)\right)^{-1} + b_2 \left(d + \frac{\sqrt{\Delta}}{\psi} \tanh\left(\frac{\sqrt{\Delta}}{A}\xi\right)\right)^{-2}.$$

Substituting Eq. (16) into Eq. (15), together with Eq. (11) and simplifying, our obtained exact solutions become (if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$) respectively:

$$u_{1_8}(\xi) = a_0 + b_1 \left(d + \frac{\sqrt{-\Delta}}{\psi} \cot\left(\frac{\sqrt{-\Delta}}{A}\xi\right)\right)^{-1} + b_2 \left((d + \frac{\sqrt{-\Delta}}{\psi} \cot\left(\frac{\sqrt{-\Delta}}{A}\xi\right)\right)^{-2}.$$

$$u_{1_9}(\xi) = a_0 + b_1 \left(d - \frac{\sqrt{-\Delta}}{\psi} \tan\left(\frac{\sqrt{-\Delta}}{A}\xi\right)\right)^{-1} + b \left(d - \frac{\sqrt{-\Delta}}{\psi} \tan\left(\frac{\sqrt{-\Delta}}{A}\xi\right)\right)^{-2}.$$

Again for Case 2, substituting Eq. (17) into Eq. (15) along with Eq. (7) and simplifying, the traveling wave solutions (if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$) respectively become:

$$u_{2_1}(\xi) = a_0 + \frac{1}{2A^2} \left\{ B^2 - \Omega \coth^2\left(\frac{\sqrt{\Omega}}{2A}\xi\right) + 4d\psi \ (B + d\psi) \right\}.$$

$$u_{2_2}(\xi) = a_0 + \frac{1}{2A^2} \left\{ B^2 - \Omega \tanh^2\left(\frac{\sqrt{\Omega}}{2A}\xi\right) + 4d\psi \ (B + d\psi) \right\},$$

where

$$\xi = x - \left\{ \frac{1}{A^2} \left(6a_0A^2 - 6A^2 + 12d^2\psi^2 + 12Bd\psi - 8E\psi + B^2 \right) \right\} t.$$

Substituting Eq. (17) into Eq. (15), along with Eq. (8) and simplifying, yields exact solutions (if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$) respectively:

$$u_{2_3} = a_0 + \frac{1}{2A^2} \left\{ B^2 + \Omega \cot^2 \left(\frac{\sqrt{-\Omega}}{2A} \xi \right) + 4d\psi \ (B + d\psi) \right\}.$$

$$u_{2_4}(\xi) = a_0 + \frac{1}{2A^2} \left\{ B^2 + \Omega \tan^2 \left(\frac{\sqrt{-\Omega}}{2A} \xi \right) + 4d\psi \ (B + d\psi) \right\}.$$

Substituting Eq. (17) into Eq. (15), along with Eq. (9) and simplifying, our obtained solution becomes:

$$u_{2_5}(\xi) = a_0 + \frac{1}{2A^2} \left\{ B^2 - \left(\frac{2\,\psi\, C_2}{C_1 + C_2\xi} \right)^2 + 4\,d\psi\,(B + d\psi) \right\},$$

Substituting Eq. (17) into Eq. (15), together with Eq. (10) and simplifying, yields following traveling wave solutions (if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$) respectively:

$$u_{2_6}(\xi) = a_0 + \frac{2}{A^2} \left\{ d\psi(B + d\psi) + \sqrt{\Delta} \left(B \coth \left(\frac{\sqrt{\Delta}}{A} \xi \right) - \sqrt{\Delta} \coth^2 \left(\frac{\sqrt{\Delta}}{A} \xi \right) \right) \right\}.$$

$$u_{2_7}(\xi) = a_0 + \frac{2}{A^2} \left\{ d\psi\,(B + d\psi) + \sqrt{\Delta} \left(B \tanh \left(\frac{\sqrt{\Delta}}{A} \xi \right) - \sqrt{\Delta} \tanh^2 \left(\frac{\sqrt{\Delta}}{A} \xi \right) \right) \right\}.$$

Substituting Eq. (17) into Eq. (15), along with Eq. (11) and simplifying, our exact solutions become (if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$) respectively:

$$u_{2_8}(\xi) = a_0 + \frac{2}{A^2} \left\{ d\psi\,(B + d\psi) + \sqrt{\Delta} \left(iB \cot \left(\frac{\sqrt{-\Delta}}{A} \xi \right) + \sqrt{\Delta} \cot^2 \left(\frac{\sqrt{-\Delta}}{A} \xi \right) \right) \right\}.$$

$$u_{2_9}(\xi) = a_0 + \frac{2}{A^2} \left\{ d\psi(B + d\psi) - \sqrt{\Delta} \left(iB \tan \left(\frac{\sqrt{-\Delta}}{A} \xi \right) - \sqrt{\Delta} \tan^2 \left(\frac{\sqrt{-\Delta}}{A} \xi \right) \right) \right\},$$

Finally, for Case 3, substituting Eq. (18) into Eq. (15), together with Eq. (7) and simplifying, yields following traveling wave solutions (if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$) respectively:

$$u_{3_1}(\xi) = a_0 - \frac{\Omega}{2A^2} \coth^2 \left(\frac{\sqrt{\Omega}}{2A} \xi \right) + \frac{4b_2\psi^2}{\Omega} \tanh^2 \left(\frac{\sqrt{\Omega}}{2A} \xi \right).$$

$$u_{3_2}(\xi) = a_0 - \frac{\Omega}{2A^2} \tanh^2 \left(\frac{\sqrt{\Omega}}{2A} \xi \right) + \frac{4b_2\psi^2}{\Omega} \coth^2 \left(\frac{\sqrt{\Omega}}{2A} \xi \right).$$

where

$$\xi = x - \left\{ \frac{2}{A^2}(3a_0A^2 - 3A^2 - B^2 - 4E\psi) \right\} t.$$

Substituting Eq. (18) into Eq. (15), along with Eq. (8) and simplifying, we obtain following solutions (if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$) respectively:

$$u_{3_3}(\xi) = a_0 + \frac{\Omega}{2A^2}\cot^2\left(\frac{\sqrt{-\Omega}}{2A}\xi\right) - \frac{4b_2\psi^2}{\Omega}\tan^2\left(\frac{\sqrt{-\Omega}}{2A}\xi\right).$$

$$u_{3_4}(\xi) = a_0 + \frac{\Omega}{2A^2}\tan^2\left(\frac{\sqrt{-\Omega}}{2A}\xi\right) - \frac{4b_2\psi^2}{\Omega}\cot^2\left(\frac{\sqrt{-\Omega}}{2A}\xi\right).$$

Substituting Eq. (18) into Eq. (15), along with Eq. (9) and simplifying, we get

$$u_{3_5}(\xi) = a_0 - \frac{2\psi^2}{A^2}\left(\frac{C_2}{C_1 + C_2\xi}\right)^2 + b_2\left(\frac{C_2}{C_1 + C_2\xi}\right)^{-2}.$$

Substituting Eq. (18) into Eq. (15), along with Eq. (10) and simplifying, yields following exact traveling wave solutions (if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$) respectively:

$$u_{3_6}(\xi) = a_0 - \frac{2\psi^2}{A^2}\left(\frac{-B}{2\psi} + \frac{\sqrt{\Delta}}{\psi}\coth(\frac{\sqrt{\Delta}}{A}\xi)\right)^2 + b_2\left(\frac{-B}{2\psi} + \frac{\sqrt{\Delta}}{\psi}\coth(\frac{\sqrt{\Delta}}{A}\xi)\right)^{-2}.$$

$$u_{3_7}(\xi) = a_0 - \frac{2\psi^2}{A^2}\left(\frac{-B}{2\psi} + \frac{\sqrt{\Delta}}{\psi}\tanh(\frac{\sqrt{\Delta}}{A}\xi)\right)^2 + b_2\left(\frac{-B}{2\psi} + \frac{\sqrt{\Delta}}{\psi}\tanh(\frac{\sqrt{\Delta}}{A}\xi)\right)^{-2}.$$

Substituting Eq. (18) into Eq. (15), along with Eq. (11) and simplifying, our obtained exact solutions become (if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$) respectively:

$$u_{3_8}(\xi) = a_0 - \frac{2\psi^2}{A^2}\left(\frac{-B}{2\psi} + \frac{\sqrt{-\Delta}}{\psi}\cot\left(\frac{\sqrt{-\Delta}}{A}\xi\right)\right)^2 + b_2\left(\frac{-B}{2\psi} + \frac{\sqrt{-\Delta}}{\psi}\cot\left(\frac{\sqrt{-\Delta}}{A}\xi\right)\right)^{-2}.$$

$$u_{3_9}(\xi) = a_0 - \frac{2\psi^2}{A^2}\left(\frac{-B}{2\psi} - \frac{\sqrt{-\Delta}}{\psi}\tan\left(\frac{\sqrt{-\Delta}}{A}\xi\right)\right)^2 + b\left(\frac{-B}{2\psi} - \frac{\sqrt{-\Delta}}{\psi}\tan\left(\frac{\sqrt{-\Delta}}{A}\xi\right)\right)^{-2}.$$

4. Discussion

Comparison of the *new* with the *basic* (G'/G)-expansion is given below followed by the advantages. In Ref. [24], Song and Ge used the linear ordinary differential

equation as auxiliary equation and traveling wave solutions presented in the form $u(\xi) = \sum_{i=0}^{m} a_i (G'/G)^i$, where $a_m \neq 0$. It is notable to point out that several of our solutions are coincided with already published results, if parameters are given particular values which verify our solutions. Furthermore, in Ref. [24], Song and Ge investigated the (3+1)-dimensional Kadomtsev–Petviashvili equation to find exact solutions via the basic (G'/G)-expansion method and achieved only three solutions (A1)-(A6) in the Appendix. In contrast we give a set of twenty seven solutions for the (3+1)-dimensional Kadomtsev–Petviashvili equation.

This reveals the immediate advantage of the *new approach* over the *basic* (G'/G)-expansion method, since it provides a large quantity of new exact solutions with some free parameters. The extra solutions of course yield more significance into the understanding of the physical phenomena studied. On top of the enriched physical significance, the NLEEs closed form solutions assist the numerical solvers to compare the correctness of their results and help them in convergence and stability analysis studies.

5. Conclusion and future work

The new generalized (G'/G)-expansion method is innovatively and lucratively used to establish traveling wave solutions of the (3+1)-dimensional Kadomtsev–Petviashvili equation. Comparing with the other methods in the literature, the new generalized (G'/G)-expansion method appears to be easier and faster, by means of the symbolic algebra packages. This article confirms that the method is direct, brief and effective. The method can be used for treating many other NLEEs of mathematical physics. Treading in this direction we plan to use the Sumudu transform to study the equations and compare its solutions to the those already obtained herein and in the literature. We plan to use a numerico-analytical hybrid approach based on the new generalized expansion, and Sumudu transform methods [37]–[46].

Appendix: Song and Ge's solutions [24]

Song and Ge [24] obtained the following exact (3+1)-dimensional Kadomtsev–Petviashvili solutions by using the *basic* (G'/G)-expansion method:

When $\lambda^2 - 4\mu > 0$,

$$(A.1) \qquad u_1 = -\tfrac{1}{2}(\lambda^2 - 4\mu)\left(\frac{C_1 \sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi\right)+C_2 \cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi\right)}{C_1 \cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi\right)+C_2 \sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi\right)}\right)^2 + \tfrac{\lambda^2}{2} - 2\mu,$$

and

$$(A.2) \quad u_2 = -\tfrac{1}{2}(\lambda^2-4\mu)\left(\frac{C_1 \sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi\right)+C_2 \cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi\right)}{C_1 \cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi\right)+C_2 \sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi\right)}\right)^2 - \tfrac{1}{6}(2\lambda^2+4\mu) + \tfrac{\lambda^2}{2},$$

where $\xi = x + y + z - (-6 + \lambda^2 - 4\mu)t$ and C_1, C_2 are arbitrary constants.

For $C_2 \neq 0$, $C_1^2 < C_2^2$, the above solutions (A.1) turns into

$$u_1 = -\tfrac{1}{2}(\lambda^2 - 4\mu)\tanh^2\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2} + \xi_0\right) + \tfrac{\lambda^2}{2} - 2\mu.$$

and the solution (A.2) turns into

$$u_2 = -\tfrac{1}{2}(\lambda^2 - 4\mu)\tanh^2\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2} + \xi_0\right) - \tfrac{1}{6}(2\lambda^2 + 4\mu) + \tfrac{\lambda^2}{2}, \xi_0 = \tanh^{-1}\left(\frac{C_1}{C_2}\right),$$

when $\lambda^2 - 4\mu < 0$,

$$(A.3) \qquad u_3 = -\tfrac{1}{2}(4\mu - \lambda^2)\left(\frac{-C_1\sin\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi\right)+C_2\cos\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi\right)}{C_1\cos\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi\right)+C_2\sin\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi\right)}\right)^2 + \tfrac{\lambda^2}{2} - 2\mu,$$

and

$$(A.4) \quad u_4 = -\tfrac{1}{2}(4\mu-\lambda^2)\left(\frac{-C_1\sin\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi\right)+C_2\cos\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi\right)}{C_1\cos\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi\right)+C_2\sin\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi\right)}\right)^2 - \tfrac{1}{6}(2\lambda^2+4\mu) + \tfrac{\lambda^2}{2},$$

where $\xi = x + y + z - (-6 - \lambda^2 + 4\mu)t$ and C_1, C_2 are arbitrary constants.

For $C_2 \neq 0$, $C_1^2 < C_2^2$, solutions (A.3) and (A.4), respectively, turn into

$$u_3 = -\tfrac{1}{2}(4\mu - \lambda^2)\cot^2\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2} + \xi_0\right) + \tfrac{\lambda^2}{2} - 2\mu,$$

and

$$u_4 = -\tfrac{1}{2}(4\mu - \lambda^2)\cot^2\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2} + \xi_0\right) - \tfrac{1}{6}(2\lambda^2 + 4\mu) + \tfrac{\lambda^2}{2}, \xi_0 = \tanh^{-1}\left(\frac{C_1}{C_2}\right).$$

when $\lambda^2 - 4\mu = 0$,

$$(A.5) \qquad u_5 = \tfrac{-2C_2^2}{(C_1+C_2\xi)^2} + \tfrac{\lambda^2}{2} - 2\mu,$$

where $\xi = x + y + z + 6t$ and C_1, C_2 are arbitrary constants.

$$(A.6) \qquad u_6 = \tfrac{-2C_2^2}{(C_1+C_2\xi)^2} - \tfrac{1}{6}(2\lambda^2 + 4\mu) + \tfrac{\lambda^2}{2},$$

where $\xi = x + y + z + 6t$ and C_1, C_2 are arbitrary constants.

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COLORING OF BIFUZZY GRAPHS

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Abstract. In this paper, we introduce coloring function of a bifuzzy edge graph $G = (V, B)$ using (α, β)-cuts of G and determine the chromatic number of G. We compute k-coloring of bifuzzy graph $\tilde{G} = (V, A, B)$ based on the family of bifuzzy sets on V. We establish strong coloring of a bifuzzy graph and compute strong chromatic number. We also present a new channel assignment problem by using a bifuzzy graph.

Keywords: bifuzzy sets, bifuzzy graphs, chromatic number, bifuzzy number.
Mathematics Subject Classification 2000: 03E72, 05C72, 05C78, 05C99.

1. Introduction

In 1983, Atanassov [7] introduced the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets [28]. Atanassov added a new component(which determines the degree of non-membership) in the definition of fuzzy set. The fuzzy sets give the degree of membership of an element in a given set (and the non-membership degree equals one minus the degree of membership), while intuitionistic fuzzy sets give both a degree of membership and a degree of non-membership which are more-or-less independent from each other, the only requirement is that the sum of these two degrees is not greater than 1. Gerstenkorn and Mańko [14] renamed the intuitionistic fuzzy sets as bifuzzy sets. The concept of bifuzzy set can be viewed as an alternative approach in case where available information is not sufficient to define the impreciseness by the conventional fuzzy set. Intuitionistic fuzzy sets are being studied and used in different fields of science including computer science, mathematics, medical and engineering [12], [15], [19]. The concept of fuzzy numbers and its arithmetic operations were first introduced and investigated by Chang and Zadeh [11]. Burillo et al. [10] proposed the definition of intuitionistic fuzzy number. Mahapatra and Mahapatra [21] defined trapezoidal intuitionistic fuzzy number and arithmetic operations of trapezoidal intuitionistic fuzzy number based on (α, β)-cut method. Based on Zadeh's fuzzy relations [29] Kaufmann defined in

[17] a fuzzy graph. Rosenfeld [26] described the structure of fuzzy graphs obtaining analogs of several graph theoretical concepts. Bhattacharya [9] gave some remarks on operations on fuzzy graphs introduced by Mordeson and Nair in [23]. Kóczy [19] discussed fuzzy graphs by considering fuzzy edge set with crisp vertex set, fuzzy vertex set with crisp edge set, fuzzy vertex and edge set or crisp vertices and edges with fuzzy connectivity and crisp graph with fuzzy weights. The concept of chromatic number of fuzzy graph was introduced by Muñoz et. al. [24]. The authors considered fuzzy graphs with crisp vertex set, i.e., fuzzy graphs $G = (\sigma, \mu)$ for which $\sigma(v) = 1 \; \forall \; v \in V$, and edges with membership degree in $[0, 1]$. Eslahchi and Onagh [13] introduced coloring of fuzzy graphs $G = (\sigma, \mu)$, that fuzzy coloring partitions the vertex set into different color classes and the number of distinct color classes is the fuzzy chromatic number. Shannon and Atanassov [27] introduced the concept of intuitionistic fuzzy graphs and investigated some of their properties. Parvathi et al. defined operations on intuitionistic fuzzy graphs in [25]. Akram et al. [1]–[4] introduced many new concepts, including strong intuitionistic fuzzy graphs, intuitionistic fuzzy trees, intuitionistic fuzzy hypergraphs, and intuitionistic fuzzy digraphs in decision support systems. In this paper, we introduce coloring function of a bifuzzy edge graph $G = (V, B)$ using (α, β)-cuts of G and determine the chromatic number of G. We compute k-coloring of bifuzzy graph $\tilde{G} = (V, A, B)$ based on the family of bifuzzy sets on V. We establish strong coloring of a bifuzzy graph and compute strong chromatic number. We also present a new channel assignment problem by using a bifuzzy graph.

2. Prelimilaries

In this section, we review some basic definitions which will be used in our next sections.

Definition 2.1 [28],[29] A *fuzzy subset* μ on a set X is a mapping $\mu : X \to [0, 1]$. A mapping $\nu : X \times X \to [0, 1]$ is called a *fuzzy relation* on X if $\nu(x, y) \leq \min\{\mu(x), \mu(y)\}$ for all $x, y \in X$.

Definition 2.2 [24] If $G = (V, \mu)$ is a fuzzy graph, where $V = \{1, 2, ..., n\}$ and μ is a fuzzy relation on the set of all subsets of $V \times V$. Assume $I = \alpha \cup \{0\}$, where $\alpha = \{\alpha_1 < \alpha_2 < \cdots < \alpha_k\}$ is the fundamental set (level set) of G. For each $\alpha \in I$, G_α denotes the crisp graph $G_\alpha = (V, E_\alpha)$, where $E_\alpha = \{ij \mid 1 \leq i < j \leq n, \mu(i, j) \geq \alpha\}$ and $\chi_\alpha = \chi(G_\alpha)$ denotes the chromatic number of crisp graph G_α. The chromatic number of fuzzy graph G is the fuzzy number $\chi(G) = \{(x, \nu(x)) \mid x \in X\}$, where $X = \{1, 2, ..., |V|\}$, $\nu(x) = \sup\{\alpha \in I \mid x \in A_\alpha\}$, $\forall x \in X$ and $A_\alpha = \{1, 2, ..., \chi_\alpha\}$, $\forall \alpha \in I$.

Definition 2.3 [13] A family $\Gamma = \{\gamma_1, \gamma_2, ..., \gamma_k\}$ of fuzzy sets on V is called a *k-fuzzy coloring* of $G = (V, \sigma, \mu)$ if

(a) $\bigvee \Gamma = \sigma$,

(b) $\gamma_i \bigwedge \gamma_j = 0$,

(c) for every strong edge xy of G, $\min\{\gamma_i(x), \gamma_i(y)\} = 0$ $(0 \leq i \leq k)$.
The least value of k for which G has a *k-fuzzy coloring*, denoted by $\chi^f(G)$, is called the *fuzzy chromatic number* of G.

Gerstenkorn and Mańko [14] re-named the intuitionistic fuzzy sets as bifuzzy sets.

Definition 2.4 [7] A mapping $A = (\mu_A, \nu_A) : X \to [0,1] \times [0,1]$ is called a *bifuzzy set* in X if $\mu_A(x) + \nu_A(x) \leq 1$ $\forall x \in X$, where the mappings $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\nu_A(x)$) of each element $x \in X$ to A, respectively.

Definition 2.5 [10] A bifuzzy set $A = \{(x, \mu_A(x)), \nu_A(x) | x \in X\}$ is called a bifuzzy number (IFN) if the following holds:

(i) A is a bifuzzy subset of real line R,

(ii) normal, i.e., there is any $x_0 \in R$ such that $\mu_A(x_0) = 1$, $\nu_A(x_0) = 0$
(x_0 is called the mean value of A),

(iii) A is convex set for the membership function $\mu_A(x)$, i.e.,
$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\mu_A(x_1), \mu_A(x_2)\},\ \forall\ x_1, x_2 \in R,\ \forall\ \lambda \in [0,1]$,

(iv) A is concave set for non-membership function $\nu_A(x)$, i.e.,
$\nu_A(\lambda x_1 + (1-\lambda)x_2) \leq \max\{\nu_A(x_1), \nu_A(x_2)\},\ \forall\ x_1, x_2 \in R, \forall\ \lambda \in [0,1]$.

Definition 2.6 [22] Let $A{=}(V, \mu_A, \nu_A)$ be a bifuzzy set and $\alpha, \beta \in [0,1]$, $\alpha{+}\beta \leq 1$. Then we call $A^{(\alpha,\beta)} = \{x \in V |\ \mu_A(x) \geq \alpha,\ \nu_A(x) \leq \beta\}$ the $(\alpha,\beta)-$*cut set* of A.

Definition 2.7 [8] Let $A = (\mu_A, \nu_A)$ be a bifuzzy set on V. The *support* of A is defined as $\text{supp}(A) = \{x \in V |\ \mu_A(x) \neq 0, \nu_A(x) \neq 0\}$.

3. Coloring of bifuzzy graph with crisp nodes

In this section, we present coloring function of bifuzzy graphs with crisp nodes and bifuzzy edges.

Definition 3.8 A *bifuzzy graph* is defined as a pair $G = (V, B)$ such that

(i) V is the crisp set of nodes,

(ii) the functions $\mu_B : E \subseteq V \times V \to [0,1]$ and $\nu_B : E \subseteq V \times V \to [0,1]$ are defined by $\mu_B(x,y) \leq \min\{\mu_A(x), \mu_A(y)\}$ and $\nu_B(x,y) \geq \max\{\nu_A(x), \nu_A(y)\}$ such that $0 \leq \mu_B(x,y) + \nu_B(x,y) \leq 1\ \forall\ \{x,y\} \in E$.

We call $B = (\mu_B, \nu_B)$ the bifuzzy edge set.

We consider bifuzzy graphs with crisp vertex set, i.e., bifuzzy graphs $G = (A, B)$ for which $\mu_A(x) = 1$, $\nu_A(x) = 0\ \forall\ x \in V$, and edges with membership and non-membership degrees in $[0,1]$.

Example 3.1 Consider a simple graph $G^* = (V, E)$ such that $V = \{u_1, u_2, u_3, u_4\}$ and $E = \{u_1u_2, u_2u_3, u_3u_4, u_4u_1\}$. Let B be a bifuzzy set on E defined by

$$B = \{(u_1u_2, 0.3, 0.1), (u_2u_3, 0.4, 0.1), (u_3u_4, 0.2, 0.1), (u_4u_1, 0.3, 0.2)\}.$$

$G = (V, B)$ is a bifuzzy graph of $G^* = (V, E)$ with crisp set of vertices and bifuzzy set of edges as shown in Figure 1.

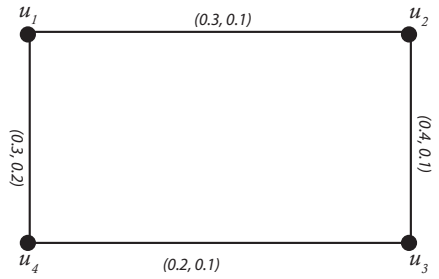

Figure 1: Bifuzzy graph $G = (V, B)$

Definition 3.9 For any $\alpha_i, \beta_i \in I$ such that $\alpha_i + \beta_i \leq 1$ and $I = \alpha \cup \beta \cup \{0\} \cup \{1\}$, where $\alpha = \{\alpha_1 < \alpha_2 < \cdots < \alpha_k\}$ and $\beta = \{\beta_1 < \beta_2 < \cdots < \beta_k\}$ gives fundamental level set of bifuzzy(IF) graph $G = (V, B)$. For each $(\alpha_i, \beta_i) \in I$, $G_{(\alpha_i,\beta_i)} = (V, B^{(\alpha_i,\beta_i)})$ denotes the crisp graph where $B^{(\alpha_i,\beta_i)} = \{(u, v) \in V \times V \mid \mu_B(u, v) \geq \alpha_i, \nu_B(u, v) \leq \beta_i\}$.

Let $\{G_{(\alpha_i,\beta_i)} = (V, B^{(\alpha_i,\beta_i)}) \mid (\alpha_i, \beta_i) \in I\}$ be the family of (α_i, β_i)-cuts sets of bifuzzy graph G. Any (crisp) $k-$coloring $C^k_{(\alpha_i,\beta_i)}$ can be defined on $G_{(\alpha_i,\beta_i)}$. The k-coloring function of bifuzzy graph is defined through these (α_i, β_i)-cuts.

Example 3.2 Consider a simple graph $G^* = (V, E)$ such that $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{v_1v_2, v_2v_3, v_3v_4, v_4v_2\}$. For bifuzzy subgraph G of G^* with crisp set of nodes and bifuzzy set of edges as shown in Figure 2.

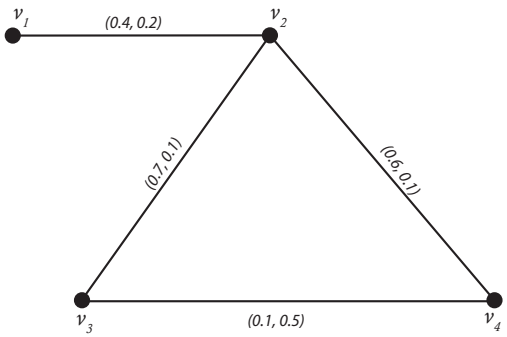

Figure 2: Bifuzzy graph $G = (V, B)$

The $(0.6, 0.1)$-cut set of G is a crisp graph $G_{(0.6,0.1)} = (V, B^{(0.6,0.1)})$, where $B^{(0.6,0.1)} = \{v_2v_3, v_4v_2\}$ as shown in Figure 3.

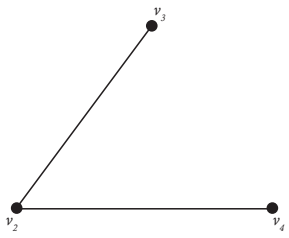

Figure 3: $G_{(0.6,0.1)}$

Definition 3.10 For a bifuzzy graph $G = (V, B)$, its *chromatic number* is a bifuzzy number

$$\chi(G) = \{(x, \nu(x), \omega(x)) \mid x \in X\},$$

where $X = \{1, 2, ..., |V|\}$, $\nu(x) = \sup\{\alpha_i \in I \mid x \in A_{\alpha_i,\beta_i}\}$, $\omega(x) = \{\beta_i \in I \mid \nu(x) = \alpha_i,\ x \in A_{\alpha_i,\beta_i}\}\ \forall x \in X$ and $A_{\alpha_i,\beta_i} = \{1, 2, ..., \chi_{(\alpha_i,\beta_i)}\}$, $\forall(\alpha_i, \beta_i) \in I$.

The chromatic number of a bifuzzy graph is a normalized bifuzzy number whose normalized value is associated with the graph having no edge. The bifuzzy graph coloring problem consists of determining the chromatic number of a bifuzzy graph and an associated coloring function. In our method, for any level (α_i, β_i), the minimum number of colors needed to color the crisp graph $G_{(\alpha_i,\beta_i)}$ will be computed. In this way, the bifuzzy chromatic number $\chi(G)$ will be defined as a bifuzzy number through (α_i, β_i)-cuts.

4. Coloring of bifuzzy graphs with bifuzzy nodes

In Section 3, we introduced the coloring function and chromatic number of bifuzzy graphs $G = (V, B)$, V is crisp set of vertices $(\mu_A(v) = 1, \nu_A(v) = 0\ \forall\ v \in V)$, B is bifuzzy set of edges. Two vertices u and v in a graph G are adjacent then the edge uv is called strong edge and weak otherwise. In the following definition, the chromatic number of bifuzzy graph is a number not a bifuzzy number.

Definition 4.11 [3] A *bifuzzy graph* with underlying set V is defined to be a pair $\tilde{G} = (V, A, B)$ such that

(i) $\mu_A : V \to [0, 1]$ and $\nu_A : V \to [0, 1]$ denote the degree of membership and non-membership of the element $x \in V$, respectively such that

$$\mu_A(x) + \nu_A(x) \leq 1,\ \forall\ x \in V,$$

(ii) the functions $\mu_B : E \subseteq V \times V \to [0, 1]$ and $\nu_B : E \subseteq V \times V \to [0, 1]$ are defined by $\mu_B(x, y) \leq \min\{\mu_A(x), \mu_A(y)\}$ and $\nu_B(x, y) \geq \max\{\nu_A(x), \nu_A(y)\}$ such that

$$0 \leq \mu_B(x, y) + \nu_B(x, y) \leq 1,\ \forall xy \in E.$$

We call A the bifuzzy vertex set of V, B the bifuzzy edge set of E, respectively. $\tilde{G} = (V, A, B)$ is a *bifuzzy graph* of $G^* = (V, E)$.

The level set of bifuzzy set A is defined as

$$L_A = \{(\alpha, \beta) \mid \mu_A(u) = (\alpha, \beta) \text{for some } u \in V\},$$

and the level set of bifuzzy set B is defined as

$$L_B = \{(\alpha, \beta) \mid \mu_B(u, v) = (\alpha, \beta) \text{ for some } (u, v) \in E\}.$$

Definition 4.12 The fundamental set of bifuzzy graph $\tilde{G} = (V, A, B)$ is defined as $L = L_A \cup L_B$, for each $(\alpha, \beta) \in L$, $\tilde{G}_{\alpha,\beta}$ denotes crisp graph $\tilde{G}_{\alpha,\beta} = (A^{\alpha,\beta}, B^{\alpha,\beta})$ where

$$A^{\alpha,\beta} = \{v \in V \mid \mu_A(v) \geq \alpha, \lambda_A(v) \leq \beta\},$$
$$B^{\alpha,\beta} = \{(u, v) \in V \times V \mid \mu_B(u, v) \geq \alpha, \lambda_B(u, v) \leq \beta\}.$$

Example 4.3 Consider a simple graph $G^* = (V, E)$ such that

$$V = \{u_1, u_2, u_3, u_4\} \text{ and } E = \{u_1u_2, u_2u_3, u_3u_4, u_4u_1\}.$$

For bifuzzy graph $\tilde{G}$ of G^*, as shown in Figure 4, $A^{(0.2,0.7)} = \{u_1, u_2, u_3, u_4\}$ and $B^{(0.2,0.7)} = \{u_1u_2, u_2u_3\}$.

So, $\tilde{G}_{(0.2,0.7)} = (A^{(0.2,0.7)}, B^{(0.2,0.7)})$ is $(0.2, 0.7)$-cut set of $\tilde{G}$.

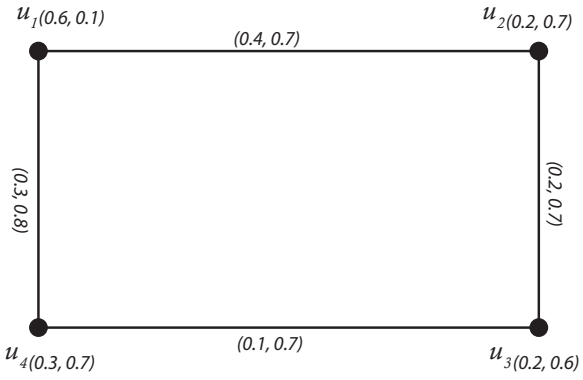

Figure 4: Bifuzzy graph $\tilde{G} = (V, A, B)$

Definition 4.13 A family $\Gamma = \{a_1, a_2, \cdots, a_k\}$ of bifuzzy sets on V is called a *k-coloring* of bifuzzy graph $\tilde{G} = (V, A, B)$ if

(a) $\bigvee \Gamma = A$,

(b) $a_i \bigwedge a_j = 0$,

(c) for every strong edge xy of $\tilde{G}$, $\min\{a_i(x), a_i(y)\} = 0$ $(0 \leq i \leq k)$.

The least value of k for which $\tilde{G}$ has a k-coloring of $\tilde{G} = (V, A, B)$, denoted by $\chi^{if}(\tilde{G})$, is called the bifuzzy chromatic number of $\tilde{G}$.

In Definition 4.3, bifuzzy coloring means that one of the vertices is having color 0 in particular color class and support set for each bifuzzy set a_i, $1 \leq i \leq k$ is an independent set of bifuzzy graph $\tilde{G}$. As in crisp coloring bifuzzy coloring partitioned the set of vertices into independent sets of vertices.

Example 4.4 Consider the following bifuzzy graph $\tilde{G} = (V, A, B)$ as shown in Figure 5.

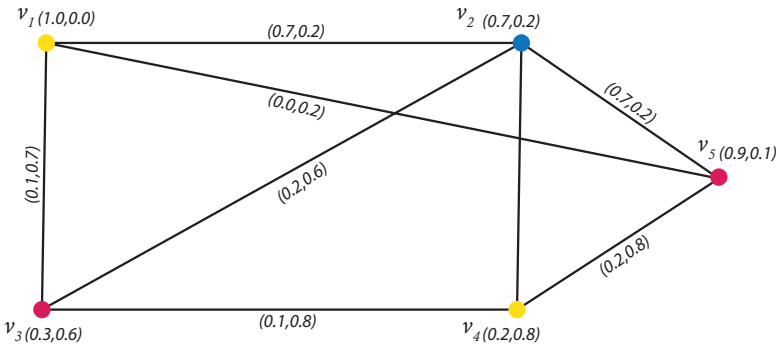

Figure 5: Intuitionistic fuzzy graph $\tilde{G} = (V, A, B)$

The bifuzzy coloring is $\Gamma = \{a_1, a_2, a_3\}$,

$$a_1(v_i) = \begin{cases} (1.0, 0.0) & \text{i=1;} \\ (0.2, 0.8) & \text{i=4;} \\ 0 & \text{otherwise,} \end{cases} \qquad a_2(v_i) = \begin{cases} (0.3, 0.6) & \text{i=3;} \\ (0.9, 0.1) & \text{i=5;} \\ 0 & \text{otherwise,} \end{cases}$$

$$a_3(v_i) = \begin{cases} (0.7, 0.2) & \text{i=2;} \\ 0 & \text{otherwise.} \end{cases} \quad \text{Hence } \chi^{if}(\tilde{G}) = 3.$$

Remark 4.1 If a bifuzzy graph $G = (V, B)$ with crisp set of vertices and bifuzzy set of edges, then there exist $(\alpha, \beta) \in I$ such that $\chi^{if}(G) = \chi_{\alpha,\beta}$. For bifuzzy graph $\tilde{G} = (V, A, B)$, it is possible that there does not exist $(\alpha, \beta) \in I$ such that $\chi^{if}(\tilde{G}) \neq \chi_{\alpha,\beta}$.

Example 4.5 Consider a bifuzzy graph $\tilde{G} = (V, A, B)$, as shown in Figure 6, such that

$$A = \{(v_1, 0.5, 0.0), (v_2, 0.7, 0.2), (v_3, 0.7, 0.0), (v_4, 0.5, 0.2)\},$$
$$B = \{(v_1v_2, 0.4, 0.2), (v_1v_4, 0.5, 0.2), (v_1v_3, 0.3, 0.1), (v_3v_4, 0.5, 0.2)\}.$$

We see that

$$\chi_{0.5,0.0} = \chi_{0.7,0.0} = \chi_{0.7,0.2} = 1, \chi_{0.3,0.1} = \chi_{0.5,0.2} = \chi_{0.4,0.2} = 2,$$

but $\chi^{if}(\tilde{G}) = 3$.

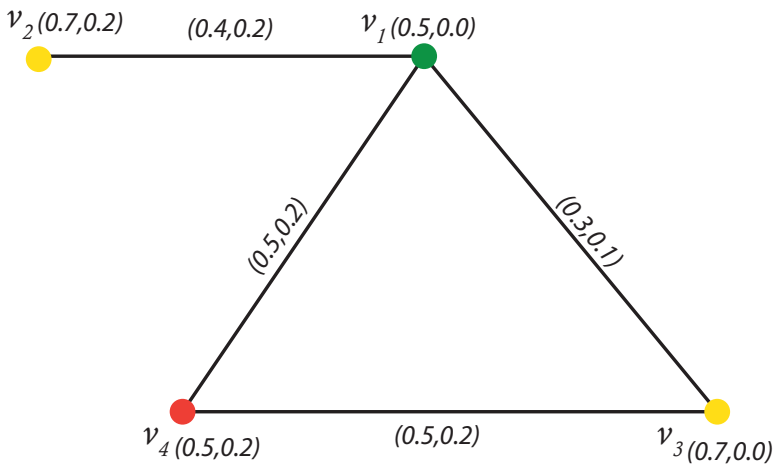

Figure 6: Bifuzzy graph $\tilde{G} = (V, A, B)$

When a fundamental set of bifuzzy set $\tilde{G}$ have some $(\alpha, \beta) \in L$ such that $\chi_{\alpha,\beta} = \chi^{if}(\tilde{G})$, then the following definition and theorem must hold.

Definition 4.14 The *chromatic number* of bifuzzy graph $\tilde{G} = (V, A, B)$ is defined as $\chi(\tilde{G}) = \max\{\chi_{\alpha,\beta} | \ (\alpha, \beta) \in L\}$, where $\chi_{\alpha,\beta} = \chi(\tilde{G}_{\alpha,\beta})$.

Theorem 4.1 *For a bifuzzy graph* $\tilde{G} = (V, A, B)$, $\chi(\tilde{G}) = \chi^{if}(\tilde{G})$.

Proof. Let $\tilde{G} = (V, A, B)$ be a bifuzzy graph on n vertices $V = \{v_1, v_2, ..., v_n\}$. Let $\chi^{if}(\tilde{G}) = k \Leftrightarrow \Gamma = \{a_1, a_2, ..., a_k\}$ is a k-coloring of bifuzzy graph $\tilde{G}$ and C_j color assigned to vertices lies in a_j^*, $j = 1, 2, ..., k \Leftrightarrow \{a_1, a_2, ..., a_k\}$ is a family of bifuzzy sets where $a_j = \{(v_j, \mu_{a_j}(v_j), \nu_{a_j}(v_j))\} \cup \{(v_i, \mu_{a_j}(v_i), \nu_{a_j}(v_i)) \mid v_j v_i \notin E, i \neq j\}$ which follows from (a) and (c) of Definition 4.3. It follows from (b) of Definition 4.3 $\bigcup_{j=1}^{k} a_j^* = V$ and $a_i^* \cap a_j^* = \emptyset, i \neq j \Leftrightarrow a_j^*$ is an independent set of vertices that is no two vertices in a_j^* are adjacent for each $j = 1, 2, ..., k \Leftrightarrow \chi(G^*) = k$, G^* is the underlying crisp graph of $\tilde{G}$.

Now, chromatic number of G^* is equal to chromatic number of some $\tilde{G}_{\alpha,\beta}$, i.e.,

$$\chi(G^*) = \chi(\tilde{G}_{\alpha,\beta}) = \chi_{\alpha,\beta} = \max\{\chi_{\alpha,\beta} \mid (\alpha, \beta) \in L\}.$$ ∎

Example 4.6 Consider bifuzzy graph $\tilde{G}$ defined on the vertex set $\{u_1, u_2, u_3, u_4\}$ and the crisp graphs $\tilde{G}_{(0.2,0.7)}$, $\tilde{G}_{(0.1,0.7)}$, $\tilde{G}_{(0.2,0.1)}$, $\tilde{G}_{(0.2,0.8)}$ as shown in Figure 7 corresponding to the level set of the fuzzy graph,

$$L = \{(0.2, 0.7), (0.1, 0.6), (0.2, 0.1), (0.2, 0.8)\}.$$

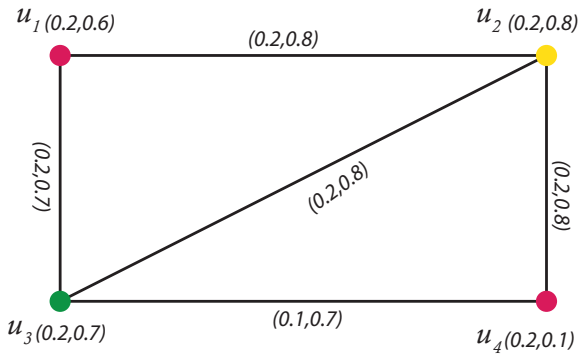

Figure 7: Bifuzzy graph $\tilde{G} = (V, A, B)$

The bifuzzy coloring is $\Gamma = \{a_1, a_2, a_3\}$,

$$a_1(v_i) = \begin{cases} (0.2, 0.6) & \text{i=1;} \\ (0.2, 0.1) & \text{i=4;} \\ 0 & \text{otherwise,} \end{cases} \qquad a_2(v_i) = \begin{cases} (0.2, 0.7) & \text{i=2;} \\ 0 & \text{otherwise,} \end{cases}$$

$$a_3(v_i) = \begin{cases} (0.2, 0.7) & \text{i=3;} \\ 0 & \text{otherwise.} \end{cases}$$

So, $\chi^{if}(\tilde{G}) = 3$ and the coloring of crisp graphs yields

$$\chi_{0.2,0.7} = \chi_{0.1,0.7} = 2, \chi_{0.2,0.1} = 1 \text{ and } \chi_{0.2,0.8} = 3.$$

Hence,

$$\chi^{if}(\tilde{G}) = \max\{\chi_{\alpha,\beta} \mid (\alpha, \beta) \in L\} = \chi_{0.2,0.8} = 3.$$

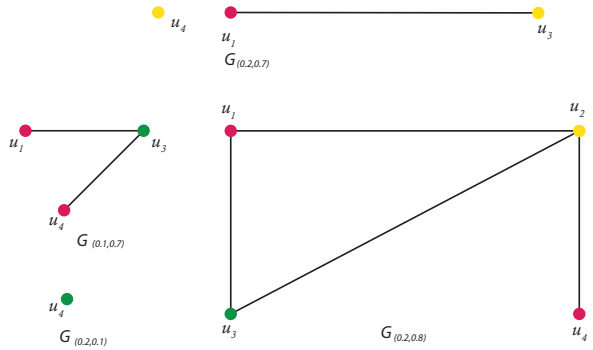

Figure 8: Crisp graphs $\tilde{G}_{(0.2,0.7)}$, $\tilde{G}_{(0.1,0.7)}$, $\tilde{G}_{(0.2,0.1)}$, $\tilde{G}_{(0.2,0.1)}$.

5. Coloring of bifuzzy graphs based on strong arcs

Coloring of bifuzzy graph will be used to solve many real life problems. Now, we define the coloring in a bifuzzy graph based on strong arcs. Moreover, the strong chromatic number of bifuzzy graph, complete bifuzzy graph and bifuzzy tree are discussed.

Definition 5.15 [16] If $u, w \in V$, the μ-strength of connectedness between u and w is

$$CONN_{\mu(B)}(u,w) = \max\{\mu_B(u,v_1) \wedge \mu_B(v_1,v_2) \wedge ... \wedge \mu_B(v_{k-1},w) \mid u, v_1, v_2, ..., v_{k-1}, w \in V,\ k = 1,2,...,n\}.$$

The ν-strength of connectedness between u and w is

$$CONN_{\nu(B)}(u,w) = \min\{\mu_B(u,v_1) \vee \mu_B(v_1,v_2) \vee ... \vee \mu_B(v_{k-1},w) \mid u, v_1, v_2, ..., v_{k-1}, w \in V,\ k = 1,2,...,n\}.$$

The μ-strength and ν-strength of connectedness between u and w in $\tilde{G}$ are denoted by $CONN_{\mu(\tilde{G})}(u,w)$ and $CONN_{\nu(\tilde{G})}(u,w)$, respectively.

Definition 5.16 Let $\tilde{G} = (V, A, B)$ be a bifuzzy graph. The *strong coloring* of a bifuzzy graph $\tilde{G}$ is a mapping $C : V(\tilde{G}) \to N$ such that $C(u) \neq C(v)$ if (u, v) is a strong arc (α-strong and β-strong) in $\tilde{G}$ is called strong coloring.

A bifuzzy graph $\tilde{G}$ is said to be $k-$strong colorable if there exist a strong coloring of G from a set of k-colors. The minimum number of colors k for which $\tilde{G}$ is $k-$strong colorable is called strong chromatic number of bifuzzy graph $\tilde{G}$ denoted by $\chi^{is}(\tilde{G})$. The end nodes of a $\delta-$arc can be assigned the same color in strong coloring of bifuzzy graph.

Example 5.7 Consider the bifuzzy graph $\tilde{G}$ as shown in Figure 9.

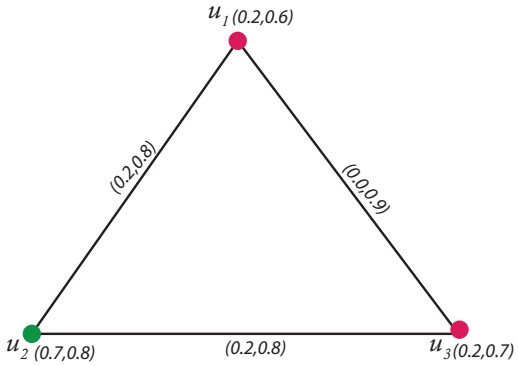

Figure 9: Strong coloring of bifuzzy graph $\tilde{G}$

We find that arc (u_1, u_2) is $\alpha-$strong arc as
$\mu_B(u_1, u_2) = 0.2 > CONN_{\mu(\tilde{G})-(u_1,u_2)}(u_1, u_2) = 0.0$ and
$\nu_B(u_1, u_2) = 0.8 < CONN_{\nu(\tilde{G})-(u_1,u_2)}(u_1, u_2) = 0.9$.
Similarly, arc (u_2, u_3) is $\alpha-$strong. But (u_1, u_3) is $\delta-$ arc as
$\mu_B(u_1, u_3) = 0.0 < CONN_{\mu(\tilde{G})-(u_1,u_3)}(u_1, u_3) = 0.2$ and
$\nu_B(u_1, u_3) = 0.9 > CONN_{\nu(\tilde{G})-(u_1,u_3)}(u_1, u_3) = 0.8$.
Hence strong coloring of bifuzzy graph gives $\chi^{is}(\tilde{G}) = 2$.

Proposition 5.1 *For a bifuzzy graph $\tilde{G} = (V, A, B)$, if $\tilde{G}$ is a complete bifuzzy graph or bifuzzy cycle then $\chi^{is}(\tilde{G}) = \chi^{if}(\tilde{G})$.*

Proof. Let $\tilde{G}$ be a complete bifuzzy graph of the complete graph G^* on n vertices, since a complete bifuzzy graph has no δ-arcs [16] and all the arcs are strong arcs, then $\chi^{is}(G) = \chi^{if}(\tilde{G}) = n$. $\tilde{G}$ is a bifuzzy cycle on n vertices of the cycle graph G^* then all the arcs in $\tilde{G}$ are strong arcs($\tilde{G}$ has at least two β-arcs) and $\tilde{G}$ has no δ-arcs in bifuzzy cycle. Thus, $\chi^{is}(\tilde{G}) = \chi^{if}(\tilde{G}) = 2$, when n is even and $\chi^{is}(\tilde{G}) = \chi^{if}(\tilde{G}) = 3$, when n is odd. ∎

Remark 5.2 The converse of the above proposition is not hold.

Example 5.8 Consider the bifuzzy graph $\tilde{G}$ as shown in Figure 10, we find that arcs (v_1, v_2), (v_1, v_5), (v_2, v_3) (v_3, v_4), (v_3, v_5) and (v_2, v_4) are strong arcs.

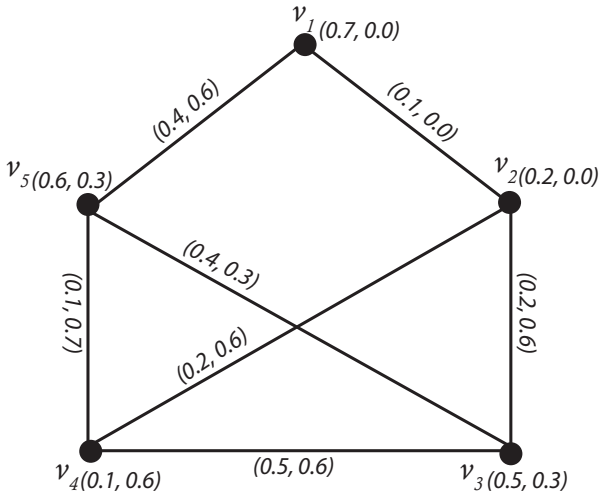

Figure 10: Bifuzzy graph $\tilde{G}$

The chromatic number and strong chromatic number of intuitionistic graph $\tilde{G}$ is 3, i.e., $\chi^{is}(\tilde{G}) = \chi^{if}(\tilde{G}) = 3$, but $\tilde{G}$ is neither complete nor is a bifuzzy cycle.

Proposition 5.2 *If $\tilde{G}$ is a bifuzzy tree, then $\chi^{is}(\tilde{G}) = \chi^{if}(T)$ where T is maximum spanning tree of $\tilde{G}$.*

Proof. Since there are no $\beta-$strong arcs in bifuzzy tree $\tilde{G}$, an arc (u, v) in a bifuzzy tree is strong if and only if (u, v) is an arc in T and the arcs of T are bifuzzy bridge. In bifuzzy graph an arc (u, v) is bifuzzy bridge if and only if (u, v) is α-strong arc [16]. Hence strong coloring of $\tilde{G}$ is exactly same as coloring of T and $\chi^{is}(\tilde{G}) = \chi^{if}(T)$. ∎

6. Application

A graph is a convenient way of representing information involving relationship between objects. The objects are represented by vertices and relations by edges. In many real world problems, we get partial information about that problem. So there is vagueness in the description of the objects or in its relationships or in both. To describe this type of relation, we need to design graph model with fission of bifuzzy set. This fission of bifuzzy set with graph is known as bifuzzy graph. For solving bifuzzy coloring problem, an algorithm which calculates the chromatic number of every crisp graph $G_{(\alpha_i,\beta_i)}$ can be used. In order to illustrate the concept of the coloring function of bifuzzy graph and its associated bifuzzy chromatic number, a bifuzzy coloring problem is presented. The problem of deciding which radio channels are best to use at different places is known as channel assignment problem. When there are n receiver and m transmitter, construct a graph in which each node represents a radio transmitter and an edge represents the joining of two nodes if the transmitters of the corresponding nodes would interfere. The interference of two nodes arises if they were assigned the same channel. An edge between two nodes represents the incompatibility of nodes. In many real life problems, such incompatibility is not crisp. The incompatibility between two nodes can be represented by different degrees. We solve this channel assignment problem with the help of bifuzzy coloring, there will be no interference between nodes.

We construct a bifuzzy graph $G = (V, B)$, as shown in Figure 11, with crisp set of vertices and bifuzzy set of edges i.e., $V = \{T_1, T_2, T_3, T_4, T_5, T_6\}$

$$\begin{aligned} B = \{&(e_1 = T_1T_2, 0.9, 0.1), (e_2 = T_1T_3, 0.7, 0.2), (e_3 = T_1T_4, 0.2, 0.8),\\ &(e_4 = T_1T_5, 0.3, 0.6), (e_5 = T_2T_3, 0.0, 0.2), (e_6 = T_2T_6, 0.1, 0.7),\\ &(e_7 = T_3T_4, 0.7, 0.1), (e_8 = T_3T_5, 0.9, 0.1), (e_9 = T_3T_6, 0.1, 0.7),\\ &(e_{10} = T_4T_5, 0.3, 0.6), (e_{11} = T_4T_6, 0.1, 0.8), (e_{12} = T_5T_6, 0.2, 0.6),\\ &(e_{13} = T_1T_6, 0.0, 0.0), (e_{14} = T_2T_4, 0.0, 0.0), (e_{15} = T_2T_5, 0.0, 0.0)\} \end{aligned}$$

In Figure 11, degree of membership and non-membership of edges represents the incompatibility and compatibility between nodes. The fundamental set of bifuzzy graph $G = (V, B)$ is defined as

$$I = \{(1, 0), (0.9, 0.1), (0.7, 0.2), (0.3, 0.6), (0.2, 0.8), (0.1, 0.7), (0.1, 0.8), (0, 1)\}.$$

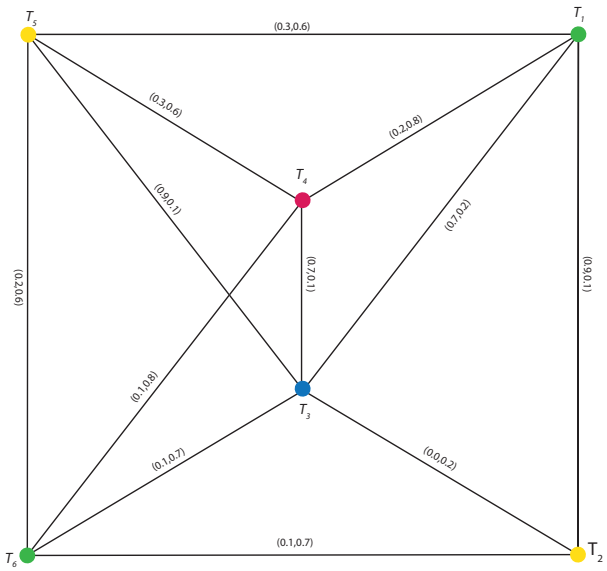

Figure 11: Bifuzzy graph $G = (V, B)$

The family of $(\alpha, \beta)-$cuts sets of G is

$$\{G_{(1,0)}, G_{(0.9,0.1)}, G_{(0.7,0.2)}, G_{(0.3,0.6)}, G_{(0.2,0.8)}, G_{(0.1,0.7)}, G_{(0.1,0.8)}, G_{(0,1)}\}.$$

For each $(\alpha, \beta) \in I$, $(\alpha, \beta)-$cuts sets of bifuzzy graph are crisp graphs, crisp coloring can be defined on each $(\alpha, \beta)-$cut, Table 1 contains the edge set $E_{\alpha,\beta}$, the chromatic number $\chi_{\alpha,\beta}$ and a $\chi_{\alpha,\beta}$-coloring $C^{\chi_{\alpha,\beta}}_{\alpha,\beta}$.

Table 1

(α, β)	$E_{\alpha,\beta}$	$\chi_{\alpha,\beta}$	$c^{\chi_{\alpha,\beta}}(T_1)$	$c^{\chi_{\alpha,\beta}}(T_2)$	$c^{\chi_{\alpha,\beta}}(T_3)$	$c^{\chi_{\alpha,\beta}}(T_4)$	$c^{\chi_{\alpha,\beta}}(T_5)$	$c^{\chi_{\alpha,\beta}}(T_6)$
$(0, 1)$	$e_1, e_2, e_3, e_4,$ $e_{13}, e_5, e_{14},$ $e_{15}, e_6, e_7,$ $e_8, e_9, e_{10},$ e_{11}, e_{12}	6	1	2	3	4	5	6
$(0.9, 0.1)$	e_1, e_8	2	1	2	1	1	1	1
$(0.7, 0.2)$	e_1, e_2, e_7, e_8	2	1	2	1	2	2	1
$(0.3, 0.6)$	$e_1, e_2, e_4,$ e_7, e_8, e_{10}	3	1	2	3	1	2	1
$(0.2, 0.8)$	$e_1, e_2, e_3, e_4,$ e_7e_8, e_{10}, e_{12}	4	1	2	2	3	4	1
$(0.1, 0.8)$	$e_1, e_2, e_3, e_4,$ $e_5, e_6, e_7, e_8,$ $e_9, e_{10}, e_{11}, e_{12}$	4	1	2	3	2	4	1
$(0.1, 0.7)$	$e_1, e_2, e_3, e_4,$ $e_5, e_6, e_7, e_8,$ $e_9, e_{10}, e_{11}, e_{12}$	4	1	2	3	2	4	1
$(1, 0)$	Φ	1	1	1	1	1	1	1

The chromatic number of bifuzzy graph $G = (V, B)$ is

$$\chi(G) = \{(1, 1, 0), (2, 0.9, 0.1), (3, 0.3, 0.6), (4, 0.2, 0.8), (5, 0, 1), (6, 0, 1)\}.$$

The analysis of chromatic number $\chi(G)$ of bifuzzy graph shows that there are more incompatible links between nodes for the lower value of α in $(\alpha, \beta)-$cut of G. Therefore more colors are needed to assign channels to the nodes in such a way nodes that are adjacent get different channels, then there will be no interference. On the other hand, there are less incompatible links between nodes for higher value of α in $(\alpha, \beta)-$cut of G.

7. Conclusion and future work

Fuzzy graph coloring has a variety of applications to problems involving scheduling and assignments. A bifuzzy model is a generalization of the fuzzy model. The bifuzzy models give more precision, flexibility and compatibility to the system as compared to fuzzy models. In this paper, we have defined the coloring function of bifuzzy graphs with crisp nodes and bifuzzy edges, the channel assignment problem solved with this coloring function, chromatic number can be defined according to the significance of bifuzzy edges of the bifuzzy graph. We defined coloring of bifuzzy graph with bifuzzy sets of nodes. We also defined strong chromatic number of a bifuzzy graph. We plan to extend our research of fuzzification to (1) m-polar fuzzy hypergraphs, (2) m-polar fuzzy soft hypergraphs, and (3) m-polar fuzzy rough hypergraphs.

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ON m-POLAR FUZZY LIE SUBALGEBRAS

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Abstract. The notion of an m-polar fuzzy set is a generalization of a bipolar fuzzy set. We apply the concept of m-polar fuzzy sets to Lie algebras. We introduce the concept of m-polar fuzzy Lie subalgebras of a Lie algebra and investigate some of their properties. We also present the homomorphisms between the Lie subalgebras of a Lie algebra and their relationship between the domains and the co-domains of the m-polar fuzzy subalgebras under these homomorphisms.

Keywords: m-polar fuzzy Lie subalgebras, m-polar fuzzy Lie ideal, homomorphisms, Cartesian product.
Subject Classification (2000): 04A72, 17B99.

1. Introduction

Lie algebras were first discovered by Sophus Lie (1842-1899) when he attempted to classify certain *smooth* subgroups of general linear groups. The groups he considered are now called Lie groups. By taking the tangent space at the identity element of such a group, he obtained the Lie algebra and hence the problems on groups can be reduced to problems on Lie algebras so that it becomes more

tractable. There are many applications of Lie algebras in many branches of mathematics and physics.

In 1965, Zadeh [12] introduced the concept of fuzzy subset of a set. A fuzzy set on a given set X is a mapping $A : X \to [0, 1]$. In 1994, Zhang [13] extended the idea of a fuzzy set and defined the notion of bipolar fuzzy set on a given set X as a mapping $A : X \to [-1, 1]$, where the membership degree 0 of an element x means that the element x is irrelevant to the corresponding property, the membership degree in $(0, 1]$ of an element x indicates that the element satisfies the property, and the membership degree in $[-1, 0)$ of an element x indicates that the element somewhat satisfies the implicit counter-property. In 2014, Chen et al. [6] introduced the notion of m-polar fuzzy sets as a generalization of bipolar fuzzy set and showed that bipolar fuzzy sets and 2-polar fuzzy sets are cryptomorphic mathematical notions and that we can obtain concisely one from the corresponding one in [6]. The idea behind this is that "multipolar information" (not just bipolar information which correspond to two-valued logic) exists because data for a real world problems are sometimes from n agents ($n \geq 2$). For example, the exact degree of telecommunication safety of mankind is a point in $[0, 1]^n (n \approx 7 \times 10^9)$ because different person has been monitored different times. There are many examples such as truth degrees of a logic formula which are based on n logic implication operators ($n \geq 2$), similarity degrees of two logic formula which are based on n logic implication operators ($n \geq 2$), ordering results of a magazine, ordering results of a university and inclusion degrees (accuracy measures, rough measures, approximation qualities, fuzziness measures, and decision preformation evaluations) of a rough set.

The notions of fuzzy ideals and fuzzy subalgebras of Lie algebras over a field were considered first in [10] by Yehia. Since then, the concepts and results of Lie algebras have been broadened to the fuzzy setting frames [1]–[5], [9]–[11]. In this paper, we introduce the concept of m-polar fuzzy Lie subalgebras of a Lie algebra and investigate some of their properties. The Cartesian product of m-polar fuzzy Lie subalgebras will be discussed. In particular, the homomorphisms between the Lie subalgebras of a Lie algebra and their relationship between the domains and the co-domains of the m-polar fuzzy subalgebras under these homomorphisms will be investigated.

2. Preliminaries

In this section, we first review some elementary aspects that are necessary for this paper. A *Lie algebra* is a vector space $\mathscr{L}$ over a field $\mathbb{F}$ (equal to $\mathbb{R}$ or $\mathbb{C}$) on which $\mathscr{L} \times \mathscr{L} \to \mathscr{L}$ denoted by $(x, y) \to [x, y]$ is defined satisfying the following axioms:

(L1) $[x, y]$ is bilinear,

(L2) $[x, x] = 0$ for all $x \in \mathscr{L}$,

(L3) $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ for all $x, y, z \in \mathscr{L}$ (Jacobi identity).

Throughout this paper, $\mathscr{L}$ is a Lie algebra and $\mathbb{F}$ is a field. We note that the multiplication in a Lie algebra is not associative, i.e., it is not true in general that $[[x,y],z] = [x,[y,z]]$. But it is *anti commutative*, i.e., $[x,y] = -[y,x]$. A subspace H of $\mathscr{L}$ closed under $[\cdot,\cdot]$ will be called a *Lie subalgebra*. A subspace I of $\mathscr{L}$ with the property $[I,\mathcal{L}] \subseteq I$ is called a *Lie ideal* of $\mathscr{L}$. Obviously, any Lie ideal is a subalgebra.

Let μ be a *fuzzy set* on $\mathscr{L}$, i.e., a map $\mu : \mathscr{L} \to [0,1]$. Then, we call a *fuzzy set* $\mu : \mathscr{L} \to [0,1]$ a *fuzzy Lie subalgebra* [10] of $\mathscr{L}$ if the following conditions are satisfied:

(a) $\mu(x+y) \geq \min\{\mu(x), \mu(y)\}$,

(b) $\mu(\alpha x) \geq \mu(x)$,

(c) $\mu([x,y]) \geq \min\{\mu(x), \mu(y)\}$,

for all $x, y \in \mathscr{L}$ and $\alpha \in \mathbb{F}$.

Definition 2.1. [6] An *m-polar fuzzy set* (or a $[0,1]^m$-set) on X is a mapping $A : \mathscr{L} \to [0,1]^m$. The set of all m-polar fuzzy sets on $\mathscr{L}$ is denoted by $m(\mathscr{L})$.

Note that $[0,1]^m$(m-power of [0, 1]) is considered a poset with the point-wise order $\leq$, where m is an arbitrary ordinal number (we make an appointment that $m = \{n \mid n < m\}$ when $m > 0$), $\leq$ is defined by $x \leq y \Leftrightarrow p_i(x) \leq p_i(y)$ for each $i \in m$ ($x, y \in [0,1]^m$), and $p_i : [0,1]^m \to [0,1]$ is the ith projection mapping ($i \in m$). $\mathbf{0} = (0,0,...,0)$ is the smallest element in $[0,1]^m$ and $\mathbf{1} = (1,1,...,1)$ is the largest element in $[0,1]^m$.

3. m-polar fuzzy Lie subalgebras

Definition 3.1. An m-polar fuzzy set C on $\mathscr{L}$ is called an *m-polar fuzzy Lie subalgebra* if the following conditions are satisfied:

(1) $C(x+y) \geq C(x) \wedge C(y))$,

(2) $C(\alpha x) \geq C(x)$,

(3) $C([x,y]) \geq C(x) \wedge C(y))$ for all $x, y \in \mathscr{L}$ and $\alpha \in \mathbb{F}$.

That is,

(1) $p_i \circ C(x+y) \geq \inf(p_i \circ C(x), p_i \circ C(y))$,

(2) $p_i \circ C(\alpha x) \geq p_i \circ C(x)$,

(3) $p_i \circ C([x,y]) \geq \inf(p_i \circ C(x), p_i \circ C(y))$

for all $x, y \in \mathscr{L}$ and $\alpha \in \mathbb{F}$, $i = 1, 2, 3, ..., m$.

Definition 3.2. An m-polar fuzzy set C on $\mathscr{L}$ is called an *m-polar fuzzy Lie ideal* if it satisfies the conditions (1),(2) and the following additional condition:

(4) $C([x,y]) \geq C(x)$

for all $x, y \in \mathscr{L}$.

From (2) it follows that:

(5) $C(0) \geq C(x)$,

(6) $C(-x) \geq C(x)$.

Example 3.3. Let $\mathbb{R}^3 = \{(x,y,z) : x,y,z \in \mathbb{R}\}$ be the set of all 3-dimensional real vectors. Then $\mathbb{R}^3$ with the bracket $[\cdot,\cdot]$ defined as the usual cross product, i.e., $[x,y] = x \times y$, forms a real Lie algebra. We also define an m-polar fuzzy set $C : \mathbb{R}^3 \to [0,1]^m$ by

$$C(x,y,z) = \begin{cases} (0.6, 0.6, ..., 0.6) & \text{if } x = y = z = 0, \\ (0.2, 0.2, ..., 0.2) & \text{otherwise.} \end{cases}$$

By routine computations, we can verify that the above m-polar fuzzy set C is an m-polar fuzzy Lie subalgebra and Lie ideal of the Lie algebra $\mathbb{R}^3$.

Proposition 3.4. *Every m-polar fuzzy Lie ideal is an m-polar fuzzy Lie subalgebra.*

We note here that the converse of Proposition 3.4 does not hold in general as can be seen in the following example.

Example 3.5. Consider $\mathbb{F} = \mathbb{R}$. Let $\mathscr{L} = \mathbb{R}^3 = \{(x,y,z) : x,y,z \in \mathbb{R}\}$ be the set of all 3-dimensional real vectors which forms a Lie algebra and define

$$\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$$
$$[x,y] \to x \times y,$$

where $\times$ is the usual cross product. We define an m-polar fuzzy set $C : \mathbb{R}^3 \to [0,1]^m$ by

$$C(x,y,z) = \begin{cases} (1, 1, ..., 1) & \text{if } x = y = z = 0, \\ (0.5, 0.5, ..., 0.5) & \text{if } x \neq 0, y = z = 0, \\ (0, 0, ..., 0) & \text{otherwise.} \end{cases}$$

Then C is an m-polar fuzzy Lie subalgebra of $\mathscr{L}$ but C is not an m-polar fuzzy Lie ideal of $\mathscr{L}$ since

$$\begin{aligned} C([(1,0,0)\ (1,1,1)]) &= C(0,-1,1) = (0,0,...,0), \\ C(1,0,0) &= (0.5, 0.5, ..., 0.5) \end{aligned}$$

That is,

$$C([(1,0,0)\ (1,1,1)]) \not\geq C(1,0,0).$$

Theorem 3.6. *Let C be an m-polar fuzzy Lie subalgebra in a Lie algebra $\mathscr{L}$. Then C is an m-polar fuzzy Lie subalgebra of $\mathscr{L}$ if and only if the non-empty upper s-level cut $C_{[s]} = U(C;s) = \{x \in \mathscr{L} \mid C(x) \geq s\}$ is a Lie subalgebra of L, for all $s \in [0,1]^m$.*

Example 3.7. Consider the group algebra $\mathbb{C}[S_3]$, where S_3 is the Symmetric group. Then $\mathbb{C}[S_3]$ assumes the structure of a Lie algebra via the bracket (commutator) operation. Clearly, the linear span of the elements $\hat{g} = g - g^{-1}$ for $g \in S_3$ is the subalgebra of $\mathbb{C}[S_3]$, which is also known as Plesken Lie algebra and denoted by $\mathscr{L}(S_3)_{\mathbb{C}}$. It is easy to see that $\mathscr{L}(S_3)_{\mathbb{C}} = Span_{\mathbb{C}}\{\widehat{(1,2,3)}\}$ and $\widehat{(1,2,3)} = (1,2,3) - (1,3,2)$.

We define an m-polar fuzzy set $C : \mathscr{L}(S_3)_{\mathbb{C}} \to [0,1]^m$ by

$$C(g) = \begin{cases} (t_1, t_2, ..., t_m), & g = \gamma(1,2,3) - \gamma(1,3,2), \text{ where } \gamma \in \mathbb{C}, g \in \mathbb{C}[S_3] \\ (s_1, s_2, ..., s_m), & \text{otherwise, where } s_i < t_i \end{cases}$$

By routine calculations, we have $\{g \in \mathbb{C}[S_3] : C(g) > (s_1, s_2, ..., s_m)\} = \mathscr{L}(S_3)_{\mathbb{C}}$. Then we see that $\mathscr{L}(S_3)_{\mathbb{C}}$ can be realized $C_{[s]}$ as an upper s_i-level cut and C is an m-polar fuzzy Lie ideal of $\mathscr{L}(S_3)_{\mathbb{C}}$.

Definition 3.8. Let C and D be two m-polar fuzzy sets of $\mathscr{L}$. We define the *sup-inf* product $[CD]$ of C and D as follows: for all x, y, $z \in \mathscr{L}$

$$[CD](x) = \begin{cases} \sup\limits_{x=[yz]} \{\inf(C(y), D(z))\} \\ \mathbf{0}, \quad \text{if x} \neq \text{[yz]}. \end{cases}$$

Let C and D be m-polar fuzzy Lie subalgebras of the Lie algebra $\mathscr{L}$. Then $[CD]$ may not be an m-polar fuzzy Lie subalgebra of $\mathscr{L}$ as this can be seen in the following counterexample:

Example 3.9. Let $\{e_1, e_2, ..., e_8\}$ be a basis of a vector space over a field $\mathbb{F}$. Then,it is not difficult to see that, by putting:

$$\begin{array}{llll} [e_1,e_2] = e_5, & [e_1,e_3] = e_6, & [e_1,e_4] = e_7, & [e_1,e_5] = -e_8, \\ [e_2,e_3] = e_8, & [e_2,e_4] = e_6, & [e_2,e_6] = -e_7, & [e_3,e_4] = -e_5, \\ [e_3,e_5] = -e_7, & [e_4,e_6] = -e_8, & [e_i,e_j] = -[e_j,e_i] & \end{array}$$

and $[e_i, e_j] = 0$ for all $i \leq j$, we can obtain a Lie algebra over a field $\mathbb{F}$. The following fuzzy sets

$$C(x) := \begin{cases} (1,1,...,1) & \text{if } x \in \{0, e_1, e_5, e_6, e_7, e_8\}, \\ (0,0,...,0) & \text{otherwise}, \end{cases}$$

$$D(x) := \begin{cases} (1,1,...,1) & \text{if } x = 0, \\ (0.5,0.5,...,0.5) & \text{if } x \in \{e_2, e_5, e_6, e_7, e_8\}, \\ (0,0,...,0) & \text{otherwise}, \end{cases}$$

are clearly fuzzy Lie subalgebras of a Lie algebra $\mathscr{L}$. Thus C and D are m-polar fuzzy Lie subalgebras of $\mathscr{L}$ because the level Lie subalgebras

$$\begin{aligned} U(C;(1,1,...,1)) &= \langle e_1, e_5, e_6, e_7, e_8\rangle, \\ U(D;(0.5,0.5,...,0.5)) &= \langle e_2, e_5, e_6, e_7, e_8\rangle \end{aligned}$$

are Lie subalgebras of $\mathscr{L}$. But $[CD]$ is not an m-polar fuzzy Lie subalgebra because the following condition does not hold:

$$[CD](e_7+e_8) \geq \inf\{[CD](e_7), [CD](e_8)\}.$$

$$(1)\ [CD](e_7) = \sup \begin{cases} \inf\{C(e_1), D(e_4)\} = (0,0,...,0), & e_7 = [e_1, e_4], \\ \inf\{C(e_2), D(e_6)\} = (0,0,...,0), & e_7 = -[e_2, e_6], \\ \inf\{C(e_3), D(e_5)\} = (0,0,...,0), & e_7 = -[e_3, e_5], \\ \inf\{C(e_4), D(e_1)\} = (0,0,...,0), & e_7 = -[e_4, e_1], \\ \inf\{C(e_6), D(e_2)\} = (0.5,0.5,...,0.5), & e_7 = [e_6, e_2], \\ \inf\{C(e_5), D(e_3)\} = (0,0,...,0), & e_7 = [e_5, e_3]. \end{cases}$$

Thus $[CD](e_7) = (0.5, 0.5, ..., 0.5)$.

(2) By using similar arguments, we can show that $[CD](e_8) = (0.5, 0.5, \cdots, 0.5)$.

(3) $[CD](e_7+e_8) = \sup\{(i)-(vi)\}$

(i) if $e_7+e_8 = [e_1(e_4-e_5)]$, then $\inf\{C(e_1), D(e_4-e_5)\} = \inf\{C(e_1), D(e_4), D(e_5)\} = (0,0,...,0)$, since $D(e_4) = (0,0,...,0)$, and if $e_7+e_8 = [(e_5-e_4)e_1]$, then, $\inf\{C(e_5-e_4), D(e_1)\} = \inf\{C(e_5), D(e_4), D(e_1)\} = (0,0,...,0)$, since $C(e_4) = (0,0,...,0)$.

By using similar method, we can also obtain the following numerical results:

(ii) If $e_7+e_8 = [e_2(e_3-e_6)]$, then $\inf(C(e_2), D(e_3-e_6)) = (0,0,...,0)$.

(iii) If $e_7+e_8 = [e_3(-e_2-e_5)]$, then $\inf(C(e_3), D(e_2-e_5)) = (0,0,...,0)$.

(iv) If $e_7+e_8 = [e_4(-e_1-e_6)]$, then $\inf(C(e_4), D(-e_3-e_1)) = (0,0,...,0)$.

(v) If $e_7+e_8 = [e_5(-e_3-e_1)]$, then $\inf(C(e_5), D(-e_3-e_1)) = (0,0,...,0)$.

(vi) If $e_7+e_8 = [e_6(-e_2-e_4)]$, then $\inf(C(e_6), D(-e_2-e_4)) = (0,0,...,0)$.

Thus, $[CD](e_7+e_8) = \sup\{(0,0,...,0),(0,0,...,0),(0,0,...,0),(0,0,...,0),(0,0,...,0),(0,0,...,0)\} = (0,0,...,0)$. Hence, we have proved that

$$[CD](e_7+e_8) \not\geq \inf\{[CD](e_7), [CD](e_8)\}.$$

We now refine the product of two m-polar fuzzy Lie subalgebras C and D of $\mathscr{L}$ to an extended form.

Definition 3.10. Let C and D be two m-polar fuzzy sets of $\mathscr{L}$. Then,we define the *sup-inf* product $\ll CD \gg$ of C and D as follows, for all x, y, $z \in \mathscr{L}$

$$\ll CD \gg (x) = \begin{cases} \sup\limits_{x=\sum\limits_{i=1}^{n}[x_i y_i]} \left\{ \inf\limits_{i\in\mathbb{N}} \{\inf(C(x_i), D(y_i))\} \right\} \\ \mathbf{0}, \quad \text{if } \mathrm{x} \neq \sum\limits_{\mathrm{i}=1}^{\mathrm{n}} [\mathrm{x_i y_i}]. \end{cases}$$

From the definitions of $[CD]$ and $\ll CD \gg$, we can easily see that $[CD] \subseteq \ll CD \gg$ and $[CD] \neq \ll CD \gg$ hold generally even if C and D are both m-polar fuzzy Lie subalgebras of $\mathscr{L}$, and in this case, $\ll CD \gg$ is also an m-polar fuzzy Lie subalgebra of $\mathscr{L}$.

Theorem 3.11. *Let C be an m-polar fuzzy Lie subalgebra of Lie algebra $\mathscr{L}$. Define a binary relation $\sim$ on $\mathscr{L}$ by $x \sim y$ if and only if $C(x-y) = C(0)$ for all x, $y \in \mathscr{L}$. Then $\sim$ is a congruence relation on $\mathscr{L}$.*

Proof. We first prove that "$\sim$" is an equivalent relation.We only need to show the transitivity of "$\sim$" because the reflectivity and symmetry of "$\sim$" hold trivially. Let x, y, $z \in \mathscr{L}$, If $x \sim y$ and $y \sim z$, then $C(x-y) = C(0)$, $C(y-z) = C(0)$. Hence it follows that

$$C(x-z) \;=\; C(x-y+y-z) \geq \inf(C(x-y), C(y-z)) = C(0),$$

Consequently $x \sim z$. We now verify that "$\sim$" is a congruence relation on $\mathscr{L}$. For this purpose,we let $x \sim y$ and $y \sim z$. Then $C(x-y) = C(0)$, $C(y-z) = C(0)$. Now, for x_1, x_2, y_1, $y_2 \in \mathscr{L}$, we have

$$\begin{aligned} C((x_1+x_2)-(y_1+y_2)) &= C((x_1-y_1)+(x_2-y_2)) \\ &\geq \inf(C(x_1-y_1), C(x_2-y_2)) = C(0), \end{aligned}$$

$$\begin{aligned} C((\alpha x_1 - \alpha y_1) &= C(\alpha(x_1-y_1)) \geq C(x_1-y_1) = C(0), \\ C([x_1,x_2]-[y_1,y_2]) &= C([x_1-y_1],[x_2-y_2]) \\ &\geq \inf\{C(x_1-y_1), C(x_2-y_2)\} = C(0). \end{aligned}$$

That is, $x_1+x_2 \sim y_1+y_2$, $\alpha x_1 \sim \alpha y_1$ and $[x_1,x_2] \sim [y_1,y_2]$. Thus, "$\sim$" is indeed a congruence relation on $\mathscr{L}$. ∎

Definition 3.12. Let C be an m -polar fuzzy set on a set $\mathscr{L}$. An *m-polar fuzzy relation* on C is an m-polar fuzzy set D of $\mathscr{L} \times \mathscr{L}$ such that $D(xy) \leq \inf(C(x), C(y))$ $\forall x, y \in \mathscr{L}$.

Definition 3.13. Let C and D be m-polar fuzzy sets on a set $\mathscr{L}$. If C is an m-polar fuzzy relation on a set $\mathscr{L}$, then C is said to be an *m-polar fuzzy relation* on D if $C(x,y) \leq \inf(D(x), D(y))$ for all x, $y \in \mathscr{L}$.

Theorem 3.14. *Let C and D be two m-polar fuzzy Lie subalgebras of a Lie algebras $\mathscr{L}$. Then $C \times D$ is an m-polar fuzzy Lie subalgebra of $\mathscr{L} \times \mathscr{L}$.*

Proof. Let $x = (x_1, x_2)$ and $y = (y_1, y_2) \in \mathscr{L} \times \mathscr{L}$. Then

$$\begin{aligned}
(C \times D)(x+y) &= (C \times D)((x_1, x_2) + (y_1, y_2)) \\
&= (C \times D)((x_1 + y_1, x_2 + y_2)) \\
&= \inf(C(x_1 + y_1), D(x_2 + y_2)) \\
&\geq \inf(\inf(C(x_1), C(y_1)), T(D(x_2), D(y_2))) \\
&= \inf(\inf(C(x_1), D(x_2)), \inf(C(y_1), D(y_2))) \\
&= \inf((C \times D)(x_1, x_2)), (C \times D)(y_1, y_2)) \\
&= \inf((C \times D)(x), (C \times D)(y)), \\
(C \times D)(\alpha x) &= (C \times D)(\alpha(x_1, x_2)) = (C \times D)((\alpha x_1, \alpha x_2)) \\
&= \inf(C(\alpha x_1), D(\alpha x_2)) \geq \inf(C(x_1), D(x_2)) \\
&= (C \times D)(x_1, x_2) = (C \times D)(x), \\
(C \times D)([x, y]) &= (C \times D)([(x_1, x_2), (y_1, y_2)]) \\
&\geq \inf(\inf(C(x_1), D(x_2)), \inf(C(y_1), D(y_2))) \\
&= \inf((C \times D)(x_1, x_2)), (C \times D)(y_1, y_2)) \\
&= \inf((C \times D)(x), (C \times D)(y)),
\end{aligned}$$

This shows that $C \times D$ is an m-polar fuzzy Lie subalgebra of $\mathscr{L} \times \mathscr{L}$. ∎

Definition 3.15. Let $\mathscr{L}_1$ and $\mathscr{L}_2$ be two Lie algebras over a field $\mathbb{F}$. Then a linear transformation $f : \mathscr{L}_1 \to \mathscr{L}_2$ is called a *Lie homomorphism* if $f([x, y]) = [f(x), f(y)]$ holds for all x, $y \in \mathscr{L}_1$.

For the Lie algebras $\mathscr{L}_1$ and $\mathscr{L}_2$, it can be easily observed that if $f : \mathscr{L}_1 \to \mathscr{L}_2$ is a Lie homomorphism and C is an m-polar fuzzy Lie subalgebra of $\mathscr{L}_2$, then the m-polar fuzzy set $f^{-1}(CA)$ of $\mathscr{L}_1$ is also an m-polar fuzzy Lie subalgebra.

Definition 3.16. Let $\mathscr{L}_1$ and $\mathscr{L}_2$ be two Lie algebras. Then, a Lie homomorphism $f : \mathscr{L}_1 \to \mathscr{L}_2$ is said to have a natural extension $f : I^{\mathscr{L}_1} \to I^{\mathscr{L}_2}$ defined by for all $C \in I^{\mathscr{L}_1}$, $y \in \mathscr{L}_2$:

$$f(C)(y) = \sup\{C(x) : x \in f^{-1}(y)\}.$$

We now call these sets the homomorphic images of the m-polar fuzzy set C.

Theorem 3.17. *The homomorphic image of an m-polar fuzzy Lie subalgebra is still an m-polar fuzzy Lie subalgebra of its co-domain.*

Proof. Let y_1, $y_2 \in \mathscr{L}_2$. Then

$$\{x \mid x \in f^{-1}(y_1 + y_2)\} \supseteq \{x_1 + x_2 \mid x_1 \in f^{-1}(y_1) \text{ and}, x_2 \in f^{-1}(y_2)\}.$$

Now, we have

$$\begin{aligned} f(C)(y_1+y_2) &= \sup\{C(x) \mid x \in f^{-1}(y_1+y_2)\} \\ &\geq \{C(x_1+x_2), \mid x_1 \in f^{-1}(y_1) \text{ and } x_2 \in f^{-1}(y_2)\} \\ &\geq \sup\{\inf\{C(x_1), C(x_2)\} \mid x_1 \in f^{-1}(y_1) \text{ and } x_2 \in f^{-1}(y_2)\} \\ &= \inf\{\sup\{C(x_1) \mid x_1 \in f^{-1}(y_1)\}, \sup\{C(x_2) \mid x_2 \in f^{-1}(y_2)\}\} \\ &= \inf\{f(C)(y_1), f(C)(y_2)\}. \end{aligned}$$

For $y \in \mathscr{L}_2$ and $\alpha \in \mathbb{F}$, we have

$$\{x \mid x \in f^{-1}(\alpha y)\} \supseteq \{\alpha x \mid x \in f^{-1}(y)\}.$$

$$\begin{aligned} f(C)(\alpha y) &= \sup\{C(\alpha x) \mid x \in f^{-1}(y)\} \\ &\geq \{C(\alpha x) \mid x \in f^{-1}(\alpha y)\} \\ &\geq \sup\{C(x) \mid x \in f^{-1}(y)\} \\ &= f(C)(y), \end{aligned}$$

If $y_1,\ y_2 \in \mathscr{L}_2$ then

$$\{x \mid x \in f^{-1}([y_1, y_2])\} \supseteq \{[x_1, x_2] \mid x_1 \in f^{-1}(y_1),\ x_2 \in f^{-1}(y_2)\}.$$

Now

$$\begin{aligned} f(C)([y_1, y_2]) &= \sup\{C(x) \mid x \in f^{-1}([y_1, y_2])\} \\ &\geq \{C([x_1, x_2]) \mid x_1 \in f^{-1}(y_1) \text{ and } x_2 \in f^{-1}(y_2)\} \\ &\geq \sup\{\inf\{C(x_1), C(x_2)\} \mid x_1 \in f^{-1}(y_1) \text{ and } x_2 \in f^{-1}(y_2)\} \\ &= \inf\{\sup\{C(x_1) \mid x_1 \in f^{-1}(y_1)\}, \sup\{C(x_2) \mid x_2 \in f^{-1}(y_2)\}\} \\ &= \inf\{f(C)(y_1), f(C)(y_2)\}. \end{aligned}$$

Thus, $f(C)$ is a fuzzy Lie algebra of $\mathscr{L}_2$. ∎

Theorem 3.18. *Let $f : \mathscr{L}_1 \to \mathscr{L}_2$ be a surjective Lie homomorphism. If C and D are m-polar fuzzy Lie subalgebras of $\mathscr{L}_1$ then $f(\ll CD \gg) = \ll f(C)f(D) \gg$.*

Proof. Assume that $f(\ll CD \gg) < \ll f(C)f(D) \gg$. Now, we choose a number $t \in [0, 1]$ such that $f(\ll CD \gg)(x) < t < \ll f(C)f(D) \gg (x)$. Then,there exist $y_i,\ z_i \in \mathscr{L}_2$ such that $x = \sum_{i=1}^{n}[y_i z_i]$ with $f(C)(y_i) > t$ and $f(D)(z_i) > t$. Since f is surjective, there exists $y \in \mathscr{L}_1$ such that $f(y) = x$ and $y = \sum_{i=1}^{n}[a_i b_i]$ for some $a_i \in f^{-1}(y_i)$, $b_i \in f^{-1}(z_i)$ with $f(a_i) = y_i$, $f(b_i) = z_i$, $C(a_i) > t$ and $D(b_i) > t$. Since $f(\sum_{i=1}^{n}[a_i b_i]) = [\sum_{i=1}^{n} f([a_i b_i]) = [\sum_{i=1}^{n}[f(a_i)f(b_i)] = [\sum_{i=1}^{n}[y_i z_i] = x$, we have $f(\ll CD \gg)(x) > t$. This is a contradiction. Similarly, for the case $f(\ll CD \gg) > \ll f(C)f(D) \gg$, we can also obtain a contradiction. Hence, $f(\ll CD \gg) = \ll f(C)f(D) \gg$. ∎

Definition 3.19. Let C and D be m-polar fuzzy subalgebras of $\mathscr{L}$. Then C is said to be of the same type of D if there exists $f \in Aut(L)$ such that $C = D \circ f$, i.e., $C(x) = D(f(x))$ for all $x \in \mathscr{L}$.

Theorem 3.20. *Let C and D be two m-polar fuzzy subalgebras of $\mathscr{L}$. Then C is an m-polar fuzzy subalgebra having the same type of D if and only if C is isomorphic to D.*

Proof. We only need to prove the necessity part because the sufficiency part is trivial. Let C be an m-polar fuzzy subalgebra having the same type of D. Then there exists $\phi \in Aut(L)$ such that $C(x) = D(\phi(x))\ \ \forall x \in \mathscr{L}$.

Let $f : C(L) \to D(L)$ be a mapping defined by $f(C(x)) = D(\phi(x))$ for all $x \in \mathscr{L}$, that is, $f(C(x)) = D(\phi(x)) \quad \forall x \in \mathscr{L}$. Then, it is clear that f is surjective. Also, f is injective because if $f(C(x)) = f(C(y))$ for all $x, y \in \mathscr{L}$, then $D(\phi(x)) = D(\phi(y))$ and hence $C(x) = D(y)$. Finally, f is a homomorphism because for $x, y \in \mathscr{L}$,

$$\begin{aligned} f(C(x+y)) &= D(\phi(x+y)) = D(\phi(x) + \phi(y)), \\ f(C(\alpha x)) &= D(\phi(\alpha x)) = \alpha D(\phi(x)), \\ f(C([x,y])) &= D(\phi([x,y])) = D([\phi(x), \phi(y)]). \end{aligned}$$

Hence C is isomorphic to D. This completes the proof. ∎

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COMMUTATIVE FUNDAMENTAL RELATION IN FUZZY HYPERSEMIGROUPS

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Abstract. We introduce and study the construction of commutative fundamental relation in fuzzy hypersemigroups and investigate its basic properties. We will determine some necessary and sufficient conditions so that this relation is transitive in fuzzy hypersemigroups. Also we show that this relation is transitive in every fuzzy hypergroup.

Keywords and phrases: fuzzy hypersemigroups, fuzzy regular relation, commutative fundamental relation.

1. Introduction

The study of fuzzy hyperstructures is an interesting research topic for fuzzy sets. There are many works on the connections between fuzzy sets and hyperstructures. This can be considered into three groups. A *first* group of papers studies *crisp* hyperoperations defined through fuzzy sets. This study was initiated by Corsini in [2], [3] and then continuated by other researchers. A *second* group of papers concerns the *fuzzy hyperalgebras*. This is a direct extension of the concept of fuzzy algebras. This was intiated by Zahedi in [12]. A *third* group was introduced by Corsini and Tofan in [5]. The basic idea in this group of papers is the following: a multioperation assigns to every pair of elements of S a nonempty subset of S, while a fuzzy multioperation assigns to every pair of elements of S a nonzero fuzzy set on S. This idea was continuated by Sen, Ameri and Chowdhury in [11] where fuzzy semihypergroups are introduced. The fundamental relations are one of the most important and interesting concepts in fuzzy hyperstructures that ordinary algebraic structures are derived from fuzzy hyperstructures by them. Fundamental relation α^* on fuzzy hypersemigroups is studied in [1]. In this paper, continuing our previous work [1], we show that the relation γ^* is the smallest strongly regular equivalence relation on a fuzzy hypersemigroup S such that the quotient $S/\gamma^*$is a commutative semigroup. The structure of this paper is follows:

After an introduction in section 2, we recall some basic notions and results on fuzzy hypersemigroups. In section 3, we introduce the relation γ^* in fuzzy hypersemigroup S and prove that S/γ^* is a commutative semigroup. In section 4, the notion of complete part of a fuzzy hypersemigroup is introduced and we determine some necessary and sufficient conditions so that γ is transitive in fuzzy hypersemigroups, then we will prove that this relation is transitive in every fuzzy hypergroup.

2. Preliminaries

Recall that for a non-empty set S, a fuzzy subset μ of S is a function from S into the real unite interval $[0,1]$. We denote the set of all nonzero fuzzy subsets of S by $F^*(S)$. Also for fuzzy subsets μ_1 and μ_2 of S, then μ_1 is *smaller* than μ_2 and write $\mu_1 \leq \mu_2$ iff for all $x \in S$, we have $\mu_1(x) \leq \mu_2(x)$. Define $\mu_1 \vee \mu_2$ and $\mu_1 \wedge \mu_2$ as follows: $\forall x \in S, \quad (\mu_1 \vee \mu_2)(x) = max\{\mu_1(x), \mu_2(x)\}$ and $(\mu_1 \wedge \mu_2)(x) = min\{\mu_1(x), \mu_2(x)\}$.

A *fuzzy hyperoperation* on S is a mapping $\circ : S \times S \mapsto F^*(S)$ written as $(a,b) \mapsto a \circ b = ab$. The couple $(S, \circ)$ is called a *fuzzy hypergropoid*.

Definition 2.1 A fuzzy hypergropoid $(S, \circ)$ is called a *fuzzy hypersemigroup* if for all $a, b, c \in S$, $(a \circ b) \circ c = a \circ (b \circ c)$, where for any fuzzy subset μ of S

$$(a \circ \mu)(r) = \begin{cases} \bigvee_{t \in S}((a \circ t)(r) \wedge \mu(t)), & \mu \neq 0 \\ 0, & \mu = 0 \end{cases}$$

$$(\mu \circ a)(r) = \begin{cases} \bigvee_{t \in S}(\mu(t) \wedge (t \circ a)(r)), & \mu \neq 0 \\ 0, & \mu = 0, \end{cases}$$

for all $r \in S$.

Definition 2.2 Let μ, ν be two fuzzy subsets of a fuzzy hypergropoid $(S, \circ)$. Then we define $\mu \circ \nu$ by $(\mu \circ \nu)(t) = \bigvee_{p,q \in S}(\mu(p) \wedge (p \circ q)(t) \wedge \nu(q))$, for all $t \in S$.

Definition 2.3 A fuzzy hypersemigroup $(S, \circ)$ is called *fuzzy hypergroup* if $x \circ S = S \circ x = \chi_S$, for all $x \in S$, where χ_S is characteristic function of S.

Example 2.4 Consider a fuzzy hyperoperation $\circ$ on a non-empty set S by $a \circ b = \chi_{\{a,b\}}$, for all $a, b \in S$. Then, $(S, \circ)$ is a fuzzy hypersemigroup and fuzzy hypergroup as well.

Theorem 2.5 *Let $(S, \circ)$ be a fuzzy hypersemigroup. Then, $\chi_a \circ \chi_b = a \circ b$, for all $a, b \in S$.*

Definition 2.6 Let $(S, \star)$ and $(T, \circ)$ be two fuzzy hypersemigroups. A mapping $\phi : S \mapsto T$ is said to be a

(i) *fuzzy homomorphism* if $\phi(a \star b) \leq \phi(a) \circ \phi(b),\ (a, b \in S)$.

(ii) *fuzzy good homomorphism* if $\phi(a \star b) = \phi(a) \circ \phi(b),\ (a, b \in S)$.

Definition 2.7 Let ρ be an equivalence relation on a fuzzy hypersemigroup $(S, \circ)$, we define two relations $\overline{\rho}$ and $\overline{\overline{\rho}}$ on $F^*(S)$ as follows: for $\mu, \nu \in F^*(S)$; $\mu \overline{\rho} \nu$ if $\mu(a) > 0$ then there exists $b \in S$ such that $\nu(b) > 0$ and $a \rho b$, also if $\nu(x) > 0$ then there exists $y \in S$, such that $\mu(y) > 0$ and $x \rho y$. $\mu \overline{\overline{\rho}} \nu$ if for all $x \in S$ such that $\mu(x) > 0$ and for all $y \in S$ such that $\nu(y) > 0$, $x \rho y$.

Definition 2.8 An equivalence relation ρ on a fuzzy hypersemigroup $(S, \circ)$ is said to be *(strongly) fuzzy regular* if $a \rho b, a' \rho b'$ implies $a \circ a'\ \overline{\rho}\ b \circ b' (a \circ a'\ \overline{\overline{\rho}}\ b \circ b')$.

3. Commutative fundamental relation

Let $(S, \circ)$ be a fuzzy hypersemigroup. For every map $f : S \to S'$ where S' is a semigroup, we define $f(a \circ b) = \{f(t) : (a \circ b)(t) > 0\}$.
If ρ is a equivalence relation on a fuzzy hypersemigroup $(S, \circ)$, then we consider the following hyperoperation on the quotient set S/ρ as follows:
for every $a\rho, b\rho \in S/\rho$,

$$a\rho \oplus b\rho = \{c\rho : (a' \circ b')(c) > 0, a\rho a', b\rho b'\}$$

Theorem 3.1 *Let $(S, \circ)$ be a fuzzy hypersemigroup and ρ be an equivalence relation on S. Then*

(i) *the relation ρ is fuzzy regular on $(S, \circ)$ iff $(S/\rho, \oplus)$ is a hypersemigroup.*

(ii) *the relation ρ is strongly fuzzy regular on $(S, \circ)$ iff $(S/\rho, \oplus)$ is a semigroup.*

(iii) *if the map $f : S \to S'$, where S' is a semigroup, satisfies in the following condition:*

$$f(a \circ b) = f(a) \circ f(b) \qquad \forall a, b \in S.$$

Then the equivalence relation associated with f is strongly fuzzy regular.

Proof. Straightforward. ∎

Definition 3.2 Let $(S, \circ)$ be a fuzzy hypersemigroup. A *commutative fundamental relation* on $(S, \circ)$ is the smallest equivalence relation ρ on S such that the quotient structure $(S/\rho, \oplus)$ is a commutative semigroup.

Let $(S, \circ)$ be a fuzzy hypersemigroup. We define the relation γ on S in the following way, $\gamma = \bigcup_{n \geq 1} \gamma_n$ where $\gamma_1 = \{(s, s) : s \in S\}$ and for every $n \geq 2$, $a \gamma_n b$ if

$$\exists x_1, ..., x_n \in H (n \in \mathbb{N}), \exists \sigma \in S_n : (x_1 \circ \ldots \circ x_n)(a) > 0 \text{ and } (x_{\sigma_1} \circ \ldots \circ x_{\sigma_n})(b) > 0.$$

It is clear that γ is symetric and reflexive. We take γ^* to be the transitive closure of γ. Then γ^* is an equivalence relation on S.

Theorem 3.3 *The relation γ^* is a strongly fuzzy regular relation.*

Proof. First we have to show that for each $x, y, a \in S$

$$x\gamma y \Rightarrow xa\overline{\overline{\gamma}}ya, \quad ax\overline{\overline{\gamma}}ay. \tag{$*$}$$

If $x\gamma y$, then there exist $n \in \mathbb{N}$, $(z_1, \ldots, z_n) \in S^n$ and $\sigma \in S_n$ such that $\prod_{i=1}^{n} z_i(x) > 0$, $\prod_{i=1}^{n} z_{\sigma(i)}(y) > 0$. For every $a \in S$, set $a = z_{n+1}$ and suppose that τ be the permutation of S_n such that

$$\tau(i) = \begin{cases} \sigma(i), & \forall i \in \{1, 2, \ldots, n\} \\ n+1, & i = n+1. \end{cases}$$

For every $v, w \in S$ such that $(xa)(v) > 0, (ya)(w) > 0$, we can write

$$\left(\prod_{i=1}^{n+1} z_i\right)(v) = \left[\left(\prod_{i=1}^{n} z_i\right) a\right](v) = \bigvee_{t \in S} \left[\left(\prod_{i=1}^{n} z_i\right)(t) \wedge (ta)(v)\right]$$

For $t = x$, $\left(\prod_{i=1}^{n} z_i\right)(x) > 0$ and $(xa)(v) > 0$, so $\left(\prod_{i=1}^{n+1} z_i\right)(v) > 0$, and

$$\begin{aligned} \left(\prod_{i=1}^{n+1} z_{\tau(i)}\right)(w) &= \left[\left(\prod_{i=1}^{n} z_{\sigma(i)}\right) z_{n+1}\right](w) = \left[\left(\prod_{i=1}^{n} z_{\sigma(i)}\right) a\right](w) \\ &= \bigvee_{t \in S} \left[\left(\prod_{i=1}^{n} z_{\sigma(i)}\right)(t) \wedge (ta)(w)\right]. \end{aligned}$$

For $t = y$, $\prod_{i=1}^{n} z_{\sigma(i)}(y) > 0$ and $(ya)(w) > 0$, hence

$$\left(\prod_{i=1}^{n+1} z_{\tau(i)}\right)(w) > 0.$$

Therefore, $v\gamma w$. Thus, $xa\overline{\overline{\gamma}}ya$. In the same way, we see $x\gamma y$ implies that $ax\overline{\overline{\gamma}}ay$.

Now, if $x\gamma^* y$, there exist $m \in \mathbb{N}$ and $w_0 = x, w_1, \ldots, w_m = y \in S$ such that $w_0 = x\gamma w_1\gamma w_2\gamma \ldots \gamma w_m = y$, by $(*)$ we have $w_0 a = xa\overline{\overline{\gamma}}w_1 a\overline{\overline{\gamma}}w_2 a\overline{\overline{\gamma}} \ldots \overline{\overline{\gamma}}w_m a = ya$. For each $v, w \in S$ such that $(xa)(v) = (w_0 a)(v) > 0$ and $(ya)(w) = (w_m a)(w) > 0$, by taking $(z_1, z_2, \ldots, z_{m-1}) \in S^{m-1}$ such that $(w_1 a)(z_1) > 0, (w_2 a)(z_2) > 0, \ldots,$ $(w_{m-1} a)(z_{m-1}) > 0$, we have $v\gamma z_1\gamma z_2\gamma \ldots \gamma z_{m-1}\gamma w$, and so $v\gamma^* w$. Therefore,

$$x\gamma^* y \Rightarrow xa\overline{\overline{\gamma^*}}ya.$$

Similarly, we can prove $x\gamma^* y$ implies that $ax\overline{\overline{\gamma^*}}ay$, hence γ^* is strongly fuzzy regular. ∎

Corollary 3.4 *Let S be a fuzzy hypersemigroup. Then the quotient S/γ^* is a commutative semigroup.*

Proof. Since γ^* is a strongly fuzzy regular relation, S/γ^* is a semigroup under the following operation

$$\gamma^*(x_1) \oplus \gamma^*(x_2) = \gamma^*(z), \forall z \in S \ s.t \ \ (x_1 x_2)(z) > 0.$$

Consider $\sigma \in S_2$, such that $\sigma(1) = 2$. Then for every $z, w \in S$ such that $(x_1x_2)(z) > 0$ and $(x_{\sigma(1)}x_{\sigma(2)})(w) > 0$, we have $z\gamma w$, thus $z\gamma^* w$ and we obtain

$$\gamma^*(x_1) \oplus \gamma^*(x_2) = \gamma^*(z) = \gamma^*(w) = \gamma^*(x_2) \oplus \gamma^*(x_1).$$

So S/γ^* is a commutative semigroup. ∎

Theorem 3.5 *The relation γ^* is the commutative fundamental relation on fuzzy hypersemigroup $(S, \circ)$.*

Proof. Let ρ be a strongly fuzzy regular equivalence relation on $(S, \circ)$ such that S/ρ is a commutative semigroup. The canonical map $\phi : S \to S/\rho$ satisfies in following condition

$$\phi(xy) = \phi(x) \oplus \phi(y) \qquad \forall x, y \in S$$

Now if $x\gamma y$, then there exist $n \in \mathbb{N}$, $(z_1, \ldots, z_n) \in S^n$ and $\sigma \in S_n$ such that

$$\left(\prod_{i=1}^{n} z_i\right)(x) > 0 \ \text{ and } \ \left(\prod_{i=1}^{n} z_{\sigma(i)}\right)(y) > 0.$$

S/ρ is a semigroup, thus

$$\phi(x) = \prod_{i=1}^{n} \phi(z_i) \ \text{ and } \ \phi(y) = \prod_{i=1}^{n} \phi(z_{\sigma(i)}).$$

By commutativity of S/ρ, we have $\phi(x) = \phi(y)$ and $x\rho y$. Thus $\gamma \subseteq \rho$. Now, if $x\gamma^* y$, there exist $m \in \mathbb{N}$ and $(u_0 = x, u_1, \ldots, u_m = y) \in S^{m+1}$ such that, $x\gamma u_1\gamma u_2\gamma \ldots \gamma u_{m-1}\gamma y$. Therefore, $x\rho u_1\rho u_2\rho \ldots \rho u_{m-1}\rho y$. Since ρ is transitive, we conclude $x\rho y$. So $\gamma^* \subseteq \rho$. ∎

4. Complete parts and transitivity of γ

In the following, we will determine some necessary and sufficient conditions so that the relation γ is transitive.

Definition 4.1 Let S be a fuzzy hypersemigroup and M be a non-empty subset of S. We say that M is a complete part of S, if it satisfies in this condition: for every $n \in \mathbb{N}$, $(z_1, \ldots, z_n) \in S^n$, when there exists $z \in M$ such that $\left(\prod_{i=1}^{n} z_i\right)(z) > 0$, then for every $w \in S - M$ and $\sigma \in S_n$, we have $\left(\prod_{i=1}^{n} z_{\sigma(i)}\right)(w) = 0$.

Lemma 4.2 *Let S be a fuzzy hypersemigroup and M be a non-empty subset of S. The following conditions are equivalent:*

(i) *M is complete part of S;*

(ii) *if $x \in M$ and $x\gamma y$, then $y \in M$;*

(iii) *if $x \in M$ and $x\gamma^* y$, then $y \in M$.*

Proof. (i) $\rightarrow$ (ii) Let $x, y \in S$, $x \in M$ and $x\gamma y$. Then, there exist $n \in \mathbb{N}$, $(z_1, ..., z_n) \in S^n$, and $\sigma \in S_n$ such that

$$\left(\prod_{i=1}^{n} z_i\right)(x) > 0 \quad \text{and} \quad (\prod_{i=1}^{n} z_{\sigma(i)})(y) > 0.$$

Since M is complete part, $\left(\prod_{i=1}^{n} z_{\sigma(i)}\right)(y) > 0$ implies that $y \in M$.

(ii) $\rightarrow$ (iii) Let $x, y \in S$, $x \in M$ and $x\gamma^* y$. Then there exist $m \in \mathbb{N}$ and $(u_0 = x, u_1, ..., u_m = y) \in S^{m+1}$ such that $x\gamma u_1 \gamma u_2 \gamma \ldots \gamma u_{m-1} \gamma y$. Since $x \in M$, by applying (ii) m times, we see $y \in M$.

(iii) $\rightarrow$ (i) Let $x \in M$ and $(z_1, ..., z_n) \in S^n$ such that $\left(\prod_{i=1}^{n} z_i\right)(x) > 0$. Then, if there exist $\sigma \in S_n$ and $y \in S$ such that $\left(\prod_{i=1}^{n} z_{\sigma(i)}\right)(y) > 0$, we have $x\gamma y$, therefore, $x\gamma^* y$ and $x \in M$. By (iii) we obtain $y \in M$, whence M is a complete part of S. ∎

Now, we are going to introduce new notations:

Let S be a fuzzy hypersemigroup and $x \in S$, we set;

$$\begin{aligned} T_n(x) &= \left\{(x_1, x_2, ..., x_n) \in S^n \mid \left(\prod_{i=1}^{n} x_i\right)(x) > 0\right\} \\ P_n(x) &= \bigcup \left\{\prod_{i=1}^{n} x_{\sigma(i)} \mid \sigma \in S_n, (x_1, x_2, ..., x_n) \in T_n(x)\right\} \\ P(x) &= \bigcup_{n \geq 1} P_n(x). \end{aligned}$$

By the above notations, we have the following results:

Lemma 4.3 *For every $x, y \in S$,*

$$x\gamma y \Leftrightarrow P(x)(y) > 0.$$

Proof.

$$\begin{aligned} x\gamma y &\Leftrightarrow \exists (n \in \mathbb{N}, (x_1, ..., x_n) \in S^n, \sigma \in S_n) : \left(\prod_{i=1}^{n} x_i\right)(x) > 0, \left(\prod_{i=1}^{n} x_{\sigma(i)}\right)(y) > 0 \\ &\Leftrightarrow \exists n \in \mathbb{N}, P_n(x)(y) > 0 \\ &\Leftrightarrow P(x)(y) > 0. \end{aligned}$$

Theorem 4.4 *If S is a fuzzy hypersemigroup, then the following conditions are equivalent:*

(i) *γ is transitive,*

(ii) *for every $x \in S, \gamma^*(x) = \{y \in S : P(x)(y) > 0\}$*

(iii) *for every $x \in S$, $\gamma(x)$ is a complete part.*

Proof. (i) $\rightarrow$ (ii) For every $(x, y) \in S^2$, by Lemma 4.3 we have

$$y \in \gamma^*(x) \Leftrightarrow x\gamma^* y \Leftrightarrow x\gamma y \Leftrightarrow P(x)(y) > 0.$$

(ii) $\rightarrow$ (iii) By Lemma 4.2, if $\varnothing \neq M \subseteq S$, then M is a complete part of S if and only if M is the union of equivalence classes modulo γ^*. In particular, $\gamma^*(x)$ is a $\gamma-$ part.

(iii) $\rightarrow$ (i) If $x\gamma y$ and $y\gamma z$, then there exist $n, m \in \mathbb{N}$, $(x_1, \ldots, x_n) \in T_n(x)$, $(y_1, \ldots, y_m) \in T_m(y)$, $\sigma \in S_n$ and $\tau \in S_m$ such that $\left(\prod_{i=1}^{n} x_{\sigma(i)}\right)(y) > 0$ and $\left(\prod_{i=1}^{m} y_{\tau(i)}\right)(z) > 0$. Since $\gamma(x)$ is a complete part, $\left(\prod_{i=1}^{n} x_{\sigma(i)}\right)(y) > 0$ implies that $y \in \gamma(x)$. We have $\left(\prod_{i=1}^{m} y_i\right)(y) > 0$ and $\gamma(x)$ is a complete part, therefore $\left(\prod_{i=1}^{m} y_{\tau(i)}\right)(z) > 0$ implies that $z \in \gamma(x)$. This means $z\gamma x$. ∎

Remark 1. If S is a commutative fuzzy hypersemigroup, then the relation γ is equal to α in [1]. We proved in Ref [1], the relation α is transitive in every fuzzy hypergroup, hence γ is transitive in every commutative fuzzy hypergroup.

Remark 2. If S is a fuzzy hypergroup, then S/γ^* is a group. We define $\omega_S = \phi^{-1}(1_{S/\gamma^*})$, in which $\phi : S \rightarrow S/\gamma^*$ is the canonical projection.

Lemma 4.5 *If S is a fuzzy hypergroup and M is a non-empty subset of S, then*

(i) $\phi^{-1}(\phi(M)) = \{x \in S : (\omega_S M)(x) > 0\} = \{x \in S : (M\omega_S)(x) > 0\}$

(ii) *If M is a complete part of S, then $\phi^{-1}(\phi(M)) = M$.*

Proof. (i) Let $x \in S$ and $(t, y) \in \omega_S \times M$ such that $(ty)(x) > 0$, so $\phi(x) = \phi(t) \oplus \phi(y) = 1_{S/\gamma^*} \oplus \phi(y) = \phi(y)$, therefore $x \in \phi^{-1}(\phi(y)) \subset \phi^{-1}(\phi(M))$. Conversely, for every $x \in \phi^{-1}(\phi(M))$, there exists $b \in M$ such that $\phi(x) = \phi(b)$. By reproducibility, $a \in S$ exists such that $(ab)(x) > 0$, so $\phi(b) = \phi(x) = \phi(a) \oplus \phi(b)$. This implies $\phi(a) = 1_{S/\gamma^*}$ and $a \in \phi^{-1}(1_{S/\gamma^*}) = \omega_S$. Therefore $(\omega_S M)(x) > 0$. In the same way, we can prove that $\phi^{-1}(\phi(M)) = \{x \in S : (M\omega_S)(x) > 0\}$.

(ii) We know $M \subseteq \phi^{-1}(\phi(M))$. If $x \in \phi^{-1}(\phi(M))$, then there exists $b \in M$ such that $\phi(x) = \phi(b)$. Therefore $x \in \gamma^*(x) = \gamma^*(b)$. Since M is a complete part of S and $b \in M$, by Lemma 4.2 we conclude $\gamma^*(b) \subseteq M$ and $x \in M$. ∎

The next theorem shows that the relation γ is transitive in every fuzzy hypergroup.

Theorem 4.6 *If S is a fuzzy hypergroup, then the relation γ is transitive.*

Proof. We know if S is a fuzzy hypergroup, then S/γ^* is a group, let $z \in \omega_S$. Now, if $x\gamma^* y$, then $x \in \phi^{-1}(\phi(y)) = \{u \in S \mid (y\omega_S)(u) > 0\}$, thus there exists $t \in \omega_S$ such that $(yt)(x) > 0$. Since $t \in \omega_S$, there exist $n \in \mathbb{N}$, $(x_1, ..., x_n) \in S^n$ and $\sigma \in S_n$ such that $\left(\prod_{i=1}^{n} x_i\right)(z) > 0$ and $\left(\prod_{i=1}^{n} x_{\sigma(i)}\right)(t) > 0$.

Now, by the reproducibility of S , there is $v \in S$ such that $(vz)(y) > 0$, and since $1_{S/\gamma^*} \oplus 1_{S/\gamma^*} = 1_{S/\gamma^*}$, there is $w \in \omega_S$ such that $(wz)(z) > 0$. Moreover, since $\{z, w\} \subset \omega_S$ and the relation α is transitive in every fuzzy hypergroup (see [1]), there is an m- tuple $(y_1, ..., y_m) \in S^n$ such that

$$\left(\prod_{i=1}^{m} y_i\right)(z) > 0 \quad \text{and} \quad \left(\prod_{i=1}^{m} y_i\right)(w) > 0.$$

Now,

$$\left(\prod_{i=1}^{m} y_i \prod_{i=1}^{n} x_i\right)(z) = \bigvee_{p,q \in S} \left[\left(\prod_{i=1}^{m} y_i\right)(p) \wedge \left(\prod_{i=1}^{n} x_i\right)(q) \wedge (pq)(z)\right].$$

Considering $p = w$ and $q = z$, we obtain $\left(\prod_{i=1}^{m} y_i \prod_{i=1}^{n} x_i\right)(z) > 0$. From this, and by setting $k = z$ in the following, we have

$$\left(w \prod_{i=1}^{m} y_i \prod_{i=1}^{n} x_i\right)(z) = \bigvee_{k \in S} \left[(wk)(z) \wedge \left(\prod_{i=1}^{m} y_i \prod_{i=1}^{n} x_i\right)(k)\right] > 0.$$

Again, by last statement, and setting $g = z$, we obtain

$$(1) \qquad \left(vw\prod_{i=1}^{m} y_i \prod_{i=1}^{n} x_i\right)(y) = \bigvee_{g\in S}\left[(vg)(y) \wedge \left(w\prod_{i=1}^{m} y_i \prod_{i=1}^{n} x_i\right)(g)\right] > 0.$$

On the other hand, if we set $p = y$ and $q = t$ in

$$\left(vw\prod_{i=1}^{m} y_i \prod_{i=1}^{n} x_{\sigma(i)}\right)(x) = \bigvee_{p,q\in S}\left[\left(vw\prod_{i=1}^{m} y_i\right)(p) \wedge \left(\prod_{i=1}^{n} x_{\sigma(i)}\right)(q) \wedge (p,q)(x)\right],$$

then we obtain

$$\left(vw\prod_{i=1}^{m} y_i\right)(y) = \bigvee_{k\in S}\left[(vk)(y) \wedge \left(w\prod_{i=1}^{m} y_i\right)(k)\right]$$

by setting $k = z$ in the above, and $f = z$ in the following, we have

$$\left(w\prod_{i=1}^{m} y_i\right)(z) = \bigvee_{f\in S}\left[(wf)(z) \wedge \left(\prod_{i=1}^{m} y_i\right)(f)\right] > 0.$$

Therefore,

$$(2) \qquad \left(vw\prod_{i=1}^{m} y_i \prod_{i=1}^{n} x_{\sigma(i)}\right)(x) > 0.$$

From (1) and (2), we conclude that $x\gamma y$, thus γ is transitive. ∎

Conclusions.

We introduced the concepts of commutative fundamental relation and complete parts of a fuzzy hypersemigroup. We determined some conditions so that this relation is transitive in fuzzy hypersemigroups and we proved that this relation is transitive in every fuzzy hypergroup. We will study about fundamental relation in fuzzy hyperrings.

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ON WEAKLY PRIME L-IDEALS

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Abstract. Let L be a complete lattice. We introduce and study the notion of weakly prime L-ideals of a commutative ring with identity and investigate their properties. In particular we characterize weakly prime L-ideals of R, where R is an integral domain and L is a chain.

Keywords and phrases: L-ideal,prime L-ideal, weakly prime L-ideal, complete lattice.

1. Introduction

Throughout this paper, R is a commutative ring with identity and L is a complete lattice. A proper ideal P of R is prime if $ab \in P$ implies that $a \in P$ or $b \in P$, or equivalently, if for ideals A and B of R, $AB \subseteq R$ implies that $A \subseteq P$ or $B \subseteq R$. Weakly prime ideals in a commutative ring with non-zero identity have been introduced and studied by D.D Anderson and E. Smith [3] and weakly primary ideals have been introduced by Sh. Ebrahimi Atani and F. Farzalipour [5]. In fact, a proper ideal P of R is said to be weakly prime if $0 \neq ab \in P$ implies that either $a \in P$ or $b \in P$ and is called weakly primary ideal if $0 \neq ab \in P$ implies that $a \in P$ or $b \in Rad(P)$(the radical of P). So a weakly prime ideal is weakly primary, but since 0 is always weakly primary, a weakly primary ideal does not need to be primary.

In the last few years a considerable amount of work has been done on fuzzy ideals in general and prime fuzzy ideals in particular, and some interesting topological properties of the spectrum of fuzzy prime ideals of a ring are obtained (see [7], [8], [9], [10], [11], [13]).

Zadeh [16] introduced the notion of a fuzzy subset of a non-empty set X as a function from X to the unit interval $I = [0, 1]$. Goguen [6], replaced $I = [0, 1]$ by a complete lattice L in the definition of fuzzy subset and introduced the notion of L-fuzzy subsets and Rosenfeld introduced introduced the notion of fuzzy groups [15].

In this paper we introduce the notion of weakly prime L-ideal of a commutative ring with non-zero identity where L is a complete lattice. In fact our definition is a generalization of the notion of ordinary weakly prime ideal. We investigate some basic properties of weakly prime L-ideals and characterize the weakly prime L-ideals of R where R is an integral domain and L is a chain.

2. Preliminaries

By an *L-subset* μ of a non-empty set X, we mean a function μ from X to L. If $L = [0, 1]$, then μ is called a *fuzzy* subset of X. L^X denotes the set of all L-subsets of X. Let $a \in L$ and $y \in M$. For all $x \in M$ define $a_y \in L^M$ as follows:

$$a_y(x) = \begin{cases} a & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

a_y is called an *L-point*. Sometimes, we use y_a instead of a_y.

For $\mu, \nu \in L^R$, we say that μ is contained in ν and we write $\mu \subseteq \nu$ if $\mu(x) \leq \nu(x)$, for all $x \in R$. For $\mu, \nu \in L^R$, the intersection and union, $\mu \cup \nu$, $\mu \cap \nu \in L^R$ are defined by

$$(\mu \cup \nu)(x) = \mu(x) \vee \nu(x) \ \textit{and} \ (\mu \cap \nu)(x) = \mu(x) \wedge \nu(x), \ \textit{for all} \ x \in R$$

Also for $\mu \in L^R$, $a \in L$ we define μ_a as follows:

$$\mu_a = \{x \in R, \mu(x) \geq a\},$$

μ_a is called *$a-$cut* (or *$a-$level subset*) of μ. In particular, we denote $\mu_{\mu(0)}$ by μ_*, of course

$$\mu_* = \{x \in R | \mu(x) = \mu(0)\}.$$

Let f be a mapping from R into R' and let $\mu \in L^R$, $\nu \in L^{R'}$. Then $f(\mu) \in L^{R'}$ and $f^{-1}(\nu) \in L^R$ are defined as follows

$$f(\mu)(y) = \begin{cases} \vee\{\mu(x) \mid x \in f^{-1}(y)\} & f^{-1}(y) \neq \phi \\ 0 & \text{otherwise} \end{cases}$$

and

$$f^{-1}(\nu)(x) = \nu(f(x)) \quad \forall x \in R.$$

Let R, R' be rings and $f : R \to N$ be a ring homomorphism. $\mu \in L^R$ is called *f-invariant* if $f(x) = f(y)$ implies that $\mu(x) = \mu(y)$ for all $x, y \in R$.

By the above hypothesis, if $\mu \in LI(R)$ has the sup property, then $f(\mu_*) = (f(\mu))_*$ and if $\nu \in LI(R')$, then $f^{-1}(\nu_*) = (f^{-1}(\nu))_*$.

We recall some definitions and theorems from the book [12].

Definition 2.1. Let $\mu \in L^R$. Then μ is called an *L-ideal* of R if, for every $x, y \in R$, the following conditions are satisfies:

1) $\mu(x-y) \geq \mu(x) \wedge \mu(y)$;

2) $\mu(xy) \geq \mu(x) \vee \mu(y)$.

The set of all L-ideals of R is denoted by $LI(R)$.

Remark 2.2. By previous Definition, it is easily seen that if μ is an L-ideal of R, then $\mu(1) < \mu(x) < \mu(0)$ for all $x \in R$.

Definition 2.3. Let $\mu \in L^R$. Let

$$\langle \mu \rangle = \bigcap \{\nu \in LI(R);\ \mu \subseteq \nu\}.$$

Then $\langle \mu \rangle$ is called the L-ideal of R generated by μ.

In [12] it has been proved that if $\xi = \langle \mu \rangle$, then

$$\xi(x) = \bigvee \{\wedge_{y \in A} \mu(y) \mid A \subseteq R,\ 1 \leq |A| < \infty,\ x \in \langle A \rangle\} \text{ for all } x \in R.$$

Also, $\langle a_x \rangle = a_{\langle x \rangle}$.

Definition 2.4. Let $\mu, \nu \in LI(R)$, We define $\mu \circ \nu \in LI(R)$ as follows:

$$\mu \circ \nu(x) = \bigvee \{\mu(y) \wedge \nu(y) \mid y, z \in R, x = yz\} \text{ for all } x \in R.$$

By the previous definition, it is seen that $\langle a_x \rangle \circ \langle a_y \rangle = \langle (xy)_{a \wedge b} \rangle$.

Definition 2.5. Let R be a ring and $\zeta \in LI(R)$. Then ζ is called *prime L-ideal* of R if ζ is non-constant and for every $\mu, \nu \in LI(R)$, $\mu \circ \nu \subseteq \zeta$ implies that $\mu \subseteq \zeta$ or $\nu \subseteq \zeta$.

Definition 2.6. Let $c \in L \backslash \{1\}$, c is called a *prime element* of L if $a \wedge b \leq c$, then $a \leq c$ or $b \leq c$, for all $a, b \in L$, and c is called a *maximal element* if there does not exist $a \in L \backslash \{1\}$ such that $c < a < 1$.

Theorem 2.7. *Let $\zeta \in L^R$. Then ζ is a prime L-ideal of R if and only if $\zeta(0) = 1$ and $\zeta = 1_{\zeta_*} \cup c_R$ such that ζ_* is a prime ideal of R and c is a prime element of L.*

3. Weakly prime L-ideals

In this section, we introduce the notion of weakly prime L-ideals and investigate some basic properties of them.

Definition 3.1. An L-ideal μ f R is called weakly prime if for every $\alpha, \beta \in LI(R)$ such that $0_R \neq \alpha \circ \beta \subseteq \mu$, then either $\alpha \subseteq \mu$ or $\beta \subseteq \mu$.

Corollary 3.2. *Let μ be a weakly prime L-ideal of R. Then $0_R \neq x_a \circ y_b \subseteq \mu$ implies that $x_a \subseteq \mu$ or $y_b \subseteq \mu$, for all $x, y \in R$ and $a, b \in L$.*

Proof. Let $0_R \neq x_a \circ y_b \subseteq \mu$ for $x, y \in R$ and $a, b \in L$. Then $a \wedge b \neq 0$ and so $\langle x_a \rangle \circ \langle y_b \rangle \neq 0_R$, (since $\langle x_a \rangle \circ \langle y_b \rangle = \langle (xy)_{a \wedge b} \rangle = (a \wedge b)_{\langle xy \rangle}$ and $((a \wedge b)_{\langle xy \rangle})(xy) = a \wedge b \neq 0$). Therefore, $0_R \neq \langle x_a \rangle \circ \langle y_b \rangle \subseteq \mu$. Thus $\langle x_a \rangle \subseteq \mu$ or $\langle y_a \rangle \subseteq \mu$. Hence $x_a \subseteq \mu$ or $y_b \subseteq \mu$. ∎

Remark 3.3. It is clear that every prime L-ideal of R is weakly prime. But the converse is not true, for example 0_R is weakly prime ideal which is not prime, because if we consider a, b in L such that $a \wedge b = 0$, then $0_a \circ 0_b \subseteq 0_R$, but neither $0_a \subseteq 0_R$ nor $0_b \subseteq 0_R$. Also, in [12] it has proved that if P is a prime ideal of R and L is regular, then 1_P is a prime L-ideal of R. But it is not true for weakly prime ideals in general. For example $\langle 0 \rangle$ is weakly prime ideal of R, but $1_{\langle 0 \rangle}$ is not weakly prime L-ideal of R.

In the next three propositions, we give some interesting properties of weakly prime L-ideals.

Proposition 3.4. *Let μ be a nonconstant weakly prime L-ideal of R such that $\mu \neq 0_R$. Then $\mu(0) = 1$.*

Proof. Let $\mu(0) < 1$. Since μ is non-constant, there exists $a \in R$ such that $\mu(a) < \mu(0)$. We define two L-subsets on R as follows:

$$\alpha(x) = \begin{cases} 1 & x \in \mu_* \\ 0 & \text{otherwise} \end{cases}$$

and

$$\beta(x) = \mu(0) \quad \forall x \in R$$

Then for $0 \neq x \in R$,

$$(\alpha \circ \beta)(x) = \bigvee \{\alpha(y) \wedge \beta(z) \mid x = yz\} = \bigvee \{\alpha(y) \wedge \mu(0) \mid x = yz\} = \mu(0) \text{ or } 0$$

and again since μ is non-constant

$$(\alpha \circ \beta)(0) = \mu(0) \neq 0.$$

Also $\alpha(0) = 1 > \mu(0)$ and $\beta(a) = \mu(0) > \mu(a)$. Thus $\alpha \not\subseteq \mu$ and $\beta \not\subseteq \mu$ which is a contradiction. Hence $\mu(0) = 1$. ∎

By the previous proposition, we get if μ is a weakly prime L-ideal of R, then

$$\mu_* = \{x \in R | \mu(x) = 1\}.$$

Proposition 3.5. *Let μ be a non-constant weakly prime L-ideal of R. Then μ_* is a weakly prime ideal of R.*

Proof. Suppose that μ is a weakly prime L-ideal of R and $0 \neq ab \in \mu_*$ for $a, b \in R$. We put $\alpha = 1_{<a>}$ and $\beta = 1_{<b>}$. Then

$$\alpha \circ \beta = 1_{<a>} \circ 1_{<b>} = 1_{<ab>} \subseteq 1_{\mu_*} \subseteq \mu$$

Also $(\alpha \circ \beta)(ab) = 1 \neq 0$. Since μ is weakly prime, then $1_{<a>} \subseteq \mu$ or $1_{<b>} \subseteq \mu$. So $\mu(a) = 1$ or $\mu(b) = 1$ and thus $a \in \mu_*$ or $b \in \mu_*$. Hence μ_* is weakly prime ideal of R. ∎

Definition 3.6. An element $t \neq 1$ in L is called a weakly prime element if $0 \neq a \wedge b \leq t$ implies that either $a \leq t$ or $b \leq t$, for all $a, b \in L$.

Theorem 3.7. *Let μ be a non-constant L-ideal of R. If μ is weakly prime, then $\mu = 1_{\mu_*} \cup c_R$ such that μ_* is a weakly prime ideal of R and c s a weakly prime element of L.*

Proof. Let μ be a weakly prime L-ideal of R. By Proposition 3.5 μ_* is a weakly prime ideal of R and $\mu(0) = 1$. We prove that $|\mu(R) = 2|$ and if $1 \neq c \in \mu(R)$, then c is a weakly prime element. μ is non-constant and $1 \in \mu(R)$, so $|\mu(R)| \geq 2$. Now let $x, y \in R \setminus \mu_*$ and $\mu(x) = c$. There are two cases:

1) $c = 0$. Then $\mu(x) \leq \mu(y)$.

2) $c \neq 0$. Then $1_{\langle x \rangle}$ and c_R are L-ideals, $c_{\langle x \rangle} = \langle x_c \rangle \subseteq \mu$ and

$$0_R \neq 1_{\langle x \rangle} \circ c_R \subseteq 1_{\langle x \rangle} \cap c_R = c_{<x>} \subseteq \mu$$

Since $1_{\langle x \rangle} \not\subseteq \mu$ and μ is weakly prime, then $c_R \subseteq \mu$. Thus, $c \leq \mu(y)$ and so $\mu(x) \leq \mu(y)$. Similarly, we can get $\mu(y) \leq \mu(x)$. Hence $\mu(x) = \mu(y)$ and then $|\mu(R)| = 2$.

Now, suppose that $0 \neq a \wedge b \leq c$ for $a, b \in L$, but $a \not\leq c$ and $b \not\leq c$. Then, a_R and b_R are L-ideals of R, $0_R \neq a_R \circ b_R = (a \wedge b)_R \subseteq \mu$, but $a_R \not\subseteq \mu$ and $b_R \not\subseteq \mu$ which is a contradiction. Therefore, $\mu = 1_{\mu_*} \cup c_R$ such that μ_* is a weakly prime ideal of R and c is a weakly prime element in L. ∎

Note that if R is an integral domain and L is a chain, then the converse of the previous theorem is true. In fact:

Theorem 3.8. *Let R be an integral domain, L be a chain and μ is an L-ideal of R. If $\mu = 1_{\mu_*} \cup c_R$ such that μ_* is a weakly prime ideal of R and c is an element of L, then μ is prime and so is weakly prime.*

Proof. Every element of L is prime, since L is chain. We prove that μ_* is a prime ideal of R. For this let $ab \in \mu_*$ for $a, b \in R$. If $ab \neq 0$, then $a \in \mu_*$ or $b \in \mu_*$. If $ab = 0$, then $a = 0$ or $b = 0$. So $a \in \mu_*$ or $b \in \mu_*$. Therefore μ_* is prime. Hence, μ is prime and so is weakly prime. ∎

In the following examples, we show that whether R is not integral domain or L is not chain, the converse of Theorem 3.7 is not in general hold.

Example 3.9. Let R be a ring which is not an integral domain. Then $\langle 0 \rangle$ is weakly prime ideal of R which is not prime. Consider $a, b \neq 0$ such that $ab = 0$ and $c \in L$. Put $\mu = 1_{<0>} \cup c_R$, $\alpha = 1_{\langle a \rangle}$ and $\beta = 1_{\langle b \rangle}$. Then $\alpha \circ \beta = 1_{\langle ab \rangle} = 1_{<0>} \neq 0_R$ and $\alpha \circ \beta \subseteq 1_{\langle 0 \rangle} \cup c_R$, but $1_{\langle a \rangle} \not\subseteq \mu$ and $1_{\langle b \rangle} \not\subseteq \mu$. It means that μ is not weakly prime.

Example 3.10. Let $R = \mathbb{Z}$ (the ring of integers) and $L = \{0, a, b, 1\}$ such that $a \wedge b = 0$. Then 0 is a weakly prime element in L. Consider I any prime ideal in $\mathbb{Z}$. Then I is weakly prime. Put $\mu = 1_I \cup 0_R$, $\alpha = 1_I \cup a_R$ and $\beta = 1_I \cup b_R$. So, $0_R \neq \alpha \circ \beta \subseteq \mu$ ($(\alpha \circ \beta)(0) = 1 \neq 0$), but $\alpha \not\subseteq \mu$ and $\beta \not\subseteq \mu$.

Corollary 3.11. *Let μ be a non-constant weakly prime L-ideal of R. Then for every $x, y \in R$ such that $xy \neq 0$, either $\mu(xy) = \mu(x)$ or $\mu(xy) = \mu(y)$*

Proposition 3.12. *Let R and S be rings, f be an epimorphism of R to S and μ be an f-invariant L-ideal of R such that $\mu(R)$ is finite. If μ_* is a weakly prime ideal of R, then $f(\mu_*)$ is a weakly prime ideal of S.*

Proof. Since $\mu(R)$ is finite, it has the sup-property. So $f(\mu_*) = f(\mu)_*$. Now, suppose that $0 \neq uv \in f(\mu_*) = f(\mu)_*$. Then there exist $x, y \in R$ such that $u = f(x)$ and $v = f(y)$. So $xy \neq 0$ and $f(xy) = uv \in f(\mu)_*$. Hence

$$f(\mu)(0) = f(\mu)(f(xy)) = \bigvee\{\mu(z) \mid f(z) = f(xy)\} = \mu(0)$$

Therefore, there exists $w \in R$ such that $\mu(w) = \mu(0)$ and $f(w) = f(xy)$. Since μ is f-invariant, $\mu(w) = \mu(xy) = \mu(0)$. Thus $0 \neq xy \in \mu_*$. So $x \in \mu_*$ or $y \in \mu_*$ and then $u \in f(\mu_*)$ or $v \in f(\mu_*)$. It means that $f(\mu_*)$ is weakly prime ideal of R. ∎

Lemma 3.13. *Let f be an epimorphism of R into S and μ be an f-invariant L-ideal of R. If μ is a non-constant weakly prime L-ideal of R,then $f(\mu) = 1_{f(\mu)_*} \cup c_R$ such that $f(\mu)_*$ is a weakly prime ideal of R and c is a weakly prime element in L.*

Proof. By Theorem 3.7, $\mu = 1_{\mu_*} \cup c_R$ such that μ_* is a weakly prime ideal of R and c is a weakly prime element in L. By Proposition 3.12 , it is seen that $f(\mu)_*$ is a weakly prime ideal of S and $f(\mu)(0) = \mu(0) = 1$. Now, for $y \in S$, there exists $x \in R$ such that $f(x) = y$. If $\mu(x) = 1$, we have

$$f(\mu)(y) = f(\mu)(f(x)) = \bigvee\{\mu(z) \mid f(z) = f(x)\} = 1$$

since μ is f-invariant.

Similarly, if $\mu(x) = c$, then $f(\mu)(y) = c$. Hence $f(\mu) = 1_{f(\mu)_*} \cup c_R$ such that $f(\mu)_*$ is a weakly prime ideal of R and c is a weakly prime element in L. ∎

Corollary 3.14. *Let f be an epimorphism from R into S and μ be a non-constant weakly prime f-invariant L-ideal of R. If L is a chain and S is an integral domain, then $f(\mu)$ is a weakly prime L-ideal of S.*

Proof. By Theorem 3.8 and Theorem 3.13. ∎

In the previous corollary, if S is not an integral domain, then the image of a weakly prime L-ideal need not be weakly prime. For an example, if L is a chain, R is an integral domain and S is any ring, then $\mu = 1_{\langle 0 \rangle} \cup c_R$ is weakly prime L-ideal of R, but $f(\mu) = 1_{\langle 0 \rangle} \cup c_R$ is not a weakly prime L-ideal of S. (By Example 3.9)

Proposition 3.15. *Let S be an integral domain, f be a homomorphism of R into S and $\nu \in LI(S)$. If ν_* is a weakly prime ideal of S, then $f^{-1}(\nu)_*$ is a prime (and so weakly prime) ideal of R.*

Proof. We have $f^{-1}(\nu_*) = f^{-1}(\nu)_*$. So it is enough to prove that $f^{-1}(\nu_*)$ is prime. For this let $xy \in f^{-1}(\nu_*)$. Then $f(x)f(y) = f(xy) \in \nu_*$. If $f(x)f(y) = 0$, then $f(x) = 0$ or $f(y) = 0$. So $x \in f^{-1}(\nu_*)$ or $y \in f^{-1}(\nu_*)$.

If $f(x)f(y) \neq 0$, then $f(x) \in \nu_*$ or $f(y) \in \nu_*$, since ν_* is weakly prime. Therefore, $x \in f^{-1}(\nu_*)$ or $y \in f^{-1}(\nu_*)$. Hence $f^{-1}(\nu_*)$ is prime ideal of R. ∎

Corollary 3.16. *Let S be an integral domain and L be a chain. Suppose that f is a homomorphism of R into S. If ν is a non-constant weakly prime L-ideal of S, then $f^{-1}(\nu)$ is a weakly prime L-ideal of R.*

Proof. First $f^{-1}(\nu)(0) = \nu(f(0)) = 1$ and $f^{-1}(\nu)_*$ is a prime ideal of R, by Proposition 3.15. Now, $\nu = 1_{\nu_*} \cup c_R$ such that ν_* is a weakly prime ideal of S. For $x \in R$, if $\nu(f(x)) = 1$, then $f^{-1}(\nu)(x) = 1$. If $\nu(f(x)) = c$, then $f^{-1}(\nu)(x) = c$. It shows that $f^{-1}(\nu) = 1_{f^{-1}(\nu)_*} \cup c_R$. Thus $f^{-1}(\nu)$ is prime and so is weakly prime L-ideal of R. ∎

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ROUGH FUZZY n-ABSORBING (n-ABSORBING PRIMARY) IDEAL IN COMMUTATIVE RINGS

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Abstract. In this paper, we introduce the notions of rough n-absorbing (n-absorbing primary) ideals and rough fuzzy n-absorbing (n-absorbing primary) ideals in a ring, and give some properties of such ideals. Also, we discuss the relations between the upper and lower rough n-absorbing (n-absorbing primary) ideals and the upper and lower approximations of their homomorphism images.

Keywords: fuzzy set, rough set, lower approximation; upper approximation, ring, ideal, prime ideal, primary ideal, n-absorbing ideal, n-absorbing primary ideal, fuzzy ideal.

1. Introduction

The notion of rough sets was introduced by Z. Pawlak in the year 1982 [13] as a formal tool for modelling and processing incomplete information in information systems. Some authors have studied the algebraic properties of rough sets. Biswas and Nanda [5] introduced the notion of rough subgroups. Kuroki in [10], introduced the notion of a rough ideal in a semigroup. Kuroki and Wang [11] gave some properties of the lower and upper approximations with respect to the normal subgroups. In [6], Davvaz concerned a relationship between rough sets and ring theory and considered a ring as a universal set and introduced the notion of rough ideals and rough subrings with respect to an ideal of a ring. In [9], Kazanci and Davvaz introduced the notions of rough prime (primary) ideals and rough fuzzy

prime (primary) ideals in a ring. Now, this paper is a continuation of ideas presented by Kazanci and Davvaz [9]. We introduce the notions of rough n-absorbing (n-absorbing primary) ideals and rough fuzzy n-absorbing (n-absorbing primary) ideals in a ring, and give some properties of such ideals.

2. Preliminaries

For the sake of completeness, we would like to recall some definitions and results proposed in this field earlier. Throughout this paper R is a ring and n be a positive integer. Recall that an equivalence relation θ on R is a reflexive, symmetric, and transitive binary relation on R. If θ is an equivalence relation on R then the equivalence class of $x \in R$ is the set $\{y \in R \mid (x, y) \in \theta\}$. We write it as $[x]_\theta$.

Definition 2.1. [9] Let θ be an equivalence relation on R, then θ is called a full congruence relation if $(a, b) \in \theta$ implies $(a + x, b + x), (ax, bx)$ and $(xa, xb) \in \theta$ for all $x \in R$.

A full congruence relation θ on R is called complete if it satisfies

$$[ab]_\theta = \{xy \mid x \in [a]_\theta, y \in [b]_\theta\}, \text{ for all } a, b \in R.$$

Definition 2.2. [9] Let θ be a full congruence relation on R and A a subset of R. Then, the sets

$$\theta_-(A) = \{x \in R \mid [x]_\theta \subseteq A\} \text{ and } \theta^-(A) = \{x \in R \mid [x]_\theta \cap A \neq \emptyset\}$$

are called, respectively, the θ-lower and θ-upper approximations of the set A.

$\theta(A) = (\theta_-(A), \theta^-(A))$ is called a rough set with respect to θ if

$$\theta_-(A) \neq \theta^-(A).$$

Definition 2.3. [9] Let θ be a full congruence relation on R and A a subset of R. A non-empty subset A of a ring R is called an upper (resp. lower) rough ideal of R if $\theta^-(A)$ (resp. $\theta_-(A)$)is an ideal of R.

Theorem 2.4. [9] *Let θ be a full congruence relation on R. If A is an ideal of R, then $\theta^-(A)$ is an ideal of R.*

Theorem 2.5. [9] *Let θ be a full congruence relation on R and A is an ideal of R. If $\theta_-(A)$ is a nonempty set, then it is equal to A.*

Theorem 2.6. [9] *Let φ be an epimorphism (an onto homomorphism) of a ring R_1 to a ring R_2 and let θ_2 be a full congruence relation on R_2. Then*

(1) *$\theta_1 = \{(a, b) \in R_1 \times R_1 \mid (\varphi(a), \varphi(b)) \in \theta_2\}$ is a full congruence relation on R_1.*

(2) *If θ_2 is complete and φ is one to one, then θ_1 is complete.*

(3) $\varphi(\theta_1^-(A)) = \theta_2^-(\varphi(A))$.

(4) $\varphi(\theta_{1-}(A)) \subseteq \theta_{2-}(\varphi(A))$. *If* φ *is one to one, then* $\varphi(\theta_{1-}(A)) = \theta_{2-}(\varphi(A))$

As it is well known in the fuzzy theory established by Zadeh [14], a fuzzy subset μ of R is defined as a map from R to the unit interval $[0,1]$. Let μ and λ be two fuzzy subsets of R. The inclusion $\lambda \subseteq \mu$ is denoted by $\lambda(x) \leq \mu(x)$ for all $x \in R$, and $\mu \cap \lambda$ is defined by $\mu \cap \lambda(x) = \mu(x) \wedge \lambda(x)$ for all $x \in R$ and the expression $\mu o \lambda$ is defined by

$$(\mu o \lambda)(x) = \bigvee_{yz=x} (\mu(y) \wedge \lambda(z)).$$

Definition 2.7. [9] Let θ be a full congruence relation on R and μ a fuzzy subset of R. Then we define the fuzzy sets $\theta_-(\mu)$ and $\theta^-(\mu)$ as follow:

$$\theta_-(\mu)(x) = \bigwedge_{a \in [x]_\theta} \mu(a) \ \ and \ \ \theta^-(\mu)(x) = \bigvee_{a \in [x]_\theta} \mu(a).$$

The fuzzy sets $\theta_-(\mu)$ and $\theta^-(\mu)$ are called, respectively the θ-lower and θ-upper approximations of the fuzzy set μ.

$\theta(\mu) = (\theta_-(\mu), \theta^-(\mu))$ is called a rough fuzzy set with respect to θ if

$$\theta_-(\mu) \neq \theta^-(\mu).$$

Liu [12] introduced and examined the notion of a fuzzy ideal of a ring.

Definition 2.8. A fuzzy subset μ of a ring R is called a fuzzy ideal of R if it has the following properties:

(i) $\mu(x-y) \geq \mu(x) \wedge \mu(y)$ for all $x, y \in R$

(ii) $\mu(xy) \geq \mu(x) \vee \mu(y)$ for all $x, y \in R$

Example 1. [9] Let I be any ideal of R and let $\alpha \leq \beta \neq 0$ be numbers in $[0,1]$ then the fuzzy subsets μ defined by

$$\mu(x) = \begin{cases} \beta & \text{if } x \in I \\ \alpha & \text{otherwise} \end{cases}$$

is a fuzzy ideal.

Let μ and λ be fuzzy ideals of a ring R. Then, $\mu \cap \lambda$ is also a fuzzy ideal of R. A fuzzy subset μ of a ring R is called an lower rough fuzzy ideal of R if $\theta_-(\mu)$ is a fuzzy ideal of R. Similarly we can define upper rough fuzzy ideal. Recall that if μ, λ are fuzzy ideals of R, then $(\theta_-(\mu \cap \lambda), \theta^-(\mu \cap \lambda))$ is a rough fuzzy ideal of R. Let μ be a fuzzy subset of R. Then the sets

$$\mu_t = \{x \in R \mid \mu(x) \geq t\}, \mu_t^s = \{x \in R \mid \mu(x) > t\},$$

where $t \in [0,1]$ are called respectively, t-level subset and t-strong level subset of μ (see [9]).

Theorem 2.9. [8] *Let μ be a fuzzy subset of R. μ is a fuzzy ideal of R if and only if μ_t and μ_t^s are, if they are nonempty, ideals of R for every $t \in [0,1]$.*

Theorem 2.10. [9] *Let θ be a full congruence relation on R. If μ is a fuzzy subset of R and $t \in [0,1]$, then*

(1) $(\theta_-(\mu))_t = \theta_-(\mu_t)$

(2) $(\theta^-(\mu))_t^s = \theta^-(\mu_t^s)$.

3. Rough n-absorbing (n-absorbing primary) ideals in commutative rings

In the rest of this paper, R is a commutative ring with identity.

Let R be a ring. Recall that a nonzero proper ideal I of R is called an n-absorbing ideal of R if whenever $x_1, ..., x_{n+1} \in R$ and $x_1 \cdots x_{n+1} \in I$, then there are n of the x_i's whose product is in I (there exist $i \in \{1, ..., n+1\}$ such that $x_1 \cdots x_{i-1} x_{i+1} \cdots x_{n+1} \in I$ (see [1], [2]).

In $\mathbb{Z}$, the ideal $< p_1 \cdots p_n >= \{p_1 \cdots p_n k | k \in \mathbb{Z}\}$, where p_i is a prime number of $\mathbb{Z}$, is n-absorbing ideal of $\mathbb{Z}$.

Every prime ideal is n-absorbing but the converse is not true. For example, let $n = 3$. Then the ideals $I_1 = 27\mathbb{Z}$, $I_2 = 18\mathbb{Z}$, and $I_3 = 30\mathbb{Z}$ are 3-absorbing, but not prime ideals of $\mathbb{Z}$.

Let θ be a congruence relation on a ring R. Then a subset A of R is called a lower rough n-absorbing ideal of R if $\theta_-(A)$ is a n-absorbing ideal of R. Similarly we can define upper rough n-absorbing ideal.

Theorem 3.1. *Let θ be a complete congruence relation on R and P an n-absorbing ideal of R such that $\theta^-(P) \neq R$, then P is an upper rough n-absorbing ideal of R.*

Proof. Since P is an ideal of R, by Theorem 2.4, we know that $\theta^-(P)$ is an ideal of R. Now, let $x_1 \cdots x_{n+1} \in \theta^-(P)$, then $[x_1 \cdots x_{n+1}]_\theta \cap P \neq \emptyset$. Since θ is complete, so there exist $a_1 \in [x_1]_\theta, ..., a_{n+1} \in [x_{n+1}]_\theta$ such that $a_1 \cdots a_{n+1} \in P$. Since P is an n-absorbing ideal, there exist $i \in \{1, ..., n+1\}$ such that $a_1 \cdots a_{i-1} a_{i+1} \cdots a_{n+1} \in P$. Since $a_1 \cdots a_{i-1} a_{i+1} \cdots a_{n+1} \in [x_1 \cdots x_{i-1} x_{i+1} \cdots x_{n+1}]_\theta$, it follows that

$$[x_1 \cdots x_{i-1} x_{i+1} \cdots x_{n+1}]_\theta \cap P \neq \emptyset.$$

Hence, we have $x_1 \cdots x_{i-1} x_{i+1} \cdots x_{n+1} \in \theta^-(P)$. Therefore, $\theta^-(P)$ is a n-absorbing ideal of R. ∎

Theorem 3.2. *Let θ be a full congruence relation on R and P be an n-absorbing ideal of R. If $\theta_-(P)$ is a nonempty set, then P is a lower rough n-absorbing ideal of R.*

Proof. The proof is straightforward by Theorem 2.5. ∎

Definition 3.3. A proper ideal I of R is n-absorbing primary if whenever $x_1, ..., x_{n+1} \in R$, $x_1 \cdots x_{n+1} \in I$ and $x_1 \cdots x_n \notin I$, there exist $i \in \{1, ..., n\}$ such that

$$(x_{n+1}x_1 \cdots x_{i-1}x_{i+1} \cdots x_n)^k \in I,$$

for some positive integer k.

Let θ be a congruence relation on a ring R. Then a subset A of R is called a lower rough n-absorbing primary ideal of R if $\theta_-(A)$ is an n-absorbing primary ideal of R. Similarly we can define upper rough n-absorbing primary ideal.

We call Q a rough n-absorbing (n-absorbing primary) ideal of R if it is both a lower and an upper rough n-absorbing (n-absorbing primary) ideal of R.

Theorem 3.4. *Let θ be a complete congruence relation on R and Q an n-absorbing primary ideal of R such that $\theta^-(Q) \neq R$. Then, Q is an upper rough n-absorbing primary ideal of R.*

Proof. Let $x_1 \cdots x_{n+1} \in \theta^-(Q)$ and $x_1 \cdots x_n \notin \theta^-(Q)$. Then $[x_1 \cdots x_{n+1}]_\theta \cap Q \neq \emptyset$. Since θ is complete, so there exist $a_1 \in [x_1]_\theta, ..., a_n \in [x_n]_\theta$ and $a_{n+1} \in [x_{n+1}]_\theta$, $a_1 \cdots a_{n+1} \in Q$. Since Q is an n-absorbing primary ideal, it follows that there exist $i \in \{1, ..., n\}$ such that $(a_{n+1}a_1 \cdots a_{i-1}a_{i+1} \cdots a_n)^k \in Q$ for some positive integer k. Since θ is complete, it follows that

$$a_{n+1}a_1 \cdots a_{i-1}a_{i+1} \cdots a_n \in [x_{n+1}x_1 \cdots x_{i-1}x_{i+1} \cdots x_n]_\theta.$$

We get

$$(a_{n+1}a_1 \cdots a_{i-1}a_{i+1} \cdots a_n)^k \in [(x_{n+1}x_1 \cdots x_{i-1}x_{i+1} \cdots x_n)^k]_\theta.$$

Hence,

$$[(x_{n+1}x_1 \cdots x_{i-1}x_{i+1} \cdots x_n)^k]_\theta \cap Q \neq \emptyset,$$

which implies that $(x_{n+1}x_1 \cdots x_{i-1}x_{i+1} \cdots x_n)^k \in \theta^-(Q)$. Therefore, $\theta^-(Q)$ is an n-absorbing primary ideal of R. ∎

Theorem 3.5. *Let θ be a full congruence relation on R and Q be an n-absorbing primary ideal of R. If $\theta_-(Q)$ is a nonempty set, then Q is a lower rough n-absorbing primary ideal of R.*

Proof. The proof is straightforward by Theorem 2.5. ∎

Theorem 3.6. *Let φ be an epimorphism of a ring R_1 to a ring R_2 and let θ_2 be a full congruence relation on R_2. Let A be a subset of R_1.*

If $\theta_1 = \{(a, b) \in R_1 \times R_1 \mid (\varphi(a), \varphi(b)) \in \theta_2\}$, then

(i) *If θ_2 is complete, $\theta_1^-(A)$ is an n-absorbing ideal of R_1 if and only if $\theta_2^-(\varphi(A))$ is an n-absorbing ideal of R_2.*

(ii) *If θ_2 is complete, $\theta_1^-(A)$ is an n-absorbing primary ideal of R_1 if and only if $\theta_2^-(\varphi(A))$ is an n-absorbing primary ideal of R_2.*

Proof. (i) Assume that $\theta_1^-(A)$ is an n-absorbing ideal of R_1. Suppose that $x_1, ..., x_{n+1} \in R_2$ such that $x_1 \cdots x_{n+1} \in \theta_2^-(\varphi(A))$. Then there exist $a_1, ..., a_{n+1} \in R_1$ such that $\varphi(a_1) = x_1, ..., \varphi(a_{n+1}) = x_{n+1}$. Thus $[\varphi(a_1) \cdots \varphi(a_{n+1})]_{\theta_2} \cap \varphi(A) \neq 0$. Since θ_2 is complete, there exists elements $\varphi(u_1) \in [\varphi(a_1)]_{\theta_2}, ..., \varphi(u_{n+1}) \in [\varphi(a_{n+1})]_{\theta_2}$ such that $\varphi(u_1) \cdots \varphi(u_{n+1}) = \varphi(u_1 \cdots u_{n+1}) \in \varphi(A)$. Then we have $u_1 \in [a_1]_{\theta_1}, ..., u_{n+1} \in [a_{n+1}]_{\theta_1}$ and there exists $c \in A$ such that $\varphi(c) = \varphi(u_1 \cdots u_{n+1})$. Hence $u_1 \cdots u_{n+1} \in [a_1 \cdots a_{n+1}]_{\theta_1}$ and $c \in [u_1 \cdots u_{n+1}]_{\theta_1}$. Thus $[a_1 \cdots a_{n+1}]_{\theta_1} \cap A \neq \emptyset$ which implies that $a_1 \cdots a_{n+1} \in \theta_1^-(A)$. Since $\theta_1^-(A)$ is an n-absorbing ideal of R_1, there exist $i \in \{1, ..., n+1\}$ such that $a_1 \cdots a_{i-1} a_{i+1} \cdots a_{n+1} \in \theta_1^-(A)$. From this and [9, Theorem 2.9], we obtain that $x_1 \cdots x_{i-1} x_{i+1} \cdots x_{n+1} = \varphi(a_1 \cdots a_{i-1} a_{i+1} \cdots a_{n+1}) \in \varphi(\theta_1^-(A)) = \theta_2^-(\varphi(A))$. This means that $\theta_2^-(\varphi(A))$ is an n-absorbing ideal of R_2.

Conversely, assume that $\theta_2^-(\varphi(A))$ is an n-absorbing ideal of R_2. Suppose that $x_1 \cdots x_{n+1} \in \theta_1^-(A)$ for some $x_1, ..., x_{n+1} \in R_1$, then by [9, Theorem 2.9], we obtain that $\varphi(x_1 \cdots x_{n+1}) = \varphi(x_1) \cdots \varphi(x_{n+1}) \in \varphi(\theta_1^-(A)) = \theta_2^-(\varphi(A))$. Since $\theta_2^-(\varphi(A))$ is an n-absorbing ideal of R_2, we have n of the $\varphi(x_i)$'s whose product is in $\in \varphi(\theta_1^-(A))$, then there exists $a \in \theta_1^-(A)$ such that $\varphi(x_1 \cdots x_{i-1} x_{i+1} \cdots x_{n+1}) = \varphi(a)$. Thus $[a]_{\theta_1} \cap A \neq \emptyset$ and $x_1 \cdots x_{i-1} x_{i+1} \cdots x_{n+1} \in [a]_{\theta_1}$. So $[x_1 \cdots x_{i-1} x_{i+1} \cdots x_{n+1}]_{\theta_1} \cap A \neq \emptyset$, which implies $x_1 \cdots x_{i-1} x_{i+1} \cdots x_{n+1} \in \theta_1^-(A)$. This means that $\theta_1^-(A)$ is an n-absorbing ideal of R_1.

(ii) The proof is similar to the proof of (i). ∎

Theorem 3.7. *Let φ be an isomorphism of a ring R_1 to a ring R_2 and let θ_2 be a complete congruence relation on R_2. Let A be a subset of R_1.*

If $\theta_1 = \{(a, b) \in R_1 \times R_1 \mid (\varphi(a), \varphi(b)) \in \theta_2\}$, then

(i) *$\theta_{1-}(A)$ is an n-absorbing ideal of R_1 if and only if $\theta_{2-}(\varphi(A))$ is an n-absorbing ideal of R_2,*

(ii) *$\theta_{1-}(A)$ is an n-absorbing primary ideal of R_1 if and only if $\theta_{2-}(\varphi(A))$ is an n-absorbing primary ideal of R_2.*

Proof. By [9, Theorem 2.9], we have $\varphi(\theta_{1-}(A)) = \theta_{2-}(\varphi(A))$. Now, the proof is similar to the proof of Theorem 3.6. ∎

4. Rough fuzzy n-absorbing(n-absorbing primary) ideals

Definition 4.1.

(i) A fuzzy ideal ρ is called n-absorbing if for all $x_1, ..., x_{n+1} \in R$, there exist $i \in \{1, ..., n+1\}$ such that $\rho(x_1 \cdots x_{n+1}) = \rho(x_1 \cdots x_{i-1} x_{i+1} \cdots x_{n+1})$.

(ii) A fuzzy ideal ρ is called n-absorbing primary if for all $x_1, ..., x_{n+1} \in R$, there exist $i \in \{1, ..., n+1\}$ such that $\rho(x_1 \cdots x_{n+1}) = \rho((x_1 \cdots x_{i-1} x_{i+1} \cdots x_{n+1})^k)$ for some integer k.

Example 2. Let I be any n-absorbing ideal of R and let $\alpha \leq \beta \neq 0$ be numbers in $[0,1]$ then the fuzzy subsets μ defined by

$$\mu(x) = \begin{cases} \beta & \text{if } x \in I \\ \alpha & \text{otherwise} \end{cases}$$

is a fuzzy n-absorbing ideal.

Theorem 4.2. *Let μ be a fuzzy ideal of a commutative ring with identity R. Then*

(i) *If μ is an n-absorbing fuzzy ideal then the set*

$$\{\alpha_i = \mu(x_1 \cdots x_{i-1}x_{i+1} \cdots x_{n+1})/x_i \in R,\ i \in \{1, ..., n+1\}\}$$

is totally ordered set with the least element $\mu(1)$ and the greatest element $\mu(0)$.

(ii) *μ is n-absorbing if and only if μ_t is n-absorbing for all $t > \mu(1)$. For $t = \mu(1)$, $\mu_t = R$.*

Proof. (i) If $x \in R$, then $\mu(0) = \mu(x - x) \geq \mu(x) = \mu(x.1) \geq \mu(1)$. Further, if $\alpha_i \in [0,1]$ with $\mu(x_1 \cdots x_{i-1}x_{i+1} \cdots x_{n+1}) = \alpha_i$, then $\mu(x_1 \cdots x_{n+1}) = \alpha_i$ for some $i \in \{1, ..., n+1\}$, then $\alpha_i = \mu(x_1 \cdots x_{n+1}) \geq \mu(x_1 \cdots x_{j-1}x_{j+1} \cdots x_{n+1}) = \alpha_j$ for all $j \in \{1, ..., n+1\}$.

(ii) Let μ_t be n-absorbing for all $t > \mu(1)$ and $x_1, ..., x_{n+1} \in R$. If $\mu(x_1 \cdots x_{n+1}) = \mu(1)$, then as $\mu(1)$ is the smallest element of $\mu(R)$, $\mu(x_1 \cdots x_{i-1}x_{i+1} \cdots x_{n+1}) = \mu(x_1 \cdots x_{n+1})$ for all $i \in \{1, ..., n+1\}\}$. If $\mu(x_1 \cdots x_{n+1}) = \alpha > \mu(1)$, then μ_t being n-absorbing, there is $i \in \{1, ..., n+1\}\}$ such that $x_1 \cdots x_{i-1}x_{i+1} \cdots x_{n+1} \in \mu_t$. Accordingly, $\mu(x_1 \cdots x_{i-1}x_{i+1} \cdots x_{n+1}) = \mu(x_1 \cdots x_{n+1})$ for some $i \in \{1, ..., n+1\}\}$. The converse is obvious. ∎

The following proposition is straightforward.

Theorem 4.3. *Let Q be an ideal of R.*

(i) *The characteristic function χ_Q is an n-absorbing primary fuzzy ideal if and only if Q is an n-absorbing primary ideal.*

(ii) *If ϱ is an n-absorbing primary fuzzy ideal, then $\varrho_t (\neq \emptyset)$ is an n-absorbing primary ideal for all $0 \leq t \leq 1$.*

(iii) *Every n-absorbing fuzzy ideal is n-absorbing primary.*

Theorem 4.4. *Let φ be an epimorphism of a ring R_1 to a ring R_2 and let θ_2 be a complete full congruence relation on R_2. Let μ be a fuzzy subset of R_1.*

If $\theta_1 = \{(a,b) \in R_1 \times R_1 \mid (\varphi(a), \varphi(b)) \in \theta_2\}$, then

(i) *$\theta_1^-(\mu)$ is a fuzzy ideal of R_1 if and only if $\theta_2^-(\varphi(\mu))$ is a fuzzy ideal of R_2.*

(ii) $\theta_1^-(\mu)$ *is a fuzzy* n*-absorbing ideal of* R_1 *if and only if* $\theta_2^-(\varphi(\mu))$ *is a fuzzy* n*-absorbing ideal of* R_2.

(iii) $\theta_1^-(\mu)$ *is a fuzzy* n*-absorbing primary ideal of* R_1 *if and only if* $\theta_2^-(\varphi(\mu))$ *is a fuzzy* n*-absorbing primary ideal of* R_2.

Moreover, if φ *is one-to-one, then we have*

(iv) $\theta_{1-}(\mu)$ *is a fuzzy ideal of* R_1 *if and only if* $\theta_2^-(\varphi(\mu))$ *is a fuzzy ideal of* R_2.

(v) $\theta_{1-}(\mu)$ *is a fuzzy* n*-absorbing ideal of* R_1 *if and only if* $\theta_2^-(\varphi(\mu))$ *is a fuzzy* n*-absorbing ideal of* R_2.

(vi) $\theta_{1-}(\mu)$ *is a fuzzy* n*-absorbing primary ideal of* R_1 *if and only if* $\theta_2^-(\varphi(\mu))$ *is a fuzzy* n*-absorbing primary ideal of* R_2.

Proof. (i) By [9, Theorem 3.6], we obtain that $\theta_1^-(\mu)$ is a fuzzy ideal of R_1 if and only if $(\theta_1^-(\mu))_t^s$ is, if it is nonempty, an ideal of R_1 for every $t \in [0,1]$. Using [9, Theorem 3.7], we have $(\theta_1^-(\mu))_t^s = (\theta_1^-(\mu_t^s))$. By Theorem 4.2, we obtain that $\theta_1^-(\mu_t^s)$ is an ideal of R_1 if and only if $\theta_2^-(\varphi(\mu_t^s))$ is an ideal of R_2. It is clear that $\varphi(\mu_t^s) = (\varphi(\mu))_t^s$. From this and [9, Theorem 3.6], we have

$$\theta_2^-(\varphi(\mu_t^s)) = \theta_2^-((\varphi(\mu))_t^s) = (\theta_2^-(\varphi(\mu)))_t^s$$

By [9, Theorem 3.6], we obtain $(\theta_2^-(\varphi(\mu)))_t^s$ is an ideal of R_2 for every $t \in [0,1]$ if and only if $\theta_2^-(\varphi(\mu))$ is a fuzzy ideal of R_2.

The proof of other parts is similar to the proof of (i). ∎

Theorem 4.5. *Let* μ *be a fuzzy* n*-absorbing ideal of* R.

(i) *If* θ *is a complete congruence relation on* R *and* $\theta_-(\mu) \neq \emptyset$, *then* μ *is a lower rough fuzzy* n*-absorbing ideal of* R.

(ii) *If* θ *is a complete congruence relation on* R, *then* μ *is an upper rough fuzzy* n*-absorbing ideal of* R.

Proof. (i) Since μ is a fuzzy n-absorbing ideal, by Theorem 4.2, we know that $\mu_t(t > \mu(1))$ is, if it is non-empty, an n-absorbing ideal of R. Then by Theorem 3.2, we obtain that $\theta_-(\mu_t)$, if it is non-empty, is an n-absorbing ideal of R. From this and [9, Theorem 3.7], we know that $(\theta_-(\mu))_t$ is an n-absorbing ideal of R. Now, by Theorem 4.2, we obtain that $\theta_-(\mu)$ is a fuzzy n-absorbing ideal of R.

(ii) It can be seen in a similar way. ∎

Theorem 4.6. *Let* θ *be a full congruence relation on* R. *Then* μ *is a lower [an upper] rough fuzzy* n*-absorbing ideal if and only if* μ_t, μ_t^s *are, if they are nonempty, lower [upper] rough* n*-absorbing ideals of* R *for every* $t \in [0,1]$.

Proof. It is straightforward. ∎

5. Conclusions

The rough sets theory is regarded as a generalization of the classical sets theory. A key notion in rough set is an equivalence relation. An equivalence is sometime difficult to be obtained in rearward problems due to vagueness and incompleteness of human knowledge. The combination of fuzzy set and rough set theory lead to various models. The relations between fuzzy sets, rough sets and ring theory have been already considered by the second author in [6], [7], [9]. In the present paper, we introduced the notions of rough n-absorbing (n-absorbing primary) ideals and rough fuzzy n-absorbing (n-absorbing primary) ideals in a commutative ring. We discussed the relations between upper (lower) rough n-absorbing and n-absorbing primary ideals and upper (lower) approximations of their homomorphism images. Also, we proved that a fuzzy set is a rough fuzzy n-absorbing ideal if and only if its level set is a rough n-absorbing ideal.

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ON THE APPROXIMATE SOLUTIONS OF SYSTEMS OF ODEs BY LEGENDRE OPERATIONAL MATRIX OF DIFFERENTIATION

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Abstract. In this article, a general framework for solving system of ordinary differential equations by implementing a relatively new numerical technique called the Legendre operational matrix of differentiation is presented for the first time. This method can be an effective procedure to obtain analytic and approximate solutions for different systems of ordinary differential equations. Different from other numerical techniques, shifted Legendre polynomials and their properties are employed for deriving a general procedure for forming this matrix. Comparisons are made between approximate solutions, exact solutions and numerical ones for several examples. Moreover, estimate error for the given algorithm is presented.

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1. Introduction

Many real life phenomena can be modelled by systems of ordinary differential equations (ODEs). For example, mathematical models of series of circuits and mechanical systems involving several springs attached in series can lead to a system of differential equations. Moreover, such systems are often encountered in chemical, ecological, biological, and engineering applications [13]. Many phenomena in chemical kinetics and engineering are modelled by stiff systems [7]. Stiff problems pose special computational difficulties because explicit numerical methods cannot solve these problems without severe limitations on the step size [2]. In control theory, system of ODEs can also exhibit chaotic behaviours [11, 10]. As is well-known, a chaotic system is a nonlinear deterministic system having complex and unpredictable behavior and exhibits sensitive dependence on initial conditions. One such system is the so-called Genesio system [6]. It is one of the paradigms of chaos since it captures many features of chaotic systems.

A system of ODEs can be expressed in the form:

$$y_i' = f_i(t, y_1, \ldots, y_n), \quad y_i(t_0) = y_{0,i}, \quad i = 1, 2, \ldots, n, \tag{1.1}$$

where f_i are (linear or non-linear) real-valued functions, t_0 and $y_{0,i}$ are real numbers. By a change of independent variable $t \to t + t_0$, systems of the form (1.1) can always be translated to the origin, and so in this paper, we focus on finding approximate solution to equations of the form

$$y_i' = f_i(t, y_1, \ldots, y_n), \quad y_i(0) = y_{0,i}, \quad i = 1, 2, \ldots, n. \tag{1.2}$$

Various numerical integration algorithms (for example, Runge-Kutta algorithms) for approximating solutions of the above of systems (1.2) have been presented in the literature. However, these algorithms offer approximate solutions at discrete points only thereby making it impossible to get continuous solutions. Approximate analytic solutions of certain classes of systems of ODEs have been given in [8], [2].

Legendre operational matrix of differentiation (LOMD), first proposed by Saadatmandi and Dehghan [12], is a powerful method for solving linear and nonlinear problems. They extended the application of Legendre polynomials to solve fractional differential equations. Pandey et al. [9] employed LOMD to solve Lane-Emden type equations. Recently, the method was applied for several modeled [1], [3]. Different from the Taylor series the Legender series approximation can update the all the previous coefficients when the new term is added. Moreover, the Legender approximation is more rapidly than the Taylor approximation with order of convergent $1/n!$ [5]. That leads to give more accurate solution. To the best of the authors' knowledge, the present work demonstrates for the first time the applicability of LOMD for obtaining the approximate analytic solutions of the system of the form (1.2). A general framework for solving system of ODEs via LOMD is presented. Several examples are studied to demonstrate the accuracy of the method.

2. Legendre polynomials and operational matrix of differentiation

The mth-order Legendre polynomials, $L_m(z)$, on the interval $[-1,1]$ are defined as

$$\begin{aligned} L_0(z) &= 1, \\ L_1(z) &= z, \\ L_{m+1}(z) &= \frac{2m+1}{m+1}\, z\, L_m(z) - \frac{m}{m+1}\, z\, L_{m-1}(z), \quad m = 1, 2, \ldots. \end{aligned}$$

These polynomials on the interval $z \in [0,1]$, so-called shifted Legendre polynomials, can be defined by introducing the change of variable $z = 2t - 1$. The shifted Legendre polynomials $L_m(2t-1)$ denoted by $P_m(t)$ can be obtained as

$$P_{m+1}(t) = \frac{(2m+1)(2t-1)}{(m+1)}\, P_m(t) - \frac{m}{m+1}\, P_{m-1}(t), \quad m = 1, 2, \ldots,$$

where $P_0(t) = 1$ and $P_1(t) = 2t - 1$. The analytic form of the shifted Legendre polynomial $P_m(t)$ of degree m is given by

$$P_m(t) = \sum_{i=1}^{m} (-1)^{m+i} \frac{(m+i)\,!\, t^i}{(m-i)\,(i\,!)^2}. \tag{2.1}$$

Note that $P_m(0) = (-1)^m$ and $P_m(1) = 1$ satisfy the orthogonality condition

$$\int_0^1 P_m(t) P_j(t)\, \mathrm{d}t = \begin{cases} \frac{1}{2m+1} & \text{for} \quad m = j, \\ 0 & \text{for} \quad m \neq j. \end{cases}$$

A function $y(t)$ square integrable in $[0,1]$, may be expressed in terms of shifted Legendre polynomials as

$$y(t) = \sum_{j=0}^{\infty} c_j P_j(t), \tag{2.2}$$

where the coefficients c_j are given by $c_j = (2j+1) \int_0^1 y(t) P_j(t)\, \mathrm{d}t,\ j = 1, 2, \ldots.$

In practice, we consider the $(m+1)$-term shifted Legendre polynomial so that

$$y(t) = \sum_{j=0}^{m} c_j P_j(t) = \mathrm{C}^{\mathrm{T}} \phi(t), \tag{2.3}$$

where the shifted Legendre coefficient vector C and the shifted Legendre vector $\phi(t)$ are given by

$$\mathrm{C}^{\mathrm{T}} = [\mathrm{c}_0, \mathrm{c}_1, \ldots, \mathrm{c}_m], \quad \phi(t) = [P_0(t), P_1(t), \ldots, P_m(t)]^{\mathrm{T}}.$$

The derivative of the vector $\phi(t)$ can be expressed as

$$\frac{\mathrm{d}\phi(t)}{\mathrm{d}t} = \mathrm{D}^1 \phi(t),\ \frac{\mathrm{d}^2\phi(t)}{\mathrm{d}t^2} = (\mathrm{D}^1)^2 \phi(t),\ \ldots,\ \frac{\mathrm{d}^n\phi(t)}{\mathrm{d}t^n} = (\mathrm{D}^1)^n \phi(t), \tag{2.4}$$

where D^1 is the $(m+1)\times(m+1)$ operational matrix of derivative. A general method of constructing such operational matrix of derivative could be presented as follows:

1. Differentiate analytically some polynomials of first degree.
2. Express these derivatives as a linear combination of polynomials of lower degree.
3. Find a general formula.

Now, the general formula of the operational matrix of derivative D^1 is given by

$$\mathrm{D}^1=(d_{ij})=\begin{cases} 2(2j+1), & \text{for } j=i-k, \\ 0 & \text{Otherwise.} \end{cases} \quad \begin{cases} k=1,3,\ldots,m, & \text{if } m \text{ odd}, \\ k=1,3,\ldots,m-1, & \text{if } m \text{ even}, \end{cases}$$

For example, for odd m we have

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 2 & 0 & 10 & 0 & \ldots & (2m-3) & 0 & 0 \\ 0 & 6 & 0 & 14 & \ldots & 0 & (2m-1) & 0 \end{pmatrix}. \tag{2.5}$$

3. Applications of the operational matrix of derivative

To solve (1.2) by means of the operational matrix of derivative [12, 4], we approximate $(y_i(t))^m$ and $g_i(t)$ by the shifted Legendre Polynomials as

$$(y_i(t))^m \simeq \left(\mathrm{C}_i^{\mathrm{T}}\phi(t)\right)^m, \tag{3.1}$$

$$g_i(t) \simeq \mathrm{G}_i^{\mathrm{T}}\phi(t), \tag{3.2}$$

where the vector $\mathrm{G}^{\mathrm{T}}=[g_0(t),\ldots,g_m(x)]^{\mathrm{T}}$ represents the non-homogenous term. By using equations (2.4), (3.1) and (3.2) we have

$$y_i'(t) \simeq \mathrm{C}_i^{\mathrm{T}}(\mathrm{D}^1)\phi(t). \tag{3.3}$$

Employing equation (3.1), the residual $\Re(x)$ for equation (1.2) can be written as

$$\Re(t)\simeq \mathrm{C}_i^{\mathrm{T}}(\mathrm{D}^1)\phi(t)-f(t,\mathrm{C}_1^{\mathrm{T}}\phi(t),\ldots,\mathrm{C}_m^{\mathrm{T}}\phi(t))-G_i^{\mathrm{T}}\phi(t). \tag{3.4}$$

Now, to find the solution $y_i(t)$ given in equation (2.3), we first collocate equation (3.4) at m points. For suitable collocation points we use the first m shifted Legendre roots of $P_{m+1}(x)$. These equations together with initial condition generate $(m+1)$ nonlinear equations which can be solved using Newton's iterative method. Consequently, $y_i(t)$ given in equation (2.3) can be calculated.

3.1. Error estimate

In this section, we give an error estimate for the LOM solution. Firstly, we define the exact and the approximate solution $y_i, \bar{y}_i$ respectively, and the error $e_i = y_i - \bar{y}_i$. The residual error of the approximate solution given by

$$R_i = \bar{y}_i' - f_i(t, \bar{y}_1, \bar{y}_2, \ldots \bar{y}_n). \tag{3.5}$$

If $R_i = 0$, then the solution is exact; else, taking (1.2)–(3.5) yields

$$e_i' - f_i(t, y_1, y_2, \ldots y_n) + f_i(t, \bar{y}_1, \bar{y}_2, \ldots \bar{y}_n) + R_i = 0,$$

which is differential equation with initial conditions $y_i(0) = 0$ for $i = 1, 2, \ldots, n$.

4. Numerical experiments

We demonstrate the efficiency of the present method through some illustrative examples.

4.1. Example 1

Let us consider the following linear system of ODEs [8]:

$$y_1'(t) = y_1(t) + y_2(t), \tag{4.1}$$
$$y_2'(t) = -y_1(t) + y_2(t), \tag{4.2}$$

with initial condition

$$y_1(0) = 0, \quad y_2(0) = 1. \tag{4.3}$$

The exact solution is $y_1(t) = e^t \sin(x), \quad y_2(t) = e^t \cos(x)$.

First, we consider the solution of the form

$$y_1(t) = \mathrm{C}_1^{\mathrm{T}} \phi(t), \tag{4.4}$$
$$y_2(t) = \mathrm{C}_2^{\mathrm{T}} \phi(t). \tag{4.5}$$

Thus, the solution will be in the form

$$y_1(t) = c_0 P_0(t) + c_1 P_1(t) + \ldots + c_m P_m(t) = \mathrm{C}_1^{\mathrm{T}} \phi(t), \tag{4.6}$$
$$y_2(t) = h_0 P_0(t) + h_1 P_1(t) + \ldots + h_m P_m(t) = \mathrm{C}_2^{\mathrm{T}} \phi(t). \tag{4.7}$$

By applying the initial conditions, we have

$$c_0 - c_1 + \ldots + c_m = 1$$
$$h_0 - h_1 + \ldots + c_m = 0.$$

The residual error of the solution can be written as

$$\Re_1 = C_1^{\mathrm{T}} D^1 \phi(t) - C_1^{\mathrm{T}} \phi(t) - C_2^{\mathrm{T}} \phi(t), \tag{4.8}$$
$$\Re_2 = C_2^{\mathrm{T}} D^1 \phi(t) + C_1^{\mathrm{T}} \phi(t) - C_2^{\mathrm{T}} \phi(t). \tag{4.9}$$

Now, taking the first m roots of $m+1$ shifted Legendre polynomials and substituted in (4.8) and (4.9) which equal to zero. By this way, we can have $2m+2$ algebraic equations with $2m+2$ constants which can be easily solved by Newton's iterative method. By taking $m = 6$, the values of C_1 and C_2 are

$$(4.10)\, C_1 = \begin{pmatrix} 0.9093308 \\ 1.134074 \\ 0.2363245 \\ 0.9902234 \times 10^{-2} \\ -0.1957476 \times 10^{-2} \\ -0.2967196 \times 10^{-3} \\ -0.1815798 \times 10^{-4} \end{pmatrix}, \quad C_2 = \begin{pmatrix} 1.378024 \\ 0.2720071 \\ -0.1402875 \\ -0.03758258 \\ -0.3404238 \times 10^{-2} \\ -0.8353643 \times 10^{-4} \\ 0.8351776 \times 10^{-5} \end{pmatrix}.$$

The computed and exact solutions for the case $m = 6$ are plotted on the same graph in Figure 1.

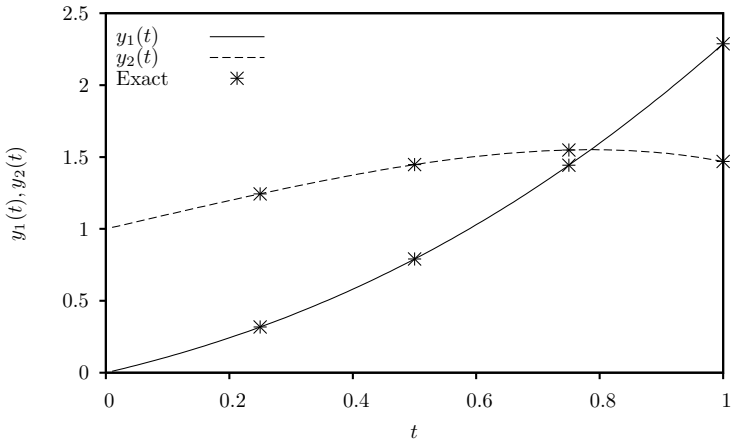

Figure 1: The LOMD solution with the exact one for Example 1 with $m = 6$.

Moreover, the absolute errors between the exact solution and the present ones for $m = 6$ and $m = 8$ are given in Figure 2.

Clearly, the figure shows that increasing m decreases the absolute errors. The estimate error can be given as

$$(4.11) \qquad e_1' - e_1 - e_2 + R_1 = 0,$$

$$(4.12) \qquad e_2' + e_1 - e_2 + R_2 = 0.$$

Subject to the initial conditions $e_1(0) = 0, e_2(0) = 0$.

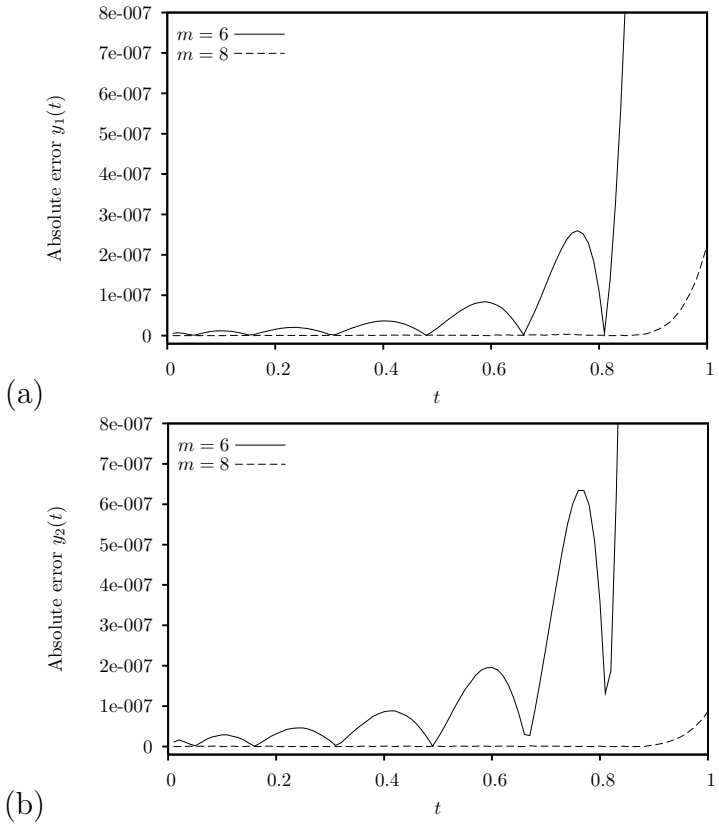

Figure 2: The absolute errors between the LOMD solutions and exact one for Example 1.

4.2. Example 2

Consider the nonlinear stiff system of ODE [2]

$$y_1'(t) = -1002y_1(t) + 1000y_2^2(t), \tag{4.13}$$

$$y_2'(t) = y_1(t) - y_2(t) - y_2^2(t), \tag{4.14}$$

subject to the initial conditions

$$y_1(0) = 1, \quad y_2(0) = 1. \tag{4.15}$$

The exact solution is

$$y_1(t) = e^{-2t}, \quad y_2(t) = e^{-t}.$$

By assuming the solution of the form

$$y_1(t) = c_0\,P_0(t) + c_1\,P_1(t) + \ldots + c_m\,P_m(t) = \mathrm{C}_1^{\mathrm{T}}\phi(t), \tag{4.16}$$

$$y_2(t) = h_0\,P_0(t) + h_1\,P_1(t) + \ldots + h_m\,P_m(t) = \mathrm{C}_2^{\mathrm{T}}\phi(t), \tag{4.17}$$

and applying the initial conditions, we have

$$\begin{aligned} c_0 - c_1 + \ldots + c_m &= 1 \\ h_0 - h_1 + \ldots + c_m &= 0. \end{aligned}$$

The residual errors are

$$\Re_1 = C_1^{\mathrm{T}} D^1\phi(t) + 1002C_1^{\mathrm{T}}\phi(t) - 1000(C_2^{\mathrm{T}}\phi(t))^2, \tag{4.18}$$

$$\Re_2 = C_2^{\mathrm{T}} D^1\phi(t) - C_1^{\mathrm{T}}\phi(t) + C_2^{\mathrm{T}}\phi(t) + (C_2^{\mathrm{T}}\phi(t))^2. \tag{4.19}$$

Using the same manner as in the previous example, we can create $2m+2$ algebraic equation with $2m+2$ constant, which can be easily solved via Newton's iterative method. By taking $m = 8$, the value of $\mathrm{C}_1^{\mathrm{T}}$ and $\mathrm{C}_2^{\mathrm{T}}$ are

$$\mathrm{C}_1 = \begin{pmatrix} 0.4323324 \\ -0.4060059 \\ 0.1316326 \\ -0.02591911 \\ 0.3665828 \times 10^{-2} \\ -0.4044962 \times 10^{-3} \\ 0.3663535 \times 10^{-4} \\ -0.2858631 \times 10^{-5} \\ 0.2868407 \times 10^{-6} \end{pmatrix}, \quad \mathrm{C}_2 = \begin{pmatrix} 0.6321206 \\ -0.3109150 \\ 0.05145307 \\ -0.5125022 \times 10^{-2} \\ 0.3650665 \times 10^{-3} \\ -0.2024552 \times 10^{-4} \\ 0.1016656 \times 10^{-5} \\ -0.3359670 \times 10^{-7} \\ 0.1173426 \times 10^{-8} \end{pmatrix}. \tag{4.20}$$

The given solution with exact one are presented in Figure 3. Moreover, the absolute errors between the exact solution and the present one for $m = 6$ and $m = 8$ are given in Figure 4. The estimate error equations given as

$$e_1' + 1002e_1 - 1000(e_2^2 + 2e_2\bar{y_2}) + R_1 = 0 \tag{4.21}$$

$$e_2' - e_1 + e_2 + e_2^2 + 2e_2\bar{y_2} + R_2 = 0 \tag{4.22}$$

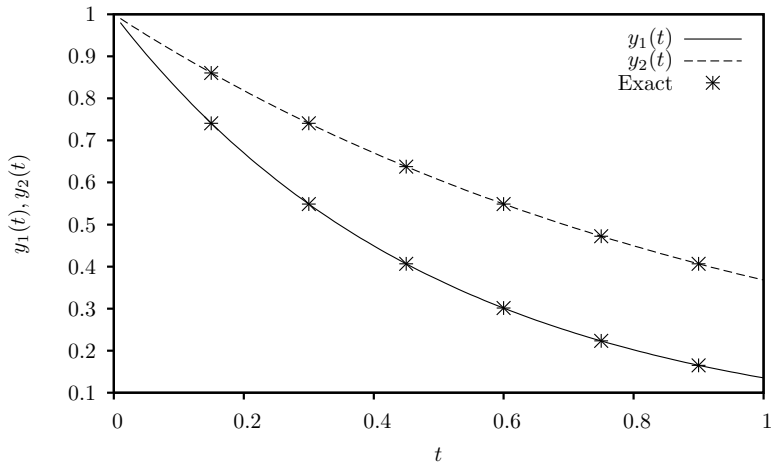

Figure 3: The LOMD solution with exact one for Example 2 with $m = 8$.

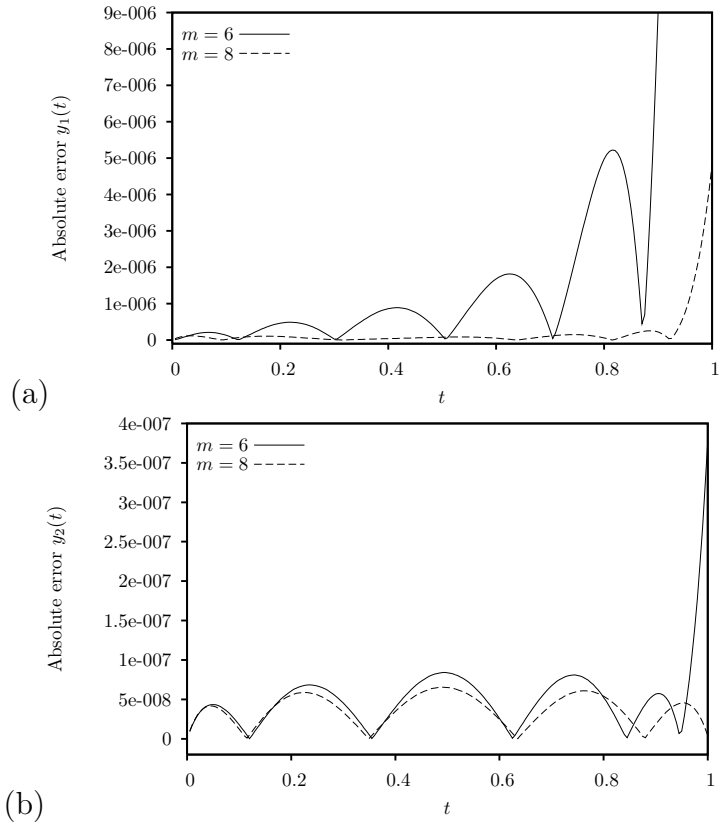

Figure 4: The absolute error between the LOMD solution and exact one for Example 2.

4.3. Example 3

Finally, we consider the nonlinear Genesio system [2]

$$y_1'(t) = y_2(t), \tag{4.23}$$
$$y_2'(t) = y_3(t), \tag{4.24}$$
$$y_3'(t) = -cy_1(t) - by_2(t) - ay_3(t) + y_1(t)^2, \tag{4.25}$$

subject to the initial conditions

$$y_1(0) = 0.2 \qquad y_2(0) = -0.3, \qquad y_3(0) = 0.1, \tag{4.26}$$

where a, b and c are positive constants, satisfying $ab < c$. The Genesio system includes a simple square part and three simple ordinary differential equations that depend on three positive real parameters [2].

According to the framework of the LOMD, the solution can be expanded as series of shifted Legendre polynomials

$$y_1(t) = c_0\,P_0(t) + c_1\,P_1(t) + \ldots + c_m\,P_m(t) = \mathrm{C}_1^{\mathrm{T}}\phi(t), \tag{4.27}$$
$$y_2(t) = h_0\,P_0(t) + h_1\,P_1(t) + \ldots + h_m\,P_m(t) = \mathrm{C}_2^{\mathrm{T}}\phi(t), \tag{4.28}$$
$$y_3(t) = s_0\,P_0(t) + s_1\,P_1(t) + \ldots + s_m\,P_m(t) = \mathrm{C}_3^{\mathrm{T}}\phi(t). \tag{4.29}$$

Substituting (4.26) into (4.27–4.29) we have

$$\begin{aligned} c_0 - c_1 + \ldots + c_m &= 0.2, \\ h_0 - h_1 + \ldots + h_m &= -0.3, \\ s_0 - s_1 + \ldots + s_m &= 0.1. \end{aligned}$$

The residual errors are then given by

$$\Re_1 = \mathrm{C}_1^{\mathrm{T}} D^1\phi(t) - \mathrm{C}_2^{\mathrm{T}}\phi(t), \tag{4.30}$$

$$\Re_2 = \mathrm{C}_2^{\mathrm{T}} D^1\phi(t) - \mathrm{C}_3^{\mathrm{T}}\phi(t), \tag{4.31}$$

$$\Re_3 = \mathrm{C}_3^{\mathrm{T}} D^1\phi(t) + c\mathrm{C}_1^{\mathrm{T}}\phi(t) + b\mathrm{C}_2^{\mathrm{T}}\phi(t) + a\mathrm{C}_3^{\mathrm{T}}\phi(t) - (\mathrm{C}_1^{\mathrm{T}}\phi(t))^2. \tag{4.32}$$

Now, taking the first m roots of $m+1$ shifted Legendre polynomials and substituting in (4.30) and (4.32) which equals to zero, we have $3m+3$ algebraic equations with $3m+3$ constants which can be easily solved by Newton's iterative method.

The values of C_1, C_2 and C_3 with $m = 8$ are

$$\mathrm{C}_1 = \begin{pmatrix} 0.06305404 \\ -0.1292413 \\ 0.9418267 \times 10^{-2} \\ 0.2318345 \times 10^{-2} \\ 0.5280416 \times 10^{-3} \\ -0.7923437 \times 10^{-4} \\ -0.2417998 \times 10^{-5} \\ 0.1845305 \times 10^{-6} \\ 0.2605319 \times 10^{-7} \end{pmatrix}, \quad \mathrm{C}_2 = \begin{pmatrix} -0.2540041 \\ 0.05966350 \\ 0.02239294 \\ 0.7359095 \times 10^{-2} \\ -0.1422908 \times 10^{-2} \\ -0.5263004 \times 10^{-4} \\ 0.4794184 \times 10^{-5} \\ 0.7757259 \times 10^{-6} \\ -0.2767657 \times 10^{-8} \end{pmatrix}, \tag{4.33}$$

$$\mathrm{C}_3 = \begin{pmatrix} 0.1339414 \\ 0.1258488 \\ 0.07307226 \\ -0.01985379 \\ -0.9335802 \times 10^{-3} \\ 0.1052177 \times 10^{-3} \\ 0.2000128 \times 10^{-4} \\ -0.2154617 \times 10^{-6} \\ -0.7247838 \times 10^{-7} \end{pmatrix}. \tag{4.34}$$

Both the LOMD and the numerical solutions obtained by the 4th-order Runge-Kutta (RK4) method for Genesio system are plotted in Figure 5.

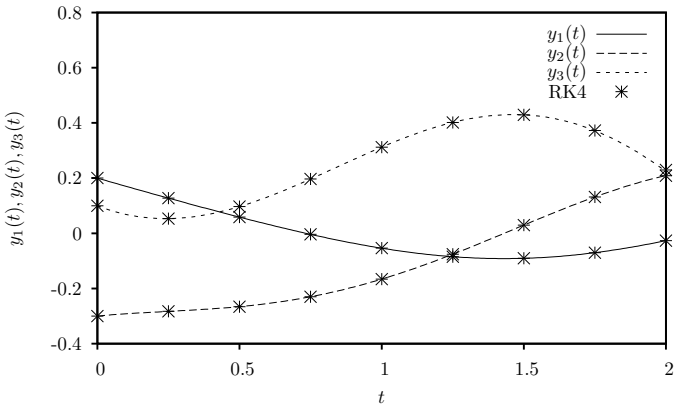

Figure 5: The solution of Genesio system with RK4 solution when $m = 10$.

It is clear that the LOMD solutions agree well with that of the RK4 solutions (see also Figure 6). The estimate error equations given as

$$e_1' - e_1 + R_1 = 0, \tag{4.35}$$

$$e_2' - e_3 + R_2 = 0, \tag{4.36}$$

$$e_3' - ce_1 + be_2 + ae_3 - e_1^2 - 2e_1\bar{y_1} + R3 = 0. \tag{4.37}$$

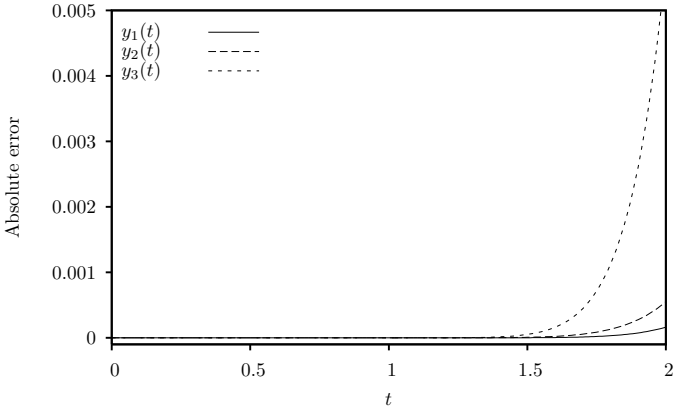

Figure 6: The absolute error between LOMD with $m = 10$ and the RK4 solution.

It is worth mentioning that the estimate error and the absolute error are closed for this example.

5. Conclusions

The Legender polynomial operational matrix of differentiation has been applied to solve linear and nonlinear systems of ordinary differential equations. The advantage of the method over others is that only small size operational matrix is required to provide the solution of high accuracy. The obtained solutions for various examples demonstrate the validity and applicability of the method compared to the other existing methods.

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MORITA EQUIVALENCE FOR SEMIRINGS WITHOUT IDENTITY[1]

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Abstract. In this paper, we study the Morita theory for semirings with slu. We characterize the equivalent functors between the subcategories of right semimodules over semirings with slu. Also, we give an equivalent condition of the Morita context of semirings (not necessarily with slu) and study the corresponding results in semirings with slu settings. Finally, we apply the results to semirings with identity.

Keywords. semiring; semimodule; adjoint pair; equivalent functors; category.
AMS Mathematics Subject Classification (2010): 16Y60.

1. Introduction

Morita equivalence theory gave a characterization of equivalences between two module categories over two rings with 1. The Hom functor and tensor product functor play an important role in studying Morita equivalence theory. The Morita equivalence theory has also been studied in many other algebraic structures, such as rings without 1, semigroups, semirings, etc. In [1], Abrams generalized the Morita equivalence theory to rings without 1. Banaschewski [2] and Knauer [13] got the Morita equivalence theory for monoids. Talwar [20] and Lawson [17] investigated the Morita equivalence theory for semigroups with local units. Laan and Marki [16] continued to study the Morita contexts in semigroup settings. On the other hand, there are very few papers related to these theory for semirings. Recently, Katsov and Nam [12] initially studied these problems for a semiring with 1. In [12], they give the Morita equivalence theory for semirings with 1. Motivated by the paper [1] and [20], it is a natural thing to consider the generalized Morita theorems for semirings without 1.

In this paper, we shall study the Morita theory for semirings without 1. The paper is constructed as follows: In Section 2, we recall some notions on semirings and semimodules; In Section 3, we give the Morita theory for semirings with slu; In Section 4, we study the Morita context for semirings (not necessarily with slu); we apply the results to semirings with 1 in Section 5.

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2. Preliminaries

A semiring is a set R with two binary operations " $+$ " and " $\cdot$ " satisfying the followings:

(1) $(R, +)$ is a commutative semigroup;

(2) $(R, \cdot)$ is a semigroup;

(3) For all $a, b, c \in R$, $a(b + c) = ab + ac, (b + c)a = ba + ca$.

If a semiring R has an element 1 such that $1r = r = r1$, for all $r \in R$, then 1 is called the identity of R.

Let R be a semiring. A commutative semigroup $(M, +)$ is called a right R-semimodule over R if there is a scalar multiplication $M \times R \to M$, denoted by $(m, r) \mapsto mr$, satisfying the followings:

(1) $m(rr^{'}) = (mr)r^{'}$;

(2) $(m + m^{'})r = mr + m^{'}r$;

(3) $m(r + r^{'}) = mr + mr^{'}$;

for all $r, r^{'} \in R$ and $m, m^{'} \in M$.

If $MR = M$, then M is called unital.

Let M and N be two R-semimodules. A map $f : M \to N$ is called R-semimodule homomorphism if f satisfies the followings:

(1) $f(m_1 + m_2) = f(m_1) + f(m_2)$,

(2) $f(mr) = f(m)r$,

for all $m, m_1, m_2 \in M,\ r \in R$.

The set of all R-semimodule homomorphisms from M to N is denoted by $\mathrm{Hom}_R(M, N)$. Let $\mathrm{End}_R(M)$ be the set of all R-semimodule homomorphisms from M to itself.

Similarly, we can define left semimodules. Let Mod-R and R-Mod be the categories of right and left semimodules, respectively.

Denote by Mod-UR (UR-Mod) the subcategory of unital right (left) R-semimodules.

Definition 1. [[11], Definition 3.1] For a right semimodule $M \in$ Mod-R and a left semimodule $N \in R$-Mod, let F be the free semigroup generated by the cartesian product $M \times N$. The tensor product $M \otimes N$ is the factor semigroup $F/\sigma^{\sharp}$, where the congruence $\sigma^{\sharp}$ is generated by the relation σ of the form

$$< (m_1 + m_2, n), (m_1, n) + (m_2, n) >,\ < (m, n_1 + n_2), (m, n_1) + (m, n_2) >,$$

and

$$< (mr, n), (m, rn) >,$$

with $m_1, m_2 \in M$, $n_1, n_2 \in N$ and $r \in R$.

Let R and S be two semirings. A commutative semigroup M is called an R-S-bisemimodule if it is both a left R-semimodule and a right S-semimodule and satisfies $(rm)s = r(ms)$, for all $r \in R, m \in M, s \in S$, denoted by ${}_RM_S$. For ${}_RM_S$ and ${}_RN_S$, if a semigroup homomorphism $f : M \to N$ satisfies $rf(m)s = f(rms)$, then f is called bisemimodule homomorphism. Let ${}_R\mathrm{Hom}_S(M, N)$ be the set of all

bisemimodule homomorphisms from M to N. Denote by R-Mod-S the category of all R-S-bisemimodules together with bisemimodule homomorphisms.

Let M be an S-R-bisemimodule and let N be a left R-semimodule. The tensor product $M \otimes_R N$ is a left S-semimodule with action

$$s \cdot \left(\sum_{i=1}^{n} m_i \otimes n_i \right) = \sum_{i=1}^{n} s \cdot m_i \otimes n_i;$$

Similarly, let M be a right R-semimodule and N be an R-S-bisemimodule, then $M \otimes_R N$ is a right S-semimodule with action

$$\left(\sum_{i=1}^{n} m_i \otimes n_i \right) \cdot s = \sum_{i=1}^{n} m_i \otimes n_i \cdot s,$$

where $s \in S, m_i \in M, n_i \in N$.

For an R-S-bisemimodule M and a left R-semimodule$N$, we have that $\mathrm{Hom}_R(M, N)$ is a left S-semimodule with action $(s \cdot f)(m) = f(m \cdot s)$; For an S-R-bisemimodule M and a right R-semimodule N, we have that $\mathrm{Hom}_R(M, N)$ is a right S-semimodule with action $(f \cdot s)(m) = f(s \cdot m)$; For a right R-semimodule and an S-R-bisemimodule N, we have that $\mathrm{Hom}_R(M, N)$ is a left S-semimodule with action $(s \cdot f)(m) = sf(m)$; For a left R-semimodule M and a R-S-bisemimodule N, we have that $\mathrm{Hom}_R(M, N)$ is a right S-semimodule with action $(f \cdot s)(m) = f(m)s$, for all $s \in S, f \in \mathrm{Hom}_R(M, N), m \in M$.

3. Morita equivalence for semirings with slu

In paper [10], Katsov proved that the tensor product functor $- \otimes_R B$ and Hom functor $\mathrm{Hom}_S(B, -)$ are adjoint pair, where R and S are semirings with identity and $B \in R$-Mod-S. In the following, we will prove that the statement also holds when R and S are semirings without identity.

Let $\mathcal{R}^\sharp$ be the smallest congruence on a semigroup S containing the relation $\mathcal{R}$. For $a, b \in S$, if there exist $u, v \in S \bigcup \{1\}$ such that

$$a = ucv, \ b = udv,$$

where $(c, d) \in \mathcal{R}$ or $(d, c) \in \mathcal{R}$, we say that a is connected to b by an elementary $\mathcal{R}$-transition [8].

Proposition 1. [8] *Let S be a semigroup and let $\mathcal{R}$ be a relation on S. For $a, b \in S$, we have $(a, b) \in \mathcal{R}^\sharp \Leftrightarrow$ one of the two conditions holds:*

(1) $a = b$.

(2) *there exists a sequence*

$$a = c_1 \to c_2 \to \cdots \to c_n = b$$

of elementary $\mathcal{R}$-transitions connecting a to b, where n is a positive integer.

Lemma 1. *Let R and S be two semirings. Assume $g : A \to Hom_S(B, C)$ is an R-semimodule homomorphism, where $A \in R\text{-}Mod, B \in S\text{-}Mod\text{-}R$ and $C \in S\text{-}Mod$. For all $a \in A$, we write $g(a) = g_a$. Then*

(1) *If (b, a) is connected to (b', a') by an elementary σ-transition, where σ is the relation defined in Definition 1, we have $g_a(b) = g_{a'}(b')$.*

(2) *We can define an S-semimodule homomorphism $g' : B \otimes_R A \to C$ by $g'\left(\sum_{i=1}^{n} b_i \otimes a_i\right) = \sum_{i=1}^{n} g_{a_i}(b_i)$.*

Proof. (1) Suppose (b, a) and (b', a') is connected by by an elementary σ-transition. Let $\mathcal{S}(B \times A)$ be the commutative free semigroup generated by $B \times A$. There exists (x, y) belongs to $\mathcal{S}(B \times A) \bigcup \{(0, 0)\}$ such that

$$\begin{aligned} (b, a) &= (x, y) + (c, d), \\ (b', a') &= (x, y) + (c', d'), \end{aligned}$$

where $((c, d), (c', d')) \in \sigma$ or $((c', d'), (c, d)) \in \sigma$.

Since $B \times A$ is the generating set and the elements in $B \times A$ are independent, we have $(x, y) = (0, 0)$. This implies that $((b, a), (b', a')) \in \sigma$ or $((b', a'), (b, a)) \in \sigma$. Without loss of generality, we assume $((b, a), (b', a')) \in \sigma$. Note that (b, a) and (b', a') are independent, there exists $r \in R$ such that $b = b'r, ra = a'$. Since g is an R-semimodule homomorphism, we have $r \cdot g(a) = g(r \cdot a)$, for all $a \in A$. Then, we get $r \cdot g_a = g_{ra}$. Hence, we have

$$g_a(b) = g_a(b'r) = (rg_a)(b') = g_{ra}(b') = g_{a'}(b').$$

(2) If $b_1 \otimes a_1 = b_2 \otimes a_2$, then $(b_1, a_1) = (b_2, a_2)$ or (b_1, a_1) is connected to (b_2, a_2) by an elementary σ-transition by Proposition 1. By part (1), we have $g_{a_1}(b_1) = g_{a_2}(b_2)$. On the other hand, if $b \otimes a = \sum_{i=1}^{n} b_i \otimes a$, then we have $\left((b, a), \left(\sum_{i=1}^{n} b_i, a\right)\right) \in \sigma$ and $b = \sum_{i=1}^{n} b_i$. This concludes that $g_a(b) = \sum_{i=1}^{n} g_a(b_i)$. Similarly, if $b \otimes a = \sum_{i=1}^{n} b \otimes a_i$, we get $g_a(b) = \sum_{i=1}^{n} g_{a_i}(b)$. Then we prove that g' is well-defined.

For all $s \in S$, we have

$$g'(s\sum_{i=1}^{n} b_i \otimes a_i) = g'\left(\sum_{i=1}^{n} sb_i \otimes a_i\right) = \sum_{i=1}^{n} g_{a_i}(sb_i) = s\sum_{i=1}^{n} g_{a_i}(b_i) = sg'\left(\sum_{i=1}^{n} b_i \otimes a_i\right).$$

Therefore, g' is an S-semimodule homomorphism. ∎

Theorem 1. *Let R and S be two semirings. For $A \in R\text{-}Mod, B \in S\text{-}Mod\text{-}R$ and $C \in S\text{-}Mod$, there is a semigroup isomorphism*

$$\Delta : Hom_S(B \otimes_R A, C) \cong Hom_R(A, Hom_S(B, C)).$$

Similarly, for $A \in$ Mod-$R, B \in R$-Mod-S and $C \in$ Mod-S, there is a semigroup isomorphism

$$\Omega : Hom_S(A \otimes_R B, C) \cong Hom_R(A, Hom_S(B, C)).$$

Proof. We only prove the first isomorphism. If $f \in \mathrm{Hom}_S(B \otimes_R A, C)$, for all $a \in A$, we can define a map $f_a : B \to C$ by $f_a(b) = f(b \otimes a)$. Then we can check that the map $\bar{f} : A \to \mathrm{Hom}_S(B, C)$ by putting $\bar{f}(a) = f_a$ is an S-semimodule homomorphism. It is a routine procedure to check that $\Delta : f \mapsto \bar{f}$ is a homomorphism.

Now, we exhibit the inverse of Δ. Define $\Delta^{-1} : \mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C)) \to \mathrm{Hom}_S(B \otimes_R A, C)$ by putting $\Delta(g) = g'$, where g' is defined as in Lemma 1. Hence, Δ is a semigroup isomorphism. ∎

By Theorem 1, we have that $(- \otimes_R B, \mathrm{Hom}_S(B, -))$ is an adjoint pair. We write $\mathcal{L} = - \otimes_R B$ and $\mathcal{K} = \mathrm{Hom}_S(B, -)$. Hence, we have the following canonical transformations

$$\upsilon : \mathcal{LK} \to Id_{\text{Mod-}S} \quad \text{and} \quad \eta : Id_{\text{Mod-}R} \to \mathcal{KL}$$

such that

$$\upsilon_C(\varphi \otimes b) = \varphi(b) \quad \text{and} \quad \eta_A(a)(b) = a \otimes b,$$

where $\varphi \in HB, b \in B$ and $a \in A$.

In particular, we take $A = B = S = R$, then we have $\upsilon_C : \mathrm{Hom}_R(R, C) \otimes_R R \to R$ by putting $\upsilon_C \left(\sum_{i=1}^{n} \varphi_i \otimes r_i \right) = \sum_{i=1}^{n} \varphi_i(r_i)$, where $C \in$ Mod-R.

Let R be a semiring and $C \in$ Mod-R. For $c \in C$, we define $\lambda_c : R \to C$ by putting $\lambda_c(r) = cr$. Then $\lambda_c \in \mathrm{Hom}_R(R, C)$. For all $\varphi \in \mathrm{Hom}_R(R, C)$, $r \in R$, since

$$(\varphi \cdot r)(a) = \varphi(ra) = \varphi(r)a = \lambda_{\varphi(r)}(a),$$

where $a \in R$, we have $\varphi \cdot r = \lambda_{\varphi(r)}$.

Definition 2. Let R be a semiring and E be an idempotents set of R. If for any finite number of elements $r_1, ..., r_n \in R$, there exists $e \in E$ such that

$$er_1 = r_1 = r_1e, ..., er_n = r_n = r_ne,$$

then E is called a set of local units of R (abbreviated slu). In this case, we say R is a semiring with slu.

Obviously, if R is a semiring with 1, then R is a semiring with slu. If R is an idempotent commutative (multiplication) semiring, by induction, we can prove that R is a semiring with slu.

Theorem 2. *Let R be a semiring with local units. Then we have $End_R(R) \otimes_R R \cong R$ as R-semimodules.*

Proof. For all $r \in R$, there exists $e \in E$ such that $er = re = r$. Define $\mu_R : R \to \mathrm{End}_R(R) \otimes_R R$ by putting $\mu_R(r) = \lambda_r \otimes e$. Suppose $f \in E$ with $rf = fr = r$. Since $\lambda_r = \lambda_e \cdot r$, we have

$$\lambda_r \otimes e = \lambda_e \cdot r \otimes e = \lambda_e \otimes re = \lambda_e \otimes r = \lambda_e \otimes rf = \lambda_e \cdot r \otimes f = \lambda_r \otimes f.$$

This proves that μ_R is independent of the choice of the idempotent. Suppose $r_1, ..., r_n \in R$. There exists $e \in E$ such that

$$er_1 = r_1, ..., er_n = r_n.$$

Then

$$\mu_R\left(\sum_{i=1}^{n} r_i\right) = \lambda_{\sum\limits_{i=1}^{n} r_i} \otimes e = \sum_{i=1}^{n} \lambda_{r_i} \otimes e = \sum_{i=1}^{n} \mu_R(r_i).$$

For all $r \in R, a \in R$, there exists $e \in E$ such that $re = er = r,\ ea = ae = a$. Then

$$\mu_R(ar) = \lambda_{ar} \otimes e = \lambda_a \cdot r \otimes e = \lambda_a \otimes re = \lambda_a \otimes er = (\lambda_a \otimes e)r = \mu_R(a)r.$$

This proves that μ_R is an R-semimodule homomorphism. Also, we can easily check that μ_R is the inverse of the map υ_R. So we have that υ_R is an isomorphism. That is, $\mathrm{End}_R(R) \otimes_R R \cong R$ as R-semimodule. ∎

Let R be a semiring with slu and $M \in$ Mod-UR. We denote by

$$\text{Mod-}FR = \{M \in \text{Mod-}UR | \upsilon_M \text{ is an isomorphism}\}.$$

Definition 3. Let R and S be two semirings with slu. We say that R and S are Morita equivalent, if the two subcategories Mod-FR and Mod-FS are equivalent.

Analogous to Theorem 6.1 in [20], we have the following.

Theorem 3. *Let R and S be two semirings with slu and G : Mod-$FR \rightleftharpoons$ Mod-FS : H are equivalent functors. Set $U = H(S)$ and $V = G(R)$. Then the following conditions hold:*

(1) $G \cong Hom_R(U, -) \otimes_S S$, $H \cong Hom_S(V, -) \otimes_R R$;

(2) $U \cong Hom_S(V, S) \otimes_R R, V \cong Hom_R(U, R) \otimes_S S$ *as bisemimodules.*

Proof. For $r \in R$, we have $\lambda_r \in \mathrm{End}_R(R)$. Then $G(\lambda_r) \in \mathrm{End}_S(G(R)) = \mathrm{End}_S(V)$. For $r \in R, v \in V$, defining $r \cdot v = G(\lambda_r)(v)$. It is easy to check that V is an R-S-bisemimodule. Similarly, we have that U is an S-R-bisemimodule.

(1) Since G and H are equivalent functors, we have

$$\mathrm{Hom}_S(S, G(M)) \cong \mathrm{Hom}_R(H(S), M) = \mathrm{Hom}_R(U, M),$$

For all $M \in$ Mod-FR. We obviously have $G(M) \in$ Mod-FS. By the definition of Mod-FS, we have

$$G(M) \cong \mathrm{Hom}_S(S, G(M)) \otimes_S S.$$

Then $G(M) \cong \mathrm{Hom}_R(U, M) \otimes_S S$. This proves that $G \cong \mathrm{Hom}_R(U, -) \otimes_S S$. Similarly, we have $H \cong \mathrm{Hom}_S(V, -) \otimes_R R$.

(2) Let $\Gamma : G \longrightarrow \mathrm{Hom}_R(U, -) \otimes_S S$ be the natural isomorphism. We know that Γ_R is a right S-isomorphism. There exists a commutative diagram:

$$\begin{array}{ccc} G(R)(=V) & \xrightarrow{\quad G(\lambda_r) \quad} & G(R)(=V) \\ {\scriptstyle \Gamma_R}\downarrow & & \downarrow{\scriptstyle \Gamma_R} \\ \mathrm{Hom}_R(U,R) \otimes_S S & \xrightarrow[\mathrm{Hom}_{R(U,\lambda_r)\otimes_S Id_S}]{} & \mathrm{Hom}_R(U,R) \otimes_S S, \end{array}$$

where

$$(\mathrm{Hom}_R(U, \lambda_r) \otimes_S Id_S)(\varphi \otimes t) = \lambda_r \varphi \otimes t.$$

Since $\lambda_r \varphi(u) = r\varphi(u) = (r \cdot \varphi)(u)$, for all $u \in U$, we have $\lambda_r \varphi = r \cdot \varphi$. Hence, we have

$$(\mathrm{Hom}_R(U, \lambda_r) \otimes_S Id_S)(\varphi \otimes t) = \lambda_r \varphi \otimes t = r \cdot \varphi \otimes t = r \cdot (\varphi \otimes t).$$

This concludes that $(\mathrm{Hom}_R(U, \lambda_r) \otimes_S Id_S)\Gamma_R(v) = r \cdot \Gamma_R(v)$, for all $v \in V$. Using the commutative diagram, we have

$$\Gamma_R(rv) = \Gamma_R(G(\lambda_r)(v)) = (\mathrm{Hom}_R(U, \lambda_r) \otimes_S Id_S)\Gamma_R(v) = r(\Gamma_R(v)).$$

It follows that Γ_R is a left R-semimodule homomorphism. This proves that $V \cong \mathrm{Hom}_R(U, R) \otimes_S S$ as S-R-bisemimodules. Similarly, we have $U \cong \mathrm{Hom}_S(V, S) \otimes_R R$ as R-S-bisemimodules. ∎

4. Morita context for semirings

In the following, we study the Morita context in semiring settings.

Definition 4. Let R and S be two semirings (not necessarily with slu). If there exist commutative semigroups U and V, such that

(1) U is an R-S-bisemimodule, V is an S-R-bisemimodule;

(2) there are bisemimodule homomorphisms $\tau : U \otimes_S V \to R$ and $\mu : V \otimes_R U \to S$ written corresponding as

$$\tau(u \otimes v) = \langle u, v \rangle, \quad \mu(v \otimes u) = [v, u]$$

such that

$$\langle u_1, v \rangle \cdot u_2 = u_1 \cdot [v, u_2], \quad [v_1, u] \cdot v_2 = v_1 \cdot \langle u, v_2 \rangle$$

for each $u, u_1, u_2 \in U,\ v, v_1, v_2 \in V$.

Then (R, S, U, V, τ, μ) is called a Morita context.

Analogous to Theorem 1 in [18], we get the following theorem.

Theorem 4. *Let U and V be two commutative semigroups. Then the following two conditions are equivalent:*

(1) *There exist two semirings R and S such that (R, S, U, V, τ, μ) is a Morita context.*

(2) *There exist semigroup homomorphisms $\Phi : U \otimes_Z V \otimes_Z U \to U$ and $\Psi : V \otimes_Z U \otimes_Z V \to V$ such that*

(I) $\Phi(\Phi(u_1 \otimes v_1 \otimes u_2) \otimes v_2 \otimes u_3) = \Phi(u_1 \otimes \Psi(v_1 \otimes u_2 \otimes v_2) \otimes u_3)$
$= \Phi(u_1 \otimes v_1 \otimes \Phi(u_2 \otimes v_2 \otimes u_3))$;

(II) $\Psi(\Psi(v_1 \otimes u_1 \otimes v_2) \otimes u_2 \otimes v_3) = \Psi(v_1 \otimes \Phi(u_1 \otimes v_2 \otimes u_2) \otimes v_3)$
$= \Psi(v_1 \otimes u_1 \otimes \Psi(v_2 \otimes u_2 \otimes v_3))$.

Proof. (1) $\Rightarrow$ (2) : Suppose that (R, S, U, V, τ, μ) is a Morita context. Define $\Phi : U \otimes_Z V \otimes_Z U \to U$ and $\Psi : V \otimes_Z U \otimes_Z V \to V$ by putting $\Phi(u_1 \otimes v_1 \otimes u_2) = \tau(u_1 \otimes v_1) \cdot u_2$ and $\Psi(v_1 \otimes u_1 \otimes v_2) = \mu(v_1 \otimes u_1) \cdot v_2$. We can easily check that Φ and Ψ satisfy the two conditions in 2).

(2) $\Rightarrow$ (1) : Define $F_a : U \to U$ by putting $F_a(u) = \Phi(a \otimes u)$ and define $G_b : V \to V$ by putting $G_b(v) = \Psi(b \otimes v)$, where $a \in U \otimes_Z V$ and $b \in V \otimes_Z U$. Then F_a and G_b are semigroup homomorphisms.

We write $\mathcal{F} = \{F_a | a \in U \otimes_Z V\}$ and $\mathcal{G} = \{R_b | b \in V \otimes_Z U\}$. For all $F_{u_1 \otimes v_1}, F_{u_2 \otimes v_2} \in \mathcal{F}$, for all $w \in U$, we have

$$\begin{aligned} F_{u_1 \otimes v_1} F_{u_2 \otimes v_2}(w) &= \Phi(u_1 \otimes v_1 \otimes \Phi(u_2 \otimes v_2 \otimes w)) \\ &= \Phi(\Phi(u_1 \otimes v_1 \otimes u_2) \otimes v_2 \otimes w) \\ &= F_{\Phi(u_1 \otimes v_1 \otimes u_2) \otimes v_2}(w). \end{aligned}$$

That is, $F_{u_1 \otimes v_1} F_{u_2 \otimes v_2} = F_{\Phi(u_1 \otimes v_1 \otimes u_2) \otimes v_2} \in \mathcal{F}$. It is easy to check that $F_a + F_b = F_{a+b} \in \mathcal{F}$ and multiplication distributes over addition from either side. Hence, $\mathcal{F}$ is a subsemiring of ${}_Z\mathrm{End}_Z(U)$. Similarly, we have that $\mathcal{G}$ is a a subsemiring of ${}_Z\mathrm{End}_Z(V)$.

For all $u \in U$, $F_a \in \mathcal{F}$, $G_b \in \mathcal{G}$, define $F_a \cdot u = \Phi(a \otimes u)$ and $u \cdot G_b = \Phi(u \otimes b)$. Then we can check that U is a $\mathcal{F}$-$\mathcal{G}$-bisemimodule. Similarly, for all $v \in V$, we can define $G_b \cdot v = \Psi(b \otimes v)$ and $v \cdot F_a = \Psi(v \otimes a)$. Then V is a $\mathcal{G}$-$\mathcal{F}$-bisemimodule.

Now, we define $\alpha : U \otimes_{\mathcal{G}} V \to \mathcal{F}$ and $\beta : V \otimes_{\mathcal{F}} U \to \mathcal{G}$ by putting $\alpha(u \otimes v) = F_{u \otimes v}$ and $\beta(v \otimes u) = G_{v \otimes u}$, where $u \in U$ and $v \in V$. It is easy to check that α and β are both bisemimodule homomorphisms. Then

$$\alpha(u_1 \otimes v) \cdot u_2 = F_{u_1 \otimes v} \cdot u_2 = \Phi(u_1 \otimes v \otimes u_2) = u_1 \cdot G_{v \otimes u_2} = u_1 \beta(v \otimes u_2).$$

Similarly, we have

$$\beta(v_1 \otimes u) v_2 = v_1 \alpha(u \otimes v_2).$$

Then $(\mathcal{F}, \mathcal{G}, U, V, \alpha, \beta)$ is a Morita context. ∎

Definition 5. Let R and S be two semirings with slu. A Morita context (R, S, U, V, τ, μ) is called unital, if U is a unital R-S-bisemimodule and V is a unital S-R-bisemimodule.

Lemma 2. *Let (R, S, U, V, τ, μ) be a unital Morita context. For any $u_1, ..., u_n \in U$ and $v_1, ..., v_m \in V$, then*

(1) *there exists idempotent $e \in R$ such that $eu_i = u_i$ and $v_j e = v_j$, for all $i = 1, \cdots, n; j = 1, ..., m$.*

(2) *there exists idempotent $f \in S$ such that $u_i f = u_i$ and $ev_j = v_j$, for all $i = 1, ..., n; j = 1, ..., m$.*

Proof. (1) Since U and V are unital, for every i and j, there exist $a_i \in U$, $b_j \in V$, $x_i \in R, y_j \in S$ such that

$$u_i = x_i a_i, \quad v_j = b_j y_j.$$

Since R is a semiring with slu, there exists an idempotent $e \in R$ such that $ex_i = x_i$ and $y_j e = y_j$. Then we can prove the result.

Similarly, we can prove (2). ∎

We extend Lemma 3.15 in [17] to semiring settings.

Lemma 3. *Let (R, S, U, V, τ, μ) be a unital Morita context. If τ and μ are surjective, then τ and μ are isomorphisms.*

Proof. Firstly, we show that τ is injective. Suppose

$$\tau\Big(\sum_i u_i \otimes v_i\Big) = \tau\Big(\sum_j u'_j \otimes v'_j\Big), \quad \text{i.e.,} \quad \sum_i \langle u_i\, v_i\rangle = \sum_j \langle u'_j, v'_j\rangle .$$

There exist idempotents $e, f \in R$ such that $eu_i = u_i$ and $v'_j f = v'_j$ by Lemma 2. Since τ is surjective, we can assume that

$$\tau\Big(\sum_k a_k \otimes b_k\Big) = e \quad \text{and} \quad \tau\Big(\sum_l c_l \otimes d_l\Big) = f.$$

Then we have

$$\begin{aligned}
\sum_i u_i \otimes v_i &= \tau\big(\sum_k a_k \otimes b_k\big)\sum_i u_i \otimes v_i = \sum_k \sum_i \langle a_k, b_k\rangle\, u_i \otimes v_i \\
&= \sum_k \sum_i a_k [b_k, u_i] \otimes v_i = \sum_k \sum_i a_k \otimes [b_k, u_i] v_i \\
&= \sum_k \sum_i a_k \otimes b_k \langle u_i, v_i\rangle = \sum_k a_k \otimes b_k \sum_i \langle u_i, v_i\rangle \\
&= \sum_k a_k \otimes b_k \sum_j \langle u'_j, v'_j\rangle = \sum_k \langle a_k, b_k\rangle \sum_j u'_j \otimes v'_j \\
&= e \sum_j u'_j \otimes v'_j.
\end{aligned}$$

Similarly, we have $\sum_j u'_j \otimes v'_j = \big(\sum_i u_i \otimes v_i\big)f$. So we have

$$\sum_i u_i \otimes v_i = e\sum_j u'_j \otimes v'_j = e\Big(\sum_i u_i \otimes v_i\Big)f = \Big(\sum_i u_i \otimes v_i\Big)f = \sum_j u'_j \otimes v'_j.$$

This proves that τ is injective. Similarly, we can prove that μ is injective. ∎

Let R be a semiring with slu. Define $\tau : R \otimes_R R \to R$ by $\tau(r_1 \otimes r_2) = r_1 r_2$, then (R, R, R, R, τ, τ) is a unital Morita context and τ is surjective. Hence, we have $R \otimes_R R \cong R$ by Lemma 3.

Theorem 5. *Let U and V be two commutative semigroups. Then the following two conditions are equivalent:*

(1) *There exist two semirings R and S with slu such that (R, S, U, V, τ, μ) is a unital Morita context and τ and μ are surjective.*

In this case, $-\otimes_R V : Mod\text{-}FR \rightleftharpoons Mod\text{-}FS : -\otimes_S U$ are equivalent functors.

(2) *There exist surjective semigroup homomorphisms $\Phi : U \otimes V \otimes U \to U$ and $\Psi : V \otimes U \otimes V \to V$ satisfy the two conditions in part (2) of Theorem 4 and*

(III) *For any finite elements $x_k \in U$, $y_l \in V$, $k = 1, 2, ..., n$, $l = 1, 2, ..., m$, there exist finite elements $u_i \in U, v_i \in V$ such that*

(i) $\Phi(\sum_i (u_i \otimes v_i) \otimes x_k) = x_k$;

(ii) $\Psi(y_l \otimes \sum_i (v_i \otimes u_i)) = y_l$;

(iii) $\sum_i \Phi(\sum_i (u_i \otimes v_i) \otimes u_i) \otimes v_i = \sum_i (u_i \otimes v_i)$.

IV) *For any finite elements $x_k \in U$, $y_l \in V, k = 1, 2, ..., n$, $l = 1, 2, ..., m$, there exist finite elements $u'_i \in U, v'_i \in V$ such that*

(i) $\Psi(\sum_i (v'_i \otimes u'_i) \otimes y) = y$;

(ii) $\Phi(x \otimes \sum_i (u'_i \otimes v'_i)) = y$;

(iii) $\sum_i \Psi(\sum_i (v'_i \otimes u'_i) \otimes v'_i) \otimes u'_i = \sum_i (v'_i \otimes u'_i)$.

Proof. (1) ⇒ (2) : For any finite elements $x_k \in U$, $y_l \in V$, $k = 1, 2, ..., n$, $l = 1, 2, ..., m$, by Lemma 2, there exists an idempotent $e \in R$ with $ex_k = x_k = x_k e$ and $ey_l = y_l = y_l e$. Since τ is surjective, there exist finite elements $u_i \in U, v_i \in V$ such that $\tau(\sum_i (u_i \otimes v_i)) = e$. Then we have

$$\Phi\Big(\sum_i (u_i \otimes v_i) \otimes x_k\Big) = \tau\Big(\sum_i u_i \otimes v_i\Big) \cdot x_k = ex_k = x_k$$

and

$$\Psi\Big(y_l \otimes \sum_i (v_i \otimes u_i)\Big) = \sum_i \mu(y_l \otimes v_i) \otimes u_i = \sum_i y_l \tau(v_i \otimes u_i) = y_l e = y_l.$$

Since $e = \tau(\sum_i u_i \otimes v_i)$ is an idempotent, we have

$$\tau\Big(e\Big(\sum_i u_i \otimes v_i\Big)\Big) = e\tau\Big(\sum_i u_i \otimes v_i\Big) = \tau\Big(\sum_i u_i \otimes v_i\Big).$$

As τ is an isomorphism, we get that $e(\sum_i u_i \otimes v_i) = \sum_i u_i \otimes v_i$. Then

$$\begin{aligned}\sum_i \Phi\Big(\sum_i (u_i \otimes v_i) \otimes u_i\Big) \otimes v_i &= \sum_i \Big(\tau\Big(\sum_i u_i \otimes v_i\Big)u_i\Big) \otimes v_i \\ &= \tau\Big(\sum_i u_i \otimes v_i\Big)\sum_i u_i \otimes v_i \\ &= e\sum_i u_i \otimes v_i = \sum_i u_i \otimes v_i.\end{aligned}$$

This proves that (III) holds. Similarly, we can prove that (IV) holds.

Suppose that (R, S, U, V, τ, μ) is a unital Morita context and τ and μ are surjective. For all $X \in$ Mod-FS, we have $\mathrm{Hom}_S(S, X) \otimes_S S \cong X$. Since S is a semirng with slu, we have $S \otimes_S S \cong S$. So we get $X \otimes_S S \cong X$. Then

$$(X \otimes_S V) \otimes_R U \cong X \otimes_S (V \otimes_R U) \cong X \otimes_S S \cong X.$$

Similarly, for all $Y \in$ Mod-FR, we have $(Y \otimes_R U) \otimes_R V \cong Y$.

Then $- \otimes_R V : \text{Mod-}FR \rightleftharpoons \text{Mod-}FS : - \otimes_S U$ are equivalent functors.

(2) $\Rightarrow$ (1) : For all $F_{a_1 \otimes b_1}, \cdots, F_{a_n \otimes b_n} \in \mathcal{F}$, by condition (III), there exist finite elements $u_i \in U, v_i \in V$ such that $\Phi(\sum_i (u_i \otimes v_i) \otimes a_k) = a_k$, $\Psi(b_l \otimes \sum_i (v_i \otimes u_i)) = b_l$ and $\sum_i \Phi(\sum_i (u_i \otimes v_i) \otimes u_i) \otimes v_i = \sum_i (u_i \otimes v_i)$. Hence,

$$\begin{aligned}F_{\sum_i (u_i \otimes v_i)} F_{a_k \otimes b_l} &= F_{\Phi(\sum_i (u_i \otimes v_i) \otimes a_k) \otimes b_l} = F_{a_k \otimes b_l}; \\ F_{a_k \otimes b_l} F_{\sum_i (u_i \otimes v_i)} &= F_{\sum_i \Phi(a_k \otimes b_l \otimes u_i) \otimes v_i} = F_{a_k \otimes \Psi(b_l \otimes \sum_i (u_i \otimes v_i))} = F_{a_k \otimes b_l}; \\ F_{\sum_i (u_i \otimes v_i)} F_{\sum_i (u_i \otimes v_i)} &= F_{\sum_i \Phi(\sum_i (u_i \otimes v_i) \otimes u_i) \otimes v_i} = F_{\sum_i (u_i \otimes v_i)}.\end{aligned}$$

This shows that $\mathcal{F}$ is a semiring with slu. Similarly, we have that $\mathcal{G}$ is a semiring with slu.

Since Φ and Ψ are surjective, we obviously have that U and V are unital as bisemimodules and α and β are surjective. Hence, $(\mathcal{F}, \mathcal{G}, U, V, \alpha, \beta)$ is a unital Morita context. ∎

5. Applications to semirings with identity

Let R be a semiring with 1. For all $M \in$ Mod-R, we have $\mathrm{Hom}_R(R, M) \otimes_R R \cong \mathrm{Hom}_R(R, M) \cong M$. Hence, Mod-$FR$ = Mod-R. By Theorem 3, we have the following statement.

Theorem 6. *Let R and S be two semirings with identity and $G : Mod\text{-}R \rightleftharpoons Mod\text{-}S : H$ are equivalent functors. Let $U = H(S)$ and $V = G(R)$. Then we have the following:*

(1) $G \cong Hom_R(U, -)$, $H \cong Hom_S(V, -)$;

(2) $U \cong Hom_S(V, S), V \cong Hom_R(U, R)$ *as bisemimodules.*

Proof. (1) Since R is a semiring with identity and U is a unital semimodule, we have $\mathrm{Hom}_R(U, N)$ is also a unital semimodule, for all $N \in$ Mod-S. This implies that

$$G \cong \mathrm{Hom}_R(U, -)\otimes_S S \cong \mathrm{Hom}_R(U, -).$$

Similarly, we have $H \cong \mathrm{Hom}_S(V, -)$.

(2) is obvious. ∎

Using Theorem 5, we can get the following theorem.

Theorem 7. *Let U and V be two commutative semigroups. Then the following two conditions are equivalent:*

(1) *There exist two semirings with identity R and S such that (R, S, U, V, τ, μ) be a unital Morita context and τ and μ are surjective. In this case, $- \otimes_R V : Mod\text{-}R \rightleftharpoons Mod\text{-}S : - \otimes_S U$ are equivalent functors.*

(2) *There exist surjective semigroup morphisms $\Phi : U \otimes_Z V \otimes_Z U \to U$ and $\Psi : V \otimes_Z U \otimes_Z V \to V$ satisfy the two conditions in part (2) of Theorem 4 and*

(III)′ *There exist finite elements $u_i \in U, v_i \in V$ such that*
$\Phi(\sum_i (u_i \otimes v_i) \otimes x) = x$ *and* $\Psi(y \otimes \sum_i (v_i \otimes u_i)) = y$, *for all* $x \in U, y \in V$.

(IV)′ *There exist finite elements $u_i' \in U, v_i' \in V$ such that*
$\Psi(\sum_i (v_i' \otimes u_i') \otimes y) = y$ *and* $\Phi(x \otimes \sum_i (u_i' \otimes v_i')) = x$, *for all* $x \in U, y \in V$.

Proof. (1) ⇒ (2) : Suppose $1_R = \tau(\sum_i u_i \otimes v_i)$. Analogous to the proof of Theorem 5, we can get that condition (III)′ is valid. Suppose $1_S = \tau(\sum_i v_i' \otimes u_i')$. We can prove that condition (IV)′ is valid.

(2) ⇒ (1) : For all $F_{u\otimes v} \in \mathcal{F}$, we have

$$F_{\sum_i (u_i\otimes v_i)} F_{u\otimes v} = F_{u\otimes v} F_{\sum_i (u_i\otimes v_i)} = F_{u\otimes v}.$$

This proves that $F_{\sum_i (u_i\otimes v_i)}$ is the identity of $\mathcal{F}$. Similarly, we can prove that $\mathcal{G}$ has identity. Using Theorem 5, we can prove the statement. ∎

The following theorem generalizes the corresponding result in semigroup theory to semiring theory.

Theorem 8. *Let R and S be two semirings with identity and the two semirings are equivalent as Theorem 6. We identify V with $U^* = Hom_R(U, R)$ and S with $End_R(U)$. Define $\tau : U^* \otimes_R U \to R$ given by $\tau(\varphi \otimes u) = \langle \varphi, u \rangle$ and $\mu : U \otimes_S U^* \to S$ given by $\mu(u \otimes \varphi) = [u, \varphi]$, where $[u, \varphi]u' = u \langle \varphi, u' \rangle$. Then (R, S, U, V, τ, μ) defines a Morita context.*

Proof. We can easily check that τ and μ are both bisemimodule homomorphisms. For all $\psi \in U^*, u, u^{'} \in U$, then

$$\begin{aligned} \left\langle \psi[u, \varphi], u^{'} \right\rangle &= \left\langle \psi, [u, \varphi]u^{'} \right\rangle = \left\langle \psi, u \left\langle \varphi, u^{'} \right\rangle \right\rangle \\ &= \langle \psi, u \rangle \left\langle \varphi, u^{'} \right\rangle = \left\langle \langle \psi, u \rangle \varphi, u^{'} \right\rangle. \end{aligned}$$

This proves that $\psi[u, \varphi] = \langle \psi, u \rangle \varphi$. Hence, (R, S, U, V, τ, μ) is a Morita context. ∎

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TRIGONOMETRIC FUNCTIONS – ONE POSSIBLE DEFINITION

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Abstract. The paper deals with trigonometric functions, presents several possibilities of their introducing and offers one definition based on geometrical ideas and using analytical means. The basic properties of such functions are derived, too.

Trigonometric functions play very important role in geometry, analysis and other branches of mathematics. But definitions of these functions can be more subtle than one might suspect.

1. Some historical remarks

Trigonometry was studied in ancient Egypt, Babylon, Chaldea and Greece. For instance Hipparchos of Nicaea (180–125 BC) tabulated values for length of the chord of some angles (called chord function). It is directly proportional to the function sine (see Fig. 1), because

$$\operatorname{chord} A = 2r \sin \frac{\alpha}{2} \cdot$$

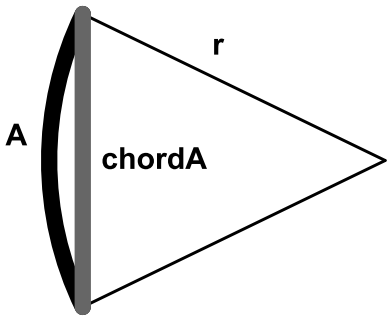

Figure 1: Chord function

Also Claudius Ptolemy (90–165 AD) solved astronomical problems linked with trigonometry.

The concepts closest to our trigonometric functions were introduced by Indian scholars at the period of Gupta dynasty, i.e. approximately from 320 to 550. In the treatise Surya Siddhanta, there are described notions $jy\bar{a}$, $koti - jy\bar{a}$ (and $utkrama - jy\bar{a}$) which correspond to the sine and cosine (see Fig. 2).

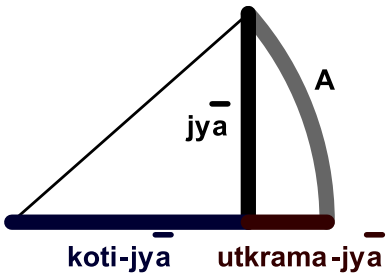

Figure 2: Trigonometric functions in Surya Siddhanta

Aryabhatiya (476–550) put together the tables of half chords which means sine. Bhaskara I. (600–680) invented the formula for approximate calculation of sine

$$\sin A = \frac{16A(\pi - A)}{5\pi^2 - 4A(\pi - A)}$$

(the formula is adjusted for radians).

Another Indian mathematician and astronomer Madhava of Sangamagrama (1340–1425) performed something like expansion of trigonometric functions into series and composed the accurate tables of sine for multiples of angle $3°45'$ (which is 1/24 of the right angle).

For more remarks about Indian ancient mathematicians see [6].

Indian mathematics was mediated to European culture usually by Arabian scholars (e.g. al-Khwarizmí) in period from 9th to 15th century. In rapid development of sciences, the trigonometric functions became the basic concept in works of many European mathematicians, let us name Leonhard Euler (1707–1783), for all, and his treatment "Introductio in analysis infinitorum". Signa sin, cos and tan themselves were introduced by Albert Girard (1595–1632).

2. Usual ways of definitions

Trigonometric functions are defined in several ways.

On elementary level, these functions are introduced geometrically by use of a rectangle triangle or a unit circle. Let us remind also one non frequented possibility – from general triangle ABC by cosine theorem

$$\cos\gamma = \frac{a^2 + b^2 - c^2}{2ab}\,.$$

In mathematical analysis, trigonometric functions are described usually by series, integrals or differential equations, see [1], [2] and especially [8]. The most often way of the definition uses the power series

$$\sin x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}, \qquad \cos x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}\,.$$

Trigonometric functions can be defined by means of differential equations. The sine is a solution of Cauchy's problem

$$y'' + y = 0 \quad \text{with conditions} \quad y(0) = 0,\ y'(0) = 1\ .$$

Or both sine and cosine are solutions of differential equation system

$$y_1' = y_2,\ y_2' = -y_1 \quad \text{with conditions} \quad y_1(0) = 0,\ y_2(0) = 1\ .$$

Another possibility how to introduce trigonometric functions is to define some of the inverse function to them by integral. We can define

$$\arctan x = \int \frac{1}{1+x^2}\,dx \quad \text{or} \quad \arcsin x = \int \frac{1}{\sqrt{1-x^2}}\,dx$$

and tangent or sine as the corresponding inverse functions which can be appropriately extended.

Similar situation is with the exponential function. It can be introduced by power series

$$\mathrm{e}^x = \sum_{n=0}^{\infty} \frac{x^n}{n}\ ,$$

as a solution of differential equation

$$y' - y = 0 \quad \text{with condition} \quad y(0) = 0$$

or as an inverse function to logarithm

$$\ln x = \int \frac{1}{x}\,dx \quad \text{for } x > 0\ .$$

One interesting definition of trigonometric functions using area of circular section is published in [7].

Each of these definitions requires quite complicated mathematical apparatus. But we often use trigonometric (or exponential) functions and their properties during studies of differential or integral calculus, including chapters on power series, differential equations or integrals. That is why we looked for some simple and sufficiently correct way of definition of trigonometric functions.

3. One simple definition of arcsine

The aim of this paper is to offer quite simple introduction of the function arcsine and then sine from it as an extension of inverse function to arcsine. This definition is based on geometrical ideas but it uses analytical means.

At first we shall calculate the longitude of curve described by (continuous and smooth) function f on $\langle a; b\rangle$ by means of so called variation. This idea comes from Pythagoras theorem.

For any partition

$$D : a = t_0 < t_1 < \cdots < t_n = b$$

of the interval $\langle a;b\rangle$ let us consider the sum

$$s(D)=\sum_{k=1}^{n}\sqrt{(t_k-t_{k-1})^2+(f(t_k)-f(t_{k-1}))^2}\ .$$

If there exists finite supremum S of such sums $s(D)$ for all partitions D of interval $\langle a;b\rangle$, we call this number S a longitude of curve f on $\langle a;b\rangle$, i.e.,

$$S=\sup\{s(D);D \text{ is a partition of interval } \langle a;b\rangle\}\ .$$

Now, we choose function describing a unit circle in the first quadrant

$$f(t)=\sqrt{1-t^2}$$

and denote by $AS(x)$ the longitude of this function on the interval $\langle 0;x\rangle$ for any $x\in\langle 0,1\rangle$. So, for any partition $D:\ 0=t_0<t_1<\cdots<t_n=x$ of interval $\langle 0;x\rangle$, we have

$$\begin{aligned}AS(x)=&\sup_D\sum_{k=1}^{n}\sqrt{(t_k-t_{k-1})^2+(f(t_k)-f(t_{k-1}))^2}=\\ =&\sup_D\sum_{k=1}^{n}\sqrt{(t_k-t_{k-1})^2+\left(\sqrt{1-t_k^2}-\sqrt{1-t_{k-1}^2}\right)^2}=\\ =&\sqrt{2}\sup_D\sum_{k=1}^{n}\sqrt{1-t_kt_{k-1}-\sqrt{1-t_k^2-t_{k-1}^2+t_k^2t_{k-1}^2}}\ .\end{aligned}$$

The function introduced in this way for $0\le x\le 1$ is well known arcussine $\arcsin x=AS(x)$.

Geometrically, it is illustrated on Fig. 3.

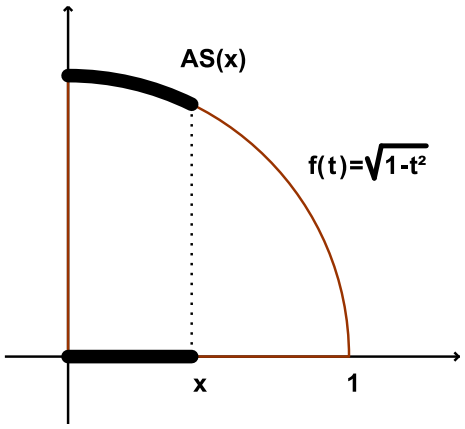

Figure 3: Definition of arcsine

For negative $-1 \leq x < 0$, arcussine can be defined by the formula

$$\arcsin x = -AS(-x).$$

Then arcussine is an odd function.

4. Properties of arcsine

Let us investigate some of arcsine properties:

1. The function AS is increasing.

   It is obvious from definition.

2. We define $\pi = 2 \cdot AS(1)$.

   Relation $AS(1) = \frac{1}{2}\pi$ corresponds with our geometrical ideas about Ludolf number π.

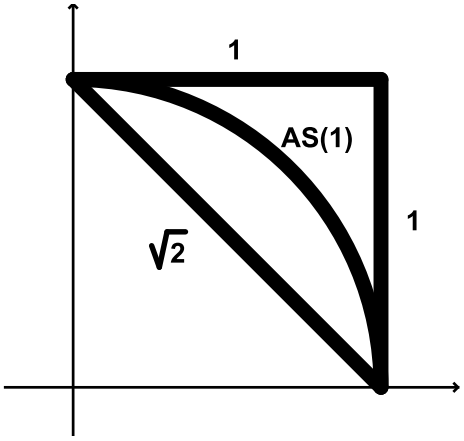

Figure 4: Second property

From Fig. 4 we can see the inequality $\sqrt{2} \leq AS(1) \leq 2$.

3. $AS(x) + AS(\sqrt{1-x^2}) = \frac{1}{2}\pi$.

   This equality is obvious from Fig. 5.

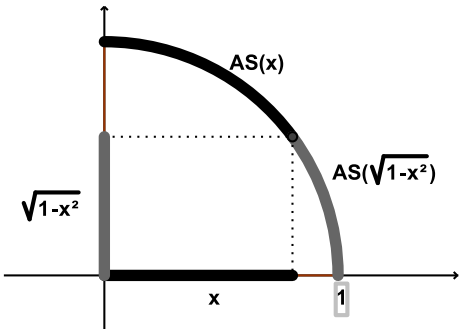

Figure 5: Third property

4. $\lim\limits_{x\to 0^+} \dfrac{AS(x)}{x} = 1.$

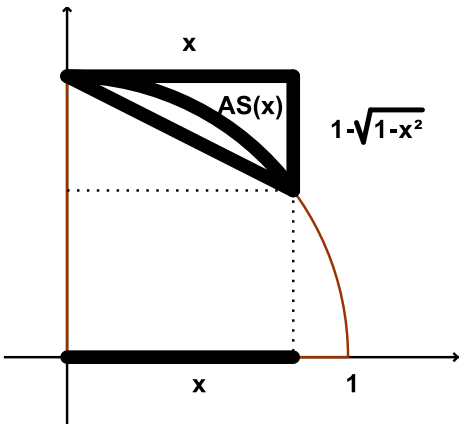

Figure 6: Fourth property

From Fig. 6 we can see the inequality

$$x + 1 - \sqrt{1-x^2} \geq AS(x) \geq \sqrt{x^2 + (1-\sqrt{1-x^2})^2} = \sqrt{2}\sqrt{1-\sqrt{1-x^2}}$$

and from that

$$\sqrt{2}\frac{\sqrt{1-\sqrt{1-x^2}}}{x} \leq \frac{AS(x)}{x} \leq 1 + \frac{1-\sqrt{1-x^2}}{x}\ .$$

Because of the limits $\lim\limits_{x\to 0} \dfrac{1-\sqrt{1-x^2}}{x^2} = \dfrac{1}{2}$ and $\lim\limits_{x\to 0} \dfrac{1-\sqrt{1-x^2}}{x} = 0$, it holds

$$\frac{AS(x)}{x} \to 1.$$

5. $\lim\limits_{x\to 1^-} \dfrac{AS(x) - AS(1)}{x-1} = \infty.$

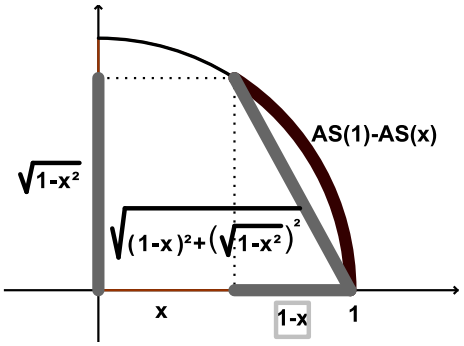

Figure 7: Fifth property

Pythagoras theorem used on triangle on Fig. 7 yields the inequality

$$AS(1) - AS(x) \geq \sqrt{(1-x)^2 + (\sqrt{1-x^2})^2} = \sqrt{2}\sqrt{1-x}$$

and from here we get

$$\frac{AS(1) - AS(x)}{1-x} \geq \frac{\sqrt{2}}{\sqrt{1-x}}.$$

Because $\lim\limits_{x \to 1^-} \dfrac{\sqrt{2}}{\sqrt{1-x}} = \infty$, our limit is also ∞.

This property is important to the possibility of a smooth extension of the inverse function.

6. $AS(y) - AS(x) = AS(x\sqrt{1-y^2} - y\sqrt{1-x^2})$.

The lengths of arc of function $f(t) = \sqrt{1-t^2}$ on interval $\langle 0; x\rangle$ and $\langle 0; y\rangle$ are presented by $AS(x)$ and $AS(y)$ respectively. Their difference $AS(y) - AS(x)$ presents the length of arc PQ of unit circle, the situation is described on Fig. 8 for $x < y$.

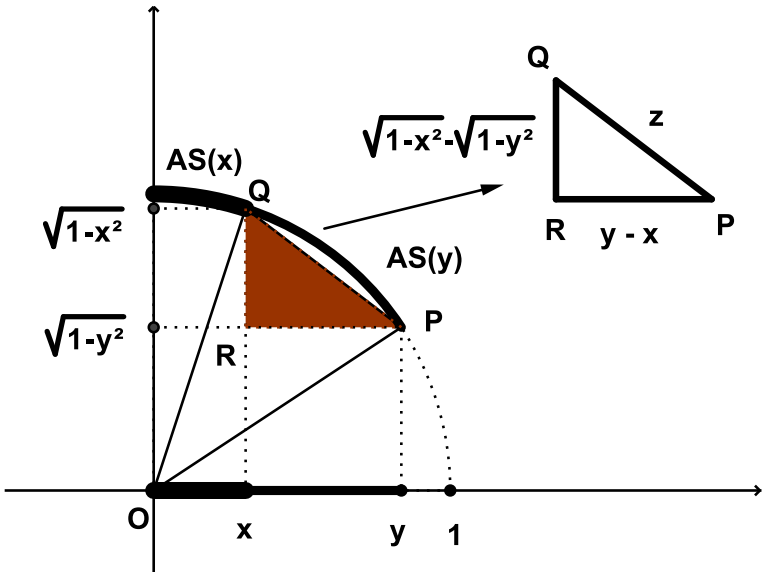

Figure 8: Sixth property

First we would like to work out the length $z = PQ$ of hypothenuse in triangle PQR. We use Pythagoras theorem and we get

$$z^2 = 2 - 2xy - 2\sqrt{1-x^2}\sqrt{1-y^2}.$$

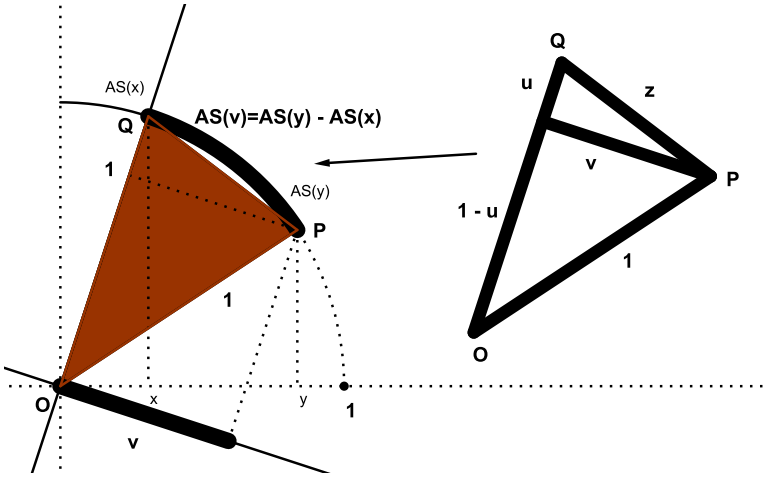

Figure 9: Sixth property

Let us consider different system of axes rotated in order that the second axis goes trough the points O and Q, it is described on Fig. 9. From this situation we see that the length of arc PQ of the unit circle is presented by $AS(v)$ for a certain v described on Fig. 9. It holds

$$AS(v) = AS(y) - AS(x) \ .$$

For deriving the required equality it remains only to express the longitude of v as a function of z and consequently as a function of x and y. We shall start from Fig. 9 and we use again Pythagoras theorem for both rectangular triangles in triangle OPQ

$$u^2 + v^2 = z^2 \quad \text{and} \quad (1-u)^2 + v^2 = 1 \ .$$

We get from these equations

$$u = \frac{z^2}{2} \quad \text{and} \quad v = \frac{z}{2}\sqrt{4 - z^2} \ .$$

Finally we calculate

$$\begin{aligned} v &= \frac{1}{2}\sqrt{2 - 2xy - 2\sqrt{1-x^2}\sqrt{1-y^2}} \cdot \sqrt{2 + 2xy + 2\sqrt{1-x^2}\sqrt{1-y^2}} \\ &= \sqrt{x^2 + y^2 - 2x^2y^2 - 2xy\sqrt{1-x^2}\sqrt{1-y^2}} \\ &= \sqrt{x^2(1-y^2) - 2xy\sqrt{1-x^2}\sqrt{1-y^2} + y^2(1-x^2)} \\ &= \sqrt{\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right)^2} \\ &= x\sqrt{1-y^2} - y\sqrt{1-x^2} \end{aligned}$$

and the equality of the sixth property is proved.

5. Definition and properties of sine and cosine

Now, we shall introduce the sin step by step for the intervals $\langle 0; \frac{\pi}{2}\rangle$, $\langle 0; \pi\rangle$, $\langle -\pi; \pi\rangle$ and $(-\infty; \infty)$.

At first we define the function $S_1 : \langle 0; \frac{\pi}{2}\rangle \to \langle 0; 1\rangle$ as the inverse function of $AS : \langle 0; 1\rangle \to \langle 0; \frac{\pi}{2}\rangle$. We obtain

$$S_1\left(\frac{\pi}{2}\right) = 1 \ , \qquad \lim_{x\to 0^+} \frac{S_1(x)}{x} = 1$$

and

$$\text{(i)} \qquad \lim_{x\to \frac{\pi}{2}^-} \frac{1 - S_1(x)}{\frac{\pi}{2} - x} = 0$$

from the properties 2, 4 and 5 of function AS.

To derive the summation formula for $x < y$, we put $u = S_1(x)$ and $v = S_1(y)$ into property 6

$$AS(v) - AS(u) = AS(u\sqrt{1 - v^2} - v\sqrt{1 - u^2}) \ .$$

Functions S_1 and AS are mutually inverse, hence $x = AS(u)$, $y = AS(v)$ and

$$S_1(AS(v) - AS(u)) = u\sqrt{1 - v^2} - v\sqrt{1 - u^2} \ .$$

From here, we obtain the formula

$$\text{(ii)} \qquad S_1(y - x) = S_1(x)\sqrt{1 - S_1^2(y)} - S_1(y)\sqrt{1 - S_1^2(x)} \ .$$

Then, we define the function S_2 on $\langle 0; \pi\rangle$ by means of S_1

$$S_2(x) = \begin{cases} S_1(x) & \text{for } x \in \langle 0; \frac{\pi}{2}\rangle, \\ S_1(\pi - x) & \text{for } x \in (\frac{\pi}{2}; \pi\rangle. \end{cases}$$

Formula (i) shows that the derivatives of S_2 at $\frac{\pi}{2}$ from the left and from the right are the same and equal to 0, hence function S_2 is smooth here.

Then, the function can be defined on $\langle -\pi; \pi\rangle$ symmetrically around the coordinates origin ($S_3(x) = -S_2(-x)$ for $-\pi \le x \le 0$) and at the end on $(\infty; \infty)$ periodically ($S_4(x) = S_3(x - 2k\pi)$ for $(2k - 1)\pi \le x \le (2k + 1)\pi$, k integers). This process again preserves the smoothness of the functions S_3 and S_4 in points $(2k + 1)\pi$ because of property 4.

The function S_4 is called sine and denoted sin. From symmetry (and periodicity), we have that it is an odd function, i.e., $\sin(-x) = -\sin x$. The second of trigonometric functions cosine is only shifted sine

$$\cos x = \sin\left(\tfrac{\pi}{2} - x\right) \ .$$

It is an even function, so $\cos(-x) = \cos x$ and its graph is symmetrical about the axis y.

From here it follows that $\cos x = \sqrt{1-\sin^2 x}$ on the interval $\langle 0; \frac{1}{2}\pi\rangle$, so we can rewrite summation formula (ii) as

$$\sin(y-x) = \sin y \cos x - \sin x \cos y$$

on this interval. Properties of introduced functions imply this relation even for all $x \in \mathbb{R}$.

Let us summarize that we have two functions defined on real numbers sin, cos and a number π for which the following properties hold:

(a) sin is increasing on $\langle 0; \frac{\pi}{2}\rangle$, $\sin(0) = 0$ and $\sin(\frac{\pi}{2}) = 1$,

(b) $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$,

(c) $\sin(y-x) = \sin y \cos x - \sin x \cos y$ for all $x, y \in \mathbb{R}$.

All other properties of these functions can be derived from here.

Let us notice that the similar way can be used for alternative introduction of generalized trigonometric functions $\sin_{p,q}$ and $\cos_{p,q}$ and number $\pi_{p,q}$, only the function

$$f_{p,q} = \sqrt[p]{1-x^q}$$

is considered (see [3], [4], [5]).

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PRICING MATURITY GUARANTEE WITH DYNAMIC WITHDRAWAL BENEFIT UNDER VASICEK INTEREST RATE MODEL

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Abstract. Motivated by Ko et al. (2010), who propose a new equity-linked product called "maturity guarantee with dynamic withdrawal benefit" (MGDWB), we consider the pricing of this product under a Vasicek stochastic interest rates framework. The explicit pricing formulas for the dynamic withdrawal benefit (DWB) payment stream and the maturity guarantee can be obtained when the DWB payment level is set to be a function of zero-coupon bond.

Keywords: maturity guarantee; dynamic withdrawal benefit; forward neutral measure.

Mathematics Subject Classification: 60G51, 60J25.

1. Introduction

Both individual and institutional investors have a basic need for protection against downside risk. The dynamic guaranteed fund has been one of the most popular investment funds in the insurance industry, recently. The fund, introduced by Gerber and his co-workers, provides a dynamic guarantee for an equity index linked portfolio to its investors with a necessary payment so that the upgraded fund unit value does not fall below a guaranteed level during the protection period. See Gerber and Pafumi (2000), Gerber and Shiu (1998, 2003), Tiong (2000) and Imai and Boyle (2001).

Researches on the dynamic guaranteed fund mainly focus on the modelling of the price of underlying assets and the design of the guarantees. See, for example, Gerber et al. (2013) investigate the valuation of variable annuities with guaranteed minimum death benefits under a jump-diffusion model. Gerber et al. (2015) consider the problem of valuing guaranteed minimum death benefits under a discrete model. Inspired by dynamic guaranteed fund and motivated by the importance of "withdrawal benefits" in enhancing sales of variable annuities, Ko et al. (2010) propose a new equity-linked product, which is called "maturity guarantee with dynamic withdrawal benefit" (MGDWB).

This new equity-linked product has two payoff features. Consider a customer paying a single premium for investing in a mutual fund or stock index. As the customer's account value attains a barrier, amounts over the barrier are automatically withdrawn from the account and paid to the customer. This payoff feature is called dynamic withdrawal benefit (DWB), which is a function of market conditions, so that it can well protect the insurers issuing variable annuities with DWB. Furthermore, the customer is provided with a guarantee that at the contract maturity date T, he will have at least a predetermined amount K. When the DWB payment level is set to be a constant or an exponential function of time and the interest-free rate is assumed to be constant, Ko et al. (2010) price such MGDWB contracts.

However, since equity-indexed annuities are usually of long maturity, the assumption of constant interest rate may not be quite adequate. In this paper, we will price a MGDWB contract in the presence of a stochastic term structure of interest rates. Assume the customer holds the contract to maturity T. Denote the time-t account value without and with DWB payments by $F(t)$ and $\hat{F}(t)$, respectively. In Ko et al. (2010), the relation between the two processes is given by

$$\hat{F}(t) = F(t) \min\left\{1, \min_{0\le s\le t} \frac{B}{F(s)}\right\}, \tag{1.1}$$

where $B \ge K$ is a constant. Obviously, the barrier B is not necessarily chosen to be a constant. See, for example, Gerber and Pafumi (2000) consider a stronger protection, where the guaranteed value of a fund unit at time t is $Ke^{\gamma t}$, for some $\gamma > 0$. In fact, γ can be explained as a guaranteed rate of return. In this paper, we extend the protection level to a stochastic level and consider the price of a MGDWB contract under a stochastic interest rate environment. However, in general, it is not easy to obtain the explicit formula for the price of the MGDWB contract under a stochastic interest rate environment. In this paper, we aim at providing a model in which the closed form formula for the price of the MGDWB contract can be obtained under a stochastic interest rate environment.

Bernard et al. (2006) propose a stochastic level with the guaranteed rate matching the return of a government bond and derive the fair value of a participating life insur- ance contract under a Vasicek interest rate environment. Therefore, following Bernard et al. (2006), we consider the relation between the two processes as follows:

$$\hat{F}(t) = F(t) \min\left\{1, \min_{0\le s\le t} \frac{BP(s,T)/P(0,T)}{F(s)}\right\}, \tag{1.2}$$

where $P(s,T)$ is the zero-coupon bond price with expiry date T at s. Assume $P(0,T) < 1$ so that $B/P(0,T) \ge K$. In fact, in (1.2), the barrier is chosen as a stochastic process given by $\frac{BP(s,T)}{P(0,T)}$. As explained in Bernard et al. (2006), the protection is equivalent to buying $B/P(0,T)$ government zero-coupon bonds having an initial value equal to $P(0,T)$ at time 0, since the guaranteed value of a fund unit at time t is equal to the value at time t of $K/P(0,T)$ zero-coupon bonds maturing

at time T. Therefore, comparing with the protection $Be^{\gamma t}$, which can be regarded as having a fixed guaranteed rate of return, the protection $BP(s,T)/P(0,T)$ can be regarded as having a guaranteed rate matching the return of a zero-coupon bond.

Let $\{\Omega, \Im, \{\Im_t\}_{0\leq t\leq T}, Q\}$ be a filtered complete probability space, where Q is the risk neutral measure and $\{\Im_t\}_{0\leq t\leq T}$ is a filtration satisfying the usual conditions of right continuity and completeness. Throughout the paper, it is assumed that all random variables and stochastic processes are well defined on this probability space and $\Im_T-$measurable. We assume that the fund process $\{F(t)\}$ follows a geometric Brownian motion under the risk-neutral probability measure Q:

$$\frac{dF(t)}{F(t)} = r_t dt + \sigma dW^Q(t). \tag{1.3}$$

It is assumed the short-term interest rate dynamics obey the relationship under Q:

$$dr_t = (a\theta - ar_t)dt + \gamma dZ_1^Q(t), a > 0, \gamma > 0.$$

Vasicek (1977) has showed the zero-coupon bond price with expiry date T at a fixed time t, $P(t,T)$, is a function of T and can be expressed as

$$P(t,T) = E^Q[e^{-\int_t^T r_s ds}|\Im_t] = e^{A(t,T)-B(t,T)r_t}, \tag{1.4}$$

where

$$\begin{cases} A(t,T) = \dfrac{(B(t,T)-T+t)}{a^2}\left(a^2\theta - \dfrac{\gamma^2}{2}\right) - \dfrac{B^2(t,T)\gamma^2}{4a}, \\ B(t,T) = \dfrac{1-e^{-a(T-t)}}{a}. \end{cases} \tag{1.5}$$

The zero-coupon bond price with expiry date T, $P(t,T)$, follows the stochastic diffusions

$$\frac{dP(t,T)}{P(t,T)} = r_t dt - \sigma_P(t,T)dZ_1^Q(t),$$

where $\sigma_P(t,T) = \frac{\gamma}{a}(1-e^{-a(T-t)})$. $Z_1^Q(t)$ and $W^Q(t)$ are two $Q-$standard Brownian motions with a correlation coefficient equal to ρ.

Consider a Brownian motion Z_2^Q independent from Z_1^Q, therefore, the Brownian motion W^Q can be expressed as

$$dW^Q(t) = \rho dZ_1^Q(t) + \sqrt{1-\rho^2}dZ_2^Q(t).$$

Then the dynamics of the fund in equation (A.8) can be rewritten as:

$$\frac{dF(t)}{F(t)} = r_t dt + \sigma\left(\rho dZ_1^Q(t) + \sqrt{1-\rho^2}dZ_2^Q(t)\right).$$

Denote by Q_T the $T-$forward-neutral measure. It is defined through its Radon-Nikodym derivative

$$\frac{dQ_T}{dQ} = e^{-\int_0^T \sigma_P(s,T)dZ_1^Q(s) - \frac{1}{2}\int_0^T \sigma_P^2(s,T)ds}.$$

From Girsanov theorem the process Z_1^Q defined by $dZ_1^{Q_T} = dZ_1^Q + \sigma_P(t,T)dt$ is a Q_T-Brownian motion.

Therefore, under the $T-$forward-neutral measure Q_T, the short-term interest rate dynamics obey the relationship

$$dr_t = a(\theta_t - r_t)dt + \gamma dZ_1^{Q_T}(t),$$

where $\theta_t = \theta - \gamma^2/a^2(1 - e^{-a(T-t)})$.

We also define $Z_2^{Q_T}$ such that $Z_1^{Q_T}$ and $Z_2^{Q_T}$ are noncorrelated Q_T-Brownian motions. Under the $T-$forward-neutral measure Q_T, the dynamics of $F(t)$ and $P(t,T)$ follow the stochastic differential equations

$$\frac{dF(t)}{F(t)} = (r_t - \rho\sigma\sigma_P(t,T))dt + \sigma\left(\rho dZ_1^{Q_T}(t) + \sqrt{1-\rho^2}dZ_2^{Q_T}(t)\right)$$

and

$$\frac{dP(t,T)}{P(t,T)} = (r_t + \sigma_P^2(t,T))dt - \sigma_P(t,T)dZ_1^{Q_T}(t).$$

By some calculations, we have

$$\frac{BP(s,T)}{F(s)P(0,T)} = \frac{B}{F_0}\exp\left\{-\int_0^s(\sigma_P(u,T)+\rho\sigma)dZ_1^{Q_T}(u) - \int_0^s \sigma\sqrt{1-\rho^2}dZ_2^{Q_T}(u) + \frac{1}{2}\int_0^s((\sigma_P(u,T)+\rho\sigma)^2 + \sigma^2(1-\rho^2))du\right\}. \tag{1.6}$$

Define the martingale H as

$$H_s = \int_0^s(\sigma_P(u,T)+\rho\sigma)dZ_1^{Q_T}(u) + \int_0^s \sigma\sqrt{1-\rho^2}dZ_2^{Q_T}(u).$$

Note that the quadratic variation of H is

$$\xi(s) = \int_0^s((\sigma_P(u,T)+\rho\sigma)^2 + \sigma^2(1-\rho^2))du = \left(\sigma^2 + \frac{\gamma^2}{a^2} + \frac{2\rho\sigma\gamma}{a}\right)s - \left(\frac{2\rho\sigma\gamma}{a^2} + \frac{2\gamma^2}{a^3}\right)(e^{-a(T-s)} - e^{-aT}) + \frac{\gamma^2}{2a^3}(e^{-2a(T-s)} - e^{-2aT}). \tag{1.7}$$

Due to Dubins-Schwarz theorem, there exists a Q_T-Brownian motion B such that

$$H_s = B_{\xi(s)}.$$

Therefore, (1.6) becomes

$$\frac{BP(s,T)}{F(s)P(0,T)} = \frac{B}{F_0}e^{-B_{\xi(s)}+\frac{1}{2}\xi(s)}, \tag{1.8}$$

from (1.8), $F(T)$ can be expressed as

$$F(T) = \frac{F_0}{P(0,T)}e^{B_{\xi(T)}-\frac{1}{2}\xi(T)}.$$

2. Contract valuation

Denote $VW(B,T)$ by the value of the DWB payment stream between time 0 and T at time 0, denote $VP(K,B,T)$ by the value of the European put option with payoff $(K-\hat{F}(T))^+$ at time 0. For simplicity, we assume the cost of the put option is to be paid at the beginning of the contract, and $0 < K \leq F_0 \leq B$. Because the customer pays $F_0 + VP(K,B,T)$, so by the fundamental theorem of asset pricing, we have

$$F_0 = VW(B,T) + E^Q[e^{-\int_0^T r_s ds}\hat{F}(T)], \tag{2.1}$$

$$VP(K,B,T) = E^Q[e^{-\int_0^T r_s ds}(K-\hat{F}(T))^+]. \tag{2.2}$$

Moving toward forward-neutral world, we obtain

$$E^Q[e^{-\int_0^T r_s ds}\hat{F}(T)] = P(0,T)E^{Q_T}[\hat{F}(T)], \tag{2.3}$$

$$E^Q[e^{-\int_0^T r_s ds}(K-\hat{F}(T))^+] = P(0,T)E^{Q_T}[(K-\hat{F}(T))^+]. \tag{2.4}$$

From (1.8), one obtains

$$\begin{aligned}\hat{F}(T) &= F(T)\min\left\{1, \min_{0\leq s\leq T}\frac{B}{F_0}e^{-B_{\xi(s)}+\frac{1}{2}\xi(s)}\right\} \\ &= F(T)\min\left\{1, \frac{B}{F_0}e^{-\max_{0\leq s\leq \xi(T)}(B_s-\frac{1}{2}s)}\right\} \qquad (2.5)\\ &= \frac{F_0}{P(0,T)}e^{B_{\xi(T)}-1/2\xi(T)}\min\left\{1, \frac{B}{F_0}e^{-\max_{0\leq s\leq \xi(T)}(B_s-\frac{1}{2}s)}\right\}.\end{aligned}$$

For simplicity, we write $X = B_{\xi(T)} - \frac{1}{2}\xi(T), M = \max_{0\leq s\leq\xi(T)}(B_s - \frac{1}{2}s)$. Then, from (2.1), (2.3) and (2.5), we obtain

$$\begin{aligned}VW(B,T) &= F_0 - P(0,T)E^{Q_T}[\hat{F}(T)] \\ &= F_0 - F_0E^{Q_T}[e^X 1\{M<b\}] - BE^{Q_T}[e^X e^{-M}1\{M>b\}],\end{aligned} \tag{2.6}$$

where $b = \ln\frac{B}{F_0} > 0$, $1\{C\}$ is the indicator of a set C. To simplify the above expression, we define a new probability $\overline{Q}$ through Radon-Nikodym derivative

$$\frac{d\overline{Q}}{dQ_T} = e^X = e^{\int_0^T(\sigma_P(u,T)+\rho\sigma)dZ_1^{Q_T}(u)+\int_0^T\sigma\sqrt{1-\rho^2}dZ_2^{Q_T}-\frac{1}{2}\xi(T)}.$$

From Girsanov theorem, we can construct two $\overline{Q}-$ standard Brownian motions $Z_1^{\overline{Q}}$ and $Z_2^{\overline{Q}}$ defined by

$$\begin{aligned} dZ_1^{\overline{Q}} &= dZ_1^{Q_T} - \int_0^s (\sigma_P(u,T) + \rho\sigma)du, \\ dZ_2^{\overline{Q}} &= dZ_2^{Q_T} - \int_0^s \sigma\sqrt{1-\rho^2}du. \end{aligned}$$

Then, under $\overline{Q}$, $F(T)$ is expressed as

$$F(T) = \frac{F_0}{P(0,T)} e^{\int_0^T (\sigma_P(s,T)+\rho\sigma)dZ_1^{\overline{Q}}(u) + \int_0^T \sigma\sqrt{1-\rho^2}dZ_2^{\overline{Q}}(u) + \frac{1}{2}\xi(T)}. \tag{2.7}$$

Similar to deriving Eq. (1.8), under $\overline{Q}$ Eq. (2.7) can be expressed as

$$F(T) = \frac{F_0}{P(0,T)} e^{\overline{B}_{\xi(T)} + \frac{1}{2}\xi(T)},$$

where $\overline{B}$ is a $\overline{Q}-$standard Brownian motion. Define $\overline{X} = \overline{B}_{\xi(T)} + \frac{1}{2}\xi(T)$ and $\overline{M} = \max\limits_{0\le s\le \xi(T)} \left(\overline{B}_s + \frac{1}{2}s\right)$. Note $E^{Q_T}[e^X] = 1$, then

$$\begin{aligned} E^{Q_T}[e^X 1\{M < b\}] &= E^{Q_T}[e^X]\overline{Q}(\overline{M} < b) = \overline{Q}(\overline{M} < b) \\ &= \Phi\left(\frac{b - 1/2\xi(T)}{\sqrt{\xi(T)}}\right) - \frac{B}{F_0}\Phi\left(\frac{-b - 1/2\xi(T)}{\sqrt{\xi(T)}}\right) \end{aligned} \tag{2.8}$$

and

$$E^{Q_T}[e^{X-M}1\{M > b\}] = E^{Q_T}[e^X]E^{\overline{Q}}[e^{-\overline{M}}1\{\overline{M} > b\}] = E^{\overline{Q}}[e^{-\overline{M}}1\{\overline{M} > b\}]. \tag{2.9}$$

Note the expectation $E^{\overline{Q}}[e^{a\overline{M}}1\{\overline{M} > b\}]$, for $a \neq -1$, has been given by Huang and Shiu (2001) and Lee (2003). To evaluate the expectation $E^{\overline{Q}}[e^{-\overline{M}}1\{\overline{M} > b\}]$, first we give the following lemma.

Lemma 2.1 *Let $X(t) = \mu t + \sigma W(t)$, $M(t) = \max\limits_{0\le s\le t} X(s)$, where $W(t)$ is a $Q-$standard Brownian motion. Then, we have*

$$Q(X(t) < k, M(t) > d)) = e^{\frac{2\mu d}{\sigma^2}} Q(X(t) \le k - 2d), b \ge 0, k \le b. \tag{2.10}$$

Proof. The proof of (2.10) can be found in Gerber and Shiu (2000) and Huang and Shiu (2001). From (2.10), we obtain

$$\begin{aligned} \overline{Q}(\overline{M} \le m) &= \overline{Q}(\overline{X} \le m) - \overline{Q}(\overline{X} \le m, \overline{M} \ge m) \\ &= \overline{Q}(\overline{X} \le m) - e^m\overline{Q}(\overline{X} \le -m), \end{aligned} \tag{2.11}$$

Using (2.11), we have

$$
\begin{aligned}
E^{\overline{Q}}[e^{-\overline{M}}1\{\overline{M}>b\}] &= \int_b^\infty e^{-m}d\overline{Q}(\overline{X}\le m) - \int_b^\infty e^{-m}d(e^m\overline{Q}(\overline{X}\le -m)) \\
&= \int_b^\infty \frac{2}{\sqrt{2\pi\xi(T)}}e^{-\frac{(m+1/2\xi(T))^2}{2\xi(T)}}dm - \int_b^\infty \overline{Q}(\overline{X}\le -m)dm \\
&= 2\Phi\left(-\frac{b+1/2\xi(T)}{\sqrt{\xi(T)}}\right) - \int_b^\infty \overline{Q}(\overline{X}\le -m)dm.
\end{aligned} \tag{2.12}
$$

Integrating the integral in (2.12) by parts, we obtain

$$
\begin{aligned}
&\int_b^\infty \overline{Q}(\overline{X}\le -m)dm \\
&= -\, b\Phi\left(-\frac{b+1/2\xi(T)}{\sqrt{\xi(T)}}\right) + \int_b^\infty \frac{m}{\sqrt{2\pi\xi(T)}}e^{-\frac{(m+1/2\xi(T))^2}{2\xi(T)}}dm \\
&= -\,(b+1/2\xi(T))\Phi\left(-\frac{b+1/2\xi(T)}{\sqrt{\xi(T)}}\right) + \sqrt{\frac{\xi(T)}{2\pi}}e^{-\frac{(b+1/2\xi(T))^2}{2\xi(T)}}.
\end{aligned} \tag{2.13}
$$

Substituting (2.13) into (2.12), we have

$$
\begin{aligned}
E^{\overline{Q}}[e^{-\overline{M}}1\{\overline{M}>b\}] &= (2+b+1/2\xi(T))\Phi\left(-\frac{b+1/2\xi(T)}{\sqrt{\xi(T)}}\right) \\
&\quad - \sqrt{\frac{\xi(T)}{2\pi}}e^{-\frac{(b+1/2\xi(T))^2}{2\xi(T)}}.
\end{aligned} \tag{2.14}
$$

Substituting (2.8)-(2.14) into (2.6), we give the explicit expression for $VW(B,T)$

$$
\begin{aligned}
VW(B,T) &= F_0\Phi\left(-\frac{b-1/2\xi(T)}{\sqrt{\xi(T)}}\right) \\
&\quad - B\left(1+\ln\frac{B}{F_0}+\frac{\xi(T)}{2}\right)\Phi\left(-\frac{b+1/2\xi(T)}{\sqrt{\xi(T)}}\right) + B\sqrt{\frac{\xi(T)}{2\pi}}e^{-\frac{(b+1/2\xi(T))^2}{2\xi(T)}}.
\end{aligned} \tag{2.15}
$$

Remark 2.1.

(1) From (1.7), we can easily obtain $\xi(T)\to+\infty$, when $T\to+\infty$, hence $VW(B,\infty)=F_0$.

(2) When B is infinity, then no withdrawal benefit payments will ever be made. Thus, $VW(\infty,T)=0$ and it is easy to check it from (2.6) and (2.8)-(2.14).

Now, we present the evaluation of the put option price $VP(B,K,T)$. We can rewrite Eq. (2.2) by using the $T-$forward-neutral equivalent martingale measure Q_T according to:

$$
VP(B,K,T) = P(0,T)E^{Q_T}[(K-\hat{F}(T))_+]. \tag{2.16}
$$

With $c = \ln \frac{KP(0,T)}{F_0} < 0$, we divide the expectation $E^{Q_T}[(K - \hat{F}(T))_+]$ into two parts:

$$
\begin{aligned}
& E^{Q_T}[(K - \hat{F}(T))_+] \\
& \quad = E^{Q_T}[(K - \hat{F}(T))_+ 1\{M < b\}] + E^{Q_T}[(K - \hat{F}(T))_+ 1\{M > b\}] \\
& \quad = E^{Q_T}\left[\left(K - \frac{F_0}{P(0,T)} e^X\right) 1\{M < b, X < c\}\right] \\
& \quad + E^{Q_T}\left[\left(K - \frac{B}{P(0,T)} e^{X-M}\right) 1\{M > b, M - X > b - c\}\right].
\end{aligned} \tag{2.17}
$$

The first term on the right hand side of (2.17) can be calculated as follows:

$$
\begin{aligned}
& E^{Q_T}\left[\left(K - \frac{F_0}{P(0,T)} e^X\right) 1\{M < b, X < c\}\right] \\
& = KQ_T(M < b, X < c) - \frac{F_0}{P(0,T)} E^{Q_T}[e^X] \overline{Q}(\overline{M} < b, \overline{X} < c) \\
& = K[Q_T(X < c) - Q_T(M \geq b, X < c)] \\
& \quad - \frac{F_0}{P(0,T)} [\overline{Q}(\overline{X} < c) - \overline{Q}(\overline{M} \geq b, \overline{X} < c)] \\
& = K\left[\Phi\left(\frac{c + 1/2\xi(T)}{\sqrt{\xi(T)}}\right) - \frac{F_0}{B}\Phi\left(\frac{c - 2b + 1/2\xi(T)}{\sqrt{\xi(T)}}\right)\right] \\
& \quad - \frac{F_0}{P(0,T)}\left[\Phi\left(\frac{c - 1/2\xi(T)}{\sqrt{\xi(T)}}\right) - \frac{B}{F_0}\Phi\left(\frac{c - 2b - 1/2\xi(T)}{\sqrt{\xi(T)}}\right)\right],
\end{aligned} \tag{2.18}
$$

where the last equality follows from (2.10). The second term on the right hand side of (2.17) is

$$
\begin{aligned}
& E^{Q_T}\left[\left(K - \frac{B}{P(0,T)} e^{X-M}\right) 1\{M > b, M - X > b - c\}\right] \\
& = KQ_T(M > b, M - X > b - c) \\
& - \frac{B}{P(0,T)} E^{Q_T}\left[e^{X-M} 1\{M > b, M - X > b - c\}\right] \\
& = \frac{B}{P(0,T)}\left[\Phi\left(\frac{c - 2b - 1/2\xi(T)}{\sqrt{\xi(T)}}\right) + \frac{F_0 K}{B}\Phi\left(\frac{c - 2b + 1/2\xi(T)}{\sqrt{\xi(T)}}\right)\right] \\
& - \frac{B}{P(0,T)}\left[(2 + 2b - c + 1/2\xi(T))\Phi\left(\frac{c - 2b - 1/2\xi(T)}{\sqrt{\xi(T)}}\right)\right. \\
& \left. - \sqrt{\frac{\xi(T)}{2\pi}} e^{-\frac{(2b - c + 1/2\xi(T))^2}{2\xi(T)}}\right],
\end{aligned} \tag{2.19}
$$

where the proof of the expressions for the probability $Q_T(M > b, M - X > b - c)$ and the expectation $E^{Q_T}[e^{X-M} 1\{M > b, M - X > b - c\}]$ are given by (A.6) and (A.8) in the Appendix.

Substituting (2.17)-(2.19) into (2.16) yields that

$$
\begin{aligned}
VP(B,K,T) =& KP(0,T)\Phi\left(\frac{c+1/2\xi(T)}{\sqrt{\xi(T)}}\right) - F_0\Phi\left(\frac{c-1/2\xi(T)}{\sqrt{\xi(T)}}\right) \\
&+ B\left(\ln\frac{KP(0,T)F_0}{B^2} - \frac{\xi(T)}{2}\right)\Phi\left(\frac{c-2b-1/2\xi(T)}{\sqrt{\xi(T)}}\right) \\
&+ B\sqrt{\frac{\xi(T)}{2\pi}}e^{-\frac{(2b-c+1/2\xi(T))^2}{2\xi(T)}}.
\end{aligned} \tag{2.20}
$$

Remark 2.2.

(1) It follows from (1.4), (1.5) that, if $a^2\theta > \frac{\gamma^2}{2}$, then $P(0,T) \to 0$ when $T \to \infty$, so we have $VP(B,K,\infty) = 0$.

(2) When $K = 0$, it can be directly obtained from (2.16) $VP(B,0,T) = 0$.

(3) When B is infinity, then no withdrawal benefit payments will ever be made. Thus, $VW(\infty,K,T)$ must be the Black-Scholes price of the European $K-$strike put option on $F(t)$ exercisable at T and it can be expressed as

$$
VP(\infty,K,T) = P(0,T)K\Phi\left(\frac{c+1/2\xi(T)}{\sqrt{\xi(T)}}\right) - F_0\Phi\left(\frac{c-1/2\xi(T)}{\sqrt{\xi(T)}}\right).
$$

In fact, $VP(\infty,K,T)$ is exactly given by the first two terms on the right hand side of (2.20). Hence, (2.20) can also be written as

$$
\begin{aligned}
&VP(B,K,T) - VP(\infty,K,T) \\
&= B\left(\ln\frac{KP(0,T)F_0}{B^2} - \frac{\xi(T)}{2}\right)\Phi\left(\frac{c-2b-1/2\xi(T)}{\sqrt{\xi(T)}}\right) \\
&+ B\sqrt{\frac{\xi(T)}{2\pi}}e^{-\frac{(2b-c+1/2\xi(T))^2}{2\xi(T)}}.
\end{aligned}
$$

Since the left hand side of (2.20) is positive, its right hand side is also positive, which is not immediately obvious.

(4) Differentiating (2.15) with respect to F_0 and simplifying, we obtain the hedge-ratio formula,

$$
\frac{\partial}{\partial F_0}VW(B,T) = \Phi\left(-\frac{b-1/2\xi(T)}{\sqrt{\xi(T)}}\right) + \frac{B}{F_0}\Phi\left(-\frac{b+1/2\xi(T)}{\sqrt{\xi(T)}}\right).
$$

Differentiating (2.20) with respect to F_0 and simplifying, we have

$$
\frac{\partial}{\partial F_0}VP(B,K,T) = -\Phi\left(\frac{c-1/2\xi(T)}{\sqrt{\xi(T)}}\right) + \frac{B}{F_0}\Phi\left(\frac{c-2b-1/2\xi(T)}{\sqrt{\xi(T)}}\right).
$$

It is interesting to note that both derivatives can be quickly obtained if we pretend that b, c are not functions of F_0 when performing the differentiation. (Similarly, differentiating the Black-Scholes formula with respect to the stock price, while pretending that d_1 and d_2 are constant, immediately yields the formula for the option's delta.)

3. Numerical analysis

In this section, we make some numerical analysis on the value of the DWB payment stream $VW(B,T)$ and the value of the European put option $VP(K,B,T)$. The price of the zero-coupon bond $P(0,T)$ is considered as a function of T and hence it is can be calculated from equation (1.6). Let $F_0 = 100,\ r_0 = 0.05,\ a = 0.4,\ \sigma = 0.1,\ \gamma = 0.008,\ \rho = 0.2,\ \theta = 0.04$.

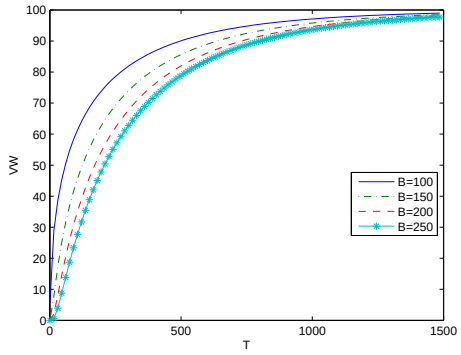

Figure 1: $VW(B,T)$ as a function of T for different B

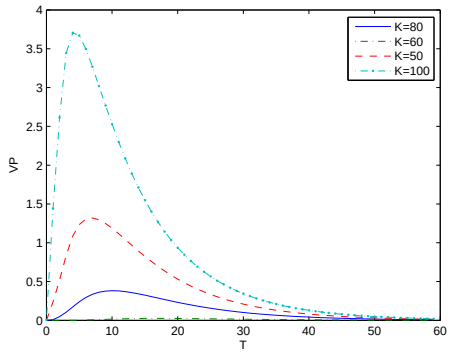

Figure 2: $VP(K,B,T)$ as a function of T for different K, $B = 120$.

Figure 1 plots $VW(B,T)$ as a function of T for different B. From Figure 1 we see, $VW(B,T)$ is a decreasing function of B. That is to say, the withdrawal benefit payments will decrease with B increasing. Figure 2 plots $VP(B,K,T)$ as a function of T for different K given $B = 120$. It is easy to see from Figure 2, $VP(B,K,T)$ is an increasing function of K. That is because $VP(B,K,T)$ is the price of put option with strike K.

4. Reflected Brownian motion

In this section, we derive the pricing formulas (2.15) and (2.20) by using the theory of Reflected Brownian motion.

$$\ln \frac{BP(t,T)}{P(0,T)\hat{F}(t)} = -\left(B_{\xi(t)} - \frac{1}{2}\xi(t)\right) + \max\left\{b, \max_{0\le s\le t}\left\{B_{\xi(s)} - \frac{1}{2}\xi(s)\right\}\right\}. \tag{4.1}$$

The process $\{\ln \frac{BP(t,T)}{P(0,T)\hat{F}(t)}\}$ is a reflected Brownian motion starting at $\ln \frac{B}{F(0)} = b$, with reflecting barrier at 0. Let $Y = \ln \frac{B}{P(0,T)\hat{F}(T)}$, then from Formula (91), Section

5.7 in Cox and Miller (1965) it follows that the probability density function of the random variable Y under Q_T is

$$\begin{aligned} p(x;b,T) &= n\left(x; b+\frac{1}{2}\xi(T), \xi(T)\right) + e^{-b} n\left(x; -b+\frac{1}{2}\xi(T), \xi(T)\right) \\ &- e^{x}\Phi\left(-\frac{x+b+1/2\xi(T)}{\sqrt{\xi(T)}}\right), x>0, \end{aligned} \tag{4.2}$$

where $n(x;\mu,\sigma^2)$ is the probability density function of the normal random variable with mean μ and variance σ^2, and $\Phi(x)$ is the standard normal distribution. This result corresponds to formula (2.5) in Ko et al. (2010). To obtain the pricing formulas (2.15) and (2.20), we also need the following formulas given in Gerber and Pafumi (2000) and Ko et al. (2010),

$$\int_a^\infty e^{cx} n(x;\mu,\sigma^2)dx = e^{\mu c+1/2\sigma^2c^2}\Phi\left(\frac{-a+\mu+c\sigma^2}{\sigma}\right), \tag{4.3}$$

where a and c are arbitrary real numbers and

$$\begin{aligned} \int_a^\infty e^{cx}\Phi\left(-\frac{x-\mu}{\sigma}\right)dx &= \frac{1}{c}e^{\mu c+1/2\sigma^2c^2}\Phi\left(\frac{-a+\mu+c\sigma^2}{\sigma}\right) \\ &- \frac{1}{c}e^{ac}\Phi\left(\frac{-a+\mu}{\sigma}\right), \end{aligned} \tag{4.4}$$

where c is an arbitrary number other than zero. By using (4.2)-(4.4), we obtain

$$\begin{aligned} VW(B,T) &= F_0 - P(0,T)E^{Q_T}[\hat{F}(T)] = F_0 - BE^{Q_T}[e^{-Y}] \\ &= F_0 - B\left[e^{-b}\Phi\left(\frac{b-1/2\xi(T)}{\sqrt{\xi(T)}}\right) + \Phi\left(-\frac{b+1/2\xi(T)}{\sqrt{\xi(T)}}\right)\right. \\ &\qquad \left. - \int_0^\infty \Phi\left(\frac{-y-b-1/2\xi(T)}{\sqrt{\xi(T)}}\right)dy\right] \\ &= F_0\Phi\left(\frac{-b+1/2\xi(T)}{\sqrt{\xi(T)}}\right) - B\Phi\left(-\frac{b+1/2\xi(T)}{\sqrt{\xi(T)}}\right) \\ &\qquad + B\int_0^\infty \Phi\left(\frac{-y-b-1/2\xi(T)}{\sqrt{\xi(T)}}\right)dy. \end{aligned} \tag{4.5}$$

Integrating the integral in (4.5) by parts, we obtain

$$\begin{aligned} &\int_0^\infty \Phi(\frac{-y-b-1/2\xi(T)}{\sqrt{\xi(T)}})dy = \int_0^\infty \frac{y}{\sqrt{2\pi\xi(T)}} e^{-\frac{(y+b+1/2\xi(T))^2}{2\xi(T)}}dy \\ &= -(\ln\frac{B}{F_0} + 1/2\xi(T))\Phi(-\frac{b+1/2\xi(T)}{\sqrt{\xi(T)}}) + B\sqrt{\frac{\xi(T)}{2\pi}}e^{-\frac{(b+1/2\xi(T))^2}{2\xi(T)}}. \end{aligned} \tag{4.6}$$

Substituting (4.6) into (4.5), we have

$$\begin{aligned} VW(B,T) &= F_0\Phi(-\frac{b-1/2\xi(T)}{\sqrt{\xi(T)}}) + B\sqrt{\frac{\xi(T)}{2\pi}}e^{-\frac{(b+1/2\xi(T))^2}{2\xi(T)}} \\ &- B(1+\ln\frac{B}{F_0}+\frac{\xi(T)}{2})\Phi(-\frac{b+1/2\xi(T)}{\sqrt{\xi(T)}}), \end{aligned} \tag{4.7}$$

which is the same as equation (2.15).

Similarly, we can obtain the explicit formula for pricing the put option.

$$\begin{aligned} VP(B,K,T) &= P(0,T)E^{Q_T}[(K-\hat{F}(T))^+] \\ &= P(0,T)E^{Q_T}\left[\left(K-\frac{B}{P(0,T)}e^{-Y}\right)^+\right] \\ &= P(0,T)E^{Q_T}\left[\left(K-\frac{B}{P(0,T)}e^{-Y}\right)1\{Y\geq b-c\}\right] \\ &= P(0,T)KQ_T(Y\geq b-c) - BE^{Q_T}[e^{-Y}1\{Y\geq b-c\}] \\ &= KP(0,T)\Phi\left(\frac{c+1/2\xi(T)}{\sqrt{\xi(T)}}\right) + B\Phi\left(\frac{c-2b-1/2\xi(T)}{\sqrt{\xi(T)}}\right) \\ &-B\left[e^{-b}\Phi\left(\frac{c-1/2\xi(T)}{\sqrt{\xi(T)}}\right)+\Phi\left(\frac{c-2b-1/2\xi(T)}{\sqrt{\xi(T)}}\right)\right. \\ &\qquad\left.-\int_{b-c}^{\infty}\Phi\left(-\frac{y+b+1/2\xi(T)}{\sqrt{\xi(T)}}\right)dy\right] \\ &= KP(0,T)\Phi\left(\frac{c+1/2\xi(T)}{\sqrt{\xi(T)}}\right) - F_0\Phi\left(\frac{c-1/2\xi(T)}{\sqrt{\xi(T)}}\right) \\ &+B\int_{b-c}^{\infty}\Phi\left(-\frac{y+b+1/2\xi(T)}{\sqrt{\xi(T)}}\right)dy. \end{aligned} \tag{4.8}$$

Integrating the integral in (4.8) by parts, we obtain

$$\begin{aligned} \int_{b-c}^{\infty}\Phi\left(-\frac{y+b+1/2\xi(T)}{\sqrt{\xi(T)}}\right)dy &= -(b-c)\Phi\left(\frac{c-2b-1/2\xi(T)}{\sqrt{\xi(T)}}\right) \\ &+\int_{b-c}^{\infty}\frac{y}{\sqrt{2\pi\xi(T)}}e^{-\frac{(y+b+1/2\xi(T))^2}{2\xi(T)}}dy \\ &= -(2b+1/2\xi(T)-c)\Phi\left(\frac{c-2b-1/2\xi(T)}{\sqrt{\xi(T)}}\right) \\ &+\sqrt{\frac{\xi(T)}{2\pi}}e^{-\frac{(2b-c+1/2\xi(T))^2}{2\xi(T)}}. \end{aligned} \tag{4.9}$$

Substituting (4.9) into (4.8) gives

$$
\begin{aligned}
VP(B,K,T) &= KP(0,T)\Phi\left(\tfrac{c+1/2\xi(T)}{\sqrt{\xi(T)}}\right) - F_0\Phi\left(\tfrac{c-1/2\xi(T)}{\sqrt{\xi(T)}}\right) \\
&+B\left(\ln\frac{KP(0,T)F_0}{B^2} - \frac{\xi(T)}{2}\right)\Phi\left(\frac{c-2b-1/2\xi(T)}{\sqrt{\xi(T)}}\right) \\
&+B\sqrt{\frac{\xi(T)}{2\pi}}e^{-\frac{(2b-c+1/2\xi(T))^2}{2\xi(T)}}.
\end{aligned} \tag{4.10}
$$

Obviously, equation (4.10) is the same with (2.20).

5. Conclusions

This article considers the pricing of the maturity guarantee with dynamic withdrawal benefit contract under a Vasicek stochastic interest rates framework. Because the DWB payment barrier is chosen to be a function of zero-coupon bond, so the closed-form formulas for the value of the DWB payment stream and the value of the European put option can be obtained by a probabilistic method or by using the theory of reflected Brownian motion.

Appendix

From (2.10), we have

$$
Q_T(M > y, X < x) = e^{-y}\Phi\left(\frac{x-2y+1/2s}{\sqrt{s}}\right), \tag{A.1}
$$

where $x \le y$, and $s = \xi(T)$. Differentiating (A.1), we obtain the joint density function of (M, X)

$$
Q_T(M \in dy, X \in dx) = \frac{1}{\sqrt{2\pi s}}e^{\frac{(x+1/2s)^2+4y^2-4xy}{2s}}\frac{4y-2x}{s}dxdy, x \le y. \tag{A.2}
$$

From equation (A.2), we have

$$
\begin{aligned}
&E^{Q_T}[1\{M > b, M - X > b - c\}] \\
&= \int_b^\infty \int_{-\infty}^{y-b+c} \frac{1}{\sqrt{2\pi s}}e^{-\frac{(x+1/2s)^2+4y^2-4xy}{2s}}\frac{4y-2x}{s}dxdy \\
&= \int_b^\infty \int_{-\infty}^{y-b+c} \frac{1}{\sqrt{2\pi s}}e^{-\frac{(x+1/2s-2y)^2}{2s}}e^{-y}\left(\frac{4y-2x-s}{s}+1\right)dxdy \\
&= \int_b^\infty \int_{-\infty}^{y-b+c} \frac{4y-2x-s}{s\sqrt{2\pi s}}e^{-y}e^{-\frac{(x+1/2s-2y)^2}{2s}}dxdy \\
&+ \int_b^\infty \Phi\left(-\frac{y+b-c-1/2s}{\sqrt{s}}\right)dy \\
&= \int_b^\infty e^{-y}I_1 dy + \int_b^\infty e^{-y}\Phi\left(-\frac{y+b-c-1/2s}{\sqrt{s}}\right)dy.
\end{aligned} \tag{A.3}
$$

Integrating I_1 by parts yields

$$I_1 = \frac{2}{\sqrt{2\pi s}} e^{-\frac{(y+b-c-1/2s)^2}{2s}}. \tag{A.4}$$

Substituting (A.4) into (A.3), we have

$$\begin{aligned} &E^{Q_T}[1\{M > b, M - X > b - c\}] \\ &= \int_b^\infty \frac{2e^{-y}}{\sqrt{2\pi s}} e^{-\frac{(y+b-c-1/2s)^2}{2s}} dy + \int_b^\infty e^{-y}\Phi\left(-\frac{y+b-c-1/2s}{\sqrt{s}}\right) dy, \end{aligned} \tag{A.5}$$

Integrating the second integral in (A.5) by parts gives

$$\begin{aligned} &E^{Q_T}[1\{M > b, M - X > b - c\}] \\ &= \int_b^\infty \frac{1}{\sqrt{2\pi s}} e^{b-c} e^{-\frac{(y+b-c+1/2s)^2}{2s}} dy + e^{-b}\Phi\left(-\frac{2b-c-1/2s}{\sqrt{s}}\right) \\ &= e^{b-c}\Phi\left(\frac{c-2b-1/2s}{\sqrt{s}}\right) + e^{-b}\Phi\left(\frac{c-2b+1/2s}{\sqrt{s}}\right). \end{aligned} \tag{A.6}$$

Similarly,

$$\begin{aligned} &E^{Q_T}[e^{X-M}1\{M > b, M - X > b - c\}] \\ &= \int_b^\infty \int_{-\infty}^{y-b+c} e^{x-y} \frac{1}{\sqrt{2\pi s}}\ e^{-\frac{(x+1/2s)^2+4y^2-4xy}{2s}}\ \frac{4y-2x}{s}\ dxdy \\ &= \int_b^\infty \int_{-\infty}^{y-b+c} \frac{4y-2x+s}{s\sqrt{2\pi s}}\ e^{-\frac{(x-1/2s-2y)^2}{2s}} dxdy \\ &\quad - \int_b^\infty \Phi\left(-\frac{y+b-c+1/2s}{\sqrt{s}}\right) dy \end{aligned} \tag{A.7}$$

We integrate the above two integrals in (A.7) by parts

$$\begin{aligned} &E^{Q_T}[e^{X-M}1\{M > b, M - X > b - c\}] \\ &= \int_b^\infty \frac{2}{\sqrt{2\pi s}}\ e^{-\frac{(y+b-c+1/2s)^2}{2s}} dy + b\Phi\left(\frac{c-2b-1/2s}{\sqrt{s}}\right) \\ &\quad - \int_b^\infty \frac{y}{\sqrt{2\pi s}}\ e^{-\frac{(y+b-c+1/2s)^2}{2s}}\ dy \\ &= (2+2b-c+1/2s)\Phi\left(\frac{c-2b-1/2s}{\sqrt{s}}\right) \\ &\quad - \int_b^\infty \frac{(y+b-c+1/2s)}{\sqrt{2\pi s}}\ e^{-\frac{(y+b-c+1/2s)^2}{2s}}\ dy \\ &= (2+2b-c+1/2s)\Phi\left(\frac{c-2b-1/2s}{\sqrt{s}}\right) - \sqrt{\frac{s}{2\pi}}\ e^{-\frac{(2b-c+1/2s)^2}{2s}}, \end{aligned} \tag{A.8}$$

where the last equality is obtained by integrating by parts.

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GRADED SEMIPRIME AND GRADED WEAKLY SEMIPRIME IDEALS[1]

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Abstract. Let G be a group and R be a commutative G-graded ring with nonzero unity 1. In this article, we define the graded semiprime ideals and the graded weakly semiprime ideals and we introduce several results concerning such ideals.

Keywords: graded semiprime ideals, graded weakly semiprime ideals, graded prime ideals, graded weakly prime ideals.

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1. Introduction and preliminaries

Recall from [2] that a proper ideal I of R ($I \neq R$) is said to be weakly prime if $a, b \in R$ and $0 \neq ab \in I$ implies $a \in I$ or $b \in I$ and recall from [5] that a proper ideal I of R is said to be weakly semiprime if $a \in R$ and $0 \neq a^2 \in I$ implies $a \in I$. The purpose of this paper is to study a generalization of weakly prime (semiprime) ideals of commutative rings to the content of commutative G-graded rings.

Let G be a group with identity e. A ring R is said to be G-graded ring if there exist additive subgroups R_g of R such that $R = \bigoplus_{g\in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. The elements of R_g are called homogeneous of degree g and R_e (the identity component of R) is a subring of R and $1 \in R_e$. For $x \in R$, x can be written uniquely as $\sum_{g\in G} x_g$ where x_g is the component of x in R_g. Also we write $h(R) = \bigcup_{g\in G} R_g$ and $supp(R, G) = \{g \in G : R_g \neq 0\}$. For more details, see [8].

Let R be a G-graded ring and I be an ideal of R. Then I is called G-graded ideal if $I = \bigoplus_{g\in G}(I \cap R_g)$, i.e., if $x \in I$ and $x = \sum_{g\in G} x_g$, then $x_g \in I$ for all $g \in G$. An ideal of a G-graded ring need not be G-graded. To see this, consider

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$R = \mathbf{Z}[i]$ and $G = \mathbf{Z}_2$. Then R is G-graded by $R_0 = \mathbf{Z}$ and $R_1 = i\mathbf{Z}$. Now, $I = \langle 1 + i\rangle$ is an ideal of R with $1 + i \in I$. If I is G-graded, then $1 \in I$, so $1 = a(1 + i)$ for some $a \in R$, i.e., $1 = (x + iy)(1 + i)$ for some $x, y \in \mathbf{Z}$. Thus $1 = x - y$ and $0 = x + y$, i.e., $2x = 1$ and hence $x = \frac{1}{2}$ a contradiction. So, I is not G-graded.

A proper graded ideal I of R is said to be graded prime if whenever $a, b \in h(R)$ such that $ab \in I$, then either $a \in I$ or $b \in I$. Also, a proper graded ideal I of R is said to be graded weakly prime if whenever $a, b \in h(R)$ such that $0 \neq ab \in I$, then either $a \in I$ or $b \in I$. Recently, various generalizations of graded prime (graded weakly prime) ideals are studied in [1], [3], [7], [9].

In this article, we define a proper graded ideal I of R to be graded semiprime if whenever $a \in h(R)$ such that $a^2 \in I$, then $a \in I$ and we define a proper graded ideal I of R to be graded weakly semiprime if whenever $a \in h(R)$ such that $0 \neq a^2 \in I$, then $a \in I$.

In this article, we give an analogous study to that is given in [6].

2. Graded semiprime ideals

In this section, we define the graded semiprime ideals and we introduce several properties of such ideal.

Definition 2.1. Let R be a graded ring and I be a proper graded ideal of R. Then I is said to be graded semiprime if whenever $a \in h(R)$ such that $a^2 \in I$, then $a \in I$.

Clearly, every graded prime ideal is a graded semiprime ideal. However, the next example shows that the converse need not be true in general.

Example 2.2. Consider $R = \mathbf{Z}[i]$ and $G = \mathbf{Z}_2$. Then R is G-graded by $R_0 = \mathbf{Z}$ and $R_1 = i\mathbf{Z}$. Consider the graded ideal $I = \langle 6\rangle \bigoplus \langle 0\rangle$. Let $(a, b)^2 \in I$. Then $(a^2, b^2) \in I$ and then 6 divides a^2 and $b = 0$. Since 2 and 3 divide 6, 2 and 3 divide a^2 and since 2 and 3 are primes, 2 and 3 divide a and since 2 and 3 are relatively prime, 6 divides a and hence $(a, b) \in I$. So, I is graded semiprime. However, I is not graded prime since $(2, 0)(3, 0) \in I$ but $(2, 0) \notin I$ and $(3, 0) \notin I$.

Theorem 2.3. *If I is a graded semiprime ideal of a graded ring R, then I_e is a semiprime ideal of R_e.*

Proof. Let $a \in R_e$ such that $a^2 \in I_e$. Since $R_e \subseteq h(R)$ and $I_e \subseteq I$, $a \in h(R)$ such that $a^2 \in I$ and since I is graded semiprime, $a \in I$ and then $a \in I \bigcap R_e = I_e$. Hence I_e is a semiprime ideal of R_e. ∎

Let R be a graded ring, I be a graded ideal of R and $a \in h(R)$. Then one can prove that $(I : a) = \{r \in R : ra \in I\}$ is an ideal of R. Clearly, if $a \in I$, then $(I : a) = R$. Moreover, we introduce the following result. Before that, we need to remember that $a \in R$ is said to be idempotent if $a^2 = a$.

Theorem 2.4. *Let R be a graded ring, I be a graded semiprime ideal of R and $a \in h(R)$. If a is an idempotent and $a \notin I$, then $(I : a)$ is a graded semiprime ideal of R.*

Proof. Since $a \notin I$, $(I : a)$ is a proper ideal of R. Firstly, we prove that $(I : a)$ is a graded ideal of R. Let $r \in (I : a)$. Then $r \in R$ such that $ra \in I$. Since R is graded,

$$r = \sum_{g \in G} r_g, \text{ where } r_g \in R_g$$

and then

$$\sum_{g \in G} r_g a = \left(\sum_{g \in G} r_g \right) a = ra \in I.$$

Since I is graded, $r_g a \in I$ for all $g \in G$ and then $r_g \in (I : a)$ for all $g \in G$ and hence $(I : a)$ is a graded ideal of R. Now, let $r \in h(R)$ such that $r^2 \in (I : a)$. Then $r^2 a \in I$ and since a is idempotent, $(ra)^2 = r^2 a^2 = r^2 a \in I$. Since $r, a \in h(R)$, $r \in R_g$ and $a \in R_h$ for some $g, h \in G$ and then $ra \in R_g R_h \subseteq R_{gh} \subseteq h(R)$ such that $(ra)^2 \in I$. Since I is graded semiprime, $ra \in I$ and then $r \in (I : a)$. Hence, $(I : a)$ is a graded semiprime ideal of R. ∎

Theorem 2.5. *Let R be a graded ring and J be a graded ideal of R. If I is a graded semiprime ideal of R such that $J^2 \subseteq I$, then $J \subseteq I$.*

Proof. Let $a \in J$. Then since J is graded, $a = \sum_{g \in G} a_g$ with $a_g \in J$ for all $g \in G$ and then $a_g \in h(R)$ such that $a_g^2 \in J^2 \subseteq I$ for all $g \in G$. Since I is graded semiprime, $a_g \in I$ for all $g \in G$ and then $a = \sum_{g \in G} a_g \in I$. Hence, $J \subseteq I$. ∎

S.E. Atani and F. Farzalipour in [4] define a proper graded ideal I of a graded ring R to be graded secondary if for every $r \in h(R)$, either $rI = I$ or $r^n I = 0$ for some integer n. We introduce the following:

Definition 2.6. *Let R be a graded ring and I be a proper graded ideal of R. Then I is said to be graded 2-secondary if for every idempotent $r \in h(R)$, either $rI = I$ or $rI = 0$.*

Theorem 2.7. *Let I be a graded 2-secondary ideal of a graded ring R and J be a graded semiprime subideal of I. Then J is a graded 2-secondary ideal of R.*

Proof. Let $r \in h(R)$ be an idempotent. Since I is graded 2-secondary, either $rI = I$ or $rI = 0$. If $rI = 0$, then $rJ \subseteq rI = 0$ and then $rJ = 0$. Suppose $rI = I$. We show that $rJ = J$. Clearly, $rJ \subseteq J$. Let $a \in J$. Since J is graded, $a = \sum_{g \in G} a_g$ with $a_g \in J$ for all $g \in G$ and then $a_g \in I = rI$ for all $g \in G$. So, for every $g \in G$, $a_g = rx_h$ for some $x_h \in I = rI$ and then $x_h = ry_k$ for some $y_k \in I$. So, $a_g = r^2 y_k = ry_k$ and then $(ry_k)^2 = a_g^2 \in J^2 \subseteq J$. Since J is graded

semiprime, $ry_k \in J$ that is $x_h \in J$ and then $a_g = rx_h \in rJ$ for all $g \in G$ and hence $a = \sum_{g\in G} a_g \in rJ$. Thus $rJ = J$ and so J is a graded 2-secondary ideal of R. ∎

The proof of the next result is straightforward by Theorem 2.7.

Corollary 2.8. *Let I be a graded 2-secondary ideal of a graded ring R and J be a graded semiprime ideal of R. Then $I \bigcap J$ is a graded 2-secondary ideal of R.*

If R is a graded ring and I is an ideal of R, then R/I is a graded ring by $(R/I)_g = (R_g + I)/I$. $a \in R$ is said to be simple nilpotent if $a^2 = 0$.

Theorem 2.9. *Let I be a graded ideal of a graded ring R. Then I is a graded semiprime ideal of R if and only if R/I has no nonzero homogeneous simple nilpotent elements.*

Proof. Suppose I is a graded semiprime ideal of R. Let $a + I \in h(R/I)$ be a simple nilpotent. Then $(a + I)^2 = 0 + I$ and then $a^2 + I = 0 + I$ that is $a^2 \in I$. Since $a + I \in h(R/I)$, $a + I \in (R/I)_g = (R_g + I)/I$ for some $g \in G$ and then $a \in R_g \subseteq h(R)$. So, $a \in h(R)$ such that $a^2 \in I$ and since I is graded semiprime, $a \in I$ that is $a + I = 0 + I$. Hence, R/I has no nonzero homogeneous simple nilpotent elements. Conversely, let $a \in h(R)$ such that $a^2 \in I$. Then $(a + I)^2 = a^2 + I = 0 + I$ and by assumption, $a + I = 0 + I$ that is $a \in I$. Hence, I is a graded semiprime ideal of R. ∎

Theorem 2.10. *Let $I \subseteq J$ be proper graded ideals of a graded ring R. Then J is a graded semiprime ideal of R if and only if J/I is a graded semiprime ideal of R/I.*

Proof. Suppose J is a graded semiprime ideal of R. Firstly, we prove that J/I is graded. Let $r + I \in J/I$. Then $r + I \in R/I$ and since R/I is graded, $r+I = \sum_{g\in G}(r+I)_g$ where $(r+I)_g \in (R/I)_g = (R_g+I)/I$ and then $\left(\sum_{g\in G} r_g\right)+I = \sum_{g\in G}(r_g + I) = \sum_{g\in G}(r + I)_g = r + I \in J/I$ and hence $\sum_{g\in G} r_g \in J$. Since J is graded, $r_g \in J$ for all $g \in G$ and then $(r + I)_g = r_g + I \in J/I$ for all $g \in G$ and hence J/I is graded ideal of R/I. Now, let $a + I \in h(R/I)$ such that $(a + I)^2 \in J/I$. Then $a \in h(R)$ such that $a^2 \in J$ and since J is graded semiprime, $a \in J$ and then $a + I \in J/I$. Hence, J/I is a graded semiprime ideal of R/I. Conversely, let $a \in h(R)$ such that $a^2 \in J$. Then $a+I \in h(R/I)$ such that $(a+I)^2 = a^2+I \in J/I$ and since J/I is graded semiprime, $a+I \in J/I$ that is $a \in J$. Hence, J is a graded semiprime ideal of R. ∎

If R and R' are two G-graded rings, then $R \times R'$ is a G-graded ring by $(R \times R')_g = R_g \times R'_g$.

Theorem 2.11. *Let R and R' be two G-graded rings and I be a graded ideal of R. Then I is a graded semiprime ideal of R if and only if $I \times R'$ is a graded semiprime ideal of $R \times R'$.*

Proof. Suppose I is a graded semiprime ideal of R. Firstly, we prove that $I \times R'$ is graded. Let $(x, y) \in I \times R'$. Then $(x, y) \in R \times R'$ and since $R \times R'$ is graded, $(x, y) = \sum_{g \in G} (x, y)_g$ where $(x, y)_g \in (R \times R')_g = R_g \times R'_g$ and then $\left(\sum_{g \in G} x_g, \sum_{g \in G} y_g\right) = \sum_{g \in G} (x_g, y_g) = \sum_{g \in G} (x, y)_g = (x, y) \in I \times R'$ and hence $\sum_{g \in G} x_g \in I$. Since I is graded, $x_g \in I$ for all $g \in G$ and then $(x, y)_g = (x_g, y_g) \in I \times R'$ for all $g \in G$ and hence $I \times R'$ is a graded ideal of $R \times R'$. Now, let $(a, b) \in h(R \times R')$ such that $(a, b)^2 \in I \times R'$. Then $a \in h(R)$ such that $a^2 \in I$ and since I is graded semiprime, $a \in I$ and then $(a, b) \in I \times R'$. Hence, $I \times R'$ is a graded semiprime ideal of $R \times R'$. Conversely, let $a \in h(R)$ such that $a^2 \in I$. Then $(a, 1') \in h(R) \times R'_e \subseteq h(R) \times h(R') = h(R \times R')$ such that $(a, 1')^2 \in I \times R'$ and since $I \times R'$ is graded semiprime, $(a, 1') \in I \times R'$ that is $a \in I$. Hence, I is a graded semiprime ideal of R. ∎

Similarly, we can prove the following:

Theorem 2.12. *Let R and R' be two G-graded rings and J be a graded ideal of R'. Then J is a graded semiprime ideal of R' if and only if $R \times J$ is a graded semiprime ideal of $R \times R'$.*

3. Graded weakly semiprime ideals

In this section, we define the graded weakly semiprime ideals and we introduce several properties of such ideal.

Definition 3.1. Let R be a graded ring and I be a proper graded ideal of R. Then I is said to be graded weakly semiprime if whenever $a \in h(R)$ such that $0 \neq a^2 \in I$, then $a \in I$.

Clearly, every graded semiprime ideal is a graded weakly semiprime ideal. However, the next example shows that the converse need not be true in general.

Example 3.2. Consider $R = M_2(K)$ (the ring of all 2×2 matrices with entries from a field K) and $G = \mathbf{Z_4}$. Then R is G-graded by

$$R_0 = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}, R_2 = \begin{pmatrix} 0 & K \\ K & 0 \end{pmatrix} \text{ and } R_1 = R_3 = 0.$$

Consider $I = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$. Then I is a graded weakly semiprime ideal of R. However, I is not graded semiprime since $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in R_2 \subseteq h(R)$ such that $A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in I$ but $A \notin I$.

Theorem 3.3. *If I is a graded weakly semiprime ideal of a graded ring R, then I_e is a weakly semiprime ideal of R_e.*

Proof. Let $a \in R_e$ such that $0 \neq a^2 \in I_e$. Since $R_e \subseteq h(R)$ and $I_e \subseteq I$, $a \in h(R)$ such that $0 \neq a^2 \in I$ and since I is graded semiprime, $a \in I$ and then $a \in I \bigcap R_e = I_e$. Hence I_e is a semiprime ideal of R_e. ∎

Let R be a graded ring, I be a graded ideal of R and $a \in h(R)$. Then we define the graded ideal $(I :_{h(R)} a) = \{r \in h(R) : ra \in I\}$.

Theorem 3.4. *Let I be a graded ideal of a graded ring R. If $(I :_{h(R)} a^2) = (0 :_{h(R)} a^2) \bigcup (I :_{h(R)} a)$ for all $a \in h(R)$, then I is a graded weakly semiprime ideal of R.*

Proof. Let $a \in h(R)$ such that $0 \neq a^2 \in I$. Then $1 \in R_e \subseteq h(R)$ such that $1.a^2 \in I$ that is $1 \in (I :_{h(R)} a^2)$. By assumption and since $a^2 \neq 0$, $1 \in (I :_{h(R)} a)$ and then $a = 1.a \in I$. Hence, I is a graded weakly semiprime ideal of R. ∎

Theorem 3.5. *Let $I \subseteq J$ be proper graded ideals of a graded ring R. If J is a graded weakly semiprime ideal of R, then J/I is a graded weakly semiprime ideal of R/I.*

Proof. By the proof of Theorem 2.10, J/I is a graded ideal of R/I. Let $a + I \in h(R/I)$ such that $0 \neq (a + I)^2 \in J/I$. Then $a \in h(R)$ such that $0 \neq a^2 \in J$ and since J is graded weakly semiprime, $a \in J$ and then $a + I \in J/I$. Hence, J/I is a graded weakly semiprime ideal of R/I. ∎

Theorem 3.6. *Let $I \subseteq J$ be proper graded ideals of a graded ring R. If I is a graded weakly semiprime ideal of R and J/I is a graded weakly semiprime ideal of R/I, then J is a graded weakly semiprime ideal of R.*

Proof. Let $a \in h(R)$ such that $0 \neq a^2 \in J$. Then $a + I \in h(R/I)$ such that $(a + I)^2 \in J/I$. If $a^2 \in I$, then since I is graded weakly semiprime, $a \in I \subseteq J$ and then it is done. Suppose $a^2 \notin I$. Then $0 \neq (a + I)^2 \in J/I$ and since J/I is graded weakly semiprime, $a + I \in J/I$ and then $a \in J$. Hence, J is a graded weakly semiprime ideal of R. ∎

Theorem 3.7. *Let I be a graded 2-secondary ideal of a graded ring R and J be a graded weakly semiprime subideal of I. If J has no nonzero homogeneous simple nilpotent elements, then J is a graded 2-secondary ideal of R.*

Proof. Let $r \in h(R)$ be an idempotent. Since I is graded 2-secondary, either $rI = I$ or $rI = 0$. If $rI = 0$, then $rJ \subseteq rI = 0$ and then $rJ = 0$. Suppose $rI = I$. We show that $rJ = J$. Clearly, $rJ \subseteq J$. Let $a \in J$. If $a = 0$, then $a = r.0 \in rJ$. Suppose $a \neq 0$. Since J is graded, $a = \sum_{g \in H} a_g$ with $a_g \in J$ for all $g \in H$ where $H = supp(R, G)$ (i.e., $a_g \neq 0$) and then $a_g \in I = rI$ for all $g \in H$. So, for every $g \in H$, $a_g = rx_h$ for some $0 \neq x_h \in I = rI$ and then $x_h = ry_k$ for some $0 \neq y_k \in I$. So, $a_g = r^2 y_k = ry_k$ and then $(ry_k)^2 = a_g^2 \in J^2 \subseteq J$. If

$(ry_k)^2 = 0$, then $a_g^2 = 0$, i.e., a_g is a homogeneous simple nilpotent element in J and then by assumption, $a_g = 0$ a contradiction. So, $(ry_k)^2 \neq 0$. Since J is graded weakly semiprime, $ry_k \in J$ that is $x_h \in J$ and then $a_g = rx_h \in rJ$ for all $g \in H$ and hence $a = \sum_{g \in H} a_g \in rJ$. Thus $rJ = J$ and so J is a graded 2-secondary ideal of R. ∎

The proof of the next result is straightforward by Theorem 3.7.

Corollary 3.8. *Let I be a graded 2-secondary ideal of a graded ring R and J be a graded weakly semiprime ideal of R. If J has no nonzero homogeneous simple nilpotent elements, then $I \bigcap J$ is a graded 2-secondary ideal of R.*

Theorem 3.9. *Let R and R' be two G-graded rings and I be a graded ideal of R. If $I \times R'$ is a graded weakly semiprime ideal of $R \times R'$, then I is a graded weakly semiprime ideal of R.*

Proof. Let $a \in h(R)$ such that $0 \neq a^2 \in I$. Then $(a, 1') \in h(R) \times R'_e \subseteq h(R) \times h(R') = h(R \times R')$ such that $0 \neq (a, 1')^2 \in I \times R'$ and since $I \times R'$ is graded semiprime, $(a, 1') \in I \times R'$ that is $a \in I$. Hence, I is a graded semiprime ideal of R. ∎

Similarly, we can prove the following:

Theorem 3.10. *Let R and R' be two G-graded rings and J be a graded ideal of R'. If $R \times J$ is a graded weakly semiprime ideal of $R \times R'$, then J is a graded weakly semiprime ideal of R'.*

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DIFFERENTIAL SANDWICH THEOREMS FOR p-VALENT FUNCTIONS ASSOCIATED WITH A CERTAIN GENERALIZED DIFFERENTIAL OPERATOR AND INTEGRAL OPERATOR

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Abstract. The purpose of this paper is to derive some subordination and superordination results for functions of the form $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k$ which are $p-$valent in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ by using certain differential operator $A^n_{\lambda,p}(\alpha, \beta, \mu)f(z)$ and integral operator $F^m_p(\rho, \vartheta)f(z)$. Some special cases are also considered.

Keywords: analytic functions, differential subordination, differential superordination, sandwich theorems.

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1. Introduction

Let $\mathcal{H}$ denote the class of functions which are *analytic* in the open unit disk

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\},$$

and let $\mathcal{H}[a, p]$ denote the subclass of the functions $f \in \mathcal{H}$ of the form:

$$f(z) = a + a_p z^p + a_{p+1} z^{p+1} + \cdots \quad (a \in \mathbb{C};\ p \in \mathbb{N} := \{1, 2, ...\}). \tag{1.1}$$

For simplicity, $\mathcal{H}[a] = \mathcal{H}[a, 1]$. Also, let $\mathcal{A}(p)$ denote the subclass of $\mathcal{H}$ consisting of functions of the form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \qquad (a_k \geq 0; p \in \mathbb{N} := \{1, 2, ...\}), \tag{1.2}$$

which are $p-$valent in $\mathbb{U}$.

Given two functions f, $g \in \mathcal{H}$. The function $f(z)$ is said to be *subordination* to $g(z)$ in $\mathbb{U}$, written $f(z) \prec g(z)$, if there exists a function $h(z)$, analytic in $\mathbb{U}$, with $h(0) = 0$ and $|h(z)| < 1$ for all $z \in \mathbb{U}$, such that $f(z) = g(h(z))$ for all $z \in \mathbb{U}$. Furthermore, if the function g is *univalent* in $\mathbb{U}$, then we have the following equivalence (see [7] and [9]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

Let $\phi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ and $\psi(z)$ be univalent in $\mathbb{U}$. If $p(z)$ is analytic in $\mathbb{U}$ and satisfies the second-order differential subordination:

$$\phi\left(p(z), zp'(z), z^2p''(z); z\right) \prec \psi(z), \tag{1.3}$$

then $p(z)$ is a solution of the differential subordination (1.3). The univalent function $q(z)$ is called a *dominant* of the solutions of the differential subordination (1.3) if $p(z) \prec q(z)$ for all $p(z)$ satisfying (1.3). A univalent dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants of (1.3) is called the *best dominant*. If $p(z)$ is univalent in $\mathbb{U}$ and satisfies the second-order differential superordination:

$$\psi(z) \prec \phi\left(p(z), zp'(z), z^2p''(z); z\right), \tag{1.4}$$

then $p(z)$ is a solution of the differential supordination (1.4). A univalent function $q(z)$ is called a *subordinant* of the solutions of the differential superordination (1.4) if $q(z) \prec p(z)$ for all $p(z)$ satisfying (1.4). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants of (1.4) is called the *best subordinant*. Using the results of Miller and Mocanu [10], Al-Hawary, Amourah and Darus [1], Aljarah and Darus [3], Bulboaca [6] considered certain classes of first-order differential superordinations as well as superordination-preserving integral operators [7]. Ali et al. [2] have used the results of Bulboaca [6] to obtain sufficient conditions for normalized analytic functions $f \in \mathcal{A}(1)$ to satisfy:

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z),$$

where q_1 and q_2 are given univalent normalized functions in $\mathbb{U}$ with $q_1(0)=q_2(0)=1$.

Also, Tuneski [14] obtained a sufficient condition for starlikeness of $f \in \mathcal{A}(1)$ in terms of the quantity $\frac{f''(z)f(z)}{(f'(z))^2}$.

Recently, Shanmugam et al. [12, 11] and [13] obtained sufficient conditions for the normalized analytic function $f \in \mathcal{A}(1)$ to satisfy:

$$q_1(z) \prec \frac{f(z)}{zf'(z)} \prec q_2(z),$$

and

$$q_1(z) \prec \frac{z^2 f'(z)}{f^2(z)} \prec q_2(z).$$

Shanmugam et al. [11] also obtained the so-called sandwich results for certain classes of analytic functions.

For the function $f \in \mathcal{A}(p)$, Faisal and Darus ([8]) defined the following differential operator:

$$A^0_{\lambda,p}(\alpha,\beta,\mu)f(z) = f(z),$$

$$A^1_{\lambda,p}(\alpha,\beta,\mu)f(z) = \left(\frac{\alpha+\beta-p(\mu+\lambda)}{\alpha+\beta}\right) f(z) + \left(\frac{\mu+\lambda}{\alpha+\beta}\right) z f'(z),$$

and for $n = 2, 3, ...,$

$$\begin{aligned} A^n_{\lambda,p}(\alpha,\beta,\mu)f(z) &= A(A^{n-1}_{\lambda,p}(\alpha,\beta,\mu)f(z)), \\ &= z^p + \sum_{k=p+1}^{\infty} \left[\frac{\alpha+(\mu+\lambda)(k-p)+\beta}{\alpha+\beta}\right]^n a_k z^k, \end{aligned} \tag{1.5}$$

for $f \in \mathcal{A}(p)$, $\alpha, \beta, \mu, \lambda \geq 0, \alpha+\beta \neq 0$ and $\mu+\lambda \neq 0$, $p \in \mathbb{N}$ and $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. It is easily verified from (1.5) that

$$\begin{aligned} \frac{\mu+\lambda}{\alpha+\beta} z (A^n_{\lambda,p}(\alpha,\beta,\mu)f(z))' &= A^{n+1}_{\lambda,p}(\alpha,\beta,\mu)f(z) \\ &\quad - \left(1 - \frac{p}{\alpha+\beta}(\mu+\lambda)\right) A^n_{\lambda,p}(\alpha,\beta,\mu)f(z). \end{aligned} \tag{1.6}$$

Also, for the function $f \in \mathcal{A}(p)$, Aouf et al. ([5]) defined the integral operator:

$$F^m_p(\rho,\vartheta)f(z) = z^p + \sum_{k=p+1}^{\infty} \left[\frac{p+\vartheta}{p+\vartheta+\rho(k-p)}\right]^m a_k z^k, \tag{1.7}$$

for $\rho > 0$, $\vartheta \geq 0$, $p \in \mathbb{N}$ and $m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. It is easily verified from (1.7) that

$$\begin{aligned} \rho z (F^{m+1}_p(\rho,\vartheta)f(z))' &= (\vartheta+p)(F^m_p(\rho,\vartheta)f(z)) \\ &\quad - [\vartheta+p(1-\rho)](F^{m+1}_p(\rho,\vartheta)f(z)). \end{aligned} \tag{1.8}$$

In this paper, we derive several subordination results, superordination results and sandwich results involving the differential operator $A^n_{\lambda,p}(\alpha,\beta,\mu)f(z)$ given by (1.5) and the integral operator $F^m_p(\rho,\vartheta)f(z)$ given by (1.7).

2. Definitions and preliminaries

In order to prove our subordinations and superordinations results, we need the following definition and lemmas.

Definition 2.1 [10] Denote by Q, the set of all functions f that are analytic and injective on $\overline{\mathbb{U}}\backslash E(f)$, where

$$E(f) = \Big\{\zeta \in \partial\mathbb{U} \lim_{z\to\zeta} f(z) = \infty\Big\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{U}\backslash E(f)$.

Lemma 2.2 [10] *Let $q(z)$ be univalent in $\mathbb{U}$ and let θ and φ be analytic in a domain D containing $q(\mathbb{U})$ with $\varphi(w) \neq 0$ when $w \in q(\mathbb{U})$. Set*

$$\psi(z) = zq'(z)\varphi(q(z)) \quad \text{and} \quad h(z) = \theta(q(z)) + \psi(z).$$

Suppose that

(i) *ψ is starlike univalent in $\mathbb{U}$,*

(ii) $\operatorname{Re}\left\{\dfrac{zh'(z)}{\psi(z)}\right\} > 0$ *for $z \in \mathbb{U}$.*

If $p(z)$ is a analytic in $\mathbb{U}$ with $p(0) = q(0)$, $p(\mathbb{U}) \subset D$ and

$$\theta(p(z)) + zp'(z)\varphi(p(z)) \prec \theta(q(z)) + zq'(z)\varphi(q(z)), \tag{2.1}$$

then $p(z) \prec q(z)$ and $q(z)$ is the best dominant of (2.1).

Lemma 2.3 [6] *Let $q(z)$ be convex univalent in $\mathbb{U}$ and let ϑ and ϕ be analytic in a domain D containing $q(\mathbb{U})$. Suppose that*

(i) $Re\left\{\dfrac{\vartheta'(q(z))}{\phi(q(z))}\right\} > 0$ *for $z \in \mathbb{U}$,*

(ii) *$\Psi(z) = zq'(z)\phi(q(z))$ is starlike univalent in $\mathbb{U}$.*

If $p(z) \in \mathcal{H}[q(0), 1] \cap Q$, with $p(\mathbb{U}) \subseteq D$, and $\vartheta(p(z)) + zp'(z)\phi(p(z))$ is univalent in $\mathbb{U}$ and

$$\vartheta(q(z)) + zq'(z)\phi(q(z)) \prec \vartheta(p(z)) + zp'(z)\phi(p(z)), \tag{2.2}$$

then $q(z) \prec p(z)$ and $q(z)$ is the best subordinant of (2.2).

3. Subordinations and superordinations results for $A^n_{\lambda,p}(\alpha, \beta, \mu)f(z)$

Unless otherwise mentioned, we assume throughout this paper that $\epsilon, \eta \in \mathbb{C}$. We begin with the following result involving differential subordination between analytic functions.

Theorem 3.1 *Let $q(z)$ be univalent in $\mathbb{U}$ with $q(0)=1$. Further, assume that*

$$\operatorname{Re}\left\{\frac{2(\delta+\kappa)q(z)}{\delta}+1+\frac{zq''(z)}{q'(z)}\right\}>0.\ (\delta\in\mathbb{C}^*=\mathbb{C}-\{0\},\kappa\geq 0) \tag{3.1}$$

If $f\in\mathcal{A}(p)$ satisfies the following subordination condition:

$$\zeta(n,\lambda,p,\beta,\delta,\alpha;z)\prec\delta zq'(z)+(\delta+\kappa)\left(q(z)\right)^2, \tag{3.2}$$

where

$$\begin{aligned}&\zeta(n,\lambda,p,\beta,\delta,\alpha;z)\\&=\frac{\delta(\alpha+\beta)}{(\mu+\lambda)}\left[\left(A^{n+1}_{\lambda,p}(\alpha,\beta,\mu)f(z)\right)^{1-\epsilon}-\epsilon\frac{\left(A^{n+2}_{\lambda,p}(\alpha,\beta,\mu)f(z)\right)\left(A^{n}_{\lambda,p}(\alpha,\beta,\mu)f(z)\right)}{\left(A^{n+1}_{\lambda,p}(\alpha,\beta,\mu)f(z)\right)^{1+\epsilon}}\right]\\&+\delta\left(\epsilon-1\right)\left(\frac{\alpha+\beta}{\mu+\lambda}-p\right)\frac{A^{n}_{\lambda,p}(\alpha,\beta,\mu)f(z)}{\left(A^{n+1}_{\lambda,p}(\alpha,\beta,\mu)f(z)\right)^{\epsilon}}+(\delta+\kappa)\frac{\left(A^{n}_{\lambda,p}(\alpha,\beta,\mu)f(z)\right)^2}{\left(A^{n+1}_{\lambda,p}(\alpha,\beta,\mu)f(z)\right)^{2\epsilon}},\end{aligned}$$

then

$$\frac{A^{n}_{\lambda,p}(\alpha,\beta,\mu)f(z)}{\left[A^{n+1}_{\lambda,p}(\alpha,\beta,\mu)f(z)\right]^{\epsilon}}\prec q(z)$$

and $q(z)$ is the best dominant.

Proof. Define a function $p(z)$ by

$$p(z)=\frac{A^{n}_{\lambda,p}(\alpha,\beta,\mu)f(z)}{\left[A^{n+1}_{\lambda,p}(\alpha,\beta,\mu)f(z)\right]^{\epsilon}},\quad(z\in\mathbb{U}). \tag{3.3}$$

Then the function $p(z)$ is analytic in $\mathbb{U}$ and $p(0)=1$. Differentiating (3.3) logarithmically with respect to z and using the identity (1.6) in the resulting equation, we get

$$\begin{aligned}\frac{p'(z)}{p(z)}&=\frac{A^{n+1}_{\lambda,p}(\alpha,\beta,\mu)f(z)-(1-\frac{p}{\alpha+\beta}(\mu+\lambda))A^{n}_{\lambda,p}(\alpha,\beta,\mu)f(z)}{\frac{\mu+\lambda}{\alpha+\beta}zA^{n}_{\lambda,p}(\alpha,\beta,\mu)f(z)}\\&\quad-\epsilon\frac{A^{n+2}_{\lambda,p}(\alpha,\beta,\mu)f(z)-(1-\frac{p}{\alpha+\beta}(\mu+\lambda))A^{n+1}_{\lambda,p}(\alpha,\beta,\mu)f(z)}{\frac{\mu+\lambda}{\alpha+\beta}zA^{n+1}_{\lambda,p}(\alpha,\beta,\mu)f(z)},\\\Rightarrow p'(z)&=p(z)\frac{\alpha+\beta}{\mu+\lambda}\left[\frac{A^{n+1}_{\lambda,p}(\alpha,\beta,\mu)f(z)}{zA^{n}_{\lambda,p}(\alpha,\beta,\mu)f(z)}-\frac{(1-\frac{p}{\alpha+\beta}(\mu+\lambda))}{z}\right.\\&\quad\left.-\epsilon\frac{A^{n+2}_{\lambda,p}(\alpha,\beta,\mu)f(z)}{zA^{n+1}_{\lambda,p}(\alpha,\beta,\mu)f(z)}+\epsilon\frac{(1-\frac{p}{\alpha+\beta}(\mu+\lambda))}{z}\right],\\&=\frac{(\alpha+\beta)}{z(\mu+\lambda)}\left[(A^{n+1}_{\lambda,p}(\alpha,\beta,\mu)f(z))^{1-\epsilon}-\epsilon\frac{(A^{n+2}_{\lambda,p}(\alpha,\beta,\mu)f(z))(A^{n}_{\lambda,p}(\alpha,\beta,\mu)f(z))}{(A^{n+1}_{\lambda,p}(\alpha,\beta,\mu)f(z))^{1+\epsilon}}\right]\\&\quad+\frac{(\epsilon-1)}{z}(\frac{\alpha+\beta}{\mu+\lambda}-p)\frac{A^{n}_{\lambda,p}(\alpha,\beta,\mu)f(z)}{(A^{n+1}_{\lambda,p}(\alpha,\beta,\mu)f(z))^{\epsilon}}.\end{aligned}$$

$$\Rightarrow \delta z p'(z) + (\delta+\kappa)\left(p(z)\right)^2$$

$$= \frac{\delta(\alpha+\beta)}{(\mu+\lambda)}\left[\left(A^{n+1}_{\lambda,p}(\alpha,\beta,\mu)f(z)\right)^{1-\epsilon} - \epsilon\frac{\left(A^{n+2}_{\lambda,p}(\alpha,\beta,\mu)f(z)\right)\left(A^{n}_{\lambda,p}(\alpha,\beta,\mu)f(z)\right)}{\left(A^{n+1}_{\lambda,p}(\alpha,\beta,\mu)f(z)\right)^{1+\epsilon}}\right]$$

$$+ \delta(\epsilon-1)\left(\frac{\alpha+\beta}{\mu+\lambda}-p\right)\frac{A^{n}_{\lambda,p}(\alpha,\beta,\mu)f(z)}{\left(A^{n+1}_{\lambda,p}(\alpha,\beta,\mu)f(z)\right)^{\epsilon}} + (\delta+\kappa)\frac{\left(A^{n}_{\lambda,p}(\alpha,\beta,\mu)f(z)\right)^2}{\left(A^{n+1}_{\lambda,p}(\alpha,\beta,\mu)f(z)\right)^{2\epsilon}}.$$

Using (3.2), we obtain

$$\delta z p'(z) + (\delta+\kappa)\left(p(z)\right)^2 \prec \delta z q'(z) + (\delta+\kappa)\left(q(z)\right)^2.$$

Therefore, now Theorem 3.1 follows by applying Lemma 2.2 by setting

$$\theta(w) = (\delta+\kappa)w^2 \quad \text{and} \quad \varphi(w) = \delta. \qquad \blacksquare$$

Corollary 3.2 *Let* $q(z) = \dfrac{1+Az}{1+Bz}$ $(-1 \le B < A \le 1)$ *in Theorem* 3.1. *Further, assume that* (3.1) *holds.*

If $f \in \mathcal{A}(p)$ *satisfies the following subordination condition:*

$$\zeta(n,\lambda,p,\beta,\delta,\alpha;z) \prec \frac{\delta(A-B)z}{(1+Bz)^2} + (\delta+\kappa)\left(\frac{1+Az}{1+Bz}\right)^2,$$

then

$$\frac{A^{n}_{\lambda,p}(\alpha,\beta,\mu)f(z)}{\left[A^{n+1}_{\lambda,p}(\alpha,\beta,\mu)f(z)\right]^{\epsilon}} \prec \frac{1+Az}{1+Bz},$$

and the function $\dfrac{1+Az}{1+Bz}$ *is the best dominant.*

Also, let $q(z) = \dfrac{1+z}{1-z}$. *Then for* $f \in \mathcal{A}(p)$ *we have*

$$\zeta(n,\lambda,p,\beta,\delta,\alpha;z) \prec \frac{2\delta z}{(1-z)^2} + (\delta+\kappa)\left(\frac{1+z}{1-z}\right)^2,$$

then

$$\frac{A^{n}_{\lambda,p}(\alpha,\beta,\mu)f(z)}{\left[A^{n+1}_{\lambda,p}(\alpha,\beta,\mu)f(z)\right]^{\epsilon}} \prec \frac{1+z}{1-z},$$

and the function $\dfrac{1+z}{1-z}$ *is the best dominant.*

Finally, by taking $q(z) = \left(\dfrac{1+z}{1-z}\right)^{\mu}$, $(0 < \mu \le 1)$, *then for* $f \in \mathcal{A}(p)$ *we have,*

$$\zeta(n,\lambda,p,\beta,\delta,\alpha;z) \prec \frac{2\delta\mu z}{(1-z)^2}\left(\frac{1+z}{1-z}\right)^{\mu-1} + (\delta+\kappa)\left(\frac{1+z}{1-z}\right)^{2\mu},$$

then

$$\frac{A^n_{\lambda,p}(\alpha,\beta,\mu)f(z)}{\left[A^{n+1}_{\lambda,p}(\alpha,\beta,\mu)f(z)\right]^{\epsilon}} \prec \left(\frac{1+z}{1-z}\right)^{\mu},$$

and the function $\left(\frac{1+z}{1-z}\right)^{\mu}$ *is the best dominant.*

Next, by appealing to Lemma 2.3, we prove the following.

Theorem 3.3 *Let $q(z)$ be convex univalent in $\mathbb{U}$ with $q(0)=1$. Assume that*

$$\operatorname{Re}\left\{\frac{2(\delta+\kappa)q(z)q'(z)}{\delta}\right\} > 0. \tag{3.4}$$

Let $f \in \mathcal{A}(p)$ such that $\frac{A^n_{\lambda,p}(\alpha,\beta,\mu)f(z)}{\left[A^{n+1}_{\lambda,p}(\alpha,\beta,\mu)f(z)\right]^{\epsilon}} \in \mathcal{H}[q(0),1]\cap Q$, $\zeta(n,\lambda,p,\beta,\delta,\alpha;z)$ is univalent in $\mathbb{U}$ and the following superordination condition

$$(\delta+\kappa)\left(q(z)\right)^2 + \delta z q'(z) \prec \zeta(n,\lambda,p,\beta,\delta,\alpha;z), \tag{3.5}$$

holds, then

$$q(z) \prec \frac{A^n_{\lambda,p}(\alpha,\beta,\mu)f(z)}{\left[A^{n+1}_{\lambda,p}(\alpha,\beta,\mu)f(z)\right]^{\epsilon}}, \tag{3.6}$$

and $q(z)$ is the best subordinant.

Proof. Let the function $p(z)$ be defined by

$$p(z) = \frac{A^n_{\lambda,p}(\alpha,\beta,\mu)f(z)}{\left[A^{n+1}_{\lambda,p}(\alpha,\beta,\mu)f(z)\right]^{\epsilon}}.$$

Then, from the assumption of Theorem 3.3, the function $p(z)$ is analytic in $\mathbb{U}$ and (3.4) holds. Hence, the subordination (3.5) is equivalent to

$$(\delta+\kappa)\left(q(z)\right)^2 + \delta z q'(z) \prec (\delta+\kappa)\left(p(z)\right)^2 + \delta z p'(z).$$

The assertion (3.6) of Theorem 3.3 now follows by an application of Lemma 2.3. ∎

Corollary 3.4 *Let $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \le B < A \le 1)$ in Theorem 3.3. Further, assume that (3.4) holds.*

If $f \in \mathcal{A}(p)$ such that $\frac{A^n_{\lambda,p}(\alpha,\beta,\mu)f(z)}{\left[A^{n+1}_{\lambda,p}(\alpha,\beta,\mu)f(z)\right]^{\epsilon}} \in \mathcal{H}[q(0),1]\cap Q$, $\zeta(n,\lambda,p,\beta,\delta,\alpha;z)$ is univalent in $\mathbb{U}$ and the following superordination condition

$$\frac{\delta(A-B)z}{(1+Bz)^2} + (\delta+\kappa)\left(\frac{1+Az}{1+Bz}\right)^2 \prec \zeta(n,\lambda,p,\beta,\delta,\alpha;z),$$

holds, then

$$\frac{1+Az}{1+Bz} \prec \frac{A^n_{\lambda,p}(\alpha,\beta,\mu)f(z)}{\left[A^{n+1}_{\lambda,p}(\alpha,\beta,\mu)f(z)\right]^{\epsilon}},$$

and $\dfrac{1+Az}{1+Bz}$ *is the best subordinant.*

Also, let $q(z)=\dfrac{1+z}{1-z}$, *then for* $f\in\mathcal{A}(p)$ *we have*

$$\frac{2\delta z}{(1-z)^2}+(\delta+\kappa)\left(\frac{1+z}{1-z}\right)^2 \prec \zeta(n,\lambda,p,\beta,\delta,\alpha;z),$$

then

$$\frac{1+z}{1-z} \prec \frac{A^n_{\lambda,p}(\alpha,\beta,\mu)f(z)}{\left[A^{n+1}_{\lambda,p}(\alpha,\beta,\mu)f(z)\right]^{\epsilon}}$$

and the function $\dfrac{1+z}{1-z}$ *is the best subordinant.*

Finally, by taking $q(z)=\left(\dfrac{1+z}{1-z}\right)^{\mu}$, $(0<\mu\leq 1)$, *then for* $f\in\mathcal{A}(p)$ *we have*

$$\frac{2\delta\mu z}{(1-z)^2}\left(\frac{1+z}{1-z}\right)^{\mu-1}+(\delta+\kappa)\left(\frac{1+z}{1-z}\right)^{2\mu} \prec \zeta(n,\lambda,p,\beta,\delta,\alpha;z),$$

then

$$\left(\frac{1+z}{1-z}\right)^{\mu} \prec \frac{A^n_{\lambda,p}(\alpha,\beta,\mu)f(z)}{\left[A^{n+1}_{\lambda,p}(\alpha,\beta,\mu)f(z)\right]^{\epsilon}},$$

and the function $\left(\dfrac{1+z}{1-z}\right)^{\mu}$ *is the best subordinant.*

Combining Theorem 3.1 and Theorem 3.3, we get the following sandwich theorem.

Theorem 3.5 *Let* q_1 *and* q_2 *be convex univalent in* $\mathbb{U}$ *with* $q_1(0)=q_2(0)=1$ *and satisfy* (3.1) *and* (3.4) *respectively.*

If $f\in\mathcal{A}(p)$ *such that* $\dfrac{A^n_{\lambda,p}(\alpha,\beta,\mu)f(z)}{\left[A^{n+1}_{\lambda,p}(\alpha,\beta,\mu)f(z)\right]^{\epsilon}}\in\mathcal{H}[q(0),1]\cap Q$, $\zeta(n,\lambda,p,\beta,\delta,\alpha;z)$ *is univalent in* $\mathbb{U}$ *and*

$$(\delta+\kappa)\left(q_1(z)\right)^2+\delta z q_1'(z) \prec \zeta(n,\lambda,p,\beta,\delta,\alpha;z) \prec (\delta+\kappa)\left(q_2(z)\right)^2+\delta z q_2'(z),$$

holds, then

$$q_1(z) \prec \frac{A^n_{\lambda,p}(\alpha,\beta,\mu)f(z)}{\left[A^{n+1}_{\lambda,p}(\alpha,\beta,\mu)f(z)\right]^{\epsilon}} \prec q_2(z),$$

and $q_1(z)$ *and* $q_2(z)$ *are, respectively, the best subordinant and the best dominant.*

Corollary 3.6 *Let* $q_i(z) = \dfrac{1+A_iz}{1+B_iz}$ $(i = 1, 2; -1 \leq B_2 < B_1 < A_1 \leq A_2 \leq 1)$ *in Theorem 3.5.*

If $f \in \mathcal{A}(p)$ *such that* $\dfrac{A^n_{\lambda,p}(\alpha,\beta,\mu)f(z)}{\left[A^{n+1}_{\lambda,p}(\alpha,\beta,\mu)f(z)\right]^{\epsilon}} \in \mathcal{H}[q(0),1]\cap Q$, $\zeta(n,\lambda,p,\beta,\delta,\alpha;z)$ *is univalent in* $\mathbb{U}$ *and*

$$\frac{\delta(A_1-B_1)z}{(1+B_1z)^2}+(\delta+\kappa)\left(\frac{1+A_1z}{1+B_1z}\right)^2 \prec \zeta(n,\lambda,p,\beta,\delta,\alpha;z) \prec \frac{\delta(A_2-B_2)z}{(1+B_2z)^2}+(\delta+\kappa)\left(\frac{1+A_2z}{1+B_2z}\right)^2,$$

holds, then

$$\frac{1+A_1z}{1+B_1z} \prec \frac{A^n_{\lambda,p}(\alpha,\beta,\mu)f(z)}{\left[A^{n+1}_{\lambda,p}(\alpha,\beta,\mu)f(z)\right]^{\epsilon}} \prec \frac{1+A_2z}{1+B_2z},$$

and $\dfrac{1+A_1z}{1+B_1z}$ *and* $\dfrac{1+A_2z}{1+B_2z}$ *are, respectively, the best subordinant and the best dominant.*

4. Subordinations and superordinations results for $F^m_p(\rho,\vartheta)f(z)$

In this section, we derive subordination and superordination results for $p-$valent functions involving the integral operator $F^m_p(\rho,\vartheta)f(z)$ given by (1.7).

Theorem 4.1 *Let* $q(z)$ *be univalent in* $\mathbb{U}$ *with* $q(0) = 1$. *Further, assume that*

$$\operatorname{Re}\left\{\frac{2(\delta+\kappa)q(z)}{\delta}+1+\frac{zq''(z)}{q'(z)}\right\} > 0. \ (\delta \in \mathbb{C}^* = \mathbb{C}-\{0\}, \kappa \geq 0), \tag{4.1}$$

If $f \in \mathcal{A}(p)$ *satisfies the following subordination condition:*

$$\Lambda(p,\vartheta,\rho;z) \prec \delta z q'(z)+(\delta+\kappa)\left(q(z)\right)^2, \tag{4.2}$$

where

$$\Lambda(p,\vartheta,\rho;z) = \frac{\delta(\vartheta+p)}{\rho}\left[\frac{\left(F^{m-1}_p(\rho,\vartheta)f(z)\right)\left(F^{m+1}_p(\rho,\vartheta)f(z)\right)-\eta\left(F^m_p(\rho,\vartheta)f(z)\right)^2}{\left(F^{m+1}_p(\rho,\vartheta)f(z)\right)^{\eta+1}}\right] + \frac{\delta(\eta-1)[\vartheta+p(1-\rho)]}{\rho}\frac{\left(F^m_p(\rho,\vartheta)f(z)\right)}{\left(F^{m+1}_p(\rho,\vartheta)f(z)\right)^{\eta}}+(\delta+\kappa)\frac{\left(F^m_p(\rho,\vartheta)f(z)\right)^2}{\left(F^{m+1}_p(\rho,\vartheta)f(z)\right)^{2\eta}},$$

then

$$\frac{F^m_p(\rho,\vartheta)f(z)}{\left[F^{m+1}_p(\rho,\vartheta)f(z)\right]^{\eta}} \prec q(z),$$

and $q(z)$ *is the best dominant.*

Proof. Define a function $p(z)$ by

$$p(z) = \frac{F_p^m(\rho,\vartheta)f(z)}{\left[F_p^{m+1}(\rho,\vartheta)f(z)\right]^\eta}, \quad (z \in \mathbb{U}). \tag{4.3}$$

Then the function $p(z)$ is analytic in $\mathbb{U}$ and $p(0) = 1$. Differentiating (4.3) logarithmically with respect to z and using the identity (1.8) in the resulting equation, we have

$$\begin{aligned} = & \frac{\delta(\vartheta+p)}{\rho}\left[\frac{\left(F_p^{m-1}(\rho,\vartheta)f(z)\right)\left(F_p^{m+1}(\rho,\vartheta)f(z)\right)-\eta\left(F_p^m(\rho,\vartheta)f(z)\right)^2}{\left(F_p^{m+1}(\rho,\vartheta)f(z)\right)^{\eta+1}}\right] \\ & +\frac{\delta\left(\eta-1\right)\left[\vartheta+p(1-\rho)\right]}{\rho}\frac{\left(F_p^m(\rho,\vartheta)f(z)\right)}{\left(F_p^{m+1}(\rho,\vartheta)f(z)\right)^\eta}+(\delta+\kappa)\frac{\left(F_p^m(\rho,\vartheta)f(z)\right)^2}{\left(F_p^{m+1}(\rho,\vartheta)f(z)\right)^{2\eta}} \\ = & \delta z p'(z)+(\delta+\kappa)\left(p(z)\right)^2, \end{aligned}$$

that is,

$$\delta z p'(z)+(\delta+\kappa)\left(p(z)\right)^2 \prec \delta z q'(z)+(\delta+\kappa)\left(q(z)\right)^2.$$

Therefore, Theorem 4.1 now follows by applying Lemma 2.2 by setting

$$\theta(w) = (\delta+\kappa)w^2 \text{ and } \varphi(w) = \delta. \qquad \blacksquare$$

Corollary 4.2 *Let* $q(z) = \dfrac{1+Az}{1+Bz}$ $(-1 \le B < A \le 1)$ *in Theorem* 4.1. *Further, assume that* (4.1) *holds.*

If $f \in \mathcal{A}(p)$ *satisfies the following subordination condition:*

$$\Lambda(p,\vartheta,\rho;z) \prec \frac{\delta(A-B)z}{(1+Bz)^2}+(\delta+\kappa)\left(\frac{1+Az}{1+Bz}\right)^2,$$

then

$$\frac{F_p^m(\rho,\vartheta)f(z)}{\left[F_p^{m+1}(\rho,\vartheta)f(z)\right]^\eta} \prec \frac{1+Az}{1+Bz},$$

and the function $\dfrac{1+Az}{1+Bz}$ *is the best dominant.*

Also, let $q(z) = \dfrac{1+z}{1-z}$. *Then, for* $f \in \mathcal{A}(p)$, *we have*

$$\Lambda(p,\vartheta,\rho;z) \prec \frac{2\delta z}{(1-z)^2}+(\delta+\kappa)\left(\frac{1+z}{1-z}\right)^2,$$

then

$$\frac{F_p^m(\rho,\vartheta)f(z)}{\left[F_p^{m+1}(\rho,\vartheta)f(z)\right]^\eta} \prec \frac{1+z}{1-z},$$

and the function $\dfrac{1+z}{1-z}$ *is the best dominant.*

Finally, by taking $q(z) = \left(\dfrac{1+z}{1-z}\right)^{\mu}$, $(0 < \mu \leq 1)$, *then for* $f \in \mathcal{A}(p)$ *we have,*

$$\Lambda(p, \vartheta, \rho; z) \prec \frac{2\delta\mu z}{(1-z)^2}\left(\frac{1+z}{1-z}\right)^{\mu-1} + (\delta+\kappa)\left(\frac{1+z}{1-z}\right)^{2\mu},$$

then

$$\frac{F_p^m(\rho, \vartheta)f(z)}{\left[F_p^{m+1}(\rho, \vartheta)f(z)\right]^{\eta}} \prec \left(\frac{1+z}{1-z}\right)^{\mu},$$

and the function $\left(\dfrac{1+z}{1-z}\right)^{\mu}$ *is the best dominant.*

Next, by appealing to Lemma 2.3 we prove the following.

Theorem 4.3 *Let* $q(z)$ *be convex univalent in* $\mathbb{U}$ *with* $q(0) = 1$. *Assume that*

$$\operatorname{Re}\left\{\frac{2(\delta+\kappa)q(z)q'(z)}{\delta}\right\} > 0. \tag{4.4}$$

Let $f \in \mathcal{A}(p)$ *such that* $\dfrac{F_p^m(\rho, \vartheta)f(z)}{\left[F_p^{m+1}(\rho, \vartheta)f(z)\right]^{\eta}} \in \mathcal{H}[q(0), 1] \cap Q$, $\Lambda(p, \vartheta, \rho; z)$ *is univalent in* $\mathbb{U}$ *and the following superordination condition*

$$(\delta+\kappa)\left(q(z)\right)^2 + \delta z q'(z) \prec \Lambda(p, \vartheta, \rho; z), \tag{4.5}$$

holds, then

$$q(z) \prec \frac{F_p^m(\rho, \vartheta)f(z)}{\left[F_p^{m+1}(\rho, \vartheta)f(z)\right]^{\eta}}, \tag{4.6}$$

and $q(z)$ *is the best subordinant.*

Proof. Let the function $p(z)$ be defined by

$$p(z) = \frac{F_p^m(\rho, \vartheta)f(z)}{\left[F_p^{m+1}(\rho, \vartheta)f(z)\right]^{\eta}}.$$

Then from the assumption of Theorem 4.3, the function $p(z)$ is analytic in $\mathbb{U}$ and (4.4) holds. Hence, the subordination (4.5) is equivalent to

$$(\delta+\kappa)\left(q(z)\right)^2 + \delta z q'(z) \prec (\delta+\kappa)\left(p(z)\right)^2 + \delta z p'(z),$$

The assertion (4.6) of Theorem 4.3 now follows by an application of Lemma 2.3. ∎

Corollary 4.4 *Let* $q(z) = \dfrac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$ *in Theorem* 4.3. *Further, assume that* (4.4) *holds.*

If $f \in \mathcal{A}(p)$ *such that* $\dfrac{F_p^m(\rho,\vartheta)f(z)}{\left[F_p^{m+1}(\rho,\vartheta)f(z)\right]^\eta} \in \mathcal{H}[q(0),1] \cap Q$, $\Lambda(p,\vartheta,\rho;z)$ *is univalent in* $\mathbb{U}$ *and the following superordination condition*

$$\frac{\delta(A-B)z}{(1+Bz)^2} + (\delta+\kappa)\left(\frac{1+Az}{1+Bz}\right)^2 \prec \Lambda(p,\vartheta,\rho;z),$$

holds, then

$$\frac{1+Az}{1+Bz} \prec \frac{F_p^m(\rho,\vartheta)f(z)}{\left[F_p^{m+1}(\rho,\vartheta)f(z)\right]^\eta},$$

and $\dfrac{1+Az}{1+Bz}$ *is the best subordinant.*

Also, let $q(z) = \dfrac{1+z}{1-z}$, *then for* $f \in \mathcal{A}(p)$ *we have*

$$\frac{2\delta z}{(1-z)^2} + (\delta+\kappa)\left(\frac{1+z}{1-z}\right)^2 \prec \Lambda(p,\vartheta,\rho;z),$$

then

$$\frac{1+z}{1-z} \prec \frac{F_p^m(\rho,\vartheta)f(z)}{\left[F_p^{m+1}(\rho,\vartheta)f(z)\right]^\eta},$$

and the function $\dfrac{1+z}{1-z}$ *is the best subordinant.*

Finally, by taking $q(z) = \left(\dfrac{1+z}{1-z}\right)^\mu$, $(0 < \mu \leq 1)$, *then for* $f \in \mathcal{A}(p)$ *we have*

$$\frac{2\delta\mu z}{(1-z)^2}\left(\frac{1+z}{1-z}\right)^{\mu-1} + (\delta+\kappa)\left(\frac{1+z}{1-z}\right)^{2\mu} \prec \Lambda(p,\vartheta,\rho;z),$$

then

$$\left(\frac{1+z}{1-z}\right)^\mu \prec \frac{F_p^m(\rho,\vartheta)f(z)}{\left[F_p^{m+1}(\rho,\vartheta)f(z)\right]^\eta},$$

and the function $\left(\dfrac{1+z}{1-z}\right)^\mu$ *is the best subordinant.*

Combining Theorem 4.1 and Theorem 4.3, we get the following sandwich theorem.

Theorem 4.5 *Let* q_1 *and* q_2 *be convex univalent in* $\mathbb{U}$ *with* $q_1(0) = q_2(0) = 1$ *and satisfy* (4.1) *and* (4.4) *respectively.*

If $f \in \mathcal{A}(p)$ *such that* $\dfrac{F_p^m(\rho,\vartheta)f(z)}{\left[F_p^{m+1}(\rho,\vartheta)f(z)\right]^\eta} \in \mathcal{H}[q(0),1] \cap Q$, $\Lambda(p,\vartheta,\rho;z)$ *is*

univalent in $\mathbb{U}$ *and*

$$(\delta+\kappa)\left(q_1(z)\right)^2+\delta z q_1'(z)\prec\Lambda(p,\vartheta,\rho;z)\prec(\delta+\kappa)\left(q_2(z)\right)^2+\delta z q_2'(z),$$

holds, then

$$q_1(z)\prec\frac{F_p^m(\rho,\vartheta)f(z)}{\left[F_p^{m+1}(\rho,\vartheta)f(z)\right]^\eta}\prec q_2(z),$$

and $q_1(z)$ *and* $q_2(z)$ *are, respectively, the best subordinant and the best dominant.*

Corollary 4.6 *Let* $q_i(z)=\dfrac{1+A_iz}{1+B_iz}$ $(i=1,2;-1\le B_2<B_1<A_1\le A_2\le 1)$ *in Theorem 4.5.*

If $f\in\mathcal{A}(p)$ *such that* $\dfrac{F_p^m(\rho,\vartheta)f(z)}{\left[F_p^{m+1}(\rho,\vartheta)f(z)\right]^\eta}\in\mathcal{H}[q(0),1]\cap Q$, $\Lambda(p,\vartheta,\rho;z)$ *is univalent in* $\mathbb{U}$ *and*

$$\frac{\delta(A_1-B_1)z}{(1+B_1z)^2}+(\delta+\kappa)\left(\frac{1+A_1z}{1+B_1z}\right)^2\prec\Lambda(p,\vartheta,\rho;z)\prec\frac{\delta(A_2-B_2)z}{(1+B_2z)^2}+(\delta+\kappa)\left(\frac{1+A_2z}{1+B_2z}\right)^2,$$

holds, then

$$\frac{1+A_1z}{1+B_1z}\prec\frac{F_p^m(\rho,\vartheta)f(z)}{\left[F_p^{m+1}(\rho,\vartheta)f(z)\right]^\eta}\prec\frac{1+A_2z}{1+B_2z},$$

and $\dfrac{1+A_1z}{1+B_1z}$ *and* $\frac{1+A_2z}{1+B_2z}$ *are, respectively, the best subordinant and the best dominant.*

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SOME REACTION DIFFUSION TYPE EQUATIONS IN QUANTUM MECHANICS

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Abstract. Many second order partial differential equations are expressed as a pair of coupled first order partial differential equations. A surprisingly large variety of problems from quantum theory to information sciences can be studied. We examine certain novel features of the solutions of equations arising from these areas of study.

Keywords: Schrödinger equation, Klein Gordon, reaction diffusion equations

1. Introduction

A reaction-diffusion equation comprises of a reaction term and a diffusion term, a typical Reaction Diffusion equation is as follows:

$$\frac{\partial u}{\partial t} = D\Delta u + f(u)$$

$u = u(x,t)$ is a variable which describes concentration of a substance or the population at time t . Δ denotes the Laplace operator. So the first term on the right hand side describes the "diffusion", including D as diffusion coefficient. The second term, $f(u)$ is a smooth function $f : \mathbb{R} \to \mathbb{R}$ representing reaction kinetics.

Instead of a scalar equation, one can also introduce systems of reaction diffusion equations, which are of the form

$$\frac{\partial u}{\partial t} = \nabla_x.(diag(d_1, d_2, ..., d_m))\nabla_x u + f(t, x, u, \nabla_x) \tag{1}$$

where $t \geq 0$ denotes time and $x \in \Omega \subset \mathbb{R}^d$ denotes position within a $d-$dimensional bounded domain Ω with a smooth boundary. The states $u = (u_1, ..., u_m)$, $i = 1, ..., m$ describe the concentrations or densities of substances or populations, $u_i = u_i(t, x)$. The functions $d_i(t, x)$ are called diffusion coefficients or diffusivities. On Ω, the bounded domain introduced earlier, boundary conditions need to be specified for equations (1). Typical conditions would include either Dirichlet, which prescribe the value at the boundary $u_i(x, t) = f_D(t, x)$, $x \in \partial\Omega$ or Neumann conditions giving the diffusion flux through the boundary $\partial\Omega$, $d_i(t, x)\nabla_x u_i(t, x) = f_N(t, x)$ or mixed boundary conditions.

One of the fascinating aspects of the natural world is the diversity of shapes that make up the animal and plant kingdoms. How these patterns arise is one of the mysteries of science. The problem that Turing addressed in his seminal paper, "The chemical basis of morphogenesis" [16] was precisely this. He presented a theory in which he proposed that cells actually respond to a chemical pre-pattern. He considered a system of morphogens reacting and diffusing in such a way that, in the absence of diffusion, they exhibited a spatially uniform steady state which would be stable. This phenomenon, termed diffusion-driven instability, has now been shown to occur in chemistry. Experimental results illustrate the formation of striped and spotted patterns, as well as more complicated patterns. This was the first example of what is now called an emergent phenomenon in the sense that the behaviour of the system, in this case a patterning instability, emerges from the components and is not part of the components. In his system, the reaction kinetics are stabilising and we know that diffusion is stabilising in the sense that it homogenises spatial patterns. Therefore, two stabilising systems interacted to produce an instability. In other words, he recognised that it was the integration of components that gave rise to the structures and behaviours we observe, rather than each behaviour being encoded in its own component. Many of these patterns can be exhibited by Turing models and there is now a vast amount of theoretical and experimental literature in this area (see [13], for a review). Turing systems have the form

$$\begin{aligned} \frac{\partial U}{\partial t} &= D_u \nabla^2 U + f(U, V) \\ \frac{\partial V}{\partial t} &= D_v \nabla^2 V + g(U, V). \end{aligned} \tag{2}$$

These equations describe the evolution of the concentrations, $U(\bar{x}, t)$, $V(\bar{x}, t)$ at spatial position $\bar{x}$ and time t, of two chemicals due to diffusion, with constant diffusion coefficients D_u, D_v, respectively, and reaction, modeled by the functions, f and g which are typically non-linear. We now examine solutions of the travelling-wave type for reaction-diffusion equations.

2. The Klein Gordon equation

Many second order partial differential equations are expressed as a pair of coupled first order partial differential equations. A surprisingly large variety of problems

from quantum theory to information sciences can be studied. We examine certain novel features of the solutions. [7]

A single particle can be described by the Klein-Gordon equation for spin 0 and the Dirac equation for spin $1/2\hbar$. It is well-known that the Dirac equation gives (within the correct limits) a relativistic wave-mechanical description of a single electron.It is significant to interpret the definition of a single particle in terms of its wave function. We seek the appropriate generalization of the nonrelativistic equation to obtain the proper relativistic equation describing a particle of a given spin.

For the spin zero particle, the nonrelativistic equation is the Schrödinger equation,

$$\sum_{k=1}^{3} \frac{1}{2m}\left[\frac{\hbar}{i}\frac{\partial}{\partial x_k} - \frac{e}{c}A_k\right]^2 \psi = \left(\frac{\hbar}{i}\frac{\partial}{\partial t} - e\phi\right)\psi, \tag{3}$$

where e is charge, m is mass and ψ is the Schrödinger wave function and A_k and ϕ are the vector and scalar potential for the electromagnetic field. The coordinates x_k will as usual be x, y, z as k goes from one to three, $x_4 = ict$, while $x_0 = ct$. We also use the symbol x to represent a vector whose components are x_k. Four vectors generally will be indicated by a Greek subscript, e.g., $x_\mu = (x_k, x_4)$. Similar notation is used for the momentum $\mathbf{P}$ and electromagnetic potentials, $A_k = (A_x, A_y, A_z), A_4 = i\phi, A_0 = \phi$, where A_k is the vector potential and ϕ, the scalar. This equation is obtained from the non-relativistic Hamiltonian describing the interaction between a charged particle and the electromagnetic field,

$$H(\mathbf{p}, \mathbf{r}) = 1/2m \sum (p_k - (e/c)A_k)^2 + e\phi \tag{4}$$

by the well-known substitutions derived, for example, by transformation theory from the commutation relations between $\mathbf{p}$ and $\mathbf{r}$,

$$p_k \to (\hbar/i)(\partial/\partial x_k), H \to \hbar i(\partial/\partial t). \tag{5}$$

To obtain the relativistic generalization of (3) we need only write the relativistic energy-momentum relation and make substitutions (5). We replace (4) by

$$\sum_{\mu=1}^{4}(p_\mu - \frac{e}{c}A_\mu)^2 + (mc)^2 = 0,$$

where

$$p_\mu = (p_k, p_4), \quad p_4 = (i/c)H,$$
$$A_\mu = (A_k, A_4), \quad A_4 = i\phi.$$

For simplicity, we now insert the substitution

$$D_\mu = \partial/\partial x_\mu - (ie/\hbar c)A_\mu \text{ and } \kappa = mc/\hbar.$$

Then (3) can be written in the coincide from

$$\sum\nolimits_\mu [D_\mu^2 - \kappa^2]\psi = 0, \tag{6}$$

we seek an approximate wave function ψ. To obtain from the above the equation

$$H\psi = i\hbar(\partial\psi/\partial t). \tag{7}$$

3. A Hamiltonian form of the Klein-Gordon equation

We look for an appropriate ψ such that ψ satisfies the Hamiltonian form for the wave equation (7). To obtain this form it is necessary to resolve ψ into the components representing the two degrees of freedom. The function ψ is then a unicolumnar matrix formed from these two components.

The obvious first step in such a development is to introduce $\partial\psi/\partial t$ as an independent component. Let

$$\psi_4 = -\kappa^{-1} D_4\psi.$$

Then the Klein-Gordon equation may be written in a form equivalent as follows:

$$\begin{aligned} D_4\psi + \kappa\psi_4 &= 0, \\ \sum\nolimits_k D_k^2\psi - \kappa D_4\psi_4 - \kappa^2\psi &= 0. \end{aligned} \tag{8}$$

These equations are already in the Hamiltonian form (7), but the combination ψ and ψ_4 does not prove to be convenient because of the symmetry of (8). Accordingly, we introduce the linear combination,

$$\begin{aligned} \psi &= 1/\sqrt{2}(\varphi + \chi), \\ \psi_4 &= 1/\sqrt{2}(\varphi - \chi). \end{aligned} \tag{9}$$

The equations for φ and χ are

$$\begin{aligned} D_4\varphi &= (1/2\kappa)\sum\nolimits_k D_k^2(\varphi + \chi) - \kappa\varphi, \\ D_4\chi &= -(1/2\kappa)\sum\nolimits_k D_k^2(\varphi + \chi) + \kappa\chi, \end{aligned}$$

which may be written more explicitly as

$$\begin{aligned} i\hbar(\partial\varphi/\partial t) &= (1/2m)(\hbar/i\nabla - e\mathbf{A}/c)^2(\varphi + \chi) + (e\phi + mc^2)\varphi, \\ i\hbar(\partial\chi/\partial t) &= -(1/2m)(\hbar/i\nabla - e\mathbf{A}/c)^2(\varphi + \chi) + (e\phi - mc^2)\chi. \end{aligned} \tag{10}$$

For particles with no charge that is taking ($e = 0$), after adjusting constants, equations (10) could be written as

$$\begin{aligned} (\partial\varphi/\partial t) &= \nabla^2(a_1\varphi + a_2\chi) + b_1\varphi \\ (\partial\chi/\partial t) &= \nabla^2(a_3\varphi + a_4\chi) + b_4\chi, \end{aligned} \tag{11}$$

where a_1,b_1,a_2,b_4 are some constants.

Choosing b_2, b_3 to be equal to zero, and setting $\bar{\psi} = \begin{bmatrix} \phi \\ \chi \end{bmatrix}$ this becomes

$$\frac{\partial \bar{\psi}}{\partial t} = \mathcal{A}\nabla^2\bar{\psi} + \mathcal{B}\bar{\psi} \tag{12}$$

$\mathcal{A}$ and $\mathcal{B}$ are 2×2 matrices, If we choose $\mathcal{A} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$ and $\mathcal{B} = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$ and a_1,a_2,a_3,a_4,b_1,b_2,b_3,b_4 are real numbers, then (12) takes on the form

$$\begin{aligned} \frac{\partial \phi}{\partial t} &= \nabla^2(a_1\phi + a_2\chi) + b_1\phi + b_2\chi \\ \frac{\partial \chi}{\partial t} &= \nabla^2(a_3\phi + a_4\chi) + b_3\phi + b_4\chi \end{aligned} \tag{13}$$

Comparing with equation (2), it is clear that these two equations although representing different physical phenomena have the same structure.

4. Dynamical analysis ([3], [2])

We try to solve equation (12) by the separation of variables [5], [9] and we take a trial solution in the form of

$$\bar{\psi} = \bar{\psi_q} e^{\sigma_q t} e^{iqx} = \begin{bmatrix} \phi_q \\ \chi_q \end{bmatrix} e^{\sigma_q t} e^{iqx} \tag{14}$$

Where $\bar{\psi_q} = \begin{bmatrix} \phi_q \\ \chi_q \end{bmatrix}$ is constant vector. Putting the Equation (14) in (12)

$$\begin{aligned} \sigma_q \bar{\psi_q} e^{\sigma_q t} e^{iqx} &= B\bar{\psi_q} e^{\sigma_q t} e^{iqx} + A(-\bar{\psi_q}) q^2 e^{\sigma_q t} e^{iqx} \\ \sigma_q \bar{\psi_q} &= B\bar{\psi_q} - A\bar{\psi_q} q^2 \end{aligned}$$

To simplify matters, we choose $a_2 = 0 = a_3$. Let us set

$$A_q = B - Aq^2 = \begin{bmatrix} b_1 - a_1 q^2 & b_2 \\ b_3 & b_4 - a_4 q^2 \end{bmatrix} \tag{15}$$

and we have

$$A_q \bar{\psi_q} = \sigma_q \bar{\psi_q} \tag{16}$$

which is an eigenvalue problem, with σ_q as the eigenvalue and $\bar{\psi_q}$ as the eigenvector of A_q [6]. This problem for a given q has generally two linearly independent eigenvectors that we will denote by $\bar{\psi_{iq}}$ for $i = 1, 2$. If the corresponding eigenvalues are σ_{iq}, the particular solution with wave number q will have the form:

$$(c_1 \bar{\psi_{1q}} e^{\sigma_{1q} t} + c_2 \bar{\psi_{2q}} e^{\sigma_{2q} t}) e^{iqx}$$

where c_1 and c_2 are constant. The solution decays for $Re(\sigma_{iq}) < 0$ for $i = 1, 2$. The characteristic polynomial for the eigenvalue problem Equation (16) can be written as

$$0 = |A_q - \sigma_q I| = \sigma_q^2 - \text{Trace}(A_q)\sigma_q + |A_q|$$

where

$$\begin{aligned} Trace(A_q) &= b_1 + b_3 - (a_1 + a_4)q^2 \\ |A_q| &= (b_1 - a_1 q^2)(b_4 - a_4 q^2) - b_2 b_3 \end{aligned} \tag{17}$$

and the eigenvalues are:

$$\sigma_q = \frac{1}{2} Trace(A_q) \pm \frac{1}{2}\sqrt{(Trace(A_q))^2 - 4|A_q|}$$

The regions of stability (both $Re(\sigma_q)$ negative) and instability (at least one $Re(\sigma_q)$ positive) is given by:

$$\begin{aligned} \text{Trace}(A_q) &< 0 \\ |A_q| &> 0 \end{aligned}$$

Linear stability is guaranteed if

$$\begin{aligned} trA_q &= (b_1 - a_1 q^2) + (b_4 - a_4 q^2) < 0 \\ detA_q &= (b_1 - a_1 q^2)(b_4 - a_4 q^2) - b_2 b_3 > 0. \end{aligned} \tag{18}$$

5. A nonlinear Schrödinger equation

We now consider the nonlinear Schrödinger equation which appears in several contexts for example [17], [4], [12]. In [4] and [12] it represents semiconductor electronics and in [10] optics in nonlinear media. In [8], photonics and in [15] the foundations of quantum mechanics. The study of these equations has catalyzed the development of mathematical concepts such as Solitons [18]. The nonlinear Schrödinger equation which we refer to here has the form

$$\begin{aligned} i\psi_t &= -\frac{1}{2}\Delta\psi + g(x)|\psi|^2\psi \\ \psi(x, 0) &= \psi_0(x) \end{aligned} \tag{19}$$

with $x \in \mathbb{R}^d$.

Taking a hint from the nonlinear Schrödinger equation (19), we can consider the following system of equations

$$\begin{aligned} \frac{\partial u}{\partial t} &= D_u \frac{\partial^2 u}{\partial x^2} + a_1 u(1 - u^2) + b_1 v \\ \frac{\partial v}{\partial t} &= D_v \frac{\partial^2 v}{\partial x^2} + a_2 u + b_2 v \end{aligned} \tag{20}$$

6. Solutions to the reaction-diffusion type nonlinear Schrödinger wave equation

We attempt to solve the equations (20) using a linearization followed by separation of variables as well as a travelling wave method. We first discuss the method of linearization of equation (20).

Linear stability analysis

The aim of this discussion is to derive, and then to understand physically, conditions that are sufficient for the real parts of all growth rates to be negative. We rewrite the system of equations (20) in a more general setting as follows:

$$\begin{aligned} \frac{\partial u}{\partial t} &= D_u \frac{\partial^2 u}{\partial x^2} + f_1(u,v) \\ \frac{\partial v}{\partial t} &= D_v \frac{\partial^2 v}{\partial x^2} + f_2(u,v) \end{aligned} \tag{21}$$

that is, setting $a_1 u(1-u^2)+b_1 v = f_1(u,v) = f_1(u,v)$ and $a_2 u + b_2 v = f_2(u,v)$ and $\mathbf{f}(u,v) = (f_1(u,v), f_2(u,v))$ We begin by assuming that there exists a stationary uniform base solution $\mathbf{u_b} = (\hat{u}, \hat{v})$ and that this satisfies the model after all partial derivatives have been set to zero giving

$$\begin{aligned} f_1(\hat{u}, \hat{v}) &= 0 \\ f_2(\hat{u}, \hat{v}) &= 0 \end{aligned} \tag{22}$$

By linearizing about the base state $\mathbf{u_b}$, for small perturbations about $\mathbf{u_b}$, that is if we let $\mathbf{u_b} = \mathbf{u_0} + \delta\mathbf{u}$, we can show that an arbitrary infinitesimal perturbation $\delta\mathbf{u_b} = (\delta\mathbf{u}, \delta\mathbf{v})$ of the base state will evolve in time according to the following linear constant-coefficient evolution equations:

$$\begin{aligned} \partial_t \delta\mathbf{u} &= a_{11}\delta u + a_{12}\delta v + D_u \partial_x^2 \delta u \\ \partial_t \delta\mathbf{v} &= a_{21}\delta u + a_{22}\delta v + D_v \partial_x^2 \delta v \end{aligned} \tag{23}$$

The constant coefficients a_{ij} come from the 2×2 Jacobian matrix $A = \frac{\partial \mathbf{f}}{\partial \mathbf{u}}$ evaluated at the constant base solution $\mathbf{u_b}$. The mathematical structure of (23) can be clarified by writing in the vector form

$$\frac{\partial}{\partial t}\delta u = \mathbf{A}\delta\mathbf{u} = \mathbf{D}\frac{\partial^2}{\partial \mathbf{x^2}}\delta\mathbf{u} \tag{24}$$

Equation (24) or its equivalent form (23) can be solved by the method of separation of variables outlined in Section (). We now approach the problem through another route, viz., The Travelling Wave method.

Travelling wave method

Our aim is to examine solutions of the form [14] $u(x,t) = \phi(x-ct)$ for (20). A travelling wave solution is a solution of the form $u(x,t) = U(z)$, $z = x+ct$, where

c is the wave speed. Assume $c \geq 0$. If a solution $u(x,t)$ represents a travelling wave the shape of the solution remains the same for all t.

$$\frac{\partial u}{\partial t} = c\frac{dU}{dz}, \frac{\partial u}{\partial x} = \frac{dU}{dz}, \frac{\partial^2 u}{\partial x^2} = \frac{d^2U}{dz^2}$$

Using the same transformation on $v(x,t)$,(20) becomes

$$c\frac{dU}{dz} = D_u\frac{d^2U}{dz^2} + a_1U(1-U^2) + b_1V$$

$$c\frac{dV}{dt} = D_v\frac{d^2V}{dz^2} + a_2U + b_2V$$

To study a simplified version, take $D_u = 0$ and $D_v = 1$

$$cU' = a_1U(1-U^2) + b_1V$$

$$cV' = V'' + a_2U + b_2V \tag{25}$$

Let $V' = W$ Equations (25) can be expressed as

$$U' = \frac{a_1}{c}U(1-U^2) + \frac{b_1}{c}V$$

$$V' = W$$

$$W' = cW - a_2U - b_2V$$

The equilibrium points are

$$\begin{aligned} eq_1 &= (0,0,0), \\ eq_2 &= \left(\sqrt{\frac{a_1b_2 - b_1a_2}{a_1b_2}}, -\frac{a_2}{b_2}\sqrt{\frac{a_1b_2 - b_1a_2}{a_1b_2}}, 0\right), \\ eq_3 &= \left(-\sqrt{\frac{a_1b_2 - b_1a_2}{a_1b_2}}, \frac{a_2}{b_2}\sqrt{\frac{a_1b_2 - b_1a_2}{a_1b_2}}, 0\right) \end{aligned}$$

The Characteristic equation for the equilibrium point $eq1$ is

$$-c\lambda^3 + (a_1 + c^2)\lambda^2 - (a_1c + cb_2)\lambda + (a_1b_2 - b_1a_2) = 0.$$

We plot the Characteristic equation to indicate two real and positive Eigenvalues in Figure 1.

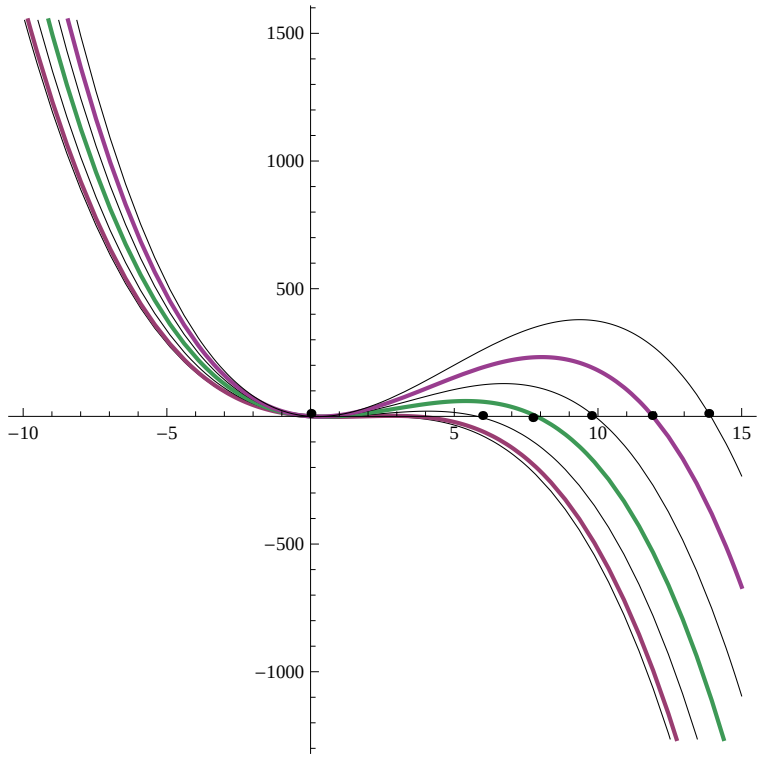

Figure 1: Graphs of the Characteristic Polynomial for $a_1 = b_1 = a_2 = b_2 = 1$, c ranging from 2 to 10, all the different characteristic polynomials showing two positive Eigenvalues, marked as dots on the X-axis

Observing Figure 1, we see that the Eigenvalues are positive, this tells us that the solutions $\to \infty$ as $t \to \infty$. Choosing different values for the constants, the Eigenvalues can be complex in which case the solutions would be bounded.

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SEARCH OF GOOD ROTATION PATTERNS THROUGH EXPONENTIAL TYPE REGRESSION ESTIMATOR IN SUCCESSIVE SAMPLING OVER TWO OCCASIONS

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Abstract. The problem of estimating the finite population mean on the second (current) occasion in successive sampling over two occasions has been discussed. Utilizing all the readily available information from first and second occasions, an efficient estimation procedure through exponential type regression estimator has been developed. It has been shown that the proposed procedure is more efficient than the one recently reported by Singh and Homa [17]. Optimum replacement policy relevant to the suggested estimation procedure has been advocated. In support of the present study a numerical illustration is given.

Keywords: exponential type regression estimator, successive sampling, variance, optimum replacement policy, efficiency.

ASM Classification: 62D05.

1. Introduction

There are many useful situations where survey needs to be repeated many times. The purpose of a repeated survey is allowing one or more items to be monitored over time. If situations that are required to be monitored are concerned with very large group of individuals (population or universe), such situations are difficult, time taking and costly affairs. To meet these requirements, rotation (successive) sampling provides a strong tool for generating the reliable estimates at different occasions. In successive sampling the use of information collected on an earlier occasion in improving the current estimates is well recognized, for instance, see Jessen [1], Patterson [2], Eckler [3], Rao and Graham [4], Das [5], Singh et al. [6], among others. Sen [7] applied this theory successfully in designing the strategies

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for estimating the population mean on the current occasion using information on two auxiliary variables readily available on variables.

Sen [8], [9] extended his work for multiple auxiliary variables. Singh et al.[10] used auxiliary information on current occasion and suggested estimators of the current population mean in two-occasion successive (rotation) sampling. Feng and Zou [11] and Biradar and Singh [12] used the auxiliary information on both the occasions for estimating the current population mean in the successive sampling. Singh [13], Singh and Vishwakarma [14], [15], [16], Singh and Homa [17], Singh and Pal [18], [19] have used the auxiliary information on both the occasions and envisaged several estimators for the estimating the population mean on current (second) occasion in two-occasion successive (rotation) sampling.

In this paper, we have made an effort to propose an exponential type regression estimator for estimating the current (second) population mean in two-occasion successive (rotation) sampling. The behavior of the suggested estimators has been examined through numerical illustration and suitable recommendations are made.

2. The suggested estimator

Let $U = (U_1, U_2, ..., U_N)$ be the finite population of N units, which has been sampled over two occasions. The character under study be denoted by $x(y)$ on the first (second) occasions respectively. We assume that information on an auxiliary variable z (stable over occasion), whose population mean $\bar{Z}$ is known, is available on both the occasions. It is assumed that the auxiliary variable z is correlated to x and y on the first and second occasions respectively. A simple random sample S_n of size n is drawn without replacement on the first occasion and a random sub sample S_m of the size m $(= n\lambda)$ units from S_n is retained (matched) for its use on the current (second) occasion. A fresh unmatched sample S_u of size $u = (n - m) = n\theta$ units is drawn on the current (second) occasion from the whole population by simple random sample (without replacement) procedure so that the sample size on the current (second) occasion remains. n, λ and θ $(\lambda + \theta = 1)$ are the fractions of the matched and fresh samples, respectively, on the current occasion. The values of λ and θ should be selected optimally. For the simplicity, we assume that the population size N is large enough so that the finite population correction (fpc) terms are ignored.

In what follows we shall use the following notations throughout this paper.

- $\bar{X}$, $\bar{Y}$, $\bar{Z}$: The population means of the variables x, y and z respectively.
- $\bar{x_n}$, $\bar{x_m}$, $\bar{y_u}$, $\bar{y_m}$, $\bar{z_m}$, $\bar{z_n}$, $\bar{z_u}$: The sample means of the respective variables based on the sample sizes indicated in subscripts.
- ρ_{yx}, ρ_{yz}, ρ_{xz}: The correlation coefficients between the variables shown in subscripts.
- S_x^2, S_y^2, S_z^2: The population mean square of x, y, z respectively.
- $\beta_{yx\cdot z}$: The population partial regression coefficient of y on x for fixed z.

- $R_x = \frac{\bar{Y}}{\bar{X}}$, $R_z = \frac{\bar{Y}}{\bar{Z}}$.
- $\beta_{yz\cdot x}$: The population partial regression coefficient of y on z for fixed x.
- S_{yx}, S_{yz}, S_{zx}: The population covariance between the variables shown in the subscripts.
- β_{yx}: The population regression coefficient of y on x.
- β_{yz}: The population regression coefficient of y on z.
- β_{zx}: The population regression coefficient of z on x.
- β_{xz}: The population regression coefficient of x on z.
- $\beta^{(m)}_{yx\cdot z}$: The sample partial regression coefficient of y on x for fixed z based on the sample size m.
- $\beta^{(m)}_{yz\cdot x}$: The sample partial regression coefficient of y on z for fixed x based on sample size m,
- $\beta^{(n)}_{yz}$: The sample regression coefficient of y on z based on sample size n.
- C_x, C_y, C_z: The population coefficient of variation of the variables given in the subscript.

To estimate the population mean $\bar{Y}$ on the current (second) occasion, two classes of estimators can be formed. First, based on sample of size ($u = n\theta$)drawn afresh on the second occasion and second based on the matched sample of side ($m = n\lambda$) common to both the occasions.

2.1. A class of estimators based on the sample drawn afresh on the second occasion

Let $(\bar{y_u}, \bar{z_u})$, be the unbiased estimators of population means $(\bar{Y}, \bar{Z})$ respectively based on u units. We consider a class of estimators of the population mean$\bar{Y}$, defined by

$$d_u = \bar{y}_u \exp\left[\frac{\alpha_0\ (\bar{Z} - \bar{z}_u)}{(\bar{Z} + \bar{z}_u)}\right], \tag{2.1}$$

where α_0 is a constant to be determined such that variance of d_u is least. For $\alpha_0 = 1$, it reduces to Singh and Homa [17] estimator.

To obtain the variance of d_u, we write

$$\bar{y}_u = \bar{Y}(1 + e_{yu}),\ \bar{z}_u = \bar{Z}(1 + e_{zu})$$

such that

$$E(e_{yu}) = E(e_{zu}) = 0$$

and

$$\begin{cases} E(e^2_{yu}) &= (C^2_y/u), \\ E(e^2_{zu}) &= (C^2_z/u), \\ E(e^2_{yu}e^2_{zu}) &= (1/u)\rho_{yz}C_yC_z. \end{cases} \tag{2.2}$$

Expressing (2.1) in terms of e_{yu} and e_{zu} we have

$$\begin{aligned} d_u &= \bar{Y}(1+e_{yu})exp[(-\alpha_0 e_{zu})/(2+e_{zu})] \\ &= \bar{Y}(1+e_{yu})exp[(-\alpha_0 e_{zu}/2)(1+(e_{zu}/2))^{-1}] \\ &\cong \bar{Y}[(1+e_{yu})-(\alpha_0 e_{zu}/2)], \end{aligned}$$

or

$$(d_u - \bar{Y}) = \bar{Y}[e_{yu} - (\alpha_0\ e_{zu}/2)]. \tag{2.3}$$

Squaring both sides of (2.3) we have

$$(d_u - \bar{Y})^2 = \bar{Y}^2(e_{yu}^2 + (\alpha_0^2 e_{zu}^2/4) - \alpha_0 e_{yu} e_{zu}). \tag{2.4}$$

Taking expectation of both sides of (2.4) we get the variance of d_u to the first degree of approximation (ignoring (fpc) term) as

$$Var(d_u) = (1/u)[S_y^2 + (\alpha_0/4)R_z S_z^2(\alpha_0 - 4(\beta_{yz}/R_z))], \tag{2.5}$$

which is minimized for

$$\alpha_0 = 2(\beta_{yz}/R_z) = \alpha_{0(opt)}. \tag{2.6}$$

Substitution of $\alpha_{0(opt)}$ in place of α_0 in (2.1) d_u yields the asymptotically optimum estimator (AOE) for $\bar{Y}$ as

$$d_u^{(0)} = \bar{y}_u \exp\left[\frac{2\beta_{yz}\ (\bar{Z} - \bar{z}_u)}{R_z(\bar{Z} + \bar{z}_u)}\right]. \tag{2.7}$$

Putting $\alpha_{0(opt)}$ in (2.5) we get the minimum variance of d_u or the variance of the AOE $d_u^{(0)}$ as

$$Min.Var(d_u) = Var(d_u^{(0)}) = (S_y^2/u)(1-\rho_{yz}^2) \tag{2.8}$$

It is to be mentioned that the $AOE d_u^{(0)}$ in (2.7) depends on the unknown parameters (β_{yz},R_z) which lacks the practical utility of $AOE\ d_u^{(0)}$. However, the values of parameters (β_{yz}, R_z) can be guessed quite accurately either through past data or experience gathered in due course of time. For discussion on this subject the reader is referred to Reddy [20], Srivenkataramana and Tracy [21], and Singh and Ruiz-Espezo [22] among others. If the guessed values of the parameters (β_{yz}, R_z) are not available, then it is worth advisable to replace (β_{yz}, R_z) by their consistent estimators $[(b_{yz}^{(u)}, \hat{R}_z^{(u)}) = (\bar{y}_u/\bar{Z}),\ (\bar{Z}$ is the known population mean of the variable $z)]$ respectively in (2.7). Thus, we get an estimator of the population mean $\bar{Y}$ based on estimated optimum value $\hat{\alpha}_{0(opt)} = b_{yz}^{(u)}\bar{Z}/\bar{y}_u$ as

$$\hat{d}_u^{(0)} = \bar{y}_u \exp\left[\frac{2b_{yz}^{(u)}\bar{Z}(\bar{Z} - \bar{z}_u)}{\bar{y}_u(\bar{Z} + \bar{z}_u)}\right]. \tag{2.9}$$

Following [Sukhatme et al. ([23], p. 238-239)], it can be shown that

$$E(b_{yz}^{(u)}) = \beta_{yz} + \left(\frac{1}{u}\right)\left(\frac{N-n}{N-2}\right)\beta_{yz}\left(\frac{\mu_{003}}{\mu_{002}} - \frac{\mu_{012}}{\mu_{011}}\right),$$

or

$$E(b_{yz}^{(u)}) = \beta_{yz} + \left(\frac{1}{u}\right)\left(\frac{N}{N-2}\right)\left(1-\frac{u}{N}\right)\beta_{yz}\left(\frac{\mu_{003}}{\mu_{002}} - \frac{\mu_{012}}{\mu_{011}}\right),$$

where $\mu_{rst} = E[(x_i - \bar{X})^r (y_i - \bar{Y})^s (z_i - \bar{Z})^t]$, (r, s, t) being non-negative integers. For large N,$(N/(N-2)) \cong 1$ and $(u/N) \cong 0$, we have

$$E(b_{yz}^{(u)}) = \beta_{yz} + \left(\frac{\beta_{yz}}{u}\right)\left(\frac{\mu_{003}}{\mu_{002}} - \frac{\mu_{012}}{\mu_{011}}\right). \tag{2.10}$$

which can be written as

$$E(b_{yz}^{(u)} = \beta_{yz} + O(u^{-1}).$$

Also from (2.10) we have

$$E\left(\frac{b_{yz}^{(u)} - \beta_{yz}}{\beta_{yz}}\right) = O(u^{-1}) \Longrightarrow E(\delta b_{yz}^{(u)}) = O(u^{-1}).$$

Now, expressing (2.9) in terms of e_{zu}, e_{yu} and $\delta b_{yz}^{(u)}$ we have

$$\begin{aligned}
\hat{d}_u^{(0)} =& \ \bar{Y}(1+e_{yu})exp\left[\frac{(-2\beta_{yz}(1+\delta\ b_{yz}^{(u)})\bar{Z}e_{zu})}{(\bar{Y}(1+e_{yz})(2+e_{zu}))}\right] \\
=& \ \bar{Y}(1+e_{yu})exp[(-\beta_{yz}\bar{Z}(1+\delta\ b_{yz}^{(u)})e_{zu})/(\bar{Y}(1+e_{yu})(1+(e_{zu}/2))^{-1}] \\
=& \ \bar{Y}(1+e_{yu})[1-\{(-\beta_{yz}\bar{Z}(1+\delta\ b_{yz}^{(u)})e_{zu})/(\bar{Y}(1+e_{yu}))(1+(e_{zu}/2))^{-1}\}+\cdots] \\
=& \ \bar{Y}(1+e_{yu}) - \beta_{yz}\bar{Z}e_{zu},
\end{aligned}$$

or

$$(\hat{d}_u^{(0)} - \bar{Y}) = (\bar{Y}e_{yu} - \beta_{yz}\bar{Z}e_{zu}). \tag{2.11}$$

Squaring both sides of (2.11) we have

$$(\hat{d}_u^{(0)} - \bar{Y})^2 = (\bar{Y}^2 e_{yu}^2 + \beta_{yz}^2 \bar{Z}^2 e_{zu}^2 - 2\beta_{yz}\bar{Y}\bar{Z}e_{yu}e_{zu}). \tag{2.12}$$

Taking expectation of both sides of (2.12), we get the variance of $\hat{d}_u^{(0)}$ to the first degree of approximation (ignoring fpc terms) as

$$Var(\hat{d}_u^{(0)}) = (1/u)(S_y^2 + \beta_{yz}^2 S_z^2 - 2\beta_{yz}^2 S_z^2) = (1/u)S_y^2(1-\rho_{yz}^2), \tag{2.13}$$

which equals to the minimum variance of d_u given by (2.8). Thus, the estimator $\hat{d}_u^{(0)}$ based on estimated optimum value can be used in practice.

2.2. A class of estimators based on the matched sample common to both occasions

We define the following class of estimators for population mean $\bar{Y}$ as

$$d_m = \bar{y}_m \exp\left[\frac{\alpha_1(\bar{x}_n - \bar{x}_m)}{(\bar{x}_n + \bar{x}_m)} + \frac{\alpha_2(\bar{z}_n - \bar{z}_m)}{(\bar{z}_n + \bar{z}_m)} + \frac{\alpha_3(\bar{Z} - \bar{z}_n)}{(\bar{Z} + \bar{z}_n)}\right] \tag{2.14}$$

where $\alpha_i' s$ $(i = 1, 2, 3)$ are suitably chosen scalars.

To obtain the variance of d_m we write $\bar{y}_m = \bar{Y}(1 + e_{ym})$, $\bar{x}_m = \bar{X}(1 + e_{xm})$, $\bar{z}_m = \bar{Z}(1 + e_{zm})$, $\bar{x}_n = \bar{X}(1 + e_{xn})$ and $\bar{z}_n = \bar{Z}(1 + e_{zn})$, such that

$$E(e_{ym}) = E(e_{xm}) = E(e_{xn}) = E(e_{zm}) = E(e_{zn}) = 0$$

and, ignoring the finite population correction (fpc) terms, we have

$$\begin{aligned}
&E(e_{ym}^2) = m^{-1}C_y^2,\ E(e_{xm}^2) = m^{-1}C_x^2,\ E(e_{zm}^2) = m^{-1}C_z^2\\
&E(e_{xn}^2) = n^{-1}C_x^2,\ E(e_{zn}^2) = n^{-1}C_z^2,\\
&E(e_{ym}e_{xm}) = m^{-1}\rho_{yx}C_yC_x,\ E(e_{ym}e_{xn}) = n^{-1}\rho_{yx}C_yC_x,\ E(e_{ym}e_{zm}) = m^{-1}\rho_{yz}C_yC_z,\\
&E(e_{xm}e_{xn}) = n^{-1}C_x^2,\ E(e_{xm}e_{zm}) = m^{-1}\rho_{xz}C_xC_z,\ E(e_{xm}e_{zn}) = n^{-1}\rho_{xz}C_xC_z,\\
&E(e_{xn}e_{zm}) = n^{-1}\rho_{xz}C_xC_z,\ E(e_{xn}e_{zn}) = n^{-1}\rho_{xz}C_xC_z \text{ and } E(e_{zm}e_{zn}) = n^{-1}C_z^2,
\end{aligned}$$

Expressing (2.14) in terms ofe_{ym}, e_{xm}, e_{zm}, e_{xn} and e_{zn} we have

$$\begin{aligned}
d_m &= \bar{Y}(1 + e_{ym})\exp[-(\alpha_1/2)(e_{xm} - e_{xn})\{1 + ((e_{xm} + e_{xn})/2)\}^{-1}\\
&\quad -(\alpha_2/2)(e_{zm} - e_{zn})\{1 + ((e_{zm} + e_{zn})/2)\}^{-1} - (\alpha_3)(e_{zn}/2)\{1 + (e_{zn}/2)\}^{-1}]\\
&\cong \bar{Y}[1 + e_{ym} - (\alpha_1/2)(e_{xm} - e_{xn}) - (\alpha_2/2)(e_{zm} - e_{zn}) - (\alpha_3/2)(e_{zn})]
\end{aligned}$$

or

$$(d_m - \bar{Y}) = \bar{Y}[e_{ym} - (\alpha_1/2)(e_{xm} - e_{xn}) - (\alpha_2/2)(e_{zm} - e_{zn}) - (\alpha_3/2)e_{zn}]. \tag{2.15}$$

Squaring both sides of (15), we have

$$\begin{aligned}
(d_m - \bar{Y})^2 = \bar{Y}^2[&e_{ym}^2 + (\alpha_1^2/4)(e_{xm} - e_{xn})^2 + (\alpha_2^2/4)(e_{zm} - e_{zn})^2\\
&+ (\alpha_3^2/4)(e_{zn}^2) - \alpha_1(e_{ym}e_{xm} - e_{ym}e_{xn})\\
&- \alpha_2(e_{ym}e_{zm} - e_{ym}e_{zn}) - \alpha_3(e_{ym}e_{zn})\\
&+ (\alpha_1\alpha_2/2)(e_{xm}e_{zm} - e_{xn}e_{zm} - e_{xm}e_{zn} + e_{xn}e_{zn})\\
&+ (\alpha_1\alpha_3/2)(e_{xm}e_{zn} - e_{xn}e_{zn}) + (\alpha_2\alpha_3/2)(e_{zm}e_{zn} - e_{zn}^2)].
\end{aligned} \tag{2.16}$$

Taking expectation of both sides (2.16), we get the variance of d_m to the first degree of approximation (ignoring fpc terms) as

$$
\begin{aligned}
Var(d_m) &= [(1/m)S_y^2 + ((1/m) - (1/n))(1/4)\{R_x^2 S_x^2 \alpha_1^2 + R_z^2 S_z^2 \alpha_2^2 \\
&- 4\alpha_1 R_x \rho_{yx} S_y S_x - 4\alpha_2 R_z \rho_{yz} S_y S_z + 2\alpha_1 \alpha_2 R_x R_z \rho_{xz} S_x S_z\} \\
&+ (1/4n)(R_z^2 S_z^2 \alpha_3^2 - 4\alpha_3 R_z S_y S_z \rho_{yz})] \\
&= [(1/m)S_y^2 + ((1/m) - (1/n))(1/4)\{R_x^2 S_x^2 \alpha_1(\alpha_1 - (4\beta_{yx}/R_x)) \\
&+ R_z^2 S_z^2 \alpha_2(\alpha_2 - (4\beta_{yz}/R_z)) + 2\alpha_1 \alpha_2 R_x R_z \rho_{xz} S_x S_z\} \\
&+ (R_z^2 S_z^2 \alpha_3/4n)(\alpha_3 - (4\beta_{yz}/R_z))].
\end{aligned} \tag{2.17}
$$

Differentiating (2.17) with respect to $\alpha_1, \alpha_2, \alpha_3$ and equating them to zero, we get

$$
\begin{cases}
\{\alpha_1 + \alpha_2 R_z(\beta_{zx}/R_x)\} = 2(\beta_{yx}/R_x), \\
\{\alpha_1 R_x(\beta_{zx}/R_z) + \alpha_2\} = 2(\beta_{yz}/R_z), \\
\alpha_3 = 2(\beta_{yz}/R_z).
\end{cases} \tag{2.18}
$$

Solving (2.18) we get the optimum values of α_1,α_2 and α_3 respectively, as

$$
\begin{cases}
\alpha_{1(opt)} = 2\{(\beta_{yx} - \beta_{yz}\beta_{zx})/R_x(1 - \beta_{xz}\beta_{zx})\} = 2(\beta_{yx.z}/R_x), \\
\alpha_{2(opt)} = 2\{(\beta_{yz} - \beta_{yx}\beta_{xz})/R_x(1 - \beta_{xz}\beta_{zx})\} = 2(\beta_{yz.x}/R_z), \\
\alpha_{3(opt)} = 2(\beta_{yz}/R_z).
\end{cases} \tag{2.19}
$$

Substitution of the optimum values $\alpha_{i(opt)}$'s $(i = 1, 2, 3)$ in (2.14) we get the asymptotically optimum estimator (AOE) in the class of estimators d_m as

$$
d_m^{(0)} = \bar{y}_m \exp\left[\frac{2\left(\frac{\beta_{yx.z}}{R_x}\right)(\bar{x}_n - \bar{x}_m)}{(\bar{x}_n + \bar{x}_m)} + \frac{2\left(\frac{\beta_{yz.x}}{R_z}\right)(\bar{z}_n - \bar{z}_m}{(\bar{z}_n + \bar{z}_m)} + \frac{2\left(\frac{\beta_{yz}}{R_z}\right)(\bar{Z} - \bar{z}_n)}{(\bar{Z} + \bar{z}_n)}\right] \tag{2.20}
$$

Substituting the optimum values $\alpha_{i(opt)}$ $(i{=}1, 2, 3)$ in the place of α_i's $(i{=}1, 2, 3)$ in (2.17) we get the minimum variance of d_m or the variance of the AOE $d_m^{(0)}$ as

$$
Min.Var(d_m) = (S_y^2/m)[1 - \rho_{yz}^2 - \theta(R_{y.xz}^2 - \rho_{yz}^2)] = Var(d_m^{(0)}), \tag{2.21}
$$

where $R_{y.xz}^2 = (\rho_{yx}^2 + \rho_{yz}^2 - 2\rho_{yx}\rho_{yz}\rho_{zx})/(1 - \rho_{xz}^2)$. It is observed from (2.20) that the AOE depends on the unknown parameters $(\beta_{yz}, \beta_{yx.z}, \beta_{yz.x}, R_x, R_z)$ which limits the practical utility of the AOE $d_m^{(0)}$. However, one can get the guessed values of the parameters$(\beta_{yz}, \beta_{yx.z}, \beta_{yz.x}, R_x, R_z)$ quite accurately either through the past data or experience gathered in due course of time. For detailed discussion on this issue the reader is referred to Murthy [24], Reddy [20], and Srivenkataramana and Tracy [21], among others. Thus, the guessed values of the parameters $(\beta_{yz}, \beta_{yx.z}, \beta_{yz.x}, R_x, R_z)$ can be obtained in practice and used in the AOE for its use in practice. Alternately, if the guessed values of the parameters $(\beta_{yz}, \beta_{yx.z}, \beta_{yz.x}, R_x, R_z)$ cannot be made available, then it is worth advisable to replace them by their consistent estimators $(b_{yz}^{(n)}, b_{yx.z}^{(m)}, b_{yz.x}^{(m)}, \hat{R}_x = (\bar{y}_m/\bar{x}_n)$,

$\hat{R}_z = (\bar{y}_m/\bar{Z})$) as respectively in (2.20). Thus, we define an estimator of the population $\bar{Y}$ (based on the estimated optimum values) as

$$\begin{aligned} \hat{d}_m^{(0)} &= \bar{y}_m \exp\left[\left\{2b_{yx.z}^{(m)} \frac{\bar{x}_n(\bar{x}_n-\bar{x}_m)}{\bar{y}_m(\bar{x}_n+\bar{x}_m)}\right\} + \left\{2b_{yz.x}^{(m)} \frac{\bar{Z}(\bar{z}_n-\bar{z}_m)}{(\bar{z}_n+\bar{z}_m)}\right\}\right. \\ &\quad \left. + \left\{2b_{yz}^{(n)} \frac{\bar{Z}(\bar{Z}-\bar{z}_n)}{\bar{y}_n(\bar{z}_n+\bar{Z})}\right\}\right], \end{aligned} \tag{2.22}$$

To obtain the variance of (2.22) we further, write

$$b_{yz}^{(n)} = \beta_{yz}(1+\delta b_{yz}^{(n)}),\ b_{yx.z}^{(m)} = \beta_{yx.z}(1+\delta b_{yx.z}^{(m)}),\ b_{yz.x}^{(m)} = \beta_{yz.x}(1+\delta b_{yz.x}^{(m)})$$

such that

$$\begin{aligned} &E(b_{yz}^{(n)}) = \beta_{yz} + O(n^{-1}) \Longrightarrow E(\delta b_{yz}^{(n)}) = O(n^{-1}), \\ &E(b_{yx.z}^{(m)}) = \beta_{yz} + O(m^{-1}) \Longrightarrow E(\delta b_{yx.z}^{(m)}) = O(m^{-1}), \\ &E(b_{yz.x}^{(n)}) = \beta_{yz} + O(m^{-1}) \Longrightarrow E(\delta b_{yz.x}^{(m)}) = O(m^{-1}). \end{aligned}$$

Expressing (2.22) in terms of $e_{ym}, e_{xm}, e_{zm}, e_{xn}, e_{zn}, \delta b_{yz}^{(n)}, \delta b_{yx.z}^{(m)}$ and $\delta b_{yz.x}^{(m)}$, we have

$$\begin{aligned} \hat{d}_m^{(0)} = \bar{Y}(1+e_{ym}) \exp\left[\left\{\frac{-(\beta_{yx.z}\bar{X}(1+\delta b_{yx.z}^{(m)})(1+e_{xn}))(e_{xm}-e_{xn})}{\bar{Y}(1+e_{ym})}\right\}\left\{1+\frac{(e_{xm}+e_{xn})}{2}\right\}^{-1}\right. \\ -\left\{\frac{(\beta_{yz.x}\bar{Z}(1+\delta b_{yz.x}^{(m)}))(e_{zm}-e_{zn})}{\bar{Y}(1+e_{ym})}\right\}\left\{1+\frac{(e_{zm}+e_{zn})}{2}\right\}^{-1} \\ \left.-\left\{\frac{(\beta_{yz}(1+\delta b_{yz}^{(n)})\bar{Z}e_{zn})}{\bar{Y}(1+e_{ym})}\right\}(1+e_{zn})^{-1}\right] \end{aligned}$$

which may be approximately written as

$$\hat{d}_m^{(0)} \cong [\bar{Y}(1+e_{ym}) - \beta_{yx.z}\bar{Z}(e_{xm}-e_{xn}) - \beta_{yz.x}\bar{Z}(e_{zm}-e_{zn}) - \beta_{yz}\bar{Z}e_{zn}],$$

or

$$(\hat{d}_m^{(0)} - \bar{Y}) \cong [\bar{Y}e_{ym} - \beta_{yx.z}\bar{Z}(e_{xm}-e_{xn}) - \beta_{yz.x}\bar{Z}(e_{zm}-e_{zn}) - \beta_{yz}\bar{Z}e_{zn}]. \tag{2.23}$$

Squaring both sides of (2.23) and then taking the expectation, we get the variance $\hat{d}_m^{(0)}$ to the first degree of approximation (ignoring fpc terms) as

$$Var(\hat{d}_m^{(0)}) = (S_y^2/m)[1-\rho_{yz}^2 - \theta(R_{y.xz}^2 - \rho_{yz}^2)], \tag{2.24}$$

which equals to the minimum variance of d_m given by (2.21). Thus, the estimator $\hat{d}_m^{(0)}$ in (2.22) is more useful in practice as it does not depend on the unknown constant.

2.3. Combined estimator of the population mean on second (current) occasion

Now, for estimating the population mean $\bar{Y}$ on second (current) occasion, we define the following estimator of $\bar{Y}$ as

$$d = [(1-\phi)\hat{d}_u^{(0)} + \phi\hat{d}_m^{(0)}], \tag{2.25}$$

where ϕ is a constant to be determined under certain criterion, and $\hat{d}_u^{(0)}$ and $\hat{d}_m^{(0)}$ are respectively defined in (2.9) and (2.22). The variance of d is given by

$$Var(d) = [(1-\phi)^2 Var(\hat{d}_u^{(0)}) + \phi^2 Var(\hat{d}_m^{(0)}) + 2Cav(\hat{d}_u^{(0)}, \hat{d}_m^{(0)})].$$

Under the supposition that population size N is large enough so that finite population correction (fpc) terms, are ignored [i.e., $(n/N) \cong 0$, $(m/N) \cong 0$, $(u/N) \cong 0$], the covariance term will be negligible,i.e., $Cav(\hat{d}_u^{(0)}, \hat{d}_m^{(0)}) \cong 0$. We note that to the first order of approximation, $Var(d) \cong MSE(d)$, where $MSE(\bullet)$ stands for mean square error $(\bullet)$.

Now, using expressions in (2.13) and (2.24), we get the variance of d to the first degree of approximation (ignoring fpc terms) as

$$\begin{aligned} Var(d) = (S_y^2/n\theta(1-\theta))[(1-\theta)(1-\rho_{yz}^2)+\phi^2\{(1-\rho_{yz}^2)-\theta^2(R_{y.xz}^2-\rho_{yz}^2)\} \\ - 2\phi(1-\theta)(1-\rho_{yz}^2)], \end{aligned} \tag{2.26}$$

which is minimum when

$$\phi = \frac{(1-\theta)(1-\rho_{yz}^2)}{\{(1-\rho_{yz}^2)-\theta^2(R_{y.xz}^2-\rho_{yz}^2)\}} = \phi_(opt), (say). \tag{2.27}$$

Thus, the resulting minimum variance of d is given by

$$Min.Var(d) = \left(\frac{S_y^2}{n}\right)\frac{[(1-\rho_{yz}^2)-\theta(R_{y.xz}^2-\rho_{yz}^2)]}{[(1-\rho_{yz}^2)-\theta^2(R_{y.xz}^2-\rho_{yz}^2)]}. \tag{2.28}$$

Singh and Homa [17] have proposed an estimator of the population mean $\bar{Y}$ as

$$d_1 = [(1-\phi_1)d_{1u} + \phi_1 d_{1m}], \tag{2.29}$$

where ϕ_1 is unknown constant to be determined so as to make the estimator d_1 is more precise,

$$d_{1u} = \bar{y}_u exp\left[\frac{(\bar{Z}-\bar{Z}_u)}{(\bar{Z}+\bar{Z}_u)}\right] \tag{2.30}$$

is an estimator of the population mean $\bar{Y}$ based on the fresh sample of size $u = (n\theta)$ drawn on the second occasion, for instance, see Bahl and Tuteja [25];

$$d_{1m} = \bar{y}_m \exp\left[\frac{(\bar{Z}-\bar{Z}_m)}{(\bar{Z}+\bar{Z}_m)}\right] + b_{yx}^{(m)}\left[\bar{x}_n \exp\frac{(\bar{Z}-\bar{Z}_n)}{(\bar{Z}+\bar{Z}_n)} - \bar{x}_m \exp\frac{(\bar{Z}-\bar{Z}_m)}{(\bar{Z}+\bar{Z}_m)}\right], \tag{2.31}$$

is an estimator based on the sample of size $m(=n(1-\theta))$ common to both the occasions. Singh and Homa [17] have derived the mean squared error / variance of d_1 up to first order of approximation under the supposition $(C_x \cong C_y \cong C_z)$.

Putting $\alpha_0 = 1$ and$(C_x \cong C_y \cong C_z)$ in (2.5), we get the variance of the estimator d_{1u} due to Singh and Homa [17] to the first degree of approximation (ignoring fpc term) as

$$Var(d_{1u}) = (S_y^2/u)\{(5/4) - \rho_{yz}\}. \tag{2.32}$$

When $(C_x \cong C_y \cong C_z)$, the correct variance expression of the estimator d_{1m} to the first degree of approximation (ignoring fpc terms) is given by

$$\begin{aligned} Var(d_{1m}) &= S_y^2[(1/m)\{(5/4) - \rho_{yz}\} \\ &-\{(1/m) - (1/n)\}\{\rho_{yx}^2((3/4) + \rho_{xz}) - \rho_{yx}(\rho_{yz} + \rho_{xz} - (1/2))\}]. \end{aligned} \tag{2.33}$$

From (2.24) and (2.33) we have

$$\begin{aligned} Var(d_{1m}) - Var(\hat{d}_m^{(0)}) &= (S_y^2/m)[(1-\theta)((1/2) - \rho_{yz})^2 \\ &+\theta\{[\rho_{yx}((1/2)-\rho_{xz})-((1/2)-\rho_{yz})]^2+\rho_{xz}^2(\rho_{yz}-\rho_{xz}\rho_{yx})^2/(1-\rho_{xz}^2)\}], \end{aligned} \tag{2.34}$$

which is positive. Thus, the proposed estimator$\hat{d}_m^{(0)}$ is more efficient than the estimator d_{1m} due to Singh and Homa [17]. So the use of $\hat{d}_m^{(0)}$ would be more profitable than the one considered by Singh and Homa [17].

It is to be noted that the variance expression of d_{1m} in (2.33) is correct while the variance expression of the estimatord_{1m} due to Singh and Homa [17] is incorrect. The variance of d_{1m} to the first degree of approximation (ignoring fpc terms) due to Singh and Homa [17] is given by

$$\begin{aligned} Var(d_{1m}) = S_y^2[(1/m)\{-\rho_{yx}^2((3/4) + \rho_{xz}) + \rho_{yx}(\rho_{yz} + \rho_{xz}-(1/2))-\rho_{yz}\} \\ + (1/n)\rho_{yx}\{((-3/4)\rho_{yx}) + (\rho_{xz}\rho_{yx}-\rho_{xz}+(1/2))\}]. \end{aligned} \tag{2.35}$$

[See Singh and Homa ([17] eq. (13), p.150), for large population size N, i.e., for large N, i.e., $(n/N) \cong 0$, $(m/N) \cong 0$, $(u/N) \cong 0$, $(1/N) \cong 0$,]

Using expressions (2.32) and (2.33), the minimum variance of Singh and Homa [17] estimator d_1 is given by

$$Min.Var(d_1) = \frac{S_y^2\delta_0(\delta_0 - \theta A)}{n(\delta_0 - \theta^2 A)}, \tag{2.36}$$

for the optimum value of ϕ_1:

$$\phi_{1(opt)} = \frac{\delta_0(1-\theta)}{(\delta_0 - \theta^2 A)}, \tag{2.37}$$

where $\delta_0 = ((5/4) - \rho_{yz})$ and $A = [\rho_{yx}^2((3/4) + \rho_{xz}) - \rho_{yx}(\rho_{yz} + \rho_{xz} - (1/2))]$. From (2.28) and (2.36) we have

$$\begin{aligned} &Min.Var(d_1) - Min.Var(d) \\ &= \frac{S_y^2[\delta_0(1-\rho_{yz}^2)A^* + A^*\{(1-\rho_{yz}^2)-\theta(R_{y.xz}^2-\rho_{yz}^2)\}((1/2)-\rho_{yz})^2]}{n(\delta_0-\theta^2 A)[(1-\rho_{yz}^2)-\theta^2(R_{y.xz}^2-\rho_{yz}^2)]} \geq 0, \end{aligned} \tag{2.38}$$

where

$$A^* = [(1-\theta)((1/2)-\rho_{yz})^2 + \theta\{[\rho_{yx}((1/2)-\rho_{xz}) - ((1/2)-\rho_{yz})]^2 + (\rho_{xz}^2(\rho_{yz}-\rho_{yx}\rho_{xz})^2/(1-\rho_{xz}^2))\}]$$

and

$$A_1^* = (\delta_0 - \theta A).$$

It follows from (2.38) that the proposed estimator d is more efficient than the Singh and Homa [17] estimator d_1.

Minimizing (2.36) with respect to θ we obtain the optimum value of θ as

$$\theta_{opt} = \frac{\delta_0 \pm \sqrt{\delta_0(\delta_0 - A)}}{A}. \tag{2.39}$$

The real values of θ_{opt} exist; if $(\delta_0 - A) \geq 0$. For any combination of correlation coefficients $(\rho_{yx}, \rho_{yz}, \rho_{zx})$ that satisfies the condition of real solution, two real values of θ_{opt} are possible. Hence, while selecting the value of θ_{opt}, it should be remembered that $0 \leq \theta_{opt} \leq 1$, all other values of are inadmissible. If both the solutions of θ_{opt} are admissible, the lowest one is the best choice as it will reduce the cost of survey. Putting the admissible value of θ_{opt}, say θ_0^*, from (2.39) in (2.36), we get the resulting value of $Min.Var(d_1)$ as

$$Min.Var(d_1)_{opt} = \frac{S_y^2\delta_0(\delta_0 - \theta_0^* A)}{n(\delta_0 - \theta_0^{*2} A)}. \tag{2.40}$$

3. Optimum replacement policy

To obtain the optimum value of θ (fraction of a sample to be taken fresh at the second occasion) so that the population mean $\bar{Y}$ may be estimated with maximum precision. Minimization of minimum variance of d in (2.28) with respect to θ yields the optimum value of θ as

$$\hat{\theta} = \frac{1 \pm \sqrt{(1-\rho_{yx.z}^2)}}{\rho_{yx.z}^2} = \theta_{(opt)}(say), \tag{3.1}$$

where

$$\rho_{yx.z} = (\rho_{yx} - \rho_{yz}\rho_{xz})/\sqrt{(1-\rho_{yx}^2)}\sqrt{(1-\rho_{yx}^2)}, \tag{3.2}$$

Since $(1-\rho_{yx.z}^2) \geq 0$, for any combination ofρ_{yx}, ρ_{yz} and ρ_{xz} two real values of $\hat{\theta}$ are possible, hence, to select a value of $\hat{\theta}$, it should be remembered that $0 \leq \hat{\theta} \leq 1$. All other values of $\hat{\theta}$ are inadmissible. In case if both the solutions of $\hat{\theta}$ are admissible, we select the minimum of these two as θ_0, where

$$\theta_0 = \frac{\left\{1 - \sqrt{(1-\rho_{yx.z}^2)}\right\}}{\rho_{yx.z}^2} \tag{3.3}$$

Putting θ_0 in (2.28) we get the resulting value of $Min.Var(d)$ as

$$Min.Var(d)_{opt} = \frac{S_y^2(1-\rho_{yx}^2)\left[1-\sqrt{(1-\rho_{yx.z}^2)}\right]}{2n}. \tag{3.4}$$

4. Efficiency comparisons

The percent relative efficiency of the proposed estimator d with respect to (i)$\bar{y}_n$, when there is no matching, (ii) $\bar{y}_\psi = (1-\psi)+\psi\bar{y}_{lm}$, when no auxiliary information is used at any occasion, where $[\bar{y}_{lm} = \bar{y}_m + b_{yx}^{(m)}(\bar{x}_n - \bar{x}_m)]$; ψ being suitably chosen scalar and (iii) Singh and Homa's [17] estimator d_1; have been obtained for different choices of ρ_{xz}, ρ_{yz} and ρ_{yx}.

To the first degree of approximation and for large population size N, i.e., $N \longrightarrow \infty$, i.e., $[(n/N) \cong 0,\ (m/N) \cong 0, (u/N) \cong 0,\ (1/N) \cong 0]$, the minimum variance of $\bar{y}_\psi$ is given by

$$Min.Var(\bar{y}_\psi) = \frac{S_y^2(1-\theta\rho_{yx}^2)}{n(1-\theta^2\rho_{yx}^2)}. \tag{4.1}$$

for the optimal value of ψ:

$$\psi_{opt} = (1-\theta)/(1-\theta^2\rho_{yx}^2). \tag{4.2}$$

Minimization of (4.1) with respect to θ yields the optimal value of θ as

$$\theta_{opt} = \left[1+\sqrt{(1-\rho_{yz}^2)}\right]^{-1}. \tag{4.3}$$

Thus, the resulting value of $Min.Var(\bar{y}_\psi$ is given by

$$Min.Var(\bar{y}_\psi)_{opt} = \frac{S_y^2}{2n}\left[1+\sqrt{(1-\rho_{yz}^2)}\ \right] \tag{4.4}$$

[see Cochran ([26], p. 346)].

The variance of $\bar{y}_n$ for large population size N, i.e., $[N \longrightarrow \infty$, i.e., $(n/N) \cong 0]$ is given by $Var(\bar{y}_n = (S_y^2/n)$.

The percent relative efficiencies (PREs) of the proposed estimator d with respect to $\bar{y}_n$, $\bar{y}_\psi$ and Singh and Homa's [17] estimator d_1 have been computed by using the formulae:

$$E_1 = \frac{2}{(1-\rho_{yz}^2)\left[1+\sqrt{(1-\rho_{yx.z}^2)}\ \right]} \times 100, \tag{4.5}$$

$$E_2 = \frac{\left[1+\sqrt{(1-\rho_{yz}^2)}\ \right]}{(1-\rho_{yz}^2)\left[1+\sqrt{(1-\rho_{yx.z}^2)}\ \right]} \times 100, \tag{4.6}$$

$$E_3 = \frac{2\delta_0(\delta_0-\theta_0^* A)}{(\delta_0-\theta_0^{*2}A)(1-\rho_{yz}^2)\left[1+\sqrt{(1-\rho_{yx.z}^2)}\ \right]} \times 100. \tag{4.7}$$

for different combinations of $(\rho_{xz}, \rho_{yz}, \rho_{yx})$ and findings are shown in Table 1. We have also computed the values of θ_0^* and θ_0 by using the formula (2.38) and (3.3) for the same combination of $(\rho_{xz},\ \rho_{yz},\ \rho_{yx})$ for which E_1, E_2, E_3 have been computed and the results are given in Table 1.

Table 1: Optimum values (θ_0, θ_0^*) and $PREs$ of the proposed estimator with respect to $\bar{y}_n, \bar{y}_\psi$ and Singh and Homa [17] estimator d_1

ρ_{xz}	ρ_{yz}	ρ_{yx}	0.3	0.4	0.5	0.6	0.7	0.8
0.9	0.65	θ_0	0.662	0.547	0.509	0.500	0.516	0.568
		θ_0^*	0.469	0.471	0.479	0.493	0.516	0.556
		E_1	229.405	189.297	176.109	173.249	178.718	196.691
		E_2	224.122	181.395	164.312	155.924	153.174	157.353
		E_3	146.608	120.535	110.406	105.486	103.840	106.213
0.8	0.75	θ_0	0.604	0.537	0.508	0.000	0.508	0.537
		θ_0^*	0.462	0.463	0.470	0.483	0.506	0.545
		E_1	276.277	245.284	232.320	228.571	232.320	245.284
		E_2	269.914	235.045	216.757	205.714	199.114	196.227
		E_3	149.350	132.411	123.661	118.262	114.719	112.479
0.7	0.85	θ_0	0.617	0.539	0.508	0.500	0.510	0.544
		θ_0^*	0.453	0.452	0.457	0.470	0.493	0.531
		E_1	444.736	388.493	366.296	360.376	367.666	392.019
		E_2	434.494	372.276	341.759	324.339	315.116	313.615
		E_3	196.469	171.903	160.134	153.300	149.285	147.648
0.6	0.90	θ_0	0.580	0.522	0.502	0.504	0.530	0.600
		θ_0^*	0.449	0.447	0.452	0.465	0.487	0.525
		E_1	610.053	549.428	528.059	530.270	557.383	631.683
		E_2	596.003	526.493	492.685	477.243	477.717	505.346
		E_3	237.947	214.914	204.234	199.644	200.309	210.472

It is observed from Table 1 that:

- for fixed values of (ρ_{xz}, ρ_{yz}), no trends for θ_0, θ_0^* and E_i $'s$ $(i = 1, 2, 3)$ are observed as the value of ρ_{yx} increases,
- for fixed values of (ρ_{yx}, ρ_{yz}), the values ofE_i's $(i = 1, 2, 3)$ increase while the value of θ_0^* decreases and no trend is observed forθ_0 as the value of ρ_{xz} decreases. Similar patterns are observed for fixed value of (ρ_{xz}, ρ_{yx}), and increasing values of ρ_{yz}.

It is further observed from Table 1 that the values of E_i's $(i = 1, 2, 3)$ are greater than 100 which follow that the proposed estimator d is more efficient than the usual unbiased estimator $\bar{y}_n$, $\bar{y}_\psi$ and the Singh and Homa [17] estimator d_1 with appreciable gain in efficiency. Thus, we recommend our proposed estimator d for its use in practice.

5. Conclusion

It is well established fact that in successive sampling, it is advantageous to utilize the entire information gathered in the previous investigation. The information on an auxiliary variable is commonly available for all the units of a finite population in survey sampling. In this paper, an exponential type regression estimator has been proposed using information on the auxiliary variable in successive (rotation) sampling over two occasions. We have studied the properties of the proposed estimator and compared with (i) $\bar{y}_n$, when there is no matching, (ii) $\bar{y}_\psi$, when no auxiliary information is used at any occasion, and (iii) the Singh and Homa's [17] estimatord_1 .It has been shown theoretically and empirically that the proposed estimator is better than $\bar{y}_n$, $\bar{y}_\psi$ and Singh and Homa's [17] estimator d_1 . Empirically it has been shown that the use of auxiliary information is highly rewarding in terms of the suggested estimator. From the values of Table 1 it is concluded that if information on moderately / highly positively correlated auxiliary variable is available in successive sampling, it is appropriate to use the envisaged estimator$d$.

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SOME NEW INTEGRAL INEQUALITIES OF HADAMARD-SIMPSON TYPE FOR EXTENDED (s,m)-PREINVEX FUNCTIONS

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Abstract. The (s,m)-preinvex function and the extended (s,m)-preinvex function are firstly introduced. An integral identity for differentiable functions is then derived. Based on this integral identity, some Hadamard-Simpson type integral inequalities are investigated, which generalizes the existing similar type integral inequalities.

Keywords: (s,m)-preinvex functions, extended (s,m)-preinvex functions, Hadamard type integral inequality, Simpson type integral inequality.

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1. Introduction

The following notation is used throughout this paper. I is an interval on the real line $\mathbb{R}$. $\mathbb{R}^n$ is used to denote a generic n-dimensional vector space, $\mathbb{R}^n_0$ denotes an n-dimensional nonnegative vector space, and $\mathbb{R}^n_+$ denotes an n-dimensional positive vector space. For any subset $K \subseteq \mathbb{R}^n$, K° is used to denote the interior of K. $\mathbb{R}_0 = [0,\infty)$. $L_1[a,b]$ is the set of integrable functions over the interval $[a,b]$. Let us firstly recall some definitions of various convex type functions.

Definition 1.1 ([6]) A function $f : I \subseteq \mathbb{R} \to \mathbb{R}_0$ is said to be a Godunova-Levin function if f is nonnegative and for all $x, y \in I$, $\lambda \in (0,1)$ we have that

$$f\big(\lambda x + (1-\lambda)y\big) \leq \frac{f(x)}{\lambda} + \frac{f(y)}{1-\lambda}.$$

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Definition 1.2 ([5]) For some $(s,m) \in (0,1]^2$, a function $f : [0,b] \to \mathbb{R}$ is said to be (s,m)-convex in the second sense if, for every $x, y \in [0,b]$ and $\lambda \in [0,1]$, we have that

$$f\big(\lambda x + m(1-\lambda)y\big) \leq \lambda^s f(x) + m(1-\lambda)^s f(y).$$

Definition 1.3 ([34]) For some $s \in [-1,1]$ and $m \in (0,1]$, a function $f : [0,b] \to \mathbb{R}_0$ is said to be extended (s,m)-convex if for all $x, y \in [0,b]$ and $\lambda \in (0,1)$ we have that

$$f\big(\lambda x + m(1-\lambda)y\big) \leq \lambda^s f(x) + m(1-\lambda)^s f(y).$$

Definition 1.4 ([1]) A set $K \subseteq \mathbb{R}^n$ is said to be invex with respect to the map $\eta : K \times K \to \mathbb{R}^n$, if $x + t\eta(y,x) \in K$ for every $x, y \in K$ and $t \in [0,1]$.

Notice that every convex set is invex with respect to the map $\eta(y,x) = y - x$, but the converse is not necessarily true. For more details please refer to [1], [35] and the references therein.

Definition 1.5 ([1]) Let $K \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : K \times K \to \mathbb{R}^n$, for every $x, y \in K$, the η-path $P_{x\nu}$ joining the points x and $\nu = x + \eta(y,x)$ is defined by

$$P_{x\nu} = \big\{z | z = x + t\eta(y,x),\ t \in [0,1]\big\}.$$

Definition 1.6 ([25]) The function f defined on the invex set $K \subseteq \mathbb{R}^n$ is said to be preinvex with respect to η if for every $x, y \in K$ and $t \in [0,1]$ we have that

$$f\Big(x + t\eta(y,x)\Big) \leq (1-t)f(x) + tf(y).$$

The concept of preinvexity is more general than convexity since every convex function is preinvex with respect to the map $\eta(y,x) = y - x$ but the converse is not true.

Definition 1.7 ([12]) The function f defined on the invex set $K \subseteq [0,b^*]$ with $b^* > 0$ is said to be m-preinvex with respect to η if for all $x, y \in K$, $t \in [0,1]$ and for some fixed $m \in (0,1]$, we have that

$$f\Big(x + t\eta(y,x)\Big) \leq (1-t)f(x) + mtf\Big(\frac{y}{m}\Big).$$

Remark 1.1. Notice that if $y \in [0,b^*]$, then for any $0 < m < 1$, $\dfrac{y}{m}$ could be greater than b^*, which is not in the domain of f. Thus, the right hand side of the inequality in this definition could be meaningless. To fix this flaw, we suggest to replace $[0,b^*]$ by the half real line $\mathbb{R}_0$.

Definition 1.8 ([15]) Let $K \subseteq \mathbb{R}_0$ be an invex set with respect to η. A function $f : K \to \mathbb{R}$ is said to be s-preinvex with respect to η, if for all $x, y \in K$, $t \in [0,1]$ and some fixed $s \in (0,1]$ we have that

$$f\Big(x + t\eta(y,x)\Big) \leq (1-t)^s f(x) + t^s f(y).$$

Now, we are ready to review some inequalities of Hermite-Hadamard and Simpson type for these kinds of convex functions mentioned above.

Theorem 1.1 ([21]) *Let $f : K = [a, a + \eta(b, a)] \to (0, \infty)$ be a preinvex function on the interval of the real numbers K° and $a, b \in K^\circ$ with $a < a + \eta(b, a)$. Then the following inequality holds:*

$$f\left(\frac{2a+\eta(b,a)}{2}\right) \leq \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Inequality (1.1) is named by the Hermite-Hadamard-Noor type inequality for preinvex functions. This result is analogous to the original Hermite-Hadamard inequalities. If $\eta(b, a) = b - a$ then inequality (1.1) reduces to the remarkable Hermite-Hadamard's inequality.

Theorem 1.2 ([4]) *If $f : [a, b] \to \mathbb{R}$ is a four times continuously differentiable mapping on the interval (a, b) and $||f^{(4)}||_\infty = \sup_{x\in(a,b)} |f^{(4)}(x)| < \infty$ then the following inequality holds:*

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2} + 2f\Big(\frac{a+b}{2}\Big)\right] - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \leq \frac{1}{2880}||f^{(4)}||_\infty (b-a)^4. \tag{1.2}$$

Theorem 1.3 ([28]) *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L_1[a, b]$, where $a, b \in I^\circ$ with $a < b$. If $\frac{1}{p} + \frac{1}{q} = 1$ and $|f'|^q$ is convex on $[a, b]$, $q > 1$, then the following inequality holds:*

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+f(b)+4f\Big(\frac{a+b}{2}\Big)\right] - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \\ &\leq \frac{b-a}{12}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}\left\{\left(\frac{3|f'(b)|^q+|f'(a)|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{|f'(b)|^q+3|f'(a)|^q}{4}\right)^{\frac{1}{q}}\right\}.\end{aligned} \tag{1.3}$$

Theorem 1.4 ([27]) *Let $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L_1[a, b]$, where $a, b \in I$ with $a < b$. If $|f'|$ is s-convex on $[a, b]$, for some fixed $s \in (0, 1]$, then*

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+f(b)+4f\Big(\frac{a+b}{2}\Big)\right] - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \\ &\leq \frac{(s-4)6^{s+1}+2\times 5^{s+2}-2\times 3^{s+2}+2}{6^{s+2}(s+1)(s+2)}(b-a)\Big[|f'(a)|+|f'(b)|\Big].\end{aligned} \tag{1.4}$$

Theorem 1.5 ([33]) *Let $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I$ with $a < b$, $f' \in L_1[a, b]$ and $0 \leq \lambda, \mu \leq 1$. If $|f'(x)|^q$ for $q \geq 1$ is extended s-convex on $[a, b]$ for some $s \in [-1, 1]$, then*

1. *when $s=-1$, the following inequality holds:*

$$\begin{aligned}&\left|f\Big(\frac{a+b}{2}\Big)-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|\\ &\le\frac{b-a}{2^{3-\frac{2}{q}}}\left\{\Big[(2\ln 2-1)|f'(a)|^q+|f'(b)|^q\Big]^{\frac{1}{q}}+\Big[|f'(a)|^q+(2\ln 2-1)|f'(b)|^q\Big]^{\frac{1}{q}}\right\};\end{aligned}\tag{1.5}$$

2. *when $q=1$ and $s=-1$, the following inequality holds:*

$$\left|f\Big(\frac{a+b}{2}\Big)-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|\le(b-a)\ln 2\Big(|f'(a)|+|f'(b)|\Big).\tag{1.6}$$

Theorem 1.6 ([2], [26]) *Let $K\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta:K\times K\to\mathbb{R}$. Suppose that $f:K\to\mathbb{R}$ is a differentiable function. If $|f'|$ is preinvex on K then for every $a,b\in K$ with $\eta(b,a)\neq 0$ we have that:*

$$\begin{aligned}&\left|\frac{f(a)+f\big(a+\eta(b,a)\big)}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\mathrm{d}x\right|\\ &\qquad\le\frac{|\eta(b,a)|}{8}\Big(|f'(a)|+|f'(b)|\Big)\end{aligned}\tag{1.7}$$

and

$$\begin{aligned}&\left|f\left(\frac{2a+\eta(b,a)}{2}\right)-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\mathrm{d}x\right|\\ &\qquad\le\frac{|\eta(b,a)|}{8}\Big(|f'(a)|+|f'(b)|\Big).\end{aligned}\tag{1.8}$$

Theorem 1.7 ([31]) *Let $A\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta:A\times A\to\mathbb{R}$. Suppose that $f:A\to\mathbb{R}$ is a differentiable function. If $q>1$, $q\ge r$, $s\ge 0$ and $|f'|$ is preinvex on A, then for every $a,b\in A$ with $\eta(a,b)\neq 0$, we have that*

$$\begin{aligned}&\left|f\left(\frac{2b+\eta(a,b)}{2}\right)-\frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)}f(x)\mathrm{d}x\right|\\ &\le\frac{|\eta(a,b)|}{4}\left\{\left(\frac{1}{r+1}\right)^{\frac{1}{q}}\left(\frac{q-1}{2q-r-1}\right)^{1-\frac{1}{q}}\left[\frac{(r+1)|f'(a)|^q+(r+3)|f'(b)|^q}{2(r+2)}\right]^{\frac{1}{q}}\right.\\ &\quad\left.+\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left(\frac{q-1}{2q-s-1}\right)^{1-\frac{1}{q}}\left[\frac{(s+3)|f'(a)|^q+(s+1)|f'(b)|^q}{2(s+2)}\right]^{\frac{1}{q}}\right\}.\end{aligned}$$

Corollary 1.1 ([31]) *Under the conditions of Theorem 1.7, when $r=s=0$, the following inequality holds:*

$$\begin{aligned}&\left|f\left(\frac{2b+\eta(a,b)}{2}\right)-\frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)}f(x)\mathrm{d}x\right|\\ &\le\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}\frac{|\eta(a,b)|}{4}\left[\left(\frac{1}{4}|f'(a)|^q+\frac{3}{4}|f'(b)|^q\right)^{\frac{1}{q}}+\left(\frac{3}{4}|f'(a)|^q+\frac{1}{4}|f'(b)|^q\right)^{\frac{1}{q}}\right].\end{aligned}\tag{1.9}$$

Currently, the Hadamard type and Simpson type inequalities concerning different kinds of preinvex functions are still interesting topics to many researchers in the field of convex analysis. For more information please refer to [3], [7]-[14], [17]-[19], [22]-[24], [29]-[30], [32] and references cited therein.

Motivated by the inspiring idea in [20], [33] and based on our previous works [16], [17], [36], in this paper we are going to introduce the (s,m)-preinvex function and the extended (s,m)-preinvex function, and then we will establish some Hadamard-Simpson-like type integral inequalities for functions related to these two new kinds of functions. In Section 2, we will introduce new definitions and an integral identity. Section 3 will be devoted of presenting the main results.

2. New definitions and an integral identity

We now introduce definitions of the (s,m)-preinvex function and the extended (s,m)-preinvex function.

Definition 2.1 Let $K = [0,b^*]$ with $b^* > 0$ be an invex set with respect to $\eta : K \times K \to \mathbb{R}_+$. For $f : K \to \mathbb{R}$ and some fixed $(s,m) \in (0,1] \times (0,1]$, if

$$f\Big(x + t\eta(y,x)\Big) \leq (1-t)^s f(x) + mt^s f\Big(\frac{y}{m}\Big) \tag{2.1}$$

is valid for all $x, y \in K$, $t \in [0,1]$, with $\frac{y}{m} \leq b^*$, then we say that $f(x)$ is an (s,m)-preinvex function with respect to η.

Definition 2.2 Let $K = [0,b^*]$ with $b^* > 0$ be an invex set with respect to $\eta : K \times K \to \mathbb{R}_+$. For $f : K \to \mathbb{R}_0$ and some fixed $(s,m) \in [-1,1] \times (0,1]$, if

$$f\Big(x + t\eta(y,x)\Big) \leq (1-t)^s f(x) + mt^s f\Big(\frac{y}{m}\Big) \tag{2.2}$$

is valid for all $x, y \in K$, $t \in (0,1)$, with $\frac{y}{m} \leq b^*$, then we say that $f(x)$ is an extended (s,m)-preinvex function with respect to η.

Remark 2.1. In Definition 2.1, if $s = 1$ then one obtains the definition of m-preinvex function. If $m = 1$ then one obtains the definition of s-preinvex function. It is also worthwhile to note that every (s,m)-preinvex function is (s,m)-convex and every extended (s,m)-preinvex functions is extended (s,m)-convex with respect to $\eta(y,x) = y - x$ respectively.

In order to establish some new Hadamard-Simpson-like type integral inequalities, we present the following integral identity.

Lemma 2.1 *Let $K \subseteq \mathbb{R}$ be an invex subset with respect to $\eta : K \times K \to \mathbb{R}$ and $a, b \in K$ with $\eta(b,a) > 0$. If $0 \leq \lambda, \mu < 1$, $f : K \to \mathbb{R}$ is a differentiable function and f' is integrable on the η-path P_{ac} for $c = a + \eta(b,a)$, then we have that*

$$
\begin{aligned}
&-\left[(1-\lambda)f(a)+(\lambda-\mu)f\left(a+\frac{\eta(b,a)}{2}\right)+\mu f(a+\eta(b,a))\right]\\
&+\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\mathrm{d}x \qquad (2.3)\\
&=\eta(b,a)\left[\int_0^{\frac{1}{2}}(1-\lambda-t)f'(a+t\eta(b,a))\mathrm{d}t+\int_{\frac{1}{2}}^1(1-\mu-t)f'(a+t\eta(b,a))\mathrm{d}t\right].
\end{aligned}
$$

Proof. Set

$$
J=\eta(b,a)\left[\int_0^{\frac{1}{2}}(1-\lambda-t)f'\Big(a+t\eta(b,a)\Big)\mathrm{d}t+\int_{\frac{1}{2}}^1(1-\mu-t)f'\Big(a+t\eta(b,a)\Big)\mathrm{d}t\right].
$$

Since $a,b\in K$ and K is an invex set with respect to η , for every $t\in[0,1]$ we have $a+t\eta(b,a)\in K$. Integrating by part yields that

$$
\begin{aligned}
J &= \eta(b,a)\left\{\frac{1}{\eta(b,a)}\left[(1-\lambda-t)f\Big(a+t\eta(b,a)\Big)\Big|_0^{\frac{1}{2}}+\int_0^{\frac{1}{2}}f\Big(a+t\eta(b,a)\Big)\mathrm{d}t\right]\right.\\
&\quad\left.+\frac{1}{\eta(b,a)}\left[(1-\mu-t)f\Big(a+t\eta(b,a)\Big)\Big|_{\frac{1}{2}}^{1}+\int_{\frac{1}{2}}^1 f\Big(a+t\eta(b,a)\Big)\mathrm{d}t\right]\right\}\\
&= -(1-\lambda)f(a)-(\lambda-\mu)f\left(a+\frac{\eta(b,a)}{2}\right)-\mu f\Big(a+\eta(b,a)\Big)\\
&\quad+\int_0^{\frac{1}{2}}f\Big(a+t\eta(b,a)\Big)\mathrm{d}t+\int_{\frac{1}{2}}^1 f\Big(a+t\eta(b,a)\Big)\mathrm{d}t\\
&= -\left[(1-\lambda)f(a)+(\lambda-\mu)f\left(a+\frac{\eta(b,a)}{2}\right)+\mu f\Big(a+\eta(b,a)\Big)\right]\\
&\quad+\int_0^1 f\Big(a+t\eta(b,a)\Big)\mathrm{d}t.
\end{aligned}
$$

Let $x=a+t\eta(b,a)$, then $\mathrm{d}x=\eta(b,a)\mathrm{d}t$ and we have

$$
\begin{aligned}
J &= -\left[(1-\lambda)f(a)+(\lambda-\mu)f\left(a+\frac{\eta(b,a)}{2}\right)+\mu f\Big(a+\eta(b,a)\Big)\right]\\
&\quad+\frac{1}{\eta(a,b)}\int_a^{a+\eta(b,a)} f(x)\mathrm{d}x,
\end{aligned}
$$

which is required.

3. Some Hadamard-Simpson-like type integral inequalities

First of all, we state the following results for (s,m)-preinvex mapping in the second sense.

Theorem 3.1 *Let $A=[0,b^*]$ with $b^*>0$ be an invex subset with respect to $\eta: A\times A\to\mathbb{R}_0$ and $a,b\in A$ with $\eta(b,a)>0$. Let $0\le\lambda,\ \mu<1$. Suppose that $f:A\to\mathbb{R}_0$ is a differentiable function and f' is integrable on the η-path P_{ac} for $c=a+\eta(b,a)$. If $|f'|$ is (s,m)-preinvex on $\left[a,\frac{b}{m}\right]$ for some fixed $s\in(0,1]$, $0\le a<b$, and $m\in(0,1]$ with $\frac{b}{m}\le b^*$, then the following inequality holds:*

$$
\begin{aligned}
&\left|(1-\lambda)f(a)+(\lambda-\mu)f\left(a+\frac{\eta(b,a)}{2}\right)+\mu f\Big(a+\eta(b,a)\Big)-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\mathrm{d}x\right| \\
&\quad\le\frac{\big|\eta(b,a)\big|}{(s+1)(s+2)}\Bigg\{\left[2\lambda^{s+2}-\frac{2s\lambda+4\lambda-s-1}{2^{s+2}}+s-s\lambda-2\lambda+1\right]\Big|f'(a)\Big| \\
&\quad\quad+m\left[2(1-\lambda)^{s+2}+\frac{2s\lambda+4\lambda-s-3}{2^{s+2}}\right]\Big|f'\Big(\frac{b}{m}\Big)\Big| \\
&\quad\quad+\left[2\mu^{s+2}-\frac{2s\mu+4\mu-s-1}{2^{s+2}}\right]\Big|f'(a)\Big| \\
&\quad\quad+m\left[2(1-\mu)^{s+2}+\frac{2s\mu+4\mu-s-3}{2^{s+2}}+s\mu+2\mu-1\right]\Big|f'\Big(\frac{b}{m}\Big)\Big|\Bigg\}.
\end{aligned}
\tag{3.1}
$$

Proof. Since A is an invex subset with respect to η, for every $t\in[0,1]$, we have $a+t\eta(b,a)\in A$. Using Lemma 2.1 and the (s,m)-preinvexity of $|f'|$ on $\left[a,\frac{b}{m}\right]$ obtains that

$$
\begin{aligned}
&\left|(1-\lambda)f(a)+(\lambda-\mu)f\left(a+\frac{\eta(b,a)}{2}\right)+\mu f\Big(a+\eta(b,a)\Big)-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\mathrm{d}x\right| \\
&\le\big|\eta(b,a)\big|\left[\int_0^{\frac12}|1-\lambda-t|\Big|f'\Big(a+t\eta(b,a)\Big)\Big|\mathrm{d}t+\int_{\frac12}^1|1-\mu-t|\Big|f'\Big(a+t\eta(b,a)\Big)\Big|\mathrm{d}t\right] \\
&\le\big|\eta(b,a)\big|\Bigg\{\int_0^{\frac12}|1-\lambda-t|(1-t)^s|f'(a)|\mathrm{d}t+m\int_0^{\frac12}|1-\lambda-t|t^s\Big|f'\Big(\frac{b}{m}\Big)\Big|\mathrm{d}t \\
&\quad+\int_{\frac12}^1|1-\mu-t|(1-t)^s|f'(a)|\mathrm{d}t+m\int_{\frac12}^1|1-\mu-t|t^s\Big|f'\Big(\frac{b}{m}\Big)\Big|\mathrm{d}t\Bigg\}.
\end{aligned}
$$

Direct computation yields that

$$
\int_0^{\frac12}|1-\lambda-t|t^s\mathrm{d}t=\frac{1}{(s+1)(s+2)}\left[2(1-\lambda)^{s+2}+\frac{2s\lambda+4\lambda-s-3}{2^{s+2}}\right],
$$

$$
\int_0^{\frac12}|1-\lambda-t|(1-t)^s\mathrm{d}t=\frac{1}{(s+1)(s+2)}\left[2\lambda^{s+2}-\frac{2s\lambda+4\lambda-s-1}{2^{s+2}}+s-s\lambda-2\lambda+1\right].
$$

Similarly, we have

$$\int_{\frac{1}{2}}^{1} |1-\mu-t|t^s \mathrm{d}t = \frac{1}{(s+1)(s+2)}\left[2(1-\mu)^{s+2} + \frac{2s\mu+4\mu-s-3}{2^{s+2}} + s\mu+2\mu-1\right],$$

$$\int_{\frac{1}{2}}^{1} |1-\mu-t|(1-t)^s \mathrm{d}t = \frac{1}{(s+1)(s+2)}\left[2\mu^{s+2} - \frac{2s\mu+4\mu-s-1}{2^{s+2}}\right].$$

Plugging these four equalities into the right hand side of the above inequality establishes the desired inequality and the proof is completed.

By elementary calculation, it is easy to get the following results.

Corollary 3.1 *Under the conditions of Theorem* 3.1*,*

1. *if* $\lambda = 1, \mu = 0$*, and* $s = m = 1$*, we have*

$$\left| f\left(\frac{2a+\eta(b,a)}{2}\right) - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\mathrm{d}x \right| \leq \frac{|\eta(b,a)|}{8}\Big(|f'(a)| + |f'(b)|\Big); \tag{3.2}$$

2. *if* $\lambda = \mu = \dfrac{1}{2}$ *and* $s = m = 1$*, we have*

$$\left| \frac{f(a)+f\big(a+\eta(b,a)\big)}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\mathrm{d}x \right| \leq \frac{|\eta(b,a)|}{8}\Big(|f'(a)| + |f'(b)|\Big); \tag{3.3}$$

3. *if* $\eta(b,a) = b-a, \lambda = \dfrac{5}{6}, \mu = \dfrac{1}{6}$ *and* $m = 1$*, we have*

$$\left| \left[f(a)+f(b)+4f\Big(\frac{a+b}{2}\Big)\right] - \frac{6}{b-a}\int_a^b f(x)\mathrm{d}x \right| \leq \frac{(s-4)6^{s+1}+2\times 5^{s+2}-2\times 3^{s+2}+2}{6^{s+1}(s+1)(s+2)}(b-a)(|f'(a)|+|f'(b)|). \tag{3.4}$$

Remark 3.1. Inequality (3.2) is the same as inequality of (1.8) established by Sarikaya in [26]. Inequality (3.3) is the same as inequality of (1.7) obtained by Barani in [2]. Inequality (3.4) is the same as inequality of (1.4) presented by Sarikaya in [27]. Thus, inequality (3.1) is a generalization of these Hadamard type inequalities.

Using Lemma 2.1 again, we are going to present and prove new integral inequalities of Hadamard-Simpson-like type for differentiable extended (s,m)-preinvex functions.

Theorem 3.2 *Let $A=[0,b^*]$ with $b^*>0$ be an invex subset with respect to $\eta: A\times A\to\mathbb{R}_0$ and $a,b\in A$ with $\eta(b,a)>0$. Assume that $0\le\lambda,\ \mu<1$. Suppose that $f: A\to\mathbb{R}_0$ is a differentiable function and f' is integrable on the η-path P_{ac} for $c=a+\eta(b,a)$. If $|f'|^q$ for $q\ge 1$ is extended (s,m)-preinvex on $[a,\frac{b}{m}]$ for some fixed $s\in[-1,1]$, $0\le a<b$, and $m\in(0,1]$ with $\frac{b}{m}\le b^*$, then*

1. *when $-1<s\le 1$, we have*

$$\begin{aligned}
&\left|(1-\lambda)f(a)+(\lambda-\mu)f\left(a+\frac{\eta(b,a)}{2}\right)+\mu f(a+\eta(b,a))-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\mathrm{d}x\right|\\
&\le\frac{|\eta(b,a)|}{[(s+1)(s+2)]^{1/q}}\left\{\left(\lambda^2-\frac{3}{2}\lambda+\frac{5}{8}\right)^{1-\frac{1}{q}}\left[\left(2\lambda^{s+2}-\frac{2s\lambda+4\lambda-s-1}{2^{s+2}}+s-s\lambda-2\lambda+1\right)\left|f'(a)\right|^q\right.\right.\\
&\qquad\left.+m\left(2(1-\lambda)^{s+2}+\frac{2s\lambda+4\lambda-s-3}{2^{s+2}}\right)\left|f'\left(\frac{b}{m}\right)\right|^q\right]^{\frac{1}{q}}\\
&\qquad+\left(\mu^2-\frac{1}{2}\mu+\frac{1}{8}\right)^{1-\frac{1}{q}}\left[\left(2\mu^{s+2}-\frac{2s\mu+4\mu-s-1}{2^{s+2}}\right)\left|f'(a)\right|^q\right.\\
&\qquad\left.\left.+m\left(2(1-\mu)^{s+2}+\frac{2s\mu+4\mu-s-3}{2^{s+2}}+s\mu+2\mu-1\right)\left|f'\left(\frac{b}{m}\right)\right|^q\right]^{\frac{1}{q}}\right\};
\end{aligned}\tag{3.5}$$

2. *when $s=-1$, $\lambda=1$ and $\mu=0$, we have*

$$\begin{aligned}
&\left|f\left(a+\frac{\eta(b,a)}{2}\right)-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\mathrm{d}x\right|\\
&\le\frac{|\eta(b,a)|}{8^{1-\frac{1}{q}}}\left\{\left[(\ln 2-\frac{1}{2})\left|f'(a)\right|^q+\frac{m}{2}\left|f'\left(\frac{b}{m}\right)\right|^q\right]^{\frac{1}{q}}\right.\\
&\qquad\left.+\left[\frac{1}{2}\left|f'(a)\right|^q+m\left(\ln 2-\frac{1}{2}\right)\left|f'\left(\frac{b}{m}\right)\right|^q\right]^{\frac{1}{q}}\right\}.
\end{aligned}\tag{3.6}$$

Proof. 1. When $-1<s\le 1$, by Lemma 2.1, we have that

$$\begin{aligned}
&\left|(1-\lambda)f(a)+(\lambda-\mu)f\left(a+\frac{\eta(b,a)}{2}\right)+\mu f\left(a+\eta(b,a)\right)-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\mathrm{d}x\right|\\
&\le|\eta(b,a)|\left[\int_0^{\frac{1}{2}}|1-\lambda-t|\left|f'(a+t\eta(b,a))\right|\mathrm{d}t+\int_{\frac{1}{2}}^1|1-\mu-t||f'(a+t\eta(b,a))|\mathrm{d}t\right].
\end{aligned}$$

Using the power-mean integral inequality, we have

$$\int_0^{\frac{1}{2}} |1-\lambda-t|\Big|f'\Big(a+t\eta(b,a)\Big)\Big|\mathrm{d}t \le \left[\int_0^{\frac{1}{2}} |1-\lambda-t|\mathrm{d}t\right]^{1-\frac{1}{q}} \left(\int_0^{\frac{1}{2}} |1-\lambda-t|\Big|f'\Big(a+t\eta(b,a)\Big)\Big|^q\mathrm{d}t\right)^{\frac{1}{q}}$$

and

$$\int_{\frac{1}{2}}^1 |1-\mu-t|\Big|f'\Big(a+t\eta(b,a)\Big)\Big|\mathrm{d}t \le \left[\int_{\frac{1}{2}}^1 |1-\mu-t|\mathrm{d}t\right]^{1-\frac{1}{q}} \left(\int_{\frac{1}{2}}^1 |1-\mu-t|\Big|f'\Big(a+t\eta(b,a)\Big)\Big|^q\mathrm{d}t\right)^{\frac{1}{q}}.$$

Now, the extended (s,m)-preinvexity of $|f'|^q$ in $[a,\frac{b}{m}]$ implies that

$$\left(\int_0^{\frac{1}{2}} |1-\lambda-t|\Big|f'\Big(a+t\eta(b,a)\Big)\Big|^q\mathrm{d}t\right)^{\frac{1}{q}} \le \left[\int_0^{\frac{1}{2}} \big|1-\lambda-t\big|\left((1-t)^s|f'(a)|^q+mt^s\Big|f'\Big(\frac{b}{m}\Big)\Big|^q\right)\mathrm{d}t\right]^{\frac{1}{q}}$$

and

$$\left(\int_{\frac{1}{2}}^1 |1-\mu-t|\Big|f'\Big(a+t\eta(b,a)\Big)\Big|^q\mathrm{d}t\right)^{\frac{1}{q}} \le \left[\int_{\frac{1}{2}}^1 \big|1-\mu-t\big|\left((1-t)^s|f'(a)|^q+mt^s\Big|f'\Big(\frac{b}{m}\Big)\Big|^q\right)\mathrm{d}t\right]^{\frac{1}{q}}.$$

Finally, the combination of these inequalities and an elementary computation establish the desired inequality.

2. When $s=-1$, $\lambda=1$ and $\mu=0$, Lemma 2.1, the power-mean integral inequality and the extended $(-1,m)$-preinvexity of $|f'(x)|^q$ on $[a,\frac{b}{m}]$ can be used in a similar way to obtain that

$$\begin{aligned}
&\left|f\Big(a+\frac{\eta(b,a)}{2}\Big)-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\mathrm{d}x\right| \\
&\le \big|\eta(b,a)\big|\left[\int_0^{\frac{1}{2}} |-t|\Big|f'\Big(a+t\eta(b,a)\Big)\Big|\mathrm{d}t+\int_{\frac{1}{2}}^1 |1-t|\Big|f'\Big(a+t\eta(b,a)\Big)\Big|\mathrm{d}t\right] \\
&\le \big|\eta(b,a)\big|\left\{\left[\int_0^{\frac{1}{2}} |-t|\mathrm{d}t\right]^{1-\frac{1}{q}}\left[\int_0^{\frac{1}{2}}\left(t(1-t)^{-1}\Big|f'(a)\Big|^q+mtt^{-1}\Big|f'\Big(\frac{b}{m}\Big)\Big|^q\right)\mathrm{d}t\right]^{\frac{1}{q}}\right. \\
&\quad \left.+\left[\int_{\frac{1}{2}}^1 |1-t|\mathrm{d}t\right]^{1-\frac{1}{q}}\left[\int_{\frac{1}{2}}^1\left((1-t)(1-t)^{-1}\Big|f'(a)\Big|^q+m(1-t)t^{-1}\Big|f'\Big(\frac{b}{m}\Big)\Big|^q\right)\mathrm{d}t\right]^{\frac{1}{q}}\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \big|\eta(b,a)\big|\Big(\frac{1}{8}\Big)^{1-\frac{1}{q}}\Bigg\{\Bigg[\int_0^{\frac{1}{2}}\Big(\frac{t}{1-t}\Big|f'(a)\Big|^q + m\Big|f'\Big(\frac{b}{m}\Big)\Big|^q\Big)\mathrm{d}t\Bigg]^{\frac{1}{q}} \\
&\quad + \Bigg[\int_{\frac{1}{2}}^1\Big(\Big|f'(a)\Big|^q + \frac{m(1-t)}{t}\Big|f'\Big(\frac{b}{m}\Big)\Big|^q\Big)\mathrm{d}t\Bigg]^{\frac{1}{q}}\Bigg\} \\
&= \big|\eta(b,a)\big|\Big(\frac{1}{8}\Big)^{1-\frac{1}{q}}\Bigg\{\Bigg[(\ln 2-\frac{1}{2})\Big|f'(a)\Big|^q + \frac{m}{2}\Big|f'\Big(\frac{b}{m}\Big)\Big|^q\Bigg]^{\frac{1}{q}} \\
&\quad + \Bigg[\frac{1}{2}\Big|f'(a)\Big|^q + m\Big(\ln 2-\frac{1}{2}\Big)\Big|f'\Big(\frac{b}{m}\Big)\Big|^q\Bigg]^{\frac{1}{q}}\Bigg\}.
\end{aligned}
$$

The proof is completed. ∎

Remark 3.2. If $\eta(b,a) = b-a$ and $m=1$, then inequality (3.6) reduces to inequality (1.5). Thus, inequality (3.6) is an extension of inequality (1.5).

Direct computation yields the following corollary.

Corollary 3.2 *Under the conditions of Theorem* 3.2,

1. *if* $q=1$ *and* $-1<s\le 1$, *then*

$$
\begin{aligned}
&\Bigg|(1-\lambda)f(a)+(\lambda-\mu)f\Big(a+\frac{\eta(b,a)}{2}\Big)+\mu f(a+\eta(b,a)) - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\mathrm{d}x\Bigg| \\
&\le \frac{\big|\eta(b,a)\big|}{(s+1)(s+2)}\Bigg\{\Bigg[2\lambda^{s+2} - \frac{2s\lambda+4\lambda-s-1}{2^{s+2}} + s - s\lambda - 2\lambda + 1\Bigg]\Big|f'(a)\Big| \\
&\qquad + m\Bigg[2(1-\lambda)^{s+2} + \frac{2s\lambda+4\lambda-s-3}{2^{s+2}}\Bigg]\Big|f'\Big(\frac{b}{m}\Big)\Big| \\
&\qquad + \Bigg[2\mu^{s+2} - \frac{2s\mu+4\mu-s-1}{2^{s+2}}\Bigg]\Big|f'(a)\Big| \\
&\qquad + m\Bigg[2(1-\mu)^{s+2} + \frac{2s\mu+4\mu-s-3}{2^{s+2}} + s\mu + 2\mu - 1\Bigg]\Big|f'\Big(\frac{b}{m}\Big)\Big|\Bigg\};
\end{aligned} \tag{3.7}
$$

2. *if* $\eta(b,a) = b-a$, $s=-1$, $\lambda=1$, $\mu=0$ *and* $q=1$, *then*

$$
\Bigg|f\Big(\frac{a+b}{2}\Big) - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x)\Bigg| \le (b-a)\ln 2\Bigg[|f'(a)| + m\Big|f'\Big(\frac{b}{m}\Big)\Big|\Bigg]; \tag{3.8}
$$

3. *if* $\lambda=\mu=\dfrac{1}{2}, s=1$ *and* $q=1$, *then*

$$
\begin{aligned}
&\Bigg|\frac{f(a)+f\big(a+\eta(b,a)\big)}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\mathrm{d}x\Bigg| \\
&\qquad \le \frac{|\eta(b,a)|}{8}\Bigg[|f'(a)| + m\Big|f'\Big(\frac{b}{m}\Big)\Big|\Bigg];
\end{aligned} \tag{3.9}
$$

4. *if* $\eta(b,a)=b-a$, $\lambda=\dfrac{5}{6},\mu=\dfrac{1}{6},-1<s\le 1$ *and* $q=1$, *then*

$$\begin{aligned}&\left|\Big[f(a)+f(b)+4f\left(\frac{a+b}{2}\right)\Big]-\frac{6}{b-a}\int_a^b f(x)\mathrm{d}x\right|\\ &\le\frac{(s-4)6^{s+1}+2\times5^{s+2}-2\times3^{s+2}+2}{6^{s+1}(s+1)(s+2)}(b-a)\left[|f'(a)|+m\left|f'\Big(\frac{b}{m}\Big)\right|\right].\end{aligned}\tag{3.10}$$

Remark 3.3. In inequalities (3.8), (3.9) and (3.10), if we let $m=1$ then we will get inequalities (1.6), (1.7) and (1.4) respectively. Thus, Theorem 3.2 and its consequences generalize the main results in [33], [2] and [27].

Theorem 3.3 *Let* $A=[0,b^*]$ *with* $b^*>0$ *be an invex subset with respect to* $\eta:A\times A\to\mathbb{R}_0$ *and* $a,b\in A$ *with* $\eta(b,a)>0$. *Assume that* $0\le\lambda,\ \mu<1$. *Suppose that* $f:A\to\mathbb{R}_0$ *is a differentiable function and* f' *is integrable on the* η*-path* P_{ac} *for* $c=a+\eta(b,a)$. *If* $\frac{1}{p}+\frac{1}{q}=1$ *and* $|f'|^q$ *for* $q>1$ *is extended* (s,m)*-preinvex on* $\left[a,\frac{b}{m}\right]$ *for some fixed* $s\in(-1,1]$, $0\le a<b$, *and* $m\in[0,1]$ *with* $\frac{b}{m}\le b^*$, *then*

$$\begin{aligned}&\left|(1-\lambda)f(a)+(\lambda-\mu)f\Big(a+\frac{\eta(b,a)}{2}\Big)+\mu f(a+\eta(b,a))-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\mathrm{d}x\right|\\ &\le\frac{|\eta(b,a)|}{(s+1)^{\frac{1}{q}}(p+1)^{\frac{1}{p}}}\Big(\frac{1}{2^{s+1}}\Big)^{\frac{1}{q}}\left\{\left[(1-\lambda)^{p+1}+\Big(\lambda-\frac{1}{2}\Big)^{p+1}\right]^{\frac{1}{p}}\left[(2^{s+1}-1)|f'(a)|^q+m\left|f'\Big(\frac{b}{m}\Big)\right|^q\right]^{\frac{1}{q}}\right.\\ &\left.+\left[\Big(\frac{1}{2}-\mu\Big)^{p+1}+\mu^{p+1}\right]^{\frac{1}{p}}\left[\left|f'(a)\right|^q+m\big(2^{s+1}-1\big)\left|f'\Big(\frac{b}{m}\Big)\right|^q\right]^{\frac{1}{q}}\right\}.\end{aligned}\tag{3.11}$$

Proof. Lemma 2.1, the Hölder integral inequality, and the extended (s,m)-preinvexity of $|f'(x)|^q$ on $[a,\frac{b}{m}]$ can be used sequently to obtain that

$$\begin{aligned}&\left|(1-\lambda)f(a)+(\lambda-\mu)f\left(a+\frac{\eta(b,a)}{2}\right)+\mu f\Big(a+\eta(b,a)\Big)-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\mathrm{d}x\right|\\ &\le\big|\eta(b,a)\big|\left[\int_0^{\frac12}|1-\lambda-t|\left|f'\Big(a+t\eta(b,a)\Big)\right|\mathrm{d}t+\int_{\frac12}^1|1-\mu-t|\left|f'\Big(a+t\eta(b,a)\Big)\right|\mathrm{d}t\right]\\ &\le\big|\eta(b,a)\big|\left[\left(\int_0^{\frac12}|1-\lambda-t|^p\mathrm{d}t\right)^{\frac1p}\left(\int_0^{\frac12}\left|f'\Big(a+t\eta(b,a)\Big)\right|^q\mathrm{d}t\right)^{\frac1q}\right.\\ &\quad\left.+\left(\int_{\frac12}^1|1-\mu-t|^p\mathrm{d}t\right)^{\frac1p}\left(\int_{\frac12}^1\left|f'\Big(a+t\eta(b,a)\Big)\right|^q\mathrm{d}t\right)^{\frac1q}\right]\\ &\le\big|\eta(b,a)\big|\left\{\left(\int_0^{\frac12}|1-\lambda-t|^p\mathrm{d}t\right)^{\frac1p}\left[\int_0^{\frac12}\left((1-t)^s\left|f'(a)\right|^q+mt^s\left|f'\Big(\frac{b}{m}\Big)\right|^q\right)\mathrm{d}t\right]^{\frac1q}\right.\\ &\quad\left.+\left(\int_{\frac12}^1|1-\mu-t|^p\mathrm{d}t\right)^{\frac1p}\left[\int_{\frac12}^1\left((1-t)^s\left|f'(a)\right|^q+mt^s\left|f'\Big(\frac{b}{m}\Big)\right|^q\right)\mathrm{d}t\right]^{\frac1q}\right\}.\end{aligned}$$

Using the fact that

$$\int_0^{\frac{1}{2}} |1-\lambda-t|^p \mathrm{d}t = \frac{(1-\lambda)^{p+1} + \left(\lambda - \frac{1}{2}\right)^{p+1}}{p+1}$$

and

$$\int_{\frac{1}{2}}^1 |1-\mu-t|^p \mathrm{d}t = \frac{\left(\frac{1}{2}-\mu\right)^{p+1} + \mu^{p+1}}{p+1},$$

we have

$$\begin{aligned}
&\left|(1-\lambda)f(a)+(\lambda-\mu)f\left(a+\frac{\eta(b,a)}{2}\right)+\mu f(a+\eta(b,a)) - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\mathrm{d}x\right| \\
&\le \frac{\left|\eta(b,a)\right|}{(s+1)^{\frac{1}{q}}(p+1)^{\frac{1}{p}}}\Bigg\{\left[(1-\lambda)^{p+1} + \left(\lambda-\frac{1}{2}\right)^{p+1}\right]^{\frac{1}{p}} \\
&\qquad\qquad \times\left[\left(1-\left(\frac{1}{2}\right)^{s+1}\right)\left|f'(a)\right|^q + \frac{m}{2^{s+1}}\left|f'\left(\frac{b}{m}\right)\right|^q\right]^{\frac{1}{q}} \\
&\quad + \left[\left(\frac{1}{2}-\mu\right)^{p+1} + \mu^{p+1}\right]^{\frac{1}{p}}\left[\frac{\left|f'(a)\right|^q}{2^{s+1}} + m\left(1-\left(\frac{1}{2}\right)^{s+1}\right)\left|f'\left(\frac{b}{m}\right)\right|^q\right]^{\frac{1}{q}}\Bigg\} \\
&= \frac{\left|\eta(b,a)\right|}{(s+1)^{\frac{1}{q}}(p+1)^{\frac{1}{p}}}\left(\frac{1}{2^{s+1}}\right)^{\frac{1}{q}}\Bigg\{\left[(1-\lambda)^{p+1} + \left(\lambda-\frac{1}{2}\right)^{p+1}\right]^{\frac{1}{p}} \\
&\qquad\qquad \times\left[\left(2^{s+1}-1\right)\left|f'(a)\right|^q + m\left|f'\left(\frac{b}{m}\right)\right|^q\right]^{\frac{1}{q}} \\
&\quad + \left[\left(\frac{1}{2}-\mu\right)^{p+1} + \mu^{p+1}\right]^{\frac{1}{p}}\left[\left|f'(a)\right|^q + m\left(2^{s+1}-1\right)\left|f'\left(\frac{b}{m}\right)\right|^q\right]^{\frac{1}{q}}\Bigg\},
\end{aligned}$$

which completes the proof. ∎

Elementary calculation provides the following corollary.

Corollary 3.3 *Under the conditions of Theorem* 3.3*,*

1. *when* $\eta(b,a) = b-a$*,* $s=1$ *and* $m=1$*, we have*

$$\begin{aligned}
&\left|(1-\lambda)f(b)+\mu f(a)+(\lambda-\mu)f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \\
&\le \frac{(b-a)}{(p+1)^{\frac{1}{p}}}\Bigg\{\left[(1-\lambda)^{p+1} + \left(\lambda-\frac{1}{2}\right)^{p+1}\right]^{\frac{1}{p}}\left[\frac{3}{8}|f'(a)|^q + \frac{1}{8}|f'(b)|^q\right]^{\frac{1}{q}} \\
&\quad + \left[\left(\frac{1}{2}-\mu\right)^{p+1} + \mu^{p+1}\right]^{\frac{1}{p}}\left[\frac{1}{8}|f'(a)|^q + \frac{3}{8}|f'(b)|^q\right]^{\frac{1}{q}}\Bigg\};
\end{aligned} \tag{3.12}$$

2. *if* $\eta(b,a)=b-a$, $\lambda=\mu, s=1$ *and* $m=1$, *we have*

$$\begin{aligned}&\left|(1-\lambda)f(b)+\lambda f(a)-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|\\&\le\frac{(b-a)}{(p+1)^{\frac{1}{p}}}\left\{\left[(1-\lambda)^{p+1}+\left(\lambda-\frac{1}{2}\right)^{p+1}\right]^{\frac{1}{p}}\left[\frac{3}{8}|f'(a)|^q+\frac{1}{8}|f'(b)|^q\right]^{\frac{1}{q}}\right.\\&\left.+\left[\left(\frac{1}{2}-\lambda\right)^{p+1}+\lambda^{p+1}\right]^{\frac{1}{p}}\left[\frac{1}{8}|f'(a)|^q+\frac{3}{8}|f'(b)|^q\right]^{\frac{1}{q}}\right\};\end{aligned}\tag{3.13}$$

3. *if* $\eta(b,a)=b-a$, $\lambda=\mu$, *we have*

$$\begin{aligned}&\left|(1-\lambda)f(b)+\lambda f(a)-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|\\&\le\frac{(b-a)}{(s+1)^{\frac{1}{q}}(p+1)^{\frac{1}{p}}}\left(\frac{1}{2^{s+1}}\right)^{\frac{1}{q}}\left\{\left[(1-\lambda)^{p+1}+\left(\lambda-\frac{1}{2}\right)^{p+1}\right]^{\frac{1}{p}}\right.\\&\times\left[\left(2^{s+1}-1\right)\left|f'(a)\right|^q+m\left|f'\left(\frac{b}{m}\right)\right|^q\right]^{\frac{1}{q}}\\&\left.+\left[\left(\frac{1}{2}-\lambda\right)^{p+1}+\lambda^{p+1}\right]^{\frac{1}{p}}\left[\left|f'(a)\right|^q+m\left(2^{s+1}-1\right)\left|f'\left(\frac{b}{m}\right)\right|^q\right]^{\frac{1}{q}}\right\};\end{aligned}\tag{3.14}$$

4. *if* $\eta(b,a)=b-a$, $\lambda=\mu=\dfrac{1}{2}, s=1$ *and* $m=1$, *we have*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|\\&\le\frac{(b-a)}{2(2p+2)^{\frac{1}{p}}}\left[\left(\frac{3}{8}|f'(a)|^q+\frac{1}{8}|f'(b)|^q\right)^{\frac{1}{q}}+\left(\frac{1}{8}|f'(a)|^q+\frac{3}{8}|f'(b)|^q\right)^{\frac{1}{q}}\right];\end{aligned}\tag{3.15}$$

5. *if* $\lambda=1,\mu=0$ *and* $s=m=1$, *we have*

$$\begin{aligned}&\left|f\left(\frac{2a+\eta(b,a)}{2}\right)-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\mathrm{d}x\right|\\&\le\frac{|\eta(b,a)|}{2(2p+2)^{\frac{1}{p}}}\left[\left(\frac{3}{8}|f'(a)|^q+\frac{1}{8}|f'(b)|^q\right)^{\frac{1}{q}}+\left(\frac{1}{8}|f'(a)|^q+\frac{3}{8}|f'(b)|^q\right)^{\frac{1}{q}}\right];\end{aligned}\tag{3.16}$$

6. *if* $\eta(b,a)=b-a$, $\lambda=\dfrac{5}{6}$ *and* $\mu=\dfrac{1}{6}$, *we have*

$$
\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+f(b)+4f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \\
&\leq \frac{(b-a)}{(s+1)^{\frac{1}{q}}(p+1)^{\frac{1}{p}}}\left(\frac{1}{2^{s+1}}\right)^{\frac{1}{q}}\left[\left(\frac{1}{3}\right)^{p+1}+\left(\frac{1}{6}\right)^{p+1}\right]^{\frac{1}{p}} \\
&\quad\times\left\{\left[\left(2^{s+1}-1\right)\left|f'(a)\right|^q+m\left|f'(\frac{b}{m})\right|^q\right]^{\frac{1}{q}}\right. \\
&\quad\left.+\left[\left|f'(a)\right|^q+m\left(2^{s+1}-1\right)\left|f'(\frac{b}{m})\right|^q\right]^{\frac{1}{q}}\right\}.
\end{aligned}
\tag{3.17}
$$

Remark 3.4. In inequality (3.16), if we let $q=\dfrac{p}{p-1}$ and exchange a and b, then we get inequality (1.9). Furthermore, when applying $s=m=1$ to inequality (3.17) then we get inequality (1.3). Thus, Theorem 3.3 and its consequences generalize some main results in [31],[28].

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A NEW CORRECTOR-PREDICTOR ALGORITHM FOR CONVEX QUADRATIC SEMIDEFINITE OPTIMIZATION

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Abstract. In this paper, we propose a new corrector-predictor algorithm for convex quadratic semidefinite optimization problem based on a new proximity measure. The search direction is obtained by an equivalent algebraic transformation of the centering equation. At each iteration, the algorithm is composed of a corrector step and a predictor step. The predictor step uses line search schemes requiring the reduction of the duality gap, while the corrector step is used to restore the iterates to the neighborhood of the central path. Finally, the algorithm has the currently best-known iteration complexity.

Keywords: convex quadratic semidefinite optimization; corrector-predictor algorithm; iteration complexity.

1. Introduction

In this paper, we consider the standard form of convex quadratic semidefinite optimization (CQSDO) problem

$$\begin{aligned}(P) \qquad \min\ & C \bullet X + \frac{1}{2} X \bullet \Omega(X) \\ & A_i \bullet X = b_i,\ i = 1, 2, \ldots, m, \\ & X \succeq 0,\end{aligned}$$

and its dual problem

$$\begin{aligned}(D) \qquad \max\ & b^T y - \frac{1}{2} X \bullet \Omega(X) \\ & \sum_{i=1}^{m} y_i A_i - \Omega(X) + S = C, \\ & X \succeq 0,\ S \succeq 0,\end{aligned}$$

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$A_i \in \mathbf{S}^n$, $i = 1, 2, \ldots, m$, $b \in \mathbb{R}^m$, $C \in \mathbf{S}^n$ and $\Omega(X) : \mathbf{S}^n \longrightarrow \mathbf{S}^n$ is a self-adjoint semidefinite linear operator, i.e., for any $A, B \in \mathbf{S}^n$, we have $\Omega(A) \bullet B = A \bullet \Omega(B)$ and $\Omega(A) \bullet A \geq 0$. Here, $\mathbf{S}^n$ denotes the vector space of $n \times n$ real symmetric matrices endowed with the inner product $A \bullet B := \mathrm{Tr}(AB)$. The sign $X \succeq 0$ ($X \succ 0$) means that matrix X is positive semidefinite (positive definite). To simplify analysis, as in [23] and [25], we will restrict ourselves to the following special case

$$\Omega(X) = \sum_{i=1}^{l} H_i^T X H_i.$$

Here, $H_i \in \mathbb{R}^{n \times n}$ is matrix and l is an integer not greater than $l \leq n^2$. It is easy to verity that $\Omega(X)$ satisfies the following conditions

$$\Omega(X) = \Omega(X)^T, \quad \Omega(X) \bullet X \geq 0, \quad X \in \mathbb{R}^{n \times n}.$$

Remark 1. If $\Omega(X) = 0$, then CQSDO problem reduces to semidefinite optimization (SDO) problem.

CQSDO problem is an extension of SDO problem and convex quadratic optimization (CQO) problem, CQSDO problem was first proposed in the researches of Kojima et al. [10]. And they also proved that the CQSDO problem can be transformation as semidefinite linear complementarity problem (SDLCP) in [11]. Additionally, the CQSDO problem has some important applications, such as the nearest Euclidean distance matrix problem [1], the nearest correlation matrix problem [20]. During the past years, there are many efficient interior-point algorithms have been proposed to solving CQSDO problem. Nie et al. [16] proposed a potential reduction algorithm for solving CQSDO problem and obtained the iteration bound as $O(\sqrt{n} \log \frac{n}{\varepsilon})$. In other article [17], they developed a predictor-corrector algorithm for CQSDO problem by using Dikin-type and Newton centering steps. For their computation they cited the conjugate gradient method. Toh [21] proposed an inexact primal-dual path-following algorithm for CQSDO problem. By numerical experiment, he also shown that the proposed algorithm is efficient and robust. Wang et al. [23] presented a new primal-dual interior point algorithm for CQSDO problem based on a parametric kernel function for large-and small-update methods. Based on a different technique for finding search direction, a new small-update interior point algorithm is proposed by Bai et al. in [4]. Later on, Lin [12] proposed an inexact spectral bundle method for CQSDO problem, where at each iteration of the corresponding algorithm, an eigenvalue problem is inexactly. Zhang et al. [24] presented a full-step interior point algorithm for CQSDO problem based on a simple univariate kernel function. At almost the same time, a large-update interior point algorithm was proposed for CQSDO problem based on a new nonself-regular function by Zhang in [25]. Kheirfam et al. [9] suggested a large-update feasible interior-point algorithm for CQSDO problem based on a parametric kernel function, and obtained the best-known iteration complexity. Finally, Mohamed et al. [13] proposed a full-NT step feasible primal-dual path-following interior point algorithm for CQSDO problem.

Recently, Potra [19] proposed two corrector-predictor interior point algorithms for solving monotone linear complementarity problem (MLCP) based on the negative infinity norm neighborhood, and shown that algorithms have $O(\sqrt{n}L)$ iteration complexity. Gurtuna et al. [8] proposed a corrector-predictor method for solving sufficient LCP that does not depend on the handicap of the matrix so that it can be applied for any sufficient LCP. Very recently, Kheirfam [6] proposed a corrector-predictor path-following algorithm for SDO problem, and presented the iteration bound as $O(\sqrt{n}\log\frac{n}{\varepsilon})$ with small-update methods. Subsequently, he generated this algorithm to convex quadratic symmetric cone optimization (CQSCO) problem [5].

The aim of this paper is to propose a corrector-predictor algorithm for CQSDO problem based on a new proximity measure. The algorithm uses corrector and predictor steps. In the corrector step, we use full-step which has the advantage that no line searches are needed, the purpose of corrector step is to restore to the neighborhood of the central path. While in the predictor step, the algorithm operates one damped NT-step, which is used to reduce the duality gap. Finally, by using new tools, the favorable iteration complexity is reported.

The remainder paper is organized as follows: In Section 2, we recall some basic concepts regards the central path and introduce the new direction. In Section 3, we present a corrector-predictor algorithm for CQSDO problem. Analysis of the algorithm is presented in Section 4. In Section 5, we derive the iteration complexity. Finally, some concluding remarks follow in Section 6.

2. Preliminaries

2.1. The central path

It is common in interior point algorithms theory to assume that (P) and (D) satisfy the interior-point condition (IPC), i.e., there exists (X^0, y^0, S^0) such that

$$A_i \bullet X^0 = b_i, \quad X^0 \succ 0, \quad i = 1, 2, ..., m, \quad \sum_{i=1}^{m} y_i^0 A_i - \Omega(X^0) + S^0 = C, \quad S^0 \succ 0,$$

and the matrices A_i, $i = 1, 2, ..., m$ are linearly independent, the same assumptions can be found in [9], [12], [13], [23], [24], [25]. Under the assumption of IPC, finding an optimal solution of (P) and (D) is equivalent to solving the following system

$$\begin{aligned} & A_i \bullet X = b_i, \quad i = 1, 2, ..., m, \quad X \succeq 0, \\ & \sum_{i=1}^{m} y_i A_i - \Omega(X) + S = C, \quad S \succeq 0, \\ & XS = 0. \end{aligned} \tag{1}$$

The basic idea of primal-dual interior point algorithms is to replace the third equation in the system (1), the so-called complementarity condition for (P) and (D), by the parameterized equation $XS = \mu E$, E denotes the $n \times n$ identity matrix.

Then, we consider the following system

$$\begin{aligned} & A_i \bullet X = b_i, \quad i = 1, 2, ..., m, \quad X \succ 0, \\ & \sum_{i=1}^{m} y_i A_i - \Omega(X) + S = C, \quad S \succ 0, \\ & XS = \mu E, \end{aligned} \tag{2}$$

where $\mu > 0$. Since the matrices A_i are linearly independent and the IPC holds, the parameterized system (2) has a unique solution, denoted as $(X(\mu), y(\mu), S(\mu))$ for each $\mu > 0$. We call $X(\mu)$ the μ-center of (P) and $(y(\mu), S(\mu))$ the μ-center of (D). The set of μ-centers (with μ running through all the positive real numbers) gives a homotopy path, which is called the central path of (P) and (D). If μ goes to zero, then the limit of the central path exists and since the limit points satisfy the complementarity condition, the limit yields an optimal solution for (P) and (D) (see [23]).

2.2. The new search direction

Here, we introduce the new search direction for CQSDO problem. In [22], Wang et al. presented a new technique for finding a class of search directions for SDO problem. He replaces the standard centering equation $XS = \mu E$ with

$$\psi\big(\frac{XS}{\mu}\big) = \psi(E),$$

where

$$\psi \in C^1, \quad \psi : \mathbb{R}^+ \to \mathbb{R}^+,$$

and require that the inverse function ψ^{-1} exists. Similar to the SDO problem case, if we replace the standard centering equation $XS = \mu E$ with $\psi(\frac{XS}{\mu}) = \psi(E)$, then system (2) can be written the following equivalent form

$$\begin{aligned} & A_i \bullet X = b_i, \quad i = 1, 2, ..., m, \quad X \succ 0, \\ & \sum_{i=1}^{m} y_i A_i - \Omega(X) + S = C, \quad S \succ 0, \\ & \psi\Big(\frac{XS}{\mu}\Big) = \psi(E). \end{aligned} \tag{3}$$

Applying Newton's method to system (3) and neglecting the term $\Delta X \Delta S$, we have

$$\begin{aligned} & A_i \bullet \Delta X = 0, \quad i = 1, 2, ..., m, \\ & \sum_{i=1}^{m} \Delta y_i A_i - \Omega(\Delta X) + \Delta S = 0, \\ & \Delta X + X \Delta S S^{-1} = \mu \Big(\psi'\Big(\frac{XS}{\mu}\Big)\Big)^{-1} \Big(\psi(E) - \psi\Big(\frac{XS}{\mu}\Big)\Big) S^{-1}. \end{aligned} \tag{4}$$

A well-known problem with the above Newton system is that ΔX is not necessarily symmetric. There are many ways are suggested for symmetrizing the third equation in system (4) such that the new system has an unique symmetric solution. In this paper, we use the Nesterov Todd (NT) symmetrization scheme in [14], [15]. Let us define

$$P := X^{\frac{1}{2}}(X^{\frac{1}{2}}SX^{\frac{1}{2}})^{-\frac{1}{2}}X^{\frac{1}{2}} = S^{-\frac{1}{2}}(S^{\frac{1}{2}}XS^{\frac{1}{2}})^{\frac{1}{2}}S^{-\frac{1}{2}}. \tag{5}$$

Then, system (4) is replaced by the system

$$\begin{aligned} & A_i \bullet \triangle X = 0, \quad i = 1, 2, ..., m, \\ & \sum_{i=1}^{m} \triangle y_i A_i - \Omega(\Delta X) + \triangle S = 0, \\ & \triangle X + P \triangle S P^T = \mu S^{-1} - X. \end{aligned} \tag{6}$$

Now, we can conclude that system (6) has a unique symmetric solution. And also define $D := P^{\frac{1}{2}}$, where for any symmetric position matrix G, the exponent $G^{\frac{1}{2}}$ denotes its symmetric square root, the role of matrix D is to scale both matrices X and S to the same matrix V by

$$V := \frac{1}{\sqrt{\mu}} D^{-1} X D^{-1} = \frac{1}{\sqrt{\mu}} DSD. \tag{7}$$

Note that the matrices D and V are symmetric and positive definite. Moreover, we have

$$V^2 := \frac{1}{\mu} D^{-1} XSD. \tag{8}$$

Let us further define

$$\begin{aligned} \bar{A}_i &:= \frac{1}{\sqrt{\mu}} DA_i D, \quad i = 1, 2, ..., m, \\ D_X &:= \frac{1}{\sqrt{\mu}} D^{-1} \triangle X D^{-1}, \\ D_S &:= \frac{1}{\sqrt{\mu}} D \triangle S D. \end{aligned} \tag{9}$$

Consequently, system (6) reduces to

$$\begin{aligned} & \bar{A}_i \bullet D_X = 0, \quad i = 1, 2, ..., m, \\ & \sum_{i=1}^{m} \triangle y_i \bar{A}_i - \bar{\Omega}(D_X) + D_S = 0, \\ & D_X + D_S := P_V, \end{aligned} \tag{10}$$

where

$$\bar{\Omega}(D_X) := \sum_{i=1}^{l} DH_i^T DD_X DD_i D,$$

$$P_V = \sqrt{\mu} D^{-1} (D\psi'(V^2)D^{-1})^{-1} (\psi(E) - D\psi(V^2)D^{-1}) S^{-1} D^{-1}.$$

Note that, if we choose a different function $\psi(t)$, then it results a different P_V, such as

- $\psi(t) = t$ yields $P_V = V^{-1} - V$, which gives the classical search direction (see [2], [3], [7], [18]);
- $\psi(t) = t^2$ yields $P_V = \frac{1}{2}(V^{-3} - V)$ (see [2], [3], [18]);
- $\psi(t) = t^{\frac{q+1}{2}}$, $q \geq 1$ yields $P_V = \frac{2}{q+1}V^{-q} - V$ (see [2], [3], [18]).

In this paper, we restrict the analysis to the case where $\psi(t) = \sqrt{t}$. This yields

$$P_V := 2(E - V).$$

For the analysis of the algorithm, we also define a new norm-based proximity measure $\delta(X, S; \mu)$ as follows

$$\delta(X, S; \mu) := \delta(V) := \frac{1}{2}\|P_V\|_F = \|E - V\|_F.$$

By using the first two equations of the system (10) and $\Omega(X)$ is a self-adjoint positive semidefinite linear operation, we have

$$D_X \bullet D_S = D_S \bullet D_X = \frac{1}{\mu}\Delta X \Omega(\Delta X) \geq 0. \tag{11}$$

Remark 2. Note that, for the SDO problem case [6], [22], we have $D_X \bullet D_S = 0$. It is the main difference between the CQSDO problem and the SDO problem in the analysis. This makes the analysis different.

3. The corrector-predictor algorithm

In this section, we propose a corrector-predictor algorithm for CQSDO problem. Firstly, we define a τ-neighborhood of the central path as follows

$$\mathcal{N}(\tau) = \Big\{(X, S) : A_i \bullet X = b_i,\ i = 1, 2, ..., m,\ X \succeq 0; \sum_{i=1}^{m} y_i A_i - \Omega(X) + S = C, \\ S \succeq 0;\ \delta(X, S; \mu) \leq \tau\Big\}.$$

The algorithm starts with (X, y, S) in the τ-neighborhood $\mathcal{N}(\tau)$, which this certainly holds at (X^0, y^0, S^0) since $\delta(X^0, S^0; \mu^0) = 0$. If, for the current iterate

(X, y, S) in the τ-neighborhood, $X \bullet S > \varepsilon$, then the algorithm performs corrector step and predictor step. In the corrector step, we define

$$\begin{aligned} D &:= \left[X^{\frac{1}{2}}(X^{\frac{1}{2}}SX^{\frac{1}{2}})^{-\frac{1}{2}}X^{\frac{1}{2}}\right]^{\frac{1}{2}} = \left[S^{-\frac{1}{2}}(S^{\frac{1}{2}}XS^{\frac{1}{2}})^{\frac{1}{2}}S^{-\frac{1}{2}}\right]^{\frac{1}{2}}, \\ V &:= \frac{1}{\sqrt{\mu}}D^{-1}XD^{-1} = \frac{1}{\sqrt{\mu}}DSD, \\ \bar{A}_i &:= \frac{1}{\sqrt{\mu}}DA_iD, \ i = 1, 2, ..., m, \end{aligned}$$

and obtain D_X, Δy and D_S by solving the following system

$$\begin{aligned} &\bar{A}_i \bullet D_X = 0, \quad i = 1, 2, ..., m, \\ &\sum_{i=1}^{m} \triangle y_i \bar{A}_i - \bar{\Omega}(D_X) + D_S = 0, \\ &D_X + D_S = 2(E - V). \end{aligned} \tag{12}$$

Then, we obtain

$$\Delta X = \sqrt{\mu}DD_XD, \quad \Delta S = \sqrt{\mu}D^{-1}D_SD^{-1},$$

and let

$$X_+ := X + \Delta X, \quad y_+ := y + \Delta y, \quad S_+ := S + \Delta S.$$

In the predictor step, we define

$$\begin{aligned} V_+ &:= \frac{1}{\sqrt{\mu}}D_+^{-1}X_+D_+^{-1} = \frac{1}{\sqrt{\mu}}D_+S_+D_+, \\ \bar{A}_i^+ &= \frac{1}{\sqrt{\mu}}D_+A_iD_+, \end{aligned}$$

and obtain D_X^p and D_S^p by solving the following system

$$\begin{aligned} &\bar{A}_i^+ \bullet D_X^p = 0, \quad i = 1, 2, ..., m, \\ &\sum_{i=1}^{m} \triangle^p y_i^+ \bar{A}_i^+ - \bar{\Omega}(D_X^p) + D_S^p = 0, \\ &D_X^p + D_S^p = -2V_+. \end{aligned} \tag{13}$$

Then, we get

$$\Delta^p X = \sqrt{\mu}D_+D_X^pD_+, \quad \Delta^p S = \sqrt{\mu}D_+^{-1}D_S^pD_+^{-1},$$

and let

$$X^p := X_+ + \theta\Delta^p X, \quad y^p := y_+ + \theta\Delta^p y, \quad S^p := S_+ + \theta\Delta^p S, \quad \mu^p := (1 - 2\theta)\mu,$$

for each $\theta \in (\theta, \frac{1}{2})$. The iterate (X^p, y^p, S^p) will be in the τ-neighborhood $\mathcal{N}(\tau)$ again. This procedure is repeated until μ is small enough, say until $n\mu \leq \varepsilon$.

The framework of corrector-predictor algorithm for CQSDO problem is shown as follows:

Algorithm: Corrector-predictor algorithm for CQSDO problem

Input:
A threshold parameter $\tau \geq 1$;
an accuracy parameter $\varepsilon > 0$;
a barrier update parameter θ, $0 < \theta < 1/2$;
a strictly feasible point (X^0, S^0) and μ^0 such that $\delta(X^0, S^0; \mu^0) \leq \tau$;
begin
$X := X^0$; $S := S^0$; $\mu := \mu^0$;
while $n\mu > \varepsilon$ do
begin
corrector step
solve system (12) and let
$(X_+, y_+, S_+) = (X, y, S) + (\Delta X, \Delta y, \Delta S)$;
predictor step
solve system (13) and let
$(X_+^p, y_+^p, S_+^p) = (X_+, y_+, S_+) + \theta(\Delta^p X, \Delta^p y, \Delta^p S)$;
μ-update
$\mu^p := (1 - 2\theta)\mu$;
$(X, y, S) := (X_+^p, y_+^p, S_+^p)$, $\mu := \mu^p$;
end
end

4. Analysis of the algorithm

In this section, we focus on the analysis of the above algorithm. Let

$$Q_V := D_X - D_S,$$

it is easy to obtain

$$D_X = \frac{P_V + Q_V}{2}, \quad D_S = \frac{P_V - Q_V}{2}, \quad D_X D_S + D_S D_X = \frac{P_V^2 - Q_V^2}{2}.$$

Note that D_X and D_S are nonorthogonal by (11), therefore

$$\begin{aligned} \|Q_V\| &= \sqrt{\mathrm{Tr}^2(D_X - D_S)} \\ &= \sqrt{\|P_V\|_F^2 - 4D_X \bullet D_S} \\ &\leq 2\delta(V). \end{aligned}$$

In the following, we recall some results without proof.

Lemma 4.1. (Lemma 6.2 and 7.3 in [7]) *Let $D_X, D_S \in \mathbf{S}^n$ be two vectors such that $Tr(D_X D_S) \geq 0$. Then we have*

$$\|D_{XS}\|_\infty \leq \frac{1}{4}\|D_X + D_S\|_F^2, \quad \|D_{XS}\|_F \leq \frac{1}{2\sqrt{2}}\|D_X + D_S\|_F^2,$$

where $D_{XS} := \frac{1}{2}(D_X D_S + D_S D_X)$.

Lemma 4.2. (Modification of Lemma 6.1 in [7]) *Let $X(\alpha) = X + \alpha\Delta X$, $S(\alpha) = S + \alpha\Delta S$ for $0 \leq \alpha \leq 1$, which $X \succ 0$ and $S \succ 0$. If one has*

$$det(X(\alpha)S(\alpha)) > 0, \ \forall \alpha \in [0, \bar{\alpha}],$$

then $X(\bar{\alpha}) \succ 0$ and $S(\bar{\alpha}) \succ 0$.

Lemma 4.3. (Lemma 4.4 in [6]) *Let $X, S \succ 0$ and $\mu > 0$. Assume that $\delta := \delta(V)$, one has*

$$\sqrt{1-\delta} \leq \lambda_i(V) \leq \sqrt{1+\delta}, \ i = 1, 2, \ldots, n \quad and \quad \|V\|_F^2 \leq n(1+\delta).$$

Lemma 4.4. (Lemma A.1 in [18]) *Let $Q \in \mathbf{S}^n_{++}$ and $M \in \mathbb{R}^{n\times n}$ be skew-symmetric (i.e., $M = -M^T$). Then* $\det(Q + M) > 0$. *Moreover, if the eigenvalues of $Q + M$ are real, then*

$$0 < \lambda_{min}(Q) \leq \lambda_{min}(Q + M) \leq \lambda_{max}(Q + M) \leq \lambda_{max}(Q),$$

which implies $Q + M \succ 0$.

4.1. Corrector step

The following lemma shows that the strict feasibility of iterates after a full NT-step.

Lemma 4.5. (Modification of Lemma 6.3 in [22]) *If $\delta := \delta(X, S; \mu) < 1$, then (X_+, y_+, S_+) are strictly feasible.*

Lemma 4.6. (Modification of Lemma 6.4 in [23]) *If $\delta := \delta(X, S; \mu) < 1$, one has*

$$\delta_+ := \delta(X_+, S_+; \mu) \leq \frac{\delta^2}{1 + \sqrt{1 - \delta^2}}.$$

Thus $\delta(X_+, S_+, \mu) \leq \delta^2$, which shows the quadratical convergence of the algorithm.

The following lemma gives an upper bound of the duality gap after a full NT-step.

Lemma 4.7. (Modification of Lemma 3.8 in [19]) *After a full NT-step, we have*

$$X_+ \bullet S_+ \leq n\mu.$$

4.2. Predictor step

Here, we present a sufficient condition for strict feasibility after a predictor step.

Lemma 4.8. *Let $X_+ \succ 0$, $S_+ \succ 0$ and $\mu > 0$ such that $\delta_+ := \delta(X_+, S_+; \mu) < 1$. Moveover, let $X^p := X_+ + \theta\Delta^p X$ and $S^p := S_+ + \theta\Delta^p S$, $\theta \in (0, \frac{1}{2})$, denote the iterates after a predictor step. Then $X^p \succ 0$ and $S^p \succ 0$ if $\mathcal{K}(\delta_+, \theta, n) > 0$, where*

$$\mathcal{K}(\delta_+, \theta, n) := (1 - \delta_+) - \frac{n\theta^2(1 + \delta_+)}{1 - 2\theta}.$$

Proof. Let us introduce

$$X^p(\alpha) := X_+ + \alpha\theta\Delta^p X, \quad S^p(\alpha) := S_+ + \alpha\theta\Delta^p S,$$

for $0 \le \alpha \le 1$. Moreover, we have

$$X^p(\alpha) = \sqrt{\mu} D_+(V_+ + \alpha\theta D_X^p)D_+, \quad S^p(\alpha) = \sqrt{\mu} D_+^{-1}(V_+ + \alpha\theta D_S^p)D_+^{-1}.$$

Therefore, we have

$$\begin{aligned} X^p(\alpha)S^p(\alpha) &= \mu D_+(V_+ + \alpha\theta D_X^p)(V_+ + \alpha\theta D_S^p)D_+^{-1} \\ &\sim \mu(V_+ + \alpha\theta D_X^p)(V_+ + \alpha\theta D_S^p), \end{aligned}$$

Using the third equation of system (13), we get

$$\begin{aligned} \frac{X^p(\alpha)S^p(\alpha)}{\mu} &\sim (V_+ + \alpha\theta D_X^p)(V_+ + \alpha\theta D_S^p) \\ &= V_+^2 + \alpha\theta(V_+ D_S^p + D_X^p V_+) + \alpha^2\theta^2 D_X^p D_S^p \\ &= V_+^2 + \alpha\theta(V_+ D_S^p + V_+ D_X^p) + \alpha\theta(D_X^p V_+ - V_+ D_X^p) + \alpha^2\theta^2 D_X^p D_S^p \\ &= V_+^2 + \alpha\theta(-2V_+^2) + \alpha\theta(D_X^p V_+ - V_+ D_X^p) + \alpha^2\theta^2 D_X^p D_S^p \\ &= (1 - 2\alpha\theta)V_+^2 + \alpha\theta(D_X^p V_+ - V_+ D_X^p) + \alpha^2\theta^2 D_X^p D_S^p. \end{aligned}$$

Furthermore, we have

$$\frac{X^p(\alpha)S^p(\alpha)}{\mu(1-2\alpha\theta)} \sim V_+^2 + \frac{\alpha^2\theta^2}{1-2\alpha\theta} D_{XS}^p + \frac{\alpha\theta}{1-2\alpha\theta}(\alpha\theta M + (1-\alpha\theta)(D_X^p V_+ - V_+ D_X^p)), \tag{14}$$

where

$$M := (D_X^p V_+ - V_+ D_X^p) + \frac{1}{2}(D_X^p D_S^p - D_S^p D_X^p).$$

One can easily verify that the matrix $\alpha\theta M + (1 - \alpha\theta)(D_X^p V_+ - V_+ D_X^p)$ is skew-symmetric. And it follows from Lemma 4.4 that

$$\begin{aligned} \lambda_{\min}\left(\frac{X^p(\alpha)S^p(\alpha)}{\mu(1-2\alpha\theta)}\right) &\ge \lambda_{\min}\left(V_+^2 + \frac{\alpha^2\theta^2}{1-2\alpha\theta} D_{XS}^p\right) \\ &\ge \lambda_{\min}(V_+^2) - \frac{\alpha^2\theta^2}{1-2\alpha\theta}\|D_{XS}^p\|_\infty \\ &\ge \lambda_{\min}(V_+^2) - \frac{\theta^2}{1-2\theta}\|D_{XS}^p\|_\infty, \end{aligned}$$

the last inequality follows by $f(\alpha) = \frac{\alpha^2\theta^2}{1-2\alpha\theta}$ is strictly increasing for $0 \le \alpha \le 1$. By Lemmas 4.1 and 4.3, we have

$$\|D^p_{XS}\|_\infty \le \frac{1}{4}\|D^p_X + D^p_S\|^2_F = \frac{1}{4}\| - 2V_+\|^2_F \le n(1+\delta_+).$$

Therefore,

$$\lambda_{\min}\left(\frac{X^p(\alpha)S^p(\alpha)}{\mu(1-2\alpha\theta)}\right) \ge (1-\delta_+) - \frac{n\theta^2(1+\delta_+)}{1-2\theta} := \mathcal{K}(\delta_+,\theta,n) > 0. \tag{15}$$

This implies that $\det(X^p(\alpha)S^p(\alpha)) > 0$ for $0 \le \alpha \le 1$.

In addition, since $X^p(0) = X_+ \succ 0$ and $S^p(0) = S_+ \succ 0$, Lemma 4.2 implies that $X^p(1) = X^p \succ 0$ and $S^p(1) = S^p \succ 0$. This proves the lemma.

In the next lemma, we investigate the effect on the proximity measure of a predictor step by an update of the parameter μ.

Lemma 4.9. *Let $\delta_+ := \delta(X_+, S_+; \mu) < 1$, $\mu^p := (1-2\theta)\mu$ for each $0 < \theta < \frac{1}{2}$, $\mathcal{K}(\delta_+,\theta,n) > 0$. Moreover, let X^p, S^p denote the iterates after a predictor step, i.e., $X^p = X_+ + \theta\Delta^p X$ and $S^p = S_+ + \theta\Delta^p S$, $\theta \in (0, \frac{1}{2})$. Then*

$$\delta^p := \delta(X^p, S^p; \mu^p) \le \delta_+ + \frac{\sqrt{2}n\theta^2(1+\delta_+)}{1-2\theta}.$$

Proof. From Lemma 4.8, we know that the predictor step is strictly feasible. Letting $\alpha = 1$, it follows from (14) that

$$(V^p)^2 \sim \frac{X^pS^p}{\mu^p} \sim V_+^2 + \frac{\theta^2}{1-2\theta}D^p_{XS} + \frac{\theta}{1-2\theta}\tilde{M}, \tag{16}$$

where $\tilde{M} := \theta M + (1-\theta)(D^p_X V_+ - V_+ D^p_X)$ is skew-symmetric. From (15), we know

$$\lambda_{min}((V^p)^2) \ge \mathcal{K}(\delta_+,\theta,n) > 0.$$

Using (16) and properties of the Frobenius norm, we get

$$\begin{aligned}
(\delta^p)^2 = \|E - (V^p)^2\|^2_F &= \sum_{i=1}^n \left(\lambda_i\left(V_+^2 + \frac{\theta^2}{1-2\theta}D^p_{XS} + \frac{\theta}{1-2\theta}\tilde{M}\right) - 1\right)^2 \\
&= \frac{1}{(1-2\theta)^2}\sum_{i=1}^n(\lambda_i((1-2\theta)V_+^2 + \theta^2 D^p_{XS} + \theta\tilde{M})^2 \\
&+ (1-2\theta)^2 - 2(1-2\theta)\lambda_i((1-2\theta)V_+^2 + \theta^2 D^p_{XS} + \theta\tilde{M})) \\
&= \frac{1}{(1-2\theta)^2}[\mathrm{Tr}(((1-2\theta)V_+^2 + \theta^2 D^p_{XS} + \theta\tilde{M})^2) \\
&+ n(1-2\theta)^2 - 2(1-2\theta)\mathrm{Tr}((1-2\theta)V_+^2 + \theta^2 D^p_{XS})] \\
&\le \frac{1}{(1-2\theta)^2}[\mathrm{Tr}(((1-2\theta)V_+^2 + \theta^2 D^p_{XS})^2) + n(1-2\theta)^2 \\
&- 2(1-2\theta)\mathrm{Tr}((1-2\theta)V_+^2 + \theta^2 D^p_{XS})]
\end{aligned}$$

$$= \frac{1}{(1-2\theta)^2}\mathrm{Tr}(((1-2\theta)(E-V_+^2)-\theta^2 D_{XS}^p)^2)$$
$$= \frac{1}{(1-2\theta)^2}\|((1-2\theta)(E-V_+^2)-\theta^2 D_{XS}^p)\|_F^2.$$

Taking the second root and using the triangle inequality, we obtain

$$\delta^p \le \|E-V_+^2\|_F + \frac{\theta^2}{1-2\theta}\|D_{XS}^p\|_F. \tag{17}$$

Furthermore, by using Lemma 4.1 and the third equation in (13) we have

$$\begin{aligned}\|D_{XS}^p\|_F &\le \frac{1}{2\sqrt{2}}\|D_X^p + D_S^p\|_F^2 \le \frac{1}{2\sqrt{2}}\|-2V_+\|_F^2 \\ &= \frac{1}{2\sqrt{2}}\sum_{i=1}^n \lambda_i(2V_+)^2 \le \sqrt{2}n(1+\delta_+).\end{aligned} \tag{18}$$

Using (17) and (18), we get

$$\delta^p \le \delta_+ + \frac{\sqrt{2}n\theta^2(1+\delta_+)}{1-2\theta}.$$

This completes the proof of lemma.

4.3. The effect on duality gap after an iteration

The following lemma gives the effect on duality gap after a main iteration.

Lemma 4.10. *After an iteration, we have*

$$X^p \bullet S^p \le \frac{n\mu^p}{1-2\theta}$$

Proof. Letting $\alpha = 1$ in (14), we have

$$\frac{X^pS^p}{\mu(1-2\theta)} \sim V_+^2 + \frac{\theta^2}{1-2\theta}D_{XS}^p + \frac{\theta}{1-2\theta}(\theta M_+ + (1-\theta)(D_X^pV_+ - V_+D_X^p)). \tag{19}$$

Note that $\theta M_+ + (1-\theta)(D_X^pV_+ - V_+D_X^p)$ is skew-symmetry, we obtain

$$\frac{1}{\mu(1-2\theta)}\mathrm{Tr}(X^pS^p) = \mathrm{Tr}(V_+^2) + \frac{\theta^2}{1-2\theta}\mathrm{Tr}(D_{XS}^p). \tag{20}$$

It follows from the third equation of (13) that

$$D_{XS}^p = 2V_+^2 - \frac{(D_X^p)^2 + (D_S^p)^2}{2}.$$

Taking trace, we get

$$\mathrm{Tr}(D_{XS}^p) = \frac{1}{2}\mathrm{Tr}\left(\frac{X_+S_+}{\mu}\right) - \frac{\|D_X^p\|_F^2 + \|D_S^p\|_F^2}{2} \le \frac{1}{2}\mathrm{Tr}\left(\frac{X_+S_+}{\mu}\right). \tag{21}$$

Using (20), (21), Lemma 4.7 and $\mu^p = (1-2\theta)\mu$, one has

$$\mathrm{Tr}(X^pS^p) \leq \left(1 - 2\theta + \frac{\theta^2}{2}\right)\mathrm{Tr}(X_+S_+) \leq \mathrm{Tr}(X_+S_+) \leq n\mu = \frac{n\mu^p}{1-2\theta},$$

which proves the desired result.

4.4. Fixing the parameters

Here, we choose the parameters τ and θ, which guarantee that after a main iteration, the proximity measure will not exceed the proximity parameter τ. Let (X, y, S) be the iterate at the start of a main iteration and $\delta := \delta(X, S; \mu) \leq \tau$. After a correct step, by Lemma 4.6, we have

$$\delta_+ = \delta(X_+, S_+; \mu) \leq \frac{\delta^2}{1 + \sqrt{1-\delta^2}}.$$

Since the right-hand side of the above inequality is monotonically increasing with respect to δ, thus

$$\delta_+ \leq \frac{\tau^2}{1 + \sqrt{1-\tau^2}}.$$

Following a predictor step and an μ-update, by Lemma 4.8, one has

$$\delta^p \leq \delta_+ + \frac{\sqrt{2}n\theta^2(1+\delta_+)}{1-2\theta}. \tag{22}$$

Note that the right-hand side of (22) is monotonically increasing with respect to δ_+, we obtain

$$\delta^p \leq \delta_+ + \frac{\sqrt{2}n\theta^2(1+\delta_+)}{1-2\theta} \leq \frac{\tau^2}{1+\sqrt{1-\tau^2}} + \frac{\sqrt{2}n\theta^2(2-\sqrt{1-\tau^2})}{1-2\theta}.$$

To keep $\delta^p \leq \tau$, if suffices to have

$$\frac{\tau^2}{1+\sqrt{1-\tau^2}} + \frac{\sqrt{2}n\theta^2(2-\sqrt{1-\tau^2})}{1-2\theta} \leq \tau.$$

At this state, we let $\tau = \frac{1}{8}$ and $\theta = \frac{1}{8\sqrt{n}}$, after some simple calculation, we have $\delta^p \leq \frac{1}{8}$. Thus, the algorithm is well defined.

5. Iteration complexity

In the present section, we derive worst-case iteration complexity of **Algorithm**.

Lemma 5.1. *Let X^0 and S^0 be strictly feasible, $n\mu^0 = X^0 \bullet S^0$ and $\delta(X^0, S^0; \mu^0) \leq \tau$. Moveover, let X^k and S^k be the iterates obtained after k iterations. Then, $X^k \bullet S^k \leq \varepsilon$ for*

$$k \geq 1 + \left\lceil \frac{1}{2\theta} \log \frac{X^0 \bullet S^0}{\varepsilon} \right\rceil.$$

Proof. It follows from Lemma 4.10 that

$$X^k \bullet S^k \leq \frac{n\mu^k}{1-2\theta} = n(1-2\theta)^{k-1}\mu^0 = (1-2\theta)^{k-1}(X^0 \bullet S^0).$$

In order to prove $X^k \bullet S^k \leq \varepsilon$, it is sufficient to prove

$$(1-2\theta)^{k-1}(X^0 \bullet S^0) \leq \varepsilon.$$

Taking natural logarithms, we get

$$(k-1)\log(1-2\theta) + \log(X^0 \bullet S^0) \leq \log\varepsilon.$$

Using $\log(1-2\theta) \leq -2\theta$ for $0 < \theta < \frac{1}{2}$, we observe that the above inequality holds if

$$-2\theta(k-1) + \log(X^0 \bullet S^0) \leq \log\varepsilon.$$

This implies the desired result.

Lemma 5.2. *Let $\tau = \frac{1}{8}$ and $\theta = \frac{1}{8\sqrt{n}}$. Then the algorithm is well-defined and the algorithm requires at most*

$$O\left(\sqrt{n}\log\frac{X^0 \bullet S^0}{\varepsilon}\right)$$

iterations. The output is a primal-dual pair (X, y, S) satisfying $X \bullet S \leq \varepsilon$.

Proof. Let $\tau = \frac{1}{8}$ and $\theta = \frac{1}{8\sqrt{n}}$, the desired result follows immediately from Lemma 5.1.

Corollary 5.3. *If one takes $X^0 = S^0 = E$, the iteration bound becomes*

$$O\left(\sqrt{n}\log\frac{n}{\varepsilon}\right),$$

which is the currently best-known iteration bound for the algorithm with small-update methods.

6. Concluding remarks

In this paper, we proposed a new corrector-predictor algorithm for CQSDO problem. Each iteration is composed of a corrector step and a predictor step. Some good properties of our algorithm are: (1) The search direction is obtained by an equivalent algebraic reformulation of the centering equation. (2) In the corrector-step, the step length need not be calculated because we have full-step. (3) The iteration complexity obtained for proposed algorithm coincides with the currently best-known complexity bound with small-update interior point algorithms. For further research, this algorithm may be possible extended to the Cartesian $P_*(\kappa)$-LCPs over symmetric cones.

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ROUGH FUZZY GROUPS AND ROUGH SOFT GROUPS

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Abstract. By means of Dubois and Prade's idea, we introduce the concept of rough fuzzy normal subgroups and investigate some related properties. Combining rough sets and soft sets, rough soft normal subgroups with respect to a normal subgroup of groups in Pawlak approximation spaces are introduced. Moreover, we investigate the relationships between lower and upper rough soft normal subgroups with respect to normal subgroups. Finally, decision making methods in rough soft sets are put forward and some related algebraic and applied examples are given.

Keywords: rough set; soft set; rough fuzzy normal subgroup; rough soft normal subgroup.

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1. Introduction

As far as known that rough set theory was firstly proposed by Pawlak [15] in 1982. In general, the Pawlak approximation operators are defined by an equivalence relation. However, the equivalence relations in Pawlak rough sets are too restrictive in applications. Based on the reason, several researcher put forward some general models (see e.g. [11], [19]). In particular, Zhang [22] researched the union and intersection operations of rough sets based on various approximations spaces. Later, by combining rough sets and fuzzy sets, the concepts of fuzzy rough sets and rough fuzzy sets were introduced by Dubois and Prade [6] in 1990. Nowadays, this theory has been applied successfully to many areas, such as machine learning, information sciences, and knowledge discovery, and so on.

At the same time, some researchers applied this theory to algebraic structures. In 1994, Biswas et al. [3] applied rough set theory to groups. Moreover, Davvaz [5] gave the notion of rough ideals with respect to an ideal of rings in 2004. After that, rough set theory in different algebraic structures was investigated by many

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authors [10, 18]. In particular, by means of the idea of Dubois and Prade [6], Zhan [20] studied rough fuzzy (fuzzy rough) strong h-ideals of hemirings and researched some important properties of them.

In 1999, Molodtsov [14] firstly introduced the concept of soft set as a new mathematical tool for dealing with uncertainties. Based on this novel idea, the research on the soft set theory is developing rapidly. The new operations on soft sets were discussed and the applications of soft set theory in decision making problems were researched by Maji et al. [12], [13]. Later, Feng [9] researched the generalized uni-int decision making schemes based on choice value soft set and Alcantud [2] put forward a novel algorithm for fuzzy soft set based decision making from multiobserver input parameter data set. Aktaş et al. [1], [17] gave the definitions of soft groups and normalistic soft groups and some related properties were derived. Moreover, soft semirings and some related notions were put forward in [7]. In particular, by combining Pawlak rough sets and soft sets, the rough soft sets were proposed by Feng et al. [8]. According to Feng's idea in [8], rough soft set theory was firstly applied to hemirings and some vital results and properties were investigated by Zhan [21].

This paper is organized as follows: In section 2, some concepts and results on groups, rough sets and soft sets were recalled. Based on the ideal of Zhan [20], we introduce the concept of rough fuzzy normal subgroups and obtain some related properties in section 3. In section 4, we give the definition of rough soft normal subgroups and investigate the relationship between lower and upper approximation operations. In section 5, we give the decision making method of rough soft groups.

2. Preliminaries

Throughout this paper, G is a group. A fuzzy subset μ of G is called a fuzzy subgroup of G if $\mu(xy) \geq \min\{\mu(x) \text{ and } \mu(y)\}$ and $\mu(x^{-1}) \geq \mu(x)$ for all $x, y \in G$. A fuzzy subgroup μ of G is said to be a fuzzy normal subgroup of G if $\mu(xyx^{-1}) \geq \mu(y)$ for all $x, y \in G$ [4, 16].

Let μ be a fuzzy set of G and $t \in [0, 1]$. Then the sets $\mu_t = \{x \in G | \mu(x) \geq t\}$ and $\mu_t^s = \{x \in G | \mu(x) > t\}$ are called t-level subset and strong t-level subset of μ, respectively.

Proposition 2.1 [4] *A fuzzy subset μ of a group G is a fuzzy normal subgroup of G if and only if for all $t \in [0, 1]$, μ_t (μ_t^s) is a normal subgroup of G.*

Molodtsov [14] defined the soft set in the following way: A pair (F, A) is called a soft set over U, where F is a mapping given by $F : A \to P(U)$ for all $A \subset E$, where U is an initial universe, E is a set of parameters and $P(U)$ is the power set of U. Let (F, A) and (H, B) be two soft sets over U. The inclusion symbol "$\subseteq$" of (F, A) and (H, B), denoted by $(F, A) \subseteq (H, B)$, is defined as $F(x) \subseteq H(x)$ for all $x \in A \subseteq B$.

Definition 2.2 [1, 17] (1) Let (F, A) be a soft set over group G. Then (F, A) is said to be a soft group over G if and only if $F(x) < G$ for all $x \in A$.

(2) Let G be a group and (F, A) be a non-null soft set over G. Then (F, A) is called a normalistic soft group over G if $F(x)$ is a normal subgroup of G for all $x \in \text{Supp}(F, A)$.

Definition 2.3 (1) Let $\mathfrak{S} = (F, A)$ and $\mathfrak{T} = (H, B)$ be two soft sets over a common group G, then the "multiplication" of $\mathfrak{S}$ and $\mathfrak{T}$, denoted by $\mathfrak{S} \cdot \mathfrak{T}$ is defined by $\mathfrak{S} \cdot \mathfrak{T} = (F, A) \cdot (H, B) = (J, A * B)$, where $J(x, y) = F(x) \cdot H(y)$ for all $(x, y) \in A * B$.

(2) Let M be a subset of G and $\mathfrak{S} = (F, A)$ a soft set over G. Then the "multiplication" of M and $\mathfrak{S}$, denoted by $M \cdot \mathfrak{S}$ is defined by $M \cdot \mathfrak{S} = M \cdot F(x)$ for all $x \in A$.

Definition 2.4 [6] Let R be an equivalence relation on the universe U and μ a fuzzy set of U. Then we define the two fuzzy sets $\underline{R}(\mu)$ and $\overline{R}(\mu)$ as follows:

$$\underline{R}(\mu)(x) = \wedge\{\mu(y)|y \in [x]_R\}$$

and

$$\overline{R}(\mu)(x) = \vee\{\mu(y)|y \in [x]_R\},$$

for all$x \in U$.

The fuzzy sets $\underline{R}(\mu)$ and $\overline{R}(\mu)$ are called, resp., the lower and upper approximations of μ. Moreover, $R(\mu) = (\underline{R}(\mu), \overline{R}(\mu))$ is called a rough fuzzy set of U if $\underline{R}(\mu) \neq \overline{R}(\mu)$.

Definition 2.5 [8] Let R be an equivalence relation on the universe U, (U, R) a Pawlak approximation space and $\mathfrak{S} = (F, A)$ a soft set over U. The lower and upper rough approximations of $\mathfrak{S}$ w.r.t. (U, R) are defined by:

$\underline{R}(\mathfrak{S}) = (\underline{F}, A)$ and $\overline{R}(\mathfrak{S}) = (\overline{F}, A)$, which are soft sets over U with $\underline{F}(x) = \underline{R}(F(x)) = \{y \in U|[y]_R \subseteq F(x)\}$ and $\overline{F}(x) = \overline{R}(F(x)) = \{y \in U|[y]_R \cap F(x) \neq \emptyset\}$, for all $x \in A$.

The soft sets $\underline{R}(\mathfrak{S})$ ($\overline{R}(\mathfrak{S})$) is called a lower (upper) approximation. Moreover, $\mathfrak{S}$ is called a rough soft set of U, if $\underline{R}(\mathfrak{S}) \neq \overline{R}(\mathfrak{S})$, for all $x \in A$.

3. Rough fuzzy normal subgroups

In this section, we introduce the concept of rough fuzzy normal subgroups of groups and investigate some related properties.

Let N be a normal subgroup of G. Then we define a relation $\equiv_N$ by $x \equiv_N y \Leftrightarrow xy^{-1} \in N$. It is clear that the relation $\equiv_N$ is a congruence relation on G. Let $[x]_N$ denote the equivalence class of x w.r.t. N. So it is easy to obtain the following lemma.

Lemma 3.1 *Let N be a normal subgroup of G. If $x, y \in G$, then*
(1) *$x \in [y]_N$ if and only if $x \in yN$,*
(2) *$[x]_N \cdot [y]_N = [xy]_N$.*

Proof. (1) If $x \in [y]_N$, then there exists $a \in N$ such that $xy^{-1} = a$. So $x = ay \in Ny = yN$. Conversely, if $x \in yN$, then there exists $a \in N$ such that $x = ya$, that is $xy^{-1} = a \in N$, hence $x \in [y]_N$.

(2) If $z \in [x]_N \cdot [y]_N$, then there exists $a \in [x]_N$ and $b \in [y]_N$ such that $z = ab$. Thus we have $z = ab \in (xN)(yN) = (xy)N$, hence $z \in [xy]_N$. Conversely, if $z \in [xy]_N$, then $z \in (xy)N = (xN)(yN)$. Thus there exists $a \in xN$ and $b \in yN$, that is, $a \in [x]_N$ and $b \in [y]_N$ such that $z = ab$, and so $z = ab \in [x]_N \cdot [y]_N$. Consequently, $[x]_N \cdot [y]_N = [xy]_N$. ∎

By means of the idea of Lemma 3.1, we introduce the concepts of rough fuzzy normal subgroups of groups.

Definition 3.2 Let N be a normal subgroup of G and μ a fuzzy set of G. Then we define two fuzzy sets $\underline{Apr}_N(\mu)$ and $\overline{Apr}_N(\mu)$ as follows:

$$\underline{Apr}_N(\mu)(x) = \bigwedge_{y \in [x]_N} \mu(y) \text{ and } \overline{Apr}_N(\mu)(x) = \bigvee_{y \in [x]_N} \mu(y), \text{ for all } x \in G.$$

The fuzzy sets $\underline{Apr}_N(\mu)$ and $\overline{Apr}_N(\mu)$ are called, resp., the lower and upper approximations of μ w.r.t. N of G. Moreover, $Apr_N(\mu) = (\underline{Apr}_N(\mu), \overline{Apr}_N(\mu))$ is called a rough fuzzy set w.r.t. N of G if $\underline{Apr}_N(\mu) \neq \overline{Apr}_N(\mu)$.

Definition 3.3 Let N be a normal subgroup of G and μ a fuzzy set of G. Then μ is called a lower (upper) rough fuzzy normal subgroup w.r.t. N of G if $\underline{Apr}_N(\mu)$ ($\overline{Apr}_N(\mu)$) is a fuzzy normal subgroup of G. Moreover, $Apr_N(\mu) = (\underline{Apr}_N(\mu), \overline{Apr}_N(\mu))$ is called a rough fuzzy normal subgroup w.r.t. N of G if both $\underline{Apr}_N(\mu)$ and $\overline{Apr}_N(\mu)$ are fuzzy fuzzy normal subgroups of G.

Example 3.4 Consider $G = \{\pm 1, \pm i\}$ be the group where $i^2 = -1$. Let $N = \{1, -1\}$, it is easy to find that N is a normal subgroup of G. Moreover, $[1]_N = N$, $[i]_N = \{i, -i\}$.

Define a fuzzy set μ of G by $\mu(1) = 0.7$ and $\mu(-1) = \mu(i) = \mu(-i) = 0.5$.

By calculations, we have

$$\underline{Apr}_N(\mu) = \frac{0.5}{1} + \frac{0.5}{-1} + \frac{0.5}{i} + \frac{0.5}{-i}$$

and

$$\overline{Apr}_N(\mu) = \frac{0.7}{1} + \frac{0.7}{-1} + \frac{0.5}{i} + \frac{0.5}{-i}.$$

This implies that $Apr_N(\mu)$ is a rough fuzzy normal subgroup w.r.t. N of G.

Proposition 3.5 *Let N and K be any two normal subgroups of G and μ a fuzzy set of G. If $N \subseteq K$, then*
(1) *$\underline{Apr}_N(\mu) \supseteq \underline{Apr}_K(\mu)$,*
(2) *$\overline{Apr}_N(\mu) \subseteq \overline{Apr}_K(\mu)$.*

Proof. For any $y \in [x]_N$, since $N \subseteq K$, then we have $y \in [x]_K$, by the Definition 3.2, hence $\underline{Apr}_N(\mu) \supseteq \underline{Apr}_K(\mu)$. Similarly, we can get $\overline{Apr}_N(\mu) \subseteq \overline{Apr}_K(\mu)$. ∎

We can get the following corollary directly from Proposition 3.5

Corollary 3.6 *Let N and K be any two normal subgroups of G and μ a fuzzy set of G. If $N \subseteq K$, then*

(1) $\underline{\underline{Apr}}_{N\cap K}(\mu) \supseteq \underline{\underline{Apr}}_N(\mu) \supseteq \underline{\underline{Apr}}_{N\cdot K}(\mu)$,
(2) $\overline{\overline{Apr}}_{N\cap K}(\mu) \subseteq \overline{\overline{Apr}}_N(\mu) \subseteq \overline{\overline{Apr}}_{N\cdot K}(\mu)$.

Proof. Since N and K are any two normal subgroups of G, so are $N \cap K$ and NK, and it is easy to obtain the inclusion relation: $N \cap K \subseteq N \subseteq N \cdot K$. ∎

Proposition 3.7 *Let N be a normal subgroup of G and μ, ν any two fuzzy sets of G. If $\mu \subseteq \nu$, then*

(1) $\underline{\underline{Apr}}_N(\mu) \subseteq \underline{\underline{Apr}}_N(\nu)$,
(2) $\overline{\overline{Apr}}_N(\mu) \subseteq \overline{\overline{Apr}}_N(\nu)$.

Proof. For any $x \in G$, by hypothesis, we have $\mu(x) \leq \nu(x)$. So $\underline{Apr}_N(\mu)(x) = \bigwedge_{y\in[x]_N} \mu(y) \leq \bigwedge_{y\in[x]_N} \nu(y) = \underline{Apr}_N(\nu)(x)$. Similarly, we can get $\overline{Apr}_N(\mu) \subseteq \overline{Apr}_N(\nu)$. This completes the proof. ∎

Now, we give the (strong) level subset of lower and upper rough approximations of a fuzzy set μ w.r.t. N of groups.

Theorem 3.8 *Let N be a normal subgroup of G. If μ is a fuzzy set of G and $t \in [0, 1]$. Then*

(1) $(\underline{\underline{Apr}}_N(\mu))_t = \underline{\underline{Apr}}_N(\mu_t)$,
(2) $(\overline{\overline{Apr}}_N(\mu))^s_t = \overline{\overline{Apr}}_N(\mu^s_t)$.

Proof. (1) For any $x \in G$,we have

$$\begin{aligned} x \in (\underline{Apr}_N(\mu))_t &\Leftrightarrow \underline{Apr}_N(\mu)(x) \geq t \\ &\Leftrightarrow \bigwedge_{y\in[x]_N} \mu(y) \geq t \\ &\Leftrightarrow \forall y \in [x]_N, \mu(y) \geq t \\ &\Leftrightarrow [x]_N \subseteq \mu_t \\ &\Leftrightarrow x \in \underline{Apr}_N(\mu_t). \end{aligned}$$

(2)

$$\begin{aligned} x \in (\overline{Apr}_N(\mu))^s_t &\Leftrightarrow \overline{Apr}_N(\mu)(x) > t \\ &\Leftrightarrow \bigvee_{y\in[x]_N} \mu(y) > t \\ &\Leftrightarrow \exists y \in [x]_N, \mu(y) > t \\ &\Leftrightarrow [x]_N \cap \mu^s_t \neq \emptyset \\ &\Leftrightarrow x \in \overline{Apr}_N(\mu^s_t). \end{aligned}$$

This completes the proof. ∎

Lemma 3.9 *Let N and A be two normal subgroups of G. Then*
(1) $\overline{Apr}_N(A) = N \cdot A$ *and* $\overline{Apr}_N(A)$ *is a normal subgroup of G;*
(2) *if* $\underline{Apr}_N(A) \neq \emptyset$, *then* $\underline{Apr}_N(A) = A$ *and* $\underline{Apr}_N(A)$ *is a normal subgroup of G.*

Proof. (1) Let $\forall x \in \overline{Apr}_N(A)$. Then $xN \cap A \neq \emptyset$, so there exists $a \in G$ such that $a \in xN \cap A$, that is, $x \in aN$ and $a \in A$, hence $x \in A \cdot N = N \cdot A$.

Conversely, if $x \in N \cdot A$. Then there exists $a \in N$ and $b \in A$ such that $x = ab$, hence $b \in xN$ and so $b \in xN \cap A$, that is, $x \in \overline{Apr}_N(A)$. Therefore, $\overline{Apr}_N(A) = N \cdot A$. Since N and A are two normal subgroups of G, so $\overline{Apr}_N(A)$ is a normal subgroup of G.
(2) By hypothesis, if $\underline{Apr}_N(A) \neq \emptyset$, so there exists $s \in \underline{Apr}_N(A) \subseteq A$, that is $aN \subseteq A$. Thus $\forall y \in N$, $\exists b \in A$ such that $b = ay$. Since A is a normal subgroup of G, we have $y \in A$ and so $N \subseteq A$. And then, for all $b \in A$, we have $bN \subseteq A$, this implies that $b \in \underline{Apr}_N(A)$, so $A \subseteq \underline{Apr}_N(A)$, hence $\underline{Apr}_N(A) = A$ and so $\underline{Apr}_N(A)$ is a normal subgroup of G. This completes the proof. ∎

Theorem 3.10 Let N be a normal subgroup of G. If μ is a fuzzy normal subgroup of G, then
(1) $\overline{Apr}_N(\mu)$ is a fuzzy normal subgroup of G,
(2) if $\underline{Apr}_N(\mu) \neq \emptyset$, then $\underline{Apr}_N(\mu)$ is a fuzzy normal subgroup of G.

Proof. (1) Let μ be a fuzzy normal subgroup of G. For any $t \in [0,1]$, by Theorem 3.8 (2), $(\overline{Apr}_N(\mu))_t^s = \overline{Apr}_N(\mu_t^s)$. By Proposition 2.1, we know that μ_t^s is a normal subgroup of G. Hence, by Lemma 3.9 (1), $\overline{Apr}_N(\mu_t^s)$ is a normal subgroup of G. Then by Proposition 2.1, $\overline{Apr}_N(\mu)$ is a fuzzy normal subgroup of G.
(2) Since $\underline{Apr}_N(\mu) \neq \emptyset$, by Theorem 3.8 (1), there exists $t \in [0,1]$ such that $(\underline{Apr}_N(\mu))_t = \underline{Apr}_N(\mu_t)$. Let t be any value that fulfills the above property. Then it is clear that $\mu_t \neq \emptyset$, and we know from Proposition 2.1 that μ_t is a normal subgroup of G. Hence, by Lemma 3.9 (2), $(\underline{Apr}_N(\mu))_t$ is a normal subgroup of G. Then, by Proposition 2.1, $\underline{Apr}_N(\mu)$ is a fuzzy normal subgroup of G. This completes the proof. ∎

By Theorem 3.10, we can obtain the following corollary.

Corollary 3.11 *Let N be a normal subgroup of G. If μ is a fuzzy normal subgroup of G and $\underline{Apr}_N(\mu) \neq \emptyset$, then $Apr_N(\mu)$ is a rough fuzzy normal subgroup of G.*

4. Rough soft normal subgroups

Definition 4.1 Let N be a normal subgroup of G, (G, N) a Pawlak approximation space and $\mathfrak{S} = (F, A)$ a soft set over G. The lower and upper rough approximations of $\mathfrak{S} = (F, A)$ w.r.t. (G, N) are denoted by:

$$\underline{Apr}_N(\mathfrak{S}) = (\underline{F}_N, A) \quad and \quad \overline{Apr}_N(\mathfrak{S}) = (\overline{F}_N, A),$$

which are soft sets over G with $\underline{F}_N(x) = \underline{Apr}_N(F(x)) = \{y \in G | yN \subseteq F(x)\}$ and $\overline{F}_N(x) = \overline{Apr}_N(F(x)) = \{y \in G | yN \cap F(x) \neq \emptyset\}$, for all $x \in A$.

(1) $\underline{Apr}_N(\mathfrak{S}) = \overline{Apr}_N(\mathfrak{S})$, the soft set $\mathfrak{S}$ is said to be definable;

(2) $\underline{Apr}_N(\mathfrak{S}) \neq \overline{Apr}_N(\mathfrak{S})$, $\underline{Apr}_N(\mathfrak{S})$ ($\overline{Apr}_N(\mathfrak{S})$) is called a lower (upper) rough soft normal subgroup w.r.t. N over G, if $\underline{F}_N(x)$ ($\overline{F}_N(x)$) is a normal subgroup of G, for all $x \in \mathrm{Supp}(\mathrm{F}, \mathrm{A})$. Moreover, $\mathfrak{S}$ is called a rough soft normal subgroup w.r.t. N over G, if $\underline{F}_N(x)$ and $\overline{F}_N(x)$ are normal subgroups of G, for all $x \in \mathrm{Supp}(\mathrm{F}, \mathrm{A})$.

Example 4.2 Let $G = \{1, 3, 5, 7\}$ be a group of non-negative integers module 8 with usual multiplication. Let $N = \{1, 3\}$, then N is a subgroup of G. Define a soft set $\mathfrak{S} = (F, A)$ over G, $A = \{a, b\}$, where $F(a) = G$, $F(b) = \{1, 3, 5\}$. By calculations, we have $\underline{F}_N(a) = G$, $\underline{F}_N(b) = \{1, 3\}$, $\overline{F}_N(a) = G$, $\overline{F}_N(b) = G$. This shows that $\underline{F}_N(x)$ and $\overline{F}_N(x)$ are normal subgroups of G for all $x \in \mathrm{Supp}(\mathrm{F}, \mathrm{A})$. So $\mathfrak{S}$ is a rough soft normal subgroup w.r.t. N over G.

Lemma 4.3 *Let N be a normal subgroup of G and $\mathfrak{S} = (F, A)$ a soft set over G. Then $\overline{Apr}_N(\mathfrak{S}) = N \cdot \mathfrak{S}$.*

Proof. Let y be any element of $\overline{F}_N(x)$. Then $yN \cap F(x) \neq \emptyset$, $\forall x \in A$, and so there exists $a \in G$ such that $a \in yN \cap F(x)$, that is, $y \in aN$ and $b \in F(x)$, hence $y \in N \cdot F(x)$.

Conversely, if $y \in N \cdot F(x)$, then there exists $a \in N$ and $b \in F(x)$ such that $y = ab$, hence $b \in yN$ and so $b \in yN \cap F(x)$, that is, $y \in \overline{F}_N(x)$. Therefore, $\overline{F}_N(x) = N \cdot F(x)$, $\forall x \in A$, that is, $\overline{Apr}_N(\mathfrak{S}) = N \cdot \mathfrak{S}$. ∎

Theorem 4.4 *Let N be a normal subgroup of G and $\mathfrak{S} = (F, A)$ a normalistic soft group over G. Then*

(1) $\mathfrak{S}$ is an upper rough soft normal subgroup w.r.t. N over G,

(2) $\underline{Apr}_N(\mathfrak{S}) \neq \emptyset \Leftrightarrow \underline{Apr}_N(\mathfrak{S}) = \mathfrak{S}$ and $\mathfrak{S}$ is a lower rough soft normal subgroup w.r.t. N over G.

Proof. (1) By Lemma 4.3, we can obtain easily.

(2) By the assumption, if $\underline{Apr}_N(\mathfrak{S}) \neq \emptyset$, in other words, for all $x \in \mathrm{Supp}(\mathrm{F}, \mathrm{A})$, there exists $a \in \overline{F}_N(x) \subseteq F(x)$, that is $aN \subseteq F(x)$. Thus for any $y \in N$, $\exists b \in F(x)$ such that $ay = b$. Since $\mathfrak{S}$ is a normalistic soft group over G, we have $y \in F(x)$ and so $N \subseteq F(x)$. And then, for $\forall c \in F(x)$, we have $cN \subseteq F(x)$, this implies that $c \in \overline{F}_N(x)$, so $F(x) \subseteq \overline{F}_N(x)$, hence $F(x) = \overline{F}_N(x)$, $\forall x \in \mathrm{Supp}(\mathrm{F}, \mathrm{A})$, that is $\underline{Apr}_N(\mathfrak{S}) = \mathfrak{S}$.

Conversely, if $\underline{Apr}_N(\mathfrak{S}) = \mathfrak{S}$, it is clear that $\underline{Apr}_N(\mathfrak{S}) \neq \emptyset$. This implies that $\underline{Apr}_N(\mathfrak{S}) \neq \emptyset \Rightarrow \underline{Apr}_N(\mathfrak{S}) = \mathfrak{S} \Rightarrow \underline{Apr}_N(\mathfrak{S}) \neq \emptyset$.

Since $\mathfrak{S} = (F, A)$ is a normalistic soft group over G, so $\mathfrak{S}$ is a lower rough soft normal subgroup w.r.t. N over G. This completes the proof. ∎

By Theorem 4.4, we have the following corollary.

Corollary 4.5 *Let N be a normal subgroup of G and $\mathfrak{S} = (F, A)$ a normalistic soft group over G such that $\underline{Apr}_N(\mathfrak{S}) \neq \emptyset$. then $\mathfrak{S}$ is a rough soft normal subgroup w.r.t. N over G.*

The following example shows that $\mathfrak{S}$ does not need to be a normalistic soft group over G if $\mathfrak{S}$ is a rough soft normal subgroup over G.

Example 4.6 In Example 4.2, we can find that $\underline{F}_N(x)$ and $\overline{F}_N(x)$ are normal subgroups of G for all $x \in \mathrm{Supp(F, A)}$, so $\mathfrak{S}$ is a rough soft normal subgroup over G, but $\mathfrak{S}$ is not a normalistic soft group over G.

Theorem 4.7 *Let N be a normal subgroup of G. If $\mathfrak{S}$ is a lower (resp. upper) rough soft normal subgroup over G, then $\underline{Apr}_N(\mathfrak{S})$ (resp. $\overline{Apr}_N(\mathfrak{S})$) is a rough soft normal subgroup over G.*

Proof. Assume that $\mathfrak{S}$ is a lower rough soft normal subgroup over G. Summing up the discussions of Theorem 4.4, $\overline{Apr}_N(\mathfrak{S})$ is a normalistic soft group over G and $\underline{Apr}_N(\mathfrak{S})$ is an upper soft normal subgroup over G. On the other hand, it is clear that $\underline{Apr}_N(\underline{Apr}_N(\mathfrak{S})) \subseteq \underline{Apr}_N(\mathfrak{S})$. Let $a \in \underline{F}_N(x)$, where $x \in \mathrm{Supp(F, A)}$. Then $aN \subseteq F(x)$, and so, $aN \cdot N \subseteq F(x)$, in other words, $a \in \underline{Apr}_N(\underline{Apr}_N(\mathfrak{S}))$. Hence $\underline{Apr}_N(\underline{Apr}_N(\mathfrak{S})) = \underline{Apr}_N(\mathfrak{S})$ and thus $\underline{Apr}_N(\mathfrak{S})$ is a lower rough soft normal subgroup over G. Therefore, $\underline{Apr}_N(\mathfrak{S})$ is a rough soft normal subgroup over G. Similarly, we can get $\overline{Apr}_N(\mathfrak{S})$ is a rough soft normal subgroup over G. This completes the proof. ∎

Proposition 4.8 *Let N be a normal subgroup of G, $\mathfrak{S} = (F, A)$ and $\mathfrak{T} = (H, B)$ any two non-null soft sets over G. Then*

(1) $\overline{Apr}_N(\mathfrak{S}) \cdot \overline{Apr}_N(\mathfrak{T}) = \overline{Apr}_N(\mathfrak{S} \cdot \mathfrak{T})$;

(2) $\underline{Apr}_N(\mathfrak{S}) \cdot \underline{Apr}_N(\mathfrak{T}) \subseteq \underline{Apr}_N(\mathfrak{S} \cdot \mathfrak{T})$.

Proof. (1) According to Lemma 4.3, $\forall x \in \mathrm{Supp(F, A)}$, $y \in \mathrm{Supp(H, B)}$, we have $\overline{F}_N(x) \cdot \overline{H}_N(y) = (N \cdot F(x))(N \cdot H(y)) = N \cdot N \cdot F(x) \cdot H(y) = N \cdot F(x) \cdot H(y) = \overline{Apr}_N(F(x) \cdot H(y))$. This completes the proof.

(2) $\forall x \in \mathrm{Supp(F, A)}$, $y \in \mathrm{Supp(H, B)}$, let $\forall c \in \underline{F}_N(x) \cdot \underline{H}_N(y)$. Then $c = ab$ for some $a \in \underline{F}_N(x)$ and $b \in \underline{H}_N(y)$. Therefore $aN \subseteq F(x)$ and $bN \subseteq H(y)$. Since $aN \cdot bN = abN = cN \subseteq F(x) \cdot H(y)$. Hence $c \in \underline{Apr}_N(F(x) \cdot H(y))$. This completes the proof. ∎

Proposition 4.9 *Let N be a normal subgroup of G, $\mathfrak{S} = (F, A)$ and $\mathfrak{T} = (H, B)$ any two normalistic soft groups over G. Then*

$$\underline{Apr}_N(\mathfrak{S}) \cdot \underline{Apr}_N(\mathfrak{T}) = \underline{Apr}_N(\mathfrak{S} \cdot \mathfrak{T}).$$

Proof. By Theorem 4.4 (2), it is obvious. ∎

Combining Proposition 4.8 and 4.9, we have the following proposition.

Proposition 4.10 *Let N be a normal subgroup of G, $\mathfrak{S} = (F, A)$ and $\mathfrak{T} = (H, B)$ any two normalistic soft groups over G, then*

(1) *$\mathfrak{S} \cdot \mathfrak{T}$ is an upper rough soft normal subgroup w.r.t. N over G,*

(2) *if $\underline{Apr}_N(\mathfrak{S}) \neq \emptyset$ and $\underline{Apr}_N(\mathfrak{T}) \neq \emptyset$, $\mathfrak{S} \cdot \mathfrak{T}$ is a lower rough soft normal subgroup w.r.t. N over G.*

Corollary 4.11 *Let N be a normal subgroup of G, $\mathfrak{S} = (F, A)$ and $\mathfrak{T} = (H, B)$ any two normalistic soft groups over G such that $\underline{Apr}_N(\mathfrak{S}) \neq \emptyset$ and $\underline{Apr}_N(\mathfrak{T}) \neq \emptyset$, then $\mathfrak{S} \cdot \mathfrak{T}$ is a rough soft normal subgroup w.r.t. N over G.*

Proposition 4.12 *Let N and K be two normal subgroups of G such that $N \subseteq K$, $\mathfrak{S} = (F, A)$ a non-null soft set over G. Then*

(1) $\underline{Apr}_N(\mathfrak{S}) \supseteq \underline{Apr}_K(\mathfrak{S})$;

(2) $\overline{Apr}_N(\mathfrak{S}) \subseteq \overline{Apr}_K(\mathfrak{S})$.

Proof. (1) For all $x \in A$, let $a \in \underline{Apr}_K(F(x)) = \underline{F}_K(x)$, then $aK \subseteq F(x)$, we have $aN \subseteq aK \subseteq F(x)$, hence $a \in \underline{Apr}_N(F(x))$, that is $\underline{Apr}_K(F(x)) \subseteq \underline{Apr}_N(F(x))$, $\forall x \in A$. This completes the proof.

(2) By Lemma 4.3, $\overline{Apr}_N(\mathfrak{S}) = N \cdot \mathfrak{S}$, $\overline{Apr}_K(\mathfrak{S}) = K \cdot \mathfrak{S}$. By hypothesis, we have $N \cdot \mathfrak{S} \subseteq K \cdot \mathfrak{S}$. Thus, $\overline{Apr}_N(\mathfrak{S}) \subseteq \overline{Apr}_K(\mathfrak{S})$. ∎

5. Applications of rough soft groups in decision making

In this section, we give a new decision making method for rough soft sets.

Decision making method. We will put forward the new method to find which is the best parameter e of a given soft set $\mathfrak{S} = (F, A)$. In other words, $F(e)$ is the nearest accurate subgroup on $\mathfrak{S}$ w.r.t. a normal subgroup of a group.

Let G be a group and E a set of related parameters. Let $A=\{e_1, e_2, ..., e_m\}\subseteq E$ and $\mathfrak{S} = (F, A)$ be an original description soft set over G. Let N be a normal subgroup of G and (G, N) a Pawlak approximation space. Then we present the decision algorithm for rough soft groups as follows:

Step 1. Input the original description group G, soft set $\mathfrak{S}$ and Pawlak approximation space (G, N), where N is a normal subgroup of G.

Step 2. Compute the lower and upper rough soft approximation operations $\underline{Apr}_N(\mathfrak{S})$ and $\overline{Apr}_N(\mathfrak{S})$ on $\mathfrak{S}$, respectively.

Step 3. Compute the different values of $\|F(e_i)\|$, where $\|F(e_i)\| = \frac{|\overline{F}_N(e_i)|-|\underline{F}_N(e_i)|}{|F(e_i)|}$.

Step 4. Find the minimum value $\|F(e_k)\|$ of $\|F(e_i)\|$, where $\|F(e_k)\|=\min_i \|F(e_i)\|$.

Step 5. The decision is $F(e_k)$.

Example 5.1 Assume that we want to find the nearest accurate group on a soft set $\mathfrak{S}$. Let $G = S_3 = \{(1), (12), (13), (23), (123), (132)\}$ be a group, then we can check that $N = \{(1), (12)\}$ is a normal subgroup of G, and define a soft set $\mathfrak{S} = (F, A)$ over S_3, the set of parameters is $A = \{e_1, e_2, e_3, e_4\}$. The tabular representation of the soft set $\mathfrak{S}$ is given in Table 1.

Table 1. Table for soft set $\mathfrak{S}$

	(1)	(12)	(13)	(23)	(123)	(132)
e_1	1	1	1	1	0	0
e_2	1	1	1	0	0	0
e_3	1	0	0	0	0	0
e_4	1	0	1	0	1	1

By Definition 4.1, we have two soft sets $\underline{Apr}_N(\mathfrak{S})$ and $\overline{Apr}_N(\mathfrak{S})$ over G. The tabular representations of the soft sets are given by Table 2 and 3,respectively.

Table 2: Table for soft set $\underline{Apr}_N(\mathfrak{S})$

	(1)	(12)	(13)	(23)	(123)	(132)
e_1	1	1	0	0	0	0
e_2	1	1	0	0	0	0
e_3	0	0	0	0	0	0
e_4	0	0	1	0	0	1

Table 3: Table for soft set $\overline{Apr}_N(\mathfrak{S})$

	(1)	(12)	(13)	(23)	(123)	(132)
e_1	1	1	1	1	1	1
e_2	1	1	1	0	0	1
e_3	1	1	0	0	0	0
e_4	1	0	1	1	1	1

By calculations, we find $\|F(e_1)\|$=1, $\|F(e_2)\|$=0.67, $\|F(e_3)\|$=2, $\|F(e_4)\|$=0.75. Thus the minimum value for $\|F(e_i)\|$ is $\|F(e_2)\|$=0.67. That is $F(e_2)$ is the nearest accurate subgroup on $\mathfrak{S}$.

Remark 5.2 Generally speaking, we always tend to get the optimal choices in the actual decision problems. In the above method, we want to obtain the best parameter e of the given soft set $\mathfrak{S} = (F, A)$. By using the decision algorithm in the step 4, we can find that the smaller the values of $\|F(e_i)\|$ are, the more our options are optimal. By the formula of $\|F(e_i)\|$, we obtain that the values of $\|F(e_i)\|$ are smaller while the difference between $|\overline{F}_N(e_i)|$ and $|\underline{F}_N(e_i)|$ are smaller, that is, the parameter e is the optimal choice and $F(e_i)$ is the nearest accurate subgroup structure.

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A NUMERICAL SOLUTION OF THE ARBITRARY ORDER WEAKLY SINGULAR INTEGRAL USING BLOCK-PULSE FUNCTIONS AND APPLICATIONS

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Abstract. The purpose of this paper is to obtain the approximation of the arbitrary order weakly singular integral using Block-Pulse functions. The obtained results can be used to solve the numerical solution of higher order linear and nonlinear weakly singular Volterra integral equation of the second kind. Furthermore, the initial equations are transformed into a system of algebraic equations. Finally, some examples are given to demonstrate the validity and applicability of this approach, results of these examples show that this new method is an efficient algorithm.

Keywords: weakly singular integral; weakly singular integral equation; Block Pulse functions; operational matrix; numerical solution.

2010 Mathematics Subject Classication: 45G05; 45F15.

1. Introduction

Many engineering and scientific problems can be reduced into integral equations. These integral equations are always singular, some are even supersingular. Especially, all of the natural boundary integral equations can result to singular integral equations [10], [19], [18]. Due to the singularity of the kernel, many quadrature rules for the singular integrals are less accurate than their counterparts for Riemann integrals. The weakly singular integral in many equations can be observed [8], [16], [4], [9], [1], [11], [12]. The weakly singular Volterra integral equations are also found in a lot of physical, chemical, and biological problems. For example, reaction-diffusion problems, crystal growth etc [5], [7], [13]. Therefore, it is necessary to study the weakly singular integral.

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As the Block-Pulse functions have many good properties [2], [14], they are used to get the quadrature formula of the arbitrary order weakly singular integral in this paper. The numerical examples show that the precision is very high. Therefore, this method and operational matrix of the Block-Pulse functions can be used to solve a class of higher order linear and nonlinear weakly singular Volterra integral equation. In this work, the operational matrix of the Block-Pulse functions are obtained using the operational matrix of Legendre wavelet. Finally, the equation is translated into a linear or nonlinear system of algebraic equations which are easier to get the solutions.

In this study, considering the following linear weakly singular Volterra integral equation of the second kind:

$$\sum_{i=0}^{n} a_i(t)y^{(i)}(t) + \lambda \int_0^t (t-s)^{-\alpha} y(s)ds = f(t). \tag{1.1}$$

where $a_i(t)$, $f(t)$ are continuous functions on $[0,1]$ and $y^{(i)}(t)$ stands for the i th-order derivative of $y(t)$. λ is a real constants.

The numerical solutions of the nonlinear weakly singular Volterra integral equation of the second kind can also got using this method:

$$\sum_{i=0}^{n} a_i(t)y^{(i)}(t) + \lambda \int_0^t (t-s)^{-\alpha} [y(s)]^p ds = f(t). \tag{1.2}$$

2. The quadrature formula of the arbitrary order weakly singular integral

The Block-Pulse functions and some properties are introduced in this part. The set of these functions, over interval $[0,T)$, is defined as [14]:

$$b_i(x) = \begin{cases} 1, & \frac{iT}{m} \le x < \frac{(i+1)T}{m}; \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

where $i = 0, 1, 2, \cdots m-1$, with a positive integer value for m. In this paper, it is assumed that $T = 1$. The useful properties of the Block-Pulse functions:

1. *Disjointness:*

$$b_i(x)b_j(x) = \begin{cases} b_i(x), & i = j; \\ 0, & i \neq j. \end{cases} \tag{2.2}$$

2. *Orthogonality:*

$$\int_0^1 b_i(x)b_j(x)dx = \begin{cases} 1/m, & i = j; \\ 0, & i \neq j. \end{cases} \tag{2.3}$$

3. *Completeness:* For any $f \in L^2([0,1])$, the sequence $\{b_i\}$ is complete if $\int b_i f = 0$ results in $f = 0$ almost everywhere. Because of completeness of $\{b_i(x)\}$, Parsevals identity holds, i.e. we have $\int_0^1 f^2(x)dx = \sum_{i=0}^{\infty} f_i^2 \|b_i(x)\|^2$, for every real bounded function $f(x) \in L^2([0,1])$ and

$$f_i = m \int_0^1 b_i(x) f(x) dx. \tag{2.4}$$

Arbitrary order weakly singular integral is shows as following formula:

$$I(t) = \int_0^t \frac{g(s)}{(t-s)^{\alpha}} ds, 0 \le t \le 1,\ 0 < \alpha < 1. \tag{2.5}$$

where $g(s) \in L^2([0,1])$.

From the orthogonality property of the Block-Pulse functions, it is possible to expand functions into their Block-Pulse series, so it can be written as:

$$g(s) \cong \sum_{i=0}^{m-1} c_i b_i(s) = c^T B_m(s). \tag{2.6}$$

where $c = (c_0, c_1, \cdots, c_{m-1})^T$, $B_m(s) = (b_0(s), b_1(s), \cdots, b_{m-1}(s))^T$.

Since equation (2.6) is substituted into equation (2.5), then, we have:

$$I(t) = \int_0^t \frac{g(s)}{(t-s)^{\alpha}} ds = c^T \int_0^t \frac{B_m(s)}{(t-s)^{\alpha}} ds = c^T D(t) \tag{2.7}$$

where

$$D(t) = \int_0^t \frac{B_m(s)}{(t-s)^{\alpha}} ds. \tag{2.8}$$

Combining equation (2.1) and equation (2.8), we can obtain

$$\begin{aligned} D(t) &= \int_0^t \frac{B_m(s)}{(t-s)^{\alpha}} ds \\ &= \int_0^{1/m} \frac{B_m(s)}{(t-s)^{\alpha}} ds + \int_{1/m}^{2/m} \frac{B_m(s)}{(t-s)^{\alpha}} ds + \cdots + \int_{i/m}^{t} \frac{B_m(s)}{(t-s)^{\alpha}} ds \\ &= \begin{pmatrix} -\frac{(t-1/m)^{1-\alpha} - t^{1-\alpha}}{1-\alpha} \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -\frac{(t-2/m)^{1-\alpha} - (t-1/m)^{1-\alpha}}{1-\alpha} \\ 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} + \cdots + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \frac{(t-i/m)^{1-\alpha}}{1-\alpha} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \\ &= \left(-\frac{(t-\frac{1}{m})^{1-\alpha} - t^{1-\alpha}}{1-\alpha} - \frac{(t-\frac{2}{m})^{1-\alpha} - (t-\frac{1}{m})^{1-\alpha} 1-\alpha}{\cdots} \frac{(t-\frac{i}{m})^{1-\alpha} 1-\alpha}{0} \cdots 0 \right)^T \end{aligned} \tag{2.9}$$

where $D(0) = (0, 0, ..., 0)^T$.

Let $t = k/m,\ k \in 1, 2, ..., m-1$. The following results can be obtained by using equation (2.10)

$$D\left(\frac{k}{m}\right) = \left(-\frac{(\frac{k}{m}-\frac{1}{m})^{1-\alpha}-(\frac{k}{m})^{1-\alpha}}{1-\alpha}, -\frac{(\frac{k}{m}-\frac{2}{m})^{1-\alpha}-(\frac{k}{m}-\frac{1}{m})^{1-\alpha}}{1-\alpha}, ..., \frac{(\frac{k}{m}-\frac{i}{m})^{1-\alpha}}{1-\alpha}, 0, ..., 0\right)^T.$$

At this time, $i = k - 1$.

Then, combining equation (2.7) and equation (2.10), the numerical solution of equation (2.5) can be obtained.

3. Applied method

Consider the following linear weakly singular Volterra integral equation:

$$\sum_{i=0}^{n} a_i(t) y^{(i)}(t) + \lambda \int_0^t (t-s)^{-\alpha} y(s) ds = f(t), \tag{3.1}$$

under the initial conditions

$$y^{(n-1)}(0) = y_{n-1}, ..., y(0) = y_0,\ 0 \le t \le 1, 0 < \alpha < 1. \tag{3.2}$$

where $a_i(t)$, $f(t)$ are continuous functions on $[0, 1]$ and $y^{(i)}(t) \in L^2([0, 1])$ stands for the i th-order derivative of $y(t)$. λ and $y_k (k = 0, 1, 2, ..., n-1)$ are real constants.

Before solving equation (3.1), the operational matrix of Block-Pulse functions can be given using the operational matrix of Legendre wavelet.

It is generally known that the Legendre wavelet in the interval $[0, 1)$ can be defined as [15]:

$$\psi_{nm}^{(k)}(x) = \begin{cases} \sqrt{2m+1}\, 2^{\frac{k}{2}} P_m(2^{k+1}x - 2n + 1), & x \in [\frac{n-1}{2^k}, \frac{n}{2^k}); \\ 0, & \text{otherwise.} \end{cases} \tag{3.3}$$

P_m is said Legendre polynomial.

Set P is the Legendre wavelet operational matrix of integration, where

$$P = \frac{1}{2^k} \begin{bmatrix} L & F & F & \cdots & F \\ O & L & F & \cdots & F \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ O & O & O & L & F \\ O & O & O & O & L \end{bmatrix}_{m \times m},$$

$$F = \begin{bmatrix} 2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}_{M \times M}$$

and

$$L=\begin{bmatrix} 1 & \frac{1}{\sqrt{3}} & 0 & 0 & \cdots & 0 & 0 & 0 \\ -\frac{1}{\sqrt{3}} & 0 & \frac{\sqrt{3}}{3\sqrt{5}} & 0 & ... & 0 & 0 & 0 \\ 0 & -\frac{\sqrt{3}}{3\sqrt{5}} & 0 & \frac{\sqrt{5}}{5\sqrt{7}} & ... & 0 & 0 & 0 \\ 0 & 0 & -\frac{\sqrt{7}}{5\sqrt{5}} & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & ... & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & ... & -\frac{\sqrt{2M-3}}{(2M-3)\sqrt{2M-5}} & 0 & \frac{\sqrt{2M-3}}{(2M-3)\sqrt{2M-1}} \\ 0 & 0 & 0 & 0 & ... & 0 & -\frac{\sqrt{2M-1}}{(2M-3)\sqrt{2M-5}} & 0 \end{bmatrix}_{M\times M}$$

where $m = 2^{k-1}M$.

Let $t_i = \frac{i-\frac{1}{2}}{2^{k-1}M}$, $i = 1, 2, ..., 2^{k-1}M$, the Legendre wavelet matrix [14] can be obtained:

$$\Phi_{m\times m} = [\Psi(t_1), \Psi(t_2), ..., \Psi(t_{2^{k-1}M})] \tag{3.4}$$

where

$$\Psi(t) = [\psi^{(k)}_{1,0}(t), ..., \psi^{(k)}_{1,M-1}(t), ..., \psi^{(k)}_{2^{k-1},0}(t), ..., \psi^{(k)}_{2^{k-1},M-1}(t)]^T.$$

There is a relation between the Block-Pulse functions and Legendre wavelet, we have found:

$$\Psi(t) = \Phi_{m\times m} B_m(t). \tag{3.5}$$

From Equation (3.5), we have

$$\begin{aligned} \int_0^t B_m(s)ds &= \int_0^t \Phi^{-1}_{m\times m}\Psi(s)ds = \Phi^{-1}_{m\times m}\int_0^t \Psi(s)ds \\ &= \Phi^{-1}_{m\times m} P\Psi(t) = \Phi^{-1}_{m\times m} P\Phi_{m\times m} B_m(t). \end{aligned} \tag{3.6}$$

Let $Q = \Phi^{-1}_{m\times m} P\Phi_{m\times m}$, Q is called the Block-Pulse operational matrix of integration, namely

$$\int_0^t B_m(s)ds = QB_m(t). \tag{3.7}$$

Since $y^{(n)}(t) \in L^2([0, 1])$ it is supposed that

$$y^{(n)}(t) \cong \sum_{i=0}^{m-1} d_i b_i(t) = d^T B_m(t). \tag{3.8}$$

Then

$$\begin{aligned} y^{(n-1)}(t) &= \int_0^t y^{(n)}(s)ds + y^{(n-1)}(0) = d^T QB_m(t) + y^{(n-1)}(0) \\ &= [d^T Q + A^T y^{(n-1)}(0)]B_m(t) \end{aligned} \tag{3.9}$$

$$y^{(n-2)}(t) = \int_0^t y^{(n-1)}(s)ds + y^{(n-2)}(0) \tag{3.10}$$
$$= [d^T Q^2 + A^T y^{(n-1)}(0)Q + A^T y^{(n-1)}(0)]B_m(t)$$
$$\vdots$$
$$y(t) = \int_0^t y'(s)ds + y(0) = [d^T Q^n + A^T y^{(n-1)}(0)Q^{n-1} \tag{3.11}$$
$$+ \cdots + A^T y'(0)Q + A^T y(0)]B_m(t),$$

and, $\forall i \in \{0, 1, 2, ..., n\}$,

$$y^{(i)}(t) = [d^T Q^{n-i} + A^T y^{(n-1)}(0)Q^{n-i-1} + \cdots + A^T y^{(i)}(0)]B_m(t) \tag{3.12}$$

where $A = m \int_0^1 B_m(t)dt$.

Substituting equation (3.12), equation (3.11) and equation (2.7) into equation (3.1), we have

$$d^T \left(\sum_{i=0}^{n} a_i(t) Q^{n-i} B_m(t) + \lambda Q^n D(t) \right)$$
$$= f(t) - \sum_{i=0}^{n} a_i(t)(A^T y^{(n-1)}(0)Q^{n-i-1} + \cdots + A^T y^{(i)}(0))B_m(t) \tag{3.13}$$
$$- \lambda(A^T y^{(n-1)}(0)Q^{n-1} + \cdots + A^T y(0))D(t)$$

Discreting equation (3.13) by taking step $\Delta = \frac{1}{m}$ of t, a linear system of algebraic equations can be easily obtained. Then d^T can be obtained. $y(t)$ can be obtained by using equation (3.11).

Consider the following nonlinear weakly singular Volterra integral equation:

$$\sum_{i=0}^{n} a_i(t) y^{(i)}(t) + \lambda \int_0^t (t-s)^{-\alpha} [y(s)]^p ds = f(t) \tag{3.14}$$

under the initial conditions equation (3.2).

Let $\beta^T = d^T Q^n + A^T y^{(n-1)}(0)Q^{n-1} + \cdots + A^T y'(0)Q + A^T y(0)$, namely

$$\beta = (\beta_0, \beta_1, \cdots, \beta_{m-1})^T.$$

Equation (3.11) can be translated into:

$$f(t) = \beta^T B_m(t). \tag{3.15}$$

According to the properties of the Block-Pulse functions, we have

$$[f(t)]^p = [\beta^p]^T B_m(t) \tag{3.16}$$

where $\beta^p = (\beta_0^p, \beta_1^p, \cdots, \beta_{m-1}^p)^T$.

Substituting equation (3.12), equation (3.16) and equation (2.7) into equation (3.14), we have:

$$\begin{aligned}(3.17)\qquad &\beta^T \sum_{i=0}^{n} a_i(t) Q^{-i} B_m(t) + \lambda [\beta^p]^T D(t) \\ &= f(t) + \sum_{i=0}^{n} a_i(t)(A^T y^{(i-1)} Q^{-1} + \cdots + A^T y(0) Q^{-i}) B_m(t)\end{aligned}$$

when $p = 1$, equation (3.17) is equation (3.13).

Discreting equation (3.17) by taking step $\Delta = \frac{1}{m}$ of t, a linear system of algebraic equations can be easily obtained. Then β can be obtained. $y(t)$ can be obtained by using equation (3.15).

4. Numerical examples

Example 1. Consider the weakly singular integral [17]:

$$(3.18)\qquad I_1(t) = \int_0^t \frac{s^n}{\sqrt{t-s}} ds$$

The exact solution is

$$\frac{\sqrt{\pi} t^{(\frac{1}{2}+n)} \Gamma(n+1)}{\Gamma(n+\frac{3}{2})}.$$

Taking $m = 16$, $m = 32$, and making use of MATLAB2011a, Fig.1 and Fig.2 are comparison of the approximations with the exact.

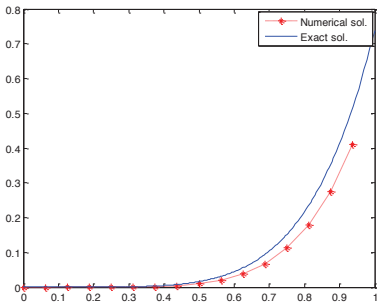

Figure 1: $m = 16$, $n = 5$

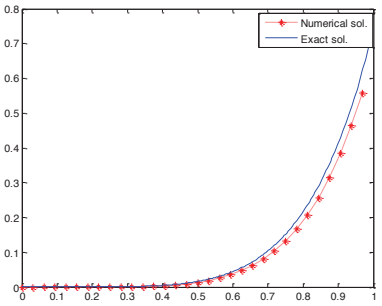

Figure 2: $m = 16$, $n = 5$

Example 2. Consider the weakly singular integral [6]:

$$(3.19)\qquad I_2(t) = \int_0^t \frac{y(s)}{\sqrt{t-s}} ds, \ 0 \le t < 1,$$

where

$$y(s) = \frac{2^{2r-1}}{\pi} r \frac{(\Gamma(r))^2}{\Gamma(2r)} s^{r-1/2}.$$

The exact solution is x^r. Making use of MATLAB2011a, Fig.3 and Fig.4 are comparison of the approximations solution with the exact in this paper.

From the above results, the approximations are in good agreement with exact solution, and the value of is bigger, the precision is higher.

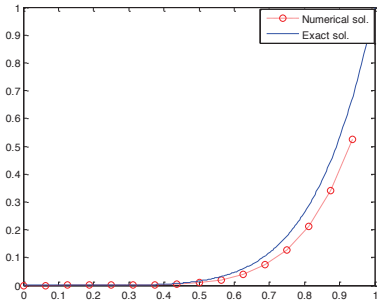

Figure 3: $m = 16$, $r = 6$

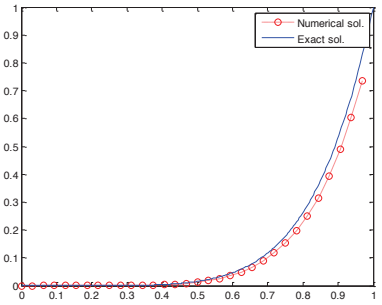

Figure 4: $m = 32$, $r = 6$

Example 3. Consider the weakly singular Volterra integral equation [3]:

$$y(t) + \int_0^t y(s)(t-s)^{-1/2} ds = \frac{1}{2}\pi t + \sqrt{t}, \ 0 \le t < 1. \tag{3.20}$$

The exact solution is $\sqrt{t}$. Table 1 can be obtained through taking $m = 16, 32, 64$, and applying the above method in solving numerical solution of this problem.

Table 1: The exact solution and absolute error for different of m.

t	$m = 16$	$m = 32$	$m = 64$	$m = 128$	Exact solution
0	0.0000	0.0000	0.0000	0.0000	0.0000
1/8	2.23e-002	1.66e-002	7.54e-003	4.24e-004	0.3536
2/8	2.58e-002	1.28e-002	6.89e-003	4.78e-004	0.5000
3/8	2.25e-002	1.09e-002	6.03e-003	6.34e-004	0.6124
4/8	1.99e-002	9.71e-003	9.25e-004	8.69e-005	0.7071
5/8	1.81e-002	8.82e-003	8.37e-004	7.36e-005	0.7906
6/8	1.67e-002	8.22e-003	8.04e-004	6.85e-005	0.8660
7/8	1.56e-002	7.62e-003	7.57e-004	5.33e-005	0.9354

Example 4. Consider the weakly singular Volterra integral equation [3]:

$$y(t) + \int_0^t y(s)(t-s)^{-1/2} ds = 1, \ 0 \le t < 1. \tag{3.21}$$

The exact solution is $y(t) = \exp(\pi t) erfc(\sqrt{\pi t})$. The numerical solution and exact solution, shown in Table 2, can be obtained through taking $m = 16, 32, 64$, and applying the above method in solving numerical solution of this problem by MATLAB2011a.

As can be seen in Table 1 and Table 2, a good approximation with the exact solution using above method is achieved. Besides, with the increase of m, the errors become smaller and smaller.

Table 2: The numerical solution and exact solution for different of m.

t	$m = 16$	$m = 32$	$m = 64$	$m = 128$	Exact solution
0	1.0000	1.0000	1.0000	1.0000	1.0000
1/8	0.5429	0.5389	0.5558	0.5560	0.5561
2/8	0.4411	0.4504	0.4605	0.4607	0.4608
3/8	0.3903	0.3980	0.4052	0.4052	0.4053
4/8	0.3556	0.3615	0.3670	0.3671	0.3671
5/8	0.3292	0.3340	0.3385	0.3385	0.3385
6/8	0.3083	0.3122	0.3160	0.3130	0.3160
7/8	0.2910	0.2943	0.2975	0.2975	0.2975

5. Conclusion

This paper uses the Block-Pulse functions and their good properties to solve the arbitrary order weakly singular integral. The numerical solution of higher order linear and nonlinear weakly singular Volterra integral equation of the second kind can be obtained using this method and the Block-Pulse functions operational matrix of integration. Furthermore, this equation is transformed into a system of algebraic equations which is easily to be solved. Numerical examples show that this new method is an efficient algorithm.

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ON P-DERIVATIONS AND P-JORDAN DERIVATIONS OF A RING

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Abstract. In this paper, we generalize ordinary derivation and Jordan derivation of a ring to the concepts of P-derivation and P-Jordan derivation of a ring respectively. Some properties, e.g. the composition of P-derivations and P-Jordan derivations, are given. Moreover, it proved that if $R/P(R)$ is 2-torsion free for a ring R and its prime radical $P(R)$, P-Jordan derivations of R are P-derivations. More mainly, some conclusions on the commutativity of R/I and $R/P(R)$ are obtained via P-derivations and P-Jordan derivations, for a prime ideal I of R.

Keywords: P-derivation, P-Jordan derivation, torsion free, prime radical, commutativity.

Mathematics Subject Classification (2010): 16N60, 16W25, 16U80, 13N15.

1. Introduction

Let R be an associative ring. Let $P(R)$ denote the *prime radical* of R, i.e., the intersection of all prime ideals of R. For any $a, b \in R$, we shall write $[a, b]$ for $ab - ba$. A ring R is said to be *n-torsion free*, where n is a positive number, if whenever $na = 0$, with $a \in R$, then $a = 0$. For a ring R, if there is a smallest positive number n such that $nR = 0$, we say R to have *characteristic* n, denoted as $charR = n$. Otherwise, we say that R has characteristic zero. A ring R is called a *semiprime ring* if $P(R) = 0$, or equivalently, $aRa = (0)$ implies $a = 0$. An additive mapping $d : R \longrightarrow R$ is called a *derivation* of R, if $d(ab) = d(a)b + ad(b)$ holds for all pairs $a, b \in R$. An additive mapping $d : R \longrightarrow R$ is called a *Jordan derivation* of R, if $d(a^2) = d(a)a + ad(a)$ holds for all $a \in R$. Obviously, every derivation is a Jordan derivation. The converse is, in general, not true. The following example

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gives a Jordan derivation, which is not a derivation. We are thankful for the first person who constructed it, though we haven't found who did this work:

Let k be an algebraic closed field and let $S = k[x]$ with the relation $x^2 = 0$. Let $R = \begin{pmatrix} S & S \\ I & S \end{pmatrix}$ where $I = kx$, the ideal of S. It is easy to see that R, with matrix addition and multiplication, is a ring. For any $r = \begin{pmatrix} a & b \\ c & f \end{pmatrix}$ where $a, b, f \in R$ and $c \in I$, define

$$d : R \longrightarrow R,$$
$$\begin{pmatrix} a & b \\ c & f \end{pmatrix} \longmapsto \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}$$

Clearly, d is additive. Observe that for any $c \in I$, $c^2 = 0$, we have

$$d(r^2) = \begin{pmatrix} 0 & (a+f)c \\ 0 & 0 \end{pmatrix} = d(r)r + rd(r)$$

Thus d is a Jordan derivation. Meanwhile, let $r_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, r_2 = \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix}$. Then $d(r_1 r_2) = 0$ and $d(r_1)r_2 + r_1 d(r_2) = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \neq 0$, hence d is not a derivation.

It has been an increasingly interested problem that wether Jordan derivations degenerate to ordinary derivations on a particular ring or algebra. Herstein [7] showed that it is true on a 2-torsion free prime ring, and another proof is given by M. Bresar in [4], moreover, the same author generalized the results to 2-torsion free semiprime rings in [3]. In recent years, a series of more general concepts of derivations and Jordan derivations are introduced and studied on semiprime rings (see [1], [2], [5]–[7], [10], [11]).

Motivated above, we may assume that for a Jordan derivation d of a particular ring R, where R is not necessarily a prime or semiprime ring, $d(ab) - d(a)b - ad(b)$ may be in an ideal of R, for any $a, b \in R$, obviously, if the ideal is zero ideal, d is an ordinary derivation. So it is interesting to find a relatively small ideal satisfying this condition. In this paper, we will verify that the prime radical satisfies this condition. Let us first introduce the following definitions:

Definition 1.1. Let R be a ring. An additive mapping $d : R \longrightarrow R$ is called a *P-derivation* if $(d(ab) - d(a)b - ad(b)) \in P(R)$ holds for all pairs $a, b \in R$.

Definition 1.2. An additive mapping $d : R \longrightarrow R$ is called a *P-Jordan derivation* if $(d(a^2) - d(a)a - ad(a)) \in P(R)$ holds for all $a \in R$.

With these definitions, we will investigate properties of rings with P-derivations and P-Jordan derivations. This paper is arranged as follow:

In Section 2, we prove that over a ring R satisfying $R/P(R)$ is 2-torsion free, its P-Jordan derivations and P-derivations coincide. Thus whenever $R/P(R)$ is 2-torsion free, the Jordan derivation d of R satisfies $d(ab) - d(a)b - ad(b) \in P(R)$,

$\forall a, b \in R$, which is a generalization of M. Breasar's result in [3]. For the proof of the main result in this section, some characterizations of P-derivations and P-Jordan derivations are also given. In Section 3, some properties of P-derivations and P-Jordan derivations are given. We provide a necessary condition for that the composition of two P-(Jordan) derivations is again a P-(Jordan) derivation. And the results of this section will be used for the proof of the conclusions in the next section. In Section 4, we consider the commutativity of the quotient rings $R/P(R)$ and R/I, where I is a prime ideal of R, as an application of conclusions in Section 2 and Section 3. In [9], Posner proved that in a prime ring R, a non-zero derivation satisfying a particular polynomial identity forces R to be commutative. We will generalize Posner's results by omitting the prime condition in [9], and give some sufficient conditions for the commutativity of R/I via P-derivations. Moreover, the main results of Section 2 enable us to discuss the commutativity of the quotient ring $R/P(R)$ and R/I via P-(Jordan) derivations of R when $R/P(R)$ or R/I is 2-torsion free.

2. Characterization of P-derivation and $P-$Jordan derivation

Lemma 2.1. *Let R be a ring with a P-Jordan derivation d, and $R/P(R)$ is 2-torsion free. Then for all $a, b, c \in R$ the following statements hold:*

(i) $d(ab+ba) - (d(a)b + ad(b) + d(b)a + bd(a)) \in P(R)$,

(ii) $d(aba) - (d(a)ba + ad(b)a + abd(a)) \in P(R)$,

(iii) $d(abc+cba) - (d(a)bc + ad(b)c + abd(c) + d(c)ba + cd(b)a + cbd(a)) \in P(R)$.

Proof. (i) Since $d(a^2) - (ad(a) + d(a)a) \in P(R)$. Then $d((a+b)^2) - (d(a+b)(a+b) + (a+b)d(a+b)) \in P(R)$, we write this as $(d(a^2) - ad(a) - d(a)a) + (d(b^2) - bd(b) - d(b)b) + (d(ab+ba) - d(a)b - ad(b) - d(b)a - bd(a)) \in P(R)$, thus $d(ab+ba) - (d(a)b + ad(b) + d(b)a + bd(a)) \in P(R), \forall a, b \in R$.

(ii) Consider $d(a(ab+ba)+(ab+ba)a) = d((a^2b+ba^2)+2aba) = d(a^2b+ba^2)+2d(aba)$, $d(a(ab+ba) + (ab+ba)a) - (d(a)(ab+ba) + ad(ab+ba) + d(ab+ba)a + (ab+ba)d(a)) \in P(R)$, by (i), we get $d(a)(ab+ba) + ad(ab+ba) + d(ab+ba)a + (ab+ba)d(a) - (d(a)ab+d(a)ba+a(d(a)b+ad(b)+d(b)a+bd(a))+(d(a)b+ad(b)+d(b)a+bd(a))a+(ab+ba)d(a)) \in P(R)$, i.e. $d(a)ab + d(a)ba + a(d(a)b + ad(b) + d(b)a+bd(a))+(d(a)b+ad(b)+d(b)a+bd(a))a+(ab+ba)d(a)-(d(a)ab+2d(a)ba+ad(a)b+a^2d(b)+2abd(a)+2ad(b)a+bad(a)+d(b)a^2+bd(a)a) = 0 \in P(R)$. On the other hand, $d(a^2b+ba^2)-d(a^2)b-a^2d(b)-d(b)a^2-bd(a^2) \in P(R)$, then by (i), we have $d((a^2b+ba^2)+2aba)-(d(a^2)b+a^2d(b)+d(b)a^2+bd(a^2)+2d(aba)) \in P(R)$ thus $d(a^2)b + a^2d(b) + d(b)a^2 + bd(a^2) + 2d(aba) - (d(a)ab + ad(a)b + a^2d(b) + d(b)a^2 + bad(a) + bd(a)a + 2d(aba)) \in P(R)$, then from (3) and (5) we get $2d(aba) - 2(d(a)ba + ad(b)a + abd(a)) = 2(d(aba) - (d(a)ba + ad(b)a + abd(a))) \in P(R)$, thus $d(aba) - (d(a)ba + ad(b)a + abd(a)) \in P(R)$.

(iii) Consider $d((a+c)b(a+c))$. $d((a+c)b(a+c)) = d(aba)+d(cbc)+d(abc+cba)$ $d(aba) - (d(a)ba + ad(b)a + abd(a)) \in P(R)$, $d(cbc) - (d(c)bc + cd(b)c + cbd(c)) \in$

$P(R)$, $d((a+c)b(a+c))-(d(a+c)b(a+c)+(a+c)d(b)(a+c)+(a+c)bd((a+c))) \in P(R)$, then we have $(d(aba)-(d(a)ba+ad(b)a+abd(a)))+((cbc)-(d(c)bc+cd(b)c+cbd(c)))+(d(abc+cba)-(d(a)bc+ad(b)c+abd(c)+d(c)ba+cd(b)a+cbd(a))) \in P(R)$, thus $d(abc+cba)-(d(a)bc+ad(b)c+abd(c)+d(c)ba+cd(b)a+cbd(a)) \in P(R)$. ∎

Remark 2.2. We should notice that if $R/P(R)$ is 2-torsion free, a linear map d of R is a P-Jordan derivation if and only if $d(ab+ba)-(d(a)b+ad(b)+d(b)a+bd(a)) \in P(R)$, $\forall a, b \in R$.

Proof. In fact, Lemma 2.1 has already verified the "only if" part. For the "if" part, we get $2d(a^2)-2d(a)a-2ad(a) \in P(R)$ by replacing b to a in $d(ab+ba)-(d(a)b+ad(b)+d(b)a+bd(a)) \in P(R)$, since $R/P(R)$ is 2-torsion free, it follows $d(a^2)-d(a)a-ad(a) \in P(R)$. ∎

Lemma 2.3. *Let R be a ring with a P-Jordan derivation d, and $R/P(R)$ is 2-torsion free. Then $(d(ab)-d(a)b-ad(b))x[a,b]+[a,b]x(d(ab)-d(a)b-ad(b))) \in P(R)$, $\forall a, b, x \in R$,*

Proof. First consider $d(abxba+baxab)=d(abxba)+d(baxab)$. By Lemma 2.1(ii), we have $d(abxba)-d(a)bxba-ad(bxb)a-abxbd(a) \in P(R)$, $d(bxb)-d(b)xb-bd(x)b-bxd(b) \in P(R)$, thus, $d(abxba)-d(a)bxba-ad(b)xba-abd(x)ba-abxd(b)a-abxbd(a) \in P(R)$, similarly, we have $d(baxab)-d(b)axab-bd(a)xab-bad(x)ab-baxd(a)b-baxad(b) \in P(R)$, and by Lemma 2.1(iii), $d(abxba+baxab)-d(ab)xba-abd(x)ba-abxd(ba)-d(ba)xab-bad(x)ab-baxd(ab) \in P(R)$, then it follows that $(d(ab)-d(a)b-ad(b))x[a,b]+[a,b]x(d(ab)-d(a)b-ad(b)) \in P(R)$. ∎

Lemma 2.4. *Let R be a ring with two elements $a, b \in R$ and $R/P(R)$ is 2-torsion free. If for all $x \in R$ the relation $axb+bxa \in P(R)$ holds. Then $axb, bxa \in P(R)$ is fulfilled for all $x \in R$.*

Proof. Let $x, y \in R$. Then, by the assumption, $axbyaxb+bxayaxb \in P(R)$, $bxayaxb+axaybxb \in P(R)$, $axaybxb+axbyaxb \in P(R)$. Thus, we have $2axbyaxb \in P(R)$, since $R/P(R)$ is 2-torsion free, it follows that $axbyaxb \in P(R)$, thus $axb \in P(R), \forall a, b, x \in R$. Similarly $bxa \in P(R), \forall a, b, x \in R$. ∎

Corollary 2.5. *Let R be a ring and $R/P(R)$ is 2-torsion free. Then for any $a, b, x \in R$, $(d(ab)-d(a)b-ad(b))x[a,b] \in P(R)$, and $[a,b]x(d(ab)-d(a)b-ad(b)) \in P(R)$.*

Proof. It follows from Lemma 2.3 and Lemma 2.4 immediately. ∎

Recall that a ring is 6-torsion free if and only if it is both 2-torsion free and 3-torsion free. The next theorem states that the converse of Lemma 2.1(2) is right when $R/P(R)$ is 6-torsion free ring:

Theorem 2.6. *Let R be a ring with a linear map d, and $R/P(R)$ is 6-torsion free. Then d is a P-Jordan derivation if and only if $d(aba)-(d(a)ba+ad(b)a+abd(a)) \in P(R), \forall a, b \in R$.*

Proof. It suffices to verify the "if" part. By the assumption, $d(x^2yx^2)-d(x^2)yx^2-x^2d(y)x^2-xyd(x^2)\in P(R),\forall x,y\in R$, On the other hand, $d(x^2yx^2)-d(x)xyx^2-xd(xyx)x-x^2yxd(x)\in P(R),\forall x,y\in R$, thus we have $d(x^2yx^2)-d(x)yx^2-xd(x)yx^2-x^2d(y)x^2-x^2yd(x)x-x^2yxd(x),\forall x,y\in R$. Let $A(x)=d(x^2)-d(x)x-xd(x)$, we have therefore $A(x)yx^2+x^2yA(x)=0,\forall x,y\in R$. By Lemma 2.4, we have $A(x)x^2,x^2A(x)\in P(R),\forall x\in R$. Let $B(x,y)=d(xy+yx)-d(x)y-d(y)x-xd(y)-yd(x)$, then from $A(x+y)(x+y)^2\in P(R),\forall x,y\in R$, we have $A(x)y^2+A(y)x^2+B(x,y)x^2+B(x,y)y^2+A(x)(xy+yx)+A(y)(xy+yx)+B(x,y)(xy+yx)\in P(R),\forall x,y\in R$, then substituting $-x$ for x in the above relation we have $A(x)y^2+A(y)x^2-B(x,y)x^2-B(x,y)y^2-A(x)(xy+yx)-A(y)(xy+yx)+B(x,y)(xy+yx)\in P(R),\forall x,y\in R$, thus $B(x,y)x^2+B(x,y)y^2+A(x)(xy+yx)+A(y)(xy+yx)\in P(R),\forall x,y\in R$, also $4B(x,y)x^2+B(x,y)y^2+4A(x)(xy+yx)+A(y)(xy+yx)\in P(R),\forall x,y\in R$, it follows that $3A(x)(xy+yx)+3B(x,y)x^2\in P(R),\forall x,y\in R$, since $R/P(R)$ is 6-torsion free. Thus it is 3-torsion free. Then $A(x)(xy+yx)+B(x,y)x^2\in P(R),\forall x,y\in R$, right multiplication by $A(x)x$, we have $A(x)xyA(x)x+A(x)yxA(x)x\in P(R),\forall x,y\in R$, notice that $xA(x)xyxA(x)x+xA(x)yx^2A(x)x\in P(R),\forall x,y\in R$, by $x^2A(x)\in P(R),\forall x\in R$, we have $xA(x)xyxA(x)x\in P(R),\forall x,y\in R$, therefore $xA(x)x\in P(R),\forall x\in R$, since $A(x)xyA(x)x+A(x)yxA(x)x\in P(R),\forall x,y\in R$, then we have $A(x)xyA(x)x\in P(R),\forall x\in R$, it follows $A(x)x\in P(R),\forall x\in R$, by the relation $A(x)(xy+yx)+B(x,y)x^2\in P(R),\forall x,y\in R$, we have $A(x)yx+B(x,y)x^2\in P(R),\forall x,y\in R$, thus $xA(x)yxA(x)+xB(x,y)x^2A(x)\in P(R),\forall x,y\in R$, so $xA(x)yxA(x)\in P(R)$ which gives $xA(x)\in P(R),\forall x,y\in R$. Notice that $A(x+y)=A(x)+A(y)+B(x,y)$, then it follows from $(x+y)A(x+y)\in P(R),\ \forall x,y\in R$ that $xA(y)+yA(x)+(x+y)B(x,y)\in P(R),\ \forall x,y\in R$, replace x by $-x$ we have $-xA(y)+yA(x)+(x-y)B(x,y)\in P(R),\forall x,y\in R$, therefore $yA(x)+xB(x,y)\in P(R),\forall x,y\in R$, left multiplying with $A(x)$ follows $A(x)yA(x)\in P(R),\forall x,y\in R$, thus $A(x)\in P(R)$, i.e d is a P-Jordan derivation. ∎

Now, we prove our main result of this section:

Theorem 2.7. *Let R be a ring with a P-Jordan derivation d, and $R/P(R)$ is 2-torsion free. Then d is a P-derivation.*

Proof. Our aim is to show that $(d(ab)-d(a)b-ad(b))\in P(R),\forall a,b\in R$. By Corollary 2.5, we have $(d(a(b+c))-d(a)(b+c)-ad(b+c))x[a,b+c]\in P(R)$, $\forall a,b,c\in R$, then $(d(ab)-d(a)b-ad(b))x[a,c]+(d(ac)-d(a)c-ad(c))x[a,b]\in P(R)$. By Corollary 2.5 again, we have $(d(ab)-d(a)b-ad(b))x[a,c]y(d(ac)-d(a)c-ad(c))x[a,b]\in P(R),\ \forall a,b,c,x,y\in R$. Then, we have $((d(ab)-d(a)b-ad(b))x[a,c])y((d(ab)-d(a)b-ad(b))x[a,c])\in P(R)$. By the definition of $P(R)$, $(d(ab)-d(a)b-ad(b))x[a,c]\in P(R),\forall a,b,c,x\in R$. Thus, replacing a by $a+z$, we get $(d((a+z)b)-d(a+z)b-(a+z)d(b))x[a+z,c]\in P(R)$, then $(d(ab)-d(a)b-ad(b))x[z,c]+(d(zb)-d(z)b-zd(b))x[a,c]\in P(R)$, and $(d(ab)-d(a)b-ad(b))x[z,c]y(d(zb)-d(z)b-zd(b))x[a,c]\in P(R)$. It follows that $((d(ab)-d(a)b-ad(b))x[z,c])y((d(ab)-d(a)b-ad(b))x[z,c])\in P(R)$, thus $(d(ab)-d(a)b-ad(b))x[z,c]\in P(R),\ \forall a,b,c,x,z\in R$.

Now, we consider $[(d(ab)-d(a)b-ad(b)),c]x[(d(ab)-d(a)b-ad(b)),c]$, by Corollary 2.5, $(d(ab)-d(a)b-ad(b))(cx)[(d(ab)-d(a)b-ad(b)),c]\in P(R)$, and $c(d(ab)-d(a)b-ad(b))x[(d(ab)-d(a)b-ad(b)),c]\in P(R)$, thus $[(d(ab)-d(a)b-ad(b)),c]x[(d(ab)-d(a)b-ad(b)),c]\in P(R)$, hence $[(d(ab)-d(a)b-ad(b)),c]\in P(R)$ $\forall a,b,c\in R$. On the other hand, $((d(ab)-d(a)b-ad(b))[z,c])x((d(ab)-d(a)b-ad(b))[z,c])\in P(R)$, hence $(d(ab)-d(a)b-ad(b))[z,c]\in P(R)$, $\forall a,b,c,z\in R$. Since $d(ab+ba)-d(a)b-ad(b)-d(b)a-bd(a)\in P(R)$, i.e. $(d(ab)-d(a)b-ad(b))+(d(ba)-d(b)a-ad(b))\in P(R)$, and $(d(ab)-d(a)b-ad(b))-(d(ba)-d(b)a-ad(b))=d[a,b]+[b,d(a)]+[d(b),a]$, multiply with $(d(ab)-d(a)b-ad(b))$, we have $(d(ab)-d(a)b-ad(b))[b,d(a)]\in P(R)$, $(d(ab)-d(a)b-ad(b))[d(b),a]\in P(R)$, thus $2(d(ab)-d(a)b-ad(b))^2-(d(ab)-d(a)b-ad(b))d[a,b]\in P(R)$. Now, consider $d((d(ab)-d(a)b-ad(b))[a,b](d(ab)-d(a)b-ad(b)))$ since $(d(ab)-d(a)b-ad(b))[a,b]\in P(R)$, and $[a,b](d(ab)-d(a)b-ad(b))\in P(R)$, then $d(d(ab)-d(a)b-ad(b))[a,b](d(ab)-d(a)b-ad(b))\in P(R)$, $(d(ab)-d(a)b-ad(b))[a,b]d(d(ab)-d(a)b-ad(b))\in P(R)$, and since $2(d(ab)-d(a)b-ad(b))^3-(d(ab)-d(a)b-ad(b))^2d[a,b]\in P(R)$, thus $2(d(ab)-d(a)b-ad(b))^3\in P(R)$, hence $(d(ab)-d(a)b-ad(b))\in P(R)$. ∎

Corollary 2.8. *Let R be a ring with a Jordan derivation d, and $R/P(R)$ is 2-torsion free. Then d is a P-derivation.*

Corollary 2.9. *Let R be a ring with a P-Jordan derivation d and a prime ideal I. If R/I is 2-torsion free. Then $d(ab)-d(a)b-ad(b)\in I$.*

Proof. By the proof of Theorem 2.7 we may find that, there exits an integer n such that $2^n(d(ab)-d(a)b-ad(b))\in P(R)\subseteq I$. Since R/I is 2-torsion free, then $d(ab)-d(a)b-ad(b)\in I$. ∎

3. The composition of two P-(Jordan)derivations

In this section, we continue with the properties of P-derivations and P-Jordan derivations. We give a necessary condition for that the composition of two P-(Jordan) derivations is again a P-(Jordan) derivation. These results will be used in the next section, while they are also independently interesting.

Lemma 3.1. *Let R be a ring with a prime ideal I and a P-derivation of d. Let a be an element of R such that $ad(x)\in I$ for all $x\in R$. Then either $a\in I$ or* $\mathrm{Im}d\subseteq I$.

Proof. Since d is a P-derivation, we have $d(xy)-d(x)y-xd(y)\in P(R)\subseteq I$. From the assumptions, $ad(xy)\in I$, for any $x,y\in R$, i.e. $ad(x)y+axd(y)\in I$, since $ad(x)y\in I$. Then $axd(y)\in I$ for all $x,y\in R$, by the definition of prime ideal, we finish the proof. ∎

Corollary 3.2. *Let R be a ring with a prime ideal I and a P-Jordan derivation of d. $R/P(R)$ is 2-torsion free. Let a be an element of R such that $ad(x)\in I$ for all $x\in R$. Then either $a\in I$ or* $\mathrm{Im}d\subseteq I$.

Proof. It follows immediately from Theorem 2.7 and Lemma 3.1. ∎

Lemma 3.3. *Let R be a ring with a prime ideal I. Let p, q, r be elements of R such that $paqar \in I$ for all $a \in R$. Then at least one of p, q, r is in I.*

Proof. If $p \in I$ or $r \in I$, then we complete this proof. Now let $p, r \notin I$. Our aim is to prove $q \in I$. Since $paqar \in I$, we have $p(a+b)q(a+b)r \in I, \forall a, b \in R$, we write this as $paqbr + pbqar \in I, \forall a, b \in R$. On the other hand, $paqart \in I, \forall t \in I$, it follows $p(aqart)qbr + pbq(aqart)r \in I, \forall b \in R$, since $p(aqart)qbr \in I, \forall a, t \in R$, we have $pbq(aqart)r \in I, \forall a, b, t \in R$, by the arbitration of b and t, we have $qaqar \in I, \forall a \in R$. Similarly, $paqaq \in I, \forall a \in R$. From $paqar \in I, \forall a \in R$, $p(a+b)q(a+b)r \in I$, and $paqbr + pbqar \in I$, then $paq(tqbqb)r + p(tqbqb)qar \in I$. Then $p(tqbqb)qar \in I$, thus we have $qbqbq \in I$ for any $b \in R$. Also $q(a+aqb)q(a+aqb)q \in I, \forall a, b \in R$, i.e. $qaqbqaq \in I, \forall a, b \in R$, thus $qaq \in I, \forall a \in R$, and it follows $q \in I$. ∎

The next theorem will provide a necessary condition for the composition of two P-derivations is again a P-derivation.

Theorem 3.4. *Let R be a ring. d_1, d_2 are P-derivations of R such that $d_1d_2(ab) - d_1d_2(a)b - ad_1d_2(b) \in I$. Then for any prime ideal I, either* $\mathrm{Im}2d_1 \subseteq I$ *or* $\mathrm{Im}2d_2 \subseteq I$.

Proof. Note that for any prime ideal I, $P(R) \subseteq I$, then $d_1d_2(a)b + ad_1d_2(b) - d_1d_2(ab) \in I, \forall a, b \in R$. $d_1d_2(ab) - d_1d_2(a)b - ad_1d_2(b) - d_1(a)d_2(b) - d_2(a)d_1(b), \forall a, b \in R$, then $d_1(a)d_2(b) + d_2(a)d_1(b) \in I, \forall a, b \in R$, then $d_1(ad_1(c))d_2(b) + d_2(ad_1(c))d_1(b) \in I, \forall a, b \in R$, thus $d_2(a)d_1(c)d_1(b) + d_1(a)d_1(c)d_2(b) \in I, \forall a, b, c \in R$. Notice that $d_1(c)d_2(b) + d_2(c)d_1(b) \in I, \forall b, c \in R$. Then $d_2(a)d_1(c)d_1(b) - d_1(a)d_2(c)d_1(b) \in I, \forall a, b, c \in R$. By Lemma 3.1, $d_2(a)d_1(c) - d_1(a)d_2(c) \in I, \forall a, c \in R$. Along with $d_1(a)d_2(c) + d_2(a)d_1(c) \in I, \forall a, c \in R$, it follows $2d_2(a)d_1(c) \in I$, if $\mathrm{Im}d_1 \nsubseteq I$. By Lemma 3.1 again, we have $2d_2(a) \in I, \forall a \in R$. Similarly, if $\mathrm{Im}d_2 \nsubseteq I$. Then $2d_1(c) \in I, \forall c \in R$. Therefore, we finish the proof. ∎

Corollary 3.5. *Let R be a ring, and I an prime ideal, and R/I is 2-torsion free. d_1, d_2 are P-derivations of R such that d_1d_2 is also a P-derivation. Then either* $\mathrm{Im}d_1 \subseteq I$ *or* $\mathrm{Im}d_2 \subseteq I$.

Proof. It can be deduced immediately from Theorem 2.7 and Theorem 3.4. ∎

Corollary 3.6. *Let R be a ring, and $R/P(R)$ is 2-torsion free and $n(\geq 2)$ is an integer. d is a P-(Jordan) derivation of R such that d^n is also a P-(Jordan) derivation. Then* $\mathrm{Im}d \subseteq P(R)$.

4. On commutativity of quotient ring via P-(Jordan) derivations

In this section, we discuss the commutativity of $R/P(R)$ as well as R/I, where I is a prime ideal of R. Posner showed in [9] that some properties of a derivation determine the commutativity of a prime ring. In other word, the author provided some sufficient conditions for the commutativity of a prime ring via its derivation.

We will generalize these results by providing sufficient conditions for R/I to be commutative via the P-derivations of R. Moreover, if $R/P(R)$ is 2-torsion free, we consider the commutativity of $R/P(R)$ via P-(Jordan) derivations. These can be regarded as an application of Section 2 and Section 3.

Theorem 4.1. *Let R be a ring with a prime ideal I. If there exists a P-derivation d of R such that:*

(i) $[a, d(a)] \in I$ *for all* $a \in R$.

(ii) $\mathrm{Im}d \nsubseteq I$.

Then R/I is commutative.

Proof. We consider the characteristic of R/I. In the case that $\mathrm{char}R/I \neq 2$, we have $[a+ax, d(a+ax)] \in I, \forall a, x \in R$, we write this as $ad(ax)-d(a)ax+axd(a)-d(ax)a \in I, \forall a, x \in R$, thus we have $a^2d(x) - d(a)xa - ad(x)a + axd(a) \in I$, $\forall a, x \in R$. Similarly, from $[a + xa, d(a + xa)] \in I$, we have $d(x)a^2 - ad(x)a - axd(a) + d(a)xa \in I, \forall a, x \in R$, so $a^2d(x) + d(x)a^2 - 2ad(x)a \in I, \forall a, x \in R$, it follows $[[d(x), a], a] \in I, \forall a, x \in R$, then $[[d(x), a + d(x)], a + d(x)] \in I, \forall a, x \in R$, therefore $[[d(x), a], d(x)] \in I$. Notice that the function $f_{d(x)} : a \mapsto [d(x), a]$ is a derivation and $f_{d(x)}(f_{d(x)}(a)) \in I, \forall a, x \in R$, thus by Corollary 3.5, $[d(x), a] \in I$, i.e., $f_a(d(x)) \in I, \forall a, x \in R$. Since $\mathrm{Im}d \nsubseteq I$, by Corollary 3.5 again, we have $f_a(R) \subseteq I, \forall a \in R$ (i.e., R/I is commutative).

Now, we consider the case that $\mathrm{char}R/I = 2$. Then from the previous discussion, we have $[d(x), a^2] \in I, \forall a, x \in R$, thus $[d(x), a + b] \in I, \forall a, b, x \in R$, and therefore $[d(x), ab + ba] \in I, \forall a, b, x \in R$, then we get $[d(x), aad(x) + ad(x)a] \in I$, $\forall a, b, x \in R$, it follows that $ad(x)ad(x) - d(x)ad(x)a \in I, \forall a, b, x \in R$. Since $\mathrm{char}R/I = 2$, we have $ad(x)ad(x) + d(x)ad(x)a \in I, \forall a, b, x \in R$. On the other hand we have $[d(x), ad(x) + d(x)a] \in I, \forall a, x \in R$, then $[d(x)^2, a] \in I, \forall a, x \in R$. Above all, we have $(ad(x) + d(x)a)^2 = ad(x)ad(x) + d(x)ad(x)a + ad(x)^2a + d(x)a^2d(x) \in I, \forall a, x \in R$. Let $z = ad(x) + d(x)a$. Then we have $[d(x), z(bz)^2 + bzbz^2] \in I, \forall b, x \in R$. Since $z^2 \in I$, we have $[d(x), z(bz)^2] \in I, \forall b, x \in R$, it follows that $(zd(x) + d(x)z)bzbz \in I, \forall b, x \in R$. By Lemma 3.3, we have $z \in I$ or $zd(x) + d(x)z \in I$, any way, $zd(x) + d(x)z \in I$ holds for all $a, x \in R$. Similarly, $[d(x), zw(bz)^2] \in I, \forall a, x, w \in R$, i.e., $(zwd(x) + d(x)zw)bzbz \in I$, $\forall b, x, w \in R$. Notice that $[z, d(x)] \in I$, we have $z[d(x), w] \in I, \forall b, x, w \in R$, namely, by Lemma 3.1, we have $z \in I$ or $[d(x), w] \in I, \forall x, w \in R$, any way $[d(x), y] \in I$, $\forall x, y \in R$. Observe that $d(b^2) = d(b)b + bd(b) \in I$. Then $d(ab^2) - d(a)b^2 \in I$ and $[d(a)b^2, x] \in I, \forall a, x \in I$. Since $\mathrm{Im}d \nsubseteq I$, i.e $\exists a \in R$ such that $d(a) \notin I$. Then from $[d(a)b^2, x] \in I$ and $[d(a), b^2] \in I$, we have $d(a)[b^2, x] \in I, \forall a, b, x \in R$. By Lemma 3.1 again, since $d(a) \notin I$, it follows that $[b^2, x] \in I, \forall a, b, x \in R$. Thus, by the discussion above, we have $[b, x] \in I, \forall b, x \in R$. Hence R/I is commutative. ∎

Corollary 4.2. *Let R be a ring with a prime ideal I. If there exists a derivation d of R such that*

(i) $[a, d(a)] \in I$ *for all* $a \in R$.

(ii) Im$d \not\subseteq I$.

Then R/I is commutative.

According to Polynomial Identity(PI) theory, the condition $[a, d(a)] \in I$ in Theorem 4.1 is actually a polynomial identity of the P-derivation d modulo the prime ideal I. It is interesting to check whether the commutativity still holds if we weaken the polynomial identity of Theorem 4.1. The next theorem will show that, under the condition $R/P(R)$ is 2-torsion free, the conclusion could be consolidated.

Theorem 4.3. *Let R be a ring and I a prime ideal, and R/I is 2-torsion free. If there exists a P-derivation d of R such that:*

(i) $[[d(a), a], x] \in P(R), \forall a, x \in R$,

(ii) *for any prime ideal I of R,* Im$d \not\subseteq I$.

Then $R/P(R)$ is commutative.

Proof. Let us assume that for any prime ideal I, Im$d \not\subseteq I$. We prove that for any prime ideal I, $[a, x] \in I, \forall a, x \in R$, thus $R/P(R)$ is commutative. By assumption we have $[[d(x), x], y] \in P(R), \forall x, y \in R$. Replace x by $a + [a, x]$ it follows $[[d(a + [a, x]), a + [a, x]], y] \in P(R), \forall a, x, y \in R$. Since $[[d(a), a], y] \in P(R), [[d([a, x]), [a, x]], y] \in P(R)$ we then get the following $[[d(a), [a, x]], y] + [[[d(a), x], a], y] + [[[d(x), a], a], y] \in P(R), \forall a, x, y \in R$ write this as $[[d(a), [a, x]], y] + [[a, [x, d(a)]], y] + [[[d(x), a], a], y] \in P(R), \forall a, x, y \in R$. Observe that $[d(a), [a, x]] + [a, [x, d(a)]] + [x, [d(a), a]] \in P(R), \forall a, x \in R$, by the assumption, $[x, [d(a), a]] \in P(R), \forall a, x \in R$, thus $[[[d(x), a], a], y] \in P(R), \forall a, x, y \in R$, i.e. $[a^2d(x) + d(x)a^2 - 2ad(x)a, y] \in P(R), \forall a, x, y \in R$, replacing y by a, we have $3ad(x)a^2 + a^3d(x) - (3a^2d(x)a + d(x)a^3) \in P(R) \subseteq I, \forall a, x \in R$. Now consider the case that charR/I=3. Then we have $[a^3, d(x)] \in I, \forall a, x \in R$. Since for any prime ideal I, Im$d \not\subseteq I$. Then by Corollary 3.5, $[a^3, x] \in I, \forall a, x \in R$. $I \ni [(a+b)^3 - a^3 - b^3, x] = [a^2b + aba + ba^2 + b^2a + bab + ab^2, x], \forall a, b, x \in R$, also we have $[a^2b + aba + ba^2 - b^2a - bab - ab^2, x] \in I$, $\forall a, b, x \in R$, and then it follows $[a^2b + aba + ba^2, x] \in I$, $\forall a, b, x \in R$, thus $[a^2(ab) + a(ab)a + (ab)a^2, x] = [a(a^2b + aba + ba^2), x] \in I$, $\forall a, b, x \in R$. Observe that $[a(a^2b + aba + ba^2), x] - (a^2b + aba + ba^2)[a, x] \in I$, $\forall a, b, x \in R$, then $[a, x](a^2b + aba + ba^2) \in I$, $\forall a, b, x \in R$. Since $[a^2b + aba + ba^2, x] \in I$, $\forall a, b, x \in R$, it follows $[a, x]R(a^2b + aba + ba^2) \subseteq I$, $\forall a, b, x \in R$. If $[a, x] \in I$, $\forall a, x \in R$. Then we finish the proof. Let us assume that there exists $a', x' \in R$ such that $[a', x'] \notin I$. Thus $a'^2b + a'ba' + ba'^2 \in I$, $\forall b \in R$, by the definition of prime ideal. Note that $[a', [a', b]] = a'^2b - 2a'ba' + ba'^2$ and since char$R/I = 3$, then $[a', [a', b]] \in I, \forall b \in R$. By Corollary 3.5 we have $[a', y] \in I$, $\forall y \in R$. It is a contradiction when $y = x'$.

Next, consider the case that char$R/I \neq 3$. Since we have already got $3ad(x)a^2 + a^3d(x) - (3a^2d(x)a + d(x)a^3) \in P(R) \subseteq I$, $\forall a, x \in R$. Thus $3ad(a)a^2 + a^3d(a) - (3a^2d(a)a + d(a)a^3) \in I, \forall a \in R$. It follows that $[a^3, d(a)] - 3a[a, d(a)]a \in I$, $\forall a \in R$. Since $[[d(a), a], x] \in I$, $\forall a, x \in R$, we have $[a^3, d(a)] - 3[a, d(a)]a^2 \in I$,

$\forall a \in R$. Since $[a, d(a)]a = ad(a)a - d(a)a^2, a[a, d(a)] = a^2d(a) - ad(a)a$, observe that $[[a, d(a)], a] \in I$, thus $2[a, d(a)]a - [a^2, d(a)] \in I$, $\forall a \in R$.

On the other hand, replacing x by $ad(x)$ in $3ad(x)a^2 + a^3d(x) - (3a^2d(x)a + d(x)a^3) \in I, \forall a, x \in R$, we have $3a^2d(d(x))a^2+a^4d(d(x))-3a^3d(d(x))a-ad(d(x))a^3 - a(3ad(d(x)))a^2 + a^3d(d(x)) - 3a^2d(d(x)) - d(d(x))a^3 \in I$, $\forall a, x \in R$, observe that $a(3ad(d(x)))a^2 + a^3d(d(x)) - 3a^2d(d(x)) - d(d(x))a^3 \in I$, $\forall a, x \in R$, thus $3a^2d(d(x))a^2 + a^4d(d(x)) - 3a^3d(d(x))a - ad(d(x))a^3 \in I, \forall a, x \in R$. Similarly, we have $3ad(a) + a^3d(a)d(x) - 3a^2d(a)d(x)a - d(a)d(x)a^3 \in I, \forall a, x \in R$, comparing with $3d(a)ad(x)a^2 + d(a)a^3d(x) - 3d(a)a^2d(x)a - d(a)d(x)a^3 \in I$, we have $3[a, d(a)]d(x)a^2 + [a^3, d(a)]d(x) - 3[a^2, d(a)]d(x)a \in I, \forall a, x \in R$, since $[a^3, d(a)] - 3[a, d(a)]a^2, 2[a, d(a)]a-[a^2, d(a)]\in I$, then $3[a, d(a)](d(x)a^2+a^2d(x)-2ad(x)a) \in I$, $\forall a, x \in R$. Because $\mathrm{char}R/I \neq 3$, it follows $[a, d(a)](d(x)a^2+a^2d(x)-2ad(x)a) \in I$, $\forall a, x \in R$. Since $[[a, d(a)], x] \in I$, $\forall a, x \in R$, so $[a, d(a)]R(d(x)a^2 + a^2d(x) - 2ad(x)a) \subseteq I$, $\forall a, x \in R$. Therefore, $[a, d(a)]\in I$ or $(d(x)a^2+a^2d(x)-2ad(x)a)\in I$. Assume $\exists a \in R$ such that $[a, d(a)] \notin I$, then $(d(x)a^2+a^2d(x)-2ad(x)a) \in I, \forall x \in R$. Replacing x by ax, we have $ad(x)a^2 + a^3d(x) - 2a^2d(x)a + d(a)xa^2 + a^2d(a)x - 2ad(a)xa \in I, \forall x \in R$. Since $d(x)a^2+a^2d(x)-2ad(x)a \in I$, thus $ad(x)a^2+a^3d(x)-2a^2d(x)a = a(d(x)a^2 + a^2d(x) - 2ad(x)a) \in I$, $\forall x \in R$. Therefore, $d(a)xa^2 + a^2d(a)x - 2ad(a)xa \in I, \forall x \in R$. We also have $d(a)a^2+a^2d(a)-2ad(a)a\in I$. Then $(d(a)a^2 + a^2d(a) - 2ad(a)a)x \in I$, $\forall x \in R$. Thus $(d(a)xa^2 + a^2d(a)x - 2ad(a)xa) - ((d(a)a^2 + a^2d(a) - 2ad(a)a)x) = d(a)a[x, a^2] - 2ad(a)a[a, x] \in I, \forall x \in R$. Replacing x by ax, we have $d(a)a[x, a^2] - 2ad(a)a[x, a] \in I$, $\forall x \in R$. On the other hand, $ad(x)[x, a^2] - 2a^2d(a)[a, x] \in I$, $\forall x \in R$. Thus we have $[a, d(a)][x, a^2] - 2a[a, d(a)][x, a] \in I$, $\forall x \in R$, then it follows $[a, d(a)]R([x, a^2] - 2a[x, a]) \in I$, $\forall x \in R$, therefore, $[x, a^2] - 2a[x, a] = xa^2 + a^2x - 2axa = [[x, a], a] \in I$, $\forall x \in R$. By Corollary 3.5, $[x, a] \in I$, $\forall x \in R$. ∎

From Theorem 4.1 and Theorem 4.3, we can see that, if we restrict R to the condition that $R/P(R)$ is 2-torsion free, the condition: $[a, d(a)] \in I, \forall a \in R$ could be weaken correspondingly. It's an interesting question that whether or not the conclusion of Theorem 4.3 is true with the condition: $R/P(R)$ is 2-torsion free omitted? Unfortunately, we haven't found the way approaching to the answer. On the other hand, we can state this question in a more universal form: could we continue to weaken the condition: $[a, d(a)] \in I, \forall a \in R$ in Theorem 4.1? This may be an interesting question for researchers of PI theory.

An immediate corollary of Theorem 4.3 could be used for the commutativity of 2-torsion free semiprime ring:

Corollary 4.4. *If a derivation d of a 2-torsion free semiprime ring R satisfies:*

(i) $[a, d(a)]$ *is in the central of R,*

(ii) *any prime ideal I,* $\mathrm{Im}d \nsubseteq I$.

Then R is commutative.

Also, a sufficient condition for the commutativity of R/I, where I is a prime ideal of R, can also be got from Theorem 4.3.

Corollary 4.5. *Let R be a ring and $R/P(R)$ is 2-torsion free. For any prime ideal I of R, if there exists a P-(Jordan) derivation d of R such that*

(i) $[[d(a), a], x] \in I, \forall a, x \in R$,

(ii) $\mathrm{Im}d \nsubseteq I$.

Then R/I is commutative.

We know that Jordan derivations are P-Jordan derivations, Corollary 4.5 also enables us to discuss the commutativity of R/I via the Jordan derivations of R.

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ON WEAKLY πg-CLOSED SETS IN TOPOLOGICAL SPACES

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Abstract. In this paper, the concepts of weakly πg-continuous functions, weakly πg-compact spaces and weakly πg-connected spaces are introduced and some of their properties are investigated.

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1. Introduction

Recently many authors have studied various classes of generalized closed sets in general topology. Dontchev and Noiri [9] have introduced the concept of πg-closed sets and studied their most fundamental properties in topological spaces. Also, recently, Ekici and Noiri [10] have introduced a generalization of πg-closed sets and πg-open sets.

In this paper, we study a new class of generalized closed sets in topological spaces. We introduce the notions of weakly πg-closed sets and weakly πg-open sets, which are weaker forms of πg-closed sets and πg-open sets, respectively. Also, the relationships among related generalized closed sets are investigated.

2. Preliminaries

Throughout this paper (X, τ), (Y, σ) and (Z, η) (or X, Y and Z) represent topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X, τ), cl(A), int(A) and A^c denote the closure of A, the interior of A and the complement of A respectively.

We recall the following definitions which are useful in the sequel.

Definition 2.1 A subset A of a space (X, τ) is called

1. semi-open set [18] if A $\subseteq$ cl(int(A));
2. preopen set [21] if A $\subseteq$ int(cl(A));
3. α-open set [23] if A $\subseteq$ int(cl(int(A)));
4. β-open set [1] (= semi-preopen [2]) if A $\subseteq$ cl(int(cl(A)));
5. b-open set [3] if A $\subseteq$ cl(int(A)) $\cup$ int(cl(A));
6. regular open set [32] if A = int(cl(A));
7. π-open set [38] if A is the finite union of regular open sets.

The complements of the above mentioned open sets are called their respective closed sets.

The semi-closure [7] (resp. α-closure [23], preclosure [25]) of a subset A of X, scl(A) (resp. αcl(A), pcl(A)), is defined to be the intersection of all semi-closed (resp. α-closed, preclosed) sets of (X, τ) containing A. It is known that scl(A) (resp. αcl(A), pcl(A)) is a semi-closed (resp. an α-closed, preclosed) set.

Definition 2.2 A subset A of a space (X, τ) is called

1. a generalized closed (briefly g-closed) set [17] if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is open in (X, τ).

   The complement of g-closed set is called g-open set;

2. a semi-generalized closed (briefly sg-closed) set [5] if scl(A) $\subseteq$ U whenever A $\subseteq$ U and U is semi-open in (X, τ).

   The complement of sg-closed set is called sg-open set;

3. an α-generalized closed (briefly αg-closed) set [19] if αcl(A) $\subseteq$ U whenever A $\subseteq$ U and U is open in (X, τ).

   The complement of αg-closed set is called αg-open set;

4. a generalized α-closed (briefly $g\alpha$-closed) set [20] if αcl(A) $\subseteq$ U whenever A $\subseteq$ U and U is α-open in (X, τ).

   The complement of $g\alpha$-closed set is called $g\alpha$-open set;

5. a $\hat{g}$-closed set [35] (= ω-closed [31]) if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is semi-open in (X, τ).

   The complement of $\hat{g}$-closed set is called $\hat{g}$-open set;

6. a $^{\star}g$-closed set [36] if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is $\hat{g}$-open in (X, τ).

   The complement of $^{\star}g$-closed set is called $^{\star}g$-open set;

7. a $^{\#}g$-semi-closed (briefly $^{\#}g$s-closed) set [37] if scl(A) $\subseteq$ U whenever A $\subseteq$ U and U is $^{\star}g$-open in (X, τ).

   The complement of $^{\#}g$s-closed set is called $^{\#}g$s-open set;

8. a $\tilde{g}$-closed set [16] if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is $^{\#}g$s-open in (X, τ).

   The complement of $\tilde{g}$-closed set is called $\tilde{g}$-open set;

9. a $\tilde{g}$-semi-closed (briefly $\tilde{g}s$-closed) set [34] if scl(A) $\subseteq$ U whenever A $\subseteq$ U and U is $^{\#}g$s-open in (X, τ).

   The complement of $\tilde{g}s$-closed set is called $\tilde{g}s$-open set;

10. a πg-closed set [9] if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is π-open in (X, τ).

    The complement of πg-closed set is called πg-open set.

Definition 2.3 A subset A of a topological space (X, τ) is called

1. a weakly g-closed (briefly wg-closed) set [33] if cl(int(A)) $\subseteq$ U whenever A $\subseteq$ U and U is open in (X, τ).

2. a weakly $\tilde{g}$-closed (briefly w$\tilde{g}$-closed) set [29] if cl(int(A)) $\subseteq$ U whenever A $\subseteq$ U and U is $^{\#}g$s-open in (X, τ).

3. a weakly ω-closed set [30] if cl(int(A)) $\subseteq$ U whenever A $\subseteq$ U and U is semi-open in (X, τ).

4. a regular weakly generalized closed (briefly rwg-closed) set [22] if cl(int(A)) $\subseteq$ U whenever A $\subseteq$ U and U is regular open in (X, τ).

Definition 2.4 Let X and Y be topological spaces. A mapping f : X $\rightarrow$ Y is called

1. completely continuous [4] (resp. R-map[6]) if f^{-1}(V) is regular open in X for each open (resp. regular open) set in Y.

2. perfectly continuous [24] if f^{-1}(V) is both open and closed in X for each open set V in Y.

3. $\hat{g}$-irresolute [35] if f^{-1}(V) is $\hat{g}$-open in X for each $\hat{g}$-open subset V in Y.

4. πg-continuous [9] if f^{-1}(V) is πg-closed in X for every closed subset V of Y.

5. π-irresolute [12] if f^{-1}(V) is π-closed in X for every π-closed subset V of Y.
6. irresolute [8] if f^{-1}(V) is semi-open in X for every semi-open subset V in Y.

Definition 2.5 [15] A space (X, τ) is called πg-$T_{1/2}$ if every πg-closed set is closed.

Definition 2.6 [12] A mapping f :X $\to$ Y is called contra πg-continuous if f^{-1}(V) is πg-closed in X for every open set V of Y.

Definition 2.7 [12] A topological space (X, τ) is said to be locally πg-indiscrete if every πg-open set of X is closed in X.

Definition 2.8 [26] A subset A of (X, τ) is said to be πgp-closed if pcl(A) $\subseteq$ U whenever A $\subseteq$ U and U is π-open in X.

Definition 2.9 [26] A space (X, τ) is called πgp-$T_{1/2}$ if every πgp-closed set is preclosed.

Definition 2.10 [14] A mapping f : $(X, \tau) \to (Y, \sigma)$ is said to be almost πgp-continuous if f^{-1}(int(cl(V))) is a πgp-open set in X for every V $\in \sigma$.

Definition 2.11 [13] A mapping f : X $\to$ Y is called (πg, s)-continuous if the inverse image of each regular open set of Y is πg-closed in X.

Definition 2.12 [27] A mapping f : $(X, \tau) \to (Y, \sigma)$ is said to be w$\tilde{g}$-continuous if the inverse image of every open set in (Y, σ) is w$\tilde{g}$-open in X.

Definition 2.13 [28] A mapping f : $(X, \tau) \to (Y, \sigma)$ is said to be $\tilde{g}$-continuous if the inverse image of every open set in (Y, σ) is $\tilde{g}$-open in X.

Definition 2.14 [15] A topological space X is called πg-compact if every cover of X by πg-open sets has finite subcover.

Definition 2.15 [11] A space X is said to be almost connected if X cannot be written as a disjoint union of two non-empty regular open sets.

Definition 2.16 [12, 13] A space X is called πg-connected if X is not the union of two disjoint nonempty πg-open sets.

Definition 2.17 [12] A mapping f : X $\to$ Y is called πg-open if image of each πg-open set is πg-open.

Definition 2.18 [15] A mapping f : X $\to$ Y is said to be πg-irresolute if f^{-1}(V) is πg-closed in (X, τ) for every πg-closed set V of (Y, σ).

Lemma 2.19 [26] *Let Y be open in X. Then*

(1) *If A is π-open in Y, then there exists a π-open set B in X such that A=B$\cap$Y.*

(2) *If A is π-open in X, then A $\cap$ Y is π-open in Y.*

3. Weakly πg-closed sets

We introduce the definition of weakly πg-closed sets in a topological space and study the relationships of such sets.

Definition 3.1 A subset A of a topological space (X, τ) is called a weakly πg-closed (briefly wπg-closed) set if cl(int(A)) $\subseteq$ U whenever A $\subseteq$ U and U is π-open in (X, τ).

Proposition 3.2 *Every πg-closed set is wπg-closed. But the converse of this implication is not true in general.*

Example 3.3 Let X = {a, b, c, d} and τ = {ϕ, {a}, {b, c}, {a, b, c}, X}. Then the set {c} is wπg-closed but not πg-closed in (X, τ).

Theorem 3.4 *Every wπg-closed set is rwg-closed but not conversely.*

Proof. Let A be any wπg-closed set and let U be regular open set containing A. Then U is a π-open set containing A. We have cl(int(A)) $\subseteq$ U. Thus A is rwg-closed.

Example 3.5 Let X = {a, b, c, d} and τ = {ϕ, {a}, {b}, {a, b}, {a, b, c}, X}. Then the set {a, b} is rwg-closed but not a wπg-closed.

Theorem 3.6 *Every wg-closed set is wπg-closed but not conversely.*

Proof. Let A be any wg-closed set and let U be π-open set containing A. Then U is an open set containing A. We have cl(int(A)) $\subseteq$ U. Thus A is wπg-closed.

Example 3.7 Let X = {a, b, c, d} and τ = {ϕ, {a}, {b}, {a, b}, {a, b, c}, X}. Then the set {a, c} is wπg-closed but not a wg-closed.

Theorem 3.8 *If a subset A of a topological space (X, τ) is both closed and αg-closed, then it is wπg-closed in (X, τ).*

Proof. Let A be an αg-closed set in (X, τ) and U be an π-open set containing A. Then U is open containing A and so U $\supseteq$ αcl(A) = A $\cup$ cl(int(cl(A))). Since A is closed, U $\supseteq$ cl(int(A)) and hence A is wπg-closed in (X, τ).

Theorem 3.9 *If a subset A of a topological space (X, τ) is both π-open and wπg-closed, then it is closed.*

Proof. Since A is both π-open and wπg-closed, A $\supseteq$ cl(int(A)) = cl(A) and hence A is closed in (X, τ).

Corollary 3.10 *If a subset A of a topological space (X, τ) is both π-open and wπg-closed, then it is both regular open and regular closed in (X, τ).*

Theorem 3.11 *Let (X, τ) be a πg-$T_{1/2}$ space and $A \subseteq X$ be π-open. Then, A is wπg-closed if and only if A is πg-closed.*

Proof. Let A be πg-closed. By Proposition 3.2, it is wπg-closed. Conversely, let A be wπg-closed. Since A is π-open, by Theorem 3.9, A is closed. Since X is πg-$T_{1/2}$, A is πg-closed.

Theorem 3.12 *A set A is wπg-closed if and only if cl(int(A)) − A contains no non-empty π- closed set.*

Proof. *Necessity.* Let F be a π-closed set such that F $\subseteq$ cl(int(A)) − A. Since F^c is π- open and A $\subseteq F^c$, from the definition of wπg-closed set it follows that cl(int(A)) $\subseteq F^c$. ie. F $\subseteq$ (cl(int(A)))^c. This implies that F $\subseteq$ (cl(int(A))) $\cap$ (cl(int(A)))$^c = \phi$.

Sufficiency. Let A$\subseteq$ G, where G is π-open set in X. If cl(int(A)) is not contained in G, then cl(int(A)) $\cap\, G^c$ is a non-empty π-closed subset of cl(int(A)) − A, we obtain a contradiction. This proves the sufficiency and hence the theorem.

Corollary 3.13 *A wπg-closed set A is regular closed if and only if cl(int(A))−A is π-closed and cl(int(A)) $\supseteq$ A.*

Proof. *Necessity.* Since the set A is regular closed, cl(int(A)) − A = ϕ is regular closed and hence π-closed.

Sufficiency. By Theorem 3.12, cl(int(A)) − A contains no non-empty π-closed set. That is cl(int(A)) − A = ϕ. Therefore A is regular closed.

Theorem 3.14 *Let (X, τ) be a topological space and B $\subseteq$ A $\subseteq$ X. If B is wπg-closed set relative to A and A is both open and wπg-closed subset of X then B is wπg-closed set relative to X.*

Proof. Let B $\subseteq$ U and U be a π-open in (X, τ). Then B $\subseteq$ A $\cap$ U. Since B is wπg-closed relative to A, $cl_A(int_A$(B)) $\subseteq$ A $\cap$ U. That is A $\cap$ cl(int(B)) $\subseteq$ A $\cap$ U. We have A $\cap$ cl(int(B)) $\subseteq$ U and then [A $\cap$ cl(int(B))] $\cup$ (cl(int(B)))^c $\subseteq$ U $\cup$ (cl(int(B)))^c. Since A is wπg-closed in (X, τ), we have cl(int(A)) $\subseteq$ U $\cup$ (cl(int(B)))^c. Therefore cl(int(B)) $\subseteq$ U since cl(int(B)) is not contained in (cl(int(B)))^c. Thus B is wπg-closed set relative to (X, τ).

Corollary 3.15 *If A is both open and wπg-closed and F is closed in a topological space (X, τ), then A $\cap$ F is wπg-closed in (X, τ).*

Proof. Since F is closed, we have A $\cap$ F is closed in A. Therefore cl_A(A $\cap$ F) = A $\cap$ F in A. Let A $\cap$ F $\subseteq$ G, where G is π-open in A. Then $cl_A(int_A$(A $\cap$ F)) $\subseteq$ G and hence A $\cap$ F is wπg-closed in A. By Theorem 3.14, A$\cap$ F is wπg-closed in (X, τ).

Theorem 3.16 *If A is wπg-closed and A $\subseteq$ B $\subseteq$ cl(int(A)), then B is wπg-closed.*

Proof. Since A$\subseteq$B, cl(int(B))−B$\subseteq$cl(int(A))−A. By Theorem 3.12, cl(int(A))−A contains no non-empty π-closed set and so cl(int(B)) − B. Again by Theorem 3.12, B is wπg-closed.

Theorem 3.17 *Let (X, τ) be a topological space and $A \subseteq Y \subseteq X$ and Y be open. If A is $w\pi g$-closed in X, then A is $w\pi g$-closed relative to Y.*

Proof. Let A $\subseteq$ Y $\cap$ G where G is π-open in (X, τ). Since A is wπg-closed in (X, τ), A $\subseteq$ G implies cl(int(A)) $\subseteq$ G. That is Y $\cap$ (cl(int(A))) $\subseteq$ Y $\cap$ G where Y $\cap$ cl(int(A)) is closure of interior of A in (Y, σ). Thus A is wπg-closed relative to (Y, σ).

Theorem 3.18 *If a subset A of a topological space (X, τ) is nowhere dense, then it is $w\pi g$-closed.*

Proof. Since int(A) $\subseteq$ int(cl(A)) and A is nowhere dense, int(A) = ϕ. Therefore cl(int(A)) = ϕ and hence A is wπg-closed in (X, τ).

The converse of Theorem 3.18 need not be true as seen in the following Example.

Example 3.19 Let X = {a, b, c} and τ = {ϕ, {a}, {b, c}, X}. Then the set {a} is wπg-closed in (X, τ) but not nowhere dense in (X, τ).

Remark 3.20 If any subsets A and B of topological space X are wπg-closed, then their intersection need not be wπg-closed.

Example 3.21 Let X = {a, b, c, d} and τ = {ϕ, {a}, {b}, {a, b}, {a, b, c}, X}. In this topological space the subsets {a, c} and {a, d} are wπg-closed but their intersection {a} is not wπg-closed in (X, τ).

Proposition 3.22 *Every $g\alpha$-closed set is $w\pi g$-closed but not conversely.*

Proof. Let A be any $g\alpha$-closed subset of (X, τ) and let U be an π-open set containing A. Then U is α-open set containing A. Now G $\supseteq$ αcl(A) $\supseteq$ cl(int(cl(A))) $\supseteq$ cl(int(A)). Thus A is wπg-closed in (X, τ).

The converse of Proposition 3.22 need not be true as seen in the following Example.

Example 3.23 Let X = {a, b, c, d} and τ = {ϕ, {a}, {a, b}, X}. Then the set {a, b, c} is wπg-closed but not $g\alpha$-closed in (X, τ).

Remark 3.24 wπg-closedness is independent of semi-closedness, β-closedness, b-closedness, sg-closedness and $\tilde{g}s$-closedness in (X, τ).

Example 3.25 Let X = {a, b, c, d} and τ = {ϕ, {a}, {b}, {a, b}, {a, b, c}, X}. Then the set {a, b, c} is wπg-closed in (X, τ) but not semi-closed, β-closed, b-closed, sg-closed and $\tilde{g}s$-closed in (X, τ).

Example 3.26 Let X = {a, b, c, d} and τ = {ϕ, {a}, {b}, {a, b}, {a, c}, {a, b, c}, X}. Then the set {a, c} is semi-closed, β-closed, b-closed, sg-closed and $\tilde{g}s$-closed in (X, τ) but not wπg-closed in (X, τ).

Remark 3.27 The following diagram shows the relationships established between wπg-closed sets and some other sets. A $\rightarrow$ B represents A implies B but not conversely.

Diagram

$$\begin{array}{ccccc} \text{closed} & \longrightarrow & \text{w}\tilde{g}\text{-closed} & \longrightarrow & \text{weakly } \omega\text{-closed} \\ \downarrow & & & & \downarrow \\ \text{rwg-closed} & \longleftarrow & \text{w}\pi g\text{-closed} & \longleftarrow & \text{wg-closed} \end{array}$$

Definition 3.28 A subset A of a topological space X is called wπg-open set if A^c is wπg-closed in X.

Proposition 3.29

1. *Every πg-open set is wπg-open;*
2. *Every g-open set is wπg-open.*

Theorem 3.30 *A subset A of a topological space X is wπg-open if G $\subseteq$ int(cl(A)) whenever G $\subseteq$ A and G is π-closed.*

Proof. Let A be any wπg-open. Then A^c is wπg-closed. Let G be a π-closed set contained in A. Then G^c is a π-open set in X containing A^c. Since A^c is wπg-closed, we have cl(int(A^c)) $\subseteq G^c$. Therefore G $\subseteq$ int(cl(A)).

Conversely, we suppose that G $\subseteq$ int(cl(A)) whenever G $\subseteq$ A and G is π-closed. Then G^c is a π-open set containing A^c and $G^c \supseteq$ (int(cl(A)))^c. It follows that $G^c \supseteq$ cl(int(A^c)). Hence A^c is wπg-closed and so A is wπg-open.

4. Weakly πg-continuous mappings

Definition 4.1 Let X and Y be topological spaces. A mapping f : X $\rightarrow$ Y is called weakly πg-continuous (briefly wπg-continuous) if f^{-1}(U) is a wπg-open set in X for each open set U in Y.

Example 4.2 Let X=Y=$\{a, b, c\}, \tau = \{\phi, \{a\}, \{b, c\}, X\}$ and $\sigma = \{\phi, \{a\}, Y\}$. The mapping $f : (X, \tau) \rightarrow (Y, \sigma)$ defined by $f(a) = b$, $f(b) = c$ and $f(c) = a$ is wπg-continuous, because every subset of X is wπg-closed.

Proposition 4.3 *Every πg-continuous mapping is wπg-continuous.*

Proof. It follows from Proposition 3.29 (1).

The converse of Proposition 4.3 need not be true as per the following example.

Example 4.4 Let X = Y = {a, b, c}, τ = {ϕ, {a}, {b}, {a, b}, X} and σ = {ϕ, {a}, Y}. Let the mapping f : $(X, \tau) \to (Y, \sigma)$ be the identity map. Then f is wπg-continuous but it is not πg-continuous.

Theorem 4.5 *A mapping $f : X \to Y$ is called wπg-continuous if and only if $f^{-1}(U)$ is a wπg-closed set in X for each closed set U in Y.*

Proof. Let U be any closed set in Y. According to the assumption $f^{-1}((U^c))$= X \ f^{-1}(U) is wπg-open in X, so f^{-1}(U) is wπg-closed in X.

The converse can be proved in a similar manner.

Theorem 4.6 *Suppose that X and Y are spaces and the family of πg-open sets of X is closed under arbitrary unions. If a mapping $f : X \to Y$ is contra πg-continuous and Y is regular, then f is wπg-continuous.*

Proof. Let f : X → Y be contra πg-continuous and Y be regular. By Theorem 16 of [12], f is πg-continuous. Hence, f is wπg-continuous.

Theorem 4.7 *Let $f : (X, \tau) \to (Y, \sigma)$ be a mapping. If f is contra πg-continuous and (X, τ) is locally πg-indiscrete, then f is wπg-continuous.*

Proof. Let f : X → Y be contra πg-continuous and (X, τ) be locally πg-indiscrete. By Theorem 21 of [12], f is continuous. Hence, f is wπg-continuous.

Theorem 4.8 *Suppose that a topological space (X, τ) is πgp-$T_{1/2}$ and submaximal and Y is regular. If f is almost πgp-continuous, then f is wπg-continuous.*

Proof. Let f be almost πgp-continuous. By Theorem 30 of [14], f is almost πg-continuous. Also, by Theorem 38 of [14], f is πg-continuous. Hence, f is wπg-continuous.

Theorem 4.9 *Let Y be a regular space and $f : X \to Y$ be a mapping. Suppose that the collection of πg-closed sets of X is closed under arbitrary intersections. Then if f is (πg, s)-continuous, f is wπg-continuous.*

Proof. Let f be $(\pi g, s)$-continuous. By Theorem 24 of [13], f is πg-continuous. Thus, f is wπg-continuous.

Remark 4.10 Every $\tilde{g}$-continuous mapping is w$\tilde{g}$-continuous and every w$\tilde{g}$-continuous mapping is wπg-continuous but not conversely as shown in [27] and in the below example.

Example 4.11 Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{c\}, \{a, c\}, \{b, c\}, X\}$ and $\sigma = \{\phi, \{a\}, \{b\}, \{a, b\}, Y\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the identity mapping. Then f is wπg-continuous but not w$\tilde{g}$-continuous.

Proposition 4.12 *If $f : X \to Y$ is perfectly continuous and π-irresolute, then it is R-map.*

Proof. Let V be any regular open subset of Y. According to the assumption, f^{-1}(V) is both π-open and closed in X. Since f^{-1}(V) is closed it is wπg-closed. Then f^{-1}(V) is both π-open and wπg-closed. Hence by Corollary 3.10 it is regular open in X, so f is R-map.

Definition 4.13 A topological space X is weakly πg-compact (briefly wπg-compact) if every wπg-open cover of X has a finite subcover.

Remark 4.14 Every wπg-compact space is πg-compact.

Theorem 4.15 *Let $f : X \to Y$ be a surjective wπg-continuous mapping. If X is wπg-compact, then Y is compact.*

Proof. Let $\{A_i : i \in I\}$ be an open cover of Y. Then $\{f^{-1}(A_i) : i \in I\}$ is a wπg-open cover of X. Since X is wπg-compact, it has a finite subcover, say $\{ f^{-1}(A_1), f^{-1} (A_2),...., f^{-1}(A_n)\}$. Since f is surjective $\{A_1, A_2, ..., A_n\}$ is a finite subcover of Y and hence Y is compact.

Definition 4.16 A topological space X is weakly πg-connected (briefly wπg-connected) if X cannot be written as the disjoint union of two non-empty wπg-open sets.

Theorem 4.17 *If a topological space X is wπg-connected, then X is almost connected and πg-connected.*

Proof. It follows from the fact that each regular open set and each πg-open set is wπg-open.

Theorem 4.18 *For a topological space X the following statements are equivalent:*

(1) *X is wπg-connected.*

(2) *The empty set ϕ and X are only subsets which are both wπg-open and wπg-closed.*

(3) *Each wπg-continuous mapping from X into a discrete space Y which has at least two points is a constant function.*

Proof. (1) $\Rightarrow$ (2). Let S $\subseteq$ X be any proper subset, which is both wπg-open and wπg-closed. Its complement X $\setminus$ S is also wπg-open and wπg-closed. Then X = S $\cup$ (X $\setminus$ S) is a disjoint union of two non-empty wπg-open sets which is a contradiction with the fact that X is wπg-connected. Hence, S = ϕ or X.

(2) $\Rightarrow$ (1). Let X = A $\cup$ B where A $\cap$ B = ϕ, A $\neq \phi$, B $\neq \phi$ and A, B are wπg-open. Since A = X $\setminus$ B, A is wπg-closed. According to the assumption A = ϕ, which is a contradiction.

(2) $\Rightarrow$ (3). Let f : X $\to$ Y be a wπg-continuous mapping where Y is a discrete space with at least two points. Then $f^{-1}(\{y\})$ is wπg-closed and wπg-open for each y $\in$Y and X = $\cup\{ f^{-1}(\{y\}) \mid y \in Y\}$. According to the assumption,

$f^{-1}(\{y\})=\phi$ or $f^{-1}(\{y\}) = X$. If $f^{-1}(\{y\}) = \phi$ for all $y \in Y$, f will not be a mapping. Also there is no exist more than one $y \in Y$ such that $f^{-1}(\{y\}) = X$. Hence, there exists only one $y \in Y$ such that $f^{-1}(\{y\}) = X$ and $f^{-1}(\{y_1\}) = \phi$ where $y \neq y_1 \in Y$. This shows that f is a constant mapping.

(3) $\Rightarrow$ (2). Let $S \neq \phi$ be both wπg-open and wπg-closed in X. Let $f : X \to Y$ be a wπg-continuous mapping defined by $f(S) = \{a\}$ and $f(X \setminus S) = \{b\}$ where $a \neq b$. Since f is constant mapping we get $S = X$.

Theorem 4.19 *Let $f : X \to Y$ be a wπg-continuous surjective mapping. If X is wπg-connected, then Y is connected.*

Proof. We suppose that Y is not connected. Then $Y = A \cup B$ where $A \cap B = \phi$, $A \neq \phi$, $B \neq \phi$ and A, B are open sets in Y. Since f is wπg-continuous surjective mapping $X = f^{-1}(A) \cup f^{-1}(B)$ are disjoint union of two non-empty wπg-open subsets. This is contradiction with the fact that X is wπg-connected.

5. Weakly πg-open mappings and weakly πg-closed mappings

Definition 5.1 Let X and Y be topological spaces. A mapping $f : X \to Y$ is called weakly πg-open (briefly wπg-open) if f(V) is a wπg-open set in Y for each open set V in X.

Remark 5.2 Every πg-open mapping is wπg-open but not conversely.

Example 5.3 Let $X = Y = \{a, b, c, d\}$, $\tau = \{\phi, \{a\}, \{a, b, d\}, X\}$ and $\sigma = \{\phi, \{a\}, \{b, c\}, \{a, b, c\}, Y\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the identity mapping. Then f is $w\pi g$-open but not πg-open.

Definition 5.4 Let X and Y be topological spaces. A mapping $f : X \to Y$ is called weakly πg-closed (briefly wπg-closed) if f(V) is a wπg-closed set in Y for each closed set V in X.

It is clear that an open mapping is wπg-open and a closed mapping is wπg-closed.

Theorem 5.5 *Let X and Y be topological spaces. A mapping $f : X \to Y$ is wπg-closed if and only if for each subset B of Y and for each open set G containing $f^{-1}(B)$ there exists a wπg-open set F of Y such that $B \subseteq F$ and $f^{-1}(F) \subseteq G$.*

Proof. Let B be any subset of Y and let G be an open subset of X such that $f^{-1}(B) \subseteq G$. Then $F = Y \setminus f(X \setminus G)$ is wπg-open set containing B and $f^{-1}(F) \subseteq G$. Conversely, let U be any closed subset of X. Then $f^{-1}(Y \setminus f(U)) \subseteq X \setminus U$ and $X \setminus U$ is open. According to the assumption, there exists a wπg-open set F of Y such that $Y \setminus f(U) \subseteq F$ and $f^{-1}(F) \subseteq X \setminus U$. Then $U \subseteq X \setminus f^{-1}(F)$. From $Y \setminus F \subseteq f(U) \subseteq f(X \setminus f^{-1}(F)) \subseteq Y \setminus F$ follows that $f(U) = Y \setminus F$, so f(U) is wπg-closed in Y. Therefore f is a wπg-closed mapping.

Remark 5.6 The composition of two wπg-closed mappings need not be wπg-closed as we can see from the following example.

Example 5.7 Let $X = Y = Z = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{b, c\}, X\}$, $\sigma = \{\phi, \{a\}, \{a, b\}, Y\}$ and $\eta = \{\phi, \{a\}, \{b\}, \{a, b\}, Z\}$. We define $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ be the identity mappings. Hence both f and g are $w\pi g$-closed mappings. For a closed set $U = \{a\}, (gof)(U) = g(f(U)) = g(\{a\}) = \{a\}$ which is not $w\pi g$-closed in Z. Hence the composition of two $w\pi g$-closed mappings need not be $w\pi g$-closed.

Theorem 5.8 *Let X, Y and Z be topological spaces. If $f : X \to Y$ be a closed mapping and $g : Y \to Z$ be a $w\pi g$-closed map, then g o $f : X \to Z$ is a $w\pi g$-closed mapping.*

Definition 5.9 A mapping f : X $\to$ Y is called a weakly πg-irresolute (briefly wπg-irresolute) mapping if f^{-1}(U) is a wπg-open set in X for each wπg-open set U in Y.

Example 5.10 Let $X = Y = \{a, b, c\}, \tau = \{\phi, \{b\}, \{a, c\}, X\}$ and $\sigma = \{\phi, \{b\}, Y\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the identity mapping. Then f is wπg-irresolute.

Remark 5.11 The following examples show that irresoluteness and wπg-irresoluteness are independent.

Example 5.12 Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, \{a\}, X\}$ and $\sigma = \{\phi, \{a\}, \{c\}, \{a, c\}, Y\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the identity mapping. Then f is wπg-irresolute but not irresolute.

Example 5.13 Let X = Y = {a, b, c}, τ = {ϕ, {a}, {b}, {a, b}, X} and σ = {ϕ, {a, b}, Y}. Let f : $(X, \tau) \to (Y, \sigma)$ be the identity mapping. Then f is irresolute but not wπg-irresolute.

Remark 5.14 Every πg-irresolute function is wπg-continuous but not conversely. Also, the concepts of πg-irresoluteness and wπg-irresoluteness are independent of each other.

Example 5.15 Let X = Y = {a, b, c, d}, τ = {ϕ, {a}, {b, c}, {a, b, c}, X} and σ = {ϕ, {a}, {a, b, d}, Y}. Let f : $(X, \tau) \to (Y, \sigma)$ be the identity mapping. Then f is wπg-continuous but not πg-irresolute.

Example 5.16 Let $X{=}Y{=}\{a, b, c, d\}$, $\tau{=}\{\phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, b, c\}, X\}$ and $\sigma = \{\phi, \{a\}, \{b, c\}, \{a, b, c\}, Y\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the identity mapping. Then f is πg-irresolute but not wπg-irresolute.

Example 5.17 Let $X = \{a, b, c, d\}$, $Y = \{p, q\}$, $\tau = \{\phi, \{d\}, \{b, c\}, \{b, c, d\}, X\}$ and $\sigma = \{\phi, \{p\}, Y\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be defined as $f(a) = f(c) = f(d) = p$ and $f(b) = q$. Then f is wπg-irresolute but not πg-irresolute.

Theorem 5.18 *The composition of two wπg-irresolute mappings is also wπg-irresolute.*

Theorem 5.19 *Let $f : X \to Y$ and $g : Y \to Z$ be mappings such that $g \circ f : X \to Z$ is wπg-closed mapping. Then the following statements hold:*

(1) *if f is continuous and injective, then g is wπg-closed.*

(2) *if g is wπg-irresolute and injective, then f is wπg-closed.*

Proof. (1) Let F be a closed set of Y. Since f^{-1}(F) is closed in X, we can conclude that (g o f)(f^{-1}(F)) is wπg-closed in Z. Hence g(F) is wπg-closed in Z. Thus g is a wπg-closed mapping.

(2) It can prove in a similar manner as (1).

Theorem 5.20 *If $f : X \to Y$ is an wπg-irresolute mapping, then it is wπg-continuous.*

Remark 5.21 The converse of the above Theorem need not be true in general. Let X = Y = {a, b, c, d}, τ = {ϕ, {a}, {b}, {a, b}, {a, c}, {a, b, c}, X} and σ = {ϕ, {d}, Y} The mapping f : X $\to$ Y defined as f(a) = d, f(b) = c, f(c) = b and f(d) = a. Then f is wπg-continuous but not wπg-irresolute. Since f^{-1}({a}) = {d} is not wπg-open in X.

Theorem 5.22 *If $f : X \to Y$ is a surjective wπg-irresolute mapping and X is wπg-compact, then Y is wπg-compact.*

Theorem 5.23 *If $f : X \to Y$ is surjective wπg-irresolute mapping and X is wπg-connected, then Y is wπg-connected.*

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UP-ALGEBRAS CHARACTERIZED BY THEIR ANTI-FUZZY UP-IDEALS AND ANTI-FUZZY UP-SUBALGEBRAS[1]

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Abstract. In this paper, anti-fuzzy UP-ideals and anti-fuzzy UP-subalgebras concepts of UP-algebras are introduced and proved some results. We also introduce the notions of Cartesian product and dot product of fuzzy sets, and then we study related properties. Further, we discuss the relation between anti-fuzzy UP-ideals (resp. anti-fuzzy UP-subalgebras) and level subsets of a fuzzy set. Anti-fuzzy UP-ideals and anti-fuzzy UP-subalgebras are also applied in the Cartesian product of UP-algebras.

Keywords: UP-algebra, anti-fuzzy UP-ideal, anti-fuzzy UP-subalgebra.

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1. Introduction and preliminaries

Among many algebraic structures, algebras of logic form important class of algebras. Examples of these are BCK-algebras [9], BCI-algebras [10], BCH-algebras [7], KU-algebras [27], SU-algebras [17] and others. They are strongly connected with logic. For example, BCI-algebras introduced by Iséki [10] in 1966 have connections with BCI-logic being the BCI-system in combinatory logic which has application in the language of functional programming. BCK and BCI-algebras are two classes of logical algebras. They were introduced by Imai and Iséki [9], [10] in 1966 and have been extensively investigated by many researchers. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras.

A fuzzy subset f of a set S is a function from S to a closed interval $[0, 1]$. The concept of a fuzzy subset of a set was first considered by Zadeh [38] in 1965. The

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fuzzy set theories developed by Zadeh and others have found many applications in the domain of mathematics and elsewhere.

After the introduction of the concept of fuzzy sets by Zadeh [38], several researches were conducted on the generalizations of the notion of fuzzy set and application to many logical algebras such as: In 2001, Lele, Wu, Weke, Mamadou and Njock [19] studied fuzzy ideals and weak ideals in BCK-algebras. Jun [11] introduced the notion of Q-fuzzy subalgebras of BCK/BCI-algebras, and provided some appropriate examples. In 2002, Jun, Roh and Kim [13] studied fuzzy B-algebras in B-algebras. Yonglin and Xiaohong [37] introduced the notion of fuzzy a-ideals in BCI-algebras, and investigated its properties. In 2004, Jun [12] introduced the concept of (α, β)-fuzzy ideals of BCK/BCI-algebras. In 2005, Akram and Dar [1] introduced the notions of T-fuzzy subalgebras and T-fuzzy H-ideals in BCI-algebras and investigated some of their properties. Jun [14] introduced the notion of (α, β)-fuzzy subalgebras of BCK/BCI-algebras. In 2007, Kordi and Moussavi [18] studied fuzzy p-ideals, fuzzy H-ideals and fuzzy BCI-positive implicative ideals of BCI-algebras. Akram and Dar [2] introduced the notion of fuzzy ideals in K-algebras. In 2008, Saeid and Jun [31] introduced the concept of anti fuzzy subalgebras of BCK/BCI-algebras by using the notion of anti fuzzy points. In 2009, Saeid and Rezvani [32] introduced the notion of fuzzy BF-algebra and fuzzy topological BF-algebras. Hadipour [6] introduced the notion of (α, β)-fuzzy BF-algebras. In 2010, Song, Jun and Lee [35] introduced the notion of fuzzy ideals in BE-algebras. Saeid [30] introduced the notion of fuzzy dot BCK-subalgebras. Karamdin [16] characterized anti fuzzy ideals in weak BCC-algebras. In 2011, Mostafa, Abd-Elnaby and Yousef [22] introduced the notion of fuzzy KU-ideals in KU-algebras. Mostafa, Omar and Marie [23] introduced the notion of anti-fuzzy sub-implicative ideal of BCI-algebras. In 2012, Sitharselvam, Priya and Ramachandran [33] introduced the notion of anti Q-fuzzy KU-ideals of KU-algebras. In 2014, Yamini and Kailasavalli [36] introduced the notion of B-ideals and fuzzy B-ideals in B-algebras. Rajam and Chandramouleeswaran [28] introduced the notion of L-fuzzy β-subalgebras on β-algebras, and investigated some of their properties. Al-Shehri [3] introduced the notion of anti fuzzy implicative ideals of BCK-algebras. In 2015, Mostafa, Abdel Naby and Elkabany [25] introduced the notion of α-anti fuzzy new ideals of PU-algebras. Kailasavalli and Yamini [15] introduced the notion of anti fuzzy B-ideals and anti fuzzy B-algebras of B-algebras.

Iampan [8] now introduced a new algebraic structure, called a UP-algebra. The notions of fuzzy subalgebras and fuzzy ideals play an important role in studying the many logical algebras. Somjanta, Thuekaew, Kumpeangkeaw and Iampan [34] introduced and studied fuzzy UP-subalgebras and fuzzy UP-ideals of UP-algebras, and investigated some of its properties. In this paper, anti-fuzzy UP-ideals and anti-fuzzy UP-subalgebras concepts of UP-algebras are introduced and proved some results. Further, we discuss the relation between anti-fuzzy UP-ideals (resp. anti-fuzzy UP-subalgebras) and level subsets of a fuzzy set. Anti-fuzzy UP-ideals and anti-fuzzy UP-subalgebras are also applied in the Cartesian product of UP-algebras.

Before we begin our study, we will introduce the definition of a UP-algebra.

Definition 1.1. [8] An algebra $A = (A; \cdot, 0)$ of type $(2, 0)$ is called a *UP-algebra* if it satisfies the following axioms: for any $x, y, z \in A$,

(UP-1) $(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0$,

(UP-2) $0 \cdot x = x$,

(UP-3) $x \cdot 0 = 0$, and

(UP-4) $x \cdot y = y \cdot x = 0$ implies $x = y$.

Example 1.2. [8] Let X be a set. Define a binary operation $\cdot$ on the power set of X by putting $A \cdot B = B \cap A'$ for all $A, B \in \mathcal{P}(X)$. Then $(\mathcal{P}(X); \cdot, \emptyset)$ is a UP-algebra.

Example 1.3. [8] Let $A = \{0, a, b, c\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	a	b	c
0	0	a	b	c
a	0	0	0	0
b	0	a	0	c
c	0	a	b	0

Then $(A; \cdot, 0)$ is a UP-algebra.

In what follows, let A and B denote UP-algebras unless otherwise specified. The following proposition is very important for the study of UP-algebras.

Proposition 1.4. [8] *In a UP-algebra A, the following properties hold: for any $x, y \in A$,*

(1) $x \cdot x = 0$,

(2) $x \cdot y = 0$ *and* $y \cdot z = 0$ *imply* $x \cdot z = 0$,

(3) $x \cdot y = 0$ *implies* $(z \cdot x) \cdot (z \cdot y) = 0$,

(4) $x \cdot y = 0$ *implies* $(y \cdot z) \cdot (x \cdot z) = 0$,

(5) $x \cdot (y \cdot x) = 0$,

(6) $(y \cdot x) \cdot x = 0$ *if and only if* $x = y \cdot x$, *and*

(7) $x \cdot (y \cdot y) = 0$.

On a UP-algebra $A = (A; \cdot, 0)$, we define a binary relation $\leq$ on A [8] as follows: for all $x, y \in A$,

$$x \leq y \text{ if and only if } x \cdot y = 0. \tag{1.1}$$

Proposition 1.5. [8] *In a UP-algebra A, the following properties hold: for any $x, y \in A$,*

(1) $x \leq x$,

(2) $x \leq y$ *and* $y \leq x$ *imply* $x = y$,

(3) $x \leq y$ *and* $y \leq z$ *imply* $x \leq z$,

(4) $x \leq y$ *implies* $z \cdot x \leq z \cdot y$,

(5) $x \leq y$ *implies* $y \cdot z \leq x \cdot z$,

(6) $x \leq y \cdot x$, *and*

(7) $x \leq y \cdot y$.

Proposition 1.6. [8] *An algebra $A = (A; \cdot, 0)$ of type $(2, 0)$ with a binary relation $\leq$ defined by (1.1) is a UP-algebra if and only if it satisfies the following conditions: for all $x, y, z \in A$,*

(1) $(y \cdot z) \leq (x \cdot y) \cdot (x \cdot z)$,

(2) $0 \cdot x = x$,

(3) $x \leq 0$, *and*

(4) $x \leq y$ *and* $y \leq x$ *imply* $x = y$.

Definition 1.7. [8] A nonempty subset B of A is called a *UP-ideal* of A if it satisfies the following properties:

(1) the constant 0 of A is in B, and

(2) for any $x, y, z \in A, x \cdot (y \cdot z) \in B$ and $y \in B$ imply $x \cdot z \in B$.

Clearly, A and $\{0\}$ are UP-ideals of A.

Theorem 1.8. [8] *Let A be a UP-algebra and $\{B_i\}_{i \in I}$ a family of UP-ideals of A. Then $\bigcap_{i \in I} B_i$ is a UP-ideal of A.*

Definition 1.9. [8] A subset S of A is called a *UP-subalgebra* of A if the constant 0 of A is in S, and $(S; \cdot, 0)$ itself forms a UP-algebra. Clearly, A and $\{0\}$ are UP-subalgebras of A.

Applying Proposition 1.4 (1), we can then easily prove the following proposition.

Proposition 1.10. [8] *A nonempty subset S of a UP-algebra $A = (A; \cdot, 0)$ is a UP-subalgebra of A if and only if S is closed under the $\cdot$ multiplication on A.*

Theorem 1.11. [8] *Let A be a UP-algebra and $\{B_i\}_{i \in I}$ a family of UP-subalgebras of A. Then $\bigcap_{i \in I} B_i$ is a UP-subalgebra of A.*

Definition 1.12. [38] A *fuzzy set* in a nonempty set X (or a fuzzy subset of X) is an arbitrary function $f\colon X \to [0,1]$ where $[0,1]$ is the unit segment of the real line.

Definition 1.13. A fuzzy set f in A is called an *anti-fuzzy UP-ideal* of A if it satisfies the following properties: for any $x, y, z \in A$,

(1) $f(0) \leq f(x)$, and

(2) $f(x \cdot z) \leq \max\{f(x \cdot (y \cdot z)), f(y)\}$.

Definition 1.14. A fuzzy set f in A is called an *anti-fuzzy UP-subalgebra* of A if for any $x, y \in A$,

$$f(x \cdot y) \leq \max\{f(x), f(y)\}.$$

Example 1.15. Let $A = \{0, 1\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1
0	0	1
1	0	0

Then $(A; \cdot, 0)$ is a UP-algebra. We define a fuzzy set f in A as follows:

$$f(0) = 0.1 \text{ and } f(1) = 0.2.$$

Using this data, we can show that f is an anti-fuzzy UP-ideal of A and an anti-fuzzy UP-subalgebra of A.

Example 1.16. Let $A = \{0, a, b\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	a	b
0	0	a	b
a	0	0	a
b	0	0	0

Then $(A; \cdot, 0)$ is a UP-algebra. We define a fuzzy set f in A as follows:

$$f(0) = 0.1, f(a) = 0.2 \text{ and } f(b) = 0.2.$$

Using this data, we can show that f is an anti-fuzzy UP-subalgebra of A.

Definition 1.17. [34] Let f be a fuzzy set in A. The fuzzy set $\overline{f}$ defined by $\overline{f}(x) = 1 - f(x)$ for all $x \in A$ is called the *complement* of f in A.

Remark 1.18. For all fuzzy set f in A, we have $f = \overline{\overline{f}}$.

Definition 1.19. [34] Let f be a fuzzy set in A. For any $t \in [0,1]$, the sets

$$U(f;t) = \{x \in A \mid f(x) \geq t\} \text{ and } U^+(f;t) = \{x \in A \mid f(x) > t\}$$

are called an *upper t-level subset* and an *upper t-strong level subset* of f, respectively. The sets

$$L(f;t) = \{x \in A \mid f(x) \leq t\} \text{ and } L^{-}(f;t) = \{x \in A \mid f(x) < t\}$$

are called a *lower t-level subset* and a *lower t-strong level subset* of f, respectively.

Definition 1.20. [21] Let f be a function from a nonempty set X to a nonempty set Y. If μ is a fuzzy set in X, then fuzzy set β in Y defined by

$$\beta(y) = \begin{cases} \inf\{\mu(t)\}_{t \in f^{-1}(y)} & \text{if } f^{-1}(y) \neq \emptyset, \\ 0 & \text{if } f^{-1}(y) = \emptyset \end{cases}$$

is said to be the *image of μ under f*. Similarly, if β is a fuzzy set in Y, then the fuzzy set $\mu = \beta \circ f$ in X (i.e., the fuzzy set defined by $\mu(x) = \beta(f(x))$ for all $x \in X$) is called the *preimage of β under f*.

Definition 1.21. [29] A fuzzy set f in A has inf *property* if for any nonempty subset T of A, there exists $t_0 \in T$ such that $f(t_0) = \inf\{f(t)\}_{t \in T}$.

Definition 1.22. [5] A *fuzzy relation* on a nonempty set X is an arbitrary function $f\colon X \times X \to [0,1]$ where $[0,1]$ is the unit segment of the real line.

Definition 1.23. [24] Let f and g be fuzzy sets in nonempty sets A and B, respectively. The *Cartesian product* of f and g is $f \times g\colon A \times B \to [0,1]$ defined by

$$(f \times g)(x,y) = \max\{f(x), g(y)\} \text{ for all } x \in A \text{ and } y \in B.$$

The *dot product* of f and g is $f \cdot g\colon A \times B \to [0,1]$ defined by

$$(f \cdot g)(x,y) = \min\{f(x), g(y)\} \text{ for all } x \in A \text{ and } y \in B.$$

Definition 1.24. [24] If f is a fuzzy set in a nonempty set X, the *strongest fuzzy relation* on X is $\mu_f\colon X \times X \to [0,1]$ defined by $\mu_f(x,y) = \max\{f(x), f(y)\}$ for all $x, y \in X$. For $x, y \in X$, we have $f(x), f(y) \in [0,1]$. Thus $\mu_f(x,y) = \max\{f(x), f(y)\} \in [0,1]$. Hence, μ_f is a fuzzy relation on X. We note that if f is a fuzzy set in a nonempty set X, then $f \times f = \mu_f$.

Definition 1.25. If f is a fuzzy set in a nonempty set X, the *weakness fuzzy relation* on X is $\beta_f\colon X \times X \to [0,1]$ defined by $\beta_f(x,y) = \min\{f(x), f(y)\}$ for all $x, y \in X$. For $x, y \in X$, we have $f(x), f(y) \in [0,1]$. Thus $\beta_f(x,y) = \min\{f(x), f(y)\} \in [0,1]$. Hence, β_f is a fuzzy relation on X. We note that if f is a fuzzy set in a nonempty set X, then $f \cdot f = \beta_f$.

Definition 1.26. [21] Let X and Y be any two nonempty sets and let $f\colon X \to Y$ be any function. A fuzzy set μ in X is called *f-invariant* if $f(x) = f(y)$ implies $\mu(x) = \mu(y)$ for all $x, y \in X$.

Definition 1.27. [8] Let $(A; \cdot, 0)$ and $(A'; \cdot', 0')$ be UP-algebras. A mapping f from A to A' is called a *UP-homomorphism* if

$$f(x \cdot y) = f(x) \cdot' f(y) \text{ for all } x, y \in A.$$

A UP-homomorphism $f\colon A \to A'$ is called a

(1) *UP-endomorphism* of A if $A' = A$,

(2) *UP-epimorphism* if f is surjective,

(3) *UP-monomorphism* if f is injective, and

(4) *UP-isomorphism* if f is bijective. Moreover, we say A is *UP-isomorphic* to A', symbolically, $A \cong A'$, if there is a UP-isomorphism from A to A'.

Let f be a mapping from A to A', and let B be a nonempty subset of A, and B' of A'. The set $\{f(x) \mid x \in B\}$ is called the *image* of B under f, denoted by $f(B)$. In particular, $f(A)$ is called the *image* of f, denoted by $\mathrm{Im}(f)$. Dually, the set $\{x \in A \mid f(x) \in B'\}$ is said the *inverse image* of B' under f, symbolically, $f^{-1}(B')$. Especially, we say $f^{-1}(\{0'\})$ is the *kernel* of f, written by $\mathrm{Ker}(f)$. That is,

$$\mathrm{Im}(f) = \{f(x) \in A' \mid x \in A\}$$

and

$$\mathrm{Ker}(f) = \{x \in A \mid f(x) = 0'\}.$$

Theorem 1.28. [8] *Let $(A; \cdot, 0_A)$ and $(B; *, 0_B)$ be UP-algebras and let $f\colon A \to B$ be a UP-homomorphism. Then the following statements hold:*

(1) $f(0_A) = 0_B$, *and*

(2) *for any $x, y \in A$, if $x \leq y$, then $f(x) \leq f(y)$.*

Lemma 1.29. [34] *Let f be a fuzzy set in A. Then the following statements hold: for any $x, y \in A$,*

(1) $1 - \max\{f(x), f(y)\} = \min\{1 - f(x), 1 - f(y)\}$, *and*

(2) $1 - \min\{f(x), f(y)\} = \max\{1 - f(x), 1 - f(y)\}$.

2. Main results

In this section, we study anti-fuzzy UP-ideals and anti-fuzzy UP-subalgebras of UP-algebras.

Theorem 2.1. *Every anti-fuzzy UP-ideal of A is an anti-fuzzy UP-subalgebra of A.*

Proof. Let f be an anti-fuzzy UP-ideal of A. Let $x, y \in A$. Then

$$\begin{aligned} f(x \cdot y) &\leq \max\{f(x \cdot (y \cdot y)), f(y)\} && \text{(Definition 1.13 (2))} \\ &= \max\{f(x \cdot 0), f(y)\} && \text{(Proposition 1.4 (1))} \\ &= \max\{f(0), f(y)\} && \text{(UP-3)} \\ &= f(y) && \text{(Definition 1.13 (1))} \\ &\leq \max\{f(x), f(y)\}. \end{aligned}$$

Hence, f is an anti-fuzzy UP-subalgebra of A. ∎

Lemma 2.2. *Let f be an anti-fuzzy UP-ideal of A. If the inequality $x \leq y \cdot z$ holds in A for all $x, y, z \in A$, then $f(z) \leq \max\{f(x), f(y)\}$ for all $x, y, z \in A$.*

Proof. Assume $x \leq y \cdot z$ for all $x, y, z \in A$. Then $x \cdot (y \cdot z) = 0$. By Definition 1.13 (2), we have

$$f(x \cdot z) \leq \max\{f(x \cdot (y \cdot z)), f(y)\}. \tag{2.1}$$

By (2.1) and (UP-2), let $x = 0$, so

$$f(z) = f(0 \cdot z) \leq \max\{f(0 \cdot (x \cdot z)), f(x)\} = \max\{f(x \cdot z), f(x)\}. \tag{2.2}$$

By (2.1) and Definition 1.13 (1), we have

$$f(x \cdot z) \leq \max\{f(x \cdot (y \cdot z)), f(y)\} = \max\{f(0), f(y)\} = f(y). \tag{2.3}$$

By (2.2) and (2.3), we have

$$f(z) \leq \max\{f(x \cdot z), f(x)\} \leq \max\{f(y), f(x)\} = \max\{f(x), f(y)\}. \qquad \blacksquare$$

Lemma 2.3. *If f is an anti-fuzzy UP-ideal of A and if $x, y \in A$ is such that $x \leq y$ in A, then $f(x) \geq f(y)$.*

Proof. Let $x, y \in A$ be such that $x \leq y$ in A. Then $x \cdot y = 0$. Thus

$$\begin{aligned}
f(y) &= f(0 \cdot y) && ((\text{UP-2})) \\
&\leq \max\{f(0 \cdot (x \cdot y)), f(x)\} && (\text{Definition 1.13 (2)}) \\
&= \max\{f(0 \cdot 0), f(x)\} \\
&= \max\{f(0), f(x)\} && ((\text{UP-2})) \\
&= f(x). && (\text{Definition 1.13 (1)})
\end{aligned}$$

$\blacksquare$

Lemma 2.4. [26] *Let f be a fuzzy set in A. For any $t \in [0, 1]$, the following properties hold:*

(1) $L(f; t) = U(\overline{f}; 1 - t)$,

(2) $L^-(f; t) = U^+(\overline{f}; 1 - t)$,

(3) $U(f; t) = L(\overline{f}; 1 - t)$, *and*

(4) $U^+(f; t) = L^-(\overline{f}; 1 - t)$.

Lemma 2.5. [20] *For any $a, b \in \mathbb{R}$ such that $a < b$, $a < \dfrac{b + a}{2} < b$.*

Theorem 2.6. *Let f be a fuzzy set in A. Then the following statements hold:*

(1) *f is an anti-fuzzy UP-ideal of A if and only if it satisfies the condition $(\star)$: for all $t \in [0, 1]$, $L(f; t) \neq \emptyset$ implies $L(f; t)$ is a UP-ideal of A,*

(2) *f is an anti-fuzzy UP-ideal of A if and only if it satisfies the condition ($\star$): for all $t \in [0,1]$, $L^-(f;t) \neq \emptyset$ implies $L^-(f;t)$ is a UP-ideal of A,*

(3) *$\overline{f}$ is an anti-fuzzy UP-ideal of A if and only if it satisfies the condition ($\star$): for all $t \in [0,1]$, $U(f;t) \neq \emptyset$ implies $U(f;t)$ is a UP-ideal of A, and*

(4) *$\overline{f}$ is an anti-fuzzy UP-ideal of A if and only if it satisfies the condition ($\star$): for all $t \in [0,1]$, $U^+(f;t) \neq \emptyset$ implies $U^+(f;t)$ is a UP-ideal of A.*

Proof. (1) Assume that f is an anti-fuzzy UP-ideal of A. Let $t \in [0,1]$ be such that $L(f;t) \neq \emptyset$. Let $x \in A$ be such that $x \in L(f;t)$. Then $f(x) \leq t$. Thus

$$\begin{aligned} f(0) &= f(x \cdot 0) && \text{(UP-3)} \\ &\leq \max\{f(x \cdot (x \cdot 0)), f(x)\} && \text{(Definition 1.13(2))} \\ &= \max\{f(x \cdot 0), f(x)\} && \text{(UP-3)} \\ &= \max\{f(0), f(x)\} && \text{(UP-3)} \\ &= f(x) && \text{(Definition 1.13(1))} \\ &\leq t. \end{aligned}$$

Thus $0 \in L(f;t)$. Now, let $x, y, z \in A$ be such that $x \cdot (y \cdot z) \in L(f;t)$ and $y \in L(f;t)$. Then $f(x \cdot (y \cdot z)) \leq t$ and $f(y) \leq t$. By Definition 1.13 (2), we have $f(x \cdot z) \leq \max\{f(x \cdot (y \cdot z)), f(y)\} \leq t$, so $x \cdot z \in L(f;t)$. Hence, $L(f;t)$ is a UP-ideal of A.

Conversely, assume that condition ($\star$) holds and suppose that Definition 1.13 (1) is false. Then there exists $x \in A$ such that $f(0) > f(x)$. Let $t = \dfrac{f(x) + f(0)}{2}$. Then $t \in [0,1]$ and by Lemma 2.5, we have $f(0) > t > f(x)$. Thus $x \in L(f;t)$, so $L(f;t) \neq \emptyset$. By assumption, $L(f;t)$ is a UP-ideal of A. Thus $0 \in L(f;t)$, so $f(0) \leq t$. This is a contradiction. Hence, $f(0) \leq f(x)$ for all $x \in A$. Suppose that Definition 1.13 (2) is false. Then there exist $x, y, z \in A$ such that $f(x \cdot z) > \max\{f(x \cdot (y \cdot z)), f(y)\}$. Let $g_0 = \dfrac{f(x \cdot z) + \max\{f(x \cdot (y \cdot z)), f(y)\}}{2}$. Then $g_0 \in [0,1]$ and by Lemma 2.5, we have $\max\{f(x \cdot (y \cdot z)), f(y)\} < g_0 < f(x \cdot z)$. Thus $x \cdot z \notin L(f;g_0)$. Since $\max\{f(x \cdot (y \cdot z)), f(y)\} < g_0$, we have $f(y) < g_0$ and $f(x \cdot (y \cdot z)) < g_0$. Thus $y \in L(f;g_0)$ and $x \cdot (y \cdot z) \in L(f;g_0)$, so $L(f;g_0) \neq \emptyset$. By assumption, we have $L(f;g_0)$ is a UP-ideal of A. Thus $x \cdot z \in L(f;g_0)$. This is a contradiction. Hence, $f(x \cdot z) \leq \max\{f(x \cdot (y \cdot z)), f(y)\}$ for all $x, y, z \in A$. Therefore, f is an anti-fuzzy UP-ideal of A.

(2) Assume that f is an anti-fuzzy UP-ideal of A. Let $t \in [0,1]$ be such that $L^-(f;t) \neq \emptyset$. Let $x \in A$ be such that $x \in L^-(f;t)$. Then $f(x) < t$. Thus

$$\begin{aligned} f(0) &= f(x \cdot 0) && \text{(UP-3)} \\ &\leq \max\{f(x \cdot (x \cdot 0)), f(x)\} && \text{(Definition 1.13(2))} \\ &= \max\{f(x \cdot 0), f(x)\} && \text{(UP-3)} \\ &= \max\{f(0), f(x)\} && \text{(UP-3)} \\ &= f(x) && \text{(Definition 1.13(1))} \\ &< t. \end{aligned}$$

Thus $0 \in L^-(f;t)$. Now, let $x, y, z \in A$ be such that $x \cdot (y \cdot z) \in L^-(f;t)$ and $y \in L^-(f;t)$. Then $f(x \cdot (y \cdot z)) < t$ and $f(y) < t$. By Definition 1.13 (2), we have $f(x \cdot z) \leq \max\{f(x \cdot (y \cdot z)), f(y)\} < t$, so $x \cdot z \in L^-(f;t)$. Hence, $L^-(f;t)$ is a UP-ideal of A.

Conversely, assume that the condition $(\star)$ holds and suppose that Definition 1.13 (1) is false. Then there exists $x \in A$ such that $f(0) > f(x)$. Let $t = \dfrac{f(x) + f(0)}{2}$. Then $t \in [0,1]$ and by Lemma 2.5, we have $f(0) > t > f(x)$. Thus $x \in L^-(f;t)$, so $L^-(f;t) \neq \emptyset$. By assumption, $L^-(f;t)$ is a UP-ideal of A. Thus $0 \in L^-(f;t)$, so $f(0) < t$. This is a contradiction. Hence, $f(0) \leq f(x)$ for all $x \in A$. Suppose that Definition 1.13 (2) is false. Then there exist $x, y, z \in A$ such that $f(x \cdot z) > \max\{f(x \cdot (y \cdot z)), f(y)\}$. Let $g_0 = \dfrac{f(x \cdot z) + \max\{f(x \cdot (y \cdot z)), f(y)\}}{2}$. Then $g_0 \in [0,1]$ and by Lemma 2.5, we have $\max\{f(x \cdot (y \cdot z)), f(y)\} < g_0 < f(x \cdot z)$. Thus $x \cdot z \notin L^-(f;g_0)$. Since $\max\{f(x \cdot (y \cdot z)), f(y)\} < g_0$, we have $f(y) < g_0$ and $f(x \cdot (y \cdot z)) < g_0$. Thus $y \in L^-(f;g_0)$ and $x \cdot (y \cdot z) \in L^-(f;g_0)$, so $L^-(f;g_0) \neq \emptyset$. By assumption, we have $L^-(f;g_0)$ is a UP-ideal of A. Thus $x \cdot z \in L^-(f;g_0)$. This is a contradiction. Hence, $f(x \cdot z) \leq \max\{f(x \cdot (y \cdot z)), f(y)\}$ for all $x, y, z \in A$. Therefore, f is an anti-fuzzy UP-ideal of A.

(3) Assume that $\overline{f}$ is an anti-fuzzy UP-ideal of A. Let $t \in [0,1]$ be such that $U(f;t) \neq \emptyset$. Let $x \in A$ be such that $x \in U(f;t)$. Then $f(x) \geq t$. Thus

$$\begin{aligned}
\overline{f}(0) &= \overline{f}(x \cdot 0) && \text{(UP-3)}\\
&\leq \max\{\overline{f}(x \cdot (x \cdot 0)), \overline{f}(x)\} && \text{(Definition 1.13 (2))}\\
&= \max\{\overline{f}(x \cdot 0), \overline{f}(x)\} && \text{(UP-3)}\\
&= \max\{\overline{f}(0), \overline{f}(x)\} && \text{(UP-3)}\\
&= \overline{f}(x). && \text{(Definition 1.13 (1))}
\end{aligned}$$

This implies that $1 - f(0) \leq 1 - f(x)$. Hence, $f(0) \geq f(x) \geq t$, so $0 \in U(f;t)$. Now, let $x, y, z \in A$ be such that $x \cdot (y \cdot z) \in U(f;t)$ and $y \in U(f;t)$. Then $f(x \cdot (y \cdot z)) \geq t$ and $f(y) \geq t$. By Definition 1.13 (2), we have $\overline{f}(x \cdot z) \leq \max\{\overline{f}(x \cdot (y \cdot z)), \overline{f}(y)\}$. Thus

$$\begin{aligned}
1 - f(x \cdot z) &\leq \max\{1 - f(x \cdot (y \cdot z)), 1 - f(y)\}\\
&= 1 - \min\{f(x \cdot (y \cdot z)), f(y)\}. && \text{(Lemma 1.29 (2))}
\end{aligned}$$

Thus $f(x \cdot z) \geq \min\{f(x \cdot (y \cdot z)), f(y)\} \geq t$, so $x \cdot z \in U(f;t)$. Hence, $U(f;t)$ is a UP-ideal of A.

Conversely, assume that condition $(\star)$ holds and suppose that Definition 1.13 (1) is false. Then there exists $x \in A$ such that $\overline{f}(0) > \overline{f}(x)$. Thus $1 - f(0) > 1 - f(x)$, so $f(0) < f(x)$. Let $t = \dfrac{f(x) + f(0)}{2}$. Then $t \in [0,1]$ and by Lemma 2.5, we have $f(0) < t < f(x)$. Thus $x \in U(f;t)$, so $U(f;t) \neq \emptyset$. By assumption, $U(f;t)$ is a UP-ideal of A. Thus $0 \in U(f;t)$, so $f(0) \geq t$. This is a contradiction. Hence, $\overline{f}(0) \leq \overline{f}(x)$ for all $x \in A$. Suppose that Definition 1.13 (2) is false. Then

there exist $x, y, z \in A$ such that $\overline{f}(x \cdot z) > \max\{\overline{f}(x \cdot (y \cdot z)), \overline{f}(y)\}$. Thus

$$\begin{aligned} 1 - f(x \cdot z) &> \max\{1 - f(x \cdot (y \cdot z)), 1 - f(y)\} \\ &= 1 - \min\{f(x \cdot (y \cdot z)), f(y)\}. \qquad \text{(Lemma 1.29 (2))} \end{aligned}$$

Thus $f(x \cdot z) < \min\{f(x \cdot (y \cdot z)), f(y)\}$. Let $g_0 = \dfrac{f(x \cdot z) + \min\{f(x \cdot (y \cdot z)), f(y)\}}{2}$. Then $g_0 \in [0, 1]$ and by Lemma 2.5, we have $f(x \cdot z) < g_0 < \min\{f(x \cdot (y \cdot z)), f(y)\}$. Thus $x \cdot z \notin U(f; g_0)$. Since $\min\{f(x \cdot (y \cdot z)), f(y)\} > g_0$, we have $f(y) > g_0$ and $f(x \cdot (y \cdot z)) > g_0$. Thus $y \in U(f; g_0)$ and $x \cdot (y \cdot z) \in U(f; g_0)$, so $U(f; g_0) \neq \emptyset$. By assumption, we have $U(f; g_0)$ is a UP-ideal of A. Thus $x \cdot z \in U(f; g_0)$. This is a contradiction. Hence, $\overline{f}(x \cdot z) \leq \max\{\overline{f}(x \cdot (y \cdot z)), \overline{f}(y)\}$ for all $x, y, z \in A$. Therefore, $\overline{f}$ is an anti-fuzzy UP-ideal of A.

(4) Assume that $\overline{f}$ is an anti-fuzzy UP-ideal of A. Let $t \in [0, 1]$ be such that $U^+(f; t) \neq \emptyset$. Let $x \in A$ be such that $x \in U^+(f; t)$. Then $f(x) > t$. Thus

$$\begin{aligned} \overline{f}(0) &= \overline{f}(x \cdot 0) && \text{(UP-3)} \\ &\leq \max\{\overline{f}(x \cdot (x \cdot 0)), \overline{f}(x)\} && \text{(Definition 1.13 (2))} \\ &= \max\{\overline{f}(x \cdot 0), \overline{f}(x)\} && \text{(UP-3)} \\ &= \max\{\overline{f}(0), \overline{f}(x)\} && \text{(UP-3)} \\ &= \overline{f}(x). && \text{(Definition 1.13 (1))} \end{aligned}$$

This implies that $1 - f(0) \leq 1 - f(x)$. Hence, $f(0) \geq f(x) > t$, so $0 \in U^+(f; t)$. Now, let $x, y, z \in A$ be such that $x \cdot (y \cdot z) \in U^+(f; t)$ and $y \in U^+(f; t)$. Then $f(x \cdot (y \cdot z)) > t$ and $f(y) > t$. By Definition 1.13 (2), we have $\overline{f}(x \cdot z) \leq \max\{\overline{f}(x \cdot (y \cdot z)), \overline{f}(y)\}$. Thus

$$\begin{aligned} 1 - f(x \cdot z) &\leq \max\{1 - f(x \cdot (y \cdot z)), 1 - f(y)\} \\ &= 1 - \min\{f(x \cdot (y \cdot z)), f(y)\}. \qquad \text{(Lemma 1.29 (2))} \end{aligned}$$

Thus $f(x \cdot z) \geq \min\{f(x \cdot (y \cdot z)), f(y)\} > t$, so $x \cdot z \in U^+(f; t)$. Hence, $U^+(f; t)$ is a UP-ideal of A.

Conversely, assume that condition $(\star)$ holds and suppose that Definition 1.13 (1) is false. Then there exists $x \in A$ such that $\overline{f}(0) > \overline{f}(x)$. Thus $1 - f(0) > 1 - f(x)$, so $f(0) < f(x)$. Let $t = \dfrac{f(x) + f(0)}{2}$. Then $t \in [0, 1]$ and by Lemma 2.5, we have $f(0) < t < f(x)$. Thus $x \in U^+(f; t)$, so $U^+(f; t) \neq \emptyset$. By assumption, $U^+(f; t)$ is a UP-ideal of A. Thus $0 \in U^+(f; t)$, so $f(0) > t$. This is a contradiction. Hence, $\overline{f}(0) \leq \overline{f}(x)$ for all $x \in A$. Suppose that Definition 1.13 (2) is false. Then there exist $x, y, z \in A$ such that $\overline{f}(x \cdot z) > \max\{\overline{f}(x \cdot (y \cdot z)), \overline{f}(y)\}$. Thus

$$\begin{aligned} 1 - f(x \cdot z) &> \max\{1 - f(x \cdot (y \cdot z)), 1 - f(y)\} \\ &= 1 - \min\{f(x \cdot (y \cdot z)), f(y)\}. \qquad \text{(Lemma 1.29 (2))} \end{aligned}$$

Thus $f(x \cdot z) < \min\{f(x \cdot (y \cdot z)), f(y)\}$. Let $g_0 = \dfrac{f(x \cdot z) + \min\{f(x \cdot (y \cdot z)), f(y)\}}{2}$. Then $g_0 \in [0, 1]$ and by Lemma 2.5, we have $f(x \cdot z) < g_0 < \min\{f(x \cdot (y \cdot z)), f(y)\}$.

Then $x \cdot z \notin U^+(f; g_0)$. Since $\min\{f(x \cdot (y \cdot z)), f(y)\} > g_0$, we have $f(y) > g_0$ and $f(x \cdot (y \cdot z)) > g_0$. Thus $y \in U^+(f; g_0)$ and $x \cdot (y \cdot z) \in U^+(f; g_0)$, so $U^+(f; g_0) \neq \emptyset$. By assumption, we have $U^+(f; g_0)$ is a UP-ideal of A. Thus $x \cdot z \in U^+(f; g_0)$. This is a contradiction. Hence, $\overline{f}(x \cdot z) \leq \max\{\overline{f}(x \cdot (y \cdot z)), \overline{f}(y)\}$ for all $x, y, z \in A$. Therefore, $\overline{f}$ is an anti-fuzzy UP-ideal of A. ∎

Theorem 2.7. *Let f be a fuzzy set in A. Then the following statements hold:*

(1) *f is an anti-fuzzy UP-subalgebra of A if and only if it satisfies the condition $(\star)$: for all $t \in [0, 1]$, $L(f; t) \neq \emptyset$ implies $L(f; t)$ is a UP-subalgebra of A,*

(2) *f is an anti-fuzzy UP-subalgebra of A if and only if it satisfies the condition $(\star)$: for all $t \in [0, 1]$, $L^-(f; t) \neq \emptyset$ implies $L^-(f; t)$ is a UP-subalgebra of A,*

(3) *$\overline{f}$ is an anti-fuzzy UP-subalgebra of A if and only if it satisfies the condition $(\star)$: for all $t \in [0, 1]$, $U(f; t) \neq \emptyset$ implies $U(f; t)$ is a UP-subalgebra of A, and*

(4) *$\overline{f}$ is an anti-fuzzy UP-subalgebra of A if and only if it satisfies the condition $(\star)$: for all $t \in [0, 1]$, $U^+(f; t) \neq \emptyset$ implies $U^+(f; t)$ is a UP-subalgebra of A.*

Proof. (1) Assume that f is an anti-fuzzy UP-subalgebra of A. Let $t \in [0, 1]$ be such that $L(f; t) \neq \emptyset$. Let $x, y \in L(f; t)$. Then $f(x) \leq t$ and $f(y) \leq t$. Since f is an anti-fuzzy UP-subalgebra of A, we have $f(x \cdot y) \leq \max\{f(x), f(y)\} \leq t$. Thus $x \cdot y \in L(f; t)$. Hence, $L(f; t)$ is a UP-subalgebra of A.

Conversely, assume that the condition $(\star)$ holds. Let $x, y \in A$ and let $t = \max\{f(x), f(y)\}$, so $t \in [0, 1]$. Thus $f(x) \leq t$ and $f(y) \leq t$, so $x, y \in L(f; t)$. Thus $L(f; t) \neq \emptyset$. By assumption, we have $L(f; t)$ is a UP-subalgebra of A. Thus $x \cdot y \in L(f; t)$, so $f(x \cdot y) \leq t = \max\{f(x), f(y)\}$. Hence, f is an anti-fuzzy UP-subalgebra of A.

(2) Assume that f is an anti-fuzzy UP-subalgebra of A. Let $t \in [0, 1]$ be such that $L^-(f; t) \neq \emptyset$. Let $x, y \in L^-(f; t)$. Then $f(x) < t$ and $f(y) < t$. Since f is an anti-fuzzy UP-subalgebra of A, we have $f(x \cdot y) \leq \max\{f(x), f(y)\} < t$. Thus $x \cdot y \in L^-(f; t)$. Hence, $L^-(f; t)$ is a UP-subalgebra of A.

Conversely, assume that the condition $(\star)$ holds. Suppose that there exist $x, y \in A$ such that $f(x \cdot y) > \max\{f(x), f(y)\}$. Then $f(x \cdot y) \in [0, 1]$. Choose $t = f(x \cdot y)$. Thus $f(x) < t$ and $f(y) < t$, so $x, y \in L^-(f; t) \neq \emptyset$. By assumption, we have $L^-(f; t)$ is a UP-subalgebra of A and so $x \cdot y \in L^-(f; t)$. Thus $f(x \cdot y) < t = f(x \cdot y)$ which is a contradiction. Hence, $f(x \cdot y) \leq \max\{f(x), f(y)\}$ for all $x, y \in A$. Therefore, f is an anti-fuzzy UP-subalgebra of A.

(3) Assume that $\overline{f}$ is an anti-fuzzy UP-subalgebra of A. Let $t \in [0, 1]$ be such that $U(f; t) \neq \emptyset$. Let $x, y \in U(f; t)$. Then $f(x) \geq t$ and $f(y) \geq t$. Since $\overline{f}$ is an anti-fuzzy UP-subalgebra of A, we have $\overline{f}(x \cdot y) \leq \max\{\overline{f}(x), \overline{f}(y)\}$. Thus

$$\begin{aligned} 1 - f(x \cdot y) &\leq \max\{1 - f(x), 1 - f(y)\} \\ &= 1 - \min\{f(x), f(y)\}. \qquad \text{(Lemma 1.29(2))} \end{aligned}$$

Thus $f(x \cdot y) \geq \min\{f(x), f(y)\} \geq t$. Thus $x \cdot y \in U(f;t)$. Hence, $U(f;t)$ is a UP-subalgebra of A.

Conversely, assume that the condition $(\star)$ holds. Let $x, y \in A$ and let $t = \min\{f(x), f(y)\}$, so $t \in [0,1]$. Thus $f(x) \geq t$ and $f(y) \geq t$, so $x, y \in U(f;t)$. Thus $U(f;t) \neq \emptyset$. By assumption, we have $U(f;t)$ is a UP-subalgebra of A. Thus $x \cdot y \in U(f;t)$, so $f(x \cdot y) \geq t = \min\{f(x), f(y)\}$. Thus $1 - f(x \cdot y) \leq 1 - \min\{f(x), f(y)\}$, so

$$\begin{aligned} \overline{f}(x \cdot y) &= 1 - f(x \cdot y) \\ &\leq 1 - \min\{f(x), f(y)\} \\ &= \max\{1 - f(x), 1 - f(y)\} && \text{(Lemma 1.29(2))} \\ &= \max\{\overline{f}(x), \overline{f}(y)\}. \end{aligned}$$

Hence, $\overline{f}$ is an anti-fuzzy UP-subalgebra of A.

(4) Assume that $\overline{f}$ is an anti-fuzzy UP-subalgebra of A. Let $t \in [0,1]$ be such that $U^+(f;t) \neq \emptyset$. Let $x, y \in U^+(f;t)$. Then $f(x) > t$ and $f(y) > t$. Since $\overline{f}$ is an anti-fuzzy UP-subalgebra of A, we have $\overline{f}(x \cdot y) \leq \max\{\overline{f}(x), \overline{f}(y)\}$. Thus

$$\begin{aligned} 1 - f(x \cdot y) &\leq \max\{1 - f(x), 1 - f(y)\} \\ &= 1 - \min\{f(x), f(y)\}. && \text{(Lemma 1.29 (2))} \end{aligned}$$

Thus $f(x \cdot y) \geq \min\{f(x), f(y)\} > t$, so $x \cdot y \in U^+(f;t)$. Hence, $U^+(f;t)$ is a UP-subalgebra of A.

Conversely, assume that the condition $(\star)$ holds. Suppose that there exist $x, y \in A$ such that $\overline{f}(x \cdot y) > \max\{\overline{f}(x), \overline{f}(y)\}$. Thus

$$\begin{aligned} 1 - f(x \cdot y) &> \max\{1 - f(x), 1 - f(y)\} \\ &= 1 - \min\{f(x), f(y)\}. && \text{(Lemma 1.29(2))} \end{aligned}$$

Thus $f(x \cdot y) < \min\{f(x), f(y)\}$. Now $f(x \cdot y) \in [0,1]$, we choose $t = f(x \cdot y)$. Thus $f(x) > t$ and $f(y) > t$, so $x, y \in U^+(f;t) \neq \emptyset$. By assumption, we have $U^+(f;t)$ is a UP-subalgebra of A and so $x \cdot y \in U^+(f;t)$. Thus $f(x \cdot y) > t = f(x \cdot y)$ which is a contradiction. Hence, $\overline{f}(x \cdot y) \leq \max\{\overline{f}(x), \overline{f}(y)\}$ for all $x, y \in A$. Therefore, $\overline{f}$ is an anti-fuzzy UP-subalgebra of A. ∎

Proposition 2.8. *If f is an anti-fuzzy UP-ideal of A, then $f(x \cdot (x \cdot y)) \leq f(y)$ for all $x, y \in A$.*

Proof. Let $x, y \in A$. Then

$$\begin{aligned} f(x \cdot (x \cdot y)) &\leq \max\{f(x \cdot ((x \cdot y) \cdot (x \cdot y))), f(x \cdot y)\} && \text{(Definition 1.13(2))} \\ &= \max\{f(x \cdot 0), f(x \cdot y)\} && \text{(Proposition 1.4(1))} \\ &= \max\{f(0), f(x \cdot y)\} && \text{(UP-3)} \\ &= f(x \cdot y) && \text{(Definition 1.13 (1))} \\ &\leq \max\{f(x \cdot (y \cdot y)), f(y)\} && \text{(Definition 1.13 (2))} \\ &= \max\{f(x \cdot 0), f(y)\} && \text{(Proposition 1.4 (1))} \\ &= \max\{f(0), f(y)\} && \text{(UP-3)} \\ &= f(y). && \text{(Definition 1.13 (1))} \end{aligned}$$

Corollary 2.9. *Let f be a fuzzy set in A. Then the following statements hold:*

(1) *if f is an anti-fuzzy UP-ideal of A, then for every $t \in \mathrm{Im}(f)$, $L(f;t)$ is a UP-ideal of A, and*

(2) *if $\overline{f}$ is an anti-fuzzy UP-ideal of A, then for every $t \in \mathrm{Im}(f)$, $U(f;t)$ is a UP-ideal of A.*

Proof. (1) Assume that f is an anti-fuzzy UP-ideal of A and let $t \in \mathrm{Im}(f)$. Then $t = f(x)$ for some $x \in A$, so $f(x) \leq t$. Thus $x \in L(f;t)$, so $L(f;t) \neq \emptyset$. By Theorem 2.6 (1), we have $L(f;t)$ is a UP-ideal of A.

(2) Assume that $\overline{f}$ is an anti-fuzzy UP-ideal of A and let $t \in \mathrm{Im}(f)$. Then $t = f(x)$ for some $x \in A$, so $f(x) \geq t$. Thus $x \in U(f;t)$, so $U(f;t) \neq \emptyset$. By Theorem 2.6 (3), we have $U(f;t)$ is a UP-ideal of A. ∎

Corollary 2.10. *Let f be a fuzzy set in A. Then the following statements hold:*

(1) *if f is an anti-fuzzy UP-subalgebra of A, then for every $t \in \mathrm{Im}(f)$, $L(f;t)$ is a UP-subalgebra of A, and*

(2) *if $\overline{f}$ is an anti-fuzzy UP-subalgebra of A, then for every $t \in \mathrm{Im}(f)$, $U(f;t)$ is a UP-subalgebra of A.*

Proof. (1) Assume that f is an anti-fuzzy UP-subalgebra of A and let $t \in \mathrm{Im}(f)$. Then $t = f(x)$ for some $x \in A$, so $f(x) \leq t$. Thus $x \in L(f;t)$, so $L(f;t) \neq \emptyset$. By Theorem 2.7 (1), we have $L(f;t)$ is a UP-subalgebra of A.

(2) Assume that $\overline{f}$ is an anti-fuzzy UP-subalgebra of A and let $t \in \mathrm{Im}(f)$. Then $t = f(x)$ for some $x \in A$, so $f(x) \geq t$. Thus $x \in U(f;t)$, so $U(f;t) \neq \emptyset$. By Theorem 2.7 (3), we have $U(f;t)$ is a UP-subalgebra of A. ∎

Corollary 2.11. *Let I be a UP-ideal of A. Then the following statements hold:*

(1) *for any $k \in (0,1]$, there exists an anti-fuzzy UP-ideal f of A such that $L(f;t) = I$ for all $t < k$ and $L(f;t) = A$ for all $t \geq k$, and*

(2) *for any $k \in [0,1)$, there exists an anti-fuzzy UP-ideal g of A such that $U(\overline{g};t) = I$ for all $t > k$ and $U(\overline{g};t) = A$ for all $t \leq k$.*

Proof. (1) Define a fuzzy set $f\colon A \to [0,1]$ by

$$f(x) = \begin{cases} 0 & \text{if } x \in I, \\ k & \text{if } x \notin I. \end{cases}$$

Case 1: To show that $L(f;t) = I$ for all $t < k$, let $t \in [0,1]$ be such that $t < k$. Let $x \in L(f;t)$. Then $f(x) \leq t < k$, so $f(x) \neq k$. Thus $f(x) = 0$, so $x \in I$. That is $L(f;t) \subseteq I$. Let $x \in I$. Then $f(x) = 0 \leq t$, so $x \in L(f;t)$. That is $I \subseteq L(f;t)$. Hence, $L(f;t) = I$ for all $t < k$.

Case 2: To show that $L(f;t) = A$ for all $t \geq k$, let $t \in [0,1]$ be such that $t \geq k$. Clearly, $L(f;t) \subseteq A$. Let $x \in A$. Then

$$f(x) = \begin{cases} 0 < t & \text{if } x \in I, \\ k \leq t & \text{if } x \notin I, \end{cases}$$

so $x \in L(f;t)$. That is $A \subseteq L(f;t)$. Hence, $L(f;t) = A$ for all $t \geq k$.

It follows from Theorem 2.6 (1) that f is an anti-fuzzy UP-ideal of A.

(2) Define a fuzzy set $f\colon A \to [0,1]$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in I, \\ k & \text{if } x \notin I. \end{cases}$$

Case 1: To show that $U(f;t) = I$ for all $t > k$, let $t \in [0,1]$ be such that $t > k$. Let $x \in U(f;t)$. Then $f(x) \geq t > k$, so $f(x) \neq k$. Thus $f(x) = 1$, so $x \in I$. That is $U(f;t) \subseteq I$. Let $x \in I$. Then $f(x) = 1 \geq t$, so $x \in U(f;t)$. That is $I \subseteq U(f;t)$. Hence, $U(f;t) = I$ for all $t > k$.

Case 2: To show that $U(f;t) = A$ for all $t \leq k$, let $t \in [0,1]$ be such that $t \leq k$. Clearly, $U(f;t) \subseteq A$. Let $x \in A$. Then

$$f(x) = \begin{cases} 1 > t & \text{if } x \in I, \\ k \geq t & \text{if } x \notin I, \end{cases}$$

so $x \in U(f;t)$. That is $A \subseteq U(f;t)$. Hence, $U(f;t) = A$ for all $t \leq k$.

It follows from Theorem 2.6 (3) that $\overline{f}$ is an anti-fuzzy UP-ideal of A. By Remark 1.18, we have $U(\overline{\overline{f}};t) = U(f;t) = I$ for all $t > k$ and $U(\overline{\overline{f}};t) = U(f;t) = A$ for all $t \leq k$. Putting $\overline{f} = g$. Then g is an anti-fuzzy UP-ideal of A such that $U(\overline{g};t) = I$ for all $t > k$ and $U(\overline{g};t) = A$ for all $t \leq k$. ∎

Corollary 2.12. *Let I be a UP-subalgebra of A. Then the following statements hold:*

(1) *for any $k \in (0,1]$, there exists an anti-fuzzy UP-subalgebra f of A such that $L(f;t) = I$ for all $t < k$ and $L(f;t) = A$ for all $t \geq k$, and*

(2) *for any $k \in [0,1)$, there exists an anti-fuzzy UP-subalgebra g of A such that $U(\overline{g};t) = I$ for all $t > k$ and $U(\overline{g};t) = A$ for all $t \leq k$.*

Proof. (1) Define a fuzzy set $f\colon A \to [0,1]$ by

$$f(x) = \begin{cases} 0 & \text{if } x \in I, \\ k & \text{if } x \notin I. \end{cases}$$

In the proof of Corollary 2.11 (1), we have $L(f;t) = I$ for all $t < k$ and $L(f;t) = A$ for all $t \geq k$.

It follows from Theorem 2.7 (1) that f is an anti-fuzzy UP-subalgebra of A.

(2) Define a fuzzy set $f\colon A \to [0,1]$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in I, \\ k & \text{if } x \notin I. \end{cases}$$

In the proof of Corollary 2.11 (2), we have $U(f;t) = I$ for all $t > k$ and $U(f;t) = A$ for all $t \leq k$.

It follows from Theorem 2.7 (3) that $\overline{f}$ is an anti-fuzzy UP-subalgebra of A. By Remark 1.18, we have $U(\overline{\overline{f}};t) = U(f;t) = I$ for all $t > k$ and $U(\overline{\overline{f}};t) = U(f;t) = A$ for all $t \leq k$. Putting $\overline{f} = g$. Then g is an anti-fuzzy UP-subalgebra of A such that $U(\overline{g};t) = I$ for all $t > k$ and $U(\overline{g};t) = A$ for all $t \leq k$. ∎

Theorem 2.13. *Let $(A;\cdot,0_A)$ and $(B;*,0_B)$ be UP-algebras and let $f\colon A \to B$ be a UP-epimorphism. Then the following statements hold:*

(1) *for every anti-fuzzy UP-ideal β of B, μ is an anti-fuzzy UP-ideal of A, and*

(2) *for every anti-fuzzy UP-subalgebra β of B, μ is an anti-fuzzy UP-subalgebra of A.*

Proof. (1) Let β be an anti-fuzzy UP-ideal of B. Let $x \in A$. Then

$$\begin{aligned}
\mu(0_A) &= (\beta \circ f)(0_A) \\
&= \beta(f(0_A)) \\
&\leq \beta(f(x)) && \text{(Proposition 1.6 (3), Theorem 1.28 (2), Lemma 2.3)} \\
&= (\beta \circ f)(x) \\
&= \mu(x).
\end{aligned}$$

Let $x, y, z \in A$. Then

$$\begin{aligned}
\mu(x \cdot z) &= (\beta \circ f)(x \cdot z) \\
&= \beta(f(x \cdot z)) \\
&= \beta(f(x) * f(z)) \\
&\leq \max\{\beta(f(x) * (f(y) * f(z))), \beta(f(y))\} && \text{(Definition 1.13 (2))} \\
&= \max\{\beta(f(x \cdot (y \cdot z))), \beta(f(y))\} \\
&= \max\{(\beta \circ f)(x \cdot (y \cdot z)), (\beta \circ f)(y)\} \\
&= \max\{\mu(x \cdot (y \cdot z)), \mu(y)\}.
\end{aligned}$$

Hence, μ is an anti-fuzzy UP-ideal of A.

(2) Let β be an anti-fuzzy UP-subalgebra of B. Let $x, y \in A$. Then

$$\begin{aligned}
\mu(x \cdot y) &= (\beta \circ f)(x \cdot y) \\
&= \beta(f(x \cdot y)) \\
&= \beta(f(x) * f(y)) \\
&\leq \max\{\beta(f(x)), \beta(f(y))\} && \text{(Definition 1.14)} \\
&= \max\{(\beta \circ f)(x), (\beta \circ f)(y)\} \\
&= \max\{\mu(x), \mu(y)\}.
\end{aligned}$$

Hence, μ is an anti-fuzzy UP-subalgebra of A. ∎

Lemma 2.14. *Let $(A;\cdot,0_A)$ and $(B;*,0_B)$ be UP-algebras and let $f\colon A\to B$ be a UP-epimorphism. Let μ be an f-invariant fuzzy set in A with* inf *property. For any $a,b\in B$, there exist $a_0\in f^{-1}(a)$ and $b_0\in f^{-1}(b)$ such that $\beta(a)=\mu(a_0)$, $\beta(b)=\mu(b_0)$ and $\beta(a\cdot b)=\mu(a_0\cdot b_0)$.*

Proof. Let $a,b\in B$. Since f is surjective, we have $f^{-1}(a), f^{-1}(b)$ and $f^{-1}(a*b)$ are nonempty subsets of A. By Definition 1.20, we obtain

$$\begin{aligned}
\beta(a) &= \inf\{\mu(t)\}_{t\in f^{-1}(a)} \\
&= \mu(a_0) \text{ for some } a_0\in f^{-1}(a), && \text{(Definition 1.21)}\\
\beta(b) &= \inf\{\mu(t)\}_{t\in f^{-1}(b)} \\
&= \mu(b_0) \text{ for some } b_0\in f^{-1}(b) && \text{(Definition 1.21)}\\
\beta(a\cdot b) &= \inf\{\mu(t)\}_{t\in f^{-1}(a\cdot b)} \\
&= \mu(c) \text{ for some } c\in f^{-1}(a*b). && \text{(Definition 1.21)}
\end{aligned}$$

Since $f(c)=a*b=f(a_0)*f(b_0)=f(a_0\cdot b_0)$ and μ is f-invariant, we have $\mu(c)=\mu(a_0\cdot b_0)$. Hence, $\beta(a\cdot b)=\mu(a_0\cdot b_0)$. ∎

Theorem 2.15. *Let $(A;\cdot,0_A)$ and $(B;*,0_B)$ be UP-algebras and let $f\colon A\to B$ be a UP-epimorphism. Then the following statements hold:*

(1) *for every f-invariant anti-fuzzy UP-ideal μ of A with* inf *property, β is an anti-fuzzy UP-ideal of B, and*

(2) *for every f-invariant anti-fuzzy UP-subalgebra μ of A with* inf *property, β is an anti-fuzzy UP-subalgebra of B.*

Proof. (1) Let μ be an f-invariant anti-fuzzy UP-ideal of A with inf property. By Definition 1.13 (1), we have $\mu(0_A)\le\mu(x)$ for all $x\in A$. By Theorem 1.28 (1), we have $0_A\in f^{-1}(0_B)$ and so $f^{-1}(0_B)\neq\emptyset$. Thus $\beta(0_B)=\inf\{\mu(t)\}_{t\in f^{-1}(0_B)}\le\mu(0_A)$. Let $y\in B$. Since f is surjective, we have $f^{-1}(y)\neq\emptyset$. By Definition 1.13 (1), we have $\mu(0_A)\le\mu(t)$ for all $t\in f^{-1}(y)$. Thus $\mu(0_A)$ is a lower bound of $\{\mu(t)\}_{t\in f^{-1}(y)}$, so $\mu(0_A)\le\inf\{\mu(t)\}_{t\in f^{-1}(y)}=\beta(y)$. By Proposition 1.5 (3), we have $\beta(0_B)\le\beta(y)$. Let $a,b,c\in B$. By Lemma 2.14, there exist $a_0\in f^{-1}(a), b_0\in f^{-1}(b)$ and $c_0\in f^{-1}(c)$ such that $\beta(b)=\mu(b_0), \beta(a*c)=\mu(a_0\cdot c_0)$ and $\beta(a*(b*c))=\mu(a_0\cdot(b_0\cdot c_0))$. Thus

$$\begin{aligned}
\beta(a*c) &= \mu(a_0\cdot c_0) \\
&\le \max\{\mu(a_0\cdot(b_0\cdot c_0)),\mu(b_0)\} && \text{(Definition 1.13 (2))}\\
&= \max\{\beta(a*(b*c)),\beta(b)\}.
\end{aligned}$$

Hence, β is an anti-fuzzy UP-ideal of B.

(2) Let μ be an f-invariant anti-fuzzy UP-subalgebra of A with inf property. Let $a,b\in B$. Since f is surjective, we have $f^{-1}(a), f^{-1}(b)$ and $f^{-1}(a*b)$ are nonempty subsets of A. By Lemma 2.14, there exist $a_0\in f^{-1}(a), b_0\in f^{-1}(b)$ such that $\beta(a)=\mu(a_0), \beta(b)=\mu(b_0)$ and $\beta(a*b)=\mu(a_0\cdot b_0)$. Thus

$$\begin{aligned}\beta(a*b) &= \mu(a_0 \cdot b_0) \\ &\leq \max\{\mu(a_0), \mu(b_0)\} && \text{(Definition 1.14)} \\ &= \max\{\beta(a), \beta(b)\}.\end{aligned}$$

Hence, β is an anti-fuzzy UP-subalgebra of B. ∎

Remark 2.16. [26] Let $(A; \cdot, 0_A)$ and $(B; *, 0_B)$ be UP-algebras. Then $A \times B$ is a UP-algebra defined by

$$(x_1, x_2) \diamond (y_1, y_2) = (x_1 \cdot y_1, x_2 * y_2)$$

for all $x_1, y_1 \in A$ and $x_2, y_2 \in B$.

Theorem 2.17. *Let f be a fuzzy set in A. Then the following statements hold:*

(1) *if μ_f is an anti-fuzzy UP-ideal of $A \times A$, then f is an anti-fuzzy UP-ideal of A, and*

(2) *if β_f is an anti-fuzzy UP-ideal of $A \times A$, then f is an anti-fuzzy UP-ideal of A.*

Proof. (1) Assume that μ_f is an anti-fuzzy UP-ideal of $A \times A$. Let $x \in A$. Then

$$\begin{aligned}f(0) &= \max\{f(0), f(0)\} \\ &= \mu_f(0,0) \\ &\leq \mu_f(x,x) && \text{(Definition 1.13 (1))} \\ &= \max\{f(x), f(x)\} \\ &= f(x).\end{aligned}$$

Let $x, y, z \in A$. Then

$$\begin{aligned}f(x \cdot z) &= \max\{f(x \cdot z), f(x \cdot z)\} \\ &= \mu_f(x \cdot z, x \cdot z) \\ &= \mu_f((x,x) \diamond (z,z)) \\ &\leq \max\{\mu_f((x,x) \diamond ((y,y) \diamond (z,z))), \mu_f(y,y)\} && \text{(Definition 1.13 (2))} \\ &= \max\{\mu_f(x \cdot (y \cdot z), x \cdot (y \cdot z)), \mu_f(y,y)\} \\ &= \max\{\max\{f(x \cdot (y \cdot z)), f(x \cdot (y \cdot z))\}, \max\{f(y), f(y)\} \\ &= \max\{f(x \cdot (y \cdot z)), f(y)\}.\end{aligned}$$

Hence, f is an anti-fuzzy UP-ideal of A.

(2) Similarly to as in the proof of (1). ∎

Proposition 2.18. *Let f be a fuzzy set in A. Then the following statements hold:*

(1) *if μ_f is an anti-fuzzy UP-subalgebra of $A \times A$, then f is an anti-fuzzy UP-subalgebra of A, and*

(2) *if β_f is an anti-fuzzy UP-subalgebra of $A \times A$, then f is an anti-fuzzy UP-subalgebra of A.*

Proof. (1) Assume that μ_f is an anti-fuzzy UP-subalgebra of $A \times A$. Let $x, y \in A$. Then

$$\begin{aligned} f(x \cdot z) &= \max\{f(x \cdot z), f(x \cdot z)\} \\ &= \mu_f(x \cdot z, x \cdot z) \\ &= \mu_f((x, x) \diamond (z, z)) \\ &\leq \max\{\mu_f((x, x), \mu_f(y, y)\} && \text{(Definition 1.14)} \\ &\leq \max\{\max\{f(x), f(x)\}, \max\{f(y), f(y)\}\} \\ &= \max\{f(x), f(y)\}. \end{aligned}$$

Hence, f is an anti-fuzzy UP-subalgebra of A.

(2) Similarly to as in the proof of (1). ∎

Lemma 2.19. [4] *For any $a, b, c, d \in \mathbb{R}$, the following properties hold:*

(1) $\max\{\max\{a, b\}, \max\{c, d\}\} = \max\{\max\{a, c\}, \max\{b, d\}\}$, *and*

(2) $\min\{\min\{a, b\}, \min\{c, d\}\} = \min\{\min\{a, c\}, \min\{b, d\}\}$.

Theorem 2.20. *Let $(A; \cdot, 0_A)$ and $(B; *, 0_B)$ be UP-algebras. Then the following statements hold:*

(1) *if f is an anti-fuzzy UP-ideal of A and g is an anti-fuzzy UP-ideal of B, then $f \times g$ is an anti-fuzzy UP-ideal of $A \times B$, and*

(2) *if f is an anti-fuzzy UP-subalgebra of A and g is an anti-fuzzy UP-subalgebra of B, then $f \times g$ is an anti-fuzzy UP-subalgebra of $A \times B$.*

Proof. (1) Assume that f is an anti-fuzzy UP-ideal of A and g is an anti-fuzzy UP-ideal of B. Let $(x, y) \in A \times B$. Then

$$\begin{aligned} (f \times g)(0, 0) &= \max\{f(0), g(0)\} \\ &\leq \max\{f(x), g(y)\} && \text{(Definition 1.13 (1))} \\ &= (f \times g)(x, y). \end{aligned}$$

Now, let $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in A \times B$. Then

$$\begin{aligned} &(f \times g)((x_1, x_2) \diamond (z_1, z_2)) \\ &\quad = (f \times g)(x_1 \cdot z_1, x_2 * z_2) \\ &\quad = \max\{f(x_1 \cdot z_1), g(x_2 * z_2)\} \\ &\quad \leq \max\{\max\{f(x_1 \cdot (y_1 \cdot z_1)), f(y_1)\}, \\ &\qquad \max\{g(x_2 * (y_2 * z_2)), g(y_2)\}\} && \text{(Definition 1.13 (2))} \\ &\quad = \max\{\max\{f(x_1 \cdot (y_1 \cdot z_1)), \{g(x_2 * (y_2 * z_2))\}\}, \\ &\qquad \max\{f(y_1), g(y_2)\}\} && \text{(Lemma 2.19 (1))} \\ &\quad = \max\{(f \times g)(x_1 \cdot (y_1 \cdot z_1), x_2 * (y_2 * z_2)), (f \times g)(y_1, y_2)\} \\ &\quad = \max\{(f \times g)((x_1, x_2) \diamond ((y_1, y_2) \diamond (z_1, z_2))), (f \times g)(y_1, y_2)\}. \end{aligned}$$

Hence, $f \times g$ is an anti-fuzzy UP-ideal of $A \times B$.

(2) Let $(x_1, x_2), (y_1, y_2) \in A \times B$. Then

$$\begin{aligned}
&(f \times g)((x_1, x_2) \diamond (y_1, y_2)) \\
&\quad = (f \times g)(x_1 \cdot y_1, x_2 * y_2) \\
&\quad = \max\{f(x_1 \cdot y_1), g(x_2 * y_2)\} \\
&\quad \leq \max\{\max\{f(x_1), f(y_1)\}, \max\{g(x_2), g(y_2)\}\} && \text{(Definition 1.14)} \\
&\quad = \max\{\max\{f(x_1), g(x_2)\}, \max\{f(y_1), g(y_2)\}\} && \text{(Lemma 2.19 (1))} \\
&\quad = \max\{(f \times g)(x_1, x_2), (f \times g)(y_1, y_2)\}.
\end{aligned}$$

Hence, $f \times g$ is an anti-fuzzy UP-subalgebra of $A \times B$. ∎

Give examples of conflict that f and g are anti-fuzzy UP-ideals (resp. anti-fuzzy UP-subalgebras) of A but $f \cdot g$ is not an anti-fuzzy UP-ideal (resp. anti-fuzzy UP-subalgebra) of $A \times A$.

Example 2.21. Let $A = \{0, 1\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1
0	0	1
1	0	0

Then $(A; \cdot, 0)$ is a UP-algebra. We define a fuzzy set f and g in A as follows:

$$f(0) = 0.1, f(1) = 0.5, g(0) = 0.2 \text{ and } g(1) = 0.5.$$

Using this data, we can show that f and g are anti-fuzzy UP-ideals of A. Let $x_1 = 0, x_2 = 0, y_1 = 1, y_2 = 0, z_1 = 1, z_1 = 1$. Then

$$\begin{gathered}
(f \cdot g)((x_1, x_2) \diamond (z_1, z_2)) = 0.5 \leq 0.2 = \\
\max\{(f \cdot g)((x_1, x_2) \diamond ((y_1, y_2) \diamond (z_1, z_2))), (f \cdot g)(y_1, y_2)\}.
\end{gathered}$$

Thus Definition 1.13 (2) is false. Hence, $f \cdot g$ is not an anti-fuzzy UP-ideal of $A \times A$.

Example 2.22. Let $A = \{0, a, b\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	a	b
0	0	a	b
a	0	0	a
b	0	0	0

Then $(A; \cdot, 0)$ is a UP-algebra. We define a fuzzy set f and g in A as follows:

$$f(0) = 0.1, f(a) = 0.2, f(b) = 0.1, g(0) = 0.1, g(a) = 0.2 \text{ and } g(b) = 0.1.$$

Using this data, we can show that f and g are anti-fuzzy UP-subalgebras of A. Let $x_1 = 0, x_2 = 1, y_1 = 1, y_2 = 2$. Then

$$(f \cdot g)((x_1, x_2) \diamond (y_1, y_2)) = 0.2 \geq 0.1 = \max\{(f \cdot g)(x_1, x_2), (f \cdot g)(y_1, y_2)\}.$$

Thus Definition 1.14 is false. Hence, $f \cdot g$ is not an anti-fuzzy UP-subalgebra of $A \times A$.

Theorem 2.23. *Let $(A; \cdot, 0_A)$ and $(B; *, 0_B)$ be UP-algebras and let f be a fuzzy set in A and g be a fuzzy set in B. If $f \times g$ is an anti-fuzzy UP-ideal of $A \times B$, then the following statements hold:*

(1) *either $f(0_A) \leq f(x)$ for all $x \in A$ or $g(0_B) \leq g(y)$ for all $y \in B$,*

(2) *if $f(0_A) \leq f(x)$ for all $x \in A$, then either $g(0_B) \leq g(y)$ for all $y \in B$ or $g(0_B) \leq f(x)$ for all $x \in A$,*

(3) *if $g(0_B) \leq g(y)$ for all $y \in B$, then either $f(0_A) \leq f(x)$ for all $x \in A$ or $f(0_A) \leq g(y)$ for all $y \in B$, and*

(4) *either f is an anti-fuzzy UP-ideal of A or g is an anti-fuzzy UP-ideal of B.*

Proof. (1) Suppose that $f(0_A) > f(x)$ for some $x \in A$ and $g(0_B) > g(y)$ for some $y \in B$. Then $(f \times g)(x, y) = \max\{f(x), g(y)\} < \max\{f(0_A), g(0_B)\} = (f \times g)(0_A, 0_B)$ which is a contradiction. Hence, either $f(0_A) \leq f(x)$ for all $x \in A$ or $g(0_B) \leq g(y)$ for all $y \in B$

(2) Assume that $f(0_A) \leq f(x)$ for all $x \in A$. Suppose that $g(0_B) > g(y)$ for some $y \in B$ and $g(0_B) > f(x)$ for some $x \in A$. Then $g(0_B) > f(x) \geq f(0_A)$. Thus

$$\begin{aligned}(f \times g)(x, y) &= \max\{f(x), g(y)\} \\ &< \max\{g(0_B), g(0_B)\} \\ &= g(0_B) \\ &= \max\{f(0_A), g(0_B)\} \\ &= (f \times g)(0_A, 0_B)\end{aligned}$$

which is a contradiction. Hence, either $g(0_B) \leq g(y)$ for all $y \in B$ or $g(0_B) \leq f(x)$ for all $x \in A$.

(3) Assume that $g(0_B) \leq g(y)$ for all $y \in B$. Suppose that $f(0_A) > f(x)$ for some $x \in A$ and $f(0_A) > g(y)$ for some $y \in B$. Then $f(0_A) > g(y) \geq g(0_B)$. Thus

$$\begin{aligned}(f \times g)(x, y) &= \max\{f(x), g(y)\} \\ &< \max\{f(0_A), f(0_A)\} \\ &= f(0_A) \\ &= \max\{f(0_A), g(0_B)\} \\ &= (f \times g)(0_A, 0_B)\end{aligned}$$

which is a contradiction. Hence, either $f(0_A) \leq f(x)$ for all $x \in A$ or $f(0_A) \leq g(y)$ for all $y \in B$.

(4) Suppose that f is not an anti-fuzzy UP-ideal of A and g is not an anti-fuzzy UP-ideal of B. By (1), assume that $f(0_A) \leq f(x)$ for all $x \in A$. Then

from (2), either $g(0_B) \leq g(y)$ for all $y \in B$ or $g(0_B) \leq f(x)$ for all $x \in A$. If $g(0_B) \leq f(x)$ for all $x \in A$, then for all $x \in A$,

$$(f \times g)(x, 0_B) = \max\{f(x), g(0_B)\} = f(x). \tag{2.4}$$

Since $f \times g$ is an anti-fuzzy UP-ideal of $A \times B$, we have for any $x, y, z \in A$,

$$\begin{aligned}
f(x \cdot z) &= (f \times g)(x \cdot z, 0_B) && ((2.4)) \\
&= (f \times g)(x \cdot z, 0_B * 0_B) && \text{(UP-3)} \\
&= (f \times g)((x, 0_B) \diamond (z, 0_B)) \\
&\leq \max\{(f \times g)((x, 0_B) \diamond [(y, 0_B) \diamond (z, 0_B)]), \\
&\quad (f \times g)(y, 0_B)\} && \text{(Definition 1.13 (2))} \\
&= \max\{(f \times g)(x \cdot (y \cdot z), 0_B * (0_B * 0_B)), (f \times g)(y, 0_B)\} \\
&= \max\{(f \times g)(x \cdot (y \cdot z), 0_B), (f \times g)(y, 0_B)\} && \text{(UP-3)} \\
&= \max\{\max\{f(x \cdot (y \cdot z)), g(0_B)\}, \max\{f(y), g(0_B)\}\} \\
&= \max\{f(x \cdot (y \cdot z)), f(y)\}.
\end{aligned}$$

Hence, f is an anti-fuzzy UP-ideal of A which is a contradiction. Assume that $g(0_B) \leq g(y)$ for all $y \in B$. Then from 2.23, either $f(0_A) \leq f(x)$ for all $x \in A$ or $f(0_A) \leq g(y)$ for all $y \in B$. If $f(0_A) \leq g(y)$ for all $y \in B$, then for all $y \in B$,

$$(f \times g)(0_A, y) = \max\{f(0_A), g(y)\} = g(y). \tag{2.5}$$

Since $f \times g$ is an anti-fuzzy UP-ideal of $A \times B$, we have for any $x, y, z \in B$,

$$\begin{aligned}
g(x * z) &= (f \times g)(0_A, x * z) && ((2.5)) \\
&= (f \times g)(0_A \cdot 0_A, x * z) && \text{(UP-3)} \\
&= (f \times g)((0_A, x) \diamond (0_A, z)) \\
&\leq \max\{(f \times g)((0_A, x) \diamond [(0_A, y) \diamond (0_A, z)]), \\
&\quad (f \times g)(0_A, y)\} && \text{(Definition 1.13 (2))} \\
&= \max\{(f \times g)(0_A \cdot (0_A \cdot 0_A)), (x * (y * z)), (f \times g)(0_A, y)\} \\
&= \max\{(f \times g)(0_A, x * (y * z)), (f \times g)(0_A, y)\} && \text{(UP-3)} \\
&= \max\{\max\{f(0_A), g(x * (y * z))\}, \max\{f(0_A), g(y)\}\} \\
&= \max\{g(x * (y * z)), g(y)\}.
\end{aligned}$$

Hence, g is an anti-fuzzy UP-ideal of B which is a contradiction. Since f is not an anti-fuzzy UP-ideal of A and g is not an anti-fuzzy UP-ideal of B, and $f(0_A) \leq f(x)$ for all $x \in A$ and $g(0_B) \leq g(y)$ for all $y \in B$, there exist $x, y, z \in A$ and $x', y', z' \in B$ such that

$$f(x \cdot z) > \max\{f(x \cdot (y \cdot z)), f(y)\}$$

and

$$g(x' * z') > \max\{g(x' * (y' * z')), g(y')\}.$$

Thus

$$\max\{f(x\cdot z), g(x'*z')\} > \max\{\max\{f(x\cdot(y\cdot z)), f(y)\}, \max\{g(x'*(y'*z')), g(y')\}\}.$$

Since $f \times g$ is an anti-fuzzy UP-ideal of $A \times B$, we have

$$\begin{aligned}
\max\{f(x\cdot z), g(x'*z')\} &= (f\times g)(x\cdot z, x'*z') \\
&= (f\times g)((x,x')\diamond(z,z')) \\
&\le \max\{(f\times g)((x,x')\diamond[(y,y')\diamond(z,z')]), \\
&\quad (f\times g)(y,y')\} \qquad \text{(Definition 1.13 (2))} \\
&= \max\{(f\times g)(x\cdot(y\cdot z), x'*(y'*z')), (f\times g)(y,y')\} \\
&= \max\{\max\{f(x\cdot(y\cdot z)), g(x'*(y'*z'))\}, \\
&\quad \max\{f(y), g(y')\}\}.
\end{aligned}$$

Thus $\max\{f(x\cdot z), g(x'*z')\} \not> \max\{\max\{f(x\cdot(y\cdot z)), f(y)\}, \max\{g(x'*(y'*z')), g(y')\}\}$ which is a contradiction. Similarly, by (1), if $g(0_B) \le g(y)$ for all $y \in B$, we have a contradiction. Hence, either f is an anti-fuzzy UP-ideal of A or g is an anti-fuzzy UP-ideal of B. ∎

Theorem 2.24. *Let $(A;\cdot,0_A)$ and $(B;*,0_B)$ be UP-algebras and let f be a fuzzy set in A and g be a fuzzy set in B. If $f\times g$ is an anti-fuzzy UP-subalgebra of $A\times B$, then either f is an anti-fuzzy UP-subalgebra of A or g is an anti-fuzzy UP-subalgebra of B.*

Proof. Suppose that f is not an anti-fuzzy UP-subalgebra of A and g is not an anti-fuzzy UP-subalgebra of B. Then there exist $x, y \in A$ and $a, b \in B$ such that

$$f(x\cdot y) > \max\{f(x), f(y)\}$$

and

$$g(a*b) > \max\{g(a), g(b)\}.$$

Thus $\max\{f(x\cdot y), g(a*b)\} > \max\{\max\{f(x), f(y)\}, \max\{g(a), g(b)\}\}$. Since $f\times g$ is an anti-fuzzy UP-subalgebra of $A\times B$, we have

$$\begin{aligned}
\max\{f(x\cdot y), g(a*b)\} &= (f\times g)(x\cdot y, a*b) \\
&= (f\times g)((x,a)\diamond(y,b)) \\
&\le \max\{(f\times g)(x,a), (f\times g)(y,b)\} \qquad \text{(Definition 1.14)} \\
&= \max\{\max\{f(x), g(a)\}, \max\{f(y), g(b)\}\} \\
&= \max\{\max\{f(x), f(y)\}, \max\{g(a), g(b)\}\}. \qquad \text{(Lemma 2.19 (1))}
\end{aligned}$$

Thus $\max\{f(x\cdot y), g(a*b)\} \not> \max\{\max\{f(x), f(y)\}, \max\{g(a), g(b)\}\}$ which is a contradiction. Hence, either f is an anti-fuzzy UP-subalgebra of A or g is an anti-fuzzy UP-subalgebra of B. ∎

Theorem 2.25. *Let f be a fuzzy set in A. Then the following statements hold:*

(1) *f is an anti-fuzzy UP-ideal of A if and only if μ_f is an anti-fuzzy UP-ideal of $A \times A$, and*

(2) *f is an anti-fuzzy UP-subalgebra of A if and only if μ_f is an anti-fuzzy UP-subalgebra of $A \times A$.*

Proof. (1) Assume that f is an anti-fuzzy UP-ideal of A. By Theorem 2.20 (1), we have $\mu_f = f \times f$ is an anti-fuzzy UP-ideal of $A \times A$.

Conversely, assume that μ_f is an anti-fuzzy UP-ideal of $A \times A$. Since $f \times f = \mu_f$, it follows from Theorem 2.23 (4) that f is an anti-fuzzy UP-ideal of A.

(2) Assume that f is an anti-fuzzy UP-subalgebra of A. By Theorem 2.20 (2), we have $\mu_f = f \times f$ is an anti-fuzzy UP-subalgebra of $A \times A$.

Conversely, assume that μ_f is an anti-fuzzy UP-subalgebra of $A \times A$. Since $f \times f = \mu_f$, it follows from Theorem 2.24 that f is an anti-fuzzy UP-subalgebra of A. ∎

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ON p-NILPOTENCY OF FINITE GROUPS

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Abstract. A subgroup H of a finite group G is said to be weakly s-supplemently embedded subgroup in G if there exists a subgroup T of G such that $G = HT$ and $H \cap T \leq H_{se}$, where H_{se} is an s-quasinormally embedded subgroup of G contained in H. In this paper we investigate the structure of G under the assumption that some subgroups of P are weakly s-supplemently embedded in G, and some new criteria are obtained.

Keywords: p-nilpotent group, Sylow subgroup, weakly s-supplemently embedded subgroup.

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1. Introduction

It is well known that the normalizer of Sylow subgroups of a group play an important role in the structure of groups. Let P be a Sylow subgroup of a group G. An interesting question is what one can say about G if some properties of the normalizer $N_G(P)$ of P are known. For example, the well known Burnside's Theorem asserts that if $N_G(P) = C_G(P)$, then G is p-nilpotent. Hall in [1] got the generalization of Burnside's theorem: if $p^{'}$-elements of $N_G(P)$ are commute

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to the elements of P and the class size of P is less than p, then G is p-nilpotent. In [2], Wielandt showed that a group G is p-nilpotent if it has a regular Sylow p-subgroup whose G-normalizer is p-nilpotent. On the other hand, local analytic theory of groups is somehow substantial in studying the structure of finite groups and normalizers play a very important role in the local analysis theory. Therefore, it is of interest to study the structure of finite groups from properties of the normalizer of a Sylow subgroup.

Let G be a group and H a subgroup of G. H is said to be s-permutable (or s-quasinormal, π-quasinormal) in G if H permutes with every Sylow subgroup of G; H is called c-normal in G if G has a normal subgroup T such that $G = HT$ and $H \cap T \leq H_G$, where H_G is the normal core of H in G; H is called weakly s-permutable in G if there is a subnormal subgroup T of G such that $G = HT$ and $H \cap T \leq H_{sG}$, where H_{sG} is the maximal s-permutable subgroup of G contained in H; H is said to be s-permutably embedded in G if for each prime p dividing $|H|$, a Sylow p-subgroup of H is also a Sylow p-subgroup of some s-permutable subgroup of G. More recently, authors in [3] introduced the following concept, which covers both weakly s-permutably embedded property and weakly s-permutability.

Definition 1.1. Let H be a subgroup of G. H is called a weakly s-supplementedly embedded subgroup of G if there is a subgroup T of G such that $G = HT$ and $H \cap T \leq H_{se}$, where H_{se} is an s-permutably embedded subgroup H_{se} of G contained in H.

Let G be a group, p a prime and P a Sylow p-subgroup of G, we introduce the two families of subgroups:

$\mathcal{H}(P) = \{H \leq P \mid P' \leq H \leq \Phi(P)\}$

$\mathcal{K}(P) = \{K \leq G \mid K$ is p-closed and the Sylow p-subgroup of K is contained in $\mathcal{H}(P)\}$.

It is obvious that $\mathcal{H}(P) \subseteq \mathcal{K}(P)$ and each element in $\mathcal{H}(P)$ is normal in P. Now, we consider the structure of G under the assumption that some subgroups of P are weakly s-supplementedly embedded in G and some new criteria are obtained. For example, we prove the following results:

Theorem 3.1. *Let P be a Sylow p-subgroup of a group G, where p is a prime divisor of $|G|$. Then G is p-nilpotent if and only if there exists $H \in \mathcal{H}(P)$ such that H is weakly s-supplementedly embedded in G and $N_G(P)$ is p-nilpotent.*

Theorem 3.5. *Let $\mathcal{F}$ be a saturated formation containing the class of all supersolvable groups $\mathcal{U}$, and assume that G is a group with a normal subgroup E satisfies $G/E \in \mathcal{F}$. Suppose that for any prime p dividing $|E|$ and $P \in Syl_p(E)$, there exists $K \in \mathcal{K}(P)$ such that K is weakly s-supplementedly embedded in G and every maximal subgroup of P or every cyclic subgroup of P with prime order and order 4 is weakly s-supplementedly embedded in $N_G(P)$, then $G \in \mathcal{F}$.*

Theorem 3.6. *Let G be a group and P a Sylow p-subgroup of G, where p is a prime of $|G|$. Then G is p-nilpotent if and only if there exists a subgroup $H \in \mathcal{H}(P)$ such that H is weakly s-supplementedly embedded in G and $N_G(H)$ is p-nilpotent.*

All groups in this paper are finite. Let H be a subgroup of a group G, H_{se} denote an s-permutably embedded subgroup of G contained in H. The other notations and terminology are standard, as in [2].

2. Preliminaries

Lemma 2.1. ([4])

(a) *An s-permutable subgroup of G is subnormal in G.*

(b) *If $H \leq K \leq G$ and H is s-permutable in G, then H is s-permutable in K.*

(c) *Let $K \trianglelefteq G$. If H is s-permutable in G, then HK/K is s-permutable in G/K.*

(d) *If P is an s-permutable p-subgroup of G for some prime p, then $N_G(P) \geq O^p(G)$.*

Lemma 2.2. ([5, Lemma 2.1]) *Suppose that U is s-permutably embedded in a group G, and that $H \leq G$ and $N \trianglelefteq G$.*

(a) *If $U \leq H$, then U is s-permutably embedded in H.*

(b) *UN is s-permutably embedded in G and UN/N is s-permutably embedded in G/N.*

Lemma 2.3. ([6, Lemma 2.5]) *Suppose that H is s-permutable in a group G, P a Sylow p-subgroup of H, where p is a prime. If either $H_G = 1$ or $P \leq O_p(G)$, then P is s-permutable in G.*

Lemma 2.4. ([3, Lemma 2.3]) *Let U be a weakly s-supplemently embedded subgroup of a group G and N a normal subgroup of G. Then:*

(a) *If $U \leq H \leq G$, then U is weakly s-supplemently embedded in H.*

(b) *If $N \leq U$, then U/N is weakly s-supplemently embedded in G/N.*

(c) *Let π be a set of primes, U a π-subgroup and N a π' -subgroup. Then UN/N is weakly s-supplemently embedded in G/N.*

Lemma 2.5. *Let P be a Sylow p-subgroup of a group G, H the normal Sylow p-subgroup of a subgroup K of G and $H \leq \Phi(P)$. If K is weakly s-supplemently embedded in G, then H is s-permutably embedded in G.*

Proof. By hypothesis, there is a subgroup A of G and an s-permutably embedded subgroup K_{se} of G contained in K such that $G = KA$ and $K \cap A \leq K_{se}$. Since $H \trianglelefteq K$, for P, there exists a Sylow p-subgroup P_1 of A such that $P = HP_1 \leq \Phi(P)P_1 \leq P_1$, so $H \leq P \leq A$ and then H is a Sylow p-subgroup of K_{se}. It follows from the definition of the s-permutably embedded subgroup that H is an s-permutably embedded subgroup of G. ∎

Lemma 2.6. ([7]) *If P is a Sylow p-subgroup of G and $N \trianglelefteq G$ such that $P \cap N \leq \Phi(P)$, then N is p-nilpotent.*

Lemma 2.7. *Let $P/\Phi(P)$ be a minimal normal subgroup of a group $G/\Phi(P)$, where p is a prime divisor of $|G|$ and P a Sylow p-subgroup of G. If every maximal subgroup of P or every cyclic subgroup of P with prime order and order 4 is weakly s-supplemently embedded in G, then P is cyclic.*

Proof. Let P_1 be a proper subgroup of P. If P_1 is weakly s-supplemently embedded in G, we claim that $P_1 \leq \Phi(P)$. Let T be a supplement of P_1 in G such that $G = P_1T$ and $P_1 \cap T \leq (P_1)_{se}$. Then $G = P_1T$ and $P = P \cap G = P \cap P_1T = P_1(P \cap T)$. Since $P/\Phi(P)$ is abelian, $(P \cap T)\Phi(P)/\Phi(P) \trianglelefteq G/\Phi(P)$, and hence $(P \cap T)\Phi(P) \trianglelefteq G$. Since $P/\Phi(P)$ be a minimal normal Sylow p-subgroup of $G/\Phi(P)$, $P \cap T \leq \Phi(P)$ or $P \cap T = P$. If $P \cap T \leq \Phi(P)$, then $P = P_1(P \cap T) = P_1$, a contradiction. Now assume that $P \cap T = P$. Then $P_1 \leq P_1 \cap T \leq (P_1)_{se} \leq O_p(G) = P$. Hence P_1 is s-permutable in G by Lemma 2.3. So $P_1\Phi(P)/\Phi(P)$ is s-permutable in $G/\Phi(P)$ follows from Lemma 2.1(c) and so $N_{G/\Phi(P)}(P_1\Phi(P)/\Phi(P)) \geq O^p(G/\Phi(P))$. Note that $P/\Phi(P)$ is abelian, $P_1\Phi(P)/\Phi(P)$ is normal in $G/\Phi(P)$. By the minimal normality of $P/\Phi(P)$ in $G/\Phi(P)$ again, we have $P_1 \leq \Phi(P)$.

If every maximal subgroup of P is weakly s-supplemently embedded in G, then by the above argument P has a unique maximal subgroup, which implies that P is cyclic.

If every cyclic subgroup of P with prime order and order 4 is weakly s-supplemently embedded in G, then we also have $|P/\Phi(P)| = p$ and then P is cyclic. Otherwise, let $K/\Phi(P)$ be any non-trivial cyclic subgroup of $P/\Phi(P)$. Let $x \in K \setminus \Phi(P)$ such that $T = \langle x\rangle\Phi(P)$. Then by the above argument, $\langle x\rangle \leq \Phi(P)$ and so $T = \Phi(P)$, a contradiction. This contradiction completes the proof the lemma. ∎

Lemma 2.8. *Let G be a group and P a normal Sylow p-subgroup of G, where $(|G|, p-1) = 1$. Then G is p-nilpotent if and only if every maximal subgroup of P or every cyclic subgroup of P with prime order and order 4 is weakly s-supplemently embedded in G.*

Proof. If G is p-nilpotent, then G has a normal p-complement T. Let P_1 be a subgroup of P. If P_1 is maximal in P, then P_1T is normal in G for $|G : P_1T| = p$. Hence P_1 is weakly s-supplemently embedded in G. Now assume that P_1 is a cyclic subgroup of P with prime order and order 4 . It follows that P_1 is a Sylow p-subgroup of P_1T. Let Q be a Sylow q-subgroup of G, where $q \neq p$ is a prime divisor of $|G|$. Then $Q \leq T$ and so $P_1TQ = QP_1T = P_1T$. By hypothesis, P is normal in G and so $P_1TP = PT$ is a subgroup of G, which implies that P_1T is an s-permutable subgroup of G and P_1 is s-permutably embedded in G. Hence P_1 is weakly s-supplemently embedded in G.

Conversely, assume that every cyclic subgroup of P with prime order and order 4 is weakly s-supplemently embedded in G and G is non-p-nilpotent. Let G be a counterexample with minimal order. Let M be a proper subgroup of G.

Then $P \cap M$ is a normal Sylow p-subgroup of M. It follows from Lemma 2.4 that every cyclic subgroup of P with prime order and order 4 is weakly s-supplemently embedded in M and so M is p-nilpotent. By [8, VI, Theorem 24.2], $P/\Phi(P)$ is a G-chief factor of P. Now by Lemma 2.7, P is cyclic and G is p-nilpotent follows from Burside's Theorem, a contradiction. Now assume that every maximal subgroup P_1 of P is weakly s-supplemently embedded in G and G is non-p-nilpotent. Let G be a counterexample with minimal order. Let N be a minimal normal subgroup of G contained in P. It is easy to see that G/N is p-nilpotent. By a routine argument, we have that $N = P$. Now by Lemma 2.7 again, we have that G is p-nilpotent, a contradiction. The lemma is proved. ∎

Lemma 2.9. *Let Q be a normal Sylow q-subgroup of a group G such that G/Q is supersolvable, where q is a prime divisor of $|G|$. If every maximal subgroup of Q or every minimal subgroup of Q with prime order and order 4 is weakly s-supplemently embedded in G, then G is supersolvable.*

Proof. Assume the result is false and let G be a counterexample with minimal order. By Lemma 2.4, it is easy to see that if every minimal subgroup of Q with prime order and order 4 is weakly s-supplemently embedded in G and G is non-supersolvable, then G is a minimal non-supersolvable, i.e., each proper subgroup of G is supersolvable and G is non-supersolvable. Now by [10], G has a normal Sylow p-subgroup P such that $G = PM$, where M is a supersolvable maximal subgroup of G and $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$. If $P \neq Q$, then $G \lesssim G/P \times G/Q$ is supersolvable, a contradiction. Hence $P = Q$. Now Q is cyclic by Lemma 2.7 and then G is supersolvable, a contradiction.

So we may assume that every maximal subgroup of Q is weakly s-supplemently embedded in G. Let N be a minimal normal subgroup of G contained in Q. Let Q_1/N be a maximal subgroup of Q/N, then Q_1 is maximal in Q and by hypothesis and Lemma 2.4, Q_1/N is weakly s-supplemently embedded in G/N. So G/N satisfies the hypothesis and by induction G/N is supersolvable. It follows that N is the unique minimal normal subgroup of G contained in Q and $N \not\leq \Phi(G)$. Hence $N = Q$. Now by Lemma 2.7 again, we have that Q is cyclic and so G is supersolvable, a contradiction. This contradiction completes the proof. ∎

Lemma 2.10. *Let G be a group and $P \in Syl_p(G)$ where $p \in \pi(G)$. If P is abelian and $N_G(P)$ is p-nilpotent, then G is p-nilpotent.*

Proof. Since $N_G(P)$ is p-nilpotent, $N_G(P) = P \times H$, where H is a normal p-complement of P in $N_G(P)$, so $H \leq C_G(P)$. On the other hand, by the hypothesis, P is abelian, hence $P \leq C_G(P)$. Thus we have $N_G(P) = C_G(P)$. By the Burnside theorem, G is p-nilpotent. ∎

3. Main results

Theorem 3.1. *Let P be a Sylow p-subgroup of a group G, where p is a prime divisor of $|G|$. Then G is p-nilpotent if and only if there exists $H \in \mathcal{H}(P)$ such that H is weakly s-supplemently embedded in G and $N_G(P)$ is p-nilpotent.*

Proof. If G is p-nilpotent, then $N_G(P)$ is p-nilpotent and G has a normal p-complement T such that $G = PT$. It follows that $P'T$ is normal in G and P' is a Sylow p-subgroup of $P'T$, which implies that P' is a weakly s-supplemently embedded subgroup of G. Obviously $P' \in \mathcal{H}(P)$. Now we prove that if there exists $H \in \mathcal{H}(P)$ such that H is weakly s-supplemently embedded in G and $N_G(P)$ is p-nilpotent, then G is p-nilpotent. Assume this is false and let G be a counterexample with minimal order. We derive a contradiction in several steps.

Step 1. H is a non-identity s-permutably embedded subgroup of G and G is not a non-abelian simple group.

By Lemma 2.5, H is an s-permutably embedded subgroup of G. If $H = 1$, then $P' \leq H = 1$ implies that P is abelian. Now by Lemma 2.10, G is p-nilpotent, a contradiction. Let A be an s-permutable subgroup of G such that H is a Sylow p-subgroup of A. So $A \neq 1$. Since $H < P$, $A < G$. Therefore A is a non-trivial abnormal subgroup of G, which implies that G is not a non-abelian simple group.

Step 2. G has a unique minimal normal subgroup N, G/N is p-nilpotent. Furthermore, $O_{p'}(G) = 1$ and $N \not\leq \Phi(G)$.

Let N be a minimal normal subgroup of G, consider the quotient group $\overline{G} = G/N$. Then $\overline{P} = PN/N$ is a Sylow p-subgroup of $\overline{G}$. Certainly, $N_{\overline{G}}(\overline{P}) = \overline{N_G(P)}$ and $(\overline{P})' \leq \overline{P'} \leq \overline{H} \leq \overline{\Phi(P)} \leq \Phi(\overline{P})$. It follows that $(\overline{P})' \leq \overline{H} \leq \Phi(\overline{P})$. Hence $\overline{H} \in \mathcal{H}(\overline{P})$. By Step 1 and Lemma 2.2, it is easy to see that G/N satisfies the hypotheses, so by induction G/N is p-nilpotent. Obviously N is the unique minimal normal subgroup of N. Furthermore, $O_{p'}(G) = 1$ and $N \not\leq \Phi(G)$.

Step 3. $O_p(G) = 1$.

Assume that $O_p(G) \neq 1$. Then $N \leq O_p(G)$ and by Step 2 $N \cap \Phi(G) = 1$, it follows that $O_p(G) \cap \Phi(G) = 1$. Now, by [9, Lemma 2.6] we have $N = O_p(G)$.

Now, we claim that $N \leq \Phi(P)$. Let A be an s-permutable subgroup of G such that H is a Sylow p-subgroup of A. If $A_G \neq 1$, then $O_p(G) = N \leq H \leq \Phi(P)$. If $A_G = 1$, then by Lemma 2.3 H is an s-permutable subgroup of G. It follows from Lemma 2.1(e) that $O^p(G) \leq N_G(H)$ and so $G = PO^p(G) \leq N_G(H)$, which implies that $H \trianglelefteq G$. Hence either $H = 1$ or $N \leq H$. If $H = 1$, then $P' = 1$ and so by Lemma 2.10 G is p-nilpotent, a contradiction. Hence $N \leq H$ and it follows that $N \leq \Phi(P)$. Assume that $G = \langle M, N\rangle$ for $M \subseteq G$. Then $G = \langle M\rangle N$. Since $N \leq P$, $P = P \cap N\langle M\rangle = N(P \cap \langle M\rangle) = P \cap \langle M\rangle$, so $P = P \cap \langle M\rangle$ and $G = \langle M\rangle$. Hence $N \leq \Phi(G)$, which contradicts Step 2. So $O_p(G) = 1$.

Step 4. N is p-nilpotent, the finial contradiction.

If $NP < G$, then NP satisfies the hypotheses and by induction NP is p-nilpotent. Therefore N is p-nilpotent, which contradicts Step 2 and Step 3. Hence $G = NP$.

By Step 1, H is a Sylow p-subgroup of an s-permutably embedded subgroup A of G. If $A_G = 1$, then by Lemma 2.3 H is s-permutable in G and so $H \leq O_p(G)$, which contradicts Step 3. So $A_G \neq 1$. It follows from the uniqueness of N that

$N \leq A_G \leq A$ and so $H \cap N$ is a Sylow p-subgroup of N. Noting that $P \cap N$ is also a Sylow p-subgroup of N and $H \cap N \leq P \cap N$, $P \cap N = H \cap N \leq H \leq \Phi(P)$. By Lemma 2.6, N is p-nilpotent, which contradicts Step 2 and Step 3. The finial contradiction completes the proof. ∎

Theorem 3.2. *Let P be a Sylow p-subgroup of a group G, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. Then G is p-nilpotent if and only if there exists $H \in \mathcal{H}(P)$ such that H is weakly s-supplemently embedded in G and every maximal subgroup of P or every cyclic subgroup of P with prime order and order 4 is weakly s-supplemently embedded in $N_G(P)$.*

Proof. In view of Lemma 2.8 and Theorem 3.1, the result is obvious. ∎

Theorem 3.3. *Let P be a Sylow p-subgroup of a group G, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. Then G is p-nilpotent if and only if there exists $K \in \mathcal{K}(P)$ such that K is weakly s-supplemently embedded in G and every maximal subgroup of P or every cyclic subgroup of P with prime order and order 4 is weakly s-supplemently embedded in $N_G(P)$.*

Proof. This theorem follows from Lemma 2.5 and Theorem 3.2. ∎

Theorem 3.4. *Let G be a group and p a prime dividing the order of G with $(|G|, p-1) = 1$. Suppose that E is a normal subgroup of G such that G/E is p-nilpotent. Let P be a Sylow p-subgroup of E. If there exists $K \in \mathcal{K}(P)$ such that K is weakly s-supplemently embedded in G and every maximal subgroup of P or every cyclic subgroup of P with prime order and order 4 is weakly s-supplemently embedded in $N_G(P)$, then G is p-nilpotent.*

Proof. Assume that the result is false and let G with subgroup E be a minimal counterexample to the theorem in respect to $|G| + |E|$. By Lemma 2.4 and Theorem 3.3, E is p-nilpotent. Let T be the normal p-complement of E, then $T \trianglelefteq G$. If $T \neq 1$, we consider G/T with subgroup E/T. It is easy to see that $E = PT$ and $(|P|, |T|) = 1$. With a similar argument as in Step 2 of Theorem 3.1, we know that the hypothesis is still true for G/T with subgroup E/T, hence the minimal choice of G implies that G/M is p-nilpotent. Thus G is p-nilpotent, a contradiction. So we may assume that $T = 1$, i.e., $E = P$ is a p-group. Let K/P be the normal p-complement of G/P, this makes sense as $G/P = G/E$ is p-nilpotent. It is clear that there exists $H \in \mathcal{H}(P)$ such that H is weakly s-supplemently embedded in K and every maximal subgroup of P or every cyclic subgroup of P with prime order and order 4 is weakly s-supplemently embedded in $N_K(P)$, whence K is p-nilpotent by Theorem 3.3, so that $K_{p'} char T \trianglelefteq G$ yielding that $K_{p'}$ is also a normal Hall p'-subgroup of G, i.e., G is p-nilpotent, a contradiction. This contradiction completes the proof. ∎

Theorem 3.5. *Let $\mathcal{F}$ be a saturated formation containing the class of all supersolvable groups $\mathcal{U}$, and assume that G is a group with a normal subgroup E satisfies $G/E \in \mathcal{F}$. Suppose that for any prime p dividing $|E|$ and $P \in Syl_p(E)$, there exists $K \in \mathcal{K}(P)$ such that K is weakly s-supplemently embedded in G and*

every maximal subgroup of P or every cyclic subgroup of P with prime order and order 4 is weakly s-supplemently embedded in $N_G(P)$, then $G \in \mathcal{F}$.

Proof. Assume that the result is false and let G be a counterexample of minimal order. By Theorem 3.3, G is a Sylow-tower group. Let $q = max\pi(G)$ and $Q \in Syl_q(G)$. Then $Q \trianglelefteq G$. By Lemmas 2.5 and 2.4, it is easy to see that G/Q satisfies the hypothesis and by induction G/Q is supersolvable. Now by Lemma 2.9, G is supersolvable. ∎

Theorem 3.6. *Let G be a group and P a Sylow p-subgroup of G, where p is a prime of $|G|$. Then G is p-nilpotent if and only if there exists a subgroup $H \in \mathcal{H}(P)$ such that H is weakly s-supplemently embedded in G and $N_G(H)$ is p-nilpotent.*

Proof. We only need to prove the sufficient part. Assume it is false and let G be a counterexample with minimal order. Then

(1) $O_{p'}(G) = 1$.

Suppose that $O_{p'}(G) \neq 1$. Consider $G/O_{p'}(G)$. Then by Lemma 2.4 and [11, Lemma 3.6.10], we have $G/O_{p'}(G)$ satisfies the hypothesis of the theorem. The choice of G yields that $G/O_{p'}(G)$ is p-nilpotent, which implies that G is p-nilpotent, a contradiction.

(2) H is an s-permutably embedded subgroup of G. Let A be an s-permutable subgroup of G such that H is a Sylow p-subgroup of A. Then $A_G \neq 1$.

By Lemma 2.5, H is an s-permutably embedded subgroup of G. If $A_G = 1$, then H is s-permutable in G by Lemma 2.3 and so $O^p(G) \leq N_G(H)$. Notice that H is normal in P, $G = PO^p(G) \leq N_G(H)$ is p-nilpotent, a contradiction. Hence $A_G \neq 1$.

(3) G is p-nilpotent, the final contradiction.

Since H is a Sylow p-subgroup of HA_G, $N_{G/A_G}(HA_G/A_G) = N_G(H)A_G/A_G$ follows from [11, Lemma 3.6.10] and $HA_G/A_G \in \mathcal{H}(PA_G/A_G)$. It is easy to see that G/A_G satisfies the hypothesis of the theorem and by the minimal choice of G, we have that G/A_G is p-nilpotent.

Since $P \cap A_G = H \cap A_G \leq \Phi(P)$, A_G is p-nilpotent by Lemma 2.6. By (1), $A_G \leq H \leq \Phi(P)$ and so $A_G \leq \Phi(G)$, which implies that G is p-nilpotent, the final contradiction. ∎

From Theorem 3.6, we can obtain

Theorem 3.7. *Let G be a group. Then G is nilpotent if and only if there exists a subgroup $H \in \mathcal{H}(P)$ such that H is weakly s-supplemently embedded in G and $N_G(H)$ is p-nilpotent for any $P \in Syl_p(G)$.*

4. Some applications

Let G be a group and P a Sylow p-subgroup of G, where $p \in \pi(G)$. If $N_G(P)$ is p-nilpotent, then it is easy to see that $P' \in Syl_p((N_G(P))')$ and $\Phi(P) \in Syl_p(\Phi(N_G(P)))$. So Theorem A has the following corollaries:

Corollary 4.1. *Let P be a Sylow p-subgroup of a group G, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. Then G is p-nilpotent if and only if P' is weakly s-supplemently embedded in G and every maximal subgroup of P or every cyclic subgroup of P with prime order and order 4 is weakly s-supplemently embedded in $N_G(P)$.*

Corollary 4.2. *Let P be a Sylow p-subgroup of a group G, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. Then G is p-nilpotent if and only if $\Phi(P)$ is weakly s-supplemently embedded in G and every maximal subgroup of P or every cyclic subgroup of P with prime order and order 4 is weakly s-supplemently embedded in $N_G(P)$.*

Corollary 4.3. *Let P be a Sylow p-subgroup of a group G, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. Then G is p-nilpotent if and only if $(N_G(P))'$ is weakly s-supplemently embedded in G and every maximal subgroup of P or every cyclic subgroup of P with prime order and order 4 is weakly s-supplemently embedded in $N_G(P)$.*

Corollary 4.4. *Let P be a Sylow p-subgroup of a group G, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. Then G is p-nilpotent if and only if $\Phi(N_G(P))$ is weakly s-supplemently embedded in G and every maximal subgroup of P or every cyclic subgroup of P with prime order and order 4 is weakly s-supplemently embedded in $N_G(P)$.*

Corollary 4.5. ([3, Theorem 3.1]) *Let G be a group and assume p is a prime dividing the order of G with $(|G|, p-1) = 1$. If there exists a Sylow p-subgroup P of G such that every maximal subgroup of P is weakly s-supplemently embedded in $N_G(P)$ and if P' is s-quasinormal in G, then G is p-nilpotent.*

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STRUCTURED KNOWLEDGE IN THE FRAME OF BAK-SNEPPEN MODELS

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Abstract. Instruction aims at the acquisition of a structured or procedural knowledge and was always torn between a frame of imitation of examples (more or less exhaustive) and a frame of theoretical or conceptual knowledge, exposed more or less systematically. Mathematics has the longest history at the level of its phylogeny, since it built in time many alternating systems of consistent relations, even if they clashed when they had to be mixed and glued together. The mathematical evolution of the individual (ontogeny) is subject to sudden jumps unlike the softer evolution of less structured disciplines. The moments of confrontation between different structures or innovation sparks need time to affect the existing structures, both in philogeny and in ontogeny. Bak-Sneppen models of evolution help to understand many social phenomena, especially when voluntary or random competition may hinder the search of optimal individual paths, but at the same time forces development. In mathematics, because of its high developed structure, knowledge waves can hinder comprehension both of pupils and of scholars. The paper discusses some paradoxes where a greater knowledge has proved to diminish efficiency and capability.

Keywords: structured knowledge, mathematical teaching, Ontogeny versus Phylogeny, Bak-Sneppen model, partitioned frames and evolution.
AMS Mathematics Subject Classification: 11A05, 34C28, 97C50, 97C80, 97D10.

Introduction

Since the most ancient times mathematics and linguistic (as rhetoric) were subjects of methodological reflection. Many teaching experiences were attempted and descriptions are available. Mathematics found a conceptual foundation, challenging but transmissible, starting from Euclides' Elements (third century b.C.). They rapidly succeeded in acquiring a leading role as a fundamental reference and kept this role for over 1500 years. On the other side, linguistic found its basis in Panini's book (fourth century b.C.) for Sanskrit, and in Cicero's and Quintilianus' works (first century b.C. and first century a.C.) for Latin, hence for a long teaching tradition in European cultural development.

Instruction and consequent acquisition of a structured or procedural knowledge always were torn between a frame of imitation of examples (more or less exhaustive) and a frame of theoretical or conceptual knowledge, exposed more or less systematically. Better reactions are usually achieved by imitation, while a stronger structure is acquired when a solid systematic basis is available, so that examples and experiments can be inserted in a solid frame. Theoretic basis alone is not sufficient, since it is difficult to be memorized and transferred into practice, while every knowledge requires to be recalled and strengthened.

The article by Rittle-Johnson et al. [35] evaluates an intuitive belief that in mathematics learning the development of conceptual knowledge precedes that of procedural knowledge. Based on this belief, many people advocate that instruction should take place in a conceptual-to-procedural order. To the contrary, the authors' review of the evidence finds that the development of both types of knowledge is a 'two-way' street, that is, the evidence supports the idea that advances in procedural knowledge can improve conceptual knowledge and vice versa. They conclude that there may be more than one way to arrange the order of instruction to enhance mathematics achievement.

In the short period of study of a young pupil (here we can speak of Ontogeny) the continuous superimposition of new adjustments over previous knowledges (right or wrong does not matter) creates very often a conflict that confounds student's ideas.

The authors chose to insert this problems in a frame of interconnected selective evolution. A simple essential model seems to be Bak-Sneppen model, even if it is not suitable for quantitative statistical analysis, but only to express tendency trends.

In this paper, we refer to some moments of mathematics development.

We are dealing with the following problem: what happens when some part of didactics seems not to be 'sufficient', and therefore must be modified? But what does it means 'sufficient?' Sufficient for who? With what purpose? Is it usuful to use techniques that are known to be corrected without proving them? Or is it useful also a proof that has no application? Hardy in [23] says:

> *A formal proof is a kind of X-ray picture of an actual or possible piece of reasoning, revealing the bones (the form) but making the flesh (the content, the meaning) invisible.*

In the long period (a biologist should speak of Philogeny) this happens when new knowledge arises, as it was with de l'Hopital's fundamental book of 1696, where for the first time some techniques of differential calculus were made explicit, so that Newton's and Leibniz's researches could become a general purpose tool. Another critical step for mathematical analysis was the long evolution of the concept of limit, what still allows different didactical choices: should the limit of sequences be introduced first, or should the limit of functions be given at once, and should Euclidean space be used or a general topological space? Another problem is given by the fact that the definition of limit does not give any hint to the way how that is to be found.

A long lasting problem was a correct general definition of integral and nowadays it is to find the best way of introducing the theory to the students of various specializations (including also pure mathematicians) so that the measures over lower size variety and the integration of differential forms become particular cases. The theorems of Gauss-Green, Divergence and Stokes, is it better to add them in the appendix or simply state and prove them just in favorable cases? Has the student of Mathematics the right and duty to know the applications of mathematics? Many mathematics and science educators are in favor of a more realistic education, where modelling activities are used to treat concepts in realistic, everyday life contexts (see [21] and [12]). Gravemeijer and Doorman in [22] note that there is a reflextive relation between the use of context problems and the development of the student's experiental reality. At one hand, the context problems are rooted in this reality at the other hand, solving these context problems helps the students to expand their reality.

Furthermore, as pointed out by Dray and Manogue in [13], in mathematics classes, integrals are viewed very differently than in physics classes. In the mathematics classes students take in preparation for physics, 1D integrals are nearly always considered to be the area under a curve, and in 3D, surfaces and volumes are always considered in terms of the Cartesian basis vectors and parametrized. On the other hand, in physics, integrals (in any number of dimensions) are considered to be sums of little bits of stuff (mass, charge, E*dA, etc.), and the coordinate system for basis vectors is chosen based on the symmetry of the problem. At the junior level, students are struggling to transition their thinking from the mathematics classrooms' view of integration to the physicists' view.

On a more general level there is the secular difference between anglosaxon and continental approach: the first starts from the observation of physical phenomena, while the second starts from the concept of function and seems to lie inside mathematics, continuing the famous querelle between Newton and Leibniz. Incidentally we can recall that Italian students seem to be confused if they start from an anglosaxon text, because they have a good training in mathematics, but no particular training in physics.

The first section summarizes the model and some interesting cases studied by the authors, namely it is recalled how sectional evolution hinders the destruction of achieved objectives, but can also prevent further improvements (turris eburnea, ivory tower effect).

The second section discusses some general behaviors that often happen in social sciences, and also in economics. It can be remarked that the acritic acceptance of models otherwhere successful happens also in education, with results that can lead to opposite effects.

The third section describes a case of mathematical long period phylogeny, namely two completely different approaches to the computation of a root, and shows how the method used in computers rather follows the fashion than a true search of efficiency.

Finally, the fourth section describes an experiment performed at the University of Udine, where different behaviors in mathematical phylogeny can be recognized. They seem to insert Kanizsa' conceptual frame into Bak-Sneppen evolution model, allowing an interpretation connected with Freudenthal theory of sudden conceptual jumps.

This paper is the mathematical development of a previous note dedicated to a different audience specialized in education sciences (see [34]).

1. Inland Bak-Sneppen models and overtaking phenomenon

The Bak-Sneppen model was originally introduced as an archetypical (or ideal) model of evolution by Per Bak and Kim Sneppen ([4]) (compare also [3] and [30]). After a comparison of many existing models that could explain (well) only some particular features of the phaenomena, ignoring many collateral aspects, they attained a synthetic model, that glues globality of information and locality of performance, but is not restricted to single cells. Their classical model, that has been developed in many more sophisticated forms, was defined as follows. There is a circle of N nodes and they are occupied by N different species, each of which has at present a random fitness. The fitness values are independent and uniformly distributed on (0; 1). At each discrete time step the system is updated by locating the species with lowest fitness and replacing its fitness by a new random variable (between 0 and 1). This would in itself reach optimization of all the species. This is the utopian dream of instruction, social support, church and many other charitable institutions. The absence of result is usually justified by the lack of economical resources. But actually Bak-Sneppen model gives a more subtle explanation: when a species changes its fitness this action perturbs the two nearest species, that in turn change their own fitness at random, even if it were already good. Therefore while usually the worst species improves, the neighbors are turned at hazard. Bak-Sneppen models can be defined on a wide range of graphs using the same update rule as above. The model shows that this random processes can result in self-organization to a critical state. Threshold of fitness rises, rapidly at first, then exponentially slows until it reaches, around 0.66, the critical state from which level extinction sweep back and forth through the ecosystem. Hence a general progress is anyhow achieved, since it becomes rare that a species falls under the critical level, while the distribution of values above is tendentially uniform.

Brunk in ([8]) suggests that self-organized criticality is the sort of process that should have great intuitive appeal to social scientists. A large bibliography of applications of self organized criticality can be found in ([39]; see also [2] and [17]) for an economic application). The authors anyhow think that a sound basis for applying Bak-Sneppen model of contact with neighbors is given by Duesenberry's demonstration effect. Its first presentation can be found in ([15]), while many application to social sciences can be found for example in ([9]).

Although the Bak-Sneppen model is extremely simple, it has not yet been completely solved in spite of numerous analytical and numerical investigations. A recent explicit solving formula for the case of 4 nodes (the smallest non obvious dimension) is to be found in ([36]) and for 5 nodes in ([37]). Motivated by the difficulty of analyzing rigorously even the one-dimensional version of the Bak-Sneppen model, Barbay and Kenyon in ([5]) proposed a simpler model with discrete fitness values. In their model each species has fitness 0 or 1, and each new fitness is drawn from the Bernoulli distribution with parameter p. Since there are typically several least fit species, the process then repeatedly chooses at random a species for mutation among the least fit species. In case of small number of nodes explicit solutions can be found using Markov chains, as it was done by the authors in ([31]). Parameter p can substitute up to a certain level a plurality of values, but it cannot explain the staircase phenomenon found by the authors in ([30])) and ([31]), and in particular the phaenomenon of overtaking in partitioned frames. Hence binary structure, though simple and appealing is not sufficient for a thorough description of what may happen. Anyhow also this case is by no means trivial, as it was shown by Meester and Znamenski in ([26]).

Changes in the structure of the contacts have been studied (also in ([31]), Table 5). In particular the classical Bak-Sneppen bilateral adjacent contact structure is somewhat more stable than other more erratic structures since random and scattered perturbations keep the average level of fitness at a lower level. This depends on the fact that in the classical model a long sequence of high values can be attacked only from its boundaries, so that it takes a long time before it undergoes a change (usually unfavourable). The natural step was to investigate systems where this structure becomes stable, what was performed in ([30]), Section 3. There the global system is split in two or more subsystems that do not have contacts with each other so that only the subsystem where a minimum is found is subject to evolution, while all the other subsystems are preserved from any change. This protection increases the average fitness of the global system, both during transitions and in the terminal states. Remark that the only stable terminal states are those in which the cells of all the subsystems but one have reached the top level of fitness. It may take a very long number of iterations (infinite if the fitness is continuous) to reach this terminal situations, and usually some form of staircase is experimentally found, so that the average fitness seems to stay for a long period and then suddendly jumps to a new higher level (Staircase behavior described in the conclusions of ([31]). When the number of fitness levels is greater than 2 a very important phenomenon may arise, namely 'overtaking', as was explained in section 5, fig.1, of ([30]): if the local minimum of a subsystem

is higher than the global minimum, but some of its subjects have not reached the maximum, this system cannot change, so that it defends its good state, but it cannot improve. On the contrary, for a subsystem where an absolute minimum exists there is the possibility not only of reaching but even of overtaking the 'good' system, even if the probability is very low. This fact explains some sudden scientific, cultural, economic and political phenomena of overtaking: the principle '*quieta non movere*' (Let quiet things stay) may lead to standstill, in the models just like very often in real life.

2. Structured knowledge: construction and deconstruction

The basic Bak-Sneppen model asks a dramatic question: why those who are involved in the process of improving the weaker species may become worse? A first answer is obvious, because someone has to bear the costs, but the aforementioned Duesenberry model provides a further explanation. It is clarified by the brutal paradox of ([29]) which is reported here:

> *Suppose you went to a doctor who said 'I am going to do an appendicectomy on you'- When you asked why, the doctor answered, 'because I did one on my last patient and it made him better'. We suspect you would hightail it out of the office, because you know that the treatment ought to fit the disease, regardless of whether or not the treatment helped the previous patient. Strangely enough, that logical thought happens more than we might care to admit in most companies.*

We took this paradox from the book of Brischi and Marzano ([7]), where the uncritical imitation of a successful process is stigmatized. In economics the presentation of case studies leads ipso facto to consider them as best practices to follow. The spread of this theory is dominant, and, depending on the model of Bak-Sneppen, should lead to an overall improvement of the general situation, but the price is to admit ups and downs around the average level, what can be disappointing.

The tendency towards the imitation of the neighbor (real or virtual) often acts in the social sciences and economics, leading to differentiation, which is often masked as innovation. Broadening main fields of action in economy can be studied in the context of INPUT-OUTPUT matrices, as was done by Droli et al. in [14] and by Chang et al. in [10] and [11]. The trend towards multi-functionality can also be recognized in the coalescence of territorial structures, as was shown by Taverna et al. in [38].

Multifunctionality has dispersive character and often ends up creating a short period of damage rather than of benefit, as Bak-Sneppen model foresees. This is in agreement with the classic vision already mentioned by Dante in his *Comedy* ([1])

> *...Follow my steps, though all such wisper of you:*
> *be as a tower of stone, its lofty crown*
> *unswayed by anything the winds may do.*

For when a man lets his attention range
toward every wisp, he loses true direction,
sapping his mind?s force with continual change... Purg. 5, 13-18

...it is no learning
to understand what one does not retain... Par. 5, 41-42

In the last verses it is already covered the need to anchor the new acquisitions, without dispersion, as is well known by experts in advertising.

Two theoretical grounds ideal for a test are language teaching and especially teaching and research in mathematics, as it can be protected from external influences, although there are educational trends that seek to expand their scope of action. Here we meet the eternal competition between the core of a knowledge system and its additions, as described by the authors in (Piccinini et al. [30]) in the examples of the third section, in particular in formula (28).

The model of overtaking, mentioned in the previous section, seems to validate the traditional teaching techniques of separation between different areas, avoiding bundles of heterogeneous notions, the effectiveness of which may be of short duration.

The networks now allow to combine multi-functionality with specialization, though the cost is that of a virtual intermediation chain, composed either by human operators (typically poorly paid), or by automatic operators consisting of expert systems suitable to filter the information through the maze of networks (see [25]). The danger is that expert systems that learn by studying the behavior of users end up with the polarization into information hubs ignoring developments useful but not fashionable. For example, the various constructions of the real numbers and the definitions associated as upper and lower bounds, limit, etc., lend themselves to network structures where the starting point can be freely selected, and then the others can be achieved by showing the equivalence. The interested reader is referred to Weiss in [41] for a comprehensive survey of most, if not all, constructions of the real numbers found in the literature. But what is the price to pay in understanding if you start from an inconvenient point?

3. Mathematical Phylogeny

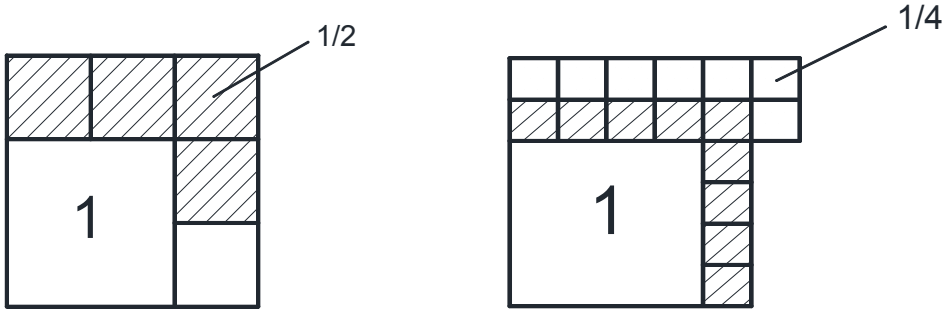
Originally, there are natural numbers and the language that expresses them. An extensive analysis of the inconsistencies and linguistic connotations that are associated with numbers was made in (Piccinini et al. [32]), with reference to ([16]), chap. 5 where he studied the loss and compensation. Mathematical philology, like all artistic and constructive activities, enters in the conceptual scheme of ([24]), especially in chap. 3 dedicated to disposal and retention. It is interesting the dual role of the mathematician. On the one hand he is a builder who discards what is no longer useful, and on the other is a philologist/archaeologist/artist, who digs up and ennobles the past. The historical co-existence of the two roles will be shown by the example of the algorithms for the calculation of the square root. Note that even an algebraic number as 'root of 2' from the operational point of view is

known only as an approximation represented by the decimal sequence 1.41421... (for a review see [6], for a recent computer procedure see, for example, [27]).

The fact that it represents the diagonal of the unit square convinces one of its life, overcoming the Pythagorean problem of its non-commensurability with the side of the square, but does not make it operationally correct, as there is no appropriate tool to measure fractions over the second or third digit. The algorithms in use are all of the iterative type. What was once taught in school, despite its complexity due to the need to avoid repetition of calculations, was based on the introduction at each new step of the highest decimal digit without exceeding the value of 2. The calculus becomes simpler when the bynary representation of a number is used since the choice is just between 0 or 1, infact for the square root of two $\sqrt{10.00 \cdot 00 \cdot 00} = 1.011...$ and the procedure is the following

$$
\begin{array}{rcccc}
 & 1. & 0 & 1 & 1 \\
\hline
\surd & 10. & 00 & 00 & 00 \\
 & 1 & & & \\
\hline
 & 1 & 00 & & \\
 & & 0 & & \\
\hline
 & 1 & 00 & 00 & \\
 & & 10 & 01 & \\
\hline
 & & 1 & 11 & 00 \\
 & & 1 & 01 & 01 \\
\hline
 & & & 1 & 11
\end{array}
$$

Geometrically, it is equivalent to covered the square 1×1 with squares of half size at each stage



(a) The first and second digits are 1 and 0 (b) The third digit is 1

The binary value 10 means two squares of side 1. The only admissible square covered is the square 1×1, hence the remainder is 1, that is 4 squares of side 1/2.

In order to get an admissible square of side $1 + 1/2$, $5(= 2 \times 2 + 1)$ squares of side 1/2 should be at disposal, what exceeds the remainder. Hence a thinner covering must be used. Since no new smaller squares are at hand, the previous 4 squares of side 1/2 become 16 squares of side 1/4.

In order to get an admissible square of side $1 + 1/4$, $9(= 4 \times 2 + 1)$ squares of side 1/2 should be at disposal, what is possible, with the remainder of 7 = 16-9

squares of side 1/4, what corresponds to 28 squares of side 1/8, that will be used for the following thinner expansions of the covering.

On the other hand, the computer uses a method that is inspired by the ancient approximation technique used by the Babylonians. The procedure is based on a geometric reasoning followed by a process of error compensation, both very natural. You are required to build a rectangle of fixed area A so that the base and height are equal. As an initial assumption suppose base is b_1. In correspondence the height is $h_1 = A/b_1$. Excluding the fortunate case in which $b_1 = h_1$ you must improve the choice, taking the base to an intermediate value between the old base and the old height, such as the average of the two, and the process is repeated. The formula becomes

$$b_{n+1} = \frac{1}{2}(h_n + b_n) = \frac{1}{2}\left(\frac{A}{b_n} + b_n\right). \tag{2.1}$$

In the case of $A = 2$, starting from the initial value 1 the first terms are $1, 1.5, 1.417, 1.41422....$ It is important to note that any initial positive value leads to a sequence that converges to the same limit $\sqrt{2}$, while starting from a negative value the sequence tends to $-\sqrt{2}$. This raises several questions; why this method, old and quite intuitive, is not normally taught in primary schools? Nowadays every micro-computer calculates the square root so that manual calculation becomes unnecessary, but it was not taught even fifty years ago when an automatic instrument was not available. The reasons are at least three: it is necessary to have knowledge of the real numbers (not only as geometric intuition), the concept of limit of a sequence is required, some statistical idea is needed for stating that the average of two numbers gives an error minor of the difference between the two original numbers. Finally it should be proved that the sequence (2.1) defined by recurrence actually converges to a finite limit, and this limit is the very root of A. The continuity, even at the level of intuition, allows to understand that the limit is given by a rectangle in which the base and height are equal. The first course in mathematical analysis provides an initial response in the case of the equation (2.1) but a deepening and a generalization to a different choice of average (which is what is actually used in computers) requires the concept of contraction. Furthermore, since the process is stopped after a finite (and small) number of iterations, it is important to be able to estimate the residual error. From this point of view, the school system of computing one decimal digit after the other has the advantage of providing immediately a maximum for the error, and therefore it appears safer. In the case of equation (2.1), on the contrary, the estimation is not evident, as the sequence b_n is a monotone decreasing sequence (starting from the second term), that therefore converges, but does not supply an immediate estimate of the error. The sequences that provide an easy estimation of the error are those nested obtained starting from a positive sequence that decreases towards 0, in which the terms are taken to alternating signs. Finally a constant, which will be the limit, is added.

In our case, equation (2.1) is replaced by

$$b_{n+1} = \frac{kN}{b_n} + (1-k)b_n. \tag{2.2}$$

If $1/2 < k < 1$, then the sequence is nested, but the convergence to the limit becomes slower as k approaches 1. Therefore in the computer k is chosen as a value simple to be represented in the binary system but not far from $1/2$. As noted before, using binary system, the school algorithm becomes extremely simple and the running time of the machine is less than that required by the elegant equation (2.2).

One explanation may be sought in the fact that in the initial years of computers the theory of contractions was very popular and was considered an advanced tool that scientifically characterized its users. It may also be noted that for the roots of higher order algorithms of type (2.1) are easy to use and to check. The basic algorithm for h-th root becomes

$$x_{n+1} = \frac{1}{h}\left(\frac{(h-1)N}{x_n^{h-1}} + x_n\right). \tag{2.3}$$

However, in view of the low frequency of use, it is in general reduced to the indirect calculation of x^y with $y = 1/h$.

4. Mathematical Ontogeny

In mathematics, the structured knowledge taught in the courses often destroys the ability to solve problems by intuition. Most general methods are learned theoretically and are not compared with the simpler, albeit less general, techniques. One obvious case is solving a quadratic equation in which, if there is suspicion that the solution is integer, it is easier to inspect directly the divisors of the known term, switching to formula only in case of failure. Experience shows that about half of the students of scientific courses ignore this shortcut, while conversely almost every person at the moment who has to solve the same problem as a game uses the simplified method, perhaps because the perception that the solution must really be integer is greater.

Vicario ([40]) recalled an experiment of Kanizsa showing that the increase in theoretical information reduced the ability to solve actual problems (for details see ([33], note 12). Strictly related is the ontogeny of the interesting case of drawing a triangle knowing the three sides. The reader should try himself before reading what follows!

> Use white paper (without lines and squares) and draw a triangle with three sides of length 4, 5 and 6. The only tool allowed is just a ruler, but neither compasses nor square.

It is possible that you decide to use analytic geometry (if you remember it) by solving an algebraic system like

$$\begin{cases} x^2 + y^2 = 25 \\ (x-6)^2 + y^2 = 16 \end{cases}$$

that is, $x = 15/4 = 3.75$, $y = 5\sqrt{7}/4 \simeq 3.307$, for which a choice of possible vertices becomes (in a suitable Cartesian coordinate system) the triad $(0,0)$, $(6,0)$ and (x,y). The available instrument is not ideal for working with the Cartesian coordinates, especially in the absence of an orthogonal network, however, a solution more or less approximate can be found. How can you verify and improve it? But it could be established that the tool is a piece of white paper where the three lengths $(4,5,6)$ are marked on the margin and nothing else. The unavailability of a metric system prevents you from using the solution of the algebraic system . What to do?

We conducted the experiment with two different cohorts of students at the University of Udine. The cohort of engineering students (first year) knew in its totality (possibly through the suggestion of neighbors) the classical Euclidean construction, made with ruler and compass. Once fixed the first two points at a distance (e.g.) 6 the probem was thus to approximate the construction of an arc of a circle of radius 5 centered at the first point using the segment on the sheet of lenght 5 and rotating, thus obtaining a simulacrum of the circle. Similarly for the other circle. Here too, the question is the same: how to verify and improve the result? At this point the construction of Euclid, without compass, becomes an unnecessary complexity. The other cohort was formed by students (mostly girls) of Educational Science (first year). Almost all students presented a very efficient solution, suitable also to be taught in primary school. From the first point A draw a segment of length (e.g.) 5 with a random angle, obtaining a provisional point C_1. From B trace the ray that passes through C_1 and mark the point C_2 at distance 4 from point B. In general C_2 does not coincide with the point C_1, otherwise the problem is solved. Draw now the half-line from A passing through the point C_2 and draw the point at distance 5 from A, and so on, alternating the starting points A and B. After two or three iterations it happens that the points are stabilized, and this is a solution (approximate) of the problem. It is therefore clearly a convergent iterative method (though the students did not have the mathematical knowledge to know this fact), let us give an example

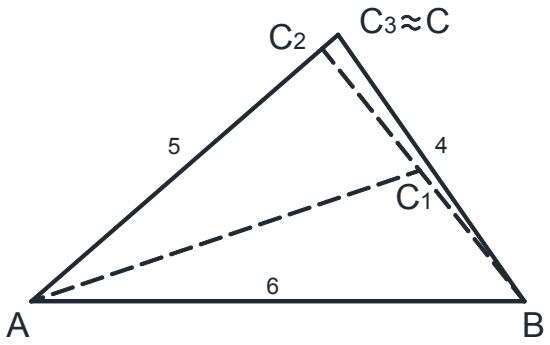


Figure 1: Triangle with lengths $4, 5, 6$

The iterative formula, wanting to simulate the process on the computer is as follows. Denote by a the distance AB, b the distance AC, c the distance BC; ϕ be the angle BAC, θ the angle ABC. Start from an arbitrary angle θ_1. The recurrence is

$$\phi_k = \arctan\left(\frac{b\sin(\theta_k)}{-a + b\cos(\theta_k)}\right),$$

$$\theta_{k+1} = \arctan\left(\frac{c\sin(\phi_k)}{a + c\cos(\phi_k)}\right).$$

The confrontation between students of the two cohorts lead engineers to understand that their trial and error process of drawing points of the circles becomes selected and streamlined through the process of students of education sciences. Two facts can be appreciated: rigorous seclusion prevents improvements and allows overtaking; in the learning of mathematics progress is tied with sudden discontinuities as fully described by Freudenthal in [20] with some examples starting from [18].

5. Conclusions

Scientific knowledge is subject to a twin evolution, since its development towards novelty creates disconnections and inconsistencies, while the need of structure requires order and method so that transmission and comprehension can be ensured. Models of biological evolution can help to understand many social and economic phenomena, where the need for progress not necessarily succeeds and the behavior of neighbors may alter the status quo. Bak-Sneppen is one of the most significant models because it balances at best explication power and simplicity. Unlike cellular automata models, Bak-Sneppen models join locality and globality.

The history of philosophy first and of science later are full of secular changes, and this happened in mathematics as well, even if it is less susceptible of perturbation. The Thucydidean $\kappa\tau\eta\mu\alpha\ \epsilon\sigma\ \alpha\iota\epsilon\iota$ (purchase forever, I, 22) holds, but still there are secular variations, especially in individual ontogeny according to Freudenthal in [19]. Two sections of this paper are devoted to mathematics, with examples simple enough to be understood also by laics. The Bak-Sneppen evolution model of species is a valuable starting point for the interpretation of these phenomena where progress in knowledge may involve a loss of efficiency. The ironic sentence of Umberto Eco's final novel (Numero Zero, 2015).

> *The losers, as the self-taught, have an increasingly extensive knowledge than the winning; if you want to win you have to know one only thing, and not to waste time to know them all; the pleasure of learning is for losers. The more things one knows, the more things did not go the right way.*

is perfectly suitable for the description of these up-and-down disappointing evolutions. Mathematics is the field of science where Bak-Sneppen evolution is clearly perceived both in the short period of ontogeny and in the long period of phylogeny.

The linguistic area, especially after structuralism, is another field where the model operates on a large scale as phylogeny, especially with respect to phenomena of contact. At the level of individual ontogeny, in glottodidactic, structured grammars and systematic thesaurus meet and sometimes clash with methods based on practice and assimilation of microsystems. The authors touched in [32] a small stretch of encounter between mathematics and linguistics, but the experiment of comparing the two subjects deserves to be resumed and expanded, keeping in mind the work of Petit-Cocorda [28] and trying to find out examples where the theoretical and methodological progress damages the capability of correct expression, creating phantom structures not sustained by the actual experience of Chomsky's 'native speakers'.

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HYPERGROUPS AND FUZZY SETS ASSOCIATED MODULO A SUBGROUP

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Abstract. This paper is the results of the ideas suggested by P. Corsini in his paper [10]. Our investigations take in account the paper [13] of I. Cristea. We study the hypergroup generated by the cosets modulo a subgroup (normal subgroup). We prove that $(G/H, \circ_4)$ is a complete hypergroup. We take many particular examples to illustrate some known results on hypergroups.

Keyword: fuzzy sets, hypergroups, complete hypergroup, hyperoperation.

1. Introduction

Since the introduction of the concept of fuzzy subgroup of a given group, by A. Rosenfeld [22]; the notions of different fuzzy algebraic structures have been introduced and intensively studied worldwide. Many results on group theory have be extended in a natural way to fuzzy groups. Later, B. Davvaz and his collaborators generalized the notion of fuzzy subgroup: They defined the concept of fuzzy subhypergroup of a hypergroup. In the same way many Other relations establishing the relationship between hyperstructures and fuzzy sets were stated by P. Corsini. He associated a join space to a fuzzy set and a fuzzy set to a join space. These connections lead to a sequence of fuzzy sets and join spaces. The sequence ends if two consecutive join spaces are isomorphic. This argument has been studied in depth by I. Cristea in her PhD thesis. Moreover in [28], M. Stefanescu together with I. Cristea studied the above sequence in the general general and particularly for a complete hypergroup. P. Corsini and I. Cristea [5] determined the fuzzy grade of a particular non-complete 1-hypergroup. In the same way I. Cristea investigated [14] the sequences of join spaces associated the case of a hypergroupoid using fuzzy sets endowed with two membership functions.

More connections between algebraic hyperstructures and fuzzy sets can be found in many recent publications [1], [21], [22], [24].

Our contribution to the development of such structure is to study as suggested by P. Corsini, the properties of a hypergroup generated by the cosets modulo a subgroup (normal subgroup). We prove that $(G/H, \circ_4)$ is a complete hypergroup. We illustrate our results by many examples.

2. Preliminaries

First of all, we shall recall some basic definitions of hypergroup theory. Let H be a nonempty set and denote by $\mathcal{P}^*(H)$ the set of all nonempty subsets of H.

Definition 1. A hyperoperation on H is a mapping $\circ : H \times H \longrightarrow \mathcal{P}^*(H)$.

A nonempty set H endowed with a hyperoperation $\circ$ is said to be a hypergroupoid.

The image of the pair $(a, b) \in H \times H$ is usually denoted by $a \circ b$ and called the hyperproduct of a and b.

If A and B are nonempty subsets of H, then we define

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b$$

and call it the hyperproduct of A and B.

Also, for any $a, b \in H$, we define

$$a/b = \{x \in H | a \in x \circ b\}.$$

Definition 2.

1. The hypergroupoid $(H, \circ)$ is called a semihypergroup if the hyperoperation $\circ$ is associative.

2. If $(H, \circ)$ satisfies the reproducibility law $a \circ H = H \circ a = H,\ \forall a \in H$, then we say that it is a quasihypergroup.

3. A hypergroup is a hypergroupoid which is both a semihypergroup and a quasihypergroup.

4. A hypergroup H is said to have a unity or to be unitary if there exists a unique element $1 \in H$ such that $1 \circ a = a \circ 1 = \{a\},\ \forall a \in H$.

Definition 3. A subset K of hypergroup $(H, \circ)$ is a subhypergroup if and only if the following conditions are satisfied:

1. $a \circ b \subset K$, for all $a, b \in K$,

2. $a \circ K = K \circ a = K$, for all $a \in K$.

Clearly, H is a subhypergroup of itself. Any other subhypergroup of H will be called a proper subhypergroup. In the sequel we recall the following theorems needed to study the fuzzy subset as a function modulo a subgroup H.

Let us introduce some definitions which will be useful.

Definition 4. Let $(G, \circ)$ be a hypergroup and μ be a fuzzy subset of G. μ is called a fuzzy subhypergroup if the following conditions are satisfied.

1. For any $a, b \in G$ one has
$$\min\{\mu(a), \mu(b)\} \leq \min\{\mu(c),\ c \in a \circ b\}.$$
2. For any $a, b \in G$ there exists $c \in G$ such that $a \in b \circ c$ and
$$\min\{\mu(a), \mu(b)\} \leq \mu(c).$$
3. For any $a, b \in G$ there exists $c \in G$ such that $a \in c \circ b$ and
$$\min\{\mu(a), \mu(b)\} \leq \mu(c).$$

If the hyperoperation is commutative, μ is a subhypergroup if the conditions 1 and 2 are satisfied.

Theorem 5 (Cauchy's theorem)**.** *If a prime integer p divides the order of a group G, the group G has at least one element of order p.*

Theorem 6 (Sylow's theorems)**.** *If a group G is of order $p^n m$, p prime and $gcd(p, m) = 1$, then:*

(1) *The group G has at least one subgroup of order p^n called a $p-$Sylow of G.*

(2) *The $p-$Sylow are conjugate.*

(3) *There exists at least one subgroup of order p^α, $\alpha \in \{1, 2, ..., n\}$.*

(4) *The number n_p of $p-$Sylow is such that:*

(a) *$n_p \equiv 1[p]$ and*

(b) *n_p divides the integer m.*

3. The main results

Let H be a subgroup of a group $(G, \cdot)$. We know that the relation, $\mathcal{R}$ defined on G by
$$x\mathcal{R}y \iff x \cdot y^{-1} \in H,$$
is an equivalence relation and for any $x \in G$ the equivalence class of x is defined by $\overline{x} = H \cdot x$.

In the sequel, we will be concerned with the relation modulo a subgroup H defined above on a group G.

Proposition 1. *If on set G we define $\circ$ by $a \circ b = \overline{a} \cup \overline{b}$, then $(G, \circ)$ is a commutative hypergroup.*

Proof. 1. The commutativity is clear.

2. Since for any $x \in G,\ \overline{x} \neq \emptyset$, we then have $x \circ y \neq \emptyset$.

3. It can easily proved that $\forall x, y, z \in G,\ (x \circ y) \circ z = \overline{x} \cup \overline{y} \cup \overline{z}$ (the union of sets is associative).

4. For any $a, x \in G$, we have $x \in a \circ x$. So $G \subset a \circ G$ and then $G = a \circ G$ and by the commutativity we also have $G = G \circ a$ for any $a \in G$. ∎

Now, for any $a \in G$, we set

$$\begin{aligned} Q(a) &= \{(x, y) \in G^2 \mid a \in x \circ y\}, \\ q(a) &= |Q(a)|, \\ A(a) &= \sum_{(x,y)\in Q(a)} \frac{1}{|x \circ y|}, \\ \mu_A(a) &= \frac{A(a)}{q(a)}. \end{aligned}$$

Proposition 2. *For any $a \in G$, we set*

$$Q(a) = \{(x, y) \in G^2 \mid a \in x \circ y\} = Ha \times G \cup G \times Ha.$$

Proof. $(x, y) \in Q(a) \iff a \in x \circ y = \overline{x} \cup \overline{y} \iff x \in \overline{a}$ or $y \in \overline{a} \iff (x, y) \in \overline{a} \times G$ or $(x, y) \in G \times \overline{a} \iff (x, y) \in (\overline{a} \times G) \cup (G \times \overline{a})$. ∎

Proposition 3. *If H is a subgroup of order n and G is of order nm, then*

$$\forall a \in G,\ q(a) = |Q(a)| = n^2(2m - 1).$$

Proof. By the above proposition, $|Q(a)| = |Ha \times G| + |G \times Ha| - |Ha \times G \cap G \times Ha| = |H| \times |G| + |G| \times |H| - |Ha \times G \cap G \times Ha|$. On the other hand, $Ha \times G \cap G \times Ha = Ha \times Ha$ so $|Ha \times G \cap G \times Ha| = |H|^2 = n^2$. Finally, $q(a) = |Q(a)| = 2nnm - n^2 = n^2(2m - 1)$. ∎

Proposition 4. *If H is a subgroup of order n and G is of order nm, then*

$$\forall a \in G,\ |A(a)| = nm = |G|.$$

Proof. Regarding the nature of elements of $Q(a)$ for an element $a \in G$, we can divide the elements in three kinds, elements of $(ha, y), y \notin Ha$, the elements of type $(x, ha), x \notin Ha$ and the elements of the type $(ha, h'a)$ where $h, h' \in H$.

1. We have exactly $|H|(|G| - |H|) = n^2(m - 1)$ elements of the first type and we have $ha \circ y = Ha \cup Hy$ and $Ha \cap Hy = \emptyset$ so $|ha \circ y| = 2|H| = 2n$.

2. Similarly, we have exactly $|H|(|G| - |H|) = n^2(m - 1)$ elements of the second kind and $x \circ ha = Hx \cup Ha$ and $Hx \cap Ha = \emptyset$ so $|x \circ ha| = 2|H| = 2n$.

3. For the last type, we have n^2 elements and $x \circ y = Ha \cup Ha = Ha$ so $|x \circ y| = n$.

Finally,

$$A(a) = \sum_{(x,y)\in Q(a)} \frac{1}{|x \circ y|} = \sum_{(x,y) \text{ of first type}} \frac{1}{|x \circ y|} + \sum_{(x,y) \text{ of second type}} \frac{1}{|x \circ y|}$$

$$+ \sum_{(x,y) \text{ of third type}} \frac{1}{|x \circ y|} = n^2(m-1) \cdot \frac{1}{2n} + n^2(m-1) \cdot \frac{1}{2n} + n^2 \cdot \frac{1}{n} = nm. \quad \blacksquare$$

Proposition 5. *If H is a subgroup of order n and G is of order nm then the fuzzy function introduced by Corsini is constant on all elements of the group G.*

Proof. For any element $a \in G$, we have the function

$$\mu(a) = \frac{A(a)}{q(a)} = \frac{mn}{n^2(2m-1)} = \frac{m}{n(2m-1)}. \quad \blacksquare$$

From Proposition 8, we can see that the function μ depends only on the cardinalities of both the group G and the subgroup H and not on a specific elements of G, so we denote it by $\mu(nm, n)$. It is easy to prove the following corollaries.

Corollary 6. *If the order n of the group G is fixed, as a function of the order m of the subgroup H, $\mu(nm, n)$ is a decreasing function (the derivative with respect to m is negative).*

Corollary 7. *If the order m of the subgroup H is fixed, as a function of the order n of the group G, $\mu(mn, m)$ is a decreasing function (the derivative with respect to n is negative).*

The above hyperoperation will be denoted by $\circ_1$. We can also introduce a new hyperoperation on a group G denoted by $\circ_2$.

Definition 8. Let H be a subgroup of a group G. For $a, b \in G$, let we define

$a \circ_2 b = \{c \in G \mid abc \in H\}$.

$a \circ_2 b \neq \emptyset$ since $b^{-1}a^{-1} \in a \circ_2 b$.

When it is important to specify the subgroup H, $\circ_2$ will be denoted $\circ_2^H$.

Definition 9. Let H be a subgroup of a group G. For $a \in G$, let we define

$$\mu(a) = \frac{1}{\min\{n \in \mathbb{N}^* \mid a^n \in H\}}.$$

If there is no integer $n \in \mathbb{N}^*$ such that $a^n \in H$, we set $\mu(a) = 0$.

Remark 1.

1. In the case where $H = \{e\}$ the above function is such that $\mu(a) = \frac{1}{ord(a)}$ and $\mu(a) = 0$ if $ord(a)$ is infinite.

2. $h \in H$ if and only if $\mu(h) = 1$.

3. Any element a of finite order of is such $\mu(a) > 0$, since at least

$$\mu(a) \geq \frac{1}{ord(a)},\ a^{ord(a)} = e \in H.$$

Let us study the case of a cyclic group G of finite order n generated by an element a. In this case the subgroup $H = \langle a^k \rangle$ with $gcd(k, n) > 1$ otherwise H becomes the whole group G.

Proposition 10. *Let G be a group (finite or infinite) generated by a and let H be the subgroup generated by a^k. Then*

1. $gcd(m, k) = 1 \Longrightarrow \mu(a^m) = \frac{1}{k}$.

2. $\mu(a^m) = \frac{1}{k_1}$, *where* $k = dk_1,\ gcd(m, k) = d$.

Proof.
1. We have

(a) $(a^m)^k = (a^k)^m \in H$.

(b) Suppose now that there exists $k_1 < k$ such that $(a^m)^{k_1} \in H$ and then there exists $r \in \mathbb{N}$ such that $(a^m)^{k_1} = (a^k)^r$ and therefore $mk_1 = kr$. The last equality implies that k divides mk_1 but $gcd(m, k) = 1$ then by Gauss's theorem k divides k_1 which implies in particular that $k_1 \geq k$ which is a contradiction. So $k = min\{r \in \mathbb{N} \mid (a^m)^r \in H\}$.

2. Now, let $d = gcd(m, k)$. Then $\exists k_1, m_1 \in \mathbb{N}$ such that $k = dk_1$, $m = dm_1$, $gcd(k_1, m_1) = 1$.

(a) $(a^m)^{k_1} = a^{mk_1} = a^{m_1 dk_1} = (a^{dk_1})^{m_1} = (a^k)^{m_1} \in H$.

(b) Suppose that there exists $k_2 < k_1$ such that $(a^m)^{k_2} \in H$. Then, we have $(a^m)^{k_2} \in H \iff (a^m)^{k_2} = (a^k)^r$ so $mk_2 = kr$, and then $k = dk_1$ divides $mk_2 = dm_1k_2$, and then k_1 divides m_1k_2 but $gcd(k_1, m_1) = 1$, then k_1 divides k_2. We get, in particular, $k_2 \geq k_1$, contradiction. ∎

Definition 11. Let G be a group (finite or infinite) generated by a and let H be the subgroup generated by a^k. We define on G a hyperoperation:

$$a^{m_1} \circ a^{m_2} = \{x \in G \mid min\{\mu(a^{m_1}), \mu(a^{m_2})\} \leq \mu(x) \leq \max\{\mu(a^{m_1}), \mu(a^{m_2})\}\}.$$

Proposition 12. *Let G, H as in Proposition* 15 *and μ as in the above definition. Then*

1. *If* $gcd(m_1, k)=gcd(m_2, k) = 1$, *then* $a^{m_1} \circ a^{m_2} = \{x \in G \mid x = a^m,\ gcd(m, k) = 1\}$.

2. *If* $gcd(m_1, k) = 1$ *and* $gcd(m_2, k) = d > 1$, *then there exists* k_1 *such that* $k = dk_1$. *In this case,* $\mu(a^{m_1}) = \frac{1}{k}$ *and* $\mu(a^{m_2}) = \frac{1}{k_1}$, *so* $(a^{m_1}) \circ (a^{m_2}) = \{a^r \mid \frac{1}{k} \leq \mu(a^r) \leq \frac{1}{k_1}\} = \{a^r \mid gcd(r, k) = 1\} \cup \{a^r \mid gcd(r, k) = d' \leq d\}$.

3. *If* $gcd(m_1, k) = d_1 > 1$, $gcd(m_2, k) = d_2 > 1$, *let* $d_1k_1 = k$ *and* $d_2k_2 = k$. *We can suppose that* $\frac{1}{k_1} \leq \frac{1}{k_2}$. *In this case* $(a^{m_1}) \circ (a^{m_2}) = \{a^r \mid \frac{1}{k_1} \leq \mu(a^r) \leq \frac{1}{k_2}\}$ $= \{a^r \mid d_1 \leq d \leq d_2,\ d = gcd(r, k)\}$.

The next proposition has been proved in [16] (Proposition 2), but we want to prove it in the present study.

Proposition 13 (16, Prop.2)**.** *Let* G, H *as in Proposition* 15 *and* μ *as in the above definition. Then* μ *is a fuzzy subhypergroup of* $(G, \circ)$.

Proof.

1. (a) If $gcd(m_1, k) = gcd(m_2, k) = 1$ then for any $a^m \in a^{m_1} \circ a^{m_2}$ we have $gcd(m, k) = 1$ so

$$min\{\mu(a^{m_1}), \mu(a^{m_2})\} = min\{\mu(c),\ c \in a^{m_1} \circ a^{m_2}\} = \frac{1}{k}.$$

   (b) If $gcd(m_1, k) = 1$ and $gcd(m_2, k) = d > 1$ and if $k = dk_1$ by the proposition 17, any element a^m of $a^{m_1} \circ a^{m_2}$ is such that

$$\frac{1}{k} \leq \mu(a^m) \leq \frac{1}{k_1}$$

   so

$$min\{\mu(a^{m_1}), \mu(a^{m_2})\} = \frac{1}{k} \leq min\{\mu(c),\ c \in a^{m_1} \circ a^{m_2}\}.$$

   (c) If $gcd(m_1, k) = d_1 > 1, gcd(m_2, k) = d_2 > 2$, to fix the ideas, we can suppose that $d_1 \geq d_2$, so there exists two natural integers $k_1 \leq k_2$ such that $k = d_1k_1 = d_2k_2$. In this case, the proposition 17 gives $\frac{1}{k_2} \leq \mu(a^m) \leq \frac{1}{k_1}$ for any $a^m \in a^{m_1} \circ a^{m_2}$ so the following inequality holds.

$$min\{\mu(a^{m_1}), \mu(a^{m_2})\} = \frac{1}{k_2} \leq min\{\mu(c),\ c \in a^{m_1} \circ a^{m_2}\}.$$

2. (a) If $gcd(m_1, k) = gcd(m_2, k) = 1$ and $c = a^{m_1}$ then $a^{m_1} \in a^{m_1} \circ a^{m_2}$ and $min\{\mu(a^{m_1}), \mu(a^{m_2})\} = \frac{1}{k} = \mu(c)$.

   (b) If $gcd(m_1, k) = 1$ and $gcd(m_2, k) = d > 1$, it suffices to take in the definition, $c = a^{m_1}$, so $a^{m_1} \in a^{m_1} \circ a^{m_2}$ and then

$$min\{\mu(a^{m_1}), \mu(a^{m_2})\} = \frac{1}{k} \leq \mu(c) = \frac{1}{k}.$$

(c) If $gcd(m_1, k) = d_1 > 1, gcd(m_2, k) = d_2 > 2$, d_1 and d_2 are such 1) (c). As in the two precedent cases It suffices to take $c = a^{m_1}$ so $\mu(c) = \frac{1}{k_1}$, $a^{m_1} \in a^{m_2} \circ a^{m_1}$ and

$$min\{\mu(a^{m_1}), \mu(a^{m_2})\} = \frac{1}{k_1} = \mu(c). \qquad \blacksquare$$

Now, we study an example of a complete hypergroup.

Proposition 14. *Let H be a normal subgroup of a group G if $\circ_4$ is defined by $\forall a, b \in G,\ a \circ_4 b = abH$, $(G, \circ_4)$ is a hypergroup.*

Proof.

1. $(G, \circ_4)$ is a hypergroup.
   (a) It is clear that for any $a, b \in G, \emptyset \neq abH \in \mathcal{P}^*(G)$.
   (b) Let $x \in (a \circ_4 b) \circ_4 c$ then there exist $h_1, h_2 \in H$ such that $x = abh_1ch_2$. But $x = abh_1ch_2 = a(bch_1)h_2 \in a \circ_4 (bch_1)$. On the other hand, $bch_1 \in b \circ_4 c$ so $x \in a \circ_4 (b \circ_4 c)$ and this implies $(a \circ_4 b) \circ_4 c \subset a \circ_4 (b \circ_4 c)$. In the same way, we can easily get the other inclusion.
   (c) For any element $a \in G$, we have $a \circ_4 G \subset G$. Let $a, x \in G$ there exists an element $y = a^{-1}xh,\ h \in H$ then $a \circ_4 y = ayH = a(a^{-1}xh)H = xH$ and $x \in xH$ so there exists $y \in G$ such that $x \in a \circ_4 y$ and $G \subset a \circ_4 G$.
2. Now, for any x in the quotient group G/H, let $A_x = x$. We then have:
   (a) $G = \bigcup_{x \in G/H} A_x$.
   (b) If $x \neq y$ then $A_x \cap A_y = x \cap y = \emptyset$
   (c) If $(a, b) \in A_x \times A_y$ then $a \circ_4 b = abH = \overline{ab} = \overline{a}\overline{b} = xy = A_{xy}$ since xy is an equivalence class (H is normal). $\blacksquare$

Remark 2. As mentioned in [12] Proposition 3.1, If $|G| = |G/H|$ which case corresponds to $H = \{e\}$, our classes A_x are singletons and the hyperoperation $\circ_4$ is the group binary operation.

Remark 3. If the subgroup H is of index 2, the group G is commutative, the order of H is great or equal than 2. So there exists $a \in G \setminus H$ such that by the above construction we have,

$$b \circ_4 c = \begin{cases} H & \text{if } \exists h \in H \text{ such that } b = ahc^{-1} \\ aH & \text{if } \forall h \in H,\ b \neq ahc^{-1}. \end{cases}$$

Definition 15. Let H be a set endowed with two hyperoperations $*$ and $\star$.
$*$ is said to be finer than $\star$ if $\forall a, b \in H, a * b \subset a \star b$.

It is easy to prove the following lemma.

Lemma 16. *Let $H_1 \subset H_2 \subset ... \subset H_n$ be a finite sequence of normal subgroups of a group G, then $xH_i \subset xH_{i+1}$.*

Proposition 17. *Let $H_1 \subset H_2 \subset ... \subset H_n$ be a finite sequence of normal subgroups of a group G. If we denote by $\circ_i$ the hyperoperation associated to G/H_i as in Proposition 20, then $\circ_i$ is finer than $\circ_{i+1}$.*

Proof. Trivial. ∎

Let $H_1 \subset H_2 \subset ... \subset H_n$ be a finite sequence of normal subgroups of a group G. For $a \in G$, let we denote by $Q_i(a) = \{(x, y) \in G \mid a \in x \circ_i y\}$. We then have.

Proposition 18. $q_i(a) = |Q_i(a)| = |G|.|H_i|,\ A_i(a) = |G|$ *and* $\mu_i(a) = \frac{1}{|H_i|}$.

Proof. $Q_i(a)=\{(x, y) \in G\times G \mid a \in x \circ_i y\}=\{(x, x^{-1}ah^{-1}) \mid h \in H_i\}=G\times x^{-1}aH_i$ and then $|Q_i(a)|=|G| \cdot |H_i|$. Now, let $(x, x^{-1}ah) \in Q_i(a)$. Then $x \circ_i x^{-1}ah=aH$ and $|x \circ_i x^{-1}ah| = |aH| = |H|$, and since there exist exactly $|G| \cdot |H_i|$ elements of this type, we deduce $A_i(a) = \frac{|G|\cdot|H_i|}{|G|} = |G|$. The value of $\mu_i(a)$ is then given by $\mu_i(a) = \frac{|G|}{|G|\cdot|H_i|} = \frac{1}{|H_i|}$. ∎

Proposition 19. *The group G endowed with $\circ_2$ is a hypergroupoid and verifies the reproduction property.*

Proof. As seen in the definition 13, it is trivial that $\circ_2$ is an hyperoperation on G.

Let $a \in G$. For any $x \in G$, one has $x \in a \circ_2 z$ for any $z = a^{-1}hx^{-1},\ h \in H$ and then $x \in \bigcup_{z\in G} a \circ_2 z = a \circ_2 G$. ∎

Remark 4. The hyperoperation $\circ_2$ is not in general associative. To see this, it suffices to take in the symmetric group S_3 the subgroup generated by the transposition $\tau_{1,2}$, that is, $H = \{id, \tau_{1,2}\}$. In this case, it is easy to prove that:

$[(\tau_{1,2} \circ_2 \tau_{2,3}] \circ_2 s_4 = \{id, \tau_{1,2}, \tau_{1,3}, s_4\}$ and $\tau_{1,2} \circ_2 [\tau_{2,3} \circ_2 s_4] = \{\tau_{1,2}, \tau_{1,3}, s_4, s_5\}$, where the permutations s_4 and s_5 are given by

$$s_4 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix},\ s_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

Definition 20. Let G be a group, for $a, b \in G$ we define $a \circ_3 b = \langle a, b\rangle$, the subgroup generated by a and b.

The following result is well known.

Proposition 21. $(G, \circ_3)$ *is a hypergroup.*

Example 22. If we take in the proposition $G = S_3$, we have: Let

$$S_3 = \left\{ id = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \tau_{1,2} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \tau_{1,3} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \right.$$
$$\left. \tau_{2,3} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, s_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, s_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \right\}$$

$\overset{\rightarrow}{\Gamma}\circ_3$	id	$\tau_{1,2}$	$\tau_{1,3}$	$\tau_{2,3}$	s_1	s_2
id	$\{id\}$	$\{id,\tau_{1,2}\}$	$\{id,\tau_{1,3}\}$	$\{id,\tau_{2,3}\}$	$\{id,s_1,s_2\}$	$\{id,s_1,s_2\}$
$\tau_{1,2}$	$\{id,\tau_{1,2}\}$	$\{id,\tau_{1,2}\}$	S_3	S_3	S_3	S_3
$\tau_{1,3}$	$\{id,\tau_{1,3}\}$	S_3	$\{id,\tau_{1,3}\}$	S_3	S_3	S_3
$\tau_{2,3}$	$\{id,\tau_{2,3}\}$	S_3	S_3	$\{id,\tau_{2,3}\}$	S_3	S_3
s_1	$\{id,s_1,s_2\}$	S_3	S_3	S_3	$\{id,s_1,s_2\}$	$\{id,s_1,s_2\}$
s_2	$\{id,s_1,s_2\}$	S_3	S_3	S_3	$\{id,s_1,s_2\}$	$\{id,s_1,s_2\}$

$Q(id) = S_3 \times S_3$.
$Q(\tau_{1,2}) = \{\tau_{1,2}\} \times S_3 \cup S_3 \times \{\tau_{1,2}\} \cup \{\tau_{1,3}\} \times \{\tau_{2,3}, s_1, s_2\} \cup \{\tau_{2,3}, s_1, s_2\} \times \{\tau_{1,3}\}$.
$Q(\tau_{1,3}) = \{\tau_{1,3}\} \times S_3 \cup S_3 \times \{\tau_{1,3}\} \cup \{\tau_{2,3}\} \times \{\tau_{1,2}, s_1, s_2\} \cup \{\tau_{2,3}, s_1, s_2\} \times \{\tau_{2,3}\}$.
$Q(\tau_{2,3}) = \{\tau_{2,3}\} \times S_3 \cup S_3 \times \{\tau_{2,3}\} \cup \{\tau_{1,2}\} \times \{\tau_{2,3}, s_1, s_2\} \cup \{\tau_{2,3}, s_1, s_2\} \times \{\tau_{1,2}\}$.
$Q(s_1) = S_3 \times S_3$.
$Q(s_2) = S_3 \times S_3$.
So $q(id) = q(s_1) = q(s_2) = 36$ and $q(\tau_{1,3}) = q(\tau_{2,3}) = q(\tau_{1,2}) = 18$.

It is easy to prove that:

1. $A(id) = \frac{67}{6}$,
2. $A(\tau_{1,2}) = A(\tau_{1,3}) = A(\tau_{2,3}) = \frac{23}{6}$,
3. $A(s_1) = A(s_2) = \frac{17}{3}$.

We then deduce that the function μ is such that:

1. $\mu(id) = \frac{67}{216}$,
2. $\mu(\tau_{1,2}) = \mu(\tau_{1,3}) = \mu(\tau_{2,3}) = \frac{23}{108}$,
3. $\mu(s_1) = \mu(s_2) = \frac{17}{108}$.

Remark 5. *We have* $\mu(id) > \mu(\tau_{1,2}) = \mu(\tau_{1,3}) = \mu(\tau_{2,3}) > \mu(s_1) = \mu(s_2)$.

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WEAKLY EQUIVALENCE PRESERVING FUNCTORS

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Abstract. Let R be a commutative ring and T be an additive covariant functor from the category of R-modules to itself. We recall that an R-module M is said to be T-acyclic (resp. T-coacyclic) if the n-th right (resp. left) derived functor of T on M is zero for any positive integer n. Assume that T is a left (resp. right) exact functor and X, Y are two bounded to the left (resp. right) R-complexes of T-acyclic (resp. T-coacyclic) R-modules such that $X \simeq Y$. We will show that $T(X) \simeq T(Y)$. As an application, we extend the main results of Sazeedeh, Divaani-Aazar and Mohammadi that provide some new ways for computing local (co)homology modules.

Keywords: complex, completion functor, section functor, weakly equivalence preserving functors.

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1. Introduction

Throughout this paper R is a commutative ring. We denote the category of R-complexes (resp. R-modules) by $\mathcal{C}(R)$ (resp. $\mathcal{C}_0(R)$) and we use the symbol $\simeq$ for denoting quasi-isomorphisms in the category $\mathcal{C}(R)$. Let T be an additive covariant functor from the category of R-modules to itself and M be an R-module. We recall that M is said to be T-*acyclic* (resp. T-*coacyclic*) if the n-th right (resp. left) derived functor of T on M is zero, i.e. $R^nT(M) = 0$ (resp. $L_nT(M) = 0$) for any positive integer n. Assume that T is a left (resp. right) exact functor and N is an R-module. We know that for computing the n-th right (resp. left) derived functor of T on N, we can use right (resp. left) resolutions of N by T-acyclic (resp. T-coacyclic) R-modules. More precisely, let I and P be an injective resolution and projective resolution of N, respectively. Assume that E (resp. Q) is a right (resp. left) resolution of N by T-acyclic (resp. T-coacyclic) R-modules. Then we have $I \simeq E$ (resp. $P \simeq Q$) and then $T(I) \simeq T(E)$ (resp. $T(P) \simeq T(Q)$). So, for computing $R^nT(N)$ (resp. $L_nT(N)$), we can use right (resp. left) resolutions of N by T-acyclic (resp. T-coacyclic) R-modules. Now, we want to replace R-modules N by R-complexes and we will show that for an R-complex X,

i) If T is a left exact functor and $H_i(X) = 0$ for all $i \gg 0$, then for any bounded to the left R-complex E of T-acyclic R-modules such that $X \simeq E$, we have $T(E) \simeq T(I)$, where I is any injective resolution of X.

ii) If T is a right exact functor and $H_i(X) = 0$ for all $i \ll 0$, then for any bounded to the right R-complex Q of T-coacyclic R-modules such that $X \simeq Q$, we have $T(Q) \simeq T(P)$, where P is any projective resolution of X.

We say that T is a weakly equivalence preserving functor over $\mathcal{C}(R)$ if for any two R-complexes X and Y such that $X \simeq Y$, then $T(X) \simeq T(Y)$. So part i) (resp. part ii)) induces that the left (resp. right) exact functor T is a weakly equivalence preserving functor over the full subcategory of bounded to the left (resp. right) R-complexes of T-acyclic (resp. T-coacyclic) R-modules. These results are very important for computing derived functors and specially, when we apply them for famous functors such as section functor and completion functor and we get some ways for computing local (co)homology modules. Our methods extend [7, Remark, Page 2890], [6, Theorem 2.5] and [4, Corollary 4.7].

2. Prerequisites

Let R be a commutative ring. An *R-complex* X is a sequence of R-modules X_n and R-homomorphisms ∂_n^X

$$X = \cdots \longrightarrow X_{n+1} \xrightarrow{\partial_{n+1}^X} X_n \xrightarrow{\partial_n^X} X_{n-1} \longrightarrow \cdots,$$

such that $\partial_n^X \partial_{n+1}^X = 0$ for all $n \in \mathbb{Z}$. The R-complex X is said to be *bounded to the left* (resp. *right*), if there is an integer u such that $X_n = 0$ for all $n > u$ (resp. $n < u$). An R-module M is thought of as the R-complex

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow M \longrightarrow 0 \longrightarrow \cdots,$$

with M in degree zero. More information about complexes can be found in [2] and [3].

We denote the category of R-complexes (resp. R-modules) by $\mathcal{C}(R)$ (resp. $\mathcal{C}_0(R)$). The full subcategory of bounded to the right (resp. left) R-complexes of projective (resp. injective) R-modules is denoted by $\mathcal{C}_{\sqsupset}^P(R)$ (resp. $\mathcal{C}_{\sqsubset}^I(R)$). Also $\mathcal{C}_{\sqsupset}^F(R)$ denotes the full subcategory of bounded to the right R-complexes of flat R-modules. Let $T : \mathcal{C}_0(R) \longrightarrow \mathcal{C}_0(R)$ be an additive functor. We denote the full subcategory of bounded to the right (resp. left) R-complexes of T-coacyclic (resp. T-acyclic) R-modules by $\mathcal{C}_{\sqsupset}^{T-coacy.}(R)$ (resp. $\mathcal{C}_{\sqsubset}^{T-acy.}(R)$).

Let $\alpha : X \to Y$ be a morphism of R-complexes. It is called *quasi-isomorphism* if the induced morphism $H(\alpha) : H(X) \to H(Y)$ is an isomorphism, and we denote it by $\alpha : X \xrightarrow{\simeq} Y$. The *mapping cone* complex of α, $\mathrm{Cone}\,(\alpha)$, is defined by the formulas $(\mathrm{Cone}(\alpha))_l = Y_l \oplus X_{l-1}$ and $\partial_l^{\mathrm{Cone}\,(\alpha)}(y_l, x_{l-1}) = (\partial_l^Y(y_l) + \alpha_{l-1}(x_{l-1}), -\partial_{l-1}^X(x_{l-1}))$, for all integers l. Also, we say that two R-complexes X

and Y are *equivalent* and we denote it by $X \simeq Y$ if and only if there exist an R-complex Z and two quasi-isomorphisms $X \xrightarrow{\simeq} Z \xleftarrow{\simeq} Y$. One can see that the relation $\simeq$ is an equivalence relation in the category $\mathcal{C}(R)$.

Assume that X is an R-complex such that $H_i(X) = 0$ for all $i \ll 0$. A *projective resolution* of X is a quasi-isomorphism $\alpha : P \xrightarrow{\simeq} X$ where $P \in \mathcal{C}^P_{\sqsupset}(R)$. By [1, Remarks 1.7 1)], the projective resolution of X exists. Also, assume that Y is an R-complex such that $H_i(Y) = 0$ for all $i \gg 0$. An *injective resolution* of Y is a quasi-isomorphism $\alpha : Y \xrightarrow{\simeq} I$ where $I \in \mathcal{C}^I_{\sqsubset}(R)$. By [1, Remarks 1.7 2)], the injective resolution of Y exists.

3. The results

We start this section by recalling the following definition of Foxby [3, 6.37].

Definition 3.1. Let ξ and ξ' be two subcategories of $\mathcal{C}(R)$. A functor $T : \xi \longrightarrow \xi'$ is said to be *equivalence preserving* if

$$X \simeq Y \Longrightarrow T(X) \simeq T(Y)$$

for all $X, Y \in \xi$.

In the following examples, we provide some equivalence preserving functors.

Example 3.2. As for any two R-modules M and N, we have

$$M \simeq N \Longleftrightarrow M \cong N,$$

then any functor $T : \mathcal{C}_0(R) \longrightarrow \mathcal{C}_0(R)$ is an equivalence preserving functor.

Example 3.3. The following functors are equivalence preserving:

i) $\mathrm{Hom}_R(P, -) : \mathcal{C}(R) \longrightarrow \mathcal{C}(R)$ for any R-complex $P \in \mathcal{C}^P_{\sqsupset}(R)$,

ii) $\mathrm{Hom}_R(-, I) : \mathcal{C}(R) \longrightarrow \mathcal{C}(R)$ for any R-complex $I \in \mathcal{C}^I_{\sqsubset}(R)$,

iii) $F \otimes_R - : \mathcal{C}(R) \longrightarrow \mathcal{C}(R)$ for any R-complex $F \in \mathcal{C}^F_{\sqsupset}(R)$,

iv) $\mathrm{Hom}_R(-, W) : \mathcal{C}^P_{\sqsupset}(R) \longrightarrow \mathcal{C}(R)$ for any R-complex W,

v) $\mathrm{Hom}_R(U, -) : \mathcal{C}^I_{\sqsubset}(R) \longrightarrow \mathcal{C}(R)$ for any R-complex U,

vi) $V \otimes_R - : \mathcal{C}^F_{\sqsupset}(R) \longrightarrow \mathcal{C}(R)$ for any R-complex V.

Proof. See [3, 6.13, 6.15, 7.10, 6.17, 6.19, 7.14]. □

Now, we introduce weakly equivalence preserving functors.

Definition 3.4. Let ξ be a subcategory of $\mathcal{C}(R)$. A functor $T : \mathcal{C}_0(R) \longrightarrow \mathcal{C}_0(R)$ is said to be *weakly equivalence preserving* on ξ if its induced functor $T : \xi \longrightarrow \mathcal{C}(R)$ is an equivalence preserving functor.

Example 3.3 gives us some weakly equivalence preserving functors, as follows.

Example 3.5.

i) The functor $\mathrm{Hom}_R(-,W)$ is a weakly equivalence preserving on $\mathcal{C}^P_{\sqsupset}(R)$ for any R-module W.

ii) The functor $\mathrm{Hom}_R(U,-)$ is a weakly equivalence preserving on $\mathcal{C}^I_{\sqsubset}(R)$ for any R-module U.

iii) The functor $V\otimes_R -$ is a weakly equivalence preserving on $\mathcal{C}^F_{\sqsupset}(R)$ for any R-module V.

We need the following lemma for proving our main result.

Lemma 3.6. *Let $T:\mathcal{C}_0(R)\longrightarrow\mathcal{C}_0(R)$ be an additive functor. Any morphism $\alpha:X\to Y$ of R-complexes yields an isomorphism $\phi:\mathrm{Cone}(T(\alpha))\to T(\mathrm{Cone}(\alpha))$ of R-complexes.*

Proof. We assume that T is covariant. The proof for the other case is similar. Assume that $\lambda^1_i: Y_i\to \mathrm{Cone}(\alpha)_i$ and $\lambda^2_i: X_i\to \mathrm{Cone}(\alpha)_{i+1}$ are the natural monomorphisms and $\pi^1_i:\mathrm{Cone}(\alpha)_i\to Y_i$ and $\pi^2_i:\mathrm{Cone}(\alpha)_{i+1}\to X_i$ are the natural epimorphisms for any integer $i\in\mathbb{Z}$. So, one has $\pi^1_i\lambda^1_i=1_{Y_i}$, $\pi^2_i\lambda^2_i=1_{X_i}$ and $\lambda^1_i\pi^1_i+\lambda^2_{i-1}\pi^2_{i-1}=1_{\mathrm{Cone}(\alpha)_i}$. Define

$$\phi_i: T(Y_i)\oplus T(X_{i-1})\to T(Y_i\oplus X_{i-1})$$

by

$$\phi_i(v,w):=T(\lambda^1_i)(v)+T(\lambda^2_{i-1})(w)$$

for all $(v,w)\in T(Y_i)\oplus T(X_{i-1})$. We claim that

$$\phi:=(\phi_i)_i:\mathrm{Cone}(T(\alpha))\to T(\mathrm{Cone}(\alpha))$$

is an isomorphism of R-complexes. For each integer i, one can easily examine that

$$\lambda^1_{i-1}\alpha_{i-1}-\lambda^2_{i-2}\partial^X_{i-1}=\partial^{\mathrm{Cone}(\alpha)}_i\lambda^2_{i-1}$$

and $\lambda^1_{i-1}\partial^Y_i=\partial^{\mathrm{Cone}(\alpha)}_i\lambda^1_i$. Now, it is straightforward to check that the diagram

$$\begin{array}{ccc}
\mathrm{Cone}(T(\alpha))_i & \xrightarrow{\partial^{\mathrm{Cone}(T(\alpha))}_i} & \mathrm{Cone}(T(\alpha))_{i-1}\\
\downarrow{\scriptstyle \phi_i} & & \downarrow{\scriptstyle \phi_{i-1}}\\
T(\mathrm{Cone}(\alpha)_i) & \xrightarrow{T(\partial^{\mathrm{Cone}(\alpha)}_i)} & T(\mathrm{Cone}(\alpha)_{i-1})
\end{array}$$

is commutative for all integers i, and so ϕ is a morphism of R-complexes. It remains to show that each ϕ_i is an isomorphism. To this end, we define

$$\psi_i: T(Y_i\oplus X_{i-1})\to T(Y_i)\oplus T(X_{i-1})$$

by

$$\psi_i(u):=(T(\pi^1_i)(u),T(\pi^2_{i-1})(u))$$

for any $u\in T(Y_i\oplus X_{i-1})$. One can see that $\psi_i\phi_i=1_{\mathrm{Cone}(T(\alpha))_i}$ and $\phi_i\psi_i=1_{T(\mathrm{Cone}(\alpha)_i)}$ for all integers i. ∎

Proposition 3.7. *Any covariant additive exact functor $T : \mathcal{C}_0(R) \longrightarrow \mathcal{C}_0(R)$ is weakly equivalence preserving on $\mathcal{C}(R)$.*

Proof. Assume that $X \simeq Y$ for two R-complexes $X, Y \in \mathcal{C}(R)$. Then there exist a third R-complex Z and two quasi-isomorphisms, $X \xrightarrow{\alpha} Z \xleftarrow{\beta} Y$. By [2, Lemma A.1.19], $\mathrm{Cone}(\alpha)$ and $\mathrm{Cone}(\beta)$ are exact R-complexes and so $T(\mathrm{Cone}(\alpha))$ and $T(\mathrm{Cone}(\beta))$ are exact R-complexes, too. Lemma 3.6 yields that $\mathrm{Cone}(T(\alpha))$ and $\mathrm{Cone}(T(\beta))$ are exact as well. Applying again [2, Lemma A.1.19] yields that $T(\alpha)$ and $T(\beta)$ are quasi-isomorphisms and so $T(X) \simeq T(Y)$. ∎

Next, we present the main result of this paper.

Theorem 3.8. *Let $T : \mathcal{C}_0(R) \longrightarrow \mathcal{C}_0(R)$ be an additive covariant functor.*

i) *Assume that T is a right exact functor. Let Q be a bounded to the right R-complex of T-coacyclic R-modules and P be a projective resolution of Q. Then $T(Q) \simeq T(P)$.*

ii) *Assume that T is a left exact functor. Let E be a bounded to the left R-complex of T-acyclic R-modules and I be an injective resolution of E. Then $T(E) \simeq T(I)$.*

Proof. The proofs of i) and ii) are entirely similar, so we prove only part i). Set $s := \inf\{i \in \mathbb{Z} | Q_i \neq 0\}$ and $t := \inf\{i \in \mathbb{Z} | P_i \neq 0\}$. Set $u := \min\{t, s\}$. Without loss of generality, we may assume that $u = s$. As P is a projective resolution of Q, there exists a quasi-isomorphism $\alpha : P \longrightarrow Q$. Therefore, the following

$$\begin{aligned}\mathrm{Cone}(\alpha) = \cdots \longrightarrow Q_i \oplus P_{i-1} &\xrightarrow{\partial_i^{\mathrm{Cone}(\alpha)}} Q_{i-1} \oplus P_{i-2} \xrightarrow{\partial_{i-1}^{\mathrm{Cone}(\alpha)}} \cdots \\ &\xrightarrow{\partial_{s+2}^{\mathrm{Cone}(\alpha)}} Q_{s+1} \oplus P_s \xrightarrow{\partial_{s+1}^{\mathrm{Cone}(\alpha)}} Q_s \longrightarrow 0,\end{aligned}$$

is an exact R-complex by [2, Lemma A.1.19]. Clearly, any projective R-module is T-coacyclic and the class of T-coacyclic R-modules is closed under finite direct sums. Because of this, the exact R-complex $\mathrm{Cone}(\alpha)$ is a R-complex of T-coacyclic R-modules. Now, by splitting $\mathrm{Cone}(\alpha)$ into short exact sequences, we have the following short exact sequences of T-coacyclic R-modules

$$0 \longrightarrow K_{s+1} \longrightarrow Q_{s+1} \oplus P_s \xrightarrow{\partial_{s+1}^{\mathrm{Cone}(\alpha)}} Q_s \longrightarrow 0,$$

and

$$0 \longrightarrow K_{i+1} \longrightarrow Q_{i+1} \oplus P_i \xrightarrow{\partial_{i+1}^{\mathrm{Cone}(\alpha)}} K_i \longrightarrow 0,$$

where $K_i = \ker(\partial_i^{\mathrm{Cone}(\alpha)})$ for any $i > s$. By applying T over the above short exact sequences and using the assumption, we obtain the following exact sequences

$$0 \longrightarrow T(K_{s+1}) \longrightarrow T(Q_{s+1} \oplus P_s) \xrightarrow{T(\partial_{s+1}^{\mathrm{Cone}(\alpha)})} T(Q_s) \longrightarrow 0,$$

and

$$0 \longrightarrow T(K_{i+1}) \longrightarrow T(Q_{i+1} \oplus P_i) \overset{T(\partial_{i+1}^{\mathrm{Cone}(\alpha)})}{\longrightarrow} T(K_i) \longrightarrow 0.$$

By adjoining the above short exact sequences to each other, one can see that $T(\mathrm{Cone}(\alpha))$ is an exact R-complex. Hence, Lemma 3.6 yields that the R-complex $\mathrm{Cone}(T(\alpha))$ is also exact and so $T(\alpha) : T(P) \longrightarrow T(Q)$ is a quasi-isomorphism by [2, Lemma A.1.19] and so $T(P) \simeq T(Q)$. ∎

In the following corollary, we extend parts ii) and iii) of Example 3.5. Also, [6, Theorem 2.5] and [4, Corollary 4.7] are special cases of the following corollary that we mention them in Corollary 3.10.

Corollary 3.9. *Let $T : \mathcal{C}_0(R) \longrightarrow \mathcal{C}_0(R)$ be an additive covariant functor.*

i) *Assume that T is a right exact functor. Then it is weakly equivalence preserving on $\mathcal{C}_{\sqsupset}^{T-coacy.}(R)$.*

ii) *Assume that T is a left exact functor. Then it is weakly equivalence preserving on $\mathcal{C}_{\sqsubset}^{T-acy.}(R)$.*

Proof. We only prove part i), because the proof of part ii) is similar. Assume that $Q \simeq Q'$ for any two R-complexes $Q, Q' \in \mathcal{C}_{\sqsupset}^{T-coacy.}(R)$. Let P be a projective resolution of Q. Then $P \simeq Q \simeq Q'$, and so P is a projective resolution of Q' by [1, 1.1.P and 1.4.P]. Hence, $T(Q') \simeq T(P) \simeq T(Q)$ by Theorem 3.8 i). ∎

As an application of our main theorem, we prove the completion (resp. section) functor is weakly equivalence preserving on the full subcategory of bounded to the right (resp. left) R-complexes of Gorenstein flat (resp. injective) R-modules. By using these, we give some methods for computing local (co)homology R-modules which are the main theorems of [6] and [4].

Before doing this, we recall that an R-module M is said to be *Gorenstein flat* if there exists an exact R-complex F of flat R-modules such that $M \cong \mathrm{im}(F_0 \longrightarrow F_{-1})$ and $F \otimes_R I$ is exact for all injective R-modules I. Any flat R-module is Gorenstein flat. Also, an R-module N is said to be *Gorenstein injective* if there exists an exact R-complex I of injective R-modules such that $N \cong \mathrm{im}(I_1 \longrightarrow I_0)$ and $\mathrm{Hom}_R(E, I)$ is exact for all injective R-modules E. Any injective R-module is Gorenstein injective.

Let $\mathcal{D}(R)$ denote the derived category of R-complexes. Assume that $\mathfrak{a}$ is an ideal of R. The left derived functor of $\mathfrak{a}$-adic completion functor, $\Lambda^{\mathfrak{a}}(-) = \varprojlim_n(R/\mathfrak{a}^n \otimes_R -)$, exists in $\mathcal{D}(R)$ and is denoted by $\mathbf{L}\Lambda^{\mathfrak{a}}(-)$. For any R-complex X with $H_i(X) = 0$ for all $i \ll 0$, we have $\mathbf{L}\Lambda^{\mathfrak{a}}(X) := \Lambda^{\mathfrak{a}}(P)$, where P is a (every) projective resolution of X, see e.g. [5]. For any integer i, the i-th local homology module of X with respect to $\mathfrak{a}$ is defined by $H_i^{\mathfrak{a}}(X) := H_i(\mathbf{L}\Lambda^{\mathfrak{a}}(X))$. We recall that the $\mathfrak{a}$-section functor, $\Gamma_{\mathfrak{a}}(-) = \varinjlim_n \mathrm{Hom}_R(R/\mathfrak{a}^n, -)$ is a left exact functor and its right derived functor exists in $\mathcal{D}(R)$ which is denoted by $\mathbf{R}\Gamma_{\mathfrak{a}}(-)$. For any R-complex X with $H_i(X) = 0$ for all $i \gg 0$, we have $\mathbf{R}\Gamma_{\mathfrak{a}}(X) := \Gamma_{\mathfrak{a}}(I)$, where I is an (every) injective resolution of X, see e.g. [5]. For any integer i, the i-th local cohomology module of X with respect to $\mathfrak{a}$ is defined by $H^i_{\mathfrak{a}}(X) := H_{-i}(\mathbf{R}\Gamma_{\mathfrak{a}}(X))$.

Corollary 3.10. *Let $\mathfrak{a}$ be an ideal of the Noetherian ring R.*

i) $\Lambda^{\mathfrak{a}}(-)$ *is weakly equivalence preserving on the full subcategory of bounded to the right R-complexes of Gorenstein flat R-modules. Hence, for any R-complex X with $H_i(X) = 0$ for all $i \ll 0$ and any bounded to the right R-complex Q of Gorenstein flat R-modules such that $X \simeq Q$, we have $\mathbf{L}\Lambda^{\mathfrak{a}}(X) \simeq \Lambda^{\mathfrak{a}}(Q)$, and so $H^{\mathfrak{a}}_i(X) \cong H_i(\Lambda^{\mathfrak{a}}(Q))$ for all $i \in \mathbb{Z}$.*

ii) $\Gamma_{\mathfrak{a}}(-)$ *is weakly equivalence preserving on the full subcategory of bounded to the left R-complexes of Gorenstein injective R-modules. Hence, for any R-complex X with $H_i(X) = 0$ for all $i \gg 0$ and any bounded to the left R-complex E of Gorenstein injective R-modules such that $X \simeq E$, we have $\mathbf{R}\Gamma_{\mathfrak{a}}(X) \simeq \Gamma_{\mathfrak{a}}(E)$, and so $H^i_{\mathfrak{a}}(X) \cong H_{-i}(\Gamma_{\mathfrak{a}}(E))$ for all $i \in \mathbb{Z}$.*

Proof. i) Since for any integer i, $H_i(\mathbf{L}H^{\mathfrak{a}}_0(-)) \cong H_i(\mathbf{L}\Lambda^{\mathfrak{a}}(-))$ and any Gorenstein flat R-module is $\Lambda^{\mathfrak{a}}$-coacyclic by [6, Lemma 2.2 i)], then every Gorenstein flat R-module is $H^{\mathfrak{a}}_0$-coacyclic. Assume that X and Y are two bounded to the right R-complexes of Gorenstein flat R-modules such that $X \simeq Y$. Corollary 3.9 i) and [6, Lemma 2.2 ii)] imply that

$$\Lambda^{\mathfrak{a}}(X) \simeq H^{\mathfrak{a}}_0(X) \simeq H^{\mathfrak{a}}_0(Y) \simeq \Lambda^{\mathfrak{a}}(Y),$$

and so $\Lambda^{\mathfrak{a}}(-)$ is weakly equivalence preserving on the full subcategory of bounded to the right R-complexes of Gorenstein flat R-modules. Now, assume that X is an R-complex such that $H_i(X) = 0$ for all $i \ll 0$ and P is a projective resolution of X. Then $P \simeq Q$ and so the first part of our proof yields that $\Lambda^{\mathfrak{a}}(P) \simeq \Lambda^{\mathfrak{a}}(Q)$, as desired.

ii) By [7, Theorem 3.1], any Gorenstein injective R-module is $\Gamma_{\mathfrak{a}}$-acyclic. Note that in the statement of that result, the ring is assumed to be Gorenstein. But, the author has not used this assumption. So, the first part of our assertion follows from Corollary 3.9 ii). Now, assume that X is an R-complex such that $H_i(X) = 0$ for all $i \gg 0$ and I is an injective resolution of X. Then $I \simeq E$ and so $\Gamma_{\mathfrak{a}}(I) \simeq \Gamma_{\mathfrak{a}}(E)$ by Corollary 3.9 ii). ∎

In the next results, we present the analogues of Theorem 3.8 and Corollary 3.9 for additive contravariant functors. We leave their proofs to the reader.

Theorem 3.11. *Let $T : \mathcal{C}_0(R) \longrightarrow \mathcal{C}_0(R)$ be an additive contravariant functor.*

i) *Assume that T is a right exact functor. Let E be a bounded to the left R-complex of T-acyclic R-modules and I be an injective resolution of E. Then $T(E) \simeq T(I)$.*

ii) *Assume that T is a left exact functor. Let Q be a bounded to the right R-complex of T-coacyclic R-modules and P be a projective resolution of Q. Then $T(Q) \simeq T(P)$.*

Part ii) of the following corollary extends Example 3.5 i).

Corollary 3.12. *Let $T : \mathcal{C}_0(R) \longrightarrow \mathcal{C}_0(R)$ be an additive contravariant functor.*

i) *Assume that T is a right exact functor. Then it is weakly equivalence preserving on $\mathcal{C}_{\sqsubset}^{T-acy.}(R)$.*

ii) *Assume that T is a left exact functor. Then it is weakly equivalence preserving on $\mathcal{C}_{\sqsupset}^{T-coacy.}(R)$.*

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OLVER'S METHOD FOR SOLVING ROOTS OF p-ADIC POLYNOMIAL EQUATIONS

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Abstract. Let $\mathbb{Z}_p[x]$ be the set of all functions whose coefficients are in the field of p-adic integers $\mathbb{Z}_p$. This work considers a problem of finding a root of a polynomial equation $P(x) = 0$ where $P(x) \in \mathbb{Z}_p[x]$. The solution is approximated through an analogue of Olver's method for finding roots of polynomial equations $P(x) = 0$ in $\mathbb{Z}_p$.

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1. Introduction

A p-adic number, which was first introduced in 1897 by Kurt Hensel, is an extension of the field of rationals $\mathbb{Q}$ such that congruences modulo powers of a fixed prime p are related to proximity in the so called *p-adic metric*. This metric provides a totally different notion of 'closeness' or absolute value. To be precise, two p-adic numbers are said to be close when their difference is divisible by a high power of p, meaning to say, the higher the power the closer they are. This notion of closeness obviously shows that the field of p-adic numbers denoted by $\mathbb{Q}_p$ extends the ordinary arithmetic in $\mathbb{Q}$ in a way different from the extension of $\mathbb{Q}$ to the real and complex number systems $\mathbb{R}$ and $\mathbb{C}$. It is worth mentioning that the creation of p-adic numbers was actually due to an attempt to bring the ideas and techniques of power series methods into number theory. In fact, when these numbers were introduced they were considered as an exotic part of pure mathematics without any application [15] (see, e.g., [10] and [18] for applications of p-adic numbers to mathematics). Nevertheless, various applications to other fields of mathematics, especially in analysis, algebraic geometry and related areas in the applied sciences (e.g., physics and bio-informatics), have been recently proposed and discovered. For instance, the field of p-adic analysis essentially provides an alternative form of calculus (cf. [2], [17]) and the p-adic numbers appear to have some applications in modeling DNA sequences and genetic codes (cf. [4]). In 1968, Monna and van der Blij proposed to apply p-adic numbers to physics and in 1972, Beltrametti and Cassinelli investigated a model of p-adic valued quantum

mechanics from the positions of quantum logic. p-adic numbers were also found to have some important applications in quantum physics since the 1980's (cf. [15]). For more applications in the physical sciences, especially in the construction of physical models (e.g., string theory, quantum mechanics, quantum cosmology and dynamical systems), we refer the readers to [3] and [17]. One may also want to consult an article of Razikov [15] (and see the references therein) for a popular introduction to the theory of p-adic numbers.

On the other hand, in a purely theoretical aspect, Hensel's lemma provides sufficient conditions for the existence of roots in $\mathbb{Z}_p$ of polynomials in $\mathbb{Z}_p[x]$. A classical application of this lemma deals with the problem of finding roots of a p-adic number a in $\mathbb{Q}_p$ and this was in fact the subject of several recent investigations about p-adic numbers. In 2010, for instance, Knapp and Xenophontos [9] showed how classical root-finding methods from numerical analysis can be used to calculate inverses of units modulo prime powers. In the same year, Zerzaihi, Kecies and Knapp [19] applied some classical root-finding methods, such as the fixed-point method, in finding square roots of p-adic numbers through Hensel's lemma. In 2011, Zerzaihi and Kecies [20] used the secant method to find the cubic roots of p-adic numbers. These authors [8] then applied the Newton method to find the cubic roots of p-adic numbers in $\mathbb{Q}_p$. A similar problem also appeared in [6] wherein Ignacio et al. computed the square roots of p-adic numbers via the Newton-Raphson method. In [1] and [14], it was observed that none of these aforementioned works have considered the problem of finding roots of a general p-adic polynomial. So, motivated by this problem, Bacani and the author proposed in [1] an analogue of Steffensen's method in finding roots of a general p-adic polynomial equation $f(x) = 0$ in $\mathbb{Z}_p$. Meanwhile, in [14], the author described an analogue of Halley's method for approximating roots of p-adic polynomial equations $f(x) = 0$ in $\mathbb{Z}_p$. A related study which examines a p-adic analogue of Newton-Raphson's method was also considered in [13]. In this work, we offer another approach in solving a root-finding problem $f(x) = 0$ in the p-adic case. In particular, we shall show that Olver's method works well in finding a root of the equation $f(x) = 0$ in $\mathbb{Z}_p$. As in our previous investigations, we shall examine the rate of convergence of the method and show that the sequence of approximants generated through the recursion for Olver's method converges to a unique root of the equation $f(x) = 0$ in $\mathbb{Z}_p$.

The rest of the paper is structured as follows. In Section 2, we provide a brief discussion about the essentials of $\mathbb{Q}_p$. Our main contribution is formally stated and proved in Section 3. This is followed by a simple example in Section 4 illustrating the method (and its convergence) in the p-adic setting and, in Section 5, a short conclusion about our present work is stated.

2. Preliminaries

In this section we discuss briefly some important properties of $\mathbb{Q}_p$. For a more detailed discussion of the topic, we refer the readers to a book of Katok on p-adic numbers [7].

2.1. The field $\mathbb{Q}_p$

The p-adic norm $|\cdot|_p$ and the p-adic valuation v_p on $\mathbb{Q}$ are formally defined as follows.

Definition 1. Let p be a fixed prime. The p-adic norm $|\cdot|_p : \mathbb{Q} \to \{p^n : n \in \mathbb{Z}\} \cup \{0\}$ is defined as follows:

$$\forall x \in \mathbb{Q} : |x|_p = \begin{cases} p^{-v_p(x)}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases}$$

where v_p is the p-adic valuation defined by $v_p(x) = \max\{r \in \mathbb{Z} : p^r \mid x\}$. This norm induces the so-called p-adic metric d_p given by

$$\begin{aligned} d_p : \mathbb{Q} \times \mathbb{Q} &\longrightarrow \mathbb{R}^+ \\ (x, y) &\longmapsto d_p(x, y) = |x - y|_p. \end{aligned}$$

The p-adic norm $|\cdot|_p$ satisfies the following important properties (cf. [7]):

(i) $|xy|_p = |x|_p |y|_p$,

(ii) $|x + y|_p \leq \max\{|x|_p, |y|_p\}$, where equality holds if $|x|_p \neq |y|_p$ and

(iii) $|x/y|_p = |x|_p / |y|_p$.

Meanwhile, the formal definition of the field of p-adic numbers $\mathbb{Q}_p$ is given as follows.

Definition 2. The field $\mathbb{Q}_p$ of p-adic numbers is the completion of $\mathbb{Q}$ with respect to the p-adic norm $|\cdot|_p$. The elements of $\mathbb{Q}_p$ are equivalence classes of Cauchy sequences in $\mathbb{Q}$ with respect to the extension of the p-adic norm defined as $|a|_p = \lim_{n\to\infty} |a_n|_p$, where $\{a_n\}$ is a Cauchy sequence of rational numbers representing $a \in \mathbb{Q}_p$.

The following theorem provides a way to write a p-adic number in a unique representation.

Theorem 1. ([5]) *Given a p-adic number $a \in \mathbb{Q}_p$, there is a unique sequence of integers $(a_n)_{n \geq N}$, with $N = v_p(a)$, such that $0 \leq a_n \leq p - 1$ for all n and*

$$a = a_N p^N + a_{N+1} p^{N+1} + \cdots + a_n p^n + \cdots = \sum_{k=N}^{\infty} a_i p^i.$$

In view of the above statement, a p-adic number is naturally defined as a number $a \in \mathbb{Q}_p$ whose canonical expansion contains only non-negative powers of p. The set of p-adic integers is denoted by $\mathbb{Z}_p$ and is given by

$$\mathbb{Z}_p = \left\{ a \in \mathbb{Q}_p : a = \sum_{i=0}^{\infty} a_i p^i,\ 0 \leq a_i \leq p - 1 \right\} = \left\{ a \in \mathbb{Q}_p : |a|_p \leq 1 \right\}.$$

Definition 3. The group of invertible elements in $\mathbb{Z}_p$ (or the group of p-adic units) denoted by $\mathbb{Z}_p^{\times}$ is given by

$$\mathbb{Z}_p^{\times} = \left\{ a \in \mathbb{Z}_p : a = \sum_{i=0}^{\infty} a_i p^i,\ a_0 \neq 0 \right\} = \left\{ a \in \mathbb{Q}_p : |a|_p = 1 \right\}.$$

By virtue of Theorem 1 we may write, in an alternative way, a p-adic number in terms of their p-adic valuation.

Corollary 2. *Let $a \in \mathbb{Q}_p$. Then, $a = p^{v_p(a)}u$ for some $u \in \mathbb{Z}_p^{\times}$.*

The following result will be use frequently in our discussion.

Lemma 3. ([7]) *Let $a, b \in \mathbb{Q}_p$. Then, $a \equiv b \pmod{p^m} \iff |a-b|_p \leq p^{-m}$.*

2.2. Functions over $\mathbb{Q}_p$

The continuity and differentiability in the p-adic setting are described in the usual fashion. Let $X \subset \mathbb{Q}_p$. A function $f : X \to \mathbb{Q}_p$ is said to be continuous at $a \in X$ if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if $|x-a|_p < \delta$, then $|f(x)-f(a)|_p < \varepsilon$. A function f is said to be continuous on $E \subseteq X$ if f is continuous for every $a \in E$. Also, let $a \in X$ be an accumulation point of X. Then, the function f is differentiable at a if the derivative of f at a, defined by $f'(a) = \lim\limits_{x \to a} \frac{f(x)-f(a)}{x-a}$ exists. So f will be differentiable on X if $f'(a)$ exists at all $a \in X$. Evidently, any polynomial function f in $\mathbb{Q}_p[x]$ is continuous and differentiable at every $a \in \mathbb{Q}_p$ (cf. [14]).

2.3. p-Adic roots

The following are some well-known results in the study of p-adic roots.

Theorem 4. (Hensel's lemma) *Let F be a polynomial of degree $q \in \mathbb{N}$ whose coefficients are p-adic integers, i.e., $F(x) = c_0 + c_1 x + c_2 x^2 + \ldots + c_q x^q \in \mathbb{Z}_p[x]$ and $F'(x) = c_1 + 2c_2 x + 3c_3 x^2 + \ldots + q c_q x^{q-1}$ be its derivative. Suppose for $\overline{a_0} \in \mathbb{Z}_p$ we have $F(\overline{a_0}) \equiv 0 \pmod{p}$ and $F'(\overline{a_0}) \not\equiv 0 \pmod{p}$. Then, there is a unique $a \in \mathbb{Z}_p$ such that $F(a) = 0$ and $a \equiv \overline{a_0} \pmod{p}$.*

The above theorem can be found, for instance, in [16, p. 48].

Theorem 5. *A polynomial with integer coefficients has a root in $\mathbb{Z}_p$ if and only if it has an integer root modulo p^m for any $m \in \mathbb{N}$.*

For the proof of the above theorems, one can consult a text on p-adic analysis by Katok [7].

Having these ideas understood, we are now in the position to state and prove our main result in the next section.

3. Statement and proof of the main result

Olver's method is a root-finding algorithm which is iteratively defined by

$$\text{(OM)} \qquad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{1}{2}\left\{ \frac{[f(x_n)]^2 f''(x_n)}{[f'(x_n)]^3} \right\}, \qquad \forall n \in \mathbb{N}_0.$$

This method is cubically convergent; that is, for every sequence $(x_n)_{n\in\mathbb{N}_0}$ generated through the recurrence (OM), we have

$$\lim_{n\to\infty}\frac{|x_{n+1}-L|}{|x_n-L|^3}=\text{const.}>0$$

where L is the limit of the sequence $(x_n)_{n\in\mathbb{N}_0}$. So it is faster compared to the well-known Newton's (sometimes called Newton-Raphson's) method. The algorithm is named after the American mathematician Frank W.J. Olver who first introduced the method in [11].

In this section, as stated in the Introduction, we are interested in finding a root of the polynomial equation $f(x)=0$ with $f(x)\in\mathbb{Z}_p[x]$ through Olver's method. Regarding the rate of convergence of the method (in the p-adic setting), we use the following definition.

Definition 4. We say that the sequence $(x_n)_{n\in\mathbb{N}_0}$ converges to its limit L (in the p-adic sense) with order γ if $|x_{n+1}-L|_p\le p^{-\gamma^n}$.

Hereon, for simplicity, we use $a\equiv_p b$ to denote the congruence relation $a\equiv b \pmod p$. Our main result is given as follows.

Theorem 6. *Let $x_0\in\mathbb{Z}$ be such that $f(x_0)\equiv_p 0$ and $f'(x_0)\not\equiv_p 0$ where $f(x)\in\mathbb{Z}_p[x]$ is a polynomial of degree $q\in\mathbb{N}$. Define the sequence $(x_n)_{n\in\mathbb{N}_0}$ recursively by Olver's iterative formula* (OM)*. Then, we have the following results:*

(i) *For all $n\in\mathbb{N}_0$, we have $x_n\in\mathbb{Z}_p$, $f'(x_n)\not\equiv_p 0$ and*

$$f(x_n)\equiv\begin{cases}0\pmod{p^{3^n}} & \text{if } p\neq 2,\\ 0\pmod{p^{4^n}} & \text{if } p=2.\end{cases}$$

(ii) *The sequence $(x_n)_{n\in\mathbb{N}_0}$ generated by the recursion* (OM) *converges to a unique zero $\xi\equiv_p x_0$ of f in $\mathbb{Z}_p$.*

(iii) *The speed of convergence in* (ii) *is cubic for $p\neq 2$ and quartic for $p=2$.*

Before we prove the above results, we first state and prove for completeness the following lemma which is a key ingredient in our proof.

Lemma 7. ([1]) *Let $a,y\in\mathbb{Z}_p$ and suppose that $y\equiv_{p^m}0$ for some positive integer m. Then,*

$$f(a+y)\equiv_{p^{km}}\sum_{j=0}^{k-1}\frac{f^{(j)}(a)}{j!}y^j,\qquad \forall k\in\{1,2,\dots,\deg(f)\},$$

where $\deg(f)$ *denotes the degree of the polynomial $f\in\mathbb{Z}_p[x]$.*

Proof. Let $f\in\mathbb{Z}_p[x]$ with $\deg(f)=q\in\mathbb{N}$ and $k\in\{1,2,\dots,q\}$. Further, assume that $a,y\in\mathbb{Z}_p$ and suppose $y\equiv_{p^m}0$ for some positive integer m. Using the well-known Taylor expansion formula (TEF) to $f(a+y)$, we get

$$f(a+y) = f(a) + \cdots + \frac{f^{(k-1)}(a)}{(k-1)!}y^{k-1} + y^k \sum_{j=0}^{q-k} \frac{f^{(j+k)}(a)}{(j+k)!}y^j.$$

But, by assumption, $y \equiv_{p^m} 0$ for some positive integer m. Hence, $y^k \equiv_{p^{km}} 0$ for every k. Thus, we have

$$f(a+y) \equiv_{p^{km}} f(a) + \cdots + \frac{f^{(k-1)}(a)}{(k-1)!}y^{k-1},$$

as desired. ∎

In view of the above lemma, one easily finds that $f(a+y) \equiv_{p^m} f(a)$, $f(a+y) \equiv_{p^{2m}} f(a) + f'(a)y$ and $f(a+y) \equiv_{p^{3m}} f(a) + f'(a)y + f''(a)y^2/2$.

Now, we are ready to prove our main result (Theorem 6). Throughout the proof, we use $a \equiv_{p^m} b$ for $a \equiv b \pmod{p^m}$ and noting this is equivalent to $|a-b|_p \leqslant p^{-m}$.

Proof of Theorem 6. Let $x_0 \in \mathbb{Z}$ be such that $f(x_0) \equiv_p 0$ and $f'(x_0) \not\equiv_p 0$. Furthermore, define the sequence $(x_n)_{n\in\mathbb{N}_0}$ recursively by (OM). We first prove (i).

Proof of (i). In this part, we need to show that $f(x_n) \equiv_{p^{3^n}} 0$ for $p \neq 2$ and $f(x_n) \equiv_{p^{4^n}} 0$ for $p = 2$ and $f'(x_n) \not\equiv_p 0$ for all $n \in \mathbb{N}_0$. The proof of this part proceeds by induction on n. First we note that since $x_0 \in \mathbb{Z}$, then $x_0 \in \mathbb{Z}_p$. So for the basis step we have, by definition,

$$f(x_1) = f\left(x_0 - \frac{f(x_0)}{f'(x_0)} - \frac{1}{2}\left\{\frac{[f(x_0)]^2 f''(x_0)}{[f'(x_0)]^3}\right\}\right) =: f(x_0+y_0).$$

In view of Lemma 7, we have

$$f(x_0+y_0) \equiv_{p^3} f(x_0) + f'(x_0)y_0 + \frac{f''(x_0)}{2}y_0^2.$$

Note that the right hand side of the above relation can be simplified as

$$\frac{[f(x_0)]^3[f''(x_0)]^2\{4[f'(x_0)]^2 + f(x_0)f''(x_0)\}}{8[f'(x_0)]^6} =: \frac{U}{V}.$$

Clearly, for $p \neq 2$ and by the fact that $f(x_0) \equiv_p 0$ and $f'(x_0) \not\equiv_p 0$, it follows that

$$f(x_1) \equiv_{p^3} f(x_0+y_0) \equiv_{p^3} 0.$$

Now, from the form of f, we have $f''(x) = \sum_{j=0}^{q-2} = (j+1)(j+2)a_{j+2}x^j$. But since $2 \mid (j+1)(j+2)$, then $f''(x) \equiv_2 0$. Thus, for $p = 2$, we see that $U \equiv_{2^7} 0$ and $V \equiv_{2^3} 0$. Hence, $f(x_1) \equiv_{p^4} 0$ for $p = 2$. Similarly, since $f'(x_1) = f'(x_0+y_0)$, we have (by Lemma 7) $f'(x_1) \equiv_p f'(x_0)$. We have just shown that the result holds for $n = 1$.

Meanwhile, for the induction hypothesis, we suppose that for some $n_0 \in \mathbb{N}$ the following results hold:

$$\forall n \geq n_0 : \; x_n \in \mathbb{Z}_p, \quad f'(x_n) \not\equiv_p 0, \quad f(x_n) \equiv \begin{cases} 0 \pmod{p^{3^n}} & \text{if } p \neq 2, \\ 0 \pmod{p^{4^n}} & \text{if } p = 2. \end{cases}$$

Let

$$y_n := -\frac{f(x_n)}{f'(x_n)} - \frac{1}{2}\left\{\frac{[f(x_n)]^2 f''(x_n)}{[f'(x_n)]^3}\right\}.$$

Since $x_{n+1} = x_n + y_n$, it follows that

$$|x_{n+1}|_p = |x_n + y_n|_p \leq \max\{|x_n|_p, |y_n|_p\}.$$

Note, however, that we have the estimate $|y_n|_p \leq p^{-3^n}$ for any fixed prime p. So $|x_{n+1}|_p \leq \max\{1, p^{-3^n}\} = 1$. Therefore, by definition of elements in $\mathbb{Z}_p$, we have $x_{n+1} \in \mathbb{Z}_p$. On the other hand, since $f'(x_{n+1}) = f'(x_n + y_n)$, then by Lemma 7 it follows that

$$f'(x_{n+1}) = f'(x_n) + y_n \sum_{j=0}^{q-1} \frac{f^{(j+2)}(x_n)}{(j+1)!} y_n^j.$$

Taking modulo p on both sides of the above equation and using Lemma 3, we get $f'(x_{n+1}) \equiv_p f'(x_n) \not\equiv_p 0$. Moreover, since $f(x_{n+1}) = f(x_n + y_n)$ and by Lemma 7, we have the equation

$$f(x_{n+1}) \equiv_{p^{3^{n+1}}} f(x_n) + f'(x_n) y_n + \frac{f''(x_n)}{2} y_n^2.$$

Expanding the right hand side of the above relation, we get

$$\frac{[f(x_n)]^3 [f''(x_n)]^2 \left\{4[f'(x_n)]^2 + f(x_n) f''(x_n)\right\}}{8[f'(x_n)]^6} =: \frac{U_n}{V_n}.$$

Similar to what we have observed earlier, we'll obtain the congruence relation $f(x_{n+1}) \equiv_{p^{3^{n+1}}} 0$ for $p \neq 2$. Similarly, for $p = 2$ we get (as $U_n \equiv_{2^7}$ and $V \equiv_{2^3} 0$) $f(x_{n+1}) \equiv_{p^{4^{n+1}}} 0$. By principle of induction, it now follows that

$$\forall n \in \mathbb{N}_0 : \; x_n \in \mathbb{Z}_p, \quad f'(x_n) \not\equiv_p 0, \quad f(x_n) \equiv \begin{cases} 0 \pmod{p^{3^n}} & \text{if } p \neq 2, \\ 0 \pmod{p^{4^n}} & \text{if } p = 2. \end{cases}$$

This validates our first result (i).

Proof of (ii). For the second part we need to show that, with the same assumption as in the first part, the sequence $(x_n)_{n\in\mathbb{N}_0}$ defined recursively by (OM) converges to a unique zero $\xi \equiv_p x_0$ of f in $\mathbb{Z}_p$. To establish this result, we first prove that $(x_n)_{n\in\mathbb{N}_0}$ is Cauchy. So we proceed as follows. Note that

$$\begin{aligned} |x_{n+1} - x_n|_p &= \left| x_n - \frac{f(x_n)}{f'(x_n)} - \frac{1}{2}\left\{\frac{[f(x_n)]^2 f''(x_n)}{[f'(x_n)]^3}\right\} - x_n \right|_p \\ &= \frac{|2f(x_n)[f'(x_n)]^2 - [f(x_n)]^2 f''(x_n)|_p}{|2[f'(x_n)]^3|_p} =: \frac{|A|_p}{|B|_p}. \end{aligned}$$

Here we distinguish two cases: (C.1) $p \neq 2$ and (C.2) $p = 2$.

Case 1. If $p \neq 2$, then $A \equiv_{p^{3^n}} 0$ since $f(x_n) \equiv_{p^{3^n}} 0$ by (i). Furthermore, $B \not\equiv_p 0$ since $f'(x_n) \not\equiv_p 0$ by (i) and $p \neq 2$. Hence, from Lemma 3, we obtain the estimate $|x_{n+1} - x_n|_p \leq p^{-3^n}$. Letting $n \to \infty$, we see that $\lim_{n\to\infty} |x_{n+1} - x_n|_p = 0$, i.e., $(x_n)_{n\in\mathbb{N}_0}$ is Cauchy. So, for the case $p \neq 2$, we have $(x_n)_{n\in\mathbb{N}_0}$ is Cauchy.

Case 2. If, on the other hand, $p = 2$, then $A \equiv_{p^{4^{n+1}}} 0$ and $B \equiv_p 0$. Thus, by Lemma 3, we have $|A|_p \leq p^{-4^n-1}$ and $|B|_p \leq p^{-1}$. Then, $|A|_p/|B|_p \leq p^{-4^n}$ and so $|x_{n+1} - x_n|_p \to 0$ as $n \to \infty$. Therefore, for $p = 2$, $(x_n)_{n\in\mathbb{N}_0}$ is again a Cauchy sequence.

We see that, in any case, the sequence $(x_n)_{n\in\mathbb{N}_0}$ is Cauchy. Thus, it must converge to some number ξ in $\mathbb{Z}_p$. Finally, the uniqueness of the zero $\xi \equiv_p x_0$ of f in $\mathbb{Z}_p$ follows directly from the uniqueness of the sequence $(x_n)_{n\in\mathbb{N}_0}$.

Proof of (iii). As we have seen from the previous item, we have the estimate

$$\forall n \in \mathbb{N}_0 : \quad |x_{n+1} - x_n| \leq \begin{cases} p^{-3^n}, & \text{if } p \neq 2, \\ p^{-4^n}, & \text{if } p = 2. \end{cases}$$

Hence, it follows immediately that the sequence of approximants $(x_n)_{n\in\mathbb{N}_0}$ converges to a unique zero $\xi \equiv_p x_0$ of f in $\mathbb{Z}_p$ cubically for $p \neq 2$ and in quartic sense for $p = 2$. This completes the proof of the theorem. ∎

4. Example

To simply illustrate the method (and its convergence), we consider the root-finding problem $f(x) = 0$ in $\mathbb{Z}_{17}$ where $f(x) = x^3 - x - 1$. We consider several values for the initial iterate x_0 and see how many iterations does it needs before the sequence of approximants $(x_n)_{n\in\mathbb{N}_0}$ converges to a unique solution $\xi \equiv_p x_0$ of the equation $f(x) = 0$ in $\mathbb{Z}_{17}$.

First, we note that the sequence of approximants $(x_n)_{n\in\mathbb{N}_0}$ with initial condition x_0 is given by the recursion

$$x_{n+1} = x_n - \frac{x_n^3 - x_n - 1}{3x_n^2 - 1} - \frac{1}{2}\left\{\frac{(x_n^3 - x_n - 1)^2(6x_n)}{(3x_n^2 - 1)^3}\right\}.$$

One can easily verify that the assumptions in Theorem 6 are satisfied. In fact, we have $f(5) = 119 \equiv 0 \pmod{17}$ and $f'(5) = 74 \not\equiv 0 \pmod{17}$. Using the initial values $x_0 = 22, 39, 56, 73, 90$ ($\equiv_{17} 5$), we get the following table.

x_0	x_1	x_2	x_3	x_4	x_4	x_6	x_7	x_8	x_9
22	1	4	8	13	6	16	**5**		
39	8	13	6	16	**5**				
56	10	7	1	4	8	13	6	16	**5**
73	8	13	6	16	**5**				
90	11	3	**5**						

Observe from the above table that the n-th approximant x_n eventually takes the value of 5 after some iteration. Note that this value is, in fact, the unique root of the polynomial equation $f(x) = 0$ in $\mathbb{Z}_{17}$. Furthermore, notice from above table that it requires (at least) 7 (resp. $4, 9, 4, 3$) iterations before the sequence of approximants $(x_n)_{n\in\mathbb{N}_0}$ converges to the unique solution $\xi = 5$ of $f(x)=x^3-x-1=0$ in $\mathbb{Z}_{17}$ given that the initial iterate is $x_0 = 22$ (resp. $39, 56, 73, 90$). These results, in addition, corroborate our findings stated in Theorem 6.

5. Summary

We have considered in this work the problem of finding a p-adic root of a general polynomial equation $P(x) = 0$ with $P(x) \in \mathbb{Z}_p[x]$. It was shown that the sequence $(x_n)_{n\in\mathbb{N}_0}$ defined recursively by the iterative formula of Olver's method converges to a unique zero $\xi \equiv_p x_0$ of P in $\mathbb{Z}_p$. Moreover, we found that the p-adic analogue of Olver's method has rates of convergence of order 3 and 4 for values of $p \neq 2$ and $p = 2$, respectively. For future studies, it might be interesting to investigate whether Olver's method is indeed more efficient in computing p-adic roots compared to other well-known root-finding algorithms such as the Newton-Raphson's method [13] and Halley's method [14]. This, in turn, provides considerable interests to further examine these methods in the p-adic setting.

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SOLITONS AND OTHER SOLUTIONS TO NONLINEAR PDEs USING $(\frac{G'}{G})$-EXPANSION METHOD

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Abstract. In this article, we apply $(\frac{G'}{G})$-expansion method to construct exact traveling wave solutions with parameters of four nonlinear PDEs having non-integer balance numbers, namely, the convection-diffusion-reaction equation with power-law nonlinearity with density-independent (or density-dependent) diffusion, and the generalized KdV-mKdV equation with any order (or with higher-order) nonlinear terms. When the parameters take up special values, the solitary wave solutions as well as the trigonometric and rational function solutions are derived from the exact traveling wave solutions.The used method in this article presents a wider applicability for handling nonlinear wave equations.

Keywords: homogeneous balance principle; $(\frac{G'}{G})$-expansion method; nonlinear PDEs; exact traveling wave solutions; solitary wave solutions; non-integer balance numbers; trigonometric function solutions; rational function solutions.

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1. Introduction

It is significant to looking for the exact traveling wave solutions for nonlinear partial differential equations (PDEs) in nonlinear sciences. These equations are widely used to describe many important phenomena and dynamic processes in physics, chemistry, biology, fluid mechanics plasma, optical fibers and other areas of engineering. As mathematical models of the phenomena, the investigation of exact traveling wave solutions of these equations will help us to understand these phenomena better. In recent decades, various effective approaches have been developed to construct the exact traveling wave solutions of these nonlinear equations, such as the exp-function methods [1]-[7], the sine-cosine method [8], [9], the homogeneous

balance method [10], [11], the tanh-sech method [12], [13], the extended tanh-coth method [14], [15], the $(\frac{G'}{G})$-expansion method [16]-[18], the modified simple equation method [19], [20], the multiple exp-function method [21], [22], the modified Kudryashov method [23], the method of soliton ansatz [24]-[46], and so on.

The objective of this article is to use $(\frac{G'}{G})$-expansion method [47]-[49] to find the exact traveling wave solutions and the solitary wave solutions of nonlinear PDEs having non-integer balance numbers, such as the convection-diffusion-reaction equation [50] with power-law nonlinearity and with density-independent (or density-dependent) diffusion, and the generalized KdV-mKdV equation [51]-[53] with any order (or with higher-order) nonlinear terms.

This article is organized as follows: In Section 2, the description of $(\frac{G'}{G})$-expansion method is given. In Section 3, we apply this method to solve the four nonlinear PDEs having non-integer balance numbers indicated above. In Section 4, physical explanations of some results are presented. In Section 5, conclusions are obtained.

2. Description of $(\frac{G'}{G})$-expansion method

Suppose that a nonlinear PDE has the form

$$F(u, u_t, u_x, u_{tt}, u_{xx}, ...) = 0, \tag{2.1}$$

where $u = u(x,t)$ is an unknown function, F is a polynomial in $u(x,t)$ and its partial derivatives in which the highest order derivatives and nonlinear terms are involved.

The main steps of the used method are described as follows [47]-[49]:

Step 1. We use the wave transformation

$$u(x,t) = U(\xi), \quad \xi = kx + wt, \tag{2.2}$$

where k and w are nonzero constants, to reduce Eq. (2.1) into the following nonlinear ordinary differential equation (ODE):

$$P(U, U', U'', ...) = 0, \tag{2.3}$$

where P is a polynomial in $U(\xi)$ and its total derivatives $U', U'', ...$, such that $U' = \frac{dU}{d\xi}$, $U'' = \frac{d^2U}{d\xi^2}$ and so on.

Step 2. We suppose that Eq. (2.3) has the formal solution:

$$U(\xi) = A\left[\frac{G'(\xi)}{G(\xi)}\right]^m, \tag{2.4}$$

and $G(\xi)$ satisfies the linear ODE:

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0, \tag{2.5}$$

where A, m, λ and μ are constants to be determined later.

Step 3. Balancing the highest order derivatives with the highest nonlinear terms in Eq. (2.3), we can find the balance number m in Eq. (2.4).

Step 4. Substituting Eq. (2.4) along with Eq. (2.5) into Eq. (2.3) and setting all the coefficients of powers of $(\frac{G'}{G})$ to zero, we have a set of algebraic equations which can be solved to find the values of A, λ, μ, k and w.

Step 5. We solve the ODE (2.5) to find the values of the ratio $(\frac{G'}{G})$. Consequently, we can find the exact traveling wave solutions and the solitary wave solutions of Eq. (2.1).

3. Applications

In this section, we will apply the method described in Sec.2, to find exact traveling wave solutions and the solitary wave and singular solitary wave solutions of the following nonlinear PDEs:

Example 3.1. The convection-diffusion-reaction equation with power-law nonlinearity and with density-independent diffusion.

This equation is well-known [54] and has the form:

$$u_t = [u(b_0 + b_1 u^p)]_x + u_{xx} + u(1-u^p)(c_0 + c_1 u^p) \ , \ p > 0, \tag{3.1}$$

where p, b_0, b_1, c_0, c_1 are real numbers. Eq. (3.1) is a general equation which can include the Fisher equation and a number of other well-known convection-diffusion-reaction equations, such as the Newell-whitehead equation, the Zeldovich equation and the Nagumo equation. Eq. (3.1) has been discussed in [54] using a simplest equation method based on the Ricatti and Bernoulli equations, where exact traveling wave solutions have been found. Eq. (3.1) has not been discussed elsewhere using the method of Sec.2. Let us now, solve Eq. (3.1) by using the method of Section 2. To this aim, we use the wave transformation

$$u(x,t) = U(\xi), \qquad \xi = x - ct, \tag{3.2}$$

where c is a nonzero constant to be determined later referred as the velocity of the traveling wave propagation, to reduce Eq. (3.1) into the following nonlinear ODE:

$$U'' + b_1(p+1)U'U^p + (c+b_0)U' + c_0 U + (c_1 - c_0)U^{p+1} - c_1 U^{2p+1} = 0. \tag{3.3}$$

Balancing U'' with U^{2p+1} yields the balance number m as follows:

$$m = \frac{1}{p}, \ \ p > 0, \tag{3.4}$$

which is non-integer. From (2.4) and (3.4), we assume that the solution of Eq. (3.3) has the form

$$U(\xi) = A\left[\frac{G'(\xi)}{G(\xi)}\right]^{\frac{1}{p}}, \ \ p > 0, \tag{3.5}$$

where $G(\xi)$ satisfies Eq. (2.5). Substituting (3.5) along with Eq. (2.5) into Eq. (3.3) and according to the homogeneous balance principle [55], collecting all the coefficients of powers of $\left[\frac{G'(\xi)}{G(\xi)}\right]$ and setting them to zero, we have the following algebraic equations:

$$(3.6)\quad \left(\frac{G'}{G}\right)^{\frac{1}{p}} : (\lambda^2+2\mu)-\lambda P(c+b_0)+c_0p^2-\mu b_1p(p+1)A^p=0,$$

$$(3.7)\quad \left(\frac{G'}{G}\right)^{\frac{1}{p}-1} : [\lambda(2-p)-p(c+b_0)]\mu=0,$$

$$(3.8)\quad \left(\frac{G'}{G}\right)^{\frac{1}{p}+1} : \lambda(2+p)-p(c+b_0)+[p^2(c_1-c_0)-p\lambda b_1(p+1)]A^p=0,$$

$$(3.9)\quad \left(\frac{G'}{G}\right)^{\frac{1}{p}-2} : \mu^2(1-p)=0,$$

$$(3.10)\quad \left(\frac{G'}{G}\right)^{\frac{1}{p}+2} : (1+p)-b_1p(p+1)A^p-c_1p^2A^{2p}=0$$

From (3.7) or (3.9) we deduce that

$$(3.11)\quad \mu=0\ .$$

From (3.8) and (3.10) we have the equation

$$(3.12)\quad \lambda(2+p)-\lambda[(p+1)-c_1p^2A^{2p}]-p(c+b_0)+p^2(c_1-c_0)A^p=0,$$

which can be simplified to become

$$(3.13)\quad \lambda\, c_1p^2A^{2p}+p^2(c_1-c_0)A^p-p(c+b_0)+\lambda=0.$$

Substituting (3.11) into (3.6) we have

$$(3.14)\quad \lambda^2-\lambda p(c+b_0)+p^2c_0=0.$$

Multiply Eq.(3.13) by λ and using Eq.(3.14) we have

$$(3.15)\quad c_1p^2\lambda^2A^{2p}+\lambda p^2A^p(c_1-c_0)-c_0p^2=0.$$

From (3.10) and (3.15) we have the condition

$$(3.16)\quad \frac{c_1p^2\lambda^2}{c_1p^2}=\frac{\lambda p^2(c_1-c_0)}{b_1p(p+1)}=\frac{c_0p^2}{p+1},$$

which gives

$$(3.17)\quad \lambda^2=\frac{\lambda p(c_1-c_0)}{b_1(p+1)}=\frac{c_0p^2}{p+1},$$

where $c_1 \neq c_0$. By solving Eq. (3.15) and using relations (3.17) we deduce that

$$(3.18)\quad A^P = \frac{c_0}{\lambda c_1} \quad or \quad A^P = \frac{-1}{\lambda},$$

where

$$(3.19)\quad \lambda = \frac{p(c_1 - c_0)}{b_1(p+1)} \quad and \quad \lambda^2 = \frac{c_0 p^2}{p+1}.$$

Now, from (3.14) and (3.19) we deduce that the velocity c is given by

$$(3.20)\quad c = \frac{b_1 c_0(p+2)}{(c_1 - c_0)} - b_0,$$

and consequently ξ is given by

$$(3.21)\quad \xi = x - ct = x - \left[\frac{c_0 b_1(p+2)}{(c_1 - c_0)} - b_0\right] t.$$

The exact traveling wave solution of Eq. (3.1) is given by

$$(3.22)\quad U(\xi) = \left[-\frac{c_0}{c_1}\left(\frac{A_1\, e^{-\lambda\xi}}{A_0 + A_1\, e^{-\lambda\xi}}\right)\right]^{\frac{1}{p}},$$

or

$$(3.23)\quad U(\xi) = \left[\frac{A_1\, e^{-\lambda\xi}}{A_0 + A_1\, e^{-\lambda\xi}}\right]^{\frac{1}{p}},$$

where A_0 and A_1 are constants of integration. Setting $\frac{A_0}{A_1} = \pm e^{-\lambda\xi_0}$, where ξ_0 is a constant, then we have the solitary wave solutions of Eq. (3.1) in the forms

$$(3.24)\quad U(\xi) = \left\{\frac{-c_0}{2c_1}\left[1 - \tanh\frac{\lambda(\xi - \xi_0)}{2}\right]\right\}^{\frac{1}{p}},$$

or

$$(3.25)\quad U(\xi) = \left\{\frac{1}{2}\left[1 - \tanh\frac{\lambda(\xi - \xi_0)}{2}\right]\right\}^{\frac{1}{p}},$$

while, the singular solitary wave solutions of Eq. (3.1) have the forms

$$(3.26)\quad U(\xi) = \left\{\frac{-c_0}{2c_1}\left[1 - \coth\frac{\lambda(\xi - \xi_0)}{2}\right]\right\}^{\frac{1}{p}},$$

or

$$(3.27)\quad U(\xi) = \left\{\frac{1}{2}\left[1 - \coth\frac{\lambda(\xi - \xi_0)}{2}\right]\right\}^{\frac{1}{p}},$$

where $c_0/c_1 < 0$.

Example 3.2. The convection-diffusion-reaction equation with power-law nonlinearity and with density-dependent diffusion

This equation is well-known [54] and has the form:

$$u_t = [u(b_0 + b_1 u^p)]_x + (u^{p+1})_{xx} + u^{1-p}(1-u^p)(c_0 + c_1 u^p),\ p > 0\ , \tag{3.28}$$

where p, b_0, b_1, c_0, c_1 are real numbers. Eq. (3.28) has been discussed in [54] using a simplest equation method based on the Riccati and Bernoulli equations. Eq. (3.28) has not been investigated elsewhere using the method of Section 2. Let us now solve Eq. (3.28) using the method of Section 2. To this aim, we use the wave transformation (3.2) to reduce Eq. (3.28) into the following ODE:

$$\begin{aligned}(p+1)U^pU'' + b_1(p+1)U'U^p + (c+b_0)U' + p(p+1)U^{p-1}U'^2 \\ + c_0U^{1-p} + (c_1-c_0)U - c_1U^{p+1} = 0.\end{aligned} \tag{3.29}$$

Balancing U^pU'' with U^{1-p} gives the balance number m as follows:

$$m = \frac{-1}{p},\ \ p > 0, \tag{3.30}$$

which is non-integer. From (2.4) and (3.30), we assume that the solution of Eq. (3.29) has the form

$$U(\xi) = A\left[\frac{G'(\xi)}{G(\xi)}\right]^{\frac{-1}{p}},\ \ p > 0, \tag{3.31}$$

where $G(\xi)$ satisfies Eq. (2.5). Substituting (3.31) along with Eq. (2.5) into Eq. (3.29) and according to the homogeneous balance principle [55], collecting all the coefficients of powers of $[\frac{G'(\xi)}{G(\xi)}]$ and setting them to zero, we have the following algebraic equations:

$$\left(\frac{G'}{G}\right)^{\frac{-1}{p}} : A^p\,[b_1p(p+1) + \lambda(p+1)(p+2)] + \lambda P(c+b_0) - (c_0-c_1)p^2 = 0, \tag{3.32}$$

$$\begin{aligned}\left(\frac{G'}{G}\right)^{\frac{-1}{p}-1} &: \mu\ p(c+b_0) - A^p[c_1\ p^2 - \lambda\ b_1p(p+1) \\ &\quad - (p+1)(\lambda^2+2\mu) - \lambda^2\ p(p+1) - 2\mu p(p+1)] = 0,\end{aligned} \tag{3.33}$$

$$\left(\frac{G'}{G}\right)^{\frac{-1}{p}+1} : p(c+b_0) + c_0\ p^2A^{-p} + (p+1)A^p = 0, \tag{3.34}$$

$$\left(\frac{G'}{G}\right)^{\frac{-1}{p}-2} : A^p\mu[b_1p(p+1) + \lambda(p+1)(p+2) + 2\lambda p(p+1)] = 0, \tag{3.35}$$

$$\left(\frac{G'}{G}\right)^{\frac{-1}{p}-3} : \mu^2(p+1)(2p+1)A^{p+1} = 0. \tag{3.36}$$

From (3.35) or (3.36) we deduce that

(3.37) $\mu = 0.$

Now, Eq. (3.33) reduces to

(3.38) $\lambda^2(p+1)^2 + \lambda b_1 p(p+1) - c_1 p^2 = 0.$

Multiply Eq. (3.34) by A^p, we get the equation

(3.39) $(p+1)A^{2p} + p(c+b_0)A^p + c_0\ p^2 = 0.$

Multiply Eq. (3.39) by λ and use Eq. (3.32) we have the equation

(3.40) $\lambda(p+1)A^{2p} + [(c_0 - c_1)\ p^2 - \{b_1 p(p+1) + \lambda(p+1)(p+2)\}A^p]A^p + \lambda c_0 p^2 = 0,$

which can be rewritten in the form

(3.41) $(p+1)[\lambda(p+1) + b_1 p]A^{2p} - (c_0 - c_1)\ p^2 A^p - \lambda c_0 p^2 = 0.$

With the aid of (3.38), we can solve Eq. (3.41) to get

(3.42) $A^p = \dfrac{\lambda c_0}{c_1} \quad or \quad A^p = -\lambda,$

where λ can be determined by solving Eq. (3.38) to get the values

(3.43) $\lambda = \dfrac{p}{2(p+1)}\left[-b_1 \pm \sqrt{b_1^2 + 4c_1}\right] \quad and \quad c_1 \neq 0.$

In the case of $A^p = \frac{\lambda c_0}{c_1}$, we deduce from (3.39) that the velocity c is given by

(3.44) $c = -\left[\dfrac{\lambda^2 c_0(p+1) + c_1^2 p^2}{\lambda c_1 p} + b_0\right],$

while in the case of $A^p = -\lambda$ we deduce from(3.39) that the velocity c is given by

(3.45) $c = \left[\dfrac{\lambda^2(p+1) + c_0 p^2}{\lambda p} - b_0\right].$

Note that the values (3.44) and (3.45) can be found also if we use the Eqs. (3.32) and (3.38). Now, the exact traveling wave solutions of Eq. (3.28) are

(3.46) $U(\xi) = \left[-\dfrac{c_1}{c_0}\left(\dfrac{A_1\ e^{-\lambda\xi_1}}{A_0 + A_1\ e^{-\lambda\xi_1}}\right)\right]^{\frac{-1}{p}},$

where ξ_1 is given by

(3.47) $\xi_1 = x - ct = x + \left[\dfrac{\lambda^2 c_0(p+1) + c_1^2 p^2}{\lambda c_1 p} + b_0\right]t,$

or

$$U(\xi)=\left[\frac{A_1\ e^{-\lambda\xi_2}}{A_0+A_1\ e^{-\lambda\xi_2}}\right]^{\frac{-1}{p}},$$

where ξ_2 is given by

$$\xi_2=x-ct=x-\left[\frac{\lambda^2(p+1)+c_0p^2}{\lambda p}-b_0\right]t, \tag{3.48}$$

and λ is given by (3.43). Setting $\frac{A_0}{A_1}=\pm e^{-\lambda\xi_0}$, where ξ_0 is a constant, then we have the solitary wave solutions of Eq. (3.28) as follows:

$$U(\xi)=\left\{\frac{-c_1}{2c_0}\left[1-\tanh\frac{\lambda(\xi_1-\xi_0)}{2}\right]\right\}^{\frac{-1}{p}}, \tag{3.49}$$

or

$$U(\xi)=\left\{\frac{1}{2}\left[1-\tanh\frac{\lambda(\xi_2-\xi_0)}{2}\right]\right\}^{\frac{-1}{p}}, \tag{3.50}$$

while the singular solitary wave solutions of Eq. (3.28)are given by:

$$U(\xi)=\left\{\frac{-c_1}{2c_0}\left[1-\coth\frac{\lambda(\xi_1-\xi_0)}{2}\right]\right\}^{\frac{-1}{p}}, \tag{3.51}$$

or

$$U(\xi)=\left\{\frac{1}{2}\left[1-\coth\frac{\lambda(\xi_2-\xi_0)}{2}\right]\right\}^{\frac{-1}{p}}, \tag{3.52}$$

where $c_1/c_0<0$.

Example 3.3. The generalized KdV-mKdV equation with any order nonlinear terms

This equation is well-known [51]-[53] and has the form:

$$u_t+(\alpha+\beta u^p+\gamma u^{2p})_{u_x}+u_{xxx}=0,\ p>0\ , \tag{3.53}$$

where α,β,γ and p are constants. Eq. (3.53) has been investigated in [51] using the homogeneous balance principle combined with a sub-ODE method and in [53] using the $(\frac{G'}{G})$-expansion method which is absolutely different from $(\frac{G'}{G})$-expansion method described in Section 2. Eq. (3.53) has not been discussed elsewhere using the method in Section 2. Let us now solve it using the method of Section 2. To this aim, we use the wave transformation (3.2) to reduce Eq. (3.53) into the following ODE:

$$(\alpha-c)U'+\beta U'U^p+\gamma U'U^{2p}+U'''=0. \tag{3.54}$$

Integrating (3.54) once with respect to ξ with vanishing the constant of integration, we get

$$(3.55)\quad (\alpha - c)U + \left(\frac{\beta}{p+1}\right)U^{p+1} + \left(\frac{\gamma}{2p+1}\right)U^{2p+1} + U'' = 0.$$

Balancing U'' with U^{2p+1} yields the balance number m as follows:

$$(3.56)\quad m = \frac{1}{p}, \quad p > 0,$$

which is non-integer. From (2.4) and (3.56), we assume that the solution of Eq. (3.55) has the form

$$(3.57)\quad U(\xi) = A\left[\frac{G'(\xi)}{G(\xi)}\right]^{\frac{1}{p}}, \quad p > 0,$$

where $G(\xi)$ satisfies Eq. (2.5). Substituting (3.57) along with Eq. (2.5) into Eq. (3.55) and according to the homogeneous balance principle [55], collecting all the coefficients of powers of $\left[\frac{G'(\xi)}{G(\xi)}\right]$ and setting them to zero, we have the following algebraic equations:

$$(3.58)\quad \left(\frac{G'}{G}\right)^{\frac{1}{p}-2} : A\mu^2(1-p^2)(1+2p) = 0,$$

$$(3.59)\quad \left(\frac{G'}{G}\right)^{\frac{1}{p}+2} : \gamma p^2 A^{2p}(1+p) + (1+2p)(p+1)^2 = 0,$$

$$(3.60)\quad \left(\frac{G'}{G}\right)^{\frac{1}{p}+1} : \beta p^2 A^p(1+2p) + \lambda(1+2p)(p+1)(p+2) = 0,$$

$$(3.61)\quad \left(\frac{G'}{G}\right)^{\frac{1}{p}} : (\alpha - c)p^2(p+1)(1+2p) + (\lambda^2 + 2\mu)(1+2p)(p+1) = 0\ ,$$

$$(3.62)\quad \left(\frac{G'}{G}\right)^{\frac{1}{p}-1} : \lambda\mu(2-p)(p+1)(1+2p) = 0.$$

From (3.58) or (3.62) we deduce that

$$(3.63)\quad \mu = 0.$$

Eq. (3.59) gives

$$(3.64)\quad A^{2p} = -\frac{(p+1)(1+2p)}{\gamma p^2}\ ,$$

while (3.60) gives

$$(3.65)\quad A^p = -\frac{\lambda(p+1)(2+p)}{\beta p^2}\ ,$$

From (3.64) and (3.65) we conclude that

$$\lambda^2 = -\frac{\beta^2 p^2 (2p+1)}{\gamma(p+1)(p+2)^2}. \tag{3.66}$$

Substituting (3.63) and (3.66) into Eq. (3.61) we have the velocity c as follows:

$$c = \alpha - \frac{\beta^2 (2p+1)}{\gamma(p+1)(p+2)^2} \ , \tag{3.67}$$

and consequently ξ is given by

$$\xi = x - ct = x - \left[\alpha - \frac{\beta^2 (2p+1)}{\gamma(p+1)(p+2)^2}\right] t. \tag{3.68}$$

From (3.65) and (3.66) we deduce that

$$A^p = \frac{\beta(2p+1)}{\lambda\gamma(p+2)}. \tag{3.69}$$

Now, the exact traveling wave solution of Eq. (3.53) is given by

$$U(\xi) = \left[-\frac{\beta(2p+1)}{\gamma(p+2)}\left(\frac{A_1\, e^{-\lambda\xi}}{A_0 + A_1\, e^{-\lambda\xi}}\right)\right]^{\frac{1}{p}}, \tag{3.70}$$

where A_0 and A_1 are arbitrary constants of integration.

Choose $\frac{A_0}{A_1} = \pm e^{-\lambda\xi_0}$, where ξ_0 is a constant, we have the solitary wave solution of Eq. (3.53) in the form

$$U(\xi) = \left\{-\frac{\beta(2p+1)}{2\gamma(p+2)}\left[1 - \tanh\frac{\lambda(\xi-\xi_0)}{2}\right]\right\}^{\frac{1}{p}}, \tag{3.71}$$

and the singular solitary wave solution of Eq. (3.53) in the form

$$U(\xi) = \left\{-\frac{\beta(2p+1)}{2\gamma(p+2)}\left[1 - \coth\frac{\lambda(\xi-\xi_0)}{2}\right]\right\}^{\frac{1}{p}}, \tag{3.72}$$

provided $\beta/\gamma < 0$, and λ, ξ are given by (3.66) and (3.68) respectively.

Example 3.4. The generalized KdV-mKdV equation with higher-order nonlinear terms

This equation is well-known [52] and has the form:

$$u_t + (\varepsilon u^{-2p} + \delta u^{-p} + \alpha + \beta u^p + \gamma u^{2p})_{u_x} + u_{xxx} = 0, \quad p > 1 \ , \tag{3.73}$$

where $\varepsilon, \delta, \alpha, \beta, \gamma$ and p are real numbers. If $\varepsilon = \delta = 0$, then Eq. (3.73) reduces to the generalized KdV-mKdV equation (3.53) with any order nonlinear terms. Eq. (3.73) has been investigated in [52] using the homogeneous balance principle combined with a subsidiary ODE method, and its solutions have been found. Eq. (3.73) has not been solved elsewhere

using the method of Section 2. Let us now solve it using the method of Section 2. To this aim, we use the wave transformation (3.2) to reduce Eq. (3.73) into the following nonlinear ODE:

$$(\alpha-c)U'+\varepsilon U'U^{-2p}+\delta U'U^{-p}+\beta U'U^{p}+\gamma U'U^{2p}+U'''=0. \tag{3.74}$$

Integrating (3.74) with respect to ξ with vanishing the constant of integration, we get

$$\begin{aligned}(\alpha-c)U &+\left(\frac{\varepsilon}{1-2p}\right)U^{1-2p}+\left(\frac{\delta}{1-p}\right)U^{1-p}+\left(\frac{\beta}{1+p}\right)U^{1+p}\\ &+\left(\frac{\gamma}{1+2p}\right)U^{1+2p}+U''=0.\end{aligned} \tag{3.75}$$

Balancing U'' with U^{2p+1} yields the balance number m as follows:

$$m=\frac{1}{p},\ \ p>1\ , \tag{3.76}$$

which is non-integer. From (2.4) and (3.76), we assume that the solution of Eq. (3.75) is given by

$$U(\xi)=A\left[\frac{G'(\xi)}{G(\xi)}\right]^{\frac{1}{p}}, \tag{3.77}$$

where $G(\xi)$ satisfies Eq. (2.5). Substituting(3.77) along with Eq. (2.5) into Eq. (3.75) and according to the homogeneous balance principle [55], collecting all the coefficients of powers of $\left[\frac{G'(\xi)}{G(\xi)}\right]$ and setting them to zero, we have the following algebraic equations:

$$\left(\frac{G'}{G}\right)^{\frac{1}{p}}:(\alpha-c)p^2+(\lambda^2+2\mu p)=0, \tag{3.78}$$

$$\left(\frac{G'}{G}\right)^{\frac{1}{p}+1}:\beta p^2A^p+\lambda(1+p)(2+p)=0, \tag{3.79}$$

$$\left(\frac{G'}{G}\right)^{\frac{1}{p}+2}:\gamma p^2A^{2p}+(1+p)(1+2p)=0, \tag{3.80}$$

$$\left(\frac{G'}{G}\right)^{\frac{1}{p}-1}:\delta p^2A^{-p}+\lambda\mu(1-p)(2-p)=0, \tag{3.81}$$

$$\left(\frac{G'}{G}\right)^{\frac{1}{p}-2}:\varepsilon p^2A^{-2p}+\mu^2(1-p)(1-2p)=0. \tag{3.82}$$

Eqs. (3.79) and (3.80) give

$$A^p=-\frac{\lambda(1+p)(2+p)}{\beta p^2}\quad and\quad A^{2p}=-\frac{(1+p)(1+2p)}{\gamma p^2}, \tag{3.83}$$

while from (3.81) and (3.82), we have

$$(3.84)\quad A^{-p}=-\frac{\lambda\mu(1-p)(2-p)}{\delta p^2}\quad and\quad A^{-2p}=-\frac{\mu^2(1-p)(1-2p)}{\varepsilon p^2}.$$

From (3.83) and (3.84), we note that the interchanges $p\leftrightarrow -p$ yields the relations

$$(3.85)\quad \delta=\mu\ \beta\quad and\quad \varepsilon=\mu^2\ \gamma\,.$$

From (3.83) we have

$$(3.86)\quad \lambda^2=-\frac{\beta^2\ p^2\ (1+2p)}{\gamma\ (1+p)(2+p)^2}\,,$$

while from (3.84) we have

$$(3.87)\quad \lambda^2=-\frac{\delta^2\ p^2\ (1-2p)}{\varepsilon\ (1-p)(2-p)^2}\,.$$

Under conditions (3.85) we see the right-hand-side of (3.86) is the same as the right-hand-side of (3.87). Now, the exact traveling wave solution of Eq. (3.73) is given by

$$(3.88)\quad U(\xi)=\left[\frac{\beta(2p+1)}{\lambda\,\gamma(p+2)}\left(\frac{G'(\xi)}{G(\xi)}\right)\right]^{\frac{1}{p}}\,,$$

where, with the aid of (3.78) and (3.86) we can determine the velocity c, and consequently we have

$$(3.89)\quad \xi=x-c\,t=x-\left[\alpha-\frac{\beta^2(1+2p)}{\gamma(1+p)(2+p)^2}+\frac{2\mu}{p}\right]t\,.$$

It is well-known that, the ratio $\left(\frac{G'}{G}\right)$ can be determined with the help of the solutions of Eq. (2.5) to get

(i) If $\lambda^2-4\mu>0$, then

$$(3.90)\quad \left(\frac{G'}{G}\right)=-\frac{\lambda}{2}+\frac{1}{2}\sqrt{\lambda^2-4\mu}\left(\frac{C_1\exp\left(\frac{\xi}{2}\sqrt{\lambda^2-4\mu}\right)-C_2\exp\left(\frac{-\xi}{2}\sqrt{\lambda^2-4\mu}\right)}{C_1\exp\left(\frac{\xi}{2}\sqrt{\lambda^2-4\mu}\right)+C_2\exp\left(\frac{-\xi}{2}\sqrt{\lambda^2-4\mu}\right)}\right),$$

where C_1 and C_2 are arbitrary constants, while ξ is given by (3.89).
From (3.88) and (3.90), the exact traveling wave solution of Eq. (3.73) is given by

(3.91)

$$U(\xi)=\left\{-\frac{\beta(2p+1)}{2\,\gamma(p+2)}\left[1-\frac{\sqrt{\lambda^2-4\mu}}{\lambda}\left(\frac{C_1\exp\left(\frac{\xi}{2}\sqrt{\lambda^2-4\mu}\right)-C_2\exp\left(\frac{-\xi}{2}\sqrt{\lambda^2-4\mu}\right)}{C_1\exp\left(\frac{\xi}{2}\sqrt{\lambda^2-4\mu}\right)+C_2\exp\left(\frac{-\xi}{2}\sqrt{\lambda^2-4\mu}\right)}\right)\right]\right\}^{\frac{1}{p}},$$

where $\beta/\gamma < 0$. Choosing $\frac{C_2}{C_1} = \pm\, e^{-\sqrt{\lambda^2-4\mu}\,\xi_0}$, where ξ_0 is a constant, then the solitary wave solution of Eq. (3.73) follows from (3.91) and has the form

$$(3.92)\quad U(\xi) = \left\{ -\frac{\beta(2p+1)}{2\,\gamma(p+2)} \left[1 - \frac{\sqrt{\lambda^2-4\mu}}{\lambda} \tanh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\,(\xi-\xi_0) \right) \right] \right\}^{\frac{1}{p}},$$

which is agreement with (3.71) if $\mu = 0$ (i.e., if $\varepsilon = \delta = 0$), while the singular solitary wave solution of Eq. (3.73) follows also from Eq. (3.91) and has the form

$$(3.93)\quad U(\xi) = \left\{ -\frac{\beta(2p+1)}{2\,\gamma(p+2)} \left[1 - \frac{\sqrt{\lambda^2-4\mu}}{\lambda} \coth\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\,(\xi-\xi_0) \right) \right] \right\}^{\frac{1}{p}},$$

which is agreement with (3.72) if $\mu = 0$ $(i.e.\ \ if\ \ \varepsilon = \delta = 0)$. Note that

$$(3.94)\quad \lambda^2 - 4\mu = -\left[\frac{\beta^2 p^2 (2p+1)}{\gamma(1+p)(p+2)^2} + 4\mu \right],$$

which is positive if $\gamma < 0$ and $\mu < 0$ and negative if $\gamma > 0$ and $\mu > 0$. Consequently, we have the following results:

(ii) If $\lambda^2 - 4\mu < 0$, then

$$(3.95)\quad \left(\frac{G'}{G}\right) = -\frac{\lambda}{2} + \frac{1}{2}\sqrt{4\mu-\lambda^2} \left(\frac{-C_1\, sin\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right) + C_2\, cos\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right)}{C_1\, cos\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right) + C_2\, sin\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right)} \right),$$

where C_1 and C_2 are arbitrary constants, while ξ is given by (3.89). From (3.88) and (3.95), we have the exact traveling wave solution of Eq. (3.73) in the form:

(3.96)

$$U(\xi) = \left\{ -\frac{\beta(2p+1)}{2\,\gamma(p+2)} \left[1 - \frac{\sqrt{4\mu-\lambda^2}}{\lambda} \left(\frac{-C_1\, sin\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right) + C_2\, cos\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right)}{C_1\, cos\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right) + C_2\, sin\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right)} \right) \right] \right\}^{\frac{1}{p}},$$

where $\beta/\gamma < 0$.

Choosing $\frac{C_2}{C_1} = \tan\,(\frac{\xi_0}{2}\sqrt{4\mu-\lambda^2})$, where ξ_0 is a constant, then we have the trigonometric function solution

$$(3.97)\quad U(\xi) = \left\{ -\frac{\beta(2p+1)}{2\,\gamma(p+2)} \left[1 + \frac{\sqrt{4\mu-\lambda^2}}{\lambda} \tan\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\,(\xi-\xi_0) \right) \right] \right\}^{\frac{1}{p}},$$

while if $\frac{C_1}{C_2} = \tan\left(\frac{\xi_0}{2}\sqrt{4\mu-\lambda^2}\right)$, where ξ_0 is a constant, then we have the trigonometric function solution

$$U(\xi) = \left\{ -\frac{\beta(2p+1)}{2\,\gamma(p+2)}\left[1 - \frac{\sqrt{4\mu-\lambda^2}}{\lambda}\cot\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\,(\xi+\xi_0)\right)\right]\right\}^{\frac{1}{p}}, \tag{3.98}$$

where ξ is given by (3.89).

(iii) If $\lambda^2 - 4\mu = 0$, then we get

$$\left(\frac{G'}{G}\right) = -\frac{\lambda}{2} + \frac{C_2}{C_1 + C_2\xi}, \tag{3.99}$$

where C_1 and C_2 are arbitrary constants. With the aid of (3.89) we deduce that

$$\xi = x - \left[\alpha - \frac{\beta^2(1+2p)}{2\gamma(1+p)(2+p)}\right]t\,. \tag{3.100}$$

Now, we have the rational function solution of Eq. (3.73) as follows:

$$U(\xi) = \left\{ -\frac{\beta(2p+1)}{2\,\gamma(p+2)}\left[1 - \frac{C_2}{\lambda(C_1+C_2\xi)}\right]\right\}^{\frac{1}{p}}, \tag{3.101}$$

where ξ is given by (3.100), and $\beta/\gamma < 0$.

4. Physical explanations for some results

In this section, we present some graphs of the solitary wave solutions by taking suitable values of the parameters to visualize the mechanism of the original equations (3.1), (3.28), (3.53) and (3.73). Their solutions (3.24), (3.49), (3.71) and (3.92) are solitary wave solutions, while their solutions (3.26), (3.51), (3.72) and (3.93) are singular solitary wave solutions respectively. Eq. (3.73) has also the trigonometric solution (3.97) and (3.98) as well as the rational function solution (3.101). For more convenience, the graphical representations of some of these solutions are shown in Figs. 1-3 as follows:

5. Conclusions

We have derived the exact traveling wave solutions including the solitary wave and singular solitary wave solutions for the nonlinear PDEs (3.1), (3.28), (3.53) and (3.73) having non-integer balance numbers by using $(\frac{G'}{G})$-expansion method with the aid of the homogeneous balance principle described in Section 2 of this paper. In view of mathematical analysis, we see that the used method is an efficient method of integrability for constructing these solutions. On comparing the used method described in Section 2 of this paper with the other methods used in [27]-[46], [51]-[54] we deduce that our method is different and to our knowledge has been used in the first time to solve the equations (3.1), (3.28), (3.53) and

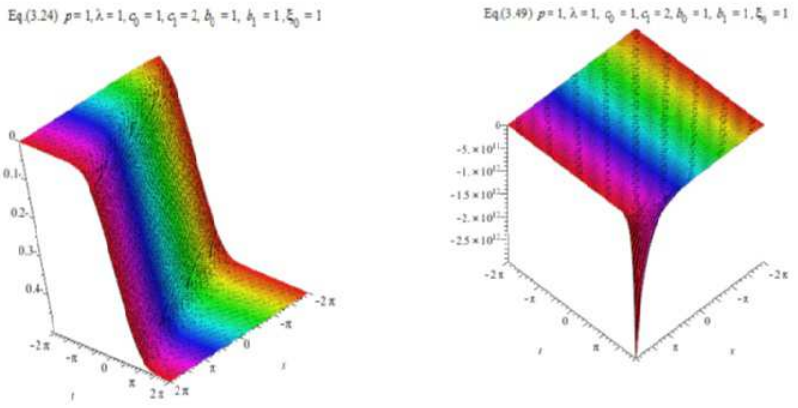

Figure 1: The solitary wave solutions (3.24) and (3.49)

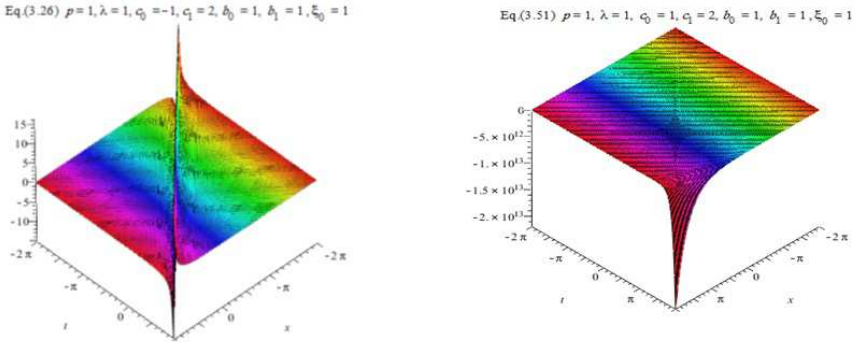

Figure 2: The singular solitary wave solutions (3.26) and (3.51)

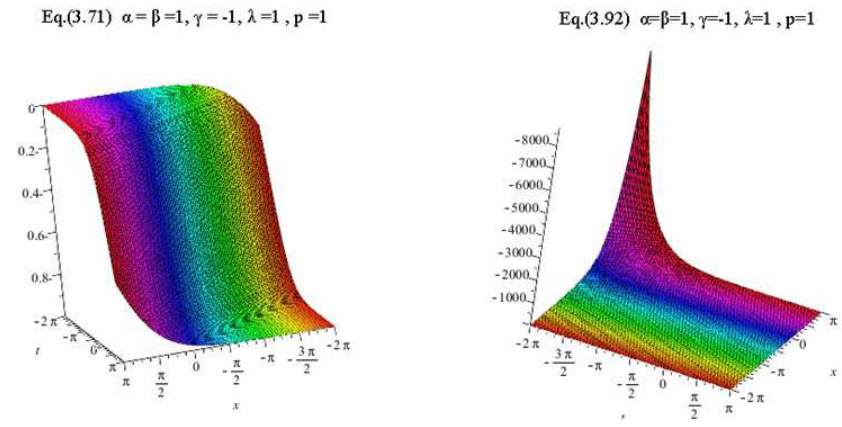

Figure 3: The solitary wave solutions (3.71) and (3.92)

(3.73). Consequently, our results in this paper are new and not reported elsewhere. The obtained solutions may be significant important for the explanations of some special physical phenomena. Finally, our results in this article have been checked by using the Maple by putting them back into the original equations (3.1), (3.28), (3.53) and (3.73).

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OSCILLATION OF THIRD-ORDER NONLINEAR NEUTRAL DIFFERENTIAL EQUATIONS WITH DISTRIBUTED TIME DELAY

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Abstract. With the development of modern society, research on properties of ordinary differential equation is becoming one of the hotspots in mathematical field. Neutral differential equation which is usually generated in natural science and engineering field is always extensively concerned by many scientific researchers for it can effectively describe multiple complex phenomena in natural world.

In recent years, differential equation and non-linear differential equation with time delay have attracted more and more attention. However, few researches concern about the properties of neutral differential equations with time delay. On account of this, we explored the oscillation of third-order nonlinear neural delay differential equations. Based on operators and integration techniques and with the help of proper comparison theorem, we established some oscillation sufficient conditions for several new solutions to such kind of equation. The obtained theorem which popularizes and improves results of the existing literature is applicable to neural differential equations. These results fully reflect the effects of time delay in oscillation of equation.

Keywords: third-order differential equation; neural; non-linear; oscillation.

1. Introduction

Owing to the demand on development of productivity and mathematicization of science in 17 and 18 century, calculus gained a further development and was gradually closely integrated with a series of subjects such as mechanics [1], physics, geometry [2] and acoustics [3]. Under such a background, differential equation theory emerged in response to the needs of times. Neutral differential equation which has been researched for a long time can be used for describing many natural phenomena. Hence studying neural differential equation is of great scientific and academic values theoretically and practically. Many interesting and important research topics deriving from study of neural differential equation have attracted much attention.

Due to abstraction and complexity [4], systems such as circuit signals, infection of epidemic disease [5] and automatic control usually have the defect of time delay. For instance, lossless transport network of switching circuit of high-speed computer can be affected by time delay feedback [6]. Time delay of dynamic equation can reflect effects of previous, current and future states of matters on change rules of matters [7] and deeply reveal the nature of matters [8]. Therefore, neural differential equation with time delay becomes the research object of scholars. Oscillation is a quite important research field in neural differential equation. With the development of science and technology, oscillation is a universal motion mode for substances [9].

In recent years, many scholars in this field have made several researches on oscillation of first-order and high-order, linear and non-linear neutral differential equation [10, 11] and obtained considerable academically valuable achievements. In 1836, Sturm proposed and deeply studied the problem of oscillation of second-order linear differential equation for the first time through studying heat conduction [12, 13], which laid a good research basis for the oscillation theory of differential equation [14]. Based on previous studies, this study explored the oscillation of third-order differential equation.

2. Theoretical basis of third-order nonlinear neutral differential equation with distributed time delay

In recent years, oscillation and asymptotic property of delay differential equation have been extensively concerned. Compared to oscillation of second-order delay differential equation, few results concern third-order delay differential equation, especially third-order or higher neural differential equation with time delay [15]. Previous researches considered asymptotic property of non-oscillatory solution of third-order nonlinear neutral differential equation with time delay [16]

$$[a(t)(b(t)(x(t)+p(t)x(\sigma(t)))')']'+q(t)x(\tau(t))=0,\ t\geq t_0 \tag{2.1}$$

as well as oscillation of solution of third-order neutral differential equation with distributed time delay [17]

$$(r(t)([x(t)+p(t)x(\sigma(t))]'')^{\alpha})'+\int_a^b q(t,\xi)f(x(g(t,\xi)))d\xi=0,\ t\geq t_0. \tag{2.2}$$

This study mainly considers oscillation of third-order neutral differential equation with more extensively distributed time delays:

$$\{r_1(t)(r_2(t)(x(t)+p(t)x(\sigma(t)))')')^{\gamma}\}'+\int_a^b q(t,\xi)f(x(\tau(t,\xi)))d\xi=0,\ t\geq t_0, \tag{2.3}$$

that is,

$$\{r_1(t)(r_2(t)(x(t)+p(t)x(\sigma(t)))')')^{\gamma}\}'+q(t)f(x(t),x(\tau(t)))=0,\ t\geq t_0. \tag{2.4}$$

Several sufficient conditions for equation oscillation can be obtained if we apply Riccati transformation and Hardy inequality method. t_0 is supposed as a real number. Besides, $I = [t_0, \infty)$ and $R^+ = (0, +\infty)$. Then, we establish the following conditions.

(H1) For all $t \in [t_0, +\infty)$, $r_1, r_2 \in C^1[t_0, +\infty)$ $(r_1(t), r_2(t) > 0$, $r_1'(t), r_2'(t) > 0$ and $\int_{t_0}^{+\infty} \frac{1}{r_2(s)} ds = +\infty$, $\int_{t_0}^{+\infty} r_1^{-\frac{1}{\gamma}}(s) ds = +\infty)$;

(H2) For all $t \in [t_0, +\infty)$, $p, q \in C[t_0, +\infty)$ $(0 \le p(t) \le b < 1$, $q(t) > 0$, b refers to constant);

(H3) $\tau, \sigma \in C[t_0, +\infty)$, $\sigma(t) \le t$, $\tau(t) \le t$, $\lim\limits_{t \to +\infty} \sigma(t) = +\infty$, $\lim\limits_{t \to +\infty} \tau(t) = +\infty$;

(H4) $f \in C(R^2, R)$. There is a positive constant k that can make $\frac{f(x,y)}{y^\gamma} \ge k$ when $y \ne 0$; $q, \tau \in C(I \times [a, b], R^+), \tau(t, \xi) \le t, \tau(t, \xi)$ is non-decreasing with regard to t and ξ and moreover $\lim\limits_{t \to +\infty} \min\limits_{\xi \in [a,b]} \tau(t, \xi) = \infty$;

(H5) $\gamma = \frac{m}{n}$ (m and n are positive odd numbers); $f(u) \in C(R, R)$, $\frac{f(u)}{u^\gamma} \ge c > 0$, $u \ne 0$.

We suppose $z(t) = x(t) + p(t)x(\sigma(t))$. It can be known from $\lim\limits_{t \to +\infty} \sigma(t) = +\infty$ and $\lim\limits_{t \to +\infty} \tau(t) = +\infty$ that, there is T that can make $\sigma(t) > t_0$ when $t > T$; besides, there is a minimum value as $\sigma(t)$ is continuous in the interval of $[t_0, +\infty)$ and it is recorded as t_1. Similarly, for $\tau(t)$, there is also a minimum value t_2 in the interval of $[t_0, +\infty)$. Hence we take the value of t_* as $\min\{t_1, t_2\}$; then we get $\tau(t) \ge t_*$ and $\sigma(t) \ge t_*$. Therefore we consider the solution of equation (2.4) has definition in the interval of $[t_*, +\infty)$ and there is also T_x that can make $x(t)$ second order and continuously differentiable and make $r_1(t)((r_2(t)(x(t) + p(t)x(\tau(t)))')')^\gamma$ first order and continuously differentiable and meet equation (2.4) when $t \in [T_x, +\infty)$. If $x(t)$ is unbounded in zero locus set of $[T_x, +\infty)$, then solution of equation (2.4), i.e., $x(t)$, is considered as oscillatory, that is, the final result is neither positive nor negative. In contrast, if $x(t)$ is bounded in zero locus set, then solution of equation (2.4), i.e., $x(t)$, is non-oscillatory, that is, solution of equation (2.4) is positive or negative. Before proof of theorem, necessary prerequisite knowledge and lemma are given in the following.

3. Prerequisite knowledge

Lemma 3.1 *Suppose* (H1)∼(H5) *are tenable and* $x(t)$ *is the final positive solution of equation* (2.3)*, then function* $z(t)$ *defined by* $z(t) = x(t) + p(t)x(\sigma(t))$ *only has one of the following properties.*

(1) $z(t) > 0$, $r_1(t)z'(t) > 0$, $(r_1(t)z'(t))' > 0$;

(2) $z(t) > 0$, $r_1(t)z'(t) < 0$, $(r_1(t)z'(t))' > 0$.

Proof. As $x(t)$ is the final positive solution of equation (2.3), $t \in [t_1, +\infty)$, we have:

$$x(t) > 0, x(\sigma(t)) > 0, x(\tau(t,\xi)) > 0.$$

It can be known from (H2) and $z(t) = x(t) + p(t)x(\sigma(t))$ that $z(t) \geq x(t) > 0$ when $t \in [t_1, +\infty)$.

It can be known from (H5) that,

$$\frac{f(x(\tau(t,\xi)))}{x\gamma(\tau(t,\xi))} \geq c > 0.$$

Hence, for all $t \in [t_1, +\infty)$, there is $\int_a^b q(t,\xi)f(x(\tau(t,\xi)))d\xi > 0$, then

$$(r_1(t)((r_2)z'(t))')^\gamma)' = -\int_a^b q(t,\xi)f(x(\tau(t,\xi)))d\xi < 0,\ t \geq t_1 \tag{3.1}$$

We can know that $r_1(t)((r_2(t)(x(t) + p(t)x(\tau(t)))')')^\gamma$ is a decreasing function on $[t_1, +\infty)$ and it is positive or negative eventually. Thus we have verified the establishment of the first condition.

Otherwise, there is constant $G > 0$ and $t_2 \geq t_1$ that can make:

$$r_1(t)((r_2(t)z'(t))')^\gamma \leq -G < 0,\ t \geq t_2,$$

is the nth-autocommutator subgroup of G.

Now, we obtain the following series of characteristic subgroups. Integrating the above equation from t_2 to t and combining H1, we have:

$$r_2(t)z'(t) \leq r_2(t_2)z'(t_2) - G^{\frac{1}{\gamma}} \int_{t_2}^t \frac{1}{r_1^{\frac{1}{\gamma}}(s)} ds \to -\infty (t \to +\infty). \tag{3.2}$$

Hence, $r_2(t)z'(t)$ is negative eventually. Besides, it can be known from $(r_2(t)z'(t))' < 0$ and $r_2(t)z'(t) < 0$ that, $z(t)$ is negative eventually, which is contradictory with $z(t) > 0$. Therefore, $((r_2(t)z'(t))')^\gamma$ is positive, i.e., $(r_2(t)z'(t))' > 0$ holds.

It can be known from $(r_2(t)z'(t))' > 0$ that, $r_2(t)z'(t)$ is monotonically increasing in the interval of $[t_1, +\infty)$; therefore $r_2(t)z'(t)$ is either negative or positive. The proof of Lemma 3.1 is over. ∎

It can be known from the above requirement of the question and monotonic boundedness theory that, there is constant $d \geq 0$ that can make $\lim\limits_{t\to\infty} z(t) = d$. Besides, due to limit existence of $z(t)$, $d = 0$. Otherwise, if $d > 0$, then for any $\varepsilon > 0$, there is

$$d < z(t) < d + \varepsilon, d < z(\sigma(t)) < d + \varepsilon, d < z(\tau(t,\xi)) < d + \varepsilon, \tag{3.3}$$

when $t_1 > t_0$

Therein, $L = \dfrac{d - p(d+\varepsilon)}{d+\varepsilon} > 0$.

It can be known from (H5) that, when $t \geq t_2$, there is $t_2 \in [t_1, +\infty)$ that can make $f(x(\tau(t,\xi))) \geq cx^{\gamma}(\tau(t,\xi)) \geq cL^{\gamma}z^{\gamma}(\tau(t,\xi))$.

Combining equation (3.3), we have:

$$(r_1(t)((r_2(t)z'(t))')^{\gamma})' \leq -cL^{\gamma}d^{\gamma}\int_a^b q(t,\xi)d\xi \tag{3.4}$$

Hence, we can conclude that function $r_1(t)((r_2(t)z'(t))')^{\gamma}$ is monotonically decreasing in the interval of $[t_2, +\infty)$. It can be known from the monotonic boundedness theory that $r_1(t)((r_2(t)z'(t))')^{\gamma} > 0$; there is a finite limit, i.e., $\lim\limits_{t\to\infty} r_1(t)((r_2(t)z'(t))')^{\gamma} \geq 0$. Integrating equation (3.4) in the interval of $[t, +\infty)$, we have:

$$r_1(t)((r_2(t)z'(t))')^{\gamma} - \lim_{t\to\infty} r_1(t)((r_2(t)z'(t))')^{\gamma} \geq -cL^{\gamma}d^{\gamma}\int_t^{+\infty}\int_a^b q(s,\xi)d\xi ds.$$

Hence, $r_1(t)((r_2(t)z'(t))')^{\gamma} \geq cL^{\gamma}d^{\gamma}\int_t^{+\infty}\int_z^b q(s,\xi)d\xi ds$, that is,

$$(r_2(t)z'(t))' \geq c^{\frac{1}{\gamma}}Ld\left(\frac{1}{r_1(t)}\int_t^{+\infty}\int_a^b q(s,\xi)d\xi ds\right)^{\frac{1}{\gamma}}. \tag{3.5}$$

It can be known from $(r_2(t)z'(t))' > 0$ that $r_2(t)z'(t)$ is monotone increasing in the interval of $[t_2, +\infty)$. Besides, $r_2(t)z'(t) < 0$. It can be known from the monotonic boundedness theory that there is a finite limit, i.e., $\lim\limits_{t\to\infty} r_2(t)z'(t) \leq 0$. Integrating equation (3.5) in the interval of $[t, +\infty)$, we have:

$$\lim_{t\to\infty} r_2(t)z'(t) - r_2(t)z'(t) \geq -c^{\frac{1}{\gamma}}Ld\int_t^{+\infty}\left(\frac{1}{r_1(u)}\int_u^{+\infty}\int_a^b q(s,\xi)d\xi ds\right)^{\frac{1}{\gamma}}du.$$

Then, we get

$$-r_2(t)z'(t) \geq c^{\frac{1}{\gamma}}Ld\int_t^{+\infty}\left(\frac{1}{r_1(u)}\int_u^{+\infty}\int_a^b q(s,\xi)d\xi ds\right)^{\frac{1}{\gamma}}du,$$

that is,

$$-z'(t) \geq \frac{c^{\frac{1}{\gamma}}Ld}{r_2(t)}\int_t^{+\infty}\left(\frac{1}{r_1(u)}\int_u^{+\infty}\int_a^b q(s,\xi)d\xi ds\right)^{\frac{1}{\gamma}}du. \tag{3.6}$$

As $z'(t)$ is monotonically decreasing in the interval of $[t_2, +\infty)$ ($z'(t) < 0$ and $z(t) > 0$), we can know that there is a finite limit, i.e., $\lim\limits_{t\to\infty} z(t) \geq 0$. Integrating equation (3.6) in the interval of $[t, +\infty)$, we have:

$$z(t_2) - \lim_{t\to\infty} z(t) \geq c^{\frac{1}{\gamma}}Ld\int_{t2}^{+\infty}\frac{1}{r_2(t)}\int_v^{+\infty}\left(\frac{1}{r_1(u)}\int_u^{+\infty}\int_a^b q(s,\xi)d\xi ds\right)^{\frac{1}{\gamma}}dudv,$$

Hence,

$$z(t_2) \geq c^{\frac{1}{\gamma}}Ld\int_{t2}^{+\infty}\frac{1}{r_2(t)}\int_v^{+\infty}\left(\frac{1}{r_1(u)}\int_u^{+\infty}\int_a^b q(s,\xi)d\xi ds\right)^{\frac{1}{\gamma}}dudv.$$

The result obtained is contradictory with equation (3.2).

Because of $0 < x(t) < z(t)$, we know that $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} z(t) = 0$. The proof of Lemma 3.1 is over. ∎

Lemma 3.2 *Suppose there is* $w(t) > 0, w'(t) > 0, w''(t) < 0, g(t) \le t$ *in the interval of* $t \in [t_0, +\infty)$. *Then, for any* $\theta \in (0,1)$, *there is* $T_\theta \ge t_0$ *that can make:*

$$w(g(t)) \ge \theta \frac{g(t)}{t} w(t),\ t \ge T_\theta.$$

Proof. Applying Lagrange's mean value theorem and considering that fact that $w'(t)$ is monotonically decreasing, we have:

$$w(t) - w(g(t)) \le w'(g(t))(t - g(t))$$

Multiplying both sides of the above inequation with $\frac{1}{w(g(t))}$, we have:

$$\frac{w(t)}{w(g(t))} \le 1 + \frac{w'(g(t))(t - g(t))}{w(g(t))}. \tag{3.7}$$

Using Lagrange's mean value theorem again, we have:

$$w(g(t)) \ge w(g(t)) - w(t_0) \ge w'(g(t))(g(t) - t_0).$$

Therefore, for any $\theta \in (0,1)$, there is $T_\theta \ge t_0$ that can make:

$$\frac{w(g(t))}{w'(g(t))} \ge g(t)\left(1 - \frac{t_0}{g(t)}\right) \ge \theta g(t). \tag{3.8}$$

Combining equation (3.7) and (3.8), we can get:

$$\frac{w(t)}{w(g(t))} \le 1 + \frac{1}{\theta g(t)}(t - g(t)) \le \frac{t}{\theta g(t)},$$

i.e., $w(g(t)) \ge \theta \frac{g(t)}{t} w(t), t \ge T_\theta$. The proof of Lemma 3.2 is over. ∎

Lemma 3.3 *If* $z(t) > 0, r_2(t)z'(t) > 0$ *and* $(r_2(t)z'(t))' > 0$ *hold in the interval of* $t \in [T_\theta, +\infty)$, *then there is* $\beta \in (0,1)$ *and* $T_\beta \ge T_\theta$ *that can make:*

$$z(t) \ge \beta t z'(t),\ t \ge T_\beta.$$

Proof. Suppose $y(t) = \int_{T_\theta}^{t} T_2(s)z'(s)ds, t \in [T_\theta, +\infty)$. Then, the following conditions with regard to $y(t)$ hold.

$$y(t) > 0, y'(t) > 0, y''(t) > 0, y'''(t) \le 0.$$

Besides, there is $\beta \in (0,1)$ and $T_\beta \ge T_\theta$ that can make $y(t) \ge \beta t y'(t), t \ge T_\beta$, that is,

$$\frac{\int_{T_\theta}^{t} r_2(s)z'(s)ds}{r_2(t)z'(t)} \ge \beta t,\ t \ge T_\beta. \tag{3.9}$$

Using the distribution integral method and taking (H1) into account, we have:

$$\int_{T_\theta}^{t} r_2(s)z'(s)ds = r_2(t)z'(t)\,\big|_{T_\theta}^{t} - \int_{T_\theta}^{t} r_2'(s)z(s)ds$$
$$= r_2(t)z(t) - r_2(T_\theta)z(T_\theta) - \int_{T_\theta}^{t} r_2'(s)z(s)ds < r_2(t)z(t).$$

Combining this with equation (3.9) yields:

$$\frac{r_2(t)z(t)}{r_2(t)z'(t)} \geq \beta t,\ t \geq T_\beta,$$

that is, $z(t) \geq \beta t z'(t), t \geq T_\beta$. The proof of Lemma 3.3 is over. ∎

Lemma 3.4 *Suppose A and B are non-negative constants. Then we have:*

$$A^\lambda - \lambda A B^{\lambda-1} + (\mu - 1)B^\lambda \geq 0,\ \lambda > 1.$$

The equation holds if and only if $A = B$.

4. Main results and proof

Theorem 4.1 *Suppose* (H1)∼(H5) *and equation* (3.2) *are tenable, and moreover*

$$\limsup_{t\to\infty} \int^{t} \left[\left(\frac{\beta\tau_1(s)R(\tau_1(s))}{r_2(\tau_1(s)} \right)^{\gamma} q_1(s) - \left(\frac{\gamma}{\gamma+1} \right)^{\gamma+1} \frac{R'(\tau 1(s))}{R(\tau 1(s))} \right] ds = +\infty \quad (4.1)$$

where β is given by Lemma 3.3. *For convenience, we write:*

$$\tau_1(s) = \tau(s, a)$$
$$R(\tau_1(s)) \int_{t0}^{\tau_1(s)} r_1^{-\frac{1}{\gamma}}(s)ds, q_1(s) = c(1-p)^\gamma \int_a^b q(s,\xi)d\xi.$$

Then, every solution of equation (2.3), *i.e., $x(t)$, either oscillate or satisfies* $\lim_{t\to+\infty} x(t) = 0$.

Proof. Suppose $x(t)$ is a non-oscillatory solution of equation (2.3) and it is positive. Then, we can know from Lemma 3.1 that, $z(t)$ can only satisfy property (1) or (2). On that condition that $z(t)$ satisfying property (1), there exists $t_1 \geq t_0$ such that

$$z(t) > 0, z(\sigma(t)) > 0, z'(t) > 0$$

when $t \geq t_1$. Then, $z(t)$ is considered monotonically increasing in the interval of $[t_1, +\infty)$. And then

$$x(t) = z(t) - p(t)x(\sigma(t)) \geq z(t) - pz(\sigma(t)) \geq (1-p)z(t).$$

Combining this with (H5), we have:

$$\begin{aligned}(r_1(t)((r_2(t)z(t))')^\gamma)' &= -\int_a^b q(t,\xi)f(x(\tau(t,\xi)))d\xi \\ &\le -c(1-p)^\gamma \int_a^b q(t,\xi)z^\gamma(\tau(t,\xi))d\xi \\ &\le -c(1-p)^\gamma(\tau(t,a))\int_a^b q(t,\xi)d\xi \\ &\le -q_1(t)z^\gamma(\tau_1(t)) < 0.\end{aligned}$$

Then, we get

$$(r_1(t)((r_2(t)z(t))')^\gamma)' = -q_1(t)z^\gamma(\tau_1(t)) < 0, \tag{4.2}$$

where $q_1(t) = c(1-p)^\gamma \int_a^b q(t,\xi)d\xi$.

$$W(t) = R^\gamma(\tau_1(t))r_1(t)\left(\frac{(r_2(t)z'(t))'}{r_2(\tau_1(t))z'(\tau_1(t))}\right)^\gamma$$

is defined.

It can be known from equation (3.1) that,

$$W(t) = R^\gamma(\tau_1(t))r_1(t)\left(\frac{(r_2(t)z'(t))'}{r_2(\tau_1(t))z'(\tau_1(t))}\right)^\gamma.$$

Thus, when $t \ge t_2$, there is $t_2 \ge t_1$ such that

$$z(t) > 0, (r_2(t)z(t))' > 0, r_2(\tau_1(t)z'(\tau_1(t)) > 0, (r_2(t)z'(t))'' < 0.$$

Then, $W(t) > 0$ $(t \ge t_2)$. Considering Lemma 3.3 and equation (4.2), we have:

$$\begin{aligned}&W'(t) \\ &= \gamma R^{\gamma-1}(\tau_1(t))R'(\tau_1(t))r_1(t)\left(\frac{(r_2(t)z'(t))'}{r_2(\tau_1(t))z'(\tau_1(t))}\right)^\gamma \\ &\quad + R^\gamma(\tau_1(t))\left(\frac{r_1(t)((r_2(t)z'(t))')^\gamma}{(r_2(\tau_1(t))z'(\tau_1(t)))^\gamma}\right) \\ &\le \gamma R^{\gamma-1}(\tau_1(t))R'(\tau_1(t))r_1(t)\left(\frac{(r_2(t)z'(t))'}{r_2(\tau_1(t))z'(\tau_1(t))}\right)^\gamma - \frac{R^\gamma(\tau_1(t))q_1(t)z'(\tau_1(t))}{(r_2(\tau_1(t))z'(\tau_1(t)))^\gamma} \\ &\quad - R^\gamma(\tau_1(t))\frac{r_1(t)((r_2(t)z'(t))')^\gamma(r_2(\tau_1(t))z'(\tau_1(t)))'\tau_1(t))}{(r_2(\tau_1(t))z'(\tau_1(t)))^{\gamma+1}} \\ &\le \gamma R^{\gamma-1}(\tau_1(t))R'(\tau_1(t))r_1(t)\left(\frac{(r_2(t)z'(t))'}{r_2(\tau_1(t))z'(\tau_1(t))}\right)^\gamma - \left(\frac{\beta\tau_1(t)R(\tau_1(t))}{r_2(\tau_1(t))}\right)^\gamma q_1(t) \\ &\quad - R^\gamma(\tau_1(t))\frac{r_1(t)z_1'(t)((a(t)z'(t))')^{\gamma+1}}{(a(\tau_1(t))z'(\tau_1(t)))^{\gamma+1}} \\ &= \frac{\gamma R'(\tau_1(t))}{R(\tau_1(t))}R(\tau_1(t))r_1(t)\left(\frac{(r_2(t)z'(t))'}{r_2(\tau_1(t))z'(\tau_1(t))}\right)^\gamma - \left(\frac{\beta\tau_1(t)R(\tau_1(t))}{r_2(\tau_1(t))}\right)^\gamma q_1(t)\end{aligned}$$

$$
\begin{aligned}
&- \frac{\gamma\tau'(t)}{R(\tau_1(t))r_1^{\frac{1}{\gamma}}(\tau_1(t))} R^{\gamma+1}(\tau_1(t)) r_1^{\frac{\gamma+1}{\gamma}}(t) \left(\frac{(r_2(t)z_1'(t))'}{(r_2(\tau_1(t))z'(\tau_1(t))} \right)^{\gamma+1} \\
&= - \left(\frac{\beta\tau_1(t)R(\tau_1(t))}{r_2(\tau_1(t))} \right)^{\gamma} q_1(t) + \left(W(t) - W^{\frac{\gamma+1}{\gamma}}(t) \right) \frac{\gamma R'(\tau_1(t))}{R(\tau_1(t))}.
\end{aligned}
$$

It can be known from Lemma 3.4 that,

$$
W'(t) \leq - \left(\frac{\beta\tau_1(t)R(\tau_1(t))}{r_2(\tau_1(t))} \right)^{\gamma} q_1(t) + \left(\frac{\gamma+1}{\gamma} \right)^{\gamma+1} \frac{R'(\tau_1(t))}{R(\tau_1(t))}.
$$

Integrating both sides of the above equation, we have:

$$
W(t) \leq W(t_2) - \int_{t_2}^{t} \left[\left(\frac{\beta\tau_1(s)R(\tau_1(s))}{r_2(\tau_1(s))} \right)^{\gamma} q_1(s) + \left(\frac{\gamma+1}{\gamma} \right)^{\gamma+1} \frac{R'(\tau_1(s))}{R(\tau_1(s))} \right] ds.
$$

Assume $t \to +\infty$. Considering equation (4.1), we know that, $W(t) \to -\infty$, which is contradictory with $W(t) > 0$. Therefore, $x(t)$ oscillates.

Then, if $z(t)$ satisfy property (2), we can know from equation (3.2) and Lemma 3.1 that $\lim\limits_{t\to+\infty} x(t) = 0$. The proof of the theorem is over. ∎

Deduction 4.2 *If equation* (4.1) *in Theorem* 4.1 *is substituted with*

$$
\liminf_{t\to+\infty} \frac{1}{\ln R(\tau_1(t))} \int_{t_1}^{t} \left(\frac{\beta\tau_1(s)R(\tau_1(s))}{r_2(\tau_1(s))} \right)^{\gamma} q_1(s)ds > \left(\frac{\gamma+1}{\gamma} \right)^{\gamma+1},
$$

then the conclusion of Theorem 4.1 *still holds.*

Proof. If we only consider $z(t)$ satisfying property (1), then for all sufficiently large t, there exists $\varepsilon > 0$ and $t_2 \geq t_1$ that can make:

$$
\frac{1}{\ln R(\tau_1(t))} \int_{t_2}^{t} \left(\frac{\beta\tau_1(s)R(\tau_1(s))}{r_2(\tau_1(s))} \right)^{\gamma} q_1(s)ds \geq \left(\frac{\gamma+1}{\gamma} \right)^{\gamma+1} + \varepsilon
$$

that is,

$$
\int_{t_2}^{t} \left(\frac{\beta\tau_1(s)R(\tau_1(s))}{r_2(\tau_1(s))} \right)^{\gamma} q_1(s)ds - \left(\frac{\gamma+1}{\gamma} \right)^{\gamma+1} \ln R(\tau_1(t)) \geq \varepsilon \ln R(\tau_1(t)).
$$

Therefore,

$$
\begin{aligned}
&\int_{t_2}^{t} \left[\left(\frac{\beta\tau_1(s)R(\tau_1(s))}{r_2(\tau_1(s))} \right)^{\gamma} q_1(s)ds - \left(\frac{\gamma+1}{\gamma} \right)^{\gamma+1} \frac{R'(\tau_1(s))}{R(\tau_1(s))} \right] ds \\
&\qquad \leq \varepsilon \ln R(\tau_1(t)) + \left(\frac{\gamma+1}{\gamma} \right)^{\gamma+1} \ln R(\tau_1(t_2))
\end{aligned} \tag{4.3}
$$

It can be known from (H1) that $\lim\limits_{t\to+\infty} R(\tau_1(t)) = \infty$. Hence, equation (4.3) ensures the holding of equation (4.1). ∎

Deduction 4.3 *If equation* (4.1) *in Theorem* 4.1 *is substituted with*

$$\liminf_{t\to+\infty}\left(\frac{\beta\tau_1(t)R(\tau_1(t))}{r_2(\tau_1(t))}\right)^{\gamma}\tau_1(t)\frac{R(\tau_1(t))}{R'(\tau_1(t))}>\left(\frac{\gamma+1}{\gamma}\right)^{\gamma+1}, \tag{4.4}$$

then the conclusion of Theorem 4.1 *holds.*

Proof. If equation (4.4) holds, then for all sufficiently large t, there exists $\varepsilon>0$ and $t_2\geq t_1$ that can make

$$\left(\frac{\beta\tau_1(t)R(\tau_1(t))}{r_2(\tau_1(t))}\right)^{\gamma}\tau_1(t)\frac{R(\tau_1(t))}{R'(\tau_1(t))}\geq\left(\frac{\gamma+1}{\gamma}\right)^{\gamma+1}+\varepsilon.$$

Multiplying both sides of the above equation with $\frac{R'(\tau_1(t))}{R(\tau_1(t))}$, we have:

$$\left(\frac{\beta\tau_1(t)R(\tau_1(t))}{r_2(\tau_1(t))}\right)^{\gamma}\tau_1(t)-\left(\frac{\gamma+1}{\gamma}\right)^{\gamma+1}\frac{R(\tau_1(t))}{R'(\tau_1(t))}\geq\varepsilon\frac{R(\tau_1(t))}{R'(\tau_1(t))}.$$

Therefore, the above equation ensures the holding of equation (4.4).

Next, we will prove the Philos oscillation criteria of equation (2.3). Considering set $D_0=\{(t,s):t>s>T_0\}$ and $D=\{(t,s):t\geq s>T_0\}$, we call the function H possesses property P if the function $H\in C(D,R)$ and satisfies the following conditions:

(I) $H(t,t)=0, t\geq t_0, H(t,s)>0,(t,s)\in D_0$;

(II) For all $(t,s)\in D_0$, $\frac{\partial H(t,s)}{\partial s}$ is continuous and non-positive.

Theorem 4.4 *Suppose* (H1)∼(H5) *and equation* (3.2) *hold. Then there is the P function* $H(t,s), h\in C(D_0,R)$ *and* $\rho\in C'(I,R^+)$ $(I=[t_0,\infty),\ R^+=(0,+\infty))$ *that can make*

$$-\frac{\partial H(t,s)}{\partial s}=h(t,s)H^{\frac{\gamma}{\gamma+1}}(t,s)+\frac{\rho'(s)}{\rho(s)}H(t,s) \tag{4.5}$$

and

$$\limsup_{t\to+\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left(H(t,s)Q(s)-\frac{\rho(s)r_1(s)}{(\gamma+1)^{\gamma+1}}\left|h(t,s)\right|^{\gamma+1}\right)ds=+\infty, \tag{4.6}$$

where

$$Q(s)=\frac{\rho(s)q_1(s)}{r_2^{\gamma}(s)}\left(\frac{\beta\theta\tau'(s)}{s}\right)^{\gamma},$$

$\beta,\theta,g_1(s)$ *and* $q_1(s)$ *are defined according to Lemmas* 3.2 *and* 3.3. *Then, every solution of equation* (2.3)*, i.e., oscillates or satisfies* $\lim_{t\to+\infty}x(t)=0$.

Proof. If $x(t)$ is a non-oscillatory solution of equation (2.3) and does not lose generality, then we can suppose $x(t) > 0$. Similar to the proof of Theorem 4.1, we consider $z(t)$ satisfies property (1) or (2). If $z(t)$ satisfies property (1), i.e., there exists $t_1 \geq t_0$ that can make

$$z(t) > 0, r_2(t)z'(t) > 0, (r_2(t)z'(t))' > 0,$$

when $t \geq t_1$.

Moreover, equation (4.2) holds. The function $W(t)$ is defined as follows.

$$W(t) = \rho(t)r_1(t)\left(\frac{(r_2(t)z'(t))'}{r_2(t)z'(t)}\right)^{\gamma}, \ t \geq t_1.$$

Then $W(t) > 0$ $(t \geq t_1)$. Combining equation (2.3) and (4.2), we have:

$$\begin{aligned}
W(t) &= \rho'(t)r_1(t)\left(\frac{(r_2(t)z'(t))'}{r_2(t)z'(t)}\right)^{\gamma} + \rho(t)\left(\frac{r_1(t)((r_2(t)z'(t))')^{\gamma}}{(r_2(t)z'(t))^{\gamma}}\right)' \\
&= \rho'(t)r_1(t)\left(\frac{(r_2(t)z'(t))'}{r_2(t)z'(t)}\right)^{\gamma} + \rho(t)\left(\frac{r_1(t)((r_2(t)z'(t))')^{\gamma}}{(r_2(t)z'(t))^{\gamma}}\right)' \\
&\quad - \rho(t)\frac{\gamma r_1(t)((r_2(t)z'(t))')^{\gamma+1}}{(r_2(t)z'(t))^{\gamma+1}} \\
&\leq \frac{\rho'(t)}{\rho(t)}\rho(t)r_1(t)\left(\frac{(r_2(t)z'(t))'}{r_2(t)z'(t)}\right)^{\gamma} - \frac{\rho'(t)q_1(t)}{r_2^{\gamma}(t)}\left(\frac{z(\tau_1(t))}{z'(t)}\right)^{\gamma} \\
&\quad - \frac{\gamma}{(\rho(t)r_1(t))^{\frac{1}{\gamma}}}\rho^{1+\frac{1}{\gamma}}(t)r_1^{1+\frac{1}{\gamma}}(t)\left(\frac{(r_2(t)z_1'(t))'}{r_2(t)z_1'(t)}\right)^{\gamma+1} \\
&= \frac{\rho'(t)}{\rho(t)}W(t) - \frac{\rho'(t)q_1(t)}{r_2^{\gamma}(t)}\left(\frac{z(\tau_1(t))}{z'(t)}\right)^{\gamma} - \frac{\gamma}{(\rho(t)r_1(t))^{\frac{1}{\gamma}}}W^{\frac{1+\gamma}{\gamma}}(t),
\end{aligned}$$

that is,

$$W'(t) \leq \frac{\rho'(t)}{\rho(t)}W(t) - \frac{\rho'(t)q_1(t)}{r_2^{\gamma}(t)}\left(\frac{z(\tau_1(t))}{z'(t)}\right)^{\gamma} - \frac{\gamma}{(\rho(t)r_1(t))^{\frac{1}{\gamma}}}W^{\frac{1+\gamma}{\gamma}}(t). \tag{4.7}$$

In Theorem 4.4, suppose $w(t) = z'(t)$. Then, we get $\frac{1}{z'(t)} \geq \frac{\theta\tau_1(t)}{tz'(\tau_1(t))}$, $t \geq T_\theta \geq t_1$. Combining this with Theorem 4, we have:

$$\frac{1}{z'(\tau_1(t))} \geq \frac{\beta\tau_1(t)}{z(\tau_1(t))}, \ t \geq T_\beta \geq T_\theta,$$

then $\frac{1}{z'(t)} \geq \frac{\theta\beta\tau_1^2(t)}{tz(\tau_1(t))}, t \geq T_\beta$. Combining this with equation (4.7), we have:

$$W'(t) \leq \frac{\rho'(t)}{\rho(t)}W(t) - \frac{\rho(t)q_1(t)}{r_2^{\gamma}(t)}\left(\frac{\beta\theta\tau_1'(t)}{t}\right)^{\gamma} - \frac{\gamma}{(\rho(t)r_1(t))^{\frac{1}{\gamma}}}W^{\frac{1+\gamma}{\gamma}}(t), \ t \geq T_\beta. \tag{4.8}$$

Both sides of equation (4.8) multiply with $H(t,s)$ and then it is integrated from $T(\geq T_\beta)$ to t. Combining the above result with equation (4.5), we get the following equation:

$$\begin{aligned}
&\int_T^t H(t,s)Q(s)ds \\
&\leq \int_T^t H(t,s)\left[-W'(s)+\frac{\rho'(s)}{\rho(s)}W(s)-\gamma(\rho(s)r_1(s))^{-\frac{1}{\gamma}}W^{\frac{\gamma+1}{\gamma}}(s)\right]ds \\
&\leq -H(t,s)W(s)\Big|_T^t + \\
&\quad \int_T^t\left[\frac{\partial H(t,s)}{\partial s}W(s)+\frac{\rho'(s)}{\rho(s)}H(t,s)W(s)-\gamma H(t,s)(\rho(s)r_1(s))^{-\frac{1}{\gamma}}W^{\frac{\gamma+1}{\gamma}}(s)\right]ds \\
&= H(t,T)W(T)+ \\
&\quad \int_T^t\left[-h(t,s)H^{\frac{\gamma+1}{\gamma}}(t,s)W(s)-\gamma H(t,s)(\rho(s)r_1(s))^{-\frac{1}{\gamma}}W^{\frac{\gamma+1}{\gamma}}(s)\right]ds \\
&\leq H(t,T)W(T)+ \\
&\quad \int_T^t\left[|h(t,s)|\,H^{\frac{\gamma+1}{\gamma}}(t,s)W(s)-\gamma H(t,s)(\rho(s)r_1(s))^{-\frac{1}{\gamma}}W^{\frac{\gamma+1}{\gamma}}(s)\right]ds \qquad (4.9)
\end{aligned}$$

Combining the above result with Theorem 5, we suppose

$$A=W(s),\ B=\frac{\rho(s)r_1(s)\,|h(t,s)|^\gamma}{(\gamma+1)^\gamma H^{\frac{\gamma}{\gamma+1}}(t,s)},\ \lambda=1+\frac{1}{\gamma}.$$

Then, we have:

$$\left(1+\frac{1}{\gamma}\right)W(s)\frac{(\rho(s)r_1(s))^{\frac{1}{\gamma}}\,|h(t,s)|^\gamma}{(\gamma+1)H^{\frac{\gamma}{\gamma+1}}(t,s)}-W^{1+\frac{1}{\gamma}}(s)\leq\frac{1}{\gamma}\frac{(\rho(s)r_1(s))^{\frac{\gamma+1}{\gamma}}\,|h(t,s)|^{\gamma+1}}{(\gamma+1)^{\gamma+1}H(t,s)}.$$

Multiplying both ends of the above equation with $\gamma H(t,s)(\rho(s)r_1(s))^{-\frac{1}{\gamma}}$, we have:

$$|h(t,s)|\,H^{1+\frac{1}{\gamma}}(t,s)W(s)-\gamma H(t,s)(\rho(s)r_1(s))^{-\frac{1}{\gamma}}W^{1+\frac{1}{\gamma}}(s)\leq\frac{\rho(s)r_1(s)\,|h(t,s)|^{\gamma+1}}{(\gamma+1)^\gamma}.$$

Then, equation (4.9) transforms into:

$$\int_T^t H(t,s)Q(s)ds\leq H(t,T)W(T)+\int_T^t\frac{\rho(s)r_1(s)\,|h(t,s)|^{\gamma+1}}{(\gamma+1)^{1+\gamma}}ds.$$

Therefore,

$$\frac{1}{H(t,T_\beta)}\int_{T_\beta}^t\left(H(t,s)Q(s)-\frac{\rho(s)r_1(s)\,|h(t,s)|^{\gamma+1}}{(\gamma+1)^{1+\gamma}}\right)ds\leq W(T_\beta),$$

which is contradictory with equation (4.6). It follows that $x(t)$ is the oscillation solution of equation (2.3).

Then, if $x(t)$ satisfies property (2), we can get $\lim_{t\to+\infty} x(t) = 0$ from equation (3.2) and the theorem. ∎

5. Some properties of autonilpotent groups

Note that when $r_2(t) = 1$, equation (2.3) transforms into:

$$\{r(t)((x(t) + p(t)x(\sigma(t)))')')^{\gamma}\}' + \int_a^b q(t,\xi)f(x(\tau(t,\xi)))d\xi = 0,\ t \geq t_0.$$

Corresponding theorem of the above equation can be obtained from the theorem of this study. Therefore, we can say, theorem of this study generalizes corresponding theorem of the above equation.

6. Conclusion

In conclusion, this study established oscillatory criteria for equation (2.3). The novelty of this study lies in the innovation of the differential equations we studied, understandable inference process and convenient use of the conclusion obtained. However, we found that, the restrictive conditions given by the equation were too strict. Whether restrictive conditions can be reduced or relaxed and whether such kind of method can be used for exploring oscillation of higher-order functional differential equation are expected to be further explored in the future.

Exploration of boundary value problem of differential equation is an important part of differential equation, which is of great significance to many problems theoretically. Due to practical condition and demand of practical engineering, more and more scholars will deeply discuss over boundary value problem of differential equation. Chinese financial industry develops rapidly currently, which increases the complexity and uncertainty of research problems. Most specialists consider that boundary value problem of random differential equation will be the hot research direction of boundary value problem in the future.

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COPROXIMINALITY RESULTS IN KÖTHE BOCHNER SPACES

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Abstract. As a counterpart to best approximation in normed linear spaces, best coapproximation was introduced by Franchetti and Furi, [3], in 1972. If X is a Banach space, $E(X)$ the Köthe Bochner function space and G is a closed subspace of X, the problem is: under what conditions the subspace $E(G)$ is proximinal (coproximinal) in $E(X)$? In this paper we prove that if G is a coproximinal separable subspace of X, then $E(G)$ is coproximinal in $E(X)$. Some other results are presented.

Keywords: best coapproximation, coproximinal set, Köthe Bochner function spaces.

Classification Numbers: Primary 46B20, Secondary 46E40.

1. Introduction

Let (T, μ) be a finite complete measure space and E denote the space of all equivalence classes of μ-measurable real-valued functions on T. For f and $g \in E$, $f \leq g$ means that $f(t) \leq g(t)$, μ-almost every where $t \in T$. For simplicity, we write: a.e $t \in T$. When E is equipped with a norm $\|\cdot\|_E$ and becomes a Banach space, it is then called (see [8] or [9]) a Köthe function space if the following conditions hold:

1. For each measurable subset A of T, with $\mu(A)$ is finite then the characteristic function $\chi_A \in E$.

2. For any two functions f and g such that $|f| \leq |g|$ and $g \in E$ imply $f \in E$ with $\|f\|_E \leq \|g\|_E$.

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A Köthe space E is said to be strictly monotone if $|f| > |g|$ then $\|f\|_E > \|g\|_E$, in other words: if $f \geq g \geq 0$ and $\|f\|_E = \|g\|_E$ imply $f = g$.

For a real Banach space $(X, \|\cdot\|_X)$ and a Köthe space E on a measure space (T, μ), let $E(X)$ be the space of all (equivalence classes of) strongly measurable functions $f : T \to X$ such that $\|f(\cdot)\|_X \in E$, equipped with the norm

$$|||f||| = \|\|f(\cdot)\|_X\|_E .$$

It is known, see [8], that $(E(X), |||\cdot|||)$ is a Banach space called the Köthe Bochner function space. The most important classes of Köthe Bochner function spaces are the Lebesgue-Bochner spaces $L^p(X)$, $(1 \leq p < \infty)$.

The concept of best coapproximation was initially introduced by Franchetti and Furi, [3], in order to study some characteristic properties of real Hilbert spaces. It was then considered by Papini and Singer, [13], and recently by many, see for example [12], [14] and [4]. This theory is largely concerned with the questions of existence, uniqueness and characterization of elements of best coapproximation.

For $x \in X$ and G a non empty subset of a normed space X, then $y_0 \in G$ is called best coapproximation point of G to x, if $\|y_0 - y\| \leq \|x - y\|, \forall y \in G$. The set G is called coproximinal in X, if for each $x \in X$, there exists at least one point of best coapproximation to x in G. The set-valued map $R_G : X \to 2^G$ defined by

$$R_G(x) = \{y_0 \in G : \|y_0 - y\| \leq \|x - y\|, \forall y \in G\},$$

is called the cometric projection. Let $R(G) = \{x \in X : R_G(x) \neq \phi\}$. Clearly, $G \subset R(G)$ and G is coproximinal in X iff $R(G) = X$. G is called densely coproximinal in X if the set $R(G)$ is dense in X. It is known [4] that if G is coproximinal subspace of X then it is closed and satisfies that $X = G \oplus \breve{G}$, where $\breve{G} = \{x \in X : \|y\| \leq \|y - x\|, \forall y \in G\}$. The set $\breve{G}$ is called the cometric complement of G.

The proximinality problem in vector valued function spaces was widely studied by many authors, see [1], [5], [7]. The problem of best coapproximation in the space of continuous functions was considered in [14, 15], while some few results on best coapproximation in $L^p(I, X), 1 \leq p < \infty$, can be found in [4], [10]. The problem of best coapproximation in the Köthe Bochner function space never been considered.

The object of this paper is to extend some of the results, on best coapproximation, in [4, 10] to Köthe Bochner function spaces and to get new ones, in order to give an answer to the following question: If G is coproximinal in X, under what conditions $E(G)$ is coproximinal in $E(X)$? Our main result is to prove that, Theorem 8, if G is a coproximinal separable subspace of X, then $E(G)$ is coproximinal in $E(X)$.

2. Coproximinality of $E(G)$ in $E(X)$

Throughout this section, (I, μ) is a finite measure space, X is a real Banach space and $E(X)$ is a Köthe Bochner function space.

Theorem 1 *Let G be a closed subset of a Banach space X and let E be a Köthe space. For f in $E(X)$ and g in $E(G)$ such that $g(t)$ are best coapproximation points in G to $f(t)$ in X, a.e. $t \in I$, then g is a best coapproximation to f.*

Proof. For $t \in I$, let $g(t)$ be a best coapproximation element in G to $f(t) \in X$. Then

$$\|g(t) - y\| \leq \|f(t) - y\|, \ \forall y \in G, \text{ a.e. } t \in I.$$

Hence, in particular, for any function h in $E(G)$, we have

$$\|g(t) - h(t)\| \leq \|f(t) - h(t)\|, \text{ a.e. } t \in I.$$

This implies,

$$\|\|g(\cdot) - h(\cdot)\|_X\|_E \leq \|\|f(\cdot) - h(\cdot)\|_X\|_E, \forall h \in E(G).$$

So, we get

$$|||g - h||| \leq |||f - h|||, \forall h \in E(G).$$

Hence, g is a best coapproximation to f. ∎

Let $A_1, ..., A_n$ be a sequence of mutually disjoint measurable subsets of I, such that $\cup A_i = I$ and let $a_1, ..., a_n$ be a finite sequence of real or complex numbers. A simple function is a function $f : I \to \mathbb{C}$ of the form

$$f(t) = \sum_{k=1}^{n} a_k \chi_{A_k}(t), \quad t \in I,$$

where for each k, χ_{A_k} is the characteristic function on A_k. A simple function in $E(X)$ is a function $f : I \to X$ of the form

$$f(t) = \sum_{k=1}^{n} x_k \chi_{A_k}(t), \quad x_k \in X \text{ and } t \in I.$$

It is known that the collection of all simple functions on a given measurable space forms a commutative algebra over $\mathbb{C}$ that is dense in the function space $E(X)$ on that space.

Lemma 2 *Let G be coproximinal set in X, then every simple function in $E(X)$ has a best coapproximation (a simple function) in $E(G)$.*

Proof. Let $f = \sum_{i=1}^{n} x_i \chi_{A_i}$ be a simple function in $E(X)$. For each $g \in E(G)$ and a given $\epsilon > 0$, there exists a simple function $\varphi_g = \sum_{i=1}^{n} y_i \chi_{A_i}$ in $E(G)$, with $y_i \in G$, such that $|||g - \varphi_g||| < \epsilon$. Since G is coproximinal, let z_i, for each i, be the best coapproximation in G to $x_i \in X$. Hence,

$$\|x_i - y_i\|_X \geq \|z_i - y_i\|_X, \quad 1 \leq i \leq n.$$

Now,

$$
\begin{aligned}
|||f-g||| &\geq |||f-\varphi_g||| - |||\varphi_g - g||| \\
&> \left\| \sum_{i=1}^{n} \|x_i - y_i\|_X \chi_{A_i}(t) \right\|_E - \epsilon \\
&\geq \left\| \sum_{i=1}^{n} \|z_i - y_i\|_X \chi_{A_i}(t) \right\|_E - \epsilon \\
&\geq |||g_0 - \varphi_g||| - \epsilon
\end{aligned} \tag{1}
$$

by taking $g_0 = \sum_{i=1}^{n} z_i \chi_{A_i}$. This implies from (1) above

$$|||f-g||| > |||g_0 - g||| - 2\epsilon$$

and since ϵ arbitrary, we get

$$|||f-g||| \geq |||g_0 - g|||$$

for all $g \in E(G)$. ∎

Theorem 3 *For any densely coproximinal set $G \subset X$, $E(G)$ is densely coproximinal in $E(X)$.*

Proof. Let $f = \sum_{i=1}^{n} x_i \chi_{A_i}$ be a simple function in $E(X)$. Since $R(G)$ is dense, given $\epsilon > 0$, we get a simple function $\varphi = \sum_{i=1}^{n} z_i \chi_{A_i}$, with $z_i \in R(G)$ and $|||f - \varphi||| < \epsilon$. For each i, let $y_i \in G$ be a best coapproximation point to z_i, then from the above Lemma, $\varphi_0 = \sum_{i=1}^{n} y_i \chi_{A_i}$ is a best coapproximation to φ in $E(G)$. ∎

In what follows, we consider coproximinality when G is separable in the Banach space X. Let G be a closed subset in the Banach space X. For each $f \in E(X)$, define the map $\pi_f : I \to 2^G$ as

$$\pi_f(t) = \{g \in G : f(t) - g \in \breve{G}\},\ t \in I,$$

which is a set-valued map, taking each element t of a measurable space I into a subset of G, precisely the set of best coapproximation points to $f(t)$. One can easily see that π_f can be written as the composition $R_G \circ f$. Certain conditions can be imposed on π_f so that it has a measurable selection. Recall that ([13]) a set-valued map on a measure space I, $\pi : I \to 2^X$ is said to be weakly measurable if the set $\{t \in I : \pi(t) \bigcap O \neq \phi\}$ is measurable, for every open subset O in X. By definition, such a mapping has a measurable selection if there exists a measurable function $h : I \to X$ such that $h(t) \in \pi(t)$ for each $t \in I$. The following lemma can be found in [13] (see also [10] or [4]).

Lemma 4 *Let $\pi : I \to 2^Y$ be a weakly measurable set-valued map. If Y is a separable Banach space then π has a measurable selection.*

Theorem 5 *Let G be a separable subspace of X such that π_f as defined above is weakly measurable. Then $E(G)$ is coproximinal in $E(X)$ if G is coproximinal in X.*

Proof. Suppose that G is coproximinal in X, and let f be in $E(X)$. The assumption that the map $\pi_f : I \to 2^G$, defined above, is weakly measurable implies that it has a measurable selection, say $g : I \to G$ such that $g(t) \in \pi_f(t)$, for all $t \in I$ (by the previous lemma), since its values are in a separable space G. Hence the result follows from Theorem 1 above if we show that $g \in E(G)$.

Now, $g : I \to G$ such that $g(t) \in \pi_f(t)$, for all $t \in I$, so $g(t) \in G$ is a best coapproximation point of $f(t)$ in X. Thus $f(t) - g(t) \in \breve{G}$, for all $t \in I$, which implies that

$$\|z\| \leq \|z - (f(t) - g(t))\|, \forall z \in G$$

taking $z = -g(t)$, we get

$$\|g(t)\| \leq \|-g(t) - (f(t) - g(t))\| = \|f(t)\|, \ a.e\ t \in I.$$

Hence, $g \in E(G)$ and $g(t)$ is a best coapproximation point to $f(t)$. ∎

3. Main result

Our main result of this paper concerns coproximanility of $E(G)$ in $E(X)$ for a separable closed subspace G in X: Theorem 7 below. For the proof, we will follow a technique used by Mendoza, [11], for proximinality. The following lemma is also needed in the proof (Lemma 3 of [16]).

Lemma 6 *Assume $\mu(I) < \infty$. Suppose (M, d) is a metric space and A is a subset of I such that $\mu^*(A) = \mu(I)$, where μ^* denotes the outer measure associated to μ. If g is a mapping from I to M with separable range, then for any $\epsilon > 0$, there exists a countable partition $\{E_n\}$ of I in measurable sets and $A_n \subset A \cap E_n$ such that $\mu^*(A_n) = \mu(E_n)$ and $diam(g(A_n)) < \epsilon$ for all n, where*

$$diam(g(A_n)) = \sup \{\|g(s) - g(t)\| : s, t \in A_n\}.$$

Theorem 7 *Let (I, μ) be a finite measure space, G be a separable coproximinal subspace of X and $f : I \to X$ be measurable function. Then there is a measurable function $g : I \to G$ such that $g(t)$ is a point of coapproximation to $f(t)$ in G, a.e. $t \in I$.*

Proof. Let $f : I \to X$ be a measurable function. So we may assume that $f(I)$ is a separable set in X. Using the fact that μ is finite we can find countable partitions $\{I_n\}_{n=1}^{\infty}$ of I in measurable sets such that $\text{diam}(f(I_n)) < \frac{1}{2}$ and $\mu(I_n) < \infty$, for all n.

Now, since G is coproximinal in X then for each $t \in I$, let $y_t \in G$ be a point of best coapproximation to $f(t)$. Define g_0 from I into G such that $g_0(t) = y_t$, as above. Applying Lemma 6 above in this section, to the mapping g_0 in each I_n, taking $\epsilon = \frac{1}{2}$ and $I = A = I_n$. We get a countable partition in each I_n and therefore a countable partition in the whole of I. Thus we get a countable partition $\{E_n\}_{n=1}^{\infty}$ of I in measurable sets and a sequence of subsets $\{A_n\}_{n=1}^{\infty}$ of I such that

$$\begin{aligned} A_n &\subseteq E_n,\ \mu^*(A_n) = \mu(E_n) < +\infty, \\ \operatorname{diam}(g_0(A_n)) &< \frac{1}{2},\ \operatorname{diam}(f(E_n)) < \frac{1}{2}. \end{aligned}$$

Let us apply again the same argument in each E_n with $\epsilon = \frac{1}{2^2}$, $I = E_n$ and $A = A_n$. For each n we get a countable partition $\{E_{n_k} : 1 \leq k < \infty\}$ of E_n in measurable sets and a sequence $\{A_{n_k} : 1 \leq k < \infty\}$ of subsets of I such that

$$\begin{aligned} A_{n_k} &\subseteq E_{n_k} \cap A_n,\ \mu^*(A_{n_k}) = \mu(E_{n_k}), \\ \operatorname{diam}(g_0(A_{n_k})) &< \frac{1}{2^2} \text{ and } \operatorname{diam}(f(E_{n_k})) < \frac{1}{2^2}, \end{aligned}$$

for all n and k. Now for each natural number k, let $\triangle_k$ be the set of k-tuples of natural numbers and let $\triangle = \bigcup_{k=1}^{\infty} \triangle_k$. On this $\triangle$ consider the partial order defined by $(m_1, m_2, ..., m_i) \leq (n_1, n_2, ..., n_j)$ if and only if $i \leq j$ and $m_k = n_k$ for $k = 1, 2, ..., i$. Then by induction for each natural number k, we can take a partition $\{E_\alpha : \alpha \in \triangle_k\}$ of measurable subsets of I and a collection $\{A_\alpha\}_{\alpha \in \triangle_k}$ of subsets of I such that

(1) $A_\alpha \subseteq E_\alpha$ and $\mu^*(A_\alpha) = \mu(E_\alpha)$ for each α.

(2) $A_\alpha \subseteq A_\beta$ and $E_\alpha \subseteq E_\beta$ if $\beta \leq \alpha$.

(3) $\operatorname{diam}(f(E_\alpha)) < \frac{1}{2^k}$ and $\operatorname{diam}(g_0(A_\alpha)) < \frac{1}{2^k}$ if $\alpha \in \triangle_k$.

We may assume that $A_\alpha \neq \emptyset$ for all α (forget the $\alpha's$ for which $A_\alpha = \emptyset$). For each $\alpha \in \triangle$, take $t_\alpha \in A_\alpha$ and define g_k from I into G by

$$g_k(\cdot) = \sum_{\alpha \in \triangle_k} g_0(t_\alpha) \chi_{E_\alpha}(\cdot).$$

Then, for each $t \in I$ and $n \leq k$, we have

$$\|g_n(t) - g_k(t)\| = \left\| \sum_{\alpha \in \triangle_n} g_0(t_\alpha) \chi_{E_\alpha}(t) - \sum_{\beta \in \triangle_k} g_0(t_\beta) \chi_{E_\beta}(t) \right\|$$

By (1) and (2) above and since $n \leq k$, we have:

$$\begin{aligned} \|g_n(t) - g_k(t)\| &\leq \left\| \sum_{\beta \in \triangle_k} \left(g_0(t_\alpha) - g_0(t_\beta) \right) \chi_{E_\beta}(t) \right\| \\ &\leq \sum_{\beta \in \triangle_k} \left\| g_0(t_\alpha) - g_0(t_\beta) \right\| \mu(E_\beta) \\ &\leq \frac{1}{2^n}. \end{aligned}$$

Therefore, $(g_k(t))$ is a Cauchy sequence in X for each $t \in I$, hence convergent. Let $g : I \to G$ be the pointwise limit of (g_k). Since g_k is measurable for each k, then g is measurable. Now, let $t \in I$ and n a natural number.

Suppose $t \in E_\alpha$. Then, for any $y \in G$, we have:

$$\begin{aligned} \|g_n(t) - y\| &= \|g_0(t_\alpha) - y\| \\ &\leq \|f(t_\alpha) - y\| \\ &\leq \|f(t) - y\| + \|f(t) - f(t_\alpha)\| \\ &\leq \|f(t) - y\| + \frac{1}{2^n}. \end{aligned}$$

Letting $n \to \infty$ we get

$$\|g(t) - y\| \leq \|f(t) - y\|$$

for all $y \in G$ and so $g(t)$ is a best coapproximation point of $f(t)$ in G. ∎

Theorem 8 *Let G be a separable subspace of X and E a Köthe space. Then $E(G)$ is coproximinal in $E(X)$ if G is coproximinal in X.*

Proof. Suppose that G is a separable coproximinal subspace in X. Let f be a function in $E(X)$, then Theorem 7 guarantees that there exists a measurable function g defined on I with values in G such that $g(t)$ is a point of best coapproximation to $f(t)$ in G for almost all t. It follows from Theorem 1 that g is a point of best coapproximation to f in $E(G)$. ∎

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INTUITIONISTIC PERMEABLE VALUES IN BCK/BCI-ALGEBRAS

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Abstract. Intuitionistic permeable values in BCK/BCI-algebras are introduced, and several properties are investigated. A relation between an intuitionistic permeable S-value and an intuitionistic permeable I-value is discussed. Conditions for the intuitionistic lower (upper) level set to be S-energetic and I-energetic are considered. Conditions for a couple of numbers to be an intuitionistic permeable S-value are studied.

Keywords: S-energetic subset, I-energetic subset, intuitionistic permeable S-value, intuitionistic permeable I-value.

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1. Introduction

The notions of energetic (resp. right vanished, right stable) subsets and (anti) permeable values in BCK/BCI-algebras are introduced by Jun et al. [2]. Using the notion of (anti) fuzzy subalgebras/ideals of BCK/BCI-algebras, they investigated relations among subalgebras/ideals, energetic subsets, (anti) permeable values, right vanished subsets and right stable subsets.

In this paper, we introduce the notion of intuitionistic permeable values in BCK/BCI-algebras, and investigate several properties. We provide a relation between an intuitionistic permeable S-value and an intuitionistic permeable I-value. We study conditions for the intuitionistic lower (upper) level set to be

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S-energetic and I-energetic. We consider conditions for a couple of numbers to be an intuitionistic permeable S-value. We show that if an intuitionistic fuzzy set $A = (f_A, g_A)$ in a BCK-algebra X is an intuitionistic fuzzy ideal of X, then the nonempty intuitionistic upper level set of $A = (f_A, g_A)$ is a right stable subset of X and the nonempty intuitionistic lower level set of $A = (f_A, g_A)$ is a right vanished subset of X.

2. Preliminaries

The BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki and has been extensively investigated by many researchers.

An algebra $(X; *, 0)$ of type $(2, 0)$ is called a *BCI-algebra* if it satisfies the following conditions:

(I) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = 0)$,

(II) $(\forall x, y \in X)\ ((x * (x * y)) * y = 0)$,

(III) $(\forall x \in X)\ (x * x = 0)$,

(IV) $(\forall x, y \in X)\ (x * y = 0, y * x = 0 \ \Rightarrow\ x = y)$.

If a BCI-algebra X satisfies the following identity

(V) $(\forall x \in X)\ (0 * x = 0)$,

then X is called a *BCK-algebra.* Any BCK/BCI-algebra X satisfies the following axioms:

$$(\forall x \in X)\,(x * 0 = x)\,, \tag{2.1}$$

$$(\forall x, y, z \in X)\,(x \le y \ \Rightarrow\ x * z \le y * z,\ z * y \le z * x)\,, \tag{2.2}$$

$$(\forall x, y, z \in X)\,((x * y) * z = (x * z) * y)\,, \tag{2.3}$$

$$(\forall x, y, z \in X)\,((x * z) * (y * z) \le x * y) \tag{2.4}$$

where $x \le y$ if and only if $x * y = 0$. A nonempty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$. A subset I of a BCK/BCI-algebra X is called an *ideal* of X if it satisfies

$$0 \in I, \tag{2.5}$$

$$(\forall x \in X)\,(\forall y \in I)\,(x * y \in I \ \Rightarrow\ x \in I)\,. \tag{2.6}$$

We refer the reader to the books [1], [4] for further information regarding BCK/BCI-algebras.

An intuitionistic fuzzy set $A = (f_A, g_A)$ in X is called an *intuitionistic fuzzy subalgebra* of X (see [3]) if the following condition is valid.

$$(\forall x, y \in X) \begin{pmatrix} f_A(x * y) \ge \min\{f_A(x), f_A(y)\} \\ g_A(x * y) \le \max\{g_A(x), g_A(y)\} \end{pmatrix}. \tag{2.7}$$

An intuitionistic fuzzy set $A = (f_A, g_A)$ in X is called an *intuitionistic fuzzy ideal* of X (see [3]) if the following conditions are valid.

$$(\forall x \in X)\,(f_A(0) \geq f_A(x),\ g_A(0) \leq g_A(x))\,, \tag{2.8}$$

$$(\forall x, y \in X)\left(\begin{array}{l} f_A(x) \geq \min\{f_A(x*y), f_A(y)\} \\ g_A(x) \leq \max\{g_A(x*y), g_A(y)\} \end{array}\right). \tag{2.9}$$

Note that every intuitionistic fuzzy ideal is an intuitionistic fuzzy subalgebra in a BCK-algebra (see [3]).

Every intuitionistic fuzzy ideal $A = (f_A, g_A)$ of X satisfies the following condition (see [3]).

$$(\forall x, y \in X)\,(x*y = 0\ \Rightarrow\ f_A(x) \geq f(y),\ g_A(x) \leq g_A(y))\,. \tag{2.10}$$

3. Intuitionistic permeable values

In what follows, let X denote a BCK/BCI-algebra unless otherwise specified.

Definition 3.1. ([2]) A non-empty subset A of X is said to be *S-energetic* if it satisfies

$$(\forall a, b \in X)\,(a*b \in A\ \Rightarrow\ \{a, b\} \cap A \neq \emptyset)\,. \tag{3.1}$$

Definition 3.2. Let $A = (f_A, g_A)$ be an intuitionistic fuzzy set in X and $(t_A, s_A) \in [0,1] \times [0,1]$.

We say that (t_A, s_A) is an *intuitionistic permeable S-value* for $A = (f_A, g_A)$ if the following assertion is valid:

$$(\forall x, y \in X)\left(\begin{array}{l} f_A(x*y) \geq t_A\ \Rightarrow \max\{f_A(x), f_A(y)\} \geq t_A \\ g_A(x*y) \leq s_A\ \Rightarrow \min\{g_A(x), g_A(y)\} \leq s_A \end{array}\right). \tag{3.2}$$

Example 3.3. (1) Let $X = \{0, 1, 2, 3\}$ be a BCK-algebra with the following Cayley table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	2	1	0

Let $A = (f_A, g_A)$ be an intuitionistic fuzzy set in X defined by

$$f_A = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0.3 & 0.5 & 0.4 & 0.6 \end{pmatrix},\quad g_A = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0.6 & 0.4 & 0.3 & 0.2 \end{pmatrix}.$$

If we take $(t_A, s_A) \in (0.3, 1] \times [0, 0.6)$, then it is easy to check that (t_A, s_A) is an intuitionistic permeable S-value for $A = (f_A, g_A)$.

(2) Let $X = \{0, 1, 2, a, b\}$ be a BCI-algebra with the following Cayley table:

$*$	0	1	2	a	b
0	0	0	0	a	a
1	1	0	1	b	a
2	2	2	0	a	a
a	a	a	a	0	0
b	b	a	b	1	0

Let $A = (f_A, g_A)$ be an intuitionistic fuzzy set in X defined by

$$f_A = \begin{pmatrix} 0 & 1 & 2 & a & b \\ 0.2 & 0.5 & 0.3 & 0.4 & 0.5 \end{pmatrix}$$

and

$$g_A = \begin{pmatrix} 0 & 1 & 2 & a & b \\ 0.7 & 0.4 & 0.6 & 0.2 & 0.3 \end{pmatrix}.$$

If we take $(t_A, s_A) \in (0.2, 1] \times (0.3, 0.7)$, then we know that (t_A, s_A) is an intuitionistic permeable S-value for $A = (f_A, g_A)$.

For an intuitionistic fuzzy set $A = (f_A, g_A)$ in X and $(t_A, s_A) \in [0, 1] \times [0, 1]$, the intuitionistic upper (resp. lower) level sets are defined as follows:

$$U(f_A; t_A) := \{x \in X \mid f_A(x) \geq t_A\},\ L(g_A; s_A) := \{x \in X \mid g_A(x) \leq s_A\},$$

Theorem 3.4. *If (t_A, s_A) is an intuitionistic permeable S-value for an intuitionistic fuzzy set $A = (f_A, g_A)$ in X, then the intuitionistic upper level set $U(f_A; t_A)$ and the intuitionistic lower level set $L(g_A; s_A)$ are S-energetic subsets of X when they are nonempty.*

Proof. Assume that $U(f_A; t_A) \neq \emptyset \neq L(g_A; s_A)$ for $(t_A, s_A) \in [0, 1] \times [0, 1]$. Let $a, b, x, y \in X$ be such that $a * b \in U(f_A; t_A)$ and $x * y \in L(g_A; s_A)$. Then $f_A(a * b) \geq t_A$ and $g_A(x * y) \leq s_A$. It follows from (3.2) that

$$\max\{f_A(a), f_A(b)\} \geq t_A, \tag{3.3}$$

$$\min\{g_A(x), g_A(y)\} \leq s_A. \tag{3.4}$$

The fact (3.3) implies that $f_A(a) \geq t_A$ or $f_A(b) \geq t_A$. Thus $\{a, b\} \cap U(f_A; t_A) \neq \emptyset$. From the fact (3.4), we have $\{x, y\} \cap L(g_A; s_A) \neq \emptyset$. Therefore $U(f_A; t_A)$ and $L(g_A; s_A)$ are S-energetic subsets of X. ∎

Note from [2, Theorem 3.3] that if A is an S-energetic subset of X, then $X \setminus A$ is a subalgebra of X. Hence we have the following corollary.

Corollary 3.5. *If (t_A, s_A) is an intuitionistic permeable S-value for an intuitionistic fuzzy set $A = (f_A, g_A)$ in X, then $X \setminus U(f_A; t_A)$ and $X \setminus L(g_A; s_A)$ are subalgebras of X when they are nonempty.*

Theorem 3.6. *If $A = (f_A, g_A)$ is an intuitionistic fuzzy subalgebra of X, then $X \setminus U(f_A; t)$ and $X \setminus L(g_A; t)$ are S-energetic subsets of X for all $t \in [0, 1]$ whenever they are nonempty.*

Proof. Let $x, y \in X$ be such that $x * y \in X \setminus U(f_A; t)$ and $x * y \in X \setminus L(g_A; t)$ for $t \in [0, 1]$. If $\{x, y\} \cap (X \setminus U(f_A; t)) = \emptyset$, then $x, y \in U(f_A; t)$ and so $x * y \in U(f_A; t)$ since $U(f_A; t)$ is a subalgebra of X. Assume that $\{x, y\} \cap (X \setminus L(g_A; t)) = \emptyset$. Then $x, y \in L(g_A; t)$ and so $x * y \in L(g_A; t)$ since $L(g_A; t)$ is a subalgebra of X. This is a contradiction. Therefore $X \setminus U(f_A; t)$ and $X \setminus L(g_A; t)$ are S-energetic subsets of X for all $t \in [0, 1]$. ∎

Corollary 3.7. *If $A = (f_A, g_A)$ is an intuitionistic fuzzy ideal of a BCK-algebra X, then $X \setminus U(f_A; t)$ and $X \setminus L(g_A; t)$ are S-energetic subsets of X for all $t \in [0, 1]$ whenever they are nonempty.*

Definition 3.8. An intuitionistic fuzzy set $A = (f_A, g_A)$ in X is called an *anti-intuitionistic fuzzy subalgebra* if the following condition is valid.

$$(\forall x, y \in X) \left(\begin{array}{l} f_A(x * y) \leq \max\{f_A(x), f_A(y)\} \\ g_A(x * y) \geq \min\{g_A(x), g_A(y)\} \end{array} \right). \tag{3.5}$$

Theorem 3.9. *For an intuitionistic fuzzy set $A = (f_A, g_A)$ in X, let $(t_A, s_A) \in [0, 1] \times [0, 1]$ be such that $U(f_A; t_A) \neq \emptyset \neq L(g_A; s_A)$. If $A = (f_A, g_A)$ is an anti-intuitionistic fuzzy subalgebra of X, then (t_A, s_A) is an intuitionistic permeable S-value for $A = (f_A, g_A)$.*

Proof. Let $x, y \in X$ be such that $f_A(x * y) \geq t_A$ and $g_A(x * y) \leq s_A$. Using (3.5), we have

$$t_A \leq f_A(x * y) \leq \max\{f_A(x), f_A(y)\}$$

and

$$s_A \geq g_A(x * y) \geq \min\{g_A(x), g_A(y)\}.$$

Therefore (t_A, s_A) is an intuitionistic permeable S-value for $A = (f_A, g_A)$. ∎

Theorem 3.10. *If $A = (f_A, g_A)$ is an anti-intuitionistic fuzzy subalgebra of X, then $U(f_A; t_A)$ and $L(g_A; s_A)$ are S-energetic subsets of X for all $(t_A, s_A) \in [0, 1] \times [0, 1]$ with $U(f_A; t_A) \neq \emptyset \neq L(g_A; s_A)$.*

Proof. Assume that $U(f_A; t_A) \neq \emptyset \neq L(g_A; s_A)$ for $(t_A, s_A) \in [0, 1] \times [0, 1]$. Let $x, y \in X$ be such that $x * y \in U(f_A; t_A)$. Then $t_A \leq f_A(x * y) \leq \max\{f_A(x), f_A(y)\}$, and so $f_A(x) \geq t_A$ or $f_A(y) \geq t_A$. Thus $\{x, y\} \cap U(f_A; t_A) \neq \emptyset$. If $a * b \in L(g_A; s_A)$ for $a, b \in X$, then $s_A \geq g_A(a * b) \geq \min\{g_A(a), g_A(b)\}$. It follows that $g_A(a) \leq s_A$ or $g_A(b) \leq s_A$, that is, $a \in L(g_A; s_A)$ or $b \in L(g_A; s_A)$. Hence $\{a, b\} \cap L(g_A; s_A) \neq \emptyset$. This completes the proof. ∎

The following example shows that the converse of Theorem 3.10 is not true in general, that is, there exist an intuitionistic fuzzy set $A = (f_A, g_A)$ in X and $(t_A, s_A) \in [0, 1] \times [0, 1]$ such that

1. $U(f_A; t_A)$ and $L(g_A; s_A)$ are S-energetic subsets of X.
2. $A = (f_A, g_A)$ is not an anti-intuitionistic fuzzy subalgebra of X.

Example 3.11. Let $X = \{0, a, b, c, d\}$ be a BCK-algebra with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	0	0
b	b	a	0	0	a
c	c	b	a	0	b
d	d	a	a	a	0

Let $A = (f_A, g_A)$ be an intuitionistic fuzzy set in X defined by

$$f_A = \begin{pmatrix} 0 & a & b & c & d \\ 0.3 & 0.4 & 0.4 & 0.2 & 0.4 \end{pmatrix}$$

and

$$g_A = \begin{pmatrix} 0 & a & b & c & d \\ 0.5 & 0.4 & 0.4 & 0.3 & 0.2 \end{pmatrix}.$$

Let $(t_A, s_A) \in (0.3, 0.4] \times [0.2, 0.4)$. Then $U(f_A; t_A) = \{a, b, d\}$ and

$$L(g_A; s_A) = \begin{cases} \{c, d\} & \text{if } s_A \in [0.3, 0.4), \\ \{d\} & \text{if } s_A \in [0.2, 0.3) \end{cases}$$

are S-energetic subsets of X. But $A = (f_A, g_A)$ is not an anti-intuitionistic fuzzy subalgebra of X.

Definition 3.12. ([2]) A non-empty subset A of X is said to be *I-energetic* if it satisfies:

$$(\forall x, y \in X)\,(y \in A \ \Rightarrow\ \{x, y * x\} \cap A \neq \emptyset)\,. \tag{3.6}$$

Definition 3.13. Let $A = (f_A, g_A)$ be an intuitionistic fuzzy set in X and $(t_A, s_A) \in [0, 1] \times [0, 1]$. We say that (t_A, s_A) is an *intuitionistic permeable I-value* for $A = (f_A, g_A)$ if the following assertion is valid:

$$(\forall x, y \in X) \begin{pmatrix} f_A(x) \geq t_A \ \Rightarrow \max\{f_A(x * y), f_A(y)\} \geq t_A \\ g_A(x) \leq s_A \ \Rightarrow \min\{g_A(x * y), g_A(y)\} \leq s_A \end{pmatrix}. \tag{3.7}$$

Example 3.14. In Example 3.3(2), the number $(t_A, s_A) \in (0.2, 1] \times (0.3, 0.7)$ is an intuitionistic permeable I-value for $A = (f_A, g_A)$.

Theorem 3.15. *If $A = (f_A, g_A)$ is an anti-intuitionistic fuzzy subalgebra of a BCK-algebra X, then every intuitionistic permeable I-value for $A = (f_A, g_A)$ is an intuitionistic permeable S-value for $A = (f_A, g_A)$.*

Proof. Note that $f_A(0) \leq f_A(x)$ and $g_A(0) \geq g_A(x)$ for all $x \in X$.

Let $(t_A, s_A) \in [0,1] \times [0,1]$ be an intuitionistic permeable I-value for $A = (f_A, g_A)$. Assume that $f_A(x * y) \geq t_A$ and $g_A(x * y) \leq s_A$ for all $x, y \in X$. Using (3.7), (2.3), (III) and (V), we have

$$\begin{aligned} t_A &\leq \max\{f_A((x * y) * x), f_A(x)\} = \max\{f_A((x * x) * y), f_A(x)\} \\ &= \max\{f_A(0 * y), f_A(x)\} = \max\{f_A(0), f_A(x)\} = f_A(x) \end{aligned}$$

and

$$\begin{aligned} s_A &\geq \min\{g_A((x * y) * x), g_A(x)\} = \min\{g_A((x * x) * y), g_A(x)\} \\ &= \min\{g_A(0 * y), g_A(x)\} = \min\{g_A(0), g_A(x)\} = g_A(x). \end{aligned}$$

It follows that

$$\max\{f_A(x), f_A(y)\} \geq f_A(x) \geq t_A \text{ and } \min\{g_A(x), g_A(y)\} \leq g_A(x) \leq s_A.$$

Therefore (t_A, s_A) is an intuitionistic permeable S-value for $A = (f_A, g_A)$. ∎

Corollary 3.16. *If $A = (f_A, g_A)$ is an anti-intuitionistic fuzzy ideal of a BCK-algebra X, then every intuitionistic permeable I-value for $A = (f_A, g_A)$ is an intuitionistic permeable S-value for $A = (f_A, g_A)$.*

The converse of Theorem 3.15 is not true in general as seen in the following example.

Example 3.17. Let $X = \{0, a, b, c\}$ be a BCK-algebra with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Let $A = (f_A, g_A)$ be an intuitionistic fuzzy set in X defined by

$$f_A = \begin{pmatrix} 0 & a & b & c \\ 0.2 & 0.4 & 0.5 & 0.5 \end{pmatrix}, \quad g_A = \begin{pmatrix} 0 & a & b & c \\ 0.3 & 0.2 & 0.4 & 0.2 \end{pmatrix}.$$

If we take $(t_A, s_A) \in (0.4, 0.5] \times [0.2, 0.3)$, then $U(f_A; t_A) = \{b, c\}$ and $L(g_A; s_A) = \{a, c\}$. It is easy to check that (t_A, s_A) is an an intuitionistic permeable S-value for $A = (f_A, g_A)$. Note that $f_A(b) = 0.5 \geq t_A$ and $g_A(a) = 0.2 \leq s_A$. Hence $\max\{f_A(b * a), f_A(a)\} = 0.4 < t_A$ and $\min\{g_A(a * b), g_A(b)\} = 0.3 > s_A$. This shows that (t_A, s_A) is not an intuitionistic permeable I-value for $A = (f_A, g_A)$.

Theorem 3.18. *If $A = (f_A, g_A)$ is an intuitionistic fuzzy ideal of X, then $X \setminus U(f_A; t_A)$ and $X \setminus L(g_A; s_A)$ are empty or I-energetic subsets of X for all $(t_A, s_A) \in [0,1] \times [0,1]$.*

Proof. Assume that $X\setminus U(f_A;t_A)$ and $X\setminus L(g_A;s_A)$ are nonempty for all $(t_A,s_A)\in[0,1]\times[0,1]$. If $y\in X\setminus U(f_A;t_A)$ and $b\in X\setminus L(g_A;s_A)$, then

$$t_A > f_A(y) \geq \min\{f_A(y*x), f_A(x)\} \tag{3.8}$$

$$s_A < g_A(b) \leq \max\{g_A(b*a), g_A(a)\} \tag{3.9}$$

for all $a,x\in X$. From (3.8), we have $f_A(y*x)<t_A$ or $f_A(x)<t_A$, that is, $y*x\in X\setminus U(f_A;t_A)$ or $x\in X\setminus U(f_A;t_A)$. Hence $\{x,y*x\}\cap(X\setminus U(f_A;t_A))\neq\emptyset$. From (3.9), we get $g_A(b*a)>s_A$ or $g_A(a)>s_A$, that is, $b*a\in X\setminus L(g_A;s_A)$ or $a\in X\setminus L(g_A;s_A)$. Thus $\{a,b*a\}\cap(X\setminus L(g_A;s_A))\neq\emptyset$. Therefore $X\setminus U(f_A;t_A)$ and $X\setminus L(g_A;s_A)$ are I-energetic subsets of X for all $(t_A,s_A)\in[0,1]\times[0,1]$. ∎

Theorem 3.19. *If (t_A,s_A) is an intuitionistic permeable I-value for an intuitionistic fuzzy set $A=(f_A,g_A)$ in X, then the intuitionistic upper level set $U(f_A;t_A)$ and the intuitionistic lower level set $L(g_A;s_A)$ are empty or I-energetic subsets of X.*

Proof. Assume that $U(f_A;t_A)$ and $L(g_A;s_A)$ are nonempty for all $(t_A,s_A)\in[0,1]\times[0,1]$. Let $y\in U(f_A;t_A)$. Then $f_A(y)\geq t_A$, and so

$$\max\{f_A(y*x), f_A(x)\}\geq t_A \text{ for all } x\in X.$$

It follows that $f_A(y*x)\geq t_A$ or $f_A(x)\geq t_A$, that is, $y*x\in U(f_A;t_A)$ or $x\in U(f_A;t_A)$. Hence $\{x,y*x\}\cap U(f_A;t_A)\neq\emptyset$. If $b\in L(g_A;s_A)$, then $g_A(b)\leq s_A$ and thus

$$\min\{g_A(b*a), g_A(a)\}\leq s_A$$

for all $a\in X$. It follows that $g_A(b*a)\leq s_A$ or $g_A(a)\leq s_A$ and so that

$$\{a,b*a\}\cap L(g_A;s_A)\neq\emptyset.$$

Therefore $U(f_A;t_A)$ and $L(g_A;s_A)$ are I-energetic subsets of X. ∎

Theorem 3.20. *Let $A=(f_A,g_A)$ be an intuitionistic fuzzy subalgebra of X which satisfies the following condition:*

$$(x*y)*z=0 \;\Rightarrow\; \begin{pmatrix} f_A(x)\geq\min\{f_A(y),f_A(z)\} \\ g_A(x)\leq\max\{g_A(y),g_A(z)\} \end{pmatrix} \tag{3.10}$$

for all $x,y,z\in X$. Then $X\setminus U(f_A;t_A)$ and $X\setminus L(g_A;s_A)$ are empty or I-energetic subsets of X for all $(t_A,s_A)\in[0,1]\times[0,1]$.

Proof. Assume that $A=(f_A,g_A)$ is an intuitionistic fuzzy subalgebra of X satisfying the condition (3.10). Note that $f_A(0)\geq f_A(x)$ and $g_A(0)\leq g_A(x)$ for all $x\in X$. Since $(x*(x*y))*y=0$ for all $x,y\in X$, it follows from (3.10) that $f_A(x)\geq\min\{f_A(x*y),f_A(y)\}$ and $g_A(x)\leq\max\{g_A(x*y),g_A(y)\}$. Hence $A=(f_A,g_A)$ is an intuitionistic fuzzy ideal of X. Using Theorem 3.18, we know that $X\setminus U(f_A;t_A)$ and $X\setminus L(g_A;s_A)$ are I-energetic subsets of X for all $(t_A,s_A)\in[0,1]\times[0,1]$ whenever they are nonempty. ∎

Definition 3.21. ([2]) Let Q be a non-empty subset of X. Then Q is said to be *right vanished* if it satisfies:

$$(\forall a, b \in X)\,(a * b \in Q \;\Rightarrow\; a \in Q)\,. \tag{3.11}$$

Q is said to be *right stable* if $Q * X := \{a * x \mid a \in Q,\ x \in X\} \subseteq Q$.

Theorem 3.22. *If $A = (f_A, g_A)$ is an intuitionistic fuzzy ideal of a BCK-algebra X, then*

(1) *The nonempty intuitionistic upper level set of $A = (f_A, g_A)$ is a right stable subset of X.*

(2) *The nonempty intuitionistic lower level set of $A = (f_A, g_A)$ is a right stable subset of X.*

Proof. (1) Let $y \in U(f_A; t_A)$ Then $f_A(y) \geq t_A$. Since $(y*x)*y = 0$ for all $x, y \in X$, it follows from (2.10) that $f_A(y * x) \geq f_A(y) \geq t_A$, that is, $y * x \in U(f_A; t_A)$ for all $x \in X$. Thus $U(f_A; t_A)$ is a right stable subset of X.

(2) If $b \in L(g_A; s_A)$, then $g_A(b) \leq s_A$ and so $g_A(b * x) \leq g_A(b) \leq s_A$ for all $x \in X$ by (2.10). Hence $b * x \in L(g_A; s_A)$ for all $x \in X$, and therefore $L(g_A; s_A)$ is a right stable subset of X. ∎

4. Conclusion

We have introduced the notion of intuitionistic permeable values in BCK/BCI-algebras, and have investigated several properties. We have discussed the relation between an intuitionistic permeable S-value and an intuitionistic permeable I-value, and have presented conditions for the intuitionistic lower (upper) level set to be S-energetic and I-energetic. We have considered conditions for a couple of numbers to be an intuitionistic permeable S-value, and have shown that if an intuitionistic fuzzy set $A = (f_A, g_A)$ in a BCK-algebra X is an intuitionistic fuzzy ideal of X, then the nonempty intuitionistic upper level set of $A = (f_A, g_A)$ is a right stable subset of X and the nonempty intuitionistic lower level set of $A = (f_A, g_A)$ is a right vanished subset of X. Based on this paper, we will apply the notion of this article to other algebraic structures, for example, semigroup theory, (pseudo) MV-algebras, (pseudo) BL-algebra, etc.

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SOME ELEGANT PROOFS IN 2-METRIC SPACE AND G-METRIC SPACE

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Abstract. Elegant analytical proofs of some fixed point theorems in 2-metric space and G-metric space are presented through elementary set theoretical notions of real numbers and repeated use of the rectangle inequality of the G-metric, without an appeal to iterations. The unique fixed points obtained are shown to be contractive fixed points in these spaces.

Keywords: 2-metric space, G-metric space, infimum property, self-map, fixed point.

Mathematics Subject Classification: 54H25.

1. Introduction

Let (X, ρ) be a metric space and $a = \rho(x, y)$, $b = \rho(y, z)$ and $c = \rho(z, x)$ be the sides of Δxyz with vertices x, y and z in the plane. Intuitively, the area of Δxyz defines a d-metric d on X, while the perimeter of Δxyz defines a G-metric G on X. The pair (X, d) then defines a 2-metric space and (X, G), a G-metric space. These two notions were introduced by Gahler [8] and Mustafa and Sims [23] respectively as natural generalizations of a metric.

To begin with, we have

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Definition 1.1. Let X be a nonempty set and $d : X \times X \times X \to [0, \infty)$ such that

(2m1) Given a pair of distinct elements $x, y \in X$, there exists a $z \in X$ such that $d(x, y, z) > 0$, and $d(x, y, z) = 0$ whenever at least two of x, y and z are equal in X,

(2m2) $d(x, y, z) = d(x, z, y) = d(y, x, z) = d(z, x, y) = d(y, z, x) = d(z, y, x)$ for all $x, y, z \in X$,

(2m3) $d(x, y, z) \leq d(x, y, w) + d(x, w, z) + d(w, y, z)$ for all $x, y, z, w \in X$.

Then d is called a 2-*metric* on X and the pair (X, d), a 2-*metric space*, and (2m2) is usually known as the *axiom of symmetry* under a permutation on x, y and z. In view of the fact that the area of a triangle face of a tetrahedron does not exceed the sum of the areas of the remaining faces, (2m3) is referred to as the *tetrahedron inequality*.

Definition 1.2. The topology induced by a 2-metric consists of all open spheres with two centres of the form $B_r(x, y) = \{z \in X : G(x, y, z) < r, r > 0\}$, and is called a 2-*metric topology*.

Remark 1.1. Given any metric space which consists of more than two points, there always exists a 2-metric compatible with the topology of the space. But the converse is not true. That is one can find a 2-metric space which does not have a countable basis associated with one of its arguments [8].

Remark 1.2. In a metric space (X, ρ), the metric ρ is always a continuous function of both x and y. In a 2-metric space (X, d), d may not be continuous in all the three variables x, y and z, though it is continuous function in any one of them. A 2-metric d is said to be *continuous* [24] if it is continuous in any two and hence in all of x, y and z.

Definition 1.3. A sequence $\langle x_n \rangle_{n=1}^{\infty} \subset X$ is said be to 2-*Cauchy* if $d(x_n, x_{n+k}, z) \to 0$ as $n \to \infty$ for each integer $k \geq 1$ and all $z \in X$.

Definition 1.4. A sequence $\langle x_n \rangle_{n=1}^{\infty} \subset X$ is said to be 2-*convergent* with limit $p \in X$ if $d(x_n, p, z) \to 0$ as $n \to \infty$ for all $z \in X$ and is denoted by $x_n \to p$ as $n \to \infty$.

Definition 1.5. A 2-metric space X is said to be *complete* if every 2-Cauchy sequence in it is 2-convergent with limit in it [24].

Remark 1.3. It is well-known that every convergent sequence is Cauchy in a metric space. But a 2-convergent sequence may fail to be 2-Cauchy, as shown in [24]. However, every 2-convergent sequence is 2-Cauchy whenever the 2-metric d is continuous.

In view of Remarks 1.1-1.3, it is appropriate to state that there is no relation between a metric and 2-metric, unlike the claim by Gahler [8] that 2-metric space is a natural generalization of a metric. For more about fixed point theorems in 2-metric spaces, one can refer to [10], [15], [16], [25], [26], [29], [31], [32], [34] and [35].

Definition 1.6. Let X be a nonempty set and $G : X \times X \times X \to \mathbb{R}$ such that

(G1) $G(x, y, z) \geq 0$ for all $x, y, z \in X$ with $G(x, y, z) = 0$ if $x = y = z$,

(G2) $G(x, x, y) > 0$ for all $x, y \in X$ with $x \neq y$,

(G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$,

(G4) $G(x, y, z) = G(x, z, y) = G(y, x, z) = G(z, x, y) = G(y, z, x) = G(z, y, x)$ for all $x, y, z \in X$

(G5) $G(x, y, z) \leq G(x, w, w) + G(w, y, z)$ for all $x, y, z, w \in X$

Then the pair (X, G) is called a G-metric space with G-metric G on X. Axioms (G4) and (G5) are referred to as the *symmetry* and the *rectangle inequality* (of G) respectively.

Given a G-metric space (X, G), define

$$\rho_G(x, y) = G(x, y, y) + G(x, x, y) \text{ for all } x, y, z \in X. \tag{1.1}$$

Then it is seen in [23] that ρ_G is a metric on X, and that the family of all G-balls $\{B_G(x, r) : x \in X, r > 0\}$ is the base topology, called the G-metric topology $\tau(G)$ on X, where $B_G(x, r) = \{y \in X : G(x, y, y) < r\}$. Further, it was shown that the G-metric topology coincides with the metric topology induced by the metric ρ_G, which allows us to readily transform many concepts from metric spaces into the setting of G-metric space.

Definition 1.7. A sequence $\langle x_n \rangle_{n=1}^{\infty}$ in a G-metric space (X, G) is said to be G-convergent with limit $p \in X$ if it converges to p in the G-metric topology, $\tau(G)$.

It is also known from [23] that $G(x, y, z)$ is jointly continuous in all the three variables x, y and z.

Definition 1.8. A sequence $\langle x_n \rangle_{n=1}^{\infty}$ in a G-metric space (X, G) is said to be G-Cauchy if for every $\varepsilon > 0$ there is a positive integer N such that $G(x_n, x_m, x_n) < \varepsilon$ for all $l, m, n \geq N$.

By Corollary 1 of Proposition 9 of [23], it follows that every G-convergent sequence in a G-metric space (X, G) is G-Cauchy.

Definition 1.9. A G-metric space (X, G) is said to be G-complete (or complete) if every G-Cauchy sequence in X converges in it.

Definition 1.10. [The authors, [28]] A fixed point p of f on a G-metric space (X, G) is a G-contractive fixed point of it if the orbital sequence

$$O_f(x) = x, fx, ..., f^n x, ...$$

at each $x \in X$ converges to p.

Example 1.1. Let $X = [0, \infty)$ with $G(x, y, z) = \begin{cases} 0 & \text{if } x = y = z \\ \max\{x, y, z\} & \text{otherwise.} \end{cases}$

Define $fx = \begin{cases} 0 & \text{if } 0 \leq x < \frac{1}{2}, \\ qx & \text{otherwise,} \end{cases}$ for all $x \in X$, where $0 \leq q < 1$. Then, we see that 0 is the unique fixed point of f and for each $x \in X$, the f-orbit $O_f(x) = x, qx, q^2x, ..., q^n, x, ...$ converges to 0. That is, 0 is a G-contractive fixed point of f.

An extensive research has been done in recent years in G-metric spaces. To mention a few, we have Abbas and Rhoades [1], Aydi et al [4], Choudary et al [5], Jleli & Samet [11], Karapinar & Paul [12], Meenakshi et al [17], Mahanta & Mohanta [18], Mustafa [19], Mustafa et al [22], Shatanawi [30] and others cited in references.

Many fixed point theorems established in the literature of 2-metric and G-metric structures followed the usual iterative procedure to obtain a fixed point. For the first time, the authors presented an elegant proofs of some fixed point theorems including the contraction mapping theorem in 2-metric space and G-metric space (See [27], [28]). In this paper, we present elegant proofs of some fixed point theorems in 2-metric space and G-metric space. In this sequel, the unique fixed points obtained are contractive fixed points in these spaces.

3. Main results

We begin with the infimum property of real numbers, as stated below:

Lemma 3.1. *Let $S \subset \mathbb{R}$ be nonempty and bounded below. Then $\alpha = \inf S$ exists.*

An immediate consequence of Lemma 3.1 is:

Lemma 3.2. *Let α be the infimum of $S \subset \mathbb{R}$. Then there exists a sequence $\langle p_n \rangle_{n=1}^{\infty}$ in S with $\lim_{n\to\infty} x_n = \alpha$.*

We have

Theorem 3.1. *Let f be a self-map on a 2-metric space (X, d) such that*

(3.1) $d(fx, fy, z) \leq \alpha d(x, fx, z) + \beta d(y, fy, z) + \gamma d(x, y, z)$ *for all $x, y, z \in X$,*

where α, β and γ are nonnegative real numbers with at least one of them positive and $\alpha + \beta + \gamma < 1$. If X is 2-complete, then f will have a unique fixed point p, which will also be a 2-contractive fixed point of f.

Proof. Note that if $\alpha = \beta = \gamma = 0$, then from (3.1) with $y = fx$, we get $d(fx, f^2x, z) \leq 0$ for all $z \in X$ so that $f^2x = fx$ or that each fx is a fixed point of f. Given $x \in X$ with $fx \neq x$, by (2m1), there exists a point $z \in X$ such that $d(x, fx, z) > 0$. Therefore, we define $S = \{d(x, fx, z) > 0 : x, z \in X\}$. In view of

Lemma 3.1, inf $S = a \geq 0$ exists. If possible suppose that $a > 0$. Then (3.1) with $y = fx$ gives

$$d(fx, f^2x, z) \leq \alpha d(x, fx, z) + \beta d(fx, f^2x, z) + \gamma d(x, fx, z)$$

or $(1-\beta)d(fx, f^2x, z) \leq (\alpha+\gamma)d(x, fx, z)$ so that $d(fx, f^2x, z) \leq cd(x, fx, z)$, where $c = \frac{\alpha+\gamma}{1-\beta}\cdot$ Now $\alpha+\beta+\gamma < 1$ implies that $c < 1$ so that from (3.1), it follows that $d(fx, f^2x, z) < a$. Since $d(fx, f^2x, z) \in S$, this shows that a cannot be a lower bound of S, which is a contradiction. Therefore, inf $S = a = 0$.

Hence, by Lemma 3.2, we can choose a sequence $x_1, x_2, ..., x_n, ...$ of points in X such that

$$\lim_{n,\to\infty} d(x_n, fx_n, z) = 0 \text{ for each } z \in X. \tag{3.2}$$

To prove that $\langle x_n\rangle_{n=1}^{\infty}$ is 2-Cauchy, in this case, we use (2m2), (2m3) and (3.1) repeatedly to get

$$\begin{aligned}
d(x_n, x_{n+k}, z) &\leq d(x_n, x_{n+k}, fx_n) + d(x_n, fx_n, z) + d(fx_n, x_{n+k}, z)\\
&\leq d(x_n, fx_n, x_{n+k}) + d(x_n, fx_n, z) + [d(fx_n, x_{n+k}, fx_{n+k})\\
&\qquad + d(fx_n, fx_{n+k}, z) + d(fx_{n+k}, x_{n+k}, z)]\\
&= d(x_n, fx_n, x_{n+k}) + d(x_n, fx_n, z) + d(fx_n, fx_{n+k}, x_{n+k})\\
&\qquad + d(fx_n, fx_{n+k}, z) + d(x_{n+k}, fx_{n+k}, z)\\
&\leq d(x_n, fx_n, x_{n+k}) + d(x_n, fx_n, z)\\
&\qquad + [\alpha d(x_n, fx_n, x_{n+k}) + \beta d(x_{n+k}, fx_{n+k}, x_{n+k})\\
&\qquad\qquad + \gamma d(x_n, x_{n+k}, x_{n+k})]\\
&\qquad + [\alpha d(x_n, fx_n, z) + \beta d(x_{n+k}, fx_{n+k}, z)\\
&\qquad\qquad + \gamma d(x_n, x_{n+k}, z)] + d(x_{n+k}, fx_{n+k}, z)]
\end{aligned}$$

so that for all $n, k \geq 1$ and all $z \in X$, we have

$$\begin{aligned}
(1-\gamma)d(x_n, x_{n+k}, z) &\leq d(x_n, fx_n, x_{n+k}) + \beta d(x_{n+k}, fx_{n+k}, x_{n+k})\\
&\qquad + \gamma[d(x_n, x_{n+k}, x_{n+k})\}\\
&\qquad + \alpha[d(x_n, fx_n, z) + \beta d(x_{n+k}, fx_{n+k}, z)\\
&\qquad\qquad + \gamma d(x_{n+k}, fx_{n+k}, z).
\end{aligned}$$

Proceeding the limit as $n \to \infty$ in this and then using (3.2), we see that

$$(1-\gamma)\lim_{n,\to\infty} d(x_n, x_{n+k}, z) = 0 \text{ or } \lim_{n,\to\infty} d(x_n, x_{n+k}, z) = 0$$

for all $k \geq 1$ and all $z \in X$, proving that $\langle x_n\rangle_{n=1}^{\infty}$ is a Cauchy sequence in X.

Since X is 2-complete, $\langle x_n\rangle_{n=1}^{\infty}$ is 2-convergent with

$$\lim_{n,\to\infty} d(x_n, p, z) = p \text{ for all } z \in X. \tag{3.3}$$

To prove that $fp = p$, again by (2m2) and (2m3), (3.1), we see that

$$\begin{aligned} d(fp,p,z) &\le d(fp,p,x_n)+d(fp,x_n,z)+d(x_n,p,z) \\ &= d(x_n,p,fp)+d(fp,x_n,z)+d(x_n,p,z) \\ &\le d(x_n,p,fp)+[d(fp,x_n,fx_n)+d(fp,fx_n,z)+d(fx_n,x_n,z)]+d(x_n,p,z) \\ &= d(x_n,p,fp)+d(fx_n,fp,x_n)+d(fx_n,fp,z)+d(x_n,fx_n,z)+d(x_n,p,z) \\ &\le d(x_n,p,fp)+[\alpha d(x_n,fx_n,x_n)+\beta d(p,fp,x_n)+\gamma d(x_n,p,x_n)] \\ &\qquad +[\alpha d(x_n,fx_n,z)+\beta d(p,fp,z)+\gamma d(x_n,p,z)] \\ &\qquad\qquad +d(x_n,fx_n,z)+d(x_n,p,z). \end{aligned}$$

As $n \to \infty$, this together with (3.3), we obtain that $d(fp,p,z) \le 0$ for all $z \in X$ or $fp = p$. That is p is a fixed point of f.

Uniqueness. Let q be another fixed point of f so that$fq = q$. Then, from (3.1), we get

$$d(p,q,z = d(fp,fq,z) \le \beta[d(p,fp,z) + d(q,fq,z)] \text{ or } (1-\alpha-\beta-\gamma)d(p,q,z) \le 0$$

for all $z \in X$ and hence $p = q$. That is, the fixed point of f is unique.

Further, all the f-orbits converge to the fixed point p.

In fact, taking $y = p = fp$ in (3.1) and simplifying, it follows that

$$\begin{aligned} d(f^n x,p,z) &= d(f^n x,f^n p,z) \\ &\le \alpha d(f^{n-1}x,f^n x,z) + \beta d(f^{n-1}p,f^n p,z) + \gamma d(f^{n-1}x,f^{n-1}p,z) \\ &\le \alpha d(f^{n-1}x,f^n x,z) + \gamma d(f^{n-1}x,f^{n-1}p,z) \\ &= \alpha d(f^{n-1}x,f^n x,z) + \gamma d(f^{n-1}x,p,z) \\ &= \alpha d(f^{n-1}x,f^n x,z)+\gamma[d(f^{n-1}x,p,f^n x)+d(f^{n-1}x,f^n x,z)+d(f^n x,p,z)] \end{aligned}$$

or

$$d(f^n x,p,z) \le \left(\frac{\alpha+\gamma}{1-\gamma}\right) d(f^{n-1}x,f^n x,z) + \left(\frac{\gamma}{1-\gamma}\right) d(f^{n-1}x,f^n x,p). \tag{3.4}$$

Again by (3.1), we have

$$d(f^{n-1}x,f^n x,z) \le \alpha d(f^{n-2}x,f^{n-1}x,z) + \beta d(f^{n-1}x,f^n x,z)\gamma d(f^{n-2}x,f^{n-1}x,z)$$

or

$$d(f^{n-1}x,f^n x,z) \le \left(\frac{\alpha+\gamma}{1-\beta}\right) d(f^{n-2}x,f^{n-1}x,z),$$

which, by induction gives

$$d(f^{n-1}x,f^n x,z) \le \left(\frac{\alpha+\gamma}{1-\beta}\right)^{n-1} d(x,fx,z) \text{ for all } z \in X. \tag{3.5}$$

In particular,

$$d(f^{n-1}x, f^n x, p) \leq \left(\frac{\alpha+\gamma}{1-\beta}\right)^{n-1} d(x, fx, p). \tag{3.6}$$

Inserting (3.5) and (3.6) in (3.4), it follows that

$$d(f^n x, p, z) \leq \left(\frac{\alpha+\gamma}{1-\gamma}\right)\left(\frac{\alpha+\gamma}{1-\beta}\right)^{n-1} d(x, fx, z) + \left(\frac{\gamma}{1-\gamma}\right)\left(\frac{\alpha+\gamma}{1-\beta}\right)^{n-1} d(x, fx, p) \text{ for all } z \in X.$$

Then, applying the limit as $n \to \infty$ and using the choice of $\alpha+\beta+\gamma < 1$ in this, finally we get

$$d(f^n x, p, z) \to 0 \text{ for all } x, z \in X,$$

that is p is a 2-contractive fixed point of f. ∎

Writing $\gamma = 0$ and $\alpha = \beta = k$ in Theorem (3.1), we get Kannan's type result as follows:

Corollary 3.1. *Let f be a self-map on a 2-metric space (X, d) such that*

$$d(fx, fy, z) \leq k[d(x, fx, z) + d(y, fy, z)] \text{ for all } x, y, z \in X \tag{3.7}$$

where $0 \leq k < \frac{1}{2}$. If X is 2-complete, then f will have a unique fixed point.

From this definition of a G-metric space, it immediately follows that

$$\text{If } x, y \in X \text{ are such that } G(x, y, y) = 0, \text{ then } x = y. \tag{3.8}$$

and that

$$G(x, y, y) \leq 2G(x, x, y) \text{ for all } x, y \in X. \tag{3.9}$$

Lemma 3.3. [Mustafa and Sims [23]] *The following statements are equivalent in a G-metric space (X, G):*

(a) $\langle x_n \rangle_{n=1}^{\infty} \subset X$ *is G-convergent with limit $p \in X$,*

(b) $\lim_{n\to\infty} G(x_n, x_n, p) = 0$,

(c) $\lim_{n\to\infty} G(x_n, p, p) = 0$.

The authors [28] presented an analytical proof of the following result of Mustafa et al [21]:

Theorem 3.2. *Suppose that (X, G) is a complete G-metric space and f, a self-map on X satisfying*

$$G(fx, fy, fz) \leq aG(x, fx, fx) + bG(y, fy, fy) + cG(z, fz, fz) + eG(x, y, z) \text{ for all } x, y, z \in X, \tag{3.10}$$

where a, b, c and e are nonnegative real numbers with $a + b + c + e < 1$. Then, f will have a unique fixed point p and f is continuous at p.

Then, writing $a = b = c = q$ and $e = 0$ in Theorem 3.2, we get

Corollary 3.2. *Suppose that (X, G) is a complete G-metric space and f, a self-map on X satisfying*

$$G(fx, fy, fy) \leq q[G(x, fx, fx) + G(y, fy, fy) + G(z, fz, fz)] \quad \text{for all } x, y, z \in X, \tag{3.11}$$

where $0 \leq q < \frac{1}{3}\cdot$ Then f will have a unique fixed point p and f is continuous at p.

Recently, Vats et al [33] established the following theorem:

Theorem 3.3. *Suppose that (X, G) is a complete G-metric space and f, a self-map on X satisfying*

$$\begin{aligned} G(fx, fy, fz) \leq k \max \{&G(x, fx, fx), G(x, fy, fy), \\ &G(x, fz, fz), G(y, fy, fy), G(y, fx, fx), \\ &G(y, fz, fz), G(z, fz, fz), G(z, fx, fx), \\ &G(z, fy, fy)\} \text{ for all } x, y, z \in X, \end{aligned} \tag{3.12}$$

where $0 \leq k < \frac{1}{2}\cdot$ Then f will have a unique fixed point p and f is continuous at p.

It may be noted that condition (3.12) is partially weaker than (3.11). In fact, since the arithmetic average of three real numbers cannot exceed their maximum, (3.11) can be written as

$$\begin{aligned} G(fx, fy, fy) &\leq 3q \left[\frac{G(x, fx, fx) + G(y, fy, fy) + G(z, fz, fz)}{3}\right] \\ &\leq k' \max \{G(x, fx, fx), G(y, fy, fy), G(z, fz, fz)\} \\ &\leq k' \max \{G(x, fx, fx), G(x, fy, fy), \\ &\quad G(x, fz, fz), G(y, fy, fy), G(y, fx, fx), \\ &\quad G(y, fz, fz), G(z, fz, fz), G(z, fx, fx), G(z, fy, fy)\}, \end{aligned} \tag{3.13}$$

which is (3.12) with $0 \leq k' = 3q < 1$. In other words, under the restricted range $\left[0, \frac{1}{2}\right)$ for k', (3.11) implies (3.12). Thus, Theorem 3.3 is a partial generalization of Theorem 3.2.

Remark 3.4. If $k = 0$, then with $z = y = fx$, (3.12) or (3.11) gives $G(fx, f^2x, f^2x) = 0$ or $f^2x = fx$ for each $x \in X$. That is, every fx is a fixed point of f or the fixed point is not unique. Therefore, the choice of $k = 0$ is excluded n this paper.

We shall give an analytical proof of Theorem 3.3, which does not need iterations to obtain a fixed point p of f. Then, we shall establish that p is, in fact, a G-contractive fixed point of f.

Proof. Let $S = \{G(x, fx, fx) : x \in X\}$. Then, S being a nonempty set of nonnegative numbers, which is bounded below, has an infimum, say $a \geq 0$, by Lemma 3.1.

If possible, let $a > 0$. From (3.12) with $y = fx$ and $z = fx$, we have

$$\begin{aligned} G(fx, f^2x, f^2x) \leq k \max \{&G(x, fx, fx), G(x, f^2x, f^2x), G(x, f^2x, f^2x), \\ &G(fx, f^2x, f^2x), G(fx, fx, fx), G(fx, f^2x, f^2x), \\ &G(fx, f^2x, f^2x), G(fx, fx, fx), G(x, f^2x, f^2x)\}, \end{aligned}$$

which implies

$$G(fx, f^2x, f^2x) \leq k \max\{G(x, fx, fx), G(x, f^2x, f^2x), G(fx, f^2x, f^2x)\}.$$

But by the rectangle inequality (G5),

$$G(x, f^2x, f^2x) \leq G(fx, f^2x, f^2x) + G(x, fx, fx).$$

Using this in the previous inequality, we get

$$G(fx, f^2x, f^2x) \leq k[G(fx, f^2x, f^2x) + G(x, fx, fx)]$$

or

$$G(fx, f^2x, f^2x) \leq \left(\frac{k}{1-k}\right) G(x, fx, fx) \text{ for all } x \in X. \tag{3.14}$$

Since $0 \leq k < \frac{1}{2}$, we see that $\frac{k}{1-k} < 1$. So, from (3.14), it would follow that $G(fx, f^2x, f^2x) < a$, where $G(fx, f^2x, f^2x) \in S$. This implies that a cannot be a lower bound of S, contradicting the choice of a.

Therefore, $\inf S = a = 0$.

By Lemma 3.2, we can choose the points $x_1, x_2, ..., x_n, ...$ in X such that

$$G(x_n, fx_n, fx_n) \in S \text{ for } n = 1, 2, 3, ...$$

and

$$\lim_{n\to\infty} G(x_n, fx_n, fx_n) = 0. \tag{3.15}$$

Now, repeatedly using the rectangle inequality (G5) and (3.12), we see that

$$\begin{aligned} G(x_n, x_m, x_m) &\leq G(x_n, fx_n, fx_n) + G(fx_n, x_m, x_m) \\ &\leq G(x_n, fx_n, fx_n) + [G(fx_n, fx_m, fx_m) + G(fx_m, x_m, x_m)] \\ &\leq G(x_n, fx_n, fx_n) + G(fx_n, fx_m, fx_m) + 2G(x_m, fx_m, fx_m) \end{aligned}$$

Now

$$\begin{aligned}G(fx_n, fx_m, fx_m) &\le k\max\{G(x_n, fx_n, fx_n), G(x_n, fx_m, fx_m), G(x_n, fx_m, fx_m),\\ &\quad G(x_m, fx_m, fx_m), G(x_m, fx_n, fx_n), G(x_m, fx_m, fx_m),\\ &\quad G(x_m, fx_m, fx_m), G(x_m, fx_n, fx_n), G(x_m, fx_m, fx_m)\}\\ &= k\max\{G(x_n, fx_n, fx_n), G(x_n, fx_m, fx_m),\\ &\qquad G(x_m, fx_m, fx_m), G(x_m, fx_n, fx_n)\}\\ &\le k\max\{G(x_n, fx_n, fx_n) + G(x_m, x_n, x_n),\\ &\qquad G(x_m, fx_m, fx_m) + G(x_n, x_m, x_m)\}\\ &\le k\max\{G(x_n, fx_n, fx_n) + 2G(x_n, x_m, x_m)],\\ &\qquad G(x_m, fx_m, fx_m) + G(x_n, x_m, x_m)\}\\ &= k[G(x_n, fx_n, fx_n) + G(x_m, fx_m, fx_m)] + 2G(x_n, x_m, x_m)]\end{aligned}$$

or

$$G(x_n, x_m, x_m) \le \left(\frac{1+k}{1-2k}\right) G(x_n, fx_n, fx_n) + \left(\frac{2+k}{1-2k}\right) G(x_m, fx_m, fx_m).$$

Employing the limit as $m, n \to \infty$ in this and using (3.15), we get

$$\lim_{n,m\to\infty} G(x_n, x_m, x_m) = 0,$$

proving that $\langle x_n\rangle_{n=1}^{\infty}$ is G-Cauchy.

Since X is G-complete, we can find a point $p \in X$ such that

$$\lim_{n\to\infty} x_n = p. \tag{3.16}$$

Again using (G5) and (3.9) in (3.12), we have

$$\begin{aligned}G(p, fp, fp) &\le G(p, fx_n, fx_n) + G(fx_n, fp, fp)\\ &\le [G(p, x_n, x_n) + G(x_n, fx_n, fx_n)] + G(fx_n, fp, fp).\end{aligned} \tag{3.17}$$

Also

$$\begin{aligned}G(fx_n, fp, fp) &\le k\max\{G(x_n, fx_n, fx_n), G(x_n, fp, fp), G(x_n, fp, fp),\\ &\qquad G(p, fp, fp), G(p, fx_n, fx_n), G(p, fp, fp),\\ &\qquad G(p, fp, fp), G(p, fx_n, fx_n), G(p, fp, fp)\}\\ &= k\max\{G(x_n, fx_n, fx_n), G(x_n, fp, fp),\\ &\qquad G(p, fp, fp), G(p, fx_n, fx_n)\}\\ &\le k\max\{G(x_n, fx_n, fx_n), G(x_n, fp, fp), G(p, fp, fp),\\ &\qquad G(x_n, fx_n, fx_n) + G(p, x_n, x_n)\}.\end{aligned} \tag{3.18}$$

From (3.17) and (3.18), we get

$$\begin{aligned}G(p, fp, fp) &\le G(p, x_n, x_n) + G(x_n, fx_n, fx_n)\\ &\quad + k\max\{G(x_n, fx_n, fx_n), G(x_n, fp, fp), G(p, fp, fp),\\ &\qquad G(x_n, fx_n, fx_n) + G(p, x_n, x_n)\}.\end{aligned}$$

Proceeding the limit as $n \to \infty$ in this, and then using (3.15) and (3.16),

$$G(p, fp, fp) \leq [0 + 0 + k \max\{0, G(p, fp, fp), G(p, fp, fp), [0 + 0]\}]$$

which implies that $(1 - \beta)d(p, fp, fp) \leq 0$ or $fp = p$.

That is, p is a fixed point of f.

Uniqueness. Suppose q is another fixed point of f so that $fq = q$. Then from (3.12) with $x = p$ and $y = z = q$, we have

$$\begin{aligned} G(p, q, q) &= G(fp, fq, fq) \\ &\leq k \max\{G(p, fp, fp), G(p, fq, fq), G(p, fq, fq), \\ &\qquad G(q, fq, fq), G(q, fp, fp), G(q, fq, fq), \\ &\qquad G(q, fq, fq), G(q, fp, fp), G(q, fq, fq)\} \\ &= k \max\{0, G(p, q, q), G(p, q, q), 0, G(q, p, p), 0, 0, G(q, p, p), 0\} \\ &= k \max\{G(p, q, q), G(q, p, p)\} \\ &\leq 2kG(p, q, q), \end{aligned}$$

which implies that $(1 - 2k)G(p, q, q) \leq 0$ or $p = q$. That is p is unique fixed point of f. The G-continuity of f at p is obtained as in [33]. ∎

Remark 3.5. An exciting feature of Theorem 3.3 is that p is a G-contractive fixed point of f, whenever $0 \leq k < \frac{1}{3}$. In fact, we write $y = z = p$ in (3.12) and then use (G5) to obtain

$$\begin{aligned} G(f^n x, fp, fp) =& \leq k \max\{G(f^{n-1}x, fx, fx), G(f^{n-1}x, fp, fp), G(f^{n-1}x, fp, fp), \\ &\qquad G(p, fp, fp), d(p, f^{n-1}x, f^{n-1}x), G(p, fp, fp), \\ &\qquad G(p, fp, fp), G(p, f^{n-1}x, f^{n-1}x), G(p, fp, fp)\} \\ &= k \max\{G(f^{n-1}x, f^n x, f^n x), G(f^{n-1}x, p, p), 0, \\ &\qquad G(p, f^n x, f^n x), 0, G(p, f^n x, f^n x)\} \\ &= k \max\{G(f^{n-1}x, f^n x, f^n x), G(f^{n-1}x, p, p), G(p, f^n x, f^n x)\} \\ &= k\{G(p, f^n x, f^n x) + G(f^{n-1}x, p, p)\} \\ &\leq k\{2G(f^n x, p, p) + G(f^{n-1}x, p, p)\}. \end{aligned}$$

or

$$G(f^n x, p, p) \leq c \cdot G(f^{n-1}x, p, p),$$

where $\frac{k}{1-2k} = c$. By induction, we have

$$G(f^n x, p, p) \leq c^n G(fx, p, p),$$

which as $n \to \infty$ implies that $G(f^n x, p, p) \to 0$ for each $x \in X$, since $c < 1$. Thus p is a G-contractive fixed point of f.

Taking $z = y$ in Theorem 3.3, we have

Corollary 3.3. *Suppose that (X, G) is a complete G-metric space and f, a self-map on X satisfying*

$$G(fx, fy, fy) \leq k \max\{G(x, fx, fx), G(x, fy, fy), G(y, fy, fy), G(y, fx, fx)\} \text{ for all } x, y, z \in X, \tag{3.19}$$

where $0 \leq k < \frac{1}{2}$. Then f will have a unique fixed point p and f is continuous at p.

Proof. We set

$$\rho_G(x, y) = \max\{(G(x, y, y), G(x, x, y)\} \text{ for all } x, y \in X, \tag{3.20}$$

where the exchange of x and y yields the symmetry of ρ_G. It can be seen that ρ_G defines a metric on X.

Now, (3.19) can be written as

$$\begin{aligned} &G(fx, fy, fy) \\ &\leq 2k \max\Big\{\frac{G(x, fx, fx) + G(x, x, fx)}{2}, \frac{G(x, fy, fy) + G(x, x, fy)}{2}, \\ &\frac{G(y, fx, fx) + G(y, y, fx)}{2}, \frac{G(y, fy, fy) + G(y, y, fy)}{2}\Big\}. \end{aligned} \tag{3.21}$$

Interchanging the roles of x and y in this, we get

$$\begin{aligned} &G(fy, fx, fx) \\ &\leq 2k \max\Big\{\frac{G(y, fy, fy) + G(y, y, fy)}{2}, \frac{G(y, fx, fx) + G(y, y, fx)}{2}, \\ &\frac{G(x, fy, fy) + G(x, x, fy)}{2}, \frac{G(x, fx, fx) + G(x, x, fx)}{2}\Big\}. \end{aligned} \tag{3.22}$$

Taking the maximum of (3.21) and (3.22), and then using (3.20), it follows that

$$\begin{aligned} \rho_G(x, y) &\leq 2k \max\Big\{\frac{G(x, fx, fx) + G(x, x, fx)}{2}, \frac{G(x, fy, fy) + G(x, x, fy)}{2}, \\ &\qquad \frac{G(y, fx, fx) + G(y, y, fx)}{2}, \frac{G(y, fy, fy) + G(y, y, fy)}{2}\Big\} \\ &\leq 2k \max\{\rho_G(x, fx), \rho_G(x, fy), \rho_G(y, fx), \rho_G(y, fy)\} \text{ for all } x, y \in X, \end{aligned}$$

which is a special case of Ciric's quasi-contraction, and a unique fixed point follows from [7] for complete metric space. In general, if any two of the three variables in the the contraction type condition (3.12) are the same, a unique fixed point can be obtained from Ciric's result in a complete metric space [7]. ∎

Writing $z = fy$ in Theorem 3.3, we have

Corollary 3.4. *Suppose that (X, G) is a complete G-metric space and f, a self-map on X satisfying*

$$\begin{aligned} G(fx, fy, f^2y) \le k \max\{&G(x, fx, fx), G(x, fy, fy), \\ &G(x, f^2y, f^2y), G(y, fy, fy), G(y, fx, fx), \\ &G(y, f^2y, f^2y), G(fy, f^2y, f^2y), G(fy, fx, fx)\}, \\ &\text{for all } x, y, z \in X, \end{aligned} \tag{3.23}$$

where $0 < k < \frac{1}{2}$. Then f will have a unique fixed point p and f is continuous at p.

Remark 3.6. Write

$$\rho_G(x, y) = \max\{(G(x, fx, f^2y), G(y, fy, f^2x)\} \text{ for all } x, y \in X. \tag{3.24}$$

It is interesting to remark that $x = y$ does not imply that $\rho_G(x, y) = 0$. In fact, from (3.24) with $y = x$, we get $\rho_G(x, x) = G(x, fx, f^2x)$, which will be positive for each $x \in X$ with $fx \neq x$. That is, ρ_G is not a metric on X. In other words, Corollary 3.4 cannot be characterized in terms of metric space to determine a fixed point of f.

With an altered form of the condition (3.12), we now prove

Theorem 3.4. *Suppose that (X, G) is a complete G-metric space and f, a self-map on X satisfying*

$$\begin{aligned} G(fx, fy, fz) \le k \max\{&G(x, x, fx), G(x, x, fy), G(x, x, fz), \\ &G(y, y, fy), G(y, y, fx), G(y, y, fz), G(z, z, fz), \\ &G(z, z, fx), G(z, z, fy)\} \text{ for all } x, y, z \in X, \end{aligned} \tag{3.25}$$

where $0 < k < \frac{1}{2}$. Then f will have a unique fixed point p and f is continuous at p. Further if $0 < k < 1/3$, then p will be its G-contractive fixed point.

Proof. Let $a \ge 0$ be the infimum of the set

$$S = \{G(x, x, fx) : x \in X\},$$

in view of Lemma 3.1.

If possible, let $a > 0$. From (3.25) with $y = x$ and $z = fx$, we have

$$\begin{aligned} G(fx, fx, f^2x) \le k \max\{&G(x, x, fx), G(x, x, fx), G(x, x, f^2x), \\ &G(x, x, fx), G(x, x, fx), G(x, x, f^2x), G(fx, fx, f^2x), \\ &G(fx, fx, fx), G(x, fx, fx)\}, \end{aligned}$$

which implies

$$G(fx, fx, f^2x) \le k \max\{G(x, x, fx), G(x, x, f^2x), G(fx, fx, f^2x)\}.$$

But, by rectangle inequality,

$$G(x, x, f^2x) \leq G(x, x, fx) + G(fx, fx, f^2x).$$

Using this in the previous inequality,

$$G(fx, fx, f^2x) \leq k[G(x, x, fx) + G(fx, fx, f^2x)] \text{ for all } x \in X.$$

or

$$d(fx, fx, f^2x) \leq \left(\frac{k}{1-k}\right) G(x, x, fx) \text{ for all } x \in X. \tag{3.26}$$

Since $\frac{k}{1-k} < 1$, from (3.26), it would follow that $G(fx, fx, f^2x) < a$ for some $x \in X$. This contradicts the choice of a. Therefore, $a = \inf S = 0$.

By Lemma 3.2, we can choose the points $x_1, x_2, ..., x_n, ...$ in X such that

$$G(x_n, x_n, fx_n) \in S \text{ for } n = 1, 2, 3, ...$$

and

$$\lim_{n\to\infty} G(x_n, x_n, fx_n) = 0. \tag{3.27}$$

Now repeatedly using the rectangle inequality (G5) and (3.25), we see that

$$\begin{aligned} G(x_m, x_m, x_n) &\leq G(x_m, fx_m, fx_m) + G(fx_m, fx_m, x_n) \\ &\leq G(x_m, x_m, fx_m) + [G(fx_m, fx_m, fx_n) \\ &\quad + G(fx_n, x_n, x_n)] \\ &\leq G(x_m, x_m, fx_m) + G(fx_m, fx_m, fx_n) \\ &\quad + 2G(x_n, fx_n, fx_n). \end{aligned} \tag{3.28}$$

Now,

$$\begin{aligned} G(fx_m, fx_m, fx_n) \leq k \max \{&G(x_m, x_m, fx_m), G(x_m, x_m, fx_m), \\ &G(x_m, x_m, fx_n), G(x_m, x_m, fx_m), G(x_m, x_m, fx_m), \\ &G(x_m, x_m, fx_n), G(x_n, x_n, fx_n), \\ &G(x_n, x_n, fx_m), G(x_n, x_n, fx_m)\} \end{aligned}$$

which can be written as

$$\begin{aligned} G(fx_m, fx_m, fx_n) &\leq k \max \{G(x_m, x_m, fx_m), G(x_m, x_m, fx_n), \\ &\quad G(x_n, x_n, fx_n), G(x_n, x_n, fx_m)\} \\ &\leq k \max\{[G(x_m, x_m, x_n) + G(x_n, x_n, fx_n)], \\ &\quad [G(x_n, x_n, x_m) + G(x_m, x_m, fx_m)]\} \\ &\leq k \max\{[G(x_m, x_m, x_n) + G(x_n, x_n, fx_n)], \\ &\quad [2G(x_m, x_m, x_n) + G(x_m, x_m, fx_m)]\} \\ &= k[2G(x_m, x_m, x_n) + G(x_n, x_n, fx_n)] \\ &\quad + G(x_m, x_m, fx_m)]. \end{aligned} \tag{3.29}$$

Substituting (3.29) in (3.28), we get

$$G(x_m, x_m, x_n) \leq G(x_m, x_m, fx_m) + 2G(x_n, x_n, fx_n) \\ +k[2G(x_m, x_m, x_n) + G(x_m, x_m, fx_m) + G(x_n, x_n, fx_n)]$$

or

$$G(x_m, x_m, x_n) \leq \left(\frac{1+k}{1-2k}\right) G(x_m, x_m, fx_m) + \left(\frac{2+k}{1-2k}\right) G(x_n, x_n, fx_n).$$

Employing the limit as $m, n \to \infty$ in this and using (3.27), we get

$$\lim_{n,m\to\infty} G(x_m, x_m, x_n) = 0,$$

proving that $\langle x_n \rangle_{n=1}^{\infty}$ is G-Cauchy.

Since X is G-complete, we can find a point $p \in X$ such that

$$\lim_{n\to\infty} x_n = p. \tag{3.30}$$

Again, by repeated application of rectangle inequality (G5), from (G4) and (3.25), we have

$$\begin{aligned} G(p, p, fp) &\leq G(p, p, fx_n) + G(fx_n, fx_n, fp) \\ &\leq [G(p, p, x_n) + G(x_n, x_n, fx_n)] + G(fx_n, fx_n, fp). \end{aligned} \tag{3.31}$$

Now,

$$\begin{aligned} G(fx_n, fx_n, fp) &\leq k \max \{G(x_n, , x_n, fx_n), G(x_n, , x_n, fx_n), \\ &\qquad G(x_n, , x_n, fp), G(x_n, , x_n, fx_n), \\ &\qquad G(x_n, , x_n, fx_n), G(x_n, , x_n, fp), \\ &\qquad G(p, p, fp), G(p, p, fx_n), G(p, p, fx_n)\} \\ &= k \max\{G(x_n, x_n, fx_n), G(x_n, x_n, fp), \\ &\qquad G(p, p, fp), G(p, p, fx_n)\} \\ &\leq k \max\{G(x_n, x_n, fx_n), G(x_n, x_n, fp), G(p, p, fp), \\ &\qquad [G(p, p, x_n) + G(x_n, x_n, fx_n)]\}. \end{aligned} \tag{3.32}$$

Substituting (3.32) in (3.31), we get

$$G(p, p, fp) \leq G(p, p, x_n) + G(x_n, x_n, fx_n) + k \max \{G(x_n, fx_n, fx_n), \\ G(x_n, x_n, fp), G(p, p, fp), [G(p, p, x_n) + G(x_n, x_n, fx_n)]\}.$$

Proceeding the limit as $n \to \infty$ in this, we get

$$G(p, p, fp) \leq [0 + 0 + k \max\{0, G(p, p, fp), G(p, p, fp), [0 + 0]\}],$$

which implies that

$$(1 - k)G(p, p, fp) \leq 0 \text{ or } fp = p.$$

That is, p is a fixed point of f. It can be easily established as in the proof of Theorem 3.3 and Remark 3.5 that the fixed point is unique, which will also be a G-contractive fixed point of f. ∎

Restricting the the terms in 3.12, we have

Corollary 3.5. *Let (X, G) is a complete G-metric space satisfying,*

$$G(fx, fy, fz) \leq k[G(x, fy, fy) + G(y, fz, fz) + G(z, fx, fx)], \quad \text{for all } x, y, z \in X. \tag{3.33}$$

Then f has a unique fixed point.

While, restricting the terms in (3.25), we have

Corollary 3.6. *Let (X, G) is a complete G-metric space satisfying,*

$$G(fx, fy, fz) \leq k[G(x, x, fy) + G(y, y, fz) + G(z, z, fx)], \quad \text{for all } x, y, z \in X. \tag{3.34}$$

Then f has a unique fixed point.

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ANALYTIC SOLUTION FOR RLC CIRCUIT OF NON-INTEGER ORDER

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Abstract. In this paper, we discuss second order fractional differential equation model for RLC circuit with order $1 < \alpha \leq 2$ and $0 < \beta \leq 1$. Further, we use Laplace transform method including convolution theorem to obtain the solution.
Keywords: Mittag-Leffler function, Laplace transform, mathematical modeling, RLC circuit.
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1. Introduction

Fractional calculus goes back to the beginning of the theory of differential calculus. Fractional calculus has become new mathematical method for solution of diverse problems in mathematics, sciences and engineering as it is generalization of classical calculus to derivatives and integrals of fractional non-integer order. The applications of fractional calculus has just emerged in last two decades, due to progress in the area of chaos that revealed subtle relationship with fractional

calculus concepts. Apparently, mathematical models involving fractional order derivatives has became a powerful and widely used tool for better modeling and control of processes in many areas of science and engineering as it reflects study of intrinsic dissipative processes that are complicated in nature, particularly for systems where memory or hereditary properties plays a significant role. Large amount of research oriented studies were developed concerning the applications of fractional calculus in the field of physics. Fractional model for electrical circuits such as RL, RC, RLC have already been proposed. Recently, Gómez *et al.* [3] considered Caputo derivatives and Numerical Laplace transform to get the solution of RL and RC circuits. Further, they also analysed RLC circuit in time domain and found solution in terms of Mittag-Leffler function. Shah *et. al* [8] has obtained analytic solution of RL electrical circuit described by a fractional differential equation of the order $0 < \alpha \leq 1$ and used the Laplace transform of the fractional derivative in the Caputo sense. In order to stimulate more interest in subject and to show its utility, this paper is devoted to new and recent application of fractional calculus i.e. RLC electrical circuit considering the second order fractional differential equation with parameters $\alpha \in (1, 2], \beta \in (0, 1]$ and obtained analytic solution in terms of Mittag-Leffler function using integral transform method.

2. Used integral transforms and special functions

Definition. In 1903, the Swedish mathematician Gósta Mittag-Leffler introduced the function

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha > 0. \tag{1}$$

The Mittag-Leffler function in two parameters $E_{\alpha,\beta}(z)$ was introduced by Agarwal [4] and reported by Goenflo *et al.* [6] in their book,

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0, \quad \beta > 0. \tag{2}$$

Its k-th derivative is given by,

$$E^{(k)}_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} \frac{(j+k)! z^j}{j!\Gamma(\alpha j + \alpha k + \beta)}, \ k = 0, 1, 2, 3, \ldots \tag{3}$$

The Mittag-Leffler type function [7] is defined by,

$$\mathcal{E}_k(z, y; \alpha, \beta) = z^{k\alpha+\beta-1} E^{(k)}_{\alpha,\beta}(yz^\alpha) \quad k = 0, 1, 2, 3, \ldots \tag{4}$$

Definition [1]. Let $f(t)$ be an arbitrary function defined on the interval $0 \leq t < \infty$. Then,

$$F(s) = \bar{f}(s) = \int_0^\infty e^{-st} f(t)\, dt \tag{5}$$

is the Laplace transform of $f(t)$, provided that the integral exists.

For existence of integral (5), the function $f(t)$ must be piecewise continuous and of exponential order α, original function $f(t)$ can be restored from the Laplace transform $F(s)$ with the help of inverse Laplace transform,

$$f(t) = \mathcal{L}^{-1}\{F(s);t\} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}F(s)ds, \quad c = Re(s). \tag{6}$$

Definition. The *Caputo's* definition of fractional derivative is given by

$${}_0^C D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^n(\tau)}{(t-\tau)^{\alpha-n+1}}d\tau, \tag{7}$$

where $\alpha \in R$ is order of fractional derivative, $n-1 < \alpha \leq n$ and $n \in N = \{1,2,3,\dots\}$, $f^n(\tau) = \frac{d^n}{dt^n}f(\tau)$ and $\Gamma(.)$ is Euler Gamma function.

Laplace transform of fractional derivative [1], [7].

The Laplace transform of Caputo fractional derivative has the form

$$\mathcal{L}\left\{{}_0^C D_t^p f(t)\right\} = s^p F(s) - \sum_{k=0}^{n-1} s^{p-k-1} f^{(k)}(0), \qquad (n-1 < p \leq n). \tag{8}$$

where $f^{(k)}$ is k-th order derivative.

Convolution theorem [1]. *If* $\mathcal{L}\{f(t)\} = \bar{f}(s)$ *and* $\mathcal{L}\{g(t)\} = \overline{g}(s)$, *then*

$$\mathcal{L}\{f(t) * g(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} = \bar{f}(s)\,\bar{g}(s), \tag{9}$$

or, equivalently,

$$\mathcal{L}^{-1}\left\{\bar{f}(s)\,\bar{g}(s)\right\} = f(t) * g(t), \tag{10}$$

where $f(t) * g(t)$ *is called the convolution of* $f(t)$ *and* $g(t)$, *and is defined by the integral*

$$f(t) * g(t) = \int_0^t f(t-\tau)g(\tau)d\tau. \tag{11}$$

The integral in (11) is often referred to as the convolution integral and is denoted simply by $(f * g)(t)$.

Laplace transform of Mittag-Leffler function [7].

A two parameter function of Mittag-Leffler type is defined by series expansion,

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad (\alpha > 0,\ \beta > 0). \tag{12}$$

Laplace transform of Mittag-Leffler function is given by [7],

$$\mathcal{L}\left[z^{k\alpha+\beta-1}E^{(k)}_{\alpha,\beta}(\pm az^{\alpha})\right] = \frac{k!\ s^{\alpha-\beta}}{(s^{\alpha}\mp a)^{k+1}} \qquad k = 0, 1, 2, 3, \ldots. \tag{13}$$

3. Mathematical model for RLC electrical circuit

The integrated process of translating real world problem into mathematical problem is termed as mathematical modeling. It includes mathematical concepts such as function, variables, constants, inequality, etc. taken from different branches of mathematics. Here, we formulate the model for electrical circuit which are widely used in branch of physics and engineering sciences.

The oscillating electrical circuit in which resistor (R), inductor (L) and capacitor (C) are connected with voltage (E) is known as RLC circuit. There are number of ways of connecting these elements across voltage supply, we consider electrical circuit where these elements are connected in series with voltage as shown in figure (1). Here, the capacitance (C), the inductance (L) and the resistor (R) are considered as positive constants.

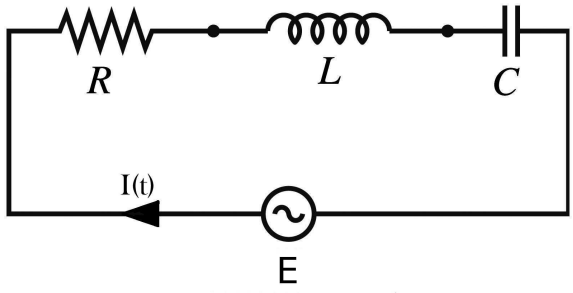

Figure 1: RLC Circuit

Nomenclature

$E(t)$	–	The voltage of the power source (measured in volts = V)
$I(t)$	–	The current in the circuit at time t (measured in amperes = A)
R	–	The resistance of the resistor (measured in ohms = V/A)
L	–	The inductance of the inductor (measured in henry = H)
C	–	The capacitance of the capacitor (measured in farads = F = C/V)

4. Formulation of fractional differential equation model and its solution

Considering the RLC circuit, the constitutive equations associated with three elements, i.e., resistor, inductor and capacitor are:

- The voltage drop across inductor, i.e., $U_L(t) = \frac{d}{dt}I(t)$.
- The voltage drop across resistor, i.e., $U_R(t) = RI(t)$.
- The voltage drop across capacitor, i.e., $U_C(t) = \frac{1}{C}\int_0^t I(\xi)d\xi$.

As per Kirchhoff's voltage law, around any loop in a circuit, the voltage rises must equal to the voltage drops. We have,

$$U_L(t) + U_R(t) + U_C(t) = E(t), \tag{14}$$

or

$$LI' + RI + \frac{1}{C}\int I dt = E\left(t\right). \tag{15}$$

to get rid of integral, we differentiate (15) with respect to t, which yields non-homogeneous second order ordinary differential equation,

$$LI'' + RI' + \frac{1}{C}I = E(t), \tag{16}$$

In this paper, we develop model for RLC circuit in the form of fractional differential equation as,

$$LD^{\alpha}I\left(t\right) + RD^{\beta}I\left(t\right) + \frac{1}{C}I\left(t\right) = E\left(t\right), \tag{17}$$

where $D^{\alpha}I(t) = \frac{d^{\alpha}I}{dt^{\alpha}}$ and $D^{\beta}I(t) = \frac{d^{\beta}I}{dt^{\beta}}$, $1 < \alpha \leq 2$, $0 < \beta \leq 1$, when $\lim\limits_{\alpha\to 2}\frac{d^{\alpha}I}{dt^{\alpha}} = \frac{d^2I}{dt^2}$ and $\lim\limits_{\beta\to 1}\frac{d^{\beta}I}{dt^{\beta}} = \frac{dI}{dt}$.

Taking the Laplace transform on both sides of (17) by considering the initial condition $I(0) = A$ and $I'(0) = B$, and further using (8), we get,

$$\mathcal{L}\left\{L\,D^{\alpha}I(t)\right\} + \mathcal{L}\left\{R\,D^{\beta}I(t)\right\} + \mathcal{L}\left\{\frac{1}{C}I(t)\right\} = \mathcal{L}\left\{E\left(t\right)\right\}.$$

Using (8), we have,

$$\begin{aligned}
&L\{s^{\alpha}I(s) - s^{\alpha-1}I(0) - s^{\alpha-2}I'(0)\} + R\{s^{\beta}I(s) - s^{\beta-1}I(0)\} + \frac{1}{C}\{I(s)\} = E(s)\\
&\Rightarrow L\left\{s^{\alpha}I\left(s\right) - s^{\alpha-1}A - s^{\alpha-2}B\right\} + R\left\{s^{\beta}I\left(s\right) - s^{\beta-1}B\right\} + \frac{1}{C}\left\{I\left(s\right)\right\} = E\left(s\right)\\
&\Rightarrow I\left(s\right) = \frac{E\left(s\right)}{\left\{Ls^{\alpha} + Rs^{\beta} + \frac{1}{C}\right\}} + LA\frac{s^{\alpha-1}}{\left\{Ls^{\alpha} + Rs^{\beta} + \frac{1}{C}\right\}}\\
&\qquad + LB\frac{s^{\alpha-2}}{\left\{Ls^{\alpha} + Rs^{\beta} + \frac{1}{C}\right\}} + RB\frac{s^{\beta-1}}{\left\{Ls^{\alpha} + Rs^{\beta} + \frac{1}{C}\right\}}
\end{aligned} \tag{18}$$

Solution of (18) is obtained by taking inverse Laplace transform and using (4), (13), we get,

$$
\begin{aligned}
I(t) = C\int_0^t E(t-u)\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{1}{(LC)^{k+1}}\mathcal{E}_k\left(u,\frac{-R}{L};\alpha-\beta,\alpha+\beta k\right)du \\
+\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{ALC}{(LC)^{k+1}}\ \mathcal{E}_k\left(t,-\frac{R}{L};\alpha-\beta,\beta k+1\right) \\
+\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{BLC}{(LC)^{k+1}}\ \mathcal{E}_k\left(t,-\frac{R}{L};\alpha-\beta,\beta k+2\right) \\
+\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{RAC}{(LC)^{k+1}}\ \mathcal{E}_k\left(t,-\frac{R}{L};\alpha-\beta,\beta(k-1)+\alpha+1\right),
\end{aligned}
\tag{19}
$$

where $1<\alpha\leq 2$ and $0<\beta\leq 1$ also $\alpha-\beta>0$.

Some cases for different values of $E(t)$ are discussed below.

Case I. When constant electromotive force is applied, i.e., $E(t)=E_0$, then (19) takes the following form,

$$
\begin{aligned}
I(t) = CE_0\int_0^t\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{1}{(LC)^{k+1}}\ \mathcal{E}_k\left(u,-\frac{R}{L};\alpha-\beta,\alpha+\beta k\right)du \\
+\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{ALC}{(LC)^{k+1}}\ \mathcal{E}_k\left(t,-\frac{R}{L};\alpha-\beta,\beta k+1\right) \\
+\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{BLC}{(LC)^{k+1}}\ \mathcal{E}_k\left(t,-\frac{R}{L};\alpha-\beta,\beta k+2\right) \\
+\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{RAC}{(LC)^{k+1}}\ \mathcal{E}_k\left(t,-\frac{R}{L};\alpha-\beta,\beta(k-1)+\alpha+1\right).
\end{aligned}
\tag{20}
$$

Case II. When periodic electromotive force is applied, i.e., $E(t)=E_0\cos\omega t$, where E_0 and ω are constants, then (19) yields,

$$
\begin{aligned}
I(t) = CE_0\int_0^t\cos\omega(t-u)\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\ \frac{1}{(LC)^{k+1}}\ \mathcal{E}_k\left(u,-\frac{R}{L};\alpha-\beta,\alpha+\beta k\right)du \\
+\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{ALC}{(LC)^{k+1}}\ \mathcal{E}_k\left(t,-\frac{R}{L};\alpha-\beta,\beta k+1\right) \\
+\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{BLC}{(LC)^{k+1}}\ \mathcal{E}_k\left(t,-\frac{R}{L};\alpha-\beta,\beta k+2\right) \\
+\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{RAC}{(LC)^{k+1}}\ \mathcal{E}_k\left(t,-\frac{R}{L};\alpha-\beta,\beta(k-1)+\alpha+1\right).
\end{aligned}
\tag{21}
$$

Case III. When periodic electromotive force is applied, i.e. $E(t)=E_0\sin\omega t$, where E_0 and ω are constants, then (19) reduces to

$$
\begin{aligned}
I(t) = CE_0 \int_0^t \sin\omega(t-u) \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{1}{(LC)^{k+1}} \mathcal{E}_k\left(u, -\frac{R}{L}; \alpha-\beta, \alpha+\beta k\right) du \\
+ \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{ALC}{(LC)^{k+1}} \ \mathcal{E}_k\left(t, -\frac{R}{L}; \alpha-\beta, \beta k+1\right) \\
+ \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{BLC}{(LC)^{k+1}} \ \mathcal{E}_k\left(t, -\frac{R}{L}; \alpha-\beta, \beta k+2\right) \\
+ \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{RAC}{(LC)^{k+1}} \ \mathcal{E}_k\left(t, -\frac{R}{L}; \alpha-\beta, \beta(k-1)+\alpha+1\right).
\end{aligned} \tag{22}
$$

Some special cases of Equation (17)**.** When we take $\beta = 1$, (17) reduces to the form,

$$
L\ D^{\alpha} I(t) + R\frac{dI}{dt} + \frac{1}{C} I(t) = E(t), \qquad 1 < \alpha \leq 2. \tag{23}
$$

On further simplification yields,

$$
\begin{aligned}
I(t) = C \int_0^t E(t-u) \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{1}{(LC)^{k+1}} \ \mathcal{E}_k\left(u, -\frac{R}{L}; \alpha-1, \alpha+k\right) du \\
+ \sum_{k=0}^{\infty} \tfrac{(-1)^k}{k!} \tfrac{ALC}{(LC)^{k+1}} \ \mathcal{E}_k\left(t, -\tfrac{R}{L}; \alpha-1, k+1\right) \\
+ \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{BLC}{(LC)^{k+1}} \ \mathcal{E}_k\left(t, -\frac{R}{L}; \alpha-1, k+2\right) \\
+ \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{RAC}{(LC)^{k+1}} \ \mathcal{E}_k\left(t, -\frac{R}{L}; \alpha-1, \alpha+k\right),
\end{aligned} \tag{24}
$$

where $1 < \alpha \leq 2 \Rightarrow \alpha - 1 > 0$.

Similarly, on setting $\alpha = 2$ in (17) with $1 < \alpha \leq 2,\ 0 < \beta \leq 1$, this reduces to following form,

$$
L\frac{d^2 I}{dt^2} + R\ D^{\beta} I(t) + \frac{1}{C} I(t) = E(t). \tag{25}
$$

Solution of this equation can also be obtained by aforesaid method, which gives,

$$
\begin{aligned}
I(t) = C \int_0^t E(t-u) \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{1}{(LC)^{k+1}} \mathcal{E}_k\left(u, -\frac{R}{L}; 2-\beta, 2+\beta k\right) du \\
+ \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{ALC}{(LC)^{k+1}} \ \mathcal{E}_k\left(t, -\frac{R}{L}; 2-\beta, \beta k+1\right) \\
+ \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{BLC}{(LC)^{k+1}} \ \mathcal{E}_k\left(t, -\frac{R}{L}; 2-\beta, 2+\beta k\right) \\
+ \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{RAC}{(LC)^{k+1}} \ \mathcal{E}_k\left(t, -\frac{R}{L}; 2-\beta, \beta(k-1)+3\right).
\end{aligned} \tag{26}
$$

4. Conclusion

Fractional calculus has been recognized as advantageous mathematical tool in modeling and control of dynamical systems. We have obtained the analytic solution of the second order fractional differential equation associated with a RLC electrical circuit in time domain using Caputo derivative in terms of Mittag-Leffler type function which can be implemented for computational study of behaviour of current.

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IDEMPOTENT ELEMENTS OF PRE-GENERALIZED HYPERSUBSTITUTIONS OF TYPE (m, n)

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Abstract. A generalized hypersubstitution of type $\tau = (m, n)$ is a mapping σ which maps the m-ary operation symbol f and n-ary operation symbol g to the term $\sigma(f)$ and $\sigma(g)$, and may not preserved arities. Each generalized hypersubstitution can be extended to a mapping $\hat{\sigma}$ on the set of all terms of type $\tau = (m, n)$. The structure $(\mathrm{Hyp}_{\mathrm{G}}(\tau); \circ_{\mathrm{G}}, \sigma_{\mathrm{id}})$ is a monoid where σ_{id} is an identity hypersubstitution. A pre-generalized hypersubstitution of type $\tau = (m, n)$, namely σ, is a generalized hypersubstitution of type $\tau = (m, n)$ where $\sigma(f)$ and $\sigma(g)$ are not variables. In this paper, we characterize idempotent pre-generalized hypersubstitutions of type $\tau = (m, n)$.

Keywords: pre-generalized hypersubstitution, idempotent elements.
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1. Introduction

Let n be a natural number. Let $X_n = \{x_1, \ldots, x_n\}$ be an n-element set. The set X_n is called an *n-element alphabet* and its elements are called *variables*. Let $\{f_i : i \in I\}$ be the set of *operation symbols*, indexed by set I. The sets X_n and $\{f_i : i \in I\}$ have to be disjoint. To every operation symbol f_i, we assign a natural number $n_i \geq 1$, called the *arity* of f_i. As in the definition of algebra, the sequence $\tau = (n_i)_{i \in I}$ of all the arities is called the *type*. The classes of algebras are described by logical expressions. This formal language is built up by variables from an n-element set. With this notation for operation symbols and variables, we can define the terms of type τ, (see [3], [4], [5]).

An n-ary terms of type τ is defined in the following inductive way:

(i) Every variable $x_i \in X_n$ is an n-ary term.

(ii) If $t_1, \ldots, t_{n_i}$ are n-ary terms and f_i is an n_i-ary operation symbol, then $f_i(t_1, \ldots, t_{n_i})$ is an n-ary term.

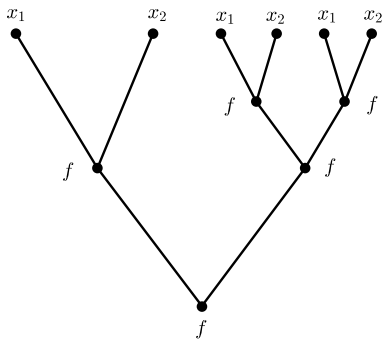
The set $W_\tau(X_n) = W_\tau(\{x_1, \ldots, x_n\})$ of all n-ary terms is the smallest set which contains $x_1, \ldots, x_n$ and is closed under finite application of (ii). We denote the set of all terms of type τ by

$$W_\tau(X) := \bigcup_{m=1}^{\infty} W_\tau(X_m).$$

Denoted by $\mathrm{var}(t)$, $\mathrm{op}(t)$, $\mathrm{ops}(t)$ and $\mathrm{firstop}(t)$ the set of all variables occurring in the term t, the number of operation symbols occurring in the term t, the set of all operation symbols occurring in the term t and the first operation symbol (from the left) occurring in the term t, respectively.

Terms can be visualized as trees, where the vertices are labeled by operation symbols and the leaves are labeled by variables (see [1]). Trees have many applications in Mathematics, Computer Science, Linguistic and in other fields. For instance, the following tree corresponds to the term:

$$f(f(x_1, x_2), f(f(x_1, x_2), f(x_1, x_2))).$$



In universal algebra, identities are used to classify algebras into collections, called *varieties* and hyperidentities are used to classify varieties into collections, called *hypervarities*. The concept of hypersubstitution was introduced by K. Denecke, D. Lau, R. Pöschel and D. Schweigert [2]. In [8], S. Leeratanavalee and K. Denecke gave the concept of *generalized superposition* of terms $S^k : (W_\tau(X))^{k+1} \to W_\tau(X)$ by the following steps: for any term $t \in W_\tau(X)$,

(i) if $t = x_j \in X_n$, then $S^k(x_j, t_1, \ldots, t_k) := t_j$;

(ii) if $t = x \in X \setminus X_k$, then $S^k(x, t_1, \ldots, t_k) := x$;

(iii) if $t = f_i(s_1, \ldots, s_{n_i})$ and assume that $S^k(s_j, t_1, \ldots, t_k)$ for $1 \leq j \leq n_i$ are already defined, then

$$S^k(f_i(s_1, \ldots, s_{n_i}), t_1, \ldots, t_k) := f_i(S^k(s_1, t_1, \ldots, t_k), \ldots, S^k(s_{n_i}, t_1, \ldots, t_k)).$$

Such operation is a tool to study generalized hypersubstitutions.

A *generalized hypersubstitution* of type τ is a mapping $\sigma:\{f_i:i \in I\}\to W_\tau(X)$ which maps operation symbols of type τ to a term of the same type, and may not preserved arity. We denoted by $\mathrm{Hyp}_{\mathrm{G}}(\tau)$ the set of all generalized hypersubstitutions of type τ. The generalized hypersubstitution σ can be extended to a mapping $\hat{\sigma} : W_\tau(X) \to W_\tau(X)$ on the set of all terms of type τ inductively defined as follows:

(i) $\hat{\sigma}[x] := x \in X$;

(ii) $\hat{\sigma}[f_i(t_1, \ldots, t_{n_i})] := S^{n_i}(\sigma(f_i), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$ for any n_i-ary operation symbol f_i and assume that $\hat{\sigma}[t_j]$ are already defined for all $1 \leq j \leq n_i$.

The concept of generalized hypersubstitutions is a generalization of the concept of hypersubstitutions.

In [8], the authors defined a binary operation $\circ_{\mathrm{G}}$ on $\mathrm{Hyp}_{\mathrm{G}}(\tau)$ by $\sigma_1 \circ_{\mathrm{G}} \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$ where $\circ$ is an usual composition. They have shown that the structure $\mathbf{Hyp}_{\mathrm{G}}(\tau) := (\mathrm{Hyp}_{\mathrm{G}}(\tau); \circ_{\mathrm{G}}, \sigma_{\mathrm{id}})$ is a monoid where σ_{id} is an identity hypersubstitution. The authors also proved the following propositions.

Proposition 1.1 ([8]). *For arbitrary terms $t, t_1, \ldots, t_n \in W_\tau(X)$ and for arbitrary generalized hypersubstitutions $\sigma, \sigma_1, \sigma_2$ we have*

(i) $S^n(\hat{\sigma}[t], \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]) = \hat{\sigma}[S^n(t, t_1, \ldots, t_n)]$,

(ii) $(\hat{\sigma}_1 \circ \sigma_2)\hat{} = \hat{\sigma}_1 \circ \hat{\sigma}_2$.

Proposition 1.2 ([8]). *The monoid $\boldsymbol{Hyp}(\tau) := (Hyp(\tau); \circ_h, \sigma_{id})$ of all arity preserving hypersubstitutions of type τ forms a submonoid of $\boldsymbol{Hyp}_G(\tau)$.*

2. Pre-generalized hypersubstitutions of type (m, n)

In [2], the authors studied M-hyperidentities and M-solid varieties based on submonoid M of the monoid $\mathbf{Hyp}(\tau)$. They defined a number of natural such monoid based on various properties of hypersubstitutions.

According to [10], the authors studied pre-generalized hypersubstitutions of type $(2, 2)$. In practice, the type of these two operation symbols that we consider need not to be $(2, 2)$. Thus, we extend this idea to consider operation symbols of type (m, n). This section we provide the definition of pre-generalized hypersubstitutions of type (m, n). We also recall the definitions of projection generalized hypersubstitutions and weak projection generalized hypersubstitutions of type (m, n).

Definition 2.1. Let f and g be operation symbols of type (m, n). We denote the generalized hypersubstitution σ with $\sigma(f) = t_1$ and $\sigma(g) = t_2$ by σ_{t_1,t_2}.

(i) A generalized hypersubstitution σ of type (m, n) is called a *projection generalized hypersubstitution* if the term $\sigma(f)$ and $\sigma(g)$ are variables. We denote the set of all projection generalized hypersubstitutions of type (m, n) by $P_{\mathrm{G}}(m, n)$.

(ii) A generalized hypersubstitution σ of type (m, n) is called a *weak projection generalized hypersubstitution* if the term $\sigma(f)$ or $\sigma(g)$ is variable. We denote the set of all weak projection generalized hypersubstitutions of type (m, n) by $WP_{\mathrm{G}}(m, n)$.

(iii) A generalized hypersubstitution σ of type (m, n) is called a *pre-generalized hypersubstitution* if the terms $\sigma(f)$ and $\sigma(g)$ are not variables. We denote the set of all pre-generalized hypersubstitutions of type (m, n) by $Pre_{\mathrm{G}}(m, n)$. That is, $Pre_{\mathrm{G}}(m, n) := \mathrm{Hyp}_{\mathrm{G}}(m, n) \setminus WP_{\mathrm{G}}(m, n)$.

In [7], S. Leeratanavalee showed that for any type τ, the set $P_{\mathrm{G}}(\tau) \cup \{\sigma_{\mathrm{id}}\}$ and $Pre_{\mathrm{G}}(\tau)$ are submonoids of $\mathbf{Hyp}_{\mathrm{G}}(\tau)$. It is easy to see that $WP_{\mathrm{G}}(\tau) \cup \{\sigma_{\mathrm{id}}\}$ is a submonoid of $\mathbf{Hyp}_{\mathrm{G}}(\tau)$, and $P_{\mathrm{G}}(\tau) \cup \{\sigma_{\mathrm{id}}\}$ forms a submonoid of $WP_{\mathrm{G}}(\tau) \cup \{\sigma_{\mathrm{id}}\}$, (see [7]).

In 2007, S. Leeratanavalee [6] characterized idempotent elements of weak projection generalized hypersubstitutions of type $(2, 2)$ which was published in 2011. Later, in 2015, N. Lekkoksung and P. Jampachon [9] characterized idempotent elements of weak projection generalized hypersubstitutions of type (m, n).

Another submonoid of the monoid of all generalized hypersubstitutions of type τ is the monoid of all pre-generalized hypersubstitutions of type τ. In 2007, W. Puninagool and S. Leeratanavalee [10] characterized idempotent elements of pre-generalized hypersubstitutions of type $(2, 2)$. In this paper, we characterize idempotent pre-generalized hypersubstitutions of type (m, n).

3. Main results

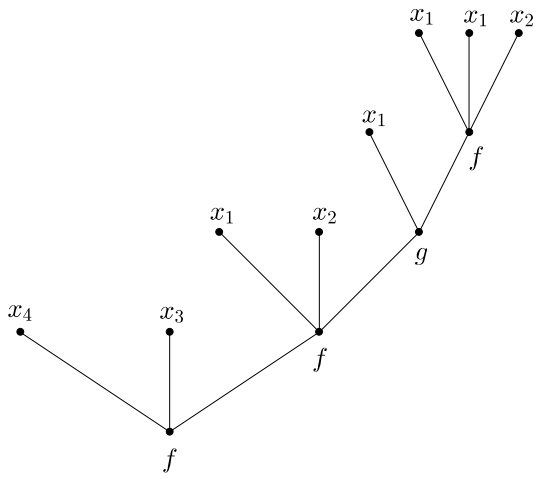
For any semigroup S, an element e of S is called an *idempotent element* of S if $ee = e$. In this section, we present a sufficient and necessary condition for elements of $Pre_{\mathrm{G}}(m, n)$ to be an idempotent element.

Let F be a variable over the two-elements alphabet $\{f, g\}$. Let $t = F(t_1, \ldots, t_j)$ where F has arity $j \in \{m, n\}$ and $i \leq \min\{m, n\}$, we define $M^i(t)$ by:

(i) if $t_i \in X$, then $M^i(t) = t_i$;

(ii) if $t_i = F'(s_1 \ldots, s_k)$ where F' has arity $k \in \{m, n\}$ and assume that $M^i(s_i)$ are already defined, then $M^i(t) = M^i(s_i)$.

For example, let f and g be operation symbols of type 3 and 2, respectively, and $t = f(x_4, x_3, f(x_1, x_2, g(x_1, f(x_1, x_1, x_2))))$. This term t can be visualized by a tree:



Then

$M^1(t) = x_4$ $\quad M^2(t) = x_3$ $\quad M^3(t)$ does not define.
$M^2(g(x_1, f(x_1, x_1, x_2)) = x_1$ $\quad M^3(f(x_1, x_1, x_2)) = x_2$.

Proposition 3.1 ([9]). *Let σ_{t_1,t_2} be a generalized hypersubstitution of type (m, n). Then the following statements are equivalent:*

(i) *σ_{t_1,t_2} is idempotent;*

(ii) *$\hat{\sigma}_{t_1,t_2}[t_1] = t_1$ and $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$.*

To characterize idempotent elements of $Pre_{\mathrm{G}}(m, n)$, we separate our consideration into three cases:

Case 1. $\mathrm{op}(t_1) = 1$ and $\mathrm{op}(t_2) = 1$;
Case 2. $\mathrm{op}(t_1) = 1$, $\mathrm{op}(t_2) > 1$ and $\mathrm{op}(t_1) > 1$, $\mathrm{op}(t_2) = 1$;
Case 3. $\mathrm{op}(t_1) > 1$ and $\mathrm{op}(t_2) > 1$.

Case 1. $\mathbf{op}(t_1) = 1$ and $\mathbf{op}(t_2) = 1$

Assume that $\hat{\sigma}_{t_1,t_2}[t_1] = t_1$, in case of $\mathrm{op}(t_1) = 1 = \mathrm{op}(t_2)$, we consider only three cases:

1. $\mathrm{firstop}(t_1) = f$ and $\mathrm{firstop}(t_2) = f$;
2. $\mathrm{firstop}(t_1) = g$ and $\mathrm{firstop}(t_2) = g$;
3. $\mathrm{firstop}(t_1) = f$ and $\mathrm{firstop}(t_2) = g$

since for the case $\mathrm{firstop}(t_1) = g$ and $\mathrm{firstop}(t_2) = f$ is impossible. Indeed, for $s_1, \ldots, s_m \in X$,

$$t_1 = \hat{\sigma}_{t_1,t_2}[t_1] = S^m(\sigma_{t_1,t_2}(g), s_1, \ldots, s_m) = S^m(t_2, s_1, \ldots, s_m).$$

This implies $\mathrm{firstop}(t_1) = f$, a contradiction.

Proposition 3.2. *Let $t_1 = f(s_1, \ldots, s_m)$ and $t_2 = f(s'_1, \ldots, s'_m)$ where $s_1, \ldots, s_m$, $s'_1, \ldots, s'_m \in X$. Then the following statements are equivalent:*

(1) *σ_{t_1,t_2} is idempotent;*

(2) *the following conditions holds:*

(i) *if $x_j \in var(t_1)$ where $1 \leq j \leq m$, then $s_j = x_j$;*

(ii) *if $s_i = x_j$ where $1 \leq i, j \leq m$, then $s'_i = s'_j$;*

(iii) *if $s_j = x \in X \setminus X_m$ where $1 \leq j \leq m$, then $s'_j = x$.*

Proof. (1) $\Rightarrow$ (2) : Since σ_{t_1,t_2} is idempotent, $\hat{\sigma}_{t_1,t_2}[t_1] = t_1$ and $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$. That is, $S^m(t_1, s_1, \ldots, s_m) = t_1$ and $S^m(t_1, s'_1, \ldots, s'_m) = t_1$.

(i) Assume that $x_j \in \mathrm{var}(t_1)$ where $1 \leq j \leq m$. We suppose that $s_j = x_i$ where $x_i \in X_m$ and $i \neq j$. Thus, we have to replace x_j in the term t_1 by x_i. This implies that $S^m(t_1, s_1, \ldots, s_m) \neq t_1$, a contradiction. Hence, $s_j = x_j$.

(ii) Assume that $s_j = x_i$ where $1 \leq i, j \leq m$. Suppose that $s'_j \neq s'_i$. From $S^m(t_1, s'_1, \ldots, s'_m)$, we have to replace s_j in the term t_1 by s'_i. Since $s'_j \neq s'_i$, $S^m(t_1, s'_1, \ldots, s'_m) \neq t_2$, this is a contradiction. Thus, $s'_j = s'_i$.

(iii) Assume that $s_j = x \in X \setminus X_m$ where $1 \leq j \leq m$. Suppose that $s'_j \neq x$. From $S^m(t_1, s'_1, \ldots, s'_m)$, we have to replace s_j in the term t_1 by x. Since $s'_j \neq x$, $S^m(t_1, s'_1, \ldots, s'_m) \neq t_2$, this is a contradiction. Thus, $s'_j = x$.

(2) $\Rightarrow$ (1) : We consider these conditions:

(i) If $x_j \in \mathrm{var}(t_1)$ where $1 \leq j \leq m$, then we replace x_j in the term t_1 by x_j.

(ii) If $s_i = x_j$ where $1 \leq i, j \leq m$, then we replace x_j in the term t_1 by s'_i.

(iii) If $s_j = x \in X \setminus X_m$ where $1 \leq j \leq m$, then we replace x in the term t_1 by x.

This shows that $\hat{\sigma}_{t_1,t_2}[t_1] = t_1$ and $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$. Therefore, σ_{t_1,t_2} is idempotent. ∎

Similarly, we obtain the following proposition.

Proposition 3.3. *Let $t_1 = g(s_1, \ldots, s_n)$ and $t_2 = g(s'_1, \ldots, s'_n)$ where $s_1, \ldots, s_n$, $s'_1, \ldots, s'_n \in X$. Then the following statements are equivalents:*

(1) *σ_{t_1,t_2} is idempotent;*

(2) *the following conditions hold:*

(i) *if $x_j \in var(t_2)$ where $1 \leq j \leq n$, then $s'_j = x_j$;*

(ii) *if $s'_i = x_j$ where $1 \leq i, j \leq n$, then $s_i = s_j$;*

(iii) *if $s'_j = x \in X \setminus X_n$ where $1 \leq j \leq n$, then $s_j = x$.*

For the last possibility, in case of $\mathrm{op}(t_1) = 1 = \mathrm{op}(t_2)$, we obtain the following proposition.

Proposition 3.4. *Let $t_1 = f(s_1, \ldots, s_m)$ and $t_2 = g(s'_1, \ldots, s'_n)$ where $s_1, \ldots, s_m$, $s'_1, \ldots, s'_n \in X$. Then the following statements are equivalents:*

(1) *σ_{t_1,t_2} is idempotent;*

(2) *the following conditions holds:*

(i) *if $x_j \in var(t_1)$ where $1 \leq j \leq m$, then $s_j = x_j$;*

(ii) *if $x_k \in var(t_2)$ where $1 \leq k \leq n$, then $s_k = x_k$.*

Proof. (1) $\Rightarrow$ (2) : Since σ_{t_1,t_2} is idempotent,

$$t_1 = \hat{\sigma}_{t_1,t_2}[t_1] = S^m(\sigma_{t_1,t_2}(f), s_1, \ldots, s_m) = S^m(t_1, s_1, \ldots, s_m) \tag{3.1}$$

$$t_2 = \hat{\sigma}_{t_1,t_2}[t_2] = S^n(\sigma_{t_1,t_2}(g), s'_1, \ldots, s'_n) = S^n(t_2, s'_1, \ldots, s'_n). \tag{3.2}$$

(i) Assume that $x_j \in \mathrm{var}(t_1)$ where $1 \leq j \leq m$. We suppose that $s_j = x_i$ where $x_i \in X$ and $i \neq j$. Then, in the right-hand-side of (3.1), we have to replace x_j in the term t_1 by x_i. This implies $S^m(t_1, s_1, \ldots, s_m) \neq t_1$, which is a contradiction. Thus, $s_j = x_j$.

(ii) Similarly to (i).

(2) $\Rightarrow$ (1) : We consider the following conditions:

(i) If $x_j \in \mathrm{var}(t_1)$ where $1 \leq j \leq m$, then we replace x_j in the term t_1 by x_j.

(ii) If $x_k \in \mathrm{var}(t_2)$ where $1 \leq j \leq n$, then we replace x_k in the term t_2 by x_k.

This shows that $\hat{\sigma}_{t_1,t_2}[t_1] = t_1$ and $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$. Therefore, σ_{t_1,t_2} is idempotent. ∎

Case 2. $\mathbf{op}(t_1) = 1$, $\mathbf{op}(t_2) > 1$ and $\mathbf{op}(t_2) = 1$, $\mathbf{op}(t_1) > 1$

At first, we consider in case that $\mathrm{op}(t_1) = 1$ and $\mathrm{op}(t_2) > 1$. We need the following lemmas before proving main results.

Lemma 3.5. *Let σ_{t_1,t_2} be a generalized hypersubstitution of type (m, n). If $op(t_1)$=1, $op(t_2) > 1$ and σ_{t_1,t_2} is idempotent, then $firstop(t_1) = f$.*

Proof. Suppose that $\mathrm{firstop}(t_1) = g$. That is, $t_1 = g(s_1, \ldots, s_n)$ where $s_1, \ldots, s_n \in W_{(m,n)}(X)$. Then

$$t_1 = \hat{\sigma}_{t_1,t_2}[t_1] = S^n(\sigma_{t_1,t_2}(g), s_1, \ldots, s_n) = S^n(t_2, s_1, \ldots, s_n) \neq t_1$$

since $\mathrm{op}(t_2) > 1$. This is a contradiction, so $\mathrm{firstop}(t_1) = f$. ∎

Lemma 3.6. *Let σ_{t_1,t_2} be a generalized hypersubstitution of type (m,n), $op(t_1)$=1, $op(t_2) > 1$ and $t_1 = f(s_1,\ldots,s_m)$ where $s_1,\ldots,s_m \in X$. Then the following conditions are equivalent:*

(1) $\hat{\sigma}_{t_1,t_2}[t_1] = t_1$;

(2) *firstop*$(t_1) = f$ *and if* $x_j \in var(t_1)$ *where* $1 \leq j \leq m$, *then* $s_j = x_j$.

Proof. (1) $\Rightarrow$ (2) : By Lemma 3.5, we have $\text{firstop}(t_1) = f$. Assume that $x_j \in \text{var}(t_1)$ where $1 \leq j \leq m$. We suppose that $s_j = x_i$ where $x_i \in X_m$ and $i \neq j$. Thus, we have to replace x_j in the term t_1 by x_i. This implies that $S^m(t_1, s_1, \ldots, s_m) \neq t_1$, a contradiction. Hence, $s_j = x_j$.

(2) $\Rightarrow$ (1) : Since $\text{op}(t_1) = 1$ and $\text{firstop}(t_1) = f$, $t_1 = f(s_1,\ldots,s_n)$ where $s_1,\ldots,s_n \in X$. Consider

$$\hat{\sigma}_{t_1,t_2}[t_1] = S^m(\sigma_{t_1,t_2}(f), s_1, \ldots, s_n) = S^m(t_1, s_1, \ldots, s_n),$$

if $x_j \in \text{var}(t_1)$ where $1 \leq j \leq m$, then we replace x_j in the term t_1 by x_j. In case that $x \in \text{var}(t_1)$ where $x \in X \setminus X_m$, then we replace x in the term t_1 by x. Therefore, $S^m(t_1, s_1, \ldots, s_n) = t_1$, so $\hat{\sigma}_{t_1,t_2}[t_1] = t_1$. ∎

By the above lemma, in the case that $\text{op}(t_1) = 1$, we have to consider the following two cases:

- $\text{firstop}(t_2) = g$;
- $\text{firstop}(t_2) = f$.

Then we obtain the following proposition.

Proposition 3.7. *Let σ_{t_1,t_2} be a generalized hypersubstitution of type (m,n), $op(t_1)$=1, $op(t_2) > 1$, t_1=$f(s'_1,\ldots,s'_m)$ and t_2=$g(s_1,\ldots,s_n)$ where $s'_1,\ldots,s'_m \in X$, $s_1,\ldots,s_n \in W_{(m,n)}(X)$. Then the following conditions are equivalent:*

(1) σ_{t_1,t_2} *is idempotent;*

(2) *the following conditions hold*

(i) *if* $x_j \in var(t_1)$ *where* $1 \leq j \leq m$, *then* $s'_j = x_j$;

(ii) *if* $x_j \in var(t_2)$ *where* $1 \leq j \leq n$, *then* $s_j = x_j$.

Proof. (1) $\Rightarrow$ (2) : Since $\hat{\sigma}_{t_1,t_2}[t_1] = t_1$, (i) is satisfied by Lemma 3.6.

Assume that $x_j \in \text{var}(t_2)$ where $1 \leq j \leq n$. We suppose that $s_j \neq x_j$ where $1 \leq j \leq n$. Then $s_j \in W_{(m,n)}(X) \setminus X$ or $s_j = x_i$ where $i \neq j$. If $s_j \in W_{(m,n)}(X) \setminus X$, then $\text{op}(t_2) < \text{op}(S^n(\sigma_{t_1,t_2}(g), \hat{\sigma}_{t_1,t_2}[s_1], \ldots, \hat{\sigma}_{t_1,t_2}[s_n]))$. If $s_j = x_i$ where $i \neq j$, then we have to replace x_j in the term t_2 by x_i. It follows that $S^n(t_2, \hat{\sigma}_{t_1,t_2}[s_1], \ldots, \hat{\sigma}_{t_1,t_2}[s_n]) \neq t_2$, which contradicts to the idempotency of σ_{t_1,t_2}. Thus, $s_j = x_j$, that is, the condition (ii) is satisfied.

(2) $\Rightarrow$ (1) : By Lemma 3.6, we have $\hat{\sigma}_{t_1,t_2}[t_1] = t_1$. We show that $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$. Since $\hat{\sigma}_{t_1,t_2}[t_2] = S^n(t_2, \hat{\sigma}_{t_1,t_2}[s_1], \ldots, \hat{\sigma}_{t_1,t_2}[s_n])$, we have to replace s_j in the term t_2

by x_j since the condition (ii) is satisfied. Thus, $S^n(t_2, \hat{\sigma}_{t_1,t_2}[s_1], \ldots, \hat{\sigma}_{t_1,t_2}[s_n]) = t_2$, so $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$. Hence, σ_{t_1,t_2} is idempotent. ∎

To prove the case $\text{firstop}(t_2) = f$, we need the following lemmas.

Lemma 3.8. *Let σ_{t_1,t_2} be an idempotent generalized hypersubstitution of type (m,n), $op(t_1) = 1$, $op(t_2) > 1$ and $t_2 = f(s_1, \ldots, s_m)$ where $s_1, \ldots, s_m \in W_{(m,n)}(X)$. If $x_j \in var(t_1)$ where $1 \leq j \leq m$, then $firstop(s_j) = f$ or $s_j \in X$.*

Proof. Assume that $x_j \in \text{var}(t_1)$. Suppose that $s_j = g(s'_1, \ldots, s'_n)$ where $s'_1, \ldots, s'_n \in W_{(m,n)}(X)$. Since $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$,

$$t_2 = S^m(t_1, \hat{\sigma}_{t_1,t_2}[s_1], \ldots, \hat{\sigma}_{t_1,t_2}[s_m]). \tag{3.3}$$

Now, $\hat{\sigma}_{t_1,t_2}[s_j] = S^n(t_2, \hat{\sigma}_{t_1,t_2}[s'_1], \ldots, \hat{\sigma}_{t_1,t_2}[s'_n])$, we replace $\hat{\sigma}_{t_1,t_2}[s_j]$ in (3.3), yield,

$$\begin{aligned} t_2 = S^m(t_1, \hat{\sigma}_{t_1,t_2}[s_1], \ldots, \hat{\sigma}_{t_1,t_2}[s_{j-1}], S^n(t_2, \hat{\sigma}_{t_1,t_2}[s'_1], \ldots, \hat{\sigma}_{t_1,t_2}[s'_n]), \\ \hat{\sigma}_{t_1,t_2}[s_{j+1}], \ldots, \hat{\sigma}_{t_1,t_2}[s_m]). \end{aligned} \tag{3.4}$$

For simplicity, we write the right-hand-side of (3.4) by A. Since $x_j \in \text{var}(t_1)$, $\text{op}(t_2) < \text{op}(A)$. This is a contradiction. Thus, $\text{firstop}(s_j) = f$. ∎

Lemma 3.9. *Let σ_{t_1,t_2} be an idempotent generalized hypersubstitution of type (m,n), $op(t_1) = 1$, $op(t_2) > 1$, $t_1 = f(x_{j_1}, \ldots, x_{j_m})$ and $t_2 = f(\overline{s}_1, \ldots, \overline{s}_m)$ where $x_{j_1}, \ldots, x_{j_m} \in X$, $\overline{s}_1, \ldots, \overline{s}_m \in W_{(m,n)}(X)$. Then $ops(t_2) = \{f\}$.*

Proof. Suppose that $g \in \text{ops}(t_2)$. Without losing of generality, let $\text{firstop}(\overline{s}_k) = g$ for some $1 \leq k \leq m$. We write $\overline{s}_k = g(s'_1, \ldots, s'_n)$ where $s'_1, \ldots, s'_n \in W_{(m,n)}(X)$. Since $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$,

$$t_2 = S^m(t_1, \hat{\sigma}_{t_1,t_2}[\overline{s}_1], \ldots, \hat{\sigma}_{t_1,t_2}[\overline{s}_m]).$$

Thus, there are two possibilities to consider.

If $x_{j_k} \in X \setminus X_m$ where $1 \leq k \leq m$, then $\hat{\sigma}_{t_1,t_2}[\overline{s}_k]$ must be x_{j_k}, otherwise $t_2 \neq S^m(t_1, \hat{\sigma}_{t_1,t_2}[\overline{s}_1], \ldots, \hat{\sigma}_{t_1,t_2}[\overline{s}_m])$.

If $x_{j_k} \in X_m$ where $1 \leq k \leq m$, then by Lemma 3.8, $\text{firstop}(\overline{s}_k) = f$ or $\overline{s}_k \in X$. Therefore, we have $\text{ops}(t_2) = \{f\}$. ∎

Lemma 3.10. Let σ_{t_1,t_2} be a generalized hypersubstitution of type (m,n), $\text{op}(t_1)$=1, $\text{op}(t_2) > 1$, $t_1 = f(x_{j_1}, \ldots, x_{j_m})$ and $t_2 = f(\overline{s}_1, \ldots, \overline{s}_m)$ where $x_{j_1}, \ldots, x_{j_m} \in X$, $\overline{s}_1, \ldots, \overline{s}_n \in W_{(m,n)}(X)$. If σ_{t_1,t_2} is idempotent, then $\hat{\sigma}_{t_1,t_2}[\overline{s}_k] = \overline{s}_k$ for all $k \in \{1, \ldots, m\}$.

Proof. Let $i \in \{1, \ldots, m\}$. Since $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$,

$$t_2 = f(\overline{s}_1, \ldots, \overline{s}_m) = S^m(t_1, \hat{\sigma}_{t_1,t_2}[\overline{s}_1], \ldots, \hat{\sigma}_{t_1,t_2}[\overline{s}_m]). \tag{3.5}$$

Thus, we replace x_{j_i} in the term t_1 by x_{j_i} if $x_{j_i} \in X \setminus X_m$. If $x_{j_i} \in X_m$, then we replace x_{j_i} in the term t_1 by $\hat{\sigma}_{t_1,t_2}[\overline{s}_{j_i}]$. By (3.5), we obtain $\hat{\sigma}_{t_1,t_2}[\overline{s}_{j_i}] = \overline{s}_i$. Then

$$\begin{aligned}\hat{\sigma}_{t_1,t_2}[\overline{s}_i] &= (\hat{\sigma}_{t_1,t_2} \circ \hat{\sigma}_{t_1,t_2})[\overline{s}_{j_i}] \\ &= (\sigma_{t_1,t_2} \circ_{\mathrm{G}} \sigma_{t_1,t_2})\widehat{\ }[\overline{s}_{j_i}] \\ &= \hat{\sigma}_{t_1,t_2}[\overline{s}_{j_i}].\end{aligned}$$

It follows that $\hat{\sigma}_{t_1,t_2}[\overline{s}_i] = \hat{\sigma}_{t_1,t_2}[\overline{s}_{j_i}] = \overline{s}_i$. ∎

Lemma 3.11. *Let σ_{t_1,t_2} be an idempotent generalized hypersubstitution of type (m,n), $op(t_1) = 1$, $op(t_2) > 1$, $t_1 = f(s_1,\ldots,s_m)$ and $t_2 = f(\overline{s}_1,\ldots,\overline{s}_m)$ where $s_1,\ldots,s_m \in X$, $\overline{s}_1,\ldots,\overline{s}_n \in W_{(m,n)}(X)$. If $s_j = x \in X \setminus X_m$ where $1 \le j \le m$, then $\overline{s}_j = x$ whenever $\overline{s}_j \in X$ and $M^j(\overline{s}_i) = x$ whenever $\overline{s_i} \notin X$ where $1 \le i \le m$.*

Proof. Assume that $s_j = x \in X \setminus X_m$ where $1 \le j \le m$.

- $\overline{s}_j \in X$: Since $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$,

  $$t_2 = S^m(t_1, \hat{\sigma}_{t_1,t_2}[\overline{s}_1],\ldots,\hat{\sigma}_{t_1,t_2}[\overline{s}_m]).$$

  Thus, we have to replace $s_j = x$ in the term t_1 by x, otherwise $t_2 \neq S^m(t_1, \hat{\sigma}_{t_1,t_2}[\overline{s}_1],\ldots,\hat{\sigma}_{t_1,t_2}[\overline{s}_m])$. That is, $\hat{\sigma}_{t_1,t_2}[\overline{s}_j] = x$, so $\overline{s}_1 = x$.

- $\overline{s}_i \notin X$ where $1 \le i \le m$: We suppose that $M^j(\overline{s}_i) \neq x$. Now, $\overline{s}_i = f(s'_1,\ldots,s'_m)$ where $s'_1,\ldots,s'_m \in W_{(m,n)}(X)$. By Lemma 3.10, $\hat{\sigma}_{t_1,t_2}[\overline{s}_i] = \overline{s}_i$. Then

  $$\overline{s}_i = f(s'_1,\ldots,s'_m) = S^m(t_1, \hat{\sigma}_{t_1,t_2}[s'_1],\ldots,\hat{\sigma}_{t_1,t_2}[s'_m]).$$

  Since $s_j \in X \setminus X_m$ we replace s_j in the term t_1 by x. Thus, $\overline{s}_i \neq S^m(t_1, \hat{\sigma}_{t_1,t_2}[s'_1],\ldots,\hat{\sigma}_{t_1,t_2}[s'_m])$ since $M^j(\overline{s}_i) \neq x$, this is a contradiction. Therefore $M^j(\overline{s}_i) = x$. ∎

Proposition 3.12. *Let σ_{t_1,t_2} be a generalized hypersubstitution of type (m,n), $op(t_1) = 1$, $op(t_2) > 1$, $t_1 = f(s_1,\ldots,s_m)$ and $t_2 = f(\overline{s}_1,\ldots,\overline{s}_m)$ where $s_1,\ldots,s_n \in X$, $\overline{s}_1,\ldots,\overline{s}_n \in W_{(m,n)}(X)$. Then the following conditions are equivalent:*

(1) *σ_{t_1,t_2} is idempotent;*

(2) *if $x_j \in var(t_1)$ where $1 \le j \le m$, then $s_j = x_j$ and the following conditions hold:*

   (i) *if $s_j = x \in X \setminus X_m$ where $1 \le j \le m$, then $\overline{s}_j = x$ and for each $\overline{s}_i \notin X$ where $1 \le i \le m$, $M^j(\overline{s}_i) = x$;*

   (ii) *if $s_j = s_l$ where $1 \le j,l \le m$, then $\overline{s}_j = \overline{s}_l$;*

   (iii) *For each subterm of t_2 which is not a variable $f(s'_1,\ldots,s'_m)$ where $s'_1,\ldots,s'_m \in W_{(m,n)}(X)$, if $s_j = s_l$ where $1 \le j,l \le m$, then $s'_j = s'_l$.*

Proof. (1) $\Rightarrow$ (2) : We now know that, by Lemma 3.9, $\mathrm{ops}(t_2) = \{f\}$. By Lemma 3.6, $x_j \in \mathrm{var}(t_1)$ implies $s_j = x_j$ where $1 \leq j \leq m$.

(i) This is clear by Lemma 3.11.

(ii) Assume that $s_j = s_l$. Then there is $x_k \in X_m$ such that $s_j = x_k = s_l$. Since σ_{t_1,t_2} is idempotent,

$$t_2 = f(\overline{s}_1, \ldots, \overline{s}_n) = S^m(t_1, \hat{\sigma}_{t_1,t_2}[\overline{s}_1], \ldots, \hat{\sigma}_{t_1,t_2}[\overline{s}_m]).$$

Thus, we have to replace s_j and s_l in the term t_1 by $\overline{s}_k$. That is, $\overline{s}_j = \overline{s}_k = \overline{s}_l$.

(iii) Similarly to (ii).

(2) $\Rightarrow$ (1) : It is clear that $\hat{\sigma}_{t_1,t_2}[t_1] = t_1$. We have to show that $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$, indeed,

$$f(\overline{s}_1, \ldots, \overline{s}_m) = S^m(t_1, \hat{\sigma}_{t_1,t_2}[\overline{s}_1], \ldots, \hat{\sigma}_{t_1,t_2}[\overline{s}_m]).$$

That is, we must replace s_i in the term t_1 by $\hat{\sigma}_{t_1,t_2}[\overline{s}_i]$ for each $i \in \{1, \ldots, m\}$.

- If $s_i = x_k$ where $1 \leq i \leq m$ and $x_k \in X_m$, then we replace s_i and s_k in the term t_1 by $\overline{s}_k = \hat{\sigma}_{t_1,t_2}[\overline{s}_k]$.
- If $s_i = x \in X \setminus X_m$ where $1 \leq i \leq m$, then we consider t_2. Since $\mathrm{op}(t_2) > 1$, $\overline{s}_j \notin X$ for some $1 \leq j \leq m$. Thus, by assumption, $M^i(\overline{s}_j) = x$. And $\overline{s}_k = x$ if $\overline{s}_k \in X$ where $1 \leq k \leq m$. It follows that we have to replace $\hat{\sigma}_{t_1,t_2}[\overline{s}_j] = \overline{s}_j$ and $\hat{\sigma}_{t_1,t_2}[\overline{s}_k] = \overline{s}_k$. Simply calculate we have $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$.

Therefore, σ_{t_1,t_2} is idempotent. ∎

On the other hand we obtain the following results.

Lemma 3.13. *Let σ_{t_1,t_2} be a generalized hypersubstitution of type (m,n). If $op(t_1) > 1$, $op(t_2) = 1$ and σ_{t_1,t_2} is idempotent, then $firstop(t_2) = g$.*

Lemma 3.14. *Let σ_{t_1,t_2} be a generalized hypersubstitution of type (m,n), $op(t_1)>1$, $op(t_2) = 1$ and $t_2 = g(s_1, \ldots, s_n)$ where $s_1, \ldots, s_n \in X$. Then the following conditions are equivalent:*

(1) $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$;

(2) *$firstop(t_2) = g$ and if $x_j \in var(t_1)$ where $1 \leq j \leq n$, then $s_j = x_j$.*

Proposition 3.15. *Let σ_{t_1,t_2} be a generalized hypersubstitution of type (m,n), $op(t_1) > 1$, $op(t_2) = 1$, $t_1 = f(s'_1, \ldots, s'_m)$ and $t_2 = g(s_1, \ldots, s_n)$ where $s'_1, \ldots, s'_m \in W_{(m,n)}(X)$, $s_1, \ldots, s_n \in X$. Then the following conditions are equivalent:*

(1) *σ_{t_1,t_2} is idempotent;*

(2) *the following conditions hold*

(i) *if $x_j \in var(t_1)$ where $1 \leq j \leq m$, then $s'_j = x_j$;*
(ii) *if $x_j \in var(t_2)$ where $1 \leq j \leq n$, then $s_j = x_j$.*

Lemma 3.16. *Let σ_{t_1,t_2} be an idempotent generalized hypersubstitution of type (m,n), $op(t_1) > 1$, $op(t_2) = 1$ and $t_1 = g(s_1,\ldots,s_n)$ where $s_1,\ldots,s_n \in W_{(m,n)}(X)$. If $x_j \in var(t_2)$ where $1 \leq j \leq n$, then $firstop(s_j) = g$ or $s_j \in X$.*

Lemma 3.17. *Let σ_{t_1,t_2} be an idempotent generalized hypersubstitution of type (m,n), $op(t_1) > 1$, $op(t_2) = 1$, $t_1 = g(s_1,\ldots,s_n)$ and $t_2 = g(x_{j_1},\ldots,x_{j_n})$ where $x_{j_1},\ldots,x_{j_n} \in X$, $s_1,\ldots,s_n \in W_{(m,n)}(X)$. Then $ops(t_1) = \{g\}$.*

Lemma 3.18. *Let σ_{t_1,t_2} be a generalized hypersubstitution of type (m,n), $op(t_1)>1$, $op(t_2) = 1$, $t_1 = g(s_1,\ldots,s_n)$ and $t_2 = g(x_{j_1},\ldots,x_{j_n})$ where $x_{j_1},\ldots,x_{j_n} \in X$, $s_1,\ldots,s_n \in W_{(m,n)}(X)$. If σ_{t_1,t_2} is idempotent, then $\hat{\sigma}_{t_1,t_2}[s_k] = s_k$ for all $k \in \{1,\ldots,n\}$.*

Lemma 3.19. *Let σ_{t_1,t_2} be an idempotent generalized hypersubstitution of type (m,n), $op(t_1) > 1$, $op(t_2) = 1$, $t_1 = g(s_1,\ldots,s_n)$ and $t_2 = g(\overline{s}_1,\ldots,\overline{s}_n)$ where $\overline{s}_1,\ldots,\overline{s}_n \in X$, $s_1,\ldots,s_n \in W_{(m,n)}(X)$. If $\overline{s}_j = x \in X \setminus X_n$ where $1 \leq j \leq n$, then $s_j = x$ whenever $s_j \in X$ and $M^j(s_i) = x$ whenever $s_i \notin X$ where $1 \leq i \leq n$.*

Proposition 3.20. *Let σ_{t_1,t_2} be a generalized hypersubstitution of type (m,n), $op(t_1) > 1$, $op(t_2) = 1$, $t_1 = g(s_1,\ldots,s_n)$ and $t_2 = g(\overline{s}_1,\ldots,\overline{s}_n)$ where $\overline{s}_1,\ldots,\overline{s}_n \in X$, $s_1,\ldots,s_n \in W_{(m,n)}(X)$. Then the following conditions are equivalent:*

(1) *σ_{t_1,t_2} is idempotent;*

(2) *if $x_j \in var(t_2)$ where $1 \leq j \leq n$, then $s_j = x_j$ and the following conditions hold:*

   (i) *if $\overline{s}_j = x \in X \setminus X_n$ where $1 \leq j \leq n$, then $s_j = x$ and for each $s_i \notin X$ where $1 \leq i \leq n$, $M^j(s_i) = x$;*

   (ii) *if $\overline{s}_j = \overline{s}_l$ where $1 \leq j,l \leq n$, then $s_j = s_l$;*

   (iii) *For each subterm of t_1 which is not a variable $g(s'_1,\ldots,s'_n)$ where $s'_1,\ldots,s'_n \in W_{(m,n)}(X)$, if $\overline{s}_j = \overline{s}_l$ where $1 \leq j,l \leq n$, then $s'_j = s'_l$.*

Case 3. $\mathbf{op}(t_1) > 1$ and $\mathbf{op}(t_2) > 1$.

Assume that $\mathrm{op}(t_1) > 1$ and $\mathrm{op}(t_2) > 1$. In this case, if σ_{t_1,t_2} is idempotent, then the case $\mathrm{firstop}(t_1) = g$ and $\mathrm{firstop}(t_2) = f$ is impossible. Therefore, we consider only the following cases:

1. $\mathrm{firstop}(t_1) = f$ and $\mathrm{firstop}(t_2) = f$;
2. $\mathrm{firstop}(t_1) = g$ and $\mathrm{firstop}(t_2) = g$ and
3. $\mathrm{firstop}(t_1) = f$ and $\mathrm{firstop}(t_2) = g$.

Proposition 3.21. *Let σ_{t_1,t_2} be a generalized hypersubstitution of type (m,n), $op(t_1) > 1$, $op(t_2) > 1$, $t_1 = f(s_1,\ldots,s_m)$ and $t_2 = f(s'_1,\ldots,s'_m)$ where $s_1,\ldots,s_m$, $s'_1,\ldots,s'_m \in W_{(m,n)}(X)$. Then the following conditions are equivalent:*

(1) *σ_{t_1,t_2} is idempotent;*

(2) *if* $x_j \in var(t_1)$ *where* $1 \leq j \leq m$, *then* $s_j = x_j$ *and* $t_2 = S^m(t_1, k_1^1, \ldots, k_m^1)$
where $k_j^1 \in X$ *or* $k_j^1 = S^m(t_1, k_1^2, \ldots, k_m^2)$
where $k_j^2 \in X$ *or* $k_j^2 = S^m(t_1, k_1^3, \ldots, k_m^3)$
$\vdots$
where $k_j^{l-1} \in X$ *or* $k_j^{l-1} = S^m(t_1, k_1^l, \ldots, k_m^l)$
where $k_j^l \in X$ *for some* $l \in \mathbb{N}$.

Proof. (1) $\Rightarrow$ (2) : Assume that $x_j \in \mathrm{var}(t_1)$ where $1 \leq j \leq m$. Suppose that $s_j \neq x_j$. If $s_j \in W_{(m,n)}(X) \setminus X$, then

$$\mathrm{op}(t_1) < \mathrm{op}(S^m(\sigma_{t_1,t_2}(f), \hat{\sigma}_{t_1,t_2}[s_1], \ldots, \hat{\sigma}_{t_1,t_2}[s_m])).$$

Otherwise, we consider,

$$t_1 = S^m(t_1\hat{\sigma}_{t_1,t_2}[s_1], \ldots, \hat{\sigma}_{t_1,t_2}[s_m]).$$

Since $x_j \in \mathrm{var}(t_1)$, we replace x_j in the term t_1 by $\hat{\sigma}_{t_1,t_2}[s_j] \neq x_j$. This implies

$$S^m(t_1\hat{\sigma}_{t_1,t_2}[s_1], \ldots, \hat{\sigma}_{t_1,t_2}[s_m]) \neq t_1,$$

this is a contradiction.

Hence, $s_j = x_j$.
Now, $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$,

$$t_2 = S^m(t_1\hat{\sigma}_{t_1,t_2}[s_1'], \ldots, \hat{\sigma}_{t_1,t_2}[s_m']).$$

Since $x_j \in \mathrm{var}(t_1)$, we have to replace s_j' in the term t_1 by $\hat{\sigma}_{t_1,t_2}[s_j']$. That is, $\hat{\sigma}_{t_1,t_2}[s_j'] = s_j'$. Hence, s_j' is a variable or $\mathrm{firstop}(s_j') = f$.

In case that $s_j' \notin X$ and $\mathrm{firstop}(s_j') = g$ is impossible.

Indeed, let $s_j' = g(s_1'', \ldots, s_n'')$ where $s_1'', \ldots, s_n'' \in W_{(m,n)}(X)$.

Then $\hat{\sigma}_{t_1,t_2}[s_j'] = S^n(t_2, \hat{\sigma}_{t_1,t_2}[s_1''], \ldots, \hat{\sigma}_{t_1,t_2}[s_n''])$, so $\mathrm{firstop}(\hat{\sigma}_{t_1,t_2}[s_j']) = f$ which contradicts to $\hat{\sigma}_{t_1,t_2}[s_j'] = s_j'$.

Therefore, $s_j' = f(s_1'', \ldots, s_m'')$ where $s_1'', \ldots, s_m'' \in W_{(m,n)}(X)$.

Now, we obtain

$$t_2 = S^m(t_1, \hat{\sigma}_{t_1,t_2}[s_1'], \ldots, \hat{\sigma}_{t_1,t_2}[s_{j-1}'], S^m(t_1, \hat{\sigma}_{t_1,t_2}[s_1''], \ldots, \hat{\sigma}_{t_1,t_2}[s_m'']), \hat{\sigma}_{t_1,t_2}[s_{j+1}'], \ldots, \hat{\sigma}_{t_1,t_2}[s_m']).$$

Similarly, $s_j'' \in X$ or $\mathrm{firstop}(s_j'') = f$. We do this step and this procedure will stop after finitely many steps. Therefore,

$$t_2 = S^m(t_1, k_1^1, \ldots, k_m^1)$$

where $k_j^1 \in X$ or $k_j^1 = S^m(t_1, k_1^2, \ldots, k_m^2)$
where $k_j^2 \in X$ or $k_j^2 = S^m(t_1, k_1^3, \ldots, k_m^3)$

$\vdots$
where $k_j^{l-1} \in X$ or $k_j^{l-1} = S^m(t_1, k_1^l, \ldots, k_m^l)$
where $k_j^l \in X$ for some $l \in \mathbb{N}$.

(2) $\Rightarrow$ (1) : Consider

$$\hat{\sigma}_{t_1,t_2}[t_1] = S^m(t_1, \hat{\sigma}_{t_1,t_2}[s_1], \ldots, \hat{\sigma}_{t_1,t_2}[s_m]).$$

Then we replace x_j in the term t_1 by $\hat{\sigma}_{t_1,t_2}[s_j]$. Since $s_j = x_j$, $\hat{\sigma}_{t_1,t_2}[s_j] = x_j$. Thus, $\hat{\sigma}_{t_1,t_2}[t_1] = t_1$. Next we show that $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$. Suppose that the construction of a term t_2 terminates at the step $l \in \mathbb{N}$. This means that

$$k_j^{l-1} = S^m(t_1, k_1^l, \ldots, k_{j-1}^l, x_j, k_{j+1}^l, \ldots, k_m^l),$$

so $\hat{\sigma}_{t_1,t_2}[k_j^{l-1}] = k_j^{l-1}$. Then

$$k_j^{l-2} = S^m(t_1, k_1^{l-1}, \ldots, k_j^{l-1}, \ldots, k_m^{l-1}).$$

We replace x_j in the term t_1 by k_j^{l-1}, and so $\hat{\sigma}_{t_1,t_2}[k_j^{l-2}] = k_j^{l-2}$. We do this steps, thus, $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$. ∎

Similarly, we can prove the following proposition.

Proposition 3.22. *Let σ_{t_1,t_2} be a generalized hypersubstitution of type (m,n), $op(t_1) > 1$, $op(t_2) > 1$, $t_1 = g(s_1, \ldots, s_n)$ and $t_2 = g(s'_1, \ldots, s'_n)$ where $s_1, \ldots, s_n$, $s'_1, \ldots, s'_n \in W_{(m,n)}(X)$. Then the following conditions are equivalent:*

(1) *σ_{t_1,t_2} is idempotent;*

(2) *if $x_j \in var(t_2)$ where $1 \leq j \leq n$, then $s_j = x_j$ and $t_1 = S^n(t_2, k_1^1, \ldots, k_n^1)$*
where $k_j^1 \in X$ or $k_j^1 = S^m(t_1, k_1^2, \ldots, k_m^2)$
where $k_j^2 \in X$ or $k_j^2 = S^m(t_1, k_1^3, \ldots, k_m^3)$
$\vdots$
where $k_j^{l-1} \in X$ or $k_j^{l-1} = S^m(t_1, k_1^l, \ldots, k_m^l)$
where $k_j^l \in X$ for some $l \in \mathbb{N}$.

For the last case, we obtain the following.

Proposition 3.23. *Let σ_{t_1,t_2} be a generalized hypersubstitution of type (m,n), $op(t_1) > 1$, $op(t_2) > 1$, $t_1 = f(s_1, \ldots, s_m)$ and $t_2 = g(s'_1, \ldots, s'_n)$ where $s_1, \ldots, s_m$, $s'_1, \ldots, s'_n \in W_{(m,n)}(X)$. Then the following conditions are equivalent:*

(1) *σ_{t_1,t_2} is idempotent.*

(2) *the following statements hold:*

(2.1) *if $x_j \in var(t_1)$ where $1 \leq j \leq m$, then $s_j = x_j$;*

(2.2) *if* $x_k \in var(t_2)$ *where* $1 \leq k \leq n$, *then* $s'_k = x_k$.

Proof. (1) $\Rightarrow$ (2) : Assume that $x_j \in \mathrm{var}(t_1)$ where $1 \leq j \leq m$. Suppose that $s_j \neq x_j$. If $s_j \in W_{(m,n)}(X) \setminus X$, then

$$\mathrm{op}(t_1) < \mathrm{op}(S^m(\sigma_{t_1,t_2}(f), \hat{\sigma}_{t_1,t_2}[s_1], \ldots, \hat{\sigma}_{t_1,t_2}[s_m])).$$

Otherwise, we consider,

$$t_1 = S^m(t_1\hat{\sigma}_{t_1,t_2}[s_1], \ldots, \hat{\sigma}_{t_1,t_2}[s_m]).$$

Since $x_j \in \mathrm{var}(t_1)$, we replace x_j in the term t_1 by $\hat{\sigma}_{t_1,t_2}[s_j] \neq x_j$. This implies

$$S^m(t_1\hat{\sigma}_{t_1,t_2}[s_1], \ldots, \hat{\sigma}_{t_1,t_2}[s_m]) \neq t_1,$$

this is a contradiction. Hence, $s_j = x_j$. Similarly, we can prove that if $x_k \in \mathrm{var}(t_2)$ where $1 \leq k \leq n$, then $s'_k = x_k$.

(2) $\Rightarrow$ (1) : Consider

$$\hat{\sigma}_{t_1,t_2}[t_1] = S^m(t_1, \hat{\sigma}_{t_1,t_2}[s_1], \ldots, \hat{\sigma}_{t_1,t_2}[s_m]).$$

Then we replace x_j in the term t_1 by $\hat{\sigma}_{t_1,t_2}[s_j]$. Since $s_j = x_j$, $\hat{\sigma}_{t_1,t_2}[s_j] = x_j$. Thus, $\hat{\sigma}_{t_1,t_2}[t_1] = t_1$. Similarly, we can show that $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$. ∎

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SOME IDENTITIES FOR DEGENERATE FROBENIUS-EULER NUMBERS ARISING FROM NONLINEAR DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we derive nonlinear differential equations from the generating function of degenerate Frobenius-Euler numbers which originate from Carlitz degenerate Euler numbers. In addition, we give some explicit identities for degenerate Frobenius-Euler numbers arising from our nonlinear differential equations.

Key words and phrases: degenerate Frobenius-Euler numbers, nonlinear differential equations.

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1. Introduction

For $u\,(\neq 1) \in \mathbb{C}$, the Frobenius-Euler numbers are defined by the generating function

$$\frac{1-u}{e^t-u} = \sum_{n=0}^{\infty} H_n(u)\frac{t^n}{n!}, \quad (\text{see [2], [12], [13]}). \tag{1.1}$$

In particular, $u=-1$, we have

$$\sum_{n=0}^{\infty} H_n(-1)\frac{t^n}{n!} = \frac{2}{e^t+1} = \sum_{n=0}^{\infty} E_n\frac{t^n}{n!}, \tag{1.2}$$

where E_n are the Euler numbers (see [6], [9]).

Thus, by (1.2), we get $H_n(-1) = E_n (n \geq 0)$. In [3], L. Carlitz considered the degenerate Euler numbers which are given by the generating function

$$\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1} = \sum_{n=0}^{\infty} \mathcal{E}_{n,\lambda}\frac{t^n}{n!}. \tag{1.3}$$

Note that

$$\lim_{\lambda\to 0} \mathcal{E}_{n,\lambda} = E_n \ (n \geq 0).$$

Recently, Kim and Kim derived nonlinear differential equations from the generating function of degenerate Euler numbers and gave some new and explicit identities for Carlitz degenerate Euler numbers arising from those nonlinear differential equations (see [9]).

In [6], the nonlinear Changhee differential equations, which are related to Euler and degenerate Euler numbers, were introduced by Kim and Kim. They also gave some new explicit relationship between Changhee and Euler numbers arising from those differential equations.

Now, we consider the degenerate Frobenius-Euler numbers which originates from Carlitz degenerate Euler numbers and are given by the generating function:

$$\frac{1-u}{(1+\lambda t)^{\frac{1}{\lambda}}-u} = \sum_{n=0}^{\infty} h_{n,\lambda}(u)\frac{t^n}{n!}, \quad (u\,(\neq 1) \in \mathbb{C}), \quad (\text{see [7]}). \tag{1.4}$$

In the special case $u=-1$, we note that

$$h_{n,\lambda}(-1) = \mathcal{E}_{n,\lambda}, \ (n \geq 0).$$

It is not difficult to show that

$$\lim_{\lambda\to 0} h_{n,\lambda}(u) = H_n(u), \ (n \geq 0).$$

Recently, several authors have studied degenerate Euler numbers and degenerate Frobenius-Euler numbers (see [1]–[14]).

In this paper, we derive nonlinear differential equations from the generating function of degenerate Frobenius-Euler numbers. In addition, we give some explicit identities for degenerate Frobenius-Euler numbers arising from our nonlinear differential equations.

2. Some identities for degenerate Frobenius-Euler numbers arising from nonlinear differential equations

Let

$$F = F(t;\lambda,u) = \frac{1}{(1+\lambda t)^{\frac{1}{\lambda}} - u}, \quad (u \neq 1). \tag{2.1}$$

Then, by (2.1), we get

$$F^{(1)} = \frac{dF}{dt}(t;\lambda,u) = \frac{(-1)}{1+\lambda t}\left(F + uF^2\right). \tag{2.2}$$

Thus, by (2.2), we have

$$uF^2 = -F - (1+\lambda t)F^{(1)}. \tag{2.3}$$

Taking the derivative of (2.3) with respect to t, we get

$$2!u^2F^3 = (-1)^2\left\{2F + (3+\lambda)(1+\lambda t)F^{(1)} + (1+\lambda t)^2F^{(2)}\right\}, \tag{2.4}$$

and

$$\begin{aligned} 3!u^3F^4 &= (-1)^3\left\{6F + (\lambda^2+6\lambda+11)(1+\lambda t)F^{(1)}\right. \\ &\quad \left. + (3\lambda+6)(1+\lambda t)^2F^{(2)} + (1+\lambda t)^3F^{(3)}\right\}. \end{aligned} \tag{2.5}$$

So, we are led to put

$$N!u^NF^{N+1} = (-1)^N\sum_{i=0}^{N} a_i(N,\lambda)(1+\lambda t)^iF^{(i)}, \quad (N = 0,1,2,\ldots), \tag{2.6}$$

where $F^{(i)} = \left(\frac{d}{dt}\right)^i F(t;\lambda,u)$.

Let us take the derivative of (2.6) with respect to t. Then we have

$$\begin{aligned} (N+1)!u^NF^NF^{(1)} &= (-1)^N\sum_{i=0}^{N} a_i(N,\lambda)\,i\,(1+\lambda t)^{i-1}\lambda F^{(i)} \\ &\quad + (-1)^N\sum_{i=0}^{N} a_i(N,\lambda)(1+\lambda t)^iF^{(i+1)}. \end{aligned} \tag{2.7}$$

By (2.2) and (2.7), we easily get

$$\begin{aligned} &(N+1)!u^NF^N(F+uF^2) \\ &= (-1)^{N+1}\left\{\sum_{i=0}^{N} i\lambda a_i(N,\lambda)(1+\lambda t)^iF^{(i)} + \sum_{i=0}^{N} a_i(N,\lambda)(1+\lambda t)^{i+1}F^{(i+1)}\right\}. \end{aligned} \tag{2.8}$$

From (2.6) and (2.8), we can derive the following equation:

$$\begin{aligned}
&(N+1)!u^{N+1}F^{N+2}\\
&=-(N+1)N!u^{N}F^{N+1}\\
&+(-1)^{N+1}\left\{\sum_{i=0}^{N} i\lambda a_i(N,\lambda)(1+\lambda t)^i F^{(i)}+\sum_{i=0}^{N} a_i(N,\lambda)(1+\lambda t)^{i+1}F^{(i+1)}\right\}\\
&=(-1)^{N+1}\left\{\sum_{i=0}^{N}(N+1+i\lambda)a_i(N,\lambda)(1+\lambda t)^i F^{(i)}\right.\\
&\left.+\sum_{i=1}^{N+1} a_{i-1}(N,\lambda)(1+\lambda t)^i F^{(i)}\right\}.
\end{aligned} \tag{2.9}$$

On the other hand, by replacing N by $N+1$ in (2.6), we get

$$(N+1)!u^{N+1}F^{N+2}=(-1)^{N+1}\sum_{i=0}^{N+1} a_i(N+1,\lambda)(1+\lambda t)^i F^{(i)}. \tag{2.10}$$

By comparing the coefficients on both sides of (2.9) and (2.10), we have

$$a_0(N+1,\lambda)=(N+1)a_0(N,\lambda),\quad a_{N+1}(N+1,\lambda)=a_N(N,\lambda), \tag{2.11}$$

and

$$a_i(N+1,\lambda)=a_{i-1}(N,\lambda)+(N+1+i\lambda)a_i(N,\lambda),\quad (1\le i\le N). \tag{2.12}$$

From $F=a_0(0,\lambda)F$ in (2.6), we have

$$a_0(0,\lambda)=1. \tag{2.13}$$

By (2.3) and (2.6), we easily get

$$-F-(1+\lambda t)F^{(1)}=uF^2=-a_0(1,\lambda)F-a_1(1,\lambda)(1+\lambda t)F^{(1)}. \tag{2.14}$$

Comparing the coefficients on both sides of (2.14), we obtain

$$a_0(1,\lambda)=1,\quad a_1(1,\lambda)=1. \tag{2.15}$$

From (2.11), we easily note that

$$\begin{aligned}
&a_0(N+1,\lambda)\\
&=(N+1)a_0(N,\lambda)=(N+1)Na_0(N-1,\lambda)=\cdots=(N+1)!,
\end{aligned} \tag{2.16}$$

and

$$\begin{aligned}
&a_{N+1}(N+1,\lambda)\\
&=a_N(N,\lambda)=a_{N-1}(N-1,\lambda)=\cdots=a_1(1,\lambda)=a_0(0,\lambda)=1.
\end{aligned} \tag{2.17}$$

For $i = 1, 2, 3$ in (2.12), we have

$$a_1(N+1,\lambda) = \sum_{k=0}^{N} (N+1+\lambda)_k \, a_0(N-k,\lambda), \tag{2.18}$$

$$a_2(N+1,\lambda) = \sum_{k=0}^{N-1} (N+1+2\lambda)_k \, a_1(N-k,\lambda), \tag{2.19}$$

and

$$a_3(N+1,\lambda) = \sum_{k=0}^{N-2} (N+1+3\lambda)_k \, a_2(N-k,\lambda), \tag{2.20}$$

where

$$(x)_n = x(x-1)\cdots(x-n+1), \quad (n \geq 1), \quad (x)_0 = 1.$$

Continuing this process, we note that

$$a_i(N+1,\lambda) = \sum_{k=0}^{N-i+1} (N+1+i\lambda)_k \, a_{i-1}(N-k,\lambda). \tag{2.21}$$

Now, we give explicit expressions for $a_i(N+1)$.

From (2.18), we note that

$$\begin{aligned} a_1(N+1,\lambda) &= \sum_{k_1=0}^{N} (N+1+\lambda)_{k_1} \, a_0(N-k_1,\lambda) \\ &= \sum_{k_1=0}^{N} (N+1+\lambda)_{k_1} (N-k_1)!, \end{aligned} \tag{2.22}$$

$$\begin{aligned} a_2(N+1,\lambda) &= \sum_{k_2=0}^{N-1} (N+1+2\lambda)_{k_2} \, a_1(N-k_2,\lambda) \\ &= \sum_{k_2=0}^{N-1} \sum_{k_1=0}^{N-1-k_2} (N+1+2\lambda)_{k_2} (N-k_2+\lambda)_{k_1} (N-k_2-k_1-1)!, \end{aligned} \tag{2.23}$$

and

$$\begin{aligned} a_3(N+1,\lambda) &= \sum_{k_3=0}^{N-2} \sum_{k_2=0}^{N-2-k_3} \sum_{k_1=0}^{N-k_3-k_2-2} (N+1+3\lambda)_{k_3} \\ &\quad \times (N-k_3+2\lambda)_{k_2} (N-k_3-k_2-1+\lambda)_{k_1} \\ &\quad \times (N-k_3-k_2-k_1-2)!. \end{aligned} \tag{2.24}$$

So we can deduce that, for $1 \leq i \leq N$,

$$\begin{aligned} & a_i\left(N+1,\lambda\right) \\ &= \sum_{k_i=0}^{N-i+1} \sum_{k_{i-1}=0}^{N-i+1-k_i} \cdots \sum_{k_1=0}^{N-i+1-k_i-\cdots-k_2} \prod_{j=1}^{i} \left(N+j+1-i-\sum_{l=j+1}^{i} k_l + j\lambda\right)_{k_j} \\ &\times \left(N+1-i-\sum_{l=1}^{i} k_l\right)!. \end{aligned} \tag{2.25}$$

Therefore, we obtain the following theorem.

Theorem 1. *The nonlinear differential equations*

$$N!u^N F^{N+1} = (-1)^N \sum_{i=0}^{N} a_i\left(N,\lambda\right)\left(1+\lambda t\right)^i F^{(i)}, \quad (N=1,2,\ldots),$$

have a solution

$$F = F\left(t;\lambda,u\right) = \frac{1}{\left(1+\lambda t\right)^{\frac{1}{\lambda}} - u},$$

where

$$\begin{aligned} a_0\left(N,\lambda\right) &= N!, \\ a_i\left(N,\lambda\right) &= \sum_{k_i=0}^{N-i} \sum_{k_{i-1}=0}^{N-i-k_i} \cdots \sum_{k_1=0}^{N-i-k_i-\cdots k_2} \left(N-i-\sum_{l=1}^{i} k_l\right)! \\ &\quad \times \prod_{j=1}^{i} \left(N+j-i-\sum_{l=j+1}^{i} k_l + j\lambda\right)_{k_j}. \end{aligned}$$

We recall that the degenerate Frobenius-Euler numbers, $h_{n,\lambda}\left(u\right)$, $(n \geq 0)$, are defined by the generating function

$$\frac{1-u}{\left(1+\lambda t\right)^{\frac{1}{\lambda}} - u} = \sum_{n=0}^{\infty} h_{n,\lambda}\left(u\right) \frac{t^n}{n!}.$$

Also, the degenerate Frobenius-Euler numbers, $h_{n,\lambda}^{(r)}\left(u\right)$, of order r are given by the genreating function

$$\left(\frac{1-u}{\left(1+\lambda t\right)^{\frac{1}{\lambda}} - u}\right)^r = \sum_{n=0}^{\infty} h_{n,\lambda}^{(r)}\left(u\right) \frac{t^n}{n!}, \quad \left(u \in \mathbb{C}, u \neq 1\right).$$

From Theorem 1, we have

$$\begin{aligned} & N!u^N \left(1-u\right)^{N+1} F^{N+1} \\ &= \left(-1\right)^N \left(1-u\right)^N \sum_{i=0}^{N} a_i\left(N,\lambda\right)\left(1+\lambda t\right)^i \left(\frac{d}{dt}\right)^i \left(\frac{1-u}{\left(1+\lambda t\right)^{\frac{1}{\lambda}} - u}\right). \end{aligned} \tag{2.26}$$

The LHS of (2.26) is given by

$$N!u^N\left(\frac{1-u}{(1+\lambda t)^{\frac{1}{\lambda}}-u}\right)^{N+1}=N!u^N\sum_{n=0}^{\infty}h_{n,\lambda}^{(N+1)}(u)\frac{t^n}{n!}. \tag{2.27}$$

It is easy to show that

$$\begin{aligned}
&(-1)^N(1-u)^N\sum_{i=0}^{N}a_i(N,\lambda)(1+\lambda t)^i\left(\frac{d}{dt}\right)^i\left(\frac{1-u}{(1+\lambda t)^{\frac{1}{\lambda}}-u}\right)\\
&=(-1)^N(1-u)^N\sum_{i=0}^{N}a_i(N,\lambda)\sum_{k=0}^{\infty}(i)_k\lambda^k\frac{t^k}{k!}\sum_{l=0}^{\infty}h_{l+i,\lambda}(u)\frac{t^l}{l!}\\
&=(-1)^N(1-u)^N\sum_{i=0}^{N}a_i(N,\lambda)\sum_{n=0}^{\infty}\sum_{k=0}^{n}(i)_k\lambda^k h_{n-k+i,\lambda}(u)\binom{n}{k}\frac{t^n}{n!}\\
&=(-1)^N(1-u)^N\sum_{n=0}^{\infty}\sum_{i=0}^{N}\sum_{k=0}^{n}\binom{n}{k}(i)_k\lambda^k a_i(N,\lambda)h_{n-k+i,\lambda}(u)\frac{t^n}{n!}.
\end{aligned} \tag{2.28}$$

By (2.26), (2.27) and (2.28), we get

$$\begin{aligned}
&N!u^N h_{n,\lambda}^{(N+1)}(u)\\
&=(-1)^N(1-u)^N\sum_{i=0}^{N}\sum_{k=0}^{n}\binom{n}{k}(i)_k\lambda^k a_i(N,\lambda)h_{n-k+i,\lambda}(u).
\end{aligned} \tag{2.29}$$

Therefore, by (2.29), we obtain the following theorem.

Theorem 2. *For n, $N=0,1,2,\ldots$, we have*

$$\begin{aligned}
&N!u^N h_{n,\lambda}^{(N+1)}(u)\\
&=(-1)^N(1-u)^N\sum_{i=0}^{N}\sum_{k=0}^{n}\binom{n}{k}(i)_k\lambda^k a_i(N,\lambda)h_{n-k+i,\lambda}(u),
\end{aligned}$$

where

$$\begin{aligned}
a_0(N,\lambda)&=N!\\
a_i(N,\lambda)&=\sum_{k_i=0}^{N-i}\sum_{k_{i-1}=0}^{N-i-k_i}\cdots\sum_{k_1=0}^{N-i-k_i-\cdots-k_2}\left(N-i-\sum_{l=1}^{i}k_l\right)!\\
&\quad\times\left(\prod_{j=1}^{i}\left(N+j-i-\sum_{l=j+1}^{i}k_l+j\lambda\right)_{k_j}\right).
\end{aligned}$$

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$(\in, \in \vee q)$-OSMOTIC VALUES WITH APPLICATIONS IN BCK/BCI-ALGEBRAS

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Abstract. The notions of $(\in, \in \vee q)$-osmotic S-value and $(\overline{\in}, \overline{\in} \vee \overline{q})$-osmotic S-value are introduced, and related properties are investigated. Relations between osmotic S-value and $(\in, \in \vee q)$-osmotic S-value ($(\overline{\in}, \overline{\in} \vee \overline{q})$-osmotic S-value) are considered. Conditions for a number in the unit interval $[0, 1]$ to be an osmotic S-value (resp., $(\in, \in \vee q)$-osmotic S-value and $(\overline{\in}, \overline{\in} \vee \overline{q})$-osmotic S-value) are provided. Conditions for level sets to be S-energetic set and/or subalgebra are discussed.

Keywords: $(\in, \in \vee q)$-osmotic S-value, $(\overline{\in}, \overline{\in} \vee \overline{q})$-osmotic S-value, S-energetic set.

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1. Introduction

To make the computers simulate beings in dealing with certainty and uncertainty in information is an important task of artificial intelligence. The role of logic in mathematics and computer science is twofold – as a tool for applications in both areas, and a technique for laying the foundations. Non-classical logic including many-valued logic, fuzzy logic, etc., takes the advantage of classical logic to handle information with various facets of uncertainty (see [8] for generalized theory of uncertainty), such as fuzziness, randomness etc. Non-classical logic has become a formal and useful tool for computer science to deal with fuzzy information and

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uncertain information. Among all kinds of uncertainties, incomparability is an important one which can be encountered in our life. The algebraic structure of set theories dealing with uncertainties has been studied by some authors. The most appropriate theory for dealing with uncertainties is the theory of fuzzy sets developed by Zadeh [7]. Murali [5] proposed a definition of a fuzzy point belonging to fuzzy subset under a natural equivalence on fuzzy subset. The idea of quasi-coincidence of a fuzzy point with a fuzzy set, which is mentioned in [6], played a vital role to generate some different types of fuzzy subsets. Jun et al. [3] introduced the notions of energetic (resp. right vanished, right stable) subsets and (anti) permeable values in BCK/BCI-algebras, and investigated several properties.

In this paper, we introduce the notions of $(\in, \in \vee q)$-osmotic S-value and $(\overline{\in}, \overline{\in} \vee \overline{q})$-osmotic S-value, and investigate related properties. We consider relations between osmotic S-value and $(\in, \in \vee q)$-osmotic S-value ($(\overline{\in}, \overline{\in} \vee \overline{q})$-osmotic S-value), and provide conditions for a number in the unit interval $[0, 1]$ to be an osmotic S-value (resp., $(\in, \in \vee q)$-osmotic S-value and $(\overline{\in}, \overline{\in} \vee \overline{q})$-osmotic S-value). We discuss conditions for level sets to be S-energetic set and/or subalgebra.

2. Preliminaries

The BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki and has been extensively investigated by many researchers.

An algebra $(X; *, 0)$ of type $(2, 0)$ is called a *BCI-algebra* if it satisfies the following conditions:

(I) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = 0)$,

(II) $(\forall x, y \in X)\ ((x * (x * y)) * y = 0)$,

(III) $(\forall x \in X)\ (x * x = 0)$,

(IV) $(\forall x, y \in X)\ (x * y = 0, y * x = 0 \Rightarrow x = y)$.

If a BCI-algebra X satisfies the following identity:

(V) $(\forall x \in X)\ (0 * x = 0)$,

then X is called a *BCK-algebra.* Any BCK/BCI-algebra X satisfies the following axioms:

$$(\forall x \in X)\,(x * 0 = x)\,, \tag{2.1}$$

$$(\forall x, y, z \in X)\,(x \le y \Rightarrow x * z \le y * z,\ z * y \le z * x)\,, \tag{2.2}$$

$$(\forall x, y, z \in X)\,((x * y) * z = (x * z) * y)\,, \tag{2.3}$$

$$(\forall x, y, z \in X)\,((x * z) * (y * z) \le x * y) \tag{2.4}$$

where $x \le y$ if and only if $x * y = 0$. A nonempty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$.

We refer the reader to the books [1, 4] for further information regarding BCK/BCI-algebras.

A fuzzy set μ in a set X of the form

$$\mu(y) := \begin{cases} t \in (0,1] & \text{if } y = x, \\ 0 & \text{if } y \neq x, \end{cases}$$

is said to be a *fuzzy point* with support x and value t and is denoted by x_t.

For a fuzzy point x_t and a fuzzy set μ in a set X, Pu and Liu [6] introduced the symbol $x_t \,\alpha\, \mu$, where $\alpha \in \{\in, q\}$. To say that $x_t \in \mu$ (resp. $x_t \,q\, \mu$), we mean $\mu(x) \geq t$ (resp. $\mu(x) + t > 1$), and in this case, x_t is said to *belong to* (resp. *be quasi-coincident with*) a fuzzy set μ. To say that $x_t \in \vee\, q\, \mu$, we mean $x_t \in \mu$ or $x_t \,q\, \mu$. To say that $x_t \,\overline{\alpha}\, \mu$, we mean $x_t \,\alpha\, \mu$ does not hold, where $\alpha \in \{\in, \mathrm{q}\}$.

A fuzzy set μ in a BCK/BCI-algebra X is called a *fuzzy subalgebra* of X if it satisfies:

$$(\forall x, y \in X)\,(\mu(x * y) \geq \min\{\mu(x), \mu(y)\})\,. \tag{2.5}$$

A fuzzy set μ in a BCK/BCI-algebra X is called an $(\in,\, \in \vee\, q)$-*fuzzy subalgebra* (see [2]) of X if it satisfies: for all $x, y \in X$ and $t_1, t_2 \in (0, 1]$,

$$x_{t_1} \in \mu,\ y_{t_2} \in \mu \ \Rightarrow (x * y)_{\min\{t_1, t_2\}} \in \vee\, q\, \mu. \tag{2.6}$$

3. $(\in,\, \in \vee\, q)$-osmotic values for a fuzzy set

In what follows, let X denote a BCK/BCI-algebra unless otherwise specified.

For a fuzzy set μ in X and $t \in [0, 1]$, the (strong) upper (resp. lower) t-level sets of μ are defined as follows:

$$U(\mu; t) := \{x \in X \mid \mu(x) \geq t\},\ \ U^*(\mu; t) := \{x \in X \mid \mu(x) > t\},$$

$$L(\mu; t) := \{x \in X \mid \mu(x) \leq t\},\ \ L^*(\mu; t) := \{x \in X \mid \mu(x) < t\}.$$

Definition 3.1 ([3])**.** A non-empty subset A of X is said to be *S-energetic* if it satisfies:

$$(\forall a, b \in X)\,(a * b \in A \ \Rightarrow\ \{a, b\} \cap A \neq \emptyset)\,. \tag{3.1}$$

Proposition 3.2. *If A and B are S-energetic subsets of X, then so are $A \cup B$ and $A \cap B$.*

Proof. Straightforward. ∎

Lemma 3.3 ([2])**.** *A fuzzy set μ in X is an $(\in,\, \in \vee\, q)$-fuzzy subalgebra of X if and only if it satisfies:*

$$(\forall x, y \in X)\,(\mu(x * y) \geq \min\{\mu(x), \mu(y), 0.5\})\,. \tag{3.2}$$

Definition 3.4. Let μ be a fuzzy set in X. A number $t \in [0,1]$ is called an *osmotic S-value* for μ (it is called an anti-permeable S-value for μ in [3]) if the following assertion is valid:

$$(\forall a, b \in X)\,(\mu(a * b) \leq t \ \Rightarrow \min\{\mu(a), \mu(b)\} \leq t)\,. \tag{3.3}$$

Theorem 3.5. *Let $t \in \alpha \subseteq [0,1]$. For a fuzzy set μ in X, if the nonempty strong upper t-level set of μ is a subalgebra of X, then t is an osmotic S-value for μ.*

Proof. Let $a, b \in X$ be such that $\mu(a * b) \leq t$ for $t \in \alpha \subseteq [0,1]$. If

$$\min\{\mu(a), \mu(b)\} \not\leq t,$$

then $\mu(a) > t$ and $\mu(b) > t$, that is, $a, b \in U^*(\mu; t)$. Since $U^*(\mu; t)$ is a subalgebra of X, it follows that $a * b \in U^*(\mu; t)$, i.e., $\mu(a * b) > t$ which is a contradiction. Hence $\min\{\mu(a), \mu(b)\} \leq t$ and t is an osmotic S-value for μ. ∎

Definition 3.6. Let μ be a fuzzy set in X. A number $t \in [0,1]$ is called an $(\in, \in \vee q)$-*osmotic S-value* for μ if the following assertion is valid:

$$(\forall x, y \in X)\,(x * y \in L(\mu; t) \ \Rightarrow \ \min\{\mu(x), \mu(y), 0.5\} \leq t)\,. \tag{3.4}$$

We note that every osmotic S-value for μ is an $(\in, \in \vee q)$-osmotic S-value for μ.

Example 3.7. Let $X = \{0, 1, 2, 3, 4\}$ be a BCK-algebra with the following Cayley table:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	1	0	1	1
3	3	3	3	0	3
4	4	4	4	4	0

Let μ be a fuzzy set in X defined by

$$\mu = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0.8 & 0.3 & 0.6 & 0.7 & 0.2 \end{pmatrix}.$$

It is routine to verify that if $t \in [0,1] \setminus [0.3, 0.6)$, then t is an $(\in, \in \vee q)$-osmotic S-value for μ. If $t \in [0.3, 0.5)$, then t is not an $(\in, \in \vee q)$-osmotic S-value for μ since $2 * 3 = 1 \in L(\mu; t)$ but $\min\{\mu(2), \mu(3), 0.5\} > t$.

Theorem 3.8. *Let μ be a fuzzy set in X and $t \in \alpha \subseteq [0,1]$. If the nonempty lower t-level set of μ is an S-energetic subset of X, then t is an osmotic S-value for μ.*

Proof. For $t \in \alpha \subseteq [0,1]$, let $x, y \in X$ be such that $\mu(x * y) \leq t$. Then $x * y \in L(\mu; t)$. Assume that $\min\{\mu(x), \mu(y)\} \not\leq t$. Then $\mu(x) > t$ and $\mu(y) > t$, which imply that $x, y \notin L(\mu; t)$. Hence $\{x, y\} \cap L(\mu; t) = \emptyset$, a contradiction. Thus $\min\{\mu(x), \mu(y)\} \leq t$ and t is an osmotic S-value for μ. ∎

Corollary 3.9. *Let μ be a fuzzy set in X and $t \in \alpha \subseteq [0,1]$. If the nonempty lower t-level set of μ is an S-energetic subset of X, then t is an $(\in, \in\vee q)$-osmotic S-value for μ.*

We consider a condition for the converse of Theorem 3.8 to be true.

Theorem 3.10. *For a fuzzy set μ in X, let $t \in [0,1]$ be an $(\in, \in\vee q)$-osmotic S-value for μ. If $t < 0.5$, then the nonempty lower t-level set of μ is an S-energetic subset of X.*

Proof. Assume that $x*y \in L(\mu;t)$ for $x, y \in X$. Then $\min\{\mu(x), \mu(y), 0.5\} \leq t$ by (3.4). Since $t < 0.5$, it follows that $\mu(x) \leq t$ or $\mu(y) \leq t$. Hence $\{x,y\} \cap L(\mu;t) \neq \emptyset$, and therefore $L(\mu;t)$ is an S-energetic subset of X. ∎

Since every osmotic S-value for μ is an $(\in, \in\vee q)$-osmotic S-value for μ, we have the following theorem by Theorem 3.5.

Theorem 3.11. *Let $t \in \alpha \subseteq [0,1]$. For a fuzzy set μ in X, if the nonempty strong upper t-level set of μ is a subalgebra of X, then t is an $(\in, \in\vee q)$-osmotic S-value for μ.*

The following example shows that the converse of Theorem 3.11 may not be true, that is, there is an $(\in, \in\vee q)$-osmotic S-value t for μ such that the strong upper t-level set of μ is not a subalgebra of X.

Example 3.12. Consider the fuzzy set μ in Example 3.7. We can check that the number $t = 0.55$ is an $(\in, \in\vee q)$-osmotic S-value for μ, but $U^*(\mu;t) = \{0,2,3\}$ is not a subalgebra of X.

We now consider conditions for the converse of Theorem 3.11 to be true.

Theorem 3.13. *For a fuzzy set μ in X, let $t \in [0,1]$ be an $(\in, \in\vee q)$-osmotic S-value for μ. If $t < 0.5$, then the strong upper t-level set of μ is a subalgebra of X.*

Proof. Let $x, y \in U^*(\mu;t)$ for $x, y \in X$ and assume that $x * y \notin U^*(\mu;t)$. Then $\mu(x) > t$, $\mu(y) > t$ and $\mu(x*y) \leq t$, that is, $x*y \in L(\mu;t)$. It follows from (3.4) that $\min\{\mu(x), \mu(y), 0.5\} \leq t$. Since $t < 0.5$, we have $\mu(x) \leq t$ or $\mu(y) \leq t$ which imply that $x \notin U^*(\mu;t)$ or $y \notin U^*(\mu;t)$. This is a contradiction, and so $x * y \in U^*(\mu;t)$. ∎

Definition 3.14. Let μ be a fuzzy set in X. A number $t \in [0,1]$ is called an $(\overline{\in}, \overline{\in}\vee\overline{q})$-*osmotic S-value* for μ if the following assertion is valid:

$$(\forall x, y \in X)\,(x, y \in U^*(\mu;t) \;\Rightarrow\; \min\{\mu(x*y), 0.5\} > t)\,. \tag{3.5}$$

Example 3.15. Let $X = \{0, 1, a, b, c\}$ be a BCI-algebra with the following Cayley table:

$*$	0	1	a	b	c
0	0	0	c	b	a
1	1	0	c	b	a
a	a	a	0	c	b
b	b	b	a	0	c
c	c	c	b	a	0

Let μ be a fuzzy set in X defined by

$$\mu = \begin{pmatrix} 0 & 1 & a & b & c \\ 0.6 & 0.7 & 0.2 & 0.4 & 0.2 \end{pmatrix}.$$

It is easy to verify that every number $t < 0.6$ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-osmotic S-value for μ. If $t \in [0.6, 0.7)$, then t is not an $(\overline{\in}, \overline{\in} \vee \overline{q})$-osmotic S-value for μ since $1 \in U^*(\mu; t)$ but $\min\{\mu(1 * 1), 0.5\} = 0.5 < t$.

Theorem 3.16. *For a fuzzy set μ in X, let $t \in [0, 1]$ be an $(\overline{\in}, \overline{\in} \vee \overline{q})$-osmotic S-value for μ. If $t \geq 0.5$, then the nonempty lower t-level set of μ is an S-energetic subset of X and the nonempty strong upper t-level set of μ is a subalgebra of X.*

Proof. Let $x * y \in L(\mu; t)$ for $x, y \in X$. If $\{x, y\} \cap L(\mu; t) = \emptyset$, then $x, y \in U^*(\mu; t)$ and so $\min\{\mu(x * y), 0.5\} > t$ by (3.5). Since $t \geq 0.5$, it follows that $\mu(x * y) > t$, that is, $x * y \notin L(\mu; t)$. This is a contradiction, and therefore $\{x, y\} \cap L(\mu; t) \neq \emptyset$. Hence $L(\mu; t)$ is an S-energetic subset of X. Now, let $x, y \in U^*(\mu; t)$ for $x, y \in X$. Then $\min\{\mu(x * y), 0.5\} > t$ by (3.5), and so $\mu(x * y) > t$ since $t \geq 0.5$. Hence $x * y \in U^*(\mu; t)$, and therefore $U^*(\mu; t)$ is a subalgebra of X. ∎

Lemma 3.17. *For $t \in \alpha \subseteq [0, 1]$, if t is an osmotic S-value for a fuzzy set μ, then the nonempty lower t-level set of μ is an S-energetic subset of X.*

Proof. Let $x * y \in L(\mu; t)$ for $x, y \in X$. Then $\mu(x * y) \leq t$, and so $\min\{\mu(x), \mu(y)\} \leq t$ by (3.3). Hence $\mu(x) \leq t$ or $\mu(y) \leq t$, that is, $x \in L(\mu; t)$ or $y \in L(\mu; t)$. Thus $\{x, y\} \cap L(\mu; t) \neq \emptyset$, and therefore $L(\mu; t)$ is an S-energetic subset of X. ∎

We know that any osmotic S-value may not be an $(\overline{\in}, \overline{\in} \vee \overline{q})$-osmotic S-value as seen in the following example.

Example 3.18. Let $X = \{0, 1, 2, 3, 4\}$ be a BCK-algebra with the following Cayley table:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	0	2
3	3	2	1	0	3
4	4	4	4	4	0

Let μ be a fuzzy set in X defined by

$$\mu = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0.8 & 0.7 & 0.4 & 0.3 & 0.1 \end{pmatrix}.$$

If we take $t \in [0.5, 0.7)$, then t is an osmotic S-value for μ. But it is not an $(\overline{\in}, \overline{\in} \vee \overline{q})$-osmotic S-value for μ since $0, 1 \in U^*(\mu; t)$ but $\min\{\mu(1 * 0), 0.5\} = 0.5 \leq t$.

We provide a condition for an osmotic S-value to be an $(\overline{\in}, \overline{\in} \vee \overline{q})$-osmotic S-value.

Theorem 3.19. *For a fuzzy set μ in X, let $t \in \alpha \subseteq [0, 1]$ be an osmotic S-value for μ. If $t < 0.5$, then it is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-osmotic S-value for μ.*

Proof. Let $x, y \in U^*(\mu; t)$ for $x, y \in X$. Then $\mu(x) > t$ and $\mu(y) > t$, which imply that $x, y \notin L(\mu; t)$. Thus $\{x, y\} \cap L(\mu; t) = \emptyset$, and so $x * y \notin L(\mu; t)$ by Lemma 3.17. Thus $x * y \in U^*(\mu; t)$, and thus $\mu(x * y) > t$. Since $t < 0.5$, it follows that $\min\{\mu(x * y), 0.5\} > t$. Therefore t is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-osmotic S-value for μ. ∎

We consider conditions for a number t in $[0, 1]$ to be an $(\overline{\in}, \overline{\in} \vee \overline{q})$-osmotic S-value.

Theorem 3.20. *For a fuzzy set μ in X and $t \in [0, 1]$, if $t < 0.5$ and the nonempty strong upper t-level set of μ is a subalgebra of X, then t is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-osmotic S-value for μ.*

Proof. Let $x, y \in U^*(\mu; t)$ and assume that $\min\{\mu(x * y), 0.5\} \leq t$ for $x, y \in X$. Then $\mu(x * y) \leq t$ since $t < 0.5$. Hence $x * y \in L(\mu; t)$ and so $x * y \notin U^*(\mu; t)$. This is impossible, and thus $\min\{\mu(x * y), 0.5\} > t$ whenever $x, y \in U^*(\mu; t)$. This shows that t is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-osmotic S-value for μ. ∎

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FUZZY INTERIOR HYPERIDEALS IN ORDERED SEMIHYPERGROUPS

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Abstract. We introduce the notions of an interior hyperideal and a fuzzy interior hyperideal of an ordered semihypergroup. Then, we present a characterization of an interior hyperideal in terms of fuzzy interior hyperideals. The notion of an intra-regular ordered semihypergroup is introduced. Then, we show that fuzzy interior hyperideals and fuzzy hyperideals coincide in regular ordered semihypergroups and intra-regular ordered semihypergroups. Finally, we give the concept of a simple ordered semihypergroup and characterize simple ordered semihypergroups by means of fuzzy hyperideals and fuzzy interior hyperideals.

Keywords: ordered semihypergroup, interior hyperideal, fuzzy interior hyperideal, regular ordered semihypergroup, intra-regular ordered semihypergroup, simple ordered semihypergroup.

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1. Introduction

The notion of algebraic hyperstructures were introduced first by Marty [7] in 1934 as generalizations of algebraic structures and applied to non-commutative groups, so called hypergroups. In 2011, Heidari and Davvaz [4] defined a semihypergroup with a partially order ralation called an ordered semihypergroup. Then, Changphas and Davvaz [1] investigated properties of hyperideals in an ordered semihypergroup. After Zadeh [12] defined the notion of a fuzzy set in 1965, Rosenfeld

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[10] applied fuzzification to groups called fuzzy subgroups and then, many researchers considered fuzzification on many algebraic structures, for example, on rings, semigroups, semirings, near-rings, ordered semigroups, semihypergroups and ordered semihypergroups. In [6], the notion of a fuzzy interior ideal of an ordered semigroup was introduced by Kehayopulu and Tsingelis and it was shown that fuzzy interior ideals and fuzzy ideals coincide in a regular ordered semigroup and in an intra-regular ordered semigroup. Then Pibaljommee, Wannatong and Davvaz [9] introduced the notions of fuzzy hyperideals, fuzzy bi-hyperideals and fuzzy quasi-hyperideals of an ordered semihypergroup and proved that in regular ordered semihypergroups, fuzzy bi-hyperideals and fuzzy quasi-hyperideals coincide. Later, Pibaljommee and Davvaz [8] characterized left and right simple ordered semihypergroups, completely regular ordered semihypergroups and strongly regular ordered semihypergroups by means of their fuzzy bi-hyperideals.

In this paper, we generalized the notion of fuzzy interior ideals of an ordered semigroup introduced by Kehayopulu and Tsingelis in [6] to fuzzy interior hyperideals of an ordered semihypergroup.

2. Preliminaries

A *hypergroupoid* consists of a non-empty set H and a mapping $\circ : H \times H \to \mathcal{P}^*(H)$ called a *hyperoperation*, where $\mathcal{P}^*(H)$ denotes the set of all non-empty subsets of H (see, e.g., [2], [3], [11]). We denote by $a \circ b$ the image of the pair (a, b) in $H \times H$.

A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if it satisfies the associative property, namely,

$$(a \circ b) \circ c = a \circ (b \circ c).$$

For any non-empty subsets A, B of H, we denote

$$A \circ B := \bigcup_{a \in A, b \in B} a \circ b.$$

Instead of $\{a\} \circ A$ and $B \circ \{a\}$, we write $a \circ A$ and $B \circ a$, respectively.

Now, we recall the notion of an ordered semihypergroup defined in [4].

Definition 2.1. Let H be a non-empty set and $\leq$ be an ordered relation on H. The tri-tuple $(H, \circ, \leq)$ is called an *ordered semihypergroup* if the following conditions are satisfied.

(1) $(H, \circ)$ is a semihypergroup.

(2) $(H, \leq)$ is a partially ordered set.

(3) For every $a, b, c \in H$, $a \leq b$ implies $a \circ c \leq b \circ c$ and $c \circ a \leq c \circ b$, where $a \circ c \leq b \circ c$ means that for every $x \in a \circ c$ there exists $y \in b \circ c$ such that $x \leq y$.

A non-empty subset A of an ordered semihypergroup $(H, \circ, \leq)$ is called a *sub-semihypergroup* of H if $(A, \circ, \leq)$ is an ordered semihypergroup.

We note that for every $a, b, c, d, e, f \in H$ with $a \circ b \leq c \circ d$ and $e \leq f$, we obtain

$$a \circ b \circ e \leq c \circ d \circ f.$$

For $K \subseteq H$, we denote

$$(K] := \{a \in H \mid a \leq k \text{ for some } k \in K\}.$$

Definition 2.2. [4] A non-empty subset A of an ordered semihypergroup $(H, \circ, \leq)$ is called a *right* (*resp. left*) *hyperideal* of H if

(1) $A \circ H \subseteq A$ (resp. $H \circ A \subseteq A$),

(2) for every $a \in H$, $b \in A$ and $a \leq b$ implies $a \in A$.

If A is both a right hyperideal and a left hyperideal of H, then A is called a *hyperideal* (or *two-sided hyperideal*) of H.

Definition 2.3. A subsemihypergroup A of an ordered semihypergroup $(H, \circ, \leq)$ is called an *interior hyperideal* of H if

(1) $H \circ A \circ H \subseteq A$,

(2) for every $a \in H$, $b \in A$ and $a \leq b$ implies $a \in A$.

Example 1. Let $H = \{a, b, c, d\}$. Define a hyperoperation $\circ$ on H by the following table:

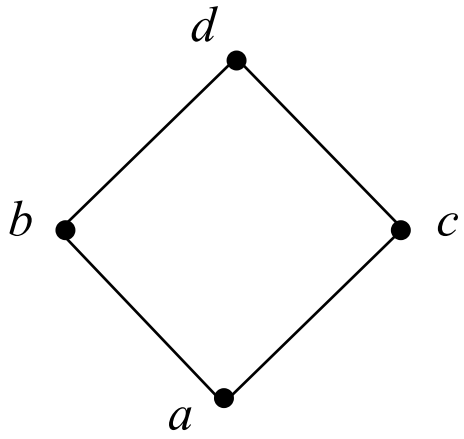
$\circ$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	a	$\{a, b\}$
d	a	a	$\{a, b\}$	$\{a, b, c\}$

We define an order relation $\leq$ as follows:

$$\leq := \{(a, a), (b, b), (c, c), (d, d), (a, b), (a, c), (a, d), (b, d), (c, d)\}.$$

The covering relation " $\prec$ " and the figure of H:

$$\prec = \{(a,b),(a,c),(b,d),(c,d)\},$$



Now, $(H, \circ, \leq)$ is an ordered semihypergroup.

It is easy to see that $\{a\}, \{a,b\}, \{a,b,c\}$ and H are all hyperideals. Moreover, $\{a,c\}$ is an interior hyperideal but not a hyperideal of H.

A fuzzy subset [12] of an ordered semihypergroup $(H, \circ, \leq)$ is a function $\mu : H \to [0,1]$. If μ satisfies the condition $\min\{\mu(a), \mu(b)\} \leq \inf\limits_{c \in a \circ b} \{\mu(c)\}$ for all $a, b \in H$, then μ is called a *fuzzy subsemihypergroup* of H.

Definition 2.4. [9] Let $(H, \circ, \leq)$ be an ordered semihypergroup. A fuzzy subset $\mu : H \to [0,1]$ is called a *fuzzy right* (*resp. left*) *hyperideal* of H if

(1) for every $a, b \in H$, $a \leq b$ implies $\mu(b) \leq \mu(a)$,

(2) for every $a, b \in H$, $\mu(a) \leq \inf\limits_{c \in a \circ b} \{\mu(c)\}$ (resp. $\mu(b) \leq \inf\limits_{c \in a \circ b} \{\mu(c)\}$).

If μ is both a fuzzy right hyperideal and a fuzzy left hyperideal of H, then μ is called a *fuzzy hyperideal* of H.

Example 2. [9] Let $H = \{a, b, c, d, e\}$. Define a hyperoperation $\circ$ on H by the following table:

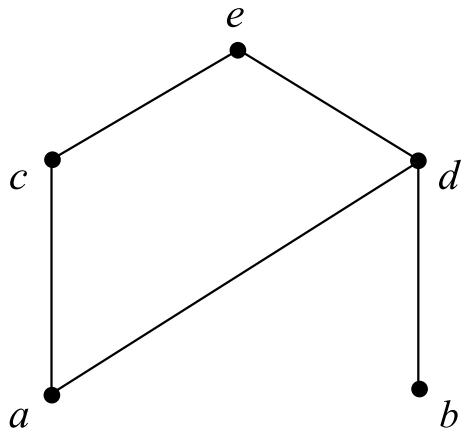
$\circ$	a	b	c	d	e
a	a	$\{a,b,d\}$	a	$\{a,b,d\}$	$\{a,b,d\}$
b	a	b	a	$\{a,b,d\}$	$\{a,b,d\}$
c	a	$\{a,b,d\}$	$\{a,c\}$	$\{a,b,d\}$	$\{a,b,c,d,e\}$
d	a	$\{a,b,d\}$	a	$\{a,b,d\}$	$\{a,b,d\}$
e	a	$\{a,b,d\}$	$\{a,c\}$	$\{a,b,d\}$	$\{a,b,c,d,e\}$

We define an order relation $\leq$ as follows:

$$\leq := \{(a,a),(b,b),(c,c),(d,d),(e,e),(a,c),(a,d),(a,e),(b,d),(b,e),(c,e),(d,e)\}.$$

We give the covering relation " $\prec$ " and the figure of H:

$$\prec = \{(a,c),(a,d),(b,d),(c,e),(d,e)\},$$



Now, $(H, \circ, \leq)$ is an ordered semihypergroup [9]. We define two fuzzy subsets μ and λ of H as follows:

$$\mu(x) := \begin{cases} 0.7 & \text{if } x = a, b, d \\ 0.3 & \text{if } x = c, e \end{cases} \quad \text{and} \quad \lambda(x) := \begin{cases} 0.9 & \text{if } x = a \\ 0.8 & \text{if } x = c \\ 0.5 & \text{if } x = b, d, e. \end{cases}$$

We can see that μ is a fuzzy hyperideal and λ is a fuzzy left hyperideal of H.

For any fuzzy subset μ of an ordered semihypergroup $(H, \circ, \leq)$ and $t \in (0, 1]$, the set

$$\mu_t = \{a \in H \mid \mu(a) \geq t\}$$

is called a *level subset* of μ.

Theorem 2.5. [9] *Let $(H, \circ, \leq)$ be an ordered semihypergroup and μ be a fuzzy subset of H. Then, μ is a fuzzy hyperideal of H if and only if for every $t \in [0, 1]$, the non-empty level subset μ_t is a hyperideal of H.*

Let $(H, \circ, \leq)$ be an ordered semihypergroup and $\emptyset \neq I \subseteq H$. Then the *characteristic function* $\mathcal{X}_I : H \to [0, 1]$ of I is defined by

$$\mathcal{X}_I := \begin{cases} 1 & \text{if } x \in I \\ 0 & \text{if } x \notin I. \end{cases}$$

Lemma 2.6. [9] *Let $(H, \circ, \leq)$ be an ordered semihypergroup and $\emptyset \neq I \subseteq H$. Then, I is a hyperideal of H if and only if the characteristic function $\mathcal{X}_I$ is a fuzzy hyperideal of H.*

3. Fuzzy interior hyperideals

In this section, we define the concept of a fuzzy interior hyperideal of an ordered semihypergroup and the concept of an intra-regular ordered semihypergroup. Then we show that every fuzzy hyperideal is a fuzzy interior hyperideal. Moreover, in regular ordered semihypergroups and intra-regular ordered semihypergroups fuzzy hyperideals and fuzzy interior hyperideals coincide.

Definition 3.1. Let $(H, \circ, \leq)$ be an ordered semihypergroup. A fuzzy subsemihypergroup $\mu : H \to [0, 1]$ is called a *fuzzy interior hyperideal* of H if the following assertions are satisfied:

(1) for every $a, b \in H, a \leq b$ implies $\mu(b) \leq \mu(a)$,

(2) for every $a, b, d \in H$, $\mu(a) \leq \inf\limits_{c \in b \circ a \circ d} \{\mu(c)\}$.

Theorem 3.2. *A fuzzy subsemihypergroup μ of an ordered semihygroup $(H, \circ, \leq)$ is a fuzzy interior hyperideal of H if and only if for every $t \in [0, 1]$, the non-empty level subset μ_t is an interior hyperideal of H.*

Proof. Assume that μ is a fuzzy interior hyperideal of H. Let $t \in [0, 1]$. Let $a \in H \circ \mu_t \circ H$. Then $a \in x \circ c \circ y$ for some $x, y \in H$, $c \in \mu_t$. By assumption, $\inf\limits_{a \in x \circ c \circ y} \{\mu(a)\} \geq \mu(c) \geq t$, we have $\mu(a) \geq t$. It follows that $H \circ \mu_t \circ H \subseteq \mu_t$. Let $x, y \in H$ be such that $x \leq y$. If $y \in \mu_t$, then $\mu(y) \geq t$. Since μ is a fuzzy interior hyperideal of H, we have $\mu(x) \geq \mu(y) \geq t$. It follows that $x \in \mu_t$. Therefore, μ_t is an interior hyperideal of H.

Assume that μ_t is an interior hyperideal of H for every $t \in (0, 1]$. Let $x, y \in H$ such that $x \leq y$. If $\mu(y) := t$, then $y \in \mu_t$. Since μ_t is an interior hyperideal of H and $x \leq y$, we have $x \in \mu_t$. So, $\mu(x) \geq t = \mu(y)$. Next, we show that $\mu(a) \leq \inf\limits_{c \in x \circ a \circ y} \{\mu(c)\}$ for every $a, x, y \in H$. Choose $\mu(a) := s$, then $a \in \mu_s$. Since μ_s is an interior hyperideal of H, we get $x \circ a \circ y \subseteq \mu_s$. Then for every $c \in x \circ a \circ y$, we have $\mu(c) \geq s$ and so $\mu(a) = s \leq \inf\limits_{c \in x \circ a \circ y} \{\mu(c)\}$. Therefore, μ is a fuzzy interior hyperideal of H. ∎

Example 3. Consider the ordered semihypergroup $(H, \circ, \leq)$ given in Example 1 and define a fuzzy subset $\mu : H \to [0, 1]$ by:

$$\mu(x) := \begin{cases} 0.7 & \text{if } x = a \\ 0.5 & \text{if } x = c \\ 0.4 & \text{if } x = b \\ 0.3 & \text{if } x = d. \end{cases}$$

Now, we have $\{a\}$, $\{a, c\}$, $\{a, b, c\}$ and H are all non-empty level subsets of H which are interior hyperideals. By Theorem 3.2, μ is a fuzzy interior hyperideal of H.

The following result can be directly proved using Theorem 3.2.

Corollary 3.3. *Let $(H, \circ, \leq)$ be an ordered semihypergroup and $\emptyset \neq I \subseteq H$. Then, I is an interior hyperideal of H if and only if the characteristic function $\mathcal{X}_I$ is a fuzzy interior hyperideal of H.*

Theorem 3.4. *Let $(H, \circ, \leq)$ be an ordered semihypergroup. Then, every fuzzy hyperideal of H is a fuzzy interior hyperideal of H.*

Proof. Assume that μ is a fuzzy hyperideal of H. Let $a, b, c \in H$. Since μ is a fuzzy left hyperideal of H and μ is a fuzzy right hyperideal of H, we have

$$\inf_{e \in c \circ a \circ b} \{\mu(e)\} \geq \inf_{s \in a \circ b} \{\mu(s)\} \geq \mu(a).$$

This implies that μ is a fuzzy interior hyperideal of H. ∎

Definition 3.5. [9] An ordered semihypergroup $(H, \circ, \leq)$ is called *regular* if for each $a \in H$ there exists $x \in H$ such that $a \leq a \circ x \circ a$.

Theorem 3.6. *Let $(H, \circ, \leq)$ be a regular ordered semihypergroup and μ be a fuzzy interior hyperideal of H. Then, μ is a fuzzy hyperideal of H.*

Proof. Assume that μ is a fuzzy interior hyperideal of H. Let $a, b \in H$. Since H is a regular ordered semihypergroup, there exists $x \in H$ such that $a \leq a \circ x \circ a$. Then $a \circ b \leq a \circ x \circ a \circ b$, that is, for every $c \in a \circ b$ there exists $e \in a \circ x \circ a \circ b$ such that $c \leq e$. Since μ is a fuzzy interior hyperideal of H, we have $\inf_{c \in a \circ b} \{\mu(c)\} \geq \mu(e)$ and $\inf_{s \in y \circ a \circ b} \{\mu(s)\} \geq \mu(a)$ for every $y \in a \circ x$. Thus,

$$\inf_{c \in a \circ b} \{\mu(c)\} \geq \mu(e) \geq \inf_{s \in y \circ a \circ b} \{\mu(s)\} \geq \mu(a).$$

This implies that μ is a fuzzy right hyperideal of H. In a similar way, we can prove that μ is a fuzzy left hyperideal of H. ∎

Corollary 3.7. *Let $(H, \circ, \leq)$ be a regular ordered semihypergroup and μ be a fuzzy subset of H. Then, μ is a fuzzy hyperideal of H if and only if μ is a fuzzy interior hyperideal of H.*

Proof. It is an immediate consequence of Theorem 3.4 and Theorem 3.6. ∎

Definition 3.8. *An ordered semihypergroup $(H, \circ, \leq)$ is called intra-regular if for each $a \in H$ there exist $x, y \in H$ such that $a \leq x \circ a^2 \circ y$.*

Example 4. Let $H = \{a, b, c, d, e\}$. Define a hyperoperation $\circ$ on H by the following table:

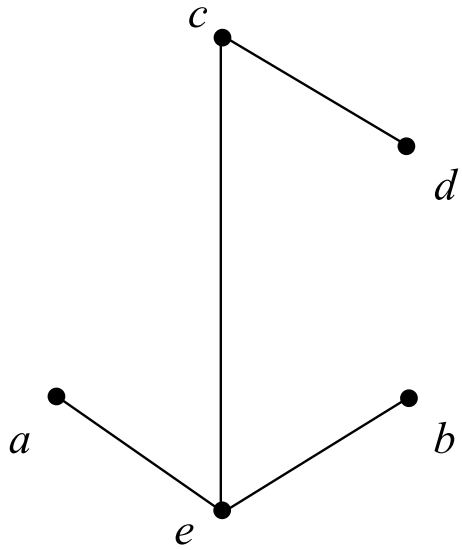
$\circ$	a	b	c	d	e
a	$\{a, e\}$	$\{b, e\}$	$\{a, b, c, e\}$	d	e
b	$\{b, e\}$	$\{a, e\}$	$\{a, b, c, e\}$	d	e
c	$\{a, b, c, e\}$	$\{a, b, c, e\}$	$\{a, b, c, e\}$	$\{a, b, c, e\}$	$\{a, b, c, e\}$
d	$\{a, b, c, e\}$	$\{a, b, c, e\}$	$\{a, b, c, e\}$	$\{a, b, c, e\}$	$\{a, b, c, e\}$
e	e	e	$\{a, b, c, e\}$	d	e

We define an order relation $\leq$ as follows:

$$\leq := \{(a,a),(b,b),(c,c),(d,d),(e,e),(e,a),(e,b),(e,c),(d,c)\}.$$

We give the covering relation " $\prec$ " and the figure of H:

$$\prec = \{(e,a),(e,b),(e,c),(d,c)\},$$



Now, $(H, \circ, \leq)$ is an ordered semihypergroup [8].

We can show that $(H, \circ, \leq)$ is an intra-regular ordered semihypergroup.

Theorem 3.9. *Let $(H, \circ, \leq)$ be an intra-regular ordered semihypergroup and μ be a fuzzy interior hyperideal of H. Then, μ is a fuzzy hyperideal of H.*

Proof. Assume that μ is a fuzzy interior hyperideal of H. Let $a, b \in H$. Since H is an intra-regular ordered semihypergroup, there exist $x, y \in H$ such that $a \leq x \circ a^2 \circ y$. Then $a \circ b \leq x \circ a^2 \circ y \circ b$, that is, for every $d \in a \circ b$ there exists $c \in x \circ a^2 \circ y \circ b$ such that $d \leq c$. Since μ is a fuzzy interior hyperideal of H, we have $\inf\limits_{d \in a \circ b} \{\mu(d)\} \geq \mu(c)$ and $\inf\limits_{s \in x \circ a \circ h} \{\mu(s)\} \geq \mu(a)$ for every $h \in a \circ y \circ b$. Therefore,

$$\inf_{d \in a \circ b} \{\mu(d)\} \geq \mu(c) \geq \inf_{s \in x \circ a \circ h} \{\mu(s)\} \geq \mu(a).$$

Hence, μ is a fuzzy right hyperideal of H. In a similar way, we can prove that μ is a fuzzy left hyperideal of H. ∎

As a consequence of Theorem 3.4 and Theorem 3.9, we have the following result.

Corollary 3.10. *Let $(H, \circ, \leq)$ be an intra-regular ordered semihypergroup and μ be a fuzzy subset of H. Then, μ is a fuzzy hyperideal of H if and only if μ is a fuzzy interior hyperideal of H.*

4. Simple ordered semihypergroups

In this section, we give a characterization of simple ordered semihypergroups by means of fuzzy hyperideals and fuzzy interior hyperideals. First, we apply the concept of a simple semihypergroup defined in [5] to define the concept of a simple ordered semihypergroup.

Definition 4.1. An ordered semihypergroup $(H, \circ, \leq)$ is called *simple* if it has no a proper hyperideal.

Example 5. Let $H = \{a, b, c\}$. Define a hyperoperation $\circ$ on H by the following table:

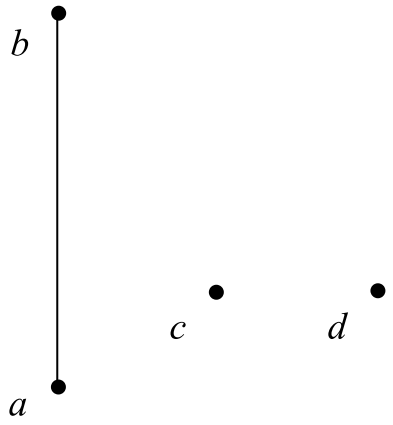
$\circ$	a	b	c
a	a	$\{a, b\}$	$\{a, c\}$
b	a	$\{a, b\}$	$\{a, c\}$
c	a	$\{a, b\}$	c

We define an order relation $\leq$ as follows:

$$\leq := \{(a, a), (b, b), (c, c), (a, b)\}.$$

We give the covering relation " $\prec$ " and the figure of H:

$$\prec = \{(a, b)\},$$



Now, $(H, \circ, \leq)$ is an ordered semihypergroup [1]. It is easy to see that $(H, \circ, \leq)$ is simple.

Let $(H, \circ, \leq)$ be an ordered semihypergroup, $a \in H$ and μ be a fuzzy subset of H. Define the set I_a as follows:

$$I_a = \{b \in H \mid \mu(b) \geq \mu(a)\}.$$

By Theorem 2.5, we obtain the following theorem.

Theorem 4.2. *Let $(H, \circ, \leq)$ be an ordered semihypergroup and μ be a fuzzy hyperideal of H. Then,*

(1) I_a *is a right hyperideal of* H *for every* $a \in H$,

(2) I_a *is a left hyperideal of* H *for every* $a \in H$,

(3) I_a *is a hyperideal of* H *for every* $a \in H$.

Theorem 4.3. *An ordered semihypergroup* $(H, \circ, \leq)$ *is a simple ordered semihypergroup if and only if every fuzzy hyperideal of* H *is a constant function.*

Proof. Assume that H is a simple ordered semihypergroup. Let μ be a fuzzy hyperideal of H and $a, b \in H$. By Theorem 4.2, we obtain I_a is a hyperideal of H. By assumption, this implies that $I_a = H$. Then $b \in I_a$, that is, $\mu(b) \geq \mu(a)$. On the other hand, we get $\mu(a) \geq \mu(b)$. Therefore, $\mu(a) = \mu(b)$.

Conversely, we assume that for every fuzzy hyperideal of H is a constant function. Let I be a hyperideal of H and $x \in H$. By Lemma 2.6, we obtain the characteristic function $\mathcal{X}_I$ is a fuzzy hyperideal of H. By assumption, $\mathcal{X}_I$ is a constant function, that is, $\mathcal{X}_I(x) = \mathcal{X}_I(b)$ for every $b \in H$. Let $a \in I$. Then $\mathcal{X}_I(x) = \mathcal{X}_I(a) = 1$, and so $x \in I$. Therefore, $H \subseteq I$. ∎

Lemma 4.4. *An ordered semihypergroup* $(H, \circ, \leq)$ *is a simple ordered semihypergroup if and only if for every* $a \in H$, $H = (H \circ a \circ H]$.

Proof. Assume that H is a simple ordered semihypergroup. We show that for every $a \in H$, $(H \circ a \circ H]$ is a hyperideal of H. Let $x \in H$ and $y \in (H \circ a \circ H]$. Then, we can write $y \leq k$ for some $k \in H \circ a \circ H$, so $x \circ y \leq x \circ k$. Since $x \circ k \subseteq H \circ a \circ H$, we have $x \circ y \subseteq (H \circ a \circ H]$. Next, let $x \in H$ and $y \in (H \circ a \circ H]$ be such that $x \leq y$. Then, we can write $x \leq y \leq k$ for some $k \in H \circ a \circ H$ so $x \in (H \circ a \circ H]$. Therefore, $(H \circ a \circ H]$ is a left hyperideal of H. In a similar way, we can prove that $(H \circ a \circ H]$ is a right hyperideal of H. Hence, $(H \circ a \circ H]$ is a hyperideal of H.

Conversely, let I be a hyperideal of H and $a \in I$. Then,

$$H = (H \circ a \circ H] \subseteq (H \circ I \circ H] \subseteq I.$$

This shows that H is simple. ∎

Theorem 4.5. *Let* $(H, \circ, \leq)$ *be an intra-regular ordered semihypergroup. Then,* H *is a simple ordered semihypergroup.*

Proof. Let $a \in H$. Then $a \leq x \circ a^2 \circ y = (x \circ a) \circ a \circ y$ for some $x, y \in H$. This follows that $a \in (H \circ a \circ H]$. By Lemma 4.4, H is simple. ∎

Theorem 4.6. *Let* $(H, \circ, \leq)$ *be an ordered semihypergroup. Then,* H *is a simple ordered semihypergroup if and only if every fuzzy interior hyperideal of* H *is a constant function.*

Proof. Assume that H is a simple ordered semihypergroup. Let μ be a fuzzy interior hyperideal of H and $a, b \in H$. By Lemma 4.4, we have $H = (H \circ b \circ H]$. So, $a \in (H \circ b \circ H]$. Then there exist $x, y \in H$ such that $a \leq x \circ b \circ y$, i.e, there

exists $e \in x \circ b \circ y$ such that $a \leq e$. Since μ is a fuzzy interior hyperideal of H, we have $\mu(a) \geq \mu(e)$ and $\inf_{s \in x \circ b \circ y} \{\mu(s)\} \geq \mu(b)$. Thus,

$$\mu(a) \geq \mu(e) \geq \inf_{s \in x \circ b \circ y} \{\mu(s)\} \geq \mu(b).$$

Hence, $\mu(a) \geq \mu(b)$. In a similar way, we can prove that $\mu(b) \geq \mu(a)$. Therefore, $\mu(a) = \mu(b)$.

Conversely, assume that every fuzzy interior hyperideal of H is a constant function. Let μ be a fuzzy hyperideal of H. By Theorem 3.4, μ is a fuzzy interior hyperideal of H. By assumption, μ is a constant function. By Theorem 4.3, H is a simple ordered semihypergroup. ∎

By Theorem 4.5 and Theorem 4.6, we have the following corollary.

Corollary 4.7. *Let $(H, \circ, \leq)$ be an intra-regular ordered semihypergroup. Then, every fuzzy interior hyperideal of H is a constant function.*

As a consequence of Theorem 4.3, Lemma 4.4 and Theorem 4.6, we present characterizations of a simple ordered semihypergroup as the following theorem.

Theorem 4.8. *Let $(H, \circ, \leq)$ be an ordered semihypergroup. Then, the following statements are equivalent:*

(1) *H is a simple ordered semihypergroup,*

(2) *$H = (H \circ a \circ H]$ for every $a \in H$,*

(3) *every fuzzy hyperideal of H is a constant function,*

(4) *every fuzzy interior hyperideal of H is a constant function.*

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ON THE OVERGROUPS OF $\mathrm{SL}(1,K)$ IN $\mathrm{GL}(r,F)$

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Abstract. For some pair of division rings K and F with $K \supset F$ and $\dim_F K = r$, we want to determine the overgroups of $K^* = \mathrm{SL}(1,K)$ in $\mathrm{GL}(r,F)$ and obtain the maximal subgroups of $\mathrm{GL}(r,F)$. Let R stand for real number field, C for complex one and Q for the skew-field of quaternions. All the overgroups of $C^* = \mathrm{SL}(1,C)$ in $\mathrm{GL}(2,R)$ and $Q^* = \mathrm{SL}(1,Q)$ in $\mathrm{GL}(2,C)$ are found in this paper.

Keywords: classical group, maximal subgroup, overgroup, quaternion.
2010 Mathematics Subject Classification: 20G15, 20E28.

1. Introduction

The subgroup structure of classical groups, especially maximal subgroups in classical groups, is one of the most important topics of group theory. By a theorem of Aschbacher(see [2]), the maximal subgroups of a classical group over finite field must be either a member of one of the classes $C_1 \sim C_8$, or an almost simple group. Under the guidance of this theorem, series of works have been done to approach the complete classification of the maximal subgroups(see [3], [8]). The second author of this paper has done much work(see [4], [5], [6], [7]) on the maximality of the subgroups in Aschbacher's classes. However, the results are for the classical groups over arbitrary fields, not necessarily finite, or sometimes over arbitrary division rings.

Let K, F be two division rings, with $K \supset F$, and $\dim_F K = r < \infty$ We can regard K as a left F-space. Write n-dimensional left K-space as $V(n,K)$, it can be regarded as an nr-dimensional left space $V = V(nr,F)$ over F. Thus the $\mathrm{GL}(n,K)$ acting on $V(n,K)$ is a subgroup of $\mathrm{GL}(nr,F)$ acting on $V(nr,F)$. In the article [5], when $n \geq 2$, the overgroups of $\mathrm{SL}(n,K)$ in $\mathrm{GL}(nr,F)$ and the overgroups of $\mathrm{Sp}(n,K,f)$ in $\mathrm{GL}(nr,F)$ were determined. As an application of the main result of the paper to the case F is a finite field and r is prime,

the maximality of the subgroups in Aschbacher's class C_3 was obtained. However, when $n = 1$, it is remained to determine the overgroups of $\mathrm{SL}(1, K)$ in $\mathrm{GL}(r, F)$. In this paper,we shall determine the overgroups of C^* in $\mathrm{GL}(2, R)$ and the overgroups of Q^* in $\mathrm{GL}(2, C)$ and obtain the maximal subgroups of $\mathrm{GL}(2, R)$ and $\mathrm{GL}(2, C)$. Our main results are the following two theorems.

Theorem 1.1 *Let R be the real number field and C the complex one. The group $\mathrm{SL}(1, C)$ is just the multiplicative group C^*, which can be written as the group $\left\{\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in \mathrm{GL}(2, R) \mid a, b \in R\right\}$. Let X be the overgroup of C^* in $\mathrm{GL}(2, R)$, $C^* = \mathrm{SL}(1, C) < X < \mathrm{GL}(2, R)$, then one of the following holds.*

- $X = C^* \rtimes \mathrm{Aut}C/R = \left\{\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \quad or \quad \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \in \mathrm{GL}(2, R) \mid a, b \in R\right\}$, *which is the normalizer of C^* in $\mathrm{GL}(2, R)$.*
- $X = \{A \in \mathrm{GL}(2, R) \mid \det A > 0\}$, *which is the group made up of all the elements in $\mathrm{GL}(2, R)$ whose determinant are positive real numbers.*

Theorem 1.2 *Let C be the complex number field and Q the skew-field of quaternions. The group $\mathrm{SL}(1, Q)$ is just the multiplicative group Q^*, which can be written as the group $\left\{\begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \in \mathrm{GL}(2, C) \mid \alpha, \beta \in C\right\}$. Let X be the overgroup of Q^* in $\mathrm{GL}(2, C)$, $Q^* = \mathrm{SL}(1, Q) < X < \mathrm{GL}(2, C)$, then one of the following holds.*

- $Q^* = \mathrm{SL}(1, Q) \lhd X \leq Q^* \rtimes \mathrm{Aut}Q/C$. *Let $e^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$, Then $Q^* \rtimes \mathrm{Aut}Q/C = \left\{\begin{pmatrix} \alpha & \beta \\ -\overline{\beta}e^{\mathrm{i}\theta} & \overline{\alpha}e^{\mathrm{i}\theta} \end{pmatrix} \in \mathrm{GL}(2, C) \mid \alpha, \beta \in C, \theta \in [0, 2\pi)\right\}$. And $Q^* \rtimes \mathrm{Aut}Q/C$ is the normalizer of Q^* in $\mathrm{GL}(2, C)$.*
- $H \lhd X < \mathrm{GL}(2, C)$. *Here $H = \{A \in \mathrm{GL}(2, C) \mid 0 < \det A \in R\}$ which is the group made up of all the elements in $\mathrm{GL}(2, C)$ whose determinant are positive real numbers. $X = H \cdot \left\langle \left\{ \begin{pmatrix} 1 & 0 \\ 0 & e^{\mathrm{i}(\mathrm{k}\alpha)} \end{pmatrix} | \alpha \in I_\alpha \right\} \right\rangle$, where I_α is a set of angles in $[0, 2\pi)$.*

2. Preliminaries

While K can be regarded as a left F-space, we can take an F-basis $\{k_1, ..., k_r\}$ of K. The left K-basis of $V(n, K)$ is marked as $\{e_1, ..., e_n\}$, thus $\{e_{ij} = k_j e_i | 1 \leq i \leq n, 1 \leq j \leq r\}$ forms an F-basis of $V(nr, F)$. Now, we write all vectors in $V = V(nr, F)$ as nr-dimensional rows and write each $g \in \mathrm{GL}(nr, F)$ as a matrix in $\mathrm{Mat}_{nr}F$, sending each $x \in V(nr, F)$ to xg. Write K as $\vec{K}$ and denote each $\vec{x} = c_1\vec{k_1} + \cdots + c_r\vec{k_r}$with all $c_i \in F$ as $\vec{x} = (c_1, ..., c_r) \in \mathrm{Mat}_{1\times r}F$ when viewing K as a left F-space. For each $\theta \in K$ we can view it as an F-linear translation by

$\vec{x} \mapsto \vec{x\theta}$ on $\vec{K}$. This transformation can be identified with the matrix $\theta^{(r)} \in \mathrm{Mat}_r F$ relative to the basis $\{k_1, ..., k_r\}$. In this point of view, we have $\mathrm{Mat}_n K \subset \mathrm{Mat}_{nr} F$. For each $\sigma \in \mathrm{Aut}K/F = \{\sigma \in \mathrm{Aut}K \mid a^\sigma = a,\ \forall a \in F\}$, it can be written as an matrix $\theta^{(r)}$ of the F-linear translation $\vec{x} \mapsto \vec{x^\sigma}$ on $\vec{K}$ relative to the basis $\{k_1, ..., k_r\}$. We point out that the normalizer of K^* in $\mathrm{GL}(r, F)$ is $K^* \rtimes \mathrm{Aut}K/F$, and regard $\mathrm{Aut}K/F$ as a subgroup of $\mathrm{GL}(nr, F)$. Each $\sigma \in \mathrm{Aut}K/F$ sends the vector $\theta_1 e_1 + \cdots + \theta_r e_r \in V(n, K)$ to $\theta_1^\sigma e_1 + \cdots + \theta_r^\sigma e_r$ with all $\theta_i \in K$, having the matrix $\sigma^{(nr)} = \mathrm{diag}(\sigma^{(r)}, ..., \sigma^{(r)})$. One can see that the normalizer of $\mathrm{SL}(n, K)$ in $\mathrm{GL}(nr, F)$ is $\Gamma = \mathrm{GL}(n, K) \rtimes \mathrm{Aut}K/F$. When $n \geq 2$, the overgroups of $\mathrm{SL}(n, K)$ in $\mathrm{GL}(nr, F)$ have been determined in the following theorem.

Theorem 2.1 ([5], Theorem 1) *Let K and F be division rings with $K \supset F$ and $\dim_F K = r < \infty$, $n \geq 2$, $N = \mathrm{SL}(n, K) \leq X \leq G = \mathrm{GL}(nr, F)$, then one of the following holds.*

- $\mathrm{SL}(nd, E) \lhd X < \Gamma = \mathrm{GL}(nd, E) \rtimes \mathrm{Aut}E/F$, *for an intermediate division ring E between F and K, where $d = \dim_E K$.*
- *$n = 2$, K is a field, $N = \mathrm{SL}(2, K) = \mathrm{Sp}(2, K, f)$ for any non-degenerate alternating K-form f, $X \trianglerighteq \mathrm{Sp}(2d, E, f_E)$ for an intermediate field E ($F \subseteq E \subseteq K$, $d = \dim_E K$) and an alternating E-form $f_E = \phi_E f$ with $0 \neq \phi_E \in \mathrm{Hom}_E(K, E)$.*
- *$N = \mathrm{SL}(2, 4) \cong A_5$ and $G = \mathrm{GL}(4, 2) \cong A_8$, $X = \mathrm{Sp}(4, 2)' \cong A_6$ or $X \cong A_7$.*

Let $1 \leq i, j \leq n$ be distinct, E_{ij} stands for $n \times n$ matrix whose (i, j)-entry is equal to 1 and zeros for all other positions.D enote identity matrix by I. $T_{ij}(a) = I + aE_{ij}$ with I $a \in F$. Then $T_{ij} = \{T_{ij}(a) | a \in F\}$ are subgroups of $\mathrm{GL}(n, F)$ which are called root subgroups of $\mathrm{GL}(n, F)$. To prove our main results,we use the following facts.

Remark 2.2 *Let F^+ be the addition group of F. Then each $T_{ij} \cong F^+$.*

$T_{ij}(a)T_{ij}(b) = T_{ij}(a + b)$, hence $T_{ij}(a)^{-1} = T_{ij}(-a)$.
With the map $t_{ij} : T_{ij}(a) \mapsto a$, we can easily get this remark.

Remark 2.3 ([1], Propositions 6.2, 6.3) $\mathrm{SL}(n, F)$ *is generated by the root subgroups T_{ij}. And the subgroups T_{ij} are conjugate in* $\mathrm{SL}(n, F)$.

Remark 2.4 *Let R stand for real number field,C for complex one and Q for the skew-field of quaternions. Then* $\mathrm{Aut}(C/R) \cong \left\{ \begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \right\}$, *while* $\mathrm{Aut}(Q/C) \cong \left\{ \begin{pmatrix} 1 & 0 \\ 0 & e^{\mathrm{i}\theta} \end{pmatrix} \mid \theta \in [0, 2\pi) \right\}$.

Take a basis $\{1, \mathrm{i}\}$ of C as a F-space, each element of C can be written as a 2-dimensional vector on $V(2, R)$. For each $a + b\mathrm{i} \in C$ with $a, b \in R$, $(a + b\mathrm{i})^\sigma = a \pm b\mathrm{i}$

with $\sigma \in \mathrm{Aut}(C/R)$. Then $\sigma : (a,b) \mapsto (a,b)\begin{pmatrix}1 & 0\\ 0 & \pm 1\end{pmatrix}$, hence $\mathrm{Aut}(C/R) \cong \left\{\begin{pmatrix}1 & 0\\ 0 & \pm 1\end{pmatrix}\right\}$. In the same way, take a basis $\{1, \mathrm{j}\}$ of Q as a C-space, the elements of Q can be written as $\alpha + \beta\mathrm{j}$ with $\alpha, \beta \in C$. For each $\sigma \in \mathrm{Aut}(Q/C), q_1, q_2 \in Q$, from $(q_1q_2)^\sigma = q_1^\sigma q_2^\sigma$ we can get

$$\mathrm{ij}^\sigma = -\mathrm{j}^\sigma \mathrm{i}, (\mathrm{j}^\sigma)^2 = -1.$$

Let $\mathrm{j}^\sigma = \alpha + \beta\mathrm{j}$. Then we can get $\alpha = 0, \beta = e^{\mathrm{i}\theta}$ and $\mathrm{j}^\sigma = e^{\mathrm{i}\theta}$. Therefore, $\mathrm{Aut}(Q/C) \cong \left\{\begin{pmatrix}1 & 0\\ 0 & e^{\mathrm{i}\theta}\end{pmatrix} \mid \theta \in [0, 2\pi)\right\}$.

3. Proofs of the main results

From Theorem 2.1 we can know that when F is a maximal skew-subfield of K, then $\mathrm{SL}(n,K) \rtimes \mathrm{Aut}K/F$ is a maximal subgroup of $\mathrm{GL}(nr,K)$ in most cases. Therefore, we can guess that $\Gamma = C^* \rtimes \mathrm{Aut}C/R$ resp. $\Lambda = Q^* \rtimes \mathrm{Aut}Q/C$ may be a maximal subgroup of $\mathrm{GL}(2,R)$ resp. $\mathrm{GL}(2,C)$. To prove this, we need the following lemma.

Lemma 3.1 *Let X be an overgroup of $\Gamma = C^* \rtimes \mathrm{Aut}C/R$ in $\mathrm{GL}(2,R)$, $A \in X \backslash \Gamma$. $\langle \Gamma, A\rangle$ refers to the group generated by Γ and A. Then $\mathrm{SL}(2,R) \lhd \langle \Gamma, A\rangle \leq X$.*

Proof. Let $A = \begin{pmatrix}a & b\\ c & d\end{pmatrix}$ with $a, b, c, d \in R$, transform A with a matrix $\begin{pmatrix}a & b\\ -b & a\end{pmatrix} \in C^*$ as follows:

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}a & b\\ -b & a\end{pmatrix}^{-1} = \begin{pmatrix}1 & 0\\ s & t\end{pmatrix},$$

where

$$s = \frac{ac+bd}{a^2+b^2}, t = \frac{ad-bc}{a^2+b^2} = \frac{\det A}{a^2+b^2}.$$

When $t = \pm 1, s \neq 0$. Or $A \in C^*$, in contradiction. Next, we give the following transformation:

$$\begin{pmatrix}1 & x\\ -x & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ s & t\end{pmatrix} = \begin{pmatrix}1+sx & tx\\ s-x & t\end{pmatrix},$$

$$\begin{pmatrix}1+sx & tx\\ s-x & t\end{pmatrix}\begin{pmatrix}1+sx & tx\\ -tx & 1+sx\end{pmatrix}^{-1} = \begin{pmatrix}1 & 0\\ f(x) & g(x)\end{pmatrix},$$

where

$$f(x) = \frac{s + (s^2+t^2-1)x - s^2x}{(1+sx)^2 + (tx)^2}, g(x) = \frac{t(1+x^2)}{(1+sx)^2+(tx)^2}, x \in R.$$

Note that $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}^{-1}, \begin{pmatrix} 1 & x \\ -x & 1 \end{pmatrix}, \begin{pmatrix} 1+sx & tx \\ -tx & 1+sx \end{pmatrix}^{-1}$ are elements in C^*, then the matrices $\begin{pmatrix} 1 & 0 \\ s & t \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ f(x) & g(x) \end{pmatrix} \in \langle \Gamma, A \rangle$. Then, the commutator

$$\begin{pmatrix} 1 & 0 \\ s & t \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ f(x) & g(x) \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ s & t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ f(x) & g(x) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ r(x) & 1 \end{pmatrix}$$
$$= T_{21}(r(x)) \in \langle \Gamma, A \rangle,$$

where

$$r(x) = \frac{[s^2 + (t-1)^2]x(1+t+sx)}{t^2(1+x^2)},\ x \in R.$$

We can find that $r(x)$ is a continuous function for $x \in R$ because $t^2(1+x^2) > 0$.

$$r'(x) = \frac{[s^2 + (t-1)^2][(1+t)(1-x^2) + 2sx]}{t^2(1+x^2)^2},\ x \in R.$$

While $s = 0$ and $t = \pm 1$,$r'(x) \equiv 0$. However, these two cases show that $A \in \Gamma$, in contradiction. Therefore, $r(x)$ has two Extreme values denoted a and b with $a < b$. Note that

$$\lim_{x \to \pm\infty} r(x) = \frac{s[s^2 + (t-1)^2]}{t^2},$$

we can get the range of the continuous function $r(x)$ is $[a, b]$.

Form Remark 2.2 we know $T_{21} = \{T_{21}(x) \mid x \in R\} \cong R^+$. Because the additive group R^+ can be generated by all the elements in $[a, b]$, the multiplicative group $T_{21} = \{T_{21}(x) \mid x \in R\}$ can be generated by all the elements in $\{T_{21}(r(x)) \mid x \in R\} = \{T_{21}(x) \mid x \in [a, b]\}$. So, for all $x \in R$, the $T_{21}(x) \in \langle \Gamma, A \rangle$. From Remark 2.3, we can find a matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \Gamma$, such that

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -r \\ 0 & 1 \end{pmatrix},$$

we can get $T_{12} < \langle \Gamma, A \rangle$. So, the special linear group $\mathrm{SL}(2, R)$ is a subgroup of $\langle \Gamma, A \rangle$ by the Remark 2.3. Note that $\mathrm{SL}(2, R)$ is just the kernel of det from $\langle \Gamma, A \rangle$ to the multiplicative group R^*, $\mathrm{SL}(2, R) \lhd \langle \Gamma, A \rangle$. ∎

From this lemma, we can get the following corollary.

Corollary 3.2 *Let X be an overgroup of C^* in $\mathrm{GL}(2, R)$, $A \in X \backslash \Gamma = C^* \rtimes \mathrm{Aut}\ C/R$. $\langle C^*, A \rangle$ refers to the group generated by C^* and A. Then, $\mathrm{SL}(2, R) \lhd \langle C^*, A \rangle \leq X$.*

Lemma 3.3 *Let X be an overgroup of $\Gamma = C^* \rtimes \mathrm{Aut}C/R$ in $\mathrm{GL}(2, R)$, $A \in X \backslash \Gamma$. $\langle \Gamma, A \rangle$ refers to the group generated by Γ and A. Then $\langle \Gamma, A \rangle = X = \mathrm{GL}(2, R)$.*

Proof. For each matrix $B \in \mathrm{GL}(2,R)$, denote $\det B = d$.We can give the following transformations:

$$\begin{cases} d>0, B\begin{pmatrix}\sqrt{d} & 0\\ 0 & \sqrt{d}\end{pmatrix}^{-1} \in \mathrm{SL}(2,R) \\ d<0, B\begin{pmatrix}\sqrt{-d} & 0\\ 0 & \sqrt{-d}\end{pmatrix}^{-1}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \in \mathrm{SL}(2,R) \end{cases}$$

Note that the matrices $\begin{pmatrix}\sqrt{d} & 0\\ 0 & \sqrt{d}\end{pmatrix}^{-1}, \begin{pmatrix}\sqrt{-d} & 0\\ 0 & \sqrt{-d}\end{pmatrix}^{-1}, \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ all are elements in Γ and $\mathrm{SL}(2,R) \lhd \langle \Gamma, A\rangle$, we can get $\mathrm{GL}(2,R) \leq \langle \Gamma, A\rangle$. Hence $\mathrm{GL}(2,R) = \langle \Gamma, A\rangle = X$. ∎

Lemma 3.4 *Let A be a matrix in $\mathrm{GL}(2,C)$ whose determinant is $\det A \in R$, $A \in X\backslash\Lambda = Q^* \rtimes \mathrm{Aut}Q/C$. $\langle C^*, A\rangle$ refers to the group generated by C^* and A. Then $\mathrm{SL}(2,C) \lhd \langle C^*, A\rangle$.*

Proof. Denote $A = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \notin \Lambda = Q^* \rtimes \mathrm{Aut}Q/C$. Science $\det A \in R$, we can transform B with a matrix $\begin{pmatrix}\alpha & \beta\\ -\overline{\beta} & \overline{\alpha}\end{pmatrix} \in Q^*$ as follow:

$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}\alpha & \beta\\ -\overline{\beta} & \overline{\alpha}\end{pmatrix}^{-1} = \begin{pmatrix}1 & 0\\ \gamma_1 & t\end{pmatrix},$$

where $\gamma_1 \in C, t \in R$ as $\det\begin{pmatrix}\alpha & \beta\\ -\overline{\beta} & \overline{\alpha}\end{pmatrix} \in R$. Science the complex number γ_1 can be written as $se^{\mathrm{i}\theta}$ where $s \in R, \theta \in [0, 2\pi), e^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$, we can transform the matrix $\begin{pmatrix}1 & 0\\ \gamma_1 & t\end{pmatrix}$ to $\begin{pmatrix}1 & 0\\ s & t\end{pmatrix}$ as follow:

$$\begin{pmatrix}e^{\mathrm{i}\frac{\theta}{2}} & 0\\ 0 & e^{-\mathrm{i}\frac{\theta}{2}}\end{pmatrix}\begin{pmatrix}1 & 0\\ \gamma_1 & t\end{pmatrix}\begin{pmatrix}e^{-\mathrm{i}\frac{\theta}{2}} & 0\\ 0 & e^{\mathrm{i}\frac{\theta}{2}}\end{pmatrix} = \begin{pmatrix}1 & 0\\ s & t\end{pmatrix}, s, t \in R.$$

Note that the matrices $\begin{pmatrix}e^{\mathrm{i}\frac{\theta}{2}} & 0\\ 0 & e^{-\mathrm{i}\frac{\theta}{2}}\end{pmatrix}, \begin{pmatrix}e^{-\mathrm{i}\frac{\theta}{2}} & 0\\ 0 & e^{\mathrm{i}\frac{\theta}{2}}\end{pmatrix} \in Q^*$, we can know $\begin{pmatrix}1 & 0\\ s & t\end{pmatrix} \notin \Lambda$. Then, from Corollary 3.2 we have $\mathrm{SL}(2,C) \lhd \langle C^*, A\rangle$. ∎

Lemma 3.5 *Let X be an overgroup of $\Lambda = Q^* \rtimes \mathrm{Aut}Q/C$ in $\mathrm{GL}(2,C)$, $A \in X\backslash\Lambda$. $\langle \Lambda, A\rangle$ refers to the group generated by Λ and A. Then $\langle \Lambda, A\rangle = X = \mathrm{GL}(2,C)$.*

Proof. For an arbitrary matrix $A \in X\backslash\Lambda$, Q^* is not normalized by A. Hence, there exists a matrix $M \in Q^*$ with $AM \neq MA$. Let $B = AMA^{-1}$, then $B \notin Q^*$, and $\det B = \det M \in R$. From Lemma 3.4 we know $\mathrm{SL}(2,C) \lhd \langle C^*, B\rangle < \langle \Lambda, B\rangle \leq X \leq \mathrm{GL}(2,C)$.

Now, we just have to prove $X = \mathrm{GL}(2, C)$. For each matrix $D \in \mathrm{GL}(2, C)$, denote $\det D = \delta = de^{\mathrm{i}\theta}$ with $d > 0$ and $\theta \in [0, 2\pi)$.W e can give the following transformations:

$$D \begin{pmatrix} \sqrt{d} & 0 \\ 0 & \sqrt{d} \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & e^{\mathrm{i}\theta} \end{pmatrix}^{-1} \in \mathrm{SL}(2, C).$$

Note that the matrices $\begin{pmatrix} \sqrt{d} & 0 \\ 0 & \sqrt{d} \end{pmatrix}^{-1}, \begin{pmatrix} 1 & 0 \\ 0 & e^{\mathrm{i}\theta} \end{pmatrix}^{-1}$ all are elements in Λ and $\mathrm{SL}(2, C) \lhd \langle \Lambda, D \rangle$, we can get $\mathrm{GL}(2, C) \leq \langle \Lambda, D \rangle$. Hence $\mathrm{GL}(2, C) = \langle \Lambda, D \rangle = X$. ∎

Proof of Theorem 1.1. The first item of Theorem 1.1 follows immediately from Lemma 3.1 and Lemma 3.3 as $C^* \rtimes \mathrm{Aut}C/R$ is a maximal subgroups of $\mathrm{GL}(2, R)$. For an arbitrary matrix $A \in X \backslash \Gamma$, C^* is not normalized by A. There exists a matrix $M \in C^*$ with $AM \neq MA$. Let $B = AMA^{-1}$, then $B \notin C^*$, and $\det B = \det M > 0$. From Corollary 3.2, $\langle C^*, B \rangle \rhd \mathrm{SL}(2, R)$. Then, for an arbitrary matrix D whose determinant is $d > 0$, we can give the following transformation: $D \begin{pmatrix} \sqrt{d} & 0 \\ 0 & \sqrt{d} \end{pmatrix}^{-1} \in \mathrm{SL}(2, R)$. That means $\langle C^*, B \rangle = \mathrm{GL}_+(2, R) := \{A \in \mathrm{GL}(2, R) | \det A > 0\}$. So, the second item of Theorem 1.1 is established. ∎

Proof of Theorem 1.2. From Lemma 3.5,we can get $Q^* \rtimes \mathrm{Aut}Q/C$ is a maximal subgroup of $\mathrm{GL}(2, C)$. From Remark 2.4, the group $Q^* \rtimes \mathrm{Aut}Q/C = \left\{ \begin{pmatrix} \alpha & \beta \\ -\overline{\beta}e^{\mathrm{i}\theta} & \overline{\alpha}e^{\mathrm{i}\theta} \end{pmatrix} \in \mathrm{GL}(2, C) \mid \alpha, \beta \in C, \theta \in [0, 2\pi) \right\}$. Then, the first item of Theorem 1.2 is established.

For an arbitrary matrix $A \in X \backslash \Lambda$, Q^* is not normalized by A. There exists a matrix $M \in Q^*$ with $AM \neq MA$. Let $B = AMA^{-1}$, then $B \notin Q^*$, and $0 < \det B = \det M \in R$. From Lemma 3.4, $\langle Q^*, B \rangle > \langle C^*, B \rangle \rhd \mathrm{SL}(2, C)$. Then, for an arbitrary matrix D whose determinant is $d > 0$, we can give the following transformation: $D \begin{pmatrix} \sqrt{d} & 0 \\ 0 & \sqrt{d} \end{pmatrix}^{-1} \in \mathrm{SL}(2, C)$. That means $\langle Q^*, B \rangle = \mathrm{GL}_+(2, C) := \{A \in \mathrm{GL}(2, C) | 0 < \det A \in R\}$. For the overgroup X between $\langle Q^*, B \rangle$ and $\mathrm{GL}(2, C)$, we can give a homomorphism

$$\Theta : X \to \left\{ \begin{pmatrix} 1 & 0 \\ 0 & e^{\mathrm{i}\theta} \end{pmatrix} \mid \theta \in [0, 2\pi) \right\}.$$

For each element $A \in X$, whose determinant is $\delta = te^{\mathrm{i}\theta}, \Theta(A) = \begin{pmatrix} 1 & 0 \\ 0 & e^{\mathrm{i}\theta} \end{pmatrix}$. The kernel of the homomorphism is just

$$H = \mathrm{GL}_+(2, C) := \{A \in \mathrm{GL}(2, C) \mid 0 < \det A \in R\}.$$

So

$$X = H \cdot \left\{ \begin{pmatrix} 1 & 0 \\ 0 & e^{\mathrm{i}\theta} \end{pmatrix} \mid \theta \in [0, 2\pi) \right\}.$$

The second item of Theorem 1.2 is finished. ∎

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QUANTUM CONDITIONAL LOGICAL ENTROPY OF DYNAMICAL SYSTEMS

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Abstract. In this paper, we will define the notion of conditional logical entropy of dynamical systems on a quantum logic. We will prove some of its properties. The relation between this measure and the notion of logical entropy of quantum dynamical systems which was defined in my paper (Open Physics, 14:1-5, 2016), will be presented.

Keywords: logical entropy, quantum logic, quantum dynamical system, conditional logical entropy dynamical system.

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1. Introduction

Birkhoff and Von Neumann in [1] have introduced the quantum logic approach. Entropy serves as a measure of information of the considered experiment and has an important application in dynamical systems ([4], [6], [7], [16]). Yuan, Khare and Roy, using the notion of state of quantum logic, introduced Shannon's entropy of finite partitions on a quantum logic ([11], [12], [20]). In my previous paper [3] the notion of relative entropy with countable partitions on a quantum logic was presented. Logical entropy is a measure on set of ordered pairs [8]. In 1982, Rao, Good, Patil and Taillie defined and studied the notion of logical entropy ([13], [14], [15]). Rao introduced precisely this concept as quadratic entropy [15] and in the years 2009 and 2013, this concept was discussed by Ellerman in ([8], [9], [10]). In [4], Ebrahimzadeh defined and studied the notion of logical entropy of quantum dynamical systems. Huls [2] and Zang [21] introduced the notion of conditional entropy of dynamical systems in the non-fuzzy sence. After that, Tok defined and studied the fuzzy conditional entropy of dynamical systems in ([17], [18]).

In the present paper, the notion of conditional logical entropy of quantum dynamical systems will be defined and some of its results will be proved.

2. Basic notions

We first, present some basic definitions and properties that will be useful in further considerations.

Definition 2.1. [20] A quantum logic QL is a σ-orthomodular lattice, i.e., a lattice L $(L, \leq, \vee, \wedge, 0, 1)$ with the smallest element 0 and the greatest element 1, an operation $' : L \to L$ such that the following properties hold for all $a, b, c \in L$:

(i) $a'' = a$, $a \leq b \Rightarrow b' \leq a'$, $a \vee a' = 1$, $a \wedge a' = 0$;

(ii) Given any finite sequence $(a_i)_{i\in I}$, $a_i \leq a_{j'}$, $i \neq j$, the join $\vee_{i\in\mathbb{N}} a_i$ exists in L;

(iii) L is orthomodular: $a \leq b \Rightarrow b = a \vee (b \wedge a')$.

Two elements $a, b \in QL$ are called orthogonal if $a \leq b'$ and denoted by $a \perp b$. A sequence $(a_i)_{i\in I}$ is said orthogonal if $a_i \perp a_j$, $\forall i \neq j$.

Definition 2.2. [20] Let L be a QL. A map $s : L \to [0, 1]$ is a state iff $s(1) = 1$ and for any orthogonal sequence $(a_i)_{i\in I}$, $s(\vee_{i\in I} a_i) = \sum_{i\in I} s(a_i)$.

Definition 2.3. [20] Let $P = \{a_1, ..., a_n\}$ be a finite system of elements of a QL. P is called to be a $\vee$-orthogonal system iff $\vee_{i=1}^{k} a_i \perp a_{k+1}$, $\forall k$.

Definition 2.4. [20] A system $P = \{a_1, ..., a_n\} \subset L$ is said to be a partition of L corresponding to a state s if:

(i) P is a $\vee$-orthogonal system;

(ii) $s(\vee_{i=1}^{n} a_i) = 1$.

Note that from Definition 2.2, we obtain $\sum_{i=1}^{n} s(a_i) = 1$.

Definition 2.5. [20] Let the system $(b_1, ..., b_m)$ be any partition corresponding to a state s and $a \in L$. The state s is said has Bayes' Property if $s(\vee_{j=1}^{m}(a \wedge b_j)) = s(a)$.

Lemma 2.6. [20] *Let $Q = (b_1, ..., b_m)$ be a partition on L, and $a \in L$, and the state s has Bayes' Property. Then* $\sum_{j=1}^{m} s(a \wedge b_j) = s(a)$.

Let $P = \{a_1, ..., a_n\}$ and $Q = \{b_1, ..., b_m\}$ be two finite partitions of a QL corresponding to a state s. The common refinement of these partitions is

$$P \vee Q = \{a_i \wedge b_j : a_i \in P, b_j \in R\}.$$

Definition 2.7. [4] Let $P = \{a_1, ..., a_n\}$ be a partition of a QL corresponding to a state s. The logical entropy of P with respect to state s is defined by

$$h_s^l(P) = \sum_{i=1}^{n} s(a_i)(1 - s(a_i)).$$

Definition 2.8. [4] Let $P = \{a_1, ..., a_n\}$ and $Q = \{b_1, ..., b_m\}$ be two partitions of a QL corresponding to a state s. The conditional logical entropy of P given R with respect to state s is defined as

$$h_s^l(P|Q) = \sum_{i=1}^{n} \sum_{j=1}^{m} s(a_i \wedge b_j)(s(b_j) - s(a_i \wedge b_j)).$$

Proposition 2.9. *Let P_1, P_2 and Q be finite partitions of a QL corresponding to a state s having Bayes' Property. Then*

(i) $h_s^l(P_1|Q) \geq 0$,

(ii) $h_s^l(P_1 \vee P_2|Q) = h_s^l(P_1|Q) + h_s^l(P_2|P_1 \vee Q)$,

(iii) $h_s^l(P_1 \vee Q) = h_s^l(Q) + h_s^l(P_1|Q)$,

(iv) $h_s^l(P_1|Q) \leq h_s^l(P_1)$.

Proof. See [4]. ∎

Definition 2.10. [4] Let $P = \{a_1, ..., a_n\}$ and $Q = \{b_1, ..., b_m\}$ be two partitions of a QL corresponding to a state s. We say Q is a s-refinement of P, denoted by $P \preceq_s Q$, if there exists a partition $I(1), ..., I(n)$ of the set $\{1, ..., m\}$ such that $a_i = \vee_{j \in I(i)} b_j$ for every $i = 1, ..., n$.

Proposition 2.11. *Let $P = \{a_1, ..., a_n\}, Q = \{b_1, ..., b_m\}$ and $R = \{c_1, ..., c_r\}$ be partitions of a QL corresponding to a state s. Then*

(i) *$P \preceq_s Q$ implies that $h_s^l(P) \leq h_s^l(Q)$;*

(ii) *If $P \preceq_s Q$ and the QL be distributive then $h_s^l(P|R) \leq h_s^l(Q|R)$.*

Proof. See [4]. ∎

3. Conditional Logical entropy of quantum dynamical systems

The assertion of the following proposition will be proved that will be useful in further propositions.

Proposition 3.1. *Let P_1, P_2 and Q be finite partitions of a QL corresponding to a state s having Bayes' Property. Then*

(i) $h_s^l(P_1|P_2 \vee Q) \leq h_s^l(P_1|P_2)$,

(ii) $h_s^l(P_1 \vee P_2|Q) \leq h_s^l(P_1|Q) + h_s^l(P_2|Q)$.

Proof. (i) Let $P_1 = \{a_1, ..., a_n\}$, $P_2 = \{b_1, ..., b_m\}$ and $Q = \{c_1, ..., c_r\}$. Since $P_1 \vee P_2$ is a partition, by Lemma 2.6, we have

$$\sum_{k=1}^{r} s(a_i \wedge b_j \wedge c_k) = s(a_i \wedge b_j)$$

Thus

$$\begin{aligned} h_s^l(P_1|P_2) &= \sum_{i=1}^{n}\sum_{j=1}^{m} s(a_i \wedge b_j)(s(b_j) - s(a_i \wedge b_j)) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{r} s(a_i \wedge b_j \wedge c_k)(\sum_{k=1}^{r} s(b_j \wedge c_k) - \sum_{k=1}^{r} s(a_i \wedge b_j \wedge c_k)) \\ &\geq \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{r} s(a_i \wedge b_j \wedge c_k)(s(b_j \wedge c_k) - s(a_i \wedge b_j \wedge c_k)) \\ &= h_s^l(P_1|P_2 \vee Q). \end{aligned}$$

(ii) From the first part of this proposition and Proposition 2.9 (ii), we may write

$$\begin{aligned} h_s^l(P_1 \vee P_2|Q) &= h_s^l(P_1|Q) + h_s^l(P_2|P_1 \vee Q) \\ &\leq h_s^l(P_1|Q) + h_s^l(P_2|Q). \end{aligned}$$

∎

Definition 3.2. [12] Let L be a QL and $\varphi : L \to L$ be a map with the following properties:

(i) $\varphi(a \vee b) = \varphi(a) \vee \varphi(b)$, $\forall a, b \in B$;

(ii) $\varphi(a \wedge b) = \varphi(a) \wedge \varphi(b)$, $\forall a, b \in B$;

(iii) $\varphi(a') = (\varphi(a))'$, $\forall a \in B$.

$\varphi : L \to L$ with respect to a state s is called state preserving if $s(\varphi(a)) = s(a)$ for every $a \in L$. Then the triple (L, s, φ) is said a quantum dynamical system where the state s having Bayes' Property.

Definition 3.3. [4] Let (L, s, φ) be a quantum dynamical system and P be a partition of (L, s). The logical entropy of T respect to P is defined by

$$h_s^l(\varphi, P) = \lim_{n\to\infty} \frac{1}{n} h_s^l(\vee_{i=1}^{n} \varphi^i P).$$

The logical entropy of φ is defined by

$$h_s^l(\varphi) = \sup_P h_s^l(\varphi, P)$$

where the supremum is taken over all finite partitions of (L, s).

In the following proposition, the existence of the limit in Definition 3.5, is shown.

Proposition 3.4. *Let (L, s, φ) be a quantum dynamical system and P, Q be partitions of (L, s). Then $\lim_{n\to\infty} \frac{1}{n} h_s^l(\vee_{i=0}^{n-1} \varphi^i P|Q)$ exists.*

Proof. Let $a_n = h_s^l(\vee_{i=0}^{n-1} \varphi^i P|Q)$. We show that for $n, t \in \mathbf{N}$, $a_{n+t} \leq a_n + a_t$ and then, by Theorem 4.9 in [19], $\lim_{n\to\infty} \frac{1}{n} a_n$ exists and equals $\inf_n \frac{a_n}{n}$. By Proposition 3.1, (ii) we have

$$\begin{aligned} a_{n+t} &= h_s^l(\vee_{i=0}^{n+t-1} \varphi^i P|Q) \\ &\leq h_s^l(\vee_{i=0}^{n-1} \varphi^i P|Q) + h_s^l(\vee_{i=n}^{n+t-1} \varphi^i P|Q) \\ &= a_n + h_s^l(\vee_{i=0}^{t-1} \varphi^{n+i} P|Q) \\ &= a_n + h_s^l(\varphi^n(\vee_{i=0}^{t-1} \varphi^i P)|Q) \\ &= a_n + h_s^l(\vee_{i=0}^{t-1} \varphi^i P|Q) \\ &= a_n + a_t. \end{aligned}$$

∎

The second stage and the final stage of the definition of the conditional logical entropy of a quantum dynamical system (L, s, φ) are given in the next definitions.

Definition 3.5. Let (L, s, φ) be a quantum dynamical system and P be a partition of (L, s). The quantum conditional logical entropy of P with respect to Q is defined by:

$$h_s^l(\varphi, P|Q) = \lim_{n\to\infty} \frac{1}{n} h_s^l(\vee_{i=0}^{n-1} \varphi^i P|Q).$$

Definition 3.6. Let (L, s, φ) be a quantum dynamical system and Q be a partition of (L, s). The quantum conditional logical entropy of φ is defined as following:

$$h_s^l(\varphi|Q) = \sup_P h_s^l(\varphi, P|Q)$$

where the supremum is taken over all finite partitions P of (L, s).

Now, the notion $h_s^l(\varphi, P|Q)$ under the relation of s-refinement will be studied.

Proposition 3.7. *Let (L, s, φ) be a quantum dynamical system and P_1, P_2 and Q be partitions of (L, s). Then*

(i) $h_s^l(\varphi, P_1|Q) \geq 0$,

(ii) $Q = \{0, 1\} \Longrightarrow h_s^l(\varphi, P_1|Q) = h_s^l(\varphi, P_1)$,

(iii) $h_s^l(\varphi, P_1|Q) \leq h_s^l(\varphi, P_1)$,

(iv) *If $P_1 \preceq_s P_2$ and L be distributive then $h_s^l(\varphi, P_1|Q) \leq h_s^l(\varphi, P_2|Q)$,*

(v) *If φ is invertible then $h_s^l(\varphi, \varphi^{-1}P_1|\varphi^{-1}Q) = h_s^l(\varphi, P_1|Q)$.*

Proof. (i), (ii) are clear.
(iii) Since for each $n \in \mathbb{N}$, $\vee_{i=0}^{n-1}\varphi^i P_1$ is a partition of (L,s), from Proposition 2.9 (iv) we get $h_s^l(\vee_{i=0}^{n-1}\varphi^i P_1|Q) \leq h_s^l(\vee_{i=0}^{n-1}\varphi^i P_1)$. So by Definitions 3.3 and 3.5 the assertion holds.
(iv) $P_1 \preceq_s P_2$ implies that for each $n \in \mathbb{N}$, $\vee_{i=0}^{n-1}\varphi^i P_1 \preceq_s \vee_{i=0}^{n-1}\varphi^i P_2$. So, by Proposition 2.11 (ii), we obtain the result.
(v) From [12], we have $\vee_{i=0}^{n-1}\varphi^i(\varphi^{-1}P_1) = \vee_{i=0}^{n-1}\varphi^{-1}(\varphi^i P_1) = \varphi^{-1}(\vee_{i=0}^{n-1}\varphi^i P_1)$. So

$$\begin{aligned} h_s^l(\varphi, \varphi^{-1}P_1|\varphi^{-1}Q) &= \lim_{n\to\infty}\frac{1}{n}h_s^l(\vee_{i=0}^{n-1}\varphi^i(\varphi^{-1}P_1)|\varphi^{-1}Q) \\ &= \lim_{n\to\infty}\frac{1}{n}h_s^l(\varphi^{-1}(\vee_{i=0}^{n-1}\varphi^i P_1)|\varphi^{-1}Q) \\ &= \lim_{n\to\infty}\frac{1}{n}h_s^l(\vee_{i=0}^{n-1}\varphi^i P_1|Q) \\ &= h_s^l(\varphi, P_1|Q). \end{aligned}$$

∎

In the next proposition, we study the notion $h_s^l(\varphi, P|Q)$ under the concept of common refinement.

Proposition 3.8. *Let (L,s,φ) be a quantum dynamical system and P_1, P_2 and Q be partitions of (L,s). Then*

$$h_s^l(\varphi, P_1 \vee P_2|Q) \leq h_s^l(\varphi, P_1|Q) + h_s^l(\varphi, P_2|Q).$$

Proof. We first show that $\vee_{i=0}^{n-1}\varphi^i(P_1 \vee P_2) = (\vee_{i=0}^{n-1}\varphi^i P_1) \vee (\vee_{i=0}^{n-1}\varphi^i P_2)$. We have

$$\begin{aligned} \varphi(P_1 \vee P_2) &= \{\varphi(a \wedge b) : a \in P_1, b \in P_2\} \\ &= \{\varphi(a) \wedge \varphi(b) : a \in P_1, b \in P_2\} \\ &= \varphi(P_1) \vee \varphi(P_2). \end{aligned}$$

Thus since $\varphi(P_1)$ and $\varphi(P_2)$ are partitions of (L,s) we get

$$\varphi^2(P_1 \vee P_2) = \varphi(\varphi(P_1 \vee P_2)) = \varphi(\varphi(P_1) \vee \varphi(P_2)) = \varphi^2(P_1) \vee \varphi^2(P_2).$$

So by induction we have $\varphi^n(P_1 \vee P_2) = \varphi^n(P_1) \vee \varphi^n(P_2)$. Hence we may write

$$\begin{aligned} \vee_{i=0}^{n-1}\varphi^i(P_1 \vee P_2) &= \varphi(P_1) \vee \varphi(P_2) \vee \varphi^2(P_1) \vee \varphi^2(P_2) \vee ...\varphi^n(P_1) \vee \varphi^n(P_2) \\ &= (\vee_{i=0}^{n-1}\varphi^i P_1) \vee (\vee_{i=0}^{n-1}\varphi^i P_2). \end{aligned}$$

Now, from Proposition 3.1 (ii), we have

$$\begin{aligned} h_s^l(\varphi, P_1 \vee P_2|Q) &= \lim_{n\to\infty}\frac{1}{n}h_s^l(\vee_{i=0}^{n-1}\varphi^i(P_1 \vee P_2)|Q) \\ &= \lim_{n\to\infty}\frac{1}{n}h_s^l((\vee_{i=0}^{n-1}\varphi^i P_1) \vee (\vee_{i=0}^{n-1}\varphi^i P_2)|Q) \\ &\leq \lim_{n\to\infty}\frac{1}{n}h_s^l(\vee_{i=0}^{n-1}\varphi^i P_1)|Q) + \lim_{n\to\infty}\frac{1}{n}h_s^l(\vee_{i=0}^{n-1}\varphi^i P_2)|Q) \\ &= h_s^l(\varphi, P_1|Q) + h_s^l(\varphi, P_2|Q). \end{aligned}$$

∎

In the following proposition, some ergodic properties of $h_s^l(\varphi|Q)$ will be studied.

Proposition 3.9. *Let (L, s, φ) be a quantum dynamical system and Q be a partition of (L, s). Then*

(i) $h_s^l(id|Q) = 0$, *where id is the identity function,*

(ii) $h_s^l(\varphi|Q) \leq h_s^l(\varphi)$,

(iii) $Q = \{0, 1\} \Longrightarrow h_s^l(\varphi|Q) = h_s^l(\varphi)$,

(iv) $h_s^l(\varphi|Q_1 \vee Q_2) \leq h_s^l(\varphi|Q_1)$.

Proof. (i) Let P be an arbitrary partition of (L, s). Since $\vee_{i=0}^{n-1}(id)^i P = P$, we have

$$h_s^l(id|Q) = \sup_P h_s^l(id, P|Q) = \lim_{n\to\infty} \frac{1}{n} h_s^l(P|Q) = 0.$$

(ii), (iii) Follow from Proposition 3.7 (ii) and (iii).

(iv) From Proposition 3.1 (i), for each partition P of (L, s) we get

$$h_s^l(\varphi, P|Q_1 \vee Q_2) \leq h_s^l(\varphi, P|Q_1).$$

So, by Definition 3.6, the assertion holds. ∎

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SOME RESULTS ON n-CAMINA PAIRS OF GROUPS

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Abstract. In [6], Lewis found a bound for $|Z(G)|$ in terms of $[G : Z(G)]$ where $(G, Z(G))$ is an Camina pair of groups. In this paper we generalize this concept to n-Camina pairs of groups with respect to the variety of nilpotent groups of class at most n, and find a bound for $\left|\frac{Z_n(G)}{Z_{n-1}(G)}\right|$ where $(G, Z_n(G))$ is an n-Camina pair of groups and G is a finite p-group.

Keywords: n-nilpotent group, n-Camina pair of groups, upper central series.

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1. Introduction

Let F be a free group freely generated by a countable set $\{x_1, x_2, ...\}$. Let $\mathcal{V}$ be the variety of groups defined by a set of laws V. It will be assumed that the reader is familiar with the notations of verbal subgroup, $V(G)$, and marginal subgroup, $V^*(G)$, associated to a variety of groups and a given group G. See, also, [4] for more information on varieties of groups.

In this paper, we always assume that $\mathcal{V}$ is the variety of groups defined by the set of laws V.

Let G be any group with a normal subgroup N, then we define $[NV^*G]$ to be the subgroup of G generated by the following set:

$$\{\nu(g_1, ..., g_{i-1}, g_i n, g_{i+1}, ..., g_r)\nu(g_1, ..., g_r)^{-1} |,\ 1 \leq i \leq r,\ \nu \in V,\ g_i \in G,\ n \in N\}.$$

It is easy to check that $[NV^*G]$ is the smallest normal subgroup T of G contained in N, such that

$$\frac{N}{T} \subseteq V^*\left(\frac{G}{T}\right). \qquad \text{(See [1]-[4], [5].)}$$

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We define the upper $\mathcal{V}$- marginal series of G to be:

$$1 = V_0^*(G) \leq V_1^*(G) = V^*(G) \leq \cdots \leq V_r^*(G) \leq \cdots,$$

where

$$\frac{V_r^*(G)}{V_{r-1}^*(G)} = V^*\left(\frac{G}{V_{r-1}^*(G)}\right), \quad \text{for } r > 0.$$

The corresponding lower $\mathcal{V}$- marginal series of G is given by

$$G = V_0(G) \geq V_1(G) = V(G) \geq \cdots \geq V_r(G) \geq \cdots,$$

where

$$V_r(G) = [V_{r-1}(G)V^*G], \quad \text{for } r > 0.$$

If H is a subgroup of G, we set $[H,_1 G] = [H, G]$ and by induction

$$[H,_n G] = [[H,_{n-1} G], G] \quad (n \geq 1).$$

In the special case when $\mathcal{V}$ is the variety of nilpotent groups of class at most n, $\mathcal{N}_n$, then $[NV^*G] = [N,_n G]$, $V_r(G) = \gamma_{rn+1}(G)$ and $V_r^*(G) = Z_{rn}(G)$, where $\gamma_i(G)$ is i-th term of lower central series and $Z_i(G)$ is i-th term of upper central series (see [8]).

Now, we have the following lemma:

Lemma 1.1. *Let $\mathcal{V}$ be a variety of groups defined by the set of laws V and G be a group and $B, N \trianglelefteq G$ and $N \leq B$, then $[BV^*G] \leq N$ if and only if $\frac{B}{N} \leq V^*\left(\frac{G}{N}\right)$.*

Proof. See Lemma 2.2 of [9]. ∎

Corollary 1.2. *Let G be a group and $B, N \trianglelefteq G$ and $N \leq B$, then $[B,_n G] \leq N$ if and only if $\frac{B}{N} \leq Z_n\left(\frac{G}{N}\right)$.*

Proof. The assertion follows immediately from Lemma 1.1 where $\mathcal{V}$=$\mathcal{N}_n$. ∎

Definition 1.3. A group G is said to be $\mathcal{V}$-nilpotent if there exists a series

$$1 = G_0 \leq G_1 \leq \cdots \leq G_r = G \qquad (*)$$

such that

$$G_i \trianglelefteq G \quad \text{and} \quad \frac{G_{i+1}}{G_i} \leq V^*\left(\frac{G}{G_i}\right), \quad \text{for } 0 \leq i \leq r-1.$$

The length of the shortest series $(*)$ is the $\mathcal{V}$-nilpotent class of G.

Theorem 1.4. *Let $1 = G_0 \leq G_1 \leq ... \leq G_r = G$ be a series in a $\mathcal{V}$-nilpotent group G, then the following properties hold*

(i) $V_r(G) = 1$,

(ii) $V_r^*(G) = G$.

Proof. See Theorem 2.4 of [9]. ∎

Definition 1.5. A group G is said to be n-nilpotent if there exists a series

$$1 = G_0 \leq G_1 \leq ... \leq G_r = G \qquad (*)$$

such that

$$G_i \trianglelefteq G \text{ and } \frac{G_{i+1}}{G_i} \leq Z_n(\frac{G}{G_i}), \quad \text{for } 0 \leq i \leq r-1.$$

The length of the shortest series $(*)$ is the n-nilpotent class of G.

The next corollary is a result of Theorem 1.4, where $\mathcal{V}$=$\mathcal{N}_n$.

Corollary 1.6. *Let $1 = G_0 \leq G_1 \leq \cdots \leq G_r = G$ be a series in a n-nilpotent group G, then $\gamma_{rn+1}(G) = 1$ and $Z_{rn}(G) = G$.*

Proof. It follows immediately from Theorem 1.4. ∎

Lemma 1.7. *Let G be a $\mathcal{V}$-nilpotent group of class c, then $V_r(G) \leq V^*_{c-r}(G)$ for $0 \leq r \leq c$.*

Proof. We use induction on n, we know $V_0(G) = G$ and $V_c^*(G) = G$. Now, let $V_r(G) \leq V^*_{c-r}(G)$, so

$$V_{r+1}(G) = [V_r(G)V^*G] \leq [V^*_{c-r}(G)V^*G] \leq V^*_{c-r-1}(G).$$ ∎

Corollary 1.8. *Let G be a n-nilpotent group of class c, then $\gamma_{rn+1}(G) \leq Z_{(c-r)n}(G)$ for $0 \leq r \leq c$.*

Proof. This follows immediately from Lemma 1.7, where $\mathcal{V}$=$\mathcal{N}_n$. ∎

Lemma 1.9. *Let G be a group and $A, B \trianglelefteq G$, then $[AB,_n G] = [A,_n G][B,_n G]$.*

Proof. It is easily obtained by induction on n. ∎

2. n-Camina pairs

Let $\mathcal{N}_n$ be the variety of nilpotent groups of class at most n and $n \geq 2$, in this section, at first we define n-Camina pairs of groups and mention some properties of them. The main theorem shows that where (G, M) is an n-Camina pair of groups, there exists a connection between M and n-lower and n-upper central series of G.

Definition 2.1. A pair (G, M) is called a n-Camina pair of groups if $1 < M < G$ is a normal subgroup of G and for every elements $g_1, ..., g_n \in G - M$ and for every element $m \in M$, there exists element $y \in G$ such that $[y, [g_1, ..., g_n]] = m$.

In the following lemma, we mention the first and the most applied property of n-Camina pairs of groups:

Lemma 2.2. *Let (G, M) be a n-Camina pair of groups, then*

$$Z_n(G) \leq M \leq \gamma_{n+1}(G).$$

Proof. By definition, we have $M \leq \gamma_{n+1}(G)$ and if we assume that $Z_n(G) \nleq M$, then there exists element $g \in Z_n(G) - M$, so, for every elements $g_1, ..., g_{n-1} \in G - M$ and for every element $m \in M$, there exists element $y \in G$ such that $[y, [g_1, ..., g_{n-1}, g]] = m$, but $m = 1$ since $[g_1, ..., g_{n-1}, g] \in [\gamma_{n-1}(G), Z_n(G)] \leq Z_1(G) = Z(G)$, which is a contradiction. ∎

Lemma 2.3. *Let (G, M) be a n-Camina pair of groups and $H \trianglelefteq G$, $H < M$, then $\left(\frac{G}{H}, \frac{M}{H}\right)$ is an n-Camina pair of groups.*

Proof. It is clear by the definition. ∎

In the next theorem, we will prove that M is a term in both n-upper and n-lower central series, where (G, M) is a n-Camina pair of groups and G is n-nilpotent of class c .

Theorem 2.4. *Let (G, M) be a n-Camina pair of groups. If G is n-nilpotent of class c, then*

(i) $M = Z_{(c-r+1)n}(G)$,

(ii) $M = \gamma_{(r-1)n+1}(G)$,

for some r where $1 < r \leq c$.

Proof. (i) By assumption, we have (G, M) is a n-Camina pair of groups, so $Z_n(G) \leq M$. If $Z_n(G) = M$, then we obtain the conclusion. Thus, we may assume that $Z_n(G) < M$, by Lemma 2.3, $\left(\frac{G}{Z_n(G)}, \frac{M}{Z_n(G)}\right)$ is an n-Camina pair of groups, therefore,

$$Z_n\left(\frac{G}{Z_n(G)}\right) \leq \frac{M}{Z_n(G)}.$$

Hence,

$$\frac{Z_{2n}(G)}{Z_n(G)} \leq \frac{M}{Z_n(G)}.$$

Therefore, $Z_{2n}(G) \leq M$. By repetition of this process, it gives $M = Z_{(c-r+1)n}(G)$ for some r with $1 < r \leq c$. Hence, if for every $1 < r \leq c$, $Z_{(c-r+1)n}(G) < M$, then $Z_{(c-1)n}(G) < M$, but

$$\frac{G}{Z_{(c-1)n}(G)} = \frac{Z_{cn}(G)}{Z_{(c-1)n}(G)} = Z_n\left(\frac{G}{Z_{(c-1)n}(G)}\right).$$

Thus, $\gamma_{n+1}(G) \leq Z_{(c-1)n}(G) < M$, which is a contradiction.

(ii) By Corollary 1.8 and (i), we have

$$\gamma_{(r-1)n+1}(G) \leq Z_{(c-r+1)n}(G) = M.$$

We claim that $\gamma_{(r-1)n+1}(G) = M$. Hence, if $\gamma_{(r-1)n+1}(G) < M$, then

$$\left(\frac{G}{\gamma_{(r-1)n+1}(G)}, \frac{M}{\gamma_{(r-1)n+1}(G)}\right)$$

is an n-Camina pair of groups, so,

$$\frac{\gamma_{(r-2)n+1}(G)}{\gamma_{(r-1)n+1}(G)} \leq Z_n\left(\frac{G}{\gamma_{(r-1)n+1}(G)}\right) \leq \frac{M}{\gamma_{(r-1)n+1}(G)}.$$

Therefore, $\gamma_{(r-2)n+1}(G) \leq M = Z_{(c-r+1)n}(G)$. Hence $\gamma_{(c-1)n+1}(G) \leq Z_0(G) = 1$, which is a contradiction. ∎

Theorem 2.5. *Let (G, M) be a n-Camina pair of groups. If $M = Z_n(G)$ and G is n-nilpotent of class c, then $\dfrac{Z_{2n}(G)}{Z_n(G)}$ has exponent p.*

Proof. At first, we show that, if $H \trianglelefteq G$ and $H \subset Z_n(G)$, then

$$Z_n\left(\frac{G}{H}\right) = \frac{Z_n(G)}{H}.$$

By [4],

$$\frac{Z_n(G)H}{H} \subseteq Z_n\left(\frac{G}{H}\right)$$

and since $H \subset Z_n(G)$, we have

$$\frac{Z_n(G)}{H} \subseteq Z_n\left(\frac{G}{H}\right).$$

On the other hand, $(G, Z_n(G))$ is an n-Camina pair of groups and $H \subset Z_n(G)$, so

$$\left(\frac{G}{H}, \frac{Z_n(G)}{H}\right)$$

is an n-Camina pair of groups, hence $Z_n(\dfrac{G}{H}) \subseteq \dfrac{Z_n(G)}{H}$. By Lemma 1.9, we have

$$[(Z_{2n}(G))^p, {}_n\, G] = [Z_{2n}(G), {}_n\, G]^p \leq (Z_n(G))^p.$$

Therefore, by Corollary 1.2, we have

$$\frac{(Z_{2n}(G))^p}{(Z_n(G))^p} \leq Z_n\left(\frac{G}{(Z_n(G))^p}\right) = \frac{Z_n(G)}{(Z_n(G))^p},$$

that is, $(Z_{2n}(G))^p \leq Z_n(G)$. Thus, $\dfrac{Z_{2n}(G)}{Z_n(G)}$ has exponent p. ∎

In closing this section, we mention two important results.

Corollary 2.6. *Let $(G, Z_n(G))$ be a n-Camina pair of groups. If G is n-nilpotent of class c, then $Z_n(G) = \gamma_{(c-1)n+1}(G)$.*

Proof. By Theorem 2.4, the result holds. ∎

Corollary 2.7. *Let $(G, Z_n(G))$ be a n-Camina pair of groups. If G is n-nilpotent of class 2, then*

(i) $Z_n(G) = \gamma_{n+1}(G)$

(ii) $\dfrac{G}{Z_n(G)}$ *has exponent p.*

Proof. (i)By the previous corollary, the result holds.

(ii) Since $Z_{2n}(G) = G$, we obtain the assertion from Theorem 2.5. ∎

3. Bounding $|Z_n(G)/Z_{n-1}(G)|$

Let $\mathcal{N}_n$ be the variety of nilpotent groups of class at most n ($n \geq 2$), and let $(G, Z_n(G))$ an n-Camina pair of groups. Then, by Lemma 2.2, we have $Z_n(G) \leq \gamma_{n+1}(G)$. Therefore, we have $Z_n(G) \leq G'$. Thus, we may assume that $Z_n(G) < G'$. In this section, we try to find a bound for $\left|\dfrac{Z_n(G)}{Z_{n-1}(G)}\right|$ in terms of $[G : Z_n(G)]$ and $[G : G']$ under some conditions.

Let G be a group. We define

$$D_G(g) = \{x \in G \mid [g, x] \in Z_n(G)\},$$

and

$$C_G(g) = \{x \in G \mid [g, x] \in Z_{n-1}(G)\},$$

which are subgroups of G and $C_G(g)$ is a subgroup of $D_G(g)$.

The following lemma is important in our study.

Lemma 3.1. *Let $(G, Z_n(G))$ be an n-Camina pair of groups. If we have $g_1, ..., g_n \in G - Z_n(G)$, then*

$$\frac{D_G([g_1, ..., g_n])}{C_G([g_1, ..., g_n])} \cong \frac{Z_n(G)}{Z_{n-1}(G)} = Z\left(\frac{G}{Z_{n-1}(G)}\right).$$

In particular, $(D_G([g_1, ..., g_n]))' \leq C_G([g_1, ..., g_n])$.

Proof. We define the map

$$f : D_G([g_1, ..., g_n]) \to \frac{Z_n(G)}{Z_{n-1}(G)},$$

where

$$f(d) = [d, [g_1, ..., g_n]]Z_{n-1}(G).$$

By definition of $D_G(g_1, ..., g_n)$, we know that f is well-defined and f is a homomorphism. Since $(G, Z_n(G))$ is an n-Camina pair of groups, for every element $Z_{n-1}(G) \in \frac{Z_n(G)}{Z_{n-1}(G)}$ there exists $y \in G$, so that

$$[y, [g_1, ..., g_n]] = z \in Z_n(G).$$

Thus, $y \in D_G([g_1, ..., g_n])$. By the definition of $D_G([g_1, ..., g_n])$, that is f is onto. Clearly, $ker(f) = C_G([g_1, ..., g_n])$. By the 1st isomorphism theorem, we have

$$\frac{D_G([g_1, ..., g_n])}{C_G([g_1, ..., g_n])} \cong \frac{Z_n(G)}{Z_{n-1}(G)}.$$ ∎

Now, we find the first bound for $\left|\frac{Z_n(G)}{Z_{n-1}(G)}\right|$ in terms of $[G : Z_n(G)]$.

Theorem 3.2. *Let $(G, Z_n(G))$ be a n-Camina pair of groups and G be a p-group. If there exist $g_1, ..., g_n \in G$ such that $o([g_1, ..., g_n]Z_n(G)) = p^k$, where $k \geq 2$ and $\frac{Z_n(G)}{Z_{n-1}(G)}$ is an elementary abelian p-group, then*

$$\left|\frac{Z_n(G)}{Z_{n-1}(G)}\right|^k p^k \leq [G : Z_n(G)].$$

Proof. At first, we show that

$$C_G([g_1, ..., g_n]^{p^{i+1}}) \geq D_G([g_1, ..., g_n]^{p^i}) \text{ for } 0 \leq i \leq k-1.$$

Suppose that $a \in D_G([g_1, ..., g_n]^{p^i})$, so $[a, [g_1, ..., g_n]^{p^i}] \in Z_n(G)$, that is,

$$[a, [g_1, ..., g_n]^{p^i}],\ Z_{n-1}(G) \in \frac{Z_n(G)}{Z_{n-1}(G)}.$$

Since $\frac{Z_n(G)}{Z_{n-1}(G)}$ is an elementary abelian p-group, we have

$$([a, [g_1, ..., g_n]^{p^i}]Z_{n-1}(G))^p = Z_{n-1}(G).$$

Therefore, $([aZ_{n-1}(G), [g_1Z_{n-1}(G), ..., g_nZ_{n-1}(G)]^{p^i}])^p = Z_{n-1}(G)$.

Since

$$[aZ_{n-1}(G), [g_1Z_{n-1}(G), ..., g_nZ_{n-1}(G)]^{p^i}] \in Z\left(\frac{G}{Z_{n-1}(G)}\right),$$

we have

$$\begin{aligned}&([aZ_{n-1}(G), [g_1Z_{n-1}(G), ..., g_nZ_{n-1}(G)]^{p^i}])^p\\ &\quad = [aZ_{n-1}(G), [g_1Z_{n-1}(G), ..., g_nZ_{n-1}(G)]^{p^{i+1}}] = Z_{n-1}(G).\end{aligned}$$

So, $[a, [g_1, ..., g_n]^{p^{i+1}}]Z_{n-1}(G) = Z_{n-1}(G)$, that is, $[a, [g_1, ..., g_n]^{p^{i+1}}] \in Z_{n-1}(G)$. This implies that $a \in C_G([g_1, ..., g_n]^{p^{i+1}})$.

Since $[g_1, ..., g_n]^{p^i} \notin Z_n(G)$ for $0 \leq i \leq k-1$, by Lemma 3.1,

$$\frac{D_G([g_1, ..., g_n]^{p^i})}{C_G([g_1, ..., g_n]^{p^i})} \cong \frac{Z_n(G)}{Z_{n-1}(G)}.$$

Clearly, $[g_1, ..., g_n] \in C_G([g_1, ..., g_n])$, hence

$$[g_1, ..., g_n]Z_n(G) \in \frac{C_G([g_1, ..., g_n])}{Z_n(G)}.$$

It follows that $p^k|[C_G([g_1, ..., g_n]) : Z_n(G)]$. We have

$$\begin{aligned}[G : Z_n(G)] &= [G : C_G([g_1, ..., g_n])][C_G([g_1, ..., g_n]) : Z_n(G)]\\ &\geq [G : C_G([g_1, ..., g_n])]p^k.\end{aligned}$$

Now, we calculate $[G : C_G([g_1, ..., g_n])]$. Clearly,

$$\begin{aligned}&[G : C_G([g_1, ..., g_n])]\\ &\quad = [G : (C_G([g_1, ..., g_n])^{p^k}] \prod_{i=0}^{k-1} [C_G([g_1, ..., g_n]^{p^{i+1}}) : C_G([g_1, ..., g_n]^{p^i})].\end{aligned}$$

Since $[g_1, ..., g_n]^{p^k} \in Z_n(G)$, we have $C_G([g_1, ..., g_n]^{p^k}) = G$, so

$$\begin{aligned}[G : C_G([g_1, ..., g_n])] &= \prod_{i=0}^{k-1} [C_G([g_1, ..., g_n]^{p^{i+1}}) : C_G([g_1, ..., g_n]^{p^i})]\\ &\geq \prod_{i=0}^{k-1} [D_G([g_1, ..., g_n]^{p^i}) : C_G([g_1, ..., g_n]^{p^i})]\\ &= \prod_{i=0}^{k-1} \left|\frac{Z_n(G)}{Z_{n-1}(G)}\right|\\ &= \left|\frac{Z_n(G)}{Z_{n-1}(G)}\right|^k.\end{aligned}$$

Finally, we have

$$[G : Z_n(G)] \geq \left|\frac{Z_n(G)}{Z_{n-1}(G)}\right|^k p^k. \qquad \blacksquare$$

Now, we mention a sufficient condition for $\frac{D_G([g_1,...,g_n])}{Z_n(G)}$ to be abelian, where $g_1,...,g_n \in G - Z_n(G)$.

Lemma 3.3. *Let $(G, Z_n(G))$ be an n-Camina pair of groups. If there exist $g_1,...,g_n \in G - Z_n(G)$ such that $C_G([g_1,...,g_n]) \cap G' = Z_n(G)$, then $\frac{D_G([g_1,...,g_n])}{Z_n(G)}$ is abelian.*

Proof. By Lemma 3.1, $(D_G([g_1,...,g_n]))' \leq C_G([g_1,...,g_n])$.

On the other hand, $(D_G([g_1,...,g_n]))' \leq G'$ so

$$(D_G([g_1,...,g_n]))' \leq C_G([g_1,...,g_n]) \cap G' = Z_n(G).$$

Thus, $\frac{D_G([g_1,...,g_n])}{Z_n(G)}$ is abelian. $\blacksquare$

In the next lemma, we give a bound for $\left|\frac{Z_n(G)}{Z_{n-1}(G)}\right|$ in terms of the index of the derived subgroup.

Lemma 3.4. *Let $(G, Z_n(G))$ be an n-Camina pair of groups where $n \geq 2$. If there exist $g_1,...,g_n \in Z_{n+1}(G) - Z_n(G)$, then*

$$\left|\frac{Z_n(G)}{Z_{n-1}(G)}\right| \leq [G : G'Z_{n+1}(G)].$$

In particular,

$$\left|\frac{Z_n(G)}{Z_{n-1}(G)}\right| \leq [G : G'].$$

Proof. For every element $a \in G$, we have

$$[[g_1,...,g_n], a] \in [[Z_{n+1}(G), G] \leq Z_n(G),$$

that is, $a \in D_G([g_1,...,g_n])$, so $G = D_G([g_1,...,g_n])$.

By Lemma 3.1, $(D_G([g_1,...,g_n]))' \leq C_G([g_1,...,g_n])$ so $G' \leq C_G([g_1,...,g_n])$. On the other hand, for every element $x \in Z_{n+1}(G)$, we have

$$[[g_1,...,g_n], x] \in [\gamma_n(G), Z_{n+1}(G)] \leq Z_1(G) \leq Z_{n-1}(G).$$

Therefore, $x \in C_G([g_1,...,g_n])$. This implies that $Z_{n+1}(G) \leq C_G([g_1,...,g_n])$. So, $G'Z_{n+1}(G) \leq C_G([g_1,...,g_n])$. Thus,

$$\begin{aligned}\left|\frac{Z_n(G)}{Z_{n-1}(G)}\right| &= [D_G([g_1,...,g_n]) : C_G([g_1,...,g_n])] \\ &= [G : C_G([g_1,...,g_n])] \leq [G : G'Z_{n+1}(G)].\end{aligned}$$ $\blacksquare$

Now, we show that $\left|\frac{Z_n(G)}{Z_{n-1}(G)}\right|$ can be bound by $[G : Z_n(G)]$ under some conditions.

Theorem 3.5. *Let $(G, Z_n(G))$ be an n-Camina pair of groups and G be a p-group. If there exist $g_1, ..., g_n \in Z_{n+1}(G) - Z_n(G)$ such that $\frac{D_G([g_1, ..., g_n])}{Z_n(G)}$ is abelian and $[G : D_G([g_1, ..., g_n])] = p$, then $\left|\frac{Z_n(G)}{Z_{n-1}(G)}\right|^2 \leq [G : Z_n(G)]$.*

Proof. Since G is a finite p-group and $[G : D_G([g_1, ..., g_n])] = p$, we have

$$D_G([g_1, ..., g_n]) \trianglelefteq G$$

and there exists $b \in G - D_G([g_1, ..., g_n])$ such that $G = \langle b \rangle D_G([g_1, ..., g_n])$. Since $b \in D_G(b)$, we have $G = D_G(b)D_G([g_1, ..., g_n])$. It follows that

$$\frac{D_G([g_1, ..., g_n]) \cap D_G(b)}{Z_n(G)} \leq \frac{Z_{n+1}(G)}{Z_n(G)},$$

so

$$D_G([g_1, ..., g_n]) \cap D_G(b) \leq Z_{n+1}(G).$$

We have

$$\begin{aligned}
&[G : D_G([g_1, ..., g_n]) \cap D_G(b)] \\
&\quad = [G : D_G([g_1, ..., g_n])][D_G([g_1, ..., g_n]) : D_G([g_1, ..., g_n]) \cap D_G(b)] \\
&\quad = p[G : D_G(b)].
\end{aligned}$$

Since $b \in G - D_G([g_1, ..., g_n])$, we have $[C_G(b) : Z_n(G)] \geq p$. Also, by Lemma 3.4,

$$\left|\frac{Z_n(G)}{Z_{n-1}(G)}\right| \leq [G : G'Z_{n+1}(G)].$$

Hence, we may conclude that

$$\begin{aligned}
\left|\frac{Z_n(G)}{Z_{n-1}(G)}\right| &\leq [G : G'Z_{n+1}(G)] \leq [G : Z_{n+1}(G)] \leq [G : D_G([g_1, ..., g_n]) \cap D_G(b)] \\
&= p[G : D_G(b)] \leq [C_G(b) : Z_n(G)][G : D_G(b)].
\end{aligned}$$

By Lemma 3.1, we deduce that $[D_G(b) : C_G(b)] = \left|\frac{Z_n(G)}{Z_{n-1}(G)}\right|$. This result implies that

$$\left|\frac{Z_n(G)}{Z_{n-1}(G)}\right|^2 \leq [G : Z_n(G)].$$ ∎

As a corollary, we obtain our bound when $[G' : Z_n(G)] = p$ and there exist $g_1, ..., g_n \in Z_{n+1}(G) - Z_n(G)$ such that $C_G([g_1, ..., g_n]) \cap G' < G'$.

Corollary 3.6. *Let $(G, Z_n(G))$ be an n-Camina pair of groups. If $[G' : Z_n(G)] = p$ and there exist $g_1, ..., g_n \in Z_{n+1}(G) - Z_n(G)$ such that $C_G([g_1, ..., g_n]) \cap G' < G'$, then*

$$\left|\frac{Z_n(G)}{Z_{n-1}(G)}\right|^2 \leq [G : Z_n(G)].$$

Proof. Since $[G' : Z_n(G)] = p$, we have

$$\left(\frac{G}{Z_n(G)}\right)' = \frac{G'}{Z_n(G)} \leq Z(\frac{G}{Z_n(G)}).$$

For any element $g \in G$, we define the map $f : G \to \frac{G'}{Z_n(G)}$, where $f(x) = [g, x]Z_n(G)$ for every element $x \in G$. It is easy to show that f is a homomorphism and $Ker(f) = D_G(g)$, so $\left|\frac{G}{D_G(g)}\right| \leq |\frac{G'}{Z_n(G)}| = p$ for every $g \in G$. Clearly, $Z_n(G) \leq C_G([g_1, ..., g_n]) \cap G'$, so

$$p = [G' : Z_n(G)] = [G' : C_G([g_1, ..., g_n]) \cap G'][C_G([g_1, ..., g_n]) \cap G' : Z_n(G)].$$

Therefore, $C_G([g_1, ..., g_n]) \cap G' = Z_n(G)$. Thus, by Lemma 3.3, $\frac{D_G([g_1, ..., g_n])}{Z_n(G)}$ is abelian. Since $C_G([g_1, ..., g_n]) \cap G' < G'$, we have $[G : D_G([g_1, ..., g_n])] = p$. So, the assertion is obtained from Theorem 3.5. ∎

Now, we define the set $\mathcal{L}(G)$, where $(G, Z_n(G))$ is an n-Camina pair of groups

$$\mathcal{L}(G) = \{x \in G \mid C_G(x) \cap G' > Z_n(G)\}.$$

If $G' = Z_n(G)$, then $\mathcal{L}(G)$ is empty, so $\mathcal{L}(G)$ is nonempty if $Z_n(G) < G'$. Clearly, G' and $C_G(G')$ are subsets of $\mathcal{L}(G)$.

In the next lemma, we give a second characterization of $\mathcal{L}(G)$.

Lemma 3.7. *Let $(G, Z_n(G))$ be an n-Camina pair of groups with $Z_n(G) < G'$. Then, we have $\mathcal{L}(G) = \bigcup_{a \in G' - Z_n(G)} C(a)$.*

Proof. If $x \in \mathcal{L}(G)$, then $C_G(x) \cap G' > Z_n(G)$, so there exists $b \in C_G(x) \cap G' - Z_n(G)$. Hence, $x \in C_G(b) \subseteq \bigcup_{a \in G' - Z_n(G)} C(a)$.

By Lemma 3.1, we deduce that $\mathcal{L}(G) \subseteq \bigcup_{a \in G' - Z_n(G)} C(a)$. This result implies that, if $x \in \bigcup_{a \in G' - Z_n(G)} C(a)$, then there exists $b \in G' - Z_n(G)$ such that $x \in C_G(b)$. It follows that $b \in (C_G(x) \cap G') - Z_n(G)$ so $Z_n(G) < C_G(x) \cap G'$, therefore, $x \in \mathcal{L}(G)$. ∎

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ON UPPER AND LOWER WEAKLY $\mathcal{I}$-CONTINUOUS MULTIFUNCTIONS

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Abstract. In this paper, we introduce and study a new generalization of $\mathcal{I}$-continuous multifunctions called weakly $\mathcal{I}$-continuous multifunctions in topological spaces.

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1. Introduction

It is well known that various types of functions play a significant role in the theory of classical point set topology. A great number of papers dealing with such functions have appeared, and a good number of them have been extended to the setting of multifunctions [2], [9], [12], [13], [14]. This implies that both, functions and multifunctions are important tools for studying other properties of spaces and for constructing new spaces from previously existing ones. The concept of ideals in topological spaces has been introduced and studied by Kuratowski [8] and Vaidyanathaswamy [17]. An ideal $\mathcal{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies (i) $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$ and (ii) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$. Given a topological space (X, τ) with an ideal $\mathcal{I}$ on X and if $\mathcal{P}(X)$ is the set of all subsets of X, a set operator $(.)^\star$: $\mathcal{P}(X) \to \mathcal{P}(X)$, called the local function [17] of A with respect to τ and $\mathcal{I}$, is defined as follows: for $A \subset X$, $A^\star(\tau, \mathcal{I}) = \{x \in X | U \cap A \notin \mathcal{I}$ for every $U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau | x \in U\}$. A Kuratowski closure operator $\mathrm{Cl}^\star(.)$ for a topology $\tau^\star(\tau, \mathcal{I})$ called the $*$-topology, finer than τ is defined by $\mathrm{Cl}^\star(A) = A \cup A^\star(\tau, \mathcal{I})$ when there is no chance of confusion, $A^\star(\mathcal{I})$ is denoted by $A^\star$. If $\mathcal{I}$ is an ideal on X, then $(X, \tau, \mathcal{I})$ is called an ideal topological space. In 1990, Jankovic and Hamlett [6] introduced the notion of $\mathcal{I}$-open sets in topological spaces. In 1992, Abd El-Monsef et al. [1] further investigated $\mathcal{I}$-open sets and $\mathcal{I}$-continuous functions. Several characterizations and properties of $\mathcal{I}$-open sets

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were provided in [1, 10]. Recently, Akdag [2] introduced and studied the concept of $\mathcal{I}$-continuous multifunctions in topological spaces. In this paper we introduce and study a new generalization of $\mathcal{I}$-continuous multifunctions called weakly $\mathcal{I}$-continuous multifunctions in topological spaces.

2. Preliminaries

Throughout this paper, (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces in which no separation axioms are assumed unless explicitly stated. Let A be a subset of a space X. For a subset A of (X, τ), $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ denote the closure of A with respect to τ and the interior of A with respect to τ, respectively. A subset S of an ideal topological space $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-open [6] if $S \subset \mathrm{Int}(S^*)$. The complement of an $\mathcal{I}$-closed set is said to be an $\mathcal{I}$-open set. The $\mathcal{I}$-closure and the $\mathcal{I}$-interior, that can be defined in the same way as $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively, will be denoted by $\mathcal{I}\,\mathrm{Cl}(A)$ and $\mathcal{I}\,\mathrm{Int}(A)$, respectively. The family of all $\mathcal{I}$-open (resp. $\mathcal{I}$-closed, regular open, regular closed) sets of (X, τ) is denoted by $\mathcal{I}O(X)$ (resp. $\mathcal{I}C(X)$, $RO(X)$, $RC(X)$). We set $\mathcal{I}O(X, x) = \{A : A \in \mathcal{I}O(X) \text{ and } x \in A\}$. By a multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$, we mean a point-to-set correspondence from X into Y, also we always assume that $F(x) \neq \varnothing$ for all $x \in X$. For a multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$, the upper and lower inverse of any subset A of Y by $F^+(A)$ and $F^-(A)$, respectively, that is $F^+(A) = \{x \in X : F(x) \subseteq A\}$ and $F^-(A) = \{x \in X : F(x) \cap A \neq \varnothing\}$. In particular, $F(y) = \{x \in X : y \in F(x)\}$ for each point $y \in Y$.

Definition 1 A multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is said to be:

1. upper weakly continuous [14], [15] if for each x of X and each open set V of Y containing $F(x)$, there exists an open set U containing x such that $F(U) \subset \mathrm{Cl}(V)$.

2. lower weakly continuous [14], [15] if for each x of X and each open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists an open set U containing x such that $F(u) \cap \mathrm{Cl}(V) \neq \emptyset$ for every $u \in U$;

3. upper $\mathcal{I}$-continuous [2] if for each $x \in X$ and each open set V of Y such that $x \in F^+(V)$, there exists $U \in \mathcal{I}O(X, x)$ such that $U \subset F^+(V)$;

4. lower $\mathcal{I}$-continuous [2] if for each $x \in X$ and each open set V of Y such that $x \in F^-(V)$, there exists $U \in \mathcal{I}O(X, x)$ such that $U \subset F^-(V)$;

5. $\mathcal{I}$-continuous [2] if it is both upper $\mathcal{I}$-continuous and lower $\mathcal{I}$-continuous;

6. lower almost $\mathcal{I}$-continuous [4] if for each $x \in X$ and for each open set V of Y such that $x \in F^-(V)$, there exists a $U \in \mathcal{I}O(X, x)$ such that $U \subset F^-(\mathrm{Int}(\mathrm{Cl}(V)))$,

7. upper almost $\mathcal{I}$-continuous [4] if for each $x \in X$ and for each open set V of Y such that $x \in F^+(V)$, there exists a $U \in \mathcal{I}O(X, x)$ such that $U \subset F^+(\mathrm{Int}(\mathrm{Cl}(V)))$;

8. almost $\mathcal{I}$-continuous [4] if it is both upper almost $\mathcal{I}$-continuous and lower almost $\mathcal{I}$-continuous;
9. lower weakly continuous [15] for each $x \in X$ and each open set V of Y such that $x \in F^-(V)$, there exists an open set U containing x such that $U \subset F^-(\mathrm{Cl}(V))$;
10. upper weakly continuous [15] for each $x \in X$ and each open set V of Y such that $x \in F^+(V)$, there exists an open set U containing x such that $U \subset F^+(\mathrm{Cl}(V))$;
11. upper slightly $\mathcal{I}$-continuous [5] at $x \in X$ if for each clopen set V of Y containing $F(x)$, there exists $U \in \mathcal{I}O(X)$ containing x such that $F(U) \subset V$;
12. lower slightly $\mathcal{I}$-continuous [5] at $x \in X$ if for each clopen set V of Y such that $F(x) \cap V \neq \varnothing$, there exists $U \in \mathcal{I}O(X)$ containing x such that $F(u) \cap V \neq \varnothing$ for every $u \in U$;
13. upper (lower) slightly $\mathcal{I}$-continuous [5] if it has this property at each point of X.

3. Weakly $\mathcal{I}$-continuous multifunctions

Definition 2 A multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is said to be:

1. upper weakly $\mathcal{I}$-continuous if for each $x \in X$ and each open set V of Y such that $x \in F^+(V)$, there exists $U \in \mathcal{I}O(X, x)$ such that $U \subset F^+(\mathrm{Cl}(V))$;
2. lower weakly $\mathcal{I}$-continuous if for each $x \in X$ and each open set V of Y such that $x \in F^-(V)$, there exists $U \in \mathcal{I}O(X, x)$ such that $U \subset F^-(\mathrm{Cl}(V))$;
3. weakly $\mathcal{I}$-continuous if it is both upper weakly $\mathcal{I}$-continuous and lower weakly $\mathcal{I}$-continuous.

The proof of the following Propositions are follows from their respective definitions and hence omitted.

Proposition 3.1 *Every upper (lower) almost $\mathcal{I}$-continuous multifunction is upper (lower) weakly $\mathcal{I}$-continuous.*

Proposition 3.2 *Every upper (lower) weakly $\mathcal{I}$-continuous multifunction is upper (lower) slightly $\mathcal{I}$-continuous.*

The following examples show that the converse of Propositions 3.1 and 3.2 is not true in general.

Example 3.3 Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a, c\}, X\}$, $\sigma = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and $\mathcal{I} = \{\emptyset, \{b\}\}$. Then the multifunction $F : (X, \tau, \mathcal{I}) \to (X, \sigma)$ defined by $F(x) = \{x\}$ for all $x \in X$ is upper weakly $\mathcal{I}$-continuous but is not upper almost $\mathcal{I}$-continuous.

Example 3.4 Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{b\}, X\}$, $\sigma = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. Then the multifunction $F : (X, \tau, \mathcal{I}) \to (X, \sigma)$ defined by $F(x) = \{x\}$ for all $x \in X$ is upper slightly $\mathcal{I}$-continuous but is not upper weakly $\mathcal{I}$-continuous.

Corollary 3.5 *Every $\mathcal{I}$-continuous multifunction is weakly $\mathcal{I}$-continuous.*

Theorem 3.6 *For a multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$, the following statements are equivalent :*

1. *F is upper weakly $\mathcal{I}$-continuous;*
2. *$F^+(V) \subset \mathcal{I}\,\mathrm{Int}(F^+(\mathrm{Cl}(V)))$ for any open set V of Y;*
3. *$\mathcal{I}\,\mathrm{Cl}(F^-(\mathrm{Int}(K))) \subset F^-(K)$ for any closed subset K of Y;*
4. *for each $x \in X$ and each open set V containing $F(x)$, there exists an $\mathcal{I}$-open set U containing x such that $F(U) \subset \mathrm{Cl}(V)$.*

Proof. (1)⇒(2): Let V be any open subset of Y and $x \in F^+(V)$. By (1), there exists an $\mathcal{I}$-open set U containing x such that $U \subset F^+(\mathrm{Cl}(V))$. Hence, $x \in \mathcal{I}\,\mathrm{Int}(F^+(\mathrm{Cl}(V)))$.
(2)⇒(1): Let V be any open subset of Y such that $x \in F^+(V)$. By (2), $x \in F^+(V) \subset \mathcal{I}\,\mathrm{Int}(F^+((\mathrm{Cl}(V))) \subset F^+(\mathrm{Cl}(V))$. Take $U = \mathcal{I}\,\mathrm{Int}(F^+(\mathrm{Cl}(V)))$. Thus, we obtain that F is upper weakly $\mathcal{I}$-continuous.
(2)⇔(3): Let K be any closed subset of Y. By (2), we have $F^+(Y \backslash K) = X \backslash F^-(K) \subset \mathcal{I}\,\mathrm{Int}(F^+(\mathrm{Cl}(Y \backslash K))) = \mathcal{I}\,\mathrm{Int}(F^+(Y \backslash \mathrm{Int}(K))) = X \backslash \mathcal{I}\,\mathrm{Cl}(F^-(\mathrm{Int}(K)))$. Thus, $\mathcal{I}\,\mathrm{Cl}(F^-(\mathrm{Int}(K))) \subset F^-(K)$. The converse is similar.
(1)⇔(2): Obvious. ∎

Theorem 3.7 *For a multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$, the following statements are equivalent :*

1. *F is lower weakly $\mathcal{I}$-continuous;*
2. *$F^-(V) \subset \mathcal{I}\,\mathrm{Int}(F^-(\mathrm{Cl}(V)))$ for any open set V of Y;*
3. *$\mathcal{I}\,\mathrm{Cl}(F^-(\mathrm{Int}(K))) \subset F^+(K)$ for any closed subset K of Y;*
4. *for each $x \in X$ and each open set V of Y such that $f(x) \cap V \neq \emptyset$, there exists an $\mathcal{I}$-open set U containing x such that $F(u) \cap \mathrm{Cl}(V) \neq \emptyset$ for each $u \in U$.*

Proof. It is similar to the proof of the Theorem 3.6. ∎

Theorem 3.8 *Let $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be a multifunction such that $F(x)$ is an open subset of Y for each $x \in X$. Then F is lower $\mathcal{I}$-continuous if and only if lower weakly $\mathcal{I}$-continuous.*

Proof. Let $x \in X$ and V be an open set of Y such that $F(x) \cap V \neq \emptyset$. Then there exists an $\mathcal{I}$-open set U containing x such that $F(u) \cap \mathrm{Cl}(V) \neq \emptyset$ for each $u \in U$. Since $F(u)$ is open, $F(u) \cap V \neq \emptyset$ for each $u \in U$ and hence F is lower $\mathcal{I}$-continuous. The converse follows by Corollary 3.5. ∎

Theorem 3.9 *Let $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be a multifunction such that $F(x)$ is closed in Y for each $x \in X$ and Y is normal. Then F is upper weakly $\mathcal{I}$-continuous if and only if F is upper $\mathcal{I}$-continuous.*

Proof. Suppose that F is upper weakly $\mathcal{I}$-continuous. Let $x \in X$ and G be an open subset of Y containing $F(x)$. Since $F(x)$ is closed in Y and Y is normal, there exist open sets V and W such that $F(x) \subset V$, $X \backslash G \subset W$ and $V \cap W = \emptyset$. We have $F(x) \subset V \subset \mathrm{Cl}(V) \subset \mathrm{Cl}(X \backslash W) = X \backslash W \subset G$. Since F is upper weakly $\mathcal{I}$-continuous, there exists an $\mathcal{I}$-open set U containing x such that $F(U) \subset \mathrm{Cl}(V) \subset G$. This shows that F is upper $\mathcal{I}$-continuous. The converse follows by Corollary 3.5. ∎

Definition 3 A subset A of a topological space (X, τ) is said to be

1. α-regular [7] if for each $a \in A$ and any open set U containing a, there exists an open set G of X such that $a \in G \subset \mathrm{Cl}(G) \subset U$;
2. α-paracompact [7] if every X-open cover A has an X-open refinement which covers A and is locally finite for each point of X.

Lemma 3.10 [7] *If A is an α-paracompact α-regular set of a topological space (X, τ) and U an open neighborhood of A, then there exists an open set G of X such that $A \subset G \subset \mathrm{Cl}(G) \subset U$.*

For a multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$, by $\mathrm{Cl}(F) : X \to Y$ we denote a multifunction as follows: $\mathrm{Cl}(F)(x) = \mathrm{Cl}(F(x))$ for each $x \in X$. Similarly, we define $\mathcal{I}\,\mathrm{Cl}\,F$.

The proof of the following two Lemmas are obvious and thus omitted.

Lemma 3.11 *If $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is a multifunction such that $F(x)$ is α-paracompact α-regular for each $x \in X$, then we have the following*

1. *$G^+(V) = F^+(V)$ for each open set V of Y,*
2. *$G^-(V) = F^-(V)$ for each closed set V of Y, where G denotes $\mathrm{Cl}\,F$ or $\mathcal{I}\,\mathrm{Cl}\,F$.*

Lemma 3.12 *For a multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$, we have the following*

1. *$G^-(V) = F^-(V)$ for each open set V of Y,*
2. *$G^+(V) = F^+(V)$ for each open set V of Y, where G denotes $\mathrm{Cl}\,F$ or $\mathcal{I}\,\mathrm{Cl}\,F$.*

Theorem 3.13 *Let $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be a multifunction such that $F(x)$ is α-regular and α-paracompact for every $x \in X$. Then the following statements are equivalent:*

1. *F is upper weakly $\mathcal{I}$-continuous;*
2. *$\mathrm{Cl}\,F$ is upper weakly $\mathcal{I}$-continuous;*
3. *$\mathcal{I}\,\mathrm{Cl}\,F$ is upper weakly $\mathcal{I}$-continuous.*

Proof. We put $G = \mathrm{Cl}(F)$ or $\mathcal{I}\,\mathrm{Cl}\,F$ in the sequel. Suppose that F is upper weakly $\mathcal{I}$-continuous. Then it follows by Theorem 3.6 and Lemmas 3.11 and 3.12 that for every open set V of Y containing $F(x)$, $G^+(V) = F^+(V) \subset \mathcal{I}\,\mathrm{Int}(F^+(\mathrm{Cl}(V))) = \mathcal{I}\,\mathrm{Int}(G^+(\mathrm{Cl}(V)))$. By Theorem 3.6, G is upper weakly $\mathcal{I}$-continuous. Conversely, Suppose that G is upper weakly $\mathcal{I}$-continuous. Then it follows by Theorem 3.6 and Lemmas 3.11 and 3.12 that for every open set V of Y containing $G(x)$, $F^+(V) = G^+(V) \subset \mathcal{I}\,\mathrm{Int}(G^+(\mathrm{Cl}(V))) = \mathcal{I}\,\mathrm{Int}(F^+(\mathrm{Cl}(V)))$. It follows by Theorem 3.6 that F is upper weakly $\mathcal{I}$-continuous. ∎

Theorem 3.14 *Let $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be a multifunction such that $F(x)$ be α-regular α-paracompact for every $x \in X$. Then the following properties are equivalent:*

1. *F is lower weakly $\mathcal{I}$-continuous;*
2. *$\mathrm{Cl}\,F$ is lower weakly $\mathcal{I}$-continuous;*
3. *$\mathcal{I}\,\mathrm{Cl}\,F$ is lower weakly $\mathcal{I}$-continuous.*

Proof. Similar to the proof of Theorem 3.27. ∎

Definition 4 Let A be a subset of a topological space (X, τ). The $\mathcal{I}$-frontier of A, denoted by $\mathcal{I}Fr(A)$, is defined by $\mathcal{I}Fr(A) = \mathcal{I}\,\mathrm{Cl}(A) \cap \mathcal{I}\,\mathrm{Cl}(X \backslash A)$.

Theorem 3.15 *Let $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be a multifunction. The set of all points x of X such that F is not upper weakly $\mathcal{I}$-continuous (resp. lower weakly $\mathcal{I}$-continuous) is identical with the union of the $\mathcal{I}$-frontiers of the upper (resp. lower) inverse images of the closure of open sets containing (resp. meeting) $F(x)$.*

Proof. Let x be a point of X at which F is not upper weakly $\mathcal{I}$-continuous. Then there exists an open set V containing $F(x)$ such that $U \cap (X \backslash F^+(\mathrm{Cl}(V))) \neq \emptyset$ for every $\mathcal{I}$-open set U containing x. Therefore, $x \in \mathcal{I}\,\mathrm{Cl}(X \backslash F^+(\mathrm{Cl}(V)))$. Since $x \in F^+(V)$, we have $x \in \mathcal{I}\,\mathrm{Cl}(F^+(\mathrm{Cl}(V)))$ and hence $x \in \mathcal{I}Fr(F^+(\mathrm{Cl}(V)))$. Conversely, if F is upper weakly $\mathcal{I}$-continuous at x, then for every open set V of Y containing $F(x)$ and there exists an $\mathcal{I}$-open set U containing x such that $F(U) \subset \mathrm{Cl}(V)$; hence $U \subset F^+(\mathrm{Cl}(V))$. Therefore, we obtain $x \in U \subset \mathcal{I}\,\mathrm{Int}(F^+(\mathrm{Cl}(V)))$. This contradicts that $x \in \mathcal{I}\,\mathrm{Int}(F^+(\mathrm{Cl}(V)))$.

The case when F is lower weakly $\mathcal{I}$-continuous is similarly shown. ∎

Definition 5 A topological space (X, τ) is said to be strongly normal if for every disjoint closed sets V_1 and V_2 of X, there exist two disjoint open sets U_1 and U_2 such that $V_1 \subset U_1$, $V_2 \subset U_2$ and $\mathrm{Cl}(U_1) \cap \mathrm{Cl}(U_2) = \emptyset$.

Recall that a multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is said to be a point closed multifunction if $F(x)$ is a closed subset of Y for all $x \in X$.

Theorem 3.16 *Suppose that the product of two $\mathcal{I}$-open sets is $\mathcal{I}$-open. If Y is a strongly normal space and $F_i : X_i \to Y$ an upper weakly $\mathcal{I}$-continuous multifunction such that F_i is point closed for $i = 1, 2$, then the set $\{(x_1, x_2) \in X_1 \times X_2$: $F_1(x_1) \cap F_2(x_2) \neq \varnothing\}$ is $\mathcal{I}$-closed in $X_1 \times X_2$.*

Proof. Let $A = \{(x_1, x_2) \in X_1 \times X_2 \colon F_1(x_1) \cap F_2(x_2) \neq \varnothing\}$ and $(x_1, x_2) \in (X_1 \times X_2) \backslash A$. Then $F_1(x_1) \cap F_2(x_2) = \varnothing$. Since Y is strongly normal and F_i is point closed for $i = 1, 2$, there exist disjoint open sets V_1, V_2 such that $F_i(x_i) \subset V_i$ for $i = 1, 2$. We have $\mathrm{Cl}(V_1) \cap \mathrm{Cl}(V_2) = \emptyset$. Since F_i is upper weakly $\mathcal{I}$-continuous, there exist $\mathcal{I}$-open sets U_1 and U_2 containing x_1 and x_2, respectively such that $F_i(U_i) \subset \mathrm{Cl}(V_i)$ for $i = 1, 2$. Put $U = U_1 \times U_2$, then U is an $\mathcal{I}$-open set and $(x_1, x_2) \in U \subset (X_1 \times X_2) \backslash A$. This shows that $(X_1 \times X_2) \backslash A$ is $\mathcal{I}$-open; hence A is $\mathcal{I}$-closed in $X_1 \times X_2$. ∎

Theorem 3.17 *Let F and G be point closed multifunctions from a topological space (X, τ) to a strongly normal space (Y, σ). If F is upper weakly $\mathcal{I}$-continuous and G is upper weakly continuous, then the set $K = \{x : F(x) \cap G(x) \neq \varnothing\}$ is $\mathcal{I}$-closed in X.*

Proof. Let $x \in X \backslash K$. Then $F(x) \cap G(x) = \varnothing$. Since F and G are point closed multifunctions and Y is a strongly normal space, there exist disjoint open sets U and V containing $F(x)$ and $G(x)$, respectively we have $\mathrm{Cl}(U) \cap \mathrm{Cl}(V) = \emptyset$. Since F and G are upper weakly $\mathcal{I}$-continuous functions, then there exist $\mathcal{I}$-open set U_1 containing x and open set U_2 containing x such that $F(U_1) \subset \mathrm{Cl}(V)$ and $F(U_2) \subset \mathrm{Cl}(V)$. Now set $H = U_1 \cap U_2$, then H is an $\mathcal{I}$-open set containing x and $H \cap K = \varnothing$; hence K is $\mathcal{I}$-closed in X. ∎

Lemma 3.18 [1] *Let A and B be subsets of a space (X, τ).*

1. *If $A \in \mathcal{I}O(X, \tau)$ and $B \in \tau$, then $A \cap B \in \mathcal{I}O(B, \tau_B)$;*
2. *If $A \in \mathcal{I}O(B, \tau_B)$ and $B \in \mathcal{I}O(X, \tau)$, then $A \in \mathcal{I}O(X, \tau)$.*

Theorem 3.19 *Let $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be a multifunction and U an open subset of X. If F is a lower (upper) weakly $\mathcal{I}$-continuous multifunction, then $F_{|_U} : U \to Y$ is a lower (upper) weakly $\mathcal{I}$-continuous multifunction.*

Proof. Let V be any open set of Y. Let $x \in U$ and $x \in F^-_{|_U}(V)$. Since F is lower weakly $\mathcal{I}$-continuous multifunction, then there exists an $\mathcal{I}$-open set G containing x such that $G \subset F^-(\mathrm{Cl}(V))$. Then $x \in G \cap U \in \mathcal{I}O(X, \tau)$ and $G \cap U \subset F^-_{|_U}(\mathrm{Cl}(V))$. This shows that $F_{|_U}$ is a lower weakly $\mathcal{I}$-continuous. The proof of the upper weakly $\mathcal{I}$-continuity of $F_{|_U}$ can be done by the similar manner. ∎

Theorem 3.20 *Let $\{A_i\}_{i \in I}$ be an open cover of a topological space X. Then a multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is upper (lower) weakly $\mathcal{I}$-continuous if and only if $F_{|_{A_i}} \colon A_i \to Y$ is a upper (lower) weakly $\mathcal{I}$-continuous for each $i \in I$.*

Proof. Let $i \in I$ and V be any open set of Y. Since F is lower weakly $\mathcal{I}$-continuous, $F^+(V) \subset \mathcal{I}\,\mathrm{Int}(F^+(\mathrm{Cl}(V)))$. We obtain $(F|A_i)^+(V) = F^+(V) \cap A_i \subset \mathcal{I}\,\mathrm{Int}(F^+(\mathrm{Cl}(V))) \cap A_i = \mathcal{I}\,\mathrm{Int}(F^+(\mathrm{Cl}(V)) \cap A_i) \subset \mathcal{I}\,\mathrm{Int}_{A_i}(F^+(\mathrm{Cl}(V)))$. Hence $F_{|_A} \colon A \to Y$ is an upper weakly $\mathcal{I}$-continuous for each $i \in I$. Conversely, Let V be any open set of Y. Since F_i is lower weakly $\mathcal{I}$-continuous for each $i \in I$, from Theorem 3.6, $F_i^+(V) \subset \mathcal{I}\,\mathrm{Int}_{A_i}(F_i^+(\mathrm{Cl}(V)))$ and since A_i is open, we have

$F^+(V) \cap A_i \subset \mathcal{I}\,\mathrm{Int}_{A_i}(F^+(\mathrm{Cl}(V)) \cap A_i)$ and $F^+(V) \cap A_i \subset \mathcal{I}\,\mathrm{Int}(F^+(\mathrm{Cl}(V))) \cap A_i$. Since $\{A_i\}_{i\in I}$ is an open cover of X, it follows that $F^+(V) \subset \mathcal{I}\,\mathrm{Int}(F^+(\mathrm{Cl}(V)))$. Hence, from Theorem 3.6, we obtain that F is upper weakly $\mathcal{I}$-continuous. The proof of the lower weakly $\mathcal{I}$-continuity of $F_{|_A}$ can be done by the similar manner.∎

Definition 6 For a multifunction, $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$, the graph multifunction $G_F : X \to X \times Y$ is defined as follows: $G_F(x) = \{x\} \times F(x)$ for every $x \in X$ and the subset $\{\{x\} \times F(x) : x \in X\} \subset X \times Y$ is called the graph multifunction of F and is denoted by $G(x)$.

Lemma 3.21 [9] *For a multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$, the following holds:*

1. $G_F^+(A \times B) = A \cap F^+(B)$;
2. $G_F^-(A \times B) = A \cap F^-(B)$

for any subset A of X and B of Y.

Theorem 3.22 *Let $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be a multifunction. If the graph multifunction of F is upper (lower) weakly $\mathcal{I}$-continuous, then F is upper (lower) weakly $\mathcal{I}$-continuous.*

Proof. Let $x \in X$ and V be any open subset of Y and containing $F(x)$. Since $X \times V$ is an open set relative to $X \times Y$ and $G_F(x) \subset X \times V$, there exists an $\mathcal{I}$-open set U containing x such that $G_F(U) \subset \mathrm{Cl}(X \times V) = X \times \mathrm{Cl}(V)$. By Lemma 3.21, we have $U \subset G_F^+(X \times \mathrm{Cl}(V)) = F^+(\mathrm{Cl}(V))$ and $F(U) \subset \mathrm{Cl}(V)$. Thus, F is upper weakly $\mathcal{I}$-continuous. The proof of the lower weakly $\mathcal{I}$-continuity of F can be done by the similar manner. ∎

Theorem 3.23 *Let $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be a multifunction such that $F(x)$ is compact for each $x \in X$. Then F is upper weakly $\mathcal{I}$-continuous if and only if $G_F : X \to X \times Y$ is upper weakly $\mathcal{I}$-continuous.*

Proof. Suppose that $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is upper weakly $\mathcal{I}$-continuous. Let $x \in X$ and W be any open set of $X \times Y$ containing $G_F(x)$. For each point $y \in F(x)$, there exist open sets $U(y) \subset X$ and $V(y) \subset Y$ such that $(x, y) \in U(y) \times V(y) \subset W$. The family $\{V(y) : y \in F(x)\}$ is an open cover of $F(x)$ and $F(x)$ is compact. Therefore, there exist a finite number of points, say, $y_1, y_2, .., y_n$ in $F(x)$ such that $F(x) \subset \cup\{V(y_i) : 1 \leq i \leq n\}$. Set $U = \cap\{U(y_i) : 1 \leq i \leq n\}$ and $V = \cup\{V(y_i) : 1 \leq i \leq n\}$. Then U and V are open sets of X and Y, respectively, and $G_F(x) = \{x\} \times F(x) \subset U \times V \subset W$. Since F is upper weakly $\mathcal{I}$-continuous, there exists $U_0 \in \mathcal{I}O(X, x)$ such that $F(U_0) \subset \mathrm{Cl}(V)$. By Lemma 3.18, $U \cap U_0 \in \mathcal{I}O(X, x)$. By Lemma 3.21, $U \cap U_0 \subset \mathrm{Cl}(U) \cap F^+(\mathrm{Cl}(V)) = G_F^+(\mathrm{Cl}(U) \times \mathrm{Cl}(V)) \subset G_F^+(\mathrm{Cl}(W))$. Therefore, we obtain $U \cap U_0 \in \mathcal{I}O(X, x)$ and $G_F(U \cap U_0) \subset \mathrm{Cl}(W)$. This shows that G_F is upper weakly $\mathcal{I}$-continuous. Conversely, suppose that $G_F : X \to X \times Y$ is upper weakly $\mathcal{I}$-continuous. Let $x \in X$ and V be any open set of Y containing $F(x)$. Since $X \times V$ is open in $X \times Y$ and $G_F(x) \subset X \times V$, there exists $U \in \mathcal{I}O(X, x)$ such that $G_F(U) \subset \mathrm{Cl}(X \times V) = X \times \mathrm{Cl}(V)$. By Lemma 3.21, we have $U \subset G_F^+(X \times \mathrm{Cl}(V)) = F^+(\mathrm{Cl}(V))$. This shows that F is upper weakly $\mathcal{I}$-continuous. ∎

Theorem 3.24 *A multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is lower weakly $\mathcal{I}$-continuous if and only if $G_F : X \to X \times Y$ is lower weakly $\mathcal{I}$-continuous.*

Proof. Suppose that F is lower weakly $\mathcal{I}$-continuous. Let $x \in X$ and W be any open set of $X \times Y$ such that $x \in G_{F^-}(W)$. Since $W \cap (\{x\} \times F(x)) \neq \emptyset$, there exists $y \in F(x)$ such that $(x, y) \in W$ and hence $(x, y) \in U \times V \subset W$ for some open sets $U \subset X$ and $V \subset Y$. Since F is lower weakly $\mathcal{I}$-continuous and $F(x) \cap V \neq \emptyset$, there exists $U_0 \in \mathcal{I}O(X, x)$ such that $U_0 \subset F^-(\mathrm{Cl}(V))$. By Lemma 3.21, we have $U \cap U_0 \subset U \cap F^-(\mathrm{Cl}(V)) = G_{F^-}(U \times \mathrm{Cl}(V)) \subset G_{F^-}(\mathrm{Cl}(W))$. Moreover, we have $U \cap U_0 \in \mathcal{I}O(X, x)$; and hence G_F is lower weakly $\mathcal{I}$-continuous. Conversely, suppose that G_F is lower weakly $\mathcal{I}$-continuous. Let $x \in X$ and V be an open set of Y such that $x \in F^-(V)$. Then $X \times V$ is open in $X \times Y$ and $G_F(x) \cap (X \times V) = (\{x\} \times F(x)) \cap (X \times V) = \{x\} \times (F(x) \cap V) \neq \emptyset$. Since G_F is lower weakly $\mathcal{I}$-continuous, there exists $U \in \mathcal{I}O(X, x)$ such that $U \subset G_{F^-}(\mathrm{Cl}(X \times V))$. By Lemma 3.21, we obtain $U \subset G_{F^-}(\mathrm{Cl}(X \times V)) = F^-(\mathrm{Cl}(V))$. This shows that F is lower weakly $\mathcal{I}$-continuous. ∎

Definition 7 A topological space (X, τ) is said to $\mathcal{I}$-T_2 [10] if for each pair of distinct points x and y in X, there exist disjoint $\mathcal{I}$-open sets U and V in X such that $x \in U$ and $y \in V$.

Theorem 3.25 *If $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is an upper weakly $\mathcal{I}$-continuous injective multifunction and point closed from a topological space X to a strongly normal space Y, then X is an $\mathcal{I}$-T_2 space.*

Proof. Let x and y be any two distinct points in X. Then we have $F(x) \cap F(y) = \varnothing$ since F is injective. Since Y is strongly normal, there exist disjoint open sets U and V containing $F(x)$ and $F(y)$, respectively such that $\mathrm{Cl}(U) \cap \mathrm{Cl}(V) = \emptyset$. Thus, there exist disjoint $\mathcal{I}$-open sets $F^+(U)$ and $F^+(V)$ containing x and y, respectively such $G \subset F^+(\mathrm{Cl}(U))$ and $W \subset F^+(\mathrm{Cl}(V))$. Therefore, we obtain $G \cap W = \emptyset$; hence X is $\mathcal{I}$-T_2. ∎

Theorem 3.26 *Suppose that (X, τ) and (X_α, τ_α) are topological spaces where X_α is connected space for each $\alpha \in J$. Let $F : X \to \prod_{\alpha \in J} X_\alpha$ be a multifunction from X to the product space $\prod_{\alpha \in J} X_\alpha$ and let $P_\alpha : \prod_{\alpha \in J} X_\alpha \to X_\alpha$ be the projection multifunction for each $\alpha \in J$ which is defined by $P_\alpha((x_\alpha)) = \{x_\alpha\}$. If F is upper (lower) weakly $\mathcal{I}$-continuous multifunction, then $P_\alpha \circ F$ is upper (lower) weakly $\mathcal{I}$-continuous multifunction for each $\alpha \in J$.*

Proof. Take any $\alpha_0 \in J$. Let V_{α_0} be an open set in $(X_{\alpha 0}, \tau_{\alpha 0})$. Then $(P_{\alpha 0} \circ F)^+(V_{\alpha 0}) = F^+(P^+_{\alpha 0}(V_{\alpha 0})) = F^+(V_{\alpha 0} \times \prod_{\alpha \neq \alpha_0} X_\alpha)$. We take $x \in (P_{\alpha_0} \circ F)^+(V_{\alpha_0})$. Since F is upper weakly $\mathcal{I}$-continuous and $V_{\alpha 0} \times \prod_{\alpha \neq \alpha_0} X_\alpha$ is an open set to $\prod_{\alpha \in J} X_\alpha$, there exists an $\mathcal{I}$-open set U containing x such $G \subset F^+(\mathrm{Cl}(V_{\alpha 0} \times \prod_{\alpha \neq \alpha_0} X_\alpha))$. Since $F^+(\mathrm{Cl}(V_{\alpha 0} \times \prod_{\alpha \neq \alpha_0} X_\alpha)) = F^+(\mathrm{Cl}(V_{\alpha 0}) \times \prod_{\alpha \neq \alpha_0} X_\alpha) = (P_{\alpha 0} \circ F)^+(\mathrm{Cl}(V_{\alpha 0}))$, $P_\alpha \circ F$ is upper weakly $\mathcal{I}$-continuous. The proof of the lower weakly $\mathcal{I}$-continuity of F can be done by the same manner. ∎

Theorem 3.27 *Suppose that for each $\alpha \in J$, (X_α, τ_α), $(Y_\alpha, \sigma_\alpha)$ are topological spaces. Let $F_\alpha : X_\alpha \to Y_\alpha$ be a multifunction for each $\alpha \in J$ and let $F : \prod_{\alpha\in J} X_\alpha \to \prod_{\alpha\in J} Y_\alpha$ be defined by $F((x_\alpha)) = \prod_{\alpha\in J} F_\alpha(x_\alpha)$ from the product space $\prod_{\alpha\in J} X_\alpha$ to the product space $\prod_{\alpha\in J} Y_\alpha$. If F is upper (lower) weakly $\mathcal{I}$-continuous multifunction, then each F_α is upper (lower) weakly $\mathcal{I}$-continuous multifunction for each $\alpha \in J$.*

Proof. Similar to the proof of Theorem 3.26. ∎

4. Additional properties

Theorem 4.1 *Let $F : X \to \prod_{i\in I} X_i$ be a multifunction from a topological space X to the product space $\prod_{i\in I} X_i$ and let $p_i : \prod_{i\in I} X_i \to X_i$ be the projection for each $i \in I$. If F is an upper (resp. lower) weakly $\mathcal{I}$-continuous multifunction, then $p_i oF$ is an upper (resp. lower) weakly $\mathcal{I}$-continuous multifunction for each $i \in I$.*

Proof. We shall prove this only for the upper continuity. Let V_{i0} be an open set in X_{i0}. We have $(p_{i0}oF)^+(V_{i0}) = F^+(p_{i0}^+(V_{i0})) = F^+(V_{i0} \times \prod_{i\in I} X_i)$. Take $x \in (p_{i0}oF)^+(V_{i0})$. Since F is upper weakly $\mathcal{I}$-continuous and $V_{i0} \times \prod_{i\in I} X_i$ is an open set, there exists an $\mathcal{I}$-open set U containing x such that $U \subset F^+(\mathrm{Cl}(V_{i0} \times \prod_{i\in I} X_i))$. Since $F^+(\mathrm{Cl}(V_{i0} \times \prod_{i\in I} X_i)) = F^+(\mathrm{Cl}(V_{i0}) \times \prod_{i\in I} X_i) = (p_{i0}oF)^+(\mathrm{Cl}(V_{i0}))$, $p_{i0}oF$ is upper weakly $\mathcal{I}$-continuous multifunction. ∎

Theorem 4.2 *Suppose that $F_1 : X \to Y$, $F_2 : X \to Z$ are multifunctions. Let $F_1 \times F_2 : X \to Y \times Z$ be a multifunction which is defined by $(F_1 \times F_2)(x) = F_1(x) \times F_2(x)$ for each $x \in X$. If $F_1 \times F_2$ is upper (lower) weakly $\mathcal{I}$-continuous multifunction, then F_1 and F_2 are upper (resp. lower) weakly $\mathcal{I}$-continuous functions.*

Proof. Let $x \in X$ and let $K \subset Y$, $H \subset Z$ be open sets such that $x \in F_1^+(K)$ and $x \in F_2^+(H)$. Then we obtain that $F_1(x) \subset K$ and $F_2(x) \subset H$ and so $F_1(x) \times F_2(x) = (F_1 \times F_2)(x) \subset K \times H$. We have $x \in (F_1 \times F_2)^+(K \times H)$. Since $F_1 \times F_2$ is upper weakly $\mathcal{I}$-continuous multifunction, there exists an $\mathcal{I}$-open set U containing x such that $U \subset (F_1 \times F_2)^+(\mathrm{Cl}(K \times H))$. We obtain that $U \subset F_1^+(\mathrm{Cl}(K))$ and $U \subset F_2^+(\mathrm{Cl}(H))$. Thus, we obtain that F_1 and F_2 are upper weakly $\mathcal{I}$-continuous multifunctions. The proof of the lower weakly $\mathcal{I}$-continuity of F_1 and F_2 is similar to that presented above. ∎

Definition 8 Let $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be a multifunction. The multigraph G_F is said to be $\mathcal{I}$-closed in $X \times Y$ if for each $(x, y) \notin G_F$, there exist $\mathcal{I}$-open set U and open set V containing x and y, respectively, such that $(U \times V) \cap G_F = \emptyset$.

Definition 9 A multifunction $F : (X, \tau) \to (Y, \sigma)$ is called punctually α-paracompact [7] if $F(x)$ is α-paracompact for each point $x \in X$.

Theorem 4.3 *Let $F : (X, \tau) \to (Y, \sigma)$ be an upper weakly $\mathcal{I}$-continuous and punctually α-paracompact multifunction into a Hausdorff space (Y, σ). Then the multigraph G_F of F is an $\mathcal{I}$-closed in $X \times Y$.*

Proof. Suppose that $(x_0, y_0) \notin G_F$. Then $y_0 \notin F(x_0)$. Since (Y, σ) is a Hausdorff space for each $y \in F(x_0)$, there exist open sets $V(y)$ and $W(y)$ containing y and y_0, respectively such that $V(y) \cap W(y) = \emptyset$. The family $\{V(y) : y \in F(x_0)\}$ is an open cover of $F(x_0)$ which is α-paracompact. Thus, it has a locally finite open refinement $\Phi = \{U_\beta : \beta \in I\}$ which covers $F(x_0)$. Let W_0 be an open neighborhood of y_0 such that W_0 intersects only finitely many members $U_{\beta 1}, U_{\beta 2}, ..., U_{\beta n}$ of Φ. Choose $y_1, y_2, .., y_n$ in $F(x_0)$ such that $U_{\beta i} \subset V(y_i)$ for each i=1,2,..n and $W = W_0 \cap (\overset{n}{\underset{i=1}{\cap}} W(y_i))$. Then W is an open neighborhood of y_0 with $W \cap (\underset{\beta \in I}{\cup} U_\beta) = \emptyset$, which implies that $W \cap \mathrm{Cl}(\underset{\beta \in I}{\cup} U_\beta) = \emptyset$. By the upper weakly $\mathcal{I}$-continuity of F, there exists $U \in \mathcal{I}O(X, x_0)$ such that $F(U) \subset \mathrm{Cl}(\underset{\beta \in I}{\cup} U_\beta)$. It follows that $(U \times W) \cap G_F = \emptyset$. Hence, the graph G_F is an $\mathcal{I}$-closed in $X \times Y$. ∎

Definition 10 A multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is said to be almost $\mathcal{I}$-open if $F(U) \subset \mathrm{Int}(\mathrm{Cl}(F(U)))$ for every $\mathcal{I}$-open set U in X.

Theorem 4.4 *If a multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is upper weakly $\mathcal{I}$-continuous and almost $\mathcal{I}$-open, then F is upper almost $\mathcal{I}$-continuous.*

Proof. Let V be an arbitrary open set containing $F(x)$. There exists an $\mathcal{I}$-open set U containing x such that $F(U) \subset \mathrm{Cl}(V)$. Since F is almost $\mathcal{I}$-open, we have $F(U) \subset \mathrm{Int}(\mathrm{Cl}(F(U))) \subset \mathrm{Int}(\mathrm{Cl}(V))$. It then follows that F is upper almost $\mathcal{I}$-continuous. ∎

Theorem 4.5 *Let $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be a multifunction such that $F(x)$ is an open set of Y for each $x \in X$. Then the following assertions are equivalent:*

1. *F is lower $\mathcal{I}$-continuous,*
2. *F is lower almost $\mathcal{I}$-continuous,*
3. *F is lower weakly $\mathcal{I}$-continuous.*

Proof. (1)⇒(2), (2)⇒(3): Obvious.
(3)⇒(1): Let $x \in X$ and V be an open set of Y such that $F(x) \cap V \neq \emptyset$. Then there exists an $\mathcal{I}$-open set U containing x such that $F(u) \cap \mathrm{Cl}(V) \neq \emptyset$ for each $u \in U$. Since $F(u)$ is open, $F(u) \cap V \neq \emptyset$ for each $u \in U$ and hence F is lower $\mathcal{I}$-continuous. ∎

Theorem 4.6 *If $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is upper weakly $\mathcal{I}$-continuous and satisfies the condition $F^+(\mathrm{Cl}(V)) \subset F^+(V)$ for every open set V of Y, then F is upper $\mathcal{I}$-continuous.*

Proof. Let V be any open set. Since F upper weakly $\mathcal{I}$-continuous, $F^+(V) \subset \mathcal{I}\,\mathrm{Int}(F^+(\mathrm{Cl}(V)))$ and hence $F^+(V) \subset \mathcal{I}\,\mathrm{Int}(F^+(\mathrm{Cl}(V))) \subset \mathcal{I}\,\mathrm{Int}(F^+(V))$. Then $F^+(V)$ is $\mathcal{I}$-open; hence F is upper $\mathcal{I}$-continuous. ∎

Theorem 4.7 *Let $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be a multifunction such that $F(x)$ is closed in Y for each $x \in X$ and Y is normal. Then the following assertions are equivalent:*

1. *F is upper weakly $\mathcal{I}$-continuous,*
2. *F is upper almost $\mathcal{I}$-continuous,*
3. *F is upper $\mathcal{I}$-continuous.*

Proof. (3)⇒(2), (2)⇒(1): Obvious.
(1)⇒(3): Suppose that F is upper weakly $\mathcal{I}$-continuous. Let $x \in X$ and G be an open set containing $F(x)$. Since $F(x)$ is closed in Y, by the normality of Y, there exists an open set V such that $F(x) \subset V \subset \mathrm{Cl}(V) \subset G$. Since F is upper weakly $\mathcal{I}$-continuous, there exists an $\mathcal{I}$-open set U containing x such that $F(U) \subset \mathrm{Cl}(V) \subset G$. This shows that F is upper $\mathcal{I}$-continuous. ∎

Definition 11 [16] A topological space X is said to be almost regular if for each $x \in X$ and each regular closed set F of X not containing x, there exist disjoint open sets U and V of X such that $x \in U$ and $F \subset V$.

Lemma 4.8 *If A is an α-almost regular α-paracompact set of a topological space X and U is a regular open neighborhood of A, then there exists an open set G of X such that $A \subset G \subset \mathrm{Cl}(G) \subset U$.*

Theorem 4.9 *If a multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is upper weakly $\mathcal{I}$-continuous and $F(x)$ is an α-almost regular and α-paracompact set for each $x \in X$, then F is upper almost $\mathcal{I}$-continuous.*

Proof. Let V be a regular open set of Y and $x \in F^+(V)$. Then $F(x) \subset V$ and by Lemma 4.8 there exists an open set W of Y such that $F(x) \subset W \subset \mathrm{Cl}(W) \subset V$. Since F is upper weakly $\mathcal{I}$-continuous, there exists $U \in \mathcal{I}O(X, x)$ such that $F(U) \subset \mathrm{Cl}(W) \subset V$. Therefore, we have $x \in U \subset F^+(V)$. This shows that $F^+(V) \in \mathcal{I}O(X, x)$. Hence F is upper almost $\mathcal{I}$-continuous. ∎

Corollary 4.10 *Let $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be an upper weakly $\mathcal{I}$-continuous multifunction such that $F(x)$ is compact for each $x \in X$ and Y is almost regular. Then F is upper almost $\mathcal{I}$-continuous.*

Lemma 4.11 *If A is an α-almost regular set of a space X, then for every regular open set U which intersects A, there exists an open set G such that $A \cap G \neq \emptyset$ and $\mathrm{Cl}(G) \subset U$.*

Theorem 4.12 *If $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is a lower weakly $\mathcal{I}$-continuous multifunction such that $F(x)$ is an α-almost regular set of Y for every $x \in X$, then F is lower almost $\mathcal{I}$-continuous.*

Proof. Let V be a regular open set of Y and $x \in F^-(V)$. Since $F(x)$ is α-almost regular, by Lemma 4.11 there exists an open set W of Y such that $F(x) \cap W \neq \emptyset$ and $\mathrm{Cl}(W) \subset V$. Since F is lower weakly $\mathcal{I}$-continuous there exists $U \in \mathcal{I}O(X, x)$ such that $F(u) \cap \mathrm{Cl}(W) \neq \emptyset$; hence $F(u) \cap V \neq \emptyset$ for every $u \in U$. Therefore, we have $x \in U \subset F^-(V)$. This shows that $F^-(V) \in \mathcal{I}O(X, x)$. Hence F is lower almost $\mathcal{I}$-continuous. ∎

Definition 12 A multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is said to be complementary continuous if for each open set V of Y, $F^-(Fr(V))$ is a closed set of X, where $Fr(V)$ denotes the frontier of V.

Theorem 4.13 *If $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is upper weakly $\mathcal{I}$-continuous and complementary continuous, then it is upper $\mathcal{I}$-continuous.*

Proof. Let $x \in X$ and V be any open set of Y such that $F(x) \subset V$. There exists $G \in \mathcal{I}O(X, x)$ such that $F(G) \subset \mathrm{Cl}(V)$. Put $U = G \cap (X \backslash F^-(Fr(V)))$. Since $F^-(Fr(V))$ is closed in X, $U \in \mathcal{I}O(X)$. Moreover, we have $F(x) \cap Fr(V) \subset [V \cap \mathrm{Cl}(V)] \cap (Y \backslash V) = \emptyset$; and hence $x \in X \backslash F^-(Fr(V))$. Thus, we obtain $U \in \mathcal{I}O(X, x)$ and $F(U) \subset V$ since $F(U) \subset F(G) \subset \mathrm{Cl}(V)$ and $F(u) \cap Fr(V) = \emptyset$ for each $u \in U$. Therefore, F is upper $\mathcal{I}$-continuous. ∎

Definition 13 A multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is said to be complementary $\mathcal{I}$-continuous if for each open set V of Y, $F^-(Fr(V))$ is an $\mathcal{I}$-closed set of X.

Theorem 4.14 *If multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is upper weakly continuous and complementary $\mathcal{I}$-continuous, then it is upper $\mathcal{I}$-continuous.*

Proof. It is similar to the proof of Theorem 4.13. ∎

Definition 14 A multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is said to be injective if $x \neq y$ implies $F(x) \cap F(y) = \emptyset$.

Theorem 4.15 *If $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is an upper weakly $\mathcal{I}$-continuous injective multifunction into a Urysohn space (Y, σ) and $F(x)$ is compact in Y for each $x \in X$, then $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-T_2.*

Proof. For any distinct points x_1, x_2 of X, we have $F(x_1) \cap F(x_2) = 0$, since F is injective. Since $F(x_i)$ is compact in a Urysohn space (Y, σ), there exist open sets V_i such that $F(x_i) \subset V_i$ for i=1,2 and $\mathrm{Cl}(V_i) \cap \mathrm{Cl}(V_2) = \emptyset$. Since F is upper weakly $\mathcal{I}$-continuous there exists $U_i \in \mathcal{I}(X, x_i)$ such that $F(U_i) \subset \mathrm{Cl}(V_i)$ for $i = 1, 2$. Therefore, we obtain $U_1 \cap U_2 = \emptyset$ and hence $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-T_2. ∎

Theorem 4.16 *If $F_1, F_2 : X \to Y$ be multifunctions into a Urysohn space (Y, σ) and $F_i(x)$ compact in Y for each $x \in X$ and each $i = 1, 2$. If $F(x_i) \cap F_2(X) \neq \emptyset$ for each $x \in X$ and F_1 is upper weakly $\mathcal{I}$-continuous, then a multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$, defined as follows $F(x) = F_1(x) \cap F_2(x)$ for each $x \in X$, is upper weakly $\mathcal{I}$-continuous.*

Proof. Let $x \in X$ and V be an open set of Y such that $F(x) \subset V$. Then, $A = F_1(x) \backslash V$ and $B = F_2(x) \backslash V$ are disjoint compact sets. Then there exist open sets V_1 and V_2 such that $A \subset V_1$ and $B \subset V_2$ and $\mathrm{Cl}(V_1) \cap \mathrm{Cl}(V_2) = \emptyset$. Since F_1 is upper weakly $\mathcal{I}$-continuous, there exists $U_1 \in \mathcal{I}O(X, x)$ such that $F_1(U_1) \subset \mathrm{Cl}(V_i \cup V)$. Since F_2 is upper weakly $\mathcal{I}$-continuous, there exists an $\mathcal{I}$-open set U_2 containing x such that $F_2(U_2) \subset \mathrm{Cl}(V_2 \cap V)$. Set $U = U_1 \cap U_2$, then $U \in \mathcal{I}O(X, x)$. If $y \in F(x_0)$ for any $x \in U$, then $y \in \mathrm{Cl}(V_1 \cup V) \cap \mathrm{Cl}(V_1 \cup V) = (\mathrm{Cl}(V_1) \cap \mathrm{Cl}(V_2)) \cup \mathrm{Cl}(V)$. Since $\mathrm{Cl}(V_1) \cap \mathrm{Cl}(V_2) = \emptyset$, we have $y \in \mathrm{Cl}(V)$ and hence $F(U) \subset \mathrm{Cl}(V)$. Therefore, F is upper weakly $\mathcal{I}$-continuous. ∎

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GELFAND THEOREM FOR BANACH HYPERALGEBRAS

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Abstract. In this paper, we prove that if A is a commutative Banach hyperalgebra then the maximal w-hyperideal space of A is in one-to-one correspondence with the set of (proper) maximal w-hyperideal in A. Also, we obtain some interesting results in direction.

Keywords: hypervectorspace, hyperalgebra, spectrum.

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1. Introduction

The concept of hyperstructure was first introduced by Marty in [6] and has been studied in the following decades to nowadays by many mathematicians. A short review of the theory of hypergroups appears in [3]. The recent books [3], [4] and [12] contain a wealth of applications. There are applications to the following subjects: geometry, hypergraphs, binary relations, combinatorics, codes, cryptography, probability, groups, rational algebraic functions and etc.

Vougiouklis in the fourth A.H.A. congress (1990)[11], introduced the notion of H_v-structures. The concept of H_v-structures constitute a generalization of the well-known algebraic hyperstructures. The principal notions on the H_v-structures can be founded in [3−5, 12]. In this paper, we introduce the hyperalgebras and the concept of hyperideals of hyperalgebras and we obtain some interesting results.

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The paper is arranged as follow. In this paper, we introduce Banach hyperalgebras and the w-hyperideals. Moreover, we define the multiplicative w-linear functionals on Banach hyperalgebra A and we define the maximal w-hyperideals space of A and give some results in this direction. In addition to, we prove that if A is a commutative Banach hyperalgebra then the maximal w-hyperideal space of A is in one-to-one correspondence with the set of (proper) maximal w-hyperideal in A.

2. Preliminaries

Let A be a non-empty set, we denote by $P^*(A)$ the set of all non-empty subsets of A. Throughout this paper the symbol $\mathbb{K}$ will be used to denote a field that is either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$.

Definition 2.1. [10] Let $\mathbb{K}$ be a field and $(A,+)$ be an abelian group. Then a quadruplet $(A,+,\cdot,\mathbb{K})$ is a hypervector space on the field $\mathbb{K}$ if the map $\cdot : \mathbb{K}\times A \longrightarrow P^*(A)$ satisfies the following conditions

(i) $\forall \lambda \in \mathbb{K}, a,b \in A; \lambda \cdot (a+b) \subseteq \lambda \cdot a + \lambda \cdot b$, (right distributivity);

(ii) $\forall \lambda,\mu \in \mathbb{K}, a \in A; (\lambda+\mu)\cdot a \subseteq \lambda \cdot a + \mu \cdot a$, (left distributivity);

(iii) $\forall \lambda,\mu \in \mathbb{K}, a \in A; \lambda \cdot (\mu \cdot a) = (\lambda\mu)\cdot a$, (associativity);

(iv) $\forall \lambda \in \mathbb{K}, a \in A; \lambda \cdot (-a) = (-\lambda)\cdot a = -(\lambda \cdot a)$;

(v) $\forall a \in A; a \in 1 \cdot a$.

Remark 2.2. In the right side of (i) and (ii) the sum is meant in the sense of Frobenius, that is,

$$\lambda \cdot a + \lambda \cdot b = \{m+n : m \in \lambda \cdot a, n \in \lambda \cdot b\}$$

$$\lambda \cdot a + \mu \cdot a = \{p+q : p \in \lambda \cdot a, q \in \mu \cdot a\}.$$

Moreover the left side of (iii) meant in the set-theoretical union of all the sets $\lambda \cdot b$, where b runs over the set $\mu \cdot a$.

Let A be a hypervector space and S be a nonempty subset of A. Then S is said to be a subhypervector space of A if itself is a hypervector space under hyperoperation " $\cdot$ " restricted to S.

Definition 2.3. [10] A semi-norm $\|\|$ on a hypervector space $(A,+,\cdot,\mathbb{K})$ is a map : $\|\| : A \longrightarrow \mathbb{R}$, of A into the positive real number such that:

(i) $\|0\| = 0$,

(ii) $\|a+b\| \leq \|a\| + \|b\| \quad \forall a,b \in A$,

(iii) $\sup \|\lambda \cdot a\| = |\lambda|\|a\| \quad \forall \lambda \in \mathbb{K}, a \in A$.

A semi-norm in A is called a norm if

$$\|a\| = 0 \quad \Longleftrightarrow a = 0.$$

A hypervector space together with a semi-norm or norm is called a semi-normed or normed hypervector space.

Definition 2.4. [8]) A hyperalgebra is a hypervector space A with a multiplication satisfying the following properties,

(i) $\forall a, b, c \in A; (ab)c = a(bc)$, (associativity);

(ii) $\forall a, b, c \in A;\ (a + b)c = ac + bc$, (left distributivity);

(iii) $\forall a, b, c \in A;\ a(b + c) = ab + ac$, (right distributivity);

(iv) $\forall a, b \in A; (-a)b = a(-b) = -(ab)$,

(v) $\forall a, b \in A$, and $\forall \lambda \in \mathbb{K};\ \lambda \cdot (ab) = (\lambda \cdot a)b = a(\lambda \cdot b)$.

The field $\mathbb{K}$ is called the scalar field of A. If $\mathbb{K} = \mathbb{R}$, A is called a real algebra, and if $\mathbb{K} = \mathbb{C}$, a complex algebra.

Example 2.5. *Let $\mathbb{C}$ be the set of all complex numbers. Then $\mathbb{C}$ is a hyperalgebra on $\mathbb{R}$, with respect usual sum and multiplication, and the following hyperoperation:*

$$\cdot : \mathbb{C} \times \mathbb{C} \longrightarrow P^*(\mathbb{C})$$

$$t \cdot z = \{re^{i\theta} : 0 \leq r \leq |t||z|, 0 \leq \theta \leq 2\pi\}, \qquad \forall t \in \mathbb{R}, z \in \mathbb{C}.$$

Remark 2.6. Let A be a hyperalgebra. An element $e \in A$ is called an identity or unit if for every $a \in A, a = ae = ea$. In this case we say that A is an unital hyperalgebra. And an element a is said to be invertible if it has an inverse in A, that is, there exists an element $a^{-1} \in A$ such that

$$aa^{-1} = a^{-1}a = e,$$

where e is the unit element of A. In this case we say that a is invertible.

It is clear that no $a \in A$ has more than one inverse. For a hyperalgebra A we let $Inv(A)$ denote the set of all invertible elements in A. The compliment of $Inv(A)$ in A is called the set of all non-invertible elements in A, and denote by $Sing(A)$. (See [9]).

Definition 2.7. [8] A semi-norm $\|\|$ on a hyperalgebra A is called a hyperalgebra semi-norm if it satisfies in definition 2.3 and the following condition:

$$\|ab\| \leq \|a\|\|b\| \qquad \forall a, b \in A.$$

A semi-norm in A is called a norm if

$$\|a\| = 0 \quad \Longleftrightarrow a = 0.$$

A hypealgebra together with a hyperalgebra semi-norm or hyperalgebra norm is called a semi-normed or normed hyperalgebra, moreover if $e \in A$ be a unit element of A, we assume that $\|e\| = 1$. A normed hypervector space or normed hyperalgebra which is complete in its norm is called a Banach hypervector space or Banach hyperalgebra, respectively.

Lemma 2.8. [8] *Let A be a hyperalgebra and $a \in A, \lambda \in \mathbb{K}$. Then there exists $z \in \lambda \cdot a$ such that $\|z\| = sup\|\lambda \cdot a\|$.*

Remark 2.9. Throughout this paper, we denote z be obtained in the Lemma 3.11 by $z_{\lambda \cdot a}$, also if $a = e$ we denote by z_λ.

Let A be a hypervector space and S be a nonempty subset of A. Then S is said to be a weak subhypervector space of A if for every $a, b \in S$ and $\lambda \in \mathbb{C}$, $a + b \in S, z_{\lambda \cdot a} \in S$. (See [8])

Definition 2.10. [8] Suppose A is a hyperalgebra and a is in A. the spectrum of a, denoted by $\sigma_A(a)$ or simply by $\sigma(a)$, is the set of all complex numbers λ such that $z_\lambda - a \in Sing(A)$, that is,

$$\sigma(a) = \{\lambda \in \mathbb{C} : z_\lambda - a \in Sing(A)\}.$$

The compliment of $\sigma(a)$ in $\mathbb{C}$ is called the resolvent set of a, and denote by $\rho(a)$.

Definition 2.11. [8] Let A be a hyperalgebra. A mapping $\varphi : A \longrightarrow \mathbb{C}$ is said to be weak linear functional (or simply, w-linear functional) if

(i) $\varphi(a + b) = \varphi(a) + \varphi(b),\ \forall a, b \in A,$

(ii) $\lambda\varphi(a) = \varphi(z_{\lambda \cdot a}),\ \forall a \in A, \lambda \in \mathbb{K}.$

Definition 2.12. [8] A w-linear functional φ on a hyperalgebra A is multiplicative if φ is nontrivial and $\varphi(ab) = \varphi(a)\varphi(b)$ for all a and b in A. We let M_A^w denote the set of all multiplicative w-linear functional on A. M_A^w is called the maximal w-hyperideal space of A for reasons that will become clear a little later.

Lemma 2.13. [8] *Let φ be a multiplicative w-linear functional on hyperalgebra A. Then $\varphi(e) = 1$.*

Definition 2.14. [8] Let A be a normed hyperalgebra, we say A is normal hyperalgebr if

(i) $\forall \lambda \in \mathbb{K}, a, b \in A; z_{\lambda \cdot (a+b)} = z_{\lambda \cdot a} + z_{\lambda \cdot b}.$

(ii) $\forall \lambda, \mu \in \mathbb{K}, a \in A; z_{(\lambda+\mu) \cdot a} = z_{\lambda \cdot a} + z_{\mu \cdot a}.$

3. Main results

Definition 2.15. Let A be a hyperalgebra over field $\mathbb{K}$. A nonempty set I of A is called

(i) a left (res. right) weak hyperideal (or simply, w-hyperideal) of hyperalgebra A if and only if

(a) $(I, +, \cdot, \mathbb{K})$ is a weak subhypervector space of $(A, +, \cdot, \mathbb{K})$; and

(b) $ab \in I$ (res. $ba \in I$), for all $a \in A$ and for all $b \in I$.

(ii) a w-hyperideal of hyperalgebra A if it is both left and right w-hyperideal of A. The hyperideal I of A denoted by $I \lhd_h^w A$.

If $I \neq A$, I is a proper w-hyperideal. Maximal w-hyperideals are proper w-hyperideals which are not contained in any larger proper w-hyperideals.

If I is a left hyperideal in A, $a \in I$, and a is let invertible, then $I = A$.

In fact, if $ba = e$, then $e \in I$ since I is a left w-hyperideal. Thus for $x \in A$, $x = xe \in I$. This forms a link between ideals and invertibility.

Lemma 2.16. *If A is a Banach hyperalgebra with identity,*

$$\begin{aligned} G_l(A) &= \{a \in A : a \text{ is left invertible}\}, \\ G_r(A) &= \{a \in A : a \text{ is right invertible}\}, \text{ and} \\ G(A) &= \{a \in A : a \text{ is invertible}\}, \end{aligned}$$

then G_l, G_l and $G(A)$ are open subsets of A.

Proof. The proof of the preceding theorem is similar case that A is a Banach algebra (see [2]). ∎

Lemma 2.17. *If A is a Banach hyperalgebra with identity, then*

(a) *the closure of a proper left, right, or two-side w-hyperideal is a proper left, right, or two-side w-hyperideal;*

(b) *a maximal left, right, or two-side w-hyperideal is closed.*

Proof. (a) Let I be a proper left w-hyperideal and let G_l be the set of left invertible elements in A. It follows that $I \cap G_l = \emptyset$. Thus $I \subseteq A \backslash G_l$. By Theorem 3.2, $A \backslash G_l$ is closed. Hence $\overline{I} \subseteq A \backslash G_l$ and thus $\overline{I} \neq A$. To see that $\overline{I}$ is a w-hyperideal, let $a \in \overline{I}, b \in A$. Therefore there is a sequence $\{a_n\}$ in I such that $\lim_{n\to\infty} a_n = a$, it follows that $\lim_{n\to\infty} ba_n = ba$, so $ba \in \overline{I}$ since I is a w-hyperideal. For every $\lambda \in K$ we have

$$\|z_{\lambda \cdot a_n} - z_{\lambda \cdot a}\| = \|z_{\lambda \cdot (a_n - a)}\| = |\lambda| \|a_n - a\|.$$

Hence $\lim_{n\to\infty} z_{\lambda \cdot a_n} = z_{\lambda \cdot a}$ and since $z_{\lambda \cdot a_n} \in I$ for $n \in N$ we obtain $z_{\lambda \cdot a} \in \overline{I}$.

(b) If M is a maximal left w-hyperideal, $\overline{M}$ is a proper left w-hyperideal by (a). Hence $M = \overline{M}$ by maximality. ∎

Lemma 2.18. [8] *If A is a hyperalgebra then every proper w-hyperideals of A is contained in a maximal w-hyperideals of A.*

Theorem 2.19. [9] *Let A be a hyperalgebra and I be a w-hyperideals in A. Then $(A/I, +, \odot, \mathbb{K})$ is a hypervector space, with respect to the following hyperoperation:*

$$\odot : \mathbb{K}K \times A/I \longrightarrow P^*(A/I)$$

$$\forall \lambda \in \mathbb{K}, \forall [a] \in A/I; \lambda \odot [a] = \{[t] : t \in \lambda \cdot a\}.$$

On the other hand,

$$\forall \lambda \in \mathbb{K},\ \forall (a+I) \in A/I;\ \lambda \odot (a+I) = \lambda \cdot a + I.$$

Lemma 2.20. *If A is a normed hyperalgebra and I is a closed w-hyperideal in A, then the quotient hypervector space A/I is a normed hypervector space with the following (quotient) norm.*

$$\|[a]\| = \inf\{\|a - b\| : b \in I\}, \quad [a] \in A/I.$$

Proof. It is clear that $\|[\underline{0}]\| = 0$ and $\|[a] + [b]\| \leq \|[a]\| + \|[b]\|$ for every $a, b \in A$. Since for every $a \in A, \lambda \in \mathbb{K}$ we have $\lambda \cdot a \subseteq \lambda \cdot a + I$, we obtain

$$|\lambda| \|[a]\| \leq |\lambda| \|a\| = \sup \|\lambda \cdot a\| \leq \sup \|\lambda \cdot a + I\| = sup \|\lambda \odot [a]\|.$$

Hence

$$|\lambda| \|[a]\| \leq \sup \|\lambda \odot [a]\|. \tag{1}$$

On the other hand, for every $t \in \lambda \cdot a$ we have

$$\|[t]\| \leq \|t\| \leq |\lambda| \|a\| = \sup \|\|\lambda \cdot a\|$$

so

$$\sup \|\lambda \odot [a]\| = \sup\{\|[t]\| : t \in \lambda \cdot a \leq \sup \|\|\lambda \cdot a\|\}. \tag{2}$$

By (1) and (2)

$$\sup \|\lambda \odot [a]\| = |\lambda| \|[a]\|. \qquad \blacksquare$$

Theorem 2.21. *If A is a Banach hyperalgebra and I be a closed w-hyperideal in A. Then A/I is a Banach hyperalgebra. If A has an identity, so does A/I.*

The proof of the preceding theorem is similar to the case where A is a Banach algebra (see [2]).

Definition 2.22. An unital hyperalgebra in which each non-zero element is invertible is a divison hyperalgebra.

Lemma 2.23. *Let A be a division normed hyperalgebra. Then $\lambda \in \sigma(a)$ if and only if $a \in \lambda \cdot e$ and $\|a\| = |\lambda|$.*

Proof. If $\lambda \in \sigma(a)$ then $(z_\lambda - a) \in Sing(A)$, it follows that $z_\lambda - a = \underline{0}$, so that $z_\lambda = a$, it follow that z_λ is unique and $a \in \lambda \cdot e$. Hence $\lambda \in \sigma(a)$ if and only if $a \in \lambda \cdot e$. ∎

Lemma 2.24. *Let A be a division normed normal hyperalgebra. Then $\sigma(a)$ has only one point.*

Proof. Note that $\sigma(a)$ is non-empty (see [8]). If $\{\lambda, \mu\} \subseteq \sigma(a)$, then

$$\|a\| = |\lambda| = |\mu|.$$

On the other hand,

$$z_{\frac{\lambda+\mu}{2}} = z_{\frac{\lambda}{2}} + z_{\frac{\mu}{2}}.$$

It follows that

$$\left|\frac{\lambda+\mu}{2}\right| \leq |\lambda| = |\mu| = \|a\|.$$

$$\begin{aligned}(\lambda+\mu)(\bar{\lambda}+\bar{\mu}) \leq 4\lambda\bar{\lambda} &\Rightarrow \lambda\bar{\mu} + \mu\bar{\lambda} \leq \lambda\bar{\lambda} + \mu\bar{\mu} \\ &\Rightarrow \lambda(\bar{\mu}-\bar{\lambda}) - \mu(\bar{\mu}-\bar{\lambda}) \leq 0 \\ &\Rightarrow (\bar{\mu}-\bar{\lambda})(\lambda-\mu) \leq 0 \\ &\Rightarrow |\lambda-\mu|^2 \leq 0 \\ &\Rightarrow |\lambda-\mu| = 0 \\ &\Rightarrow \lambda = \mu.\end{aligned}$$

This implies that $\sigma(a)$ contains only one point, which we call λ_a. ∎

Theorem 2.25. *If A is a normed normal hyperalgebra in which every non-zero element is invertible, Then A is isometrically isomorphic to $\mathbb{C}$.*

Proof. Let $a \in A$. By Lemma 3.10, $\sigma(a)$ is non-empty and contains only one point, which we call λ_a. Define $\phi : A \longrightarrow \mathbb{C}$ by $\phi(a) = \lambda_a$. If $a, b \in A$, then there exists $\lambda_a, \lambda_b \in \mathbb{C}$ such that $a \in \lambda_a \cdot e, a = z_{\lambda_a}$ and $b \in \lambda_b \cdot e, b = z_{\lambda_b}$ (see Lemma 3.10). Since $z_{\lambda_a+\lambda_b} = z_{\lambda_a} + z_{\lambda_b}$, it follows $\lambda_a + \lambda_b \in \sigma(a+b)$, that is, $\varphi(a+b) = \lambda_a + \lambda_b = \varphi(a) + \varphi(b)$.

For every $\lambda \in K$, we have $\lambda \cdot a \subseteq (\lambda\lambda_a) \cdot e$ and obviously $z_{\lambda\cdot a} = z_{\lambda\lambda_a}$, hence $\varphi(z_{\lambda\cdot a}) = \varphi(z_{\lambda\lambda_a}) = \lambda\lambda_a = \lambda\varphi(a)$.

Since $\sigma(ab)$ is non- empty then there exist $z_{\lambda_{ab}} \in \lambda_{ab} \cdot e$ such that $z_{\lambda_{ab}} = ab$, $\|ab\| = |\lambda_{ab}|$. It follows that $e \in \lambda_{ab}^{-1} \cdot (ab)$.

On the other hand, $z_{\lambda_a} z_{\lambda_b} = ab \in (\lambda_a\lambda_b) \cdot e$ and $|\lambda_a||\lambda_b| = \|ab\|$. Hence $\lambda_a\lambda_b \in \sigma(ab)$. Since in division Banach hyperalgebra z_λ is unique, therefore, $z_{\lambda_{ab}} = z_{\lambda_a} z_{\lambda_b}$. It follows that $\varphi(ab) = \varphi(a)\varphi(b)$. Obviously ϕ is one-to-one and onto. Thus ϕ is an isometrical isomorphism. ∎

Theorem 2.26. *If A is a commutative normal Banach hyperalgebra. Then M_A^w is in one-to-one correspondence with the set of (proper) maximal w-hyperideals in A.*

Proof. First assume that φ is a multiplicative w-linear functional on A. We show that $ker\varphi$ is a proper maximal w-hyperideal in A. In fact, it is clear that $ker\varphi$ is a proper w-hyperideal because if $a \in ker\varphi$ and $\lambda \in \mathbb{C}$ then $\varphi(z_{\lambda \cdot a}) = \lambda\varphi(a) = 0$. To see that it is maximal, let $a \in A \ker \varphi$. We have

$$e \in (e - z_{(\varphi(a))^{-1} \cdot a}) + z_{(\varphi(a))^{-1} \cdot a}.$$

The set in the parentheses contain in $ker\varphi$. We see that the linear span of a and $ker\varphi$ contains e, and hence any w-hyperideal containing both $ker\varphi$ and a must be the whole hyperalgebra. Thus $ker\varphi$ is maximal.

Next, assume that M be a maximal w-hyperideal of A. Then A/M is a hyperalgebra. Choose $x \in A, x \notin M$, and put

$$J = \{ax + y : a \in A, y \in M\}.$$

It is clear that J is a w-hyperideal in A which is larger than M, since $x \in J$ (take a=e, y=0).

Thus $J = A$. Then $e \in J$, and $e = ax + y$ for some $a \in A$ and $y \in M$.

If $\pi : A \to A/M$ is the quotient map. It follows $\pi(e) = \pi(a)\pi(x)$. Therefore every nonzero element $\pi(x)$ of the Banach hyperalgebra A/M is invertible in A/M. By the Theorem 3.11, there is an isomorphism ϕ of A/M onto $\mathcal{C}$. Put $\varphi = \phi \circ \pi$. Then $\varphi \in M_A$, and M is the null space of φ.

Finally, to see that the above correspondence between multiplicative w-linear functionals and maximal w-hyperideals is one-to-one, let φ_1 and φ_2 be multiplicative w-linear functionals on A with a common kernel M. We must show that $\varphi_1 = \varphi_2$. For any a in A, we can write

$$z_{\varphi_1(a) - \varphi_2(a)} = z_{\varphi_1(a)} - z_{\varphi_2(a)} = (a - z_{\varphi_2(a)}) - (a - z_{\varphi_1(a)})$$

The first term on the right is in $ker\varphi_2 = M$ and the second term is in $ker\varphi_1 = M$. Thus $z_{\varphi_1(a) - \varphi_2(a)} \in M$ and we must have $\varphi_1(a) = \varphi_2(a)$, completing the proof of the theorem. ∎

Theorem 2.27. *Let A be a commutative normal Banach hyperalgebra. Then $a \in A$ is invertible if and only if $\varphi(a) \neq 0$ for every $\varphi \in M_A^w$.*

Proof. Let $a \in A$ be invertible, then for every $\varphi \in M_A^w$ we have

$$1 = \varphi(e) = \varphi(a)\varphi(a^{-1})$$

so $\varphi(a)\varphi(a^{-1}) \neq 0$ and $\varphi(a) \neq 0$. Conversely, if a is not invertible, then the set $J = \{xa \mid x \in A\}$ dose not contain e, hence is a proper w-hyperideal in A, since

$$(\lambda \cdot x)a = \{ta : t \in \lambda \cdot x\} \quad \forall x \in A.$$

Thus $J \subseteq M$ for some maximal w-hyperideal M. By theorem 3.12, $M = ker\varphi$ for some $\varphi \in M_A^w$, then $\varphi(a) = 0$. ∎

Theorem 2.28. *Let A be a commutative normal Banach hyperalgebra and $a \in A$. Then*

$$\sigma(a) = \{\varphi(a) : \varphi \in M_A^w\}.$$

Proof. If $\varphi \in M_A^w$, then $(z_{\varphi(a)} - a) \in Sing(A)$ because, if $(z_{\varphi(a)} - a) \in Inv(A)$, then there exists $b \in A$ such that $b(z_{\varphi(a)} - a) = (z_{\varphi(a)} - a)b = e$ which implies $0 = \varphi(b)0 = 0\varphi(z) = \varphi(e) = 1$, this is a contradiction.

Conversely, if $(z_\lambda - a) \in Sing(A)$, then $(z_\lambda - a)A$ is a w-hyperideal which is, by Lemma 3.4, contained in some maximal w-hyperideal M. By Theorem 3.12, there exists $\varphi \in M_A^w$ such that $M = ker\varphi$ so $0 = \varphi(z_\lambda - a)e = \varphi(z_\lambda - a) = \varphi(z_\lambda) - \varphi(a)$, so we obtain $\varphi(z_\lambda) = \lambda$ that is $\varphi(a) = \lambda$. ∎

Remark 2.29. In particular, Theorem 3.14 implies that the set all multiplicative linear functionals of A is not empty.

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ON ALMOST 2-ABSORBING SUBMODULES

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Abstract. Let R be a commutative ring with identity and let M be a unitary R-module. A proper submodule N of an R-module M will be called almost 2-absorbing submodule if $a, b \in R$ and $m \in M$ with $abm \in N-(N : M)N$ implies that $ab \in (N : M)$ or $am \in N$, or $bm \in N$. Also a proper ideal I of R will be called almost 2-absorbing ideal if $a, b, c \in R$ with $abc \in I - I^2$ implies that $ab \in I$ or $ac \in I$, or $bc \in I$. These concepts are generalizations of the notions of 2-absorbing submodules and ideals respectively, which have been studied. In this paper we give several results concerning almost 2-absorbing submodules.

Keywords: 2-absorbing ideal, 2-absorbing submodule, Almost 2-absorbing ideal, Almost 2-absorbing submodule.

Mathematics Subject Classification: 13A15, 13C99, 13F05, 13F15.

1. Introduction

Through out this paper all rings are commutative with identity and all modules are unitary. Prime ideals play a central rule in commutative ring theory. A proper ideal P of R is said to be prime ideal if $ab \in P$ implies that $a \in P$ or $b \in P$ where $a, b \in R$. Various generalizations of prime ideals have been studied as well as extending ideals to submodules. For example, a proper ideal I of R is said to be almost prime provided that $a, b \in R$ with $ab \in I - I^2$ imply that $a \in I$ or $b \in I$(see [4], [2]). A number of generalizations of prime ideals in commutative

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rings can be found in [5], [6]. Also a proper ideal I of R is said to be 2-absorbing if whenever $a, b, c \in R$ and $abc \in I$, then either $ab \in I$ or $ac \in I$ or $bc \in I$ (see [1]). A proper ideal I of R is said to be almost 2-absorbing ideal if whenever $a, b, c \in R$ with $abc \in I - I^2$, then either $ab \in I$ or $ac \in I$, or $bc \in I$ (see [8]). A proper submodule N of M is called a 2-absorbing (resp. weakly 2-absorbing) submodule of M if whenever $a, b \in R$, $m \in M$ and $abm \in N$ (resp. $0 \neq abm \in N$), then $ab \in (N :_R M)$ or $am \in N$, or $bm \in N$ (see [8]).

Clearly, if N is a prime R-submodule of M, then N is weakly prime and then it is weakly 2-absorbing and hence it is an almost 2-absorbing R-submodule of M. If N is an almost 2-absorbing R-submodule of M, then N need not to be weakly 2-absorbing; for example $N = \langle 8 \rangle$ in the $\mathbf{Z}$-module $M = \mathbf{Z}_{40}$. So, almost 2-absorbing submodules are generalization of weakly 2-absorbing submodules. We show that if A, B are two ideals of R, N is an almost 2-absorbing R-submodule of M and L is an R-submodule of M such that $ABL \subseteq N - (N : M)N$ and $8(AB + (A + B)(N : M))(L + N) \nsubseteq (N : M)N$, then $AB \subseteq (N : M)$ or $AL \subseteq N$ or $BL \subseteq N$.

The purpose of this paper is to introduce the notion of almost 2-absorbing submodules as a new generalization of both 2-absorbing and almost prime ideals.

2. Almost 2-absorbing submodules

Following [3], a proper submodule N of an R-module M is weakly 2-absorbing if $a, b \in R$, $m \in M$ and $0 \neq abm \in N$ then $ab \in (N : M)$ or $am \in N$, or $bm \in N$. Now we introduce the concept of almost 2-absorbing submodule and generalize the concept of almost prime submodules (see [5]) to an almost 2-absorbing submodules.

Definition 2.1 Let R be an integral domain, M be an R-module and N a proper submodule of M. N is called an almost 2-absorbing submodule of M if $a, b \in R$ and $m \in M$ with $abm \in N - (N : M)N$ then either $ab \in (N : M)$ or $am \in N$, or $bm \in N$.

Let R be a commutative ring and M an R-module, then every weakly 2-absorbing submodule N of M is an almost 2-absorbing submodule of M. But the converse need not to be true. For example, consider $R = Z$, $M = Z_{40}$ and $N = \langle \bar{8} \rangle$ then $(N : M)N = N$. So N is almost prime submodule of M and hence N is an almost 2-absorbing submodule of M. On the other hand, $0 \neq 2.2(\bar{2}) = \bar{8} \in N$ while $2.2 \notin (N : M)$ and $2.\bar{2} \notin N$, so N is not weakly 2-absorbing submodule of M. So almost 2-absorbing submodules are generalization of both almost prime submodules and weakly 2-absorbing submodules.

Theorem 2.2 *Let M be an R-module and N, K be a proper submodules of M with $K \subseteq N$. Then N is an almost 2-absorbing submodule of M if and only if N/K is an almost 2-absorbing submodule of M/K.*

Proof. ($\Rightarrow$) Let N be an almost 2-absorbing submodule of M. Now, for any $a, b \in R$ and $m \in M$ assume that $ab(m+K) \in N/K - (N/K : M/K)N/K$. It follows that $ab(m+K) \in N/K - (N : M)N/K$ and so $abm \in N - (N : M)N$. Since N is an almost 2-absorbing then $ab \in (N : M)$ or $am \in N$, or $bm \in N$. Hence $ab \in (N/K : M/K)$ or $a(m+K) \in N/K$, or $b(m+K) \in N/K$. Thus N/K is an almost 2-absorbing submodule of M/K.

($\Leftarrow$) Let N/K be an almost 2-absorbing submodule of M/K, $a, b \in R$ and $m \in M$ with $abm \in N - (N : M)N$. It follows that $ab(m+K) \in N/K - (N/K : M/K)N/K$. Since N/K is an almost 2-absorbing submodule of M/K then $ab \in (N/K : M/K)$ or $a(m+K) \in N/K$, or $b(m+K) \in N/K$. So $ab \in (N : M)$ or $am \in N$, or $bm \in N$. ∎

Proposition 2.3 *Let M be an R-module and N be a submodule of M. Then the following are equivalent.*

(i) *N is an almost 2-absorbing submodule of M.*

(ii) *For $r_1r_2 \in R - (N :_R M)$,*
$(N :_M \langle r_1r_2\rangle) = (N :_M \langle r_1\rangle) \cup (N :_M \langle r_2\rangle) \cup ((N : M)N :_M \langle r_1r_2\rangle)$.

(iii) *For $r_1r_2 \in R - (N : M)$,*
$(N :_M \langle r_1r_2\rangle) = (N :_M \langle r_1\rangle)$ *or* $(N :_M \langle r_1r_2\rangle) = (N :_M \langle r_2\rangle)$, *or*
$(N :_M \langle r_1r_2\rangle) = ((N : M)N :_M \langle r_1r_2\rangle)$.

Proof. $(i) \Rightarrow (ii)$ If $r_1r_2 \in R-(N :_R M)$ and $m \in (N :_M \langle r_1r_2\rangle)$ then $r_1r_2m \in N$. But if $r_1r_2 \notin R - (N :_R M)N$ then $r_1r_2m \in N - (N :_R M)N$. So $r_1m \in N$ or $r_2m \in N$. Hence $m \in (N :_M \langle r_1\rangle)$ or $m \in (N :_M \langle r_2\rangle)$.

(ii) $\Rightarrow$ (iii) It is well known that if a module equals to the union of two modules then it is one of them.

(iii) $\Rightarrow$ (i) Let $a, b \in R$ and $m \in M$ with $abm \in N - (N : M)N$. Now assume that $ab \notin (N : M)$, we show that $am \in N$ or $bm \in N$. By (ii) $(N :_M \langle ab\rangle) = (N :_M \langle a\rangle)$ or $(N :_M \langle ab\rangle) = (N :_M \langle b\rangle)$, or $(N :_M \langle ab\rangle) = ((N : M)N :_M \langle ab\rangle)$. Since $abm \notin (N : M)N$ then $m \notin ((N : M)N :_M \langle ab\rangle)$. Therefore $m \in (N :_M \langle a\rangle)$ or $m \in (N :_M \langle b\rangle)$. Hence $am \in N$ or $bm \in N$. ∎

Lemma 2.4 *Let M be an R-module, N be an almost 2-absorbing R-submodule of M and $r, s \in R$. If L is an R-submodule of M such that $rsL \subseteq (N : M)N$ and $2rsL \nsubseteq (N : M)$, then $rs \in (N : M)$ or $rL \subseteq N$ or $sL \subseteq N$.*

Proof. Assume $K = (N : M)$ and suppose $rs \notin K$. It is enough to prove $L \subseteq (N :_M r) \cup (N :_M s)$. Let $x \in L$. If $rsx \notin (N : M)N$, then since N is almost 2-absorbing and $rs \notin K$, either $rx \in N$ or $sx \in N$ and then $x \in (N :_M r) \cup (N :_M s)$. Suppose $rsx \in (N : M)N$. Since $2rsL \nsubseteq (N : M)$, there exists $z \in L$ such that $2rsz \notin (N : M)N$ and then $2rsz \in N-(N : M)N$. Since N is almost 2-absorbing and $rs \notin K$, either $rz \in N$ or $sz \in N$. Let $m = z+x$. Then $rsm \in N-(N : M)N$ and since $rs \notin K$, either $rm \in N$ or $sm \in N$. We study three cases:

Case 1. $rz \in N$ and $sz \in N$. Clearly, $rm \in N$ or $sm \in N$ and then either $rx \in N$ or $sx \in N$.

Case 2. $rz \in N$ and $sz \notin N$. Suppose $rx \notin N$. Then $rm \in N$ and then $sm \in N$ which implies that $r(m+z) \notin N$ and $s(m+z) \notin N$. Since N is almost 2-absorbing and $rs \notin K$, $rs(m+z) \in (N : M)N$ which is a contradiction. Hence, $rx \in N$.

Case 3. Similarly, as Case 2. ∎

Lemma 2.5 *Let M be an R-module, N be an almost 2-absorbing R-submodule of M and $r \in R$. If I is an ideal of R and L is an R-submodule of M such that $rIL \subseteq N$ and $4rIL \nsubseteq (N : M)N$, then $rI \subseteq (N : M)$ or $rL \subseteq N$ or $IL \subseteq N$.*

Proof. Assume $K = (N : M)$ and suppose $rI \nsubseteq K$. The there exists $s \in I$ such that $rs \notin K$. We show that there exists $t \in I$ such that $4rtL \nsubseteq (N : M)N$ and $rt \notin K$. Since $4rIL \nsubseteq (N : M)N$, there exists $\alpha \in I$ such that $4r\alpha L \nsubseteq (N : M)N$. If $r\alpha \notin K$ or $4rsL \nsubseteq (N : M)N$, then by taking $t = \alpha$ or $t = s$, we obtain our result. Suppose $r\alpha \in K$ and $4rsL \subseteq (N : M)N$. Then $4r(s+\alpha)L \subseteq N - (N : M)N$ and $r(s+\alpha) \notin K$ and then by taking $t = s + \alpha$, we obtain our result. Hence, $2rtL \nsubseteq (N : M)N$ and then by Lemma 2.4, $L \subseteq (N :_M r) \bigcup (N :_M t)$. If $rL \subseteq N$, we are done. Suppose $rL \nsubseteq N$. Then $tL \subseteq N$. We show that $I \subseteq (K : r) \cup (N : L)$. Let $\beta \in I$. If $2r\beta L \nsubseteq (N : M)N$, then by Lemma 2.4, $r\beta \in K$ or $rL \subseteq N$ or $\beta L \subseteq N$. Since $rL \nsubseteq N$, $\beta \in (K : r) \cup (N : L)$. Suppose $2r\beta L \subseteq (N : M)N$. Then $2r(t+\beta)L \subseteq N - (N : M)N$ and then Lemma 2.4, $r(t+\beta) \in K$ or $rL \subseteq N$ or $(t+\beta)L \subseteq N$. Since $rL \nsubseteq N$, $(t+\beta) \in (K : r) \cup (N : L)$. If $t + \beta \in (N : L)$, then since $t \in (N : L)$, $\beta \in (N : L)$. Suppose $t + \beta \in (K : r) - (N : L)$. Now, $2r(2t+\beta)L \nsubseteq (N : M)N$ and $2r(2t+\beta)L \subseteq N$. Since $rt \notin K$ and $r(t+\beta) \in K$, $r(2t+\beta) \notin K$. So, by Lemma 2.4, $L \subseteq (N :_M r) \cup (N :_M 2t + \beta)$. Since $t + \beta \notin (N : L)$ and $t \in (N : L)$, $2t + \beta \notin (N : L)$ and then $L \subseteq (N : r)$ which is a contradiction. So, $t + \beta \in (N : L)$ and since $t \in (N : L)$, $\beta \in (N : L)$. Thus, $I \subseteq (K : r) \cup (N : L)$ and since $rI \nsubseteq K$, $IL \subseteq N$. ∎

Theorem 2.6 *Let A, B be two ideals of R. Suppose M is an R-module and N be an almost 2-absorbing R-submodule of M. If L is an R-submodule of M such that $ABL \subseteq N - (N : M)N$ and $8(AB + (A+B)(N : M))(L+N) \nsubseteq (N : M)N$, then $AB \subseteq (N : M)$ or $AL \subseteq N$ or $BL \subseteq N$.*

Proof. We study three cases:

Case 1. $8ABL \nsubseteq (N : M)N$. Then there exists $r \in B$ such that $8rAL \nsubseteq (N : M)N$ and then $4rAL \nsubseteq (N : M)N$. By Lemma 2.4, $rA \subseteq (N : M) = K$ or $rL \subseteq N$ or $AL \subseteq N$. If $AL \subseteq N$, then we are done. Suppose $AL \nsubseteq N$. Then $r \in (K : A) \cup (N : L)$. We show that $B \subseteq (K : A) \cup (N : L)$. Let $s \in B$. If $4sAL \nsubseteq (N : M)N$, then by Lemma 2.4, $s \in (K : A) \cup (N : L)$ as $AL \nsubseteq N$. Suppose $4sAL \subseteq (N : M)N$. Then $4(r+s)AL \subseteq N - (N : M)N$ and then by Lemma 2.4, $r + s \in (K : A) \bigcup (N : L)$ as $AL \nsubseteq N$. We study four subcases:

Subcase 1. $r+s \in (K:A)$ and $r \in (K:A)$. Then $s \in (K:A)$.

Subcase 2. $r+s \in (N:L)$ and $r \in (N:L)$. Then $s \in (N:L)$.

Subcase 3. $r \in (K:A)-(N:L)$ and $r+s \in (N:L)-(K:A)$. Then $2r+s \notin (K:A)$ and $2r+s \notin (N:L)$ and then $2r+s \notin (K:A) \cup (N:L)$. Now, $4(2r+s)AL \nsubseteq (N:M)N$ and hence by Lemma 2.4, $2r+s \in (K:A) \cup (N:L)$ as $AL \nsubseteq N$, which is a contradiction. As $r \in (K:A) \cup (N:L)$ and $r+s \in (K:A) \cup (N:L)$, we have two situations:

(1) $r \in (N:L)$ and $r+s \in (N:L)-(K:A)$, so $s \in (N:L)$.

(2) $r \in (K:A)-(N:L)$ and $r+s \in (K:A)$, so $s \in (K:A)$.

Subcase 4. $r+s \in (K:A)-(N:L)$ and $r \in (N:L)-(K:A)$. Then, as Subcase 3, $s \in (K:A) \cup (N:L)$. Thus, $B \subseteq (K:A) \cup (N:L)$.

Case 2. $8ABN \nsubseteq (N:M)N$ and $8ABL \subseteq (N:M)N$. Then $8AB(L+N) \subseteq N-(N:M)N$ and then, by Case 1, $BA \subseteq (N:M)$ or $B(L+N) \subseteq N$ or $A(L+N) \subseteq N$ and then $BA \subseteq (N:M)$ or $BL \subseteq N$ or $AL \subseteq N$.

Case 3. $8B(N:M)L \nsubseteq (N:M)N$ and $8ABL \subseteq (N:M)N$. Then, $8B(A+(N:M))L \nsubseteq (N:M)N$ and then, by Case 1, $8(A+(N:M)) \subseteq (N:M)$ or $BL \subseteq N$ or $(A+(N:M))L \subseteq N$ which implies that $BA \subseteq (N:M)$ or $BL \subseteq N$ or $AL \subseteq N$. Similarly, if $8A(N:M)L \nsubseteq (N:M)N$. ∎

Theorem 2.7 *Let M be an R-module and N be an almost 2-absorbing R-submodule of M. Suppose $r,s \in R$ and $m \in M$ such that $rsm \in (N:M)N$, $rs \notin (N:M)$, $rm \notin N$ and $sm \notin N$. Then $rsN \subseteq (N:M)N$.*

Proof. Suppose $rsN \nsubseteq (N:M)N$. Then there exists $n \in N$ such that $rsn \notin (N:M)N$ and then $rs(m+n) \in N-(N:M)N$. Since N is almost 2-absorbing, $rs \in (N:M)$ or $r(m+n) \in N$ or $s(m+n) \in N$ and then $rs \in (N:M)$ or $rm \in N$ or $sm \in N$ which are contradiction. Hence, $rsN \subseteq (N:M)N$. ∎

Proposition 2.8 *Let (R,M) be a quasi-local ring. Then every proper ideal of R is almost 2-absorbing ideal if $M^3=0$.*

Proof. Let $a,b,c \in M$ with $abc \neq 0$. Since $abc \neq 0$ then $abc \in M-M^2$. So $ab \in M$ or $ac \in M$, or $bc \in M$ then a or b, or c is not in M, which is a contradiction. Hence $abc=0$. ∎

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KUPY-NEEV HASH FUNCTION

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Abstract. Kupyna has been approved as a new cryptographic standard hash function of Ukrain in 2015 (Ukrainian standard DSTU 7564:2014). It is built on the transformations of Kalyna block cipher (Ukrainian standard DSTU 7624:2014). The design of compression function of Kupyna is nearly identical with Grøstl that makes it vulnerable to the similar attacks those were introduced on Grøstl. It is adapted for its highly secured design and its efficiency but lately Mendel et al. [4] and Zou et al. [14] mounted rebound attack on compression function of Kupyna-256 and reduced round collision attack on Kupyna hash function respectively. These attacks are applied on Kupyna because of its permutations based on AES. In this paper we propose Kupy-Neev hash function in which we change those permutations with the permutations based on *Neeva Hash* [1].

Keywords: Davis-Meyer mode, Grøstl, Kupyna, Neeva hash function, Rebound attack.

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1. Introduction

Kupyna is announced as a new standard cryptographic hash function for Ukrain in 2015. It has been chosen over GOST R 34.11-94 [6] which was used earlier for all security applications in nationwide before 2015. GOST R 34.11-94 was not efficient enough and couldn't provide required security hence a new hash standard was needed. Kupyna is quite efficient and gives high security. Due to these features, it had been started using in every practical scenario for cryptographic hash function. Kupyna uses Davis-Meyer compression function based on Even-Mansour scheme and its internal permutations are similar to the transformations of the Kalyna block cipher [11].

The compression functions of Kupyna and Grøstl [5] are very much similar which allow us to use the same cryptanalytic tools used in the analysis of Grøstl. Grøstl was chosen as one of the five finalists of SHA-3 competition because of its unique design and efficient software implementation. It is based on wide trail strategy. Mendel[10], [9] and Jean[7] had applied rebound attacks and collision attack on 5 rounds of Grøstl due to similarity of its permutations with AES and also applied collision attack on 5 rounds of Grøstl. The same attack strategy is used for Kupyna which makes it vulnerable to use in near future. This is the main reason to find an alternate of Kupyna. Thus, we propose a Kupy-Neev hash function[2] which has a different set of permutations and can provide better security.

This paper is organized in the following manner: In Section 2, Kupyna hash function is discussed briefly, we propose our Kupy-Neev hash function in Section 3 and in Section 4, the analysis of the proposed scheme is shown.

2. Kupyna hash function

Kupyna is an iterated hash function approved as Ukrain standard hash function. There are two permutations T^{+} and $T^{\oplus}$ used in the compression function of Kupyna which is very much similar to AES. In the following subsection we define the design of this hash function and the underlying permutations.

2.1. The hash function

Kupyna has two variants, Kupyna-256 and Kupyna-512 which give output of length 256 and 512 respectively. We will discuss here Kupyna-256 only. The input message M is first padded and then divided into the blocks $m_1, m_2, \ldots, m_k$ of length 512-bit. Here, Message is processed by $f(h_{i-1}, m_i)$ iteratively till all the message blocks of 512-bit get exhausted where f is the compression function and h_i is the chaining variable for $1 \leq i \leq k$. Once the updated 512-bit register h_t is obtained and then output transformation $\Omega(h_t)$ to calculate the final hash h. So,

$$\begin{aligned} h_0 &= \mathcal{IV} \\ h_i &= f(h_{i-1}, m_i) \quad \text{for } 1 \leq i \leq k \\ h &= \Omega(h_t) \end{aligned}$$

[2] *Kupy-Neev* is the proposed hash function in which the mode of iteration is same as Kupyna but underlying compression function is based on Neeva hash function [1]

The compression function f is based on two permutations T^{+} and $T^{\oplus}$ and is defined as follows:

$$f(h_{i-1}, m_i) = T^{\oplus}(h_{i-1} \oplus m_i) \oplus T^{+}(m_i) \oplus h_{i-1}$$

The output transformation Ω applied on h_t to find the final hash value h of size 256 where $trunc_n(x)$ is taking 'n' most significant bits of x.

$$\Omega(h_t) = trunc_{256}(T^{\oplus}(h_t) \oplus h_t)$$

The following figure is taken from the reference [12] shows the design of Kupyna hash function.

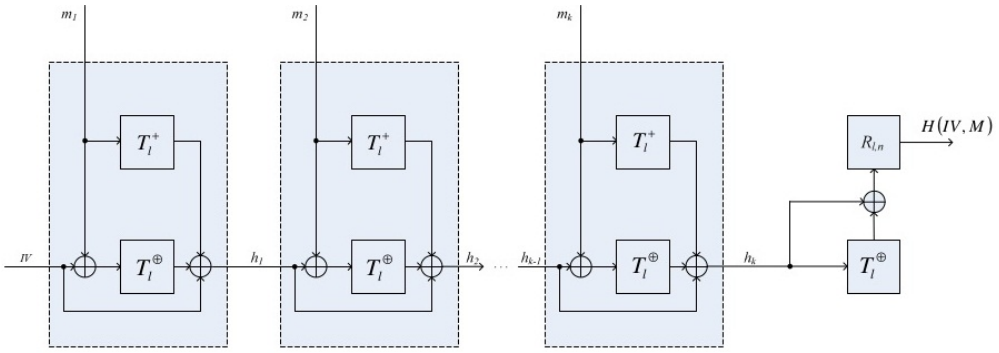

Figure 1: Kupyna hash function

2.2. The permutations T^{+} and $T^{\oplus}$

The structure of both permutations T^{+} and $T^{\oplus}$ is very similar to each other. In Kupyna-256 permutation updates on 8×8 state of 64 bytes in 10 rounds. In each round, the round transformations updates the state by means of the sequence of transformations viz., AddConstant, SubBytes, RotateBytes and MixBytes [4].

$$\textbf{MixBytes} \circ \textbf{RotateBytes} \circ \textbf{SubBytes} \circ \textbf{AddConstant}(\cdot)$$

The permutations are explained in detail with all transformations in the main document of standardization of Kupyna [11]. We describe it briefly in the following manner:

(i) **AddConstant:** In this transformation, the state is added by round constants. The column-wise modular addition mod 2^{64} is done with round constant which provides non-linearity in T^{+} but not in $T^{\oplus}$ (as round constant is xored column wise to the state). This is the only difference in both of the permutations. The round constants for $T^{\oplus}$ are defined as follows for round r, $1 \leq r \leq 10$ and column j, $1 \leq j \leq 8$:

$$\omega_j^{(r)} = ((j \ll 4) \oplus r, 00, 00, 00, 00, 00, 00, 00)^T$$

The round-independent constants for T^{+} for column j are given by:

$$\zeta_j^{(r)} = (F3, F0, F0, F0, F0, F0, F0, (7 - j) \ll 4)^T.$$

(ii) **SubBytes:** In this transformation, the S-boxes are applied to each byte of the state. S-box of 8×8 (defined in [12]) is used for decent cryptographic properties. The detailed specifications is given in the main document. This step is same for both of the permutations T^{+} and $T^{\oplus}$ but it is the only non-linear component in $T^{\oplus}$.

(iii) **RotateBytes:** In this transformation, the rows are cyclically shifted in such a manner that row j is shifted rightward by j byte positions, $0 \leqslant j \leqslant 8$. Even this transposition is same in both of the permutations.

(iv) **MixBytes:** In this transformation, the columns of the state get operated. A 8×8 circulant MDS matrix over $\mathbb{F}_{2^8}$. The coefficient of the matrix is chosen in such a way that the branch number of the MixBytes is 9 [12]. This step is also same for both the permutations T^{+} and $T^{\oplus}$.

Permutations T^{+} and $T^{\oplus}$ are almost same with all the transformations except AddConstants. This is the only difference in both of the permutations and this variation will make the structure different from AES and can resist all the direct attacks of AES like rebound and many more [4], [14], [9], [7].

3. Proposed Design: Kupy-Neev Hash

In this section, we present Kupy-Neev hash in which compression function is changed, keeping the mode of iteration of Kupyna intact. This is done because the majority of attacks applied on Kupyna [4, 14] are basically targeting the compression function, precisely its permutations i.e., T^{+} and $T^{\oplus}$. So, in the proposed design we have changed those permutations with N and N^{*}. In this design, message is divided into the blocks of 512-bit and it has two more inputs i.e., initial variable and a counter. It will be called KN-hash now onwards. Now, it will be discussed in detail.

3.1. Padding

The padding procedure for the KN-hash is same as in Kupyna hash function. It takes a message m of b bits as an input. Each message follows an unambiguous padding rule irrespective of its length. Append 1 to the end of the message and add d '0' bits in such a way that d will be the smallest non-negative solution of the congruence relation $b+1+d+96 \equiv 0 \mod 512$, i.e., $b+1+d \equiv 416 \mod 512$ where the last 96-bit represents b in bits. The maximum length of the message is restricted up to $2^{96} - 1$ bits.

3.2. Parsing

Once message m is padded, it will be divided into the 't' blocks of 512-bit length.

$$m' = m_1||m_2||\cdots||m_t,$$

where m' is the padded message.

3.3. Initial values

In the proposed scheme we need to have an initial value $\mathcal{IV}$ and a counter C_0 of lengths 512-bit each.

$$\mathcal{IV} = 1 \ll 510, \quad \text{i.e., } \mathcal{IV} = 2^{510}$$

The counter C_0 is

$$C_0 = \underbrace{000\cdots000}_{127-times}1.$$

3.4. Hash construction

An initial value $\mathcal{IV}$, a counter C_0 and the first message block m_1, each of 512-bit are taken as an input of compression function f and processed as following:

$$\begin{array}{rcll} h_0 & = & \mathcal{IV} & \\ C_i & = & C_{i-1} \boxplus_{512} 1 & \text{for } 1 \le i \le t \\ h_i & = & f(h_{i-1}, m_i, C_{i-1}) & \text{for } 1 \le i \le t \\ h & = & \Omega(h_t) & \\ \Omega(h_t) & = & trunc_{256}(N(h_t) \oplus h_t) & \end{array}$$

where N is the permutation in KN-hash and h is hash digest of 512-bit which will be used as a initial value for next message block. The hash construction of KN-hash is almost same as hash design of Kupyna. The only difference is the extra input, i.e., a counter C_0 which wasn't there initially.

3.5. Compression function

Suppose we have a message, m and we need to evaluate KN-hash of this message. Firstly, it is to be padded and divided into the 't' blocks of 512-bit, i.e., $m_1||m_2||\cdots||m_t$. Message block m_1 xored with $\mathcal{IV}$, fed to permutation N. Also, m_1 xored with C_0, fed to N^*. After applying permutations N and N^* on those 512-bit registers, the output of N and N^* are xored with initial value as in the case Davis-Meyer construction [3]. This 512-bit output will be used as chaining variable for next message block and the whole process will be repeated. It will happen till all the message blocks get exhausted. The final 512-bit output h_t, after all message blocks are used up, is xored with $N(h_t)$ to get an updated register of 512-bit. The most significant '256'-bit of this register is taken as a hash digest of the message m. The following figure represents the compression function for the proposed scheme.

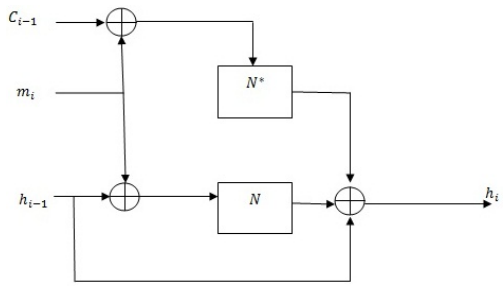

Figure 2: Kupy-Neev compression function

3.5.1. The permutations N and N^*

The N and N^* permutations act on 512-bit register. Both of these permutations are almost same but differs in Feistel type transformation. We will now describe the structure of N and N^*.

The permutation N is basically same as the permutation used in the *Neeva-hash* [1]. In N, firstly Present S-box (4×4) is applied on the register in parallel, then the register is divided in sixteen 32-bit words on which Feistel structure is applied on every 128-bit in parallel. In this Feistel type structure, the first, second and third word are updated by xoring with the fourth word and keeping the fourth word unchanged. This unbalanced Feistel structure is applied in parallel on four 128-bit strings of the register to provide word wise diffusion. After this a 16-bit left rotation is given and round constants are added under modulo 2^{32} to each round. The whole set of these transformations is named as u. It is applied 32 times for one application of N. Mathematically, we can write

$$\begin{aligned} u(x) &= (rotl_{16}(F(S(x)))) \boxplus_{2^{32}} RC_j \\ N(x) &= \underbrace{u \circ u \cdots \circ u}_{32-times}(x), \end{aligned}$$

where S is the Present S-box (4×4) acting in parallel, F is the Feistel structure defined above, $rotl_{16}$ represents the 16-bit left rotation and RC_j, $0 \leq j \leq 31$ are the 32 round constants written in the end of the paper. The permutation N is described in Figure 3.

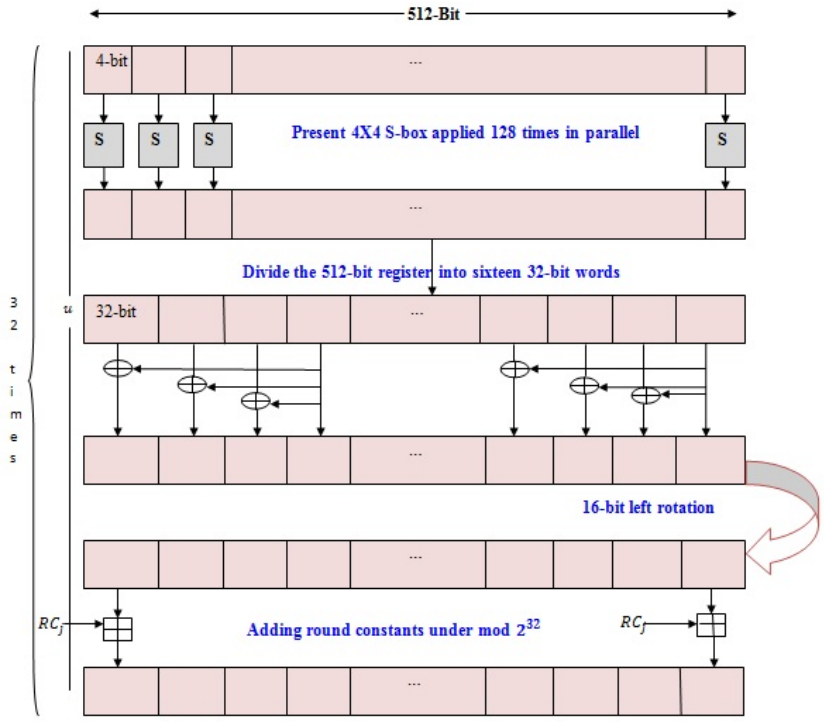

Figure 3: N Permutation

The permutation N^* is same as N except one step i.e., the Feistel type transformation is changed. Message block, m_1 is xored with C_0 and fed to N^*. After S-box application, the register is divided in sixteen 32-bit words and Feistel structure is applied on every 128-bit of 512-bit of register in parallel. In the four words structure, the second and fourth words are same and we just shift their places to first and third respectively whereas the first and third words are xored with second and fourth respectively then move to fourth and second place after a round. This balanced Feistel will act on the whole register four times in parallel. Again a 16-bit left rotation and modular addition are applied in every round till 32 rounds. The whole set of these transformation is named as v. N^* comprises of 32 times of v. Mathematically,

$$v(x) = (rotl_{16}(F'(S(x)))) \boxplus_{2^{32}} RC'_j$$

$$N^*(x) = \underbrace{v \circ v \cdots \circ v}_{32-times}(x)$$

where S is the Present S-box (4×4) acting in parallel, F' is the Feistal structure defined above, $rotl_{16}$ represents the 16-bit left rotation and RC'_j, $0 \leq j \leq 31$ are the 32 round constants written in the end of the paper. The permutation N^* is described in Figure 4.

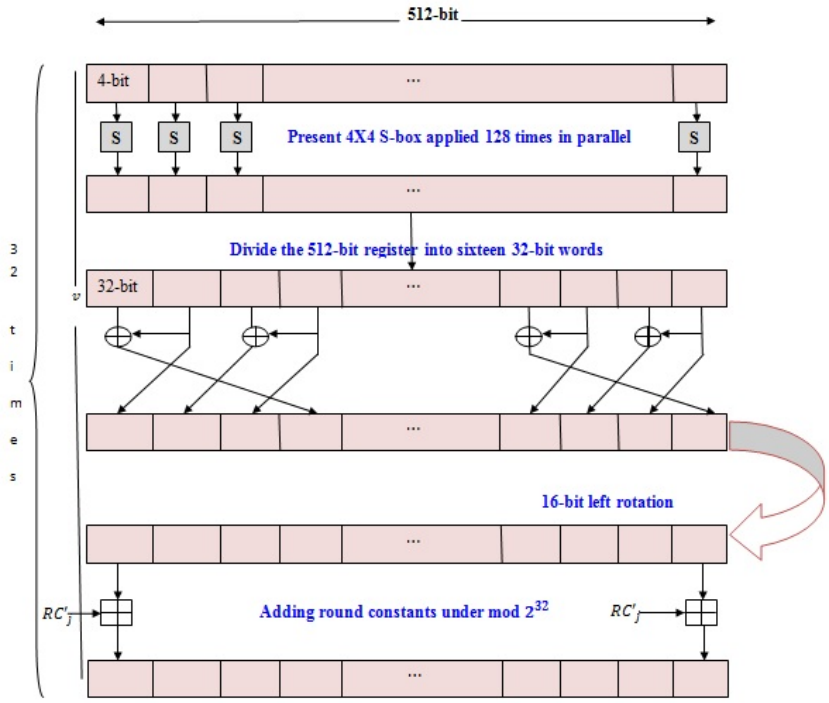

Figure 4: N^* Permutation

The step wise presentation of the proposed design is given in Algorithm 1.

input : $m_1, m_2, \ldots, m_t$
for $i = 1$ *to* t **do**
  $A_i \leftarrow h_{i-1} \oplus m_i$, $B_i \leftarrow C_{i-1} \oplus m_i$, where $h_0 = \mathcal{IV}$, C_0 *is counter*
  for $j = 0$ *to* 31 **do**
    $D_i^{(j)} \leftarrow S^{(j)}(A_i)$, $E_i^{(j)} \leftarrow S^{(j)}(B_i)$, $S^{(j)}$ represents application of Present S-box 128 times in parallel
    $D_i^{(j)} = d_0^{(j)} || d_1^{(j)} || \cdots || d_{15}^{(j)}$, $d_m^{(j)} \rightarrow$ 32-bit words, $0 \leq m \leq 15$
    $E_i^{(j)} = e_0^{(j)} || e_1^{(j)} || \cdots || e_{15}^{(j)}$, $e_m^{(j)} \rightarrow$ 32-bit words, $0 \leq m \leq 15$
    for $k = 0$ *to* 3 **do**
      for $\ell = 0$ *to* 2 **do**
        $d_{4k+\ell}^{(j)} \leftarrow d_{4k+\ell}^{(j)} \oplus d_{4k+3}^{(j)}$,
        $D_i^{(j)} \leftarrow d_0^{(j)} || d_1^{(j)} || \cdots || d_{15}^{(j)}$,
      end
    end
    for $k = 0$ *to* 3 **do**
      for $\ell = 0$ *to* 1 **do**
        $e_{4k+2\ell}^{(j)} \leftarrow e_{4k+2\ell+1}^{(j)}$,
        $e_{4k+2\ell+1}^{(j)} \leftarrow e_{4k-2\ell+2}^{(j)} \oplus e_{4k-2\ell+3}^{(j)}$
        $E_i^{(j)} \leftarrow e_0^{(j)} || e_0^{(j)} || \cdots || e_{15}^{(j)}$
      end
    end
    $D_i^{(j)} \leftarrow rotl_{16}(D_i^{(j)})$, $E_i^{(j)} \leftarrow rotl_{16}(E_i^{(j)})$
    $D_i^{(j)} \leftarrow D_i^{(j)} \boxplus_{2^{32}} RC_j$, $E_i^{(j)} \leftarrow E_i^{(j)} \boxplus_{2^{32}} RC'_j$
  end
  $h_i \leftarrow D_i^{(31)} \oplus E_i^{(31)} \oplus h_{i-1}$
end
return $A_t \leftarrow D_i^{(31)}$, $B_t \leftarrow E_t^{(31)}$ *and* $h_t \leftarrow A_t \oplus B_t \oplus h_{t-1}$
$h \leftarrow \Omega(h_t)$, $\Omega(h_t) = trunc_n(N(h_t) \oplus (h_t))$ where N is the permutation acting on A_i register.

Algorithm 1: Kupy-Neev hash function

3.5. Test value of KN-hash function

Test values of the three inputs are given below:

$$
\begin{aligned}
\textit{KN-hash}(a) &= f9bd54c7 \;\; 0220e770 \;\; ebabcc0f \;\; 87f93de4 \\
&\quad\;\; 2fc762fa \;\; 10e83eb2 \;\; bc3513ca \;\; 88a825fd \\
\textit{KN-hash}(ab) &= 967e0808 \;\; 383147f4 \;\; ac97bf2c \;\; 4f5f2d36 \\
&\quad\;\; a4fab471 \;\; 94aee45f \;\; 052e169e \;\; f5c8991a \\
\textit{KN-hash}(abc) &= 17097fab \;\; e50bf15f \;\; 53868e47 \;\; 609da983 \\
&\quad\;\; 466f24c9 \;\; 8eba82dd \;\; a3ceaf82 \;\; 0b6023a2
\end{aligned}
$$

4. Analysis of KN-hash

In this section, we show the analysis of the proposed scheme.

4.1. Efficiency

In this subsection, we discuss the efficiency of the proposed scheme. The following table provides the efficiency of KN-hash on an Intel core 2 duo 32-bit OS E8400 @ 3 Ghz processor with 1 GB RAM.

File Size (in MB)	**KN-hash** (in Sec)
1	2.758
5	13.636
10	27.677

4.2. Avanlanche effect

A 1024-bit input file M is taken and KN-hash of M, i.e., $h(M)$ is calculated. Now changing the ith bit of the message M, 1024 new files M_i is generated where $1 \leq i \leq 1024$.

Also, the corresponding KN-hash of each of the M_i, $h(M_i)$ is calculated for $1 \leq i \leq 1024$. The hamming distance of each of the M_i from the M is exactly 1 and now we compute the hamming distance of $h(M_i)$ from $h(M)$, i.e., the hamming weight d_i such that $d_i = wt(h(M) \oplus h(M_i))$. The hamming distances in the corresponding 32-bit words of the hash value $h(M_i)$ and $h(M)$ is calculated for $1 \leq i \leq 1024$.

The results have been shown in the following table with the maximum, the minimum, the mode and the average value of distances mentioned above.

Changes	W_1	W_2	W_3	W_4	W_5	W_6	W_7	W_8	*KN-Hash*
Max	26	26	25	24	26	24	25	24	155
Min	8	7	8	8	6	8	9	7	107
Mode	16	16	15	16	16	16	16	16	129
Mean	16.00	16.08	16.14	16.01	15.85	15.92	16.01	15.96	128.06

Table 1: Hamming Distances

To satisfy the strict avalanche effect, change in one input bit should change the final hash value by = 50%, i.e., each d_i should be 128 for $1 \leq i \leq 1024$. It has been noticed from the above results that d_i's are lying between 107 and 155 and the most frequent value is $d_i = 129$ for $1 \leq i \leq 1024$. The mean value is very near to 128 (i.e., 128.06).

The following graph and table show the distribution of the 1024 files with respect to their differences (distance) in bits.

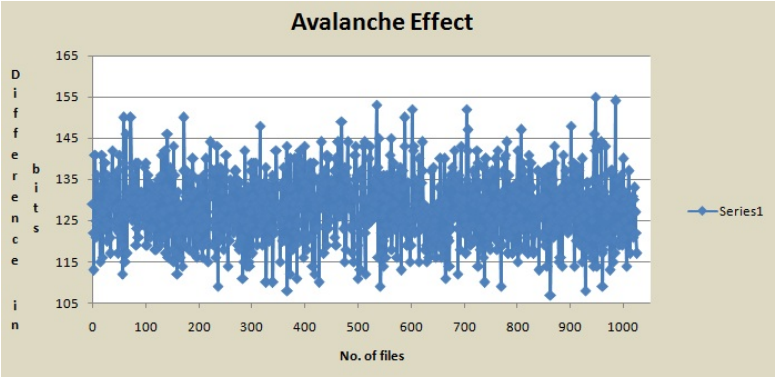

Figure 5: Hamming Distances range of the 1024 files

Range of Distance	No. of Files	% ***KN-hash***
128 ± 5	527	**51.46**
128 ± 10	831	**81.15**
128 ± 15	976	**95.31**
128 ± 20	1013	**98.92**

Table 2: Hamming Distances range of distances

4.3. Bit-variance test

The bit-variance test signifies the impact of change in input bits on the hash digest bits. Given a message, all the changes of the input message is taken up and then we evaluate the corresponding KN-hash digest of all the changes. Then, for each digest bit the probabilities of taking on the values of 1 and 0 are measured considering all the digests produced by applying input message bit changes. Bit-variance test measures the uniformity of each bit of the output. If the probability $P_i(1) = P_i(0) = 1/2$ for all digest bits $i = 1, \ldots, 256$ the KN-hash function has achieved maximum performance with respect to bit-variance test [8].

If we consider the 1025 files which we have used in testing the avalanche effect, viz. $M, M_1, M_2, \ldots, M_{1024}$ which we have generated for conducting avalanche effect, the following results can be found:

Digest length = 256
Number of digests = 1025
Mean frequency of 1s (expected) = 512.50
Mean frequency of 1s (calculated) = 512.08

The observed mean of 1025 files is found to be almost equal to expected mean. Hence, this hash function passes the bit-variance test. The following graph shows the probability of each of the bit (256-bit) to be '1' is approximately 0.50.

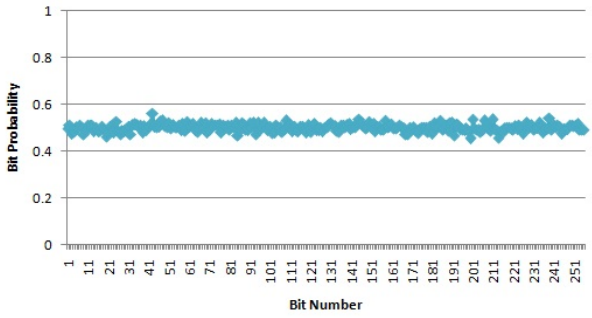

Figure 6: The probability of the bit position

4.4. Near-collision resistant

A hash function is called near-collision resistant if it is computationally hard to find two different inputs with hash outputs differ in small number of bits i.e., for this case if we have two different messages M and M', their KN-hash values shouldn't be almost same. Near-collision has been found in a hash function H means hamming weight of $(H(M) \oplus H(M'))$ (for $M \neq M'$) is relatively small (upto 16-bit).

To check with this, we have taken 100,000 files and evaluated xor of hamming distance of *KN-hash* of two random files from the lot. Two files are randomly chosen from 100,000 by $\binom{100000}{2}$ ways, i.e. 4,999,950,000 files. Analysis shows the minimum and maximum hamming weight of $H(M) \oplus H(M')$ is 79 and 179 respectively. The hamming weight 129 comes maximum number of times i.e. 249,065,436.

No. of files having the difference between and

$$(108 \leq \#files \leq 148) \ = 4,945,683,150 \ (i.e., \ 98.91\%)$$

It means almost 99% of files are between hamming distance 128±20 which is a good estimation with respect to the fact that they won't give a near-collision attack as for near-collision the hamming distance of two files needs to be really small viz. upto 16-bit. Hence analysis shows KN-hash function resists near-collision attack.

4.5. Differential characteristics

The differential attack [2] exploits input differences so the output can be controlled which can eventually help in finding a reduced round collision. It is a chosen plaintext attack where differences in input pairs are chosen to propagate a high probable differential trail. We try to find the expected number of input pairs that satisfy the differential trail which can be propagate to full fledged attack. Here, we try to find an upper-bound for the probability of such a differential trail.

In this proposed design Present S-box is used. Maximum differential probability for arbitrary input difference producing a output difference in a single S-box application is $\frac{4}{16} = 2^{-2}$ [13]. It means for only one active S-box in each round, differential attack still requires 2^{64} chosen plaintexts to distinguish the first 32-bit of KN-hash.

Now, we try to apply differential attack on KN-hash function. The compression function of KN-hash has two permutations. For computing differential characteristics, we replace modular additions in both the permutations by xor which is a weaker version of proposed scheme. Hence calculating the bound for this simplified hash would imply the minimum number of active S-boxes for KN-hash function. There are two independent permutations of the compression function of KN-hash. First we will calculate the number of active S-boxes in permutation N and then in N^* when one bit input difference causes the active S-box at the most significant byte of the word of updated register. In N, one bit input difference causes one active S-box at MSB in the first round and one more active S-box after second round. The total number of active S-box is 12 after 9 rounds. If we keep continuing in this manner till 32 rounds, the number of active S-box would reach about 50. In N^*, one bit input difference causes one active S-box at the end of first round and one more active S-box after second round. The total number of active S-box is 30 after 9 rounds. It has been noticed by computation, at the end of 32 rounds, the number of active S-boxes have come close to 280. After the 32 rounds, both the permutations are getting xored with chaining variable. The first 31 rounds of both the permutations are independent so their active S-boxes won't dependent on each other but the output of last round would get xored. Hence, the common active S-boxes will get cancelled in the final round. So, the minimum number of active S-boxes is $50 + 280 = 330$. Hence the maximal probability of finding a differential characteristic is $(2^{-2})^{330}$, i.e., 2^{-660}. That means, we require, 2^{660} chosen plaintexts to distinguish the most significant 32-bit of the output which is impossible to achieve. With this high complexity, it can be said that differential cryptanalysis is not applicable to simplified version of Kupy-Neev hash hence it will not applicable on KN-hash function.

Hence all the attacks which were applied earlier on Kupyna [4, 14] can not applied on KN-hash due to high complexity in finding a differential trail for rebound attack. We may say those attacks are not applicable on KN-hash function.

5. Conclusion

In this paper, we propose a modified version of Kupyna hash function, i.e., Kupy-Neev which provides a better security with respect to rebound attacks described in [4, 14](the complexity of getting a differential trail for rebound attack is very high making it impossible to achieve). It passes all the statistical tests viz. avalanche effect, bit-variance test and also ensures that differential attack can not be applied on it. It shows resistance to near-collision. It seems to be good substitute for Kupyna-256 and can be used for many of its applications.

Round Constants for permutation N

$$
\begin{aligned}
RC_1 &= \texttt{2caff5a02f51df53603c5024c7aa47e8743f5e5c007f58784e838c3f6443ded4}\\
&\quad \texttt{86af048638676d354c1fe6775b977a92fb55b40321ec466457493abaf5aa2c92}\\
RC_2 &= \texttt{1abd61f1e854cd0081f848475c483623534dd75f52f3d3c026ec386eba9e2c0c}\\
&\quad \texttt{00654ecd261459d63fbb55298f21cfe49a27871e67606f7ec6d2152d38b1a2c1}\\
RC_3 &= \texttt{53034f24df8df1aac35d55ff8efae984207dc07ede317bf1a665331895b7066c}\\
&\quad \texttt{03e9b11a541d70b79702652a6ccd97ecfcb161022e5884ad763e86fe0abaff8f}\\
RC_4 &= \texttt{d8b60e2e1e754b46b5af065da4d92e4a5ab3c7e86aa4a7517e5a0e197651994e}\\
&\quad \texttt{6d63ff57caeede4290a17a42c80964e79c71033eb5dfb7938b727e1dfc25dafa}\\
RC_5 &= \texttt{9b85887a7b663471090a3d0f4624e7bc7134163499968645a6af1d219122cdd7}\\
&\quad \texttt{e5b8b484f42fb301328ac3d258209238a40c3a284e10d7f3cb6cb7b6f654dffc}\\
RC_6 &= \texttt{06ad769f77cd63f5c84e327a21ce6c514748ae4b426d7195d6976396eff5b44b}\\
&\quad \texttt{4ef29290fe7ca48b4d0995e13a6ab6423127434a8976267179debbef850c2f36}\\
RC_7 &= \texttt{063ee0437d6c66c276e172a1cbd67a61a90b596c5d83952ec22e0092f3e4911c}\\
&\quad \texttt{725a1b9712a76fcf85fd54b4b718521e4b17203998ac4bfd91f0d7f4fb25c8ca}\\
RC_8 &= \texttt{c39a7ad78e81e9242ccb41d1b1574574da8dbfaaa330359177677fbe0468b636}\\
&\quad \texttt{5308191d09aeffbb2e1f891cb8461290e5501950e3b576c5fd8573de3a5a7874}\\
RC_9 &= \texttt{ca2892ac52c4584c8e3630280db645997954e76247f5680d1859535c3a84e8f2}\\
&\quad \texttt{8a24d90247189c067007b6795de89f1d1c37fc0fcd711da9e40b9a60335c9f90}\\
RC_{10} &= \texttt{49c6a9016e7853a0f40a494a01bc372cb49209d8dad27fde52349733c0e7b2c8}\\
&\quad \texttt{a82638cda99c3129f0115849feb8eaf7edbc2bef5035eef4d9f56f93b685474c}\\
RC_{11} &= \texttt{34c22f4dcdebbcd2cd9beb565c5a69b698ff63fd9c070077c30b5dc0951bd993}\\
&\quad \texttt{d35c537a4c09943704407476af25befb8555191a2a51d7ed10c9c3a380f5a325}\\
RC_{12} &= \texttt{c5fdc78c1ce337a04bccf1fb7df02469dbd2efd30b342ee205baa132e14ded71}\\
&\quad \texttt{9a12c4af77346d795af044b006ccf1d4f8ac3433d6334eb4c4dde7387974fc33}\\
RC_{13} &= \texttt{37e3e6892fa89038918f05740a94e8a1cc41335ea7d326b304e13310d5c08714}\\
&\quad \texttt{07e85f2eccb417119998c0efa433689b34f06d1f87ddb37f7ae4c3b789e6bee4}\\
RC_{14} &= \texttt{8da343159c772050af3fb3d5a0bcba355da09a454e3335bf16c985ab0f137e30}\\
&\quad \texttt{0a9dfd52b490473b0d58a1674f2b8255dbfdad86731891362726f02ebdf18a7e}\\
RC_{15} &= \texttt{d3cfbb1f842b45e797b5742154159f1306389ad4eac215cc617d8385e685d4de}\\
&\quad \texttt{0b69427d64880996d9d06e87cf6a24d8529b373be56b066b04362c4c7b90f8ba}\\
RC_{16} &= \texttt{ded56147b6eefe88b732a54a78964f81a4bc0febc447b7473df9b570aad7acd1}\\
&\quad \texttt{ae86a32afb47b5c79ff468dd9e1e6c5bd58aece5c87ac2ff11a39c583e2a06bd}\\
RC_{17} &= \texttt{4b5ec16d05b88072669af65097954a415a905096b3a114f77ac737987e9352c4}\\
&\quad \texttt{660f3a7f2e204ce20df75d30b35d21e312381a59c9d65debf7cc82300d8563f1}\\
RC_{18} &= \texttt{343718f2bf5baaaa76cdd610d946bebacbd95986c2985410413bc37ffbdd409d}\\
&\quad \texttt{aeb4c1406f620f25c48ca325db43205d4b38a75ab5c598ccd1027ab2fc5e27eb}\\
RC_{19} &= \texttt{114b44054fb96f4a8e664b24e92f2a2f1a1a6a5e81b3e63459d6527e763fb90e}\\
&\quad \texttt{27d85c423761b40778b9f19bb351b8a756b1d3a2eb304a08b4747b68d8b22416}\\
RC_{20} &= \texttt{5c831813899762d920a441f1a62525a9a545ba96e0d71666e518248349b63c85}\\
&\quad \texttt{008aeba8c112abfc93cd31c00f5835e9105f698b6033100a0591179568d8e3ce}\\
RC_{21} &= \texttt{f0bc90231e3443db2a9f54edf4365d8a0b2a25ddd5ccff0e0b8791f53656df51}\\
&\quad \texttt{a983eaa3b2e22fdb57060dc59eb02bdcc39946f5fb2f283043b80c9fc726d61c}\\
RC_{22} &= \texttt{ccdda6b43c4d3a163df49e8a451c6221146582965a146cd63e4940ffa98b3860}\\
&\quad \texttt{45a06da94ddf2b56c65684a5571078bf62050e229650c88c981c6f09b5a7556f}\\
RC_{23} &= \texttt{d79c0cb92356316b733ee306fb43afddbd5a8d2acd73e5ec611f2e2699611ca4}\\
&\quad \texttt{42b9d41bbf7f1fe683d7c9687e2105d3fec8a8c86c6b4fbe8a3b959bef19752d}\\
RC_{24} &= \texttt{22a0dd925bc3c47112cadf83b5517d41e4680f82e12cab8a0c0113e2d0bcc2c4}\\
&\quad \texttt{c533f5878b0fe6c4e94c2cc07a4edc96a0049dba50a93ee9b31f7b8555d54d56}\\
RC_{25} &= \texttt{f7cd8d5dc9bb363f32ba80f767981a49e03f15ce47ea7b1f81bd7c12f68ccf7b}\\
&\quad \texttt{1984586276fdfd57e48c56c1835e8c7e86df86fff2819b0724c3f3ffe0d7987c}\\
RC_{26} &= \texttt{8bd0f2eaab2cafab5764d382877195ac463a7b0a74876e886497168f74669164}\\
&\quad \texttt{8d24ab227d92b420458e47fd7c32e95b2a5298120e34b87d5be2dc977ee5b03e}\\
RC_{27} &= \texttt{8b28210e7343e8c090600f6477e1e78e0de05da55bf7475cff48f737b6a0bdaa}\\
&\quad \texttt{11db94422310a0357777311e10ba51110641da043640b0ee9b057a796dd81e92}\\
RC_{28} &= \texttt{550e1c62c3515afa955c7d8e5331b71980e2f651476436bf06c581c37384cfed}\\
&\quad \texttt{35a1c76e41628dbc02badeffadb5268c975df8dfcb169953931aca87828bedce}\\
RC_{29} &= \texttt{34979f57c626c8876a89f053067ff126ff0212db7b40adc8378b327f02dd079f}\\
&\quad \texttt{51e61a064a419166684975f76a3797c8d30b53afe1dca48be231c72ff4b4f02b}\\
RC_{30} &= \texttt{5ff130189f819a6ed9ec077a66564f97bcfef8f77d7ae0166f4f4c5c4fd64aa4}\\
&\quad \texttt{fb22707a028838e8984b04dd18547be14a8e65f9ef6a4ec020fdd3ecf704d419}\\
RC_{31} &= \texttt{295f8412ef720ab64196b275a7322cdd488de46b9e9b9f16f3bbdbaa30aecc4a}\\
&\quad \texttt{7081d36afde5c9844bc55105108600297a1f4cce42bb2f8a328841bc84f78df7}\\
RC_{32} &= \texttt{7fa1c85e639e2abda286fc8fd3e9301ee440c058c4bc20de4f67c21bfb4c8c4f}\\
&\quad \texttt{3206bf0a4637a18cd3e47cb8c7a57c008d849a966c31ce92d37ba84477325279}
\end{aligned}
$$

Round Constants for permutation N^*

RC'_1 = 5d046a2baf8903c2c0796fa29af2b7cafab0747519f905d8c68cd1bf489b12ac
d33d5537869d31f76cac00cc6cbba30aeb67f04e63d241a09b13a10ed1fd7e6a

RC'_2 = d25c6530098a276988f9819ab2cd1ac144db4a288945aff231b24b849f5baf5f
a17f3262275680a2f31b6be63663f1e6e33b8aff712bfdacffbb7e2901be9114

RC'_3 = a25749ffe4bf76edb812eac99d8d8bfaae147202115cebdb56e9d2299804ceaa
5e5d19b652c6eb6468ebf0df5ebe4e47d21143fd68a1f002d2f1fdd70f1a61fa

RC'_4 = 8fc6f48a1cf4c21c3f3f98143c8f59718d674a0ad7a93a62cf3ef3dcc7129579
9687589ea64c7bf47e91651a27f62593503dd5242ac33da26c3c410245d8e1c4

RC'_5 = 6d86d2f53b56674719f84bd16d3815be975ccea1020f62e11f2565866ad7af93
c8f2b6a49aca4c7c550827fb6b96521bc628c9ad18930d8f79f2e3a21abf5328

RC'_6 = d2c1875ad58639a87923a6213c753a18fc4d0d46a5f741015c9ed00539a9b145
62a59d725a307154fbce7af0dd0baf86da89846be1d445cc21fce5a08517a660

RC'_7 = 0221d0cc755151351edb0b5d51d3c1a07f8fc42d47ca4f95b57cae52a215acad
1cdc0258e4df1aede455da4212901ff9f18df3c613cf8dea37a48f2b4f666912

RC'_8 = 239d297416bb2ce3d812fb4aef59800b15df5ed99e05d3b6d3f07c74a07fba13
730364437c412b9dc52a0f4c1aef045dbe459fdb75857cda57cbdbd3113ddf0a

RC'_9 = a0b2865c9fd45d09b046c1b67b381066f46677c6a6f70822e889587f14b26731
99fb9c5c6ec4956324696ac9d3caad443ab50ac29a01618c5675ce5a6f51931d

RC'_{10} = db42e8fbf9565f052b02e02dd6ab60729fdfcc216324f538822d33b991be439e
fca0943bd1d9b69e6405c3fe9a7a979aa6a477c8eec930b7ca26d71bce8ceb12

RC'_{11} = fa0df8d26ba2e9d6e047f123640dac1ecc85ba10b6efa98edeb5f41982ea98f8
35a26d838997ce646bead97ddb40cc46261e513a9cbf7238dfb672c73e7d5872

RC'_{12} = 041f785a090566b19d90a4c574f6543ef28b088f996b5819929b25cb9c630b04
416ce2218fdbac3a2b9639313e7178bcb2a6957724d4eacae80ee6563081e018

RC'_{13} = efd9d5f241dca04aa090aca4392d35d4337d1c8b36eae3b3afb240832025e007
6811b601c5593d15cc6aa5a75d254256b4403e08c67e2c1346cde0544ceda29c

RC'_{14} = e043b75fa236ab026afdce5b864422d3c40d8f951d0e4936a39bbef9137fe0f7
8623e4f2b4f4c93080d30ddb26589d1a664603ede1fe262607c2168c0725cd83

RC'_{15} = cd749fe0f6b17d01ec8c68a870948b997f313f1e3b1f30131f04c665cd64abca
e6cf4f2f00a632747a7fa1fa0540c9b75db887173e3a2a38adde17cec958ab18

RC'_{16} = 97d1270874df951188ac3cc51ec172b891774f165150c6e86c147aa7e70195d7
318a467e09b8d359faa1d85d7bbe1d85a60b7f59d90d9c42654e37a7b63e5197

RC'_{17} = 5282219265d0fd08efddfdd6a1df2fdbddb3b10df401e99083aecd996c45c85c
8a85cf10c512dc0107d77bf9cba1492ed191d34b09221de22a86c3f234801f18

RC'_{18} = 67f9280d67a31a356001771d48278681c33238d980574689ab3874283da8b41c
74c17614365f9b9b0be5e52cdb3e5a9cf8f5e948074cd7059c6d197570c77966

RC'_{19} = 5dc4434aba58dbca52f0dea0bf1ee7df4b9c52400b955298c8892030c3d6bcb3
3eaa484994b3075dd4cc358bc68391208943ea7466959241a80759c772c2da5d

RC'_{20} = 7e64e8bfa0221b2bbb8cc97ab13933d116b11fc5ff5e37551e5292efe9b3541e
98b57d5d92a84ad6b8a541f51ce8674dd37f88a2a605ff1b7167e6b925e5beb4

RC'_{21} = a42fdb4232327996be8449fdebbc4a67d59ae460a5c46fc02edc6a75c5efc454
fd33720e6fc50e66c847ac1b7accdeb7d44e9e913732c7c1c1f9c4361455c566

RC'_{22} = c16c0f5e2162475cef9a47a14519a45bef52579e437f3a0eefdbd09eedc51f9b
3a9bcf5cbc592d387e8c9705320aa38611ab6475dbbc3b75d63274a4a3117246

RC'_{23} = eb3e1c4786e7aab3010d31c8293eeff0ca0fac0c804360d4bb2c8133a72f2706
2e7f6ba885cd03067bf9e2eb792ecf24cb641ffecf0c7231c128b026a67b1572

RC'_{24} = 58eef3b65d891e515986ae9eab7da8bd21d5d3274776d2fcb23e6e79a06f7e2e
0ed7bd93afe9e2777208820b8f2b5645fe4083524e48abad20a0d76bfdeb4b20

RC'_{25} = b29f4c02aa1f90d9c8c29f4521358641526b2ac83da18d8d0d628fc18ff49f7e
bbdac636c6f945cd2ad68c30580d77f526fb92300f07090578dec15aefc46d52

RC'_{26} = 81c87e3026252418a1b0cb1f1c0c17012fd21e85428855c1179f567b96fb4ca5
8c25dcab75ba157b2d26cc47cd8d5986e874f26503d632fdc29a74662993b266

RC'_{27} = 650422cab1172d382927dac42dc8f5adae3d51ee4b772519896c48bc5b3a1947
0074d912ff4ea04d570e1ab927b8b8ca0c07aebc48ee5fd68c091b34d439d711

RC'_{28} = 5771e887a1b39882d45a47c75c70087976064ca213de7291db73cc4b6ad3d205
0adab0a1369e7c230c3a95624794596656c3905aa5dc86b12b5e8b8953b951ab

RC'_{29} = 53d8a133925ce4e33a7f1d570e2e494482ae750c59617de406905a5d2ca8b99e
39b3af8eb08695f8f67e3bb39e1bd08aeae3dcfcf5b9d16982024bb50e917b52

RC'_{30} = 429e558e968d4b38fdf18ec099f1ed41c089c1c450798ff8d1125c48880648a3
86cd56c906620f6534e05b0528a97c1f005b8badf6406336f2734d1e52d2f2e7

RC'_{31} = 7eb11ffbf4d71d488b4e0d81dc7b03411c28ad0dfcb62db5ae4be8769c4a0542
fb175654eb6082c3b4836f41f192334429be0cf2ae81655ffb2ea6ee2f0d0a9e

RC'_{32} = 5ae98f187afdc72060f3e9e4997385ad1e3aa44a02420454d2430966dc3648bd
7237fc80b1ce14902b9aca152fdd8545fbaf40cde4aa7daf9e0f7034e4a9cb1d

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DYNAMIC BEHAVIOR OF TRAVELING WAVE SOLUTIONS FOR A CLASS FOR THE TIME-SPACE COUPLED FRACTIONAL kdV SYSTEM WITH TIME-DEPENDENT COEFFICIENTS

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Abstract. In this paper, a simplified bilinear method combined with a fractional transform has been used to obtain a new multiple soliton solutions for the Fractional coupled fractional kdV equations with variable coefficients. These systems appear in biology, engineering, mechanics, complex physics phenomena economics, signal image processing, notably control theory, groundwater problems and chemistry. Dispersion relations on the effects of the inhomogeneities of the model "due to the variable coefficients" are derived and interpreted for deterministic of the characteristic-line and velocity of each obtained soliton waves.

Keywords: simplified bilinear method; N-soliton solutions; modified Riemann-Liouville derivative; fractional coupled KdV systems.

1. Introduction

Fractional differential equations (FDEs) are the generalizations of classical differential equations with integer orders. The search of multiple soliton solutions to coupled systems of nonlinear fractional partial differential equations is of great importance, because these systems appear in biology, engineering, mechanics, complex physics phenomena economics, signal image processing, notably control theory, groundwater problems and chemistry. A variety of powerful and direct methods have been developed in this field. The analytical and approximate solutions of deterministic fractional partial differential equations attracted great attention and became a considerably interesting subject in mathematical physics. There are many methods for calculating the approximate solutions for nonlinear

fractional partial differential equations such as Adomian decomposition method [1, 2], the expansion-function method [3], the variational iterations method [4] and the homotopy perturbation method [5], [6]. The exact solutions for nonlinear FDEs are still of great interest. Li and He [7] introduced complex transform for reducing FDEs into ordinary differential equations (ODEs), so that all analytical methods for advanced calculus can be easily applied to fractional calculus [8], [9]. The coupled FKdV equations are very complicated and not easy to solve by direct integration method. To the best of my knowledge, all the numerical or analytical methods that deals with these systems had found only one soliton solutions, such as in [10].

Our aim in this paper is to find N- soliton solutions and multiple singular soliton solutions for the generalized space-time coupled fractional Korteweg–de Vries (CFKdV) equations with variable coefficients of the form

$$\begin{aligned} &D_t^{\alpha}u_{_1} + f(t)D_x^{3\beta}u_1 + g(t)u_1D_x^{\beta}u_1 + h(t)u_2D_x^{\beta}u_2 = 0,\\ &D_t^{\alpha}u_{_2} + f(t)D_x^{3\beta}u_2 + k(t)u_1D_x^{\beta}u_2 = 0, \end{aligned} \tag{1.1}$$

where D_t^{α} and D_x^{β} denote the modified Riemann-Liouville derivatives with respect to the time $t > 0$ and the horizontal coordinate space x, respectively, of the the amplitude of the relevant wave models $u_1(x,t)$ and $u_2(x,t)$, and $f(t), g(t), h(t)$ and $k(t)$ are time-dependent functions, and $0 < \alpha, \beta \leq 1$ are parameters describing the order of the fractional derivatives of u_1 and u_2. The corresponding constraint conditions in this paper for (1.1), will be $f(t) = c_1k(t)$, and $k(t) - g(t) = c_2h(t)$, where c_1 and c_2 are arbitrary constants. To our knowledge, there have been no discussions on (1.1) under these conditions. We will apply a new simplified bilinear method with fractional transforms that transform the fractional partial differential equation into a classical one, then we investigate the N-soliton solutions for equataion (1.1). In this work, a new form of the simplified bilinear method is adopted. It is well-known that the Hirota bilinear form provides an efficient tool to solve nonlinear differential equations of mathematical physics [11]–[14], [15], [16].

This paper is organized as follows: some necessary definition and notation of the modified Riemann–Liouville derivative are given. In Section 2, a new simplified bilinear method based on a transformation method combined with the Hirota's bilinear sense is introduced. The N-soliton solutions and multiple singular soliton solutions for equation (1.1) are constructed in Sections 3 and 4. Section 5 is devoted for discussion and concluding remarks regards the effects of the variable coefficients and the collision behavior and propagation properties.

2. Preliminaries

There are many definitions of fractional derivative; for more details see [17]. In this paper, we will consider the modified Riemann–Liouville derivative defined by Jumarie [18], [19].

Definition 2.1 A real function $f(x)$, $x > 0$, is said to be in the space C_α; $\alpha \in \mathbb{R}$; if there exist a real number $p > \alpha$ such that $f(x) = x^p f_1(x)$, where $f_1(x)$ is continuous in $[0, \infty)$, and is said to be in the space C_α^m, $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, if $f^{(m)} \in C_\alpha$.

Definition 2.2 Assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuous (but not necessarily differentiable) function, the Jumarie's modified Riemann-Liouville fractional derivative of order α is defined by

$$D_x^\alpha f(x) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)} \displaystyle\int_0^x \frac{f(\tau) - f(0)}{(x-\tau)^{\alpha+1}} d\tau, & \alpha < 0 \\ \dfrac{1}{\Gamma(1-\alpha)} \dfrac{d}{dx} \displaystyle\int_0^x \frac{f(\tau) - f(0)}{(x-\tau)^{\alpha}} d\tau, & 0 < \alpha < 1 \\ \dfrac{d^n}{dx^n} \left(f^{(\alpha-n)}(x) \right), & n \le \alpha \le n+1, n \ge 1. \end{cases} \tag{2.1}$$

Some properties of Jumarie's modified Riemann-Liouville fractional derivative (2.1) are listed below [18], [19].

For $f, g \in C_\alpha, \alpha \in \mathbb{R}$,

$$D_x^0 f(x) = f(x).$$

i.e. $D_x^0 = I$ is the identity operator.

$$D_x^\alpha \left(k_1 f(x) + k_2 g(x) \right) = k_1 D_x^\alpha f(x) + k_2 D_x^\alpha g(x), \qquad k_1, k_2 \in \mathbb{C}.$$

In Jumarie's sense, the fractional derivative of a constant function is zero, which means that it has merits over the original one: For $c \in \mathbb{R}, \alpha \ge 0$,

$$D_x^\alpha c = 0.$$

If $f(x) = x^r$, then

$$D_x^\alpha f(x) = D_x^\alpha \left(x^r \right) = \frac{\Gamma(r+1)}{\Gamma(r-\alpha+1)} x^{r-\alpha}.$$

Jumarie's fractional derivative of the product is as the same as the usual rule

$$D_x^\alpha \left(f(x) g(x) \right) = f(x) D_x^\alpha \left(g(x) \right) + g(x) D_x^\alpha \left(f(x) \right).$$

Finally,

$$D_x^\alpha f\left(g(x) \right) = \frac{df}{dg} D_x^\alpha g(x) = \left(\frac{dg}{dx} \right)^\alpha D_g^\alpha f\left(g(x) \right).$$

3. Multiple soliton solutions for the fractional coupled kdV system

In order to derive the N-soliton solutions of any completely integrable equation, we usually use the Hirota's direct method, which depends on a transformation for considered equation to a bilinear form. These bilinear forms are usually used to

derive the auxiliary function. It is worth to mention that it is not easy to find the bilinear form for many equations, it sometimes requires to introduce a new dependent variables, or even independent variables. To avoid using the bilinear forms, Hereman et al. [20], [21], formally introduced the simplified algorithm to derive the auxiliary functions by assuming that the N-soliton solutions can be expressed as polynomials of exponential functions. The Cole-Hopf transformation method combined with the simplified Hirota's sense is a powerful method to determine multiple soliton solutions and multiple singular soliton solutions for integrable systems [22]–[30].

In this section, we apply simplified bilinear method, to construct soliton solutions of space-time coupled fractional Korteweg–de Vries (1.1):

Consider the following fractional transforms

$$\begin{aligned} T &= \frac{p_1 t^{\alpha}}{\Gamma(1+\alpha)}, \\ X &= \frac{p_2 x^{\beta}}{\Gamma(1+\beta)}, \end{aligned} \tag{3.1}$$

where p_1 and p_2 are constants. Using (3.1), we can convert fractional derivatives into the following classical derivatives:

$$\begin{aligned} \frac{\partial^{\alpha} u_j}{\partial t^{\alpha}} &= p_1 \frac{\partial u}{\partial T}, \\ \frac{\partial^{\beta} u_j}{\partial x^{\beta}} &= p_2 \frac{\partial u}{\partial X}, \ j = 1, 2, \end{aligned}$$

see [23]. Therefore, (1.1) becomes

$$\begin{aligned} &\frac{\partial u_{_1}}{\partial T} + F(T)\frac{\partial^3 u_{_1}}{\partial X^3} + G(T)u_1\frac{\partial u_{_1}}{\partial X} + H(T)u_2\frac{\partial u_{_2}}{\partial X} = 0, \\ &\frac{\partial u_{_2}}{\partial T} + F(T)\frac{\partial^3 u_{_2}}{\partial X^3} + K(T)u_1\frac{\partial u_{_2}}{\partial X} = 0, \end{aligned} \tag{3.2}$$

If we substitute

$$u_j(X,T) = e^{\theta_i(X,T)}, \ j = 1, 2, \quad \text{with} \ \ \theta_i(X,T) = h_i X - \omega_i(T), \tag{3.3}$$

into the linear terms of equation (3.2), we get the dispersion relation as follows

$$\omega_i(T) = \int h_i^3 F(T) dT. \tag{3.4}$$

Thus, we obtain

$$\theta_i(X,T) = h_i X - \int h_i^3 F(T) dT. \tag{3.5}$$

Assuming the single soliton solutions of (3.2) as

$$u_j(X,T) = R_j(\ln a(X,T))_{XX}, \tag{3.6}$$

where $a(X,T)$ is the auxiliary function given by

$$a(X,T) = 1 + e^{\theta_1(X,T)} = 1 + e^{h_1 X - \int h_1^3 F(T)dT}. \tag{3.7}$$

Substituting (3.6) and (3.7) into (3.2), then solving for R_1, R_2 and R_3, we find two sets of solutions as

$$R_1 = \frac{12F(T)}{K(T)}, \quad R_2 = \frac{\pm 12F(T)}{K(T)}\sqrt{\frac{K(T)-G(T)}{H(T)}}. \tag{3.8}$$

To obtain a numerical value of R_j, we set the constraints $F(T) = c_1 K(T)$, and $K(T) - G(T) = c_2 H(T)$, where c_1 and c_2 are arbitrary constants

Now, substituting (3.8) into (3.6), we get the following sets of single soliton solutions for (3.2)

$$\begin{aligned} u_1(X,T) &= \frac{12F(T)}{K(T)}h_1^2\frac{e^{\theta_1(X,T)}}{(1+e^{\theta_1(X,T)})^2} = \frac{3F(T)}{K(T)}h_1^2 \sec h^2\left(\frac{\theta_1(X,T)}{2}\right), \\ u_2(X,T) &= \frac{\pm 12F(T)}{K(T)}\sqrt{\frac{K(T)-G(T)}{H(T)}}h_1^2\frac{e^{\theta_1(X,T)}}{(1+e^{\theta_1(X,T)})^2} \\ &= \frac{\pm 3(F(T))}{K(T)}\sqrt{\frac{K(T)-G(T)}{H(T)}}h_1^2 \sec h^2\left(\frac{\theta_1(X,T)}{2}\right), \end{aligned} \tag{3.9}$$

where

$$\theta_1(X,T) = h_1 X - \int h_1^3 F(T)dT.$$

Using the fractional transforms in (3.1), we obtain the following sets of single soliton solutions $u_j(x,t)$ for (1.1)

$$\begin{aligned} u_1(x,t) &= \frac{12f(t)}{k(t)}h_1^2\frac{e^{\frac{h_1 x^\beta}{\Gamma(1+\beta)} - \frac{\alpha h_1^3 \int f(t)t^{\alpha-1}dt}{\Gamma(1+\alpha)}}}{\left(1+e^{\frac{h_1 x^\beta}{\Gamma(1+\beta)} - \frac{\alpha h_1^3 \int f(t)t^{\alpha-1}dt}{\Gamma(1+\alpha)}}\right)^2} \\ &= \frac{3f(t)}{k(t)}h_1^2 \sec h^2\left(\frac{\theta_1(x,t)}{2}\right), \end{aligned} \tag{3.10}$$

$$\begin{aligned} u_2(x,t) &= \frac{\pm 12f(t)}{k(t)}\sqrt{\frac{k(t)-g(t)}{h(t)}}h_1^2\frac{e^{\frac{h_1 x^\beta}{\Gamma(1+\beta)} - \frac{\alpha h_1^3 \int f(t)t^{\alpha-1}dt}{\Gamma(1+\alpha)}}}{\left(1+e^{\frac{h_1 x^\beta}{\Gamma(1+\beta)} - \frac{\alpha h_1^3 \int f(t)t^{\alpha-1}dt}{\Gamma(1+\alpha)}}\right)^2} \\ &= \frac{\pm 12f(t)}{k(t)}\sqrt{\frac{k(t)-g(t)}{h(t)}}h_1^2 \sec h\left(\frac{\theta_1(x,t)}{2}\right), \end{aligned} \tag{3.11}$$

where

$$\theta_1(x,t) = \frac{h_1 x^\beta}{\Gamma(1+\beta)} - \frac{\alpha h_1^3 \int f(t) t^{\alpha-1} dt}{\Gamma(1+\alpha)}, \tag{3.12}$$

with the constraints $f(t) = c_1 k(t)$, and $k(t) - g(t) = c_2 h(t)$, where c_1 and c_2 are arbitrary constants.

To obtain the two-soliton solutions, we let

$$a(X,T) = 1 + e^{\theta_1(X,T)} + e^{\theta_2(X,T)} + b_{12} e^{\theta_1(X,T)+\theta_2(X,T)}\Big\}, \tag{3.13}$$

where θ_1 and θ_2 are defined in (3.5). Using (3.13) in (3.6) and substituting the results in (3.2), we obtained the phase shift as follows

$$b_{12} = \frac{(h_1 - h_2)^2}{(h_1 + h_2)^2}. \tag{3.14}$$

Substituting (3.14), (3.13) and (3.8) into (3.6), we obtain sets of two solitons solutions for (3.2), then by using (3.1) and

$$\theta_i(x,t) = \frac{h_i x^\beta}{\Gamma(1+\beta)} - \frac{\alpha h_i^3 \int f(t) t^{\alpha-1} dt}{\Gamma(1+\alpha)}, \quad i = 1, 2,$$

we can find two solitons solutions for (1.1).

This shows that (1.1) does not show any resonant phenomenon [15] because the phase shift term b_{12} in (3.14) cannot be 0 or ∞ for $|h_1| \neq |h_2|$. It is well known that a two-soliton solution can degenerate into a resonant trial under the conditions

$$b_{12} = 0 \quad \text{or} \quad (b_{12})^{-1} = 0, \quad \text{for} \quad |h_1| \neq |h_1|.$$

The three-soliton solution is determined by

$$\begin{aligned} a(X,T) = 1 &+ e^{\theta_1(X,T)} + e^{\theta_2(X,T)} + e^{\theta_3(X,T)} + b_{12} e^{\theta_1(X,T)+\theta_2(X,T)} \\ &+ b_{13} e^{\theta_1(X,T)+\theta_3(X,T)} + b_{23} e^{\theta_2(X,T)+\theta_3(X,T)} \\ &+ b_{123} e^{\theta_1(X,T)+\theta_2(X,T)+\theta_3(X,T)}, \end{aligned} \tag{3.15}$$

where

$$b_{ij} = \frac{(h_i - h_j)^2}{(h_i + h_j)^2}, \qquad 1 \leq i < j \leq 3. \tag{3.16}$$

Proceeding as before we find

$$b_{123} = b_{12} b_{13} b_{23}. \tag{3.17}$$

Since (3.17) holds, we can find three-soliton solutions for transformed coupled mkdv system (3.2), and with the fractional transformation (3.1) and using

$$\theta_i(x,t) = \frac{h_i x^\beta}{\Gamma(1+\beta)} - \frac{\alpha h_i^3 \int f(t) t^{\alpha-1} dt}{\Gamma(1+\alpha)}, i = 1, 2, 3,$$

we can obtain three solitons solutions for (1.1), which implies that N-soliton solutions exist for any order N [11], [12].

4. Multiple singular soliton solutions for the fractional coupled kdV system

In order to obtain a single singular soliton solution, we substitute

$$u_j(X,T) = e^{h_i X - \omega_i(T)}, \tag{4.1}$$

into the linear part of (3.2), concluding the dispersion relation (3.4), then, as a result we get

$$w_i(t) = \int h_i^3 F(T) dT. \tag{4.2}$$

The singular single soliton solution of (3.2) is assumed to be

$$u_j(X,T) = R_j \left(\ln(a(X,T))\right)_{XX}, \quad j = 1,2, \tag{4.3}$$

where $a(X,T)$ is given by

$$a(X,T) = 1 - e^{\theta_1(X,T)} = 1 - e^{h_1 X - \int h_1^3 F(T) dT}. \tag{4.4}$$

When we substitute (4.4) into (3.2), and solving for R_j, we found

$$R_1 = \frac{12F(T)}{K(T)}, R_2 = \frac{\pm 12F(T)}{K(T)}\sqrt{\frac{K(T)-G(T)}{H(T)}}. \tag{4.5}$$

Putting these values of R_j, we get the following sets of singular soliton solutions for (3.2)

$$u_1(X,T) = \frac{12F(T)}{K(T)} h_1^2 \frac{e^{\theta_1(X,T)}}{\left(1-e^{\theta_1(X,T)}\right)^2} = \frac{3F(T)}{K(T)} h_1^2 \csc h^2\left(\frac{\theta_1(X,T)}{2}\right), \tag{4.6}$$

$$\begin{aligned} u_2(X,T) &= \frac{\pm 12F(T)}{K(T)}\sqrt{\frac{K(T)-G(T)}{H(T)}} h_1^2 \frac{e^{\theta_1(X,T)}}{\left(1-e^{\theta_1(X,T)}\right)^2} \\ &= \frac{\pm 3(F(T))}{K(T)}\sqrt{\frac{K(T)-G(T)}{H(T)}} h_1^2 \csc h^2\left(\frac{\theta_1(X,T)}{2}\right), \end{aligned} \tag{4.7}$$

where

$$\theta_1(X,T) = h_1 X - \int h_1^3 F(T) dT.$$

Using the fractional transforms in (3.1), we obtain the following sets of singular soliton solutions $u_j(x,t)$ for (1.1)

$$\begin{aligned} u_1(x,t) &= \frac{12f(t)}{k(t)} h_1^2 \frac{e^{\frac{h_1 x^\beta}{\Gamma(1+\beta)} - \frac{\alpha h_1^3 \int f(t) t^{\alpha-1} dt}{\Gamma(1+\alpha)}}}{\left(1 - e^{\frac{h_1 x^\beta}{\Gamma(1+\beta)} - \frac{\alpha h_1^3 \int f(t) t^{\alpha-1} dt}{\Gamma(1+\alpha)}}\right)^2} \\ &= \frac{3f(t)}{k(t)} h_1^2 \csc h^2\left(\frac{\theta_1(x,t)}{2}\right), \end{aligned} \tag{4.8}$$

$$u_2(x,t) = \frac{\pm 12 f(t)}{k(t)}\sqrt{\frac{k(t)-g(t)}{h(t)}}h_1^2 \frac{e^{\frac{h_1 x^\beta}{\Gamma(1+\beta)}-\frac{\alpha h_1^3 \int f(t)t^{\alpha-1}dt}{\Gamma(1+\alpha)}}}{\left(1-e^{\frac{h_1 x^\beta}{\Gamma(1+\beta)}-\frac{\alpha h_1^3 \int f(t)t^{\alpha-1}dt}{\Gamma(1+\alpha)}}\right)^2} \tag{4.9}$$

$$= \frac{\pm 12 f(t)}{k(t)}\sqrt{\frac{k(t)-g(t)}{h(t)}}h_1^2 \operatorname{csc} h(\frac{\theta_1(x,t)}{2}),$$

where

$$\theta_1(x,t) = \frac{h_1 x^\beta}{\Gamma(1+\beta)} - \frac{\alpha h_1^3 \int f(t)t^{\alpha-1}dt}{\Gamma(1+\alpha)}. \tag{4.10}$$

Similarly we can find the constraints $f(t) = c_1 k(t)$, and $k(t) - g(t) = c_2 h(t)$, to obtain a numerical value of R_j where c_1 and c_2 are arbitrary constants.

The two singular soliton solutions are obtained by setting

$$a(X,T) = 1 - e^{\theta_1(X,T)} - e^{\theta_2(X,T)} + b_{12}e^{\theta_1(X,T)+\theta_2(X,T)}, \tag{4.11}$$

where $\theta_1(X,T)$ and $\theta_2(X,T)$ are defined in (3.5). Now, by substituting (4.11) into (4.3), then in (3.2), to obtain the phase shift b_{12} as in (3.16). In general,

$$b_{ij} = \frac{(h_i - h_j)^2}{(h_i + h_j)^2}, \quad 1 \le i < j \le 3. \tag{4.12}$$

Substituting (4.12), (4.11) and (4.5) into (4.3), a sets of two solitons solutions for (3.2) are obtained and then, by using (3.1), we can find two solitons solutions for (1.1).

For the singular three-soliton solutions we use

$$\begin{aligned} a(X,T) = {} & 1 - e^{\theta_1(X,T)} - e^{\theta_2(X,T)} - e^{\theta_3(X,T)} + b_{12}e^{\theta_1(X,T)+\theta_2(X,T)} \\ & + b_{13}e^{\theta_1(X,T)+\theta_3(X,T)} + b_{23}e^{\theta_2(X,T)+\theta_3(X,T)} \\ & - b_{123}e^{\theta_1(X,T)+\theta_2(X,T)+\theta_3(X,T)}, \end{aligned} \tag{4.13}$$

Proceeding as before, we can obtain the singular three-soliton solutions for (3.2), then by using (3.1) we can find three singular solitons solutions for (1.1).

5. Discussion and concluding remarks

In this section we will discuss the effects of the inhomogeneities, namely, variable coefficients. We will see that the dispersion relation plays a crucial role in the analysis, since it can be used to obtain the characteristic line and velocity v for each soliton.

We obtain sets of single soliton solutions (3.10) and (3.11) with distinct soliton amplitudes *amp* for $u_1(x,t)$ and $u_2(x,t)$ which can be expressed as

$$amp\ of\ u_j(x,t) = \begin{cases} \left|\dfrac{3f(t)}{k(t)}h_1^2\right|, & j=1, \\ \left|\dfrac{3f(t)}{k(t)}\sqrt{\dfrac{k(t)-g(t)}{h(t)}}h_1^2\right|, & j=2. \end{cases}$$

With the characteristic-line method [31], [32], the characteristic wedge for each solitary wave for $u_j(x,t)$ defined by

$$x^\beta = \frac{\alpha\Gamma(1+\beta)}{\Gamma(1+\alpha)}\int h_i^2 f(t)t^{\alpha-1}dt.$$

The velocity v of each solitary wave for $u_j(x,t),\ j=1,2,$

$$v_i = \frac{1}{\beta}\left(\frac{\alpha\Gamma(1+\beta)}{\Gamma(1+\alpha)}\int h_i^2 f(t)t^{\alpha-1}dt\right)^{\frac{1}{\beta}-1}\frac{\alpha\Gamma(1+\beta)h_i^2 f(t)t^{\alpha-1}}{\Gamma(1+\alpha)}.$$

The analysis is illustrated by studying couplings of the kdV equations with time-fractional derivatives (FkdV)

$$\begin{aligned} &D_t^\alpha u_1 + f(t)\frac{\partial^3 u_1}{\partial x^3} + g(t)u_1\frac{\partial u_1}{\partial x} + h(t)u_2\frac{\partial u_2}{\partial x} = 0, \\ &D_t^\alpha u_2 + f(t)\frac{\partial^3 u_2}{\partial x^3} + k(t)u_1\frac{\partial u_2}{\partial x} = 0, \end{aligned} \tag{5.1}$$

From (3.10) and (3.11) we obtain the singular soliton solution for (5.1) where

$$\theta_i = h_i x - \frac{\alpha}{\Gamma(1+\alpha)}\int h_i^3 f(t)t^{\alpha-1}dt. \tag{5.2}$$

The characteristic face can be derived from relation (5.2) for each solitary wave by

$$x = \frac{\alpha}{\Gamma(1+\alpha)}\int h_i^2 f(t)t^{\alpha-1}dt. \tag{5.3}$$

Then the velocity of the wave is

$$v_x = \frac{\alpha}{\Gamma(1+\alpha)}h_i^2 t^{\alpha-1}f(t). \tag{5.4}$$

The soliton amplitude of $u_1(x,t)$ is dependent of the variable coefficients $f(t)$ and $k(t)$ and independent of the variable coefficients $g(t)$ and $h(t)$, but the soliton amplitude of $u_2(x,t)$ is dependent of the variable coefficients $f(t)$, $g(t), h(t)$ and $k(t)$. The propagation velocity of the solitary wave is dependent just of the coefficient functions $f(t)$.

Moreover, we see that from (5.4) the soliton travels along the positive x-axis direction when the following inequality holds

$$\frac{\alpha}{\Gamma(1+\alpha)} h_i^2 t^{\alpha-1} f(t) > 0. \tag{5.5}$$

Fig. 1 shows that the solitonic amplitude of $u_1(x,t)$ increases with an increase in the ratio $f(t)/k(t)$.

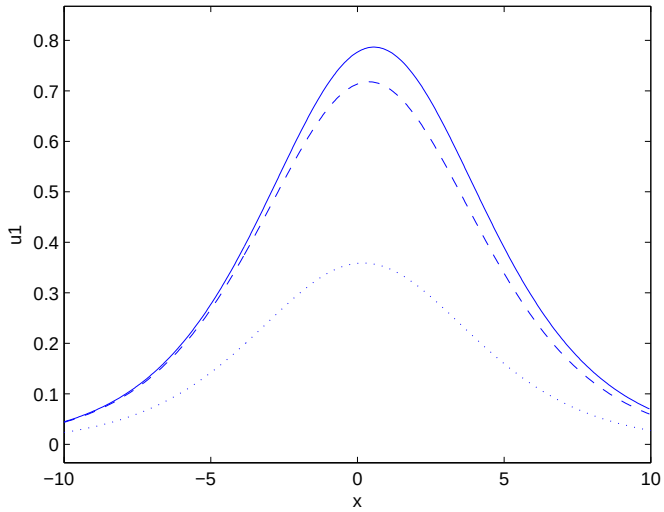

Figure 1: Evolution plots of the one-soliton solution given by expression (3.10) with $h_1 = 0.4$, $t = 2$, $\alpha = 0.8$, $f(t) = \lambda t^{2.4}$, $g(t) = t^{2.4}$, $\lambda = 1.1$ (solid line), $\lambda = 0.8$ (dashed line line), $\lambda = 0.4$ (dot line).

However, expression (5.4) indicates that the propagation velocity of the solitary wave is influenced by the coefficient functions $f(t)$, while it is independent of the coefficient function $g(t), g(t)$ and $k(t)$.

In Fig. 2, we choose $\alpha = 0.7, h_1 = 0.6, h_2 = 1, f(t) = 2t^{2.3}$ and $k(t) = t^{2.3}$, then the characteristic curve (5.3) given by

$$x - \frac{7h_i^2}{30\Gamma(1.7)} t^3 + \eta = 0.$$

Then the soliton reveals the cubical type propagation trajectory with the unalterable amplitude but continuously changeable velocity.

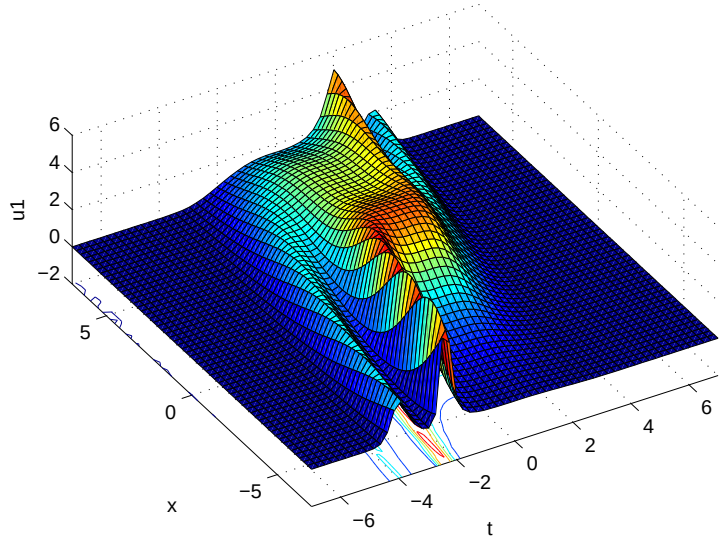

Figure 2: The profile figure for two soliton solution of $u_1(x,t)$ with $\alpha = 0.7$, $f(t) = 2t^{2.3}$, $k(t) = t^{2.3}$, $h_1 = 0.6$, $h_2 = 1$.

In Fig. 3, we choose $\alpha = 0.9, h_1 = 0,9, h_2 = 1, f(t) = t^{0.1}\sin 8t + t^{1.1}$ and $k(t) = 2t^{0.1}\sin 8t + 2t^{1.1}$.

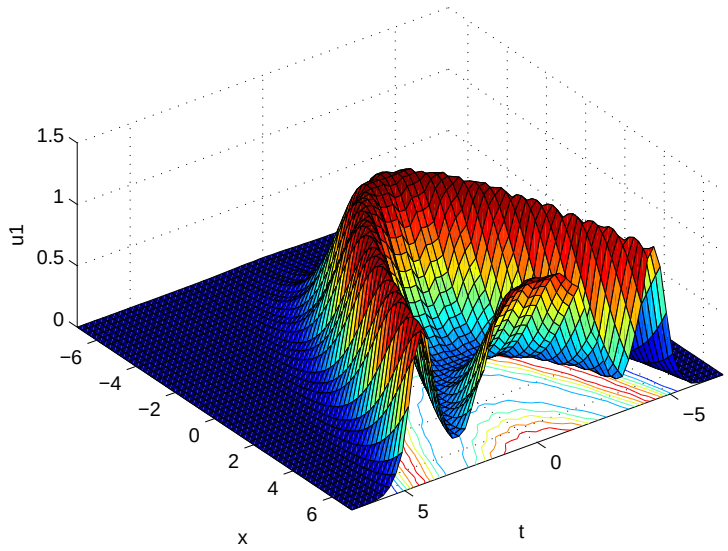

Figure 3: The profile figure for two soliton solution of $u_1(x,t)$ with $\alpha = 0.9$, $f(t) = t^{0.1}\sin 8t + t^{1.1}$, $k(t) = 2t^{0.1}\sin 8t + 2t^{1.1}$, $h_1 = 0.9$, $h_2 = 1$.

Then, in the characteristic curve (5.3), given by

$$x + \frac{9h_i^2}{10\Gamma(1.9)}\left[\frac{\cos 8t}{8} - \frac{t^2}{2}\right] + \eta = 0,$$

we note that when t approaches zero, i.e., $\cos(8t) \gg t^2$, the trajectory is snake-like type with periodic oscillation. Otherwise, when t is far from the origin, the trajectory is parabolic-like type one. From which we conclude that, besides the periodic oscillation of the solitons in the local region, the large scale propagation trajectories for such a structure are the parabolic-typed curves. Likewise, if the variable coefficients are taken as the other forms, the corresponding characteristic curves will present different characters.

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THE SOLUTIONS OF PAULI EQUATION IN DE SITTER SPACE BACKGROUND AND HOMOGENEOUS MANIFOLD $SU(2)/U(1)$

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Abstract. In this paper, we consider a particle with spin $\frac{1}{2}$ in de Sitter space-time. Procedure for transition to the Pauli approximation is conducted in the equation in the variable (t, r), obtained after separating the angular dependence of (θ, φ) from the wave function. We make the suitable second order equation corresponding to de Sitter space time for particle spine $\frac{1}{2}$ we then compare this equation to Jacobi polynomial and obtain the wave function and eigenvalues (energy spectrum) which is important for the corresponding system.
Also, by taking the advantage from weight and main function in Jacobi polynomial and obtain the corresponding algebra.

Keywords: factorization method; Dirac equation; Jacobi polynomial; raising and lowering operators.

1. Introduction

De Sitter and Anti de Sitter models have taken attraction in the context of development of the quantum theory in curved space-time for a long time. The most attention about such curved space-time was to solve the wave function for field with different spin in relativistic [1]– [5].

Now, we want to make the equation corresponding to spine $\frac{1}{2}$ particle nonrelativistic in de Sitter space-time. This equation lead us to compare it with Jacobi polynomial [6], [7]. By comparing two equation order by order, one can obtain the exact solution

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for the wave function and eigenvalues (energy spectra). Finally, we take advantage from weight and main function from Jacobi polynomial [9], [10], and obtain the generator of algebra for the corresponding system. This form of algebra gives the symmetry of the system which is important in physics and mathematics point of view. For this reason, first we have to make the Pauli equation in de Sitter space-time.

2. The Pauli equation in de Sitter space

The de Sitter space-time Dirac equation in orthogonal coordinates is,

$$\left[i\gamma^k\left(e^{\alpha}_{(k)}\partial_\alpha + A_k\right) - m\right]\Psi = 0, \tag{1}$$

where

$$A_k(x) = \frac{1}{2}e^{\alpha}_{(k);\alpha}(x) = \frac{1}{2}\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\alpha}\sqrt{-g}e^{\alpha}_{(k)}, \tag{2}$$

In order to arrange the $A_k(x)$, we have to introduce the de Sitter metric background,

$$dS^2 = dt^2 - \cosh^2 t dr^2 - \cosh^2 t \sin^2 r d\theta^2 - \cosh^2 t \sin^2 r \sin^2\theta d\phi^2, \tag{3}$$

where

$$\begin{aligned} &g_{00} = 1, && g_{11} = -\cosh^2 t, \\ &g_{22} = -\cosh^2 t \sin^2 r, && g_{33} = -\cosh^2 t \sin^2 r \sin^2\theta, \end{aligned} \tag{4}$$

and

$$g = det(g_{i,j}), \qquad i,j = 0,..,3,$$

Also, the corresponding tetrad,

$$\begin{aligned} &e^{\alpha}_{(0)} = (1,0,0,0)\,, && e^{\alpha}_{(1)} = \left(0,0,\frac{1}{\cosh t \sin r},0\right), \\ &e^{\alpha}_{(2)} = \left(0,0,0,\frac{1}{\cosh t \sin r \sin\theta}\right), && e^{\alpha}_{(3)} = \left(0,\frac{1}{\cosh t},0,0\right), \end{aligned} \tag{5}$$

So, by using (2), (3), (4) and (5) we can rewrite equation (1) as,

$$\left[i\,\gamma^0\cosh t\left(\partial_t + \frac{3}{2}\tanh t\right) + \frac{1}{\sin r}\Sigma_{\theta,\phi} + i\,\gamma^3\left(\partial_r + \frac{1}{\tan r}\right) - m\cosh t\right]\Psi = 0, \tag{6}$$

where $\Sigma_{\theta,\phi} = i\,\gamma^1\partial_\theta + \gamma^2\frac{i\,\partial_\phi + i\,\sigma^{12}\cos\theta}{\sin\theta}$.

Now, we choose the change of variable for the wave function which is given by

$$\Psi(x) = \frac{1}{\sin r \cosh^{\frac{3}{2}} t}\varphi(x), \tag{7}$$

we obtain a simpler equation

$$\left[i\,\gamma^0\cosh t\frac{\partial}{\partial t}+i\,\gamma^3\frac{\partial}{\partial r}+\frac{1}{\sin r}\Sigma_{\theta\phi}-m\cosh t\right]\varphi=0, \tag{8}$$

Here, we need to describe the φ in spin system, as we know the φ have a three indexes as j, m and δ. We use the Wigners function j which define spin as $\nu = j + \frac{1}{2}$ and m angular momentum and δ is parity. So, we consider the following expression

$$\varphi_{jm\delta}=\left|\begin{array}{c} f_1(t,r)D_{-\frac{1}{2}} \\ f_2(t,r)D_{+\frac{1}{2}} \\ f_3(t,r)D_{-\frac{1}{2}} \\ f_4(t,r)D_{+\frac{1}{2}} \end{array}\right| = \left|\begin{array}{c} f_1(t,r)D_{-\frac{1}{2}} \\ f_2(t,r)D_{+\frac{1}{2}} \\ \delta f_2(t,r)D_{-\frac{1}{2}} \\ \delta f_1(t,r)D_{+\frac{1}{2}} \end{array}\right| \tag{9}$$

$$\delta=\pm 1, \qquad f_4=\delta f_1, \qquad f_3=\delta f_2,$$

Here we take $\delta = 1$, so the corresponding equation will he as,

$$\begin{aligned} &\left(\frac{\partial}{\partial r}+\frac{\nu}{\sin r}\right)f+\left(i\,\cosh t\frac{\partial}{\partial t}+\delta m\cosh t\right)g=0, \\ &\left(\frac{\partial}{\partial r}-\frac{\nu}{\sin r}\right)g-\left(i\,\cosh t\frac{\partial}{\partial t}-\delta m\cosh t\right)f=0, \end{aligned} \tag{10}$$

Insted of $f_1(t, r)$ and $f_2(t, r)$, we use the linear combination which are given by,

$$f(t,r)=\frac{f_1+f_2}{\sqrt{2}}, \qquad g(t,r)=\frac{f_1-f_2}{i\sqrt{2}}.$$

First, we separate the rest energy by the formal change,

$$i\,\frac{\partial}{\partial t} \qquad \Rightarrow \qquad M+i\,\frac{\partial}{\partial t}.$$

consider just $\delta = 1$,

$$\begin{aligned} &\frac{1}{\cosh t}\left(\frac{\partial}{\partial r}+\frac{\nu}{\sin r}\right)f+\left(M+i\,\frac{\partial}{\partial t}+M\right)g=0, \\ &\frac{1}{\cosh t}\left(\frac{\partial}{\partial r}-\frac{\nu}{\sin r}\right)g-\left(M+i\,\frac{\partial}{\partial t}-M\right)f=0, \end{aligned} \tag{11}$$

We solve the up system, therefore, non-relativistic functions $f(t, r)$ and $g(t, r)$ will obtain of equation following [8],

$$i\frac{\partial f}{\partial t}=-\frac{1}{2M}\frac{1}{\cosh^2 t}\left(\frac{\partial^2}{\partial r^2}-\frac{\nu^2+\nu\cos r}{\sin^2 r}\right)f, \tag{12}$$

and

$$g=-\frac{1}{2M}\frac{1}{\cosh t}\left(\frac{\partial}{\partial r}+\frac{\nu}{\sin r}\right)f, \tag{13}$$

so, we solve equation (12) with the following separation of variable [8],

$$f(t,r) = e^{-iE\tanh t} f(r), \tag{14}$$

where

$$\left(\frac{d^2}{dr^2} - \frac{\nu^2 + \nu\cos r}{\sin^2 r} + 2ME\right) f(r) = 0. \tag{15}$$

Now, we use a new variable, $z = -\cos r, z \in (-1,+1)$ in equation (15), so we have

$$(1-z^2)\frac{d^2 f}{dz^2} - z\frac{df}{dz} + \left(\nu\frac{z}{1-z^2} - \frac{\nu^2}{1-z^2} + 2ME\right) f = 0. \tag{16}$$

In order to solve equation (16), we use of factorization method and following change of variable

$$f(r) = U(z)P(z) \tag{17}$$

Substituting (17) in (16) leads to

$$\begin{aligned}&(1-z^2)P''(z) + \left(2\frac{U'}{U}(1-z^2) - z\right) P'(z)\\ &+ \left((1-z^2)\frac{U''}{U} - z\frac{U'}{U} + \frac{\nu z}{1-z^2} - \frac{\nu^2}{1-z^2} + 2ME\right) P(z) = 0.\end{aligned} \tag{18}$$

In order to obtain function U, we compared (18) with the following associated Jacobi differential equation

$$\begin{aligned}&(1-z^2)P_{n,m}^{\prime\prime\,(\alpha,\beta)}(z) - [\alpha-\beta+(\alpha+\beta+2)z]P_{n,m}^{\prime\,(\alpha,\beta)}(z)\\ &+\Big[n(\alpha+\beta+n+1) - \frac{m(\alpha+\beta+m+(\alpha-\beta)z)}{1-z^2}\Big]P_{n,m}^{(\alpha,\beta)}(z) = 0.\end{aligned} \tag{19}$$

So we obtain $U(z)$ as,

$$U(z) = N_0(1+z)^{\frac{2\beta+1}{4}}(1-z)^{\frac{2\alpha+1}{4}}, \tag{20}$$

Therefore, the corresponding eigenfunction will be as,

$$f(z) = N_0(1+z)^{\frac{2\beta+1}{4}}(1-z)^{\frac{2\alpha+1}{4}} P_{n,m}^{(\alpha,\beta)}(z).$$

By comparing multiples $P_{n,m}^{(\alpha,\beta)}$ and P in equations (18) and (19), we can obtain α, β and E as

$$\alpha = \frac{m-j}{2m-1}, \qquad \beta = -1-\alpha, \qquad 2ME = n^2. \tag{21}$$

Finally, we have

$$f(r) = N_0\left(\cot\frac{r}{2}\right)^{\frac{2\alpha+1}{2}} \psi_{n,m}^{(\alpha)}(r),$$

where

$$\psi^{(\alpha)}_{n,m}(r) = P^{(\alpha,-1-\alpha)}_{n,m}(-\cos\theta) = a_n(-1)^m$$
$$(22)\quad \left[(1-x)^{-\frac{2m+2\alpha-1}{4}}(1+x)^{-\frac{2m-2\alpha-3}{4}}\left(\frac{d}{dx}\right)^{n-m}(1-x)^{n+\alpha}(1+x)^{n-1-\alpha}\right]_{x=-\cos\theta}.$$

Also, in view of normalization condition in quantum mechanics, we obtain the normalization coefficient N_0 as,

$$\int_{-\infty}^{\infty} f(r)f^*(r)dz = 1, \qquad N_0^2\int_{-\infty}^{\infty}(\psi^{(\alpha)}{}_{n,m}(r))^2U^2(r)d(r) = 1,$$

Thus, solutions of the Pauli equation in non-static de Sitter metrics have been constructed as follow

$$\Psi^{Pauli}_{Ejm}(t,r,\theta,\phi) = e^{iE\tanh t}f(r)\begin{vmatrix} D^j_{-m,-\frac{1}{2}}(\phi,\theta,0) \\ D^j_{-m,+\frac{1}{2}}(\phi,\theta,0)\end{vmatrix}$$

Spectral parameter E (it does not represent the energy of stationary states) is quantized according to the rule (the same for state with different parities) of quantum mechanic

$$(23)\qquad 2ME = n^2.$$

Here, we can obtain the raising and lowering operators corresponding to equation (15) with corresponding α and β [6]. So, we factorized equation (21) as

$$B(m)A(m)\psi^{(\alpha)}_{n,m}(r) = E(n,m)\psi^{(\alpha)}_{n,m}(r),\ \ A(m)B(m)\psi^{(\alpha)}_{n,m-1}(r) = E(n,m)\psi^{(\alpha)}_{n,m-1}(r)$$

where

$$(24)\quad B(m) = \frac{d}{dr} + W_m(r),\ A(m) = -\frac{d}{dr} + W_m(r),\ E(n,m) = n^2-(m-1)^2,$$

where the super potential $W_m(r)$ will be as $W_m(r) = (1-m)\cot r - \frac{1}{2}(1+2\alpha)\csc r$. By using (24), we obtain operators $J_\pm(m)$ that describe the shape invariance symmetry on the homogeneous manifold $SU(2)/U(1)$ parametrized by $\{\theta,\varphi\}$ which are ([11])

$$(25)\qquad J_\pm(m) \ = \ \pm\frac{\partial}{\partial\theta} - \frac{\imath}{\sin\theta}\frac{\partial}{\partial\varphi} - \frac{m-1}{2\tan\theta}.$$

So, the shape invariance equations on the homogeneous manifold $SU(2)/U(1)$ are obtained by following equation,

$$(26)\qquad \begin{aligned} J_+(m)J_-(m)\phi_{n,m}(\theta,\varphi) &= E(n,m)\phi_{n,m}(\theta,\varphi), \\ J_-(m)J_+(m)\phi_{n,m-1}(\theta,\varphi) &= E(n,m)\phi_{n,m-1}(\theta,\varphi),\end{aligned}$$

where spectrum $E(n,m)$ have already been introduced in equations (24) and the bases of representation will be as,

$$(27)\qquad \phi_{n,m}(\theta,\varphi) = e^{\frac{\imath\varphi(1+2\alpha)}{2}}P^{(\alpha,-1-\alpha)}_{n,m}(-\cos\theta).$$

3. Conclusion

In this paper, we studied the Dirac system with spin $\frac{1}{2}$ equation. On the other hand we introduced the Jacobi equation and obtained the first order operators for such equation. We then compered the Dirac equation with Jacobi second order equation and obtained the exact solution for the wave function and energy spectrum. Finally we take advantage from Jacobi equation and factorized the corresponding Dirac system with first order equations. These first order equation in physics point of view will be form of raising and lowering operators. In here we had two cases, the first one we factorized second order Dirac equation in terms of indices n and m . In second case just was factorized in terms of indices m. In future, two operators can be applied for different system for the coherent states. Also, such operators help us to discuss the generators of supersymmetry as super charges. Also, these operators for the partner potential and partner Hamiltonian may be play important role.

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ON MARGINAL AUTOMORPHISM OF INFINITE GROUPS

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Abstract. Let W be a non-empty subset of a free group and G be any group. An automorphism α of G is called marginal automorphism if $g^{-1}\alpha(g) \in W^*(G)$ for any $g \in G$ where $W^*(G)$ is marginal subgroup of G. In this paper we study some properties of marginal automorphism of infinite group G.

Keywords: marginal automorphism, purely non-abelian group, finitely generated group.

1. Introduction

Let F be a free group on a countably infinite set $\{x_1, x_2, ...\}$ and let W be a nonempty subset of F. If $w = {x_1}^{l_1}...{x_r}^{l_r} \in W$ and $g_1, ..., g_r$ are elements of a arbitrary group G, the *value* of the word w at $(g_1...., g_r)$ is $w(g_1, ..., g_r) = {g_1}^{l_1}...{g_r}^{l_r}$. We define

$$W(G) =< w(g_1, g_2, ...)|g_i \in G, w \in W > .$$

$$W^*(G) = \{g \in G|w(g_1, ..., g_{i-1}g_ig, g_{i+1}, ..., g_r) = w(g_1, ..., ..., g_r) \text{ for all } g_i \in G, \\ a \in N \text{ and all } w(x_1, x_2, ..., x_r) \in W\}.$$

$W(G)$ and $W^*(G)$ are the verbal subgroup and marginal subgroup of G with respect to W, respectively. Clearly that $W(G)$ is fully-invariant and $W^*(G)$ is always characteristic subgroup of G (see [9] for more information).

An automorphism σ of G is called marginal with respect to W if $g^{-1}\sigma(g) \in W^*(G)$ for all $g \in G$. The set of all marginal automorphism of G is normal subgroup of $\mathrm{Aut}(G)$ which is denoted by $\mathrm{Aut}_{W^*}(G)$.

An automorphism α of G is called verbal automorphism with respect to W if $g^{-1}\sigma(g) \in W(G)$ for all $g \in G$. The set of all verbal automorphism of G, denoted by $\mathrm{Aut}_W(G)$, is a normal subgroup of $\mathrm{Aut}(G)$.

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A variety is an equationally defined class of groups. More precisely the class of all groups G such that $W(G) = 1$, or equivalently $W^*(G) = G$ is called the variety ν determined by W. In particular, if $W = \{[x_1, x_2]\}$ where $[x_1, x_2] = {x_1}^{-1}{x_2}^{-1}x_1x_2$, then variety ν is the class of abelian groups and $W^*(G) = Z(G)$ and $W(G) = G'$. In abelian variety, $\mathrm{Aut}_W(G) = \mathrm{IA}(G)$, and it is called IA-automorphism of G, also $\mathrm{Aut}_{W^*}(G) = \mathrm{Autcent}(G)$, and it is called central automorphism of G, see also [1]-[3], [6], [7].

Franciosi, De Giovanni and M.L. Newell in [3], proved that if G be a purely non-abelian such that $Z(G)$ or $\frac{G}{G'}$ is torsion with finite abelian section rank, then $\gamma : \mathrm{Autcent}(G) \longrightarrow \mathrm{Hom}(G, Z(G))$ is a bijection map. Also they characterized such groups that have nilpotent central automorphism.

Rai in [8] showed that for a finite p-group G, $C_{IA(G)}Z(G) = \mathrm{Autcent}\ (G)$ if and only if $G' = Z(G)$. ($C_{IA(G)}Z(G)$ is the set of all IA-automorphism of G fixing $Z(G)$ element-wise.)

Jamali and Mosavi in [6] using the concept of isoclinic obtained the relation between central automorphism of two isoclinic groups. They proved that if G is a group such that $Z(G) \leq G'$ and H be a group isoclinic to G, then there is a monomorphism from $\mathrm{Aut}_c(G)$ into $\mathrm{Aut}_c(H)$.

In the present paper, we generalize the above result to marginal automorphism and prove the following theorems:

Theorem 1.1 *Let ν be a variety and $\emptyset \neq W \subseteq F$ and let G, H be two ν-isologism groups such that $W^*(G) \leq W(G)$ and $W^*(H) \leq W(H)$. Then*

$$Aut_{W^*}(G) \simeq Aut_{W^*}(H).$$

Theorem 1.2 *Let G be a purely non-abelian group and $\emptyset \neq W \subseteq F$ such that $W^*(G) \leq Z(G)$. If*

(i) *$W^*(G)$ is torsion group with finite abelian section rank, or*

(ii) *$\frac{G}{W(G)}$ is torsion group with finite abelian section rank and $W(G) \leq G'$.*

Then there is a bijection map between $Aut_{W^*}(G)$ and Hom $\Big(\frac{G}{W(G)}, W^*(G)\Big)$.

Theorem 1.3 *Let G be a finitely generated purely non-abelian group and $\emptyset \neq W \subseteq F$ such that $G' \leq W(G) \leq W^*(G) \leq Z(G)$ and $W^*(G)$ is torsion with finite abelian section rank. Then $C_{Aut_W(G)}(W^*(G)) = Aut_{W^*}(G)$ if and only if $W^*(G) = W(G)$.*

Theorem 1.4 *Let G be a purely non-abelian group such that $W^*(G)$ or $\frac{G}{W(G)}$ is finite and $W^*(G) \leq Z(G)$. Then $Aut_{W^*}(G)$ is a nilpotent group of finite exponent.*

2. Preliminary Lemmas

Notation in this paper is standard. Let G be a group, by $Z(G)$, G',Aut(G) we denoted the center, the commutator subgroup and the group of all automorphism, respectively. Also a non abelian group G that has no non-trivial abelian direct factor is said to be purely non-abelian group.

Let G be an abelian group, the element of finite order of G form a subgroup $T(G)$, the torsion subgroup of G. Moreover, elements with order some power of a fixed prime p likewise form a subgroup G_p, the p-primary component of G. By The Primary Decomposition Theorem, the torsion-subgroup T is the direct sum of the primary components of G.

In this section, we collect all results that will be used in Section 3. We need the following results of Attar [7] that are useful in our investigations.

Lemma 2.5 ([7]) *Let G be a group and $\emptyset \neq W \subseteq F$. Every marginal automorphism of G leaves fixed every element of the verbal subgroup $W(G)$.*

Proposition 2.6 ([7])*Let G be a group and and $\emptyset \neq W \subseteq F$. If $W^*(G)$ is abelian, then $Hom\big(G, W^*(G)\big) \simeq Hom\big(\frac{G}{W(G)}, W^*(G)\big)$.*

The next theorem, determines structure of $\mathrm{Aut}_W(G)$ and the set of all verbal automorphism of G fixing $W^*(G)$ element-wise.

Theorem 2.7 *Let G be a group and $\emptyset \neq W \subseteq F$ such that $W(G) \leq W^*(G) \cap Z(G)$. Then*

(i) $Aut_W(G) \simeq Hom\Big(\frac{G}{W(G)}, W(G)\Big)$.

(ii) $C_{Aut_W(G)}(W^*(G)) \simeq Hom\Big(\frac{G}{W^*(G)}, W(G)\Big)$.

Proof. (i) Let $\sigma : \mathrm{Aut}_W(G) \longrightarrow \mathrm{Hom}\Big(\frac{G}{W(G)}, W(G)\Big)$ defined by $\sigma(f) = \sigma_f$, where $\sigma_f\big(gW(G)\big) = g^{-1}f(g)$, for each $f \in \mathrm{Aut}_W(G)$ and $g \in G$. Since $g^{-1}f(g) \in W(G) \leq Z(G)$, it is clear that $\sigma_f \in \mathrm{Hom}\Big(\frac{G}{W(G)}, W(G)\Big)$, i.e., σ is well-defined.

Let $f_1, f_2 \in \mathrm{Aut}_W(G)$ since here $W(G) \leq W^*(G)$, so $\mathrm{Aut}_W(G) \leq \mathrm{Aut}_{W^*}(G)$ and by Lemma 2.5,

$$\sigma_{f_1 f_2}\big(gW(G)\big) = g^{-1}f_1\big(gg^{-1}f_2(g)\big) = g^{-1}f_1(g)g^{-1}f_2(g)$$

Therefore, $\sigma(f_1 f_2) = \sigma(f_1)\sigma(f_2)$ and σ is a homomorphism. Clearly, σ is injective, so it remains to show that σ is surjective. Let $h \in Hom\Big(\frac{G}{W(G)}, W(G)\Big)$, we define the map $f : G \longrightarrow G$ by $f(g) = gh\big(gW(G)\big)$ for all $g \in G$. It is obvious that f is a monomorphism. Now, $g = gh\big(gW(G)\big)\big(h(gW(G))\big)^{-1}$ for every $g \in G$. Since

$h\big(gW(G)\big) \in W(G)$, so $f\big(h(gW(G))\big)^{-1} = \big(h(gW(G))\big)^{-1}$, which implies that f is surjective. Also we have $g^{-1}f(g) = g^{-1}gh\big(gW(G)\big) \in W(G)$, so $f \in Aut_W(G)$ such that $\sigma(f) = h$. Therefore, σ is an isomorphism.

(ii) If we utilize the above method for $C_{Aut_W(G)}(W^*(G))$, the result is obtained. ∎

Similar to Theorem 2.7, we obtain the following lemma, used in proof of Theorem 1.4.

Lemma 2.8 *Let G be a group and $\emptyset \neq W \subseteq F$ such that $W^*(G) \leq Z(G)$. Then*

(i) $C_{Aut_{W^*}(G)}(W^*(G)) \simeq Hom\Big(\dfrac{G}{W^*(G)W(G)}, W^*(G)\Big)$.

(ii) $C_{Aut_{W^*}(G)}\Big(\dfrac{G}{W(G)}\Big) \simeq Hom\Big(\dfrac{G}{W(G)}, W(G) \cap W^*(G)\Big)$.

The items of the following lemma are well-known and easily verified.

Lemma 2.9 *Let U, V and W be abelian groups. Then*

(i) $Hom(U \times V, W) \simeq Hom(U, W) \times Hom(V, W)$.

(ii) $Hom(U, V \times W) \simeq Hom(U, V) \times Hom(U, W)$.

(iv) *If U is torsion-free of rank n, then $Hom(U, V) \simeq V^n$.*

(v) *If U is a torsion and V is a torsion-free group, then $Hom(U, V) = 1$.*

The following concept was first introduced by Hall [4].

Definition 2.10 Let ν be a variety of groups defined by the set of laws W and G and H be groups. A ν-isologism between G and H is a pair of isomorphisms (α, β) with $\alpha : \dfrac{G}{W^*(G)} \longrightarrow \dfrac{H}{W^*(H)}$ and $\beta : W(G) \longrightarrow W(H)$, such that for all $s > 0$, all $w(x_1, ..., x_s) \in V(F_s)$ and all $g_1, ..., g_s \in G$, it holds that $\beta\big(w(g_1, ..., g_s)\big) = w(h_1..., h_s)$, whenever $h_i \in \alpha\big(g_i W^*(G)\big)$ $(i = 1, ..., s)$. We write $G \underset{\nu}{\sim} H$ and we will say that G and H are ν-isologic.

In abelian variety, two group G and H in above definition are said to be isoclinic.

Lemma 2.11 ([5]) *Let G and H be two groups and $\emptyset \neq W \subseteq F$. Let (α, β) be a ν-isologism between G and H. Let $v \in W(G)$. Then the following hold.*

(i) $\alpha\big(vW^*(G)\big) = \beta(v)W^*(H)$.

(ii) *If $g \in G$ and $h \in \alpha(gW^*(G))$, then $\beta(v^g) = \beta(v)^h$.*

Corollary 2.12 *Let ν be a variety and $\emptyset \neq W \subseteq F$ and let (α, β) be a ν-isologism between G and H. Then β induces an isomorphism from $W^*(G) \cap W(G)$ onto $W^*(H) \cap W(H)$.*

3. Main results

Proof of Theorem 1.1. Let (α, β) be a ν-isologism between G and H. Suppose $\sigma \in Aut_{W^*}(G)$ and $h \in H$, since α is surjective, there exists $g \in G$ such that $\alpha(gW^*(G)) = hW^*(H)$. Define a map $f_\sigma : H \longrightarrow H$ by $f_\sigma(h) = h\beta(g^{-1}\sigma(g))$ for any $h \in H$.

(i) f_σ is well-defined:
Let $h_1, h_2 \in H$ and $\alpha(g_iW^*(G)) = h_iW^*(H)$ for $i = 1, 2$. If $h_1 = h_2$ then $\alpha(g_1W^*(G)) = \alpha(g_2W^*(G))$ so $g_1{g_2}^{-1} \in W^*(G) \leq W(G)$, since σ fixes $W(G)$ element-wise hence ${g_1}^{-1}\sigma(g_1) = {g_2}^{-1}\sigma(g_2)$ and $f_\sigma(h_1) = f_\sigma(h_2)$

(ii) f_σ is homomorphism:
Let $h_1, h_2 \in H$ and $\alpha(g_iW^*(G)) = h_iW^*(H)$ for $i = 1, 2$. Then,

$$\begin{aligned} f_\sigma(h_1h_2) &= h_1h_2\beta\big((g_1g_2)^{-1}\sigma(g_1g_2)\big) \\ &= h_1h_2\beta\big({g_2}^{-1}{g_1}^{-1}\sigma(g_1)g_2{g_2}^{-1}\sigma(g_2)\big) \\ &= h_1\Big(\beta\big(({g_1}^{-1}\sigma(g_1))^{g_2}\big)\Big)^{{h_2}^{-1}} h_2\beta\big({g_2}^{-1}\sigma(g_2)\big). \end{aligned}$$

Applying Lemma 2.11, we obtain $\beta\big(({g_1}^{-1}\sigma(g_1))^{g_2}\big) = \beta\big({g_1}^{-1}\sigma(g_1)\big)^{h_2}$ so $f_\sigma(h_1h_2) = h_1\beta\big({g_1}^{-1}\sigma(g_1))h_2\beta({g_2}^{-1}\sigma(g_2)\big) = f_\sigma(h_1)f_\sigma(h_2)$.

(iii) f_σ is injective:
Let $f_\sigma(h) = 1$ where $\alpha(gW^*(G)) = hW^*(H)$, so $\beta(g^{-1}\sigma(g)) = h^{-1}$. By Lemma 2.11, $\alpha\big(g^{-1}\sigma(g)W^*(G)\big) = \beta\big(g^{-1}\sigma(g)\big)W^*(H) = h^{-1}W^*(H)$,

$$g^{-1}\sigma(g)W^*(G) = \alpha^{-1}(h^{-1}W^*(H)) = \alpha^{-1}(hW^*(H))^{-1} = g^{-1}W^*(G).$$

Hence $\sigma(g) \in W^*(G)$, since $W^*(G)$ is characteristic in G, $g \in W^*(G)$ that Lemma 2.5, implies $\sigma(g) = g$, so $h = 1$.

(iv) f_σ is surjective:
Let $h \in H$ where $\alpha(gW^*(G)) = hW^*(H)$. $h = \underbrace{h\beta(g^{-1}\sigma(g))}_{\in Imf_\sigma}(\beta(g^{-1}\sigma(g)))^{-1}$.
By Lemma 2.11, $\beta(g^{-1}\sigma(g))W^*(H) = \alpha\Big(\big(g^{-1}\sigma(g)\big)W^*(G)\Big)$.
Since $g^{-1}\sigma(g) \in W^*(G) \leq W(G)$ and σ fixes $W(G)$ element-wise.

$$\begin{aligned} f_\sigma\big(\beta(g^{-1}\sigma(g))\big) &= \beta(g^{-1}\sigma(g))\beta(g^{-1}\sigma(g))^{-1}\beta(\sigma(g^{-1}\sigma(g))) \\ &= \beta\big(\sigma(g^{-1}\sigma(g))\big) \\ &= \beta\big(g^{-1}\sigma(g)\big). \end{aligned}$$

Hence $H \leq Imf_\sigma$.

(v) $h^{-1}f_\sigma(h) = \beta(g^{-1}\sigma(g))$, for any $h \in H$ where $\alpha(gW^*(G)) = hW^*(H)$. Since $g^{-1}\sigma(g) \in W^*(G)$ and $W^*(G) \cap W(G) = W^*(G)$ by Lemma 2.12, $\beta(g^{-1}\sigma(g)) \in W^*(H) \cap W(H) = W^*(H)$. Hence, $f_\sigma \in Aut_{W^*}(H)$.

Let $\gamma : Aut_{W^*}(G) \longrightarrow Aut_{W^*}(H)$ such that $\gamma(\sigma) = f_\sigma$. We show that γ is an isomorphism. Suppose $\sigma_1, \sigma_2 \in \mathrm{Aut}_{W^*}(G)$ and $h \in H$ such that $\alpha(gW^*(G)) = hW^*(H)$. Then,

$$\begin{aligned}\gamma(\sigma_1\sigma_2) = f_{\sigma_1\sigma_2}(h) &= h\beta(g^{-1}\sigma_1\sigma_2(g)) \\ &= h\beta\big(g^{-1}\sigma_2(g)(\sigma_2(g))^{-1}\sigma_1(\sigma_2(g))\big) \\ &= h\beta\big(g^{-1}\sigma_2(g)\big)\beta\big((\sigma_2(g)^{-1}\sigma_1(\sigma_2(g))\big).\end{aligned}$$

Since $g^{-1}\sigma_2(g) \in W^*(G) \leq W(G)$, with use of Lemma 2.11, we have

$$\begin{aligned}\alpha^{-1}\Big(h\beta(g^{-1}\sigma_2(g))W^*(H)\Big) &= \alpha^{-1}\big(hW^*(H)\big)\alpha^{-1}\Big(\beta(g^{-1}\sigma_2(g))W^*(H)\Big) \\ &= gW^*(G)\alpha^{-1}\Big(\alpha\big(g^{-1}\sigma_2(g)W^*(G)\big)\Big) \\ &= \sigma_2(g)W^*(G).\end{aligned}$$

Therefore,

$$\gamma(\sigma_1\sigma_2) = f_{\sigma_1}\big(h\beta\big(g^{-1}\sigma_2(g)\big)\big) = f_{\sigma_1}f_{\sigma_2}(h) = \gamma(\sigma_1)\gamma(\sigma_2)$$

It remains to prove that γ is a bijection. Notice that $(\alpha^{-1}, \beta^{-1})$ is a ν-isologism between H and G. Let $f \in \mathrm{Aut}_{W^*}(H)$, we define $\sigma_f(g) = g\beta^{-1}(h^{-1}f(h))$ for any $g \in G$, where $\alpha^{-1}(hW^*(H)) = gW^*(G)$.

In a similar proof as above, $\sigma_f \in Aut_{W^*}(G)$.

Now, we define $\delta : Aut_{W^*}(H) \longrightarrow Aut_{W^*}(G)$ as $\delta(f) = \sigma_f$.

Let $g \in G$ where $\alpha(gW^*(G)) = hW^*(H)$. Then

$$\delta\gamma(\sigma)(g) = \delta(f_\sigma)(h) = g\beta^{-1}(h^{-1}f_\sigma(h)) = g\beta^{-1}(h^{-1}h\beta(g^{-1}\sigma(g)) = \sigma(g).$$

So $\delta\gamma(\sigma) = \sigma$ i.e. $\delta\gamma = 1$, similarly $\gamma\delta = 1$. Thus γ is an isomorphism. ∎

Let $\sigma \in \mathrm{Autcent}(G)$ and $x \in G$ so the map $x \longrightarrow x^{-1}\sigma(x)$ defines a homomorphism from G into $Z(G)$. On the other hand the map $x \longrightarrow xf(x)$ defines an endomorphism of G for every f in $\mathrm{Hom}(G, Z(G))$. If G is a finite group, this endomorphism is a central automorphism if and only if $f(x) \neq x^{-1}$ for every $1 \neq x \in G$. In [1], Adney and Yen show that a necessary and sufficient condition for a finite group G to have such a bijection between $\mathrm{Autcent}(G)$ and $\mathrm{Hom}(G, Z(G))$ is that G has no abelian direct factors. Franciosi, De Giovanni and M.L. Newell in [3], with use of following lemma extend this result to a wider class containing all periodic groups with finite abelian section rank.

Lemma 3.13 ([3]) *Let G be a group and $\alpha \in Hom(G, Z(G))$ such that $\alpha(g) = g^{-1}$ for some $g \neq 1$. If $Im\alpha$ is the direct product of a torsion group with finite abelian section rank and a free abelian group, then G has a non-trivial abelian direct factor.*

Here we want to obtain the corresponding statement of Franciosi and De Giovanni theorem in arbitrary variety of group. If $W^*(G)$ is contained in the center of G then we have same result of Lemma 3.13 for $\alpha \in \mathrm{Hom}(G, W^*(G))$.

Proof of Theorem 1.2. By Proposition 2.6, it is enough to show that there is a bijection map between $\mathrm{Aut}_{W^*}(G)$ and $\mathrm{Hom}\big(G, W^*(G)\big)$. Let $\alpha \in \mathrm{Aut}_{W^*}(G)$ and $g \in G$ now we define $\alpha^* : G \longrightarrow W^*(G)$ such that $\alpha^*(g) = g^{-1}\alpha(g)$. From $g^{-1}\alpha(g) \in W^*(G) \leq Z(G)$ is obtained that α^* is a homomorphism so the map

$$\begin{aligned} \psi : \mathrm{Aut}_{W^*}(G) &\longrightarrow \mathrm{Hom}\big(G, W^*(G)\big) \\ \alpha &\longrightarrow \alpha^* \\ &\qquad \alpha^*(g) = g^{-1}\alpha(g) \end{aligned}$$

is well-defined. It is clear that ψ is a one-to-one map. Our map is surjective, for this let $\gamma \in \mathrm{Hom}\big(G, W^*(G)\big)$, we prove the map $\alpha : G \longrightarrow G$ where $\alpha(g) = g\gamma(g)$ is a marginal automorphism of G such that $\psi(\alpha) = \gamma$. Since $\gamma(g) \in W^*(G) \leq Z(G)$, α is an endomorphism of G. With let composition of the γ and incidence mapping $W^*(G) \longrightarrow Z(G)$, we have $\dfrac{G}{ker\gamma} \simeq Im\gamma \leq W^*(G) \leq Z(G)$ and $G' \leq ker\gamma$. Here by hypothesis $Im\gamma$ is torsion with finite abelian section rank. If $g \in ker\alpha$ then $\alpha(g) = 1$ so $\gamma(g) = g^{-1}$ that by Lemma 3.13, $g = 1$. Now, we show that α is onto and this concludes the proof of theorem. Let $T = Im\gamma$, the p-component T_p of T satisfies the minimal condition on subgroups for each prime p. Since $T_p \simeq \alpha(T_p)$, it follows that $\alpha(T_p) = T_p$, and T is contained in $Im\alpha$. Now $g = g\gamma(g)(\gamma(g))^{-1} \in Im\alpha$ for any $g \in G$, so $Im\alpha = G$. ∎

By the previous theorem, we can study the relation between $C_{Aut_{W(G)}}(W^*(G))$ and $\mathrm{Aut}_{W^*}(G)$. Let G be any group and $W^*(G) \leq Z(G)$ and $\dfrac{G}{W(G)}$ be torsion group so, for any $\sigma \in \mathrm{Aut}_{W^*}(G)$ and $g \in G$, there exsits $n \in \mathbb{N}$ such that $g^n \in W(G)$ and by Lemma 2.5, $\sigma(g^n) = g^n$. Since $g^{-1}\sigma(g) \in W^*(G) \leq Z(G)$, $(g^{-1}\sigma(g))^n = 1$. Now, if $W^*(G)$ is torsion-free then $\mathrm{Aut}_{W^*}(G) = 1$. Moreover, if $W(G) \leq W^*(G)$ then $\dfrac{G}{W^*(G)}$ is torsion and $W(G)$ is torsion-free so, by Lemmas 2.8 and 2.9, $C_{Aut_{W(G)}}(W^*(G)) = 1$. Therefore, we have Theorem 1.3.

Proof of Theorem 1.3. Let $W^*(G) = W(G)$ so, by Lemma 2.5, result is obtained clearly.

Conversely, suppose $C_{Aut_{W(G)}}(W^*(G)) = \mathrm{Aut}_{W^*}(G)$ and $W(G) \neq W^*(G)$ then $W(G) < W^*(G)$. By Theorems 1.2 and 2.7, there is a bijection between $\mathrm{Hom}\Big(\dfrac{G}{W^*(G)}, W(G)\Big)$ and $\mathrm{Hom}\Big(\dfrac{G}{W(G)}, W^*(G)\Big)$.

G is finitely generated and $G' \leq W(G) \leq W^*(G)$ and $W^*(G)$ is torsion so $W^*(G), W(G)$ are finite and $\dfrac{G}{W(G)}, \dfrac{G}{W^*(G)}$ are finitely generated abelian groups. Let $\dfrac{G}{W(G)} = T\big(\dfrac{G}{W(G)}\big) \times \mathbb{Z}^r$ and $\dfrac{G}{W^*(G)} = T\big(\dfrac{G}{W^*(G)}\big) \times \mathbb{Z}^s$ for $r, s \geqslant 0$.

By Lemma 2.9,

$$\left|\mathrm{Hom}\left(T\left(\frac{G}{W^*(G)}\right),W(G)\right)\right|\cdot|W(G)|^s=\left|\mathrm{Hom}\left(T\left(\frac{G}{W(G)}\right),W^*(G)\right)\right|\cdot|W^*(G)|^r$$

$$\frac{G/W(G)}{W^*(G)/W(G)}\simeq\frac{G}{W^*(G)}\ \text{ so }\ \frac{T\big(G/W(G)\big)}{W^*(G)/W(G)}\times\mathbb{Z}^r\simeq T\big(G/W^*(G)\big)\times\mathbb{Z}^s.$$

Hence $r=s$ and $\dfrac{|T\big(G/W(G)\big)|}{|T\big(G/W^*(G)\big)|}=\dfrac{|W^*(G)|}{|W(G)|}$ and, since $W(G)\neq W^*(G)$, we get $\left|T\left(\dfrac{G}{W^*(G)}\right)\right|<\left|T\left(\dfrac{G}{W(G)}\right)\right|$. Now, by Lemma 2-8 of [2],

$$\left|\mathrm{Hom}\left(T\left(\frac{G}{W^*(G)}\right),W(G)\right)\right|<\left|\mathrm{Hom}\left(T\left(\frac{G}{W(G)}\right),W^*(G)\right)\right|,$$

which is a contradiction. ∎

To determine a group that its marginal automorphism is nilpotent, we need the following lemma of [3].

Lemma 3.14 ([3]) *Let G he a purely non-abelian group, and let α be an automorphism of G such that $\alpha^n=1$ for some positive integer n. If M is a torsion central subgroup of G such that $[G,\alpha]\leq M$, then $[G,\alpha]\leq M_\pi$, where π is the set of prime divisors of n.*

Lemma 3.15 *Let G be a purely non-abelian group and $\emptyset\neq W\subseteq F$ such that $W^*(G)\leq Z(G)$. If $[G,Aut_{W^*}(G)]$ is torsion. Then every torsion subgroup of $Aut_{W^*}(G)$ is the direct product of its Sylow subgroups.*

Proof. Let $T=[G,\mathrm{Aut}_{W^*}(G)]$ that is an abelian torsion group so T is the direct sum of the primary components of G. Let $\alpha\in\mathrm{Aut}_{W^*}(G)$ and $g\in G$ and $\alpha^*:G\longrightarrow T$ be a map where $g\longrightarrow g^{-1}\alpha(g)$. Since $W^*(G)\leq Z(G)$, the map α^* is a homomorphism. We define $\alpha_p:G\longrightarrow G$ such that $\alpha_p(g)=g\pi_p\alpha^*(g)$ which $\pi_p:T\longrightarrow T_p$ be the natural projection. It is strightforward to see that $\alpha_p\in\mathrm{Aut}_{W^*}(G)$. Let $\theta:\mathrm{Aut}_{W^*}(G)\longrightarrow\mathrm{Aut}_{W^*}(G)$ defined by $\theta(\alpha)=\alpha_p$ for each $\alpha\in\mathrm{Aut}_{W^*}(G)$, now θ is a homomorphism: for if $\alpha,\beta\in\mathrm{Aut}_{W^*}(G)$ and $g\in G$, then

$$\begin{aligned}\alpha_p\beta_p(g)&=\alpha_p\Big(g\pi_p(\beta^*(g))\Big)=g\pi_p(\beta^*(g))\pi_p\Big(\alpha^*\big(g\pi_p\beta^*(g)\big)\Big)\\&=g\pi_p(\beta^*(g))\pi_p(\alpha^*(g))\pi_p\Big(\alpha^*\big(\pi_p\beta^*(g)\big)\Big)\\&=g\pi_p(\alpha^*(g))\pi_p(\beta^*(g))\pi_p\Big(\alpha^*\big(\beta^*(g)\big)\Big)\\&=g\pi_p(\alpha^*(g))\pi_p\Big(\beta^*(g)\alpha^*\big(\beta^*(g)\big)\Big)\\&=g\pi_p(\alpha^*(g))\pi_p\Big(\alpha\big(\beta^*(g)\big)\Big)=g\pi_p\big(g^{-1}\alpha(g)\alpha\big(g^{-1}\beta(g)\big)\big)\\&=g\pi_p\big(g^{-1}\alpha\beta(g)\big)=g\pi_p(\alpha\beta)^*(g)=(\alpha\beta)_p(g).\end{aligned}$$

Let A be a torsion subgroup of $\mathrm{Aut}_{W^*}(G)$ so $\theta(A) = A_p$ is also torsion and since $[G, A_p] \in T_p$, with use of Lemma 3.14, A_p is a p-group. Let group C is generated by A_p's and prove C is an abelian group. Let p, q are distinct primes and show $[A_p, A_q] = 1$

$$\alpha_p(\alpha_q(g)) = \alpha_p(gg^{-1}\alpha_q(g)) = \alpha_p(g)\alpha_p\big(g^{-1}\alpha_q(g)\big) = gg^{-1}\alpha_p(g)\alpha_p\big(g^{-1}\alpha_q(g)\big).$$

Since $g^{-1}\alpha_q(g) \in T_q \leq W^*(G) \leq Z(G)$ and $[T_q, A_p] = [T_p, A_q] = 1$ so $\alpha_p(\alpha_q(g)) = g\alpha_p\big(g^{-1}\alpha_q(g)\big)g^{-1}\alpha_p(g) = \alpha_q(g)g^{-1}\alpha_p(g) = \alpha_q(g)\alpha_q\big(g^{-1}\alpha_p(g)\big) = \alpha_q(\alpha_p(g))$.

Now, C is a torsion abelian group so it is direct product of its Sylow subgroups. If we show A is contained in C, the proof is complete.

Let $\gamma \in A$ so γ has finite order n let $p_1, ..., p_r$ be prime divisors of n, hence by Lemma 3.14, $[G, \gamma] \leq T_{p_i}$ for $1 \leq i \leq r$ so $[G, \gamma] \leq T_{p_1} \times ... \times T_{p_r}$. Let g be any element of G with use of definition of maps γ_{p_i} and γ^* we have

$$\begin{aligned}\gamma(g) = gg^{-1}\gamma(g) = g\gamma^*(g) &= g\pi_{p_1}(\gamma^*(g))\pi_{p_2}(\gamma^*(g))...\pi_{p_r}(\gamma^*(g)) \\ &= \gamma_{p_r}(\gamma_{p_{r-1}}(...\gamma_{p_1}(g))).\end{aligned}$$

Therefore, $\gamma = \gamma_{p_r}\big(\gamma_{P_{r-1}}(...(\gamma_{p_1}))\big) \in C$, which implies $A \leq C$. ∎

Lemma 3.16 *Let G be a purely non-abelian group and $\emptyset \neq W \subseteq F$ such that $W^*(G)$ or $\dfrac{G}{W(G)}$ is torsion and $W^*(G) \leq Z(G)$. Then every torsion subgroup of $Aut_{W^*}(G)$ is the direct product of its Sylow subgroups and the set $\pi(Aut_{W^*}G)$ is contained in $\pi(W^*(G))$. In particular, every finite subgroup of $Aut_{W^*}(G)$ is nilpotent.*

Proof. By use of Lemma 3.15, it is enough to show that $T = [G,\mathrm{Aut}_{W^*}(G)]$ is torsion. We first assume that $W^*(G)$ is torsion. Then $T \leq W^*(G)$ is torsion. Now, suppose that $\dfrac{G}{W(G)}$ is torsion. Let $\alpha \in \mathrm{Aut}_{W^*}(G)$ and define the map $f : G \longrightarrow [G, \alpha]$, it is clearly that f is an epimorphism with $C_G(\alpha)$ as its kernel so $\dfrac{G}{C_G(\alpha)} \simeq [G, \alpha]$. Let $\dfrac{G}{C_G(\alpha)}$ is torsion-free and $gC_G(\alpha)$ be any non trivial element of it, i.e., $g \notin C_G(\alpha)$ since by Lemma 2.5, $W(G) \leq C_G(\alpha)$, $gC_G(\alpha)$ is nontrival element of $\dfrac{G}{W(G)}$ so there exist $n \in N$ such that $g^n \in W(G) \leq C_G(\alpha)$ then $\big(gC_G(\alpha)\big)^n = 1$ that it is a contradiction. Hence $\dfrac{G}{C_G(\alpha)} \simeq [G, \alpha]$ is torsion and since α be arbitrary element of $\mathrm{Aut}_{W^*}(G)$ so T is torsion.

Now, if α be a marginal automorphism of order p, then by Lemma 3.14, $[G, \alpha] \in T_p$ so $1 \neq T_p \leq W^*(G)$ and $p \in \pi(W^*(G))$.

Since every finite subgroup of $\mathrm{Aut}_{W^*}(G)$ is the direct product of its Sylow subgroups, so it is nilpotent. ∎

As a consequence of Lemma 3.16, we obtain the following corollary.

Corollary 3.17 *Let G be a purely non-abelian p-group (p prime) and $\emptyset \neq W \subseteq F$ such that $W^*(G) \leq Z(G)$. Then every torsion subgroup of $Aut_{W^*}(G)$ is a p-group.*

Proof of Theorem 1.4. Let $W^*(G)$ is torsion. Since $W^*(G)$ is characteristic in G, there is a homomorphism $\mathrm{Aut}_{W^*}(G) \longrightarrow \mathrm{Aut}\ W^*(G)$ with kernel $C_{Aut_{W^*(G)}}W^*(G)$, that by Theorem 2.8, its kernel is an abelian group of finite exponent. Hence, $\mathrm{Aut}_{W^*}(G)$ is an abelian-by-finite group of finite exponent which its Sylow subgroups by Lemma 6-34 of [10] are nilpotent. Now, use of Lemma 3.16, implies that $\mathrm{Aut}_{W^*}(G)$ is nilpotent group.

Suppose now that $\dfrac{G}{W(G)}$ is finite, so there is a homomorphism

$$\begin{aligned} \psi : \mathrm{Aut}_{W^*}(G) &\longrightarrow \mathrm{Aut}\left(\frac{G}{W(G)}\right) \\ \alpha &\longrightarrow \alpha^* \\ & \quad \alpha^*(gW(G)) = \alpha(g)W(G) \end{aligned}$$

Since α is an automorphism, α^* is epimorphism and if $gW(G) \in ker\alpha^*$ then $\alpha(g) \in W(G)$ but $W(G)$ is fully-invariant subgroup G so $g \in W(G)$ and α^* is one-to-one. Therefore, ψ is well-define. It is clear that ψ is homomorphism with kernel $C_{Aut_{W^*(G)}}\left(\dfrac{G}{W(G)}\right)$. The result with use of Lemma 2.8 is obtained in the previous case. ∎

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ON THE ORDER AND LOWER ORDER OF LAPLACE-STIELTJES TRANSFORMATIONS WITH INDEX PAIR (p, q)

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Abstract. In this paper, in order to study the precise growth of entire functions represented by Laplace-Stieltjes transformations, we have introduced the concept of (p, q)-order and lower (p, q)-order and obtained their coefficient characterizations.

Keywords: Laplace-Stieltjes transformations, entire function, order, type.
AMS Subject Classification: 30B50, 30D10.

1. Introduction

Consider the Laplace-Stieltjes transformation defined by

$$G(s) = \int_0^{\infty} \exp(-sx)\, d\alpha(x) \tag{1.1}$$

where $\alpha(x)$ is a function of bounded variation on any finite interval $[0, X]$, $(0 < X < +\infty)$, $s = \sigma + it$, σ and t are real variables. We choose a monotonic increasing sequence of real numbers $\{\lambda_n\}$ satisfying the following conditions:

$$0 = \lambda_1 < \lambda_2 < \lambda_3 < \cdot\ \cdot\ \cdot < \lambda_n \uparrow +\infty, \tag{1.2}$$

$$\limsup_{n\to\infty} (\lambda_{n+1} - \lambda_n) < +\infty,\ \limsup_{n\to\infty} \frac{\ln n}{\lambda_n} = D < +\infty. \tag{1.3}$$

We put

$$K_n^* = \sup_{\substack{\lambda_n < x \le \lambda_{n+1},\\ -\infty < t < +\infty}} \left| \int_{\lambda_n}^{x} e^{-ity} d\alpha(y) \right|.$$

In [2], Jiarong obtained the following Valiron-Knopp-Bohr formula:

Theorem A. *Suppose that the Laplace-Stieltjes transformation* (1.1) *satisfies*

$$\limsup_{n\to\infty} (\lambda_{n+1}-\lambda_n) < +\infty \quad \textit{and} \quad \limsup_{n\to\infty} \frac{\ln n}{\lambda_n} < +\infty,$$

and σ_μ^G *denotes the abscissa of uniform convergence of the integral in* (1.1). *Then*

$$\limsup_{n\to\infty} \frac{\ln K_n^*}{\lambda_n} \le \sigma_\mu^G \le \limsup_{n\to\infty} \frac{\ln K_n^*}{\lambda_n} + \limsup_{n\to\infty} \frac{\ln n}{\lambda_n}. \tag{1.4}$$

Suppose that

$$\limsup_{n\to\infty} \frac{\ln K_n^*}{\lambda_n} = 0. \tag{1.5}$$

If $D = 0$, *then, by* (1.4) *and* (1.5), *it follows that* $\sigma_\mu^G = 0$ *and* $F(s)$ *is analytic in the right half plane* $\sigma > 0$.

Various authors have obtained the growth properties of the analytic function defined by (1.1). Yinying and Daochun [6] introduced a type function using the proximate orders. Similarly, Hong et al [1] also obtained some growth properties using the proximate order and type function. Kong and Yang [7] considered the Laplace-Stieltjes transformations given by (1.1) converging uniformly in the whole complex plane $Re(s) > -\infty$ and studied their growth properties. Yingying and Hong [8] have also studied the properties of Laplace-Stieltjes transformations. In 2012, Luo Xi and Kong Yinying [5] defined Laplace-Stieltjes transformations in a different manner by taking positive exponents in the integral in (1.1). Thus they defined Laplace-Stieltjes transformations as given below:

$$F(s) = \int_0^{+\infty} \exp(sy) d\alpha(y), (s = \sigma + it), \tag{1.6}$$

where $\alpha(y)$ satisfies same conditions as stated earlier. Let the sequence $\{\lambda_n\}$ satisfy both conditions stated in (1.3). We put

$$A_n^* = \sup_{\substack{\lambda_n < x \le \lambda_{n+1}, \\ -\infty < t < +\infty}} \left| \int_{\lambda_n}^{x} e^{ity} d\alpha(y) \right|.$$

Then the result of Theorem A can be proved for (1.6) also. We assume that the function $F(s)$ given by (1.6) satisfies the condition $\liminf_{n\to\infty} \frac{\ln (A_n^*)^{-1}}{\lambda_n} = +\infty$ i.e. $F(s)$ represents an entire function.

We now give

Definition 1. The maximum modulus, the maximum term and central index of $F(s)$ given by (1.6) are defined as

$$\begin{aligned} M(\sigma,F) &= \sup_{-\infty<t<+\infty} |F(\sigma+it)|, \\ M_\mu(\sigma,F) &= \sup_{\substack{0<x\leqslant+\infty,\\ -\infty<t<+\infty}} \left|\int_0^x e^{sy}d\alpha(y)\right|, s=\sigma+it, \sigma>0, \\ \mu(\sigma,F) &= \max_{1\leq n<N}\left\{A_n^* e^{\lambda_n\sigma}\right\}, \sigma>0, \\ N(\sigma,F) &= \max\left\{n;\mu(\sigma,F)=A_n^* e^{\lambda_n\sigma}\right\}. \end{aligned}$$

Using Lemma 1 [5], we can easily show for our case that for $\varepsilon>0$ and σ sufficiently large,

$$\frac{1}{2}\mu(\sigma,F) \leq M_\mu(\sigma,F) \leq \mu(\sigma+D+\varepsilon,F). \tag{1.7}$$

In [7], Kong and Yang have studied the growth properties of L-S transform representing entire function using generalized functions. In this paper we have defined (p,q)-order and lower (p,q)-order of Laplace-Stieltjes transformations and obtained their coefficient characterizations. These results are more explicit and depict the relation between growth parameters and behaviour of the sequence $\{A_n^*\}$.

Following the definitions given by Juneja et al [4] for classical Dirichlet series, we define the index pair (p,q) of entire functions represented by L-S transformation and their order and lower order. We use the following notations: $\exp^{[0]}x=\ln^{[0]}x=x$; $\exp^{[m]}x = \ln^{[-m]}x = \exp(\exp^{[m-1]}x) = \ln(\ln^{[-m-1]})$, $m=\pm1,\pm2,...$ $E_{[m]}(x) = \prod_{i=0}^{m}\exp^{[i]}x$; $\Lambda_{[m]}(x) = \prod_{i=0}^{m}\ln^{[i]}x$, $E_{[-m]}(x) = \frac{x}{\Lambda_{[m-1]}(x)}$, $\Lambda_{[-m]}(x) = \frac{x}{E_{[m-1]}(x)}$, $m=0,\pm1,...$

We define the (p,q)-order and lower (p,q)-order with index pair (p,q) of entire functions represented by Laplace-Stieltjes transformations $F(s)$ given by (1.6) above. Hence we put

$$\rho(\lambda) = \limsup_{\sigma\to\infty}(\inf)\frac{\ln^{[p]}M_\mu(\sigma,F)}{\ln^{[q]}\sigma}, \tag{1.8}$$

where p and q are integers such that $p>q\geq0$. It can be easily seen that $0\leq\rho(p,q)\leq\infty$ if $p>q+1$ and $1\leq\rho(p,q)\leq\infty$ if $p=q+1$. Further, in view of (1.7) we have the equivalent formula for order ρ and lower order λ i.e.,

$$\rho(\lambda) = \limsup_{\sigma\to\infty}(\inf)\frac{\ln^{[p]}\mu(\sigma,F)}{\ln^{[q]}\sigma}. \tag{1.9}$$

We also put for $\alpha\geq0$,

$$P_\alpha(x) = \begin{cases} x & \text{if } q+1<p<\infty \\ x+\alpha & \text{if } p=q+1=2 \\ \max(1,x) & \text{if } 3\leq q+1=p<\infty \\ \infty & \text{if } p=q=\infty. \end{cases}$$

We put $P_1(x)=P(x)$. For more details about the notations, we shall refer to [4].

2. Main results

We now prove

Theorem 1. *Let* $F(s) = \int_0^\infty \exp(sy)\mathrm{d}\alpha(y)$ *be Laplace-Stieltjes transformation and let for a pair of integers* (p,q), $p \geq 2, q \geq 0$, *the* (p,q)*-order* ρ *of* $F(s)$ *be defined by* (1.8). *Then*

$$\rho = P(L), \tag{2.1}$$

where

$$L \equiv L(p,q) = \limsup_{n\to\infty} \frac{\ln^{[p-1]}\lambda_n}{\ln^{[q]}\left\{(1/\lambda_n)\ln\left(A_n^*\right)^{-1}\right\}}. \tag{2.2}$$

Proof. Let us assume that $\rho = \rho(p,q) < \infty$, then from the definition of ρ, for any $\varepsilon > 0$ and all $\sigma > \sigma_0(\varepsilon)$, we get

$$\ln M_\mu(\sigma, F) < \exp^{[p-1]}\left\{(\rho+\varepsilon)\ln^{[q]}\sigma\right\}.$$

From (1.7) and the definition of $\mu(\sigma, F)$, we have

$$\ln A_n^* + \lambda_n\sigma \;\leq\; \ln\mu(\sigma,F) \;\leq\; \ln M_\mu(\sigma,F) + o(1).$$

Hence

$$\ln A_n^* < \exp^{[p-1]}\left\{(\rho+\varepsilon)\ln^{[q]}\sigma\right\} - \lambda_n\sigma. \tag{2.3}$$

For $(p,q) \neq (2,1)$, we choose

$$\sigma = \exp^{[q-1]}\left\{\ln^{[p-2]}(\lambda_n)\right\}^{1/(\rho+\varepsilon)}.$$

For $p \geq 2$ and $q \geq 0$, from (2.3) we have

$$\ln A_n^* < \;\lambda_n - \lambda_n \exp^{[q-1]}\left\{\ln^{[p-2]}\lambda_n\right\}^{1/(\rho+\varepsilon)}.$$

Now, if $p = q+1$, then $\rho \geq 1$. Therefore, for $p = q+1 \geq 3$, the above inequality gives

$$\frac{1}{\lambda_n}\ln\left(A_n^*\right)^{-1} > \exp^{[q-1]}\left\{\ln^{[p-2]}\lambda_n\right\}^{1/(\rho+\varepsilon)} - 1,$$

or

$$\ln^{[q]}\left\{\frac{1}{\lambda_n}\ln\left(A_n^*\right)^{-1}\right\} > \frac{1}{\rho+\varepsilon}\left[\ln^{[p-1]}\lambda_n\right](1+o(1)),$$

or

$$\rho+\varepsilon > \frac{\left[\ln^{[p-1]}\lambda_n\right]}{\ln^{[q]}\left\{(1/\lambda_n)\ln\left(A_n^*\right)^{-1}\right\}}(1+o(1)).$$

By proceeding to limits, we get

$$\rho\,(p,p-1) \geq \max\,(1, L\,(p,p-1))\,. \tag{2.4}$$

From the above inequality, we also have for $p > q+1$ and $(p,q) \neq (2,1)$,

$$\rho\,(p,q) \geq L\,(p,q)\,.$$

Thus (2.1) is proved for all pairs (p,q) except for $(p,q) = (2,1)$.

Now, for $(p,q) = (2,1)$, we choose

$$\sigma = \ \lambda_n^{\ 1/(\rho+\varepsilon)}.$$

After substituting this value of σ in (2.3), we get for $(p,q) = (2,1)$

$$\ln A_n^* < \lambda_n - \lambda_n(\lambda_n)^{1/(\rho+\varepsilon)},$$

or

$$\ln\left\{\frac{1}{\lambda_n}\ln\,(A_n^*)^{-1}\right\} > \frac{1}{(\rho+\varepsilon)}\ln\,(\lambda_n) + o\,(1)\,.$$

By proceeding to limits, we get

$$\limsup_{n\to\infty}\ \frac{\ln\lambda_n}{\ln\left\{(1/\lambda_n)\ln\,(A_n^*)^{-1}\right\}}\ \leq\ \rho\,.$$

Hence, for all index pair (p,q),

$$\rho\,(p,q) \geq L\,(p,q)\,. \tag{2.5}$$

To prove the reverse inequality, we assume that $L(p,q) < \infty$. For any $\varepsilon > 0$, there is a positive integer n_o such that for $n > n_o$, we have

$$A_n^* < \exp\left\{-\lambda_n \exp^{[q-1]}\left(\ln^{[p-2]}\lambda_n\right)^{1/L+\varepsilon}\right\}. \tag{2.6}$$

Also, for a number $\sigma_o > 1$ and $\sigma > \sigma_o$, we can find an integer S such that

$$\lambda_S < \exp^{[p-2]}\left(\ln^{[q]}2e^{\sigma}\right)^{(L+\varepsilon)} < \lambda_{S+1}.$$

From the definition, we have $M_\mu\,(\sigma,F) \leq \sum\limits_{n=0}^{\infty} A_n^* \exp\,(\sigma\lambda_n)\,.$ Then

$$\begin{aligned} M_\mu\,(\sigma,F) \leq O(1) &+ \sum_{n=n_o+1}^{S} A_n^* \exp\,(\sigma\lambda_n) \\ &+ \sum_{n=S+1}^{\infty} \exp\left\{\sigma\lambda_n - \lambda_n \exp^{[q-1]}\left(\ln^{[p-2]}\lambda_n\right)^{1/(L+\varepsilon)}\right\}. \end{aligned} \tag{2.7}$$

Hence, using (2.6) we obtain

$$\begin{aligned}\sum_{n=n_o+1}^{S} A_n^* \exp(\sigma\lambda_n) &< \exp(\sigma\lambda_S) \sum_{n=n_o+1}^{S} \exp\left\{-\lambda_n \exp^{[q-1]}\left(\ln^{[p-2]}\lambda_n\right)^{1/L+\varepsilon}\right\} \\ &< \exp(\sigma\lambda_S) \sum_{n=n_o+1}^{\infty} \exp\left\{-\lambda_{n_o} \exp^{[q-1]}\left(\ln^{[p-2]}\lambda_n\right)^{1/L+\varepsilon}\right\} \\ &< B\exp(\sigma\lambda_S)\end{aligned} \tag{2.8}$$

since the series on right hand side is convergent. Similarly, we have

$$\begin{aligned}&\sum_{n=S+1}^{\infty} \exp\left\{\sigma\lambda_n - \lambda_n \exp^{[q-1]}\left(\ln^{[p-2]}\lambda_S\right)^{1/(L+\varepsilon)}\right\} \\ &\quad \leq \sum_{n=S+1}^{\infty} \exp\{\sigma\lambda_n - \lambda_n \ln(2e^{\sigma})\} \;\leq \sum_{n=0}^{\infty} 2^{-\lambda_n} = C < \infty.\end{aligned} \tag{2.9}$$

Combining the estimates obtained in (2.8) and (2.9), we get

$$M_{\mu}(\sigma, F) \leq A + B\exp\left\{\sigma\exp^{[p-2]}\left(\ln^{[q]}2e^{\sigma}\right)^{(L+\varepsilon)}\right\} + C$$

or

$$\ln^{[2]} M_{\mu}(\sigma, F) \leq \exp^{[p-3]}\left(\ln^{[q]}2e^{\sigma}\right)^{(L+\varepsilon)} + o(1)$$

or

$$\begin{aligned}\ln^{[p]} M_{\mu}(\sigma, F) &\leq \ln\left(\ln^{[q]}2e^{\sigma}\right)^{(L+\varepsilon)}(1+o(1)) \\ &= \left\{(L+\varepsilon)\ln^{[q]}\ln(2e^{\sigma})\right\}(1+o(1)) \\ &\simeq (L+\varepsilon)\ln^{[q]}(\sigma).\end{aligned}$$

Proceeding to limits, we obtain

$$\limsup_{\sigma\to\infty} \frac{\ln^{[p]} M_{\mu}(\sigma, F)}{\ln^{[q]}\sigma} \;=\; \rho(p,q) \leq L(p,q). \tag{2.10}$$

Thus the above inequality is true for all pairs of (p, q). From (2.5) and (2.10), we get (2.1) and the proof of the Theorem 1 is complete. ∎

Next, we prove

Lemma 1. *Let $F(s) = \int_0^{\infty} \exp(sy)\mathrm{d}\alpha(y)$ be Laplace-Stieltjes transformation having (p, q)-order ρ and lower (p, q)-order λ. Then*

$$\limsup_{\sigma\to\infty}(\inf) \frac{\ln^{[p-1]}\lambda_{N(\sigma,F)}}{\ln^{[q]}\sigma} \;=\; \rho(\lambda). \tag{2.11}$$

Proof. From a result of Jingjing [3, p.7], we have

$$\ln \mu(\sigma, F) = \ln \mu(\sigma', F) + \int_{\sigma'}^{\sigma} \lambda_{N(x,F)}\, dx, \quad \text{where } 0 < \sigma' < \sigma < \infty.$$

Since $\lambda_{N(\sigma,F)}$ is nondecreasing, therefore we have

$$\ln \mu(\sigma, F) \leq \ln \mu(\sigma', F) + \lambda_{N(\sigma,F)}(\sigma - \sigma')$$

or

$$(1 - o(1))\ \ln \ln \mu(\sigma, F) \leq (1 + o(1))\ \ln \lambda_{N(\sigma,F)},$$

or

$$\ln^{[p]} \mu(\sigma, F) \leq \ln^{[p-1]} \lambda_{N(\sigma,F)}.$$

Hence,

$$\limsup_{\sigma\to\infty}(\inf)\ \frac{\ln^{[p]} \mu(\sigma, F)}{\ln^{[q]} \sigma} \leq \limsup_{\sigma\to\infty}(\inf)\ \frac{\ln^{[p-1]} \lambda_{N(\sigma,F)}}{\ln^{[q]} \sigma}.$$

Conversely, we have

$$\ln \mu(2\sigma, F) = \ln \mu(\sigma', F) + \int_{\sigma'}^{2\sigma} \lambda_{N(x,F)}\, dx \geq O(1) + \int_{\sigma}^{2\sigma} \lambda_{N(x,F)}\, dx \geq \sigma\, \lambda_{N(\sigma,F)}.$$

Following as above, we obtain

$$\limsup_{\sigma\to\infty}(\inf)\ \frac{\ln^{[p]} \mu(\sigma, F)}{\ln^{[q]} \sigma} \geq \limsup_{\sigma\to\infty}(\inf)\ \frac{\ln^{[p-1]} \lambda_{N(\sigma,F)}}{\ln^{[q]} \sigma}.$$

Combining the two inequalities obtained above, we get (2.11). ∎

Now, we obtain some lower bounds for the lower (p, q)-order λ.

Lemma 2. *Let $F(s) = \int_0^{\infty} \exp(sy) \mathrm{d}\alpha(y)$ be Laplace-Stieltjes transformation having lower (p, q)-order λ and let $\{n_k\}_{k=1}^{\infty}$ be an increasing sequence of natural numbers. Then*

$$\lambda \geq P_{\alpha}(\Im) \tag{2.12}$$

where

$$\alpha \equiv \alpha(\{n_k\}) = \lim_{k\to\infty} \inf \frac{\ln \lambda_{n_{k-1}}}{\ln \lambda_{n_k}},$$

and

$$\Im \equiv \Im(\{n_k\}, p, q) = \lim_{k\to\infty} \inf \frac{\ln^{[p-1]} \lambda_{n_{k-1}}}{\ln^{[q]} \left\{1/\lambda_{n_k} \ln\left(A^*_{n_k}\right)^{-1}\right\}}. \tag{2.13}$$

Proof. We assume that $0 < \Im < \infty$. Then, for a given ε, $\Im > \varepsilon > 0$, we have from (2.13)

$$\left(A^*_{n_k}\right) > \exp\left[-\lambda_{n_k}\exp^{[q]}\left\{\ln^{[p-1]}\lambda_{n_{k-1}}\Big/(\Im-\varepsilon)\right\}\right]. \tag{2.14}$$

Since addition of an exponential polynomial does not affect the growth of $F(s)$, we can assume that (2.14) holds for all k. Now for the pair $(p,q) \neq (2,1)$, we choose a sequence

$$\sigma_k = 2 + \exp^{[q]}\left\{\ln^{[p-1]}\lambda_{n_{k-1}}\Big/(\Im-\varepsilon)\right\}, k = 1, 2, ... \tag{2.15}$$

Then $\{\sigma_k\}$ is monotonic increasing and $\sigma_k \to \infty$ as $k \to \infty$. If $\sigma_k \leq \sigma < \sigma_{k+1}$, then for $(p,q) \neq (2,1)$, from (2.14) we get

$$\ln M_\mu(\sigma, F) \geq \ln A^*_{n_k} + \lambda_{n_k}\sigma_k > \quad \sigma_k\lambda_{n_k} - \lambda_{n_k}\exp^{[q]}\left\{\ln^{[p-1]}\lambda_{n_{k-1}}\Big/(\Im-\varepsilon)\right\}.$$

Now, substituting for σ_k, from (2.15), we get

$$\begin{aligned}\ln M_\mu(\sigma, F) \geq 2\lambda_{n_k} &= 2\exp^{[p-1]}\left\{(\Im-\varepsilon)\ln^{[q]}(\sigma_{k+1}-2)\right\}\\ &\geq 2\exp^{[p-1]}\left\{(\Im-\varepsilon)\ln^{[q]}(\sigma-2)\right\}.\end{aligned}$$

For sufficiently large value of σ, we have

$$\ln^{[p]} M_\mu(\sigma, F) \geq (\Im-\varepsilon)\ln^{[q]}\sigma + o(1).$$

For $p = q+1 \geq 3$, we have $\lambda \geq 1$, so we have, by the above inequality,

$$\liminf_{\sigma\to\infty} \frac{\ln^{[p]} M_\mu(\sigma, F)}{\ln^{[q]}\sigma} \geq \max(1, \Im). \tag{2.16}$$

For $p > q+1$, the same inequality implies

$$\liminf_{\sigma\to\infty} \frac{\ln^{[p]} M_\mu(\sigma, F)}{\ln^{[q]}\sigma} \geq \Im. \tag{2.17}$$

Now, for $(p,q) = (2,1)$, as $1 \leq \Im < \infty$, we choose

$$\sigma_k = 2\left(\lambda_{n_{k-1}}\right)^{1/(\Im-\varepsilon)}, k = 1, 2, ... \tag{2.18}$$

If $\sigma_k \leq \sigma < \sigma_{k+1}$, after proceeding as in the previous case, we get

$$\ln M_\mu(\sigma, F) \geq \lambda_{n_k}\left(\lambda_{n_{k-1}}\right)^{1/(\Im-\varepsilon)},$$

which gives

$$\frac{\ln\ln M_\mu(\sigma, F)}{\ln\sigma} \geq \frac{\ln\lambda_{n_k}}{\ln\sigma_{k+1}} + \frac{1}{(\Im-\varepsilon)}\,\frac{\ln\lambda_{n_{k-1}}}{\ln\sigma_{k+1}}.$$

After substituting the value of σ_{k+1}, from (2.18) and proceeding to limits as $k \to \infty$, we get

$$\lambda \geq \Im + \chi. \tag{2.19}$$

After combining (2.16), (2.17) and (2.19) we get the desired result for $0 < \Im < \infty$. If $\Im = 0$ (2.12) is trivially true and if $\Im = \infty$, then, repeating the above arguments with an arbitrarily large number in place of $(\Im - \varepsilon)$, we get the desired result. This completes the proof of the Lemma 2. ∎

In the next result, we characterize the lower $(p,q)-$ order in terms of the ratio $\left\{A^*_{n-1}\Big/A^*_n\right\}$. We prove

Lemma 3. *Let $F(s) = \int_0^\infty \exp(sy)\mathrm{d}\alpha(y)$ be Laplace-Stieltjes transformation having lower (p,q)-order λ and let $\{n_k\}_{k=1}^\infty$ be an increasing sequence of natural numbers, then*

$$\lambda \geq C_\chi(\Im^*) \tag{2.20}$$

where

$$\Im^* \equiv \Im^*(\{n_k\}, p, q) = \liminf_{k\to\infty} \frac{\ln^{[p-1]} \lambda_{n_{k-1}}}{\ln^{[q]}\left\{\left(\ln A^*_{n_{k-1}}\Big/A^*_{n_k}\right)\Big/\left(\lambda_{n_k} - \lambda_{n_{k-1}}\right)\right\}}, \tag{2.21}$$

and χ is as defined in Lemma 2.

Proof. If $\Im^*$ is zero then (2.20) is trivially true, therefore it is sufficient to consider the case when $0 < \Im^* \leq \infty$. For any ε such that $\Im^* > \varepsilon > 0$ and for all $k > k_1 = k_1(\varepsilon)$, from (2.21), we have

$$\left(A^*_{n_{k-1}}\Big/A^*_{n_k}\right) < \exp\left[\left(\lambda_{n_k} - \lambda_{n_{k-1}}\right)\exp^{[q]}\left\{\ln^{[p-1]}\lambda_{n_{k-1}}\Big/(\Im^* - \varepsilon)\right\}\right].$$

Writing the above inequality for $m = k_1 + 1, ..., k$ and multiplying side by side, we get

$$A^*_{n_k} > A^*_{n_{k_1}} \prod_{m=k_1+1}^{k} \exp\left[-\left(\lambda_{n_m} - \lambda_{n_{m-1}}\right)\exp^{[q]}\left\{\ln^{[p-1]}\lambda_{n_{m-1}}\Big/(\Im^* - \varepsilon)\right\}\right]. \tag{2.22}$$

Let the sequence $\{\sigma_k\}$ be chosen as in equations (2.15) and (2.18) for the two cases and let $\sigma_k \leq \sigma < \sigma_{k+1}$. Then for all $\sigma > \sigma_o = \sigma_o(\varepsilon)$, we have by (2.22)

$$\begin{aligned}
\ln M_\mu(\sigma, F) &\geq \ln A^*_{n_k} + \lambda_{n_k}\sigma_k \\
&> \ln A^*_{n_{k_1}} - \sum_{m=k_1+1}^{k}\left(\lambda_{n_m} - \lambda_{n_{m-1}}\right)\exp^{[q]}\left\{1/(\Im^* - \varepsilon)\ln^{[p-1]}\lambda_{n_{m-1}}\right\} + \lambda_{n_k}\sigma_k \\
&= \ln A^*_{n_{k_1}} - \lambda_{n_k}\exp^{[q]}\left\{1/(\Im^* - \varepsilon)\ln^{[p-1]}\lambda_{n_{m-1}}\right\} \\
&+ \sum_{m=k_1}^{k-1}\lambda_{n_m}\left[\exp^{[q]}\left\{1/(\Im^* - \varepsilon)\ln^{[p-1]}\lambda_{n_m}\right\} - \exp^{[q]}\left\{1/(\Im^* - \varepsilon)\ln^{[p-1]}\lambda_{n_{m-1}}\right\}\right] \\
&> \ln A^*_{n_{k_1}} - \lambda_{n_k}\exp^{[q]}\left\{1/(\Im^* - \varepsilon)\ln^{[p-1]}\lambda_{n_{k-1}}\right\} + \lambda_{n_k}\sigma_k.
\end{aligned}$$

Now, choosing the sequence $\{\sigma_k\}$ for the two cases and proceeding as in Lemma 2 we get the desired result (2.20). ∎

In the next result, we obtain the reverse estimates for the lower (p,q) order. We have

Lemma 4. *Let* $F(s) = \int_0^\infty \exp(sy)\mathrm{d}\alpha(y)$ *be Laplace-Stieltjes transformation having lower* (p,q)*-order* λ *such that* $\theta(n) \equiv \frac{\ln\left(A_n^*/A_{n+1}^*\right)}{\lambda_{n+1}-\lambda_n}$ *forms a nondecreasing function of* n *for* $n > n_o$*. Then*

$$\lambda \leq C\left(\Im_o\right) \tag{2.23}$$

and

$$\lambda \leq C\left(\Im_o^*\right) \tag{2.24}$$

where

$$\Im_o \equiv \Im_o(p,q) = \liminf_{n\to\infty} \frac{\ln^{[p-1]}\lambda_{n-1}}{\ln^{[q]}\left\{1/\lambda_n \ln\left(A_n^*\right)^{-1}\right\}}, \tag{2.25}$$

and

$$\Im_o^* \equiv \Im_o^*(p,q) = \liminf_{n\to\infty} \frac{\ln^{[p-1]}\lambda_{n-1}}{\ln^{[q]}\left\{1/(\lambda_n-\lambda_{n-1})\ln\left(A_{n-1}^*/A_n^*\right)\right\}}. \qquad ∎$$

Proof. We know that, if $\mu(\sigma,F) = A_n^* e^{\lambda_n\sigma}$ is the maximum term for a given σ, then

$$A_{n-1}^* e^{\lambda_{n-1}\sigma} \leq A_n^* e^{\lambda_n\sigma} > A_{n+1}^* e^{\lambda_{n+1}\sigma}$$

or

$$\theta(n-1) \leq \sigma < \theta(n).$$

Hence, for infinitely many values of n, $\theta(n) > \theta(n-1)$ and $\theta(n) \to \infty$ as $n \to \infty$. When $\theta(n) > \theta(n-1)$, we have $\mu(\sigma,F) = \max\limits_{n\in N}\left\{A_n^* e^{\lambda_n\sigma}\right\}$, $\sigma > 0$ and $N(\sigma,F) = n$ for $\theta(n-1) \leq \sigma < \theta(n)$.

Now, let

$$\phi = \liminf_{\sigma\to\infty} \frac{\ln^{[p-1]}\lambda_{N(\sigma,F)}}{\ln^{[q]}\sigma}.$$

First, we assume that $\phi > 0$. For any ε such that $\phi > \varepsilon > 0$, and for all $\sigma > \sigma_o = \sigma_o(\varepsilon)$, we have

$$\lambda_{N(\sigma,F)} > \exp^{[p-1]}\left\{(\phi-\varepsilon)\ln^{[q]}\sigma\right\}.$$

Now, let $A_{n_1}^*\exp(\lambda_{n_1}s)$ and $A_{n_2}^*\exp(\lambda_{n_2}s)$, $(n_1 > n_o, \theta(n_1-1) > \sigma_o)$, be two consecutive maximum terms of $F(s)$ so that $n_1 \leq n_2 - 1$. Let $n_1 < n \leq n_2$. Since $A_{n_1}^*\exp(\lambda_{n_1}s)$ is the maximum term we have $N(\sigma,F) = n_1$ for $\theta(n_1-1) \leq \sigma < \theta(n_1)$. For σ in this interval,

$$\lambda_{n_1} > \exp^{[p-1]}\left\{(\phi-\varepsilon)\ln^{[q]}\sigma\right\}.$$

Since $\theta(n_1) = \theta(n_1+1) = ... = \theta(n-1)$, we have

$$\lambda_{n-1} \geq \lambda_{n_1} > \exp^{[p-1]}\left\{(\phi-\varepsilon)\ln^{[q]}(\theta(n-1)-\delta)\right\} \tag{2.26}$$

where $\delta = \max\{1, [\theta(n_1) - \theta(n_1-1)]/2\}$. Since $\theta(n)$ is nondecreasing, we have

$$\ln\left|A^*_{n_o}/A^*_{n_{o+1}}\right| + ... + \ln\left|A^*_{n-1}/A^*_n\right| = \ln\left|A^*_{n_o}/A^*_n\right| = \sum_{k=n_0}^{n-1}(\lambda_{k+1}-\lambda_k)\theta(k)$$
$$\leq (\lambda_n - \lambda_{n_o})\theta(n-1), (n > n_o),$$

and hence

$$\ln^{[q]}\left(\frac{1}{\lambda_n}\ln(A^*_n)^{-1}\right) \leq \ln^{[q]}\theta(n-1) + o(1). \tag{2.27}$$

From (2.26), we have

$$\ln^{[p-1]}\lambda_{n-1} \geq \left\{(\phi-\varepsilon)\ln^{[q]}(\theta(n-1)-\delta)\right\}. \tag{2.28}$$

Combining (2.27) and (2.28), we get

$$\frac{\ln^{[q]}\left(\frac{1}{\lambda_n}\ln(A^*_n)^{-1}\right)}{\ln^{[p-1]}\lambda_{n-1}} \leq \frac{1}{(\phi-\varepsilon)} + o(1).$$

By passing to limits, the above inequality leads to

$$\phi \leq \Im_o. \tag{2.29}$$

When $\phi = 0$, the above inequality is evidently true. It follows from (2.29) that $\lambda = P(\phi) \leq P(\Im_o)$ [using Lemma 1]. This proves (2.23). To prove (2.24), we find from (2.28) that for sufficiently large n, and for any ε such that $\phi > \varepsilon > 0$,

$$\frac{\ln^{[p-1]}\lambda_{n-1}}{\ln^{[q]}(\theta(n-1)-\delta)} \geq \phi - \varepsilon.$$

After substituting the value of $\theta(n-1)$ and passing to limits, we get

$$\phi \leq \Im^*_o.$$

The inequality is obvious if $\phi = 0$.

So, from the last inequality, we get $\lambda = C(\phi) \leq C(\Im^*_o)$ [Lemma 1] and this completes the proof of Lemma 4. ∎

Combining the results of Lemmas 2, 3 and 4, we obtain the following characterization of lower (p,q) order.

Theorem 2. *Let* $F(s) = \int_0^\infty \exp(sy)\mathrm{d}\alpha(y)$ *be Laplace-Stieltjes transformation having lower* (p,q)*-order* λ *such that* $\theta(n) \equiv \frac{\ln(A_n^*/A_{n+1}^*)}{\lambda_{n+1}-\lambda_n}$ *forms a nondecreasing function of* n *for* $n > n_o$ *and* (1.3) *holds. Then, for* $(p,q) \neq (2,1)$,

$$\lambda = C(\Im_o) = C(\Im_0^*), \tag{2.30}$$

where $\Im_o$ *and* $\Im_0^*$ *are defined as in Lemma* 3.

Further, (2.30) *holds for* $(p,q) = (2,1)$ *also if* $\ln\lambda_n \simeq \ln\lambda_{n+1}$, *as* $\mathrm{n} \to \infty$.

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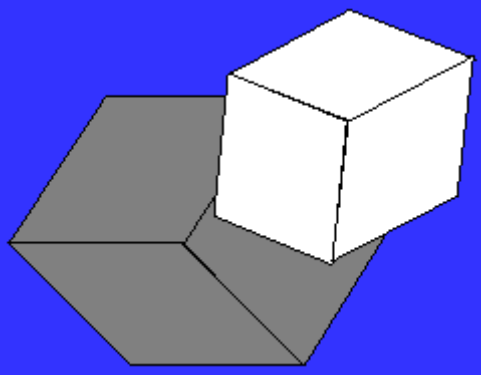

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