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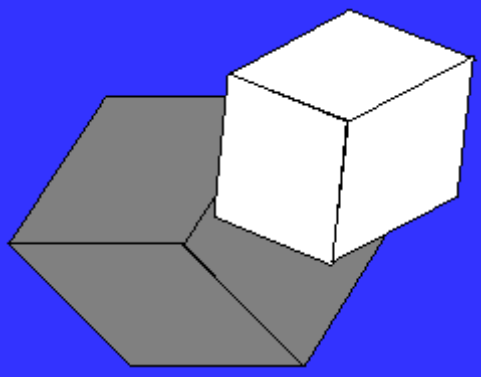
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56.	Collectanea Mathematica - Barcelona	E
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59.	Colloquium Mathematicum - Instytut Matematyczny - Warszawa	PL
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SOME PROPERTIES OF AUTONILPOTENT GROUPS

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Abstract. In this paper, we introduce the notion of autonilpotent groups and study the basic properties of this new notion. Among other results, it is shown that autonilpotency property is stronger than the usual nilpotency of groups. We also classify all finite abelian groups, which are autonilpotent.

Keywords and phrases: autocommutator subgroup, autocentral series, autonilpotent group.

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1. Introduction

Let G be a group and $Aut(G)$ denote the automorphisms group of G. For any $g \in G$ and $\alpha \in Aut(G)$,

$$[g, \alpha] = g^{-1}g^{\alpha}$$

is the *autocommutator* of g and α. Inductively, for all $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n \in$ Aut(G),

$$[g, \alpha_1, \alpha_2, \ldots \alpha_n] = [[g, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}], \alpha_n]$$

is the autocommutator of $g, \alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n \in$ of weight n+1 ($n \geq 1$).

The following definition is vital in our investigations.

Definition 1.1.

(a) Let G be a group and A denotes the automorphism group of G. Then

$$K_1(G) = K(G) = [G, A] = \langle [g, \alpha] \mid g \in G, \alpha \in A \rangle$$

is the autocommutator subgroup of G and inductively,

$$K_n(G) = K(G) = [\ G, \underbrace{A, ..., A}_{n-times}\ ] = \langle [g, \alpha_1, ..., \alpha_n] \mid g \in G, \alpha_1, ..., \alpha_n \in A \rangle.$$

is the n th-autocommutator subgroup of G. Now, we obtain the following series of characteristic subgroups

$$G = K_0(G) \supseteq K_1(G) = K(G) \supseteq K_2(G) \supseteq ... \supseteq K_n(G) \supseteq ... ,$$

which we call the lower autocentral series of the group G.

In particular, if we restrict ourselves to inner automorphisms group of G, $Inn(G)$, then the above series will be the usual lower central series of G.

(b) We call the set of elements

$$L(G) = \{g \in G \mid [g, \alpha] = 1 \text{ or } g^{\alpha} = g, \ \forall \alpha \in A\}$$

the autocentre of G. Clearly, it is a characteristic subgroup of G (see [1] for more information) and, if $A = Inn(G)$, then $L(G) = Z(G)$ is the centre of G.

Now, we define the upper autocentral series of G in the following way:

$$\langle 1 \rangle = L_0(G) \subseteq L_1(G) = L(G) \subseteq L_2(G) \subseteq ... \subseteq L_n(G) \subseteq ...$$

where

$$\frac{L_n(G)}{L_{n-1}G)} = L\left(\frac{G}{L_{n-1}(G)}\right),$$

or, equivalently,

$$L_n(G) = \pi_G^{-1}\left(L\left(\frac{G}{L_{n-1}(G)}\right)\right),$$

for all $n \geq 2$, in which $\pi_G : G \to \dfrac{G}{L_{n-1}(G)}$ is a homomorphism.

In particular, if we take the group of inner automorphisms we obtain the usual upper central series of G. A group G is said to be autonilpotent group of class at most n if $L_n(G) = G$, for some natural number $n \in N$.

In [1], [2], some properties of autocommutator subgroup of a finite group are studied. In [3], we showed that every finite abelian group is the nth autocentral subgroup of some finite abelian group, for any natural number n.

The main purpose of this paper is to determine all finite abelian groups, which are autonilpotent.

The following lemma follows easily from the definition.

Lemma 1.2. *Let G be a group and $x \in L_n(G)$, for some $n \geq 1$, then for all $\alpha_1, ..., \alpha_n \in Aut(G)$,*

$$[x, \alpha_1, ..., \alpha_n] = 1.$$

Corollary 1.3. *If G is an autonilpotent group of class n, then $K_n(G) = \langle 1 \rangle$.*

Remark 1.4. For any group G and each natural number n,

$$K_n(G) \geq \gamma_n(G),$$

and

$$L_n(G) \leq Z_n(G).$$

Note that the above inequalities will be attained, when the group G is taken to be the symmetric group S_3, since $Aut(S_3) = Inn(S_3)$.

One observes that autonilpotent groups are nilpotent, but the converse is not true in general.

Example 1.5. One can easily check that

$$L(Z_2) = Z_2; \quad L(Z_3) = \langle 1 \rangle; \quad L_2(Z_4) = Z_4; \quad L(Z_6) = \{e, x^3\}; \quad L_2(Z_6) = Z_6.$$

Hence the cyclic groups of orders 2 and 4 are autonilpotent and the ones of orders 3 and 6 are not, while they are nilpotent in the usual sense.

It is known that the symmetric group S_3 is not nilpotent and it is easily checked that $L(S_3) = \langle 1 \rangle$, hence it is not autonilpotent as well.

The following property for autonilpotent groups is similar to the one in the usual nilpotent groups.

Proposition 1.6. *If G is a non-trivial autonilpotent group, then*

$$L(G) \neq \langle 1 \rangle.$$

Proof. By the assumption there exists $n \geq 1$ such that $L_n(G) = G$. Now, if $L(G) = \langle 1 \rangle$ then by the definition $L_i(G)$ must be trivial for $i \geq 1$, which gives a contradiction.

2. Some properties of autonilpotent groups

In this section, it is shown that some of the known results on nilpotent groups can be carried over to autonilpotent groups.

In the view of Example 1.5 in the previous section, we show that all cyclic groups of order 2^n, $n > 1$, are autonilpotent, while it is not the case for arbitrary cyclic groups.

Remark 2.1. The cyclic group Z_p, of odd prime order p is not autonilpotent, while it is nilpotent, since $Aut(Z_p) = U_{p-1}$ is a cyclic group of order $p - 1$ and so it can not fix any element of Z_p.

It will be shown that this is held for all cyclic groups of order p^n, when $p \neq 2$ (see Remark 2.8, below). On the other hand, if p = 2 the following result shows that the cyclic group of order 2n is autonilpotent.

Theorem 2.2. *The cyclic group, Z_{2^n}, of order 2^n $(n > 1)$, is an autonilpotent group.*

Proof. Let $Z_{2^n} = \langle x \mid x^{2^n} = 1\rangle$ be the cyclic group of order 2^n $(n > 1)$. Clearly, if $r = 2t + 1$ is an odd number the map $\alpha : x \mapsto x^r$ is an automorphism. So

$$r2^{n-1} = (2t + 1)2^{n-1} \equiv 2^{n-1} \ \ (mod\ 2^n).$$

Hence $(x^{2^{n-1}})^\alpha = x^{r2^n-1} = x^{2^{n-1}}$. Now, if for each $\alpha \in Aut(Z_{2^n})$ and $s \in N$,

$$(x^s)^\alpha = x^s,$$

then $x^{rs} = x^s$ and so $x^{r(s-1)} = 1$, which implies that $2^n \mid r(s-1)$, i.e. $s = 2^{n-1}$. Thus $L(Z_{2^n}) = \{e, x^{2^{n-1}}\}$ and hence $L\left(\dfrac{Z_{2^n}}{L(Z_{2^n})}\right)$ is a cyclic group of order 2^{n-1}. Continuing in this way, we get $L_n(Z_{2^n}) = Z_{2^n}$, after n-steps, which proves the claim.

In the following lemma, we determine the autonilpotency property of abelian 2-groups, except cyclic 2-groups.

Lemma 2.3. *If $Z_{2^m} = \langle x \mid x^{2^m} = 1\rangle$ is a cyclic group of order 2^m, then*

$$L(Z_{2^m} \times Z_{2^m}) = \langle 1\rangle.$$

Proof. (i) Let $Z_{2^m} \times Z_{2^m} = \langle x\rangle \times \langle y\rangle$, in which x and y are both of orders 2^m. Clearly, the following map is an automorphism of $Z_{2^m} \times Z_{2^m}$.

$$\alpha_j : \begin{cases} x & \to & x^i y^j \\ y & \to & x^{i_1} y^{j_1} \end{cases}$$

where $0 \le i, i_1, j, j_1 < 2^m$ and are all odd, but not $i = i_1$ and $j = j_1$.

Now, if $x^r y^s \in L(Z_{2^m} \times Z_{2^m})$ then it must be fixed under all the automorphisms such as α is given above. So, $\alpha(x^r y^s) = x^r y^s$, and it follows that $x^{r_i} y^{r_j} x^{si_1} y^{sj_1} = x^r y^s$. This gives $x^{ri-r+si_1} y^{rj-s+sj_1} = 1$ or $x^{r(i-1)+si_1} y^{rj+s(j_1-1)} = 1$. Hence 2^m divides $r(i-1)+si_1$ and $rj+s(j_1-1)$. Now, if $i = i_1 = 1$ or $j = j_1 = 1$ then 2^m must divide r and s. Hence $r = s = 0$ and so $x^r y^s = e$ which implies that $L(Z_{2^m} \times Z_{2^m}) = \langle 1\rangle$ and thus the group $Z_{2^m} \times Z_{2^m}$ can not be autonilpotent.

Lemma 2.4. *If $G = Z_{2^n} \times Z_{2^m}$, where $n > m > 1$, then G cannot be autonilpotent.*

Proof. Let

$$G = Z_{2^n} \times Z_{2^m} = \langle x \mid x^{2^n} = 1\rangle \times \langle y \mid y^{2^n} = 1\rangle.$$

Clearly, if i and j are odd numbers such that $i < 2^n$ and $j < 2^m$, then $o(x^i) = 2^n$ and $o(y^j x^{2^{n-m}}) = 2^m$. Hence the following map

$$\alpha : \begin{cases} x & \to & x^i \\ y & \to & y^j x^k, \quad k = 0, \text{ or } 2^{n-m},\ j \neq 0, \end{cases}$$

is an automorphism of the group G.

Now, if the element $x^r y^s$ of G is in $L(G)$, then for each i, j and k with the above conditions, we have $\alpha(x^r y^s) = x^r y^s$ and hence $x^{ir} x^{sk} y^{sj} = x^r y^s$. So $x^{r(i-1)+sk} y^{s(j-1)} = 1$ or $y^{s(j-1)} = 1$ and $x^{r(i-1)+sk} = 1$. These follow that $2^m | s(j-1)$ and $2^n | r(i-1) + sk$. By the choice of i and j as above, we may assume $i = 2^n - 1$ and $j = 3$. Then we obtain $s = 2^m k'$ and $r = 2^n - 1$, and so $x^r y^s = x^{2^{n-1}}$. It follows that

$$L_1(Z_{2^n} \times Z_{2^m}) = \langle x^{2^{n-1}} \rangle = Z_2,$$

and, by the definition, we have

$$\frac{L_2(Z_{2^n} \times Z_{2^m})}{Z_2} = L\left(\frac{Z_{2^n} \times Z_{2^m}}{Z_2}\right) = L\left(Z_{2^{n-1}} \times Z_{2^m}\right) = \langle x^{2^{n-1}} \rangle \cong Z_2.$$

Continuing the above procedure and using Lemma 2.3, we obtain

$$L_{n-m+1}(Z_{2^n} \times Z_{2^m}) = L\left(\frac{Z_{2^n} \times Z_{2^m}}{Z_{2^{n-m}}}\right) = L(Z_{2^m} \times Z_{2^m}) = \langle 1 \rangle,$$

and hence $Z_{2^n} \times Z_{2^m}$ is not autonilpotent.

Theorem 2.5. *Let G be a finite abelian group such that there exists a natural number n with n and $n-1$ are coprime to $|G|$. Then $K(G) = G$.*

Proof. The property $(n, |G|) = 1$ implies that $G^n = G$ and so the map $\alpha : G \to G$ given by $x \to x^n$ is an automorphism. Therefore, for all $x \in G$, we have

$$[x^n, \alpha] = x^{-n}\alpha(x^n) = x^{-n}x^{n^2} = x^{(n^2-n)} = (x^n)^{n-1} = (x^{n-1})^n \in G^n$$

. This implies that $G \leq K(G) \leq G$ and hence $K(G) = G$.

Corollary 2.6. *If G is a finite abelian group of odd order, then $K(G) = G$.*

Lemma 2.7. *Let $G = Z_{p^k} \times ... \times Z_{p^k}$ be the direct product of $m (m > 1)$ copies of cyclic groups of order a prime p. Then $K(G) = G$.*

Proof. If the prime p is odd, then the result follows by the above corollary. Now, let $p = 2$ and $G = \langle a_1 \mid a_1^{2^k} = 1 \rangle \times \ldots \times \langle a_m \mid a_m^{2^k} = 1 \rangle$. Clearly, the following maps are all automorphism:

$$\alpha_j : \begin{cases} a_1 \to a_1 a_j, & j \neq 1, \\ a_i \to a_i, & i \geq 2, \end{cases}$$

and

$$\phi : \begin{cases} a_1 \to a_1, \\ a_2 \to a_1 a_2, \\ a_i \to a_i, & i \geq 3. \end{cases}$$

Then we have

$$[a_1, \alpha_j] = a_1^{-1}\alpha_j(a_1) = a_1^{-1}a_1 a_j = a_j \in K(G)$$

and

$$[a_2, \phi] = a_2^{-1}\phi(a_2) = a_2^{-1}a_1a_2 = a_1 \in K(G).$$

Hence $G \subseteq K(G)$ and so $K(G) = G$.

Clearly, Corollary 1.3 shows that the group $G = Z_{p^k} \times ... \times Z_{p^k}$ is not autonilpotent.

Remark 2.8. Note that using similar methods as in Lemmas 2.6, 2.4 and 2.3, one can show that if

$$G = \underbrace{Z_{2^n} \times \ldots \times Z_{2^n}}_{r-times} \times Z_{2^{m_1}} \times \ldots \times Z_{2^{m_k}}$$

where $n > m_1 \geq ... \geq m_k$ and $r > 1$. Then $K(G) = G$ and hence, by Corollary 1.3, the group G can not be autonilpotent.

The following example shows that there are nilpotent groups, which are not autonilpotent.

Example 2.9. Consider $D_8 = \langle a, b | a^4 = b^2 = 1, bab = a^{-1} \rangle$, the dihedral group of order 8. Clearly, the group of automorphisms of D_8 is of order 8 and one may check that

$$L(D_8) = \{e, a^2\} \cong Z_2$$

and

$$\frac{L_2(D_8)}{L(D_8)} = L\left(\frac{D_8}{L(D_8)}\right) = L(Z_2 \times Z_2) = \langle 1 \rangle.$$

Hence $L_2(D_8) = L(D_8)$, which implies that D_8 is not an autonilpotent group, while it is nilpotent.

The following result gives the basic step in proving our main goal which says that; the abelian autonilpotent groups, are the only cyclic groups of order 2^n, for $n \geq 1$.

Theorem 2.10. *If the group $G = H \times K$ is the direct product of its characteristic subgroups H and K, then for all $n \geq 1$,*

$$L_n(H \times K) = L_n(H) \times L_n(K).$$

Proof. We proceed by induction on n. If $n = 1$, then by Theorem 1.1 [1], $K_1(H \times K) = K_1(H) \times K_1(K)$ and hence $L_1(H \times K) = L_1(H) \times L_1(K)$.

Assume the result holds for $n - 1$ and consider the canonical epimorphisms

$$\pi_H : H \longrightarrow \frac{H}{L_{n-1}(H)},$$

$$\pi_K : K \longrightarrow \frac{K}{L_{n-1}(K)}.$$

Then using induction hypothesis, we may define

$$\begin{aligned}\pi &= \pi_H \times \pi_K : H \times K \longrightarrow \frac{H}{L_{n-1}(H)} \times \frac{K}{L_{n-1}(K)} \overset{\psi}{\cong} \frac{H \times K}{L_{n-1}(H) \times L_{n-1}(K)} \\ &= \frac{H \times K}{L_{n-1}(H \times K)},\end{aligned}$$

So, we obtain an epimorphism

$$\varphi = \psi o \pi : H \times K \longrightarrow \frac{H \times K}{L_{n-1}(H \times K)} .$$

Hence, we have

$$\begin{aligned}L_n(H \times K) &= \varphi^{-1}\left(L\left(\frac{H \times K}{L_{n-1}(H \times K)}\right)\right) \\ &= \pi^{-1}\psi^{-1}\left(L\left(\frac{H \times K}{L_{n-1}(H) \times L_{n-1}(K)}\right)\right) \\ &= \pi^{-1}\left(L\left(\frac{H}{L_{n-1}(H)}\right) \times L\left(\frac{K}{L_{n-1}(K)}\right)\right) \\ &= L_n(H) \times L_n(K),\end{aligned}$$

which gives the result.

The following corollaries are immediate consequences of the above theorem.

Corollary 2.11. *The above theorem holds, when the subgroups H and K are of co-prime orders.*

Corollary 2.12. *If $G = H \times K$, is the direct product of its characteristic subgroups such that H or K is not autonilpotent, then so is not G.*

Theorem 2.13. *A finite abelian group is autonilpotent if and only if it is a cyclic 2-group.*

Proof. The necessity condition follows from Theorem 2.2. Now, for the reverse conclusion, we assume that G is not a cyclic 2-group. So it is either abelian 2-group or G has a direct summand Z_{p^t}, where p is an odd prime number and $t \geq 1$. In the first case, Lemmas 2.3, 2.4 and Remark 2.8 imply that the group G is not autonilpotent. In the second case, Corollary 1.3 shows that Z_{p^t} can not be autonilpotent. Thus, Corollary 2.12 gives the result.

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SEMIGROUP COMPACTIFICATION IN AN INTUITIONISTIC FUZZY CONVERGENCE TOPOLOGICAL SPACE

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Abstract. The purpose of this paper is to discuss the problem of intuitionistic fuzzy lim semigroup compactification on intuitionistic fuzzy convergence topological semigroup and prove that the set of all intuitionistic fuzzy lim semigroup compactification of an intuitionistic fuzzy convergence topological semigroup is an upper complete semi Lattice.

Keywords: intuitionistic fuzzy convergence topological space, intuitionistic fuzzy convergence topological semigroup,intuitionistic fuzzy Bohr lim compactification, intuitionistic fuzzy lim semigroup compactification.

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1. Introduction

The concept of fuzzy set was introduced by Zadeh [15]. Since then the concept has invaded nearly all branches of Mathematics. Chang [3] introduced and developed the theory of fuzzy topological spaces and since then the various notions in classical topology have been extended to fuzzy topological spaces. Fuzzy sets have applications in many fields such as information [12] and control [11]. Atanassov [1] generalized fuzzy sets to intuitionistic fuzzy sets. Coker [5] introduced the notions of an intuitionistic fuzzy topological space, intuitionistic fuzzy continuous function

and some related concepts. K. Geetha [7] discussed F-Semigroup compactification and Lowen et al [8] defined the notion of a fuzzy convergence space. The main objective of this paper is to discuss the problem of intuitionistic fuzzy lim semigroup compactification on intuitionistic fuzzy convergence topological semigroup and prove that the set of all intuitionistic fuzzy lim semigroup compactification of an intuitionistic fuzzy convergence topological semigroup is an upper complete semi Lattice.

2. Preliminaries

Definition 2.1. [1] Let X be a non empty fixed set and I be the closed interval $[0, 1]$. An intuitionistic fuzzy set (IFS) A is an object of the following form $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$ where the function $\mu_A : X \to I$ and $\gamma_A : X \to I$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non membership (namely $\gamma_A(x)$) for each element $x \in X$ to the set A respectively and $0 \leq \mu_A(x)+\gamma_A(x) \leq 1$ for each x ∈ X. Obviously, every fuzzy set A on a nonempty set X is an IFS of the following form $A = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle : x \in X\}$. For the sake of simplicity, we shall use the symbol $A = \langle x, \mu_A(x), \gamma_A(x)\rangle$ for the intuitionistic fuzzy set $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$.

Definition 2.2. [5] An intuitionistic fuzzy topology (IFT) in Coker's sense on a non empty set X is a family τ of IFSs in X satisfying the following axioms.

(T_1) $0_\sim$,$1_\sim \in \tau$

(T_2) $G_1 \cap G_2 \in \tau$ for any $G_1, G_2 \in \tau$

(T_3) $\cup G_i \in \tau$ for arbitrary family $\{G_i/i \in I\} \subseteq \tau$

Definition 2.3. [4] Let X be a non empty set and $x \in X$ a fixed element in X. If $r \in I_0$, $s \in I_1$ are fixed real numbers such that $r + s \leq 1$, then the intuitionistic fuzzy set $x_{r,s} = \langle x, x_r, 1 - x_{1-s}\rangle$ is said to be an intuitionistic fuzzy point (IFP) in X, where r denotes the degree of membership of $x_{r,s}$ and $x \in X$ the support of $x_{r,s}$.

The IFP $x_{r,s}$ is contained in the IFS A ($x_{r,s} \in A$) if and only if $r < \mu_A(x)$ and $s > \gamma_A(x)$.

Definition 2.4. [13] Let S be a non empty set and $\circ$ be a binary operation on S. The algebraic system $(S, \circ)$ is said to be a semigroup if the operation $\circ$ is associative.

Definition 2.5. [13] Let $\langle S, *\rangle$ be a semigroup and $T \subseteq S$. If the set T is closed under the operation $*$, then $\langle T, *\rangle$ is said to be a subsemigroup of $\langle S, *\rangle$.

Definition 2.6. [13] Let $(S, \circ)$ and $(Y, *)$ be two algebraic systems of the same type in the sense that both $\circ$ and $\star$ are binary operations. A function $g : X \to Y$ is said to be a homomorphism from $(S, \circ)$ and $(Y, *)$ if for any $x_1, x_2 \in X$, $g(x_1 \circ x_2) = g(x_1) \star g(x_2)$.

Definition 2.7. [13] Let g be a homomorphism from $(X, \circ)$ to $(Y, *)$. If $g : X \to Y$ is one to one and onto then g is said to be an isomorphism.

Definition 2.8. [6] A semigroup is a non empty set S together with an associative multiplication $(x, y) \to xy$ from $S \times S$ into S. If S has a Hausdroff topology such that $(x, y) \to xy$ is continuous with the product topology on $S \times S$, then S is said to be topological semigroup.

Definition 2.9. [9] If S is a topological semigroup, then a Bohr compactification of S is a pair (β, B) where B is a compact topological semigroup, $\beta : S \to B$ is a continuous homomorphism of S into a compact topological semigroup B and if $g : S \to T$ is a continuous homomorphism of S into a compact topological semigroup T, then there exists a unique continuous homomorphism $f : B \to T$, such that $f \circ \beta = g$.

Definition 2.10. [8] Given a set X, the pair $(X, \lim)$ is said to be a fuzzy convergence space, where $\lim : \mathbb{F}(X) \to I^X$, provided:

(i) $\lim \mathfrak{F} = \inf_{\mathfrak{G} \in P_m(\mathfrak{F})} \lim \mathfrak{G}, \mathfrak{F} \in \mathbb{F}(X)$

(ii) $\lim \mathfrak{F} \leq c(\mathfrak{F}), \mathfrak{F} \in \mathbb{F}_P(X)$,

(iii) $\lim \mathfrak{G} \leq \lim \mathfrak{F}$ when $\mathfrak{F} \subseteq \mathfrak{G}, \mathfrak{F}, \mathfrak{G} \in \mathbb{F}_P(X)$,

(iv) $\lim \alpha \dot{1}_x \geq \alpha 1_x$ for $0 < \alpha \leq 1$ and $x \in X$.

3. Intuitionistic fuzzy convergence topological semigroup

Note 3.1.

(i) ζ^X is the set of all intuitionistic fuzzy sets on X.

(ii) $(\zeta^X, \subseteq \cap, \cup)$ is an intuitionistic fuzzy lattice with minimum element $0_\sim$ and maximum element $1_\sim$.

Definition 3.1. A nonempty collection $\mathfrak{F}$ of elements in the lattice $(\zeta^X, \subseteq \cap, \cup)$ is said to be an intuitionistic fuzzy filter on X if

(i) $0_\sim \notin \mathfrak{F}$.

(ii) $A, B \in \mathfrak{F}$ implies $A \cap B \in \mathfrak{F}$.

(iii) $C \in \mathfrak{F}$ when $C \supseteq A$ and $A \in \mathfrak{F}$.

Definition 3.2. An intuitionistic fuzzy filter $\mathfrak{F}$ is said to be an intuitionistic fuzzy ultra filter if there is no other intuitionistic fuzzy filter finer than $\mathfrak{F}$.

Definition 3.3. An intuitionistic fuzzy filter $\mathfrak{F}$ is said to be an intuitionistic fuzzy prime filter (or) prime intuitionistic fuzzy filter if $A \cup B \in \mathfrak{F}$, then $A \in \mathfrak{F}$ or $B \in \mathfrak{F}$.

Note 3.2. Minimal prime intuitionistic fuzzy filter is the smallest prime intuitionistic fuzzy filter.

Note 3.3.

(i) $U(X)$: Set of all intuitionistic fuzzy ultrafilters on X.

(ii) $\mathbb{IF}(X)$: Set of all intuitionistic fuzzy filters on X.

(iii) $\mathbb{IF}_p(X)$: Set of all intuitionistic fuzzy prime filters on X.

(iv) $\mathbb{IF}_m(\mathfrak{F})$: Set of all minimal prime intuitionistic fuzzy filters finer than $\mathfrak{F}$.

Definition 3.4. A base for an intuitionistic fuzzy filter is a nonempty subset $\mathfrak{B}$ of ζ^X obeying

(i) $0_\sim \notin \mathfrak{B}$,

(ii) $A, B \in \mathfrak{B}$ implies $A \cap B \supseteq C$ for some $C \in \mathfrak{B}$.

The intuitionistic fuzzy filter generated by $\mathfrak{B}$ is denoted by

$$[\mathfrak{B}] = \{B \in \zeta^X / B \supseteq A \text{ and } A \in \mathfrak{B}\}.$$

Definition 3.5. Let X be a nonempty set and $A = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle : x \in X\}$ be an intuitionistic fuzzy set on X. The characteristic set of $\mathfrak{F} \in \mathbb{IF}(X)$ is defined by

$$c(\mathfrak{F}) = \bigcup_{A \in \mathfrak{F}} A.$$

Definition 3.6. Let $x_{r,s}$ be an intuitionistic fuzzy point. An intuitionistic fuzzy characteristic function of $x_{r,s}$ is denoted by

$$1_{x_{r,s}} = \langle x, \mu_{x_{r,s}}, \gamma_{x_{r,s}}\rangle.$$

Definition 3.7. An intuitionistic fuzzy set $\alpha 1_{x_{r,s}}$ is of the form

$$\alpha 1_{x_{r,s}} = \langle y, \alpha\mu_{x_{r,s}}(y), \alpha\gamma_{x_{r,s}}(y)\rangle,$$

where $r + s \leq 1$ and $\alpha \in (0, 1)$.

Note 3.4. The collection of all intuitionistic fuzzy sets of the form $\alpha 1_{x_{r,s}}$ is denoted by $\mathfrak{E}$.

Definition 3.8. An intuitionistic fuzzy filter generated by $\mathfrak{E}$ is denoted and defined by $[\mathfrak{E}] = \{B \in \zeta^X / B \supseteq A, A \in \mathfrak{E}\}$, however $[\mathfrak{E}]$ is written as $\alpha \dot{1}_{x_{r,s}}$.

Definition 3.9. Let X be a non empty set and the ordered pair $(X, \lim)$ is said to be an intuitionistic fuzzy convergence space if the intuitionistic fuzzy function $\lim : \mathbb{IF}(X) \to \zeta^X$ satisfies the following properties.

(i) For every $\mathfrak{F} \in \mathbb{IF}(X)$, $\lim \mathfrak{F} = \bigcap_{\mathfrak{G} \in \mathbb{IF}_m(\mathfrak{F})} \lim \mathfrak{G}$.

(ii) For every $\mathfrak{F} \in \mathbb{IF}_p(X)$, $\lim \mathfrak{F} \subseteq c(\mathfrak{F})$.

(iii) If $\mathfrak{F} \subseteq \mathfrak{G}$, then $\lim \mathfrak{G} \subseteq \lim \mathfrak{F}$ for every $\mathfrak{F}, \mathfrak{G} \in \mathbb{IF}_p(X)$.

(iv) For every $\alpha \in (0,1)$ and $\alpha 1_{x_{r,s}} \in \mathfrak{E}$, $\lim(\alpha \dot{1}_{x_{r,s}}) \supseteq \alpha 1_{x_{r,s}}$.

Definition 3.10. Let $(X, \lim)$ be an intuitionistic fuzzy convergence space. Then the operator int : $\zeta^X \to \zeta^X$ is defined by $\text{int}(\lim \mathfrak{F}) = \cup\{\lim \mathfrak{G} / \lim \mathfrak{G} \subseteq \lim \mathfrak{F}$, for every $\mathfrak{G} \in U(X)$ and $\mathfrak{F} \in \mathbb{IF}(X)\}$.

Definition 3.11. Let $(X, \lim)$ be an intuitionistic fuzzy convergence space and $T_{\lim} = \{\lim \mathfrak{F} / \text{int}(\lim \mathfrak{F}) = \lim(\mathfrak{F})$, for every $\mathfrak{F} \in \mathbb{IF}(X)\}$. Then the ordered pair $(X, T_{\lim})$ is said to be an intuitionistic fuzzy convergence topological space and the members of $T_{\lim}$ are said to be an intuitionistic fuzzy lim open sets (in short, IFLOS).

The complement of an intuitionistic fuzzy lim open sets are intuitionistic fuzzy lim closed sets (in short, IFLCS).

Note 3.5. $I_0 = (0,1]$ and $I_1 = [0,1)$.

Definition 3.12. An intuitionistic fuzzy net S is a function $S : \mathfrak{D} \to \mathfrak{I}$ where $\mathfrak{D}$ is a directed set with order relation $\leq$ and $\mathfrak{I}$ the collection of all intuitionistic fuzzy points in X. Thus for each $n \in \mathfrak{D}$, S(n) is an intuitionistic fuzzy point belonging to $\mathfrak{I}$. Let its value in $I_0 \times I_1$ be denoted as an intuitionistic fuzzy pair $\langle r_n, s_n \rangle$ such that $r_n \in I_0$ and $s_n \in I_1$ with $r_n + s_n \leq 1$. Thus, the crisp net is denoted as $V(S) = \{\langle r_n, s_n \rangle / n \in \mathfrak{D}\}$ in [0,1]. If $\{r_n\}$ converges to α and $\{s_n\}$ converges to β where $\alpha \in (0,1]$ and $\beta \in [0,1)$ such that $\alpha + \beta \leq 1$ then V(S) is said to be an intuitionistic fuzzy (α, β) net.

Definition 3.13. Let $(X, T_{\lim})$ be an intuitionistic fuzzy convergence topological space, $x_{r,s}$ be an intuitionistic fuzzy point for any $r \in I_0$, $s \in I_1$ with $r + s \leq 1$ and $\lim \mathfrak{P} = \{\langle x, \mu_{\lim \mathfrak{P}}(x), \gamma_{\lim \mathfrak{P}}(x) \rangle : x \in X\}$ be an intuitionistic fuzzy lim closed set in $(X, T_{\lim})$. Then $\lim \mathfrak{P}$ is said to be an intuitionistic fuzzy remoted lim neighborhood (in short., $IFR \lim -nbd$) of $x_{r,s}$ if $x_{r,s} \notin \lim \mathfrak{P}$.

Definition 3.14. An intuitionistic fuzzy point $x_{r,s}$ of an intuitionistic fuzzy convergence topological space $(X, T_{\lim})$ is said to be an intuitionistic fuzzy cluster point of an intuitionistic fuzzy net $S : \mathfrak{D} \to \mathfrak{I}$ if for each intuitionistic fuzzy remoted lim neighbourhood $\lim \mathfrak{P}$ of $x_{r,s}$, $S(n)$ frequently does not belong to $\lim \mathfrak{P}$.

Definition 3.15. Let $(X, T_{\lim})$ be an intuitionistic fuzzy convergence topological space. Then $(X, T_{\lim})$ is said to be a T_2 intuitionistic fuzzy convergence topological space if for every pair of distinct intuitionistic fuzzy points $x_{r,s}$ and $y_{l,m}$ with distinct supports x and y there are intuitionistic fuzzy remoted lim neighborhoods $\lim \mathfrak{P} = \{\langle x, \mu_{\lim \mathfrak{P}}(x), \gamma_{\lim \mathfrak{P}}(x) \rangle : x \in X\}$ and $\lim \mathfrak{Q} = \{\langle x, \mu_{\lim \mathfrak{Q}}(x), \gamma_{\lim \mathfrak{Q}}(x) \rangle : x \in X\}$ of $x_{r,s}$ and $y_{l,m}$ respectively so that $\lim \mathfrak{P} \cup \lim \mathfrak{Q} = 1_\sim$.

4. Intuitionistic fuzzy convergence topological semigroup

Definition 4.1. Let $(X, T_{\lim})$ and $(Y, S_{\lim})$ be any two intuitionistic fuzzy convergence topological spaces and $f : (X, T_{\lim}) \to (Y, S_{\lim})$ be an intuitionistic fuzzy function. Then f is said to be an intuitionistic fuzzy lim continuous function if for every intuitionistic fuzzy lim open set $\lim \mathfrak{F}$ in $(Y, S_{\lim})$, $f^{-1}(\lim \mathfrak{F})$ is an intuitionistic fuzzy lim open set in $(X, T_{\lim})$.

Definition 4.2. Let $(X, T_{\lim})$ be an T_2 intuitionistic fuzzy convergence topological space. Then $(X, T_{\lim})$ is said to be an intuitionistic fuzzy convergence topological semigroup if for every $x, y \in X$ the intuitionistic fuzzy function $f : (x, y) \to (xy)$ of $(X, T_{\lim}) \times (X, T_{\lim})$ into $(X, T_{\lim})$ is an intuitionistic fuzzy lim continuous function.

Note 4.1. Let $\lim \mathfrak{F} = \{\langle x, \mu_{\lim \mathfrak{F}}(x), \gamma_{\lim \mathfrak{F}}(x)\rangle : x \in X\}$. Then

$$(\lim \mathfrak{F})^{-1} = \{\langle x, \mu^{-1}_{lim\mathfrak{F}}(x), \gamma^{-1}_{\lim \mathfrak{F}}(x)\rangle : x \in X\}.$$

Definition 4.3. An intuitionistic topological group is a set X which carries a group structure, an intuitionistic topology and satisfies the following axioms.

(i) The function $(x, y) \to xy$ of $X \times X$ into X is continuous.

(ii) The function $x \to x^{-1}$ of X into X is continuous.

Definition 4.4. Let (X, T) be an intuitionistic topological space. Then $f : (X, T) \to [0.1]$ is said to be a lower semi intuitionistic continuous function if $f^{-1}((\alpha, 1))$ is intuitionistic open in X for some $\alpha \in I$.

Note 4.2. $i(T_{\lim})$ is the smallest intuitionistic topology for X which makes every member of $T_{\lim}$ a lower semi intuitionistic continuous function.

Remark 4.1. An association between intuitionistic fuzzy convergence topological semigroup and topological semigroup are discussed in Propositions 4.1 and 4.2.

Proposition 4.1. *A topological semigroup X is an intuitionistic fuzzy convergence topological semigroup with an intuitionistic fuzzy convergence topology*

$$T_{\lim} = \{\lim \mathfrak{F} / \lim \mathfrak{F} \text{ is a lower semi continuous function } \}.$$

Proof. Let $f : (x, y) \to (xy)$ of $(X, T_{\lim}) \times (X, T_{\lim})$ into $(X, T_{\lim})$ be an intuitionistic fuzzy function. Let $\lim \mathfrak{F} \in T_{\lim}$. Since $\lim \mathfrak{F}$ is a lower semi intuitionistic continuous function $(\lim \mathfrak{F})^{-1}(\alpha, 1)$ is intuitionistic open in X. Since X is an intuitionistic topological semigroup, $f^{-1}((\lim \mathfrak{F})^{-1}(\alpha, 1)) = (\lim \mathfrak{F} \circ f)^{-1}(\alpha, 1)$ is intuitionistic open in $(X, T_{\lim}) \times (X, T_{\lim})$. Therefore $(\lim \mathfrak{F} \circ f)$ is a lower semi intuitionistic continuous function. Hence, $f^{-1}(\lim \mathfrak{F})$ is an intuitionistic fuzzy lim open set in $(X, T_{\lim}) \times (X, T_{\lim})$. Therefore, f is an intuitionistic fuzzy lim continuous function. Hence, $(X, T_{\lim})$ is an intuitionistic fuzzy convergence topological semigroup.

Proposition 4.2. *If $(X, T_{\lim})$ is an intuitionistic fuzzy convergence topological semigroup, then $(X, i(T_{\lim}))$ is an intuitionistic topological semigroup.*

Proof. Let $(X, T_{\lim})$ is an intuitionistic fuzzy convergence topological semigroup and $f : (x, y) \to (xy)$ of $(X, i(T_{\lim})) \times (X, i(T_{\lim}))$ into $(X, i(T_{\lim}))$. Suppose that $W \in i(T_{\lim})$ be a subbasic intuitionistic open set in X. Then, there exists $\lim \mathfrak{g} \in T_{\lim}$ such that $W = (\lim \mathfrak{g})^{-1}(\alpha, 1)$ for $\alpha \in I$. Since $(X, T_{\lim})$ is an intuitionistic fuzzy convergence topological semigroup, $\lim \mathfrak{g} \in T_{\lim}$ implies $f^{-1}(\lim \mathfrak{g}) \in T_{\lim} \times T_{\lim}$. That is, $\lim \mathfrak{g} \circ f \in T_{\lim} \times T_{\lim}$. That is, $f^{-1}((\lim \mathfrak{g})^{-1}(\alpha, 1)) \in i(T_{\lim} \times T_{\lim})$. Hence, $f^{-1}(W) \in i(T_{\lim} \times T_{\lim})$. Hence, f is intuitionistic continuous. Therefore, $(X, i(T_{\lim}))$ is a topological semigroup.

Definition 4.5. Let $(X, T_{\lim})$ be an intuitionistic fuzzy convergence topological semigroup. Then $(X, T_{\lim})$ is said to be topologically generated provided that the intuitionistic fuzzy convergence topology $T_{\lim}$ is the set of all lower semi continuous function from $(X, i(T_{\lim}))$ into I.

Definition 4.6. Let $(X, T_{\lim})$ be an intuitionistic fuzzy convergence topological space. If $S \subset X$, then the characteristic function χ_S is defined as

$$\chi_S(x) = \begin{cases} 1_\sim & x \in S \\ 0_\sim & x \notin S \end{cases}$$

for all $x \in X$.

Note 4.3. Let X be a non empty set and $A \subseteq X$. Then an intuitionistic fuzzy set χ_A^* is of the form $\langle x, \chi_A, 1 - \chi_A \rangle$.

Definition 4.7. Let $(X, T_{\lim})$ be an intuitionistic fuzzy convergence topological space. Then the intuitionistic fuzzy lim closure * of A is denoted and defined as $IFcl^*_{\lim}(A) = \cap\{\lim \mathfrak{G} / \lim \mathfrak{G} \supseteq A, \lim \mathfrak{G}$ is an intuitionistic fuzzy lim closed set$\}$.

Definition 4.8. Let $(X, T_{\lim})$ be an intuitionistic fuzzy convergence topological space and A be an intuitionistic fuzzy set in $(X, T_{\lim})$. Then A is said to be an intuitionistic fuzzy lim dense in $(X, T_{\lim})$ if $IFcl^*_{\lim}(A) = \chi_X^*$.

Definition 4.9. Let $(X, T_{\lim})$ be an intuitionistic fuzzy convergence topological space and Y be any subset of X. Then $T_{\lim}/Y = \{\lim \mathfrak{A}/Y / \lim \mathfrak{A} \in T_{\lim}\}$ is an intuitionistic fuzzy convergence topology on Y and is said to be an induced or relative intuitionistic fuzzy convergence topology. The ordered pair $(Y, T_{\lim}/Y)$ is said to be an intuitionistic fuzzy convergence subspace of $(X, T_{\lim})$.

Definition 4.10. Let $(X, T_{\lim})$ be an intuitionistic fuzzy convergence topological space and $(Y, S_{\lim})$ be an intuitionistic fuzzy convergence subspace of $(X, T_{\lim})$. Then $(Y, S_{\lim})$ is intuitionistic fuzzy lim dense in $(X, T_{\lim})$ if χ_Y^* is an intuitionistic fuzzy lim dense in $(X, T_{\lim})$.

5. Intuitionistic fuzzy Bohr lim compactification

Definition 5.1. Let $(X, T_{\lim})$ and $(Y, S_{\lim})$ be any two intuitionistic fuzzy convergence topological spaces and $f : (X, T_{\lim}) \to (Y, S_{\lim})$ be an intuitionistic fuzzy function. Then f is said to be an intuitionistic fuzzy lim open if for every intuitionistic fuzzy lim open set $\lim \mathfrak{F}$ in $(X, T_{\lim})$, $f(\lim \mathfrak{F})$ is an intuitionistic fuzzy lim open set in $(Y, S_{\lim})$.

Definition 5.2. Let $(X, T_{\lim})$ and $(Y, S_{\lim})$ be any two intuitionistic fuzzy convergence topological spaces and $f : (X, T_{\lim}) \to (Y, S_{\lim})$ be an intuitionistic fuzzy function. Then f is said to be an intuitionistic fuzzy lim homeomorphism if f is an intuitionistic fuzzy bijective function. That is, f is both an intuitionistic fuzzy lim continuous function and an intuitionistic fuzzy lim open function.

Definition 5.3. Let $(X, T_{\lim})$ and (Y, S_{lim}) be any two intuitionistic fuzzy convergence topological semi groups and $\theta : (X, T_{\lim}) \to (Y, S_{\lim})$ be an intuitionistic fuzzy function. Then θ is said to be an intuitionistic fuzzy lim morphism if

(i) θ is an algebraic homomorphism of X into Y.

(ii) θ is an intuitionistic fuzzy lim continuous function of $(X, T_{\lim})$ into $(Y, S_{\lim})$.

A bijective function θ of $(X, T_{\lim})$ into $(Y, S_{\lim})$ is an intuitionistic fuzzy lim isomorphism if

(i) θ is an algebraic isomorphism of X into Y.

(ii) θ is an intuitionistic fuzzy lim homeomorphism of $(X, T_{\lim})$ into $(Y, S_{\lim})$.

Definition 5.4. Let $(X, T_{\lim})$ be an intuitionistic fuzzy convergence topological space and $\lim \mathfrak{F}$ be an intuitionistic fuzzy set in $(X, T_{\lim})$. Then $\lim \mathfrak{F}$ is said to be an intuitionitic fuzzy N-lim compact if each intuitionistic fuzzy (α, β) net contained in $\lim \mathfrak{F}$ has at least a cluster point $x_{\alpha,\beta}$ with value $\langle \alpha, \beta \rangle$ where $0 < \alpha \leq 1$, $0 \leq \beta < 1$ and $\alpha + \beta \leq 1$.

Definition 5.5. Let $(X, T_{\lim})$ be an intuitionistic fuzzy convergence topological space. If $(X, T_{\lim})$ is topologically generated, then $(X, T_{\lim})$ is said to be an intuitionistic fuzzy N-lim compact.

Definition 5.6. Let $(X, T_{\lim})$ be an intuitionistic fuzzy convergence topological semigroup. The intuitionistic fuzzy Bohr lim compactification of $(X, T_{\lim})$ is a pair (α, Y) such that Y is an intuitionistic fuzzy N-lim compact convergence topological semigroup and $\alpha : X \to Y$ is an intuitionistic fuzzy lim morphism provided whenever $\beta : X \to Z$ is an intuitionistic fuzzy lim morphism of X into an intuitionistic fuzzy N-lim compact convergence topological semigroup Z, then there exists a unique intuitionistic fuzzy lim morphism $f : Y \to Z$ such that $f \circ \alpha = \beta$.

Proposition 5.1. *Let $(X, T_{\lim})$ be a topologically generated intuitionistic fuzzy convergence topological semigroup. Then $(X, i(T_{\lim}))$ is an intuitionistic topological semigroup.*

Proof. The proof is simple by Proposition 4.2.

Definition 5.7. Let $(X, T_{\lim})$ and $(Y, S_{\lim})$ be any two intuitionistic fuzzy convergence topological spaces and $\theta : (X, T_{\lim}) \to (Y, S_{\lim})$ be an intuitionistic fuzzy function. Then $(X, T_{\lim})$ is *dense in $(Y, S_{\lim})$ if θ is an intuitionistic fuzzy lim morphism.

Proposition 5.2. *Let (β, B) be the Bohr compactification of $(X, i(T_{\lim}))$ where B is a N-compact topological semigroup and β a continuous homomorphism of X into B. Let $T_{\lim_B} = \{g | g$ is a lower semi continuous function from B into I such that $g|X \in T_{\lim}\}$. Then $(B, T_{\lim_B})$ is an intuitionistic fuzzy N-lim compact convergence topological semigroup and $(X, T_{\lim})$ is *dense in $(B, T_{\lim_B})$.*

Proof. Let (β, B) be the Bohr compactification of $(X, i(T_{\lim}))$ where B is a compact topological semigroup and β a continuous homomorphism of X into B. Define an intuitionistic fuzzy convergence topology as in the hypothesis. Then, by the Proposition 4.2, and by Definition 5.2, $(B, T_{\lim_B})$ is an intuitionistic fuzzy N-lim compact intuitionistic fuzzy covergence topological semigroup.

Let $g \in T_{lim_B}$. Then there exists $V \in i(T_{\lim_B})$ such that $g^{-1}(\alpha, 1) = V$ for $\alpha \in I$. Since β is a continuous function, $\beta^{-1}(V)$ is open in X. Therefore, $\beta^{-1}(g^{-1}(\alpha, 1))$ is an open set in X. That is, $g \circ \beta$ is a lower semi continuous function. Thus, $\beta^{-1}(g) \in T_{\lim}$. Therefore, β is an intuitionistic fuzzy lim continuous function. Hence, β is an intuitionistic fuzzy lim morphism of $(X, T_{\lim})$ into $(B, T_{\lim_B})$. Hence, $(X, T_{\lim})$ is *dense in $(B, T_{\lim_B})$.

Proposition 5.3. Let $(X, T_{\lim})$ be a topologically generated intuitionistic fuzzy convergence topological semigroup and let (f, Y), (g, Z) be any two intuitionistic fuzzy Bohr lim compactifications of X. Then there exists an intuitionistic fuzzy lim isomorphism $h : Y \to Z$ such that $h \circ f = g$.

Proof. Since (f, Y) is an intuitionistic fuzzy Bohr lim compactification of X, there exists an intuitionistic fuzzy lim morphism $h : (Y, S) \to (Z, R)$ such that $h \circ f = g$. Similarly, there exists a intuitionistic fuzzy lim morphism $\pi : Z \to Y$ such that $\pi \circ g = f$. It is clear that $\pi \circ h = I_Y$ and $h \circ \pi = I_Z$. That is, $\pi \circ h$ and $h \circ \pi$ are identity intuitionistic fuzzy lim morphism of Y and Z respectively. Therefore, h is an intuitionistic fuzzy lim homeomorphism. Hence, h is an intuitionistic fuzzy lim isomorphism of Y onto Z.

6. Intuitionistic fuzzy lim semigroup compactification

Definition 6.1. Let $(X, T_{\lim})$ and $(Y, S_{\lim})$ be any two intuitionistic fuzzy convergence topological spaces and $\theta : (X, T_{\lim}) \to (Y, S_{\lim})$ be an Intuitionistic fuzzy lim morphism. Then θ is said to be an intuitionistic fuzzy dense lim morphism if it has a intuitionistic fuzzy lim dense image set.

Definition 6.2. Let X be an intuitionistic fuzzy convergence topological semigroup. Then intuitionistic fuzzy lim semigroup compactification of $(X, T_{\lim})$ is an

ordered pair (α, A) where A is a intuitionistic fuzzy N-lim compact intuitionistic fuzzy convergence topological semigroup and $\alpha : X \to A$ is an intuitionistic fuzzy dense lim morphism.

Definition 6.3. Let (α, A) and (β, B) be any two intuitionistic fuzzy lim semigroup compactifications of an intuitionistic fuzzy convergence topological semigroup X. Then (α, A) and (β, B) are said to be equivalent if there exists an intuitionistic fuzzy lim isomorphism $\nu : A \to B$ such that $\nu \circ \alpha = \beta$.

Definition 6.4. Let (α, A) and (β, B) be any two intuitionistic fuzzy lim semigroup compactifications of an intuitionistic fuzzy convergence topological semigroup X. Then $(\alpha, A) \supseteq (\beta, B)$ if there exists an onto intuitionistic fuzzy lim morphism $\nu : A \to B$ such that $\nu \circ \alpha = \beta$.

Definition 6.5. Let (α, A) and (β, B) be any two intuitionistic fuzzy lim semigroup compactifications of an intuitionistic fuzzy convergence topological semigroup X. Then (α, A) and (β, B) are said to be equivalent if $(\alpha, A) \supseteq (\beta, B)$ and $(\beta, B) \supseteq (\alpha, A)$.

Definition 6.6. Let $(X_i, T_{\lim_i})$ be an intuitionistic fuzzy convergence topological space for each $i \in I$. Let X be the cartesian product of $\{X_i / i \in I\}$ and P_i be the intuitionistic fuzzy projection of the product X into the i th co-ordinate set X_i. Let $\beta = \bigcap_{i \in I_1} \{P_i^{-1}(\lim \mathfrak{F}_i) / \lim \mathfrak{F}_i \in T_{\lim_i}$ where$\mathfrak{F}_i \in \mathbb{IF}_i \ i \in I_1$, where I_1 is any finite subset of I.

Then β is a basis for an intuitionistic fuzzy convergence topology $T_{\lim}$ on X said to be the product intuitionistic fuzzy convergence topology and the ordered pair $(X, T_{\lim})$ is said to be the product space of an intuitionistic fuzzy convergence topological space $(X_i, T_{\lim_i})$ for $i \in I$.

Proposition 6.1. *Let $(X, T_{\lim})$ be the product space of the intuitionistic fuzzy convergence topological semigroups $(X_i, T_{\lim_i})$ for $i \in I$ and P_i be the intuitionistic fuzzy projection of the product X into the i-th co-ordinate set X_i. Then the following are valid.*

(i) *For each $i \in I$, the intuitionistic fuzzy projection P_i is an intuitionistic fuzzy* lim *continuous function.*

(ii) *The intuitionistic fuzzy function $f : (Y, S_{\lim}) \to (X, T_{\lim})$ is an intuitionistic fuzzy lim continuous function if and only if for every $i \in I$, $P_i \circ f$ is intuitionistic fuzzy* lim *continuous function where $(Y, S_{\lim})$ is an intuitionistic fuzzy convergence topological semigroup.*

Proof. (i) By Definition 6.6, P_i is an intuitionistic fuzzy lim continuous function.

(ii) Let $f : (Y, S_{lim}) \to (X, T_{\lim})$ be an intuitionistic fuzzy lim continuous function. Since each P_i is an intuitionistic fuzzy lim continuous function, the composition $P_i \circ f$ is an intuitionistic fuzzy lim continuous function.

Conversely suppose that $P_i \circ f$ is an intuitionistic fuzzy lim continuous function for every $i \in I$. Then

$$(P_i \circ f)^{-1}(\lim \mathfrak{F}_i) = f^{-1}(P_i^{-1}(\lim \mathfrak{F}_i)) \in S_{\lim} \text{ for every } i \in I.$$

Put $\lim \mathfrak{G} = (P_i^{-1}(\lim \mathfrak{F}_i))$. Now, $f^{-1}(\lim \mathfrak{G}) \in S_{\lim}$. Hence, f is an intuitionistic fuzzy lim continuous function.

Proposition 6.2. *Let $\{X_i/i \in I\}$ be a collection of intuitionistic fuzzy convergence topological semigroups and $X = \prod_{i\in I} X_i$. Then X with co-ordinate wise multiplication is an intuitionistic fuzzy convergence topological semigroup and each intuitionistic fuzzy projection $P_i : X \to X_i$ is an intuitionistic fuzzy lim morphism.*

Proof. Let X be a semigroup under co-ordinate wise multiplication, where the associativity of multiplication follows from that of the semigroups in the collection $\{X_i/i \in I\}$. Since each X_i is an intuitionistic fuzzy convergence topological semigroup for each $x_{r,s}, y_{r,s} \in X_i$, the intuitionistic fuzzy function $g_i : (x_{r,s}, y_{r,s}) \to (x_{r,s}y_{r,s})$ is an intuitionistic fuzzy lim continuous function.

Let $g : (x, y) \to (xy)$ of $(X, T_{\lim}) \times (X, T_{\lim})$ into $(X, T_{\lim})$ be an intuitionistic fuzzy function. Let $x_{r,s}, y_{r,s} \in X$ be any two intuitionistic fuzzy points in X. Then, $g(x_{r,s}, y_{r,s}) = (x_{r,s}y_{r,s}) = g_i(x_{r,s}, y_{r,s})$ and, therefore, $P_i \circ g = g_i$. Since each g_i is an intuitionistic fuzzy lim continuous function, g is also an intuitionistic fuzzy lim continuous function. Hence, $\prod_{i\in I} X_i$ is an intuitionistic fuzzy convergence topological semigroup. Hence, from the definition of multiplication on X each P_i is an intuitionistic fuzzy lim morphism.

Proposition 6.3. *Let $(X, T_{\lim})$ be an intuitionistic fuzzy convergence topological semigroup and A be a subsemigroup of X. Then the subspace $(A, T_{\lim}/A)$ is also an intuitionistic fuzzy convergence topological semigroup.*

Proof. Let $T_{\lim}/A = \{\lim \mathfrak{F}/A = \lim \mathfrak{F} \cap \chi_A^* \text{ where } \lim \mathfrak{F} \in T_{\lim}\}$, $x, y \in A$ and $f : (x, y) \to (xy)$ of $(A, T_{\lim}/A) \times (A, T_{\lim}/A)$ into $(A, T_{\lim}/A)$ be an injective intuitionistic fuzzy function. Consider $\lim \mathfrak{G} \in T_{\lim}/A$. That is, there exists $\lim \mathfrak{F} \in T_{\lim}$ such that $lim\mathfrak{F}/A \in T_{\lim}/A$. Since $(X, T_{\lim})$ is an intuitionistic fuzzy convergence topological semigroup, $f : (X, T_{\lim}) \times (X, T_{\lim}) \to (X, T_{\lim})$ is an intuitionistic fuzzy lim continuous function. Hence, $f^{-1}(\lim \mathfrak{F})$ is an intuitionistic fuzzy lim open in $T_{\lim} \times T_{\lim}$. Now, $\lim \mathfrak{G} = \lim \mathfrak{F} \cap \chi_A \in T_{\lim}/A$. Then $f^{-1}(\lim \mathfrak{G}) = f^{-1}(\lim \mathfrak{F} \cap \chi_A^*) \in T_{\lim}/A \times T_{\lim}/A$, which is an intuitionistic fuzzy lim open set. Hence, f/A is an intuitionistic fuzzy lim continuous function. Hence, $(A, T_{\lim}/A)$ is an intuitionistic fuzzy convergence topological semigroup.

Note 6.1. Let $(X, T_{\lim})$ be an intuitionistic fuzzy convergence topological semigroup and Y be a subsemigroup of X such that $Y = \overline{Y} = X - Y$. Then

(i) Y is said to be a closed subsemigroup of X.

(ii) the intuitionistic fuzzy subspace $(Y, T_{\lim}/Y)$ is said to be a closed intuitionistic fuzzy convergence topological semigroup.

Definition 6.7. Let $(X, T_{\lim})$ be an intuitionistic fuzzy convergence topological semigroup. If $(X, T_{\lim})$ is an intuitionistic fuzzy $N-$lim compact convergence topological semigroup, then each closed intuitionistic fuzzy convergence topological semigroup is an intuitionistic fuzzy $N-$lim compact convergence topological semigroup.

Definition 6.8. Let $(X_i, T_{\lim_i})$ be intuitionistic fuzzy convergence topological spaces for every $i \in I$ and X be the cartesian product of $X_i/i \in J$. If each X_i is an intuitionistic fuzzy $N-$lim compact convergence topological semigroup, then X is also an intuitionistic fuzzy $N-$lim compact convergence topological semigroup.

Notation 6.1. $T_{\lim}(X)$ denotes the set of all equivalence classes of intuitionistic fuzzy lim semigroup compactification of $(X, T_{\lim})$ where $(X, T_{\lim})$ is an intuitionistic fuzzy convergence topological semigroup.

Definition 6.9. Let $(\zeta^X, \subseteq \cap, \cup)$ be an intuitionistic fuzzy lattice. Then $(\zeta^X, \subseteq \cap, \cup)$ is said to be intuitionistic fuzzy complete if each of its non empty subsets has a least upper bound and a greatest lower bound.

Definition 6.10. Intuitionistic fuzzy upper semi lattice $(\zeta^X, \subseteq \cap, \cup)$ is a partially ordered set that has a least upper bound for any non empty finite subset.

Proposition 6.4. *The partially ordered set $(T(X), \subseteq)$ is an upper complete semi lattice.*

Proof. Let $\{\alpha_i X/i \in I\}$ be a subset of $T_{\lim}$(X). Consider the product $\prod_{i\in I}\{\alpha_i X\}$. Since each $\alpha_i X$ is an intuitionistic fuzzy N-lim compact convergence topological semigroup, by Definition 6.8 and Proposition 6.2, $\prod_{i\in I}\{\alpha_i X\}$ is an intuitionistic fuzzy N-lim compact convergence topological semigroup. Define a function $\theta : X \to \prod_{i\in I} \alpha_i X$ by $\theta((x))_i = \alpha_i(x)$. Also define the intuitionistic fuzzy functions $\pi_i : \prod_{i\in I} \alpha_i X \to \alpha_i X$ and $\alpha_i : X \to \alpha_i X$. Clearly $\pi_i \circ \theta = \alpha_i$. Since each α_i is an intuitionistic fuzzy lim continuous function, for every i, $\theta((xy))_i = \alpha_i(xy) = \alpha_i(x)\alpha_i(y) = (\theta(x))_i(\theta(y))_i$. That is, θ is algebraically a homomorphism. Hence, θ is an intuitionistic fuzzy lim morphism of X into $\prod_{i\in I} \alpha_i X$.

Let θX denote $\overline{\theta(X)}$. Then, by Note 6.1, Proposition 6.3, and by Definition 6.7, θX is an intuitionistic fuzzy N-lim compact convergence topological semigroup. Also, $\theta : X \to \theta X$ is a intuitionistic fuzzy dense lim morphism. Therefore, $(\theta X, \theta)$ is an intuitionistic fuzzy lim semigroup compactification of X.

For each $i \in I$, let $f_i : \theta X \to \alpha_i X$ be the restriction of the projection to θX. Consider $(f_i \circ \theta)x = f_i \circ \theta(x) = (\theta(x))_i = \alpha_i(x)$. That is, $f_i \circ \theta = \alpha_i$. Therefore, $(\theta X, \theta) \supseteq (\alpha_i X, \alpha_i)$ for every $i \in I$.

Suppose that $(\theta_0 X, \theta_0) \supseteq (\alpha_i X, \alpha_i)$ for every i. Now, $g_i : \theta_0 X \to \alpha_i X$ such that $g_i \circ \theta_0 = \alpha_i$. Define $f : \theta_0 X \to \prod_{i\in I} \alpha_i X$ by $(f(y))_i = g_i(y)$. Then $\pi_i \circ f = g_i$ where π_i and g_i are IF lim morphisms. Therefore, f is also an intuitionistic fuzzy lim morphism.

Consider $f(\theta_0(x))_i = g_i \circ \theta_0(x) = \alpha_i(x) = (\theta(x))_i$. Therefore, $f \circ \theta_0 = \theta$. Now, $f(\theta_0 x) = f(\theta_0(x)) = \theta X$. Then $\overline{f(\theta_0 X)} = \theta X$. Therefore, f is a onto intuitionistic fuzzy dense lim morphism and $f \circ \theta_0 = \theta$. Hence, $(\theta X, \theta) \subseteq (\theta_0 X, \theta_0)$ and thus θX is the least upper bound of $\{\alpha_i, X\}$ for each $i \in I$. Hence, the partially ordered set $(T_{\lim}(X), \subseteq)$ is an upper complete semi lattice.

Remark 6.1. From Propositions 6.4., $(\theta, \theta X)$ is an intuitionistic fuzzy lim semigroup compactification of an intuitionistic fuzzy convergence topological semigroup $(X, T_{\lim})$.

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THE NONLOCAL BOUNDARY VALUE PROBLEMS FOR STRONGLY SINGULAR HIGHER-ORDER NONLINEAR FUNCTIONAL-DIFFERENTIAL EQUATIONS

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Abstract. A priori boundedness principle is proven for the nonlocal boundary value problems for strongly singular higher-order nonlinear functional-differential equations. Several sufficient conditions of solvability of the Dirichlet problem under consideration are derived from the a priori boundedness principle. The proof of the a priori boundedness principle is based on the Agarwal–Kiguradze type theorems, which guarantee the existence of the Fredholm property for strongly singular higher-order linear differential equations with argument deviations under the nonlocal boundary conditions.

Key words and phrases: Higher order functional-differential equations, Dirichlet boundary value problem, strong singularity, Fredholm property, a priori boundedness principle.

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1. Statement of the main results

1.1. Statement of the problem and a survey of the literature. Consider the functional differential equation

$$u^{(2m+1)}(t) = F(u)(t) \tag{1.1}$$

with the boundary conditions

$$\int_a^b u(s)d\varphi(s) = 0 \tag{1.2}$$

where $\varphi(b) - \varphi(a) \neq 0,\ u^{(i)}(a) = 0,\ u^{(i)}(b) = 0\ (i = 1, ..., m)$.

Here, $-\infty < a < b < +\infty$, $\varphi : [a, b] \to R$ is a function of bounded variation, and the operator F acting from the set of $(m-1)$-th time continuously differentiable on $]a, b[$ functions, to the set $L_{loc}(]a, b[)$. By $u^{(i)}(a)$ $(u^{(i)}(b))$, we denote the right (the left) limit of the function $u^{(i)}$ at the point $a\,(b)$.

The problem is singular in the sense that for an arbitrary x the right-hand side of equation (1.1) may have nonintegrable singularities at the points a and b.

Throughout the paper we use the following notations:

$R^+ = [0, +\infty[$;

$[x]_+$ the positive part of number x, that is $[x]_+ = \frac{x+|x|}{2}$;

$L_{loc}(]a, b[)$ $(L_{loc}(]a, b]))$ is the space of functions $y :]a,\ b[\to R$, which are integrable on $[a + \varepsilon,\ b - \varepsilon]$ for arbitrary small $\varepsilon > 0$;

$L_{\alpha,\beta}(]a, b[)$ $(L^2_{\alpha,\beta}(]a, b[))$ is the space of integrable (square integrable) with the weight $(t-a)^\alpha(b-t)^\beta$ functions $y :]a, b[\to R$, with the norm

$$||y||_{L_{\alpha,\beta}} = \int_a^b (s-a)^\alpha (b-s)^\beta |y(s)|ds \quad \Big(||y||_{L^2_{\alpha,\beta}} = \Big(\int_a^b (s-a)^\alpha (b-s)^\beta y^2(s)ds \Big)^{1/2}\Big);$$

$L([a, b]) = L_{0,0}(]a, b[),\ L^2([a, b]) = L^2_{0,0}(]a, b[)$;

$M(]a, b[)$ is the set of the measurable functions $\tau :]a, b[\to]a, b[$;

$\widetilde{L}^2_{\alpha,\beta}(]a, b[)$ $(\widetilde{L}^2_{\alpha}(]a, b]))$ is the Banach space of $y \in L_{loc}(]a, b[)$ $(L_{loc}(]a, b]))$ functions, with the norm

$$||y||_{\widetilde{L}^2_{\alpha,\beta}} \equiv \max\Big\{ \Big[\int_a^t (s-a)^\alpha \Big(\int_s^t y(\xi)d\xi \Big)^2 ds \Big]^{1/2} : a \le t \le \frac{a+b}{2} \Big\}$$
$$+ \max\Big\{ \Big[\int_t^b (b-s)^\beta \Big(\int_t^s y(\xi)d\xi \Big)^2 ds \Big]^{1/2} : \frac{a+b}{2} \le t \le b \Big\} < +\infty.$$

$L_m(]a, b[)$ is the Banach space of $y \in L_{loc}(]a, b[)$ functions, with the norm

$$||y||_{L_m} = \sup\Big\{ [(s-a)(b-t)]^{m-1/2} \int_s^t |y(\xi)|d\xi : \quad a < s \le t < b \Big\} < +\infty.$$

$C^n_{loc}(]a, b[)$, $(\widetilde{C}^{n-1}_{loc}(]a, b[))$ is the space of the functions $y :]a, b[\to R$, which are continuous (absolutely continuous) together with $y', y'', ..., y^{(n-1)}$ on $[a + \varepsilon,\ b - \varepsilon]$ for arbitrarily small $\varepsilon > 0$.

$\widetilde{C}^{n,m}(]a, b[)$ $(m \le n)$ is the space of the functions $y \in \widetilde{C}^n_{loc}(]a, b[)$, such that

$$\int_a^b |x^{(m)}(s)|^2 ds < +\infty. \tag{1.3}$$

$C_2^m(]a,b[)$ is the Banach space of the functions $y \in C_{loc}^m(]a,b[)$, such that

$$\begin{aligned} &\limsup_{t\to a} \frac{|x^{(i)}(t)|}{(t-a)^{m-i+1/2}} < +\infty \ (i=1,...,m), \\ &\limsup_{t\to b} \frac{|x^{(i)}(t)|}{(b-t)^{m-i+1/2}} < +\infty \ (i=1,...,m), \end{aligned} \tag{1.4}$$

with the norm:

$$||x||_{C_2^m} = ||x||_C + \sum_{i=1}^{m} \sup\Big\{\frac{|x^{(i)}(t)|}{\alpha_i(t)} : a<t<b\Big\},$$

where $\alpha_i(t) = (t-a)^{m-i+1/2}(b-t)^{m-i+1/2}$.

$\widetilde{C}_2^m(]a,b[)$ is the Banach space of the functions $y \in \widetilde{C}_{loc}^m(]a,b[)$, such that conditions $\Big(\int\limits_a^b |x^{(m+1)}(s)|^2 ds\Big)^{1/2} < +\infty$ and (1.4) hold, with the norm:

$$||x||_{\widetilde{C}_2^m} = ||x||_{C_1^m} + \Big(\int\limits_a^b |x^{(m+1)}(s)|^2 ds\Big)^{1/2}.$$

$D_n(]a,b[\times R^+)$ is the set of such functions $\delta :]a,b[\times R^+ \to L_n(]a,b[)$ that $\delta(t,\cdot) : R^+ \to R^+$ is nondecreasing for every $t \in]a,b[$, and $\delta(\cdot,\rho) \in L_n(]a,b[)$ for any $\rho \in R^+$.

$D_{2m-2,2m-2}(]a,b[\times R^+)$ is the set of such functions $\delta :]a,b[\times R^+ \to \widetilde{L}^2_{2m-2,2m-2}(]a,b[)$ that $\delta(t,\cdot) : R^+ \to R^+$ is nondecreasing for every $t \in]a,b[$, and $\delta(\cdot,\rho) \in \widetilde{L}^2_{2m-2,2m-2}(]a,b[)$ for any $\rho \in R^+$.

A solution of problem (1.1), (1.2) is sought in the space $\widetilde{C}^{2m,m+1}(]a,b[)$.

The singular ordinary differential and functional-differential equations, have been studied with sufficient completeness under different boundary conditions, see for example [1], [2], [4] – [14], [17], [25]-[29] and the references cited therein. But the equation (1.1), under the boundary condition (1.2), is not studied even in the case when equation (1.1) has the form

$$x^{(2m+1)}(t) = \sum_{j=0}^{m} p_j(t) x^{(j)}(\tau_j(t)) + f(x)(t), \tag{1.5}$$

where the singularity of the functions $p_j : L_{loc}([a,\, b])$ be such that the inequalities

$$\begin{aligned} &\int\limits_a^b (s-a)^{2m-1}(b-s)^{2m-1}[(-1)^{n-m}p_1(s)]_+ ds < +\infty, \\ &\int\limits_a^b (s-a)^{2m-j}(b-s)^{2m-j}|p_j(s)|ds < +\infty \quad (j=2,...,m), \end{aligned} \tag{1.6}$$

are not fulfilled (in this case we sad that the linear part of the operator F is a strongly singular), the operator f continuously acting from $C_2^m(]a,\,b[)$ to $L_{\widetilde{L}^2_{2m-2,2m-2}}(]a,b[)$, and the inclusion

$$\sup\{f(x)(t) : ||x||_{C_2^m} \leq \rho\} \in \widetilde{L}^2_{2m-2,2m-2}(]a,b[). \tag{1.7}$$

holds. The first step in studying of the differential equations with strong singularities was made by R.P. Agarwal and I. Kiguradze in the article [3], where the linear ordinary differential equations under conditions (1.2), in the case when the functions p_j have strong singularities at the points a and b, are studied. Also the ordinary differential equations with strong singularities under two-point boundary conditions are studied in the articles of I. Kiguradze [15], [16], and N. Partsvania [24]. In the papers [20], [21] these results are generalized for linear differential equation with deviating arguments i.e., are proven the Agarwal-Kiguradze type theorems, which guarantee Fredholm's property for linear differential equation with deviating arguments.

In this paper, on the basis of articles [3] and [19], we prove a priori boundedness principle for the problem (1.1), (1.2) in the case where equation (1.1) is in form (1.5).

Now, we introduce some results from articles [20], [21], which we need for this work. Consider the equation

$$u^{(2m+1)}(t) = \sum_{j=1}^{m} p_j(t)u^{(j-1)}(\tau_j(t)) + q(t) \qquad \text{for} \qquad a < t < b. \tag{1.8}$$

By $h_j :]a,b[\times]a,b[\to R_+$ and $f_j : [a,b]\times M(]a,b[) \to C_{loc}(]a,b[\times]a,b[)$ $(j = 1, ..., m)$ we denote the functions and operator, respectively defined by the equalities

$$\begin{aligned} h_1(t,s) &= \Big|\int_s^t (\xi - a)^{n-2m}[(-1)^{n-m}p_1(\xi)]_+ d\xi\Big|, \\ h_j(t,s) &= \Big|\int_s^t (\xi - a)^{n-2m}p_j(\xi)d\xi\Big| \quad (j = 2, ..., m), \end{aligned} \tag{1.9}$$

and

$$f_j(c,\tau_j)(t,s) = \Big|\int_s^t (\xi - a)^{n-2m}|p_j(\xi)|\Big|\int_\xi^{\tau_j(\xi)} (\xi_1 - c)^{2(m-j)}d\xi_1\Big|^{1/2} d\xi\Big|, \tag{1.10}$$

and also we put that

$$f_0(t,s) = \Big|\int_s^t |p_0(\xi)|d\xi\Big|.$$

Let $k = 2k_1 + 1$ ($k_1 \in N$), then

$$k!! = \begin{cases} 1 & \text{for } k \leq 0, \\ 1 \cdot 3 \cdot 5 \cdots k & \text{for } k \geq 1. \end{cases}$$

Now, we can introduce the main theorem of the paper [20].

Theorem 1.1. *Let there exist numbers $t^* \in]a, b[$, $l_{k0} > 0$, $l_{kj} > 0$, $\bar{l}_{kj} \geq 0$, and $\gamma_{kj} > 0$ ($k = 0, 1$; $j = 1, ..., m$) such that along with*

$$\begin{aligned} B_0 \equiv & \bar{l}_{00}\Big(\frac{2^{m-1}}{(2m-3)!!}\Big)^2 \frac{(b-a)^{m-1/2}}{(2m-1)^{1/2}} \frac{(t^*-a)^{\gamma_{00}}}{\sqrt{2\gamma_{00}}} \int_a^b \frac{|\varphi(\xi)-\varphi(a)|+|\varphi(\xi)-\varphi(b)|}{|\varphi(b)-\varphi(a)|} d\xi \\ & + \sum_{j=1}^{m} \Big(\frac{(2m-j)2^{2m-j+1}\, l_{0j}}{(2m-1)!!(2m-2j+1)!!} + \frac{2^{2m-j-1}(t^*-a)^{\gamma_{0j}}\, \bar{l}_{0j}}{(2m-2j-1)!!(2m-3)!!\sqrt{2\gamma_{0j}}} \Big) < \frac{1}{2}, \end{aligned} \tag{1.11}$$

$$\begin{aligned} B_1 \equiv & \bar{l}_{10}\Big(\frac{2^{m-1}}{(2m-3)!!}\Big)^2 \frac{(b-a)^{m-1/2}}{(2m-1)^{1/2}} \frac{(b-t^*)^{\gamma_{10}}}{\sqrt{2\gamma_{10}}} \int_a^b \frac{|\varphi(\xi)-\varphi(a)|+|\varphi(\xi)-\varphi(b)|}{|\varphi(b)-\varphi(a)|} d\xi \\ & + \sum_{j=1}^{m} \Big(\frac{(2m-j)2^{2m-j+1}\, l_{1j}}{(2m-1)!!(2m-2j+1)!!} + \frac{2^{2m-j-1}(b-t^*)^{\gamma_{0j}}\, \bar{l}_{1j}}{(2m-2j-1)!!(2m-3)!!\sqrt{2\gamma_{1j}}} \Big) < \frac{1}{2}, \end{aligned} \tag{1.12}$$

the conditions

$$\begin{aligned} & (t-a)^{m-\gamma_{00}-1/2} f_0(t,s) \leq \bar{l}_{00}, \qquad (t-a)^{2m-j} h_{j_1}(t,s) \leq l_{0j_1}, \\ & (t-a)^{m-\gamma_{0j}-1/2} f_j(a,\tau_j)(t,s) \leq \bar{l}_{0j} \qquad \text{for } a<t\leq s\leq t^*, \end{aligned} \tag{1.13}$$

$$\begin{aligned} & (b-t)^{m-\gamma_{10}-1/2} f_0(t,s) \leq \bar{l}_{10}, \qquad (b-t)^{2m-j} h_{j_1}(t,s) \leq l_{1j_1}, \\ & (b-t)^{m-\gamma_{1j}-1/2} f_j(b,\tau_j)(t,s) \leq \bar{l}_{1j} \qquad \text{for } t^*\leq s\leq t<b \end{aligned} \tag{1.14}$$

$j = 1, \cdots, m$ hold. Then for every $q \in \widetilde{L}^2_{2m-2,\,2m-2}(]a, b[)$ problem (1.8), (1.2) *is uniquely solvable in the space $\widetilde{C}^{2m,\,m+1}(]a,\, b[)$.*

Also, in [21], the following theorem is proven:

Theorem 1.2. *Let all the conditions of Theorem* 1.5 *are satisfied. Then the unique solution u of problem* (1.8), (1.2) *for every $q \in \widetilde{L}^2_{2m-2,2m-2}(]a, b[)$ admits the estimate*

$$||u^{(m+1)}||_{L^2} \leq r||q||_{\widetilde{L}^2_{2m-2,\,2m-2}}, \tag{1.15}$$

with

$$r = \frac{2^m}{(1 - 2\max\{B_0, B_1\})(2m-1)!!},$$

and thus constant $r > 0$ depends only on the numbers l_{kj}, $\bar{l}_{kj}, \gamma_{kj}$ ($k = 0, 1$; $j = 0, ..., m$), and a, b, t^*.

Remark 1.1. Under conditions of Theorem 1.2, for every $q \in \widetilde{L}^2_{2m-2,\,2m-2}(]a,b[)$ the unique solution u of problem (1.8), (1.2) admits the estimate

$$||u^{(m+1)}||_{\widetilde{C}^{2m}_2} \le r_{m,\varphi}||q||_{\widetilde{L}^2_{2m-2,\,2m-2}}, \tag{1.16}$$

with

$$r_{m,\varphi} = \Big(1+\int_a^b \frac{|\varphi(s)-\varphi(a)|+|\varphi(s)-\varphi(b)|}{|\varphi(b)-\varphi(a)|}ds\frac{(b-a)^{m-1/2}}{(m-1)!(2m-1)^{1/2}2^{m-1/2}}$$
$$+\sum_{j=1}^{m}\frac{1}{(m-i)!(2m-2i+1)^{1/2}}\Big(\frac{2}{b-a}\Big)^{m-i+1/2}\Big)\frac{2^m}{(1-2\max\{B_0,B_1\})(2m-1)!!}.$$

1.2. Theorems on a solvability of problem (1.1), (1.2). Define the operator $P : C^m_2(]a,b[) \times C^m_2(]a,b[) \to L_{loc}(]a,b[)$, by the equality

$$P(x,y)(t) = \sum_{j=0}^{m} p_j(x)(t)y^{(j)}(\tau_j(t)) \qquad \text{for} \quad a < t < b \tag{1.17}$$

where $p_j : C^m_2(]a,\,b[) \to L_{loc}(]a,\;b[)$, and $\tau_j \in M(]a,\,b[)$. Also, for any $\gamma > 0$, define the set A_γ by the relation

$$A_\gamma = \{x \in \widetilde{C}^m_2(]a,\,b[) : ||x||_{\widetilde{C}^m_2} \le \gamma\}. \tag{1.18}$$

To formulate this a priori boundedness principle, we have to introduce

Definition 1.1. Let γ_0 and γ be the positive numbers. We said that the continuous operator $P : C^m_2(]a,b[) \times C^m_2(]a,b[) \to L_m(]a,b[)$ is γ_0,γ consistent with boundary condition (1.2) if:

(i) for any $x \in A_{\gamma_0}$ and almost all $t \in]a,b[$, the inequality

$$\sum_{j=0}^{m} |p_j(x)(t)x^{(j)}(\tau_j(t))| \le \delta(t,\,||x||_{\widetilde{C}^m_2})||x||_{\widetilde{C}^m_2} \tag{1.19}$$

holds, where $\delta \in D_{2m}(]a,b[\times R^+)$.

(ii) for any $x \in A_{\gamma_0}$ and $q \in \widetilde{L}^2_{2m-2,2m-2}(]a,b[)$, the equation

$$y^{(2m+1)}(t) = \sum_{j=0}^{m} p_j(x)(t)y^{(j)}(\tau_j(t)) + q(t) \tag{1.20}$$

under boundary conditions (1.2), has the unique solution y in the space $\widetilde{C}^{2m,\,m+1}(]a,\;b[)$ and

$$||y||_{\widetilde{C}^m_2} \le \gamma||q||_{\widetilde{L}^2_{2m-2,\,2m-2}}. \tag{1.21}$$

Definition 1.2. We said that the operator P is γ consistent with boundary condition (1.2), if the operator P is γ_0, γ consistent with boundary condition (1.2) for any $\gamma_0 > 0$.

In the sequel, it will always be assumed that the operator F_p, defined by the equality

$$F_p(x)(t) = |F(x)(t) - \sum_{j=0}^{m} p_j(x)(t)x^{(j)}(\tau_j(t))(t)|,$$

continuously acting from $C_2^m(]a,b[)$ to $L_{\widetilde{L}^2_{2m-2,2m-2}}(]a,b[)$, and

$$\widetilde{F}_p(t,\rho) \equiv \sup\{F_p(x)(t) : ||x||_{C_2^m} \leq \rho\} \in \widetilde{L}^2_{2m-2,2m-2}(]a,\ b[) \tag{1.22}$$

for each $\rho \in [0, +\infty[$.

Then, the following theorem is valid.

Theorem 1.3. *Let the operator P be γ_0, γ consistent with boundary condition* (1.2)*, and there exist a positive number $\rho_0 \leq \gamma_0$, such that*

$$||\widetilde{F}_p(\cdot, \min\{2\rho_0, \gamma_0\})||_{\widetilde{L}^2_{2m-2,2m-2}} \leq \frac{\gamma_0}{\gamma}. \tag{1.23}$$

Moreover, for any $\lambda \in]0,1[$, let an arbitrary solution $x \in A_{\gamma_0}$ of the equation

$$x^{(2m+1)}(t) = (1-\lambda)P(x,x)(t) + \lambda F(x)(t) \tag{1.24}$$

under conditions (1.2)*, admits the estimate*

$$||x||_{\widetilde{C}_2^m} \leq \rho_0. \tag{1.25}$$

Then problem (1.1), (1.2) *is solvable in the space $\widetilde{C}^{2m,m+1}(]a,b[)$.*

From Theorem 1.3 with $\rho_0 = \gamma_0$ immediately follows

Corollary 1.1. *Let the operator P be γ_0, γ consistent with the boundary condition* (1.2)*, and*

$$|F(x)(t) - \sum_{j=1}^{m} p_j(x)(t)x^{(j-1)}(\tau_j(t))(t)| \leq \eta(t,\ ||x||_{\widetilde{C}_1^{m-1}}) \tag{1.26}$$

for $x \in A_{\gamma_0}$ and almost all $t \in]a,\ b[$, and

$$||\eta(\cdot,\ \gamma_0)||_{\widetilde{L}^2_{2m-2,\ 2m-2}} \leq \frac{\gamma_0}{\gamma}, \tag{1.27}$$

where $\eta \in D_{2m-2,\,2m-2}(]a,\ b[\times R^+)$. Then problem (1.1), (1.2) *is solvable in the space $\widetilde{C}^{2m,m+1}(]a,\ b[)$.*

Corollary 1.2. *Let the operator P be γ consistent with the boundary condition* (1.2)*, inequality* (1.26) *holds for $x \in \widetilde{C}_2^m(]a,\, b[)$ and almost all $t \in]a,\, b[$, where $\eta(\cdot,\, \rho) \in \widetilde{L}^2_{2m-2,\,2m-2}(]a,\, b[)$ for any $\rho \in R^+$, and*

$$\limsup_{\rho\to+\infty} \frac{1}{\rho} ||\eta(\cdot,\, \rho)||_{\widetilde{L}^2_{2m-2,\,2m-2}} < \frac{1}{\gamma}. \tag{1.28}$$

Then, problem (1.1), (1.2) *is solvable in the space $\widetilde{C}^{2m,m+1}(]a, b[)$.*

Now define the operators

$$h_j : C_1^{m-1}(]a, b[)\times]a, b[\times]a, b[\to R_+,$$
$$f_j : C_1^{m-1}(]a, b[) \times [a, b] \times M(]a, b[) \to R,\ (j = 1, ..., m)$$

by the equalities

$$\begin{aligned} h_1(x,t,s) &= \Big|\int_s^t (\xi-a)^{n-2m}[(-1)^{n-m}p_1(x)(\xi)]_+ d\xi\Big|, \\ h_j(x,t,s) &= \Big|\int_s^t (\xi-a)^{n-2m}p_j(x)(\xi)d\xi\Big| \quad (j=2,...,m), \end{aligned} \tag{1.29}$$

and

$$f_j(x,c,\tau_j)(t,s) = \Big|\int_s^t (\xi-a)^{n-2m}|p_j(x)(\xi)|\Big|\int_\xi^{\tau_j(\xi)} (\xi_1-c)^{2(m-j)}d\xi_1\Big|^{1/2}d\xi\Big|. \tag{1.30}$$

Theorem 1.4. *Let the continuous operator $P : C_1^{m-1}(]a, b[) \times C_1^{m-1}(]a, b[) \to L_n(]a,\, b[)$ admit to condition* (1.19)*, where $\delta \in D_n(]a, b[\times R^+)$, $\tau_j \in M(]a,\, b[)$ and the numbers $\gamma_0, t^* \in]a, b[$, $l_{kj} > 0$, $\bar{l}_{kj} > 0$, $\gamma_{kj} > 0$ $(k = 1, 2;\ j = 1, ..., m)$, be such that the inequalities*

$$(t-a)^{2m-j}h_j(x,t,s) \le l_{0j}, \quad \limsup_{t\to a}(t-a)^{m-\frac{1}{2}-\gamma_{0j}}f_j(x,a,\tau_j)(t,s) \le \bar{l}_{0j} \tag{1.31}$$

for $a < t \le s \le t^*$, $||x||_{\widetilde{C}_1^{m-1}} \le \gamma_0$,

$$(b-t)^{2m-j}h_j(x,t,s) \le l_{1j}, \quad \limsup_{t\to b}(b-t)^{m-\frac{1}{2}-\gamma_{1j}}f_j(x,b,\tau_j)(t,s) \le \bar{l}_{1j} \tag{1.32}$$

for $t^* \le s \le t < b$, $||x||_{\widetilde{C}_1^{m-1}} \le \gamma_0$, and conditions (1.11), (1.12) *hold. Moreover, let the operator F and the function $\eta \in D_{2n-2m-2,2m-2}(]a, b[\times R^+)$ be such that condition* (1.26) *and the inequality*

$$||\eta(\cdot,\gamma_0)||_{\widetilde{L}^2_{2n-2m-2,2m-2}} < \frac{\gamma_0}{r_n}, \tag{1.33}$$

be fulfilled, where

$$r_n=\Big(1+\sum_{j=1}^{m}\frac{2^{m-j+1/2}}{(m-j)!(2m-2j+1)^{1/2}(b-a)^{m-j+1/2}}\Big)\frac{2^m(1+b-a)(2n-2m-1)}{(\nu_n-2\max\{B_0,B_1\})(2m-1)!!}.$$

Then, problem (1.1), (1.2) *is solvable in the space* $\widetilde{C}^{n-1,m}(]a,b[)$.

Theorem 1.5. *Let the operator* F *and function* η *are such that condition* (1.26), (1.28) *hold and the continuous operator* $P: C_1^{m-1}(]a,b[)\times C_1^{m-1}(]a,b[)\to L_n(]a,b[)$ *admits condition* (1.19) *where* $\delta\in D_n(]a,b[\times R^+)$. *Let moreover the measurable functions* $\tau_j\in M(]a,b[)$ *and the numbers* $t^*\in]a,b[$, $l_{kj}>0$, $\bar{l}_{kj}>0$, $\gamma_{kj}>0$, $(k=1,2;\ j=1,...,m)$ *be such that the inequalities*

$$(t-a)^{2m-j}h_j(x,t,s)\le l_{0j},\quad \limsup_{t\to a}(t-a)^{m-\frac{1}{2}-\gamma_{0j}}f_j(x,a,\tau_j)(t,s)\le\bar{l}_{0j} \tag{1.34}$$

for $a<t\le s\le t^*$, $x\in\widetilde{C}_1^{m-1}(]a,b[)$,

$$(b-t)^{2m-j}h_j(x,t,s)\le l_{1j},\quad \limsup_{t\to b}(b-t)^{m-\frac{1}{2}-\gamma_{1j}}f_j(x,b,\tau_j)(t,s)\le\bar{l}_{1j} \tag{1.35}$$

for $t^*\le s\le t<b$, $x\in\widetilde{C}_1^{m-1}(]a,b[)$, *and conditions* (1.11), (1.12) *hold. Then, problem* (1.1), (1.2) *is solvable in the space* $\widetilde{C}^{n-1,m}(]a,b[)$.

Remark 1.2. Let $\gamma_0>0$, operators $\alpha_j(t)p_j(x)(t)$ $(j=1,...,m)$ continuously acting from the space $C_1^{m-1}(]a,b[)$ to the space $L_n(]a,b[)$, exist the function $\delta_j\in D_n(]a,b[)$ such that for any $x\in A_{\gamma_0}$

$$|p_j(x)(t)|\alpha_j(t)\le\delta_j(t,\,||x||_{\widetilde{C}_1^{m-1}})\quad\text{for } a<t<b, \tag{1.36}$$

and exists constants $\kappa>0,\ \varepsilon>0$ such that

$$\begin{array}{ll}|\tau_j(t)-t|\le\kappa(t-a)\ \ (j=1,\cdots,m) & \text{for } a<t<a+\varepsilon,\\ |\tau_j(t)-t|\le\kappa(b-t)\ \ (j=1,\cdots,m) & \text{for } b-\varepsilon<t<b,\end{array} \tag{1.37}$$

Then, the operator P defined by equality (1.17), continuously acting from A_{γ_0} to the space $L_n(]a,b[)$, and there exists the function $\delta\in D_n(]a,b[)$ such that item (ii) of Definition 1.1 holds.

Now, consider the equation with deviating arguments

$$u^{(n)}(t)=f(t,u(\tau_1(t)),u'(\tau_2(t)),\cdots,u^{(m-1)}(\tau_m(t)))\quad\text{for}\quad a<t<b, \tag{1.38}$$

where $-\infty<a<b<+\infty$, $f:]a,\ b[\times R^m\to R$ is a function, satisfying the local Carathéodory conditions and $\tau_j\in M(]a,b[)$ $(j=0,...,n-1)$ are measurable functions.

Corollary 1.3. *Let the functions* $\tau_j \in M(]a,\, b[)$ *and the numbers* $t^* \in]a, b[$, $\kappa \geq 0$, $\varepsilon > 0$, $l_{kj} > 0$, $\bar{l}_{kj} > 0$, $\gamma_{kj} > 0$, $(k = 1, 2;\ j = 1, ..., m)$ *be such that the conditions* (1.11)-(1.14), (1.37) *and the inclusions*

$$\alpha_j p_j \in L_n(]a, b[)\ (j = 1, ..., m) \tag{1.39}$$

are fulfilled. Moreover, let

$$\begin{aligned}\Big|f(t, x(\tau_1(t)), x'(\tau_2(t)), ..., x^{(m-1)}(\tau_m(t))) - \sum_{j=1}^{m} p_j(t) x^{(j-1)}(\tau_j(t))(t)\Big| \\ \leq \eta(t,\, ||x||_{\widetilde{C}_1^{m-1}})\end{aligned} \tag{1.40}$$

for $x \in \widetilde{C}_1^{m-1}(]a,\, b[)$ *and almost all* $t \in]a, b[$, *where* $\eta(\cdot, \rho) \in \widetilde{L}^2_{2n-2m-2,2m-2}(]a, b[)$ *for any* $\rho \in R^+$, *and condition* (1.28) *holds. Then problem* (1.38), (1.2) *is solvable in the space* $\widetilde{C}^{n-1,m}(]a, b[)$.

Remark 1.3. From conditions (1.39), conditions (1.6) do not follow.

Now, to illustrate our results, consider on $]a,\, b[$ the second order functional-differential equations

$$u''(t) = -\frac{\lambda |u(t)|^k}{[(t-a)(b-t)]^{2+k/2}} u(\tau(t)) + q(x)(t), \tag{1.41}$$

$$u''(t) = -\frac{\lambda |\sin u^k(t)|}{[(t-a)(b-t)]^2} u(\tau(t)) + q(x)(t), \tag{1.42}$$

where $\lambda, k \in R^+$, the function $\tau \in M(]a, b[)$, the operator $q : C_1^{m-1}(]a, b[) \to \widetilde{L}^2_{0,0}(]a, b[)$ is continuous and

$$\eta(t, \rho) \equiv \sup\{|q(x)(t)| : ||x||_{\widetilde{C}_1^{m-1}} \leq \rho\} \in \widetilde{L}^2_{0,0}(]a, b[).$$

Then, from Theorems 1.4 and 1.5, it follows

Corollary 1.4. *Let the function* $\tau \in M(]a, b[)$, *the continuous operator* $q : C_1^{m-1}(]a, b[) \to \widetilde{L}^2_{0,0}(]a, b[)$, *and the numbers* $\gamma_0 > 0,\ \lambda \geq 0,\ k > 0$, *be such that*

$$|\tau(t) - t| \leq \begin{cases} (t-a)^{3/2} & \text{for } \ a < t \leq (a+b)/2 \\ (b-t)^{3/2} & \text{for } \ (a+b)/2 \leq t < b \end{cases}, \tag{1.43}$$

$$||\eta(t,\, \gamma_0)||_{\widetilde{L}^2_{0,\,0}} \leq \Big(1 + \sqrt{\frac{2}{b-a}}\Big)^{-1} \frac{(b-a)^2 - 16\lambda\gamma_0^k(1 + [2(b-a)]^{1/4})}{2(1+b-a)(b-a)^2}, \tag{1.44}$$

and

$$\lambda < \frac{(b-a)^2}{32\gamma_0^k(1 + [2(b-a)]^{1/4})}. \tag{1.45}$$

Then, problem (1.41), (1.2) *is solvable.*

Corollary 1.5. *Let the function* $\tau \in M(]a,\, b[)$, *continuous operator* $q : C_1^{m-1}(]a, b[) \to \widetilde{L}^2_{0,0}(]a, b[)$, *and the number* $\lambda \geq 0$ *by such, that inequalities* (1.28) *with* $n = 2$, (1.43) *and*

$$\lambda < \frac{(b-a)^2}{32(1+[2(b-a)]^{1/4})}, \tag{1.46}$$

hold. Then, problem (1.42), (1.2) *is solvable.*

2. Auxiliary propositions

2.1. Lemmas on some properties of the equation $x^{(2m)}(t) = \lambda(t)$. First, we introduce two lemmas without proofs. The first Lemma is proved in [3].

Lemma 2.1. *Let* $i \in 1, 2,\ \ x \in \widetilde{C}^{m-1}_{loc}(]t_0,\, t_1[)$ *and*

$$x^{(j-1)}(t_i) = 0 \quad (j = 1, \ldots, m), \qquad \int_{t_0}^{t_1} |x^{(m)}(s)|^2 ds < +\infty. \tag{2.1}$$

Then

$$\Big| \int_{t_i}^{t} \frac{(x^{(j-1)}(s))^2}{(s-t_i)^{2m-2j+2}} ds \Big|^{1/2} \leq \frac{2^{m-j+1}}{(2m-2j+1)!!} \Big| \int_{t_i}^{t} |x^{(m)}(s)|^2 ds \Big|^{1/2} \tag{2.2}$$

for $t_0 \leq t \leq t_1$.

This second lemma is a particular case of Lemma 4.1 in [9]

Lemma 2.2. *If* $x \in C^{2m-1}_{loc}(]a, a_1])$, *then for any* $s, t \in]a, a_1]$, *we have the equality*

$$(-1)^m \int_s^t x^{(n)}(\xi) x(\xi) d\xi = w_{2m}(x)(t) - w_{2m}(x)(s) + \int_s^t |x^{(m)}(\xi)|^2 d\xi$$

$$w_{2m}(x)(t) = \sum_{j=1}^{m} (-1)^{m+j-1} x^{(2m-j)}(t) x(t).$$

Lemma 2.3. *Let the numbers* $a_1 \in]a, b[,\ t_{0k} \in]a, a_1[,$ *and* $\varepsilon_{ik}, \varepsilon_i, \beta_k, \beta \in R^+$, $k \in N$, $i = 1, ..., m$ *be such that*

$$\lim_{k \to +\infty} t_{0\,k} = a, \quad \lim_{k \to +\infty} \beta_k = \beta, \quad \lim_{k \to +\infty} \varepsilon_{i,k} = \varepsilon_i. \tag{2.3}$$

Moreover, let

$$\lambda \in \widetilde{L}^2_{2m-2,0}(]a, a_1]), \tag{2.4}$$

be a nonnegative function, $x_k \in \widetilde{C}^{2m-1,m}(]a,b[)$ *be a solution of the problem*

$$x^{(2m)}(t) = \beta_k \lambda(t), \tag{2.5}$$

$$x^{(i-1)}(t_{0\,k}) = 0 \ \ (i = 1, ..., m), \qquad x^{(i-1)}(a_1) = \varepsilon_{i,k} \ \ (i = 1, ..., m), \tag{2.6}$$

and $x \in \widetilde{C}^{2m-1,m}(]a,b[)$ *be a solution of the problem*

$$x^{(2m)}(t) = \beta \lambda(t), \tag{2.7}$$

$$x^{(i-1)}(a) = 0 \ \ (i = 1, ..., m), \qquad x^{(i-1)}(a_1) = \varepsilon_i \ \ (i = 1, ..., m). \tag{2.8}$$

Then

$$\lim_{k \to +\infty} x_k^{(j-1)}(t) = x^{(j-1)}(t) \quad (j = 1, ..., 2m) \quad \text{uniformly in} \quad]a, a_1]. \tag{2.9}$$

Analogously one can prove

Lemma 2.4. *Let the numbers* $b_1 \in]a,b[$, $t_{0k} \in]b_1,b[$, *and* $\varepsilon_{ik}, \varepsilon_i$, $\beta_k, \beta \in R^+$, $k \in N$, $i = 1, ..., n-m$ *be such that*

$$\lim_{k \to +\infty} t_{0\,k} = b, \quad \lim_{k \to +\infty} \beta_k = \beta, \quad \lim_{k \to +\infty} \varepsilon_{i,k} = \varepsilon_i.$$

Moreover, let $\lambda \in \widetilde{L}^2_{0,2m-2}(]b_1,b])$ *is a nonnegative function,* $x_k \in \widetilde{C}^{n-1,m}(]a,b[)$ *be a solution of problem* (2.5) *under the conditions*

$$x^{(i-1)}(b_1) = \varepsilon_{i,k} \ \ (i = 1, \cdots, m), \qquad x^{(i-1)}(t_{0\,k}) = 0 \ \ (i = 1, ..., n-m),$$

and $x \in \widetilde{C}^{n-1,\,m}(]a,b[)$ *be a solution of equation* (2.7) *under the conditions*

$$x^{(i-1)}(b_1) = \varepsilon_i \ \ (i = 1, ..., m), \qquad x^{(i-1)}(b) = 0 \ \ (i = 1, ..., n-m). \tag{2.10}$$

Then, equalities (2.9) *hold.*

Lemma 2.5. *Let* $a < a_1 < b_1 < b$, $\varepsilon_i \in R^+$ *and*

$$\lambda \in \widetilde{L}^2_{2n-2m-2,\,0}(]a,\ a_1]) \quad (\lambda \in \widetilde{L}^2_{0,\,2m-2}(]b_1,b])$$

be nonnegative function. Then for the solution $x \in \widetilde{C}^{n-1,\,m}(]a,\,b[)$ *of problem* (2.7), (2.8) ((2.7), (2.10)) *with* $\beta = 1$, *the estimate*

$$\int_a^{a_1} |x^{(m)}(s)|^2 ds \le \Theta_1(x, a_1, \lambda) \ \Big(\int_{b_1}^{b} |x^{(m)}(s)|^2 ds \le \Theta_2(x, b_1, \lambda) \Big) \ (k \in N) \tag{2.11}$$

is valid, where

$$\begin{gathered} \Theta_1(x, a_1, \lambda) = 2|w_n(x)(a_1)| + \gamma_1 ||\lambda||^2_{\widetilde{L}^2_{2n-2m-2,0}(]a,a_1])}, \\ \Big(\Theta_2(x, b_1, \lambda) = 2|w_n(x)(b_1)| + \gamma_2 ||\lambda||^2_{\widetilde{L}^2_{0,2m-2}(]b_1,b])} \Big), \end{gathered} \tag{2.12}$$

and

$$\gamma_1 = \Big(\frac{2^{m-1}(2m+1)}{(2m-1)!!} \Big)^2, \ \ \gamma_2 = \Big(\frac{2^{m-1}(2m+1)(b-a+1)}{(2m-1)!!} \Big)^2.$$

2.2. Lemmas on the Banach space $\widetilde{C}^{m-1}_{1,\Gamma}(]a,\,b[)$. Let the bounded linear operator $\Gamma : \widetilde{C}^{m-1}_1 \to \widetilde{C}^m_2$ be defined by the equality

$$\Gamma(x)(t) = \int_a^b G(t,s)x(s)ds,$$

and

$$G(t,s) = \frac{1}{\varphi(b)-\varphi(a)} \times \begin{cases} \varphi(s)-\varphi(b) & \text{for } \ s \geq t \\ \varphi(s)-\varphi(a) & \text{for } \ s < t \end{cases} \tag{2.13}$$

is the Green's function of the problem:

$$w'(t) = 0, \qquad \int_a^b w(s)d\varphi(s) = 0. \tag{2.14}$$

where $\varphi : [a,b] \to R$ is a function of bounded variation and $\varphi(b)-\varphi(a) \neq 0$.

The problem (2.14) has only the trivial solution, thus $\Gamma(x) = 0$ if and only if $x = 0$, and we can define the Banach spaces:

$C^{m-1}_{1,\Gamma}(]a,\,b[)$ is the Banach space of the functions $x \in C^{m-1}_{loc}(]a,\,b[)$, such that

$$\begin{aligned} &\limsup_{t\to a} \frac{|x^{(i-1)}(t)|}{(t-a)^{m-i+1/2}} < +\infty \ (i = 1, \cdots, m), \\ &\limsup_{t\to b} \frac{|x^{(i-1)}(t)|}{(b-t)^{m-i+1/2}} < +\infty \ (i = 1, \cdots, m), \end{aligned} \tag{2.15}$$

with the norm:

$$||x||_{C^{m-1}_{1,\Gamma}} = ||\Gamma(x)||_C + \sum_{i=1}^m \sup\Big\{\frac{|x^{(i-1)}(t)|}{\alpha_i(t)} \ : \ a < t < b\Big\},$$

where $\alpha_i(t) = (t-a)^{m-i+1/2}(b-t)^{m-i+1/2}$.

$\widetilde{C}^{m-1}_{1,\Gamma}(]a,\,b[)$ is the Banach space of the functions $x \in \widetilde{C}^{m-1}_{loc}(]a,\,b[)$, such that conditions (1.3) and (2.15) hold, with the norm:

$$||x||_{\widetilde{C}^{m-1}_{1,\Gamma}} = ||\Gamma(x)||_C + ||x||_{C^m_1} + \Big(\int_a^b |x^{(m)}(s)|^2 ds\Big)^{1/2}.$$

Definition 2.3. Let $\rho \in R^+$ and the function $\eta \in L_{loc}(]a,b[)$ be nonnegative. Then $S(\rho,\eta)$ is a set of such $z \in C^{n-1}_{loc}(]a,b[)$ that

$$\Big|z^{(i-1)}\Big(\tfrac{a+b}{2}\Big)\Big| \leq \rho \qquad (i = 1, ..., n), \tag{2.16}$$

$$|z^{(n-1)}(t) - z^{(n-1)}(s)| \leq \int_s^t \eta(\xi)d\xi \qquad \text{for } a < s \leq t < b, \tag{2.17}$$

$$z^{(i-1)}(a) = 0 \ (i = 1, ..., m), \qquad z^{(i-1)}(b) = 0 \ (i = 1, ..., m). \tag{2.18}$$

Lemma 2.6. *For the function $z \in \widetilde{C}^{2m-1,m}(]a,\ b[)$, let conditions* (2.18) *be satisfied. Then, $z \in \widetilde{C}^{m-1}_{1,\Gamma}(]a,b[)$ and the estimates*

$$|z^{(i-1)}(t)| \leq \frac{|t-c_k|^{m-i+1/2}}{(m-i)!(2m-2i+1)^{1/2}}\Big|\int_{c_k}^{t}|z^{(m)}(s)|^2ds\Big|^{1/2} \quad \text{for } a<t<b, \tag{2.19}$$

$i = 1, ..., m$, hold for $k = 1, 2$, where $c_1 = a$, $c_2 = b$.

Proof. First, not that in view of inclusion $z \in \widetilde{C}^{2m-1,m}(]a,b[)$, the equality

$$z^{(i-1)}(t) = \sum_{j=i}^{l}\frac{(t-c)^{j-i}}{(j-i)!}z^{(j-1)}(c) + \frac{1}{(l-i)!}\int_{c}^{t}(t-s)^{l-i}z^{(l)}(s)ds \tag{2.20}$$

for $a < t < b$, $i = 1, ..., l$, $l = 1, ..., 2m$, holds, where

$$\begin{aligned} &1.\quad c \in [a,b] \quad \text{if} \quad l \leq m;\\ &2.\quad c \in]a,b[\quad \text{if} \quad l > m, \end{aligned}$$

and exists $r > 0$ such that

$$\int_{a}^{b}|z^{(m)}(s)|^2ds \leq r. \tag{2.21}$$

Equality (2.20), with $l = m,\ c = a$ and with $l = m,\ c = b$ by conditions (2.18), (2.21) and the Schwartz inequality yields (2.19). But, from (2.19) with $i = 1$, we have that $||z||_C < +\infty$, and then, by the inequality

$$|\Gamma(z)(t)| \leq \int_{a}^{b}\frac{|\varphi(s)-\varphi(a)|+|\varphi(s)-\varphi(b)|}{|\varphi(b)-\varphi(a)|}ds||z||_C, \tag{2.22}$$

we have

$$||\Gamma(z)||_C < +\infty. \tag{2.23}$$

From (2.19), (2.21) and (2.23) it is clear that $z \in \widetilde{C}^{m-1}_{1,\Gamma}(]a,b[)$. ∎

Lemma 2.7. *Let $\rho \in R^+$, and $\eta \in \widetilde{L}^2_{2m-2,\,2m-2}(]a,b[)$ is a nonnegative function. Then $S(\rho,\eta)$ is a compact subset of the space $\widetilde{C}^{m-1}_{1,\Gamma}(]a,\ b[)$.*

Proof. Condition (2.17) yields the inequality $|z^{(2m)}(t)| \leq \eta(t)$. Thus there exists such a function $\eta_1 \in \widetilde{L}^2_{2m-2,\,2m-2}(]a,b[)$ that

$$z^{(2m)}(t) = \eta_1(t), \quad \text{for} \quad a<t<b \tag{2.24}$$

$$|\eta_1(t)| \le \eta(t) \quad \text{for} \quad a < t < b \tag{2.25}$$

From Theorem 1.1, it follows that problem (2.24), (2.18) has a unique solution $z \in C^{2m-1,m}(]a,b[)$, i.e., there exists $r > 0$ such that the inequality (2.21) holds.

For any $z \in S(\rho,\eta)$, from equality (2.20) with $l = 2m$, by (2.16), (2.24) and (2.25)we get

$$|z^{(i-1)}(t)| \le \gamma_i(t) \quad \text{for} \quad a < t < b, \quad (i = 1, ..., 2m), \tag{2.26}$$

where

$$\gamma_i(t) = \rho_i + \frac{1}{(2m-i)!}\Big| \int_c^t (t-s)^{n-i}\eta(s)ds\Big| \quad (i = 1, ..., 2m).$$

Now, let $z_k \in S(\rho,\eta)$ $(k \in N)$. By virtue of the Arzelà-Ascoli lemma and conditions (2.17), (2.26), the sequence $\{z_k\}_{k=1}^{+\infty}$ contains a subsequence $\{z_{k_\ell}\}_{\ell=1}^{+\infty}$ such that $\{z_{k_\ell}^{(i-1)}\}_{\ell=1}^{+\infty}$ $(i = 1, ..., 2m)$ are uniformly convergent on $]a,b[$. Thus, without loss of generality, we can assume that $\{z_k^{(i-1)}\}_{k=1}^{+\infty}$ $(i = 1, ..., 2m-1)$ are uniformly convergent on $]a,\ b[$. Let $\lim\limits_{k\to+\infty} z_k(t) = z_0(t)$, then $z_0 \in \widetilde{C}_{loc}^{2m-1}(]a,\ b[)$ and

$$\lim_{k\to+\infty} z_k^{(i-1)}(t) = z_0^{(i-1)}(t) \quad (i = 1, ..., 2m) \quad \text{uniformly on} \quad]a,\ b[. \tag{2.27}$$

From (2.27), in view of the inclusions $z_k \in S(\rho,\eta)$, immediately it follows that

$$\Big|z_0^{(i-1)}\Big(\tfrac{a+b}{2}\Big)\Big| \le \rho \qquad (i = 1, ..., 2m), \tag{2.28}$$

$$z_0^{(i-1)}(a) = 0 \ (j = 1, \cdots, m), \quad z_0^{(i-1)}(b) = 0 \ (j = 1, ..., m). \tag{2.29}$$

$$|z_0^{(2m-1)}(t) - z_0^{(2m-1)}(s)| \le \int_s^t \eta(\xi)d\xi \qquad \text{for } a < s \le t < b. \tag{2.30}$$

From (2.28)-(2.30), it is clear that $z_0 \in S(\rho,\eta)$. To finish the proof, we must show that

$$\lim_{k\to+\infty} ||z_k(t) - z_0(t)||_{\widetilde{C}_{1,\Gamma}^{m-1}} = 0, \tag{2.31}$$

$$S(\rho,\eta) \subset \widetilde{C}_{1,\Gamma}^{m-1}(]a,b[). \tag{2.32}$$

First, note that, from (2.27), by the conditions $z_k \in S(\rho,\eta)$ and (2.29), we have

$$\lim_{k\to+\infty} ||z_k - z_0||_C = 0,$$

from which, by (2.22), we get

$$\lim_{k\to+\infty} ||\Gamma(z_k - z_0)||_C = 0. \tag{2.33}$$

Let, $x_k = z_0 - z_k$, and $a_1 \in]a,b[,\ b_1 \in]a_1,b[$. Then, it is clear that $x_k \in S(\rho',\eta')$, where $\rho' = 2\rho$, $\eta' = 2\eta$. Thus, for any x_k, there exists $\eta_k \in \widetilde{L}^2_{2m-2,2m-2}(]a,b[)$ such that

$$x_k^{(2m)}(t) = \eta_k(t), \tag{2.34}$$

$$x_k^{(i-1)}(a) = 0 \ \ (i = 1, ..., m), \qquad x_k^{(i-1)}(b) = 0 \ \ (i = 1, ..., m), \tag{2.35}$$

where

$$|\eta_k(t)| \le 2\eta(t) \quad \text{for } a < t < b \quad (k \in N). \tag{2.36}$$

On the other hand, from (2.19) with $z = x_k$, in view of (2.35), we get

$$\begin{aligned} |x_k^{(i-1)}(t)| &\le \Big(\int_a^t |x_k^{(m)}(s)|^2 ds \Big)^{1/2} (t-a)^{m-i+1/2} \quad \text{for} \quad a < t < a_1, \\ |x_k^{(i-1)}(t)| &\le \Big(\int_t^b |x_k^{(m)}(s)|^2 ds \Big)^{1/2} (b-t)^{m-i+1/2} \quad \text{for} \quad b_1 < t < b, \end{aligned} \tag{2.37}$$

for $i = 1, \dots, m$.

Now, let w_{2m} be the operator defined in Lemma 2.2 and Θ_1, Θ_2 are functions defined by (2.12) with $\lambda = \eta_k$. Then, conditions (2.27) yields

$$\lim_{k \to +\infty} w_{2m}(x_k)(a_1) = 0, \qquad \lim_{k \to +\infty} w_{2m}(x_k)(b_1) = 0 \ \ (k \in N), \tag{2.38}$$

and, from the definition of the norm $||\cdot||_{\widetilde{L}^2_{\alpha,\beta}}$, (2.36) and (2.38), it follows that, for any $\varepsilon > 0$, we can choose $a_1 \in]a,\ \min\{a+1,\ b\}[$, $b_1 \in]\max\{b-1, b\}, b[$ and $k_0 \in N$, such that

$$\begin{aligned} \Theta_1(x_k, a_1, 2\eta) &\le \frac{\varepsilon}{6}(b-b_1)^{m-1/2} \quad (k \ge k_0), \\ \Theta_2(x_k, b_1, 2\eta) &\le \frac{\varepsilon}{6}(a_1-a)^{m-1/2} \quad (k \ge k_0). \end{aligned} \tag{2.39}$$

By using Lemma 2.5 for x_k, in view of (2.37) and (2.39), we get

$$\int_a^{a_1} |x_k^{(m)}(s)|^2 ds \le \frac{\varepsilon}{6} \qquad \int_{b_1}^b |x_k^{(m)}(s)|^2 ds \le \frac{\varepsilon}{6} \ \ (k \ge k_0), \tag{2.40}$$

$$\frac{|x_k^{(i-1)}(t)|}{\alpha_i(t)} \le \frac{\varepsilon}{2m} \qquad \text{for} \quad t \in]a,\ a_1] \cup [b_1,\ b[, \ \ (1 \le i \le m, \ \ k \ge k_0). \tag{2.41}$$

Also, in view of (2.27) without loss of generality we can assume that

$$\frac{|x_k^{(i-1)}(t)|}{\alpha_i(t)} \le \frac{\varepsilon}{2m} \quad \text{for} \quad a_1 \le t \le b_1, \quad (1 \le i \le n-1, \ k \ge k_0), \tag{2.42}$$

$$\int_{a_1}^{b_1} |x_k^{(m)}(s)|^2 ds \le \frac{\varepsilon}{6} \quad (k \ge k_0). \tag{2.43}$$

From (2.33), (2.40)-(2.43), equality (2.31) immediately follows.

Let, now $z \in S(\rho, \eta)$ and $z_k = \delta_k z$, where $\lim_{k \to +\infty} \delta_k = 0$. Then, by (2.27), it is clear that $z_0 \equiv 0$ and then, from (2.31) it follows $z \in \widetilde{C}^{m-1}_{1,\Gamma}(]a, b[)$, i.e., inclusion (2.32) holds. ∎

Lemma 2.8. *Let $\tau_j \in M(]a,b[),\ \alpha \geq 0,\ \beta \geq 0$ and exists $\delta \in]0, b-a[$ such that*

$$|\tau_j(t) - t| \leq k_1(t-a)^{\beta} \quad for \quad a < t \leq a+\delta. \tag{2.44}$$

Then

$$\Big| \int_t^{\tau(t)} (s-a)^{\alpha} ds \Big| \leq \begin{cases} k_1[1+k_1\delta^{\beta-1}]^{\alpha}(t-a)^{\alpha+\beta} & for \ \ \beta \geq 1 \\ k_1[\delta^{1-\beta}+k_1]^{\alpha}(t-a)^{\alpha\beta+\beta} & for \ \ 0 \leq \beta < 1 \end{cases},$$

for $a < t \leq a+\delta$.

Proof. First note that

$$\Big| \int_t^{\tau(t)} (s-a)^{\alpha} ds \Big| \leq (\max\{\tau(t), t\} - a)^{\alpha} |\tau(t) - t| \quad \text{for} \quad a \leq t \leq a+\delta,$$

and $\max\{\tau(t), t\} \leq t + |\tau(t) - t| \quad \text{for} \quad a \leq t \leq a+\delta$. Then, in view of condition (2.44), we get

$$\Big| \int_t^{\tau(t)} (s-a)^{\alpha} ds \Big| \leq k_1[(t-a) + k_1(t-a)^{\beta}]^{\alpha}(t-a)^{\beta} \quad \text{for} \quad a \leq t \leq a+\delta.$$

The last inequality yields the validity of our lemma. ∎

Analogously, one can prove

Lemma 2.9. *Let $\tau_j \in M(]a,b[),\ \alpha \geq 0,\ \beta \geq 0$ and exists $\delta \in]0, b-a[$ such that*

$$|\tau_j(t) - t| \leq k_1(b-t)^{\beta} \quad for \quad b-\delta \leq t < b. \tag{2.45}$$

Then

$$\Big| \int_t^{\tau(t)} (b-t)^{\alpha} ds \Big| \leq \begin{cases} k_1[1+k_1\delta^{\beta-1}]^{\alpha}(b-t)^{\alpha+\beta} & for \ \ \beta \geq 1 \\ k_1[\delta^{1-\beta}+k_1]^{\alpha}(b-t)^{\alpha\beta+\beta} & for \ \ 0 \leq \beta < 1 \end{cases},$$

for $b-\delta \leq t < b$.

2.3. Lemmas on the solutions of auxiliary problems.

Lemma 2.10. *For any function $x \in C_1^{m-1}(]a,\, b[)$ ($x \in \widetilde{C}_1^{m-1}(]a,\, b[)$) the equality*

$$||\Gamma(x)||_{C_2^m} = ||x||_{C_{1,\Gamma}^{m-1}}, \qquad (||\Gamma(x)||_{\widetilde{C}_2^m} = ||x||_{\widetilde{C}_{1,\Gamma}^{m-1}})$$

holds

Proof. From the definition of the norms $||\cdot||_{\widetilde{C}_2^m}$ and $||\cdot||_{\widetilde{C}_{1,\Gamma}^{m-1}}$ we have

$$\begin{aligned}||\Gamma(x)||_{\widetilde{C}_2^m} &= ||\Gamma(x)||_C + \sum_{i=1}^m \sup\Big\{\frac{|\Gamma^{(i)}(x)(t)|}{\alpha_i(t)} \ : \ a<t<b\Big\} \\ &\quad + \Big(\int_a^b |\Gamma^{(m+1)}(x)(s)|^2 ds\Big)^{1/2} \\ &= ||\Gamma(x)||_C + \sum_{i=1}^m \sup\Big\{\frac{|x^{(i-1)}(t)|}{\alpha_i(t)} \ : \ a<t<b\Big\} \\ &\quad + \Big(\int_a^b |x^{(m)}(s)|^2 ds\Big)^{1/2} = ||x||_{\widetilde{C}_{1,\Gamma}^{m-1}}. \end{aligned}$$ ∎

Now, define the continuous operators $P_\Gamma : C_{1,\Gamma}^{m-1}(]a,b[)\times C_{1,\Gamma}^{m-1}(]a,b[) \to L_n(]a,b[)$ as

$$P_\Gamma(x,y)(t) = P(\Gamma(x),\Gamma(y))(t) \qquad \text{for} \qquad a<t<b,$$

i.e.,

$$P_\Gamma(x,y)(t)=\sum_{j=1}^m p_{j,\Gamma}(x)(t)\Gamma^{(j)}(y)(\tau_j(t))+p_{0,\Gamma}(x)(t)\int_a^b G(\tau_0(t),s)\Gamma(y)(s)ds \tag{2.46}$$

where the operators $p_{j,\Gamma} : C_1^{m-1}(]a,\ b[) \to L_n(]a,b[)$ $(j=0,...,m)$ are defined by the equalities

$$p_{j,\Gamma}(x)(t) = p_j(\Gamma(x))(t),$$

and $F_{p,\Gamma} : C_{1,\Gamma}^{m-1}(]a,b[) \to \widetilde{L}^2_{2m-2,2m-2}(]a,\ b[)$ as

$$F_{p,\Gamma}(x)(t) = F_{p,\Gamma}(\Gamma(x))(t) = |F(\Gamma(x))(t) - P_\Gamma(x,x)(t)|. \tag{2.47}$$

Lemma 2.11. *If the operator $P : C_2^m(]a,b[)\times C_2^m(]a,b[) \to L_n(]a,\ b[)$ is consistent with the numbers γ_0,γ, and the set $A_{\gamma_0,\Gamma} \subset \widetilde{C}_{1,\Gamma}^{m-1}(]a,b[)$ is defined by the equality*

$$A_{\gamma_0,\Gamma} = \{x \in \widetilde{C}_{1,\Gamma}^{m-1}(]a,\ b[) : ||x||_{\widetilde{C}_{1,\Gamma}^{m-1}} \le \gamma_0\},$$

then:

(i) *for any $x_0 \in A_{\gamma_0,\Gamma}$ and almost all $t \in]a,\ b[$, the inequality*

$$\begin{aligned}\sum_{j=1}^m |p_{j,\Gamma}(x_0)(t)x_0^{(j-1)}(\tau_j(t))| + |p_{0,\Gamma}(x_0)(t)\int_a^b G(\tau_0(t),s)x_0(s)ds| \\ \le \eta(t,\ ||x_0||_{\widetilde{C}_{1,\Gamma}^{m-1}})||x_0||_{\widetilde{C}_{1,\Gamma}^{m-1}}.\end{aligned} \tag{2.48}$$

holds, where $\delta \in D_n(]a,\ b[\times R^+)$.

(ii) *for any* $x_0 \in A_{\gamma_0,\Gamma}$ *and* $q \in \widetilde{L}^2_{2m-2,\,2m-2}(]a,\,b[)$, *the equation*

$$z^{(2m)}(t)=\sum_{j=1}^{m} p_{j,\Gamma}(x_0)(t)z^{(j-1)}(\tau_j(t))+p_{0,\Gamma}(x_0)(t)\int_a^b G(\tau_0(t),s)z(s)ds+q(t) \tag{2.49}$$

under the boundary conditions (2.18), *has the unique solution* $z \in \widetilde{C}^{2m-1,m}(]a,\,b[)$, *and*

$$||z||_{\widetilde{C}^{m-1}_{1,\Gamma}} \le \gamma ||q||_{\widetilde{L}^2_{2m-2,\,2m-2}} \tag{2.50}$$

Proof. In view of Lemma 2.10, for any $x_0 \in A_{\gamma_0,\Gamma}$ we have $y = \Gamma(x_0) \in A_{\gamma_0}$, where A_{γ_0} is defined by (1.18), and then, from inequality (1.19), we get

$$\begin{aligned}
&\sum_{j=1}^{m} |p_{j,\,\Gamma}(x_0)(t)x_0^{(j-1)}(\tau_j(t))| + |p_{0,\,\Gamma}(x_0)(t)\int_a^b G(\tau_0(t),s)x_0(s)ds| \\
&\quad = \sum_{j=1}^{m} |p_j(\Gamma(x_0))(t)\Gamma^{(j)}(x_0)(\tau_j(t))| + |p_0(\Gamma(x_0))(t)\Gamma(x_0)(t)| \\
&\quad = \sum_{j=0}^{m} |p_j(y)(t)y^{(j)}(\tau_j(t))| \le \eta(t,\,||y||_{\widetilde{C}^m_2})||y||_{\widetilde{C}^m_2} \\
&\quad = \eta(t,\,||\Gamma(x_0)||_{\widetilde{C}^m_2})||\Gamma(x_0)||_{\widetilde{C}^m_2} = \eta(t,\,||x_0||_{\widetilde{C}^{m-1}_{1,\Gamma}})||x_0||_{\widetilde{C}^{m-1}_{1,\Gamma}}.
\end{aligned}$$

Now, let $x_0 \in A_{\gamma_0,\Gamma}$ and $q \in \widetilde{L}^2_{2m-2,2m-2}(]a,b[)$, the function $y \in \widetilde{C}^{2m,m+1}(]a,b[)$ is a solution of problem (1.20), (1.2), for $x = \Gamma(x_0)$. Let also $z = y'$, then $z \in \widetilde{C}^{2m-1,m}(]a,b[)$ and in view of (1.2) we have the representation $y = \Gamma(z)$, and from (1.20) and (1.21) it follows

$$\begin{aligned}
z^{(2m)}(t) &= \sum_{j=0}^{m} p_j(\Gamma(x_0))(t)\Gamma^{(j)}(z)(\tau_j(t)) + q(t) \\
&= \sum_{j=1}^{m} p_{j,\,\Gamma}(x_0)(t)z^{(j-1)}(\tau_j(t)) + p_{0,\,\Gamma}(x_0)(t)\int_a^b G(\tau_0(t),s)z(s)ds + q(t)
\end{aligned}$$

and

$$||z||_{\widetilde{C}^{m-1}_{1,\Gamma}} = ||\Gamma(z)||_{\widetilde{C}^m_2} = ||y||_{\widetilde{C}^m_2} \le \gamma ||q||_{\widetilde{L}^2_{2m-2,\,2m-2}}.$$

Thus, relations (2.48), (2.48), and (2.48) are valid. ∎

Lemma 2.12. *Let* (1.22) *and all the conditions of Theorem* 1.3 *hold. Then*

$$\widetilde{F}_{p,\Gamma}(t,\,\rho) \equiv \sup\{F_{p,\Gamma}(x)(t) : ||x||_{C^{m-1}_{1,\Gamma}} \le \rho\} \in \widetilde{L}^2_{2m-2,\,2m-2}(]a,\,b[), \tag{2.51}$$

$$||\widetilde{F}_{p,\Gamma}(\cdot,\,\min\{2\rho_0,\,\gamma_0\})||_{\widetilde{L}^2_{2m-2,\,2m-2}} \le \frac{\gamma_0}{\gamma}, \tag{2.52}$$

and, for any $\lambda \in]0,1[$, *an arbitrary solution* $z \in A_{\gamma_0,\Gamma}$ *of the equation*

$$z^{(2m)}(t) = (1-\lambda)P_\Gamma(z,\, z)(t) + \lambda F(\Gamma(z))(t) \tag{2.53}$$

under condition (2.18), *admits the estimate*

$$||z||_{C^{m-1}_{1,\Gamma}} \le \rho_0, \tag{2.54}$$

Proof. In view of Lemma 2.10, and (2.47), by the notation $y \equiv \Gamma(x) \in C^m_2(]a,\, b[)$ and (1.22), we get

$$\begin{aligned}\sup\{F_{p,\Gamma}(x)(t) : ||x||_{C^{m-1}_{1,\Gamma}} \le \rho\} &= \sup\{F_p(\Gamma(x))(t) : ||x||_{C^{m-1}_{1,\Gamma}} \le \rho\}\\ &= \sup\{F_p(\Gamma(x))(t) : ||\Gamma(x)||_{C^m_2} \le \rho\}\\ &= \sup\{F_p(y)(t) : ||y||_{C^m_2} \le \rho\} \in \widetilde{L}^2_{2m-2,\,2m-2}(]a,\, b[).\end{aligned}$$

Thus (2.51) holds.

From the last relation with $\rho = \min\{2\rho_0,\, \gamma_0\}$, in view of (1.23), we get (2.52).

Let now $z \in A_{\gamma_0,\Gamma}$ is a solution of problem (2.53), (2.18), and $x \equiv \Gamma(z)$. Then

$$x^{(2m+1)}(t) = \Gamma^{(2m+1)}(z)(t) = z^{(2m)}(t) = (1-\lambda)P_\Gamma(z,\, z)(t) + \lambda F(\Gamma(z))(t)$$

$$= (1-\lambda)P(\Gamma(z),\, \Gamma(z))(t) + \lambda F(\Gamma(z))(t) = (1-\lambda)P(x,\, x)(t) + \lambda F(x)(t),$$

i.e., x is a solution of problem (1.24), (1.2) end then the estimate (1.25) holds, from which by Lemma 2.10 we get (2.54). ∎

Now, let the operator $q : C^{m-1}_{1,\Gamma}(]a,b[) \to \widetilde{L}^2_{2m-2,2m-2}(]a,b[)$ be continuous and for any $x \in \widetilde{C}^{m-1}_{1,\Gamma}(]a,b[) \subset C^{m-1}_{1,\Gamma}(]a,b[)$ consider the nonhomogeneous equation

$$z^{(2m)}(t) = \sum_{j=1}^{m} p_{j,\Gamma}(x_0)(t) z^{(j-1)}(\tau_j(t)) + p_{0,\Gamma}(x_0)(t) \int_a^b G(\tau_0(t), s) z(s) ds + q(x)(t) \tag{2.55}$$

and the corresponding homogeneous equation

$$z^{(2m)}(t) = \sum_{j=1}^{m} p_{j,\,\Gamma}(x_0)(t) z^{(j-1)}(\tau_j(t)) + p_{0,\,\Gamma}(x_0)(t) \int_a^b G(\tau_0(t), s) z(s) ds, \tag{2.56}$$

and let E^n be a set of the solutions of problem (2.55), (2.18). Assume that the operator $P : C^m_2(]a,b[) \times C^m_2(]a,b[) \to L_n(]a,b[)$ be γ_0, γ consistent with the boundary condition (1.2), and operator $q : C^m_2(]a,b[) \to \widetilde{L}^2_{2m-2,2m-2}(]a,b[)$, be continuous. Then, by Lemma 2.11, problem (2.55), (2.18) has the unique solution $z \in \widetilde{C}^{2m-1,m}(]a,\, b[)$. But, in view of Lemma 2.6, it is clear that $z \in \widetilde{C}^{m-1}_{1,\Gamma}(]a,\, b[)$. Thus, $E^n \cap \widetilde{C}^{m-1}_{1,\Gamma}(]a,\, b[) \ne \emptyset$, and there exists the operator $U : \widetilde{C}^{m-1}_{1,\Gamma}(]a,\, b[) \to E^n \cap \widetilde{C}^{m-1}_{1,\Gamma}(]a,\, b[)$ defined by the equality

$$U(x)(t) = z(t).$$

Lemma 2.13. *$U : \widetilde{C}^{m-1}_{1,\Gamma}(]a,b[) \to E^n \cap \widetilde{C}^{m-1}_{1,\Gamma}(]a,b[)$ is a continuous operator.*

Proof. Let $x_k \in \widetilde{C}^{m-1}_{1,\Gamma}(]a,\, b[)$ and $z_k(t) = U(x_k)(t)$ $(k = 1, 2)$, $y = y_2 - y_1$, and the operator P is defined by (1.17). Then

$$z^{(2m)}(t) = P_\Gamma(x_2,\, z)(t) + q_0(x_1,\, x_2)(t)$$

where $q_0(x_1,\, x_2)(t) = P_\Gamma(x_2,\, y_1)(t) - P_\Gamma(x_1,\, y_1)(t) + q(x_2)(t) - q(x_1)(t)$. Hence, by item *ii.* of Lemma 2.11 we have

$$||U(x_2) - U(x_1)||_{\widetilde{C}^{m-1}_{1,\Gamma}} \le \gamma ||q_{0,\Gamma}(x_1,\, x_2)||_{\widetilde{L}^2_{2m-2,\,2m-2}}.$$

Since the operators P_Γ and q are continuous, this estimate implies the continuity of the operator U. ∎

3. Proofs

Proof of Remark 1.1. Let z be a solution of problem (2.49), (2.18) with $p_{j,\Gamma}(x_0) = p_j$ $(j = 0, ..., m)$, then $u = \Gamma(z)$ is a solution of problem (1.8), (1.2), and from inequalities (2.19) it follows the estimate

$$|z^{(i-1)}(t)| \le \frac{[(b-t)(t-a)]^{m-i+1/2}}{(m-i)!(2m-2i+1)^{1/2}} \Big(\frac{2}{b-a}\Big)^{m-i+1/2} ||z^{(m)}||_{L^2},$$

for $a \le t \le b$, and

$$||z||_C \le \frac{(b-a)^{m-1/2}}{(m-1)!(2m-1)^{1/2} 2^{m-1/2}} ||z^{(m)}||_{L^2}.$$

From the last estimates, by the definition of the norm $||\cdot||_{\widetilde{C}^{m-1}_{1,\Gamma}}$ and (2.22), we have

$$||z||_{\widetilde{C}^{m-1}_{1,\Gamma}} \le \int_a^b \frac{|\varphi(s) - \varphi(a)| + |\varphi(s) - \varphi(b)|}{|\varphi(b) - \varphi(a)|} ds \frac{(b-a)^{m-1/2}}{(m-1)!(2m-1)^{1/2} 2^{m-1/2}} ||z^{(m)}||_{L^2}$$

$$+ \Big(1 + \sum_{j=1}^{m} \frac{1}{(m-i)!(2m-2i+1)^{1/2}} \Big(\frac{2}{b-a}\Big)^{m-i+1/2}\Big) ||z^{(m)}||_{L^2},$$

from which, by Lemma 2.10, estimate (1.15) and the fact that $u' = z$, (1.16) immediately follows. ∎

Proof of Theorem 1.3. Let δ and λ are the functions and numbers appearing in Definition 1.1, and the operator $\widetilde{F}_{p,\Gamma}(t, \rho)$ be defined by (2.51). We set

$$\eta(t) = \delta(t,\ \gamma_0)\gamma_0 + \widetilde{F}_{p,\Gamma}(t,\, \min\{2\rho_0, \gamma_0\}), \tag{3.57}$$

$$\chi(s)=\begin{cases}1 & \text{for } 0\le s\le\rho_0\\ 2-s/\rho_0 & \text{for } \rho_0<s<2\rho_0\\ 0 & \text{for } s\ge 2\rho_0\end{cases}, \tag{3.58}$$

$$q(x)(t)=\chi(||x||_{\widetilde{C}^{m-1}_{1,\Gamma}})F_{p,\Gamma}(x)(t). \tag{3.59}$$

From (1.22) it is clear that the nonnegative functions $\widetilde{F}_{p,\Gamma}$, η, admits the inclusion

$$\widetilde{F}_{p,\Gamma}(\cdot,\ \min\{2\rho_0,\gamma_0\}),\ \eta\in\widetilde{L}^2_{2m-2,\,2m-2}(]a,\ b[), \tag{3.60}$$

and, for every $x\in A_{\gamma_0,\Gamma}\subset\widetilde{C}^{m-1}_{1,\Gamma}(]a,b[)$ and almost all $t\in]a,b[$, the inequality

$$|q(x)(t)|\le\widetilde{F}_{p,\Gamma}(t,\ \min\{2\rho_0,\gamma_0\})\qquad\text{for } a<t<b \tag{3.61}$$

holds.

Let $U:A_{\gamma_0,\Gamma}\to E^n\cap\widetilde{C}^{m-1}_{1,\Gamma}(]a,\ b[)$ is a operator appeared in Lemma 2.13, from which it follows that U is a continuous operator. From (i) and (ii) of Definition 1.1, i.e., by Lemma 2.11, from (2.48) and (2.50) it is clear that, for each $x\in A_{\gamma_0,\Gamma}$, the conditions

$$||z||_{\widetilde{C}^{m-1}_{1,\Gamma}}\le\gamma_0,\quad |z^{(n-1)}(t)-z^{(n-1)}(s)|\le\int_s^t\eta(\xi)d\xi\qquad\text{for}\qquad a<t<b$$

hold. Thus, in view of Definition 2.3, the operator U maps the ball $A_{\gamma_0,\Gamma}$ into its own subset $S(\rho_1,\,\eta)$. From Lemma 2.2, it follows that $S(\rho_1,\,\eta)$ is the compact subset of the ball $A_{\gamma_0,\Gamma}\subset\widetilde{C}^{m-1}_{1,\Gamma}(]a,\ b[)$, i.e., the operator u maps the ball $A_{\gamma_0,\Gamma}$ into its own compact subset. Therefore, owing to Schauders's principle, there exists $z\in S(\rho_1,\,\eta)\subset A_{\gamma_0,\Gamma}$, such that

$$z(t)=U(z)(t)\quad\text{for}\quad a<t<b.$$

Thus, by (2.55) and notation (3.59), the function $z\,(z\in A_{\gamma_0})$ is a solution of problem (2.53), (2.18), where

$$\lambda=\chi(||z||_{\widetilde{C}^{m-1}_{1,\Gamma}}). \tag{3.62}$$

If $\gamma_0=\rho_0$, then, in view of condition $z\in A_{\gamma_0,\Gamma}$, by (3.58) we have that $\lambda=1$, and then, in view of (2.53) and (3.59), the function z is a solution of the equation

$$z^{(2m)}(t)=F(\Gamma(z))(t) \tag{3.63}$$

under boundary conditions (2.18). Thus the function $u=\Gamma(z)$ (where $u'=z$) is a solution of problem (1.1), (1.2).

Let us show now, that z admits estimate (2.50) in the case when $\rho_0<\gamma_0$. Assume the contrary. Then either

$$\rho_0<||z||_{\widetilde{C}^{m-1}_1}<2\rho_0, \tag{3.64}$$

or

$$||z||_{\widetilde{C}_1^{m-1}} \geq 2\rho_0. \tag{3.65}$$

If condition (3.64) holds, then by virtue of (3.58) and (3.62), z is a solution of equation (2.53) with $\lambda \in]0, 1[$, and then (2.54) holds. But this contradicts (3.64).

Assume now that (3.65) is fulfilled. Then, by virtue of (3.58) and (3.62), we have that $\lambda = 0$. Therefore, $x \in A_{\gamma_0,\Gamma}$ is a solution of problem (2.56), (2.18). Thus, from (ii) of Lemma 2.11, it is obvious that $z \equiv 0$, because problem (2.49), (2.18) has only a trivial solution. But this contradicts condition (3.65), i.e., estimate (2.54) is valid. From estimate (2.54) and (3.58), we have that $\lambda = 1$, and then, in view of (2.55) and (3.59), the function z is a solution of problem (3.63), (2.18), which admits to the estimate (2.50). Thus, the function $u = \Gamma(z)$ (where $u' = z$) is a solution of problem (1.1), (1.2). ∎

Proof of Corollary 1.2. First, note that, in view of condition (1.28), there exists such a $\gamma_0 > 2\rho_0$ that condition (1.23) holds, and, in view of Definition 1.2, the operator P is γ_0, γ consistent.

On the other hand, from (1.28), it follows the existence of the number ρ_0, such that

$$\gamma||\eta(\cdot, \rho)||_{\widetilde{L}^2_{2n-2m-2,\,2m-2}} < \rho \quad \text{for} \qquad \rho > \rho_0. \tag{3.66}$$

Let x be a solution of problem (1.24), (1.2) for some $\lambda \in]0, 1[$. Then $y = x$ is also a solution of problem (1.20), (1.2), where $q(t) = \lambda\Big(F(x)(t) - P(x, x)(t)\Big)$. Now, let $\rho = ||x||_{\widetilde{C}_1^{m-1}}$ and assume that

$$\rho > \rho_0. \tag{3.67}$$

holds. Then, in view of the $\gamma-$consistency of the operator p with boundary conditions (1.2), inequality (1.21) holds and thus, by condition (1.26), we have

$$\rho = ||x||_{\widetilde{C}_1^{m-1}} \leq \gamma||q(x)||_{\widetilde{L}^2_{2n-2m-2,\,2m-2}} \leq \gamma||\eta(\cdot, \rho)||_{\widetilde{L}^2_{2n-2m-2,\,2m-2}}.$$

But the last inequality contradicts (3.66). Thus, assumption (3.67) is not valid and $\rho \leq \rho_0$. Therefore, for any $\lambda \in]0, 1[$, an arbitrary solution of problem (1.24), (1.2) admits the estimate (1.25). Therefore, all the conditions of Theorem 1.3 are fulfilled, from which the solvability of problem (1.1), (1.2) follows. ∎

Proof of Theorem 1.4. Let r_n be the constant defined in Remark 1.1. First, let us prove that the operator P is γ_0, r_n consistent with boundary conditions (1.2). From the conditions of our theorem, it is obvious that the item (i) of Definition 1.1 is satisfied. Let now x be an arbitrary fixed function from the set A_{γ_0} and let $p_j(t) \equiv p_j(x)(t)$. Thus in view of (1.31), (1.32) all the assumptions of Theorem 1.1 are satisfied, and then for any $q \in \widetilde{L}^2_{2n-2m-2,\,2m-2}(]a, b[)$ the problem (1.20), (1.2) has unique solution y. Also in view of Remark 1.1 there exists the constant $r_n > 0$,

(which depends only on the numbers l_{kj}, $\bar{l}_{kj}, \gamma_{kj}$ $(k = 1, 2;\ j = 1, \cdots, m)$, and a, b, t^*, n) such that estimate (1.21) holds with $\gamma = r_n$. I.e., the operator P is γ_0, r_n consistent with boundary conditions (1.2). Therefore all the assumptions of Corollary 1.1 are fulfilled, from which the solvability of problem (1.1), (1.2) follows. ∎

Proof of Theorem 1.5. Let r_n be the constant defined in Remark 1.1. First prove that operator P is r_n consistent with boundary conditions (1.2). From the conditions of our theorem it is obvious that the item (i) of Definition 1.1 is satisfied. Let now γ_0 be an arbitrary nonnegative number, x be arbitrary fixed function from the space A_{γ_0} and let $p_j(t) \equiv p_j(x)(t)$. Then, in view of (1.34), (1.35), all the assumptions of Theorem 1.1 are satisfied and then, for any $q \in \widetilde{L}^2_{2n-2m-2,2m-2}(]a, b[)$, problem (1.20), (1.2) has a unique solution y. Also, in view of Remark 1.1, there exists the constant $r_n > 0$, (which depends only on the numbers $l_{kj}, \bar{l}_{kj}, \gamma_{kj}$ $(k = 1, 2;\ j = 1, ..., m)$, and $a, b, t^*, n,$) such that estimate (1.21) holds with $\gamma = r_n$. I.e., the operator P is γ_0, r_n consistent with boundary conditions (1.2) for arbitrary $\gamma_0 > 0$. Thus, by Definition 1.1, the operator P is r_n consistent with boundary conditions (1.2). Therefore, all the assumptions of Corollary 1.2 are fulfilled, from which the solvability of problem (1.1), (1.2) follows. ∎

Proof of Remark 1.2. By Schwartz's inequality, the definition of the norm $||y||_{\widetilde{C}_1^{m-1}}$ and inequalities (1.36), (2.2) for any $x, y \in A_{\gamma_0}$ and $z = y - x$, we have

$$
\begin{aligned}
&|p_j(y)(t)z^{(j-1)}(\tau_j(t))| = |p_j(y)(t)z^{(j-1)}(t)| + |p_j(y)(t)|\Big|\int_t^{\tau_j(t)} z^{(j)}(\psi)d\psi\Big| \\
&\le ||z||_{\widetilde{C}_1^{m-1}}|p_j(y)(t)|\alpha_j(t)\Big(1 + \frac{1}{\alpha_j(t)}\Big(\int_t^{\tau_j(t)} (\psi - a)^{2m-2j}d\psi\Big)^{1/2}\Big)
\end{aligned} \tag{3.68}
$$

for $a < t < b$. On the other hand, from conditions (1.37), by Lemmas 2.8 and 2.9, it is cleat that

$$
\begin{aligned}
&\alpha_j^{-1}(t)\Big(\int_s^{\tau_j(s)} (\xi - a)^{2m-2j}d\xi\Big)^{1/2} \le \frac{\sqrt{\kappa(1+\kappa)}}{\varepsilon^{m-j+1/2}} \quad \text{for } s \in]a,\ a + \varepsilon] \cup [b - \varepsilon, b[, \\
&\alpha_j^{-1}(t)\Big(\int_s^{\tau_j(s)} (\xi - a)^{2m-2j}d\xi\Big)^{1/2} \le \varepsilon^{-2m+2j-1}\Big(\int_a^b (\xi - a)^{2m-2j}d\xi\Big)^{1/2} \\
&= \frac{(b-a)^{m-j+1/2}}{\sqrt{2m-2j+1}\varepsilon^{2m-2j+1}} \quad \text{for } s \in]a + \varepsilon, b - \varepsilon[.
\end{aligned}
$$

Then, if we put

$$
\kappa_0 = \min\Big\{\frac{\sqrt{\kappa(1+\kappa)}}{\varepsilon^{m-j+1/2}},\ \frac{(b-a)^{m-j+1/2}}{\sqrt{2m-2j+1}\varepsilon^{2m-2j+1}}\Big\}, \tag{3.69}
$$

from (3.68), by the last estimates, we get the inequality

$$
\begin{aligned}
|p_j(y)(t)z^{(j-1)}(\tau_j(t))| &\le ||z||_{\widetilde{C}_1^{m-1}}(1+\kappa_0)|p_j(y)(t)|\alpha_j(t) \\
&\le ||z||_{\widetilde{C}_1^{m-1}}(1+\kappa_0)\delta_j(t,\, ||z||_{\widetilde{C}_1^{m-1}})
\end{aligned} \tag{3.70}
$$

for $a < t < b$. Analogously, we get that

$$
|(p_j(y)(t) - p_j(x)(t))x^{(j-1)}(\tau_j(t))| \le ||x||_{\widetilde{C}_1^{m-1}}(1+\kappa_0)|p_j(y)(t) - p_j(x)(t)|\alpha_j(t)
$$

for $a < t < b$. From (3.70) and the last inequality, it is obvious that the operator P defined by equality (1.17) continuously acting from A_{γ_0} to the space $L_n(]a, b[)$, and the item (ii) of Definition 1.1 holds, with $\delta(t,\rho) = (1+\kappa_0)\sum\limits_{j=1}^{m}\delta_j(t,\rho)$. ∎

Proof of Corollary 1.3. From conditions (1.40) and (1.37), by Remark 1.2, we obtain that the operator P defined by equality (1.17) with $p_j(x)(t) = p_j(t)$, continuously acting from A_{γ_0} to the space $L_n(]a, b[)$, for any $\gamma_0 > 0$, i.e., continuously acting from $\widetilde{C}_1^{m-1}(]a, b[)$ to the space $L_n(]a,\, b[)$. Therefore, it is clear that all the conditions of Theorem 1.5 would be satisfied with

$$
F(x)(t) = f(t, x(\tau_1(t)), x'(\tau_2(t)), \cdots, x^{(m-1)}(\tau_m(t))), \quad \delta(t,\rho) = (1+\kappa_0)\sum_{j=1}^{m}|p_j(t)|,
$$

where the constant κ_0 is defined by equality (3.69). Thus problem (1.38), (1.2) is solvable. ∎

Proof of Corollary 1.4. Let the operators $F, p_1 : C^{m-1}(]a,\, b[) \to L_{loc}(]a,\ b[)$, and the function $\eta :]a,\, b[\times R^+ \to R^+$ be defined by equalities

$$
F(x)(t) = -\frac{\lambda|x(t)|^k}{[(t-a)(b-t)]^{2+k/2}}x(\tau(t))+q(x)(t), \quad p_1(x)(t) = -\frac{\lambda|x(t)|^k}{[(t-a)(b-t)]^{2+k/2}}.
$$

Then, it is easy to verify that in view of (1.43)-(1.45), conditions (1.11), (1.12), (1.26), (1.31)–(1.40) are satisfied with

$$
\begin{aligned}
&\delta(t,\, \rho) = \frac{\rho^k\lambda}{[(t-a)(b-t)]^2}, \quad l_{01} = \frac{4\gamma_0^k\lambda}{(b-a)^2}, \quad \bar{l}_{01} = \frac{16\gamma_0^k\lambda}{(b-a)^2}, \\
&r_2 = \Big(1+\sqrt{\frac{2}{b-a}}\Big)\frac{2(1+b-a)(b-a)^2}{(b-a)^2 - 16\lambda\gamma_0^k(1+[2(b-a)]^{1/4})}, \\
&B_0 = B_1 = \frac{16\lambda\gamma_0^k}{(b-a)^2}(1+[2(b-a)]^{1/4}), \quad t^* = (a+b)/2, \quad \gamma_{01} = \gamma_{11} = \frac{1}{4}.
\end{aligned} \tag{3.71}
$$

Thus all the condition of Theorem 1.4 are satisfied, from which follows solvability of problem (1.41), (1.2). ∎

Proof of Corollary 1.5. Let the operators $F, p_1 : C^{m-1}(]a, b[) \to L_{loc}(]a, b[)$, and the function $\eta :]a, b[\times R^+ \to R^+$ be defined by equalities

$$
F(x)(t) = -\frac{\lambda|\sin x^k(t)|}{[(t-a)(b-t)]^2}x(\tau(t)) + q(x)(t), \quad p_1(x)(t) = -\frac{\lambda|\sin x^k(t)|}{[(t-a)(b-t)]^2}.
$$

Then it is easy to verify that in view of (1.28), (1.43), and (1.46), all the conditions of Theorem 1.5 follow, where δ, l_{01}, $\bar{l}_{01}$, r_2, B_0, B_1, t^*, γ_{01}, γ_{11}, are defined by (3.71) with $\rho = 1$, $\gamma_0 = 1$, from which the solvability of problem (1.41), (1.2) follows. ∎

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A COUPLED FIXED POINT THEOREM AND t-NORM OF HADŽÍC TYPE IN FUZZY METRIC SPACE

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Abstract. In this paper, we introduce a coupled fixed point theorem and t-Norm of Hadžíc type in fuzzy metric space and we also give an application of our result.

Keywords: coupled fixed points, fuzzy metric space, occasionally weakly compatible maps.
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1. Introduction

Zadeh's [10] investigation of the notion of fuzzy set has led to a rich growth of fuzzy Mathematics. Many authors have introduced the concept of fuzzy metric in different ways. There are many view points of the notion of the fuzzy metric spaces in fuzzy topology (see Deng [1], Erceg [2], George and Veeramani [3], Kaleva and Seikkala [5], Kramosil and Michalek [6]). In 2009, Lakshmikantham et.al. [7] introduced coupled fixed point theorems in partially ordered metric spaces. Motivated by them, we introduce a coupled fixed point theorem and t-Norm of Hadžíc type in fuzzy metric space.

2. Preliminaries

Definition 2.1. A t-norm t is a 2-place function $t : [0,1] \times [0,1] \to [0,1]$ which satisfies the following conditions:

(i) $t(0,0) = 0$;
(ii) $t(0,1) = 1$;
(iii) $t(a,b) = t(b,a)$;
(iv) if $a \leq c$ and $b \leq d$, then $t(a,b) \leq t(c,d)$;
(v) $t(t(a,b),c) = t(a,t(b,c))$,

for all $a, b, c, d \in [0,1]$.

Definition 2.2. Let $\sup_{0<t<1} \Delta(t,t) = 1$. A t-norm Δ is said to be of Hadžíc-type (H-type) if the family of functions $\left\{\Delta^m(t)\right\}_{m=1}^{\infty}$ is equi-continuous at $t = 1$, where

$$\Delta^1(t) = t,\ \Delta^{m+1}(t) = \Delta(\Delta^m(t)),\ m = 1, 2, 3, \ldots,\ t \in [0,1].$$

The t-norm $\Delta_M = \min$ is an example of H-type.

Remark 2.1. Δ is an H-type if and only if, for any $\lambda \in (0,1)$, there exists $\delta(\lambda) \in (0,1)$ such that following implications hold:

$$t > 1 - \delta(\lambda) \text{ implies } \Delta^m(t) > 1 - \lambda.$$

Remark 2.2. Every t-norm Δ_M is of Hadžíc-type but converse need not be true (see [4]).

Definition 2.3. The 3-tuple (X, M, t) is called fuzzy metric space in the sense of Kramosil and Michalek [6] if X is an arbitrary set, t is a continuous t-norm and M is a fuzzy set on $X^2 \times [0, \infty)$ satisfying the following conditions:

(i) $M(x, y, 0) = 0$;
(ii) $M(x, y, t) = 1$ if and only if $x = y$;
(iii) $M(x, y, t) = M(y, x, t)$;
(iv) $M(x, z, t + s) \geq t(M(x, y, t), M(y, z, s))$;
(v) $M(x, y, .) : [0, \infty) \to [0, 1]$ is left continuous;

for all $x, y, z \in X$ and $s, t > 0$.

Note that the function $M(x, y, t)$ can be thought of as the degree of nearness between x and y with respect to t.

In our theory, we consider the following conditions:

(vi) $\lim_{t\to\infty} M(x, y, t) = 1$ for all x, y in X.

Definition 2.4. A sequence $\{x_n\}$ in a fuzzy metric space (X, M, t) is said to be

(i) convergent to x in X if, for every $\varepsilon > 0$ and $\lambda > 0$, there is an integer $n_0 \in N$ such that $M(x_n, x, \varepsilon) > 1 - \lambda$, for all $n \geq n_0$.

(ii) Cauchy sequence in X if, for every $\varepsilon > 0$ and $\lambda > 0$, there is an integer $n_0 \in N$ such that $M(x_n, x_m, \varepsilon) > 1 - \lambda$, for all $m, n \geq n_0$.

(iii) complete if every Cauchy sequence in X is convergent in X.

Definition 2.5. Define $\Phi = \{\phi : R^+ \to R^+\}$, where $R^+ = [0, +\infty)$ and each $\phi \in \Phi$ satisfying the following conditions:

(ϕ_1) ϕ is non-decreasing.

(ϕ_2) ϕ is upper semi-continuous from the right.

(ϕ_3) $\sum_{n=0}^{\infty} \phi^n(t) < +\infty$ for all $t > 0$, where $\phi^{n+1}(t) = \phi(\phi^n(t)), n \in N$.

Clearly, if $\phi \in \Phi$, then $\phi(t) < t$ for all $t > 0$.

Definition 2.6. An element $x \in X$ is called the common fixed point of the mappings $f : X \times X \to X$ and $g : X \to X$ if $x = f(x, x) = g(x)$.

Definition 2.7. An element $(x, y) \in X \times X$ is called a

(i) coupled fixed point of the mapping $f : X \times X \to X$ if $f(x, y) = x,\ f(y, x) = y$.

(ii) coupled coincidence point of the mappings $f : X \times X \to X$ and $g : X \to X$ if $f(x, y) = g(x),\ f(y, x) = g(y)$.

(iii) common coupled fixed point of the mappings $f : X \times X \to X$ and $g : X \to X$ if $x = f(x, y) = g(x),\ y = f(y, x) = g(y)$.

Definition 2.8. The mappings $f : X \times X \to X$ and $g : X \to X$ are called commutative if $gf(x, y) = f(gx, gy)$ for all $x, y \in X$.

Definition 2.9. The mappings $f : X \times X \to X$ and $g : X \to X$ are called weakly compatible if $f(x, y) = g(x)$, $f(y, x) = g(y)$ implies $gf(x, y) = f(gx, gy)$, $gf(y, x) = f(g(y), g(x))$ for all $x, y \in X$.

Remark 2.3. For convenience, we denote

$$\big[M(x, y, t)\big]^n = \underbrace{M(x, y, t) * M(x, y, t) * \cdots * M(x, y, t)}_{n}, \text{ for all } n \in N. \tag{2.1}$$

3. Main result

Theorem 3.1. *Let $(X, M, *)$ be a fuzzy metric space, $*$ being continuous t-norm of H-type. Let $f : X \times X \to X$ and $g : X \to X$ be two mappings and there exists $\phi \in \Phi$ such that*

$$M(f(x, y), f(u, v), \phi(t)) \geq \psi[M(gx, gu, t) * M(gy, gv, t)], \tag{3.1}$$

for all $x, y, u, v \in X$ and $t > 0$, where $\psi : [0,1] \to [0,1]$ is continuous function such that $\psi(t) \geq t$ for all $t \in [0,1]$.

Suppose that $f(X \times X) \subseteq g(X)$, f and g are weakly compatible, range space of one of the mappings f or g is complete. Then f and g have a coupled coincidence point. Moreover, there exists a unique point x in X such that $x = f(x,x) = g(x)$.

Proof. Let x_0, y_0 be two arbitrary points in X. Since $f(X \times X) \subseteq g(X)$, we can choose x_1, y_1 in X such that $g(x_1) = f(x_0, y_0), g(y_1) = f(y_0, x_0)$.

Continuing in this way, we can construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $g(x_{n+1}) = f(x_n, y_n)$ and $g(y_{n+1}) = f(y_n, x_n)$ for all $n \geq 0$.

Step I. We first show that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences.

Since $*$ is a t-norm of H-type, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\underbrace{(1-\delta) * (1-\delta) * \cdots * (1-\delta)}_{p} \geq (1-\varepsilon), \text{ for all } p \in N. \tag{3.2}$$

Since $\lim_{t\to\infty} M(x,y,t) = 1$, for all x, y in X, there exists $t_0 > 0$ such that

$$M(gx_0, gx_1, t_0) \geq (1-\delta) \text{ and } M(gy_0, gy_1, t_0) \geq (1-\delta).$$

Also, since $\phi \in \Phi$, using condition (ϕ_3), we have

$$\sum_{n=1}^{\infty} \phi^n(t_0) < \infty.$$

Then for any $t > 0$ there exists $n_0 \in N$ such that

$$t > \sum_{k=n_0}^{\infty} \phi^k(t_0). \tag{3.3}$$

Using condition (3.1), we have

$$\begin{aligned} M(gx_1, gx_2, \phi(t_0)) &= M(f(x_0,y_0), f(x_1,y_1), \phi(t_0)) \\ &\geq \psi[M(gx_0, gx_1, t_0) * M(gy_0, gy_1, t_0)] \\ &\geq M(gx_0, gx_1, t_0) * M(gy_0, gy_1, t_0) \end{aligned}$$

$$\begin{aligned} M(gy_1, gy_2, \phi(t_0)) &= M(f(y_0,x_0), f(y_1,x_1), \phi(t_0)) \\ &\geq \psi[M(gy_0, gy_1, t_0) * M(gx_0, gx_1, t_0)] \\ &\geq M(gy_0, gy_1, t_0) * M(gx_0, gx_1, t_0). \end{aligned}$$

Similarly, we can also get

$$\begin{aligned} M(gx_2, gx_2, \phi^2(t_0)) &= M(f(x_1, y_1), f(x_2, y_2), \phi^2(t_0)) \\ &\geq \psi[M(gx_1, gx_2, t_0) * M(gy_1, gy_2, t_0)] \\ &\geq [M(gx_0, gx_1, t_0)]^2 * [M(gy_0, gy_1, t_0]^2 \end{aligned}$$

$$\begin{aligned} M(gy_2, gy_3, \phi^2(t_0)) &= M(f(y_1, x_1), f(y_2, x_2), \phi^2(t_0)) \\ &\geq [M(gy_0, gy_1, t_0)]^2 * [M(gx_0, gx_1, t_0)]^2. \end{aligned}$$

Continuing in this way, we can get

$$\begin{aligned} M(gx_n, gx_{n+1}, \phi^n(t_0)) &\geq [M(gx_0, gx_1, t_0)]^{2^{n-1}} * [M(gy_0, gy_1, t_0)]^{2^{n-1}} \\ M(gy_n, gy_{n+1}, \phi^n(t_0)) &\geq [M(gy_0, gy_1, t_0)]^{2^{n-1}} * [M(gx_0, gx_1, t_0)]^{2^{n-1}}. \end{aligned}$$

So, from (3.2) and (3.3), for $m > n \geq n_0$, we have

$$\begin{aligned} M(gx_n, gx_m, t) &\geq M\left(gx_n, gx_m, \sum_{k=n_0}^{\infty} \phi^k(t_0)\right) \\ &\geq M\left(gx_n, gx_m, \sum_{k=n}^{m-1} \phi^k(t_0)\right) \\ &\geq M\left(gx_n, gx_{n+1}, \phi^n(t_0)\right) * M\left(gx_{n+1}, gx_{n+2}, \phi^{n+1}(t_0)\right) \\ &\quad * \cdots * M\left(gx_{m-1}, gx_m, \phi^{m-1}(t_0)\right) \\ &\geq \{[M(gx_0, gx_1, t_0)]^{2^{n-1}} * [M(gy_0, gy_1, t_0)]^{2^{n-1}}\} \\ &\quad * \{[M(gx_0, gx_1, t_0)]^{2^n} * [M(gy_0, gy_1, t_0)]^{2^n}\} * \cdots \\ &\quad * \{[M(gx_0, gx_1, t_0)]^{2^{m-2}} * [M(gy_0, gy_1, t_0)]^{2^{m-2}}\} \\ &= [M(gx_0, gx_1, t_0)]^{2^{n-1}(2^{m-n}-1)} * [M(gy_0, gy_1, t_0)]^{2^{n-1}(2^{m-n}-1)} \\ &\geq \underbrace{(1-\delta) * (1-\delta) * \cdots * (1-\delta)}_{2^{n-1}(2^{m-n}-1)} \geq (1-\varepsilon), \end{aligned}$$

which implies that

$$M(gx_n, gx_m, t) \geq (1-\varepsilon) \text{ for all } m, n \in N \text{ with } m > n \geq n_0 \text{ and } t > 0.$$

Therefore, $\{gx_n\}$ is a Cauchy sequence in X.
Similarly, $\{gy_n\}$ is also a Cauchy sequence in X.

Step II. Now, we show that f and g have coupled coincidence point. Without loss of generality, we can assume that $g(X)$ is complete, there exist points x, y in $g(X)$ so that $\lim_{n\to\infty} g(x_{n+1}) = x$ and $\lim_{n\to\infty} g(y_{n+1}) = y$.

For $x, y \in g(X)$ implies the existence of p, q in X such that $g(p) = x,\ g(q) = y$ and hence

$$\lim_{n\to\infty} g(x_{n+1}) = \lim_{n\to\infty} f(x_n, y_n) = g(p) = x,$$
$$\lim_{n\to\infty} g(y_{n+1}) = \lim_{n\to\infty} f(y_n, x_n) = g(q) = y.$$

From (3.1), we have

$$\begin{aligned} M(f(x_n, y_n), f(p, q), \phi(t)) &\geq \psi[M(gx_n, g(p), t) * M(gy_n, g(q), t)] \\ &\geq [M(gx_n, g(p), t) * M(gy_n, g(q), t)]. \end{aligned}$$

Taking limit as $n \to \infty$, we have

$$M(g(p), f(p, q), \phi(t)) = 1, \text{ i.e., } f(p, q) = g(p) = x.$$

Similarly, $f(q, p) = g(q) = y$.
Since, f and g are weakly compatible, so $f(p, q) = g(p) = x$ and $f(q, p) = g(q) = y$ implies $gf(p, q) = f(g(p), g(q))$ and $gf(q, p) = f(g(q), g(p))$, i.e., $g(x) = f(x, y)$ and $g(y) = f(y, x)$.
Thus, f and g have a coupled coincidence point.

Step III. Now we show that $g(x) = x$ and $g(y) = y$.

Since $*$ is a t-norm of H-type, then for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\underbrace{(1-\delta) * (1-\delta) * \cdots * (1-\delta)}_{p} \geq (1-\varepsilon), \text{ for all } p \in N.$$

Also, since $\phi \in \Phi$, using condition (ϕ_3), we have

$$\sum_{n=1}^{\infty} \phi^n(t_0) < \infty.$$

Then for any $t > 0$ there exists $n_0 \in N$ such that

$$t > \sum_{k=n_0}^{\infty} \phi^k(t_0).$$

Using condition (3.1), we have

$$\begin{aligned} M(gx, x, \phi(t_0)) &= M(f(x, y), f(p, q), \phi(t_0)) \\ &\geq \psi[M(gx, gp, t_0) * M(gy, gq, t_0)] \\ &\geq M(gx, gp, t_0) * M(gy, gq, t_0) \end{aligned}$$

Similarly,

$$M(gy, y, \phi(t_0)) \geq M(gy, gq, t_0) * M(gx, gp, t_0).$$

Continuing in the same way, for all $n \in N$

$$M(gx, x, \phi^n(t_0)) \geq [M(gx, gx, t_0)]^{2^{n-1}} * [M(gy, y, t_0)]^{2^{n-1}}.$$

Thus, we have

$$\begin{aligned} M(gx, x, t) &\geq M\left(gx, x, \sum_{k=n_0}^{\infty} \phi^k(t_0)\right) \\ &\geq M\left(gx, x, \phi^{n_0}(t_0)\right) \\ &\geq \left[M(gx, x, t_0)\right]^{2^{n_0-1}} * \left[M(gy, y, t_0)\right]^{2^{n_0-1}} \\ \geq \underbrace{(1-\delta) * (1-\delta) * \cdots * (1-\delta)}_{2^{n_0}} &\geq (1-\varepsilon), \text{ for all } n_0 \in N. \end{aligned}$$

So, for any $\varepsilon > 0$, we have

$$M(gx, y, t) \geq (1-\varepsilon), \text{ for all } t > 0.$$

This implies that $g(x) = x$. Similarly, $g(y) = y$.

Step IV. Now we show that $x = y$.

Since $*$ is a t-norm of H-type for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\underbrace{(1-\delta) * (1-\delta) * \cdots * (1-\delta)}_{P} \geq (1-\varepsilon), \text{ for all } p \in N.$$

Since $\lim_{t\to\infty} M(x, y, t) = 1$, for all x, y in X, there exists $t_0 > 0$ such that

$$M(x, y, t_0) \geq (1-\delta).$$

Also, since $\phi \in \Phi$, using condition (ϕ_3), we have

$$\sum_{n=1}^{\infty} \phi^n(t_0) < \infty.$$

Then for any $t > 0$ there exists $n_0 \in N$ such that

$$t > \sum_{k=n_0}^{\infty} \phi^k(t_0).$$

Using condition (3.1), we have

$$\begin{aligned} M(x, y, \phi(t_0)) &= M(f(p, q), f(q, p), \phi(t_0)) \\ &\geq \psi[M(gp, gq, t_0) * M(gq, gp, t_0)] \\ &\geq M(gp, gq, t_0) * M(gq, gp, t_0) \\ &= [M(x, y, t_0)]^2. \end{aligned}$$

Continuing likewise, we have for all $n \in N$, that

$$M(x, y, \phi^n(t_0)) \geq [M(x, y, t_0)]^{2^n}.$$

Thus, we have

$$\begin{aligned} M(x, y, t) &\geq M\left(x, y, \sum_{k=n_0}^{\infty} \phi^k(t_0)\right) \\ &\geq M\left(x, y, \phi^{n_0}(t_0)\right) \\ &\geq \left[M(x, y, t_0)\right]^{2^{n_0}}. \end{aligned}$$

$$\geq \underbrace{(1-\delta) * (1-\delta) * \cdots * (1-\delta)}_{2^{n_0}} \geq (1-\varepsilon), \text{ which implies that } x = y.$$

Thus, f and g have common fixed point x in X.

Step V. Uniqueness:

Let z be any point in X such that $z \neq x$ with $g(z) = z = f(z, z)$.
Since $*$ is a t-norm of H-type for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\underbrace{(1-\delta) * (1-\delta) * \cdots * (1-\delta)}_{p} \geq (1-\varepsilon), \text{ for all } p \in N.$$

Since $\lim_{t\to\infty} M(x, y, t) = 1$, for all x, y in X, there exists $t_0 > 0$ such that

$$M(x, y, t_0) \geq (1-\delta).$$

Also, since $\phi \in \Phi$, using condition (ϕ_3), we have

$$\sum_{n=1}^{\infty} \phi^n(t_0) < \infty.$$

Then for any $t > 0$ there exists $n_0 \in N$ such that

$$t > \sum_{k=n_0}^{\infty} \phi^k(t_0).$$

Using condition (3.1), we have

$$\begin{aligned} M(x, z, \phi(t_0)) &= M(f(x, x), f(z, z), \phi(t_0)) \\ &\geq \psi[M(g(x), g(z), t_0) * M(g(x), g(z), t_0)] \\ &\geq M(g(x), g(z), t_0) * M(g(x), g(z), t_0) \\ &= [M(x, z, t_0)]^2. \end{aligned}$$

Thus, we have

$$\begin{aligned} M(x,z,t) &\geq M\left(x,z,\sum_{k=n_0}^{\infty}\phi^k(t_0)\right) \\ &\geq M\left(x,z,\phi^{n_0}(t_0)\right) \\ &\geq \left(\left[M(x,z,t_0)\right]^{2^{n_0-1}}\right)^2 \\ &\geq \left(M(x,z,t_0)\right)^{2^{n_0}} \\ &\geq \underbrace{(1-\delta)*(1-\delta)*\cdots*(1-\delta)}_{2^{n_0}} \geq (1-\varepsilon), \end{aligned}$$

which implies that $x = z$. Thus, f and g have a unique common fixed point in X.

4. An application

Theorem 4.1. *Let $(X, M, *)$ be a fuzzy metric space, $*$ being continuous t-norm define by $a * b = \min\{a, b\}$ for all a, b in X. Let A, B be weakly compatible self maps on X satisfying the following condition:*

(4.1) $M(X) \subseteq N(X)$,

(4.2) *there exists $\phi \in \Phi$ such that $M(Ax, Ay, \phi(t)) \geq \psi[M(Bx, By, t)]$, for all $x, y, \in X$ and $t > 0$, where $\psi : [0,1] \to [0,1]$ is continuous function such that $\psi(t) \geq t$ for all $t \in [0,1]$.*

If range of any one of the maps A or B is complete, then A and B have a unique common fixed point in X.

Proof. Assuming $f(x, y) = A(x)$ and $g(x) = A(x)$ for all $x, y \in X$ in Theorem 3.1, we get the proof.

Taking $\phi(t) = kt,\ k \in (0,1)$ and $\psi(t) = t$, we have the following:

Corollary 4.1. *Let $(X, M, *)$ be a fuzzy metric space, $*$ being continuous t-norm define by $a * b = \min\{a, b\}$ for all a, b in X. Let A, B be weakly compatible self maps on X satisfying (4.1) and the following condition:*

(4.3) *there exists $k \in (0,1)$ such that $M(Ax, Ay, kt) \geq M(Bx, By, t)$, for all $x, y \in X$ and $t > 0$.*

If range of any one of the maps A or B is complete, then A and B have a unique common fixed point in X.

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ON WEAKLY SS-QUASINORMAL SUBGROUPS OF FINITE GROUPS

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Abstract. Let H be a subgroup of a finite group G, then we say that H is weakly SS-quasinormal in G, if there exists a normal subgroup T of G such that HT is s-permutable and $H \cap T$ is SS-quasinormal in G. In this paper, we investigate the influence of some weakly SS-quasinormal subgroups on the structure of G. Some new criterias about the p-nilpotency and supersolubility of a finite group were obtained. We also generalized some known results about formations.

Keywords: s-permutable subgroup, SS-quasinormal subgroup, weakly SS-quasinormal subgroup, p-nilpotent group, formation.

Mathematical Subject Classification (2010): 20D10, 20D20.

1. Introduction

All groups considered in this paper will be finite and we use conventional notions and notation, as in D. Gorenstein [5]. Let $\mathcal{F}$ denote a formation, we use $\mathcal{U}$ to stand for the class of all supersoluble groups. Let H be a subgroup of G, $T \leq G$ is said to be a supplement of H in G if $HT = G$. A subgroup H of G is said to be $\mathcal{F}$-supplemented in G if there exists a subgroup $L \in \mathcal{F}$ such that $G = HL$. In this case, we say that L is an $\mathcal{F}$-supplement of H in G.

Recall that a subgroup H of G is said to be s-permutable [11] (or s-quasinormal [3]) in G, if H permutes with every Sylow subgroup P of G. Following Wang in [18], a subgroup H is c-normal in G if G has a normal subgroup T such that $G = HT$ and $H \cap T \leq H_G$, where H_G is the normal core of H in G. By assuming that some subgroups of G satisfying the s-permutability or c-normality, many interesting results have been derived (see for example, [16], [1], [15], [12], [20], [22], [2], [10]). As a development, recently in [7], the concept of S-embedded subgroup

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was introduced: a subgroup H is said to be S-embedded in G if there exists a normal subgroup N such that HN is s-permutable in G and $H \cap N \leq H_{sG}$, where H_{sG} is the largest s-permutable subgroup of G contained in H. In [7], the authors obtained that:

Theorem C and D. *Let $\mathcal{F}$ be a saturated formation containing all supersoluble groups and G a group with a normal subgroup E such that $G/E \in \mathcal{F}$. Suppose that for every non-cyclic Sylow subgroup P of E (or $F^*(E)$, respectively), every maximal subgroup of P or every cyclic subgroup H of P with prime order or order 4 (if P is a non-abelian 2-group and $H \nsubseteq Z_\infty(G)$) is S-embedded in G. Then $G \in \mathcal{F}$.*

These two theorems generalized a lot of meaningful results. As another generalizations of the s-permutability, in [14] the authors introduced that: a subgroup H of G is said to be an SS-quasinormal subgroup (Supplement-Sylow-quasinormal subgroup) of G if there is a supplement B of H to G such that H permutes with every Sylow subgroup of B. In this paper, we integrated the above concepts and introduce that:

Definition 1.1 A subgroup H of a group G is said to be weakly SS-quasinormal in G, if there exists a normal subgroup T of G such that HT is s-permutable and $H \cap T$ is SS-quasinormal in G.

Remark. Obviously, every S-embedded subgroup and SS-quasinormal subgroup of G is weakly SS-quasinormal in G. In general, a weakly SS-quasinormal subgroup of G need not be S-embedded or SS-quasinormal in G. For instance:

Example 1. Let $G = S_5$ be the symmetric group of degree 5, $H = S_4$ and P a Sylow 5-subgroup of G. Since P is a supplement of H to G and H permutes with P, H is SS-quasinormal and thus weakly SS-quasinormal in G. But neither H nor $H \cap A_5 = A_4$ is s-permutable in G, because they are not subnormal subgroups of G. Since the only normal subgroups of G are A_5 and G itself, $H = S_4$ is not S-embedded in G.

Example 2. Let $G = S_5$, $K = \langle (12) \rangle$ and $T = A_5$. Since $T \trianglelefteq G$ is a complement of K, K is weakly SS-quasinormal in G. But the only supplement of K to G are A_5 and G itself and $K\langle (12345) \rangle \neq \langle (12345) \rangle K$, thus we know K is not SS-quasinormal in G.

In this paper, we investigate the influence of some weakly SS-quasinormal subgroups on the structure of a finite group G. Our main result is:

Main results. Let $\mathcal{F}$ be a saturated formation containing all supersoluble groups $\mathcal{U}$. Then a group $G \in \mathcal{F}$ if and only if G has a normal subgroup E such that $G/E \in \mathcal{F}$ and for every non-cyclic Sylow subgroup P of E (or $F^*(E)$, respectively), every maximal subgroup of P not having a supersoluble supplement in G or every cyclic subgroup H of P with prime order or order 4 (if P is a non-abelian 2-group and $H \nsubseteq Z_\infty(G)$) without a supersoluble supplement in G is weakly SS-quasinormal in G.

2. Preliminaries

We list here some basic results which will be used in the sequel.

Lemma 2.1 ([11]) *Suppose that H is an s-permutable subgroup of G, $K \leq G$ and $N \trianglelefteq G$. Then the following statements hold:*

(1) *If $K \leq G$, then $H \cap K$ is s-permutable in K.*

(2) *HN and $H \cap N$ are s-permutable in G, HN/N is s-permutable in G/N.*

(3) *H is subnormal in G.*

(4) *If H is a p-group for some prime p, then $N_G(H) \geq O^p(G)$.*

Lemma 2.2 ([14, Lemma 2.1]) *Suppose that H is SS-quasinormal in a group G, $K \leq G$ and N is a normal subgroup of G. Then*

(1) *If $H \leq K$, then H is SS-quasinormal in K.*

(2) *HN/N is SS-quasinormal in G/N.*

Lemma 2.3 ([14, Lemma 2.2]) *Let P be a p-subgroup of G, where p is a prime. Then the following statements are equivalent:*

(1) *P is s-permutable in G.*

(2) *$P \leq O_p(G)$ and P is SS-quasinormal in G.*

Lemma 2.4 ([14, Lemma 2.5]) *If a p-subgroup P of G is SS-quasinormal in G, then P permutes with every Sylow q-subgroup of G with $q \neq p$.*

Now, we can prove that:

Lemma 2.5 *Suppose that H is weakly SS-quasinormal in a group G, $K \leq G$ and $N \trianglelefteq G$.*

(1) *If $H \leq K$, then H is weakly SS-quasinormal in K.*

(2) *If $N \leq H$, then H/N is weakly SS-quasinormal in G/N.*

(3) *Let π be a set of primes, H a π-subgroup and N a normal π'-subgroup of G. Then HN/N is weakly SS-quasinormal in G/N.*

(4) *If $K \trianglelefteq G$ and $H \leq K$, then G has a normal subgroup L contained in K such that HL is s-permutable and $H \cap L$ is SS-quasinormal in G.*

(5) *If $H \leq O_p(G)$, then H is S-embedded in G.*

Proof. By hypothesis, there exists a normal subgroup T of G such that HT is s-permutable and $H \cap T$ is SS-quasinormal in G.

(1) First, we have $K \cap T \trianglelefteq K$. By Lemma 2.1(1) and Lemma 2.2(1), we can see that $H(K \cap T) = K \cap HT$ is s-permutable and $H \cap (K \cap T) = H \cap T$ is SS-quasinormal in K, respectively. Hence H is weakly SS-quasinormal in K.

(2) Clearly, we have $TN/N \trianglelefteq G/N$, $(H/N)(TN/N) = HT/N$ is s-permutable in G/N and $(H/N) \cap (TN/N) = (H \cap TN)/N = (H \cap T)N/N$. By Lemma 2.2(2), $(H \cap T)N/N$ is SS-quasinormal in G/N. Hence H/N is weakly SS-quasinormal in G/N.

(3) It is easy to see that $TN/N \trianglelefteq G/N$, $(HN/N)(TN/N) = HTN/N$ is s-permutable in G/N. Since H is a π-group and N a π'-group,

$$|H \cap TN| = \frac{|H| \cdot |TN|_\pi}{|HTN|_\pi} = \frac{|H| \cdot |T|_\pi}{|HT|_\pi} = |H \cap T|_\pi = |H \cap T|.$$

This implies that $H \cap TN = H \cap T$, so $(HN/N) \cap (TN/N) = (HN \cap TN)/N = (H \cap TN)N/N = (H \cap T)N/N$ which is SS-quasinormal in G/N by Lemma 2.2(2). Hence HN/N is weakly SS-quasinormal in G/N.

(4) Let $L = K \cap T$, then it is easy to see that $L \trianglelefteq G$, $HL = K \cap HT$ is s-permutable in G and $H \cap L = H \cap T$ is SS-quasinormal in G.

(5) From Lemma 2.3, it is clear. ∎

Lemma 2.6 *Let G be a group and p a prime dividing $|G|$ with $(|G|, p-1) = 1$.*

(1) *If N is normal in G of order p, then N lies in $Z(G)$.*

(2) *If G has cyclic Sylow p-subgroups, then G is p-nilpotent.*

(3) *If M is a subgroup of G with index p, then M is normal in G.*

Lemma 2.7 ([4, A, Lemma 1.2]) *Let U, V and W be subgroups of a group G. Then the following statements are equivalent:*

(1) $U \cap VW = (U \cap V)(U \cap W)$;

(2) $UV \cap UW = U(V \cap W)$.

Lemma 2.8 ([17, Lemma 2.20]) *Let A be a p'-group of automorphisms of the p-group P of odd order. Assume that every subgroup of P with prime order is A-invariant, then A is cyclic.*

3. Main results

Theorem 3.1 *Let P be a Sylow p-subgroup of a group G, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. If every maximal subgroup of P not having a p-nilpotent supplement in G, or every cyclic subgroup H of P with prime order or order 4 (if P is a non-abelian 2-group and $H \nsubseteq Z_\infty(G)$) without a p-nilpotent supplement in G is weakly SS-quasinormal in G, then G is p-nilpotent.*

Proof. Suppose that the result is false and let G be a counterexample of minimal order. We treat with the following two cases:

Case 1. Every cyclic subgroup H of P with prime order or order 4 (if P is a non-abelian 2-group and $H \nsubseteq Z_\infty(G)$) without a p-nilpotent supplement in G is weakly SS-quasinormal in G.

Let K be a proper subgroup of G and $P_0 \in Syl_p(K)$. Then there exists some $x \in G$ such that $P_0^x \leq P$. Since K is p-nilpotent if and only if K^x is p-nilpotent. Without loss of generality, we may assume that $P_0 \leq P$. Since $Z_\infty(G) \cap K \leq Z_\infty(K)$, by hypothesis and Lemma 2.5(1) every cyclic subgroup H of P_0 with prime order or order 4 (if P_0 is a non-abelian 2-group and $H \nsubseteq Z_\infty(K)$) is weakly SS-quasinormal in K. Thus K is p-nilpotent by induction. Therefore, G is a minimal non-p-nilpotent group. Then we have: (i) $G = [P]Q$, where P is a normal Sylow p-subgroup and Q a non-normal cyclic Sylow q-subgroup of G; (ii) $P/\Phi(P)$ is a chief factor of G; (iii) the exponent of P is p or 4.

Let $X/\Phi(P)$ be a minimal subgroup of $P/\Phi(P)$, then there exists $x \in X \backslash \Phi(P)$ such that $X/\Phi(P) = \langle x\rangle\Phi(P)/\Phi(P)$ and $|\langle x\rangle| = p$ or 4. If $\langle x\rangle$ has a p-nilpotent supplement B in G, then $B\Phi(P)/\Phi(P)$ is a p-nilpotent supplement of $X/\Phi(P)$ in $G/\Phi(P)$ and $|G/\Phi(P) : B\Phi(P)/\Phi(P)| \leq p$. Thus we have $B\Phi(P)/\Phi(P) \trianglelefteq G/\Phi(P)$ and the normal p-complement of $B\Phi(P)/\Phi(P)$ is also a normal p-complement of $G/\Phi(P)$. Since the class of all p-nilpotent groups formed a saturated formation and $\Phi(P) \leq \Phi(G)$, G is p-nilpotent, which is a contradiction. Therefore, by hypothesis either $\langle x\rangle \subseteq Z_\infty(G)$ or $\langle x\rangle$ is weakly SS-quasinormal in G. In the former case, we have $P \cap Z_\infty(G) \nsubseteq \Phi(P)$. Then by the fact that $P/\Phi(P)$ is a chief factor of G, we have $(P \cap Z_\infty(G))\Phi(P) = P$ and hence $P \leq Z_\infty(G)$. In this case, it is easy to see that G is nilpotent, which is a contradiction. Next, we suppose that $\langle x\rangle$ is weakly SS-quasinormal in G. By Lemma 2.5(4), there are some s-permutable subgroup C and normal subgroup T of G such that $\langle x\rangle T = C \leq P$ and $\langle x\rangle \cap T$ is SS-quasinormal in G. If $X/\Phi(P)$ is s-permutable in $G/\Phi(P)$, then we can easily deduce that $X/\Phi(P) \trianglelefteq G/\Phi(P)$ since $P/\Phi(P)$ is a chief factor of G. Therefore, $P/\Phi(P) = X/\Phi(P)$ is a cyclic group. Hence P is cyclic and G is p-nilpotent. This contradiction shows that $X/\Phi(P)$ is not s-permutable in $G/\Phi(P)$. Therefore, $\langle x\rangle$ is not s-permutable in G. Since $\langle x\rangle \cap T$ is s-permutable in G by Lemma 2.3, we have $1 < T < P$. Hence $T\Phi(P) \neq P$, which implies that $T \leq \Phi(P)$. But then $X/\Phi(P) = \langle x\rangle\Phi(P)/\Phi(P) = \langle x\rangle T\Phi(P)/\Phi(P) = C\Phi(P)/\Phi(P)$ is an s-permutable subgroup of $G/\Phi(P)$, a contradiction.

Case 2. Every maximal subgroup of P not having a p-nilpotent supplement in G is weakly SS-quasinormal in G.

In this case, we break the proof into the following six steps:

(1) P is not cyclic and every maximal subgroup of P has no p-nilpotent supplement in G.

By Lemma 2.6(2), we may assume that P is not cyclic. Suppose that H is a maximal subgroup of P which has a p-nilpotent supplement T in G, we prove that G is p-nilpotent. If not, let K be a non-p-nilpotent subgroup of G which contains P and is such that every proper subgroup of K is p-nilpotent. Then by [8, IV, Theorem 5.4], K is a minimal non-nilpotent group and the following hold: (i) $K = [P]K_q$, where P is a normal Sylow p-subgroup and K_q a cyclic Sylow q-subgroup of K; (ii) $P/\Phi(P)$ is a chief factor of K.

Since $G = HT$, $K = K\cap HT = H(K\cap T)$. The facts $K\cap T \leq T$ is p-nilpotent but K is not p-nilpotent implies that $L = K\cap T$ is a proper subgroup of K. Hence L is nilpotent. Let $L = L_p \times L_q$. Obviously, L_q is also a Sylow q-subgroup of K. Since $P = HL_p$, L_p is not contained in $\Phi = \Phi(P)$. Now we consider the factor group K/Φ. The fact $L_q \leq N_K(L_p)$ implies that $L_q\Phi/\Phi \leq N_{K/\Phi}(L_p\Phi/\Phi)$. On the other hand, since P/Φ is an elementary abelian group, we have $L_p\Phi/\Phi \trianglelefteq P/\Phi$. Hence $L_p\Phi/\Phi \trianglelefteq \langle L_q\Phi/\Phi, P/\Phi\rangle = K/\Phi$. Since $L_p\Phi/\Phi \neq 1$ and P/Φ is a chief factor of K, $L_p\Phi/\Phi = P/\Phi$. It follows that $L_p = P$. Consequently, we get that $L = K$. This contradiction completes the proof of (1).

(2) G is not a non-abelian simple group.

Assume that G is a non-abelian simple group. Let P_1 be a maximal subgroup of P, by (1) we know P_1 is weakly SS-quasinormal in G. Then there exists a normal subgroup T of G such that P_1T is s-permutable and $P_1 \cap T$ is SS-quasinormal in G. Note that $T = 1$ or G since G is a simple group. If $T = 1$, then $P_1 = P_1T$ is s-permutable in G. Hence P_1 is a proper subnormal subgroup of G, a contradiction. Thus $T = G$ and therefore $P_1 = P_1 \cap T$ is SS-quasinormal in G. Then there exists some supplement B of P_1 such that P_1 permutes with every Sylow subgroup of B. From $G = P_1B$, we know $|B : P_1 \cap B|_p = |G : P_1|_p = p$. Hence $P_1 \cap B$ is of index p in B_p, a Sylow p-subgroup of B containing $P_1 \cap B$. Thus $S \nsubseteq P_1$ for all $S \in Syl_p(B)$ and $P_1S = SP_1$ is a Sylow p-subgroup of G. By comparison of orders, we know that $S \cap P_1 = B \cap P_1$ holds for each $S \in Syl_p(B)$. So $B \cap P_1 = \bigcap_{b\in B}(S^b \cap P_1) \leq \bigcap_{b\in B} S^b = O_p(B)$. Since $|O_p(B) : B \cap P_1| = p$ or 1, $|B/O_p(B)|_p = p$ or 1. Then by Lemma 2.6(2), $B/O_p(B)$ is p-nilpotent, and so B is p-soluble. Hence B has a Hall p'-subgroup K. It is clear that K is also a Hall p'-subgroup of G. Thus, P_1 permutes with every Sylow subgroup of K and so P_1K is a subgroup of G. Since $|G : P_1K| = p$, P_1K is normal in G by Lemma 2.6(3), which is a contradiction. Therefore, G is not a non-abelian simple group.

(3) G has a unique minimal normal subgroup N, G/N is p-nilpotent and $\Phi(G)$=1.

Let N be a minimal normal subgroup of G. By Lemma 2.6(2), we may assume that PN/N is non-cyclic and $|PN/N| \geq p^2$. Let M/N be a maximal subgroup of PN/N, then $M = P_1N$ for some maximal subgroup P_1 of P and $P \cap N = P_1 \cap N \in Syl_p(N)$. By (1), we know P_1 is weakly SS-quasinormal in G. Then there exists a normal subgroup T of G such that P_1T is s-permutable and $P_1\cap T$ is SS-quasinormal in G. Clearly, TN/N is a normal subgroup of G/N and $P_1N/N \cdot TN/N = P_1TN/N$ is s-permutable in G/N. Moreover, since $P_1 \cap N$ is a Sylow

p-subgroup of N, $|(P_1 \cap N)(T \cap N)|_p = |P_1 \cap N| = |N|_p = |N \cap P_1T|_p$. Since P_1 is a p-group,

$$|N \cap P_1T|_{p'} = \frac{|N|_{p'} \cdot |P_1T|_{p'}}{|NP_1T|_{p'}} = \frac{|N|_{p'} \cdot |T|_{p'}}{|NT|_{p'}} = |N \cap T|_{p'} = |(P_1 \cap N)(T \cap N)|_{p'}.$$

This implies that $(N \cap P_1)(N \cap T) = N \cap P_1T$. Thus by Lemma 2.7, we have $P_1N \cap TN = (P_1 \cap T)N$. It follows from Lemma 2.2(2) that $P_1N/N \cap TN/N = (P_1 \cap T)N/N$ is SS-quasinormal in G/N. Hence M/N is weakly SS-quasinormal in G/N. Therefore, G/N satisfies the hypothesis and so it is p-nilpotent by the minimal choice of G. Since the class of all p-nilpotent groups formed a saturated formation, N is the unique minimal normal subgroup of G and $\Phi(G) = 1$.

(4) $O_{p'}(G) = 1$.

If $O_{p'}(G) \neq 1$, then $N \leq O_{p'}(G)$ and $G/O_{p'}(G)$ is p-nilpotent by (3). Hence G is p-nilpotent, a contradiction.

(5) $O_p(G) = 1$ and N is not p-nilpotent.

If $O_p(G) \neq 1$, then $N \leq O_p(G)$. Since $\Phi(G) = 1$, G has a maximal subgroup M such that $G = [N]M$. Since $O_p(G) \leq F(G) \leq C_G(N)$ and $C_G(N) \cap M \trianglelefteq G$, the uniqueness of N yields that $N = O_p(G)$. Since $P = N(P \cap M)$ and $N \cap M = 1$, $P \cap M$ is a Sylow p-subgroup of M and there exists a maximal subgroup P_1 of P such that $P \cap M \leq P_1$ and $P = NP_1$. By (1), P_1 is weakly SS-quasinormal in G. Then G has a normal subgroup T such that P_1T is s-permutable in G, and there exists some supplement B of $P_1 \cap T$ to G such that $(P_1 \cap T)B_q = B_q(P_1 \cap T)$ for any $B_q \in Syl_q(B)$. If $T = 1$, then $P_1 = P_1T$ is s-permutable in G. It follows that $P_1 \leq O_p(G) = N$ and so $P = P_1N = N$ is a minimal normal subgroup of G. Since $N_G(P_1) \geq O^p(G)$ by Lemma 2.1(4) and $P_1 \trianglelefteq P$, P_1 is a proper normal subgroup of G contained in $P = O_p(G)$, a contradiction. Thus we have $T \neq 1$ and then $N \leq T$. In this case, $N \cap P_1 = N \cap P_1 \cap T = N \cap (P_1 \cap T)B_q \trianglelefteq (P_1 \cap T)B_q$ for any $B_q \in Syl_q(B)$ with $q \neq p$. Hence $B_q \leq N_G(N \cap P_1)$ holds for any $q \neq p$. Since $N \cap P_1 \trianglelefteq P$, it is normal in G. Thus $N \cap P_1 = 1$ and $|N| = p$. By Lemma 2.6(1), $N \leq Z(G)$. Since G/N is p-nilpotent, G is also p-nilpotent, a contradiction.

If N is p-nilpotent, then $N_{p'}$ *char* $N \trianglelefteq G$, so $N_{p'} \leq O_{p'}(G) = 1$ by (4). Thus N is a p-group and hence $N \leq O_p(G) = 1$, a contradiction.

(6) The final contradiction.

Since N is not soluble, $N = S_1 \times S_2 \times \cdots \times S_k$, where S_i are isomorphic non-abelian simple groups. Let $S_p \in Syl_p(S_1)$, we now prove that $S_p \leq P_1$ for some maximal subgroup P_1 of P. If $P \cap S_1 < P$, it is clear. If $P \leq S_1$, then by hypothesis and Lemma 2.5(1), we know that every maximal subgroup of P is weakly SS-quasinormal in S_1. With a similar argument as in (2), we can get a contradiction. Thus $S_p \leq P_1$, where P_1 is a maximal subgroup of P. Then there exists a normal

subgroup T of G such that P_1T is s-permutable in G, and there is a supplement B of $P_1 \cap T$ to G such that $P_1 \cap T$ permutes with every Sylow subgroup of B.

If $T = 1$, then P_1 is s-permutable in G and so $O_p(G) \neq 1$, which contradicts with (5). Thus $T \neq 1$ and so $N \leq T$. If $P_1 \cap T = 1$, then $|T|_p \leq p$. Hence T is p-nilpotent by Lemma 2.6(2), N is also p-nilpotent. This contradiction shows that $P_1 \cap T \neq 1$. Let B_q be a Sylow q-subgroup of B, where $q \neq p$. Then

$$|B_q \cap P_1T| = \frac{|B_q| \cdot |P_1T|_q}{|B_qP_1T|_q} = \frac{|B_q| \cdot |T|_q}{|B_qT|_q} = |B_q \cap T| = |(B_q \cap P_1)(B_q \cap T)|.$$

This implies that $B_q \cap P_1T = (B_q \cap P_1)(B_q \cap T)$. Thus by Lemma 2.7, we have $B_qP_1 \cap B_qT = B_q(P_1 \cap T)$. Therefore, $N \cap P_1B_q = N \cap (P_1B_q \cap TB_q) = N \cap (P_1 \cap T)B_q$. Then we can conclude that $S_1 \cap (P_1 \cap T) = S_1 \cap P_1 = S_p$ is a Sylow p-subgroup of S_1. This means that for any prime q $(\neq p)$, $S_1 \cap (P_1 \cap T)B_q$ is a Hall $\{p, q\}$-subgroup of S_1. Since N is non-abelian, $p = 2$. Then for any prime divisor q $(q \neq 2)$ of $|S_1|$, the non-abelian simple group S_1 has a Hall $\{2, q\}$-subgroup, which contradicts with [13, Lemma 2.6]. This contradiction completes the proof of the theorem. ∎

Since a supersoluble group G is p-nilpotent for the minimal prime divisor p of $|G|$, every subgroup H of G not having a p-nilpotent supplement in G also has no supersoluble supplement in G. Thus, from Theorem 3.1 we can easily deduce that:

Corollary 3.2 *Let P be a Sylow p-subgroup of a group G, where $p = min\pi(G)$. If every maximal subgroup of P not having a supersoluble supplement in G, or every cyclic subgroup H of P with prime order or order 4 (if P is a non-abelian 2-group and $H \not\subseteq Z_\infty(G)$) without a supersoluble supplement in G is weakly SS-quasinormal in G, then G is p-nilpotent.*

Next, by using the weakly SS-quasinormal properties of some subgroups, we give out some new criteria for the supersolubility of a group G.

Theorem 3.3 *Let $\mathcal{F}$ be a saturated formation containing the class of all supersoluble groups $\mathcal{U}$. Then a group $G \in \mathcal{F}$ if and only if G has a normal subgroup E such that $G/E \in \mathcal{F}$ and for every non-cyclic Sylow subgroup P of E, every maximal subgroup of P having no supersoluble supplement in G or every cyclic subgroup H of P with prime order or order 4 (if P is a non-abelian 2-group and $H \not\subseteq Z_\infty(G)$) without a supersoluble supplement in G is weakly SS-quasinormal in G.*

Proof. We need to prove only the sufficiency. Suppose that the result is false and consider a counterexample (G, E) for which $|G||E|$ is minimal. Let p be the smallest prime divisor of $|E|$ and P a Sylow p-subgroup of E. Then

(1) E is p-nilpotent.

We may assume that P is not cyclic. Since $Z_\infty(G) \cap E \leq Z_\infty(E)$ and every subgroup of E having no supersoluble supplement in E also has no supersoluble supplement in G. By hypothesis and Lemma 2.5(1), we know that every maximal subgroup of P having no supersoluble supplement in E or every cyclic subgroup H of P with prime order or order 4 (if P is a non-abelian 2-group and $H \nleq Z_\infty(E)$) without a supersoluble supplement in E is weakly SS-quasinormal in E. Thus Corollary 3.2 implies that E is p-nilpotent.

(2) $E = P$ is not cyclic.

By (1), E is p-nilpotent. Suppose that $P < E$ and let T be a non-trivial normal p-complement of E. Then $T \trianglelefteq G$ and from Lemma 2.5(3), we can easily deduce that the hypothesis holds for G/T (with respect to E/T). Hence $G/T \in \mathcal{F}$ by the choice of G. This implies that the hypothesis is still true for (G, T). Thus $T = E$ by the choice of (G, E), a contradiction. Hence $P = E$. Since $G/E \in \mathcal{F}$, by [17, Lemma 2.16] we may suppose that E is not cyclic.

(3) Every cyclic subgroup H of P with prime order or order 4 (if P is a non-abelian 2-group and $H \nleq Z_\infty(G)$) having no supersoluble supplement in G is weakly SS-quasinormal in G.

Suppose that every maximal subgroup of P having no supersoluble supplement in G is weakly SS-quasinormal in G. We first prove that $P = G^{\mathcal{F}}$ is a minimal normal subgroup of G. Indeed, let N be a minimal normal subgroup of G contained in P. By Lemma 2.5, the hypothesis holds for G/N and so $G/N \in \mathcal{F}$ by the minimal choice of G. This implies that N is the only minimal normal subgroup of G contained in P and $N \nleq \Phi(G)$. Let M be a maximal subgroup of G such that $G = [N]M$. Then $P = P \cap NM = N(P \cap M)$. Since $P \leq F(G) \leq C_G(N)$, $P \cap M$ is normal in G and hence $P \cap M = 1$. It follows that $P = N = G^{\mathcal{F}}$ is a minimal normal subgroup of G and $|P| > p$. Let P_1 be a maximal subgroup of $P = N$. If P_1 has a supersoluble supplement K in G, then $PK = G$ and $1 \neq P \cap K \trianglelefteq G$. Thus $P \cap K = P$ and so $G = K$ is supersoluble, which is a contradiction. Next, we assume that every maximal subgroup P_1 of P is weakly SS-quasinormal in G. Then by hypothesis and Lemma 2.5(4), G has a normal subgroup T contained in P such that P_1T is s-permutable and $P_1 \cap T$ is SS-quasinormal in G. Since P is a minimal normal subgroup of G, $T = 1$ or $T = P$. If $T = 1$, then $P_1 = P_1T$ is s-permutable in G. If $T = P$, then $P_1 = P_1 \cap T$ is SS-quasinormal in G. By Lemma 2.3, we can also conclude that P_1 is s-permutable in G. By Lemma 2.1(4), we have $O^p(G) \leq N_G(P_1)$. This means that for any maximal subgroup P_1 of P, we have $|G : N_G(P_1)| = p^a$ for some integer a. Let $M_1, M_2, \cdots, M_t$ be the set of all maximal subgroups of P. Then p divides t, which contradicts with [8, III, 8.5(d)].

(4) The final contradiction.

By (3), we know that every cyclic subgroup H of P with prime order or order 4 (if P is a non-abelian 2-group and $H \nsubseteq Z_\infty(G)$) having no supersoluble supplement in G is weakly SS-quasinormal in G. By the choice of (G, E), we know $P = G^{\mathcal{F}}$. Let M be any maximal subgroup of G not containing P, then $M/M \cap P \cong G/P \in \mathcal{F}$. Hence by Lemma 2.5, the hypothesis holds for M. Then the minimal choice of G implies that $M \in \mathcal{F}$. This shows that every maximal subgroup of G not containing P belongs to $\mathcal{F}$. Thus by [6, Theorem 3.4.2], the following statements hold:

(i) $P/\Phi(P)$ is a G-chief factor of P;

(ii) P is a group of exponent p or exponent 4 (if $p = 2$ and P is non-abelian);

(iii) if P is abelian, then $\Phi(P) = 1$.

If every minimal subgroup of $P/\Phi(P)$ is s-permutable in $G/\Phi(P)$, then every maximal subgroup of $P/\Phi(P)$ is s-permutable in $G/\Phi(P)$. With a similar argument as in the proof of Theorem 3.1 case 1, we can get a contradiction. Now we choose $X/\Phi(P)$ to be a minimal subgroup of $P/\Phi(P)$ which is not s-permutable in $G/\Phi(P)$. Pick an $x \in X \backslash \Phi(P)$ and let $L = \langle x \rangle$, then $|L| = p$ or 4. If L has a supersoluble supplement K in G, then $P = P \cap LK = L(P \cap K)$ and $(P \cap K)\Phi(P)/\Phi(P) \trianglelefteq G/\Phi(P)$. If $P \cap K \leq \Phi(P)$, then $P = L$ is cyclic, which contradicts (2). If $P \cap K = P$, then $G = K$ is supersoluble, which is a contradiction too. Thus by hypothesis either $L \subseteq Z_\infty(G)$ or L is weakly SS-quasinormal in G. If $L \subseteq Z_\infty(G)$, then $P \cap Z_\infty(G) \nsubseteq \Phi(P)$ and so $(P \cap Z_\infty(G))\Phi(P) = P$, i.e., $P \leq Z_\infty(G)$. Therefore, from (ii) we obtain that $|P/\Phi(P)| = p$ and so P is a cyclic group, which contradicts with (2). Now suppose that L is weakly SS-quasinormal in G. Since $X/\Phi(P)$ is not s-permutable in $G/\Phi(P)$, L is not s-permutable in G. Thus by Lemma 2.5(4), there exists a non-identity normal subgroup T of G contained in P such that LT is s-permutable and $L \cap T$ is SS-quasinormal in G. Since L is not s-permutable in G, by Lemma 2.3 it is clear that $T \neq P$. Hence $T\Phi(P) \neq P$, which implies that $T \leq \Phi(P)$. But then $X/\Phi(P) = L\Phi(P)/\Phi(P) = LT\Phi(P)/\Phi(P)$ is s-permutable in $G/\Phi(P)$. This final contradiction completes the proof of the theorem. ∎

From Theorem 3.3, we know that:

Corollary 3.4 *A finite group G is supersoluble if and only if G has a normal subgroup E such that G/E is supersoluble and for every non-cyclic Sylow subgroup P of E, at least one of the following holds:*

(1) *Every maximal subgroup of P either has a supersoluble supplement in G or is weakly SS-quasinormal in G.*

(2) *Every cyclic subgroup H of P with prime order or order 4 (if P is a non-abelian 2-group and $H \nsubseteq Z_\infty(G)$) either has a supersoluble supplement in G or is weakly SS-quasinormal in G.*

Theorem 3.5 *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Then a group $G \in \mathcal{F}$ if and only if G has a normal subgroup E such that $G/E \in \mathcal{F}$ and for every non-cyclic Sylow subgroup P of the generalized Fitting subgroup $F^*(E)$ of E, every maximal subgroup of P not having a supersoluble supplement in G or every cyclic subgroup H of P with prime order or order 4 (if P is a non-abelian 2-group and $H \nsubseteq Z_\infty(G)$) without a supersoluble supplement in G is weakly SS-quasinormal in G.*

Proof. The necessity is evident and we need to prove only the sufficiency. Assume that the theorem is false and let (G, E) be a counterexample with $|G||E|$ minimal. Let $F = F(E)$, $p = min\pi(F(E))$ and $P \in Syl_p(F(E))$. Then

(1) Each subgroup of $F^*(E)$ has no supersoluble supplement in G.

If some subgroup H of $F^*(E)$ has a supersoluble supplement in G, then $F^*(E)$ also has a supersoluble supplement in G and so $G/F^*(E)$ is supersoluble. Thereby, $G/F^*(E)$ belongs to $\mathcal{F}$. Then Theorem 3.3 implies that $G \in \mathcal{F}$. Thus, we may assume that none of the subgroup of $F^*(E)$ has a supersoluble supplement in G.

(2) E is soluble, $F^*(E) = F$ and $C_E(F) = C_E(F^*) \leq F$.

By (1) and Lemma 2.5(2), we know that for every non-cyclic Sylow subgroup P of $F^*(E)$, every maximal subgroup of P or every cyclic subgroup H of P with prime order or order 4 (if P is a non-abelian 2-group and $H \nsubseteq Z_\infty(E)$) is S-embedded in E. Thus, we can deduce that E is soluble by [7, Theorem B]. Hence $F^*(E) = F(E) = F$. By [9, X, Theorem 13.11], we have $C_E(F) = C_E(F^*) \leq F$.

(3) Let $V/P = F(E/P)$ and $Q \in Syl_q(V)$, where $q||V/P|$. Then $q \neq p$ and either $Q \leq F$ or $p > q$ and $C_Q(P) = 1$.

Since V/P is nilpotent and $QP/P \in Syl_q(V/P)$, QP/P is a characteristic subgroup of V/P and so $QP \trianglelefteq E$. Thus $q \neq p$. By Theorem 3.3, we know QP is supersoluble. Assume that $q > p$, then Q is normal in QP and so $Q \leq F = F(E)$. If $p > q$, then $p > 2$ and F is a q'-group since p is the smallest prime divisor of $|F|$. Now let U be a Sylow r-subgroup of F, where $r \neq p$. Then $r \neq q$ and so $[U, Q] \leq P$. Assume that for some $x \in Q$ we have $x \in C_E(P)$. Since V/P is nilpotent, by [5, V, Theorem 3.6] we know $[U, \langle x\rangle] = [U, \langle x\rangle, \langle x\rangle] = 1$ and so $x \in C_E(F)$. Since $C_E(F) \leq F$ by (2), $C_Q(P) = 1$.

(4) $p > 2$.

Assume that $p = 2$, then by (3) we have $F^*(E/P) = F(E/P) = F/P$. Since $(G/P)/(E/P) \cong G/E \in \mathcal{F}$, by Lemma 2.5(3) we know that the hypothesis is still true for $(G/P, E/P)$. Therefore, $G/P \in \mathcal{F}$ by induction. Hence $G \in \mathcal{F}$ by Theorem 3.3, a contradiction.

(5) The final contradiction.

Let $V/P = F(E/P)$ and $Q \in Syl_q(V)$, where $q||V/P|$. Then by (3), either $Q \leq F$ or $p > q$ and $C_Q(P) = 1$. In the second case, Q is cyclic by (4) and Lemma 2.8. Hence every Sylow subgroup of $F^*(E/P) = F(E/P)$ either is cyclic or has the form QP/P, where Q is a Sylow subgroup of $F^*(E) = F$. Thus by Lemma 2.5(3), we know for each non-cyclic Sylow subgroup RP/P of $F^*(E/P)$, every maximal subgroup of RP/P or every cyclic subgroup HP/P of RP/P with prime order or order 4 (if R is a non-abelian 2-group and $H \nsubseteq Z_\infty(G)$) is weakly SS-quasinormal in G/P. Therefore, $G/P \in \mathcal{F}$ by induction. It is clear that G/P satisfies the hypothesis of the theorem. Since $F^*(P) = P$, by Theorem 3.3 we have $G \in \mathcal{F}$, as required. ∎

From our Theorem 3.5, we can conclude that:

Corollary 3.6 *A finite group G is supersoluble if and only if G has a normal subgroup E such that G/E is supersoluble and for every non-cyclic Sylow subgroup P of $F^*(E)$, at least one of the following holds:*

(1) *Every maximal subgroup of P either has a supersoluble supplement in G or is weakly SS-quasinormal in G.*

(2) *Every cyclic subgroup H of P with prime order or order 4 (if P is a non-abelian 2-group and $H \nsubseteq Z_\infty(G)$) either has a supersoluble supplement in G or is weakly SS-quasinormal in G.*

Remarks. Since all normal, quasinormal, s-permutable, c-normal, SS-quasinormal, nearly s-normal [19] and S-embedded subgroups of G are all weakly SS-quasinormal in G, our theorems 3.3 and 3.5 generalized many meaningful results.

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$\mathcal{I}_{g^\star}$-NORMAL AND $\mathcal{I}_{g^\star}$-REGULAR SPACES

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Abstract. $\mathcal{I}_{g^\star}$-normal and $\mathcal{I}_{g^\star}$-regular spaces are introduced and various characterizations and properties are given. Characterizations of normal, mildly normal, $g^\star$-normal and regular spaces are also given.

Keywords: $\mathcal{I}_{g^\star}$-closed and $\mathcal{I}_{g^\star}$-open set, completely codense ideal, $g^\star$-closed and $g^\star$-open set, $g^\star$-normal space, mildly normal space, regular space.
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1. Introduction and preliminaries

By a space, we always mean a topological space (X, τ) with no separation properties assumed. If A$\subseteq$X, cl(A) and int(A) will, respectively, denote the closure and interior of A in (X, τ). A subset A of a space (X, τ) is said to be regular

open [18] if A=int(cl(A)) and A is said to be regular closed [18] if A=cl(int(A)). A subset A of a space (X, τ) is said to be an α-open [11] (resp. preopen [8]) if A$\subseteq$int(cl(int(A))) (resp. A$\subseteq$int(cl(A))). The complement of α-open set is α-closed [9]. The α-closure [9] of a subset A of X, denoted by αcl(A), is defined to be the intersection of all α-closed sets containing A. The α-interior [9] of a subset A of X, denoted by αint(A), is defined to be the union of all α-open sets contained in A. The family of all α-open sets in (X, τ), denoted by τ^α, is a topology on X finer than τ. The interior of a subset A in (X, τ^α) is denoted by int_α(A). The closure of a subset A in (X, τ^α) is denoted by cl_α(A). A subset A of a space (X, τ) is said to be g-closed [6] if cl(A)$\subseteq$U whenever A$\subseteq$U and U is open. The complement of g-closed set is g-open. A subset A of a space (X, τ) is said to be $g^\#\alpha$-closed [13] (resp. rαg-closed [12]) if cl_α(A)$\subseteq$U whenever A$\subseteq$U and U is g-open (resp. regular open). A is said to be $g^\#\alpha$-open (resp. rαg-open) if X$-$A is $g^\#\alpha$-closed (resp. rαg-closed). A subset A of a space (X, τ) is said to be $g^\star$-closed [20] if cl(A)$\subseteq$U whenever A$\subseteq$U and U is g-open. A space (X, τ) is said to be $g^\star$-normal, if for every disjoint $g^\star$-closed sets A and B, there exist disjoint open sets U and V such that A$\subseteq$U and B$\subseteq$V.

An ideal $\mathcal{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies (i) A$\in\mathcal{I}$ and B$\subseteq$A imply B$\in\mathcal{I}$ and (ii) A$\in\mathcal{I}$ and B$\in\mathcal{I}$ imply A$\cup$B$\in\mathcal{I}$. Given a topological space (X, τ) with an ideal $\mathcal{I}$ on X and if $\wp$(X) is the set of all subsets of X, a set operator (.)* : $\wp$(X) $\to$ $\wp$(X), called a local function [5] of A with respect to τ and $\mathcal{I}$ is defined as follows: for A$\subseteq$X, A*($\mathcal{I}$, τ)=$\{$x$\in$X : U$\cap$A$\notin\mathcal{I}$ for every U$\in\tau$(x)$\}$ where τ(x)=$\{$U$\in\tau$: x$\in$U$\}$. We will make use of the basic facts about the local functions [[4], Theorem 2.3] without mentioning it explicitly. A Kuratowski closure operator cl*(.) for a topology τ^*($\mathcal{I}$, τ), called the $\star$-topology, finer than τ is defined by cl*(A)=A$\cup$A*($\mathcal{I}$, τ) [4]. When there is no chance for confusion, we will simply write A* for A*($\mathcal{I}$, τ) and τ^* for τ^*($\mathcal{I}$, τ). int*(A) will denote the interior of A in (X, τ^*). If $\mathcal{I}$ is an ideal on X, then (X, τ, $\mathcal{I}$) is called an ideal topological space. N is the ideal of all nowhere dense subsets in (X, τ). A subset A of an ideal topological space (X, τ, $\mathcal{I}$) is τ^*-closed [4] or $\star$-closed (resp. $*$-dense in itself [3]) if A$^*\subseteq$A (resp. A$\subseteq$A*). A subset A of an ideal topological space (X, τ, $\mathcal{I}$) is $\mathcal{I}_{g^\star}$-closed [15] if A$^*\subseteq$U whenever U is g-open and A$\subseteq$U. By Theorem 2.3 of [15], every $\star$-closed and hence every closed set is $\mathcal{I}_{g^\star}$-closed. A subset A of an ideal topological space (X, τ, $\mathcal{I}$) is said to be $\mathcal{I}_{g^\star}$-open [15] if X$-$A is $\mathcal{I}_{g^\star}$-closed. In this paper, we define $\mathcal{I}_{g^\star}$-normal, ${}_{g^\star}\mathcal{I}$-normal and $\mathcal{I}_{g^\star}$-regular spaces using $\mathcal{I}_{g^\star}$-open sets and give characterizations and properties of such spaces. Also, characterizations of normal, mildly normal, $g^\star$-normal and regular spaces are given.

An ideal $\mathcal{I}$ is said to be codense [2] if $\tau\cap\mathcal{I}$=$\{\emptyset\}$. $\mathcal{I}$ is said to be completely codense [2] if PO(X)$\cap\mathcal{I}$=$\{\emptyset\}$, where PO(X) is the family of all preopen sets in (X, τ). Every completely codense ideal is codense but not conversely [2]. The following lemmas will be useful in the sequel.

Lemma 1.1 ([16], Theorem 6) *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If $\mathcal{I}$ is completely codense, then $\tau^* \subseteq \tau^\alpha$.*

Lemma 1.2 ([15], Theorem 2.16) *Let $(X, \tau, \mathcal{I})$ be an ideal topological space where $\mathcal{I}$ is completely codense. Then the following are equivalent.*

1. *X is normal.*
2. *For any disjoint closed sets A and B, there exist disjoint $\mathcal{I}_{g^\star}$-open sets U and V such that $A \subseteq U$, $B \subseteq V$.*
3. *For any closed set A and open set V containing A, there exists an $\mathcal{I}_{g^\star}$-open set U such that $A \subseteq U \subseteq cl^\star(U) \subseteq V$.*

Lemma 1.3 [15] *If $(X, \tau, \mathcal{I})$ is an ideal topological space and $A \subseteq X$, then the following hold.*

1. *If $\mathcal{I} = \{\emptyset\}$, then A is $\mathcal{I}_{g^\star}$-closed if and only if A is $g^\star$-closed.*
2. *If $\mathcal{I} = N$, then A is $\mathcal{I}_{g^\star}$-closed if and only if A is $g^{\#}\alpha$-closed.*

Lemma 1.4 ([15], Theorem 2.2) *If $(X, \tau, \mathcal{I})$ is an ideal topological space and $A \subseteq X$, then the following are equivalent.*

1. *A is $\mathcal{I}_{g^\star}$-closed.*
2. *$cl^\star(A) \subseteq U$ whenever $A \subseteq U$ and U is g-open in X.*

Lemma 1.5 ([15], Theorem 2.12) *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subseteq X$. Then A is $\mathcal{I}_{g^\star}$-open if and only if $F \subseteq int^\star(A)$ whenever F is g-closed and $F \subseteq A$.*

Lemma 1.6 ([15], Theorem 2.15) *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then every subset of X is $\mathcal{I}_{g^\star}$-closed if and only if every g-open set is $\star$-closed.*

Proposition 1.7 [6] *Every open set is g-open but not conversely.*

2. $\mathcal{I}_{g^\star}$-normal and ${}_{g^\star}\mathcal{I}$-normal spaces

An ideal topological space $(X, \tau, \mathcal{I})$ is said to be an $\mathcal{I}_{g^\star}$-normal space if for every pair of disjoint closed sets A and B, there exist disjoint $\mathcal{I}_{g^\star}$-open sets U and V such that $A \subseteq U$ and $B \subseteq V$. Since every open set is an $\mathcal{I}_{g^\star}$-open set, every normal space is $\mathcal{I}_{g^\star}$-normal. The following Example 2.1 shows that an $\mathcal{I}_{g^\star}$-normal space is not necessarily a normal space. Theorem 2.2 below gives characterizations of $\mathcal{I}_{g^\star}$-normal spaces. Theorem 2.3 below shows that the two concepts coincide for completely codense ideal topological spaces.

Example 2.1 Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{b\}, \{a, b\}, \{b, c\}, X\}$ and $\mathcal{I} = \{\emptyset, \{b\}\}$. Then $\emptyset^* = \emptyset$, $(\{a, b\})^* = \{a\}$, $(\{b, c\})^* = \{c\}$, $(\{b\})^* = \emptyset$ and $X^* = \{a, c\}$. Here every g-open set is $\star$-closed and so, by Lemma 1.6, every subset of X is $\mathcal{I}_{g^\star}$-closed and hence every subset of X is $\mathcal{I}_{g^\star}$-open. This implies that $(X, \tau, \mathcal{I})$ is $\mathcal{I}_{g^\star}$-normal. Now $\{a\}$ and $\{c\}$ are disjoint closed subsets of X which are not separated by disjoint open sets and so (X, τ) is not normal.

Theorem 2.2 *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then the following are equivalent.*

1. *X is $\mathcal{I}_{g^\star}$-normal.*
2. *For every closed set A and an open set V containing A, there exists an $\mathcal{I}_{g^\star}$-open set U such that $A \subseteq U \subseteq cl^\star(U) \subseteq V$.*

Proof. (1)⇒(2). Let A be a closed set and V be an open set containing A. Since A and $X - V$ are disjoint closed sets, there exist disjoint $\mathcal{I}_{g^\star}$-open sets U and W such that $A \subseteq U$ and $X - V \subseteq W$. Again, $U \cap W = \emptyset$ implies that $U \cap \text{int}^\star(W) = \emptyset$ and so $\text{cl}^\star(U) \subseteq X - \text{int}^\star(W)$. Since $X - V$ is g-closed and W is $\mathcal{I}_{g^\star}$-open, $X - V \subseteq W$ implies that $X - V \subseteq \text{int}^\star(W)$ and so $X - \text{int}^\star(W) \subseteq V$. Thus, we have $A \subseteq U \subseteq \text{cl}^\star(U) \subseteq X - \text{int}^\star(W) \subseteq V$ which proves (2).

(2)⇒(1). Let A and B be two disjoint closed subsets of X. By hypothesis, there exists an $\mathcal{I}_{g^\star}$-open set U such that $A \subseteq U \subseteq \text{cl}^\star(U) \subseteq X - B$. If $W = X - \text{cl}^\star(U)$, then U and W are the required disjoint $\mathcal{I}_{g^\star}$-open sets containing A and B respectively. So, $(X, \tau, \mathcal{I})$ is $\mathcal{I}_{g^\star}$-normal.

Theorem 2.3 *Let $(X, \tau, \mathcal{I})$ be an ideal topological space where $\mathcal{I}$ is completely codense. If $(X, \tau, \mathcal{I})$ is $\mathcal{I}_{g^\star}$-normal, then it is a normal space.*

Proof. It is obvious from Theorem 2.2 and Lemma 1.2.

Theorem 2.4 *Let $(X, \tau, \mathcal{I})$ be an $\mathcal{I}_{g^\star}$-normal space. If F is closed and A is a $g^\star$-closed set such that $A \cap F = \emptyset$, then there exist disjoint $\mathcal{I}_{g^\star}$-open sets U and V such that $A \subseteq U$ and $F \subseteq V$.*

Proof. Since $A \cap F = \emptyset$, $A \subseteq X - F$ where $X - F$ is g-open. Therefore, by hypothesis, $\text{cl}(A) \subseteq X - F$. Since $\text{cl}(A) \cap F = \emptyset$ and X is $\mathcal{I}_{g^\star}$-normal, there exist disjoint $\mathcal{I}_{g^\star}$-open sets U and V such that $\text{cl}(A) \subseteq U$ and $F \subseteq V$. Thus $A \subseteq U$ and $F \subseteq V$.

The following Corollaries 2.5 and 2.6 give properties of normal spaces. If $\mathcal{I}=\{\emptyset\}$ in Theorem 2.4, then we have the following Corollary 2.5, the proof of which follows from Theorem 2.3 and Lemma 1.3, since $\{\emptyset\}$ is a completely codense ideal. If $\mathcal{I}$=N in Theorem 2.4, then we have the Corollary 2.6 below, since $\tau^*(N) = \tau^\alpha$ and $\mathcal{I}_{g^\star}$-open sets coincide with $g^\#\alpha$-open sets.

Corollary 2.5 *Let (X, τ) be a normal space with $\mathcal{I} = \{\emptyset\}$. If F is a closed set and A is a $g^\star$-closed set disjoint from F, then there exist disjoint $g^\star$-open sets U and V such that $A \subseteq U$ and $F \subseteq V$.*

Corollary 2.6 *Let $(X, \tau, \mathcal{I})$ be a normal ideal topological space where $\mathcal{I}$=N. If F is a closed set and A is a $g^\star$-closed set disjoint from F, then there exist disjoint $g^\#\alpha$-open sets U and V such that $A \subseteq U$ and $F \subseteq V$.*

Theorem 2.7 *Let $(X, \tau, \mathcal{I})$ be an ideal topological space which is $\mathcal{I}_{g^\star}$-normal. Then the following hold.*

1. *For every closed set A and every $g^\star$-open set B containing A, there exists an $\mathcal{I}_{g^\star}$-open set U such that $A \subseteq int^\star(U) \subseteq U \subseteq B$.*
2. *For every $g^\star$-closed set A and every open set B containing A, there exists an $\mathcal{I}_{g^\star}$-closed set U such that $A \subseteq U \subseteq cl^\star(U) \subseteq B$.*

Proof. (1) Let A be a closed set and B be a $g^\star$-open set containing A. Then $A \cap (X - B) = \emptyset$, where A is closed and $X - B$ is $g^\star$-closed. By Theorem 2.4, there exist disjoint $\mathcal{I}_{g^\star}$-open sets U and V such that $A \subseteq U$ and $X - B \subseteq V$. Since $U \cap V = \emptyset$, we have $U \subseteq X - V$. By Lemma 1.5, $A \subseteq \text{int}^\star(U)$. Therefore, $A \subseteq \text{int}^\star(U) \subseteq U \subseteq X - V \subseteq B$. This proves (1).

(2) Let A be a $g^\star$-closed set and B be an open set containing A. Then $X-B$ is a closed set contained in the $g^\star$-open set $X-A$. By (1), there exists an $\mathcal{I}_{g^\star}$-open set V such that $X-B \subseteq \text{int}^\star(V) \subseteq V \subseteq X-A$. Therefore, $A \subseteq X-V \subseteq \text{cl}^\star(X-V) \subseteq B$. If $U = X - V$, then $A \subseteq U \subseteq \text{cl}^\star(U) \subseteq B$ and so U is the required $\mathcal{I}_{g^\star}$-closed set.

The following Corollaries 2.8 and 2.9 give some properties of normal spaces. If $\mathcal{I} = \{\emptyset\}$ in Theorem 2.7, then we have the following Corollary 2.8. If $\mathcal{I} = N$ in Theorem 2.7, then we have the Corollary 2.9 below.

Corollary 2.8 *Let (X, τ) be a normal space with $\mathcal{I} = \{\emptyset\}$. Then the following hold.*

1. *For every closed set A and every $g^\star$-open set B containing A, there exists a $g^\star$-open set U such that $A \subseteq \text{int}(U) \subseteq U \subseteq B$.*
2. *For every $g^\star$-closed set A and every open set B containing A, there exists a $g^\star$-closed set U such that $A \subseteq U \subseteq \text{cl}(U) \subseteq B$.*

Corollary 2.9 *Let (X, τ) be a normal space with $\mathcal{I} = N$. Then the following hold.*

1. *For every closed set A and every $g^\star$-open set B containing A, there exists an $g^\#\alpha$-open set U such that $A \subseteq \text{int}_\alpha(U) \subseteq U \subseteq B$.*
2. *For every $g^\star$-closed set A and every open set B containing A, there exists an $g^\#\alpha$-closed set U such that $A \subseteq U \subseteq \text{cl}_\alpha(U) \subseteq B$.*

An ideal topological space $(X, \tau, \mathcal{I})$ is said to be ${}_{g^\star}\mathcal{I}$-normal if for each pair of disjoint $\mathcal{I}_{g^\star}$-closed sets A and B, there exist disjoint open sets U and V in X such that $A \subseteq U$ and $B \subseteq V$. Since every closed set is $\mathcal{I}_{g^\star}$-closed, every ${}_{g^\star}\mathcal{I}$-normal space is normal. But a normal space need not be ${}_{g^\star}\mathcal{I}$-normal as the following Example 2.10 shows. Theorems 2.11 and 2.13 below give characterizations of ${}_{g^\star}\mathcal{I}$-normal spaces.

Example 2.10 Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b, c\}\}$ and $\mathcal{I} = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$. Every g-open set is $\star$-closed and so every subset of X is $\mathcal{I}_{g^\star}$-closed. Now $A = \{a, b\}$ and $B = \{c\}$ are disjoint $\mathcal{I}_{g^\star}$-closed sets, but they are not separated by disjoint open sets. So $(X, \tau, \mathcal{I})$ is not ${}_{g^\star}\mathcal{I}$-normal. But $(X, \tau, \mathcal{I})$ is normal.

Theorem 2.11 *In an ideal topological space $(X, \tau, \mathcal{I})$, the following are equivalent.*

1. *X is ${}_{g^\star}\mathcal{I}$-normal.*
2. *For every $\mathcal{I}_{g^\star}$-closed set A and every $\mathcal{I}_{g^\star}$-open set B containing A, there exists an open set U of X such that $A \subseteq U \subseteq \mathrm{cl}(U) \subseteq B$.*

Proof. It is similar to the proof of Theorem 2.2.

If $\mathcal{I} = \{\emptyset\}$, then ${}_{g^\star}\mathcal{I}$-normal spaces coincide with $g^\star$-normal spaces and so if we take $\mathcal{I} = \{\emptyset\}$, in Theorem 2.11, then we have the following characterization for $g^\star$-normal spaces.

Corollary 2.12 *In a space (X, τ), the following are equivalent.*

1. *X is $g^\star$-normal.*
2. *For every $g^\star$-closed set A and every $g^\star$-open set B containing A, there exists an open set U of X such that $A \subseteq U \subseteq \mathrm{cl}(U) \subseteq B$.*

Theorem 2.13 *In an ideal topological space $(X, \tau, \mathcal{I})$, the following are equivalent.*

1. *X is ${}_{g^\star}\mathcal{I}$-normal.*
2. *For each pair of disjoint $\mathcal{I}_{g^\star}$-closed subsets A and B of X, there exists an open set U of X containing A such that $\mathrm{cl}(U) \cap B = \emptyset$.*
3. *For each pair of disjoint $\mathcal{I}_{g^\star}$-closed subsets A and B of X, there exist an open set U containing A and an open set V containing B such that $\mathrm{cl}(U) \cap \mathrm{cl}(V) = \emptyset$.*

Proof. (1)⇒(2). Suppose that A and B are disjoint $\mathcal{I}_{g^\star}$-closed subsets of X. Then the $\mathcal{I}_{g^\star}$-closed set A is contained in the $\mathcal{I}_{g^\star}$-open set $X - B$. By Theorem 2.11, there exists an open set U such that $A \subseteq U \subseteq \mathrm{cl}(U) \subseteq X - B$. Therefore, U is the required open set containing A such that $\mathrm{cl}(U) \cap B = \emptyset$.

(2)⇒(3). Let A and B be two disjoint $\mathcal{I}_{g^\star}$-closed subsets of X. By hypothesis, there exists an open set U of X containing A such that $\mathrm{cl}(U) \cap B = \emptyset$. Also, $\mathrm{cl}(U)$ and B are disjoint $\mathcal{I}_{g^\star}$-closed sets of X. By hypothesis, there exists an open set V of X containing B such that $\mathrm{cl}(U) \cap \mathrm{cl}(V) = \emptyset$.

(3)⇒(1). The proof is clear.

If $\mathcal{I} = \{\emptyset\}$, in Theorem 2.13, then we have the following characterizations for $g^\star$-normal spaces.

Corollary 2.14 *Let (X, τ) be a space. Then the following are equivalent.*

1. *X is $g^\star$-normal.*
2. *For each pair of disjoint $g^\star$-closed subsets A and B of X, there exists an open set U of X containing A such that $\mathrm{cl}(U) \cap B = \emptyset$.*

3. *For each pair of disjoint $g^\star$-closed subsets A and B of X, there exist an open set U containing A and an open set V containing B such that* $\mathrm{cl}(U) \cap \mathrm{cl}(V) = \emptyset$.

Theorem 2.15 *Let $(X, \tau, \mathcal{I})$ be an ${}_{g^\star}\mathcal{I}$-normal space. If A and B are disjoint $\mathcal{I}_{g^\star}$-closed subsets of X, then there exist disjoint open sets U and V such that* $\mathrm{cl}^\star(A) \subseteq U$ *and* $\mathrm{cl}^\star(B) \subseteq V$.

Proof. Suppose that A and B are disjoint $\mathcal{I}_{g^\star}$-closed sets. By Theorem 2.13(3), there exist an open set U containing A and an open set V containing B such that $\mathrm{cl}(U) \cap \mathrm{cl}(V) = \emptyset$. Since A is $\mathcal{I}_{g^\star}$-closed, $A \subseteq U$ implies that $\mathrm{cl}^\star(A) \subseteq U$. Similarly, $\mathrm{cl}^\star(B) \subseteq V$.

If $\mathcal{I} = \{\emptyset\}$, in Theorem 2.15, then we have the following property of disjoint $g^\star$-closed sets in $g^\star$-normal spaces.

Corollary 2.16 *Let (X, τ) be a $g^\star$-normal space. If A and B are disjoint $g^\star$-closed subsets of X, then there exist disjoint open sets U and V such that* $\mathrm{cl}(A) \subseteq U$ *and* $\mathrm{cl}(B) \subseteq V$.

Theorem 2.17 *Let $(X, \tau, \mathcal{I})$ be an ${}_{g^\star}\mathcal{I}$-normal space. If A is an $\mathcal{I}_{g^\star}$-closed set and B is an $\mathcal{I}_{g^\star}$-open set containing A, then there exists an open set U such that* $A \subseteq \mathrm{cl}^\star(A) \subseteq U \subseteq \mathrm{int}^\star(B) \subseteq B$.

Proof. Suppose A is an $\mathcal{I}_{g^\star}$-closed set and B is an $\mathcal{I}_{g^\star}$-open set containing A. Since A and $X - B$ are disjoint $\mathcal{I}_{g^\star}$-closed sets, by Theorem 2.15, there exist disjoint open sets U and V such that $\mathrm{cl}^\star(A) \subseteq U$ and $\mathrm{cl}^\star(X - B) \subseteq V$. Now, $X - \mathrm{int}^\star(B) = \mathrm{cl}^\star(X - B) \subseteq V$ implies that $X - V \subseteq \mathrm{int}^\star(B)$. Again, $U \cap V = \emptyset$ implies $U \subseteq X - V$ and so $A \subseteq \mathrm{cl}^\star(A) \subseteq U \subseteq X - V \subseteq \mathrm{int}^\star(B) \subseteq B$.

If $\mathcal{I} = \{\emptyset\}$, in Theorem 2.17, then we have the following Corollary 2.18.

Corollary 2.18 *Let (X, τ) be a $g^\star$-normal space. If A is a $g^\star$-closed set and B is a $g^\star$-open set containing A, then there exists an open set U such that* $A \subseteq \mathrm{cl}(A) \subseteq U \subseteq \mathrm{int}(B) \subseteq B$.

The following Theorem 2.19 gives a characterization of normal spaces in terms of $g^\star$-open sets which follows from Lemma 1.2 if $\mathcal{I} = \{\emptyset\}$.

Theorem 2.19 *Let (X, τ) be a space. Then the following are equivalent.*

1. *X is normal.*
2. *For any disjoint closed sets A and B, there exist disjoint $g^\star$-open sets U and V such that $A \subseteq U$ and $B \subseteq V$.*
3. *For any closed set A and open set V containing A, there exists a $g^\star$-open set U such that $A \subseteq U \subseteq \mathrm{cl}(U) \subseteq V$.*

The rest of the section is devoted to the study of mildly normal spaces in terms of $\mathcal{I}_{g^\star}$-open sets, $\mathcal{I}_g$-open sets and $\mathcal{I}_{rg}$-open sets. A space (X, τ) is said to be a mildly normal space [17] if disjoint regular closed sets are separated by disjoint open sets. A subset A of a space (X, τ) is said to be αg-closed [7] if $\mathrm{cl}_\alpha(A) \subseteq U$ whenever $A \subseteq U$ and U is open. A subset A of a space (X, τ) is said to be rg-closed [14] if cl(A)⊆U whenever A⊆U and U is regular open in X. The complements of the above said closed sets are called their respective open sets.

A subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be $\mathcal{I}_g$-closed [1] if A*⊆U whenever A⊆U and U is open. A subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be a regular generalized closed set with respect to an ideal $\mathcal{I}$ ($\mathcal{I}_{rg}$-closed) [10] if A*⊆U whenever A⊆U and U is regular open. A is called $\mathcal{I}_g$-open (resp. $\mathcal{I}_{rg}$-open) if X−A is $\mathcal{I}_g$-closed (resp. $\mathcal{I}_{rg}$-closed). Clearly, every $\mathcal{I}_{g^\star}$-closed set is $\mathcal{I}_g$-closed and every $\mathcal{I}_g$-closed set is $\mathcal{I}_{rg}$-closed but the separate converses are not true. Theorem 2.21 below gives characterizations of mildly normal spaces. Corollary 2.22 below gives characterizations of mildly normal spaces in terms of $g^{\#}\alpha$-open, αg-open and rαg-open sets. Corollary 2.23 below gives characterizations of mildly normal spaces in terms of $g^\star$-open, g-open and rg-open sets. The following Lemma 2.20 is essential to prove Theorem 2.21.

Lemma 2.20 [10] *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. A subset A⊆X is $\mathcal{I}_{rg}$-open if and only if F⊆int*(A) whenever F is regular closed and F⊆A.*

Theorem 2.21 *Let $(X, \tau, \mathcal{I})$ be an ideal topological space where $\mathcal{I}$ is completely codense. Then the following are equivalent.*

1. *X is mildly normal.*
2. *For disjoint regular closed sets A and B, there exist disjoint $\mathcal{I}_{g^\star}$-open sets U and V such that A⊆U and B⊆V.*
3. *For disjoint regular closed sets A and B, there exist disjoint $\mathcal{I}_g$-open sets U and V such that A⊆U and B⊆V.*
4. *For disjoint regular closed sets A and B, there exist disjoint $\mathcal{I}_{rg}$-open sets U and V such that A⊆U and B⊆V.*
5. *For a regular closed set A and a regular open set V containing A, there exists an $\mathcal{I}_{rg}$-open set U of X such that A⊆U⊆cl*(U)⊆V.*
6. *For a regular closed set A and a regular open set V containing A, there exists an $\star$-open set U of X such that A⊆U⊆cl*(U)⊆V.*
7. *For disjoint regular closed sets A and B, there exist disjoint $\star$-open sets U and V such that A⊆U and B⊆V.*

Proof. (1)⇒(2). Suppose that A and B are disjoint regular closed sets. Since X is mildly normal, there exist disjoint open sets U and V such that A⊆U and B⊆V. But every open set is an $\mathcal{I}_{g^\star}$-open set. This proves (2).

(2)⇒(3). The proof follows from the fact that every $\mathcal{I}_{g^\star}$-open set is an $\mathcal{I}_g$-open set.

(3)⇒(4). The proof follows from the fact that every $\mathcal{I}_g$-open set is an $\mathcal{I}_{rg}$-open set.

(4)⇒(5). Suppose A is a regular closed and B is a regular open set containing A. Then A and X−B are disjoint regular closed sets. By hypothesis, there exist disjoint $\mathcal{I}_{rg}$-open sets U and V such that A⊆U and X−B⊆V. Since X−B is regular closed and V is $\mathcal{I}_{rg}$-open, by Lemma 2.20, X−B⊆int*(V) and so X−int*(V)⊆B. Again, U∩V=∅ implies that U∩int*(V)=∅ and so cl*(U)⊆X−int*(V)⊆B. Hence U is the required $\mathcal{I}_{rg}$-open set such that A⊆U⊆cl*(U)⊆B.

(5)⇒(6). Let A be a regular closed set and V be a regular open set containing A. Then there exists an $\mathcal{I}_{rg}$-open set G of X such that $A \subseteq G \subseteq \mathrm{cl}^\star(G) \subseteq V$. By Lemma 2.20, $A \subseteq \mathrm{int}^\star(G)$. If $U = \mathrm{int}^\star(G)$, then U is an $\star$-open set and $A \subseteq U \subseteq \mathrm{cl}^\star(U) \subseteq \mathrm{cl}^\star(G) \subseteq V$. Therefore, $A \subseteq U \subseteq \mathrm{cl}^\star(U) \subseteq V$.

(6)⇒(7). Let A and B be disjoint regular closed subsets of X. Then X−B is a regular open set containing A. By hypothesis, there exists an $\star$-open set U of X such that A⊆U⊆cl*(U)⊆X−B. If V=X−cl*(U), then U and V are disjoint $\star$-open sets of X such that A⊆U and B⊆V.

(7)⇒(1). Let A and B be disjoint regular closed sets of X. Then there exist disjoint $\star$-open sets U and V such that A⊆U and B⊆V. Since $\mathcal{I}$ is completely codense, by Lemma 1.1, $\tau^* \subseteq \tau^\alpha$ and so U, V $\in \tau^\alpha$. Hence A⊆U⊆int(cl(int(U)))=G and B⊆V⊆int(cl(int(V)))=H. G and H are the required disjoint open sets containing A and B respectively. This proves (1).

If $\mathcal{I}$=N, in the above Theorem 2.21, then $\mathcal{I}_{rg}$-closed sets coincide with rαg-closed sets and so we have the following Corollary 2.22.

Corollary 2.22 *Let (X, τ) be a space. Then the following are equivalent.*

1. *X is mildly normal.*
2. *For disjoint regular closed sets A and B, there exist disjoint $g^{\#}\alpha$-open sets U and V such that $A \subseteq U$ and $B \subseteq V$.*
3. *For disjoint regular closed sets A and B, there exist disjoint αg-open sets U and V such that $A \subseteq U$ and $B \subseteq V$.*
4. *For disjoint regular closed sets A and B, there exist disjoint $r\alpha g$-open sets U and V such that $A \subseteq U$ and $B \subseteq V$.*
5. *For a regular closed set A and a regular open set V containing A, there exists an $r\alpha g$-open set U of X such that $A \subseteq U \subseteq cl_\alpha(U) \subseteq V$.*
6. *For a regular closed set A and a regular open set V containing A, there exists an α-open set U of X such that $A \subseteq U \subseteq cl_\alpha(U) \subseteq V$.*
7. *For disjoint regular closed sets A and B, there exist disjoint α-open sets U and V such that $A \subseteq U$ and $B \subseteq V$.*

If $\mathcal{I} = \{\emptyset\}$ in the above Theorem 2.21, we get the following Corollary 2.23.

Corollary 2.23 *Let (X, τ) be a space. Then the following are equivalent.*

1. *X is mildly normal.*
2. *For disjoint regular closed sets A and B, there exist disjoint $g^\star$-open sets U and V such that A⊆U and B⊆V.*
3. *For disjoint regular closed sets A and B, there exist disjoint g-open sets U and V such that A⊆U and B⊆V.*
4. *For disjoint regular closed sets A and B, there exist disjoint rg-open sets U and V such that A⊆U and B⊆V.*
5. *For a regular closed set A and a regular open set V containing A, there exists an rg-open set U of X such that A⊆U⊆cl(U)⊆V.*
6. *For a regular closed set A and a regular open set V containing A, there exists an open set U of X such that A⊆U⊆cl(U)⊆V.*
7. *For disjoint regular closed sets A and B, there exist disjoint open sets U and V such that A⊆U and B⊆V.*

3. $\mathcal{I}_{g^\star}$-regular spaces

An ideal topological space $(X, \tau, \mathcal{I})$ is said to be an $\mathcal{I}_{g^\star}$-regular space if for each pair consisting of a point x and a closed set B not containing x, there exist disjoint $\mathcal{I}_{g^\star}$-open sets U and V such that x∈U and B⊆V. Every regular space is $\mathcal{I}_{g^\star}$-regular, since every open set is $\mathcal{I}_{g^\star}$-open. The following Example 3.1 shows that an $\mathcal{I}_{g^\star}$-regular space need not be regular. Theorem 3.2 gives a characterization of $\mathcal{I}_{g^\star}$-regular spaces.

Example 3.1 Consider the ideal topological space $(X, \tau, \mathcal{I})$ of Example 2.1. Then ∅*=∅, ({b})*=∅, ({a, b})*={a}, ({b, c})*={c} and X*={a, c}. Since every *g*-open set is ⋆-closed, every subset of X is $\mathcal{I}_{g^\star}$-closed and so every subset of X is $\mathcal{I}_{g^\star}$-open. This implies that $(X, \tau, \mathcal{I})$ is $\mathcal{I}_{g^\star}$-regular. Now, {c} is a closed set not containing a∈X, {c} and a are not separated by disjoint open sets. So $(X, \tau, \mathcal{I})$ is not regular.

Theorem 3.2 *In an ideal topological space $(X, \tau, \mathcal{I})$, the following are equivalent.*

1. *X is $\mathcal{I}_{g^\star}$-regular.*
2. *For every open set V containing x∈X, there exists an $\mathcal{I}_{g^\star}$-open set U of X such that x∈U⊆cl⋆(U)⊆V.*

Proof. (1)⇒(2). Let V be an open subset such that $x \in V$. Then $X - V$ is a closed set not containing x. Therefore, there exist disjoint $\mathcal{I}_{g^\star}$-open sets U and W such that $x \in U$ and $X - V \subseteq W$. Now, $X - V \subseteq W$ implies that $X - V \subseteq \text{int}^\star(W)$ and so $X - \text{int}^\star(W) \subseteq V$. Again, $U \cap W = \emptyset$ implies that $U \cap \text{int}^\star(W) = \emptyset$ and so $\text{cl}^\star(U) \subseteq X - \text{int}^\star(W)$. Therefore, $x \in U \subseteq \text{cl}^\star(U) \subseteq V$. This proves (2).

(2)⇒(1). Let B be a closed set not containing x. By hypothesis, there exists an $\mathcal{I}_{g^\star}$-open set U such that x∈U⊆cl⋆(U)⊆X−B. If W=X−cl⋆(U), then U and W are disjoint $\mathcal{I}_{g^\star}$-open sets such that x∈U and B⊆W. This proves (1).

Theorem 3.3 *If $(X, \tau, \mathcal{I})$ is an $\mathcal{I}_{g^\star}$-regular, T_1-space where $\mathcal{I}$ is completely codense, then X is regular.*

Proof. Let B be a closed set not containing x∈X. By Theorem 3.2, there exists an $\mathcal{I}_{g^\star}$-open set U of X such that x∈U⊆cl*(U)⊆X−B. Since X is a T_1-space, {x} is g-closed and so {x}⊆int*(U), by Lemma 1.5. Since $\mathcal{I}$ is completely codense, $\tau^* \subseteq \tau^\alpha$ and so int*(U) and X−cl*(U) are α-open sets. Now, x∈int*(U)⊆int(cl(int(int*(U))))=G and B⊆X−cl*(U)⊆int(cl(int(X−cl*(U))))=H. Then G and H are disjoint open sets containing x and B respectively. Therefore, X is regular.

If $\mathcal{I}$=N in Theorem 3.2, then we have the following Corollary 3.4 which gives characterizations of regular spaces, the proof of which follows from Theorem 3.3.

Corollary 3.4 *If (X, τ) is a T_1-space, then the following are equivalent.*

1. *X is regular.*
2. *For every open set V containing x∈X, there exists an $g^\#\alpha$-open set U of X such that $x \in U \subseteq \mathrm{cl}_\alpha(U) \subseteq V$.*

If $\mathcal{I} = \{\emptyset\}$ in Theorem 3.2, then we have the following Corollary 3.5 which gives characterizations of regular spaces, the proof of which follows from Theorem 3.3.

Corollary 3.5 *If (X, τ) is a T_1-space, then the following are equivalent.*

1. *X is regular.*
2. *For every open set V containing x∈X, there exists a $g^\star$-open set U of X such that $x \in U \subseteq cl(U) \subseteq V$.*

Theorem 3.6 *If every g-open subset of an ideal topological space $(X, \tau, \mathcal{I})$ is $\star$-closed, then $(X, \tau, \mathcal{I})$ is $\mathcal{I}_{g^\star}$-regular.*

Proof. Suppose every g-open subset of X is $\star$-closed. Then by Lemma 1.6, every subset of X is $\mathcal{I}_{g^\star}$-closed and hence every subset of X is $\mathcal{I}_{g^\star}$-open. If B is a closed set not containing x, then {x} and B are the required disjoint $\mathcal{I}_{g^\star}$-open sets containing x and B respectively. Therefore, $(X, \tau, \mathcal{I})$ is $\mathcal{I}_{g^\star}$-regular.

The following Example 3.7 shows that the reverse direction of the above Theorem 3.6 is not true.

Example 3.7 Consider the real line $\mathcal{R}$ with the usual topology with $\mathcal{I} = \{\emptyset\}$. Since $\mathcal{R}$ is regular, $\mathcal{R}$ is $\mathcal{I}_{g^\star}$-regular. Obviously U = (0,1) is g-open being open in $\mathcal{R}$. But U is not $\star$-closed because, when $\mathcal{I} = \{\emptyset\}$, cl*(U) = cl(U) = [0,1] ≠ U.

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SOME MEAN INEQUALITIES FOR POSITIVE LINEAR MAPS

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Abstract. If $m_1^2 \leq A \leq M_1^2 A$ and $m_2^2 \leq B \leq M_2^2$ for some positive real numbers $m_1 \leq M_1$, $m_2 \leq M_2$ and σ is a mean between geometric and arithmetic means, then for positive unital linear map Φ

$$\begin{aligned}
\Phi\left(\frac{M_2 m_2}{M_1 m_1} A\right) \sigma \Phi(B) &\leq \frac{\frac{M_2}{m_1} + \frac{m_2}{M_1}}{2} \Phi(A\sigma B), \\
\Phi^{-\frac{1}{2}}(A\sigma B)\Phi(B)\Phi^{-\frac{1}{2}}(A\sigma B) - \Phi^{\frac{1}{2}}(A\sigma B)\Phi^{-1}(A)\Phi^{\frac{1}{2}}(A\sigma B) &\leq \left(\sqrt{\frac{M_2}{m_1}} - \sqrt{\frac{m_2}{M_1}}\right)^2, \\
\left(\frac{1}{M_1^2 m_1^2}\Phi(A)\right) \sigma \Phi(A^{-1}) &\leq \frac{M_1^2 + m_1^2}{2 M_1^2 m_1^2} \Phi(A\sigma A^{-1}).
\end{aligned}$$

Keywords: σ mean; geometric mean; arithmetic mean; positive linear maps.

1. Introduction

We use the same notation as in [4]. The operator norm is denoted by $\|\cdot\|$. Let M, m be scalars and I be the identity operator. Other capital letters are used to denote the general elements of the C^* -algebra $\mathcal{B}(\mathcal{H})$ of all bounded linear operators acting on a Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$. We write $A \geq 0$ to mean that the operator A is positive. If $A - B \geq 0$ $(A - B \leq 0)$, then we write $A \geq B$ $(A \leq B)$. A linear map Φ is positive if $\Phi(A) \geq 0$ whenever $A \geq 0$. It is said to be unital if $\Phi(I) = I$.

σ is an arbitrary means between geometric and arithmetic means. A connection σ is a binary operation $A\sigma B$, which is monotone and continuous from above in each variable and satisfies the transformer inequality, in the sense

$$C(A\sigma B)C \leqq (CAC)\sigma(CBC).$$

Though Bhagwat and Subramanian [3] introduced power means, for instance, $\{\frac{1}{2}(A^p + B^p)\}^{1/p}$, these means are not monotone in general, nor satisfy the transformer inequality.

A connection σ is a mean if it is normalized, i.e., $1\sigma 1 = 1$ where 1 denotes the identity operator. Such an axiomatic approach is already found in the paper of Nishio and Ando [1], in which parallel addition is given an axiomatic characterization. In addition, there exists an affine order isomorphism between the class of operator means and the class of positive operator monotone functions f defined on $(0, \infty)$ with $f(1) = 1$ via $f(t)I = I\sigma(tI)$ $(t > 0)$. $A\sigma B = A^{\frac{1}{2}} f(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}}$ for all strictly positive operators A, B. The operator monotone function f is called the representing function of σ.

For $A, B > 0$, the geometric mean $A\sharp B$ and arithmetic $A\nabla B$ mean are defined $A\sharp B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$ and $A\nabla B = \dfrac{A+B}{2}$, respectively.

It is well-known that the arithmetic mean ∇ is the biggest among symmetric means (see [2]). In this paper, we suppose that $\sharp \leq \sigma \leq \nabla$.

Let $m_1^2 \leq A \leq M_1^2$ and $m_2^2 \leq B \leq M_2^2$ for positive real numbers $m_1 < M_1$, $m_2 < M_2$ and σ be a mean between geometric and arithmetic means. Moslehian et al. [5] proved the following inequalities for every positive unital linear map Φ:

• Operator Pólya-Szegö inequality:

$$\Phi(A)\sharp\Phi(B) \leq \frac{1}{2}\left(\sqrt{\frac{M_1M_2}{m_1m_2}} + \sqrt{\frac{m_1m_2}{M_1M_2}}\right)\Phi(A\sharp B); \tag{1.1}$$

• Operator Shisha-Mond inequality:

$$\begin{aligned}\Phi^{-\frac{1}{2}}(A\sharp B)\Phi(B)\Phi^{-\frac{1}{2}}(A\sharp B) - \Phi^{\frac{1}{2}}(A\sharp B)\Phi^{-1}(A)\Phi^{\frac{1}{2}}(A\sharp B)\\ \leq \left(\sqrt{\frac{M_2}{m_1}} - \sqrt{\frac{m_2}{M_1}}\right)^2;\end{aligned} \tag{1.2}$$

• Operator Kantorovich inequality:

$$\Phi(A)\sharp\Phi(A^{-1}) \leq \frac{M_1^2 + m_1^2}{2M_1m_1}. \tag{1.3}$$

In this paper, we will present some inequalities for the matrix mean σ between geometric and arithmetic means which are generalizations of (1.1)∼(1.3).

2. Main results

Next, we give the main theorem.

Theorem 1 *Let $A, B \in \mathcal{B}(\mathcal{H})$ be positive invertible operators and $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a positive unital linear map. If $m_1^2 \leq A \leq M_1^2 A$ and $m_2^2 \leq B \leq M_2^2$ for some positive real numbers $m_1 \leq M_1$, $m_2 \leq M_2$ and σ is a mean between geometric and arithmetic means, then*

$$\Phi(\frac{M_2m_2}{M_1m_1}A)\sigma\Phi(B) \leq \frac{\frac{M_2}{m_1}+\frac{m_2}{M_1}}{2}\Phi(A\sigma B), \tag{2.1}$$

$$\begin{aligned}&\Phi^{-\frac{1}{2}}(A\sigma B)\Phi(B)\Phi^{-\frac{1}{2}}(A\sigma B)-\Phi^{\frac{1}{2}}(A\sigma B)\Phi^{-1}(A)\Phi^{\frac{1}{2}}(A\sigma B)\\ &\qquad\leq\left(\sqrt{\frac{M_2}{m_1}}-\sqrt{\frac{m_2}{M_1}}\right)^2,\end{aligned} \tag{2.2}$$

$$\left(\frac{1}{M_1^2m_1^2}\Phi(A)\right)\sigma\Phi(A^{-1}) \leq \frac{M_1^2+m_1^2}{2M_1^2m_1^2}\Phi(A\sigma A^{-1}). \tag{2.3}$$

Proof : By computation, we have

$$\left(\frac{M_2}{m_1}-(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}\right)\left((A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}-\frac{m_2}{M_1}\right) \geq 0,$$

whence

$$\frac{M_2m_2}{M_1m_1}A+B \leq \left(\frac{M_2}{m_1}+\frac{m_2}{M_1}\right)A\sharp B. \tag{2.4}$$

Since Φ is a positive linear map, (2.4) becomes the inequality as follows

$$\frac{M_2m_2}{M_1m_1}\Phi(A)+\Phi(B) \leq \left(\frac{M_2}{m_1}+\frac{m_2}{M_1}\right)\Phi(A\sharp B).$$

By $\nabla\geq\sigma\geq\sharp$,

$$\begin{aligned}\left(\frac{M_2m_2}{M_1m_1}\Phi(A)\right)\sigma\Phi(B) &\leq \left(\frac{M_2m_2}{M_1m_1}\Phi(A)\right)\nabla\Phi(B)\\ &\leq \frac{\frac{M_2}{m_1}+\frac{m_2}{M_1}}{2}\Phi(A\sharp B)\leq\frac{\frac{M_2}{m_1}+\frac{m_2}{M_1}}{2}\Phi(A\sigma B).\end{aligned} \tag{2.5}$$

By (2.5), we have

$$\left(\frac{M_2m_2}{M_1m_1}\Phi(A)\right)\sigma\Phi(B) \leq \frac{\frac{M_2}{m_1}+\frac{m_2}{M_1}}{2}\Phi(A\sigma B). \tag{2.6}$$

Thus (2.1) holds. It follows from (2.6) that

$$\begin{aligned}&\Phi^{-\frac{1}{2}}(A\sigma B)\Phi(B)\Phi^{-\frac{1}{2}}(A\sigma B)-\Phi^{\frac{1}{2}}(A\sigma B)\Phi^{-1}(A)\Phi^{\frac{1}{2}}(A\sigma B)\\ &\leq\left(\frac{M_2}{m_1}+\frac{m_2}{M_1}\right)-\frac{M_2m_2}{M_1m_1}\Phi^{-\frac{1}{2}}(A\sigma B)\Phi(A)\Phi^{-\frac{1}{2}}(A\sigma B)-\Phi^{\frac{1}{2}}(A\sigma B)\Phi^{-1}(A)\Phi^{\frac{1}{2}}(A\sigma B)\\ &\leq\left(\frac{M_2}{m_1}+\frac{m_2}{M_1}\right)-2\sqrt{\frac{M_2m_2}{M_1m_1}}\\ &-\left(\sqrt{\frac{M_2m_2}{M_1m_1}}(\Phi^{-\frac{1}{2}}(A\sigma B)\Phi(A)\Phi^{-\frac{1}{2}}(A\sigma B))^{\frac{1}{2}}-(\Phi^{\frac{1}{2}}(A\sigma B)\Phi^{-1}(A)\Phi^{\frac{1}{2}}(A\sigma B))^{\frac{1}{2}}\right)^2\\ &\leq\left(\sqrt{\frac{M_2}{m_1}}-\sqrt{\frac{m_2}{M_1}}\right)^2,\end{aligned}$$

which means that

$$\Phi^{-\frac{1}{2}}(A\sigma B)\Phi(B)\Phi^{-\frac{1}{2}}(A\sigma B)-\Phi^{\frac{1}{2}}(A\sigma B)\Phi^{-1}(A)\Phi^{\frac{1}{2}}(A\sigma B)\leq\left(\sqrt{\frac{M_2}{m_1}}-\sqrt{\frac{m_2}{M_1}}\right)^2.$$

So (2.2) holds.

Suppose $m_2^2=\dfrac{1}{M_1^2}\leq B=A^{-1}\leq\dfrac{1}{m_1^2}=M_2^2$ in (2.6). Thus, (2.6) becomes

$$\left(\frac{1}{M_1^2m_1^2}\Phi(A)\right)\sigma\Phi(A^{-1})\ \leq\ \frac{M_1^2+m_1^2}{2M_1^2m_1^2}\Phi(A\sigma A^{-1}),$$

which is (2.3).

Remark 2 When $\sigma=\sharp$, (2.1)–(2.3) are the operator Pólya-Szegö inequality (1.1), the operator Shisha-Mond inequality (1.2) and the operator Kantorovich inequality (1.3), respectively. Thus (2.1)–(2.3) are the generalizations of inequalities (1.1)–(1.3).

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EQUITABLE COLORINGS OF CARTESIAN PRODUCTS OF FANS WITH BIPARTITE GRAPHS[1]

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Abstract. In this paper, by the sorting method of vertices, it is obtained that the equitable chromatic number and the equitable chromatic threshold of the Cartesian products of fans with bipartite graphs.

Keywords: Cartesian product; equitable coloring; equitable chromatic number; equitable chromatic threshold.

MR (2000) Subject Classification: 05C70, 05C15.

2. Introduction

All graphs considered in this paper are finite, undirected, loopless and without multiple edges. For a positive integer k and a real number x, let $[k] = \{1, 2, ..., k\}$, $\lceil x \rceil$ and $\lfloor x \rfloor$ denote the smallest integer not less than x and the largest integer not greater than x, respectively.

A graph G is said to be k-colorable if there is a map $c : V(G) \to [k]$ such that adjacent vertices are mapped to distinct numbers. The map c is called a proper k-coloring of G, and all pre-images of a fixed number form a so-called color class. No two vertices are adjacent in each color class. The chromatic number of G, denoted by $\chi(G)$, is the smallest number k such that G is k-colorable.

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A graph $G = (V, E)$ is said to be equitably k-colorable if $V(G)$ can be divided into k independent sets $V_1, V_2, ..., V_k$ such that $||V_i| - |V_j|| \leq 1$ for all $i, j \in [k]$. The smallest integer k for which G is equitably k-colorable is called equitable chromatic number of a graph G, and denoted by $\chi_=(G)$. The equitable chromatic threshold of a graph G, denoted by $\chi^*_=(G)$, is the minimum t such that G is equitably k-colorable for all $k \geq t$. It is evident from the definition that

$$\chi(G) \leq \chi_=(G) \leq \chi^*_=(G)$$

for any graph G.

Equitable coloring as a special vertex coloring on graphs was first introduced by Meyer[1]. His motivation came from the problem of assigning one of the six days of the work week to each garbage collection route. Here, the vertices represent garbage collection routes and two such vertices are joined by an edge when the corresponding routes should not be run on the same day. The problem of assigning one of the six days of the work week to each route becomes the problem of 6-coloring of G. On practical grounds it might also be desirable to have an approximately equal number of routes run on each of the six days, so we have to color the graph in the equitable way.

Another application of equitable coloring is in scheduling and timetabling. Consider, for example, a problem of constructing university timetables. It is known that this problem can be modeled as coloring the vertices of a graph G whose nodes correspond to classes, edges correspond to time conflicts between classes, and colors to hours. If the set of available rooms is restricted, then we may be forced to partition the vertex set into independent subsets of as near equal size as possible, since then the room usage is the highest. For applications of equitable coloring such as scheduling and constructing timetables, please see [1], [5], [11], [12], [13].

In [3], by Lin and Chang, it is obtained that the exact values or upper bounds of the equitable chromatic number on Kronecker products of G and H, when G and H are complete graphs, bipartite graphs, paths or cycles, and in [4], it is studied that the equitable colorings of Cartesian product of paths and cycles, respectively, with bipartite graphs. In [16], Lih and Wu studied the equitable colorings of bipartite graphs, and in [17], Lih gave a good survey for this coloring. In [23], Zhu gave a survey for Hedetniemi's conjecture about the tensor product of graphs. The general problem of deciding if $\chi_=(G) \leq 3$ is NP-complete [10]. If, however, G has a regular or simplified structure we are sometimes able to provide a polynomial algorithm coloring it in the equitable way. For more details about this coloring, please see [1], [2], [6], [7], [8], [14], [20], [21], [22].

The Cartesian product of graphs $G = (V_1, E_1)$ and $H = (V_2, E_2)$ is the graph $G \square H$ with vertex set $\{(u, x) \,|u \in V_1, x \in V_2\}$ and edge set

$$\{(u, x)\ (v, y) \,|u = v \ with \ xy \in E_2 \ or \ x = y \ with \ uv \in E_1\}.$$

Graph products are interesting and useful in many situations. For example, Sabidussi [19] showed that any graph has the unique decomposition into prime

factors under the Cartesian product. The complexity of many problems, also equitable coloring, that deal with very large and complicated graphs is reduced greatly if one is able to fully characterize the properties of less complicated prime factors.

In the present paper, we study the equitable colorings of Cartesian products of fans with complete bipartite graphs.

2. Main results

In the following, let s, l, m, n, n' be all nonnegative integers, $F_{n'+1}$ represent the Fan with vertex set $V(F_{n'+1}) = \{x, x_1, x_2, \ldots, x_{n'}\}$, and H represent a complete bipartite graph with two parts $Y = \{y_1, y_2, ..., y_m\}$ and $Z = \{z_1, z_2, ..., z_n\}$ where $m \geq n$. We will study the equitable chromatic number and the equitable chromatic threshold of the Cartesian product $F_{n'+1}\Box H$ according to the parity of n' and m.

On the other hand, if $n = 1$, then H is a star and denoted by $K_{m,1}$. If $n'+1 = 2$ or $n' + 1 = 3$, then $F_{n'+1}$ is a path or a cycle. In this paper, we always suppose that $n' + 1 > 3$. Clearly, $\chi_=(F_{n'+1}\Box H) \geq \chi(F_{n'+1}) = 3$.

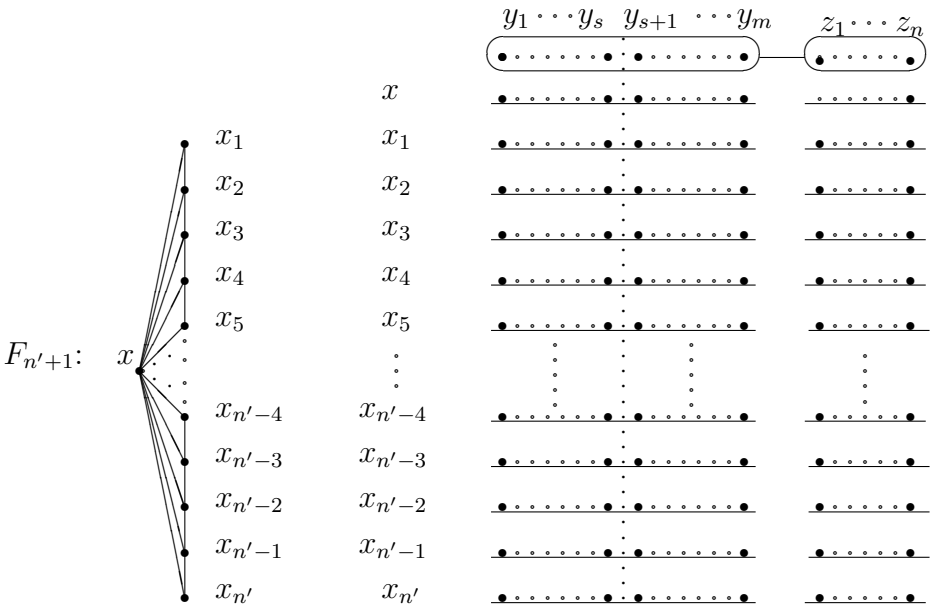


Figure 1. The Cartesian product $F_{n'+1}\Box H$

Theorem 2.1. *Suppose that $m \geq n \geq 1$, $l \geq 2$ and $k \geq 4$. If $n' = 2l$, then $F_{n'+1}\Box H$ is equitably k-colorable.*

Proof. The structure of the Cartesian product graph $F_{n'+1}\Box H$ is represented in Figure 1.

For $m = 2p+1$, $p \geq 0$ and $s = p+1$ (or for $m = 2p$, $p \geq 1$ and $s = p$), we will sort the vertices of $F_{n'+1} \square H$ as following.

$(x, y_1), (x, y_2), \ldots, (x, y_s), (x_1, z_1), (x_3, z_1), \ldots, (x_{2h-1}, z_1), \ldots, (x_{n'-1}, z_1),$
$(x_1, z_2), (x_3, z_2), \ldots, (x_{2h-1}, z_2), \ldots, (x_{n'-1}, z_2), \ldots, (x_1, z_n), (x_3, z_n), \ldots,$
$(x_{2h-1}, z_n), \ldots, (x_{n'-1}, z_n), (x_2, y_{s+1}), (x_4, y_{s+1}), \ldots, (x_{2h}, y_{s+1}), \ldots, (x_{n'}, y_{s+1}),$
$(x_2, y_{s+2}), (x_4, y_{s+2}), \ldots, (x_{2h}, y_{s+2}), \ldots, (x'_n, y_{s+2}), \ldots, (x_2, y_m), (x_4, y_m), \ldots,$
$(x_{2h}, y_m), \ldots, (x_{n'}, y_m), (x_2, y_1), (x_2, y_2) \ldots, (x_2, y_s), (x_4, y_1), (x_4, y_2) \ldots, (x_4, y_s),$
$\ldots, (x_{2h}, y_1), (x_{2h}, y_2) \ldots, (x_{2h}, y_s), \ldots, (x_{n'}, y_1), (x_{n'}, y_2), \ldots, (x_{n'}, y_s),$
$(x, z_1), (x, z_2), \ldots, (x, z_n), (x_1, y_{s+1}), (x_3, y_{s+1}), \ldots,$
$(x_{2h-1}, y_{s+1}), \ldots, (x_{n'-1}, y_{s+1}), (x_1, y_{s+2}), (x_3, y_{s+2}), \ldots,$
$(x_{2h-1}, y_{s+2}), \ldots, (x_{n'-1}, y_{s+2}), \ldots, (x_1, y_m), (x_3, y_m), \ldots,$
$(x_{2h-1}, y_m), \ldots, (x_{n'-1}, y_m), (x_1, y_1), (x_1, y_2) \ldots, (x_1, y_s),$
$(x_3, y_1), (x_3, y_2) \ldots, (x_3, y_s), \ldots, (x_{2h-1}, y_1), (x_{2h-1}, y_2) \ldots, (x_{2h-1}, y_s),$
$\ldots, (x_{n'-1}, y_1), (x_{n'-1}, y_2), \ldots, (x_{n'-1}, y_s), (x_2, z_1), (x_2, z_2), \ldots, (x_2, z_n),$
$(x_4, z_1), (x_4, z_2), \ldots, (x_4, z_n), \ldots, (x_{2h}, z_1), (x_{2h}, z_2), \ldots, (x_{2h}, z_n), \ldots,$
$(x_{n'}, z_1), (x_{n'}, z_2), \ldots, (x_{n'}, z_n), (x, y_{s+1}), (x, y_{s+2}), \ldots, (x, y_m),$

where h is a positive integer and $1 \leq h \leq l$. It is not difficult to verify that the smallest cardinality of independent set consisting of consecutive vertices is at least $\min\{lp + nl + p + 1, 2pl + l + n\}$ for $m = 2p+1$ and $\min\{2lp + n, nl + pl + p\}$ for $m = 2p$.

Let

$$\sigma_t = \left\lfloor \frac{(2l+1)(m+n) + t - 1}{k} \right\rfloor,$$

where $t \in [k]$. By $l \geq 2$ and $k \geq 4$, we have

$$\begin{aligned} \sigma_1 &= \left\lfloor \frac{(2l+1)(m+n)}{k} \right\rfloor \leq \sigma_k = \left\lfloor \frac{(2l+1)(m+n) + k - 1}{k} \right\rfloor \\ &= \left\lceil \frac{(2l+1)(m+n)}{k} \right\rceil \leq \left\lceil \frac{(2l+1)(m+n)}{4} \right\rceil. \end{aligned}$$

If $m = 2p+1$, then

$$\sigma_t \leq \left\lceil \frac{(2l+1)(m+n)}{4} \right\rceil \leq \min\{lp + nl + p + 1, 2pl + l + n\}.$$

If $m = 2p$, then

$$\sigma_t \leq \left\lceil \frac{(2l+1)(m+n)}{4} \right\rceil < \min\{2lp + n, nl + pl + p\}.$$

Hence, according to the vertex sorting above, the vertex set of $F_{n'+1} \square H$ can be partitioned into k independent sets with cardinality $\sigma_1, \sigma_2, \ldots, \sigma_k$, respectively. Therefore $F_{n'+1} \square H$ is equitably k-colorable. ∎

Theorem 2.2. *Suppose that $m \geq n \geq 1$.*

(1) *For* $n' = 4$ *and* $2n - m \in \{0, \pm 1, \pm 2\}$, *or*

(2) *for* $n' = 2l$, $l \geq 3$ *and* $m = 2n$,

we have that $F_{n'+1} \square H$ *is equitably* 3*-colorable, and then* $\chi_{=}^{*}(F_{n'+1} \square H) = 3$.

Proof. (1) Assume that s is a nonnegative integer, then we can partition the vertex set $V(F_{n'+1} \square H)$ into the following three parts:

$$V_1 = \left\{ \begin{array}{l} (x, y_1), (x, y_2), \ldots, (x, y_s), (x_1, z_1), (x_3, z_1), (x_1, z_2), (x_3, z_2), \ldots, (x_1, z_n), \\ (x_3, z_n), (x_2, y_{s+1}), (x_4, y_{s+1}), (x_2, y_{s+2}), (x_4, y_{s+2}), \ldots, (x_2, y_m), (x_4, y_m) \end{array} \right\},$$

$$V_2 = \left\{ \begin{array}{l} (x_1, y_1), (x_1, y_2) \ldots, (x_1, y_s), (x_3, y_1), (x_3, y_2) \ldots, (x_3, y_s), (x_2, z_1), (x_2, z_2), \\ \ldots, (x_2, z_n), (x_4, z_1), (x_4, z_2), \ldots, (x_4, z_n), (x, y_{s+1}), (x, y_{s+2}), \ldots, (x, y_m) \end{array} \right\},$$

and

$$V_3 = \left\{ \begin{array}{l} (x_2, y_1), (x_2, y_2) \ldots, (x_2, y_s), (x_4, y_1), (x_4, y_2) \ldots, (x_4, y_s), (x, z_1), (x, z_2), \ldots, \\ (x, z_n), (x_1, y_{s+1}), (x_3, y_{s+1}), (x_1, y_{s+2}), (x_3, y_{s+2}), \ldots, (x_1, y_m), (x_3, y_m) \end{array} \right\},$$

where $|V_1| = 2n + 2m - s$, $|V_2| = 2n + m + s$, and $|V_3| = n + 2m$.

If $m = 2n + 1$ and $s = n + 1$, then $|V_1| = 2n + 2m - s = 5n + 1$, $|V_2| = 2n + m + s = 5n + 2$, and $|V_3| = n + 2m = 5n + 2$.If $m = 2n - 1$ and $s = n$, then $|V_1| = 5n - 2$, $|V_2| = 5n - 1$, and $|V_3| = 5n - 2$.

If $m = 2n$ and $s = n$, then $|V_1| = 2n + 2m - s = 5n$, $|V_2| = 2n + m + s = 5n$, and $|V_3| = n + 2m = 5n$.

If $m = 2n + 2$ and $s = n + 1$, then $|V_1| = 2n + 2m - s = 5n + 3$, $|V_2| = 2n + m + s = 5n + 3$, and $|V_3| = n + 2m = 5n + 4$.

If $m = 2n - 2$ and $s = n - 1$, then $|V_1| = 2n + 2m - s = 5n - 3$, $|V_2| = 2n + m + s = 5n - 3$, and $|V_3| = n + 2m = 5n - 4$.

Therefore, $F_{n'+1} \square H$ is equitably 3-colorable. By Theorem 2.1, we have $\chi_{=}^{*}(F_{n'+1} \square H) = 3$.

(2) Assume that $s = n$ and $1 \leq h \leq l$, then we will partition the vertex set $V(F_{n'+1} \square H)$ into the following three parts:

$$V_1 = \left\{ \begin{array}{l} (x, y_1), (x, y_2), \ldots, (x, y_s), (x_1, z_1), (x_3, z_1), \ldots, (x_{2h-1}, z_1), \ldots, \\ (x_{n'-1}, z_1), (x_1, z_2), (x_3, z_2), \ldots, (x_{2h-1}, z_2), \ldots, (x_{n'-1}, z_2), \ldots, \\ (x_1, z_n), (x_3, z_n), \ldots, (x_{2h-1}, z_n), \ldots, (x_{n'-1}, z_n), (x_2, y_{s+1}), (x_4, y_{s+1}), \\ \ldots, (x_{2h}, y_{s+1}), \ldots, (x_{n'}, y_{s+1}), (x_2, y_{s+2}), (x_4, y_{s+2}), \ldots, (x_{2h}, y_{s+2}), \\ \ldots, (x_{n'}, y_{s+2}), \ldots, (x_2, y_m), (x_4, y_m), \ldots, (x_{2h}, y_m), \ldots, (x_{n'}, y_m) \end{array} \right\},$$

$$V_2 = \left\{ \begin{array}{l} (x_1, y_1), (x_1, y_2) \ldots, (x_1, y_s), (x_3, y_1), (x_3, y_2) \ldots, (x_3, y_s), \\ \ldots, (x_{2h-1}, y_1), (x_{2h-1}, y_2), \ldots, (x_{2h-1}, y_s), \ldots, (x_{n'-1}, y_1), \\ (x_{n'-1}, y_2) \ldots, (x_{n'-1}, y_s), (x_2, z_1), (x_2, z_2), \ldots, (x_2, z_n), \\ (x_4, z_1), (x_4, z_2), \ldots, (x_4, z_n), \ldots, (x_{2h}, z_1), (x_{2h}, z_2), \ldots, (x_{2h}, z_n), \\ \ldots, (x_{n'}, z_1), (x_{n'}, z_2), \ldots, (x_{n'}, z_n), (x, y_{s+1}), (x, y_{s+2}), \ldots, (x, y_m) \end{array} \right\},$$

and

$$V_3 = \left\{ \begin{array}{l} (x_2, y_1), (x_2, y_2) \ldots, (x_2, y_s), (x_4, y_1), (x_4, y_2) \ldots, (x_4, y_s), \ldots, (x_{2h}, y_1), \\ (x_{2h}, y_2), \ldots, (x_{2h}, y_s), \ldots, (x_{n'}, y_1), (x_{n'}, y_2), \ldots, (x_{n'}, y_s), (x, z_1), \\ (x, z_2), \ldots, (x, z_n), (x_1, y_{s+1}), (x_3, y_{s+1}), \ldots, (x_{2h-1}, y_{s+1}), \\ \ldots, (x_{n'-1}, y_{s+1}), (x_1, y_{s+2}), (x_3, y_{s+2}), \ldots, (x_{2h-1}, y_{s+2}), \ldots, \\ (x_{n'-1}, y_{s+2}), \ldots, (x_1, y_m), (x_3, y_m), \ldots, (x_{2h-1}, y_m), \ldots, (x_{n'-1}, y_m) \end{array} \right\}.$$

It is easy to see that $|V_1| = (2l+1)n$, $|V_2| = (2l+1)n$, and $|V_3| = (2l+1)n$. Hence $F_{n'+1} \square H$ is equitably 3-colorable. By Theorem 2.1, we have $\chi^*_=(F_{n'+1} \square H) = 3$. ∎

Theorem 2.3. *Suppose that $m \geq n \geq 1$, $l \geq 1$ and $k \geq 4$. If $n' = 2l+1$, then $F_{n'+1} \square H$ is equitably k-colorable.*

Proof. For $m = 2p+1$, $p \geq 0$ and $s = p+1$ (or $m = 2p$, $p \geq 1$ and $s = p$), we sort the vertex set of $F_{n'+1} \square H$ as following:

$$\begin{array}{l} (x, y_1), (x, y_2), \ldots, (x, y_s), (x_2, z_1), (x_4, z_1), \ldots, (x_{2h+2}, z_1), \ldots, (x_{n'-1}, z_1), \\ (x_2, z_2), (x_4, z_2), \ldots, (x_{2h+2}, z_2), \ldots, (x_{n'-1}, z_2), \ldots, (x_2, z_n), (x_4, z_n), \\ \ldots, (x_{2h+2}, z_n), \ldots, (x_{n'-1}, z_n), (x_1, y_{s+1}), (x_3, y_{s+1}), \ldots, (x_{2h'+1}, y_{s+1}), \\ \ldots, (x_{n'}, y_{s+1}), (x_1, y_{s+2}), (x_3, y_{s+2}), \ldots, (x_{2h'+1}, y_{s+2}), \ldots, \\ (x_{n'}, y_{s+2}), \ldots, (x_1, y_m), (x_3, y_m), \ldots, (x_{2h'+1}, y_m), \ldots, (x_{n'}, y_m), \\ (x_1, y_1), (x_1, y_2) \ldots, (x_1, y_s), (x_3, y_1), (x_3, y_2) \ldots, (x_3, y_s), \ldots, \\ (x_{2h'+1}, y_1), (x_{2h'+1}, y_2) \ldots, (x_{2h'+1}, y_s), \ldots, (x_{n'}, y_1), (x_{n'}, y_2), \\ \ldots, (x_{n'}, y_s), (x, z_1), (x, z_2), \ldots, (x, z_n), (x_2, y_{s+1}), (x_4, y_{s+1}), \ldots, \\ (x_{2h+2}, y_{s+1}), \ldots, (x_{n'-1}, y_{s+1}), (x_2, y_{s+2}), (x_4, y_{s+2}), \ldots, (x_{2h+2}, y_{s+2}), \\ \ldots, (x_{n'-1}, y_{s+2}), \ldots, (x_2, y_m), (x_4, y_m), \ldots, (x_{2h+2}, y_m), \ldots, (x_{n'-1}, y_m), \\ (x_2, y_1), (x_2, y_2) \ldots, (x_2, y_s), (x_4, y_1), (x_4, y_2) \ldots, (x_4, y_s), \ldots, (x_{2h+2}, y_1), \\ (x_{2h+2}, y_2) \ldots, (x_{2h+2}, y_s), \ldots, (x_{n'-1}, y_1), (x_{n'-1}, y_2) \ldots, (x_{n'-1}, y_s), \\ (x_1, z_1), (x_1, z_2), \ldots, (x_1, z_n), (x_3, z_1), (x_3, z_2), \ldots, (x_3, z_n), \ldots, \\ (x_{2h'+1}, z_1), (x_{2h'+1}, z_2), \ldots, (x_{2h'+1}, z_n), \ldots, (x_{n'}, z_1), (x_{n'}, z_2), \ldots, \\ (x_{n'}, z_n), (x, y_{s+1}), (x, y_{s+2}), \ldots, (x, y_m), \end{array}$$

where h, h' are all nonnegative integers, $0 \leq h \leq l-1$, and $0 \leq h' \leq l$. It is obvious that the smallest cardinality of independent set consisting of consecutive vertices in the order above is at least

$$\min\{lp + nl + 2p + 1, 2lp + l + n, pl + nl + p + l + n\}$$

when $m = 2p+1$ and

$$\min\{2lp + n, nl + p(l+2), (l+1)p + nl + n\}$$

when $m = 2p$.

Let

$$\sigma_t = \left\lfloor \frac{(2l+2)(m+n) + t - 1}{k} \right\rfloor,$$

where $t \in [k]$. By $l \geq 1$ and $k \geq 4$, we can obtain that

$$\sigma_1 = \left\lfloor \frac{(2l+2)(m+n)}{k} \right\rfloor$$

$$\leq \sigma_k = \left\lfloor \frac{(2l+2)(m+n)+k-1}{k} \right\rfloor = \left\lceil \frac{(2l+2)(m+n)}{k} \right\rceil \leq \left\lceil \frac{(2l+2)(m+n)}{4} \right\rceil.$$

If $m = 2p+1$, then

$$\sigma_t \leq \left\lceil \frac{(2l+2)(m+n)}{4} \right\rceil \leq \min\{lp+nl+2p+1, 2lp+l+n, pl+nl+p+l+n\}.$$

If $m = 2p$, then

$$\sigma_t \leq \left\lceil \frac{(2l+2)(m+n)}{4} \right\rceil \leq \min\{2lp+n, nl+p(l+2), (l+1)p+nl+n\}.$$

Therefore, according to the vertex ordering above, the vertex set of $F_{n'+1}\square H$ can be partitioned into k independent sets with cardinality $\sigma_1, \sigma_2, \ldots, \sigma_k$, respectively. Hence $F_{n'+1}\square H$ is equitably k-colorable for $k \geq 4$. □

Theorem 2.4. *Suppose that $m \geq n \geq 1$.*
(1) If $n' = 3$, and $2n - m \in \{0, \pm 1, \pm 2\}$, or
(2) if $l \geq 2$, $n' = 2l+1$ and $m = 2n$,
then $F_{n'+1}\square H$ is equitably 3-colorable, and $\chi^*_=(F_{n'+1}\square H) = 3$.

Proof. (1) Assume that s is a nonnegative integer, then we can partition the vertex set $V(F_{n'+1}\square H)$ into the following three parts:

$$V_1 = \left\{ \begin{array}{l} (x,y_1),(x,y_2),\ldots,(x,y_s),(x_2,z_1),(x_2,z_2),\ldots,(x_2,z_n), \\ (x_1,y_{s+1}),(x_3,y_{s+1}),(x_1,y_{s+2}),(x_3,y_{s+2}),\ldots,(x_1,y_m),(x_3,y_m) \end{array} \right\},$$

$$V_2 = \left\{ \begin{array}{l} (x_2,y_1),(x_2,y_2)\ldots,(x_2,y_s),(x_1,z_1),(x_1,z_2),\ldots,(x_1,z_n), \\ (x_3,z_1),(x_3,z_2),\ldots,(x_3,z_n),(x,y_{s+1}),(x,y_{s+2}),\ldots,(x,y_m) \end{array} \right\},$$

and

$$V_3 = \left\{ \begin{array}{l} (x_1,y_1),(x_1,y_2)\ldots,(x_1,y_s),(x_3,y_1),(x_3,y_2)\ldots,(x_3,y_s), \\ (x,z_1),(x,z_2),\ldots,(x,z_n),(x_2,y_{s+1}),(x_2,y_{s+2}),\ldots,(x_2,y_m) \end{array} \right\},$$

where $|V_1| = n+2m-s$, $|V_2| = 2n+m$, and $|V_3| = n+m+s$.

If $m = 2n+1$ and $s = n+1$, then $|V_1| = n+2m-s = 4n+1$, $|V_2| = 2n+m = 4n+1$, and $|V_3| = n+m+s = 4n+2$. If $m = 2n-1$ and $s = n$, then $|V_1| = 4n-2$, $|V_2| = 4n-1$, and $|V_3| = 4n-1$.

If $m = 2n$ and $s = n$, then $|V_1| = n+2m-s = 4n$, $|V_2| = 2n+m = 4n$, and $|V_3| = n+m+s = 4n$. If $m = 2n-2$ and $s = n-1$, then $|V_1| = 4n-3$, $|V_2| = 4n-2$, and $|V_3| = 4n-3$. If $m = 2n+2$ and $s = n+1$, then $|V_1| = 4n+3$, $|V_2| = 4n+2$, and $|V_3| = 4n+3$.

Hence $F_{n'+1}\Box H$ is equitably 3-colorable. Applying Theorem 2.3, we have $\chi^*_=(F_{n'+1}\Box H) = 3$.

(2) Assume that $s = n$, $0 \le h \le l-1$, and $0 \le h' \le l$. Then we can partition the vertex set $V(F_{n'+1}\Box H)$ into the following three parts:

$$V_1 = \left\{ \begin{array}{l} (x,y_1),(x,y_2),\ldots,(x,y_s),(x_2,z_1),(x_4,z_1),\ldots,(x_{2h+2},z_1),\ldots, \\ (x_{n'-1},z_1),(x_2,z_2),(x_4,z_2),\ldots,(x_{2h+2},z_2),\ldots,(x_{n'-1},z_2),\ldots, \\ (x_2,z_n),(x_4,z_n),\ldots,(x_{2h+2},z_n),\ldots,(x_{n'-1},z_n),(x_1,y_{s+1}), \\ (x_3,y_{s+1}),\ldots,(x_{2h'+1},y_{s+1}),\ldots,(x_{n'},y_{s+1}),(x_1,y_{s+2}),(x_3,y_{s+2}), \\ \ldots,(x_{2h'+1},y_{s+2}),\ldots,(x_{n'},y_{s+2}),\ldots,(x_1,y_m),(x_3,y_m), \\ \ldots,(x_{2h'+1},y_m),\ldots,(x_{n'},y_m) \end{array} \right\},$$

$$V_2 = \left\{ \begin{array}{l} (x_2,y_1),(x_2,y_2)\ldots,(x_2,y_s),(x_4,y_1),(x_4,y_2)\ldots,(x_4,y_s),\ldots, \\ (x_{2h+2},y_1),(x_{2h+2},y_2)\ldots,(x_{2h+2},y_s),\ldots,(x_{n'-1},y_1),(x_{n'-1},y_2), \\ \ldots,(x_{n'-1},y_s),(x_1,z_1),(x_1,z_2),\ldots,(x_1,z_n),(x_3,z_1),(x_3,z_2), \\ \ldots,(x_3,z_n),\ldots,(x_{2h'+1},z_1),(x_{2h'+1},z_2),\ldots,(x_{2h'+1},z_n),\ldots, \\ (x_{n'},z_1),(x_{n'},z_2),\ldots,(x_{n'},z_n),(x,y_{s+1}),(x,y_{s+2}),\ldots,(x,y_m) \end{array} \right\},$$

and

$$V_3 = \left\{ \begin{array}{l} (x_1,y_1),(x_1,y_2),\ldots,(x_1,y_s),(x_3,y_1),(x_3,y_2)\ldots,(x_3,y_s), \\ \ldots,(x_{2h'+1},y_1),(x_{2h'+1},y_2),\ldots,(x_{2h'+1},y_s),\ldots, \\ (x_{n'},y_1),(x_{n'},y_2),\ldots,(x_{n'},y_s),(x,z_1),(x,z_2),\ldots,(x,z_n), \\ (x_2,y_{s+1}),(x_4,y_{s+1}),\ldots,(x_{2h+2},y_{s+1}),\ldots,(x_{n'-1},y_{s+1}), \\ (x_2,y_{s+2}),(x_4,y_{s+2}),\ldots,(x_{2h+2},y_{s+2}),\ldots,(x_{n'-1},y_{s+2}), \\ \ldots,(x_2,y_m),(x_4,y_m),\ldots,(x_{2h+2},y_m),\ldots,(x_{n'-1},y_m) \end{array} \right\}.$$

It is obvious that

$$|V_1| = |V_2| = |V_3| = (2l+2)n.$$

Hence $F_{n'+1}\Box H$ is equitably 3-colorable. By Theorem 2.3, we have

$$\chi^*_=(F_{n'+1}\Box H)=3. \qquad \blacksquare$$

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VERTEX-TO-EDGE CENTERS W.R.T. D-DISTANCE

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Abstract. Between any two vertices of a graph we can define many distances, the usual distance, detour distance, superior distance, signal distance, degree distance etc. In some of these distances only the lengths of various paths were considered. By considering the degrees of various vertices present in a path, in addition to the length of the path, in an earlier article we introduced the concept of D-distance, $d^D(u, v)$, in graphs. In this article we study D-distance between a vertex and an edge of a graph and determine the eccentrics, radius and diameters of some classes of graphs. We also obtain relations between their eccentrics and also determine centers of graphs. Further, we prove that for any graph G either $C^D(G) \subseteq C_1^D(G)$ or $C_1^D(G) \subseteq C^D(G)$.

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1. Introduction

By a graph $G = (V, E)$ we mean a finite undirected connected graph without loops and multiple edges. The concept of distance is one of the important concepts in study of graphs. It is used in isomorphism testing, graph operations, hamiltonicity problems, extremal problems on connectivity and diameter, convexity in graphs etc. Distance is the basis of many concepts of symmetry in graphs.

In addition to the usual distance, $d(u, v)$, between two vertices $u, v \in V$ we have detour distance, superior distance, signal distance, degree distance etc.

In an earlier article [3], the authors have introduced the concept of D-distance by considering not only path length between vertices, but also the degrees of all the vertices present in a path.

The concept of distance between vertex and edge was introduced by Slater [4]. Later the concept of centers in graphs with respect to edges etc. was studied by A.P. Santhakumaran [2]. In this article we study vertex-to-edge centers with respect to D-distance.

In this article, we prove for any graph G either $C^D(G) \subseteq C_1^D(G)$ or $C_1^D(G) \subseteq C^D(G)$. We also prove vertex-to-edge center of a graph G is a subgraph of some block of G and the vertex set of every graph G is the vertex-to-edge center of some connected graph.

2. Preliminaries

For reader convenience we recall the definition of D-distance which the authors have introduced in [3].

Definition 2.1. If u, v are vertices of a connected graph G, the D-length of a connected $u - v$ path s is defined as $l^D(s) = l(s) + deg(v) + deg(u) + \sum deg(w)$ where sum runs over all intermediate vertices w of s and $l(s)$ is the length of the path.

Definition 2.2 (D-distance)**.** The *D-distance*, $d^D(u, v)$, between two vertices u, v of a connected graph G is defined as $d^D(u, v) = \min\{l(s)\}$ if u, v are distinct and $d^D(u, v) = 0$ if $u = v$, where the minimum is taken over all $u - v$ paths s in G. In other words,

$$d^D(u,v) = \begin{cases} \min\limits_s \left\{ d(u,v) + deg(u) + deg(v) + \sum deg(w) \right\} & \text{if } u \neq v \\ 0 & \text{if } u = v \end{cases}$$

where the sum runs over all intermediate vertices w in s and minimum is taken over all $u - v$ paths s in G.

Some Properties of D-distance:

1. The D-distance is a metric on the set of all vertices of a connected graph G.

2. In a connected graph, two distinct vertices u, v are *adjacent* if and only if $d^D(u, v) = deg(u) + deg(v) + 1$.

3. For any two vertices u, v we have $d^D(u, v) \leq d^D(u, w) + d^D(w, v) - deg(w)$ for all $w \in G$.

We can define D-eccentricity, D-radius and D-diameter etc. in a natural way.

Definition 2.3. The *D-eccentricity* of any vertex v, $e^D(v)$, is defined as the maximum distance from v to any other vertex, i.e., $e^D(v) = \max\{d^D(u,v) : u \in V\}$. Any vertex u for which $e^D(v) = d^D(u,v)$ is called the D-eccentric vertex of v. Further, a vertex u is said to be D-eccentric vertex of G if it is the D-eccentric vertex of some vertex.

Definition 2.4. The *D-radius*, $r^D(G)$, is the minimum D-eccentricity among all vertices of G and *D-diameter*, $d^D(G)$, is the maximum D-eccentricity i.e., $r^D(G) = \min\{e^D(u) : u \in V\}$ and $d^D(G) = \max\{e^D(u) : u \in V\}$. We have

$$r^D(G) \leq d^D(G) \leq 2r^D(G).$$

Futher,

Definition 2.5. The *D-centre* of G, $C^D(G)$, is the subgraph induced by the set of all vertices of minimum D-eccentricity. A graph is called *D-self-centered* if $C^D(G) = V$ or equivalently $d^D(G) = r^D(G)$. Similarly, the set of all vertices of maximum D-eccentricity is the *D-periphery* of G .

Next, we define the *vertex-to-edge D-distance*.

Definition 2.6. If v is a vertex and $e = xy$ is any edge (joining the vertices x and y) we define the distance between them as $d^D(v,e) = min\{d^D(v,x), d^D(v,y)\}$. Similarly, for any two subsets S, T of V we have $d^D(v,S) = min\{d^D(v,s) : s \in S\}$ and $d^D(S,T) = min\{d^D(s,t) : s \in S, t \in T\}$.

One more,

Definition 2.7. For any vertex v of G the *vertex-to-edge eccentricity*, e_1^D, of v is given by $e_1^D(v) = \max\{d^D(v,e) : e \in E\}$. A vertex v for which $e_1^D(v)$ is minimum is called a *vertex-to-edge central vertex* and the set of all vertex-to-edge central vertices of G is the *vertex-to-edge center* of G. This is denoted by $C_1^D(G)$.

3. Results on centers

We begin this section with an important result.

Theorem 3.1. *For any vertex v in a graph G we have $e^D(v) - deg(v) - 1 \leq e_1^D(v) \leq e^D(v)$. Further $e^D(v) = e_1^D(v)$ if and only if both the vertices of the eccentric edge of v are eccentric vertices of v.*

Proof. Let v be any vertex of G and let u be an eccentric vertex of v. Let $p = u = v_0, v_1, v_2, \cdots, v_k = v$ be a $u - v$ minimum D-path. Then $d^D(u,v) = k + deg(v_0)+deg(v_1)+deg(v_2)+\cdots+deg(v_k) = e^D(v)$. Let $e = v_{k-1}v_k$, since $d^D(v,e) = k-1+deg(v_0)+deg(v_1)+deg(v_2)+\cdots+deg(v_{k-1}) = e^D(v)-deg(v)-1$. It follows that $e_1^D(v) \geq e^D(v) - deg(v) - 1$. Also since $e_1^D(v) = \min\{d^D(v,e) : e \in E\}$ it follows that that $e_1^D(v) \leq d^D(v,u) = e^D(v)$. Thus $e^D(v) - deg(v) - 1 \leq e_1^D(v) \leq e^D(v)$.

Now suppose that $e_1^D(v) = e^D(v)$, for some vertex v of G. Then for any eccentric edge $e = xy$ of v we have $d^D(v,e) = e_1^D = e^D(v) \geq d^D(v,x)$. Thus $d^D(v,x) \leq d^D(v,e) \leq d^D(v,y)$.

Similarly we can prove that $d^D(v,y) \leq d^D(v,e)$ it follows that $d^D(v,y) = d^D(v,x)$ and both are equal to $e_1^D(v)$ so x and y are eccentric vertices of v.

Conversely, if both x and y are eccentric vertices of v, then $e^D(v) = d^D(v,x) = d^D(v,y)$. Hence $e_1^D(v) = e^D(v)$. ∎

Theorem 3.2. *For any graph either $C^D(G) \subseteq C_1^D(G)$ or $C_1^D(G) \subseteq C^D(G)$.*

Proof. Suppose that the result is false. Then there exit vertices u and v such that $u \in C_1^D(G) - C^D(G)$ and $v \in C^D(G) - C_1^D(G)$. Thus

$$e^D(v) < e^D(u) \tag{3.1}$$

and

$$e_1^D(u) < e_1^D(v) \tag{3.2}$$

If $e_1^D(u) = e^D(u)$ it follows from (3.1) and (3.2) that $e^D(v) < e_1^D(v)$ which is contradiction to Theorem(3.1). Similarly if $e_1^D(v) = e^D(v) - deg(v) - 1$ it follows from (3.1) and (3.2) that $e_1^D(u) < e^D(u) - deg(u) - 1$ which is again a contradiction to theorem(3.1). Hence it follows from theorem that

$$e_1^D(u) = e^D(u) - deg(u) - 1 \tag{3.3}$$

and

$$e_1^D(v) = e^D(v) \tag{3.4}$$

Now, it follows from (3.2), (3.3) and (3.4) that $e^D(u) - deg(u) - 1 < e^D(v)$. This gives $e^D(u) \leq e^D(v)$, which is a contradiction to (3.1). Hence the proof. ∎

Now, let us look at some examples.

Example 3.3. Consider the $(5,5)$ graph kite (K), shown in Figure 1. For this, we calculate the D-eccentricities of the vertices.

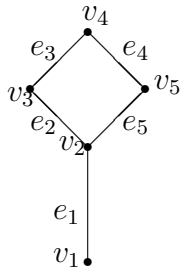


Figure 1: Kite

The following table gives the D-eccentricities of vertices of kite, K.

	v_1	v_2	v_3	v_4	v_5
$e^D(v)$	11	9	8	11	8
$e_1^D(v)$	8	6	6	9	6

Example 3.4. Consider the $(5, 6)$ graph hut H, shown in Figure 2. For this, we calculate the D-eccentricities of the vertices.

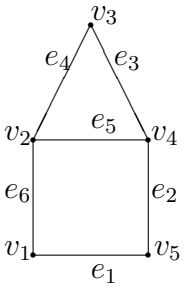


Figure 2: Hut

The following table gives the D-eccentricities of vertices of hut, H.

	v_1	v_2	v_3	v_4	v_5
$e^D(v)$	9	9	9	9	9
$e_1^D(v)$	9	7	9	7	9

Example 3.5. Consider the $(6, 8)$ graph tent (T), shown in Figure 3. For this, we calculate the D-eccentricities of the vertices.

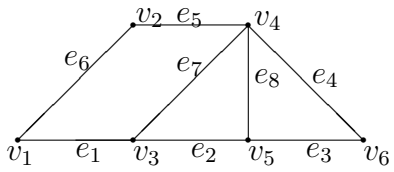


Figure 3: Tent

The following table gives the D-eccentricities of vertices in tent.

	v_1	v_2	v_3	v_4	v_5	v_6
$e^D(v)$	13	11	10	10	11	13
$e_1^D(v)$	10	10	8	8	10	10

After these examples, now we have

Remark 1. From these examples, it is clear that $C^D(K) \subseteq C_1^D(K), C^D(H) \supseteq C_1^D(H)$ and $C^D(T) = C_1^D(T)$.

Now, we have one of the result mentioned in the introduction.

Theorem 3.6. *The vertex set of every graph G, with at least two vertices, is the vertex-to-edge center of some connected graph.*

Proof. Let G be a given graph with n vertices. We construct a connected graph H such that vertex-to-edge center of H is the vertex set of G. First, if G is not complete make G a complete graph, by considering all the remaining edges. Then introduce two new vertices u and v to $V(G)$ and join them to every vertex of G but not each other as shown below. Next introduce two more vertices p and q and join p to u and q to v. The resulting graph H is given in Figure 4.

Then it is clear that

$$e_1^D(p) = e_1^D(q) = 3n + 7, e_1^D(u) = e_1^D(v) = 3n + 5 \text{ and } e_1^D(w) = 2n + 3$$

for every vertex w in G. Therefore, every w of G has minimum D-eccentricity. Hence the vertex set of G is the vertex-to-edge center of H.

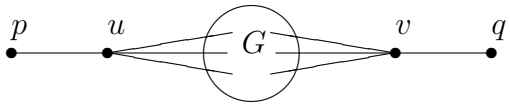


Figure 4: Construction of H

After this result, we prove a result on bipartite graphs.

Theorem 3.7. *For any vertex v of a bipartite graph G, $e^D(v) - 3 \leq e_1^D(v)$.*

Proof. Let $e = ab$ be an eccentric edge of v. If both a and b are eccentric vertices of v then $e^D(v) = d^D(v, a) = d^D(v, b)$. Let M and N be $v - a$ and $v - b$ geodesics respectively. If u is the last vertex that is common to both M and N, then the $u - a$ section of M followed by the edge ab and the $b - u$ section of N^{-1} forms an odd cycle in G, which is a contradiction to the fact that G containing even cycles. Hence it follows from Theorem 3.1 that $e^D(v) - 3 \leq e_1^D(v)$. ∎

We recall a definition before proving the next result.

Definition 3.8. A *block* of a graph is a maximal connected sub graph having no cut vertices. A graph G with all its blocks complete is called a *block graph.*

Theorem 3.9. *The vertex-to-edge center, $C_1^D(G)$, of every connected graph is a sub graph of some block of G.*

Proof. If possible that assume the center, $C_1^D(G)$, of a connected graph G lies in more than one block. Then G contains a cut vertex v such that $G - v$ has components G_1 and G_2, each of which contain a vertex of $C_1^D(G)$. Let $e = pq$ be an eccentric edge of v then $e_1(v) = d^D(v, e) = d^D(v, p)$ (say). Let M be a $v - p$ geodesic. At least one of G_1 and G_2 contains no vertices of M, say G_2 contains no vertices of M. Let w be a vertex of $C_1^D(G)$ that belongs to G_2 and let N be a $w - p$ geodesic in G. Now N followed by M gives a $w - p$ geodesic whose D-length is greater than that of M. It follows that $e_1^D(w) > e_1^D(v)$ which is contradiction

because $w \in C_1^D(G)$. Thus the vertex-to-edge center C_1^D of every connected graph G is a subgraph of some block of G. ∎

4. D-radius and D-diameter for vertex-to-edge D-distance

In this section we study some classes of graphs w.r.t. D-distance.

For cycles, we have

$$r_1^D(C_{2n}) = r_1^D(C_{2n-1}) = d_1^D(C_{2n}) = d_1^D(C_{2n-1}) = 3n - 1 \text{ for } n \geq 1.$$

Hence cycles are self centered graphs. Thus we have

Proposition 4.1. *For any cycle, C_n, we have $C^D(C_n) = C_1^D(C_n) = V(C_n)$.*

Proposition 4.2. *In cycle, C_n, the difference between the two eccentricities, $e^D(v)$ and $e_1^D(v)$ is 3 for all vertices V of C_n.*

For complete graphs, we have $r_1^D(K_n) = d_1^D(K_n) = 2n - 1$ for all $n \geq 3$.

Proposition 4.3. *Any complete graph, K_n, is self centered and further $C^D(K_n) = C_1^D(K_n) = V(K_n)$.*

Proposition 4.4. *If G is a complete graph, K_n, then the two eccentricities $e^D(v)$ and $e_1^D(v)$ are equal for all vertices v.*

Proof. This follows from the fact that $d^D(v, e) = 2n-1$ for any vertex v and any edge e of K_n. ∎

Proposition 4.5. *For wheel graph, $W_{n,1}$, we have $r_1^D(W_{n,1}) = n + 4 \ \forall \ n \geq 5$, $r_1^D(W_{3,1}) = 7$, $r_1^D(W_{4,1}) = 7$ and $d_1^D(W_{n,1}) = 2n + 1 \ \forall \ n \geq 5$, $d_1^D(W_{3,1}) = 7$, $d_1^D(W_{4,1}) = 8$.*

Proposition 4.6. *For star graph, $St_{n,1}$, we have $r_1^D(St_{n,1}) = 0$ and $d_1^D(St_{n,1}) = n + 2 \ \forall \ n \geq 2$.*

Proposition 4.7. *For any path graph, we have $r_1^D(P_{2n}) = r_1^D(P_{2n+1}) = 3n - 1 \ \forall \ n \geq 2$, $r_1^D(P_2) = r_1^D(P_3) = 0$. Similarly, $d_1^D(P_n) = 3n - 5 \ \forall \ n \geq 3$.*

Proposition 4.8. *For any complete bipartite graph, $K_{m,n}$, we have $r_1^D(K_{m,n}) = d_1^D(K_{m,n}) = m + n + 1 \ \forall \ m, n \geq 2$.*

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PRIME IDEALS IN RIGHT TERNARY NEAR-RINGS AND RIGHT TERNARY N-GROUPS

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Abstract. A Right Ternary Near-Ring (RTNR) is an algebraic system which is a group under binary addition and a ternary semigroup under ternary multiplication satisfying the right distributive law. In this paper ν-prime ideals where $\nu \in \{0, 1, 2, 3\}$ and equiprime ideals of an RTNR are defined and the relationship among them are discussed. A comparison between ν-primitive ideals and ν-prime ideals where $\nu \in \{0, 1, 2\}$ is given. If N is an RTNR then ν-prime ideals where $\nu \in \{0, 1, 2, 3, e, c\}$ in right ternary N-groups are defined and their relationships are studied. If Δ is an ideal of a right-lateral ternary N-group Γ, then the interrelationship between the ideal $(\Delta : \Gamma)$ of N and Δ are obtained. If $\nu = 0, 1, 2, 3, c$ then ν-semiprime ideals in RTNR and right ternary N-groups are also studied.

Keywords: Zero-symmetric RTNR, IFP-ideal, c-prime ideal, ν-primitive ideal, right ternary N-group.

AMS Subject Classification: 20N10, 16Y30, 16Y99.

1. Introduction

A right ternary near-ring is a generalisation of a near-ring in ternary context. Algebraic structures when studied in n-ary context reveal their fundamental properties in depth.

Prime ideals in near-rings have been studied by several authors. In 1970, Holcombe [2] defined ν-prime ideals where $\nu \in \{0, 1, 2, 3\}$. Booth and Groenewald [1] presented another generalization of prime rings, called equiprime or e-prime. Juglal, Groenewald and Lee [3] generalized the various notions of primeness that were defined in N to the N-group Γ. Tasdemir et al. [5] have studied about different prime N-ideals and IFP ideals in N-groups. The authors have defined prime ideal and c-prime ideal of an RTNR in [9].

In this paper, ν-prime ideals, where $\nu \in \{0, 1, 2, 3\}$, and equiprime ideals of an RTNR are defined and the relationship among them are studied. If N is an RTNR such that $[x\ y\ z] = [y\ x\ z]\ \forall\ x, y, z \in N$ then it is established that ν-primitive ideals of N are ν-prime ideals of N where $\nu = 0, 1, 2$. If N is an RTNR then ν-prime ideals where $\nu \in \{0, 1, 2, 3, c, e\}$ in right ternary N-groups are defined and their properties are studied. If Δ is an ideal of a right-lateral ternary N-group Γ, then the interrelationship between the ideal $(\Delta : \Gamma)$ of N and Δ are obtained. If $\nu = 0, 1, 2, 3, c$ then ν-semiprime ideals in RTNR and right ternary N-groups are also studied.

2. Preliminaries

In this section, the basic definitions and results needed for the rest of the sections are given.

Definition 2.1. [11]

(a) Let N be a non-empty set together with a binary operation $+$ and a ternary operation $[\ ] : N \times N \times N \to N$. Then $(N, +, [\ ])$ is a right ternary near-ring (RTNR) if (i) $(N, +)$ is a group (ii) $[[x\ y\ z]\ u\ v] = [x\ [y\ z\ u]\ v] = [x\ y\ [z\ u\ v]] = [x\ y\ z\ u\ v]$ for every x, y, z, u, $v \in N$ (iii) $[(x + y)\ z\ w] = [x\ z\ w] + [y\ z\ w]$ for every $x, y, z, w \in N$. Similarly left ternary near-ring and lateral ternary near ring can be defined.

(b) Let N be a right ternary near-ring. Let I be a normal subgroup $(N, +)$. Then I is called (i) a right ideal of N if $[I\ N\ N] \subseteq I$ (ii) a left ideal if $[t\ t'(t'' + i)] - [t\ t'\ t''] \in I$ (iii) a lateral ideal if $[t\ (t' + i)\ t''] - [t\ t'\ t''] \in I$ for every $t, t', t'' \in N$, $i \in I$. I is called a two-sided ideal if it is a left and right ideal of N and I is an ideal of N if it is a left, right and lateral ideal of N.

Definition 2.2. [10]

(a) A non-empty subset M of an RTNR N is called a right ternary subnear-ring (RTSNR) if (i) $(M, +)$ is a subgroup of N and (ii) $[M\ M\ M] \subseteq M$.

(b) A non-empty subset H of N is called an N-subgroup of N if (i) H is a subgroup of $(R, +)$ (ii) $[N\ N\ H] \subseteq H$ (iii) $[N\ H\ N] \subseteq H$ (iv) $[H\ N\ N] \subseteq H$. If (i) and (ii) hold, then H is called a left N-subgroup. If (i) and (iii) hold, then H is called a lateral N-subgroup. If (i) and (iv) hold, then H is called a right N-subgroup.

(c) An RTNR N is called a trio-RTNR if every one-sided N-subgroup is an N-subgroup of N.

Definition 2.3. [9]

(a) Let N be an RTNR. Then $N_0 = \{n \in N | [n\ 0\ 0] = 0\}$ is the zero-symmetric part of N. If $N = N_0$ then N is called a zero-symmetric RTNR.

(b) Let N be an RTNR. Then an ideal P of N is

(i) a prime ideal if there are ideals X, Y, Z in N such that $[X\ Y\ Z] \subseteq P \Rightarrow$ either $X \subseteq P$ or $Y \subseteq P$ or $Z \subseteq P$.

(ii) a completely prime ideal of N if for $x, y, z \in N$, $[x\ y\ z] \in P \Rightarrow$ either $x \in P$ or $y \in P$ or $z \in P$.

(iii) a completely semi-prime ideal if $x^3 = [x\ x\ x] \in P \Rightarrow x \in P$.

Definition 2.4. [8]

(a) N is a zero RTNR if $N^3 = [N\ N\ N] = \{0\}$.

(b) If N is an RTNR then an element $e \in N$ is a left (resp. right, lateral) unital element if $[e\ e\ x] = x$ ($[x\ e\ e] = x$, $[e\ x\ e] = x$) for every $x \in N$. If $[e\ e\ x] = x = [x\ e\ e]$ then e is called a bi-unital element.

(c) If N is an RTNR then $N_c = \{t \in N | [t\ 0\ 0] = t\}$ is called the constant part of N and N is called a constant RTNR if $N = N_c$.

(d) An ideal I of N is an IFP ideal if $[a\ b\ c] \in I \Rightarrow [a\ u\ b\ v\ c] \in I\ \ \forall\ \ u, v \in N$.

Theorem 2.5. [8] *An ideal I of N is maximal iff N/I is simple.*

Definition 2.6. [7]

(a) Let $(N, +, [\ ])$ be an RTNR and $(\Gamma, +)$ be a group with additive identity o. Then Γ is said to be a right ternary N-group if there exists a mapping $[\ ]_\Gamma : N \times N \times \Gamma \to \Gamma$ such that (i) $[n + m\ x\ \gamma]_\Gamma = [n\ x\ \gamma]_\Gamma + [m\ x\ \gamma]_\Gamma$ (ii) $[[n\ m\ u]\ x\ \gamma]_\Gamma = [n\ [m\ u\ x]\ \gamma]_\Gamma = [n\ m\ [u\ x\ \gamma]_\Gamma]_\Gamma$ for all $\gamma \in \Gamma$ and $n, m, u \in N$.

(b) A subgroup Δ of ${}_N\Gamma$ is said to be

(i) an N-subgroup of ${}_N\Gamma$ if $[N\ N\ \Delta]_\Gamma \subseteq \Delta$.

(ii) a normal subgroup of ${}_N\Gamma$ if $\gamma + \delta - \gamma \in \Delta\ \forall\ \gamma \in \Gamma$, $\delta \in \Delta$.

(iii) an N-ideal of ${}_N\Gamma$ if Δ is a normal subgroup of ${}_N\Gamma$ and $[n\ x\ (\gamma + \delta)]_\Gamma - [n\ x\ \gamma]_\Gamma \in \Delta\ \forall\ \gamma \in \Gamma$, $\forall\ \delta \in \Delta$ and $\forall\ n, x \in N$.

(c) A right ternary N-group ${}_N\Gamma$ is said to be

(i) simple if $\{o\}$ and Γ are the only ideals of ${}_N\Gamma$ where o is the identity element of Γ.

(ii) N-simple if Ω and Γ are the only N-subgroups of ${}_N\Gamma$, where $\Omega = [N_c\ 0\ o]$.

(iii) faithful if $(o : \Gamma) = \{0\}$.

(d) Let ${}_N\Gamma$ be a right ternary N-group of an RTNR N. Then for $x \in N$ there exists $\gamma \in \Gamma$, N is monogenic by γ w.r.to x if $[N\ x\ \gamma]_\Gamma = \Gamma$ and N is monogenic by γ if there exists $\gamma \in \Gamma$ and for every $x \in N$, $[N\ x\ \gamma]_\Gamma = \Gamma$ and Γ is a generator of Γ. A right ternary N-group ${}_N\Gamma$ is strongly monogenic if Γ is monogenic and $[N\ x\ \gamma]_\Gamma = \{o\}$ or Γ for every $x \in N$ and $\gamma \in \Gamma$.

(e) A monogenic N-group Γ with $\Gamma \neq \{o\}$ is said to be of type 0 if Γ is simple, type 1 if Γ is simple and strongly monogenic and type 2 if Γ is N_0-simple.

(f) If $[x\ y\ \gamma]_\Gamma = 0 \Rightarrow [x\ u\ y\ v\ \gamma]_\Gamma = 0\ \ \forall\ \ x, y, u, v \in N$, $\gamma \in \Gamma$ then ${}_N\Gamma$ is called an IFP RTNR.

Theorem 2.7. [7]

(i) *If ${}_N\Gamma$ is monogenic by γ_0 w.r. to $x \in N$ and L is a left ideal of N, then $[L\ x\ \gamma_0]_\Gamma$ is an ideal of ${}_N\Gamma$*

(ii) *If ${}_N\Gamma$ is monogenic by γ_0 and if e is a left unital element of N, then $\forall\ \gamma \in \Gamma$, $[e\ e\ \gamma]_\Gamma = \gamma$.*

Proposition 2.8. [7] *Let Γ be a right ternary N-group. Then ${}_N\Gamma$ is of type 2 $\Rightarrow$ ${}_N\Gamma$ is of type 1 $\Rightarrow$ ${}_N\Gamma$ is of type 0.*

Definition 2.9. [6] An ideal I of an RTNR N is called a direct summand of N if there exists an ideal J of N such that $N = I + J$. The ideal J is called a direct complement of I in N. N is called ν-primitive on ${}_N\Gamma$ if $(o : \Gamma) = \{0\}$ i.e., Γ is faithful right ternary N-group and Γ is of type ν. N is ν-primitive if there exists an N-group ${}_N\Gamma$ such that N is ν-primitive on ${}_N\Gamma$. If in N $[x\ y\ z] = [y\ x\ z]$ then an ideal I of N is called ν-primitive ideal of N if N/I is ν-primitive.

Definition 2.10. [6]

(a) Let N be a right and a lateral ternary near ring. Let I be an ideal of N. Then a group $(\Gamma, +)$ is called a right-lateral ternary N-group if $[\ ]_\Gamma : N \times N \times \Gamma \to \Gamma$ such that

(i) $[n + m\ x\ \gamma]_\Gamma = [n\ x\ \gamma]_\Gamma + [m\ x\ \gamma]_\Gamma$

(ii) $[n\ m + x\ \gamma]_\Gamma = [n\ m\ \gamma]_\Gamma + [n\ x\ \gamma]_\Gamma$

(iii) $[[n\ m\ u]\ x\ \gamma]_\Gamma = [n\ [m\ u\ x]\ \gamma]_\Gamma = [n\ m\ [u\ x\ \gamma]_\Gamma]_\Gamma$ for every $n, m, x, u \in N$ and $\gamma \in \Gamma$.

(b) If Δ_1 and Δ_2 are any two non-empty subsets of a right-lateral ternary N-group then $(\Delta_1 : \Delta_2) = \{n \in N | [n\ x\ \delta_2]_\Gamma \in \Delta_1$ and $[x\ n\ \delta_2]_\Gamma \in \Delta_1\ \forall\ x \in N$, $\delta_2 \in \Delta_2\}$. Also if for every $x, y, z \in N$, $[x\ y\ z] = [y\ x\ z]$ and Δ is an N-subgroup then $(o : \Delta)$ is an ideal of N.

3. Prime ideals in RTNR

In this section, ν-prime ideals where $\nu \in \{0, 1, 2, 3, c\}$ and equiprime ideals are defined and the relationship among them are studied. In an RTNR with $[x\ y\ z] = [y\ x\ z]\ \forall\ x, y, z \in N$, it is shown that ν-primitive ideals of N are ν-prime ideals of N where $\nu = 0, 1, 2$.

Definition 3.1. Let P be an ideal of an RTNR N. Then

(a) if $[A\ B\ C] \subseteq P \Rightarrow$ either $A \subseteq P$ or $B \subseteq P$ or $C \subseteq P$ holds for all

  (i) ideals A, B, C of N then P is 0-prime.

  (ii) left ideals A, B, C of N then P is 1-prime.

  (iii) N-subgroups A, B, C of N then P is 2-prime.

(b) P is 3-prime if $a, b, c \in N$ and $[a\ N\ b\ N\ c] \subseteq P \Rightarrow$ either $a \in P$ or $b \in P$ or $c \in P$.

(c) P is completely prime (c-prime) if $a, b, c \in N$ and $[a\ b\ c] \subseteq P \Rightarrow$ either $a \in P$ or $b \in P$ or $c \in P$.

(d) P is equiprime (e-prime) if $a \in N - P$ and $x, y \in N$ with $[a\ n\ x] - [a\ n\ y] \in P$ $\forall\ \ n \in N \Rightarrow x - y \in P$.

  Equivalently if $a \notin P$, $b \notin P$ and $[a\ n\ b\ m\ x] - [a\ n\ b\ m\ y] \in P\ \ \forall\ \ n, m \in N \Rightarrow x - y \in P$ then P is e-prime.

Definition 3.2. An RTNR N is said to be ν-prime if $\{0\}$ is a ν-prime ideal where $\nu \in \{0, 1, 2, 3, c, e\}$.

Example 3.3. Let $N = \{0, x, y, z\}$ be as given in [4, Scheme 1, p. 408]. Define $+$ as in Table 1 and the ternary operation $[\ ]$ on N by $[x\ y\ z] = (x.y).z$ for every $x, y, z \in N$ where $\cdot$ is defined as in Table 2.

Table 1:

$+$	0	x	y	z
0	0	x	y	z
x	x	0	z	y
y	y	z	0	x
z	z	y	z	0

Table 2:

$\cdot$	0	x	y	z
0	0	0	0	0
x	0	x	x	x
y	0	y	y	y
z	0	z	z	z

Since $[x\ y\ z] = 0$ holds only if either $x = 0$ or $y = 0$ or $z = 0$ and $[A\ B\ C] = 0$ holds only if either $A = \{0\}$ or $B = \{0\}$ or $C = \{0\}$, N is ν-prime where $\nu = 0, 1, 2, c$.

Table 3:

$+$	0	x	y
0	0	x	y
x	x	y	0
y	y	0	x

Table 4:

$\cdot$	0	x	y
0	0	0	0
x	0	x	x
y	0	y	y

Example 3.4. Let $N = \{0, x, y\}$ be as given in [4, Scheme 3, p. 407]. Define $+$ as in Table 3 and the ternary operation $[\ \ ]$ on N by $[xyz] = (x.y).z$ for every $x, y, z \in N$ where $\cdot$ defined as in Table 4.

Since $[a\ N\ b\ N\ c] \subseteq \{0\}$ holds only if either $a = 0$ or $b = 0$ or $c = 0$, N is a 3-prime RTNR.

Example 3.5. Let $\Gamma = \{0, 1, 2\} = \mathbb{Z}_3$ and $N = M_0(\Gamma) = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9\}$ where α_i's are defined as given in Table 5.

Table 5:

	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9
0	0	0	0	0	0	0	0	0	0
1	0	0	0	1	2	1	2	1	2
2	0	1	2	0	0	1	2	2	1

Note that $\{\alpha_1\}$ (α_1 is the zero mapping) is an e-prime ideal of N. For if $\alpha_j - \alpha_k \notin \{\alpha_1\}$ i.e., $\alpha_j \neq \alpha_k$ then for some $t \in \Gamma$, $\alpha_j(t) \neq \alpha_k(t)$ and also for any $0 \neq \alpha_l \in N$ there exists $0 \neq x \in \Gamma$ such that $\alpha_l(x) \neq 0$. Define $\theta : \Gamma \to \Gamma$ by $\theta(z) = x$ if $z = \alpha_j(t)$ and 0 otherwise. Then $\theta \in N$ and also $[\alpha_l\ \theta\ \alpha_j] \neq [\alpha_l\ \theta\ \alpha_k]$ i.e., $[\alpha_l\ \theta\ \alpha_j] - [\alpha_l\ \theta\ \alpha_k] \notin \{\alpha_1\}$. Thus $\{\alpha_1\}$ is an e-prime ideal of N. Hence $(N.+, [\ ])$ is an e-prime RTNR.

Proposition 3.6. *If P is a left ideal of an RTNR N then P is a left N-subgroup of N iff $[n\ x\ 0] \in P$ for all $n, x \in N$.*

Proof. It is obvious that if P is a left N-subgroup of N then for all $n, x \in N$, $[n\ x\ 0] \in P$. Conversely let $[n\ x\ 0] \in P$ for all $n, x \in N$. Now $[n\ x\ i] = [n\ x\ i+0] - [n\ x\ 0] + [n\ x\ 0] \Rightarrow [n\ x\ i] \in P\ \ \forall\ \ i \in P$. Hence P is a left N-subgroup of N. ∎

Proposition 3.7. *If P is an e-prime ideal then P is a left N-subgroup of N.*

Proof. In view of the above proposition it suffices to prove that $[n\ x\ 0] \in P$ for all $n, x \in N$. Now $[[n\ 0\ 0]\ m\ [n\ x\ 0]] - [[n\ 0\ 0]\ m\ 0] = [n\ 0\ 0] - [n\ 0\ 0] = 0 \in P$ by associativity in N. Since P is an e-prime ideal $[n\ x\ 0] - 0 \in P$. i.e., $[n\ x\ 0] \in P$. Hence the proof. ∎

The following lemmas describe the relationship between any two ν-prime ideals where $\nu \in \{0, 1, 2, 3, e, c\}$.

Lemma 3.8. *Every e-prime ideal of an RTNR N is a 3-prime ideal of N.*

Proof. Let P be an e-prime ideal of an RTNR N. Suppose for $a, b, c \in N$, $[a\ N\ b\ N\ c] \subseteq P$ but $a \notin P$, $b \notin P$ and $c \notin P$. Since $[a\ [n\ b\ m]\ c] \in P$ and $[a\ [n\ b\ m]\ 0] \in P\ \ \forall\ \ n, m \in N$, $[a\ [n\ b\ m]\ c]$ - $[a\ [n\ b\ m]\ 0] \in P \Rightarrow c - 0 \in P$, which contradicts the hypothesis. Hence P is a 3-prime ideal of N. ∎

Remark 3.9. The converse of the above lemma is not true. For if N is as in Example 3.4 then $\{0\}$ is 3-prime but is not e-prime as $[x\ y\ x] - [x\ y\ y] \in \{0\}$ but $x - y \notin \{0\}$.

Lemma 3.10. *Let N be an RTNR. Then every e-prime ideal is a c-prime ideal of N.*

Proof. Let $a, b, c \in N$ be such that $[a\ b\ c] \in P$. Suppose $a \notin P$ and $b \notin P$. Since P is e-prime, $[a\ b\ c] \in P$. Hence $[a\ b\ c] - [a\ b\ 0] \in P \Rightarrow c \in P$, proving that P is a c-prime ideal. ∎

Lemma 3.11. *If N is an RTNR with $[x\ y\ z] = [y\ x\ z]\ \ \forall\ \ x, y, z \in N$ then every c-prime ideal is an e-prime ideal of N.*

Proof. Let P be a c-prime ideal of N. Let $a \in N - P$ and $x, y \in N$ be such that $[a\ n\ x] - [a\ n\ y] \in P$ for all $n \in N$. Let $q \in N - P$. Now for $t \in N$, $([a\ n\ x] - [a\ n\ y])\ t\ q] \in P \Rightarrow [[x\ n\ t]\ a\ q] - [[y\ n\ t]\ a\ q] \in P$ by associativity and hypothesis. This implies that $[(x - y)\ n\ t] \in P$ as $a, q \notin P$. Hence $x - y \in P$, for otherwise $[(x - y)\ n\ t] \notin P$, a contradiction. Hence P is an e-prime ideal of N. ∎

Lemma 3.12. *Let N be an RTNR. Then every 3-prime ideal is a 2-prime ideal of N.*

Proof. Let P be a 3-prime ideal of an RTNR N. Let $[A\ B\ C] \subseteq P$ for all N-subgroups A, B, C of N. If either $A \subseteq P$ or $B \subseteq P$ or $C \subseteq P$ then the lemma follows. Suppose not. Then there exists $a \in A$, $b \in B$, $c \in C$ and $a, b, c \notin P$ with $[a\ N\ b\ N\ c] \subseteq [a\ N\ B\ N\ c] \subseteq [a\ B\ C] \subseteq [A\ B\ C] \subseteq P$. Since P is a 3-prime ideal this implies that $a \in P$ or $b \in P$ or $c \in P$, a contradiction. Hence P is a 2-prime ideal of N. ∎

Remark 3.13. The converse of the above lemma is not true. For if $N = \{0, x, y\}$ as in [4, Scheme 2, p. 407] and $+$ is defined as given in Table 6 and the ternary multiplication $[\ \ ]$ is defined as $[a\ b\ c] = (a \cdot b) \cdot c$ where $\cdot$ is defined as in Table 7.

Table 6:

$+$	0	x	y
0	0	x	y
x	x	y	0
y	y	0	x

Table 7:

$\cdot$	0	x	y
0	0	0	0
x	0	0	x
y	0	0	y

Then N is an RTNR and $\{0\}$ is a 2-prime ideal but is not a 3-prime ideal as $[x\ N\ x\ N\ x] \subseteq \{0\}$ but $x \neq 0$.

Lemma 3.14. *If N is an RTNR with unital element e then every 2-prime ideal is a 3-prime ideal.*

Proof. Let P be a 2-prime ideal of N and $[a\ N\ b\ N\ c] \subseteq P\ \ \forall\ \ a, b, c \in N$. Then

$$[N\ N\ [a\ N\ b\ N\ c]] \subseteq [N\ N\ P] \subseteq P$$

Now

$$\begin{aligned} &[N\ e\ a\ N\ b\ c\ N\ e\ c] \subseteq [N\ n\ a\ N\ b\ N\ c] \subseteq P \\ &\Rightarrow\ [N\ e\ a] \subseteq P\ \text{ or }\ [N\ b\ e] \subseteq P\ \text{ or }\ [N\ e\ c] \subseteq P \\ &\Rightarrow\ [e\ e\ a] \in P\ \text{ or }\ [e\ b\ e] \in P\ \text{ or }\ [e\ e\ c] \in P \\ &\Rightarrow\ a \in P\ \text{ or }\ b \in P\ \text{ or }\ c \in P \end{aligned}$$

Hence P is a 3-prime ideal of N. ∎

Lemma 3.15. *If N is a trio-RTNR then every 2-prime ideal is a 1-prime ideal of N.*

Proof. Let N be a trio-RTNR and P be a 2-prime ideal of N. Then N is zero-symmetric. Let A, B, C be left ideals of N such that $[A\ B\ C] \subseteq P$. Then A, B, C are left N-subgroups of N and hence are N-subgroups of N as N is a trio-RTNR. Since P is 2-prime ideal and $[A\ B\ C] \subseteq P$, $A \subseteq P$ or $B \subseteq P$ or $C \subseteq P$. Thus P is a 1-prime ideal of N. ∎

Remark 3.16. The converse of the above lemma is not true. For if $N = \{0, x, y, z\}$ as given in [ ] where $+$ is defined as in Table 8 and the ternary multiplication [ ] is defined as $[a\ b\ c] = (a \cdot b) \cdot c$ where $\cdot$ is as given in Table 9. Then N is a trio RTNR and $\{0\}$ is a 1-prime ideal but is not a 2-prime ideal as $\{0, x\} = A$ (say) is an N-subgroup of N such that $[A\ A\ A] \subseteq \{0\}$ but $A \not\subseteq \{0\}$.

Table 8:

$+$	0	x	y	z
0	0	x	y	z
x	x	0	z	y
y	y	z	0	x
z	z	y	z	0

Table 9:

$\cdot$	0	x	y	z
0	0	0	0	0
x	0	0	0	x
y	0	0	0	y
z	0	0	0	z

Lemma 3.17. *If N is an RTNR then every 1-prime ideal of N is a 0-prime ideal of N.*

Proof. Let A, B, C be ideals of N such that $[A\ B\ C] \subseteq P$. Since every ideal is a left ideal and P is 1-prime $[A\ B\ C] \subseteq P \Rightarrow A \subseteq P$ or $B \subseteq P$ or $C \subseteq P$. Thus P is a 0-prime ideal of N. ∎

Remark 3.18. The converse of the above lemma holds if N is commutative.

Lemma 3.19. *If N is an RTNR then every c-prime ideal is a 3-prime ideal of N.*

Proof. Let P be a c-prime ideal of an RTNR N. Let $[a\ N\ b\ N\ c] \subseteq P$. Then $[a\ u\ b\ v\ c] \in P$ for all $u, v \in N$. Since P is a c-prime ideal,

$$\begin{aligned}[a\ u\ b\ v\ c] \in P &\Rightarrow [a\ u\ b] \in P \text{ or } v \in P \text{ or } c \in P \\ &\Rightarrow a \in P \text{ or } u \in P \text{ or } b \in P \text{ or } v \in P \text{ or } c \in P \\ &\Rightarrow a \in P \text{ or } b \in P \text{ or } c \in P\end{aligned}$$

as $N \not\subseteq P$. Hence P is a 3-prime ideal of N. ∎

Remark 3.20. The converse holds if P is an IFP ideal of N.

Lemma 3.21. *If N is a zero-symmetric RTNR then every 2-prime ideal is a 0-prime ideal of N.*

Proof. Let P be a 2-prime ideal of a zero-symmetric RTNR. Let A, B, C be ideals of N such that $[A\ B\ C] \subset P$. Then A, B, C are N-subgroups of N and as P is a 2-prime ideal $[A\ B\ C] \subseteq P \Rightarrow A \subseteq P$ or $B \subseteq P$ or $C \subseteq P$. Hence P is a 0-prime ideal. ∎

Proposition 3.22. *If P is an e-prime ideal of an RTNR N and H is an N-subgroup of N then $H \cap P$ is an e-prime ideal of H.*

Proof. If P is an ideal and H is an N-subgroup of N then it can be easily seen that $H \cap P$ is an ideal of H. Now let $x, y \in H$ and $h \notin H \cap P$ be such that $[h\ a\ x] - [h\ a\ y] \in H \cap P\ \forall\ a \in H$. If $x - y \notin H \cap P$ then $[h\ v\ x] - [h\ v\ y] \notin P\ \forall\ v \in N$ as P is e-prime in N. This implies that $[h\ u\ [h\ v\ x]] - [h\ u\ [h\ v\ y]] \notin P$ or $[h\ [u\ h\ v]\ x] - [h\ [u\ h\ v]\ y] \notin P$ which contradicts the assumption. Therefore $x - y \in H \cap P$. Hence $H \cap P$ is e-prime in H. ∎

Remark 3.23.

(i) If N is an e-prime RTNR then every N-subgroup is an e-prime RTSNR.

(ii) If N is zero-symmetric e-prime RTNR then every ideal is an e-prime RTSNR.

Theorem 3.24. *Let N be an RTNR such that $[x\ y\ z] = [y\ x\ z]\ \ \forall\ \ x, y, z \in N$. Then*

(i) *every 0-primitive ideal of N is a 0-prime ideal of N.*

(ii) *every 1-primitive ideal of N is a 1-prime ideal of N.*

(iii) *every 2-primitive ideal of N is a 2-prime ideal of N.*

Proof. (i) Let P be a 0-primitive ideal of N. Then N/P is 0-primitive. This implies that there exists a right-lateral ternary N-group ${}_N\Gamma$ such that $(o : \Gamma) = P$ and Γ is of type 0. Also for $\gamma_0 \in \Gamma$, $\Gamma = [N\ x\ \gamma_0]_\Gamma \neq \{o\}\ \ \forall\ \ x \in N$. Suppose for ideals A, B, C of N, $[A\ B\ C] \subseteq P$ but $A \not\subseteq P$, $B \not\subseteq P$ and $C \not\subseteq P$. It is noted that as C is an ideal of $N\ \ \forall\ \ x \in N$, $[C\ x\ \Gamma]_\Gamma = [C\ x\ [N\ x\ \Gamma_0]_\Gamma]_\Gamma \subseteq [C\ x\ \gamma_0]_\Gamma \subseteq [C\ x\ \Gamma]_\Gamma\ \ \forall\ \ x \in N$ and hence $[C\ x\ \Gamma]_\Gamma = [C\ x\ \gamma_0]_\Gamma$. Now as

$C \not\subseteq (o : \Gamma)$, $[C\ x\ \Gamma]_\Gamma = [C\ x\ \Gamma_0]_\Gamma \neq \{0\}$. Since C is an ideal $[C\ x\ \gamma_0]_\Gamma$ is an ideal of ${}_N\Gamma$ and ${}_N\Gamma$ is type 0 it follows that $[C\ x\ \Gamma]_\Gamma = [C\ x\ \gamma_0]_\Gamma = \Gamma$. Thus, $[A\ B\ [C\ x\ \Gamma]_\Gamma]_\Gamma = [A\ B\ \Gamma]_\Gamma = \Gamma \neq \{o\}$. $\Rightarrow [A\ B\ C] \not\subseteq (o : \Gamma) = P$ which contradicts the assumption that $[A\ B\ C] \subseteq P$. Hence P is a 0-prime ideal of N.

(ii) Let P be a 1-primitive ideal of N. Then there exists a right-lateral ternary N-group such that $P = (o : \Gamma)$ where Γ is of type 1. If A, B, C are left ideal of N such that $[A\ B\ C] \subseteq P$ with $A \not\subseteq P$ and $B \not\subseteq P$ and $C \not\subseteq P$. Then $[C\ x\ \Gamma]_\Gamma \neq \{0\}$ and Γ is strongly monogenic $[C\ x\ \Gamma]_\Gamma = \Gamma$. Hence $[A\ B\ [C\ x\ \Gamma]_\Gamma]_\Gamma = [A\ B\ \Gamma]_\Gamma = \Gamma \neq \{o\} \Rightarrow [A\ B\ C] \not\subseteq P$, a contradiction to the assumption. Hence P is a 1-prime ideal.

(iii) Let P be a 2-primitive ideal of N. Then there exists a right-lateral ternary N-group ${}_N\Gamma$ such that $(o : \Gamma) = P$ where Γ is of type 2. Since Γ is of type 2, $\Gamma = [N\ x\ \gamma_0]_\Gamma \neq \{o\}\ \forall\ x \in N$. Suppose for N-subgroups A, B, C of N, $[A\ B\ C] \subseteq P$ but $A \not\subseteq P$, $B \not\subseteq P$ and $C \not\subseteq P$. Then as $C \not\subseteq \Gamma$, $[C\ x\ \Gamma]_\Gamma \neq \{o\}$. Also $[C\ x\ \Gamma]_\Gamma = [C\ x\ [N\ x\ \gamma_0]_\Gamma]_\Gamma \subseteq [C\ x\ \gamma_0]_\Gamma \subseteq [C\ x\ \Gamma]_\Gamma$. Thus $[C\ x\ \Gamma]_\Gamma = [C\ x\ \gamma_0]_\Gamma \neq \{o\}$. Since C is an N-subgroup of N, $[C\ x\ \gamma_0]_\Gamma$ is an N-subgroup of ${}_\Gamma N$ and since Γ is N_0-simple, $[C\ x\ \Gamma]_\Gamma = \Gamma$. Thus $[A\ B\ C\ x\ \Gamma]_\Gamma = [A\ B\ \Gamma]_\Gamma = \Gamma \neq \{o\}$ which contradicts $[A\ B\ C] \subseteq P$. Hence P is of type 2. ∎

Proposition 3.25. *If an ideal I of N is a direct summand and P is a ν-prime ideal in N then $P \cap I$ is a ν-prime ideal in I, where $\nu = 0$.*

Proof. Let A, B, C be ideals of I. Then as I is a direct summand of N, A, B, C are ideals of N. Now let $[A\ B\ C] \subseteq P \cap I \Rightarrow [A\ B\ C] \subseteq P$ and $[A\ B\ C] \subseteq I \Rightarrow A \subseteq P$ or $B \subseteq P$ or $C \subseteq P$. Thus $A \subseteq P \cap I$ or $B \subseteq P \cap I$ or $C \subseteq P \cap I$ and hence $P \cap I$ is a 0-prime ideal in I. A similar argument holds for 1-prime ideal in N. ∎

Proposition 3.26. *If $N = N_c$ then every normal subgroup of $(N, +)$ is a 0-prime ideal of N.*

Proof. Every normal subgroup of $(N_c, +)$ is clearly an ideal of N_c. Let P be an ideal of N such that $[A\ B\ C] \subseteq P$ for all ideals A, B, C of N. Consider, $[a\ b\ c] = [a\ b\ c\ 0\ 0] = [a\ b\ 0] = a$, where $a \in A$, $b \in B$, $c \in C$. Thus $A \subseteq P$. Hence P is a 0-prime ideal of N. ∎

Corollary 3.27. *N_c is a 0-prime RTSNR.*

Proof. Since $\{0\}$ is a normal subgroup of N_c and is an ideal of N_c, by the above proposition N_c is 0-prime RTSNR. ∎

Proposition 3.28. *If N is simple then either N is 0-prime RTNR or N is a zero RTNR.*

Proof. If N is simple then $\{0\}$ and N are the only ideals of N. If $\{0\}$ is a prime ideal of N then N is a prime RTNR. If $\{0\}$ is not a prime ideal of N then $N^3 \subseteq \{0\}$ but $N \not\subseteq \{0\}$. Thus $N^3 = \{0\}$ and hence N is a zero RTNR. **Proof.**

Proposition 3.29. *If I is a maximal ideal in N then I is 0-prime or $N^3 \subseteq I$.*

Proof. If I is a maximal ideal in N then N/I is simple. This implies that N/I is 0-prime RTNR or N/I is zero-RTNR. i.e., I is 0-prime ideal or $N^3 \subseteq I$. In the following the definition of ν-semiprime ideals of an RTNR N is given. ∎

Definition 3.30. Let N be an RTNR and I be an ideal of N. Then

(a) $A^3 \subseteq I \Rightarrow A \subseteq I$ holds for all

  (i) ideals A of N then I is 0-semiprime

  (ii) left ideals A of N then I is 1-semiprime

  (iii) N-subgroups A of N then I is 2-semiprime

(b) I is 3-semiprime if $[a\ N\ a\ N\ a] \subseteq I \Rightarrow a \in I$ for all $a \in I$

(c) I is c-semiprime if $a^3 \in I \Rightarrow a \in I$ for all $a \in I$

Definition 3.31. An RTNR N is called a semiprime RTNR if $\{0\}$ is a semiprime ideal of N.

Note 3.32. In an RTNR N, every ν-prime ideal is a ν-semiprime ideal where $\nu \in \{0, 1, 2, 3, c\}$.

Proposition 3.33. *The intersection of ν-prime ideals of an RTNR N is a ν-semiprime ideal of N, $\nu \in \{0, 1, 2\}$.*

Proof. Let H denote the intersection of ν-prime ideals P. Then H is an ideal of N and for any ideal A (or N-subgoup or left ideal) of N, $A^3 \subseteq H \Rightarrow A^3 \subseteq P$ for every $P \Rightarrow A \subseteq P$. Hence H is ν-semiprime ideal of N. ∎

Proposition 3.34. *If an ideal I of N is a direct summand of N and S is a ν-semiprime ideal of N then $S \cap I$ is a ν-semiprime ideal in I where $\nu \in \{0, 1\}$.*

Proof. Let A be an ideal (left ideal) of I. Then as I is a direct summand of N, A is an ideal (left ideal) of N. Now let $A^3 \subseteq S \cap I \Rightarrow A^3 \subseteq S$ and $A^3 \subseteq I \Rightarrow A \subseteq S \cap I$. Thus $S \cap I$ is a ν-semiprime ideal of N where $\nu = 0, 1$. ∎

4. Prime ideals in right ternary N-groups

In this section ν-prime ideals in right ternary N-groups are defined and their interrelationship are studied.

Definition 4.1. Let N be an RTNR and ${}_N\Gamma$ be a right ternary N-group. Let Δ be an N-ideal of Γ such that $[N\ N\ \Gamma]_\Gamma \not\subseteq \Delta$. Then

(a) if $[A\ B\ \Delta']_\Gamma \subseteq \Delta \Rightarrow$ either $[A\ x\ \Gamma]_\Gamma \subseteq \Delta$ or $[B\ x\ \Gamma]_\Gamma \subseteq \Delta$ or $\Delta' \subseteq \Delta\ \forall\ x \in N$ holds for all

  (i) ideals A, B of N and an N-ideal Δ' of Γ then Δ is called 0-prime.

(ii) left ideals A, B of N and an N-ideal Δ' of Γ then Δ is called 1-prime.

(iii) N-subgroups A, B of N and an N-subgroup Δ' of Γ then Δ is called 2-prime.

(b) Δ is 3-prime if $[a\ N\ b\ N\ \gamma]_\Gamma \subseteq \Delta \Rightarrow$ either $[a\ x\ \Gamma]_\Gamma \subseteq \Delta$ or $[b\ x\ \Gamma]_\Gamma \subseteq \Delta$ or $\gamma \in \Gamma\ \forall\ a, b \in N$ and $\gamma \in \Gamma$

(c) Δ is c-prime (completely prime) if $[a\ b\ \gamma]_\Gamma \in \Delta \Rightarrow$ either $[a\ x\ \Gamma]_\Gamma \subseteq \Delta$ or $[b\ x\ \Gamma]_\Gamma \subseteq \Delta$ or $\gamma \in \Delta\ \forall\ a, b \in N$ and $\gamma \in \Gamma$

(d) Δ is e-prime (equiprime) if $a \in N$, $[a\ x\ \Gamma]_\Gamma \not\subseteq \Delta\ \forall\ x \in N$ and $\gamma_1, \gamma_2 \in \Gamma$ then $[a\ n\ \gamma_1]_\Gamma - [a\ n\ \gamma_2]_\Gamma \in \Delta\ \forall\ n \in N \Rightarrow \gamma_1 - \gamma_2 \in \Delta$.

Definition 4.2. A right ternary N-group ${}_N\Gamma$ is said to be ν-prime where $\nu \in \{0, 1, 2, 3, c, e\}$ if $[N\ N\ \Gamma]_\Gamma \neq \{o\}$ and $\{o\}$ is a ν-prime ideal of ${}_N\Gamma$.

Proposition 4.3. *If Δ is an ideal of a right ternary N-group ${}_N\Gamma$ then Δ is an N-subgroup of ${}_N\Gamma$ iff $[n\ x\ o]_\Gamma \in \Delta$ for all $n, x \in N$.*

Proof. It is obvious that if Δ is an N-subgroup of ${}_N\Gamma$ then for all $n, x \in N$, $[n\ x\ o]_\Gamma \in \Delta$. Conversely let $[n\ x\ o]_\Gamma \in \Delta$ for all $n, x \in N$. Now $[n\ x\ \delta] = [n\ x\ \delta + o]_\Gamma - [n\ x\ o]_\Gamma + [n\ x\ o]_\Gamma \Rightarrow [n\ x\ \delta]_\Gamma \in \Delta$. Hence Δ is an N-subgroup of ${}_N\Gamma$. ∎

Proposition 4.4. *If Δ is an e-prime ideal of ${}_N\Gamma$ then Δ is an N-subgroup of ${}_N\Gamma$.*

Proof. In view of the above proposition it suffices to prove that $[n\ x\ o]_\Gamma \in \Delta$ for all $n, x \in N$. Now $[[n\ 0\ 0]\ m[n\ x\ o]_\Gamma]_\Gamma - [[n\ 0\ 0]\ m\ o]_\Gamma = [n\ 0\ o]_\Gamma - [n\ 0\ o]_\Gamma = o \in \Delta$. Since Δ is an e-prime ideal, $[n\ x\ o]_\Gamma - o \in \Delta$. i.e., $[n\ x\ o]_\Gamma \in \Delta$. Hence the proof. ∎

Theorem 4.5. *Let Δ be an N-ideal of ${}_N\Gamma$ such that $[N\ N\ \Gamma]_\Gamma \not\subseteq \Delta\ \forall\ x \in N$ and $\gamma \in \Gamma$. Then*

(i) *Δ is e-prime $\Rightarrow$ Δ is 3-prime*

(ii) *Δ is e-prime $\Rightarrow$ Δ is c-prime*

(iii) *Δ is c-prime $\Rightarrow$ Δ is 3-prime*

(iv) *Δ is 3-prime $\Rightarrow$ Δ is 2-prime*

(v) *Δ is 2-prime $\Rightarrow$ Δ is 0-prime if N is zero-symmetric*

(vi) *Δ is 2-prime $\Rightarrow$ Δ is 1-prime if N is a trio-RTNR*

(vii) *Δ is 1-prime $\Rightarrow$ Δ is 0-prime*

(viii) *Δ is 3-prime $\Rightarrow$ Δ is c-prime if Δ is an IFP ideal of ${}_N\Gamma$*

(ix) *Δ is 0-prime $\Rightarrow$ Δ is 1-prime if N is commutative.*

Proof. Let Δ be an N-ideal of ${}_N\Gamma$ such that $[N\ N\ \Gamma]_\Gamma \not\subseteq \Delta$

(i) Let $[a\ N\ b\ N\ \gamma]_\Gamma \subseteq \Delta$ where $a, b \in N$, $\gamma \in \Gamma$. Then if either $[a\ x\ \Gamma]_\Gamma \subseteq \Delta$ or $[b\ x\ \Gamma]_\Gamma \subseteq \Delta$ or $\gamma \in \Delta$ then there is nothing to prove. Suppose not. Then since Δ is e-prime, $[a\,u\,b\,v\,\gamma]_\Gamma - [a\,u\,b\,v\,0]_\Gamma \in \Delta \Rightarrow [b\ v\ \gamma]_\Gamma - [b\ v\ 0]_\Gamma \in \Delta \Rightarrow \gamma \in \Gamma$ which is not true. Hence Δ is 3-prime.

(ii) For $a, b \in N$ let $[a\ b\ \gamma]_\Gamma \in \Delta$. Then if either $[a\ x\ \Gamma]_\Gamma \subseteq \Delta$ or $[b\ x\ \Gamma]_\Gamma \subseteq \Delta$ or $\gamma \in \Delta$ then there is nothing to prove. Suppose not. Then as $[a\ b\ \gamma]_\Gamma - [a\ b\ 0]_\Gamma \in \Delta$ and Δ is e-prime, $\gamma \in \Delta$ a contradiction. Hence Δ is c-prime.

(iii) For $a, b \in N$ and $\gamma \in \Gamma$ consider $[a\ N\ b\ N\ \gamma]_\Gamma \subseteq \Delta$. Then $[a\ u\ b\ v\ \gamma]_\Gamma \in \Delta \forall\ u, v \in N$. Since Δ is c-prime, $[a[u\ b\ v]\gamma]_\Gamma \in \Delta$, either $[a\,x\,\Gamma]_\Gamma \subseteq \Delta$ or $[b\,x\,\Gamma]_\Gamma \subseteq \Delta$ or $\gamma \in \Delta\ \forall\ x \in N$ as $[N\,N\,\Gamma]_\Gamma \not\subseteq \Delta$. Thus Δ is 3-prime.

(iv) Let Δ be 3-prime in an RTNR N. Let A, B be N-subgroups of N. Let $[A\ B\ \Delta']_\Gamma \subseteq \Delta$. Now for every $a \in A, b \in B$ and $\delta' \in \Delta'$, $[a\ N\ b\ N\ \delta']_\Gamma = [a\ [N\ b\ N]\ \delta']_\Gamma \subseteq [A\ B\ \Delta']_\Gamma \subseteq \Delta$. Since Δ is 3-prime this implies that either $[a\ x\ \Gamma]_\Gamma \subseteq \Delta$ or $[b\ x\ \Gamma]_\Gamma \subseteq \Delta$ or $\delta' \in \Delta$ i.e., $[A\ x\ \Gamma]_\Gamma \subseteq \Delta$ or $[B\ x\ \Gamma]_\Gamma \subseteq \Delta$ or $\Delta' \subseteq \Delta$. Hence Δ is 2-prime.

(v) Let A and B be ideals of a zero-symmetric RTNR N and Δ' be an ideal of ${}_N\Gamma$ such that $[A\ B\ \Delta']_\Gamma \subseteq \Delta$. Then by hypothesis it follows that $[A\ x\ \Gamma]_\Gamma \subseteq \Delta$ or $[B\ x\ \Gamma]_\Gamma \subseteq \Delta$ or $\Delta' \subseteq \Delta\ \forall\ x \in N$. Hence Δ is 0-prime.

(vi) Let N be a trio-RTNR. Then N is zero-symmetric. Let A and B be left ideals of N and Δ' be an ideal of N such that $[A\ B\ \Delta']_\Gamma \subseteq \Delta$. Then $[A\ x\ \Gamma]_\Gamma \subseteq \Delta$ or $[B\ x\ \Gamma]_\Gamma \subseteq \Delta$ or $\Delta' \subseteq \Delta$ as Δ is 2-prime and N is zero-symmetric. Hence Δ is 1-prime.

(vii) Let A and B be ideals of an RTNR N such that $[A\ B\ \Delta']_\Gamma \subseteq \Delta$. Since A, B are left ideals and Δ is 1-prime, either $[A\ x\ \Gamma]_\Gamma \subseteq \Delta$ or $[b\ x\ \Gamma]_\Gamma \subseteq \Delta$ or $\Delta' \subseteq \Delta$. Hence Δ is 0-prime.

(viii) Let $a, b \in N$ and $\gamma \in \Gamma$ such that $[a\ b\ \gamma]_\Gamma \in \Delta \Rightarrow [a\ u\ b\ v\ \gamma]_\Gamma \in \Delta$ for every $u, v \in N$. Thus $[a\ N\ b\ N\ \gamma]_\Gamma \subseteq \Delta \Rightarrow [a\ x\ \Gamma]_\Gamma \subseteq \Delta$ or $[a\ x\ \Gamma]_\Gamma \subseteq \Delta$ or $\gamma \in \Gamma$. Hence Δ is c-prime.

(ix) Let A and B be left ideals of N and Δ' be an N-ideal of ${}_N\Gamma$ such that $[a\ b\ \Delta']_\Gamma \subseteq \Delta$. Since N is commutative every left ideal of N is an ideal of N and since Δ is 0-prime, either $[A\ x\ \Gamma]_\Gamma \subseteq \Delta$ or $[B\ x\ \Gamma]_\Gamma \subseteq \Delta$ or $\Delta' \subseteq \Delta$, $\forall\ \ x \in N$. Hence Δ is 1-prime. ∎

Proposition 4.6. *If N is an RTNR with unital element e and ${}_N\Gamma$ is monogenic then every 2-prime ideal is a 3-prime ideal.*

Proof. Let Δ be a 2-prime ideal of ${}_N\Gamma$ and $[a\ N\ b\ N\ \gamma]_\Gamma \subseteq \Delta\ \forall\ \gamma \in \Gamma$. Consider

$$\begin{aligned}[N\ e\ a\ N\ b\ c\ N\ e\ \gamma]_\Gamma &\subseteq [N\ N\ a\ N\ b\ N\ \gamma]_\Gamma \subseteq \Delta \\ &\Rightarrow [[N\ e\ a]\ x\ \Gamma]_\Gamma \subseteq \Delta \text{ or } [[N\ b\ e]\ x\ \Gamma]_\Gamma \subseteq \Delta \text{ or } [N\ e\ \gamma]_\Gamma \subseteq \Delta \\ &\Rightarrow [[e\ e\ a]\ x\ \Gamma]_\Gamma \subseteq \Delta \text{ or } [[e\ b\ e]\ x\ \Gamma]_\Gamma \subseteq \Delta \text{ or } [e\ e\ \gamma]_\Gamma \in \Delta \\ &\Rightarrow [a\ x\ \Gamma]_\Gamma \subseteq \Delta \text{ or } [b\ x\ \Gamma]_\Gamma \subseteq \Delta \text{ or } \gamma \in \Delta, \text{ by Theorem 2.7 (ii).}\end{aligned}$$

Hence P is a 3-prime ideal of N. ∎

Proposition 4.7. *Let N be an RTNR such that $[x\ y\ z] = [y\ x\ z]\ \forall\ x, y, z \in N$. Let Δ be an N-ideal of a right-lateral ternary N-group. Then $Q = (\Delta : \Gamma) = \{x \in N \mid [x\ n\ \gamma]_\Gamma \in \Delta$ and $[n\ x\ \gamma]_\Gamma \in \Delta,\ \forall\ \gamma \in \Gamma\}$ is an ideal of N.*

Proof. Let N be an RTNR such that $[x\ y\ z] = [y\ x\ z]\ \forall\ x, y, z \in N$. Then N is a right and lateral ternary near-ring and hence $[0\ n\ \gamma]_\Gamma = o = [n\ 0\ \gamma]_\Gamma$. Thus $0 \in Q$. As Δ is a subgroup of Γ, Q is a subgroup of N. Let $u \in N$ and $x \in Q$. Then $[u + x - u\ n\ \gamma]_\Gamma = [u\ n\ \gamma]_\Gamma + [x\ n\ \gamma]_\Gamma - [u\ n\ \gamma]_\Gamma$ and $[n\ u + x - u\ \gamma] = [n\ u\ \gamma]_\Gamma + [n\ x\ \gamma]_\Gamma - [n\ u\ \gamma]_\Gamma\ \ \forall\ \ \gamma \in \Gamma$ are in Q, showing that Q is normal in N. Let $H = [Q\ N\ N]$. Then $h = [x\ w\ v]$ where $x \in Q$ and $w, v \in N$. Now $[h\ n\ \gamma]_\Gamma = [x\ w\ v\ n\ \gamma]_\Gamma = [x\ w\ \gamma']_\Gamma \in \Delta$ and $[n\ h\ \gamma]_\Gamma = [n\ x\ w\ v\ \gamma]_\Gamma = [n\ x\ \gamma']_\Gamma \in \Delta$. Thus Q is a right ideal of N.

Since $[x\ y\ z] = [y\ x\ z]\ \forall\ x, y, z \in N$ and $[Q\ N\ N] \subseteq Q$ it follows that $[N\ Q\ N] \subseteq Q$. Thus Q is a lateral ideal of N. For $u, v, w \in N$ and $x \in Q$

$$\begin{aligned}&[[u\ v\ w + x] - [u\ v\ w]\ n\ \gamma]_\Gamma \\ &\quad = [u\ v\ [w + x\ n\ \gamma]_\Gamma]_\Gamma - [u\ v\ w\ n\ \gamma]_\Gamma \\ &\quad = [u\ v\ [w\ n\ \gamma]_\Gamma + [x\ n\ \gamma]_\Gamma]_\Gamma - [u\ v\ [w\ n\ \gamma]_\Gamma]_\Gamma \in \Delta,\end{aligned}$$

as Δ is an ideal of ${}_N\Gamma$. Also

$$\begin{aligned}[n\ ([u\ v\ w + x] - [u\ v\ w])\ \gamma]_\Gamma &= [[n\ u\ v]\ w + x\ \gamma]_\Gamma - [n\ u\ v\ w\ \gamma]_\Gamma \\ &= [[n\ u\ v]\ w\ \gamma]_\Gamma + [[n\ u\ v]\ x\ \gamma]_\Gamma - [[n\ u\ v]\ w\ \gamma]_\Gamma \in \Delta,\end{aligned}$$

as $x \in Q$ and Δ is normal in Γ. Thus Q is an ideal of N. ∎

In the following lemma, the interrelationship between Δ and the ideal quotient Q is given.

Lemma 4.8. *Let N be an RTNR such that $[x\ y\ z] = [y\ x\ z]\ \forall\ x, y, z \in N$. Let Δ be an N-ideal of a right-lateral ternary N-group. Then*

(i) *Δ is 2-prime $\Rightarrow$ Q is 2-prime*

(ii) *Δ is 3-prime $\Rightarrow$ Q is 3-prime*

(iii) *Δ is c-prime $\Rightarrow$ Q is c-prime*

(iv) *Δ is e-prime $\Rightarrow$ Q is e-prime*

(v) *Δ is 0-prime $\Rightarrow$ Q is 0-prime if ${}_N\Gamma$ is monogenic.*

Proof.

(i) Let A, B, C be N-subgroups of N such that $[A\ B\ C] \subseteq Q$. If either $A \subseteq Q$ or $B \subseteq Q$ or $C \subseteq Q$ then (i) follows. Suppose $[A\ x\ \Gamma]_\Gamma \not\subseteq Q$, $[B\ x\ \Gamma]_\Gamma \not\subseteq Q$ and $[C\ x\ \Gamma]_\Gamma \not\subseteq Q$. Since $[[A\ B\ C]\ x\ \Gamma]_\Gamma \subseteq \Delta$, $[A\ B\ [C\ x\ \gamma]\ \Gamma]_\Gamma \subseteq \Delta\ \forall\ \gamma \in \Gamma$ and $x \in N$. Also as Δ is 2-prime either $[A\ x\ \Gamma]_\Gamma \subseteq \Delta$, or $[B\ x\ \Gamma]_\Gamma \subseteq \Delta$ or $[C\ x\ \gamma]_\Gamma \subseteq \Delta\ \forall\ \gamma \in \Gamma$ which contradicts the assumption. Hence Q is 2-prime.

(ii) Let $a, b, c \in N$ be such that $[a\ N\ b\ N\ c] \subseteq Q$. If $a \in Q$ or $b \in Q$ or $c \in Q$ then (ii) follows. Suppose not. Also $[a\ N\ b\ N\ [c\ x\ \gamma]_\Gamma]_\Gamma \in \Delta \Rightarrow [a\ x\ \Gamma]_\Gamma \subseteq \Delta$ or $[b\ x\ \Gamma]_\Gamma \subseteq \Delta$ or $[c\ x\ \gamma]_\Gamma \in \Delta\ \ \forall\ \ x \in N, \gamma \in \Gamma.\ \Rightarrow a \in Q$ or $b \in Q$ or $c \in Q$, a contradiction to the assumption and hence Q is 3-prime.

(iii) Let $a, b, c \in N$ be such that $[a\ b\ c] \in Q$. Suppose $a \notin Q$, $b \notin Q$ and $c \notin Q$. Then for $x \in N$, $[a\ x\ \Gamma]_\Gamma \not\subseteq \Delta$, $[b\ x\ \Gamma]_\Gamma \not\subseteq \Delta$ and $[c\ x\ \Gamma]_\Gamma \not\subseteq \Delta$.

Now $[a\ b\ c] \in Q \Rightarrow [a\ b\ c\ x\ \Gamma]_\Gamma \subseteq \Delta \Rightarrow [a\ b\ c\ x\ \gamma]_\Gamma \in \Delta\ \forall\ \gamma \in \Gamma \Rightarrow [a\ x\ \Gamma]_\Gamma \subseteq \Delta$ or $[b\ x\ \Gamma]_\Gamma \subseteq \Delta$ or $[c\ x\ \gamma]_\Gamma \subseteq \Delta$ which is a contradiction to the assumption. Hence Q is c-prime.

(iv) Let for $a, x, y \in N$ with $a \notin Q$, $[a\ n\ x] - [a\ n\ y] \in Q\ \forall\ n \in N$.
$\Rightarrow [[a\ n\ x] - [a\ n\ y]\ t\ \gamma]_\Gamma \in \Delta,\ \forall\ t \in N \Rightarrow [a\ n\ [x\ t\ \gamma]_\Gamma]_\Gamma - [a\ n [y\ t\ \gamma]_\Gamma]_\Gamma \in \Delta$
$\Rightarrow [a\ n\ \gamma_1]_\Gamma - [a\ n\ \gamma_2]_\Gamma \in \Delta$ where $\gamma_1 = [x\ t\ \gamma]_\Gamma, \gamma_2 = [y\ t\ \gamma]_\Gamma$.
$\Rightarrow \gamma_1 - \gamma_2 \in \Delta$ as $[a\ x\ \Gamma]_\Gamma \not\subseteq \Delta \Rightarrow [x\ t\ \gamma]_\Gamma - [y\ t\ \gamma]_\Gamma \in \Delta$
$\Rightarrow [x - y\ t\ \gamma]_\Gamma \in \Delta \Rightarrow x - y \in Q$.
Hence Q is an e-prime ideal of N.

(v) Let A, B, C be ideals of N such that $[A\ B\ C] \subseteq Q$ if either $[A\ x\ \Gamma]_\Gamma \subseteq \Delta$ or $[B\ x\ \Gamma]_\Gamma \subseteq \Delta$ or $[C\ x\ \Gamma]_\Gamma \subseteq \Delta$ then there is nothing to prove. Suppose $A \not\subseteq Q$, $B \not\subseteq Q$ and $C \not\subseteq Q$. Since Γ is monogenic there exists $\gamma_0 \in \Gamma$ such that $[N\ x\ \gamma_0]_\Gamma = \Gamma$. Since $[C\ x\ \Gamma]_\Gamma \not\subseteq \Delta$, $[C\ x\ \gamma_0]_\Gamma \not\subseteq \Delta$. Since $[C\ x\ \gamma_0]_\Gamma$ is an ideal of Γ with $[C\ x\ \gamma_0]_\Gamma \not\subseteq \Delta$ and $[A\ x\ \Gamma]_\Gamma \not\subseteq \Delta$ and $[B\ x\ \Gamma]_\Gamma \not\subseteq \Delta$ and Δ is 0-prime it follows that $[A\ B\ [C\ x\ \gamma_0]_\Gamma]_\Gamma \not\subseteq \Delta$. This implies that $[A\ B\ C] \not\subseteq Q$ which contradicts the assumption. Thus Q is 0-prime. ∎

Proposition 4.9. *Let N be a zero-symmetric RTNR such that $[x\ y\ z] = [y\ x\ z]$ $\forall x, y, z \in N$ and let the right lateral ternary N-group ${}_N\Gamma$ be monogenic. Then an N-ideal Δ of ${}_N\Gamma$ is an IFP N-ideal of ${}_N\Gamma$.*

Proof. Let $a, x \in N$ and $\gamma \in \Gamma$ be such that $[a\ x\ \gamma]_\Gamma \in \Delta$. Since ${}_N\Gamma$ is monogenic there exists $\gamma_0 \in \Gamma$ and $u \in N$ such that $[N\ y\ \gamma_0]_\Gamma = \Gamma \Rightarrow \gamma = [n\ y\ \gamma_0]_\Gamma$ for some $n \in N$. Now $[a\ u\ x\ v\ \gamma]_\Gamma = [a\ u\ x\ v\ n\ y\ \gamma_0]_\Gamma = [u\ a\ v\ x\ n\ y\ \gamma_0]_\Gamma = [u\ a\ v\ x\ \gamma]_\Gamma = [u\ v\ [a\ x\ \gamma]_\Gamma]_\Gamma \in [N\ N\ \Delta]_\Gamma \subseteq \Delta$. Thus $[a\ x\ \gamma]_\Gamma \in \Delta \Rightarrow [a\ u\ x\ v\ \gamma]_\Gamma \in \Delta\ \forall u, v \in N$. Hence Δ is an IFP N-ideal of ${}_N\Gamma$. ∎

Theorem 4.10. *Let N be a zero-symmetric RTNR such that $[x\ y\ z] = [y\ x\ z]$ $\forall\ x, y, z \in N$. Let Δ be an N-ideal of right-lateral monogenic ternary N-group ${}_N\Gamma$. Then Δ is e-prime ideal iff Δ is c-prime ideal of ${}_N\Gamma$.*

Proof. If Δ is e-prime, by Theorem 4.5 (i) and (viii) and the above proposition Δ is c-prime.

Conversely let Δ be c-prime. Let $a \in N$ with $[a\ x\ \Gamma]_\Gamma \not\subseteq \Delta$ and $\gamma_1, \gamma_2 \in \Gamma$ be such that $[a\ n\ \gamma_1]_\Gamma - [a\ n\ \gamma_2]_\Gamma \in \Delta\ \forall\ n \in N$. Since Γ is monogenic there exists $\gamma_0 \in \Gamma$ such that $[N\ x\ \gamma_0]_\Gamma = \Gamma \Rightarrow u, v \in N$ such that $\gamma_1 = [u\ x\ \gamma_0]_\Gamma$ and $\gamma_2 = [v\ x\ \gamma_0]_\Gamma$. Consider, $[a\ n\ \gamma_1]_\Gamma - [a\ n\ \gamma_2]_\Gamma = [([n\ a\ u] - [n\ a\ v])\ x\ \gamma_0]_\Gamma \in \Delta \Rightarrow [([n\ a\ u] - [n\ a\ v])\ y\ \Gamma]_\Gamma \subseteq \Delta$ or $\gamma_0 \in \Delta\ \forall\ y \in N$ by the hypothesis.

If $\gamma_0 \in \Delta$ then $\Gamma = [N\ x\ \gamma_0]_\Gamma \subseteq [N\ N\ \Delta]_\Gamma \subseteq \Delta$ which is not true as $[N\ N\ \Gamma]_\Gamma \not\subseteq \Delta$. Thus $[[n\ a\ u] - [n\ a\ v]\ y\ \gamma]_\Gamma \subseteq \Delta \Rightarrow [n\ a\ u] - [n\ a\ v] \in (\Delta : \Gamma)$.

Also since $(\Delta : \Gamma) \neq N$ there exists $q \in N$ and $q \notin (\Delta : \Gamma)$. Consider $[[n\ a\ u] - [n\ a\ v]\ z\ q] \in (\Delta : \Gamma) \Rightarrow [[n\ u\ z] - [n\ v\ z]\ a\ q] \in (\Delta : \Gamma)$, by the given condition.

Since Δ is c-prime by Lemma 4.8 (iii) $(\Delta : \Gamma)$ is c-prime.

Hence $[n\ u\ z] - [n\ v\ z] \in (\Delta : \Gamma)$ or $a \in (\Delta : \Gamma)$ or $q \in (\Delta : \Gamma)$. Since $q \notin (\Delta : \Gamma)$ and $[a\ x\ \Gamma]_\Gamma \not\subseteq \Delta$, $[n\ u\ z] - [n\ v\ z] \in (\Delta : \Gamma)$. This implies that $[[n\ u\ z]\ t\ q] - [[n\ v\ z]\ t\ q] \in (\Delta : \Gamma)\ \forall\ t \in N \Rightarrow [u\ n\ z\ t\ q] - [v\ n\ z\ t\ q] \in (\Delta : \Gamma) \Rightarrow [u - v\ n\ z\ t\ q] \in (\Delta : \Gamma) \Rightarrow u - v \in (\Delta : \Gamma)$ or $[n\ z\ t] \in (\Delta : \Gamma)$ or $q \in (\Delta : \Gamma)$. If $[n\ z\ t] \subseteq (\Delta : \Gamma)$ then $N \subseteq (\Delta : \Gamma)$ which is not true. Also $q \notin (\Delta : \Gamma)$. Hence $u - v \in (\Delta : \Gamma) \Rightarrow [u - v\ x\ \gamma_0]_\Gamma \in \Delta \Rightarrow [u\ x\ \gamma_0]_\Gamma - [v\ x\ \gamma_0]_\Gamma \in \Delta \Rightarrow \gamma_1 - \gamma_2 \in \Delta$. Thus Δ is e-prime. Hence the proof. ∎

Proposition 4.11. *Let N be an RTNR such that $[x\ y\ z] = [y\ x\ z]\ \forall\ x, y, z \in N$. Let Δ be an IFP N-ideal of a right-lateral ternary N-group ${}_N\Gamma$. Then $(\Delta : \Gamma)$ is an IFP ideal of N.*

Proof. Let $[a\ b\ c] \in (\Delta : \Gamma)$ for $a, b, c \in N$.
$\Rightarrow [[a\ b\ c]\ x\ \gamma]_\Gamma \in \Delta\ \forall\ \gamma \in \Gamma,\ x \in N \Rightarrow [a\ u\ b\ v\ [c\ x\ \gamma]_\Gamma]_\Gamma \in \Delta\ \forall\ u, v \in N$
$\Rightarrow [[a\ u\ b\ v\ c]\ x\ \gamma]_\Gamma \in \Delta \Rightarrow [a\ u\ b\ v\ c] \in (\Delta : \Gamma)$.
Thus $(\Delta : \Gamma)$ is an IFP ideal of N. ∎

Lemma 4.12. *Let N be an RTNR such that $[x\ y\ z] = [y\ x\ z]\ \forall\ x, y, z \in N$. Let Δ be an N-ideal of right-lateral monogenic ternary N-group such that $Q = (\Delta : \Gamma)$ is 3-prime. Then Δ is 3-prime in ${}_N\Gamma$.*

Proof. For $a, b \in N$ and $\gamma \in \Gamma$ let $[a\ N\ b\ N\ \gamma]_\Gamma \subseteq Q$. Suppose Δ is not 3-prime. Then $a \notin Q$, $b \notin Q$ and $\gamma \notin \Delta$. Since ${}_N\Gamma$ is monogenic there exists $\gamma_0 \in \Gamma$, $x \in N$ such that $[N\ x\ \gamma_0]_\Gamma = \Gamma \Rightarrow \gamma = [n\ x\ \gamma_0]_\Gamma$ for some $n \in N$. Since $\gamma \notin \Delta$, $[n\ x\ \gamma_0]_\Gamma \notin \Delta \Rightarrow n \notin (\Delta : \Gamma)$. Also since $a \notin Q$, $b \notin Q$ and $n \notin Q$,

$$\begin{aligned}
[a\ N\ b\ N\ n] \not\subseteq Q &\Rightarrow [a\ N\ b\ N\ n] \not\subseteq [\Delta : [N\ x\ \gamma_0]_\Gamma] \\
&\Rightarrow [a\ N\ b\ N\ n\ t\ [u\ x\ \gamma_0]_\Gamma]_\Gamma \not\subseteq \Delta \text{ for all } u \in N \\
&\Rightarrow [a\ N\ b\ N[t\ u\ \gamma]_\Gamma]_\Gamma \not\subseteq \Delta, \text{ as } [x\ y\ z] = [y\ x\ z] \\
&\Rightarrow [a\ N\ b\ N\ \gamma]_\Gamma \not\subseteq Q,
\end{aligned}$$

which contradicts the hypothesis. Hence Δ is 3-prime in ${}_N\Gamma$. ∎

Corollary 4.13. *If N is an RTNR such that $[x\ y\ z] = [y\ x\ z]\ \forall\ x, y, z \in N$ and Δ is an ideal of right-lateral monogenic N-group such that $(\Delta : \Gamma)$ is c-prime then Δ is 3-prime in ${}_N\Gamma$.*

Proof. If Q is c-prime then by Lemma 3.19 Q is 3-prime and hence by the above lemma the corollary follows. ∎

5. Semiprime ideals in right ternary N-groups

In this section, ν-semiprime ideals of a right ternary N-group are defined where $\nu \in \{0, 1, 2, 3, c\}$ and their properties are discussed.

Definition 5.1. Let N be an RTNR and Δ be an N-ideal of a right ternary N-group ${}_N\Gamma$ such that $[N\ N\ \Gamma]_\Gamma \not\subseteq \Delta$. Then

(a) if $[A^3\ x\ \Gamma]_\Gamma \subseteq \Delta \Rightarrow [A\ x\ \Gamma]_\Gamma \subseteq \Delta\ \forall\ x \in N$ holds for all

  (i) ideals A of N then Δ is 0-semiprime.

  (ii) left ideals A of N then Δ is 1-semiprime.

  (iii) N-subgroups A of N then Δ is 2-semiprime.

(b) Δ is 3-semiprime if $[a\ N\ a\ N\ \gamma]_\Gamma \subseteq \Delta \Rightarrow [a\ x\ \gamma]_\Gamma \in \Delta\ \forall\ a, x \in N$ and $\gamma \in \Gamma$

(c) Δ is c-semiprime if $[a\ a\ [a\ a\ \gamma]_\Gamma]_\Gamma \in \Delta \Rightarrow [a\ x\ \gamma]_\Gamma \in \Delta\ \forall\ x, a \in N,\ \gamma \in \Gamma$.

Definition 5.2. A right ternary N-group ${}_N\Gamma$ is called ν-semiprime ($\nu = 0, 1, 2, 3, c$) if $[N\ N\ \Gamma]_\Gamma \neq \{o\}$ and $\{o\}$ is ν-semiprime ideal of ${}_N\Gamma$.

Lemma 5.3. *Let N be an RTNR and ${}_N\Gamma$ be a right ternary N-group. Then every 0-prime N-ideal of ${}_N\Gamma$ is 0-semiprime N-ideal of ${}_N\Gamma$.*

Proof. Let A be an ideal of N such that $[A^3\ x\ \Gamma]_\Gamma \subseteq \Delta$. Suppose $[A\ x\ \Gamma]_\Gamma \not\subseteq \Delta\ \forall\ x \in N$. Then there exists $\gamma \in \Gamma$ and $\gamma \notin \Delta$ such that $[A\ x\ \gamma]_\Gamma \not\subseteq \Delta$.

Since $[A\ x\ \gamma]_\Gamma$ is an N-ideal of ${}_N\Gamma$ and $[A\ A\ [A\ x\ \gamma]_\Gamma]_\Gamma \subseteq \Delta$ and Δ is 0-prime, $[A\ x\ \Gamma]_\Gamma \subseteq \Delta$ which contradicts the assumption. Hence ${}_N\Gamma$ is 0-semiprime N-ideal of ${}_N\Gamma$. ∎

Remark 5.4. Using a similar argument given in the proof of the above lemma it can be easily proved that every ν-prime N-ideal of a right ternary N-group is ν-semiprime N-ideal if $\nu = 1, 2$. The case for $\nu = 3$, c follows from the definition ν-prime and ν semiprime N-ideal of ${}_N\Gamma$.

Proposition 5.5. *Let Δ be an N-ideal of a ternary N-group ${}_N\Gamma$. Then*

(i) *Δ is c-semiprime $\Rightarrow$ Δ is 3-semiprime*

(ii) *Δ is 3-semiprime $\Rightarrow$ Δ is 2-semiprime*

(iii) *Δ is 2-semiprime $\Rightarrow$ Δ is 1-semiprime*

(iv) *Δ is 1-semiprime $\Rightarrow$ Δ is 0-semiprime if N is a zero-symmetric RTNR.*

Proof.

(i) Let Δ be a c-semiprime N-ideal. Let $a \in N$ and $\gamma \in \Gamma$ be such that $[a\ N\ a\ N\ \gamma]_\Gamma \subseteq \Delta$. Then $[a\ a\ [a\ a\ \gamma]_\Gamma]_\Gamma \in \Delta$. Since Δ is c-semiprime, $[a\ x\ \gamma]_\Gamma \in \Delta\ \forall\ x \in N$. Thus Δ is 3- semiprime.

(ii) Suppose Δ is 3-semiprime. Let A be an N-subgroup of N such that $[A^3\, x\, \Gamma] \subseteq \Delta$. Then $[a^3\ x\ \gamma]_\Gamma \in \Delta\ \forall\ a \in A, \gamma \in M$.

Consider $[a\ N\ a\ N\ \gamma]_\Gamma \subseteq [A\ N\ A\ N\ \Gamma]_\Gamma \subseteq [A\ A\ \Gamma]_\Gamma \subseteq \Delta$. Since Δ is 3-semiprime this implies that $[a\ x\ \gamma]_\Gamma \in \Delta\ \forall\ x \in N$. Hence $[A\ x\ \Gamma]_\Gamma \subseteq \Delta$. Thus Δ is 2-semiprime.

(iii) Suppose Δ is 2-semiprime. Let A be a left ideal of N such that $[A^3\ x\ \Gamma]_\Gamma \subseteq \Delta$. Suppose $[A\ x\ \Gamma]_\Gamma \not\subseteq \Delta\ \forall\ x \in N$. Then there exists $\gamma \in \Gamma$ such that $[A\ x\ \gamma]_\Gamma \not\subseteq \Delta$. Since A is left ideal of N then a is a left N-subgroup of N and hence $[A\ x\ \gamma]_\Gamma$ is an N-subgroup of ${}_\Gamma$. Also $[A\ A\ [A\ x\ \gamma]_\Gamma]_\Gamma \subseteq \Delta$ and as Δ is 2-semiprime $\Rightarrow [A\ x\ \Gamma]_\Gamma \subseteq \Delta$ which contradicts the assumption. Hence Δ is 1-semiprime.

(iv) Let Δ be 1-semiprime. Let A be an ideal of N such that $[A^3\ x\ \Gamma]_\Gamma \subseteq \Delta$ $\forall\ x \in N$. Suppose $[A\ x\ \Gamma]_\Gamma \not\subseteq\ \forall\ x \in N$. Then there exists $\gamma \in \Gamma$ such that $[A\ x\ \gamma]_\Gamma \not\subseteq \Delta$. Since $[A\ x\ \gamma]_\Gamma$ is a left ideal of ${}_N\Gamma$ and $[A\ A\ [A\ x\ \gamma]_\Gamma]_\Gamma \subseteq \Delta$ and Δ is 1-semiprime, $[A\ x\ \Gamma]_\Gamma \subseteq \Delta$ which contradicts the assumption. Hence Δ is 0-semiprime. ∎

Proposition 5.6. *If Δ is an IFP N-ideal of ${}_N\Gamma$ then Δ is 3-semiprime $\Rightarrow \Delta$ is c-semiprime.*

Proof. Let $a \in N$ and $\gamma \in \Gamma$ be such that $[a^3\ x\ \Gamma]_\Gamma \in \Delta \Rightarrow [a^3\ u\ x\ v\ \gamma]_\Gamma \in \Delta$ for every $u, v \in N$. Thus $[a^3\ N\ x\ N\ \gamma]_\Gamma \subseteq \Delta \Rightarrow [a\ [m\ a\ n\ a\ u]\ x\ v\ x\ \gamma]_\Gamma \subseteq \Delta$ or $[a\ N\ a\ N\ \gamma]_\Gamma \subseteq \Delta \Rightarrow [a\ x\ \Gamma]_\Gamma \subseteq \Delta \Rightarrow [a\ x\ \gamma]_\Gamma \in \Delta$. Hence Δ is c-semiprime. ∎

Lemma 5.7. *Let N be an RTNR such that $[x\ y\ z] = [y\ x\ z]\ \forall\ x, y, z \in N$. Let Δ be an N-ideal of a right ternary N-group of ${}_N\Gamma$. Let $Q = (\Delta : \Gamma)$. Then*

(i) *Δ is a 0-semiprime N-ideal of ${}_N\Gamma \Leftrightarrow Q$ is a 0-semiprime ideal of N*

(ii) *Δ is a 1-semiprime N-ideal of ${}_N\Gamma \Leftrightarrow Q$ is a 1-semiprime ideal of N*

(iii) *Δ is a 2-semiprime N-ideal of ${}_N\Gamma \Leftrightarrow Q$ is a 2-semiprime ideal of N*

(iv) *If Δ is a 3-semiprime N-ideal of ${}_N\Gamma$ then $Q = (\Delta : \Gamma)$ is a 3-semiprime ideal of N.*

(v) *If Δ is a c-semiprime N-ideal of ${}_N\Gamma$ then Q is a c-semiprime ideal of ${}_N\Gamma$.*

Proof. (i) Let A be an ideal of N such that $A^3 \subseteq Q \Rightarrow [A^3\ x\ \Gamma]_\Gamma \subseteq \Delta \Rightarrow [A\ A\ [A\ x\ \Gamma]_\Gamma]_\Gamma \subseteq \Delta$. Since Δ is 0-semiprime $[A\ x\ \Gamma]_\Gamma \subseteq \Delta \Rightarrow A \subseteq Q$. Conversely

let A be an ideal of N such that $[A^3\ x\ \Gamma]_\Gamma \subseteq \Delta\ \forall\ x \in N$, $A^3 \subseteq Q$. Since A is 0-semiprime ideal $A \subseteq Q$. This implies that $[A\ x\ \Gamma]_\Gamma \subseteq \Delta \Rightarrow \Delta$ is 0-semiprime.

(ii) and (iii) A similar argument as in (i) can be used to prove (ii) and (iii).

(iv) Let $a \in N$ be such that $[a\,N\,a\,N\,a] \subseteq Q$. Then $[a\,N\,a\,N\,a\,x\,\gamma]_\Gamma \subseteq \Delta$ $\forall\, x \in N$ and $\gamma \in \Gamma$. This implies that $[a\ N\ a\ N\ \gamma']_\Gamma \subseteq \Delta\ \forall\ \gamma' \in \Gamma$. Since Δ is 3-semiprime, $[a\ x\ \Gamma]_\Gamma \subseteq \Delta \Rightarrow a \in Q$. Hence Q is 3-semiprime ideal of N.

(v) Let $a \in N$ be such that $a^3 \in Q$. Then $[a^3\ x\ \gamma]_\Gamma \in \Delta\ \forall\ x \in N$ and $\gamma \in \Gamma$. Now by hypothesis $[a\ x\ \gamma]_\Gamma \in \Delta\ \forall\ \gamma \in \Gamma$. This implies that $[a\ x\ \Gamma]_\Gamma \subseteq \Delta \Rightarrow a \in Q$. Thus Q is a c-semiprime ideal of N. ∎

6. Conclusion

In this paper different prime idals in RTNR and right ternary N-groups were defined and their structural properties were investigated. Semiprime ideals were also defined in both the contexts. The interconnection between $(\Delta : \Gamma)$ and Δ were discussed. For future work radicals of an RTNR may be considered and their properties can be studied. Furthermore multiplication modules as in the binary case may be defined in the ternary context and the converse of the results involving $(\Delta : \Gamma)$ may be explored.

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$\alpha A_{\mathcal{I}}^*$-SETS, $\alpha \mathbf{C}_{\mathcal{I}}$-SETS, $\alpha C_{\mathcal{I}}^*$-SETS AND DECOMPOSITIONS OF α-$\mathcal{I}$-CONTINUITY

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Abstract. The aim of this paper is to introduce and study the notions of $\alpha A_{\mathcal{I}}^*$-sets, $\alpha C_{\mathcal{I}}$-sets and $\alpha C_{\mathcal{I}}^*$-sets in ideal topological spaces. Properties of $\alpha A_{\mathcal{I}}^*$-sets, $\alpha C_{\mathcal{I}}$-sets and $\alpha C_{\mathcal{I}}^*$-sets are investigated. Moreover, decompositions of α-$\mathcal{I}$-continuous functions and decompositions of $\alpha A_{\mathcal{I}}^*$-continuous functions via $\alpha A_{\mathcal{I}}^*$-sets, $\alpha C_{\mathcal{I}}$-sets and $\alpha C_{\mathcal{I}}^*$-sets in ideal topological spaces are established.

Keywords: $\alpha A_{\mathcal{I}}^*$-set, $\alpha C_{\mathcal{I}}$-set, $\alpha C_{\mathcal{I}}^*$-set, pre-$\mathcal{I}$-regular set, ideal topological space, decomposition, $\star$-extremally disconnected ideal space, $\star$-hyperconnected ideal space, $\mathcal{I}$-submaximal ideal space.

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1. Introduction and preliminaries

In this paper, $\alpha A^*_{\mathcal{I}}$-sets, $\alpha C_{\mathcal{I}}$-sets and $\alpha C^*_{\mathcal{I}}$-sets in ideal topological spaces are introduced and studied. The relationships and properties of $\alpha A^*_{\mathcal{I}}$-sets, $\alpha C_{\mathcal{I}}$-sets and $\alpha C^*_{\mathcal{I}}$-sets are investigated. Furthermore, decompositions of α-$\mathcal{I}$-continuous functions and decompositions of $\alpha A^*_{\mathcal{I}}$-continuous functions via $\alpha A^*_{\mathcal{I}}$-sets, $\alpha C_{\mathcal{I}}$-sets and $\alpha C^*_{\mathcal{I}}$-sets in ideal topological spaces are provided.

Throughout this paper (X, τ), (Y, σ) (or simply X, Y) denote topological spaces on which no separation axioms are assumed unless explicitly stated. For a subset A of a space X, the closure and interior of A with respect to τ are denoted by cl(A) and int(A) respectively.

An ideal $\mathcal{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies

1. A$\in\mathcal{I}$ and B$\subseteq$A$\Rightarrow$B$\in\mathcal{I}$ and
2. A$\in\mathcal{I}$ and B$\in\mathcal{I}\Rightarrow$A$\cup$B$\in\mathcal{I}$ [16].

If $\mathcal{I}$ is an ideal on X and X$\notin\mathcal{I}$, then $\mathcal{F} = \{X\backslash G : G\in\mathcal{I}\}$ is a filter [14]. Given a topological space (X, τ) with an ideal $\mathcal{I}$ on X and if $\wp$(X) is the set of all subsets of X, a set operator $(.)^* : \wp(X)\rightarrow\wp(X)$, called a local function [16] of A with respect to τ and $\mathcal{I}$ is defined as follows: for A$\subseteq$X, $A^*(\mathcal{I},\tau)=\{x\in X \mid U\cap A\notin\mathcal{I}$ for every $U\in\tau(x)\}$ where $\tau(x)=\{U\in\tau \mid x\in U\}$. A Kuratowski closure operator $cl^*(.)$ for a topology $\tau^*(\mathcal{I},\tau)$, called the $\star$-topology, finer than τ is defined by $cl^*(A)=A\cup A^*(\mathcal{I},\tau)$ [14]. When there is no chance for confusion, we will simply write A^* for $A^*(\mathcal{I},\tau)$ and τ^* for $\tau^*(\mathcal{I},\tau)$. $int^*(A)$ will denote the interior of A in $(X, \tau^*, \mathcal{I})$.

Remark 1.1 [14] The $\star$-topology is generated by τ and by the filter $\mathcal{F}$. Also the family $\{H \cap G : H \in\tau, G\in\mathcal{F}\}$ is a basis for this topology.

Lemma 1.2 [13] *Let A be a subset of an ideal topological space $(X, \tau, \mathcal{I})$. If N is open, then $N \cap cl^*(A) \subseteq cl^*(N \cap A)$.*

Definition 1.3 A subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be

1. pre-$\mathcal{I}$-open [4] if A $\subseteq$ int(cl*(A)).
2. semi-$\mathcal{I}$-open [11] if A $\subseteq$ cl*(int(A)).
3. α-$\mathcal{I}$-open [11] if A $\subseteq$ int(cl*(int(A))).
4. strongly β-$\mathcal{I}$-open [12] if A $\subseteq$ cl*(int(cl*(A))).
5. $\star$-dense [5] if cl*(A) = X.
6. t-$\mathcal{I}$-set [11] if int(A) = int(cl*(A)).
7. semi*-$\mathcal{I}$-open [8, 9] if A $\subseteq$ cl(int*(A)).

The family of all α-$\mathcal{I}$-open (resp. pre-$\mathcal{I}$-open) sets in an ideal topological space $(X, \tau, \mathcal{I})$ is denoted by $\alpha\mathcal{I}$O(X) (resp. P$\mathcal{I}$O(X)).

Remark 1.4 [8] For several subsets defined above, we have the following implications.

$$\begin{array}{ccccc} & & \text{pre-}\mathcal{I}\text{-open set} & \longrightarrow & \text{strongly } \beta\text{-}\mathcal{I}\text{-open set} \\ & & \uparrow & & \uparrow \\ \text{open set} & \longrightarrow & \alpha\text{-}\mathcal{I}\text{-open set} & \longrightarrow & \text{semi-}\mathcal{I}\text{-open set} \end{array}$$

The reverse implications are not true.

Lemma 1.5 [9] *Every semi-$\mathcal{I}$-open set is semi*-$\mathcal{I}$-open in an ideal topological space.*

Remark 1.6 The reverse implication of the above Lemma is not true in general as shown in [9].

Definition 1.7 The complement of a pre-$\mathcal{I}$-open (resp. semi-$\mathcal{I}$-open, α-$\mathcal{I}$-open, semi*-$\mathcal{I}$-open) set is called pre-$\mathcal{I}$-closed [4](resp. semi-$\mathcal{I}$-closed [11], α-$\mathcal{I}$-closed [11], semi*-$\mathcal{I}$-closed [8, 9]).

Definition 1.8 [9] The pre-$\mathcal{I}$-closure of a subset A of an ideal topological space $(X, \tau, \mathcal{I})$, denoted by $\mathrm{p}_{\mathcal{I}}\mathrm{cl}(\mathrm{A})$, is defined as the intersection of all pre-$\mathcal{I}$-closed sets of X containing A.

Lemma 1.9 [9] *For a subset A of an ideal topological space $(X, \tau, \mathcal{I})$, $p_{\mathcal{I}}cl(A) = A \cup cl(int^*(A))$.*

Definition 1.10 A function $\mathrm{f} : (X, \tau, \mathcal{I}) \to (\mathrm{Y}, \sigma)$ is said to be pre-$\mathcal{I}$-continuous [4] (resp. semi-$\mathcal{I}$-continuous [11], α-$\mathcal{I}$-continuous [11]) if $\mathrm{f}^{-1}(\mathrm{V})$ is pre-$\mathcal{I}$-open (resp. semi-$\mathcal{I}$-open, α-$\mathcal{I}$-open) in X for each open set V in Y.

Definition 1.11 A subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be

1. an $\eta\zeta$-set [17] if $\mathrm{A} = \mathrm{L} \cap \mathrm{M}$, where L is open and M is clopen in X.
2. locally closed [3] if $\mathrm{A} = \mathrm{L} \cap \mathrm{M}$, where L is open and M is closed in X.
3. $\alpha_{\mathcal{I}}\mathrm{N}_3$-set [2] if $\mathrm{A} = \mathrm{U} \cap \mathrm{V}$, where $\mathrm{U} \in \alpha\mathcal{I}\mathrm{O}(\mathrm{X})$ and $\mathrm{int}(\mathrm{cl}^*(\mathrm{V})) = \mathrm{int}(\mathrm{V})$.
4. semi-$\mathcal{I}$-regular [15] if A is a t-$\mathcal{I}$-set and semi-$\mathcal{I}$-open in X.
5. $\mathrm{B}_{\mathcal{I}}$-set [11] if $\mathrm{A} = \mathrm{U} \cap \mathrm{V}$, where U is open and V is a t-$\mathcal{I}$-set.

The family of all $\mathrm{B}_{\mathcal{I}}$-sets (resp. $\alpha_{\mathcal{I}}\mathrm{N}_3$-sets) of X is denoted by $\mathrm{B}_{\mathcal{I}}(\mathrm{X})$ (resp. $\alpha_{\mathcal{I}}\mathrm{N}_3(\mathrm{X})$).

Proposition 1.12 [1] *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If $V \in P\mathcal{I}O(X)$ and $A \in \alpha\mathcal{I}O(X)$, then $V \cap A \in P\mathcal{I}O(X)$.*

Definition 1.13 An ideal topological space $(X, \tau, \mathcal{I})$ is called

1. $\mathcal{I}$-submaximal if every $\star$-dense subset of X is open in X; [10]
2. $\star$-extremally disconnected if $\star$-closure of every open subset of X is open. [8]

Lemma 1.14 [8] *A subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is semi*-$\mathcal{I}$-open if and only if cl(A) = cl(int*(A)).*

Lemma 1.15 [7] *For a subset A of an ideal topological space, $p_{\mathcal{I}}\text{int}(A) = A \cap \text{int}(\text{cl}^*(A))$.*

Theorem 1.16 [2] *For an ideal topological space $(X, \tau, \mathcal{I})$, we have $\alpha\mathcal{I}O(X) = P\mathcal{I}O(X) \cap \alpha_{\mathcal{I}}N_3(X)$.*

Definition 1.17 [7] A subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is called pre-$\mathcal{I}$-regular if A is pre-$\mathcal{I}$-open and pre-$\mathcal{I}$-closed in $(X, \tau, \mathcal{I})$.

Definition 1.18 [6], [7] A subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is called $A^*_{\mathcal{I}}$-set if A = L ∩ M, where L is an open and M = cl(int*(M)).

Proposition 1.19 [11] *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. For a subset A of $(X, \tau, \mathcal{I})$, the following conditions are equivalent:*

1. *A is open.*
2. *A is pre-$\mathcal{I}$-open and a $B_{\mathcal{I}}$-set.*

Lemma 1.20 [1] *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. A subset A of X is α-$\mathcal{I}$-open if and only if it is semi-$\mathcal{I}$-open and pre-$\mathcal{I}$-open.*

Theorem 1.21 [10] *For an ideal topological space $(X, \tau, \mathcal{I})$, then the following properties are equivalent.*

1. *X is $\mathcal{I}$-submaximal.*
2. *Every pre-$\mathcal{I}$-open set is open.*
3. *Every pre-$\mathcal{I}$-open set is semi-$\mathcal{I}$-open and every α-$\mathcal{I}$-open set is open.*

Theorem 1.22 [7] *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $K \subseteq X$. The following properties are equivalent.*

1. *K is an open set.*
2. *K is an α-$\mathcal{I}$-open set and an $A^*_{\mathcal{I}}$-set.*
3. *K is a pre-$\mathcal{I}$-open set and an $A^*_{\mathcal{I}}$-set.*

2. $\alpha A_{\mathcal{I}}^*$-sets, $\alpha C_{\mathcal{I}}$-sets and $\alpha C_{\mathcal{I}}^*$-sets

Definition 2.1 Let $(X, \tau, \mathcal{I})$ be an ideal topological space and A $\subseteq$ X. A is said to be

1. an $\alpha C_{\mathcal{I}}^*$-set if A = L $\cap$ M, where L is an α-$\mathcal{I}$-open and M is a pre-$\mathcal{I}$-regular set in X.
2. an αA$B_{\mathcal{I}}$-set if A = L $\cap$ M, where L is α-$\mathcal{I}$-open and M is semi-$\mathcal{I}$-regular set in X.

Theorem 2.2 *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then each $\alpha C_{\mathcal{I}}^*$-set in X is a pre-$\mathcal{I}$-open set.*

Proof. Let A be an $\alpha C_{\mathcal{I}}^*$-set in X. It follows that A = L $\cap$ M, where L is an α-$\mathcal{I}$-open set and M is a pre-$\mathcal{I}$-regular set in X. Since M is a pre-$\mathcal{I}$-open set, then by Proposition 1.12, A = L $\cap$ M is a pre-$\mathcal{I}$-open set in X.

Remark 2.3 The converse of the Theorem 2.2 need not be true in general as shown in the following Example.

Example 2.4 Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, X\}$ and $\mathcal{I}$=$\{\emptyset, \{a\}\}$. Then A=$\{a, b\}$ is a pre-$\mathcal{I}$-open set but not $\alpha C_{\mathcal{I}}^*$-set.

Remark 2.5 In an ideal topological space, every α-$\mathcal{I}$-open set and every pre-$\mathcal{I}$-regular set is an $\alpha C_{\mathcal{I}}^*$-set. The converses are not true in general as shown in the following Examples.

Example 2.6 In Example 2.4, A = {a} is $\alpha C_{\mathcal{I}}^*$-set but not a pre-$\mathcal{I}$-regular set.

Example 2.7 Let X=$\{a, b, c\}$, τ=$\{\emptyset, \{a, b\}, X\}$ and $\mathcal{I}$=$\{\emptyset\}$. Then A=$\{a\}$ is $\alpha C_{\mathcal{I}}^*$-set but not an α-$\mathcal{I}$-open set.

Remark 2.8 By Remark 2.5 and Theorem 2.2, the following diagram holds for a subset A of an ideal topological space $(X, \tau, \mathcal{I})$.

$$\begin{array}{ccc} & & \text{pre-}\mathcal{I}\text{-open set} \\ & & \uparrow \\ \text{pre-}\mathcal{I}\text{-regular set} & \longrightarrow & \alpha\text{C}_{\mathcal{I}}^*\text{-set} \end{array}$$

Definition 2.9 A subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be

1. an $\alpha C_{\mathcal{I}}$-set if A = L $\cap$ M, where L is an α-$\mathcal{I}$-open set and M is a pre-$\mathcal{I}$-closed set in X.
2. an $\alpha\eta_{\mathcal{I}}$-set if A = L $\cap$ M, where L is an α-$\mathcal{I}$-open set and M is an α-$\mathcal{I}$-closed set in X.
3. an $\alpha A_{\mathcal{I}}^*$-set if A = L $\cap$ M, where L is an α-$\mathcal{I}$-open set and M = cl(int*(M)).

Remark 2.10 Let $(X, \tau, \mathcal{I})$ be an ideal topological space and A $\subseteq$ X. The following diagram holds for A.

$$\begin{array}{ccccc} & & \alpha C^*_{\mathcal{I}}\text{-set} & \longrightarrow & \alpha C_{\mathcal{I}}\text{-set} \\ & & & & \uparrow \\ A^*_{\mathcal{I}}\text{-set} & \longrightarrow & \alpha A^*_{\mathcal{I}}\text{-set} & \longrightarrow & \alpha\eta_{\mathcal{I}}\text{-set} \end{array}$$

The following Examples show that these implications are not reversible in general.

Example 2.11 Let X = {a, b, c, d}, τ = {$\emptyset$, {a}, {b}, {a, b}, {a, b, c}, X} and $\mathcal{I}$ = {$\emptyset$}. Then A = {a, b, d} is $\alpha A^*_{\mathcal{I}}$-set but not an $A^*_{\mathcal{I}}$-set.

Example 2.12 In Example 2.11, A = {c} is $\alpha\eta_{\mathcal{I}}$-set but not an $\alpha A^*_{\mathcal{I}}$-set.

Example 2.13 In Example 2.11, A = {c} is $\alpha C_{\mathcal{I}}$-set but not an $\alpha C^*_{\mathcal{I}}$-set.

Example 2.14 In Example 2.4, A = {c} is $\alpha C_{\mathcal{I}}$-set but not an $\alpha\eta_{\mathcal{I}}$-set.

Theorem 2.15 *For a subset A of an ideal topological space $(X, \tau, \mathcal{I})$, the following properties are equivalent.*

1. *A is an $\alpha C_{\mathcal{I}}$-set and a semi*-$\mathcal{I}$-open set in X.*
2. *A = L $\cap$ cl(int*(A)) for an α-$\mathcal{I}$-open set L.*

Proof. (1) $\Rightarrow$ (2): Suppose that A is an $\alpha C_{\mathcal{I}}$-set and a semi*-$\mathcal{I}$-open set in X. Since A is $\alpha C_{\mathcal{I}}$-set, then we have A = L $\cap$ M, where L is an α-$\mathcal{I}$-open set and M is a pre-$\mathcal{I}$-closed set in X. We have A $\subseteq$ M, so cl(int*(A)) $\subseteq$ cl(int*(M)). Since M is a pre-$\mathcal{I}$-closed set in X, we have cl(int*(M)) $\subseteq$ M. Since A is a semi*-$\mathcal{I}$-open set in X, We have A $\subseteq$ cl(int*(A)). It follows that A = A $\cap$ cl(int*(A))=L $\cap$ M $\cap$ cl(int*(A)) = L $\cap$ cl(int*(A)).

(2) $\Rightarrow$ (1): Let A = L $\cap$ cl(int*(A)) for an α-$\mathcal{I}$-open set L. We have A $\subseteq$ cl(int*(A)). It follows that A is a semi*-$\mathcal{I}$-open set in X. Since cl(int*(A)) is a closed set, then cl(int*(A)) is a pre-$\mathcal{I}$-closed set in X. Hence, A is an $\alpha C_{\mathcal{I}}$-set in X.

Theorem 2.16 *For a subset A of an ideal topological space $(X, \tau, \mathcal{I})$, the following properties are equivalent.*

1. *A is an $\alpha A^*_{\mathcal{I}}$-set in X.*
2. *A is an $\alpha\eta_{\mathcal{I}}$-set and a semi*-$\mathcal{I}$-open set in X.*
3. *A is an $\alpha C_{\mathcal{I}}$-set and a semi*-$\mathcal{I}$-open set in X.*

Proof. (1) ⇒ (2): Suppose that A is an $\alpha A_{\mathcal{I}}^*$-set in X. It follows that $A{=}L \cap M$, where L is an α-$\mathcal{I}$-open set and M = cl(int*(M)). This implies $A = L \cap M$ = cl[int(cl*(int(L))) ∩ int*(M)] ⊆ cl[cl*(int(L)) ∩ int*(M)] ⊆ cl[cl*(int(L) ∩ int*(M))] ⊆ cl[cl(int(L) ∩ int*(M))] ⊆ cl[int(L) ∩ int*(M)] ⊆ cl[int*(L) ∩ int*(M)] = cl(int*(L ∩ M)). Thus A ⊆ cl(int*(A)) and hence A is a semi*-$\mathcal{I}$-open set in X. Moreover, by Remark 2.10, A is an $\alpha\eta_{\mathcal{I}}$-set in X.

(2) ⇒ (3): It follows from the fact that every $\alpha\eta_{\mathcal{I}}$-set is an $\alpha C_{\mathcal{I}}$-set in X by Remark 2.10.

(3) ⇒ (1): Suppose that A is an $\alpha C_{\mathcal{I}}$-set and a semi*-$\mathcal{I}$-open set in X. By Theorem 2.15, A = L ∩ cl(int*(A)) for an α-$\mathcal{I}$-open set L. We have cl(int*(cl(int*(A)))) = cl(int*(A)). It follows that A is an $\alpha A_{\mathcal{I}}^*$-set in X.

Remark 2.17

1. The notions of $\alpha\eta_{\mathcal{I}}$-set and semi*-$\mathcal{I}$-open set are independent of each other.
2. The notions of $\alpha C_{\mathcal{I}}$-set and semi*-$\mathcal{I}$-open set are independent of each other.

Example 2.18

1. In Example 2.11, A = {c, d} is $\alpha C_{\mathcal{I}}$-set as well as $\alpha\eta_{\mathcal{I}}$-set but not semi*-$\mathcal{I}$-open set.
2. In Example 2.4, A = {a, c} is a semi*-$\mathcal{I}$-open set but it is neither $\alpha C_{\mathcal{I}}$-set nor $\alpha\eta_{\mathcal{I}}$-set.

Theorem 2.19 [7] *A subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is semi*-$\mathcal{I}$-closed if and only if A is a t-$\mathcal{I}$-set.*

Theorem 2.20 *Let $(X, \tau, \mathcal{I})$ be an $\mathcal{I}$-submaximal and $\star$-extremally disconnected ideal topological space. Then $B_{\mathcal{I}}(X) = \alpha_{\mathcal{I}} N_3(X)$.*

Proof. It follows from Lemma 1.14 and Theorem 1.21.

Theorem 2.21 *Let $(X, \tau, \mathcal{I})$ be an $\mathcal{I}$-submaximal and $\star$-extremally disconnected ideal topological space and $A \subseteq X$. The following properties are equivalent.*

1. *A is an open set in X.*
2. *A is an α-$\mathcal{I}$-open set and a $A_{\mathcal{I}}^*$-set.*
3. *A is a pre-$\mathcal{I}$-open and an $\alpha A_{\mathcal{I}}^*$-set.*

Proof. (1) ⇔ (2): It follows from Theorem 1.22.

(2) ⇒ (3): It follows from the fact that every α-$\mathcal{I}$-open set is pre-$\mathcal{I}$-open and every $A_{\mathcal{I}}^*$-set is $\alpha A_{\mathcal{I}}^*$-set.

(3) ⇒ (1): Suppose that A is a pre-$\mathcal{I}$-open set and an $\alpha A_{\mathcal{I}}^*$-set. Since A is an $\alpha A_{\mathcal{I}}^*$-set, then we have A = L ∩ M, where L is an α-$\mathcal{I}$-open set and M = cl(int*(M)). It follows that int(cl*(M)) ⊆ cl*(M) ⊆ cl(M) = cl(int*(M)) = M.

Since int(cl*(M)) $\subseteq$ M, then M is a semi*-$\mathcal{I}$-closed set. By Theorem 2.19, M is a t-$\mathcal{I}$-set. Hence, A is an $\alpha_{\mathcal{I}}$N$_3$-set. Since A is an $\alpha_{\mathcal{I}}$N$_3$-set and a pre-$\mathcal{I}$-open set, then by Theorem 2.20, A is a B$_{\mathcal{I}}$-set and a pre-$\mathcal{I}$-open set. By Proposition 1.19, A is an open set in X.

Theorem 2.22 *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subseteq X$. The following properties are equivalent.*

1. *A is an α-$\mathcal{I}$-open set in X.*
2. *A is a pre-$\mathcal{I}$-open and an $\alpha A^*_{\mathcal{I}}$-set.*

Proof. (1) $\Rightarrow$ (2): It follows from the fact that every α-$\mathcal{I}$-open set is pre-$\mathcal{I}$-open and every α-$\mathcal{I}$-open set is an $\alpha A^*_{\mathcal{I}}$-set.

(2) $\Rightarrow$ (1): Suppose that A is a pre-$\mathcal{I}$-open set and an $\alpha A^*_{\mathcal{I}}$-set. Since A is an $\alpha A^*_{\mathcal{I}}$-set, then we have A = L $\cap$ M, where L is an α-$\mathcal{I}$-open set and M = cl(int*(M)). It follows that int(cl*(M)) $\subseteq$ cl*(M) $\subseteq$ cl(M) = cl(int*(M)) = M. Since int(cl*(M)) $\subseteq$ M, then M is a semi*-$\mathcal{I}$-closed set. By Theorem 2.19, M is a t-$\mathcal{I}$-set. Hence, A is an $\alpha_{\mathcal{I}}$N$_3$-set. Since A is an $\alpha_{\mathcal{I}}$N$_3$-set and a pre-$\mathcal{I}$-open set, then by Theorem 1.16, A is an α-$\mathcal{I}$-open set in X.

Remark 2.23 The notions of pre-$\mathcal{I}$-open set and $\alpha A^*_{\mathcal{I}}$-set are independent of each other.

Example 2.24

1. In Example 2.4, A = {b, c} is $\alpha A^*_{\mathcal{I}}$-set but not a pre-$\mathcal{I}$-open set.
2. In Example 2.4, A = {a, c} is pre-$\mathcal{I}$-open set but not an $\alpha A^*_{\mathcal{I}}$-set.

Theorem 2.25 *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subseteq X$. The following properties are equivalent.*

1. *A is an α-$\mathcal{I}$-open set.*
2. *A is an $\alpha C^*_{\mathcal{I}}$-set and a semi*-$\mathcal{I}$-open set.*

Proof. (1) $\Rightarrow$ (2): It follows from the fact that every α-$\mathcal{I}$-open set is an $\alpha C^*_{\mathcal{I}}$-set and a semi*-$\mathcal{I}$-open set by Remark 1.4 and Lemma 1.5.

(2) $\Rightarrow$ (1): Let A be an $\alpha C^*_{\mathcal{I}}$-set and a semi*-$\mathcal{I}$-open set. Since A is an $\alpha C^*_{\mathcal{I}}$-set, then A is an $\alpha C_{\mathcal{I}}$-set. Since A is an $\alpha C_{\mathcal{I}}$-set and a semi*-$\mathcal{I}$-open set in X, then by Theorem 2.16, A is an $\alpha A^*_{\mathcal{I}}$-set. Moreover, since A is an $\alpha C^*_{\mathcal{I}}$-set, then A is a pre-$\mathcal{I}$-open by Theorem 2.2. Hence, by Theorem 2.22, A is an α-$\mathcal{I}$-open set in X.

Remark 2.26 The notions of $\alpha C^*_{\mathcal{I}}$-set and semi*-$\mathcal{I}$-open set are independent of each other.

Example 2.27

1. Let X=$\{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and $\mathcal{I}$=$\{\emptyset\}$. Then A=$\{a, c\}$ is semi*-$\mathcal{I}$-open set but not an $\alpha C^*_{\mathcal{I}}$-set.

2. In Example 2.7, A = {a, c} is $\alpha C^*_{\mathcal{I}}$-set but not semi*-$\mathcal{I}$-open set.

Theorem 2.28 *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subseteq X$. The following properties are equivalent.*

1. *A is an α-$\mathcal{I}$-open set.*

2. *A is a semi-$\mathcal{I}$-open set and an $\alpha C^*_{\mathcal{I}}$-set.*

3. *A is a semi-$\mathcal{I}$-open set and a pre-$\mathcal{I}$-open set.*

Proof. (1) $\Rightarrow$ (2): It is obvious.
(2) $\Rightarrow$ (3): It follows from the fact that every $\alpha C^*_{\mathcal{I}}$-set is a pre-$\mathcal{I}$-open set by Theorem 2.2.
(3) $\Rightarrow$ (1): It follows from Lemma 1.20.

Remark 2.29 The notions of semi-$\mathcal{I}$-open set and $\alpha C^*_{\mathcal{I}}$-set are independent of each other.

Example 2.30

1. In Example 2.27(1), A = {a, c} is semi-$\mathcal{I}$-open set but not an $\alpha C^*_{\mathcal{I}}$-set.

2. In Example 2.7, A = {a, c} is $\alpha C^*_{\mathcal{I}}$-set but not a semi-$\mathcal{I}$-open set.

Definition 2.31 A subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be $\alpha\text{gp}_{\mathcal{I}}$-open if N $\subseteq$ $\text{p}_{\mathcal{I}}$int(A) whenever N $\subseteq$ A and N is an α-$\mathcal{I}$-closed set in X.

Definition 2.32 A subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be α-generalized pre-$\mathcal{I}$-closed ($\alpha\text{gp}_{\mathcal{I}}$-closed) in X if X $\setminus$ A is $\alpha\text{gp}_{\mathcal{I}}$-open.

Theorem 2.33 *For a subset A of an ideal topological space $(X, \tau, \mathcal{I})$, A is $\alpha gp_{\mathcal{I}}$-closed if and only if $p_{\mathcal{I}}cl(A) \subseteq N$ whenever $A \subseteq N$ and N is an α-$\mathcal{I}$-open set in $(X, \tau, \mathcal{I})$.*

Proof. Let A be an $\alpha\text{gp}_{\mathcal{I}}$-closed set in X. Suppose that A $\subseteq$ N and N is an α-$\mathcal{I}$-open set in $(X, \tau, \mathcal{I})$. Then $X \setminus A$ is $\alpha\text{gp}_{\mathcal{I}}$-open and $X \setminus N \subseteq X \setminus A$ where $X \setminus N$ is α-$\mathcal{I}$-closed. Since $X \setminus A$ is $\alpha\text{gp}_{\mathcal{I}}$-open, then we have $X \setminus N \subseteq p_{\mathcal{I}}\text{int}(X \setminus A)$, where $p_{\mathcal{I}}\text{int}(X \setminus A) = (X \setminus A) \cap \text{int}(\text{cl}^*(X \setminus A))$. Since $(X \setminus A) \cap \text{int}(\text{cl}^*(X \setminus A))$ $= (X \setminus A) \cap (X \setminus \text{cl}(\text{int}^*(A))) = X \setminus (A \cup \text{cl}(\text{int}^*(A)))$, then by Lemma 1.9, $(X \setminus A) \cap \text{int}(\text{cl}^*(X \setminus A)) = X \setminus (A \cup \text{cl}(\text{int}^*(A))) = X \setminus p_{\mathcal{I}}\text{cl}(A)$. It follows that $p_{\mathcal{I}}\text{int}(X \setminus A) = X \setminus p_{\mathcal{I}}\text{cl}(A)$. Thus $p_{\mathcal{I}}\text{cl}(A) = X \setminus p_{\mathcal{I}}\text{int}(X \setminus A) \subseteq N$ and hence $p_{\mathcal{I}}\text{cl}(A) \subseteq N$. The converse is similar.

Theorem 2.34 *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $V \subseteq X$. Then V is an $\alpha C_{\mathcal{I}}$-set in X if and only if $V = G \cap p_{\mathcal{I}}cl(V)$ for an α-$\mathcal{I}$-open set G in X.*

Proof. If V is an $\alpha C_{\mathcal{I}}$-set, then V = G ∩ M for an α-$\mathcal{I}$-open set G and a pre-$\mathcal{I}$-closed set M. But then V ⊆ M and so V ⊆ $p_{\mathcal{I}}$cl(V)⊆ M. It follows that V = V ∩ $p_{\mathcal{I}}$cl(V) = G ∩ M ∩ $p_{\mathcal{I}}$cl(V) = G ∩ $p_{\mathcal{I}}$cl(V). Conversely, it is enough to prove that $p_{\mathcal{I}}$cl(V) is a pre-$\mathcal{I}$-closed set. But $p_{\mathcal{I}}$cl(V) ⊆ M, for any pre-$\mathcal{I}$-closed set M containing V. So, cl(int*($p_{\mathcal{I}}$cl(V))) ⊆ cl(int*(M)) ⊆ M. It follows that cl(int*($p_{\mathcal{I}}$cl(V))) ⊆ $\cap_{V \subseteq M,\ M \text{ is } pre-\mathcal{I}-closed}$ M = $p_{\mathcal{I}}$cl(V).

Theorem 2.35 *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subseteq X$. The following properties are equivalent.*

1. *A is a pre-$\mathcal{I}$-closed set in X.*
2. *A is an $\alpha C_{\mathcal{I}}$-set and an $\alpha gp_{\mathcal{I}}$-closed set in X.*

Proof. (1) ⇒ (2): It follows from the fact that any pre-$\mathcal{I}$-closed set in X is an $\alpha C_{\mathcal{I}}$-set and an $\alpha gp_{\mathcal{I}}$-closed set in X.

(2) ⇒ (1): Suppose that A is an $\alpha C_{\mathcal{I}}$-set and an $\alpha gp_{\mathcal{I}}$-closed set in X. Since A is an $\alpha C_{\mathcal{I}}$-set, then by Theorem 2.34, A = G ∩ $p_{\mathcal{I}}$cl(A) for an α-$\mathcal{I}$-open set G in $(X, \tau, \mathcal{I})$. Since A ⊆ G and A is $\alpha gp_{\mathcal{I}}$-closed set in X, then $p_{\mathcal{I}}$cl(A) ⊆ G. It follows that $p_{\mathcal{I}}$cl(A) ⊆ G ∩ $p_{\mathcal{I}}$cl(A) = A. Thus, A = $p_{\mathcal{I}}$cl(A) and hence A is pre-$\mathcal{I}$-closed.

Theorem 2.36 *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subseteq X$. If A is an $\alpha C_{\mathcal{I}}$-set in X, then $p_{\mathcal{I}}cl(A) \setminus A$ is a pre-$\mathcal{I}$-closed set and $A \cup (X \setminus p_{\mathcal{I}}cl(A))$ is a pre-$\mathcal{I}$-open set in X.*

Proof. Suppose that A is an $\alpha C_{\mathcal{I}}$-set in X. By Theorem 2.34, we have A = L ∩ $p_{\mathcal{I}}$cl(A) for an α-$\mathcal{I}$-open set L in X. It follows that $p_{\mathcal{I}}$cl(A) \ A = $p_{\mathcal{I}}$cl(A) \ (L ∩ $p_{\mathcal{I}}$cl(A)) = $p_{\mathcal{I}}$cl(A) ∩ (X \ (L ∩ $p_{\mathcal{I}}$cl(A))) = $p_{\mathcal{I}}$cl(A) ∩ ((X \ L) ∪ (X \ $p_{\mathcal{I}}$cl(A))) = ($p_{\mathcal{I}}$cl(A) ∩ (X \ L)) ∪ ($p_{\mathcal{I}}$cl(A) ∩ (X \ $p_{\mathcal{I}}$cl(A))) = ($p_{\mathcal{I}}$cl(A) ∩ (X \ L)) ∪ ϕ = $p_{\mathcal{I}}$cl(A) ∩ (X \ L). Thus $p_{\mathcal{I}}$cl(A) \ A = $p_{\mathcal{I}}$cl(A) ∩ (X \ L) and hence $p_{\mathcal{I}}$cl(A) \ A is pre-$\mathcal{I}$-closed set. Moreover, since $p_{\mathcal{I}}$cl(A) \ A is a pre-$\mathcal{I}$-closed set in X, then X \ ($p_{\mathcal{I}}$cl(A) \ A) = (X \ ($p_{\mathcal{I}}$cl(A) ∩ (X \ A)) = (X \ $p_{\mathcal{I}}$cl(A)) ∪ A is a pre-$\mathcal{I}$-open set. Thus, X \ ($p_{\mathcal{I}}$cl(A) \ A) = (X \ $p_{\mathcal{I}}$cl(A)) ∪ A is a pre-$\mathcal{I}$-open set in X.

3. Further properties

Definition 3.1 [7] Let $(X, \tau, \mathcal{I})$ be an ideal topological space. $(X, \tau, \mathcal{I})$ is said to be pre-$\mathcal{I}$-connected if X can not be expressed as the disjoint union of two nonvoid pre-$\mathcal{I}$-open sets.

Theorem 3.2 *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. The following properties are equivalent.*

1. $(X, \tau, \mathcal{I})$ *is pre-*$\mathcal{I}$*-connected.*

2. $(X, \tau, \mathcal{I})$ *can not be expressed as the disjoint union of two nonvoid* $\alpha C^*_{\mathcal{I}}$*-sets.*

Proof. (1) $\Rightarrow$ (2): Suppose that $(X, \tau, \mathcal{I})$ can be expressed as the disjoint union of two nonvoid $\alpha C^*_{\mathcal{I}}$-sets. Since any $\alpha C^*_{\mathcal{I}}$-set is a pre-$\mathcal{I}$-open set, then $(X, \tau, \mathcal{I})$ can be expressed as the disjoint union of two nonvoid pre-$\mathcal{I}$-open sets. So, $(X, \tau, \mathcal{I})$ is not pre-$\mathcal{I}$-connected. This is a contradiction.

(2) $\Rightarrow$ (1): Suppose that $(X, \tau, \mathcal{I})$ is not pre-$\mathcal{I}$-connected. Then, X can be expressed as the disjoint union of two nonvoid pre-$\mathcal{I}$-open sets. It follows that X has a nontrivial pre-$\mathcal{I}$-regular subset A. Moreover, A and B = X $\setminus$ A are pre-$\mathcal{I}$-regular. Then A and B are $\alpha C^*_{\mathcal{I}}$-sets. Hence $(X, \tau, \mathcal{I})$ can be expressed as the disjoint union of two nonvoid $\alpha C^*_{\mathcal{I}}$-sets. This is a contradiction.

Theorem 3.3 *In an* $\mathcal{I}$*-submaximal ideal space* $(X, \tau, \mathcal{I})$*, the following properties holds.*

1. *Any* $\alpha C^*_{\mathcal{I}}$*-set is an* $\eta\zeta$*-set and an* $\alpha AB_{\mathcal{I}}$*-set.*

2. *Any* $\alpha\eta_{\mathcal{I}}$*-set is a locally closed set.*

Proof. (1) Suppose that A is an $\alpha C^*_{\mathcal{I}}$-set in X. It follows that A = L $\cap$ M, where L is an α-$\mathcal{I}$-open set and M is a pre-$\mathcal{I}$-regular set in X. By Theorem 1.21, M is semi-$\mathcal{I}$-open and semi-$\mathcal{I}$-closed. It follows from Lemma 1.5 that M is semi-$\mathcal{I}$-open and semi*-$\mathcal{I}$-closed. By Theorem 2.19, M is semi-$\mathcal{I}$-open and a t-$\mathcal{I}$-set in X. Hence M is semi-$\mathcal{I}$-regular set. Thus, A is an αA$B_{\mathcal{I}}$-set in X. Furthermore, by Theorem 1.21, A is an $\eta\zeta$-set.

(2) It follows from Theorem 1.21.

Definition 3.4 [9] An ideal topological space $(X, \tau, \mathcal{I})$ is said to be $\star$-hyperconnected if A is $\star$-dense for every open subset A $\neq \phi$ of X.

Theorem 3.5 [9] *The following properties are equivalent for an ideal topological space* $(X, \tau, \mathcal{I})$ *.*

1. *X is* $\star$*-hyperconnected.*

2. *A is* $\star$*-dense for every strongly* β*-*$\mathcal{I}$*-open subset* $\phi \neq A \subseteq X$*.*

Theorem 3.6 *For an ideal topological space* $(X, \tau, \mathcal{I})$*, the following properties are equivalent.*

1. $(X, \tau, \mathcal{I})$ *is* $\star$*-hyperconnected.*

2. *any* $\alpha C^*_{\mathcal{I}}$*-set in X is* $\star$*-dense.*

Proof. (1) ⇒ (2): Let A be an $\alpha C_{\mathcal{I}}^*$-set in X. By Theorem 2.2, A is pre-$\mathcal{I}$-open. By Remark 1.4, A is strongly β-$\mathcal{I}$-open set. Since $(X, \tau, \mathcal{I})$ is a $\star$-hyperconnected ideal topological space, then by Theorem 3.5, A is $\star$-dense.

(2) ⇒ (1): Suppose that any $\alpha C_{\mathcal{I}}^*$-set in $(X, \tau, \mathcal{I})$ is $\star$-dense in X. Since an open set A in X is an α-$\mathcal{I}$-open set and every α-$\mathcal{I}$-open set A is an $\alpha C_{\mathcal{I}}^*$-set, then A is $\star$-dense. Thus, $(X, \tau, \mathcal{I})$ is $\star$-hyperconnected.

4. Decompositions of α-$\mathcal{I}$-continuity and $\alpha A_{\mathcal{I}}^*$-continuity

Definition 4.1 A function f : $(X, \tau, \mathcal{I}) \to$ (Y, σ) is said to be

1. $\alpha C_{\mathcal{I}}^*$-continuous if f^{-1}(A) is an $\alpha C_{\mathcal{I}}^*$-set in X for every open set A in Y.
2. P$R_{\mathcal{I}}$-continuous [7] if f^{-1}(A) is a pre-$\mathcal{I}$-regular set in X for every open set A in Y.

Remark 4.2 For a function f : $(X, \tau, \mathcal{I}) \to$ (Y, σ), the following diagram holds. The reverses of these implications are not true in general as shown in the following Examples.

$$\begin{array}{ccc} & & \text{pre-}\mathcal{I}\text{-continuity} \\ & & \uparrow \\ \alpha C_{\mathcal{I}}^*\text{-continuity} & \longleftarrow & \text{P}R_{\mathcal{I}}\text{-continuity} \end{array}$$

Example 4.3 Let X = {a, b, c}, τ = {$\emptyset$, {a}, X}, Y = {p, q, r}, σ = {$\emptyset$, {p}, {p, q}, {p, r}, Y}, $\mathcal{I}$ = {$\emptyset$} and $\mathcal{J}$ = {$\emptyset$}. Define f : $(X, \tau, \mathcal{I}) \to$ (Y, σ, $\mathcal{J}$) by f(a) = p; f(b) = q and f(c) = r. Then f is pre-$\mathcal{I}$-continuous but not $\alpha C_{\mathcal{I}}^*$-continuous.

Example 4.4 Let X = {a, b, c}, τ = {$\emptyset$, {a}, X}, Y = {p, q, r}, σ = {$\emptyset$, {p}, Y}, $\mathcal{I}$ = {$\emptyset$} and $\mathcal{J}$ = {$\emptyset$}. Define f : $(X, \tau, \mathcal{I}) \to$ (Y, σ, $\mathcal{J}$) by f(a) = p; f(b) = q and f(c) = r. Then f is $\alpha C_{\mathcal{I}}^*$-continuous but not P$R_{\mathcal{I}}$-continuous.

Definition 4.5 A function f : $(X, \tau, \mathcal{I}) \to$ (Y, σ) is said to be

1. $\alpha C_{\mathcal{I}}$-continuous if f^{-1}(A) is an $\alpha C_{\mathcal{I}}$-set in X for every open set A in Y.
2. $\alpha A_{\mathcal{I}}^*$-continuous if f^{-1}(A) is an $\alpha A_{\mathcal{I}}^*$-set in X for every open set A in Y.
3. $\alpha\eta_{\mathcal{I}}$-continuous if f^{-1}(A) is an $\alpha\eta_{\mathcal{I}}$-set in X for every open set A in Y.
4. $A_{\mathcal{I}}^*$-continuous [7] if f^{-1}(A) is an $A_{\mathcal{I}}^*$-set in X for every open set A in Y.

Remark 4.6 For a function f : $(X, \tau, \mathcal{I}) \to$ (Y, σ), the following diagram holds. The reverses of these implications are not true in general as shown in the following Examples.

$$\begin{array}{ccccc} & & \alpha C_{\mathcal{I}}\text{-continuity} & \longleftarrow & \alpha C_{\mathcal{I}}^*\text{-continuity} \\ & & \uparrow & & \\ \alpha\eta_{\mathcal{I}}\text{-continuity} & \longleftarrow & \alpha A_{\mathcal{I}}^*\text{-continuity} & \longleftarrow & A_{\mathcal{I}}^*\text{-continuity} \end{array}$$

Example 4.7 Let X=$\{a, b, c, d\}$, τ=$\{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$, Y=$\{p, q, r, s\}$, $\sigma = \{\emptyset, \{p\}, \{q\}, \{p, q, s\}, Y\}$, $\mathcal{I} = \{\emptyset\}$ and $\mathcal{J} = \{\emptyset\}$. Define $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ by $f(a) = p$, $f(b) = q$, $f(c) = r$ and $f(d) = s$. Then f is $\alpha A_{\mathcal{I}}^*$-continuous but not $A_{\mathcal{I}}^*$-continuous.

Example 4.8 Let X = {a, b, c, d}, τ = {$\emptyset$, {a}, {b}, {a, b}, {a, b, c}, X}, Y = {p, q, r, s}, σ = {$\emptyset$, {r}, {s}, {r, s}, Y}, $\mathcal{I}$ = {$\emptyset$} and $\mathcal{J}$ = {$\emptyset$}. Define f : $(X, \tau, \mathcal{I}) \to$ (Y, σ, $\mathcal{J}$) by f(a) = p, f(b) = q, f(c) = r and f(d) = s. Then f is $\alpha\eta_{\mathcal{I}}$-continuous but not $\alpha A_{\mathcal{I}}^*$-continuous.

Example 4.9 In Example 4.8, f is $\alpha C_{\mathcal{I}}$-continuous but not $\alpha C_{\mathcal{I}}^*$-continuous.

Example 4.10 Let X=$\{a, b, c\}$, τ=$\{\emptyset, \{a\}, X\}$, Y=$\{p, q, r\}$, σ=$\{\emptyset, \{q\}, Y\}$, $\mathcal{I}$=$\{\emptyset\}$ and $\mathcal{J}$=$\{\emptyset\}$. Define $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ by $f(a) = p$; $f(b) = q$ and $f(c) = r$. Then f is $\alpha C_{\mathcal{I}}$-continuous but not $\alpha\eta_{\mathcal{I}}$-continuous.

Definition 4.11 [7] A function f : $(X, \tau, \mathcal{I}) \to$ (Y, σ) is said to be semi*-$\mathcal{I}$-continuous if f^{-1}(V) is a semi*-$\mathcal{I}$-open set in X for every open set V in Y.

Theorem 4.12 *The following properties are equivalent for a function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$:*

1. *f is $\alpha A_{\mathcal{I}}^*$-continuous.*
2. *f is $\alpha\eta_{\mathcal{I}}$-continuous and semi*-$\mathcal{I}$-continuous.*
3. *f is $\alpha C_{\mathcal{I}}$-continuous and semi*-$\mathcal{I}$-continuous.*

Proof. *It follows from Theorem 2.16.*

Theorem 4.13 *The following properties are equivalent for a function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$:*

1. *f is α-$\mathcal{I}$-continuous.*
2. *f is pre-$\mathcal{I}$-continuous and $\alpha A_{\mathcal{I}}^*$-continuous.*
3. *f is semi*-$\mathcal{I}$-continuous and $\alpha C_{\mathcal{I}}^*$-continuous.*
4. *f is semi-$\mathcal{I}$-continuous and $\alpha C_{\mathcal{I}}^*$-continuous.*
5. *f is semi-$\mathcal{I}$-continuous and pre-$\mathcal{I}$-continuous.*

Proof. It follows from Theorems 2.22, 2.25 and 2.28.

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APPROXIMATE SOLUTION OF CONVECTION-DIFFUSION EQUATIONS USING A HAAR WAVELET METHOD

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Abstract. We present here a Haar wavelet method for numerical solution of convection-diffusion equations. Haar wavelet is a powerful mathematical tool used to solve various type of partial differential equations. The solutions obtained by Haar wavelet are more accurate and efficient.

Keywords: convection-diffusion equation, Haar wavelet method, function approximation.

AMS Subject Classification: 65M.

1. Introduction

It has been observed from the literature that many researchers are developing fast, accurate and efficient numerical schemes to handle the different problems arising in several areas of engineering and sciences. In the last few decades wavelet approaches are becoming more popular in the field of numerical approximations. Wavelet, being a powerful mathematical tool, has been widely used in image digital processing, quantum field theory and numerical analysis. Beginning from 1980s, wavelet has been used to find the numerical solution of partial differential equations. Wavelet methods attracts the interest of researchers to find the numerical solution of partial differential equations. Haar wavelet which are Daubechies wavelets of order 1 consists of piecewise constant functions. A drawback of Haar wavelet is their discontinuity. Since the derivatives do not exist in the breaking points, so it is impossible to apply the Haar wavelet for the solution of partial differential equation directly. Several definitions, concepts and modifications of Haar functions and various generalizations have been published and used. Galerkin and collocation techniques were applied to solve partial differential

equations using Haar wavelet. The Haar wavelet method for solving partial differential equations has many advantages features such as: it has very high accuracy, small computational costs, sprase representation, fast transformation, simplicity and possibility of implementation of fast algorithms especially if matrix representation is used. Also, it is very convenient for solving boundary value problems. Convection-diffusion equation is a special partial differential equation occur in numerous engineering problems. Convection-diffusion equation is a second order parabolic partial differential equation.

Consider the partial differential equation of the form

$$\frac{\partial u}{\partial t} + \varepsilon \frac{\partial u}{\partial x} = \gamma \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1, \ t > 0, \tag{1}$$

subject to the initial conditions $u(x,0) = f(x)$, $0 \leq x \leq 1$ and boundary conditions $u(0,t) = 0$, $u(1,t) = 0$, $t \geq 0$, where γ represents the viscosity coefficient and ε is the phase speed and both are assumed to be positive. The given function f is sufficiently smooth.

Wavelet analysis ia a new branch of mathematics and widely applied in differential equations. Several methods have been proposed to find the numerical solution of different linear and nonlinear partial differential equations. A survey of numerical solutions of differential equation is presented in Hariharan [7]. In 1910, Hungarian mathematicians Alfred Haar [6] introduced a function, known as Haar function which presents a rectangular pulse pair. There are many wavelet families such as Haar wavelet [6], Daubechies wavelet [5], Hermite-type trigonometric wavelet and many more. Among all these wavelet families, Haar is the simplest orthonormal wavelet with compact support. The pioneer work was done by Chen and Hsiao [3] in system analysis with Haar wavelet, who first derive a Haar operational matrix for the integral of the Haar function vector and also presented wavelet approach for optimising dynamic systems in [4]. Haar wavelet has many attractive applications in image coding, edge extraction and binary logic design. Hariharan et al. [10] presented the numerical solution of Fisher's equation using Haar wavelet. Also, Hariharan and Kannan [9] presented the numerical solution of Fitzhugh-Nagumo equation using Haar wavelet. Berwal et al. [1] presented the numerical solution of Telegraph equation. Celik [2] presented the numerical solution of generalized Burger-Huxley equation with Haar wavelet method. Lepik [12, 13] presented the numerical solution of differential and integral equation with Haar wavelet method. Kheiri and Ghafouri [11] presented Haar and Legendre wavelets collocation methods for the numerical solution of Schrondinger and wave equation. Haar wavelet method for solving some nonlinear parabolic equation is presented in [8].

In Section 2, we briefly describe the Haar wavelet method. In Section 3, we have given function approximation. Section 4, a method for solving convection-diffusion equation has been presented, and in Section 5, numerical examples have been solved using the Haar wavelet method to illustrate the efficiency and accuracy of the present method.

2. Haar wavelet method

The Haar functions are an orthogonal family of switched rectangular waveforms where amplitudes can differ from one function to another. They are defined in the interval $[0, 1]$.

$$h_i(x) = \begin{cases} 1, & \alpha \le x < \beta, \\ -1, & \beta \le x < \gamma, \\ 0, & \text{otherwise}, \end{cases} \tag{2}$$

where $\alpha = \frac{k}{m}$, $\beta = \frac{k+0.5}{m}$ and $\gamma = \frac{k+1}{m}$. Integer $m = 2^j$, $(j = 0, 1, 2, 3, 4, ..., J)$ indicates the level of the wavelet, and $k = 0, 1, 2, 3, ..., m-1$ is the translation parameter. Maximal level of resolution is J. The index i is calculated according the formula $i = m + k + 1$. In the case of minimal values, $m = 1$, $k = 0$ we have $i = 2$. The maximal value of i is $i = 2M$, where $M = 2^J$. It is assumed that the value $i = 1$, corresponding to the scaling function in $[0, 1]$

$$h_1(x) = \begin{cases} 1, & 0 \le x \le 1, \\ 0, & \text{otherwise}. \end{cases} \tag{3}$$

Let us define the collocation points $x_l = \frac{(l-0.5)}{2M}$, where $l = 1, 2, 3, ..., 2M$ and discretize the Haar function $h_i(x)$. Using the following four notations of Haar functions:

$$h_1(x) = [1, 1, 1, 1],\ h_2(x) = [1, 1, -1, -1],\ h_3 = [1, -1, 0, 0],\ h_4 = [0, 0, 1, -1],$$

we introduce the following notation:

$$H_4(x) = [h_1(x), h_2(x), h_3(x), h_4(x)]^T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \tag{4}$$

Here $H_4(x)$ is called Haar coefficient matrix. It is a square matrix of order 4, and is defined as $H(i, l) = (h_i(x_l))$, which has dimension $2M \times 2M$.

Let us integrate equation (2). We get

$$P_{1,i}(x) = \begin{cases} x - \alpha, & x\epsilon[\alpha, \beta), \\ \gamma - x, & x\epsilon[\beta, \gamma), \\ 0, & \text{elsewhere}. \end{cases} \tag{5}$$

$$P_{2,i}(x) = \begin{cases} \frac{1}{2}(x-\alpha)^2, & x\epsilon[\alpha, \beta), \\ \frac{1}{4m^2} - \frac{1}{2}(\gamma - x)^2, & x\epsilon[\beta, \gamma), \\ \frac{1}{4m^2}, & x\epsilon[\gamma, 1), \\ 0, & \text{elsewhere}. \end{cases} \tag{6}$$

3. Function approximation

Any square integrable function $y(x)$ in the interval $[0,1]$ can be expanded by a Haar series of infinite terms:

$$y(x) = \sum_{i=1}^{\infty} c_i h_i(x), \tag{7}$$

where the Haar coefficients c_i are determined as:

$$c_0 = \int_0^1 y(x) h_0(x) dx, \tag{8}$$

$$c_i = 2^j \int_0^1 y(x) h_i(x) dx, \tag{9}$$

where $i = 2^j + k$, $j \geq 0$ and $0 \leq k < 2^j$, $x \in [0,1]$ such that the following integral square error ε is minimized:

$$\varepsilon = \int_0^1 \left[y(x) - \sum_{i=1}^{m} c_i h_i(x) \right]^2 dx, \tag{10}$$

where $m = 2^j$ and $j = 0, 1, 2, 3, ...$ Usually, the series expansion of (7) contains infinite terms. If $y(x)$ is piecewise constant by itself or may be approximated as piecewise constant during each subinterval, then $y(x)$ will be terminated at finite m terms. This means

$$y(x) \cong \sum_{i=1}^{m} c_i h_i(x) = {c_m}^T h_m(x), \tag{11}$$

where the coefficients ${c_m}^T$ and the Haar function vectors $h_m(x)$ are defined as: ${c_m}^T = [c_1, c_2, ..., c_m]$ and $h_m(x) = [h_1(x), h_2(x), ..., h_m(x)]^T$,

4. Method for solving the convection-diffusion equation

Consider the convection-diffusion equation (1) with initial conditions $u(x,0) = f(x)$ and boundary conditions are $u(0,t) = g_0(t)$, $u(1,t) = g_1(t)$. Assume that $\dot{u}''(x,t)$ can be expanded in term of Haar wavelets as follows:

$$\dot{u}''(x,t) = \sum_{i=1}^{2M} a_i h_i(x), \quad t\epsilon(t_s, t_{s+1}]. \tag{12}$$

Integrating the above equation with respect to t from t_s to t, and twice with respect to x, from 0 to x, we get

$$u''(x,t) = (t-t_s)\sum_{i=1}^{2M} a_i h_i(x) + u''(x,t_s), \tag{13}$$

$$u'(x,t) = (t-t_s)\sum_{i=1}^{2M} a_i P_{1,i}(x) + u'(x,t_s) - u'(0,t_s) + u'(0,t), \tag{14}$$

$$u(x,t) = (t-t_s)\sum_{i=1}^{2M} a_i P_{2,i}(x) + u(x,t_s) - u(0,t_s) - x[u'(0,t_s) - u'(0,t)] + u(0,t), \tag{15}$$

$$\dot{u}(x,t) = \sum_{i=1}^{2M} a_i P_{2,i}(x) + x\dot{u}'(0,t) + \dot{u}(0,t). \tag{16}$$

From the initial and boundary conditions, we have the following equations as:

$$u(x,0) = f(x), \quad u(0,t) = g_0(t), \quad u(1,t) = g_1(t), \quad u(0,t_s) = g_0(t_s),$$
$$u(1,t_s) = g_1(t_s), \quad \dot{u}(0,t) = \dot{g}_0(t), \quad \dot{u}(1,t) = \dot{g}_1(t).$$

At $x = 1$ in the formulae (15) and (16) and by using conditions, we have

$$u'(0,t) - u'(0,t_s) = -(t-t_s)\sum_{i=1}^{2M} a_i P_{2,i}(1) + g_1(t) - g_1(t_s) + g_0(t_s) - g_0(t), \tag{17}$$

$$\dot{u}'(0,t) = -\sum_{i=1}^{2M} a_i P_{2,i}(1) - \dot{g}_0(t) + \dot{g}_1(t). \tag{18}$$

If the equations (17) and (18) are substituted into equations $(13) - (16)$, and the results are discretized by assuming $x \to x_l$ and $t \to t_{s+1}$, we obtain

$$u''(x_l,t_{s+1}) = (t_{s+1}-t_s)\sum_{i=1}^{2M} a_i h_i(x_l) + u''(x_l,t_s), \tag{19}$$

$$u'(x_l,t_{s+1}) = (t_{s+1}-t_s)\sum_{i=1}^{2M} a_i P_{1,i}(x_l) + g_1(t_{s+1}) - g_1(t_s) + g_0(t_s) - g_0(t_{s+1}) - (t_{s+1}-t_s)\sum_{i=1}^{2M} a_i P_{2,i}(1) + u'(x_l,t_s), \tag{20}$$

$$u(x_l,t_{s+1}) = (t_{s+1}-t_s)\sum_{i=1}^{2M} a_i P_{2,i}(x_l) + u(x_l,t_s) + g_0(t_{s+1}) - g_0(t_s) + x_l\Big[-(t_{s+1}-t_s)\sum_{i=1}^{2M} a_i P_{2,i}(1) + g_1(t_{s+1}) - g_0(t_{s+1}) - g_1(t_s) + g_0(t_s)\Big], \tag{21}$$

$$\begin{aligned}\dot{u}(x_l,t_{s+1}) &= \sum_{i=1}^{2M} a_i P_{2,i}(x_l) \\ &\quad + x_l\Big[-\sum_{i=1}^{2M} a_i P_{2,i}(1) - \dot{g}_0(t_{s+1}) + \dot{g}_1(t_{s+1})\Big] + \dot{g}_0(t_{s+1}).\end{aligned} \tag{22}$$

But, we know that

$$P_{2,i}(1) = \begin{cases} \dfrac{1}{2}, & i = 1, \\ \dfrac{1}{4m^2}, & i > 1. \end{cases} \tag{23}$$

In the given scheme

$$\dot{u}(x_l,t_{s+1}) + \varepsilon u'(x_l,t_{s+1}) = \gamma u''(x_l,t_{s+1}), \tag{24}$$

which leads us from the time layer t_s to t_{s+1} is used. From here, wavelet coefficients are calculated and solution of wave equation is obtained.

5. Numerical examples

We present here, some numerical examples of Convection-diffusion equation, which shows the accuracy and efficiency of Haar wavelet method.

Example 1: Consider the convection-diffusion equation of the form

$$\frac{\partial u}{\partial t} + 0.1\frac{\partial u}{\partial x} = 0.01\frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 1, \quad t > 0, \tag{25}$$

subject to the initial conditions $u(x,0) = e^{5x} sin\pi x, \quad 0 \le x \le 1$ and boundary conditions $u(0,t) = 0$, $u(1,t) = 0$, $t \ge 0$. The exact solution of this problem is:

$$u(x,t) = e^{5x-(0.25-0.01\pi^2)t} sin\pi x. \tag{26}$$

The process is started with $u(x_l,0) = f(x_l)$, $u'(x_l,0) = f'(x_l)$, $u''(x_l,0) = f''(x_l)$. Numerical results are presented graphically in Figure 1 for $t = 0.1, 0.2, 0.3$ and 5.0. The value of $\Delta t = 0.00001$. Absolute errors for $t = 0.1$, $t = 0.2$ and $t = 0.3$ are presented in Table 1.

xL/64	t= 0.1	t=0.2	t=0.3
1	1.0E-006	3.2E-006	5.9E-006
3	2.2E-006	7.7E-006	1.5E-005
5	2.7E-006	1.0E-005	2.1E-005
7	3.0E-006	1.1E-005	2.4E-005
9	3.3E-006	1.2E-005	2.7E-005
11	3.4E-006	1.3E-005	2.9E-005
13	3.6E-006	1.3E-005	3.0E-005
15	3.6E-006	1.3E-005	3.0E-005
17	3.5E-006	1.3E-005	2.9E-005
29	3.0E-006	1.2E-005	2.8E-005
31	5.8E-006	2.4E-005	5.3E-005
33	9.5E-006	3.8E-005	8.6E-005
59	2.1E-004	8.1E-004	1.7E-003
61	2.3E-004	8.1E-004	1.6E-003
63	1.4E-004	4.4E-004	8.5E-004

Table 1: Comparison of maximum absolute errors for Example 1.

Example 2: Consider the convection-diffusion equation of the form

$$\frac{\partial u}{\partial t} + 0.22\frac{\partial u}{\partial x} = 0.5\frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 1, \quad t > 0, \tag{27}$$

subject to the initial conditions $u(x,0) = e^{0.22x} sin\pi x, \quad 0 \leq x \leq 1$ and boundary conditions $u(0,t) = 0$, $u(1,t) = 0$, $t \geq 0$. The exact solution of this problem is:

$$u(x,t) = e^{0.22x-(0.0242-0.5\pi^2)t} sin\pi x. \tag{28}$$

The process is started with $u(x_l,0) = f(x_l)$, $u'(x_l,0) = f'(x_l)$, $u''(x_l,0) = f''(x_l)$. Numerical results are presented graphically in Figure 2 for $t = 0.1, 0.2, 0.3$. The value of $\Delta t = 0.00001$. Absolute errors for $t = 0.1$, $t = 0.2$ and $t = 0.3$ are presented in Table 2.

Example 3: Consider the convection-diffusion equation of the form

$$\frac{\partial u}{\partial t} + 0.1\frac{\partial u}{\partial x} = 0.2\frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 1, \quad t > 0, \tag{29}$$

subject to the initial conditions $u(x,0) = e^{0.25x} sin\pi x, \quad 0 \leq x \leq 1$ and boundary conditions $u(0,t) = 0$, $u(1,t) = 0$, $t \geq 0$. The exact solution of this problem is:

$$u(x,t) = e^{0.25x-(0.0125-0.2\pi^2)t} sin\pi x. \tag{30}$$

The process is started with $u(x_l,0) = f(x_l)$, $u'(x_l,0) = f'(x_l)$, $u''(x_l,0) = f''(x_l)$. Numerical results are presented graphically in Figure 3 for $t = 0.1, 0.2, 0.3$. The value of $\Delta t = 0.00001$. Absolute errors for $t = 0.1$, $t = 0.2$ and $t = 0.3$ are presented in Table 3.

xL/64	t= 0.1	t=0.2	t=0.3
1	7.0E-007	2.6E-006	5.0E-006
3	2.3E-006	8.4E-006	1.5E-005
5	4.5E-006	1.5E-005	2.7E-005
7	7.2E-006	2.2E-005	4.0E-005
9	1.0E-005	3.0E-005	5.3E-005
11	1.3E-005	3.9E-005	6.7E-005
13	1.6E-005	4.8E-005	8.2E-005
15	2.0E-005	5.7E-005	9.6E-005
17	2.3E-005	6.6E-005	1.1E-004
29	4.2E-005	1.1E-004	1.8E-004
31	4.4E-005	1.1E-004	1.8E-004
33	4.6E-005	1.2E-004	1.9E-004
59	1.8E-005	4.5E-005	6.9E-005
61	1.1E-005	2.8E-005	4.3E-005
63	4.1E-006	9.7E-006	1.4E-005

Table 2: Comparison of maximum absolute errors for Example 2.

xL/64	t= 0.1	t=0.2	t=0.3
1	6.0E-008	9.4E-008	5.2E-007
3	5.0E-008	5.4E-007	1.9E-006
5	1.9E-007	1.4E-006	4.0E-006
7	6.1E-007	2.7E-006	6.7E-006
9	1.1E-006	4.4E-006	9.9E-006
11	1.8E-006	6.3E-006	1.3E-005
13	2.5E-006	8.4E-006	1.7E-005
15	3.3E-006	1.0E-005	2.1E-005
17	4.1E-006	1.3E-005	2.5E-005
29	8.3E-006	2.6E-005	4.9E-005
31	8.8E-006	2.7E-005	5.2E-005
33	9.2E-006	2.9E-005	5.5E-005
59	4.6E-006	1.3E-005	2.4E-005
61	3.0E-006	8.6E-006	1.5E-005
63	1.0E-006	3.0E-006	5.3E-006

Table 3: Comparison of maximum absolute errors for Example 3.

Conclusion

It is conclude that Haar wavelet method is more accurate, simple, fast and computationally attractive than other known methods to solve convection-diffusion equation. The above examples demonstrates the simplicity of the Haar wavelet solution. For getting the necessary accuracy the number of collocation points and the value of M in Equation (12) may be increased.

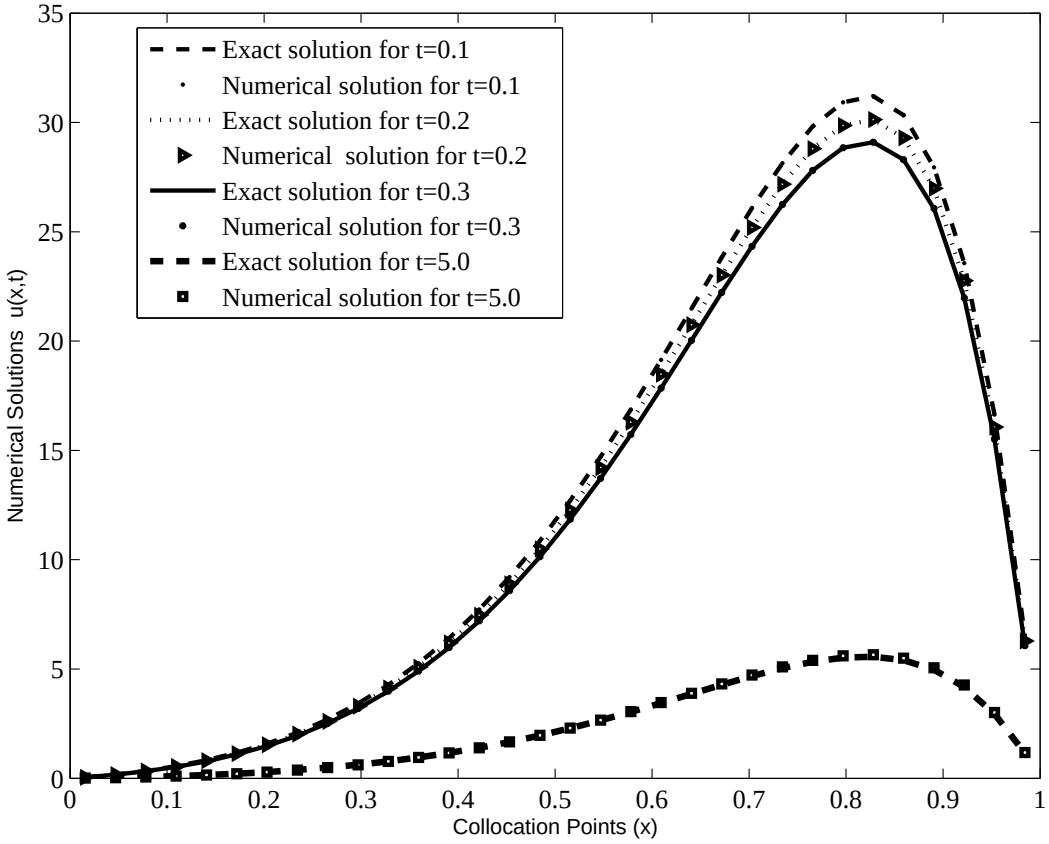


Figure 1: Comparison of numerical and exact solution for Example 1 at $t = 0.1, 0.2, 0.3, 5.0$.

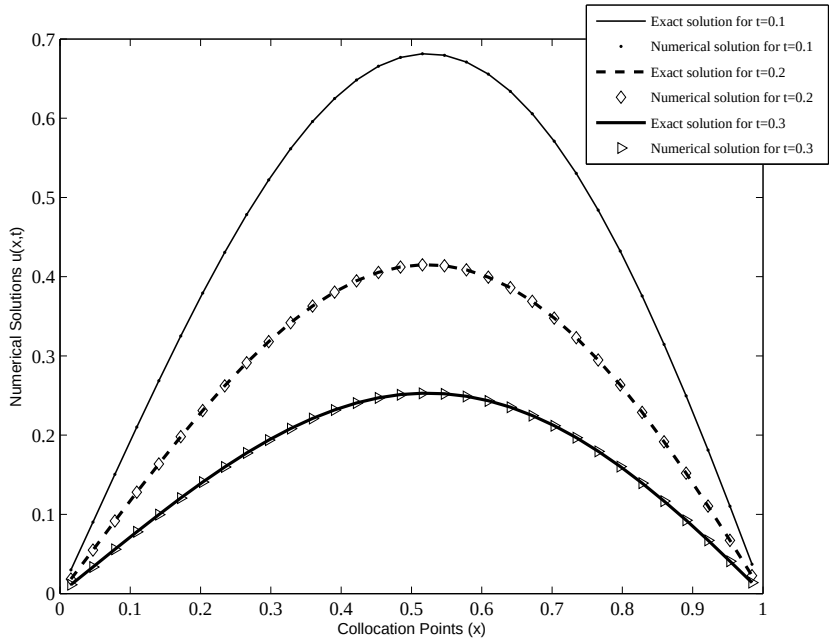


Figure 2: Comparison of numerical and exact solution for Example 2 at $t = 0.1, 0.2, 0.3$.

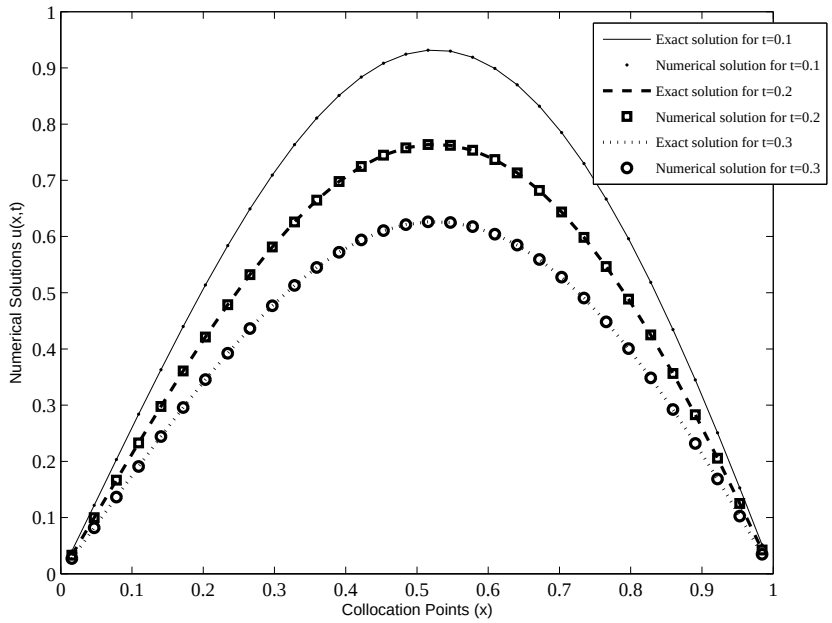


Figure 3: Comparison of numerical and exact solution for Example 3 at $t = 0.1, 0.2, 0.3$.

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MINIMAL INTUITIONISTIC GENERAL L-FUZZY AUTOMATA

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Abstract. In this paper we present an intuitionistic general L-fuzzy automaton (IGLFA) based on lattice valued intuitionistic fuzzy sets [2]. In this note, we define (α, β)-language, (α, β)-complete, (α, β)-accessible, (α, β)-reduced for an IGLFA over a bounded complete lattice L, where $\alpha, \beta \in L$ and $\alpha \leq_L N(\beta)$. In particular, we prove a theorem which is generalization of Myhill-Nerode theorem in ordinary deterministic automata. In other words for any recognizable (α, β)-language over a bounded complete lattice L, there exist minimal (α, β)-complete and deterministic IGLFA, which preserve (α, β)-language, where $\alpha, \beta \in L$ and $\alpha \leq_L N(\beta)$. Also, we show that for any given (α, β)-language $\mathcal{L}$, the minimal (α, β)-complete and deterministic IGLFA recognizing $\mathcal{L}$ is isomorphic with threshold (α, β) to any (α, β)-complete, (α, β)-accessible, deterministic, (α, β)-reduced IGLFA recognizing $\mathcal{L}$. Moreover, we give some examples to clarify these notions. Finally, by using these notions, we give some theorems and obtain some results.

Keywords: Max-min intuitionistic general L-fuzzy automata; (α, β)-language; (α, β)-reduced; Minimal intuitionistic general L-fuzzy automata; (α, β)-isomorphic.

1. Introduction

The theory of fuzzy sets was introduced by L.A. Zadeh in 1965 [40]. W.G. Wee [35] introduced the idea of fuzzy automata. E.T. Lee and L.A. Zadeh [21] in 1969 gave the concept of fuzzy finite state automata. Thereafter, there were a considerable number of authors, such as Mordeson and Malik [22], [23], Topencharov and Peeva [33] and others having contributed to this field. Fuzzy finite automata have many important applications such as in learning system, pattern recognition, neural networks and data base theory [14], [15], [22], [25], [26], [29], [36]. Atanassove [1] has extended the notion of fuzzy sets to the intuitionistic fuzzy sets (IFS) by

adding non-membership value, which may express more accurate and flexible information as compared with fuzzy sets. Intuitionistic fuzzy set theory has many applications in several subject, see [5], [6], [9], [11], [16], [18], [19], [34], [7]. Using the notion of intuitionistic fuzzy sets, W.L. Jun [17] introduced the notion of intuitionistic fuzzy finite state machines as a generalization of fuzzy finite state machines. Based on the papers [17], [18], Zhang and Li [41] discussed intuitionistic fuzzy recognizers and intuitionistic fuzzy finite automata. K. Atanassov and S. Stoeva generalized the concept of IFS to intuitionistic L-fuzzy sets [2] where L is an appropriate lattice. A. Tepavcevic and T. Gerstenkorn gave a new definition of lattice valued intuitionistic fuzzy sets in [32]. Thus, on the basis of lattice valued intuitionistic fuzzy sets, Yang et al. [38] introduced the concepts of lattice-valued intuitionistic fuzzy finite state machines. In 2004, M. Doostfatemeh and S.C. Kremer [10] extended the notion of fuzzy automata and gave the notion of general fuzzy automata. In 2014, M. Shamsizadeh and M.M. Zahedi [31] gave the notion of max-min intuitionistic general fuzzy automata. We will use bounded complete lattices as the structures of truth values. Note that usually the real unit interval $[0, 1]$ is used in the literature (the reader not familiar with lattices may, without any harm, substitute $[0, 1]$ for bounded complete lattices throughout the paper). Our paper deals with intuitionistic general fuzzy automata over a bounded complete lattice L, where endowed with a t-norm T, a t-conorm S, the least element 0 and the greatest element 1, denoted by $L = (L, \leq_L, T, S, 0, 1)$. State minimization is a fundamental problem in automata theory. There are many papers on the minimization problem of fuzzy finite automata. For example, minimization of mealy type of fuzzy finite automata in discussed in [4], minimization of fuzzy finite automata with crisp final states without outputs in studied in [3], minimizing the deterministic finite automaton with fuzzy (final) states in [24] and minimization of fuzzy machines becomes the subject of [28], [27], [30], [33]. Myhill-Nerode's theorem has been extended to fuzzy regular language and also an algorithm is given for minimizing the deterministic finite automaton with fuzzy (final) states in [20], [24]. It is important to find the minimal intuitionistic general L-fuzzy automata that recognizes the same language as a given language. In this note, for a given complete lattice $L = (L, \leq_L, T, S, 0, 1)$, we define an (α, β)-language, where $\alpha, \beta \in L, \alpha <_L N(\beta)$. Furthermore, we show that for any max-min IGLFA $\tilde{F}^*$, there exist (α, β)-complete, (α, β)-accessible and deterministic max-min IGLFA recognizing $\mathcal{L}(\tilde{F}^*)$, where $\alpha, \beta \in L, \alpha <_L N(\beta)$. Also, we prove a theorem which is generalization of Myhill-Nerode theorem in ordinary deterministic automata. In other words, we have shown that for any (α, β)-language, there exist minimal (α, β)-complete and deterministic intuitionistic general L-fuzzy automata (IGLFA). Also, we define an (α, β)-reduced IGLFA. We show that for any given (α, β)-language $\mathcal{L}$, the minimal (α, β)-complete and deterministic IGLFA recognizing $\mathcal{L}$ is isomorphic with threshold (α, β) to any (α, β)-complete, (α, β)-accessible, deterministic, (α, β)-reduced IGLFA recognizing $\mathcal{L}$. Moreover we give some new notions and results as mentioned in the abstract and some examples to clarify these new notions.

2. Preliminaries

In this section we give some definitions that we need in the sequel. Assume that E is an universal set. A fuzzy set A on E is characterized by the same symbol A as a function $A : E \to [0,1]$ where $A(u) \in [0,1]$ is the membership degree of the element $u \in E$ [40].

Definition 2.1 [1] Let A be a given subset on E. An intuitionistic fuzzy set (IFS) A^+ on E is an object of the following form

$$A^+ = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in E\},$$

where the functions $\mu_A : E \to [0,1]$ and $\nu_A : E \to [0,1]$ define the value of membership and the value of non-membership of the element $x \in E$ to the set A, respectively, and for every $x \in E$, $0 \leq \mu_A(x) + \nu_A(x) \leq 1$.

Obviously, every ordinary fuzzy set $\{(x, \mu_A(x)) \mid x \in E\}$ has an intuitionistic form $\{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle \mid x \in E\}$. If $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$, then $\pi_A(x)$ is the value of non-determinacy (uncertainty) of the membership of element $x \in E$ to the set A. In the case of ordinary fuzzy sets, where $\nu_A(x) = 1 - \mu_A(x)$, we have $\pi_A(x) = 0$ for every $x \in E$.

In the rest of this paper, we denote the set A^+ by A.

Let $L = (L, \leq_L, 0, 1)$ be a bounded (complete) lattice. By an L-fuzzy set A on E we mean a function $A : E \to L$ [12].

Definition 2.2 [37] Let $L = (L, \leq_L, 0, 1)$ be a bounded lattice. A binary operation $T : L \times L \to L$ is a lattice t-norm (Lt-norm) if it satisfies the following conditions:

1. $T(1, x) = x$,
2. $T(x, y) = T(y, x)$,
3. $T(x, T(y, z)) = T(T(x, y), z)$,
4. if $w \leq_L x$ and $y \leq_L z$, then $T(w, y) \leq_L T(x, z)$.

Definition 2.3 [37] Let $L = (L, \leq_L, 0, 1)$ be a bounded lattice. A binary operation $S : L \times L \to L$ is a lattice t-conorm (Lt-conorm) if it satisfies the following conditions:

1. $S(0, x) = x$,
2. $S(x, y) = S(y, x)$,
3. $S(x, S(y, z)) = S(S(x, y), z)$,
4. if $w \leq_L x$ and $y \leq_L z$, then $S(w, y) \leq_L S(x, z)$.

In the rest of this paper, we denote $D(x_1, D^{n-1}(x_2, ..., x_{n+1}))$ by $D^n(x_1, x_2, ..., x_{n+1})$ where D is an Lt-norm or an Lt-conorm on lattice L, $D^0(x) = x$ and $D^1(x_1, x_2) = D(x_1, x_2)$, for any $n \geq 1$.

We recall from [8] that $L^* : \{(x, y) \in [0, 1]^2 \big| 0 \leq x+y \leq 1\}$ is a complete lattice with the order defined by

$$(x_1, x_2) \leq (y_1, y_2) \text{ if and only if } x_1 \leq y_1 \text{ and } y_2 \leq x_2.$$

Definition 2.4 [2] Let X be a nonempty set and L be a complete lattice with an involutive order reversing unary operation $N : L \to L$. An intuitionistic L-fuzzy set is an object of the form $A = \{(x, \mu(x), \nu(x)) \mid x \in E\}$, where μ and ν are functions $\mu : E \to L, \nu : E \to L$, such that for all $x \in X$, $\mu(x) \leq N(\nu(x))$. We use the abbreviation ILFS for intuitionistic L-fuzzy set.

Definition 2.5 [13] Let $L \subseteq X^*$ and consider the relation $\sim_L$ on X, where $x \sim_L y$ if and only if, for all $z \in X^*, xz \in L \Longleftrightarrow yz \in L$.

From now on, we let $L = (L, \leq_L, T, S, 0, 1)$ be a complete lattice, where endowed with an Lt-norm T, an Lt-conorm S, the least element element 0 and the greatest element 1, also with an involutive order reversing unary operation $N : L \to L$.

Note 2.6 Let $A, B \in L$. In this note we assume that $A <_L B$ if and only if $A \leq_L B$ and $A \neq B$. We also assume that $A \geq_L B$ if and only if $B \leq_L A$.

3. Intuitionistic general L-fuzzy automata

Definition 1.1 An intuitionistic general L-fuzzy automaton (IGLFA) $\tilde{F}$ is a ten-tuple machine denoted by $\tilde{F} = (Q, X, \tilde{R}, Z, \tilde{\delta}, \tilde{\omega}, F_1, F_2, F_3, F_4)$, where

- Q is a set of states,
- X is a finite set of input symbols, $X = \{a_1, a_2, ..., a_m\}$,
- $\tilde{R}$ is the ILFS of start states, $\tilde{R} = \{(q, \mu^{t_0}(q), \nu^{t_0}(q)) \mid q \in R\}$, where R is a finite subset of Q,
- Z is a finite set of output symbols, $Z = \{b_1, b_2, ..., b_l\}$,
- $\tilde{\delta} : (Q \times L \times L) \times X \times Q \to L \times L$ is the augmented transition function,
- $\tilde{\omega} : (Q \times L \times L) \times Z \to L \times L$ is the output function,
- $F_1 = (F_1^T, F_1^S)$, where $F_1^T : L \times L \to L$ is an Lt-norm and it is called the membership assignment function. $F_1^T(\mu, \delta_1)$ as is seen, is motivated by two parameters μ and δ_1, where μ is the membership value of a predecessor and δ_1 is the membership value of a transition. Moreover, $F_1^S : L \times L \to L$ is an Lt-conorm, where is the dual of F_1^T respect to the involutive negation and it is called non-membership assignment function. $F_1^S(\nu, \delta_2)$ as is seen, is motivated by two

parameters ν and δ_2, where ν is the non-membership value of a predecessor and δ_2 is the non-membership value of a transition.

In this definition, the process that takes place upon the transition from the state q_i to q_j on an input a_k is given by:

$$\mu^{t+1}(q_j) = \tilde{\delta}_1((q_i, \mu^t(q_i), \nu^t(q_i)), a_k, q_j) = F_1^T(\mu^t(q_i), \delta_1(q_i, a_k, q_j)), \tag{1.1}$$

$$\nu^{t+1}(q_j) = \tilde{\delta}_2((q_i, \mu^t(q_i), \nu^t(q_i)), a_k, q_j) = F_1^S(\nu^t(q_i), \delta_2(q_i, a_k, q_j)), \tag{1.2}$$

thus

$$\begin{aligned}&\tilde{\delta}((q_i, \mu^t(q_i), \nu^t(q_i)), a_k, q_j)\\ &\quad = (\tilde{\delta}_1((q_i, \mu^t(q_i), \nu^t(q_i)), a_k, q_j), \tilde{\delta}_2((q_i, \mu^t(q_i), \nu^t(q_i)), a_k, q_j)),\end{aligned} \tag{1.3}$$

where, $\delta(q_i, a_k, q_j) = (\delta_1(q_i, a_k, q_j), \delta_2(q_i, a_k, q_j))$, which δ is an ILFS. It means that the membership value of the state q_j at time $t+1$ is computed by the function F_1^T using both the membership value of q_i at time t and the membership value of the transition. Also, the non-membership value of the state q_j at time $t+1$ is computed by function F_1^S using both the non-membership value of q_i at time t and the non-membership value of the transition.

Considering (1.1), (1.2) and Definitions 2.2, 2.3 and 2.4, $\tilde{\delta}$ is an ILFS.

• $F_2 = (F_2^T, F_2^S)$, where $F_2^T : L \times L \to L$ is an Lt-norm and it is called the membership assignment output function. $F_2^T(\mu, \omega_1)$ as is seen, is motivated by two parameters μ and ω_1, where μ is the membership value of present state and ω_1 is the membership value of an output function. Moreover, $F_2^S : L \times L \to L$ is an Lt-conorm where it is the dual of F_2^T respect to the involutive negation and is called non-membership assignment output function. $F_2^S(\nu, \omega_2)$ as is seen, is motivated by two parameters ν and ω_2, where ν is the non-membership value of present state and ω_2 is the non-membership value of an output function. Then, we have

$$\tilde{\omega}_1((q_i, \mu^t(q_i), \nu^t(q_i)), b_k) = F_2^T(\mu^t(q_i), \omega_1(q_i, b_k)), \tag{1.4}$$

$$\tilde{\omega}_2((q_i, \mu^t(q_i), \nu^t(q_i)), b_k) = F_2^S(\nu^t(q_i), \omega_2(q_i, b_k)), \tag{1.5}$$

thus,

$$\begin{aligned}&\tilde{\omega}((q_i, \mu^t(q_i), \nu^t(q_i)), b_k)\\ &\quad = (\tilde{\omega}_1((q_i, \mu^t(q_i), \nu^t(q_i)), b_k), \tilde{\omega}_2((q_i, \mu^t(q_i), \nu^t(q_i)), b_k)),\end{aligned} \tag{1.6}$$

where, $\omega(q_i, b_k) = (\omega_1(q_i, b_k), \omega_2(q_i, b_k))$, which ω is an ILFS. It means that the output membership value of the state q_i at time t is computed by the function F_2^T using both the membership value of q_i at time t and the membership value of the output function, also output non-membership value of the state q_i at time t is computed by function F_2^S using both the non-membership value of q_i at time t and the non-membership value of the output function.

Considering (1.4), (1.5) and Definitions 2.2, 2.3 and 2.4, $\tilde{\omega}$ is an ILFS.

• $F_3 = (F_3^{TS}, F_3^{ST})$, where $F_3^{ST} : L^* \to L$ is an Lt-norm and it is called the multi-non-membership function. The multi- non-membership resolution function resolves the multi-non-membership active states and assigns a single non-membership value to them. Moreover, $F_3^{TS} : L^* \to L$ is an Lt-conorm, where it is the dual of F_3^{ST} respect to the involutive negation and it is called the multi-membership function. The multi-membership resolution function resolves the multi-membership active states and assigns a single membership value to them.

• $F_4 = (F_4^{TS}, F_4^{ST})$, where $F_4^{ST} : L^* \to L$, is an Lt-norm and it is called the multi-non-membership output function. The multi-non-membership resolution output function resolves the output multi-non-membership active state and assigns a single output non-membership value to it. Moreover, $F_4^{TS} : L^* \to L$, is an Lt-conorm, where it is the dual of F_4^{ST} respect to the involutive negation and is multi-membership output function. The multi-membership resolution output function resolves the output multi-membership active state and assigns a single output membership value to it.

Let $Q_{act}(t_i)$ be the set of all active states at time t_i for all $i \geq 0$. We have $Q_{act}(t_0)=\tilde{R}$ and $Q_{act}(t_i)=\{(q, \mu^{t_i}(q), \nu^{t_i}(q)) \mid \exists (q', \mu^{t_{i-1}}(q'), \nu^{t_{i-1}}(q')) \in Q_{act}(t_{i-1}), \exists a \in X, \delta(q', a, q) \in \Delta, \mu^{t_i}(q) >_L 0\}$ for all positive integer i.

Since $Q_{act}(t_i)$ is an ILFS, to say that a state q belongs to $Q_{act}(t_i)$, we write $q \in Domain(Q_{act}(t_i))$ and for simplicity of notation we denote it by $q \in Q_{act}(t_i)$. The combination of the operations of functions F_1^T and F_3^{TS} (F_1^S and F_3^{ST}) on a multi-membership (multi- non-membership) state q_j will lead to the multi-membership (multi-non-membership) resolution algorithm. Also, the set of all transition of IGLFA $\tilde{F}$ is denoted by Δ.

Algorithm: Multi-membership resolution (multi-non-membership resolution) for **transition function**.

If there are several simultaneous transitions to the active state q_j at time $t+1$, then the following algorithm will assign a unified membership value (non-membership value) to that

1. Each transition membership value (transition non-membership value) $\delta_1(q_i, a_k, q_j)$ ($\delta_2(q_i, a_k, q_j)$) together with $\mu^t(q_i)$ ($\nu^t(q_i)$), will be processed by the membership (non-membership) assignment function F_1^T (F_1^S) and will produce a new membership value (non-membership value) as follows:

$$\begin{aligned} \tilde{\delta}_1((q_i, \mu^t(q_i), \nu^t(q_i)), a_k, q_j) &= F_1^T(\mu^t(q_i), \delta_1(q_i, a_k, q_j)), \\ \big(\tilde{\delta}_2((q_i, \mu^t(q_i), \nu^t(q_i)), a_k, q_j) &= F_1^S(\nu^t(q_i), \delta_2(q_i, a_k, q_j))\big). \end{aligned}$$

2. These new membership (non-membership) values are not necessarily equal. Hence, they will be processed by F_3^{TS} (F_3^{ST}), which is called the multi-membership (multi-non-membership) resolution function. By some manipulation on the product results by F_1^T (F_1^S), we obtain just one element, say

the instantaneous membership value (non-membership value) of the active state q_j

$$\begin{aligned} \mu^{t+1}(q_j) &= (F_3^{TS})^{n-1}(x_1, x_2, ..., x_n), \\ (\nu^{t+1}(q_j) &= (F_3^{ST})^{n-1}(x_1, x_2, ..., x_n)), \end{aligned}$$

where

- n is the number of simultaneous transitions to the active state q_j at time $t+1$ and $x_i = F_1^T(\mu^t(q_i), \delta_1(q_i, a_k, q_j))(x_i = F_1^S(\nu^t(q_i), \delta_2(q_i, a_k, q_j)))$, $1 \leq i \leq n$,
- $\delta_1(q_i, a_k, q_j)(\delta_2(q_i, a_k, q_j))$ is the membership (non-membership) value of transition from q_i to q_j upon input a_k,
- $\mu^t(q_i)$ $(\nu^t(q_i))$ is the membership (non-membership) value of q_i at time t,

Algorithm: Multi-membership resolution (Multi-non-membership resolution) for **output function**

If there are several simultaneous output to the active state q_i at time t, the following algorithm will assign a unified membership value (non-membership value) to that

1. Each output membership value (non-membership value) $\omega_1(q_i, b_k)$ $(\omega_2(q_i, b_k))$ together with $\mu^t(q_i)$ $(\nu^t(q_i))$, will be processed by the membership (non-membership) assignment function F_2^T (F_2^S) and will produce a new output membership value (non-membership value) as follows:

$$\begin{aligned} \tilde{\omega}_1((q_i, \mu^t(q_i), \nu^t(q_i)), b_k) &= F_2^T(\mu^t(q_i), \omega_1(q_i, b_k)), \\ \big(\tilde{\omega}_2((q_i, \mu^t(q_i), \nu^t(q_i)), b_k) &= F_2^S(\nu^t(q_i), \omega_2(q_i, b_k))\big). \end{aligned}$$

2. These new output membership (non-membership) values are not necessarily equal. Hence, they will be processed by F_4^{TS} (F_4^{ST}), which is called the output multi-membership (multi-non-membership) resolution function. By some manipulation on the product results by F_2^T (F_2^S), we obtain just one element, say the instantaneous output membership value (non-membership value) of the active state q_i.

$$\begin{aligned} \omega_1^t(q_i) &= (F_4^{TS})^{n-1}(x_1, x_2, ..., x_n), \\ (\omega_2^t(q_i) &= (F_4^{ST})^{n-1}(x_1, x_2, ..., x_n)), \end{aligned}$$

where

- n is the number of simultaneous outputs to the active state q_i at time t, $x_j = F_2^T(\mu^t(q_i), \omega_1(q_i, b_j))(x_j = F_2^S(\nu^t(q_i), \omega_2(q_i, b_j)))$, $1 \leq j \leq n$, for some $a_k \in X$ and q_i is an active state at time t,
- $\omega_1(q_i, b_j)(\omega_2(q_i, b_j))$ is the membership (non-membership) value of output from q_i on b_k,

- $\mu^t(q_i)$ ($\nu^t(q_i)$) is the membership (non-membership) value of q_i at time t,
- $\omega_1^t(q_i)$ ($\omega_2^t(q_i)$) is the output membership (non-membership) value of q_i at time t.

Remark 1.2 We let for all $q \in Q$ such that $q \notin \tilde{R}, \mu^{t_0}(q) = 0$ and $\nu^{t_0}(q) = 1$ and for all $q \in \tilde{R}, \mu^{t_0}(q) >_L 0$.

Note 1.3 In this paper, we assume that the max-min IGLFA has a finite number of states.

Definition 1.4 Let $\tilde{F} = (Q, X, \tilde{R}, Z, \tilde{\delta}, \tilde{\omega}, F_1, F_2, F_3, F_4)$ be an IGLFA. We define the max-min intuitionistic general L-fuzzy automaton $\tilde{F}^* = (Q, X, \tilde{R}, Z, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4)$ such that $\tilde{\delta}^* : Q_{act} \times X^* \times Q \to L \times L$, where $Q_{act} = \{Q_{act}(t_0), Q_{act}(t_1), Q_{act}(t_2), ...\}$ and for every $i \geq 0$,

$$\tilde{\delta}_1^*((q, \mu^{t_i}(q), \nu^{t_i}(q)), \Lambda, p) = \begin{cases} 1 & \text{if } p = q, \\ 0 & \text{otherwise,} \end{cases} \tag{1.7}$$

and

$$\tilde{\delta}_2^*((q, \mu^{t_i}(q), \nu^{t_i}(q)), \Lambda, p) = \begin{cases} 0 & \text{if } p = q, \\ 1 & \text{otherwise.} \end{cases} \tag{1.8}$$

Also for every $i \geq 0$, $\tilde{\delta}_1^*((q, \mu^{t_i}(q), \nu^{t_i}(q)), u_{i+1}, p) = \tilde{\delta}_1((q, \mu^{t_i}(q), \nu^{t_i}(q)), u_{i+1}, p)$ and $\tilde{\delta}_2^*((q, \mu^{t_i}(q), \nu^{t_i}(q)), u_{i+1}, p) = \tilde{\delta}_2((q, \mu^{t_i}(q), \nu^{t_i}(q)), u_{i+1}, p)$ and recursively,

$$\begin{aligned} &\tilde{\delta}_1^*((q, \mu^{t_0}(q), \nu^{t_0}(q)), u_1 u_2 ... u_n, p) = \\ &\quad \vee \{\tilde{\delta}_1((q, \mu^{t_0}(q), \nu^{t_0}(q)), u_1, p_1) \wedge \tilde{\delta}_1((p_1, \mu^{t_1}(p_1), \nu^{t_1}(p_1)), u_2, p_2) \wedge ... \\ &\quad \wedge \tilde{\delta}_1((p_{n-1}, \mu^{t_{n-1}}(p_{n-1}), \nu^{t_{n-1}}(p_{n-1})), u_n, p) \mid \\ &\qquad p_1 \in Q_{act}(t_1), ..., p_{n-1} \in Q_{act}(t_{n-1})\}, \end{aligned} \tag{1.9}$$

$$\begin{aligned} &\tilde{\delta}_2^*((q, \mu^{t_0}(q), \nu^{t_0}(q)), u_1 u_2 ... u_n, p) = \\ &\quad \wedge \{\tilde{\delta}_2((q, \mu^{t_0}(q), \nu^{t_0}(q)), u_1, p_1) \vee \tilde{\delta}_2((p_1, \mu^{t_1}(p_1), \nu^{t_1}(p_1)), u_2, p_2) \vee ... \\ &\quad \vee \tilde{\delta}_2((p_{n-1}, \mu^{t_{n-1}}(p_{n-1}), \nu^{t_{n-1}}(p_{n-1})), u_n, p) \mid \\ &\qquad p_1 \in Q_{act}(t_1), ..., p_{n-1} \in Q_{act}(t_{n-1})\}, \end{aligned} \tag{1.10}$$

in which $u_i \in X$ for all $1 \leq i \leq n$ and assume that u_{i+1} is the entered input at time t_i, for all $0 \leq i \leq n-1$.

4. Minimal intuitionistic general L-fuzzy automata

Definition 4.1 Let $\tilde{F}^* = (Q, X, \tilde{R}, Z, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4)$ be a max-min IGLFA. Suppose that $\alpha, \beta \in L$ and $\alpha \leq_L N(\beta)$. Then we say that

1. $\tilde{F}^*$ is (α, β)-complete, if for any $q \in Q, a \in X$ there exists $p \in Q$ such that $\delta_1(q, a, p) >_L \alpha$ and $\delta_2(q, a, p) <_L \beta$,

2. $q \in Q$ is (α, β)-accessible if there exist $p \in \tilde{R}, x \in X^*$ such that $\tilde{\delta}_1^*((p, \mu^{t_0}(p), \nu^{t_0}(p)), x, q) >_L \alpha$, and $\tilde{\delta}_2^*((p, \mu^{t_0}(p), \nu^{t_0}(p)), x, q) <_L \beta$,

3. $\tilde{F}^*$ is (α, β)-accessible if for any $q \in Q$, q is an (α, β)-accessible.

Definition 4.2 Let $\tilde{F}^* = (Q, X, \tilde{R}, Z, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4)$ be a max-min IGLFA. Then we say that $\tilde{F}^*$ is deterministic if there exists a unique $p_0 \in \tilde{R}$ such that $\mu^{t_0}(p_0) >_L 0$, and for any $q \in Q, a \in X$ there exists at most one $p \in Q$ such that $\delta_2(q, a, p) <_L 1$.

Example 4.3 Consider the complete lattice $L = (L, \leq_L, T, S, 0, 1)$ as in Figure 1, where $L = \{0, a, b, c, d, 1\}$ and $N(0) = 1$, $N(1) = 0$, $N(a) = b$, $N(b) = a$, $N(c) = d$, $N(d) = c$.

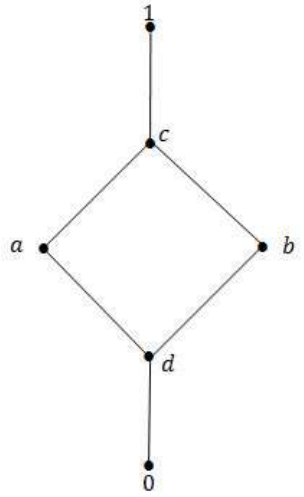


Figure 1: The complete lattice L of Example 4.3

Let the max-min IGLFA $\tilde{F}^* = (Q, X, \tilde{R}, Z, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4)$ as in Figure 2, where $Q = \{q_0, q_1, q_2, q_3\}, X = \{u, v\}, \tilde{R} = \{(q_0, 1, 0)\}, Z = \{o\}$ and $\delta : Q \times X \times Q : L \times L$ is defined as follows:

$$\begin{array}{lcllcl} \delta(q_0, u, q_1) & = & (a, 0) & \delta(q_1, v, q_3) & = & (a, b) \\ \delta(q_2, u, q_1) & = & (1, 0) & \delta(q_2, v, q_0) & = & (c, 0) \end{array}$$

$\delta(q, x, q') = (0, 1)$ for all other $(q, x, q') \in Q \times X \times Q$ and $\omega : Q \times Z : L \times L$ is defined by: $\omega(q_1, o) = (0, 1)$ and $\omega(q, e) = (1, 0)$ for all other $(q, e) \in Q \times Z$. q_1 is $(0, \alpha)$-accessible, (d, β)-accessible, where $\alpha \in \{a, b, c, d, 1\}$ and $\beta \in \{a, b, c, d\}$, q_3 is $(0, 1)$- accessible, $(0, c)$- accessible, (d, c)- accessible but q_3 is not $(d, 1)$- accessible, since $d \not\leq N(1) = 0$. $\tilde{F}^*$ is not (α, β)-accessible, not (α, β)- complete, for any $(\alpha, \beta) \in L$ in which $\alpha \leq_L N(\beta)$. Clearly, it is deterministic.

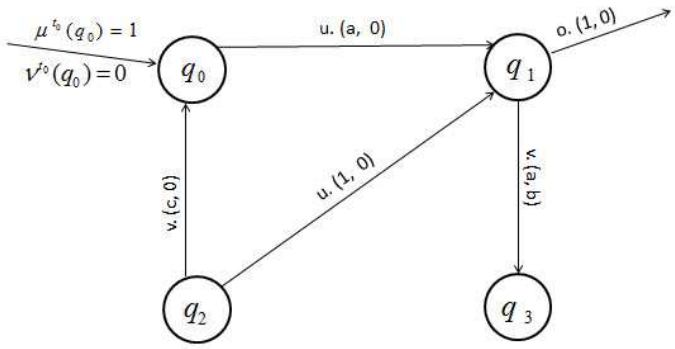


Figure 2: The IGLFA of Example 4.3

Theorem 4.4 *Let $\tilde{F}^* = (Q, X, \{p\}, Z, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4)$ be a deterministic max-min IGLFA. If*

$$\tilde{\delta}_1^*((p, \mu^{t_0}(p), \nu^{t_0}(p)), x, q) \wedge \tilde{\delta}_1^*((p, \mu^{t_0}(p), \nu^{t_0}(p)), x, q') >_L \alpha,$$

and

$$\tilde{\delta}_2^*((p, \mu^{t_0}(p), \nu^{t_0}(p)), x, r) \vee \tilde{\delta}_1^*((p, \mu^{t_0}(p), \nu^{t_0}(p)), x, r') <_L \beta,$$

then $q = q' = r = r'$, where $q, q', r, r' \in Q, x \in X^*, \alpha, \beta \in L$ and $\alpha \leq_L N(\beta)$.

Proof. First, let $x = \Lambda$. Then we have

$$\tilde{\delta}_1((p, \mu^{t_0}(p), \nu^{t_0}(p)), \Lambda, q) \wedge \tilde{\delta}_1((p, \mu^{t_0}(p), \nu^{t_0}(p)), \Lambda, q') >_L \alpha, \quad \text{and}$$
$$\tilde{\delta}_2((p, \mu^{t_0}(p), \nu^{t_0}(p)), \Lambda, r) \vee \tilde{\delta}_2((p, \mu^{t_0}(p), \nu^{t_0}(p)), \Lambda, r') <_L \beta,$$

which implies that $p = q' = q = r = r'$. Thus the theorem holds for $x = \Lambda$.

Now, we continue the proof for any $x \in X^*$ and $x \notin \Lambda$ by induction on $|x|$. Suppose that $|x| = 1$. Then

$$\tilde{\delta}_1((p, \mu^{t_0}(p), \nu^{t_0}(p)), x, q) = F_1^T(\mu^{t_0}(p), \delta_1(p, x, q)) >_L \alpha \quad \text{and}$$
$$\tilde{\delta}_1((p, \mu^{t_0}(p), \nu^{t_0}(p)), x, q') = F_1^T(\mu^{t_0}(p), \delta_1(p, x, q')) >_L \alpha$$

and

$$\tilde{\delta}_2((p, \mu^{t_0}(p), \nu^{t_0}(p)), x, r) = F_1^S(\nu^{t_0}(p), \delta_1(p, x, r)) <_L \beta \quad \text{and}$$
$$\tilde{\delta}_2((p, \mu^{t_0}(p), \nu^{t_0}(p)), x, r') = F_1^S(\nu^{t_0}(p), \delta_1(p, x, r')) <_L \beta.$$

These imply that

$$\delta_1(p, x, q) \wedge \delta_1(p, x, q') >_L \alpha \text{ and } \delta_2(p, x, r) \vee \delta_2(p, x, r') <_L \beta.$$

Since $\delta_1(p, x, q) \wedge \delta_1(p, x, q') >_L \alpha$, then $\delta_2(p, x, q) \vee \delta_2(p, x, q') <_L N(\alpha) \leq_L 1$. Then by Definition 4.2, $q = q' = r = r'$.

Now, suppose the claim holds for all $y \in X^*$ such that $|y| = n - 1$ and $n > 1$. Let $x = ya$, where $y \in X^*, a \in X$ and $|y| = n - 1$. Then

$$\tilde{\delta}_1^*((p, \mu^{t_0}(p), \nu^{t_0}(p)), x, q) = \vee\{\tilde{\delta}_1^*((p, \mu^{t_0}(p), \nu^{t_0}(p)), y, p') \wedge \tilde{\delta}_1((p', \mu^{t_0+n-1}(p'), \nu^{t_0+n-1}(p')), a, q) | p' \in Q\} >_L \alpha,$$

and

$$\tilde{\delta}_1^*((p,\mu^{t_0}(p),\nu^{t_0}(p)),x,q') = \vee\{\tilde{\delta}_1^*((p,\mu^{t_0}(p),\nu^{t_0}(p)),y,p') \wedge\tilde{\delta}_1((p',\mu^{t_0+n-1}(p'),\nu^{t_0+n-1}(p')),a,q') \mid p'\in Q\} >_L \alpha.$$

So, there exist $d,d'\in Q$ such that

$$\vee\{\tilde{\delta}_1^*((p,\mu^{t_0}(p),\nu^{t_0}(p)),y,p')\wedge\tilde{\delta}_1((p',\mu^{t_0+n-1}(p'),\nu^{t_0+n-1}(p')),a,q) \mid p'\in Q\} = \tilde{\delta}_1^*((p,\mu^{t_0}(p),\nu^{t_0}(p)),y,d)\wedge\tilde{\delta}_1((d,\mu^{t_0+n-1}(d),\nu^{t_0+n-1}(d)),a,q) >_L \alpha,$$

and

$$\vee\{\tilde{\delta}_1^*((p,\mu^{t_0}(p),\nu^{t_0}(p)),y,p')\wedge\tilde{\delta}_1((p',\mu^{t_0+n-1}(p'),\nu^{t_0+n-1}(p')),a,q') \mid p'\in Q\} = \tilde{\delta}_1^*((p,\mu^{t_0}(p),\nu^{t_0}(p)),y,d')\wedge\tilde{\delta}_1((d',\mu^{t_0+n-1}(d'),\nu^{t_0+n-1}(d')),a,q') >_L \alpha.$$

Also there exist $s,s'\in Q$ such that

$$\begin{aligned}&\tilde{\delta}_2^*((p,\mu^{t_0}(p),\nu^{t_0}(p)),x,r)\\ &= \wedge\{\tilde{\delta}_2^*((p,\mu^{t_0}(p),\nu^{t_0}(p)),y,p')\vee\tilde{\delta}_2((p',\mu^{t_0+n-1}(p'),\nu^{t_0+n-1}(p')),a,r) \mid p'\in Q\}\\ &= \tilde{\delta}_2^*((p,\mu^{t_0}(p),\nu^{t_0}(p)),y,s)\vee\tilde{\delta}_2((s,\mu^{t_0+n-1}(s),\nu^{t_0+n-1}(s)),a,r) <_L \beta,\end{aligned}$$

and

$$\begin{aligned}&\tilde{\delta}_2^*((p,\mu^{t_0}(p),\nu^{t_0}(p)),x,r')\\ &= \wedge\{\tilde{\delta}_2^*((p,\mu^{t_0}(p),\nu^{t_0}(p)),y,p')\vee\tilde{\delta}_2((p',\mu^{t_0+n-1}(p'),\nu^{t_0+n-1}(p')),a,r') \mid p'\in Q\}\\ &= \tilde{\delta}_2^*((p,\mu^{t_0}(p),\nu^{t_0}(p)),y,s')\vee\tilde{\delta}_2((s',\mu^{t_0+n-1}(s'),\nu^{t_0+n-1}(s')),a,r') <_L \beta.\end{aligned}$$

Therefore, by the induction hypothesis, $s=s'=d=d'$. Hence

$$\tilde{\delta}_1((d,\mu^{t_0+n-1}(d),\nu^{t_0+n-1}(d)),a,q)\wedge\tilde{\delta}_1((d,\mu^{t_0+n-1}(d),\nu^{t_0+n-1}(d)),a,q') >_L \alpha,$$

and

$$\tilde{\delta}_2((s,\mu^{t_0+n-1}(s),\nu^{t_0+n-1}(s)),a,r)\vee\tilde{\delta}_2((s,\mu^{t_0+n-1}(s),\nu^{t_0+n-1}(s)),a,r') <_L \beta,$$

imply that $\delta_1(d,a,q)\wedge\delta_1(d,a,q') >_L \alpha$ and $\delta_2(s,a,r)\vee\delta_2(s,a,r') <_L \beta$. Then $q=q'=r=r'$. Hence the claim is hold. ∎

Corollary 4.5 *Let $\tilde{F}^* = (Q,X,\tilde{R},Z,\tilde{\delta}^*,\tilde{\omega},F_1,F_2,F_3,F_4)$ be a max-min IGLFA. Notice that if $\tilde{F}^*$ is an (α,β)-complete and deterministic max-min IGLFA, then for any $p\in Q, a\in X$ there exists exactly one state $q\in Q$ such that $\delta_1(p,a,q) >_L \alpha$ and $\delta_2(p,a,q) <_L \beta$, where $\alpha,\beta\in L$ and $\alpha\leq_L N(\beta)$.*

Example 4.6 Consider the complete lattice $L=(L,\leq_L,T,S,0,1)$ as in Example 4.3, where $L=\{0,a,b,c,d,1\}$. Let the max-min IGLFA $\tilde{F}^*=(Q,X,\tilde{R},Z,\tilde{\delta}^*,\tilde{\omega},F_1,F_2,F_3,F_4)$ as in Figure 3, where $Q=\{q_0,q_1,q_2,q_3\}, X=\{u\}, \tilde{R}=\{(q_0,a,b)\}$, $Z=\{o\}$ and $\delta: Q\times X\times Q : L\times L$ is defined as follows:

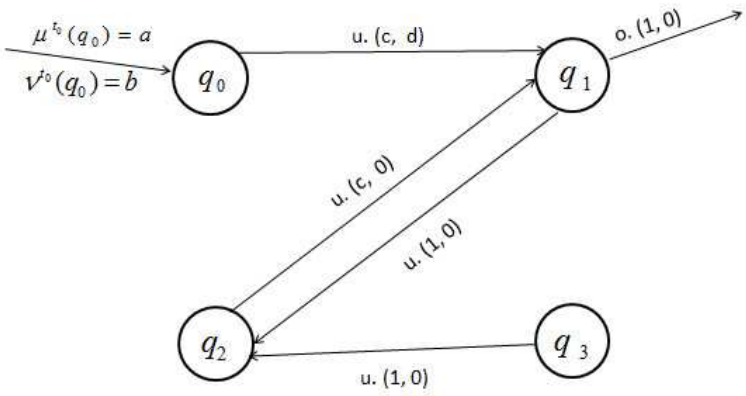


Figure 3: The IGLFA of Example 4.6

$$\delta(q_0, u, q_1) = (c, d) \qquad \delta(q_1, u, q_2) = (1, 0)$$
$$\delta(q_2, u, q_1) = (c, 0) \qquad \delta(q_3, u, q_2) = (1, 0)$$

$\delta(q, x, q') = (0, 1)$ for all other $(q, x, q') \in Q \times X \times Q$ and $\omega : Q \times Z \to L \times L$ is defined by: $\omega(q_1, o) = (1, 0)$ and $\omega(q, e) = (0, 1)$ for all other $(q, e) \in Q \times Z$. Then $\tilde{F}^*$ is (a, b)-complete and deterministic.

Definition 4.7 Let $\tilde{F}^* = (Q, X, \tilde{R}, Z, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4)$ be a max-min IGLFA. Then the (α, β)-language recognized by $\tilde{F}^*$ is a subset of X^* defined by:

$$\begin{aligned} \mathcal{L}^{\alpha,\beta}(\tilde{F}^*) = \{x \in X^* \mid\ & \tilde{\delta}_1^*((p, \mu^{t_0}(p), \nu^{t_0}(p)), x, q) \\ & \wedge \tilde{\omega}_1((q, \mu^{t_0+|x|}(q), \nu^{t_0+|x|}(q)), b) >_L \alpha, \\ & \tilde{\delta}_2^*((p, \mu^{t_0}(p), \nu^{t_0}(p)), x, q) \\ & \vee \tilde{\omega}_2((q, \mu^{t_0+|x|}(q), \nu^{t_0+|x|}(q)), b') <_L \beta, \\ & \text{for some } p \in \tilde{R}, q \in Q, b, b' \in Z\}. \end{aligned} \tag{4.1}$$

Definition 4.8 Let X be a nonempty finite set. Then subset $\mathcal{L}$ of X^* is called recognizable (α, β)- language, if there exists a max-min IGLFA $\tilde{F}^*$ such that $\mathcal{L} = \mathcal{L}^{\alpha,\beta}(\tilde{F}^*)$, where $\alpha, \beta \in L$ and $\alpha \leq_L N(\beta)$.

Theorem 4.9 *Let $\tilde{F}^* = (Q, X, \tilde{R}, Z, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4)$ be a max-min IGLFA. Then there exists an (α, β)- complete IGLFA $\tilde{F}^{*c}$ such that $\mathcal{L}^{\alpha,\beta}(\tilde{F}^*) = \mathcal{L}^{\alpha,\beta}(\tilde{F}^{*c})$, where $\alpha, \beta \in L$ and $\alpha \leq_L N(\beta)$.*

Proof. Let $\tilde{F}^* = (Q, X, \tilde{R}, Z, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4)$ does not be an (α, β)-complete max-min IGLFA. Let $Q^c = Q \cup \{t\}$, where t is an element such that $t \notin Q$. Consider $\gamma, \eta \in L$, where $\gamma >_L \alpha$, $\eta <_L \beta$ and $\gamma \leq_L N(\eta)$. We now give an algorithm in which the output is an (α, β)-complete max-min IGLFA for a given (α, β)-incomplete max-min IGLFA as input.

Algorithm: (to make (α, β)-complete)

Step 1 input incomplete $\tilde{F}^* = (Q, X, \tilde{R}, Z, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4)$, where $Q = \{q_1, q_2, ..., q_n\}, X = \{a_1, a_2, ..., a_m\}$,

Step 2 $i = 1$,

Step 3 $j = 1$, if $i \leq n$ go to the next step, else go to Step 7,

Step 4 if $j \leq m$, then $T = \{q \mid \delta_1(q_i, a_j, q) >_L \alpha\}$ else $i = i + 1$ go to Step 3,

Step 5 if $T = \emptyset$, then $\delta_1^c(q_i, a_j, t) = \gamma, \delta_2^c(q_i, a_j, t) = \eta, j = j + 1$, go to Step 4, else go to Step 6,

Step 6 for $q \in T$ if $\delta_2(q_i, a_j, q) <_L \beta$, then $j = j + 1$ and go to Step 4, else $T = T - \{q\}$ and go to Step 5,

Step 7 $\delta_1^c(p, a, q) = \delta_1(p, a, q)$, and $\delta_2^c(p, a, q) = \delta_2(p, a, q)$, for all $p, q \in Q, a \in X$, $\delta_1^c(t, a, t) = \gamma$, $\delta_2^c(t, a, t) = \eta$ for all $a \in X$,

Step 8 $Q^c = Q \cup \{t\}$, $\omega_1^c(p, b) = \begin{cases} \omega_1(p, b) & if\ p \neq t, \\ 0 & if\ p = t, \end{cases}$ and

$$\omega_2^c(p, b) = \begin{cases} \omega_2(p, b) & if\ p \neq t, \\ 1 & if\ p = t, \end{cases}$$

Step 9 output $\tilde{F}^{*c} = (Q^c, X, \tilde{R}, Z, \tilde{\delta}^{*c}, \tilde{\omega}^c, F_1, F_2, F_3, F_4)$.

It is easy to see that the max-min IGLFA $\tilde{F}^{*c} = (Q^c, X, \tilde{R}, Z, \tilde{\delta}^{*c}, \tilde{\omega}^c, F_1, F_2, F_3, F_4)$ is (α, β)- complete. It is clear that $\mathcal{L}^{\alpha,\beta}(\tilde{F}^*) \subseteq \mathcal{L}^{\alpha,\beta}(\tilde{F}^{*c})$. Let $x = u_1 u_2 ... u_{k+1} \in L^{\alpha,\beta}(\tilde{F}^{*c})$. Then there exist $p \in \tilde{R}, q \in Q^c, b, b' \in Z$ such that

$$\tilde{\delta}_1^{*c}((p, \mu^{t_0}(p), \nu^{t_0}(p)), x, q) \wedge \tilde{\omega}_1^c((q, \mu^{t_0+|x|}(q), \nu^{t_0+|x|}(q)), b) >_L \alpha,$$

and

$$\tilde{\delta}_2^{*c}((p, \mu^{t_0}(p), \nu^{t_0}(p)), x, q) \vee \tilde{\omega}_2^c((q, \mu^{t_0+|x|}(q), \nu^{t_0+|x|}(q)), b') <_L \beta.$$

These imply that $q \in Q$ and there exist $p_1, p_2, ..., p_k, p_1', p_2', ..., p_k' \in Q$ such that

$$(4.2)\quad \tilde{\delta}_1^c((p, \mu^{t_0}(p), \nu^{t_0}(p)), u_1, p_1) \wedge ... \wedge \tilde{\delta}_1^c((p_k, \mu^{t_0+|x|}(p_k), \nu^{t_0+|x|}(p_k)), u_k, q) >_L \alpha.$$

and

$$(4.3)\quad \tilde{\delta}_2^c((p, \mu^{t_0}(p), \nu^{t_0}(p)), u_1, p_1') \vee ... \vee \tilde{\delta}_2^c((p_k', \mu^{t_0+|x|}(p_k'), \nu^{t_0+|x|}(p_k')), u_k, q) <_L \beta.$$

Definition 2.2 and (4.2) imply that $\mu^{t_0}(p) >_L \alpha, \delta_1^c(p, u_1, p_1) >_L \alpha, \delta_1^c(p_1, u_2, p_2) >_L \alpha, ..., \delta_1^c(p_k, u_k, q) >_L \alpha$. Now, suppose that $p_j, 1 \leq j \leq k$, be the first state that $\delta_1^c(p_j, u_j, p_{j+1}) >_L \alpha$ and $\delta_1(p_j, u_j, p_{j+1})$ was undefined. Then $p_{j+1} = t$. Therefore $p_{j+1} = p_{j+2} = ... = q = t$, which is a contradiction. Hence

$$\tilde{\delta}_1^*((p, \mu^{t_0}(p), \nu^{t_0}(p)), x, q) \wedge \tilde{\omega}_1((q, \mu^{t_0+|x|}(q), \nu^{t_0+|x|}(q)), b) >_L \alpha.$$

In a similar manner we obtain

$$\tilde{\delta}_2^*((p,\mu^{t_0}(p),\nu^{t_0}(p)),x,q)\vee\tilde{\omega}_2((q,\mu^{t_0+|x|}(q),\nu^{t_0+|x|}(q)),b')<_L\beta.$$

Then the claim is hold. ∎

Example 4.10 Consider the complete lattice $L=(L,\leq_L,T,S,0,1)$ defined in Example 4.3, and max-min IGLFA $\tilde{F}^*=(Q,X,\tilde{R},Z,\tilde{\delta}^*,\tilde{\omega},F_1,F_2,F_3,F_4)$ as in Figure 4,

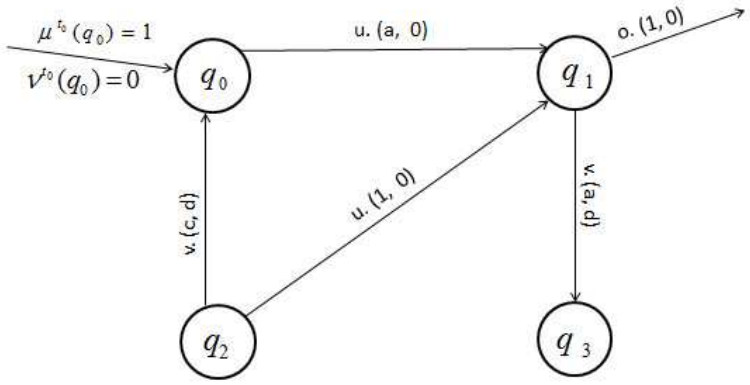


Figure 4: The IGLFA of Example 4.10

where $Q=\{q_0,q_1,q_2,q_3\},X=\{a_0=u,a_1=v\},\tilde{R}=\{(q_0,1,0)\},Z=\{o\}$ and $\delta:Q\times X\times Q\to L\times L$ is defined as follows:

$$\begin{aligned}\delta(q_0,u,q_1)&=(a,0) & \delta(q_1,v,q_3)&=(a,d)\\ \delta(q_2,u,q_1)&=(1,0) & \delta(q_2,v,q_0)&=(c,d)\end{aligned}$$

$\delta(q,x,q')=(0,1)$ for all other $(q,x,q')\in Q\times X\times Q$ and $\omega:Q\times Z\to L\times L$ is defined by: $\omega(q_1,o)=(1,0)$ and $\omega(q,e)=(0,1)$ for all other $(q,e)\in Q\times Z$.

Now, considering the complete algorithm in the proof of Theorem 4.9, and $\alpha=a$, $\beta=d$, we have

Stage 1 Let $i=0,j=0$. Then $T=\emptyset$ and $\delta_1^c(q_0,u,t)=c,\delta_2^c(q_0,u,t)=0$.

Stage 2 Let $i=0,j=1$. Then $T=\emptyset$ and $\delta_1^c(q_0,v,t)=c,\delta_2^c(q_0,v,t)=0$.

Stage 3 Let $i=1,j=0$. Then $T=\emptyset$ and $\delta_1^c(q_1,u,t)=c,\delta_2^c(q_1,u,t)=0$.

Stage 4 Let $i=1,j=1$. Then $T=\emptyset$ and $\delta_1^c(q_1,v,t)=c,\delta_2^c(q_1,u,t)=0$.

Stage 5 Let $i=2,j=0$. Then $T=\{q_1\}$ and we have $\delta_2(q_2,u,q_1)=0<_L d$.

Stage 6 Let $i=2,j=1$. Then $T=\{q_0\}$. Since $\delta_2(q_2,v,q_0)=d$, thus $T=T-\{q_0\}=\emptyset$. Then $\delta_1^c(q_2,v,t)=c,\delta_2^c(q_2,v,t)=0$.

Stage 7 Let $i=3,j=0$. Then $T=\emptyset$ and $\delta_1^c(q_3,u,t)=c,\delta_2^c(q_3,u,t)=0$.

Stage 8 Let $i=3,j=1$. Then $T=\emptyset$ and $\delta_1^c(q_3,v,t)=c,\delta_2^c(q_3,v,t)=0$.

Therefore, we have an (a, d)-complete $\tilde{F}^{*c} = (Q^c, X, \tilde{R}, Z, \tilde{\delta}^{*c}, \tilde{\omega}^c, F_1, F_2, F_3, F_4)$ as in Figure 5.

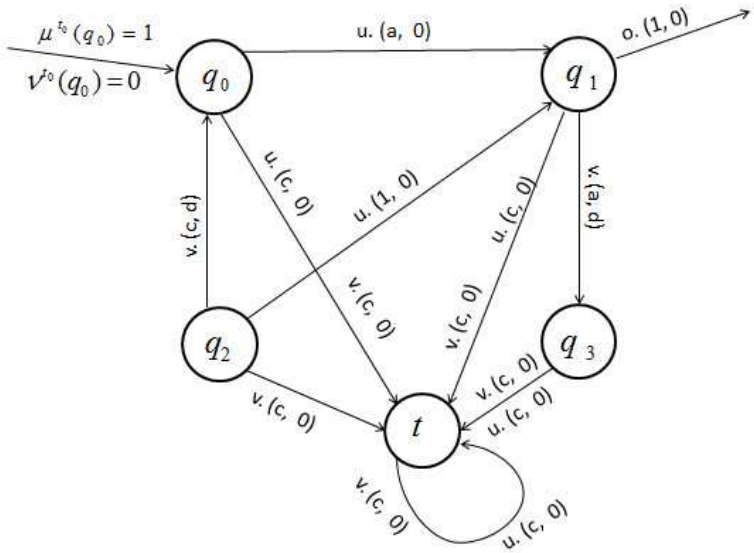


Figure 5: The (a, d)-complete $\tilde{F}^{*c}$ of Example 4.10

Theorem 4.11 *Let $\tilde{F}^* = (Q, X, \tilde{R}, Z, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4)$ be a max-min IGLFA and $\tilde{R} \neq \emptyset$. Then there exists a deterministic max-min IGLFA $\tilde{F}_d^*$ such that $\mathcal{L}^{\alpha,\beta}(\tilde{F}^*) = \mathcal{L}^{\alpha,\beta}(\tilde{F}_d^*)$, where $\alpha, \beta \in L$ and $\alpha \leq_L N(\beta)$.*

Proof. Let

$$I_x = \{q' \in Q \mid \exists q \in \tilde{R} \text{ such that } \tilde{\delta}_1^*((p, \mu^{t_0}(p), \nu^{t_0}(p)), x, q') >_L \alpha \quad \tilde{\delta}_2^*((p, \mu^{t_0}(p), \nu^{t_0}(p)), x, q') <_L \beta\}, \tag{4.4}$$

for all $x \in X^*$. Then $I_\Lambda = \{q' \in Q \mid q' \in \tilde{R}\}$.

Let $Q_d = \{I_x \mid x \in X^*\}$. Define $\delta_d : Q_d \times X \times Q_d \to L \times L$, where

$$\delta_{d1}(I_y, a, I_x) = \begin{cases} \gamma_1 & \text{if } I_x = I_{ya}, \\ 0 & \text{otherwise,} \end{cases}$$

$$\delta_{d2}(I_y, a, I_x) = \begin{cases} \eta_1 & \text{if } I_x = I_{ya}, \\ 1 & \text{otherwise,} \end{cases}$$

and $\omega_d : Q_d \times Z_d \to L \times L$, where

$$\omega_{d1}(I_x, e) = \begin{cases} \gamma_2 & \text{if } x \in \mathcal{L}^{\alpha,\beta}(\tilde{F}^*), \\ 0 & \text{otherwise,} \end{cases} \tag{4.5}$$

$$\omega_{d2}(I_x, e) = \begin{cases} \eta_2 & \text{if } x \in \mathcal{L}^{\alpha,\beta}(\tilde{F}^*), \\ 1 & \text{otherwise,} \end{cases} \tag{4.6}$$

where $\gamma_1, \eta_1, \gamma_2, \eta_2 \in L, \gamma_1 \wedge \gamma_2 >_L \alpha, \eta_1 \vee \eta_2 <_L \beta$, $\gamma_1 \leq_L N(\eta_1)$ and $\gamma_2 \leq_L N(\eta_2)$. Consider $\mu^{t_0}(I_\Lambda) = \vee\{\mu^{t_0}(q) \mid q \in I_\Lambda\}, \nu^{t_0}(I_\Lambda) = \wedge\{\nu^{t_0}(q) \mid q \in I_\Lambda\}$ and $Z_d = \{e\}$. Now, suppose that $\tilde{F}_d^* = (Q_d, X, I_\Lambda, Z_d, \tilde{\delta}_d^*, \tilde{\omega}_d, F_1, F_2, F_3, F_4)$. It is clear that δ_d is well defined. Now, we show that ω_d is well defined. Let $I_x = I_y$ and $e \in Z_d$. If $x \in \mathcal{L}^{\alpha,\beta}(\tilde{F}^*)$, then there exist $q \in \tilde{R}, p \in Q, b, b' \in Z$ such that

$$\tilde{\delta}_1^*((q, \mu^{t_0}(q), \nu^{t_0}(q)), x, p) \wedge \tilde{\omega}_1((p, \mu^{t_0+|x|}(p), \nu^{t_0+|x|}(p)), b) >_L \alpha,$$

and

$$\tilde{\delta}_2^*((q, \mu^{t_0}(q), \nu^{t_0}(q)), x, p) \vee \tilde{\omega}_2((p, \mu^{t_0+|x|}(p), \nu^{t_0+|x|}(p)), b') <_L \beta.$$

Thus $p \in I_x = I_y$. Therefore

$$\tilde{\delta}_1^*((q, \mu^{t_0}(q), \nu^{t_0}(q)), y, p) >_L \alpha \text{ and } \tilde{\delta}_2^*((q, \mu^{t_0}(q), \nu^{t_0}(q)), y, p) <_L \beta$$

for some $q \in \tilde{R}$. Also $\omega_1(p, b) >_L \alpha$ and $\omega_2(p, b) <_L \beta$. Hence $y \in \mathcal{L}^{\alpha,\beta}(\tilde{F}^*)$. In a similar way, if $y \in \mathcal{L}^{\alpha,\beta}(\tilde{F}^*)$, then $x \in \mathcal{L}^{\alpha,\beta}(\tilde{F}^*)$. Hence $\omega_d(I_x, e) = \omega_d(I_y, e)$.

Since there exists $q \in \tilde{R}$ such that $\mu^{t_0}(q) >_L 0$, then $\mu^{t_0}(I_\Lambda) >_L 0$. It is easy to see that the max-min IGLFA $\tilde{F}_d^*$ is deterministic. We show that $\mathcal{L}^{\alpha,\beta}(\tilde{F}^*) = \mathcal{L}^{\alpha,\beta}(\tilde{F}_{sd}^*)$. Let $x = u_1 u_2 ... u_{k+1} \in \mathcal{L}^{\alpha,\beta}(\tilde{F}^*)$. Then there exist $q \in \tilde{R}$, $p \in Q$, $b, b' \in Z$ such that

$$\tilde{\delta}_1^*((q, \mu^{t_0}(q), \nu^{t_0}(q)), x, p) \wedge \tilde{\omega}_1((p, \mu^{t_0+|x|}(p), \nu^{t_0+|x|}(p)), b) >_L \alpha,$$

and

$$\tilde{\delta}_2^*((q, \mu^{t_0}(q), \nu^{t_0}(q)), x, p) \vee \tilde{\omega}_2((p, \mu^{t_0+|x|}(p), \nu^{t_0+|x|}(p)), b') <_L \beta.$$

Since

$$\tilde{\delta}_1^*((q, \mu^{t_0}(q), \nu^{t_0}(q)), u_1 ... u_{k+1}, p) >_L \alpha \text{ and } \tilde{\delta}_2^*((q, \mu^{t_0}(q), \nu^{t_0}(q)), u_1 ... u_{k+1}, p) <_L \beta,$$

then $\mu^{t_0}(q) >_L \alpha, \nu^{t_0}(q) <_L \beta$. Thus $\mu^{t_0}(I_\Lambda) >_L \alpha$ and $\nu^{t_0}(I_\Lambda) <_L \beta$. Also

$$\tilde{\delta}_{d1}^*((I_\Lambda, \mu^{t_0}(I_\Lambda), \nu^{t_0}(I_\Lambda)), u_1, I_{u_1}) = F_1^T(\mu^{t_0}(I_\Lambda), \delta_{d1}(I_\Lambda, u_1, I_{u_1})) \geq_L \alpha,$$

thus $\mu^{t_1}(I_{u_1}) \geq_L \alpha$. Also we have

$$\tilde{\delta}_{d1}((I_{u_1}, \mu^{t_1}(I_{u_1}), \nu^{t_1}(I_{u_1})), u_2, I_{u_1 u_2}) = F_1^T(\mu^{t_1}(I_{u_1}), \delta_{d1}(I_{u_1}, u_2, I_{u_1 u_2})) \geq_L \alpha.$$

So if we continue this process, then by some manipulation we get that

$$\tilde{\delta}_{d1}^*((I_\Lambda, \mu^{t_0}(I_\Lambda), \nu^{t_0}(I_\Lambda)), x, I_x) \geq_L \alpha.$$

Also, we have

$$\tilde{\delta}_{d2}^*((I_\Lambda, \mu^{t_0}(I_\Lambda), \nu^{t_0}(I_\Lambda)), u_1, I_{u_1}) = F_1^S(\nu^{t_0}(I_\Lambda), \delta_{d2}(I_\Lambda, u_1, I_{u_1})) \leq_L \beta,$$

then $\nu^{t_1}(I_{u_1}) \leq_L \beta$. Therefore

$$\tilde{\delta}_{d2}((I_{u_1}, \mu^{t_1}(I_{u_1}), \nu^{t_1}(I_{u_1})), u_2, I_{u_1 u_2}) \leq_L \beta.$$

By continuing this process we obtain that

$$\tilde{\delta}^*_{d2}((I_\Lambda, \mu^{t_0}(I_\Lambda), \nu^{t_0}(I_\Lambda)), x, I_x) \leq_L \beta.$$

Also we have $\omega_{d1}(I_x, e) = \gamma_2$ and $\omega_{d2}(I_x, e) = \eta_2$. Hence $x \in \mathcal{L}^{\alpha,\beta}(\tilde{F}^*_d)$. Now, suppose that $x \in \mathcal{L}^{\alpha,\beta}(\tilde{F}^*_d)$. Then

$$\tilde{\delta}^*_{d1}((I_\Lambda, \mu^{t_0}(I_\Lambda), \nu^{t_0}(I_\Lambda)), x, I_x) \wedge \tilde{\omega}_{d1}((I_x, \mu^{t_0+|x|}(I_x), \nu^{t_0+|x|}(I_x)), e) >_L \alpha,$$

and

$$\tilde{\delta}^*_{d2}((I_\Lambda, \mu^{t_0}(I_\Lambda), \nu^{t_0}(I_\Lambda)), x, I_x) \vee \tilde{\omega}_{d2}((I_x, \mu^{t_0+|x|}(I_x), \nu^{t_0+|x|}(I_x)), e) <_L \beta.$$

Since $\omega_{d1}(I_x, e) >_L \alpha$ and $\omega_{d2}(I_x, e) <_L \beta$, then by (4.5) and (4.6) $x \in \mathcal{L}^{\alpha,\beta}(\tilde{F}^*)$. ∎

Theorem 4.12 *Let $\tilde{F}^* = (Q, X, \tilde{R}, Z, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4)$ be a max-min IGLFA and $\tilde{R} \neq \emptyset$. Then there exists an (α, β)-accessible max-min IGLFA $\tilde{F}^*_a$ such that $\mathcal{L}^{\alpha,\beta}(\tilde{F}^*) = \mathcal{L}^{\alpha,\beta}(\tilde{F}^*_a)$, where $\alpha, \beta \in L$ and $\alpha \leq_L N(\beta)$.*

Proof. By Theorem 4.11, without loss of generality we assume that $\tilde{F}^*$ is deterministic. Let S=$\{q \in Q \mid q$ be an (α, β)-accessible state$\}$, $\tilde{R}_a$=$\{(q, \mu^{t_0}(q), \nu^{t_0}(q)) \mid q \in S|_{\tilde{R}}\}$, $Z_a = Z$, $\delta_a = \delta|_{S\times X\times S}$, and $\omega_a = \omega|_{S\times Z}$, i.e., δ_a is the restriction of δ to $S \times X \times S$ and ω_a is the restriction of ω to $S \times Z$. Then the max-min IGLFA $\tilde{F}^*_a$ is (α, β)-accessible. It is clear that $\mathcal{L}^{\alpha,\beta}(\tilde{F}^*_a) \subseteq \mathcal{L}^{\alpha,\beta}(\tilde{F}^*)$. Now, let $x = u_1u_2...u_{k+1} \in \mathcal{L}^{\alpha,\beta}(\tilde{F}^*)$. Then there exist $q \in \tilde{R}, p \in Q, b \in Z$ such that

$$\tilde{\delta}^*_1((q, \mu^{t_0}(q), \nu^{t_0}(q)), x, p) \wedge \tilde{\omega}_1((p, \mu^{t_0+|x|}(p), \nu^{t_0+|x|}(p)), b) >_L \alpha,$$

and

$$\tilde{\delta}^*_2((q, \mu^{t_0}(q), \nu^{t_0}(q)), x, p) \vee \tilde{\omega}_2((p, \mu^{t_0+|x|}(p), \nu^{t_0+|x|}(p)), b') <_L \beta.$$

Therefore, there exist $p_1, p_2, ..., p_k, p'_1, p'_2, ..., p'_k \in Q$ such that

$$\tilde{\delta}^*_1((q, \mu^{t_0}(q), \nu^{t_0}(q)), u_1, p_1) \wedge ... \wedge \tilde{\delta}^*_1((p_k, \mu^{t_k}(p_k), \nu^{t_k}(p_k)), u_{k+1}, p) >_L \alpha,$$

and

$$\tilde{\delta}^*_2((q, \mu^{t_0}(q), \nu^{t_0}(q)), u_1, p'_1) \vee ... \vee \tilde{\delta}^*_2((p'_k, \mu^{t_k}(p'_k), \nu^{t_k}(p'_k)), u_{k+1}, p) <_L \beta.$$

These imply that $q \in \tilde{R}_a$ and, since $\tilde{F}^*$ is a deterministic, then

$$p_1 = p'_1, p_2 = p'_2, ..., p_k = p'_k.$$

Thus $p_1, p_2, ..., p_k$, $p \in S$. Hence

$$\tilde{\delta}^*_{a1}((q, \mu^{t_0}(q), \nu^{t_0}(q)), x, p) \wedge \tilde{\omega}_{a1}((p, \mu^{t_0+|x|}(p), \nu^{t_0+|x|}(p)), b) >_L \alpha,$$

and

$$\tilde{\delta}^*_{a2}((q, \mu^{t_0}(q), \nu^{t_0}(q)), x, p) \vee \tilde{\omega}_{a2}((p, \mu^{t_0+|x|}(p), \nu^{t_0+|x|}(p)), b') <_L \beta.$$

Then $x \in \mathcal{L}^{\alpha,\beta}(\tilde{F}^*_a)$. Hence $\mathcal{L}^{\alpha,\beta}(\tilde{F}^*) \subseteq \mathcal{L}^{\alpha,\beta}(\tilde{F}^*_a)$. Thus the claim is hold. ∎

Example 4.13 Consider the complete lattice $L = (L, \leq_L, T, S, 0, 1)$ defined in Example 4.3, and max-min IGLFA $\tilde{F}^* = (Q, X, \tilde{R}, Z, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4)$ as in Figure 6,

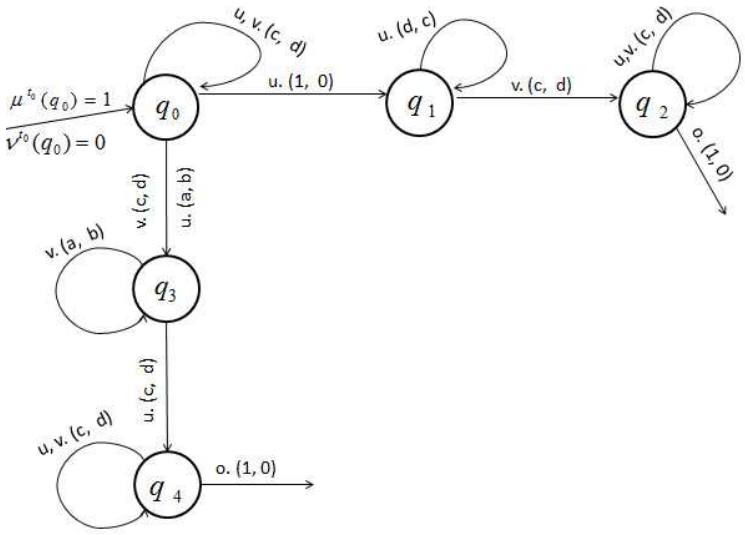


Figure 6: The IGLFA of Example 4.13

where $Q = \{q_0, q_1, q_2, q_3, q_4\}, X = \{u, v\}, \tilde{R} = \{(q_0, 1, 0)\}, Z = \{o\}$. It is clear that $\mathcal{L}^{a,b}(\tilde{F}^*) = \{u, v\}^* uv\{u, v\} \cup \{u, v\}^* vu\{u, v\}$. Considering the proof of Theorems 4.9, 4.11, 4.12, we obtain an (a, b)-complete, deterministic and (a, b)-accessible max-min IGLFA $\tilde{F}^*_{cda}$ as in Figure 7, such that $\mathcal{L}^{a,b}(\tilde{F}^*_{cda}) = \mathcal{L}^{a,b}(\tilde{F}^*)$.

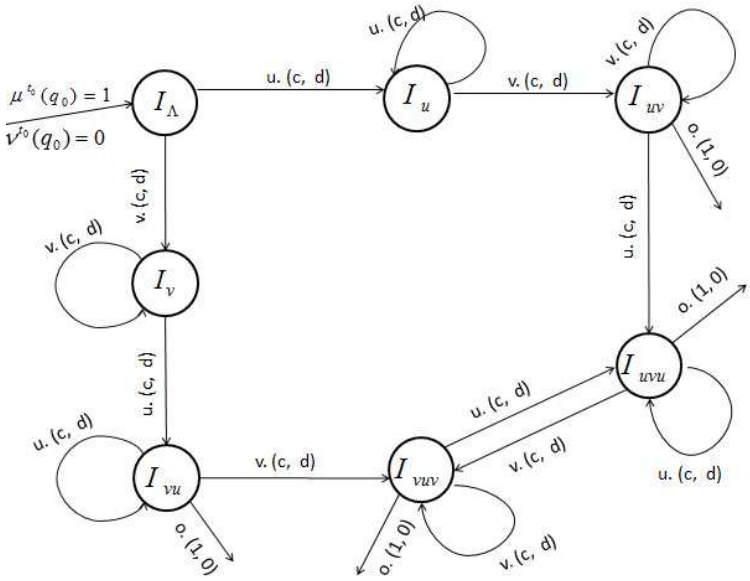


Figure 7: The (a, b)-complete, deterministic and (a, b)-accessible $\tilde{F}^*_{cda}$ of Example 4.13

Definition 4.14 Let $\tilde{F}^* = (Q, X, \{q_0\}, Z, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4)$ be an (α, β)-accessible, (α, β)-complete, deterministic max-min IGLFA, where $\alpha, \beta \in L$ and $\alpha \leq_L N(\beta)$. We define a relation on Q by $q_1 \rho^{\alpha,\beta} q_2$, if and only if the following two sets are equal

(i) $\{w \in X^* \mid \tilde{\delta}_1^*((q_1, \mu^{t_i}(q_1), \nu^{t_i}(q_1)), w, q) \wedge \tilde{\omega}_1((q, \mu^{t_i+|w|}(q), \nu^{t_i+|w|}(q)), b) >_L \alpha,$
$\tilde{\delta}_2^*((q_1, \mu^{t_i}(q_1), \nu^{t_i}(q_1)), w, q) \vee \tilde{\omega}_2((q, \mu^{t_i+|w|}(q), \nu^{t_i+|w|}(q)), b') <_L \beta,$
for some $b, b' \in Z, q \in Q\}$

and

(ii) $\{w \in X^* \mid \tilde{\delta}_1^*((q_2, \mu^{t_j}(q_2), \nu^{t_j}(q_2)), w, q) \wedge \tilde{\omega}_1((q, \mu^{t_j+|w|}(q), \nu^{t_j+|w|}(q)), b) >_L \alpha,$
$\tilde{\delta}_2^*((q_2, \mu^{t_j}(q_2), \nu^{t_j}(q_2)), w, q) \vee \tilde{\omega}_2((q, \mu^{t_j+|w|}(q), \nu^{t_j+|w|}(q)), b') <_L \beta,$
for some $b, b' \in Z, q \in Q\}$,

where $q_1 \in Q_{act}(t_i)$ $q_2 \in Q_{act}(t_j)$. It is clear that $\rho^{\alpha,\beta}$ is an equivalence relation.

Definition 4.15 We say that the (α, β)-accessible, (α, β)-complete, deterministic max-min IGLFA $\tilde{F}^*$, where $\alpha, \beta \in L$ and $\alpha \leq_L N(\beta)$, is (α, β)-reduced if $q_1 \rho^{\alpha,\beta} q_2$ implies that $q_1 = q_2$, for any $q_1, q_2 \in Q$.

Let $\tilde{F}^* = (Q, X, \{q_0\}, Z, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4)$ be an (α, β)-accessible, (α, β)-complete, deterministic max-min IGLFA, where $\alpha, \beta \in L$ and $\alpha \leq_L N(\beta)$. Now, suppose that $\rho^{\alpha,\beta}$ be the equivalence relation defined in Definition 4.14. Consider $Q/\rho^{\alpha,\beta} = \{q\rho^{\alpha,\beta} \mid q \in Q\}$ and $\tilde{R}/\rho^{\alpha,\beta} = q_0\rho^{\alpha,\beta}, \mu^{t_0}(q_0\rho^{\alpha,\beta}) = \mu^{t_0}(q_0), \nu^{t_0}(q_0\rho^{\alpha,\beta}) = \nu^{t_0}(q_0)$. We define $\delta_\rho : Q/\rho^{\alpha,\beta} \times X \times Q/\rho^{\alpha,\beta} \to L \times L$ by:

$$\delta_{\rho 1}(q_1\rho^{\alpha,\beta}, a, q_2\rho^{\alpha,\beta}) = \begin{cases} \gamma_1 & if \delta_1(q_1, a, q_2') >_L \alpha \ \& \ \delta_2(q_1, a, q_2') <_L \beta, \\ & \text{where } q_2'\rho^{\alpha,\beta} q_2 \\ 0 & \text{otherwise,} \end{cases}$$

$$\delta_{\rho 2}(q_1\rho^{\alpha,\beta}, a, q_2\rho^{\alpha,\beta}) = \begin{cases} \eta_1 & \text{if } \delta_1(q_1, a, q_2') >_L \alpha \ \& \ \delta_2(q_1, a, q_2') <_L \beta, \\ & \text{where } q_2'\rho^{\alpha,\beta} q_2 \\ 1 & \text{otherwise,} \end{cases}$$

where $\gamma_1, \eta_1 \in L, \gamma_1 >_L \alpha, \eta_1 <_L \beta$ and $\gamma_1 \leq_L N(\eta_1)$.

Define $\omega_\rho : Q/\rho^{\alpha,\beta} \times Z \to L \times L$ by:

$$\omega_{\rho 1}(q_1\rho^{\alpha,\beta}, b) = \begin{cases} \gamma_2 & if \omega_1(q_1, b') >_L \alpha \ \& \ \delta_2(q_1, b'') <_L \beta, \\ 0 & \text{otherwise,} \end{cases} \tag{4.7}$$

$$\omega_{\rho 2}(q_1\rho^{\alpha,\beta}, b) = \begin{cases} \eta_2 & if \omega_1(q_1, b') >_L \alpha \ \& \ \delta_2(q_1, b'') <_L \beta, \\ 1 & \text{otherwise,} \end{cases} \tag{4.8}$$

for some $b', b'' \in Z$ and $\gamma_2, \eta_2 \in L, \gamma_2 >_L \alpha, \eta_2 <_L \beta$ and $\gamma_2 \leq_L N(\eta_2)$.

Theorem 4.16 *Suppose that $\alpha, \beta \in L, \alpha \leq_L N(\beta)$. Then the following properties hold:*

1. *δ_ρ and ω_ρ are well defined,*
2. *$\tilde{F}^*/\rho^{\alpha,\beta} = (Q/\rho^{\alpha,\beta}, X, q_0\rho^{\alpha,\beta}, Z, \tilde{\delta}^*_\rho, \tilde{\omega}_\rho, F_1, F_2, F_3, F_4)$ is an (α,β)-reduced max-min IGLFA,*
3. *$\mathcal{L}^{\alpha,\beta}(\tilde{F}^*/\rho^{\alpha,\beta}) = \mathcal{L}^{\alpha,\beta}(\tilde{F}^*)$.*

Proof. 1. Let $q_1\rho^{\alpha,\beta}, q_2\rho^{\alpha,\beta}, p_1\rho^{\alpha,\beta}, p_2\rho^{\alpha,\beta} \in Q/\rho^{\alpha,\beta}$, $q_1\rho^{\alpha,\beta} = p_1\rho^{\alpha,\beta}$ and $q_2\rho^{\alpha,\beta} = p_2\rho^{\alpha,\beta}$. Then $q_1\rho^{\alpha,\beta}p_1$ and $q_2\rho^{\alpha,\beta}p_2$. Therefore

$$\begin{aligned}A = \{w \in X^* \mid\ & \tilde{\delta}^*_1((q_1, \mu^{t_i}(q_1), \nu^{t_i}(q_1)), w, q) \wedge \tilde{\omega}_1((q, \mu^{t_i+|w|}(q), \nu^{t_i+|w|}(q)), b) >_L \alpha,\\ & \tilde{\delta}^*_2((q_1, \mu^{t_i}(q_1), \nu^{t_i}(q_1)), w, q) \vee \tilde{\omega}_2((q, \mu^{t_i+|w|}(q), \nu^{t_i+|w|}(q)), b') <_L \beta,\\ & \text{for some } b, b' \in Z, q \in Q\} =\\ \{w \in X^* \mid\ & \tilde{\delta}^*_1((p_1, \mu^{t_j}(p_1), \nu^{t_j}(p_1)), w, q) \wedge \tilde{\omega}_1((q, \mu^{t_j+|w|}(q), \nu^{t_j+|w|}(q)), b) >_L \alpha,\\ & \tilde{\delta}^*_2((p_1, \mu^{t_j}(p_1), \nu^{t_j}(p_1)), w, q) \vee \tilde{\omega}_2((q, \mu^{t_j+|w|}(q), \nu^{t_j+|w|}(q)), b') <_L \beta,\\ & \text{for some } b, b' \in Z, q \in Q\} = B.\end{aligned}$$

Let $\delta_{\rho 1}(q_1\rho^{\alpha,\beta}, a, q_2\rho^{\alpha,\beta}) = \gamma$ and $\delta_{\rho 2}(q_1\rho^{\alpha,\beta}, a, q_2\rho^{\alpha,\beta}) = \eta$, where $\gamma, \eta \in L$ and $\gamma \leq_L N(\eta)$. Then there exists $q_2' \in Q$ such that $q_2'\rho^{\alpha,\beta}q_2$ and $\delta_1(q_1, a, q_2') >_L \alpha$ and $\delta_2(q_1, a, q_2') <_L \beta$. We have $w \in A$ if and only $w \in B$, for all $w \in X^*$. This holds in particular for all $w \in aX^*$. Then

$$\begin{aligned}\{aw \in aX^* \mid\ & \tilde{\delta}_1((q_1, \mu^{t_i}(q_1), \nu^{t_i}(q_1)), a, q_2') \wedge \tilde{\delta}^*_1((q_2', \mu^{t_{i+1}}(q_2'), \nu^{t_{i+1}}(q_2')), w, q)\\ & \wedge \tilde{\omega}_1((q, \mu^{t_i+|w|}(q), \nu^{t_i+|w|}(q)), b) >_L \alpha,\\ & \tilde{\delta}_2((q_1, \mu^{t_i}(q_1), \nu^{t_i}(q_1)), a, q_2') \vee \tilde{\delta}^*_2((q_2', \mu^{t_{i+1}}(q_2'), \nu^{t_{i+1}}(q_2')), w, q)\\ & \vee \tilde{\omega}_2((q, \mu^{t_i+|w|}(q), \nu^{t_i+|w|}(q)), b') <_L \beta,\\ & \text{for some } b, b' \in Z, q, q_2' \in Q\} =\\ \{aw \in aX^* \mid\ & \tilde{\delta}^*_1((p_1, \mu^{t_j}(p_1), \nu^{t_j}(p_1)), a, p') \wedge \tilde{\delta}^*_1((p', \mu^{t_{j+1}}(p'), \nu^{t_{j+1}}(p')), w, q)\\ & \wedge \tilde{\omega}_1((q, \mu^{t_j+|w|}(q), \nu^{t_j+|w|}(q)), b) >_L \alpha,\\ & \tilde{\delta}_2((p_1, \mu^{t_j}(p_1), \nu^{t_j}(p_1)), a, p') \vee \tilde{\delta}^*_2((p', \mu^{t_{j+1}}(p'), \nu^{t_{j+1}}(p')), w, q)\\ & \vee \tilde{\omega}_2((q, \mu^{t_j+|w|}(q), \nu^{t_j+|w|}(q)), b') <_L \beta,\\ & \text{for some } b, b' \in Z, q, p' \in Q\}.\end{aligned}$$

These imply that

$$\begin{aligned}\{w \in X^* \mid\ & \tilde{\delta}^*_1((q_2', \mu^{t_{i+1}}(q_2'), \nu^{t_{i+1}}(q_2')), w, q) \wedge \tilde{\omega}_1((q, \mu^{t_i+|w|}(q), \nu^{t_i+|w|}(q)), b) >_L \alpha,\\ & \tilde{\delta}^*_2((q_2', \mu^{t_{i+1}}(q_2'), \nu^{t_{i+1}}(q_2')), w, q) \vee \tilde{\omega}_2((q, \mu^{t_i+|w|}(q), \nu^{t_i+|w|}(q)), b') <_L \beta,\\ & \text{for some } b, b' \in Z, q \in Q\} =\\ \{w \in X^* \mid\ & \tilde{\delta}^*_1((p', \mu^{t_{j+1}}(p'), \nu^{t_{j+1}}(p')), w, q) \wedge \tilde{\omega}_1((q, \mu^{t_j+|w|}(q), \nu^{t_j+|w|}(q)), b) >_L \alpha,\\ & \tilde{\delta}^*_2((p', \mu^{t_{j+1}}(p'), \nu^{t_{j+1}}(p')), w, q) \vee \tilde{\omega}_2((q, \mu^{t_j+|w|}(q), \nu^{t_j+|w|}(q)), b') <_L \beta,\\ & \text{for some } b, b' \in Z, q \in Q\}.\end{aligned}$$

Thus $p'\rho^{\alpha,\beta}q_2'\rho^{\alpha,\beta}q_2\rho^{\alpha,\beta}p_2$ i.e., $p'\rho^{\alpha,\beta}p_2$. Also, $\tilde{F}^*$ is (α,β)-complete and deterministic, therefore $\delta_1(p_1,a,p') >_L \alpha$ and $\delta_2(p_1,a,p') <_L \beta$. Hence

$$\delta_{\rho 1}(p_1\rho^{\alpha,\beta},a,p_2\rho^{\alpha,\beta}) = \gamma \text{ and } \delta_{\rho 2}(p_1\rho^{\alpha,\beta},a,p_2\rho^{\alpha,\beta}) = \eta.$$

So δ_ρ is well defined.

Now, let $q_1\rho^{\alpha,\beta} = p_1\rho^{\alpha,\beta}$. If $\omega_{\rho 1}(q_1\rho^{\alpha,\beta},b) = \gamma'$ and $\omega_{\rho 2}(q_1\rho^{\alpha,\beta},b) = \eta'$, where $\gamma',\eta' \in L$ and $\gamma' \leq_L N(\eta')$, then $\omega_1(q_1,b') >_L \alpha$ and $\omega_2(q_1,b'') <_L \beta$, for some $b \in Z_\rho, b', b" \in Z$. Since $\tilde{F}^*$ is (α,β)-accessible, then there exists a positive integer j such that $\mu^{t_j}(q_1) >_L \alpha$ and $\nu^{t_j}(q_1) <_L \beta$. These imply that $\Lambda \in A$. Therefore $\Lambda \in B$. Then $\omega_1(q_2,b') >_L \alpha$ and $\omega_2(q_2,b'') <_L \beta$, for some $b', b" \in Z$. Hence

$$\omega_{\rho 1}(q_2\rho^{\alpha,\beta},b) = \gamma' \text{ and } \omega_{\rho 2}(q_2\rho^{\alpha,\beta},b) = \eta'.$$

Then ω_ρ is well defined.

2. Let $(q_1\rho^{\alpha,\beta})\rho^{\alpha,\beta}(q_2\rho^{\alpha,\beta})$. Now, we have to show that $q_1\rho^{\alpha,\beta} = q_2\rho^{\alpha,\beta}$. It suffices to show that $q_1\rho^{\alpha,\beta}q_2$.

Let for $w \in X^*$

$$\tilde{\delta}_1^*((q_1,\mu^{t_i}(q_1),\nu^{t_i}(q_1)),w,p_1) \wedge \tilde{\omega}_1((p_1,\mu^{t_i+|w|}(p_1),\nu^{t_i+|w|}(p_1)),b) >_L \alpha,$$
$$\tilde{\delta}_2^*((q_1,\mu^{t_i}(q_1),\nu^{t_i}(q_1)),w,p_1) \vee \tilde{\omega}_2((p_1,\mu^{t_i+|w|}(p_1),\nu^{t_i+|w|}(p_1)),b') <_L \beta,$$

for some $b,b' \in Z$. Then

$$\tilde{\delta}_{\rho 1}^*((q_1\rho^{\alpha,\beta},\mu^{t_i}(q_1\rho^{\alpha,\beta}),\nu^{t_i}(q_1\rho^{\alpha,\beta})),w,p_1\rho^{\alpha,\beta}) \wedge \tilde{\omega}_{\rho 1}((p_1\rho^{\alpha,\beta},\mu^{t_i+|w|}(p_1\rho^{\alpha,\beta}),\nu^{t_i+|w|}(p_1\rho^{\alpha,\beta})),b) >_L \alpha,$$
$$\tilde{\delta}_{\rho 2}^*((q_1\rho^{\alpha,\beta},\mu^{t_i}(q_1\rho^{\alpha,\beta}),\nu^{t_i}(q_1\rho^{\alpha,\beta})),w,p_1\rho^{\alpha,\beta}) \vee \tilde{\omega}_{\rho 2}((p_1\rho^{\alpha,\beta},\mu^{t_i+|w|}(p_1\rho^{\alpha,\beta}),\nu^{t_i+|w|}(p_1\rho^{\alpha,\beta})),b') <_L \beta,$$

for some $b,b' \in Z$. Therefore

$$\tilde{\delta}_{\rho 1}^*((q_2\rho^{\alpha,\beta},\mu^{t_j}(q_2\rho^{\alpha,\beta}),\nu^{t_j}(q_2\rho^{\alpha,\beta})),w,p_1'\rho^{\alpha,\beta}) \wedge \tilde{\omega}_{\rho 1}((p_1'\rho^{\alpha,\beta},\mu^{t_j+|w|}(p_1'\rho^{\alpha,\beta}),\nu^{t_j+|w|}(p_1'\rho^{\alpha,\beta})),b) >_L \alpha,$$
$$\tilde{\delta}_{\rho 2}^*((q_2\rho^{\alpha,\beta},\mu^{t_j}(q_2\rho^{\alpha,\beta}),\nu^{t_j}(q_2\rho^{\alpha,\beta})),w,p_1'\rho^{\alpha,\beta}) \vee \tilde{\omega}_{\rho 2}((p_1'\rho^{\alpha,\beta},\mu^{t_j+|w|}(p_1'\rho^{\alpha,\beta}),\nu^{t_j+|w|}(p_1'\rho^{\alpha,\beta})),b') <_L \beta,$$

$p_1'\rho^{\alpha,\beta} \in Q/\rho^{\alpha,\beta}$ for some $b,b' \in Z$. Since $\tilde{F}^*$ is (α,β)-accessible, then there exists a positive integer j' such that $\mu^{t_{j'}}(q_2) >_L \alpha$ and $\nu^{t_{j'}}(q_2) <_L \beta$. Thus

$$\tilde{\delta}_1^*((q_2,\mu^{t_{j'}}(q_2),\nu^{t_{j'}}(q_2)),w,p) \wedge \tilde{\omega}_1((p,\mu^{t_{j'}+|w|}(p),\nu^{t_{j'}+|w|}(p)),b) >_L \alpha,$$
$$\tilde{\delta}_2^*((q_2,\mu^{t_{j'}}(q_2),\nu^{t_{j'}}(q_2)),w,p) \vee \tilde{\omega}_2((p,\mu^{t_{j'}+|w|}(p),\nu^{t_{j'}+|w|}(p)),b') <_L \beta,$$

where $p\rho^{\alpha,\beta}p'_1$ for some $b, b' \in Z$. The converse follows in a similar manner. Hence

$$\{w \in X^* \mid \tilde{\delta}_1^*((q_1, \mu^{t_i}(q_1), \nu^{t_i}(q_1)), w, q) \wedge \tilde{\omega}_1((q, \mu^{t_i+|w|}(q), \nu^{t_i+|w|}(q)), b) >_L \alpha,$$
$$\tilde{\delta}_2^*((q_1, \mu^{t_i}(q_1), \nu^{t_i}(q_1)), w, q) \vee \tilde{\omega}_2((q, \mu^{t_i+|w|}(q), \nu^{t_i+|w|}(q)), b') <_L \beta,$$
$$\text{for some } b, b' \in Z, q \in Q\} =$$
$$\{w \in X^* \mid \tilde{\delta}_1^*((q_2, \mu^{t_j}(q_2), \nu^{t_j}(q_2)), w, q) \wedge \tilde{\omega}_1((q, \mu^{t_j+|w|}(q), \nu^{t_j+|w|}(q)), b) >_L \alpha,$$
$$\tilde{\delta}_2^*((q_2, \mu^{t_j}(q_2), \nu^{t_j}(q_2)), w, q) \vee \tilde{\omega}_2((q, \mu^{t_j+|w|}(q), \nu^{t_j+|w|}(q)), b') <_L \beta,$$
$$\text{for some } b, b' \in Z, q \in Q\}.$$

Hence the claim is hold.

3. Considering the proof of 2, it is trivial. ∎

Example 4.17 Consider the complete lattice $L = (L, \leq_L, T, S, 0, 1)$ defined in Example 4.3, and the max-min IGLFA $\tilde{F}^*_{cda} = (Q, X, \tilde{R}, Z, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4)$ as in Figure 7. By the Definition 4.14, $I_{uv}\rho^{a,b}I_{uvu}\rho^{a,b}I_{vuv}\rho^{a,b}I_{vu}$. Then the max-min IGLFA $\tilde{F}^*/\rho^{a,b}$ has the state diagram as in Figure 8.

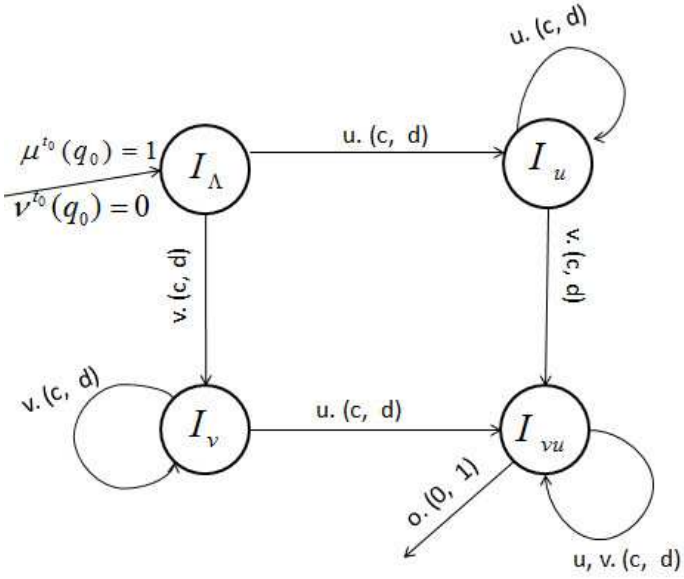


Figure 8: The reduced $\tilde{F}^*/\rho^{a,b}$ of Example 4.17

Definition 4.18 Let

$$\tilde{F}_1^* = (Q_1, X, \tilde{R}_1, Z_1, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4) \quad \text{and}$$
$$\tilde{F}_2^* = (Q_2, X, \tilde{R}_2, Z_2, \tilde{\delta}'^*, \tilde{\omega}', F_1, F_2, F_3, F_4)$$

be two max-min IGLFA. A homomorphism from $\tilde{F}_1^*$ onto $\tilde{F}_2^*$ with threshold (α, β), where $\alpha, \beta \in L$ and $\alpha \leq_L N(\beta)$, is a function ξ from Q_1 onto Q_2 such that for every $q', q'' \in Q_1$, $u \in X$ and $b_1, b_2 \in Z$ the following conditions hold:

1. $(\mu^{t_0}_{Q_1}(q') >_L \alpha \ \&\ \nu^{t_0}_{Q_1}(q') <_L \beta)$
   if and only if $(\mu^{t_0}_{Q_2}(\xi(q')) >_L \alpha \ \&\ \nu^{t_0}_{Q_2}(\xi(q')) <_L \beta)$,
2. $(\delta_1(q', u, q") >_L \alpha \ \&\ \delta_2(q', u, q") <_L \beta)$
   if and only if $(\delta'_1(\xi(q'), u, \xi(q")) >_L \alpha \ \&\ \delta'_2(\xi(q'), u, \xi(q")) <_L \beta)$,
3. $(\omega_1(q', b_1) >_L \alpha \ \&\ \omega_2(q', b_2) <_L \beta)$
   implies $(\omega'_1(\xi(q'), b) >_L \alpha \ \&\ \omega_2(\xi(q'), b_2) <_L \beta)$ for some $b, b' \in Z'$.

We say that ξ is an isomorphism with threshold (α, β) if and only if ξ is a homomorphism with threshold (α, β) that is one-one and $(\omega_1(q', b_1) >_L \alpha$ and $\omega_2(q', b_2) <_L \beta)$ if and only if $(\omega'_1(\xi(q'), b) >_L \alpha$ and $\omega'_2(\xi(q'), b_2) <_L \beta)$, for some $b, b' \in Z'$.

Definition 4.19 Let $\mathcal{L}$ be an (α, β)-language, where $\alpha, \beta \in L$ and $\alpha \leq_L N(\beta)$. A relation $R_{\mathcal{L}}$ on X^* is defined by:

for any two strings x and y in X^*, $xR_{\mathcal{L}}y$ if, for all $z \in X^*$,
either $xz, yz \in \mathcal{L}$ or $xz, yz \notin \mathcal{L}$.

Definition 4.20 Let $\tilde{F}^* = (Q, X, \{q_0\}, Z, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4)$ be a deterministic max-min IGLFA. Then for any $\alpha, \beta \in L$, where $\alpha \leq_L N(\beta)$, define a relation $R_F^{\alpha,\beta}$ on X^*. For strings x and y in X^*, $xR_F^{\alpha,\beta}y$ if and only if there exists $q \in Q$ such that

$$\tilde{\delta}^*_1((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), x, q) >_L \alpha \quad \text{and} \quad \tilde{\delta}^*_2((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), x, q) <_L \beta,$$

iff

$$\tilde{\delta}^*_1((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), y, q) >_L \alpha \quad \text{and} \quad \tilde{\delta}^*_2((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), y, q) <_L \beta.$$

Now, we show that $R_F^{\alpha,\beta}$ is an equivalence relation. It is clear that $xR_F^{\alpha,\beta}x$ and if $xR_F^{\alpha,\beta}y$, then $yR_F^{\alpha,\beta}x$.

Now, let $xR_F^{\alpha,\beta}y$ and $yR_F^{\alpha,\beta}z$. Suppose that there exists $q \in Q$ such that

$$\tilde{\delta}^*_1((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), x, q) >_L \alpha \text{ and } \tilde{\delta}^*_2((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), x, q) <_L \beta.$$

Then

$$\tilde{\delta}^*_1((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), y, q) >_L \alpha \ \text{ and } \ \tilde{\delta}^*_2((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), y, q) <_L \beta,$$

and, since $yR_F^{\alpha,\beta}z$, then

$$\tilde{\delta}^*_1((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), z, q) >_L \alpha \ \text{ and } \ \tilde{\delta}^*_2((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), z, q) <_L \beta.$$

Similarity we can obtain the converse, i.e.,

$$\tilde{\delta}^*_1((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), z, q) >_L \alpha \ \text{ and } \ \tilde{\delta}^*_2((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), z, q) <_L \beta,$$

implies that

$$\tilde{\delta}^*_1((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), x, q) >_L \alpha \text{ and } \tilde{\delta}^*_2((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), x, q) <_L \beta.$$

Then $xR_F^{\alpha,\beta}z$. Hence the claim is hold.

Note 4.21 By Definition 4.20, for any (α,β)-complete and deterministic max-min IGLFA $\tilde{F}^*$, the number of classes of equivalence relation $R_F^{\alpha,\beta}$ is less than or equal to the number of states of $\tilde{F}^*$.

Theorem 4.22 *Let $\alpha,\beta \in L$, where $\alpha \leq_L N(\beta)$. Suppose that $\mathcal{L}$ be a recognizable (α,β)-language where $\mathcal{L} = \mathcal{L}^{\alpha,\beta}(\tilde{F}^*)$, for some (α,β)-complete and deterministic max-min IGLFA $\tilde{F}^* = (Q, X, \{q_0\}, Z, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4)$. Then for a given equivalence class $[w]_{R_F^{\alpha,\beta}}$ of $R_F^{\alpha,\beta}$ there is an equivalence class $[w]_{R_\mathcal{L}}$ of $R_\mathcal{L}$ such that $[w]_{R_F^{\alpha,\beta}} \subseteq [w]_{R_\mathcal{L}}$. Each equivalence class $[w]_{R_\mathcal{L}}$ of the relation $R_\mathcal{L}$ is a finite union of equivalence classes of $R_F^{\alpha,\beta}$.*

Proof. Let $[w]_{R_F^{\alpha,\beta}}$ be an equivalence class of $R_F^{\alpha,\beta}$. Now, suppose that $x \in [w]_{R_F^{\alpha,\beta}}$. Since $\tilde{F}^*$ is an (α,β)-complete, then there exists $q \in Q$ such that

$$\tilde{\delta}_1^*((q_0,\mu^{t_0}(q_0),\nu^{t_0}(q_0)),x,q) >_L \alpha \text{ and } \tilde{\delta}_2^*((q_0,\mu^{t_0}(q_0),\nu^{t_0}(q_0)),x,q) <_L \beta.$$

Therefore

$$\tilde{\delta}_1^*((q_0,\mu^{t_0}(q_0),\nu^{t_0}(q_0)),w,q) >_L \alpha \text{ and } \tilde{\delta}_2^*((q_0,\mu^{t_0}(q_0),\nu^{t_0}(q_0)),w,q) <_L \beta.$$

By the (α,β)-complete property of $\tilde{F}^*$, for any $y \in X^*$, there exists $q' \in Q$ such that

$$\tilde{\delta}_1^*((q_0,\mu^{t_0}(q_0),\nu^{t_0}(q_0)),xy,q') \wedge \tilde{\delta}_1^*((q_0,\mu^{t_0}(q_0),\nu^{t_0}(q_0)),wy,q') >_L \alpha,$$

and

$$\tilde{\delta}_2^*((q_0,\mu^{t_0}(q_0),\nu^{t_0}(q_0)),xy,q') \vee \tilde{\delta}_2^*((q_0,\mu^{t_0}(q_0),\nu^{t_0}(q_0)),wy,q') <_L \beta.$$

If there exist $b,b' \in Z$ such that $\omega_1(q',b) >_L \alpha$ and $\omega_2(q',b) <_L \beta$, then $xy, wy \in \mathcal{L}$ otherwise $xy, wy \notin \mathcal{L}$. Then $xR_\mathcal{L}w$. Thus $x \in [w]_{R_\mathcal{L}}$ and $[w]_{R_F^{\alpha,\beta}} \subseteq [w]_{R_\mathcal{L}}$. Suppose that $[w]_{R_\mathcal{L}}$ is an equivalence class of $R_\mathcal{L}$. It is obvious that if $w \notin \mathcal{L}$, then for any $x \in [w]_{R_\mathcal{L}}$, $x \notin \mathcal{L}$ and if $w \in \mathcal{L}$, then for any $x \in [w]_{R_\mathcal{L}}$, $x \in \mathcal{L}$. If $w \notin \mathcal{L}$, then for any $q \in Q, b, b' \in Z$ we have

$$\tilde{\delta}_1^*((q_0,\mu^{t_0}(q_0),\nu^{t_0}(q_0)),w,q) \wedge \tilde{\omega}_1((q,\mu^{t_0+|w|}(q),\nu^{t_0+|w|}(q)),b) \leq_L \alpha,$$

or

$$\tilde{\delta}_2^*((q_0,\mu^{t_0}(q_0),\nu^{t_0}(q_0)),w,q) \vee \tilde{\omega}_2((q,\mu^{t_0+|w|}(q),\nu^{t_0+|w|}(q)),b') \geq_L \beta.$$

If $\tilde{\delta}_1^*((q_0,\mu^{t_0}(q_0),\nu^{t_0}(q_0)),w,q) >_L \alpha$ and $\tilde{\delta}_2^*((q_0,\mu^{t_0}(q_0),\nu^{t_0}(q_0)),w,q) <_L \beta$, then

$$\omega_1(q,b) \leq_L \alpha \quad \text{or} \quad \omega_2(q,b) \geq_L \beta.$$

The set

$$\begin{aligned} S = \{q \in Q \mid & \tilde{\delta}_1^*((q_0,\mu^{t_0}(q_0),\nu^{t_0}(q_0)),x,q) >_L \alpha, \text{ and} \\ & \tilde{\delta}_2^*((q_0,\mu^{t_0}(q_0),\nu^{t_0}(q_0)),x,q) <_L \beta, x \in [\omega]_{R_L}\}, \end{aligned} \tag{4.9}$$

is finite. Then we have $\omega_1(q,b) \leq_L \alpha$ or $\omega_2(q,b) \geq_L \beta$ for all $q \in S$ and $b \in Z$, or for all $q \in S$ there exist $b,b' \in Z$ such that $\omega_1(q,b) >_L \alpha$ and $\omega_2(q,b') <_L \beta$. Therefore the equivalence class $[w]_{R_\mathcal{L}}$ of $R_\mathcal{L}$ is a finite union of the equivalence classes $[w]_{R_F^{\alpha,\beta}}$ of $R_F^{\alpha,\beta}$, where $q \in S$. ∎

Theorem 4.23 *Let $\mathcal{L}$ be a recognizable (α,β)-language. Then there is an (α,β)-complete and deterministic IGLFA $\tilde{F}_m^*$ such that $\mathcal{L}^{\alpha,\beta}(\tilde{F}_m^*) = \mathcal{L}$ and $\tilde{F}_m^*$ is a minimal automaton, where $\alpha,\beta \in L$ and $\alpha \leq_L N(\beta)$.*

Proof. Let $\tilde{F}^* = (Q, X, \tilde{R}, Z, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4)$ be an (α,β)-complete and deterministic max-min IGLFA such that $\mathcal{L} = \mathcal{L}^{\alpha,\beta}(\tilde{F}^*)$. By Theorem 4.22, we have that the number of equivalence classes of $R_{\mathcal{L}}$ is finite. Let Q_m be the set of equivalence classes of $R_{\mathcal{L}}$ i.e., $Q_m = \{[w] \mid [w] \text{ is an equivalence class of } R_{\mathcal{L}}\}$. Let $\tilde{R}_m = \{([\Lambda], \gamma_1, \gamma_2)\}$, where $\gamma_1, \gamma_2 \in L, \gamma_1 >_L \alpha, \gamma_2 <_L \beta$ and $\gamma_1 \leq_L N(\gamma_2)$. Define $\delta_m : Q_m \times X \times Q_m \to L \times L$ by

$$\delta_{m1}([z], a, [x]) = \begin{cases} \gamma_1 & \text{if } [za] = [x], \\ 0 & \text{otherwise,} \end{cases} \tag{4.10}$$

and

$$\delta_{m2}([z], a, [x]) = \begin{cases} \gamma_2 & \text{if } [za] = [x], \\ 1 & \text{otherwise,} \end{cases} \tag{4.11}$$

Also define $\omega_m : Q_m \times \{b\} \to L \times L$ by

$$\omega_{m1}([w], b) = \begin{cases} \gamma_1 & \text{if } w \in \mathcal{L}, \\ 0 & \text{otherwise,} \end{cases} \tag{4.12}$$

and

$$\omega_{m2}([w], b) = \begin{cases} \gamma_2 & \text{if } w \in \mathcal{L}, \\ 1 & \text{otherwise.} \end{cases} \tag{4.13}$$

It is clear that $\tilde{F}_m^* = (Q_m, X, \tilde{R}_m, Z' = \{b\}, \tilde{\delta}_m^*, \tilde{\omega}_m, F_1, F_2, F_3, F_4)$ is an (α,β)-complete and deterministic IGLFA. Clearly, δ_m and ω_m are well defined.

Let $w \in \mathcal{L}$. Then, we have

$$\delta_{m1}^*(([\Lambda], \mu^{t_0}([\Lambda]), \nu^{t_0}([\Lambda])), w, [w]) >_L \alpha \quad \text{and}$$
$$\delta_{m2}^*(([\Lambda], \mu^{t_0}([\Lambda]), \nu^{t_0}([\Lambda])), w, [w]) <_L \beta,$$

and

$$\omega_{m1}([w], b) = \gamma_1 >_L \alpha, \ \omega_{m2}([w], b) = \gamma_2 <_L \beta.$$

These imply that $w \in \mathcal{L}^{\alpha,\beta}(\tilde{F}_m^*)$.

Now, suppose that $w \in \mathcal{L}^{\alpha,\beta}(\tilde{F}_m^*)$. Then there exist $[z] \in Q_m, b \in Z$ such that

$$\tilde{\delta}_{m1}^*(([\Lambda], \mu^{t_0}([\Lambda]), \nu^{t_0}([\Lambda])), w, [z]) \wedge \tilde{\omega}_{m1}(([z], \mu^{t_0+|w|}([z]), \nu^{t_0+|w|}([z])), b) >_L \alpha, \tag{4.14}$$

and

$$\tilde{\delta}_{m2}^*(([\Lambda], \mu^{t_0}([\Lambda]), \nu^{t_0}([\Lambda])), w, [z]) \vee \tilde{\omega}_{m2}(([z], \mu^{t_0+|w|}([z]), \nu^{t_0+|w|}([z])), b) <_L \beta. \tag{4.15}$$

Considering (4.10), (4.11), (4.14) and (4.15) we have $[z] = [w]$. Then $\omega_{m1}([w], b) >_L \alpha$ and $\omega_{m2}([w], b) <_L \beta$. Then $w \in \mathcal{L}$. Hence $\mathcal{L}^{\alpha}(\tilde{F}^*_m) = \mathcal{L}$. By Theorem 4.22, we have that the number of equivalence classes of $R_{\mathcal{L}}$ is less than or equal to that of equivalence classes of $R_F^{\alpha,\beta}$, and by Note 4.21, the number of classes of equivalence relation $R_F^{\alpha,\beta}$ is less than or equal to the number of states of $\tilde{F}^*$. Hence the claim holds. ∎

Example 4.24 Let the complete lattice $L = (L, \leq_L, T, S, 0, 1)$ defined in Example 4.3, and the max-min IGLFA $\tilde{F}^* = (Q, X, \tilde{R}, Z, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4)$ as in Figure 6. Considering Definition 4.19, we find $[u]$, $[v]$, $[uv] = [vu]$, $[u^2] = [u]$, $[v^2] = [v]$, $[uv^2] = [uv]$ and $[uvu] = [uv]$. Then we have $\tilde{F}^*_m$ as in Figure 9. It is

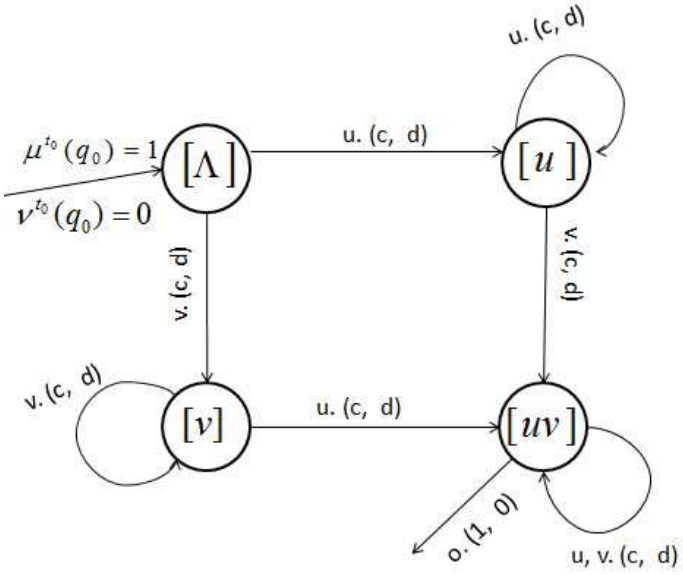


Figure 9: The minimal $\tilde{F}^*_m$ of Example 4.24

obvious that $\tilde{F}^*/\rho^{a,b}$ as in Figure 8, and $\tilde{F}^*_m$ as in Figure 9, have the same number of states.

Theorem 4.25 *For every recognizable $(\alpha, \beta)-$language $\mathcal{L}$, the minimal max-min IGLFA $\tilde{F}^*$ defined in proof of Theorem 4.23, is (α, β)-reduced, where $\alpha, \beta \in L$ and $\alpha \leq_L N(\beta)$.*

Proof. Let $\tilde{F}^*$ be an (α, β)-accessible, (α, β)-complete, deterministic max-min IGLFA. Suppose that $q_1 = [u_1], q_2 = [u_2]$ be equivalence classes of $R_{\mathcal{L}}$. Now, let $q_1 \rho^{\alpha,\beta} q_2$. Then

$$\begin{aligned} A=\{w \in X^* | & \tilde{\delta}^*_1(([u_1], \mu^{t_i}([u_1]), \nu^{t_i}([u_1])), w, q) \wedge \tilde{\omega}_1((q, \mu^{t_i+|w|}(q), \nu^{t_i+|w|}(q)), b) >_L \alpha, \\ & \tilde{\delta}^*_2(([u_1], \mu^{t_i}([u_1]), \nu^{t_i}([u_1])), w, q) \vee \tilde{\omega}_2((q, \mu^{t_i+|w|}(q), \nu^{t_i+|w|}(q)), b') <_L \beta, \\ & \text{for some } b, b' \in Z, q \text{ is an equivalence classes of } R_{\mathcal{L}}\} \\ =\{w \in X^* | & \tilde{\delta}^*_1(([u_2], \mu^{t_j}([u_2]), \nu^{t_j}([u_2])), w, q) \wedge \tilde{\omega}_1((q, \mu^{t_j+|w|}(q), \nu^{t_j+|w|}(q)), b) >_L \alpha, \\ & \tilde{\delta}^*_2(([u_2], \mu^{t_j}([u_2]), \nu^{t_j}([u_2])), w, q) \vee \tilde{\omega}_2((q, \mu^{t_j+|w|}(q), \nu^{t_j+|w|}(q)), b') <_L \beta, \end{aligned}$$

for some $b, b' \in Z, q$ is an equivalence classes of $R_{\mathcal{L}}\} = B$.

Therefore $w \in A \iff w \in B$, for all $w \in X^*$, this implies that $u_1 w \in \mathcal{L} \iff u_2 w \in \mathcal{L}$, for all $w \in X^*$. Then $[u_1] = [u_2]$. Hence $\tilde{F}^*$ is an (α, β)-reduced. ∎

Theorem 4.26 *Let $\mathcal{L}$ be a recognizable (α, β)-language, where $\alpha, \beta \in L$ and $\alpha \leq_L N(\beta)$. Let $\tilde{F}_m^*$ be the max-min IGLFA defined in the proof of Theorem* 4.23*, and $\tilde{F}^*$ be an (α, β)-complete, (α, β)-accessible, deterministic and (α, β)-reduced max-min IGLFA. Then $\tilde{F}_m^*$ and $\tilde{F}^*$ are isomorphism with threshold (α, β).*

Proof. Let $\tilde{F}_m^* = (Q_,, X, \{[\Lambda]\}, \{b\}, \tilde{\delta}_m^*, \tilde{\omega}_m, F_1, F_2, F_3, F_4)$ and $\tilde{F}^* = (Q, X, \{q_0\}, Z, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4)$. Define $\xi : Q \to Q_m$ by $\xi(q) = [u]$, where

$$\tilde{\delta}_1^*((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), u, q) >_L \alpha \text{ and } \tilde{\delta}_2^*((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), u, q) <_L \beta.$$

Since $\tilde{F}^*$ is (α, β)-accessible, then $\mu_Q^{t_0}(q_0) >_L \alpha$ and $\nu_Q^{t_0}(q_0) <_L \beta$.
Also, $\mu_{Q_m}^{t_0}([\Lambda]) >_L \alpha$ and $\nu_{Q_m}^{t_0}([\Lambda]) <_L \beta$.
Let $q_1, q_2 \in Q$ and $q_1 = q_2$. Then $q_1 \rho^{\alpha,\beta} q_2$. Therefore

$$\begin{aligned}
\{w \in X^* \mid\ & \tilde{\delta}_1^*((q_1, \mu^{t_i}(q_1), \nu^{t_i}(q_1)), w, q) \wedge \tilde{\omega}_1((q, \mu^{t_i+|w|}(q), \nu^{t_i+|w|}(q)), b) >_L \alpha, \\
& \tilde{\delta}_2^*((q_1, \mu^{t_i}(q_1), \nu^{t_i}(q_1)), w, q) \vee \tilde{\omega}_2((q, \mu^{t_i+|w|}(q), \nu^{t_i+|w|}(q)), b') <_L \beta, \\
& \text{for some } b, b' \in Z, q \in Q\} = \\
\{w \in X^* \mid\ & \tilde{\delta}_1^*((q_2, \mu^{t_j}(q_2), \nu^{t_j}(q_2)), w, q) \wedge \tilde{\omega}_1((q, \mu^{t_j+|w|}(q), \nu^{t_j+|w|}(q)), b) >_L \alpha, \\
& \tilde{\delta}_2^*((q_2, \mu^{t_j}(q_2), \nu^{t_j}(q_2)), w, q) \vee \tilde{\omega}_2((q, \mu^{t_j+|w|}(q), \nu^{t_j+|w|}(q)), b') <_L \beta, \\
& \text{for some } b, b' \in Z, q \in Q\}.
\end{aligned}$$

Since $\mu^{t_i}(q_1) >_L \alpha, \nu^{t_i}(q_1) <_L \beta$ and $\mu^{t_j}(q_2) >_L \alpha, \nu^{t_j}(q_2) <_L \beta$, then there exist $u, v \in X^*$, where $|u| = i, |v| = j$, and

$$\tilde{\delta}_1^*((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), u, q_1) >_L \alpha \text{ and } \tilde{\delta}_2^*((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), u, q_1) <_L \beta,$$

and

$$\tilde{\delta}_1^*((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), v, q_2) >_L \alpha \text{ and } \tilde{\delta}_2^*((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), v, q_2) <_L \beta.$$

Thus $uz \in \mathcal{L}$ if and only if $vz \in \mathcal{L}$ for all $z \in X^*$. Then $[u] = [v]$, i.e., $\xi(q_1) = \xi(q_2)$. Hence ξ is well defined.

Let $[u] \in Q_m$. By the (α, β)-complete property of $\tilde{F}^*$, there exists $q \in Q$ such that

$$\tilde{\delta}_1^*((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), u, q) >_L \alpha \text{ and } \tilde{\delta}_2^*((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), u, q) <_L \beta.$$

Then $\xi(q) = [u]$. Therefore ξ is surjective.

Now, let $\delta_1(q, a, q') >_L \alpha$ and $\delta_2(q, a, q') <_L \beta$. Since $\tilde{F}^*$ is (α, β)- accessible, then there exists $u \in X^*$ such that

$$\tilde{\delta}_1^*((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), u, q) >_L \alpha \text{ and } \tilde{\delta}_2^*((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), u, q) <_L \beta.$$

Therefore

$$\tilde{\delta}_1^*((q_0,\mu^{t_0}(q_0),\nu^{t_0}(q_0)),ua,q') >_L \alpha \text{ and } \tilde{\delta}_2^*((q_0,\mu^{t_0}(q_0),\nu^{t_0}(q_0)),ua,q') <_L \beta.$$

Hence $\xi(q)=[u]$ and $\xi(q')=[ua]$ and $\delta_{m1}(\xi(q),a,\xi(q')) >_L \alpha$, $\delta_{m2}(\xi(q),a,\xi(q')) <_L \beta$.

Let $\delta_{m1}(\xi(q),a,\xi(q')) >_L \alpha$ and $\delta_{m2}(\xi(q),a,\xi(q')) <_L \beta$, where $\xi(q)=[u]$ and $\xi(q')=[v]$. Then $[ua]=[v]$. Thus

$$\tilde{\delta}_1^*((q_0,\mu^{t_0}(q_0),\nu^{t_0}(q_0)),u,q) >_L \alpha \text{ and } \tilde{\delta}_2^*((q_0,\mu^{t_0}(q_0),\nu^{t_0}(q_0)),u,q) <_L \beta,$$

and

$$\tilde{\delta}_1^*((q_0,\mu^{t_0}(q_0),\nu^{t_0}(q_0)),ua,q') >_L \alpha \text{ and } \tilde{\delta}_2^*((q_0,\mu^{t_0}(q_0),\nu^{t_0}(q_0)),ua,q') <_L \beta.$$

Since $\tilde{F}^*$ is deterministic, then $\delta_1(q,a,q') >_L \alpha$ and $\delta_2(q,a,q') <_L \beta$.

Let $q \in Q$ and $\omega_1(q,b) >_L \alpha$ and $\omega_2(q,b') <_L \beta$ for some $b,b' \in Z$. By the (α,β)-accessibility property

$$\tilde{\delta}_1^*((q_0,\mu^{t_0}(q_0),\nu^{t_0}(q_0)),u,q) >_L \alpha \text{ and } \tilde{\delta}_2^*((q_0,\mu^{t_0}(q_0),\nu^{t_0}(q_0)),u,q) <_L \beta.$$

Then $u \in \mathcal{L}$. Therefore $\omega_{m1}([u],b) >_L \alpha$ and $\omega_{m2}([u],b') <_L \beta$ for some $b,b' \in Z$. Hence $\omega_{m1}(\xi(q),b) >_L \alpha$ and $\omega_{m2}(\xi(q),b') <_L \beta$ for some $b,b' \in Z$.

Then $\tilde{F}_m^*$ and $\tilde{F}^*$ are homomorphism with threshold (α,β). Now, let $q_1,q_2 \in Q$ and $\xi(q_1)=\xi(q_2)$. Then there exist $x,y \in X^*$ such that $[x]=\xi(q_1)=\xi(q_2)=[y]$. Therefore $q_1\rho^{\alpha,\beta}q_2$. Since $\tilde{F}^*$ is (α,β)-reduced, then $q_1=q_2$. Thus ξ is one-one.

Now, let $\omega_{m1}(\xi(q),b) >_L \alpha$ and $\omega_{m2}(\xi(q),b') <_L \beta$ for some $b,b' \in Z$, where $\xi(q)=[u]$. On the other hand, $\omega_{m1}([u],b) >_L \alpha$ and $\omega_{m2}([u],b') <_L \beta$ for some $b,b' \in Z$. These imply that $u \in \mathcal{L}$. Then

$$\tilde{\delta}_1^*((q_0,\mu^{t_0}(q_0),\nu^{t_0}(q_0)),u,q') \wedge \tilde{\omega}_1((q',\mu^{t_0+|u|}(q'),\nu^{t_0+|u|}(q')),b) >_L \alpha,$$
$$\tilde{\delta}_2^*((q_0,\mu^{t_0}(q_0),\nu^{t_0}(q_0)),u,q') \vee \tilde{\omega}_2((q',\mu^{t_0+|u|}(q'),\nu^{t_0+|u|}(q')),b') <_L \beta,$$

for some $b,b' \in Z$. Also, we have

$$\tilde{\delta}_1^*((q_0,\mu^{t_0}(q_0),\nu^{t_0}(q_0)),u,q) >_L \alpha \text{ and } \tilde{\delta}_2^*((q_0,\mu^{t_0}(q_0),\nu^{t_0}(q_0)),u,q) <_L \beta.$$

Since $\tilde{F}^*$ is deterministic, then $q=q'$. Therefore $\omega_1(q,b) >_L \alpha$ and $\omega_2(q,b') <_L \beta$ for some $b,b' \in Z$.

Hence $\tilde{F}_m^*$ and $\tilde{F}^*$ are isomorphism with threshold (α,β). ∎

5. Conclusion

In this paper, we present the notions of intuitionistic general L-fuzzy automata based on lattice valued intuitionistic fuzzy sets, (α,β)-language, (α,β)-accessible, (α,β)-reduced, (α,β)-complete, deterministic and isomorphic with threshold (α,β). Afterthat, it is shown that for any recognizable (α,β)-language over a bounded lattice L, there exists a minimal (α,β)-complete and deterministic IGLFA, which

preserve (α, β)-language. Finally, we show that for any given (α, β)-language $\mathcal{L}$, the minimal (α, β)-complete and deterministic IGLFA recognizing $\mathcal{L}$ is isomorphic with threshold (α, β) to any (α, β)-complete, (α, β)-accessible, deterministic, (α, β)-reduced IGLFA recognizing $\mathcal{L}$.

Now, there is an important question:

> Suppose that there is a max-min IGLFA, say $\tilde{F}^*$. Is there a minimal intuitionistic general L-fuzzy automata with more that one initial state, in which it has the same (α, β)-language as the (α, β)-language of $\tilde{F}^*$?

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RECOGNIZING CHEVALLEY GROUPS $G_2(q)$ BY nse

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Abstract. Let $G_2(q)$ be a Chevalley group over a finite field $K = F_q$ of characteristic p. For a group G, let $\mathrm{nse}(G) = \{s_k \big| k \in \omega(G)\}$ where $\omega(G)$ is the set of element orders of G and s_k is the number of elements of order k in G. In this note, we give a new characterization of some special Chevalley groups $G_2(q)$ by nse.

Keywords and Phrases: Element order, Alternating group, Thompson's conjecture, Conjugacy classes, Simple group.

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1. Introduction

For a finite group G, let $\omega(G)$ be the set of element orders of G. If $k \in \omega(G)$ and s_k be the number of elements of order k in G, then let $\mathrm{nse}(G) = \{s_k \big| k \in \omega(G)\}$. J. G. Thompson put forward a very interesting problem related to algebraic number fields as follows (see [22], for instance).

Thompson's Problem. *Let $T(G) = \{(n, s_n) \big| n \in \omega(G)$ and $s_n \in \mathrm{nse}(G)\}$, where s_n is the number of elements with order n. Suppose that $T(G) = T(H)$. If G is a finite solvable group, is it true that H is also necessarily solvable?*

So, some authors consider these cases (see [14], [19], and [20]).

If we consider only $\mathrm{nse}(G)$, then whether can it characterize finite simple groups? Some groups $PSL(2, r)$ and S_r, where r is a prime and S_r is a symmetric group of degree r, are proved valid by nse (see [2], [3]). Recently, alternating

groups A_p of special class under condition that $p \mid |G|$ but $p^2 \nmid |G|$, are characterized by nse only (see [1]). In this note, we give a new characterization of Chevalley groups $G_2(q)$ where $q \equiv -1 \mod 3$ and q is odd. In fact, we have the following.

Main theorem. *Let p be a prime and $p = q^2 + q + 1$ such that q is odd and $q \equiv -1 \mod 3$. Assume that $p||G|$. If $\mathrm{nse}(G) = \mathrm{nse}(G_2(q))$, then G is isomorphic to $G_2(q)$.*

Some notations are introduced. All groups considered are finite and p is a prime. For a natural number n, let $\pi(n)$ be the set of prime divisors of n. We denote by $\pi(G)$ the set of prime divisors of $|G|$ and by $\omega(G)$ the set of element orders of G. We also denote the number of elements of order k in G by s_k and let $\mathrm{nse}(G) := \{s_k : s_k \in \omega(G)\}$. If there is no ambiguity, we write s_k instead of $s_k(G)$. Let $GK(G)$ be a graph with vertex set $\pi(G)$ such that two primes p and q in $\pi(G)$ are joined by an edge if G has an element of order $p \cdot q$. We set $s(G)$ denote the number of connected components of the prime graph $GK(G)$ and let $m_1, m_2, \cdots, m_{s(G)}$ be the connected components of $GK(G)$. If $2 \in \pi(G)$, we assume that $2 \in m_1(G)$. $|G|$ can be expressed as a product of co-prime positive integers $OC_i, i = 1, 2, \cdots, s(G)$. The sets of order components of finite simple groups with disconnected prime graph can be obtained by [12] and [26]. Let r be a prime. Then we denote the number of the Sylow r-subgroup G_r of G by $n_r(G)$ or n_r. Also, $|x^G|$ denotes the order of conjugacy class of x in G. The other notations are standard (see [7], for instance).

2. Some preliminary results

In this section, we give some lemmas which will be used to prove the main theorem.

Lemma 1. *Let G be a finite group and m be a positive integer dividing $|G|$. If $L_m(G) = \{g \in G \big| g^m = 1\}$, then $m \big| |L_m(G)|$.*

Proof. See [8]. ∎

Lemma 2. *Let G be a group containing more than two elements. If the maximal numbers of elements of the same order in G is finite, then G is finite and $|G| \leq s(s^2 - 1)$.*

Proof. See [21]. ∎

Lemma 3. *Let G be a group and P be a cyclic Sylow p-subgroup of G of order p^a. If there is a prime r such that $p^a r \in \omega(G)$, then $s_{p^a r} = s_r(C_G(P))s_{p^a}$. In particular, $\phi(r)s_{p^a} \mid s_{p^a r}$, where $\phi(r)$ is the Euler function of r.*

Proof. See [18]. ∎

Lemma 4. *Let $q > 1$ be an integer, m be a nature number, and p be an odd prime. If p divides $q - 1$, then $(q^m - 1)_p = m_p \cdot (q - 1)_p$.*

Proof. See Lemma 8(1) of [9]. ∎

Lemma 5. *Let G be a finite group and $p \in \pi(G)$ be odd. Suppose that P is a Sylow p-subgroup of G and $n = p^s m$ with $(p, m) = 1$. If P is not cyclic and $s > 1$, then the number of elements of order n is always a multiple of p^s.*

Proof. See [15]. ∎

Lemma 6. *Let a, b and n be positive integers such that $(a, b) = 1$. Then there exists a prime p with the following properties:*

- *p divides $a^n - b^n$,*
- *p does not divide $a^k - b^k$ for all $k < n$,*

with the following exceptions: $a = 2, b = 1$; $n = 6$ and $a + b = 2^k$; $n = 2$.

Proof. See [24]. ∎

Remark 7. *If $b = 1$, the prime p is called the Zsigmondy prime. If p is a Zsigmnody of $a^n - 1$, then Fermat's little theorem shows that $n \mid p - 1$. Put $Z_n(a) = \{p : p$ is a Zsigmondy prime of $a^n - 1\}$. If $r \in Z_n(a)$ and $r \mid a^m - 1$, then $n \mid m$.*

Lemma 8. *If G is a finite group such that $t(G) \geq 2$, then G has one of the following structures:*

(1) *G is a Frobenius group or 2-Frobenius group;*

(2) *G has a normal series $1 \leq H \leq K \leq G$ such that $\pi(G/K) \cup \pi(H) \subseteq \pi_1$ and K/H is a non-abelian simple group. In particular, H is nilpotent, $G/K \lesssim \mathrm{Out}(K/H)$ and the odd order components of G are the odd order components of K/H.*

Proof. See [23]. ∎

Lemma 9. *Let G be a Frobenius group of even order with kernel K and complement H. Then $s(G) = 2$, the prime graph components of G are $\pi(H)$ and $\pi(K)$ and the following assertions hold:*

(1) *K is nilpotent;*

(2) *$|K| \equiv 1 (mod \quad |H|)$.*

Proof. See [6]. ∎

Lemma 10. *Let G be a 2-Frobenius group, i.e., G is a finite group and has a normal series $1 \leq H \leq K \leq G$ such that K and G/H are Frobenius groups with kernels H and K/H, respectively. Then:*

(1) *$t(G) = 2$, $\pi_1 = \pi(G/K) \cup \pi(H)$ and $\pi_2 = \pi(K/H)$;*

(2) *G/K and K/H are cyclic, $|G/K| \mid (|K/H| - 1)$ and $G/K \leq \mathrm{Aut}(K/H)$.*

Proof. See [6]. ∎

Lemma 11. *Let G be a finite non-abelian simple group and p is the largest prime divisor of $|G|$ with $p \mid |G|$. Then $p \nmid |\mathrm{Out}(G)|$.*

Proof. See [13]. ∎

3. Some information for $G_2(q)$

Let $\sum = \{\pm\xi_i, \xi_i - \xi_j \mid 1 \leq i,\ j \leq 3,\ i \neq j\}$ (where $\xi_1 + \xi_2 + \xi_3 = 0$) be the root system of type G_2, and choose $a = \xi_2, b = \xi_1 - \xi_2$ for a fundamental system of roots. Let χ be a homomorphism of the root module P_0 (i. e. P_0 is the additive group generated by the roots) into the multiplicative group K^*. Put $\chi(\xi) = z_i, i = 1, 2, 3$. Then the element $h(\chi_1)$ of Cartan subgroup H associated with $\mathfrak{H}$ will be denoted by $h(z_1, z_2, z_3)$. For each $r \in \sum$, there exists uniquely a homomorphism ϕ_r of $SL(2, K)$ into G such that $\phi_r \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = x_r(t), \phi_r \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = x_{-r}(t)$ for every $t \in K$. Let ω be an element of order 3 in K and $y_i \in P$ where P is a parabolic subgroup of G. Let $2 < q \equiv \epsilon \mod 3$ and assume that q is odd.

We know that $m_1(G_2(q)) = q^6(q^3-\epsilon)(q^2-1)(q+\epsilon)$ and $m_2(G_2(q)) = q^2-\epsilon q+1$.

Lemma 1. *In case $p \neq 2, 3$, the representatives and the orders of the centralizers in $G_2(q)$ are given by Table 1.*

Proof. See [5]. ∎

Lemma 2. *Let $L = G_2(q)$ with that q is odd and $q \equiv -1 \mod 3$. Let $p = q^2+q+1$ be a prime. Then*

$$s_p(L) = \sum_{|x|=p, x\neq 1} |x^L| = \frac{q^2+q}{6(q^2+q+1)}|L|.$$

Proof. Let $p = q^2 + q + 1$. Then from Lemma 1, we know that the order of the centralizers of an p-element in L is of order p and the number of the conjugacy classes of p-elements in L is $\frac{1}{6}(q^2 + q - 1 - \epsilon)$. Let $q \equiv -1 \mod 3$. Since $s_p(L) = \sum_{|x|=p, x\neq 1} |x^L|$, and $\epsilon = -1$, then

$$s_p(L) = \frac{q^2+q}{6(q^2+q+1)}|L|.$$ ∎

Lemma 3. *For $u \neq p$, then for every $u \in \omega(L)$, $p \mid s_u(L)$.*

Proof. It is easy to get from Lemma 1, that $p \mid s_u(L)$ for all $u \in \omega(L)$ such that $u \neq p$. ∎

Let n denote the number of conjugacy classes and Z the order of centralizers of elements of G, respectively.

Table 1. The representatives and the orders of the centralizers in case $p \neq 2, 3$

Representatives	n	Z
1	1	$q^6(q^2-1)(q^6-1)$
$h(-1,-1,1)$	1	$q^2(q^2-1)^2$
$h(\omega,\omega,\omega)$	1	$q^3(q^2-1)(q^3+\epsilon)$
$h(z,z^{-1},1), z^{q-1}=1, z^2\neq 1$	$\frac{q-3}{2}$	$q(q-1)^2(q+1)$
$h(z^{-1},z^{-1},z^2), z^{q-1}=1, z^2\neq 1, z^3\neq 1$	$\frac{q-4-\epsilon}{2}$	$q(q-1)^2(q+1)$
$h(z_1,z_2,z_3), z_i^{q-1}=1, z_iZ_j^{\pm 1}$	$\frac{q^2-8q+17+2\epsilon}{12}$	$(q-1)^2$
$h(z^{-1},z^{-1},z^2), z^{q+1}=1, z^2\neq 1, z^3\neq 1$	$\frac{q-2+\epsilon}{2}$	$q(q-1)(q+1)^2$
$h(z,z^{-1},1), z^{q+1}=1, z^2\neq 1$	$\frac{q-1}{2}$	$q(q-1)(q+1)^2$
$h(z_1,z_2,z_3), z_i^{q+1}=1, z_iZ_j^{\pm 1}$	$\frac{q^2-4q+5-2\epsilon}{12}$	$(q+1)^2$
$h(z,z^q,z^{-q-1}), z^{q^2-1}=1, z^{q\pm 1}\neq 1$	$\frac{(q-1)^2}{4}$	q^2-1
$h(z,z^{-q},z^{q-1}), z^{q^2-1}=1, z^{q\pm 1}\neq 1$	$\frac{(q-1)^2}{4}$	q^2-1
$h(z,z^q,z^{q^2}), z^{q^2+q+1}=1, z^3\neq 1$	$\frac{q^2+q-1-\epsilon}{6}$	q^2+q+1
$h(z,z^{-q},z^{q^2}), z^{q^2-q+1}=1, z^3\neq 1$	$\frac{q^2-q-1-\epsilon}{6}$	q^2-q+1
$h(-1,-1,-1)x_b(1)$	1	$q^2(q^2-1)$
$h(-1,-1,-1)x_c(1)$	1	$q^2(q^2-1)$
$h(-1,-1,-1)x_b(1)x_c(1)$	1	$2q^2$
$h(-1,-1,-1)x_b(1)x_c(\lambda)$	1	$2q^2$
$h(\omega,\omega,\omega)y, y\in\Phi_b$ or Φ_b^*	1	$q^3(q-\epsilon)$
$h(\omega,\omega,\omega)y, y\in\Psi$ or $\Psi_2\cap\Gamma(\omega_2)$	3	$3q^3$
$h(z,z^{-1},1)x_c(1), z^{q-1}=1, z^2\neq 1$	$\frac{q-3}{2}$	$q(q-1)$
$h(z^{-1},z^2,z^{-1})x_b(1), z^{q-1}=1, z^2\neq 1, z^3\neq 1$	$\frac{q-1-\epsilon}{2}$	$q(q-1)$
$h(z,z^{-1},1)x_{2a+b}(1), z^{q+1}=1, z\neq 1$	$\frac{q-1}{2}$	$q(q+1)$
$h(z^{-1},z^2,z^{-1})x_{3a+2b}(1), z^{q+1}=1, z^3\neq 1$	$\frac{q-1+\epsilon}{2}$	$q(q+1)$

Lemma 4. *Assume that $L = G_2(q)$ where q is odd and $q \equiv -1 \mod 3$. Let $p = q^2+q+1$ be a prime. Then $s_2 = \frac{1}{q^2(q^2-1)}|L|$, in particular, $p \mid s_2$.*

Proof. It is known that L has one class of involutions (see [12, 17]). By Lemma 1, the order of the centralizers and the numbers of conjugacy classes of the elements x of order 2 in L are $q^2(q^2-1)$ and 1, respectively. Since $s_2 = \sum_{|x|=2, x\neq 1} |x^L|$, then $s_2 = \frac{|L|}{q^2(q^2-1)}$ In particular, $p \mid s_2$. ∎

We also need the following due to Nosratpur and Darafsheh.

Lemma 5. *Let $L = G_2(q)$, where q is odd and $q \equiv -1 \mod 3$, and $d(q) = q^2+q+1$.*

(1) *If $p \in \pi(L)$, then $|L_p| \leq q^6$, where $L_p \in \mathrm{Syl}_p(L)$.*

(2) *If $p \in \pi_1(L)$, $p^\alpha \mid |L|$, and $p^\alpha - 1 \equiv 0 \mid d(q)$, then $p^\alpha = q^3$ or $p^\alpha = q^6$.*

Proof. See Lemma 3.2 of [16]. ∎

4. Proof of the main theorem

In the proof of main theorem, we use the order components of finite simple groups that are listed as in Tables 1, 2 and 3 (see [10] and [23]).

Let G be a group such that $\mathrm{nse}(G) = \mathrm{nse}(G_2(q))$, and s_n be the number of elements of order n. By Lemma 2 We note that $s_n = k\phi(n)$, where k is the number of cyclic subgroups of order n. Also we note that if $n > 2$, then $\phi(n)$ is even. If $m \in \omega(G)$, then by Lemma 1 and the above discussion, we have

$$\begin{cases} \phi(m) \mid s_m \\ m \mid \sum_{d \mid m} s_d. \end{cases} \tag{4.1}$$

Proof of the main theorem **Proof.** In the proof of the main theorem, we always assume that $\mathrm{nse}(G) = \mathrm{nse}(G_2(q))$ and $L = G_2(q)$ where q is odd and $q \equiv -1 \mod 3$. We divide the proof into the following series of Lemmas.

Lemma 1. *Let $p \neq u \in \pi(L)$. Then $p \mid s_u(L)$.*

Proof. It is easy to see that $p \mid s_u$ by Lemma 1. In fact, we can show this by the following.

By [4, Proposition 7], the maximal torus T of $G_2(q)$ has the order $(q \pm 1)^2$, $q^2 - 1$, $q^2 \pm q + 1$. Then there is an element $x \in L$ and some torus T such that $p \neq |x| = u$ and $T \leq C_L(x)$. It follows that $|x^L|$ is the multiple of $\frac{|L|}{|T|}$ for some T. But $s_u(L) = \sum_{|x|=u, x\neq 1} |x^L|$. Hence $p \mid s_u(L)$. ∎

Lemma 2. *$s_2(G) = s_2(L)$. In particular, $p \mid s_2$.*

Proof. We know that if $2 < n \in \omega(G)$, then s_n is even. By Lemma 1, $2 \mid 1 + s_2(L)$. On the other hand, in G, the only odd number in $\mathrm{nse}(G)\backslash\{1\}$ is $s_2(G)$. Hence we have $s_2(G) = s_2(L)$. By Lemma 1, $p \mid s_2$. ∎

Lemma 3. *For $u \in \omega(G)$, $p \nmid s_u(G)$ if and only if $s_u(G) = s_p(G)$. In particular $s_p(G) = s_p(L)$, in particular, $p \nmid s_p$.*

Proof. By Lemmas 2 and 3, $s_u(G) = s_p(L)$ if and only if $p \nmid s_u(G)$. By Lemma 1, $p \mid 1 + s_p(G)$ and so $p \nmid s_p(G)$. Therefore, we have $s_p(G) = s_p(L)$, in particular, $p \nmid s_p$. ∎

Lemma 4. *The Sylow p-subgroup of G is of order p.*

Proof. Let $G_p \in \mathrm{Syl}_p(G)$. Then by Lemma 1, $|G_p| \mid 1 + s_p + \cdots + s_{p^i}$ for some integer i. If $\exp(G_p) \geq p^3$, then by (4.1), $\phi(p^3) \mid s_{p^3}$ and hence, $p^2 \mid s_{p^3}$. But there is no number from $\mathrm{nse}(G)$ which is divisible by p^2. Thus $\exp(G_p) = p$ or p^2.

Let $\exp(G_p) = p^2$. Then there is an element of order p^2 with $\phi(p^2) \mid s_{p^2}$. Hence $p(p-1) \mid s_{p^2}$.

If $|G| = p^2$, then G_p is cyclic and so $n_p = \frac{s_{p^2}}{p(p-1)} = t$ for some integer t. We know that the intersection of any two Sylow p-subgroups of G may lie in a subgroup of order p. Therefore the number of cyclic subgroups of order p is something between 1 and the number of Sylow p-subgroups of G. It follows that $\frac{q^2+q}{6(q^2+q+1)}|L| = s_p \leq (p-1)n_p(G) = (p-1)t$ and so $n_p \geq \frac{|L|}{6(q^2+q+1)}$. Thus $s_{p^2} \geq \frac{p-1}{6}|L| > |L|$ since q is odd, $p = q^2+q+1$ with $q \equiv -1 \mod 3$.

If $|G_p| \geq p^3$, then by Lemma 5, $p^2 \mid s_{p^2}$ and so $s_{p^2} = p^2t$ for some integer t. But the equation $s_{p^2} = p^2t$ has no solution in nse(G), a contradiction.

Let $\exp(G_p) = p$. Then by Lemmas 2 and 3, $s_p = \frac{q^2+q}{6(q^2+q+1)}|L|$. Since

$$1+s_p = \frac{1}{6}p^2[(q^{10}-2q^9-q^8+7q^7-7q^6-5q^5+13q^4-8q^3-6q^2+15q-9)+\frac{13q+10}{q^2+q+1}],$$

then $|G_p| = p$. ∎

Lemma 5. *The prime divisor of $|G|$ is the same as $|L|$.*

Proof. By Lemma 3, $s_p = \frac{q^2+q}{6(q^2+q+1)}|L|$. Also by Lemma 4, $|G_p| = p = q^2+q+1$. It follows from Sylow's theorem, that $n_p = \frac{s_p}{\phi(p)} = \frac{|L|}{6p}$. Therefore, $\pi(G) = \pi(L)$. ∎

Lemma 6. *Let $r \in \pi(G) - \{p\}$. Then $r \cdot p \notin \omega(G)$.*

Proof. If $2 \cdot p \in \omega(G)$, then by Lemma 1, $2 \cdot p \mid 1+s_2+s_p+s_{2\cdot p}$ and also $p \mid 1+s_p$.

By Lemma 2, $p \mid s_2$. It follows that $p \mid s_{2\cdot p}$.

By Lemma 4, we have that $|G_p| = |G|_p = p$. Then by Lemma 3, $s_{2\cdot p} = s_p \cdot t$ for some integer t and so $p \mid t$.

Let $t = pk$. Then $(k, p) = 1$ and hence $s_{2\cdot p} = s_p \cdot pk = \frac{q(q+1)|L|}{6}k$. We know that $|L| = \sum_{n \in \text{nse}(G)} n$ and so $s_{2\cdot p} > 2|L|$, a contradiction. It follows that $s_{2\cdot p} = s_p \cdot pk \notin \text{nse}(G)$. Therefore $s_{2\cdot p} = s_p$, also a contradiction since $p \nmid s_p$.

Therefore $2 \cdot p \notin \omega(G)$. Similarly we can prove that $r \cdot p \notin \omega(G)$ for $r \in \pi(G) \backslash \{2, p\}$. ∎

Lemma 7. $|G| \mid \frac{q^2+q}{6}|L|$.

Proof. By Lemma 6, $r \cdot p \notin \omega(G)$ for any prime $r \in \pi(G) - \{p\}$. It follows that the Sylow r-subgroup G_r of G acts fixed freely on the set of elements of order p and so $|G_r| \mid s_p$. Therefore $|G| \mid \frac{q^2+q}{6}|L|$. ∎

Lemma 8. *There is a normal series $1 \leq H \leq K \leq G$ such that K/H is isomorphic to $G_2(q)$.*

Proof. By Lemma 6, $s(G) \geq 2$. Then by Lemma 8, we have the following:

(1) G is a Frobenius group or 2-Frobenius group;

(2) G has a normal series $1 \leq H \leq K \leq G$ such that $\pi(G/K) \cup \pi(H) \subseteq \pi_1$ and K/H is a non-abelian simple group. In particular, H is nilpotent, $G/K \lesssim Out(K/H)$ and the odd order components of G are the odd order components of K/H.

First we prove that G is neither a Frobenius group nor a 2-Frobenius group.

Let G be a Frobenius group of even order with kernel H and complement K. Then by Lemma 9, $s(G) = 2$, $\pi(G) = \{\pi(H), \pi(K)\}$. By Lemma 3, $\pi(H) = \{p\}$ or $\pi(K) = \{p\}$. If $\pi(H) = \{p\}$, then since H is nilpotent, H_p is characteristic in H. By hypothesis, we have that $H_p^g = G_p$ for some g and hence, G_p is normal in G. It follows that $s_p(G) = p - 1$, contradicting Lemma 3. Now let $\pi(K) = \{p\}$. By Lemmas 4 and 3, $n_p \mid |G|$ and by Lemma 7, $|G| \mid \frac{(q^2+q)|L|}{6}$. So by Lemma 6, there is a prime $r \in Z_{n-1}(2) \cap \pi(G)$ and so $|L_r| = |L|_r = |q^2 - 1|_r = |q+1|_r$. Since $GK(G) = \{\pi(K), \pi(H)\}$, then $r \in \pi(H)$. Since K is nilpotent, G_r is normal in G. It follows from Lemma 6, that the Sylow p-subgroup of G acts fixed point freely on the set of elements of order r and so $p \leq |G_r| \leq q + 1 < q^2 + q + 1 = p$, a contradiction.

Let G be a 2-Frobenius group. Then G has a normal series $1 \leq H \leq K \leq G$ such that $\pi(G/K) \cup \pi(H) \subseteq \pi_1$ and K/H is a cyclic group of order p and $|G/K| \mid (p-1)$. Similarly as the argument for the Frobenius group, we get a contradiction.

Therefore G has a normal series $1 \leq H \leq K \leq G$ such that $\pi(G/K) \cup \pi(H) \subseteq \pi_1$ and K/H is a non-abelian simple group. In particular, H is nilpotent, $G/K \lesssim Out(K/H)$ and the odd order components of G are the odd order components of K/H.

According to classification theorem of finite simple groups, K/H is an alternating group, sporadic group or simple group of Lie type. By Lemma 6, $s(K/H) \geq 2$.

Let $K/H \cong A_m$ with $m \geq 5$, then $m \geq p$. On the other hand, by Lemma 1 of [11] there is a prime $r \in \pi(A_m) \cap \pi(G)$ such that $q^2 - 1 \leq \frac{q^6-1}{2} < r < q^6 - 1$ and $r \in \pi(A_m)$ but $r \nmid |G|$ by Lemma 5, a contradiction.

Let K/H be sporadic simple groups, we can rule out this case by considering their odd order component since the odd components of K/H is $p = q^2 + q + 1$ with q odd and $q \equiv -1 \mod 3$.

Therefore K/H is isomorphic to a simple group of Lie type. We consider the following cases.

Case 1: Let $s(K/H) = 2$. Then we have that $OC_2(K/H) = p = q^2 + q + 1$ with q odd and $q \equiv -1 \mod 3$.

1.1. Let $K/H \cong A_{p'-1}(q')$ with $(p', q') \neq (3, 2), (3, 4)$.

Then $\frac{q'^{p'-1}-1}{(q'-1)(p',q'-1)} = q^2 + q + 1$ and so $p' - 1 = 2$, $q' = q$. Then by Lemma 7, $(q^4 - 1)(q^3 - 1)(q^2 - 1) \mid \frac{q^2+q}{6}|L| = \frac{q^2+q}{6} \cdot q^6(q^2 - 1)(q^6 - 1)$, a contradiction. Similarly, we can rule out these cases "$K/H \cong^2 A_{p'-1}(q')$ and $K/H \cong^2 A_{p'}(q')$".

1.2. Let $K/H \cong B_{p'}(3)$. Then $\frac{3^{p'}-1}{2} = q^2 + q + 1$. But by Lemma 6, the equation has no solution in $\mathbb{N}$.

1.3. Let $K/H \cong C_n(q')$ with $n = 2^m \geq 2$, q' odd. Then $\frac{q'^n+1}{(2,q-1)} = q^2 + q + 1$ and so by Lemma 6, the equation has no solution. Similarly, we can rule out "$K/H \cong C_{p'}(q')$".

1.4. Let $K/H \cong D_{p'}(q')$ with $p' \geq 5, q' = 2, 3, 5$. Then $\frac{q'^{p'}-1}{(2,q'-1)} = q^2+q+1$ and so $q \geq 2q'$. By Lemma 7, $q'^{p'} \mid q^7$ and hence, $q'^{p'} < 2^7 q'^7$. If $q' = 2$, then $p' < 14$. It follows that $p' = 5, 6, 7, \cdots, 13$. Thus order consideration rules out these cases. Similarly, we can rule out $K/H \cong D_{p'+1}(q')$ with $q' = 2, 3$.

1.5. Let $K/H \cong^2 D_n(q')$, with $n = 2^m \geq 4$, then $\frac{q'^n+1}{(2,q'+1)} = q^2+q+1$. If q' is odd, then by Lemma 6, the equation has no solution. If q' is even, then $q'^n = q(q+1)$, the equation has no solution. Similarly, we can rule out these cases "$K/H \cong^2 D_n(2)$ with $n = 2^m+1 \geq 5$, $K/H \cong^2 D_p(3)$ with $5 \leq p \neq 2^m+1$ and $K/H \cong^2 D_n(3)$ with $9 \leq 2^m+1 \neq p$".

1.6. Let $K/H \cong G_2(q')$, with $2 < q' \equiv \epsilon \mod 3, \epsilon = \pm 1$, then $q'^2 - \epsilon q' + 1 = q^2+q+1$ and hence, $q'(q'-\epsilon) = q(q+1)$. If $\epsilon = -1$, then we have the desired result. If $\epsilon = 1$, then $q' = 3, q = 2$. Order consideration rule out this case.

1.7. Let $K/H \cong^3 D_4(q')$, then $q'^4 + q'^2 + 1 = q^2+q+1$ and hence, $q'^2(q'^2+1) = q(q+1)$. Thus $q'^2 = q$, and so $(q^3-1)(q^6-1) \mid (q^2-1)(q^6-1)$, a contradiction.

1.8. Let $K/H \cong F_4(q')$ with q' odd, then $q'^4 - q'^2 + 1 = q^2+q = 1$ and hence, $q'^2(q'^2-1) = q(q+1)$, a contradiction.

1.9. Let $K/H \cong E_6(q')$, then $\frac{q'^6+q'^3+1}{(3,q'-1)} = q^2+q+1$. If q' is odd, then $q = q'^3$, and so $q^{12} \mid q^7$, a contradiction. If q' is even and $q' \equiv 1 \mod 3$, then $q'^3(q'^3+1) = 3(q^2+q)+2 = 2(1+3\frac{q^2+q}{2})$, the equation has no solution. Similarly, $K/H \ncong^2 E_6(q')$ with $q' > 2$.

Case 2: Let $s(L/K) = 3$. Then $q^2+q+1 \in \{OC_2(K/H), OC_3(K/H)\}$.

2.1. Let $K/H \cong L_2(q')$, where $4 \mid q'+1$. Then $q' = q^2+q+1$ or $\frac{q'-1}{2} = q^2+q+1$.

> If $q' = q^2+q+1$, then $4 \mid q'+1 = q^2+q+2$. So there is a prime r such that $p < r < p^6$ and $r \in \pi(H)$ or $r \in \pi(G) \setminus \pi(L_2(q'))$ contradicting the maximality of p. If $r \in \pi(H)$, then $r \sim p$, a contradiction. If the latter, $r \mid \mathrm{Out}(L_2(q'))$, contradicting Lemma 11.
>
> If $\frac{q'-1}{2} = q^2+q+1$, then $q' = 2p+1$. Similarly, we can rule out this case as above.

Similarly, $K/H \ncong L_2(q)$, with $4 \mid q-1$ and $K/H \ncong L_2(q)$, with $q > 2$ and q even.

2.2. Let $K/H \cong^2 D_{p'}(3)$ with $5 \leq p' = 2^m+1$. Then $p = \frac{3^{p'-1}+1}{2}$ or $p = \frac{3^{p'}+1}{4}$. In both cases, by Lemma 6, there is no solution. Similarly, $K/H \ncong^2 D_{p'+1}(2)$ with $n \geq 2, p' = 2^n-1$.

2.3. Let $K/H \cong G_2(q')$ with $q' \equiv 0 \mod 3$. Then $q'^2 - q' + 1 = q^2+q+1$ or $q'^2+q'+1 = q^2+q+1$. If the former, then $q'(q'-1) = q(q+1)$ and so $q' = 3, q = 2$. Order consideration rules out this case. If the latter, $q = q'$ and so $(q^2-1)^3 \mid (q^2-1)^2$, a contradiction. Similarly we can rule out "$K/H \cong^2 G_2(q')$ with $q' = 3^{2m+1} > 3$".

2.4. Let $K/H \cong F_4(q')$, where q' is even. Then $q'^4+1 = q^2+q+1$ or $q'^4 - q'^2+1 = q^2+q+1$. If the former, $q'^4 = q(q+1)$, a contradiction. If the latter, $q'^2(q'^2-1) = q(q+1)$, but the equation has no solution in $\mathbb{N}$. Similarly, $K/H \ncong^2 F_4(q')$, where $q' = 2^{2m+1} > 2$.

Case 3: $s(K/H) \in \{4,5\}$. Then $p = q^2+q+1 \in \{OC_2(K/H), OC_3(K/H), OC_4(K/H), OC_5(K/H)\}$.

3.1. Let $K/H \cong L_3(4)$ or ${}^2E_6(2)$. Then $2^n-1 = 7$ or $2^n-1 = 19$. If the former, $n = 3$, then order consideration rules out. If the latter, it is impossible.

3.2. Let $K/H \cong^2 B_2(q')$, with $q' = 2^{2m+1}$. Then $p = q^2+q+1 \in \{q'-1, q' \pm \sqrt{2q'+1}\}$. If the former, then $q^2+q+1 = p = q'-1$. Order consideration rules out. If the latter, then $\sqrt{q'}(\sqrt{q'}+\sqrt{2}) = q(q+1)$ or $\sqrt{q'}(\sqrt{q'}-\sqrt{2}) = q(q+1)$, the equation has no solution in $\mathbb{N}$.

3.3. Let $K/H \cong E_8(q')$. If $\frac{q'^{10}-q'^5+1}{q'^2-q'+1} = q^2+q+1$, then $q'^{30} \equiv 1 \mod d(q)$. By Lemma 5, $q'^{30} = q^6$ and so $q'^{120} = q^{24} > q^6$, a contradiction. If $\frac{q'^{10}+q'^5+1}{q'^2+q'+1} = q^2+q+1$, then $q'^{15} \equiv 1 \mod d(q)$. By Lemma 5, $q'^{15} = q^6$ and so $q'^{120} = q^{48} > q^6$, a contradiction. If $q'^8+q'^4+1 = q^2+q+1$, then $q'^4(q'^4+1) = q(q+1)$ and so $q'^4 = q$. But $q'^{120} = q^{30} > q^6$, a contradiction.

Hence $K/H \cong G_2(q)$ with $q \equiv -1 \mod 3$ and q odd. ∎

Lemma 9. $\pi(H) \subsetneqq \pi(q^2+q)$.

Proof. Let $r \in \pi(H)$. Then $r \neq p$ and by Lemma 8, H is nilpotent. It follows that H_r is normal in G. By Lemma 6, $r \cdot p \notin \omega(G)$, and hence, the Sylow p-subgroup of G acts fixed point freely on the set of elements of order r. Thus $p \mid |H_r|-1$. If $r \notin \pi(q(q+1))$, then by Lemma 7, $|H_r| \mid (q^2+q)_r|L|_r$. By the proof of Lemma 8, we know that $r \nmid q(q+1)$ and hence, $|H_r| \mid |L|_r$. On the other hand, $H_r \leq q^t-1$ for some integer $t \in \{2,6\}$ and so $p \leq |H_r| \leq q^2+q+1 = p$, a contradiction. Thus $\pi(H) \subsetneqq \pi(q^2+q)$. ∎

Lemma 10. *G is isomorphic to $G_2(q)$ with $q \equiv -1 \mod 3$ and q odd.*

Proof. By Lemma 8, we have that K/H is isomorphic to $G_2(q)$. We will prove that $H = 1$. Assume the contrary, then by Lemma 9, H is a π-group with $\pi = \pi(q^2+q)$. By Lemma 6, $2 \cdot p \notin \omega(G)$ and hence, the Sylow p-subgroup acts fixed point freely on H_2-1 and so $p \mid |H|-1 = (q+1)_2-1 < q+1 < q^2+q+1 = p$, where n_r denotes the number such that $r^m \mid n$ but $r^{m+1} \nmid n$ for some integer m, a contradiction. If $2 \neq r \in \pi(q(q+1))$, then we also have $r \cdot p \in \omega(G)$. It follows that the Sylow p-subgroup acts fixed point freely on H_r-1 and so $p \mid |H|-1 \leq q+1 < q^2+q+1 = p$, a contradiction. It follows that $H = 1$ and so $K \cong G_2(q)$. We refine the normal series into $1 < K < G$ and so $K < G < \mathrm{Aut}(K)$. If $G \cong G_2(q)$, it is the desired result. If $G \cong \mathrm{Aut}(G_2(q))$, then there is a prime $r \in \pi(f)$ where $q = r'^f$ (by [7, pp. xvi], $|\mathrm{Out}(G_2(q))| = f$) such that $s_r(Aut(G_2(q))) \notin \mathrm{nse}(G)$, a contradiction. This completes the proof of the Lemma. ∎

This completes the proof. ∎

5. Some applications

On Thompson's conjecture, if G and H are of the same order type, then $\mathrm{nse}(G) = \mathrm{nse}(H)$ and $|G| = |H|$.

Corollary 1. *Let $p = q^2+q+1$ be a prime and $p|||G|$ with $q \equiv -1 \mod 3$. Then $G \cong G_2(q)$ if and only if $|G| = |G_2(q)|$ and $\mathrm{nse}(G) = \mathrm{nse}(G_2(q))$.*

Shi gave the following conjecture.

Conjecture. [22] *Let G be a group and H a finite simple group. Then $G \cong H$ if and only if (a) $\omega(G) = \omega(H)$ and (b) $|G| = |H|$.*

Then we have the following corollary.

Corollary 2. *Let $p = q^2+q+1$ be a prime and $p|||G|$ with $q \equiv -1 \mod 3$. Then $G \cong G_2(q)$ if and only if $\omega(G) = \omega(G_2(q))$ and $|G| = |G_2(q)|$.*

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MORE ON UPPER AND LOWER FAINTLY ω-CONTINUOUS MULTIFUNCTIONS

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Abstract. The aim of this paper is to find under what minimal conditions the class of faintly ω-continuous multifunctions and the class of ω-continuous multifunctions agree. Similarly, we investigate when the class of faintly ω-continuous multifunctions and the class of faintly continuous multifunctions are the same.

Keywords: ω-open sets, ω-closed sets, faintly ω-continuous multifunctions.

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1. Introduction

It is well known that the class of faintly ω-continuous multifunctions contains properly the class of ω-continuous multifunctions and faintly continuous multifunctions. Recently, Zorlutuna et al. [12] introduced and studied the concept

of upper (lower) faintly continuous multifunctions and obtain some characterizations. Also, some relationships between the concept of faint continuity and known concepts of continuity and weak continuity are given. Noiri et al. [8] introduced a new class of functions called a faintly m-continuous function, which is a generalization of both faintly continuous functions and m-continuous functions. Carpintero et al. [3] introduced and studied upper (lower) faintly ω-continuous multifunctions and obtain some characterizations and properties. Carpintero et al. [2] introduced and studied a new generalization of ω-continuous multifunctions called weakly ω-continuous multifunctions and obtain some characterizations. Also, some relationships between the concept of faint ω-continuity and known concepts of ω-continuity and weak ω-continuity are given. In this paper, we find some minimal conditions in order to obtain when the class of faintly ω-continuous multifunctions and the class of ω-continuous multifunctions agree, in the same form the class of faintly ω-continuous multifunctions and the class of faintly continuous multifunctions are the same.

2. Preliminaries

Throughout this paper, (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces in which no separation axioms are assumed unless explicitly stated. Let A be a subset of a space X. For a subset A of (X, τ), $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ denote the closure of A with respect to τ and the interior of A with respect to τ, respectively. A subset A is said to be regular closed [10] if $A = \mathrm{Cl}(\mathrm{Int}(A)$. Recently, as generalization of closed sets, the notion of ω-closed sets were introduced and studied by Hdeib [6]. A point $x \in X$ is called a condensation point of A if for each $U \in \tau$ with $x \in U$, the set $U \cap A$ is uncountable. A subset A is said to be ω-closed [6] if it contains all of its condensation points. The complement of an ω-closed set is said to be an ω-open set. It is well known that a subset W of a space (X, τ) is ω-open if and only if for each $x \in W$, there exists $U \in \tau$ such that $x \in U$ and $U \backslash W$ is countable. The family of all ω-open subsets of a topological space (X, τ) denoted by $\omega O(X, \tau)$, forms a topology on X which is finer than τ. The ω-closure and the ω-interior, that can be defined in the same way as $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively, will be denoted by $\omega\,\mathrm{Cl}(A)$ and $\omega\,\mathrm{Int}(A)$, respectively. We set $\omega O(X, x) = \{A : A \in \omega O(X, \tau) \text{ and } x \in A\}$. A subset N of a topological space (X, τ) is said to be ω-neighborhood of a point $x \in X$, if there exists an ω-open set V such that $x \in V \subset N$. A point $x \in X$ is called a θ-cluster point of A [11] if $\mathrm{Cl}(V) \cap A \neq \varnothing$ for every open set V of X containing x. The set of all θ-cluster points of A is called the θ-closure of A and is denoted by $\mathrm{Cl}_\theta(A)$. If $A = \mathrm{Cl}_\theta(A)$, then A is said to be θ-closed [11]. The complement of a θ-closed set is said to be a θ-open set [11]. The union of all θ-open sets contained in $A \subset X$ is called the θ-interior of A and is denoted by $\mathrm{Int}_\theta(A)$. It follows from [11] that the collection of θ-open sets in a topological space (X, τ) forms a topology τ_θ on X. By a multifunction $F : (X, \tau) \to (Y, \sigma)$, we mean a point-to-set correspondence from X into Y, also we always assume that $F(x) \neq \varnothing$ for all $x \in X$. For a multifunction

$F : (X, \tau) \to (Y, \sigma)$, the upper and lower inverse of any subset A of Y are denoted by $F^+(A)$ and $F^-(A)$, respectively, that is $F^+(A) = \{x \in X : F(x) \subseteq A\}$ and $F^-(A) = \{x \in X : F(x) \cap A \neq \varnothing\}$. In particular, $F^-(y) = \{x \in X : y \in F(x)\}$ for each point $y \in Y$. A multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be surjective if $F(X) = Y$. A multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be lower ω-continuous [13] (resp. upper ω-continuous) multifunction if $F^-(V) \in \omega O(X, \tau)$ (resp. $F^+(V) \in \omega O(X, \tau)$) for every $V \in \sigma$. A multifunction F is said to be upper(lower) faintly continuous [12] if for each $x \in X$ and for each open set V of Y such that $x \in F^+(V)$ ($x \in F^-(V)$), there exists an open set U of X containing x such that $U \subset F^+(\mathrm{Int}(\mathrm{Cl}(V)))$ ($U \subset F^-(\mathrm{Int}(\mathrm{Cl}(V)))$). A multifunction F is faintly continuous [12] if it is both upper faintly continuous and lower faintly continuous. More generally F is said to be upper(lower) almost ω-continuous [5] if for each $x \in X$ and for each open set V of Y such that $x \in F^+(V)$ ($x \in F^-(V)$), there exists a $U \in \omega O(X, x)$ such that $U \subset F^+(\mathrm{Int}(\mathrm{Cl}(V)))$ ($U \subset F^-(\mathrm{Int}(\mathrm{Cl}(V)))$). A multifunction F is almost ω-continuous [5] if it is both upper almost ω-continuous and lower almost ω-continuous.

3. Faintly ω-continuous multifunctions

Definition 1 [3] A multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be:

1. upper faintly ω-continuous at $x \in X$ if for each θ-open set V of Y containing $F(x)$, there exists $U \in \omega O(X)$ containing x such that $F(U) \subset V$;

2. lower faintly ω-continuous at $x \in X$ if for each θ-open set V of Y such that $F(x) \cap V \neq \varnothing$, there exists $U \in \omega O(X)$ containing x such that $F(u) \cap V \neq \varnothing$ for every $u \in U$;

3. upper (lower) faintly ω-continuous if it has this property at each point of X.

4. faintly ω-continuous if it is both upper faintly ω-continuous and lower faintly ω-continuous.

Remark 3.1 It is clear that the class of faintly ω-continuous multifunctions contains properly the classes of ω-continuous multifunctions and faintly continuous multifunctions. As we can seen in the following examples.

Example 3.2 Let $X = \mathbb{R}$ with the topology $\tau = \{\emptyset, \mathbb{R}, \mathbb{R} - \mathbb{Q}\}$. Define a multifunction $F : (\mathbb{R}, \tau) \to (\mathbb{R}, \tau)$ as follows:

$$F(x) = \begin{cases} \mathbb{Q} & \text{if } x \in \mathbb{R} - \mathbb{Q} \\ \mathbb{R} - \mathbb{Q} & \text{if } x \in \mathbb{Q}. \end{cases}$$

Then F is upper faintly ω-continuous but is not upper ω-continuous.

In a similar form, we can find a multifunction G that is lower faintly ω-continuous but is not lower ω-continuous.

Example 3.3 Let $X = \mathbb{R}$ with the following topologies: $\tau_1 = \{\emptyset, \mathbb{R}, \mathbb{Q}\}$ and $\tau_2 = \{\emptyset, \mathbb{R}, \mathbb{R} - \mathbb{Q}, \mathbb{Q}\}$. Define a multifunction $F : (\mathbb{R}, \tau_1) \to (\mathbb{R}, \tau_2)$ as follows:

$$F(x) = \begin{cases} \mathbb{Q} & \text{if } x \in \mathbb{Q} \\ \mathbb{R} - \mathbb{Q} & \text{if } x \in \mathbb{R} - \mathbb{Q}. \end{cases}$$

Then F is upper faintly ω-continuous but is not upper faintly-continuous.

In a similar form, we can find a multifunction G that is lower faintly ω-continuous but is not lower faintly-continuous.

Remark 3.4 There exist any relations between the classes of ω-continuous multifunctions and faintly continuous multifunctions? In general there are no relations between them, as we can see in the following examples.

Example 3.5 Let $X = \mathbb{R}$ with the usual topology τ and $Y = \{a, b, c\}$ with topology $\tau_Y = \{\emptyset, Y, \{a\}\}$. Define a multifunction $F : (\mathbb{R}, \tau) \to (Y, \tau_Y)$ as follows:

$$F(x) = \begin{cases} \{a\} & \text{if } x < 0 \\ \{a, b\} & \text{if } x = 0 \\ \{c\} & \text{if } x > 0. \end{cases}$$

Then F is lower faintly continuous but is not lower ω-continuous.

In a similar form, we can find a multifunction G that is upper faintly continuous but is not upper ω-continuous.

Example 3.6 Let $X = \mathbb{R}$ with the following topologies: $\tau_1 = \{\emptyset, \mathbb{R}, \mathbb{Q}\}$ and $\tau_2 = \{\emptyset, \mathbb{R}, \mathbb{R} - \mathbb{Q}, \mathbb{Q}\}$. Define a multifunction $F : (\mathbb{R}, \tau_1) \to (\mathbb{R}, \tau_2)$ as follows:

$$F(x) = \begin{cases} \mathbb{Q} & \text{if } x \in \mathbb{Q} \\ \mathbb{R} - \mathbb{Q} & \text{if } x \in \mathbb{R} - \mathbb{Q} \end{cases}$$

Then F is upper ω-continuous but is not upper faintly continuous.

In a similar form, we can find a multifunction G that is lower ω-continuous but is not lower faintly continuous.

It is possible to give some additional condition to a multifunction $F : (X, \tau) \to (Y, \sigma)$ that is upper (lower) faintly ω-continuous in order to obtain that F is upper (lower) faintly continuous? In the same form to a multifunction $F : (X, \tau) \to (Y, \sigma)$ that is upper (lower) faintly ω-continuous in order to obtain that F is upper (lower) ω-continuous? Under what condition are equivalents upper (lower) faintly continuous and upper (lower) ω-continuous multifunctions?

Definition 2 [2] A multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be:

1. lower weakly ω-continuous, if for each $x \in X$ and each open set V of Y such that $x \in F^-(V)$, there exists $U \in \omega O(X, x)$ such that $U \subset F^-(\mathrm{Cl}(V))$,

2. upper weakly ω-continuous, if for each $x \in X$ and each open set V of Y such that $x \in F^+(V)$, there exists $U \in \omega O(X, x)$ such that $U \subset F^+(\mathrm{Cl}(V))$,

3. weakly ω-continuous, if it is both upper weakly ω-continuous and lower weakly ω-continuous.

Remark 3.7 It is clear that the class of weakly ω-continuous multifunctions contains properly the classes of almost ω-continuous multifunctions.

Theorem 3.8 *If $F : (X, \tau) \to (Y, \sigma)$ is lower (upper) weakly ω-continuous then F is lower (upper) faintly ω-continuous. But the converse is not true in general.*

Example 3.9 Let $X = \mathbb{R}$ with topology $\tau = \{\emptyset, \mathbb{R}, \mathbb{R} - \mathbb{Q}, \mathbb{Q}\}$ and $Y = \{a, b, c\}$ with topology $\sigma = \{\emptyset, Y, \{a\}, \{b\}, \{a, b\}\}$. Take $A \subset \mathbb{Q}$ and define $F : (\mathbb{R}, \tau) \to (Y, \sigma)$ as follows:

$$F(x) = \begin{cases} \{a\}, & \text{if } x \in \mathbb{Q} - A \\ \{b\}, & \text{if } x \in \mathbb{R} - \mathbb{Q} \\ \{c\}, & \text{if } x \in A. \end{cases}$$

Then F is upper faintly ω-continuous but is not upper weakly ω-continuous.

In a similar form, we can find a multifunction G that is lower faintly-ω-continuous but is not lower-ω-continuous.

Theorem 3.10 *Let $F : (X, \tau) \to (Y, \sigma)$ be a multifunction such that $F(x)$ is an open set of Y for each $x \in X$. Then the following assertions are equivalent:*

(1) *F is lower ω-continuous,*

(2) *F is lower weakly ω-continuous.*

Proof. (1)⇒(2): Obvious.

(2)⇒(1): Let $x \in X$ and V be an open set of Y such that $F(x) \cap V \neq \emptyset$. Then there exists an ω-open set U containing x such that $F(u) \cap \mathrm{Cl}(V) \neq \emptyset$ for each $u \in U$. Since $F(u)$ is open, $F(u) \cap V \neq \emptyset$ for each $u \in U$ and hence F is lower ω-continuous. ∎

Theorem 3.11 *Let $F : (X, \tau) \to (Y, \sigma)$ be a multifunction such that $F(x)$ is closed in Y for each $x \in X$ and Y is normal. Then the following assertions are equivalent:*

(1) *F is upper weakly ω-continuous,*

(2) *F is upper ω-continuous.*

Proof. (2)⇒(1): Obvious.

(1)⇒(2): Let $x \in X$ and G be an open set containing $F(x)$. Since $F(x)$ is closed in Y, by the normality of Y, there exists an open set V such that $F(x) \subset V \subset \mathrm{Cl}(V) \subset G$. Since F is upper weakly ω-continuous, there exists an ω-open set U containing x such that $F(U) \subset \mathrm{Cl}(V) \subset G$. This shows that F is upper ω-continuous. ∎

Definition 3 A topological space X is said to be almost regular if for each regular closed set F of X and each point $x \notin F$, there exist disjoint open sets U and V of X such that $x \in U$ and $F \subset V$.

The following characterization of almost regular spaces obtained in [7] will be useful in the sequel.

Theorem 3.12 *A topological space X is almost regular if and only if, for every open set U in X, $\mathrm{Int}_\theta(\mathrm{Cl}(U))$ is a θ-open set.*

Theorem 3.13 *Let $F : (X, \tau) \to (Y, \sigma)$ be an upper (lower) faintly ω-continuous multifunction and (Y, σ) be an almost regular space, then F is an upper (lower) weakly ω-continuous multifunction.*

Proof. We proof the theorem for upper (lower) weakly ω-continuous multifunctions. Let $x \in X$ and V any open set in Y with $x \in F^+(V)$ ($x \in F^-(V)$). There exists $y \in F(x) \subset V$ ($y \in F(x) \cap V$), since $y \in V$ and $V \subset \mathrm{Cl}(V)$, $y \in \mathrm{Int}_\theta(\mathrm{Cl}(V))$. Using the fact that (Y, σ) is almost regular, $\mathrm{Int}_\theta(\mathrm{Cl}(V))$ is a θ-open set in Y and $x \in F^+(\mathrm{Int}_\theta(\mathrm{Cl}(V)))$ ($x \in F^-(\mathrm{Int}_\theta(\mathrm{Cl}(V))))$. Since F is upper (lower) faintly ω-continuous, there exists an ω-open set U of X containing x, such that $F(U) \subset \mathrm{Int}_\theta(\mathrm{Cl}(V))$ ($z \in U$ implies $F(z) \cap \mathrm{Int}_\theta(\mathrm{Cl}(V)) \neq \emptyset$). Hence $F(U) \subset \mathrm{Cl}(V)$ ($F(U) \cap \mathrm{Int}_\theta(\mathrm{Cl}(V)) \subseteq F(U) \cap \mathrm{Cl}(V) \neq \emptyset$). Therefore, F is upper (lower) weakly ω-continuous. ∎

The following definition is a particular case of Definition 3.4 of [9].

Definition 4 [4] A multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be:

1. upper slightly ω-continuous at $x \in X$ if for each clopen set V of Y containing $F(x)$, there exists $U \in \omega O(X)$ containing x such that $F(U) \subset V$;

2. lower slightly ω-continuous at $x \in X$ if for each clopen set V of Y such that $F(x) \cap V \neq \varnothing$, there exists $U \in \omega O(X)$ containing x such that $F(u) \cap V \neq \varnothing$ for every $u \in U$;

3. upper (lower) slightly ω-continuous if it has this property at each point of X.

Remark 3.14 It is clear that every upper ω-continuous multifunction is upper slightly ω-continuous. But the converse is not true in general, as the following example shows.

Example 3.15 Let $X = \mathbb{R}$ with the topology $\tau = \{\emptyset, \mathbb{R}, \mathbb{R} - \mathbb{Q}\}$. Define a multifunction $F : (\mathbb{R}, \tau) \to (\mathbb{R}, \tau)$ as follows:

$$F(x) = \begin{cases} \mathbb{Q} & \text{if } x \in \mathbb{R} - \mathbb{Q} \\ \mathbb{R} - \mathbb{Q} & \text{if } x \in \mathbb{Q}. \end{cases}$$

Then F is upper slightly ω-continuous but is not upper ω-continuous.

Theorem 3.16 *Every upper (lower) weakly ω-continuous multifunction is upper (lower) slightly ω-continuous. But the converse is not true in general.*

Example 3.17 The multifunction defined in Example 3.9 is upper (lower) slightly ω-continuous but not upper (lower) weakly ω-continuous multifunction.

Theorem 3.18 *Every upper (lower) faintly ω-continuous multifunction is upper (lower) slightly ω-continuous. But the converse is not true in general.*

Proof. Since every clopen set is a θ-open set, the proof follows. ∎

The following example shows that the converse of Theorem 3.18 is not true in general.

Example 3.19 Let $X = \mathbb{R}$ with topology $\tau = \{\emptyset, \mathbb{R}, \mathbb{R} - \mathbb{Q}\}$ and $Y = \mathbb{R}$ with topology σ as the usual topology. Define $F : (\mathbb{R}, \tau) \to (\mathbb{R}, \sigma)$ as $F(x) = \{x\}$ for all $x \in \mathbb{R}$. It is easy to see that F is upper (lower) slightly ω-continuous but not upper (lower) faintly ω-continuous.

Definition 5 A topological space (X, τ) is said to be an ω-space [1] if every ω-open set of X is open.

Theorem 3.20 *Let $F : (X, \tau) \to (Y, \sigma)$ be a multifunction and (X, τ) be an ω-space. Then F is faintly ω-continuous if and only if F is faintly continuous.*

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SEMIGROUP IDEALS AND PERMUTING 3-GENERALIZED DERIVATIONS IN PRIME NEAR RINGS

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Abstract. The purpose of the present paper is to prove some commutativity theorems in the setting of a semigroup ideal of a 3-prime near ring admitting a permuting generalized 3-derivation, thereby extending some known results of derivations, biderivations and 3-derivations.

Keywords: 3-prime near ring, semigroup ideal, permuting 3-derivation, permuting 3-generalized derivation.

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1. Introduction

Throughout the paper, N denotes a zero-symmetric left near ring with multiplicative centre Z; and for any pair of elements $x, y \in N$, $[x, y]$ denotes the commutator $xy - yx$ while the symbol (x, y) denotes the additive commutator $x + y - x - y$. A near ring N is called zero-symmetric if $0x = 0$, for all $x \in N$ (recall that

left distributivity yields that $x0 = 0$). The near ring N is said to be 3-prime if $xNy = \{0\}$ for $x, y \in N$ implies that $x = 0$ or $y = 0$. A near ring N is called $n-$torsion-free, where n is a positive integer, if $(N, +)$ has no element of order n. For $x \in N$, $C(x) = \{a \in N \mid ax = xa\}$ denotes the centralizer of x in N. A nonempty subset U of N is called a semigroup right (resp. semigroup left) ideal if $UN \subseteq U$ (resp. $NU \subseteq U$) and if U is both a semigroup right ideal and a semigroup left ideal, it is called a semigroup ideal.

A mapping $\Delta : N \times N \times N \longrightarrow N$ is said to be permuting if $\Delta(x, y, z) = \Delta(y, x, z) = \Delta(z, y, x) = \Delta(x, z, y) = \Delta(y, z, x) = \Delta(z, x, y)$ for all $x, y, z \in N$. A mapping $\delta : N \longrightarrow N$ defined by $\delta(x) = \Delta(x, x, x)$ is called trace of Δ, where $\Delta : N \times N \times N \longrightarrow N$ is a permuting mapping. It is obvious that, if $\Delta : N \times N \times N \longrightarrow N$ is a permuting mapping which is also 3-additive (i.e., additive in all arguments), then the trace δ of Δ satisfies the relation

$$\delta(x + y) = \delta(x) + 2\Delta(x, x, y) + \Delta(x, y, y) + \Delta(x, x, y) + 2\Delta(x, y, y) + \delta(y)$$

for all $x, y \in N$.

A permuting 3-additive mapping $\Delta : N \times N \times N \longrightarrow N$ is called a permuting 3-derivation if $\Delta(xw, y, z) = \Delta(x, y, z)w + x\Delta(w, y, z)$ is fulfilled for all $x, y, z, w \in N$.

Motivated by the notion of permuting 3-derivations in near rings [10], Park and Jung [9] defined permuting 3-generalized derivations in near rings. Let $\Delta : N \times N \times N \longrightarrow N$ be a permuting 3-derivation of N. A permuting 3-additive mapping $F : N \times N \times N \longrightarrow N$ is said to be a permuting 3-right (resp. left) generalized derivation of N associated with Δ if $F(xw, y, z) = F(x, y, z)w + x\Delta(w, y, z)$ (resp. $F(xw, y, z) = \Delta(x, y, z)w + xF(w, y, z)$) for all $x, y, z, w \in N$. Also, F is said to be a permuting 3-generalized derivation of N associated with Δ if it is both a permuting 3-right and a permuting 3-left generalized derivation associated with Δ. In case of rings and near rings derivations and biderivations have received significant attention in recent years (see [1], [2], [3], [4], [12], [13], [14]). The purpose of the present paper is to prove some commutativity theorems in the setting of a semigroup ideal of a near ring admitting a permuting 3-generalized derivation, thereby extending some known results on derivations and biderivations.

Example 1.1. Let S be a commutative near-ring and let

$$N = \left\{ \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} \mid a, b \in S \right\}.$$

Then N is a near-ring under matrix addition and matrix multiplication. Define the maps $\Delta, F : N \times N \times N \longrightarrow N$ by

$$\left(\begin{pmatrix} 0 & a_1 \\ 0 & b_1 \end{pmatrix}, \begin{pmatrix} 0 & a_2 \\ 0 & b_2 \end{pmatrix}, \begin{pmatrix} 0 & a_3 \\ 0 & b_3 \end{pmatrix} \right) \longmapsto \begin{pmatrix} 0 & a_1a_2a_3 \\ 0 & 0 \end{pmatrix} \text{ and}$$

$$\left(\begin{pmatrix} 0 & a_1 \\ 0 & b_1 \end{pmatrix}, \begin{pmatrix} 0 & a_2 \\ 0 & b_2 \end{pmatrix}, \begin{pmatrix} 0 & a_3 \\ 0 & b_3 \end{pmatrix} \right) \longmapsto \begin{pmatrix} 0 & 0 \\ 0 & b_1b_2b_3 \end{pmatrix} \text{ respectively.}$$

It can be easily verified that Δ is a 3-derivation of N and F is a permuting 3-right generalized derivation of N with associated 3-derivation Δ.

Example 1.2 Let S be a commutative near-ring and let

$$N = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in S \right\}.$$

Then N is a near-ring under matrix addition and matrix multiplication. Define the maps $\Delta, F : N \times N \times N \longrightarrow N$ by

$$\left(\begin{pmatrix} a_1 & b_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a_3 & b_3 \\ 0 & 0 \end{pmatrix} \right) \longmapsto \begin{pmatrix} 0 & b_1b_2b_3 \\ 0 & 0 \end{pmatrix} \text{ and}$$

$$\left(\begin{pmatrix} a_1 & b_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a_3 & b_3 \\ 0 & 0 \end{pmatrix} \right) \longmapsto \begin{pmatrix} a_1a_2a_3 & 0 \\ 0 & 0 \end{pmatrix} \text{ respectively.}$$

It can be verified that Δ is a 3-derivation of N and F is a 3-left generalized derivation of N with associated 3-derivation Δ.

Example 1.3 Let S be a commutative near-ring and let

$$N = \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \mid a, b, c \in S \right\}.$$

Then N is a near-ring under matrix addition and matrix multiplication. Define maps $\Delta, F : N \times N \times N \longrightarrow N$ by

$$\left(\begin{pmatrix} a_1 & 0 \\ b_1 & c_1 \end{pmatrix}, \begin{pmatrix} a_2 & 0 \\ b_2 & c_2 \end{pmatrix}, \begin{pmatrix} a_3 & 0 \\ b_3 & c_3 \end{pmatrix} \right) \longmapsto \begin{pmatrix} 0 & 0 \\ b_1b_2b_3 & 0 \end{pmatrix} \text{ and}$$

$$\left(\begin{pmatrix} a_1 & 0 \\ b_1 & c_1 \end{pmatrix}, \begin{pmatrix} a_2 & 0 \\ b_2 & c_2 \end{pmatrix}, \begin{pmatrix} a_3 & 0 \\ b_3 & c_3 \end{pmatrix} \right) \longmapsto \begin{pmatrix} 0 & 0 \\ b_1b_2b_3 & 0 \end{pmatrix} \text{ respectively.}$$

It can be verified that Δ is a 3-derivation of N and F is a permuting 3-right generalized derivation and a permuting 3-left generalized derivation of N with associated 3-derivation Δ and hence F is a permuting 3-generalized derivation of N.

2. Preliminary results

We begin with the following Lemmas. Lemmas 2.7, 2.9 and 2.10 are proved using same techniques with some variations as those of Lemmas 7, 8 and 10 in [8].

Lemma 2.1 [2, Lemma 1.2] *Let N be a 3-prime near ring.*

(i) *If $z \in Z \setminus \{0\}$, then z is not a zero divisor.*

(ii) *If $Z \setminus \{0\}$ contains an element z for which $z + z \in Z$, then $(N, +)$ is abelian.*

(iii) *If $z \in Z \setminus \{0\}$ and x is an element of N such that $xz \in Z$ or $zx \in Z$, then $x \in Z$.*

Lemma 2.2 [2, Lemma 1.3] *Let N be a 3-prime near ring and U be a nonzero semigroup ideal of N.*

(i) *If* $x, y \in N$ *and* $xUy = \{0\}$, *then* $x = 0$ *or* $y = 0$.

(ii) *If* $x \in N$ *and* $xU = \{0\}$ *or* $Ux = \{0\}$, *then* $x = 0$.

Lemma 2.3 [2, Lemma 1.5] *If* N *is a* 3*-prime near ring and* Z *contains a nonzero semigroup left ideal or a semigroup right ideal, then* N *is a commutative ring.*

Lemma 2.4 [10, Lemma 2.4] *Let* N *be a near ring and let* $\Delta : N \times N \times N \longrightarrow N$ *be a permuting 3-derivation. Then we have*

$$[\Delta(x, z, w)y + x\Delta(y, z, w)]v = \Delta(x, z, w)yv + x\Delta(y, z, w)v \text{ for all } v, w, x, y, z \in N.$$

Lemma 2.5 *Let* N *be a* 3*-prime near ring and* U *be a nonzero semigroup ideal of* N. *If* Δ *is a nonzero* 3*-derivation on* N, *then* $\Delta \neq 0$ *on* U.

Proof. Suppose that $\Delta(U, U, U) = \{0\}$. For any $u, v, w \in U$, we have

$$\Delta(u, v, w) = 0. \tag{2.1}$$

Substituting ux for u in (2.1), we get

$$\Delta(u, v, w)x + u\Delta(x, v, w) = 0 \text{ for all } u, v, w \in U \text{ and } x \in N.$$

Using (2.1), we get $U\Delta(x, v, w) = \{0\}$. Invoking Lemma 2.2 (ii), we have

$$\Delta(x, v, w) = 0 \text{ for all } v, w \in U \text{ and } x \in N. \tag{2.2}$$

Substituting vy for v in (2.2), we get

$$\Delta(x, v, w)y + v\Delta(x, y, w) = 0 \text{ for all } v, w \in U \text{ and } x, y \in N.$$

Using (2.2), we find $U\Delta(x, y, w) = \{0\}$ and Lemma 2.2 (ii), yields that

$$\Delta(x, y, w) = 0 \text{ for all } w \in U \text{ and } x, y \in N. \tag{2.3}$$

Substituting wz for w in (2.3), we obtain $U\Delta(x, y, z) = \{0\}$. Another appeal to Lemma 2.2 (ii), yields that $\Delta(x, y, z) = 0$, for all $x, y, z \in N$, which is a contradiction. ∎

Lemma 2.6 *Let* N *be a* 3!*-torsion free near ring and* U *be a nonzero additive subgroup of* N. *If* Δ *is a permuting 3-additive map with trace* δ *such that* $\delta(x) = 0$ *for all* $x \in U$, *then* $\Delta = 0$ *on* U.

Proof. For any $x, y \in U$, we have the relation

$$\delta(x + y) = \delta(x) + 2\Delta(x, x, y) + \Delta(x, y, y) + \Delta(x, x, y) + 2\Delta(x, y, y) + \delta(y)$$

and so, by the hypothesis, we get

$$2\Delta(x, x, y) + \Delta(x, y, y) + \Delta(x, x, y) + 2\Delta(x, y, y) = 0 \text{ for all } x, y \in U. \tag{2.4}$$

Substituting $-x$ for x in (2.4), we obtain

$$2\Delta(x,x,y) - \Delta(x,y,y) + \Delta(x,x,y) - 2\Delta(x,y,y) = 0 \text{ for all } x,y \in U. \tag{2.5}$$

On the other hand, for any $x, y \in U$,

$$\delta(y+x) = \delta(y) + 2\Delta(y,y,x) + \Delta(y,x,x) + \Delta(y,y,x) + 2\Delta(y,x,x) + \delta(x)$$

and thus, by the hypothesis and using the fact that Δ is permuting, we have

$$2\Delta(x,y,y) + \Delta(x,x,y) + \Delta(x,y,y) + 2\Delta(x,x,y) = 0 \text{ for all } x,y \in U. \tag{2.6}$$

Comparing (2.4) and (2.5), we get

$$2\Delta(x,y,y) + \Delta(x,x,y) + \Delta(x,y,y) = \Delta(x,x,y) - 3\Delta(x,y,y)$$

which implies that

$$\begin{aligned} 2\Delta(x,y,y) + \Delta(x,x,y) + \Delta(x,y,y) + 2\Delta(x,x,y) \\ = \Delta(x,x,y) - 3\Delta(x,y,y) + 2\Delta(x,x,y). \end{aligned}$$

Hence it follows from (2.6) that

$$\Delta(x,x,y) - 3\Delta(x,y,y) + 2\Delta(x,x,y) = 0 \text{ for all } x,y \in U. \tag{2.7}$$

Substituting $-x$ for x in (2.7), we find

$$\Delta(x,x,y) + 3\Delta(x,y,y) + 2\Delta(x,x,y) = 0 \text{ for all } x,y \in U. \tag{2.8}$$

Comparing (2.7) and (2.8), we obtain

$$6\Delta(x,y,y) = 0 \text{ for all } x,y \in U.$$

Since N is 3!-torsion free, we get

$$\Delta(x,y,y) = 0 \text{ for all } x,y \in U. \tag{2.9}$$

Substituting $y+z$ for y in (2.9) and linearizing (2.9) we obtain

$$\Delta(x,y,z) = 0 \text{ for all } x,y \in U,$$

i.e., $\Delta = 0$ on U which completes the proof. ∎

Lemma 2.7 *Let N be a 3-prime near ring and U be a nonzero semigroup ideal of N. Let $\Delta : N \times N \times N \longrightarrow N$ be a 3-derivation. If for $x \in N$, $\Delta(U,U,U)x = \{0\}$ (or $x\Delta(U,U,U) = \{0\}$), then either $x = 0$ or $\Delta = 0$ on U.*

Proof. Let $\Delta(y,z,w)x = 0$ for all $y,z,w \in U$. Substituting $y = vy$, we get $\Delta(v,z,w)yx = 0$, for all $y,z,v,w \in U$. Hence $\Delta(v,z,w)Ux = \{0\}$ implies that either $x = 0$ or $\Delta = 0$ by Lemma 2.2 (i). ∎

Lemma 2.8 *Let N be a 3!-torsion free 3-prime near ring and U be a nonzero additive subgroup and a semigroup ideal of N. If Δ is a permuting 3-derivation with trace δ and $x \in N$ such that $x\delta(y) = 0$ for all $y \in U$, then either $x = 0$ or $\Delta = 0$ on U.*

Proof. For any $y, z \in U$, we have

$$\delta(y+z) = \delta(y) + 2\Delta(y,y,z) + \Delta(y,z,z) + \Delta(y,y,z) + 2\Delta(y,z,z) + \delta(z).$$

By hypothesis

$$2x\Delta(y,y,z) + x\Delta(y,z,z) + x\Delta(y,y,z) + 2x\Delta(y,z,z) = 0 \text{ for all } y,z \in U. \quad (2.10)$$

Substituting $-y$ for y in (2.10), it follows that

$$2x\Delta(y,y,z) - x\Delta(y,z,z) + x\Delta(y,y,z) - 2x\Delta(y,z,z) = 0 \text{ for all } y,z \in U. \quad (2.11)$$

On the other hand,

$$\delta(z+y) = \delta(z) + 2\Delta(z,z,y) + \Delta(z,y,y) + \Delta(z,z,y) + 2\Delta(z,y,y) + \delta(y)$$

Again using hypothesis, we have

$$2x\Delta(z,z,y) + x\Delta(z,y,y) + x\Delta(z,z,y) + 2x\Delta(z,y,y) = 0.$$

Since Δ is permuting, we get

$$2x\Delta(y,z,z) + x\Delta(y,y,z) + x\Delta(y,z,z) + 2x\Delta(y,y,z) = 0 \text{ for all } y,z \in U. \quad (2.12)$$

Comparing (2.10) and (2.11), we get

$$2x\Delta(y,z,z) + x\Delta(y,y,z) + x\Delta(y,z,z) = x\Delta(y,y,z) - 3x\Delta(y,z,z)$$

i.e.,

$$\begin{aligned} 2x\Delta(y,z,z) + x\Delta(y,y,z) + x\Delta(y,z,z) + 2x\Delta(y,y,z) \\ = x\Delta(y,y,z) - 3x\Delta(y,z,z) + 2x\Delta(y,y,z). \end{aligned}$$

Now, from (2.12), we obtain

$$x\Delta(y,y,z) - 3x\Delta(y,z,z) + 2x\Delta(y,y,z) = 0 \text{ for all } y,z \in U. \quad (2.13)$$

Substituting $-y$ for y in (2.13), we find

$$x\Delta(y,y,z) + 3x\Delta(y,z,z) + 2x\Delta(y,y,z) = 0 \text{ for all } y,z \in U. \quad (2.14)$$

Comparing (2.13) and (2.14), we obtain

$$6x\Delta(y,z,z) = 0 \text{ for all } y,z \in U.$$

Since N is 3!-torsion free, we get

$$x\Delta(y,z,z) = 0 \text{ for all } y,z \in U. \quad (2.15)$$

Substituting $z + w$ for z in (2.15), we have

$$x\Delta(w, y, z) = 0 \text{ for all } w, y, z \in U. \tag{2.16}$$

Hence by Lemma 2.7 either $x = 0$ or $\Delta = 0$ on U. ∎

Lemma 2.9 *Let N be a 3!-torsion free 3-prime near ring and U be a nonzero additive subgroup and a semigroup ideal of N. Let $\Delta : N \times N \times N \longrightarrow N$ be a 3-derivation.*

(i) *If F is a permuting 3-right generalized derivation of N associated with Δ and $x \in N$ such that $xf(y) = 0$ for all $y \in U$, then either $x = 0$ or $\Delta = 0$ on U.*

(ii) *If F is a permuting 3-generalized derivation of N associated with Δ and $x \in N$ such that $xf(y) = 0$ for all $y \in U$, then either $x = 0$ or $F = 0$ on U.*

Proof. (i) For any $y, z \in U$, we have

$$f(y + z) = f(y) + 2F(y, y, z) + F(y, z, z) + F(y, y, z) + 2F(y, z, z) + f(z).$$

Using hypothesis, we obtain

$$2xF(y, y, z) + xF(y, z, z) + xF(y, y, z) + 2xF(y, z, z) = 0 \text{ for all } y, z \in U. \tag{2.17}$$

Substituting $-y$ for y in (2.17), it follows that

$$2xF(y, y, z) - xF(y, z, z) + xF(y, y, z) - 2xF(y, z, z) = 0 \text{ for all } y, z \in U. \tag{2.18}$$

On the other hand, for any $y, z \in U$,

$$f(z + y) = f(z) + 2F(z, z, y) + F(z, y, y) + F(z, z, y) + 2F(z, y, y) + f(y)$$

and so, by the hypothesis, we have

$$2xF(z, z, y) + xF(z, y, y) + xF(z, z, y) + 2xF(z, y, y) = 0$$

Since F is permuting, we get

$$2xF(y, z, z) + xF(y, y, z) + xF(y, z, z) + 2xF(y, y, z) = 0 \text{ for all } y, z \in U. \tag{2.19}$$

Comparing (2.17) and (2.18), we get

$$2xF(y, z, z) + xF(y, y, z) + xF(y, z, z) = xF(y, y, z) - 3xF(y, z, z)$$

which implies that

$$\begin{aligned} &2xF(y, z, z) + xF(y, y, z) + xF(y, z, z) + 2xF(y, y, z) \\ &\qquad = xF(y, y, z) - 3xF(y, z, z) + 2xF(y, y, z). \end{aligned}$$

Now, from (2.19), we obtain

$$xF(y,y,z) - 3xF(y,z,z) + 2xF(y,y,z) = 0 \text{ for all } y,z \in U. \tag{2.20}$$

Substituting $-y$ for y in (2.20), we have

$$xF(y,y,z) + 3xF(y,z,z) + 2xF(y,y,z) = 0 \text{ for all } y,z \in U. \tag{2.21}$$

Combining (2.20) and (2.21), we obtain

$$6xF(y,z,z) = 0 \text{ for all } y,z \in U.$$

Since N is 3!-torsion free, we get

$$xF(y,z,z) = 0 \text{ for all } y,z \in U. \tag{2.22}$$

Substituting $z+w$ for z in (2.22), we find that

$$xF(w,y,z) = 0 \text{ for all } w,y,z \in U. \tag{2.23}$$

Replacing y by yv in (2.27) and using (2.23), we get $xy\Delta(v,z,w)=0$ for all $u,v,w,x,z \in U$. Hence $xU\Delta(v,z,w)=0$ implies that either $x=0$ or $\Delta(v,z,w)=0$ for all $u,v,w \in U$ by Lemma 2.2 (i).

(ii) Let x be a nonzero element of N. Thus from (i) we obtain $\Delta=0$. Replacing y by yv in (2.23) and using hypothesis, we get $xyF(v,z,w)=0$, for all $v,w,x,y,z \in U$. Hence $xUF(v,z,w) = \{0\}$, again application of Lemma 2.2 (i) yields that $F=0$ on U. ∎

Lemma 2.10 *Let N be a near ring and let $\Delta : N\times N\times N \longrightarrow N$ be a 3-derivation.*

(i) *If F is a 3-right generalized derivation of N associated with Δ, then*

$$[F(x,z,w)y + x\Delta(y,z,w)]v = F(x,z,w)yv + x\Delta(y,z,w)v, \quad \text{for all } v,w,x,y,z \in N.$$

(ii) *If F is a 3-left generalized derivation of N associated with Δ , then*

$$[\Delta(x,z,w)y + xF(y,z,w)]v = \Delta(x,z,w)yv + xF(y,z,w)v, \quad \text{for all } v,w,x,y,z \in N.$$

Proof. (i) Let F be a 3-right generalized derivation of N associated with Δ. Then

$$\begin{aligned} F((xy)v,z,w) &= F(xy,z,w)v + xy\Delta(v,z,w) \\ &= [F(x,z,w)y + x\Delta(y,z,w)]v + xy\Delta(v,z,w) \qquad (2.24) \\ &\qquad \text{for all } v,w,x,y,z \in N. \end{aligned}$$

On the other hand,

$$\begin{aligned} F(x(yv),z,w) &= F(x,z,w)yv + x\Delta(yv,z,w) \\ &= F(x,z,w)yv + x[\Delta(y,z,w)v + y\Delta(v,z,w)] \\ &= F(x,z,w)yv + x\Delta(y,z,w)v + xy\Delta(v,z,w) \\ &\qquad \text{for all } v,w,x,y,z \in N. \end{aligned} \tag{2.25}$$

Comparing (2.24) and (2.25), we get

$[F(x,z,w)y + x\Delta(y,z,w)]v = F(x,z,w)yv + x\Delta(y,z,w)v$ for all $v,w,x,y,z \in N$.

(ii) Let F be a 3-left generalized derivation of N associated with Δ. Then

$$\begin{aligned} F((xy)v,z,w) &= \Delta(xy,z,w)v + xyF(v,z,w) \\ &= [\Delta(x,z,w)y + x\Delta(y,z,w)]v + xyF(v,z,w) \\ &\qquad \text{for all } v,w,x,y,z \in N. \end{aligned} \tag{2.26}$$

On the other hand,

$$\begin{aligned} F(x(yv),z,w) &= \Delta(x,z,w)yv + xF(yv,z,w) \\ &= \Delta(x,z,w)yv + x[\Delta(y,z,w)v + yF(v,z,w)] \\ &= \Delta(x,z,w)yv + x\Delta(y,z,w)v + xyF(v,z,w) \\ &\qquad \text{for all } v,w,x,y,z \in N. \end{aligned} \tag{2.27}$$

Comparing (2.26) and (2.27), we get

$[\Delta(x,z,w)y + xF(y,z,w)]v = \Delta(x,z,w)yv + xF(y,z,w)v$ for all $v,w,x,y,z \in N$.

Lemma 2.11 *Let N be a 3-prime near ring and U be a nonzero semigroup ideal of N. If F is a nonzero 3-right (or left) generalized derivation of N associated with a nonzero 3-derivation Δ, then $F \neq 0$ on U.*

Proof. Let F be a nonzero 3-right generalized derivation of N such that

$$F(U,U,U) = \{0\}.$$

Then for all $x,y,z \in U$, we have $F(x,y,z) = 0$. Replacing x by xw, we get $F(x,y,z)w + x\Delta(w,y,z) = 0$ for $w,x,y,z \in U$. Hence $U\Delta(w,y,z) = \{0\}$ and Lemma 2.2 (ii) gives that $\Delta(w,y,z) = 0$, for all $w,y,z \in U$, a contradiction by Lemma 2.5 which completes the proof. ∎

3. The condition $F(U,U,U) \subseteq Z$

The theorems that we prove in this section will extend the results proved in [1, Theorem 2.1, 3.1 and 3.3], [2, Theorem 2.1 and 4.1], [3, Theorem 2 and 3], [8, Theorem 4 and 5] and [9, Theorem 3.1 and 3.2].

Theorem 3.1 *Let N be a 3-prime near ring and U be a nonzero semigroup ideal of N which is closed under addition. Suppose F is a nonzero 3-right generalized derivation of N associated with a nonzero 3-derivation Δ. If $F(U,U,U) \subseteq Z$, then $(N,+)$ is abelian.*

Proof. Suppose that $x, y, z \in U$ such that $0 \neq F(x,y,z) \in Z \setminus \{0\}$. Then $F(x,y,z+z) = F(x,y,z) + F(x,y,z) \in Z$, since F is nonzero on U by Lemma 2.11. Hence $(N,+)$ is abelian by Lemma 2.1 (ii). ∎

Theorem 3.2 *Let N be a 3!-torsion free 3-prime near ring and U be a nonzero additive subgroup and a semigroup ideal of N. Suppose Δ is a permuting 3-derivation and F is a nonzero permuting 3-generalized derivation of N associated with Δ such that $f(U) \subseteq U$, where f is the trace of F. If $F(U,U,U) \subseteq Z$, then N is a commutative ring.*

Proof. Suppose that $\Delta = 0$ on U. Then $F(xy,z,w) = F(x,z,w)y \in Z$ for all $w, x, y, z \in U$ gives that $F(x,z,w)yv = vF(x,z,w)y$ for all $w, x, y, z \in U$ and $v \in N$ this implies that $[y,v]f(x) = 0$ for all $v, x, y \in U$. Applying Lemma 2.9 (ii), we get $[y,v] = 0$ for all $y \in U$ and $v \in N$ and $U \subseteq Z$. Hence N is a commutative ring by Lemma 2.3.

Now, let us consider the case $\Delta \neq 0$ on U. Using the hypothesis, we get $F(x,y,z)w = wF(x,y,z)$ for all $x, y, z \in U$ and $w \in N$. Replacing x by xv, we obtain

$$F(x,y,z)vw + x\Delta(v,y,z)w = wF(x,y,z)v + wx\Delta(v,y,z) \quad \text{for all } v, x, y, z \in U \text{ and } w \in N. \tag{3.1}$$

Replacing x by $rf(x)$ in (3.1) and using (3.1), we have

$$r\Delta(f(x),y,z)[v,w] = 0 \text{ for all } r, v, x, y, z \in U \text{ and } w \in N. \tag{3.2}$$

Thus $U\Delta(f(x),y,z)[v,w] = 0$ and by Lemma 2.2 (ii) $\Delta(f(x),y,z)[v,w] = 0$. Substituting ry for y, we obtain $\Delta(f(x),r,z)U[v,w] = \{0\}$. Using Lemma 2.2 (i), we get either $\Delta(f(x),y,z) = 0$ or $[v,w] = 0$. Later yields that N is a commutative ring by Lemma 2.3.

Now, suppose that

$$\Delta(f(x),y,z) = 0 \text{ for all } x, y, z \in U. \tag{3.3}$$

Substituting $x + v$ for x in (3.3) and using the fact that N is 3!-torsion free, we get that

$$\Delta(F(x,v,v),y,z) = 0 \text{ for all } v, x, y, z \in U. \tag{3.4}$$

Replacing v by $v + r$ in (3.4), we get

$$\Delta(F(x,v,r),y,z) = 0 \text{ for all } r, v, x, y, z \in U. \tag{3.5}$$

Taking xt instead of x in (3.5) and using (3.5), we have

$$F(x,v,r)\Delta(t,y,z)+\Delta(x,y,z)\Delta(t,v,r)+x\Delta(\Delta(t,v,r),y,z)=0 \quad \text{for all } r,t,v,x,y,z\in U. \tag{3.6}$$

Substituting $f(r)$ for r in (3.6) and using (3.3), we get

$$F(f(r),x,v)\Delta(t,y,z)=0 \text{ for all } r,v,t,x,y,z\in U.$$

Substituting uv for v, we obtain

$$F(f(r),x,u)U\Delta(t,y,z)=\{0\}.$$

Again, by Lemma 2.2 (i), we get either

$$F(f(r),x,u)=0 \text{ or } \Delta(t,y,z)=0.$$

If $\Delta(t,y,z)=0$, for all $t,y,z\in U$, then $\Delta=0$ on U, a contradiction. Thus we find that

$$F(f(r),x,v)=0 \text{ for all } r,v,x\in U. \tag{3.7}$$

Taking $r+s$ instead of r in (3.7) and using (3.7) and the fact that N is 3!-torsion free, we obtain

$$F(F(r,r,s),x,v)=0 \text{ for all } r,s,x,v\in U. \tag{3.8}$$

Substituting $r+z$ for r in (3.8) and using (3.8), we have

$$F(F(r,z,s),x,v)=0 \text{ for all } r,s,v,x,z\in U. \tag{3.9}$$

Replacing r by rt in (3.9) and using (3.9), we get

$$F(r,z,s)\Delta(t,z,v)+F(r,x,v)\Delta(t,z,s)+r\Delta(\Delta(t,z,s),x,v)=0 \quad \text{for all } r,s,v,t,x,z\in U. \tag{3.10}$$

Substituting $f(r)$ for r in (3.10) and using (3.7), we get

$$f(r)\Delta(\Delta(t,z,s),x,v)=0 \text{ for all } r,s,v,t,x,z\in U,$$

i.e.,

$$\Delta(\Delta(t,z,s),x,v)f(r)=0 \text{ for all } r,s,v,t,x,z\in U.$$

Applying Lemma 2.9 (ii), we obtain

$$\Delta(\Delta(t,z,s),x,v)=0 \text{ for all } s,v,t,x,z\in U. \tag{3.11}$$

Taking wy instead of w in (3.11) and using (3.11), we get

$$\Delta(t,z,s)\Delta(y,x,v)+\Delta(t,x,v)\Delta(y,z,s)=0 \text{ for all } s,v,t,x,y,z\in U. \tag{3.12}$$

Replacing x,z,s,v by t in (3.12), we get

$$\delta(t)\Delta(y,t,t)=0 \text{ for all } t,y\in U. \tag{3.13}$$

Taking yr instead of y in (3.13) and using (3.13), we get

$$\delta(t)y\Delta(r,t,t)=0$$

and so

$$\delta(t)U\delta(t)=\{0\} \text{ for all } t\in U.$$

By Lemma 2.2 (i), we get $\delta(t)=0$, for all $t\in U$. Applying Lemma 2.6, we get $\Delta=0$ on U, a contradiction. ∎

4. Traces of permuting 3-generalized derivations

Ozturk and Yazarli [7, Theorem 3] proved that if N is a 2-torsion free 3-prime near ring and D_1,D_2 are nonzero symmetric (σ,τ)-biderivations of N with trace d_1 and d_2 respectively such that $d_2(y),d_2(y)+d_2(y)\in C(D_1(x,z))$, the centralizer of $(D_1(x,z)$, for all $x,y,z\in N$, then $(N,+)$ is abelian and $d_2(N)\subseteq Z$. Later in [8, Theorem 5] they obtained that if N is 2, 3-torsion free 3-prime near ring and F is a nonzero permuting 3-right generalized derivation of N with associated 3-derivation D such that $f(x),f(x)+f(x)\in C(D(y,z,w))$ for all $w,x,y,z\in N$, where f is the trace of F, then $(N,+)$ is abelian and $f(x)\in Z$ for all $x\in N$. Now we prove the following:

Theorem 4.1 *Let N be a 3!-torsion free 3-prime near ring and U be a nonzero additive subgroup and a semigroup ideal of N. Suppose Δ is a nonzero 3-derivation of N and F is a nonzero permuting 3-right generalized derivation of N associated with Δ. If $f(x),f(x)+f(x)\in C(\Delta(y,z,w))$, for all $w,x,y,z\in U$, where f is the trace of F, then $(N,+)$ is abelian. Moreover, if $\delta(U)\subseteq U$, where δ is the trace of Δ, then $f(U)\subseteq Z$.*

Proof. For all $v,w,x,y,z\in U$

$$\begin{aligned}
&\Delta(v+y,z,w)(f(x)+f(x))\\
&\quad=(f(x)+f(x))\Delta(v+y,z,w)\\
&\quad=(f(x)+f(x))[\Delta(v,z,w)+\Delta(y,z,w)]\\
&\quad=(f(x)+f(x))\Delta(v,z,w)+(f(x)+f(x))\Delta(y,z,w) \qquad (4.1)\\
&\quad=\Delta(v,z,w)(f(x)+f(x))+\Delta(y,z,w)(f(x)+f(x))\\
&\quad=\Delta(v,z,w)f(x)+\Delta(v,z,w)f(x)+\Delta(y,z,w)f(x)+\Delta(y,z,w)f(x)\\
&\quad=f(x)\Delta(v,z,w)+f(x)\Delta(v,z,w)+f(x)\Delta(y,z,w)+f(x)\Delta(y,z,w).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\Delta(v+y,z,w)(f(x)+f(x)) \\
&\quad = \Delta(v+y,z,w)f(x)+\Delta(v+y,z,w)f(x) \\
&\quad = f(x)\Delta(v+y,z,w)+f(x)\Delta(v+y,z,w) \\
&\quad = f(x)[\Delta(v,z,w)+\Delta(y,z,w)]+f(x)[\Delta(v,z,w)+\Delta(y,z,w)] \\
&\quad = f(x)\Delta(v,z,w)+f(x)\Delta(y,z,w)+f(x)\Delta(v,z,w)+f(x)\Delta(y,z,w), \\
&\quad \text{for all } v,w,x,y,z\in U.
\end{aligned} \tag{4.2}$$

Comparing (4.1) and (4.2), we obtain

$$f(x)\Delta((v,y),z,w)=0 \text{ for all } v,w,x,y,z\in U.$$

By hypothesis we get

$$\Delta((v,y),z,w)f(x)=0 \text{ for all } v,w,x,y,z\in U.$$

Hence it follows from Lemma 2.9 (i), that

$$\Delta((v,y),z,w)=0 \text{ for all } v,w,x,y,z\in U.$$

Replacing v by sv and y by sy, we get

$$\Delta(s,z,w)(v,y)=0, \text{ for all } s\in U.$$

Replacing v by vr and y by vp, we get

$$\Delta(s,z,w)v(r,p)=0, \text{ for all } s,v,w,z\in U \text{ and } r,p\in N,$$

i.e.,

$$\Delta(s,z,w)U(r,p)=\{0\}.$$

Using Lemma 2.2 (i) we get either $\Delta(s,z,w)=0$ or $(r,p)=0$. If $\Delta(s,z,w)=0$, for all $s,z,w\in U$, then $\Delta=0$ on U, a contradiction by Lemma 2.4. Hence $(r,p)=0$, for $r,p\in N$ and $(N,+)$ is abelian.

Since $f(x)\in C(\Delta(y,z,w))$, for all $x,y,z,w\in U$, we have

$$f(x)\Delta(y,z,w)=\Delta(y,z,w)f(x) \text{ for all } x,y,z,w\in U. \tag{4.3}$$

Replacing y by yv in (4.3), we obtain

$$f(x)\Delta(y,z,w)v+f(x)y\Delta(v,z,w)=\Delta(y,z,w)vf(x)+y\Delta(v,z,w)f(x) \tag{4.4}$$

Replacing y by $\delta(y)$ in (4.4), and using the hypothesis, we get

$$\Delta(\delta(y),z,w)[v,f(x)]=0 \text{ for all } v,w,x,y,z\in U. \tag{4.5}$$

Substituting zt for z in (4.5), we get

$$\Delta(\delta(y), z, w)t[v, f(x)] = 0 \text{ for all } t, v, w, x, y, z \in U$$

i.e.,

$$\Delta(\delta(y), z, w)U[v, f(x)] = 0 \text{ for all } v, w, x, y, z \in U.$$

By Lemma 2.2 (i), we get either $\Delta(\delta(y), z, w)=0$ or $[f(x), v]=0$ for $v, w, x, y, z \in U$. Suppose that $\Delta(\delta(y), z, w) = 0$ for all $y, z, w \in U$. Taking $y+v$ instead of y, we get

$$\Delta(\delta(y), z, w) + \Delta(\delta(v), z, w) + 3\Delta(\Delta(y, y, v), z, w) + 3\Delta(\Delta(y, v, v), z, w) = 0.$$

Since $\Delta(\delta(y), z, w) = 0$ and N is 3!-torsion free, we get

$$\Delta(\Delta(y, y, v), z, w) + \Delta(\Delta(y, v, v), z, w) = 0 \text{ for all } v, w, y, z \in U. \tag{4.6}$$

Replacing y by $-y$ in (4.6), we have

$$\Delta(\Delta(y, y, v), z, w) - \Delta(\Delta(y, v, v), z, w) = 0 \text{ for all } v, w, y, z \in U. \tag{4.7}$$

Combining (4.6) and (4.7), we obtain

$$\Delta(\Delta(y, v, v), z, w) = 0 \text{ for all } v, w, y, z \in U. \tag{4.8}$$

Replacing y by yx in (4.8) and using (4.8), we have

$$\Delta(y, v, v)\Delta(x, z, w) + \Delta(y, z, w)\Delta(x, v, v) = 0 \text{ for all } v, w, x, y, z \in U.$$

Taking xt instead of x in the above relation, we get

$$\Delta(y, v, v)x\Delta(t, z, w) + \Delta(y, z, w)x\Delta(t, v, v) = 0 \text{ for all } t, v, w, y, z \in U. \tag{4.9}$$

Replacing t, w, y, z by v in (4.9), we get $\delta(v)x\delta(v) = 0$, for all $x, v \in U$ i.e., $\delta(v)U\delta(v) = \{0\}$. By Lemma 2.2 (i), we get $\delta(v) = 0$, for all $v \in U$. Lemma 2.6 yields that $\Delta = 0$ on U, a contradiction by Lemma 2.5. Thus $[f(x), v] = 0$, for all $v, x \in U$. Replacing v by vr, for all $r \in N$, we get $v[f(x), r] = 0$, i.e., $U[f(x), r] = 0$ for all $x \in U$ and $r \in N$. Again by Lemma 2.2 (ii), we get $[f(x), r] = 0$ for all $x \in U$ and $r \in N$. Hence $f(U) \subseteq Z$. ∎

Theorem 4.2 *Let N be a 3!-torsion free 3-prime near ring and U be a nonzero additive subgroup and a semigroup ideal of N. Let Δ be a 3-derivation on N. If F is a nonzero permuting 3-generalized derivation of N associated with Δ such that $f(x), f(x) + f(x) \in C(F(u, v, w))$, for all $u, v, w, x \in U$, where f is the trace of F, then $(N, +)$ is abelian.*

Proof. For all $p, u, v, w, x \in U$

$$\begin{aligned}
&F(u+p, v, w)(f(x)+f(x)) \\
&\quad = (f(x)+f(x))F(u+p, v, w) \\
&\quad = (f(x)+f(x))[F(u, v, w)+F(p, v, w)] \\
&\quad = (f(x)+f(x))F(u, v, w)+(f(x)+f(x))F(p, v, w) \qquad (4.10)\\
&\quad = F(u, v, w)(f(x)+f(x))+F(p, v, w)(f(x)+f(x)) \\
&\quad = F(u, v, w)f(x)+F(u, v, w)f(x)+F(p, v, w)f(x)+F(p, v, w)f(x) \\
&\quad = f(x)F(u, v, w)+f(x)F(u, v, w)+f(x)F(p, v, w)+f(x)F(p, v, w).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&F(u+p, v, w)(f(x)+f(x)) \\
&\quad = F(u+p, v, w)f(x)+F(u+p, v, w)f(x) \\
&\quad = f(x)F(u+p, v, w)+f(x)F(u+p, v, w) \\
&\quad = f(x)[F(u, v, w)+F(p, v, w)]+f(x)[F(u, v, w)+F(p, v, w)] \qquad (4.11)\\
&\quad = f(x)F(u, v, w)+f(x)F(p, v, w)+f(x)F(u, v, w)+f(x)F(p, v, w) \\
&\quad \text{for all } p, u, v, w, x \in U.
\end{aligned}$$

Comparing (4.10) and (4.11), we obtain

$$f(x)F((u, p), v, w) = 0 \text{ for all } p, u, v, w, x \in U.$$

By hypothesis we get

$$F((u, p), v, w)f(x) = 0 \text{ for all } p, u, v, w, x \in U.$$

Hence it follows from Lemma 2.10 (ii), that

$$F((u, p), v, w) = 0 \text{ for all } p, u, v, w \in U. \qquad (4.12)$$

First, let us consider the case $\Delta = 0$. Substituting uz for u and up for p in (4.12), we get

$$F(u, v, w)(z, p)+u\Delta((z, p), v, w) = 0 \text{ for all } p, u, v, w, z \in U. \qquad (4.13)$$

Thus

$$F(u, v, w)(z, p) = 0 \text{ for all } p, u, v, w, z \in U. \qquad (4.14)$$

Replacing z by zr and p by zs in (4.14), we get

$$F(u,v,w)z(r,s)=0 \text{ for all } u,v,w,z\in U \text{ and } r,s\in N$$

i.e.,

$$F(u,v,w)U(r,s)=0 \text{ for all } u,v,w\in U \text{ and } r,s\in N. \tag{4.15}$$

Invoking Lemma 2.2 (i), either $(r,s)=0$ or $F(u,v,w)=0$. Later yields a contradiction by Lemma 2.11. Hence $(r,s)=0$ for all $r,s\in N$ and $(N,+)$ is abelian.

Now, let us consider the case $\Delta\neq 0$. Again substituting uz for u and up for p in (4.12), we get

$$\Delta(u,v,w)(z,p)+uF((z,p),v,w)=0 \text{ for all } p,u,v,w,z\in U.$$

Using (4.12) we have

$$\Delta(u,v,w)(z,p)=0 \text{ for all } p,u,v,w,z\in U. \tag{4.16}$$

Replacing z by zr and p by zs in (4.16), we obtain

$$\Delta(u,v,w)z(r,s)=0 \text{ for all } u,v,w,z\in U \text{ and } r,s\in N$$

i.e.,

$$\Delta(u,v,w)U(r,s)=\{0\} \text{ for all } u,v,w\in U \text{ and } r,s\in N. \tag{4.17}$$

Applying Lemma 2.2 (i), we get either $(r,s)=0$ or $\Delta(u,v,w)=0$. Later yields a contradiction by Lemma 2.5. Hence $(r,s)=0$ for all $r,s\in N$ and $(N,+)$ is abelian. ∎

Theorem 4.3 *Let N be a 3!-torsion free 3-prime near ring and U be a nonzero additive subgroup and a semigroup ideal of N. Suppose Δ is a 3-derivation on N and F is a nonzero permuting 3-generalized derivation of N associated with Δ such that $f(U)\subseteq U$ and $\delta(U)\subseteq U$, where f and δ are the trace of F and Δ respectively. If $f(x), f(x)+f(x)\in C(F(u,v,w))$, for all $u,v,w,x\in U$, then N is a commutative ring.*

Proof. First, let us consider the case $\Delta=0$. Then

$$F(xy,z,t)=F(x,z,t)y+x\Delta(y,z,t)$$

and we have

$$F(xy,z,t)=F(x,z,t)y \text{ for all } t,x,y,z\in U. \tag{4.18}$$

On the other hand,

$$F(xy,z,t)=\Delta(x,z,t)y+xF(y,z,t),$$

and we have

$$F(xy,z,t)=xF(y,z,t) \text{ for all } t,x,y,z\in U. \tag{4.19}$$

Comparing (4.18) and (4.19) we obtain

$$F(x,z,t)y = xF(y,z,t) \text{ for all } t,x,y,z \in U. \tag{4.20}$$

Replacing x by y in (4.20), we obtain

$$[F(y,z,t),y] = 0 \text{ for all } t,y,z \in U. \tag{4.21}$$

Substituting zr for z in (4.21), we get

$$[F(y,z,t)r,y] = 0 \text{ for all } t,y,z \in U \text{ and } r \in N.$$

Using (4.21), we find

$$F(y,z,t)[r,y] = 0 \text{ for all } t,y,z \in U \text{ and } r \in N. \tag{4.22}$$

Substituting rs for r in (4.22) and using (4.22), we get

$$F(y,z,t)r[s,y] = 0 \text{ for all } t,y,z \in U \text{ and } r,s \in N,$$

i.e.,

$$F(y,z,t)N[s,y] = 0 \text{ for all } t,y,z \in U \text{ and } s \in N.$$

Since N is a 3-prime near ring, then either $[s,y] = 0$ or $F(y,z,t) = 0$. Later yields a contradiction by Lemma 2.11. Thus $[s,y] = 0$ for all $y \in U$ and $s \in N$. Hence $U \subseteq Z$ and N is a commutative ring by Lemma 2.3.

Now, let $\Delta \neq 0$. By hypothesis

$$[f(x),F(u,v,w)] = 0 \text{ for all } u,v,w,x \in U. \tag{4.23}$$

Replacing x by $x+y$ in (4.23) and applying Theorem 4.2, we obtain

$$[F(x,x,y),F(u,v,w)]+[F(x,y,y),F(u,v,w)] = 0 \text{ for all } u,v,w,x,y \in U. \tag{4.24}$$

Setting $y = -y$ in (4.24) and comparing the result with (4.24), we obtain

$$[F(x,y,y),F(u,v,w)] = 0 \text{ for all } u,v,w,x,y \in U. \tag{4.25}$$

Replacing y by $y+z$ in (4.25) and using (4.25) and the fact that F is permuting, we have

$$[F(x,y,z),F(u,v,w)] = 0 \text{ for all } u,v,w,x,y,z \in U$$

-i.e.,

$$F(x,y,z)F(u,v,w) = F(u,v,w)F(x,y,z) \text{ for all } u,v,w,x,y,z \in U. \tag{4.26}$$

Substituting ut for u in (4.26) and applying Lemma 2.7 (ii), we obtain

$$\Delta(u,v,w)tF(x,y,z) - F(x,y,z)\Delta(u,v,w)t + uF(t,v,w)F(x,y,z)$$
$$-F(x,y,z)uF(t,v,w) = 0 \text{ for all } t,u,v,w,x,y,z \in U. \tag{4.27}$$

Substituting $f(u)$ for u in (4.27) and then using hypothesis and (4.27), we get

$$\Delta(f(u),v,w)tF(x,y,z) - F(x,y,z)\Delta(f(u),v,w)t = 0 \tag{4.28}$$

Now replacing t by $f(t)$, we have

$$\Delta(f(u),v,w)f(t)F(x,y,z) - F(x,y,z)\Delta(f(u),v,w)f(t) = 0 - i.e.,$$

$$\Delta(f(u),v,w)F(x,y,z)f(t) - F(x,y,z)\Delta(f(u),v,w)f(t) = 0.$$

Applying Lemma 2.4, we get

$$[\Delta(f(u),v,w)F(x,y,z) - F(x,y,z)\Delta(f(u),v,w)]f(t) = 0$$

Using Lemma 2.9 (ii)

$$\Delta(f(u),v,w)F(x,y,z) = F(x,y,z)\Delta(f(u),v,w) \text{ for all } u,v,w,x,y,z \in U$$

and (4.28) yields that

$$\Delta(f(u),v,w)[t,F(x,y,z)] = 0 \text{ for all } t,u,v,w,x,y,z \in U. \tag{4.29}$$

Replacing w by ww' in (4.29), we have

$$\Delta(f(u),v,w)w'[t,F(x,y,z)] = 0 \text{ for all } t,u,v,w,w',x,y,z \in U$$

i.e.,

$$\Delta(f(u),v,w)U[t,F(x,y,z)] = 0 \text{ for all } t,u,v,w,x,y,z \in U.$$

From Lemma 2.2 (i), either $\Delta(f(u),v,w) = 0$ or $[t,F(x,y,z)] = 0$. If $\Delta(f(u),v,w) = 0$ for all $u,v,w \in U$, then arguing in the similar manner as in the proof of Theorem 3.2, we arrive at a contradiction. Thus

$$[t,F(x,y,z)] = 0 \text{ for all } t,x,y,z \in U. \tag{4.30}$$

Substituting tr' for t in (4.30), we get

$$t[r',F(x,y,z)] = 0 \text{ for all } t,x,y,z \in U \text{ and } r' \in N$$

i.e.,

$$U[r',F(x,y,z)] = 0 \text{ for all } x,y,z \in U \text{ and } r' \in N. \tag{4.31}$$

By Lemma 2.2 (ii), we get $[r',F(x,y,z)] = 0$ for all $x,y,z \in U$ and $r' \in N$ and $F(x,y,z) \in Z$ for all $x,y,z \in U$ and N is a commutative ring by Theorem 3.2. ∎

The following examples show that the conditions in the hypothesis of Theorems 4.1, 4.2 and 4.3 are not superfluous.

Example 4.1 In Example 1.1 if we take U to be the set of all matrices of the type $\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$, then U is an additive subgroup but not a semigroup ideal of N. N

is not a 3-prime near ring. It can be easily verified that $\delta(U) \subseteq U$, $f(x), f(x)+f(x) \in C(\Delta(u,v,w))$, for all $u,v,w,x \in U$. But $(N,+)$ is not abelian and $f(U) \nsubseteq Z$. Hence N to be 3-prime near ring and U to be a semigroup ideal are essential in the hypothesis of Theorem 4.1.

Example 4.2 In Example 1.3 if we take U to be the set of all matrices of the type$\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$. Then U is an additive subgroup but not a semigroup ideal of N. N is not a 3-prime near ring and $\delta(U) \subseteq U$, $f(U) \subseteq U$, $f(x), f(x)+f(x) \in C(F(u,v,w))$, for all $u,v,w,x \in U$ and $F(U,U,U) \subseteq Z$. But neither $(N,+)$ is abelian nor N under multiplication is commutative. Hence N to be 3-prime near ring and U to be a semigroup ideal are essential in the hypothesis of Theorems 4.2 and 4.3.

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NOTES ON THE ARITHMETIC-GEOMETRIC MEAN INEQUALITY FOR SINGULAR VALUES

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Abstract. In this short note, we present a generalization and a new equivalent form of the arithmetic-geometric mean inequality for singular values. Meanwhile, we also show some remarks related to generalizations of the arithmetic-geometric mean inequality for singular values.

Keywords: arithmetic-geometric mean inequality, positive semidefinite matrices, singular values.

MSC (2010) Subject Classification: 15A42, 47A63, 47B15.

1. Introduction

Let M_n be the space of $n \times n$ complex matrices. We shall always denote the singular values of A by $s_1(A) \geq ... \geq s_n(A) \geq 0$, that is, the eigenvalues of the positive semidefinite matrix $|A| = (AA^*)^{\frac{1}{2}}$, arranged in decreasing order and repeated according to multiplicity. For $A \in M_n$, let $A^+ = \dfrac{|A| + A}{2}, A^- = \dfrac{|A| - A}{2}$. Let $A, B \in M_n$ be Hermitian, the order relation $A \geq B$ means, as usual, that $A - B$ is positive semidefinite. We use the direct sum notation $A \oplus B$ for the block-diagonal operator $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ defined on $M_n \oplus M_n$.

The well-known arithmetic-geometric mean inequality for singular values due to Bhatia and Kittaneh [1] says that

$$2s_j(AB^*) \leq s_j(A^*A + B^*B)$$

for any $A,\ B \in M_n$ and $j = 1, ..., n$. We state this in another form: Let $A,\ B \in M_n$ be positive semidefinite, then

$$2s_j(AB) \leq s_j\left(A^2 + B^2\right) \tag{1.1}$$

for $j = 1, ..., n$. For more information on singular values and unitarily invariant norms inequalities the reader is referred to [2]-[9].

In section 2, we first give a generalization of inequality (1.1). After that, we give a new equivalent form of inequality (1.1). Section 3 contains some remarks.

2. Main results

To generalize inequality (1.1), we need the following result [5, Theorem 1].

Lemma 2.1. *Let $A,\ X,\ B \in M_n$ such that A and B are positive semidefinite. Then*

$$s_j\left(AX - XB\right) \leq \|X\| \, s_j\left(A \oplus B\right)$$

for $j = 1, ..., n$, where $\|\cdot\|$ denotes the usual operator norm on M_n.

Theorem 2.1. *Let $A,\ B \in M_n$ be positive semidefinite and suppose that $f\left(t\right),\ g\left(t\right)$ are polynomials. Then*

$$s_j\left(ABg\left(B\right) + f\left(A\right)AB\right) \leq \max\left(\left\|f\left(A\right)\right\|, \left\|g\left(B\right)\right\|\right) s_j\left(A^2 + B^2\right)$$

for $j = 1, ..., n$.

Proof. Let

$$K_1 = \begin{bmatrix} A & 0 \\ B & 0 \end{bmatrix} \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} = \begin{bmatrix} A^2X_1 & ABX_2 \\ BAX_1 & B^2X_2 \end{bmatrix}$$

and

$$K_2 = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} \begin{bmatrix} A & 0 \\ -B & 0 \end{bmatrix} \begin{bmatrix} A & -B \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} X_1A^2 & -X_1AB \\ -X_2BA & X_2B^2 \end{bmatrix}.$$

It follows that

$$\begin{aligned} K_1 - K_2 &= \begin{bmatrix} A^2X_1 & ABX_2 \\ BAX_1 & B^2X_2 \end{bmatrix} - \begin{bmatrix} X_1A^2 & -X_1AB \\ -X_2BA & X_2B^2 \end{bmatrix} \\ &= \begin{bmatrix} A^2X_1 - X_1A^2 & ABX_2 + X_1AB \\ BAX_1 + X_2BA & B^2X_2 - X_2B^2 \end{bmatrix}. \end{aligned}$$

So, by Lemma 2.1, we have

$$\begin{aligned} s_j\left(K_1 - K_2\right) &\leq \|X_1 \oplus X_2\| \, s_j\left(\begin{bmatrix} A^2 & AB \\ BA & B^2 \end{bmatrix} \oplus \begin{bmatrix} A^2 & -AB \\ -BA & B^2 \end{bmatrix}\right) \\ &= \max\left(\|X_1\|, \|X_2\|\right) s_j\left(\begin{bmatrix} A^2 & AB \\ BA & B^2 \end{bmatrix} \oplus \begin{bmatrix} A^2 & -AB \\ -BA & B^2 \end{bmatrix}\right) \qquad (2.1) \\ &= \max\left(\|X_1\|, \|X_2\|\right) s_j\left(\begin{bmatrix} A^2 & AB \\ BA & B^2 \end{bmatrix} \oplus \begin{bmatrix} A^2 & AB \\ BA & B^2 \end{bmatrix}\right) \\ &= \max\left(\|X_1\|, \|X_2\|\right) s_j\left(\left(A^2 + B^2\right) \oplus \left(A^2 + B^2\right)\right) \end{aligned}$$

for $j = 1, ..., n$. Let

$$X_1 = f(A), X_2 = g(B), K = ABg(B) + f(A)AB.$$

Then, we have

$$K_1 - K_2 = \begin{bmatrix} 0 & K \\ K^* & 0 \end{bmatrix}.$$

Note that

$$s_j \begin{bmatrix} 0 & K \\ K^* & 0 \end{bmatrix} = s_j \begin{bmatrix} K & 0 \\ 0 & K^* \end{bmatrix},$$

so, by inequality (2.1), we get

$$s_j \begin{bmatrix} K & 0 \\ 0 & K^* \end{bmatrix} \leq \max\left(\|f(A)\|, \|g(B)\|\right) s_j \left(\left(A^2+B^2\right) \oplus \left(A^2+B^2\right)\right)$$

for $j = 1, ..., n$. Thus, we obtain

$$s_j\left(ABg(B) + f(A)AB\right) \leq \max\left(\|f(A)\|, \|g(B)\|\right) s_j\left(A^2+B^2\right)$$

for $j = 1, ..., n$. This completes the proof. ∎

Theorem 2.2. *The following statements are equivalent:*

(i) *Let $A,\ B \in M_n$ be positive semidefinite. Then*

$$s_j(A - B) \leq s_j(A \oplus B) \tag{2.2}$$

for $j = 1, ..., n$.

(ii) *Let $A,\ B \in M_n$ be positive semidefinite. Then*

$$2s_j(AB) \leq s_j\left(A^2 + B^2\right)$$

for $j = 1, ..., n$.

(iii) *Let $A, B, X \in M_n$ such that $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \geq 0$. Then*

$$2s_j(X) \leq s_j \begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$$

for $j = 1, ..., n$. Let $A, B \in M_n$ such that A is self-adjoint, $B \geqslant 0$, and $\pm A \leqslant B$. Then

$$2s_j(A) \leq s_j((B + A) \oplus (B - A))$$

for $j = 1, ..., n$.

(iv) *Let $A, B \in M_n$ such that A and B are self-adjoint. Then*

$$s_j(A + B) \leq s_j\left(\left(A^+ + B^+\right) \oplus \left(A^- + B^-\right)\right) \tag{2.3}$$

for $j = 1, ..., n$.

Proof. Note that (i), (ii), (iii), and (iv) are equivalent by [4, Theorem 2.6]. We will prove that (i) is equivalent to (v), and this will complete the proof of this theorem.

(i) $\Rightarrow$ (v). Since A and B are Hermitian, it follows that

$$\pm (A+B) \leq |A| + |B| .$$

Let

$$Y_1 = A+B,\ Y_2 = |A| + |B| .$$

Now, applying inequality (2.2) to the matrices $Y_2 + Y_1$ and $Y_2 - Y_1$, we have

$$s_j ((Y_2+Y_1) - (Y_2-Y_1)) \leq s_j ((Y_2+Y_1) \oplus (Y_2-Y_1)) ,$$

which is equivalent to

$$s_j (A+B) \leq s_j \left(\left(A^+ + B^+\right) \oplus \left(A^- + B^-\right) \right)$$

for $j = 1, ..., n$. So, we know that inequality (2.2) implies inequality (2.3).

(v) $\Rightarrow$ (i). Audeh and Kittaneh [4, p.2521] pointed out that inequality (2.3) implies inequality (2.2). So, (i) is equivalent to (v). This completes the proof. ∎

3. Remarks

Remark 3.1. Let $f(t) = g(t) = 1$ in Theorem 2.1, we obtain inequality (1.1). Let $f(t) = g(t) = t$ in Theorem 2.1, we get

$$s_j (A(A+B)B) \leq \max (\|A\|, \|B\|)\, s_j \left(A^2 + B^2\right)$$

for $j = 1, ..., n$. This is a matrix version of the following inequality:

$$a(a+b)b \leq \max(a,b) \left(a^2 + b^2\right), a \geq 0, b \geq 0.$$

Remark 3.2. Let $A,\ B,\ X \in M_n$ such that A and B are positive semidefinite. A natural extension of inequality (1.1) is

$$2 s_j (AXB) \leq s_j \left(A^2 X + X B^2\right)$$

for $j = 1, ..., n$. This is not always true. For example, if we choose

$$A = \begin{bmatrix} 0.7389 & 0.6634 & 0.7481 \\ 0.6634 & 1.0265 & 0.7836 \\ 0.7481 & 0.7836 & 1.0908 \end{bmatrix},$$

$$X = \begin{bmatrix} 0.1820 & 0.2744 & 0.8627 \\ 0.8027 & 0.3597 & 0.5781 \\ 0.0095 & 0.5384 & 0.2392 \end{bmatrix},$$

$$B = \begin{bmatrix} 1.7006 & 0.9132 & 0.3608 \\ 0.9132 & 0.8945 & 0.5280 \\ 0.3608 & 0.5280 & 0.9509 \end{bmatrix},$$

then, we have

$$2s_3(AXB) = 0.0289 \geq 0.0106 = s_3\left(A^2X+XB^2\right).$$

Remark 3.3. Let $A,\ B,\ X \in M_n$ such that A and B are positive semidefinite. Another possible extension of inequality (1.1) is

$$2s_j(AXB) \leq \|X\| s_j\left(A^2+B^2\right)$$

for $j = 1, ..., n$. This is refused by

$$A = \begin{bmatrix} 0.4327 & 0.6051 \\ 0.6051 & 0.9762 \end{bmatrix},\ X = \begin{bmatrix} 0.5730 & 0.2149 \\ 0.6816 & 0.1522 \end{bmatrix},\ B = \begin{bmatrix} 1.1111 & 0.1476 \\ 0.1476 & 0.0361 \end{bmatrix}.$$

In fact, we have

$$2s_1(AXB) = 2.8102 \geq 2.4311 = \|X\| s_1\left(A^2+B^2\right).$$

It should be note that the inequality

$$2s_j(AXB) \leq \|X\| s_j\left(A^2+B^2\right)$$

holds when X is positive semidefinite [10].

Remark 3.4. Let $A,\ B,\ X,\ Y \in M_n$ such that A and B are positive semidefinite. A possible generalization of Theorem 2.1 is

$$s_j(ABX+YAB) \leq \max\left(\|X\|, \|Y\|\right) s_j\left(A^2+B^2\right)$$

for $j = 1, ..., n$. This is not always true. For example, if we choose

$$A = \begin{bmatrix} 0.0286 & 0.0414 & 0.0472 \\ 0.0414 & 0.7134 & 0.5265 \\ 0.0472 & 0.5265 & 0.6030 \end{bmatrix},\quad X = \begin{bmatrix} 0.5992 & 0.7265 & 0.6529 \\ 0.4480 & 0.0524 & 0.8624 \\ 0.1893 & 0.6981 & 0.8502 \end{bmatrix}$$

$$Y = \begin{bmatrix} 0.2547 & 0.9273 & 0.1140 \\ 0.7466 & 0.1075 & 0.5175 \\ 0.1717 & 0.2464 & 0.2433 \end{bmatrix},\quad B = \begin{bmatrix} 0.3508 & 0.2981 & 0.3669 \\ 0.2981 & 1.0854 & 0.9269 \\ 0.3669 & 0.9269 & 1.2541 \end{bmatrix},$$

then, we have

$$s_2(ABX+YAB) = 0.3924 \geq 0.2220 = \max\left(\|X\|, \|Y\|\right) s_2\left(A^2+B^2\right).$$

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ON STRONG AND WEAK HYPER KS-IDEALS OF DECOMPOSABLE HYPER KS-SEMIGROUPS

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Abstract. This paper extends the study of hyper KS-semigroups, an algebraic hyperstructure which is a combination of hyper BCK-algebra and semihypergroups with respect to some hyperoperations to the concept of decomposable hyper KS-semigroups. We introduce the concept of strong and weak hyper KS-ideals and present some of their properties. Furthermore, we establish the idea of decomposable hyper KS-semigroups in the context of strong, weak and hyper KS-ideals, give conditions for decomposable hyper KS-semigroups, investigate its properties and structure and provide some characterizations.

Keywords: strong hyper KS-ideals, weak hyper KS-ideal, decomposable hyper KS-semigroups.

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1. Introduction

In 1966, Y. Imai and K. Iséki [4] initiated the study of BCK-algebra as a generalization of the concept of set-theoretic difference and propositional calculi. In 2006, K.H. Kim [8] introduced a new class of algebra called KS-semigroups, which is a combination of BCK-algebra and semigroup. He characterized the KS-semigroups from its ideals up to the first isomorphism theorem while M.P. Cawi and J.P. Vilela [3], proved the second and third isomorphism theorems for KS-semigroups and gave other characterizations parallel to ring theory.

The hyperstructure theory (or multialgebras) was introduced by F. Marty [9] at the 8th Congress of Scandinavian Mathematicians in 1934. In a classical algebraic structure, the composition of two elements is an element, while in an

algebraic hyperstructure, the composition of two elements is a set. In 2000, Y.B. Jun et al. applied hyperstructure theory to BCK-algebras and introduced the notion of hyper BCK-algebras as a generalization of BCK-algebra and investigated some of its properties. In 2014, A.L. Vicedo et al. [10] applied hyperstructure theory in KS-semigroups and introduced the notion of hyper KS-semigroups, a combination of concepts of hyper BCK-algebra and semihypergroups and investigated its structural properties.

This paper extends the study of hyper KS-semigroups to the concept of decomposable hyper KS-semigroups. We establish the idea of decomposable hyper KS-semigroups in the context of strong, weak and hyper KS-ideals, investigate its properties and structure and provide some characterizations. Moreover, we provide conditions for decomposable hyper KS-semigroups.

2. Preliminaries

Let H be a nonempty set endowed with a hyperoperation " $*$ ", that is, " $*$ " is a function from $H \times H$ to $P^*(H) = P(H) \setminus \{\varnothing\}$. For two nonempty subsets A and B of H, $A * B = \bigcup_{a\in A, b\in B} a * b$. We shall use $x * y$ instead of $x * \{y\}, \{x\} * y$ or $\{x\} * \{y\}$. When A is a nonempty subset of H and $x \in H$, we agree to write $A * x$ instead of $A * \{x\}$. Similarly, we write $x * A$ for $\{x\} * A$. In effect, $A * x = \bigcup_{a\in A} a * x$ and $x * A = \bigcup_{a\in A} x * a$. A set H endowed with a family Γ of hyperoperations is called a *hyperstructure*. If Γ is singleton, that is, $\Gamma = \{f\}$, then the hyperstructure is called a *hypergroupoid*. A *semihypergroup* is a hypergroupoid $(H, \cdot)$ such that for all $x, y, z \in H, (x \cdot y) \cdot z = x \cdot (y \cdot z)$, that is, $\bigcup_{u\in x\cdot y} u \cdot z = \bigcup_{v\in y\cdot z} x \cdot v$.

Definition 2.1 [5] A *hyper BCK-algebra* is a nonempty set H endowed with a hyperoperation " $*$ " and a constant 0 satisfying the following axioms: for all $x, y, z \in H$,

(H1) $(x * z) * (y * z) \ll x * y$,

(H2) $(x * y) * z = (x * z) * y$,

(H3) $x * H \ll x$,

(H4) $x \ll y$ and $y \ll x$ imply $x = y$,

where (a) $x \ll y$ is defined by $0 \in x * y$, and (b) for every $A, B \subseteq H$, $A \ll B$ is defined as follows: for all $a \in A$, there exists $b \in B$ such that $a \ll b$. In such case, we call " $\ll$ " the *hyper order* in H.

Example 2.2 [1] Let $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ and consider the hyperoperation $*$ on $\mathbb{N}$ defined as follows: for all $x, y \in \mathbb{N}$,

$$x * y = \begin{cases} \{0, x\}, & \text{if } x \leq y, \\ \{x\}, & \text{if } x > y. \end{cases}$$

Then $(\mathbb{N}, *, 0)$ is a hyper BCK-algebra.

Theorem 2.3 [5], [6] *Let H be a hyper BCK-algebra. Then for all $x \in H$ and for all nonempty subset A of H,*

(i) $x * H \ll x$ *if and only if* $x * y \ll x$ *for all* $y \in H$.

(ii) *if I is a hyper BCK-ideal of H and $A \ll I$, then $A \subseteq I$.*

Definition 2.4 [7] Let I be a nonempty subset of a hyper BCK-algebra H and $0 \in I$. Then I is a strong hyper BCK-ideal of H if $(x * y) \cap I \neq \varnothing$ and $y \in I$ imply $x \in I$ for all $x, y \in H$.

Definition 2.5 [2] A hyper BCK-algebra H is called *decomposable* if there exists a nontrivial family $\{A_i\}_{i\in\Lambda}$ of hyper BCK-ideals of H such that

(i) $H \neq A_i \neq \{0\}$ for all $i \in \Lambda$,

(ii) $H = \bigcup_{i\in\Lambda} A_i$,

(iii) $A_i \cap A_j = \{0\}$ for all $i \neq j \in \Lambda$.

In this case, we say that $H = \bigcup_{i\in\Lambda} A_i$ is a decomposition of H and we write $H = \bigoplus_{i\in\Lambda} A_i$.

Theorem 2.6 [2] *Let H be a decomposable hyper BCK-algebra with decomposition $H = \bigoplus_{i\in\Lambda} A_i$. Then A_i is a strong hyper BCK-ideal of H for all $i \in \Lambda$.*

Definition 2.7 [2] Let H be a hyper BCK-algebra and $\varnothing \neq A \subset H$. Then the subset I of H is called a *weak hyper* (resp. *hyper*) *BCK-ideal of H related to A* if

(i) $0 \in I$; and

(ii) $x * y \subseteq I$ (resp. $x * y \ll I$) and $y \in I$ imply $x \in I$ for all $x \in A$.

Theorem 2.8 [2] *Let H be decomposable hyper BCK-algebra with decomposition $H = X \bigoplus Y$, where X and Y are strong hyper BCK-ideals of H and $I \subseteq X$. If I is a weak hyper (resp. hyper) BCK-ideal of H related to X, then I is a weak hyper (resp. hyper) BCK-ideal of H.*

Definition 2.9 [10] A *hyper KS-semigroup* is a nonempty set H together with two hyperoperations " $*$ " and " $\cdot$ " and a constant 0 satisfying the following conditions:

(i) $(H, *, 0)$ is a hyper BCK-algebra.

(ii) $(H, \cdot)$ is a semihypergroup having zero as a bilaterally absorbing element, that is, $x \cdot 0 = 0 \cdot x = \{0\}$ for all $x \in H$; and

(iii) " $\cdot$ " is left and right distributive over " $*$ ", that is, for any $x, y, z \in H$, $x \cdot (y * z) = (x \cdot y) * (x \cdot z)$ and $(x * y) \cdot z = (x \cdot z) * (y \cdot z)$.

Example 2.10 [10] Consider the hyper BCK-algebra $(\mathbb{N}, *, 0)$ in Example 2.2 and the semihypergroup $(\mathbb{N}, \cdot)$ defined by $x \cdot y = \{0, \min\{x, y\}\}$ for all $x, y \in \mathbb{N}$. Then $(\mathbb{N}, *, \cdot, 0)$ is a hyper KS-semigroup.

Lemma 2.11 [10] *Let H be a hyper KS-semigroup and A, B and C be nonempty subsets of H. Then $A \ll B$ and $B \subseteq C$ imply $A \ll C$.*

In what follows, H shall mean a hyper KS-semigroup $(H, *, \cdot, 0)$ and we write $x \cdot y$ as xy for all $x, y \in H$.

Definition 2.12 [10] Let I be a nonempty subset of a hyper KS-semigroup H. Then I is a *hyper subKS-semigroup* of H if for all $x, y \in I$, $x * y \subseteq I$ and $xy \subseteq I$. I is a *hyper left* (resp. *hyper right*) *stable* if $xa \subseteq I$ (resp. $ax \subseteq I$) for all $x \in H$ and for all $a \in I$. I is a *hyper stable* if I is both hyper left and right stable. I is a *hyper left* (resp. *hyper right*) *KS-ideal* if I is a hyper left (resp. hyper right) stable and for any $x, y \in H$, $x * y \ll I$ and $y \in I$ imply that $x \in I$. I is a *hyper KS-ideal* if I is both a hyper left and a hyper right KS-ideal.

Theorem 2.13 [10] *Let $\{A_i : i \in \mathscr{I}\}$ be a nonempty collection of nonempty subsets of a hyper KS-semigroup H. If A_i is a hyper KS-ideal of H for all $i \in \mathscr{I}$, then so is $\bigcap_{i \in \mathscr{I}} A_i$.*

Theorem 2.14 [10] *The hyper product of two hyper KS-semigroups is a hyper KS-semigroup.*

3. On Decomposable Hyper KS-semigroups

In this section, we introduce the notion of decomposable hyper KS-semigroups and investigate some of its properties.

Definition 3.1 A hyper KS-semigroup H is called *decomposable* if there exists a nontrivial family $\{A_i\}_{i \in \Lambda}$ of hyper KS-ideals of H such that

(i) $H \neq A_i \neq \{0\}$ for all $i \in \Lambda$,

(ii) $H = \bigcup_{i \in \Lambda} A_i$,

(iii) $A_i \cap A_j = \{0\}$ for all $i \neq j \in \Lambda$.

In this case, we say that $H = \bigcup_{i \in \Lambda} A_i$ is a decomposition of H and we write $H = \bigoplus_{i \in \Lambda} A_i$.

Example 3.2 Consider the hyper hyper KS-semigroup H with hyperoperations given in the following Cayley's tables.

$*$	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$	$\{1\}$
2	$\{2\}$	$\{2\}$	$\{0,2\}$	$\{2\}$
3	$\{3\}$	$\{3\}$	$\{3\}$	$\{0,3\}$

$\cdot$	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{0\}$	$\{1\}$	$\{0\}$	$\{0\}$
2	$\{0\}$	$\{0\}$	$\{0,2\}$	$\{0\}$
3	$\{0\}$	$\{0\}$	$\{0\}$	$\{0,3\}$

Observe that $A_1 = \{0,1\}$, $A_2 = \{0,2\}$ and $A_3 = \{0,3\}$ are nontrivial hyper KS-ideals of H such that $A_1 \cup A_2 \cup A_3 = H$ and for all $i \neq j$, $A_i \cap A_j = \{0\}$. Thus, H is a decomposable hyper KS-semigroup.

Theorem 3.3 *Let H be a decomposable hyper KS-semigroup and $N \subseteq H$. If N is a hyper KS-ideal of H, then N is decomposable.*

Proof. Let H be a decomposable hyper KS-semigroup and suppose $N \subseteq H$. Then there exist a family of nontrivial hyper KS-ideals A_i for all $i \in \Lambda$ such that $H = \bigcup_{i\in\Lambda} A_i$ and $A_i \cap A_j = \{0\}$ for $i \neq j$. By Theorem 2.13, $N \cap A_i$ is a hyper KS-ideal of H for all $i \in \Lambda$. Now, we first show that $N = \bigcup_{i\in\Lambda} (N \cap A_i)$. Clearly, $\bigcup_{i\in\Lambda} (N \cap A_i) \subseteq N$. Let $n \in N$. Then there exist $j \in \Lambda$ such that $n \in A_j$. Thus, $n \in N \cap A_j \subseteq \bigcup_{i\in\Lambda} (N \cap A_i)$. This implies that $N \subseteq \bigcup_{i\in\Lambda} (N \cap A_i)$. Next, if $i \neq j$, then $(N \cap A_i) \cap (N \cap A_j) = N \cap (A_i \cap A_j) = N \cap \{0\} = \{0\}$. Therefore, N is decomposable. ∎

Theorem 3.4 *Let H_1 and H_2 be both decomposable hyper KS-semigroups. Then $H_1 \times H_2$ is a decomposable hyper KS-semigroup.*

Proof. Let H_1 and H_2 be decomposable hyper KS-semigroups. By Theorem 2.14, $H_1 \times H_2$ is a hyper KS-semigroup. Then there exist nontrivial families $\{A_i\}_{i\in\Lambda}$ and $\{B_s\}_{s\in\Phi}$ of hyper KS-ideals of H_1 and H_2, respectively. Now, since $H_1 \neq A_i \neq \{0\}$ for all $i \in \Lambda$ and $H_2 \neq B_s \neq \{0\}$ for all $j \in \Phi$, it follows that $H_1 \times H_2 \neq A_i \times B_s \neq \{0\}$. Moreover, since $H_1 = \bigcup_{i\in\Lambda} A_i$ and $H_2 = \bigcup_{s\in\Phi} B_s$, we have

$$H_1 \times H_2 = \bigcup_{i\in\Lambda} A_i \times \bigcup_{s\in\Phi} B_s = \bigcup_{i\in\Lambda, s\in\Phi} (A_i \times B_s).$$

Finally, since $A_i \cap A_j = \{0\}$ for $i \neq j$ and $B_s \cap B_t = \{0\}$ for $s \neq t$, then we show that $(A_i \times B_s) \cap (A_j \times B_t) = \{0\}$. Let $z \in (A_i \times B_s) \cap (A_j \times B_t)$. Then $z \in (A_i \times B_s)$ and $z \in (A_j \times B_t)$. Thus, $z = (a,b)$ with $a \in A_i, A_j$ and $b \in B_s, B_t$. This implies that $a \in A_i \cap A_j = \{0\}$ and $b \in B_s \cap B_t$. Hence, $z = 0$ and so the proof is complete. ∎

Definition 3.5 Let I be a nonempty subset of a hyper KS-semigroup H and $0 \in I$. Then I is said to be a *strong hyper left* (resp. *strong hyper right*) *KS-ideal* of H if

(i) I is a hyper left (resp. hyper right) stable;

(ii) for all $x, y \in H, (x * y) \cap I \neq \varnothing$ and $y \in I$ imply that $x \in I$.

I is a *strong hyper KS-ideal* of H if I is both strong hyper left KS-ideal and strong hyper right KS-ideal of H.

Example 3.6 Consider the hyper KS-semigroup $\mathbb{N} = \{0, 1, 2, ...\}$ in Example 2.10 and take $I = \{0, 1\}$. Then I is a strong hyper KS-ideal of $\mathbb{N}$.

Remark 3.7 Every strong hyper KS-ideal of a hyper KS-semigroup H is a hyper KS-ideal of H but the converse is not true, in general.

Example 3.8 Consider the hyper KS-semigroup $H = \{0, 1, 2\}$ in [10] given by the Cayley's tables below

$*$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0, 1\}$	$\{0, 1\}$
2	$\{2\}$	$\{1, 2\}$	$\{0, 1, 2\}$

$\cdot$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{0\}$	$\{1\}$	$\{0, 1\}$
2	$\{0\}$	$\{0, 1\}$	$\{0, 1, 2\}$

It was shown in [10] that $I = \{0, 1\}$ is a hyper KS-ideal of H. However, I is not a strong hyper KS-ideal of H since $\{1\} = (2 * 1) \cap I \neq \varnothing$ and $1 \in I$ but $2 \notin I$.

Example 3.9 Consider the decomposable hyper KS-semigroup $H = \{0, 1, 2, 3\}$ in Example 3.2 with hyper KS-ideals $A_1 = \{0, 1\}$, $A_2 = \{0, 2\}$ and $A_3 = \{0, 3\}$. Then A_1, A_2 and A_3 are strong hyper KS-ideals of H.

In Example 3.9, H is a decomposable hyper KS-semigroup and a hyper KS-ideal of H is a strong hyper KS-ideal H. This shows that the converse of Remark 3.7 holds whenever H is decomposable. We deduce this notion by the following corollary which follows from Theorem 2.6 and the fact that a hyper KS-ideal is hyper stable.

Corollary 3.10 *Let H be a decomposable hyper KS-semigroup with decomposition $H = \bigoplus_{i\in\Lambda} A_i$. Then A_i is a strong hyper KS-ideal of H for all $i \in \Lambda$.*

The following corollary is immediate from Remark 3.7 and Corollary 3.10.

Corollary 3.11 *Let H be a hyper KS-semigroup. Then H is decomposable if and only if $H = \bigoplus_{i\in\Lambda} A_i$ with A_i is a strong hyper KS-ideal of H for all $i \in \Lambda$.*

Theorem 3.12 *Let $\{A_i : i \in \mathscr{I}\}$ be a nonempty collection of nonempty strong hyper KS-ideal of a hyper KS-semigroup H. Then $\bigcap_{i\in\mathscr{I}} A_i$ is a strong hyper KS-ideal of H.*

Proof. Suppose that A_i is strong hyper KS-ideal of H for all $i \in \mathscr{I}$. Then $0 \in A_i$ for all $i \in \mathscr{I}$ by Definition 3.5. So, $0 \in \bigcap_{i\in\mathscr{I}} A_i$. Thus, $\bigcap_{i\in\mathscr{I}} A_i \neq \varnothing$. Clearly, $\bigcap_{i\in\mathscr{I}} A_i \subseteq A_i$ for all $i \in \mathscr{I}$. Let $a \in \bigcap_{i\in\mathscr{I}} A_i$ and $h \in H$. Then $a \in A_i$ for all $i \in \mathscr{I}$. Since A_i is hyper stable, it follows that $ah, ha \subseteq A_i$ for all $i \in \mathscr{I}$. Thus, $ah, ha \subseteq \bigcap_{i\in\mathscr{I}} A_i$. This means that $\bigcap_{i\in\mathscr{I}} A_i$ is hyper stable.

Suppose that $x, y \in H$ such that $(x * y) \cap \left(\bigcap_{i\in\mathscr{I}} A_i\right) \neq \varnothing$ and $y \in \bigcap_{i\in\mathscr{I}} A_i$. Then $\bigcap_{i\in\mathscr{I}} [(x * y) \cap A_i] \neq \varnothing$. Thus, $(x * y) \cap A_i \neq \varnothing$ for all $i \in \mathscr{I}$. Since A_i is a strong hyper KS-ideal of H and $y \in A_i$ for all $i \in \mathscr{I}$, we have $x \in A_i$ for all $i \in \mathscr{I}$. Hence, $x \in \bigcap_{i\in\mathscr{I}} A_i$. Therefore, $\bigcap_{i\in\mathscr{I}} A_i$ is a strong hyper KS-ideal of H. ∎

Theorem 3.13 *Let H be a decomposable hyper KS-semigroup with decomposition $H = \bigoplus_{i\in\Lambda} A_i$. Then $A_i \cup A_j$ is a strong hyper KS-ideal of H for all $i, j \in \Lambda$.*

Proof. Let $h \in H$ and $a \in A_i \cup A_j$. Then $a \in A_i$ or $a \in A_j$. Since A_i and A_j are strong hyper KS-ideals of H, it follows that $ah, ha \subseteq A_i$ or $ah, ha \subseteq A_j$. Thus, $ah, ha \subseteq A_i \cup A_j$, that is, $A_i \cup A_j$ is hyper stable. Let $i, j \in \Lambda$ and $x, y \in H$ such that $(x * y) \cap (A_i \cup A_j) \neq \varnothing$ and $y \in A_i \cup A_j$. Without loss of generality, suppose that $y \in A_i$. Since $(x * y) \cap (A_i \cup A_j) \neq \varnothing$, we have $(x * y) \cap A_i \neq \varnothing$ or $(x * y) \cap A_j \neq \varnothing$. If $(x * y) \cap A_i \neq \varnothing$, then $x \in A_i$ since A_i is a strong hyper KS-ideal of H and $y \in A_i$. Suppose $(x * y) \cap A_j \neq \varnothing$. Then there exists $t \in (x * y) \cap A_j$. Since $x \in H = \bigcup_{i\in\Lambda} A_i$, there exists $k \in \Lambda$ such that $x \in A_k$. By Definition 2.1(H3), $x * H \ll x$. By Theorem 2.3(i), $x * y \ll x \in A_k$. By Lemma 2.11, $x * y \ll A_k$ so that by Theorem 2.3(ii), $x * y \subseteq A_k$. Thus, $t \in A_k$ and so $t \in A_j \cap A_k$. If $j = k$, then $A_j = A_k$ which implies that $x \in A_j \subseteq A_i \cup A_j$. Suppose $j \neq k$. Then $t \in A_j \cap A_k = \{0\}$. Hence, $0 \in x * y$ and so $x \ll y$. Since $x \ll y \in A_i$ and A_i is a hyper BCK-ideal of H, it follows that $x \in A_i \subseteq A_i \cup A_j$. Therefore, $A_i \cup A_j$ is a strong hyper KS-ideal of H. ∎

The following corollary is immediate from Theorem 3.13.

Corollary 3.14 *Let H be a decomposable hyper KS-semigroup with decomposition $H = \bigoplus_{i\in\Lambda} A_i$. Then $\bigcup_{i\in\Omega} A_i$ is a strong hyper KS-ideal of H for a nonempty finite subset Ω of Λ.*

Theorem 3.15 *Let H be a hyper KS-semigroup. Then H is decomposable if and only if there exist nontrivial strong hyper KS-ideals $X, Y \subseteq H$ such that $H = X \cup Y$ and $X \cap Y = \{0\}$.*

Proof. Let H be a hyper KS-semigroup. The necessary condition follows directly from Corollary 3.11 and Theorem 3.13. Suppose there exist nontrivial strong hyper KS-ideals $X, Y \subseteq H$ such that $H = X \cup Y$ and $X \cap Y = \{0\}$. By Remark 3.7, X and Y are nontrivial hyper KS-ideals of H. Thus, by Definition 3.1, H is decomposable. ∎

Definition 3.16 Let H be a hyper KS-semigroup and $\varnothing \neq X \subset H$. Then a nonempty subset I of H is called

(a) a *weak hyper left (*resp. *weak hyper right) KS-ideal* of H if

(i) $0 \in I$;

(ii) I is a hyper left (resp. hyper right) stable; and

(iii) for all $x, y \in H, x * y \subseteq I$ and $y \in I$ imply that $x \in I$.

I is a *weak hyper KS-ideal* of H if I is both weak hyper left KS-ideal and weak hyper right KS-ideal of H.

(b) a *weak hyper left (*resp. *weak hyper right) KS-ideal of H related to X* if

(i) $0 \in I$;

(ii) I is a hyper left (resp. hyper right) stable related to X, that is, $xa \subseteq I$ (resp. $ax \subseteq I$) for all $x \in X$ and for all $a \in I$; and

(iii) $x * y \subseteq I$ and $y \in I$ imply that $x \in I$ for all $x \in X$.

I is a *weak hyper KS-ideal of H related to X* if I is both weak hyper left KS-ideal of H related to X and weak hyper right KS-ideal of H related to X.

Example 3.17 Consider the hyper KS-semigroup $\mathbb{N} = \{0, 1, 2, ...\}$ in Example 2.10 and take $I = \{0, 1\}$. Then I is a weak hyper KS-ideal of $\mathbb{N}$.

The following remark follows from Definition 3.16.

Remark 3.18 Let H be a hyper KS-semigroup and $\varnothing \neq X \subset H$. If I is a weak hyper KS-ideal of H, then I is a weak hyper KS-ideal of H related to X but the converse is not true, in general.

Example 3.19 Consider the hyper BCK-algebra $(H, *, 0)$ in [2] and the semihypergroup $(H, \cdot)$ with the following Cayley's tables:

$*$	0	1	2	3
0	{0}	{0}	{0}	{0}
1	{1}	{0}	{0}	{0}
2	{2}	{1}	{0}	{0}
3	{3}	{3}	{3}	{0, 3}

$\cdot$	0	1	2	3
0	{0}	{0}	{0}	{0}
1	{0}	{0}	{0}	{0}
2	{0}	{0}	{0}	{0}
3	{0}	{0}	{0}	{0, 3}

By routine calculations, we can see that H is a hyper KS-semigroup. Consider $I = \{0, 2\}$ and $X = \{0, 2, 3\}$. Then I is a weak hyper KS-ideal of H related to X. However, I is not a weak hyper KS-ideal of H since $\{0\} = 1 * 2 \subseteq I$ and $2 \in I$ but $1 \notin I$.

Example 3.20 Consider the decomposable hyper KS-semigroup H=$\{0, 1, 2, 3\}$ in Example 3.2. Take the strong hyper KS-ideal A_2=$\{0, 2\}$ and X=$\{0, 2, 3\}$. Then A_2 is a weak hyper KS-ideal of H related to X and A_2 is a weak hyper KS-ideal of H.

In Example 3.20, H is a decomposable hyper KS-semigroup. This shows that the converse of Remark 3.18 holds whenever H is decomposable. We deduce this notion by the following theorem.

Theorem 3.21 *Let H be decomposable with decomposition $H = X \bigoplus Y$, where X and Y are strong hyper KS-ideals of H and $I \subseteq X$. If I is a weak hyper KS-ideal of H related to X, then I is a weak hyper KS-ideal of H.*

Proof. Suppose $I \subseteq X$ and I is a weak hyper KS-ideal of H related to X. Let $h \in H$ and $a \in I$. If $h \in X$, then by hypothesis, $ah, ha \subseteq I$. Now, assume that $h \in Y$. Since X and Y are strong hyper KS-ideals of H, it follows that $ah, ha \subseteq Y$ and $ah, ha \subseteq X$. Thus, $ah, ha \subseteq X \cap Y = \{0\}$. So, $ah = ha = \{0\} \subseteq I$. By Theorem 2.8, I is a weak hyper BCK-ideal of H. Therefore, I is a weak hyper KS-ideal of H. ∎

The following corollary follows from Remark 3.18 and Theorem 3.21:

Corollary 3.22 *Let H be a decomposable hyper KS-semigroup with decomposition $H = X \bigoplus Y$, where X and Y are strong hyper KS-ideals of H and $I \subseteq X$. Then I is a weak hyper KS-ideal of H related to X if and only if I is a weak hyper KS-ideal of H.*

Definition 3.23 Let H be a hyper KS-semigroup and $\varnothing \neq X \subset H$. Then the subset I of H is called a *hyper left* (resp. *hyper right*) *KS-ideal of H related to X* if

(i) $0 \in I$;

(ii) I is a hyper left (resp. hyper right) stable related to X, that is, $xa \subseteq I$ (resp. $ax \subseteq I$) for all $x \in X$ and for all $a \in I$; and

(iii) $x * y \ll I$ and $y \in I$ imply that $x \in I$ for all $x \in X$.

I is a *hyper KS-ideal of H related to X* if I is both hyper left KS-ideal of H related to X and hyper right KS-ideal of H related to X.

The following remark follows from Definition 3.23.

Remark 3.24 Let H be a hyper KS-semigroup and $\varnothing \neq X \subset H$. If I is a hyper KS-ideal of H, then I is a hyper KS-ideal of H related to X but the converse is not true, in general.

Example 3.25 Consider the hyper KS-semigroup $H = \{0, 1, 2\}$ in Example 3.19. Take $J = \{0, 1\}$ and $Y = \{0, 1, 3\}$. Then J is a hyper KS-ideal of H related to Y. However, J is not a hyper KS-ideal of H since $\{1\} = 2 * 1 \ll J$ and $1 \in J$ but $2 \notin J$.

Example 3.26 Consider the decomposable hyper KS-semigroup $H = \{0, 1, 2, 3\}$ in Example 3.2. Take the strong hyper KS-ideal $A_1 = \{0, 1\}$ and $Y = \{0, 1, 3\}$. Then A_1 is a hyper KS-ideal of H related to Y.

In Example 3.26, H is a decomposable hyper KS-semigroup. This shows that the converse of Remark 3.24 holds whenever H is decomposable. We deduce this notion by the following theorem.

Theorem 3.27 *Let H be decomposable with decomposition $H = X \bigoplus Y$, where X and Y are strong hyper KS-ideals of H and $I \subseteq X$. If I is a hyper KS-ideal of H related to X, then I is a hyper KS-ideal of H.*

Proof. Suppose $I \subseteq X$ and I is a hyper KS-ideal of H related to X. Let $h \in H$ and $a \in I$. If $h \in X$, then by hypothesis, $ah, ha \subseteq I$. Suppose that $h \in Y$. Since X and Y strong hyper KS-ideals of H, it follows that $ah, ha \subseteq Y$ and $ah, ha \subseteq X$. Thus, $ah, ha \subseteq X \cap Y = \{0\}$. Hence, $ah = ha = \{0\} \subseteq I$. By Theorem 2.8, I is a hyper BCK-ideal of H. Therefore, I is a hyper KS-ideal of H. ∎

The following corollary follows from Remark 3.24 and Theorem 3.27:

Corollary 3.28 *Let H be a decomposable hyper KS-semigroup with decomposition $H = X \bigoplus Y$, where X and Y are strong hyper KS-ideals of H and $I \subseteq X$. Then I is a hyper KS-ideal of H related to X if and only if I is a hyper KS-ideal of H.*

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MORE ON KRASNER (m,n)-HYPERRINGS

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Abstract. In this paper, we introduce (idealistic) soft Krasner (m,n)-hyperrings based on the soft set theory, and derive three isomorphism theorems of soft Krasner (m,n)-hyperrings. Furthermore, we consider T-fuzzy hyperideals and falling fuzzy hyperideals of Krasner (m,n)-hyperrings by using triangular norms and the theory of falling shadows, respectively, and discuss the relationships among idealistic soft Krasner (m,n)-hyperrings, fuzzy hyperideals, T-fuzzy hyperideals and falling fuzzy hyperideals.

Keywords: Krasner (m,n)-hyperring, soft set, fuzzy set, t-norm, falling shadow.

1. Introduction

The theory of algebraic hyperstructures, introduced by Marty in 1934 [26], is a natural generalization of the theory of algebraic structures. In recent years, algebraic hyperstructures have attracted wide attention and had applications in many areas such as probabilities, fuzzy sets, rough sets, cryptography, automata, artificial intelligence, and so on [10]. n-ary hyperstructures, as a generalization of algebraic hyperstructures, introduced by Davvaz and Vougiouklis [13], were studied by many researchers [23], [42], [48]. Recently, the concept of Krasner (m,n)-ary hyperrings was defined by Mirvakili and Davvaz [27], which is a generalization of Krasner hyperrings [22]. Furthermore, Ameri and Norouzi [6] studied the prime and primary hyperideals in Krasner (m,n)-hyperrings. Also, they introduced multiplication (m,n)-hypermodules (with canonical (m,n)-hypergroups) over a given Krasner (m,n)-hyperring [7].

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After the introduction of fuzzy sets by Zadeh in 1965 [44], the theory was developed into many directions [45]. In algebraic structures, Rosenfeld [31] defined the fuzzy subgroup of a group in 1971, which initiated the study of fuzzy algebraic structures [9], [17]. In 1979, Anthony and Sherwood [8] redefined the fuzzy subgroup by using the notion of statistical triangular norms, which was proposed by Schweizer and Sklar [32] in order to generalize the ordinary triangular inequality in a metric space to the more general probabilistic metric space. Furthermore, Yuan and Lee [43] established a theoretical approach of the fuzzy algebraic systems based on the theory of falling shadows [36]. The study of connections between the fuzzy sets and algebraic hyperstructures was initiated by Zahedi et al. [46], and continued by many researchers [24], [47]. For further development, Davvaz and Corsini [12] applied the fuzzy set theory to n-ary hyperstructures and defined the concept of fuzzy n-ary subhypergroup of an n-ary hypergroup, as a generalization of the concept of fuzzy subhypergroup. As a continuation of ideas presented in [12], Davvaz [11] introduced the notion of fuzzy hyperideals of a Krasner (m, n)-hyperring and extended the fuzzy results to Krasner (m, n)-hyperrings. Moreover, Zhan et al. [48] introduced the T-fuzzy n-ary subhypergroups of an n-ary hypergroup by using triangular norms, and considered the probabilistic version of n-ary hypergroups by using random sets.

On the other hand, soft set theory, introduced by Molodtsov [28] in 1999, has been considered as an effective mathematical tool for modeling uncertainties. Different from traditional mathematical tools for dealing with uncertainties, such as probability theory, fuzzy set theory [44], vague set theory [16] and rough set theory [29], soft set theory is free from the inadequacy of the parametrization tools of these theories [28]. Molodtsov demonstrated that soft set theory has potential applications in many directions, including Riemann integration, function smoothness, game theory, measurement theory and operations research [28]. Also, soft set theory has been applied to association rules mining [18], forecasting [39], [40], mobile cloud computing [35], and decision making [14], [21, 50]. Recently, many researchers studied the algebraic structures of soft sets. Aktaş and Çağman [3], [4] defined soft groups and showed that fuzzy groups can be considered a special case of soft groups. Moreover, some basic properties of soft semirings [15], soft rings [2] and soft modules [33] were introduced. Also, Jun et al. [19], [20], [49] considered the applications of soft sets in BCK/BCI-algebras, BCH-algebras and BL-algebras. Furthermore, Yamak et al. [41] introduced soft hypergroupoids, and we presented the soft polygroups [37] and soft hypermodules [38].

In this paper, we define (idealistic) soft Krasner (m, n)-hyperrings, and investigate some related properties. Furthermore, we introduce T-fuzzy hyperideals by using triangular norms, and discuss the relationships among idealistic soft Krasner (m, n)-hyperrings, fuzzy hyperideals and T-Fuzzy hyperideals. Moreover, we give the concept of falling fuzzy hyperideals of Krasner (m, n)-hyperrings based on the theory of falling shadows, and consider the relationships among idealistic soft Krasner (m, n)-hyperrings, T-fuzzy hyperideals and falling fuzzy hyperideals.

2. Preliminaries

In this section, we review some notions and results concerning n-ary hypergroups, Krasner (m,n)-hyperrings, and soft sets.

Let H be a non-empty set and f be a mapping defined by $f : H\times H \to \mathscr{P}^*(H)$, where $\mathscr{P}^*(H)$ is the set of all non-empty subsets of H. Then f is called a binary hyperoperation on H. Let H^n denote the cartesian product $\underbrace{H \times ... \times H}_{n}$. A mapping $f : H^n \to \mathscr{P}^*(H)$ is called an n-ary hyperoperation. A non-empty set H with an n-ary hyperoperation f, denoted by (H, f), is called an n-ary hypergroupoid. If $A_1, ..., A_n$ are subsets of H, then $f(A_1, ..., A_n) = \cup\{f(a_1, ..., a_n) \mid a_i \in A_i, i = 1, ..., n\}$ [13]. For convenience, a_i^j is used to denote the sequence $a_i, a_{i+1}, ..., a_j$. When $j < i$, a_i^j is the empty set.

An n-ary hypergroupoid (H, f) is called an n-ary semihypergroup [13] if the following associative axiom are satisfied:

$$f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}) = f(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1})$$

for every $i, j \in \{1, 2, ..., n\}$ and $x_1, x_2, ..., x_{2n-1} \in H$. If for all $x_1^n \in H$, the set $f(x_1, x_2, ..., x_n)$ is a singleton, then f is called an n-ary operation and (H, f) is called an n-ary groupoid (respectively, n-ary semigroup if the n-ary operation is associative).

An n-ary semihypergroup (H, f) in which the equation $b \in f(a_1^{i-1}, x_i, a_{i+1}^n)$ has a solution $x_i \in H$ for every $a_1^{i-1}, a_{i+1}^n, b \in H$ and $1 \le i \le n$, is called an n-ary hypergroup [13]. An n-ary hypergroup (H, f) is commutative, if for every permutation $\sigma \in \mathbb{S}_n$ and for all $a_1^n \in H$ we have $f(a_1, ..., a_n) = f(a_{\sigma(1)}, ..., a_{\sigma(n)})$ [11].

A commutative n-ary hypergroup (H, f) [23] is called canonical n-ary hypergroup if

(1) there exists a unique $e \in H$, such that for all $x \in H$,

$$f(x, \underbrace{e, ..., e}_{n-1}) = x;$$

(2) for all $x \in H$ there exists a unique $x^{-1} \in H$, such that

$$e \in f(x, x^{-1}, \underbrace{e, ..., e}_{n-2});$$

(3) if $x \in f(x_1^n)$, then for all $1 \le i \le n$, we have

$$x_i \in f(x, x_1^{-1}, ..., x_{i-1}^{-1}, x_{i+1}^{-1}, ..., x_n^{-1}).$$

e is the scalar identity of (H, f) and x^{-1} is the inverse of x. The inverse of e is e.

A Krasner (m,n)-hyperring [27] is an algebraic hyperstructure (R, f, g) which satisfies the following axioms:

(1) (R, f) is a canonical m-ary hypergroup;

(2) (R, g) is an n-ary semigroup;

(3) the n-ary operation g is distributive with respect to the m-ary hyperoperation f, i.e., for every $a_1^{i-1}, a_{i+1}^n, x_1^m \in R$, $1 \leq i \leq n$,

$$g(a_1^{i-1}, f(x_1^m), a_{i+1}^n) = f(g(a_1^{i-1}, x_1, a_{i+1}^n), ..., g(a_1^{i-1}, x_m, a_{i+1}^n));$$

(4) 0 is a zero element (absorbing element) of the n-ary operation g, i.e., for every $x_2^n \in R$ we have

$$g(0, x_2^n) = g(x_2, 0, x_3^n) = ... = g(x_2^n, 0) = 0.$$

It is clear that every Krasner hyperring [22] is a Krasner (2, 2)-hyperring. Also, every Krasner $(m, 0)$-hyperring is a canonical m-ary hypergroup and every Krasner $(0, n)$-hyperring is an n-ary semigroup. Let S be a non-empty subset of a Krasner (m, n)-hyperring (R, f, g). If (S, f, g) is a Krasner (m, n)-hyperring, then S called a subhyperring of R.

Let I be a non-empty subset of a Krasner (m, n)-hyperring R and $1 \leq i \leq n$. We call I an i-hyperideal of R [27] if

(1) I is a subhypergroup of the canonical m-ary hypergroup (R, f), i.e., (I, f) is a canonical m-ary hypergroup;

(2) for every $x_1^n \in R$, $g(x_1^{i-1}, I, x_{i+1}^n) \subseteq I$.

If for every $1 \leq i \leq n$, I is an i-hyperideal, then I called a hyperideal of R. Clearly, every hyperideal of R is a subhyperring of R. Furthermore, we have that a non-empty subset I of a Krasner (m, n)-hyperring R is a hyperideal if

(1) for every $x_1^m \in I$, $f(x_1^m) \subseteq I$;

(2) for every $x \in I$, $-x \in I$;

(3) for every $x_1^n \in R$ and $1 \leq i \leq n$, $g(x_1^{i-1}, I, x_{i+1}^n) \subseteq I$.

A hyperideal I of a Krasner (m, n)-hyperring R is called normal if and only if for every $r \in R$, $f(-r, I, r, \overset{(m-3)}{0}) \subseteq I$. Suppose that I is a normal hyperideal of R, then the relation I^* defined by

$$xI^*y \text{ if and only if } f(x, -y, \overset{(m-2)}{0}) \cap I \neq \emptyset, \forall x, y \in R$$

is an equivalence relation on R [27]. Let $I^*[x]$ be the equivalence class of the element $x \in R$, then $I^*[x] = f(I, x, \overset{(m-2)}{0})$. Also, we have $I^*[(f(a_1^m))] = I^*[a]$ for all $a \in f(a_1^m)$ and $a_1^m \in R$. The set of all equivalence classes $[R : I^*] = \{I^*[x] | x \in R\}$

is a Krasner (m,n)-hyperring with the m-ary hyperoperation f/I and the n-ary operation g/I defined by

$$f/I(I^*[x_1],...,I^*[x_m]) = \{I^*[z] \mid z \in f(I^*[x_1],...,I^*[x_m])\}, \forall x_1^m \in R,$$

$$g/I(I^*[x_1],...,I^*[x_n]) = I^*[g(x_1^n)], \forall x_1^n \in R.$$

Let (R_1, f_1, g_1) and (R_2, f_2, g_2) be two Krasner (m,n)-hyperrings. A mapping $\varphi : R_1 \to R_2$ is called a homomorphism if for all $x_1^m \in R_1$ and $y_1^n \in R_1$, $\varphi(f_1(x_1,...,x_m)) = f_2(\varphi(x_1),...,\varphi(x_m))$, and $\varphi(g_1(y_1,...,y_n)) = g_2(\varphi(y_1),...,\varphi(y_n))$. A homomorphism φ is an isomorphism if φ is injective and surjective. $R_1 \cong R_2$ is used to denote that R_1 is isomorphic to R_2 [27].

Let $\varphi : R_1 \to R_2$ be a homomorphism, then (1) $\varphi(0_{R_1}) = 0_{R_2}$; (2) for all $x \in R$, $\varphi(-x) = -\varphi(x)$; (3) let $ker\varphi = \{x \in R_1 | \varphi(x) = 0_{R_2}\}$, then φ is injective if and only if $ker\varphi = \{0_{R_1}\}$.

Now, we review some notions about soft sets. See [5], [15], [20], [25], [28] for the following definitions. Let U be an initial universe set and E be a set of parameters. $\mathscr{P}(U)$ denotes the power set of U and $A \subseteq E$.

A pair (F, A) is called a soft set over U, where F is a mapping given by $F : A \to \mathscr{P}(U)$. In fact, a soft set over U is a parameterized family of subsets of the universe U. For $e \in A$, $F(e)$ may be considered as the set of e-approximate elements of the soft set (F, A). There are several examples in [25]. For a soft set (F, A) over U, the set $\text{Supp}(F, A) = \{x \in A \mid F(x) \neq \varnothing\}$ is called the support of the soft set (F, A). A soft set (F, A) is non-null if $\text{Supp}(F, A) \neq \varnothing$. For two soft sets (F, A) and (G, B) over U, we say that (F, A) is a soft subset of (G, B), denoted by $(F, A)\widetilde{\subseteq}(G, B)$, if $A \subseteq B$ and for all $e \in A$, $F(e) \subseteq G(e)$. Two soft sets (F, A) and (G, B) over U are called soft equal if $(F, A)\widetilde{\subseteq}(G, B)$ and $(G, B)\widetilde{\subseteq}(F, A)$.

Let $(F_i, A_i)_{i\in\Lambda}$ be a non-empty family of soft sets over U, then

(1) the extended intersection (union) of $(F_i, A_i)_{i\in\Lambda}$ is the soft set $(G, B) = (\bigcap_{\mathscr{E}})_{i\in\Lambda}(F_i, A_i)$ $((\bigcup_{\mathscr{E}})_{i\in\Lambda}(F_i, A_i))$, where $B = \bigcup_{i\in\Lambda} A_i$, and for all $e \in B$, $G(e) = \bigcap_{i\in\Lambda(e)} F_i(e)$ $(\bigcup_{i\in\Lambda(e)} F_i(e))$ and $\Lambda(e) = \{i \in \Lambda \mid e \in A_i\}$;

(2) the restricted intersection (union) of $(F_i, A_i)_{i\in\Lambda}$ is the soft set $(G, B) = (\bigcap_{\mathscr{R}})_{i\in\Lambda}(F_i, A_i)$ $((\bigcup_{\mathscr{R}})_{i\in\Lambda}(F_i, A_i))$, where $B = \bigcap_{i\in\Lambda} A_i \neq \varnothing$ and for all $e \in B$, $G(e) = \bigcap_{i\in\Lambda} F_i(e)$ $(\bigcup_{i\in\Lambda} F_i(e))$;

(3) the $\wedge$-intersection ($\vee$-union) of $(F_i, A_i)_{i\in\Lambda}$ is the soft set $(G, B) = \widetilde{\bigwedge}_{i\in\Lambda}(F_i, A_i)$ $(\widetilde{\bigvee}_{i\in\Lambda}(F_i, A_i))$, where $B = \prod_{i\in\Lambda} A_i$ and for all $e = (e_i)_{i\in\Lambda} \in B$, $G(e) = \bigcap_{i\in\Lambda} F_i(e_i)$ $(\bigcup_{i\in\Lambda} F_i(e_i))$;

(4) the cartesian product of $(F_i, A_i)_{i\in\Lambda}$ is the soft set $(G, B) = \prod_{i\in\Lambda}(F_i, A_i)$, where $B = \prod_{i\in\Lambda} A_i$ and for all $e = (e_i)_{i\in\Lambda} \in B$, $G(e) = \prod_{i\in\Lambda} F_i(e_i)$.

3. Soft Krasner (m,n)-hyperrings

In this section, we firstly introduce the concepts of soft Krasner (m,n)-hyperring and idealistic soft Krasner (m,n)-hyperring. Then, we consider isomorphism theorems of soft Krasner (m,n)-hyperrings. In what follows, let R be a Krasner (m,n)-hyperring.

Definition 3.1. Let (F,A) be a non-null soft set over R. Then (F,A) is called an (idealistic) soft Krasner (m,n)-hyperring over R if $F(x)$ is a subhyperring (hyperideal) of R for all $x \in \mathrm{Supp}(F,A)$.

Obviously, every idealistic soft Krasner (m,n)-hyperring over R is a soft Krasner (m,n)-hyperring over R. Let (F,A) be an idealistic soft Krasner (m,n)-hyperring over R, then (F,A) is called a trivial (whole) idealistic soft Krasner (m,n)-hyperring over R if $F(x)=\{0\}$ $(F(x)=R)$ for all $x \in A$.

Example 3.2. Let $R=\{0,1,2,3\}$ be a set with a 2-ary hyperoperation $+$ and an n-ary operation g defined by

$+$	0	1	2	3
0	0	1	2	3
1	1	$\{0,1\}$	3	$\{2,3\}$
2	2	3	0	1
3	3	$\{2,3\}$	1	$\{0,1\}$

$$g(x_1^n)=\begin{cases} 2 & \text{if } x_1^n \in \{2,3\},\\ 0 & \text{else.}\end{cases}$$

It follows that $(R,+,g)$ is a Krasner $(2,n)$-hyperring [27]. Let $\mathbb{N}$ be the set of natural numbers and (F,A) be the soft set over R, where $A=\mathbb{N}$ and $F: A \to \mathscr{P}(R)$ is the set-valued function defined by

$$F(x)=\begin{cases} R & \text{if } 2 \mid x,\\ \{0,2\} & \text{otherwise}\end{cases}$$

for all $x \in A$, then (F,A) is an idealistic soft Krasner $(2,n)$-hyperring over R. Also, (F,A) is a soft Krasner $(2,n)$-hyperring over R.

Theorem 3.3. *Let $(F_i,A_i)_{i\in\Lambda}$ be a non-empty family of (idealistic) soft Krasner (m,n)-hyperrings over R, then*

(1) *the extended intersection $(\bigcap_{\mathscr{E}})_{i\in\Lambda}(F_i,A_i)$ is an (idealistic) soft Krasner (m,n)-hyperring over R if it is non-null;*

(2) *the restricted intersection $(\bigcap_{\mathscr{R}})_{i\in\Lambda}(F_i,A_i)$ is an (idealistic) soft Krasner (m,n)-hyperring over R if it is non-null;*

(3) *if* $A_i \cap A_j = \emptyset$ *for all* $i, j \in \Lambda$ *and* $i \neq j$, *then the extended union* $(\bigcup_{\mathscr{E}})_{i\in\Lambda}(F_i, A_i)$ *is an* (*idealistic*) *soft Krasner* (m, n)*-hyperring over* R;

(4) *the* $\wedge$*-intersection* $\widetilde{\bigwedge}_{i\in\Lambda}(F_i, A_i)$ *is an* (*idealistic*) *soft Krasner* (m, n)*-hyperring over* R;

(5) *the cartesian product* $\prod_{i\in\Lambda}(F_i, A_i)$ *is an* (*idealistic*) *soft Krasner* (m, n)*-hyperring over* R.

Proof. We only prove (1), and the other proofs are similar. We can write $(\bigcap_{\mathscr{E}})_{i\in\Lambda}(F_i, A_i) = (G, B)$, where $B = \bigcup_{i\in\Lambda} A_i$ and $G(x) = \bigcap_{i\in\Lambda(x)} F_i(x)$ for all $x \in B$. For all $x \in \operatorname{supp}(G, B)$, $G(x) = \bigcap_{i\in\Lambda(x)} F_i(x) \neq \emptyset$, that is, $F_i(x) \neq \emptyset$ for all $i \in \Lambda(x)$. Also, $F_i(x)$ is a subhyperring (hyperideal) of R, since $(F_i, A_i)_{i\in\Lambda}$ be a non-empty family of (idealistic) soft Krasner (m, n)-hyperrings over R. Therefore, we have that $G(x) = \bigcap_{i\in\Lambda(x)} F_i(x)$ is a subhyperring (hyperideal) of R for all $x \in \operatorname{supp}(G, B)$. So $(\bigcap_{\mathscr{E}})_{i\in\Lambda}(F_i, A_i) = (G, B)$ is an (idealistic) soft Krasner (m, n)-hyperring over R. ∎

Definition 3.4. Let R_1 and R_2 be two Krasner (m, n)-hyperrings, (F, A) and (G, B) be soft Krasner (m, n)-hyperrings over R_1 and R_2, respectively, and $\varphi : R_1 \to R_2$ and $\psi : A \to B$ be two mappings. If φ is a homomorphism, and $\varphi(F(x)) = G(\psi(x))$ for all $x \in A$, then (φ, ψ) is called a soft homomorphism, and (F, A) is soft homomorphic to (G, B), denoted by $(F, A) \sim (G, B)$.

In Definition 3.4, if φ is a monomorphism (resp. epimorphism, isomorphism) and ψ is an injective (resp. surjective, bijective) mapping, then (φ, ψ) is called a soft monomorphism (resp. epimorphism, isomorphism), and (F, A) is soft monomorphic (resp. epimorphic, isomorphic) to (G, B). $(F, A) \simeq (G, B)$ is used to denote that (F, A) is soft isomorphic to (G, B).

Example 3.5. Let R be the Krasner $(2, n)$-hyperring and (F, A) be the soft Krasner $(2, n)$-hyperring over R described in Example 3.2. Let (G, B) be the soft set over R, where $B = \mathbb{N}$ and $G : B \to \mathscr{P}(R)$ is the set-valued function defined by

$$G(x) = \begin{cases} \{0, 2\} & \text{if } 2 \mid x, \\ \emptyset & \text{otherwise} \end{cases}$$

for all $x \in B$, then (G, B) is a soft Krasner $(2, n)$-hyperring over R.

Let $\varphi : R \to R$ be the mapping defined by

$$\varphi(x) = \begin{cases} 2 & \text{if } x \in \{2, 3\}, \\ 0 & \text{otherwise} \end{cases}$$

for all $x \in R$, then φ is a homomorphism. Define the mapping $\psi : A \to B$ by $\psi(x) = 2x$ for all $x \in A$. It is easy to check that $\varphi(F(x)) = G(\psi(x))$ for all $x \in A$. Therefore, (φ, ψ) is a soft homomorphism and $(F, A) \sim (G, B)$.

Next, we establish three isomorphism theorems of soft Krasner (m,n)-hyperrings. In the following theorems, we write $([F : I^*], A)$, which means that $[F : I^*](x) = [F(x) : I^*]$ for all $x \in A$, $I \subseteq F(x)$ for $x \in \text{Supp}(F, A)$ and $[F : I^*](x) = \emptyset$ for $x \in A - \text{Supp}(F, A)$, where (F, A) is a soft Krasner (m,n)-hyperring over R, and I is a normal hyperideal of R.

Theorem 3.6. (First Isomorphism Theorem) *Let (F, A) and (G, B) be soft Krasner (m,n)-hyperrings over Krasner (m,n)-hyperrings R_1 and R_2, respectively. If (φ, ψ) is a soft epimorphism from (F, A) to (G, B) such that $I = ker\varphi$ is a normal hyperideal of R_1 and $I \subseteq F(x)$ for $x \in Supp(F, A)$, then*

(1) $([F : I^*], A) \simeq (\varphi(F), A)$;

(2) *if ψ is bijective, then $([F : I^*], A) \simeq (G, B)$,*

where $\varphi(F)(x) = \varphi(F(x))$ for all $x \in A$.

Proof. (1) Clearly, $([F : I^*], A)$ and $(\varphi(F), A)$ are soft Krasner (m,n)-hyperrings over $[R_1 : I^*]$ and R_2, respectively. Define $\overline{\varphi} : [R_1 : I^*] \to R_2$ by $\overline{\varphi}(I^*[x]) = \varphi(x)$, for all $x \in R_1$. We show that $\overline{\varphi}$ is well-defined. Suppose that xI^*y, then we have $f_1(x, -y, \overset{(m-2)}{0_{R_1}}) \cap I \neq \emptyset$, that is, there exists $z \in f_1(x, -y, \overset{(m-2)}{0_{R_1}}) \cap I$. Thus, we have $\varphi(z) = 0_{R_2}$ and $\varphi(z) \in \varphi(f_1(x, -y, \overset{(m-2)}{0_{R_1}}))$, which imply that $0_{R_2} \in f_2(\varphi(x), -\varphi(y), \overset{(m-2)}{0_{R_2}}))$, i.e., $\varphi(x) = \varphi(y)$. We can check easily that $\overline{\varphi}$ is surjective, since φ is surjective. If $\varphi(x) = \varphi(y)$, we have $0_{R_2} \in f_2(\varphi(x), -\varphi(y), \overset{(m-2)}{0_{R_2}})$, which implies that there exists $z \in f_1(x, -y, \overset{(m-2)}{0_{R_1}})$ such that $z \in ker\varphi$, i.e., $f_1(x, -y, \overset{(m-2)}{0_{R_1}}) \cap I \neq \emptyset$. It follows that $I^*[x] = I^*[y]$. So $\overline{\varphi}$ is injective. Also, we have

$$\begin{aligned}
\overline{\varphi}(f_1/I(I^*[x_1], ..., I^*[x_m])) &= \overline{\varphi}(\{I^*[z] | z \in f_1(I^*[x_1], ..., I^*[x_m])\}) \\
&= \overline{\varphi}(\{I^*[z] | z \in f_1(I, f_1(x_1^m), \overset{(m-2)}{0_{R_1}})\}) \\
&= \varphi(f_1(I, f_1(x_1^m), \overset{(m-2)}{0_{R_1}})) \\
&= f_2(\varphi(I), \varphi(f_1(x_1^m)), \overset{(m-2)}{\varphi(0_{R_1})}) \\
&= f_2(0_{R_2}, f_2(\varphi(x_1), ..., \varphi(x_m)), \overset{(m-2)}{0_{R_2}}) \\
&= f_2(\varphi(x_1), ..., \varphi(x_m)) \\
&= f_2(\overline{\varphi}(I^*[x_1]), ..., \overline{\varphi}(I^*[x_m])),
\end{aligned}$$

$$\begin{aligned}
\overline{\varphi}(g_1/I(I^*[x_1], ..., I^*[x_n])) &= \overline{\varphi}(\{I^*[g_1(I^*[x_1], ..., I^*[x_n])]\}) = \varphi(g_1(I^*[x_1], ..., I^*[x_n])) \\
&= g_2(\varphi(I^*[x_1]), ..., \varphi(I^*[x_n])) \\
&= g_2(f_2(0_{R_2}, \varphi(x_1), \overset{(m-2)}{0_{R_2}}), ..., f_2(0_{R_2}, \varphi(x_n), \overset{(m-2)}{0_{R_2}})) \\
&= g_2(\varphi(x_1), ..., \varphi(x_n)) = g_2(\overline{\varphi}(I^*[x_1]), ..., \overline{\varphi}(I^*[x_n])).
\end{aligned}$$

So $\overline{\varphi}$ is an isomorphism. Next, we define $\overline{\psi} : A \to A$ by $\overline{\psi}(x) = x$ for all $x \in A$. Clearly, $\overline{\psi}$ is a bijective mapping. For all $x \in A$, we have $\overline{\varphi}([F(x) : I^*]) = \varphi(F(x)) = \varphi(F(\psi(x))) = \varphi(F)(\psi(x))$. It follows that $(\overline{\varphi}, \overline{\psi})$ is a soft isomorphism and $([F : I^*], A) \simeq (\varphi(F), A)$.

(2) From the proof above, we have $\overline{\varphi}$ is an isomorphism. ψ is bijective and $\overline{\varphi}([F(x) : I^*]) = \varphi(F(x)) = G(\psi(x))$ for all $x \in A$. Therefore, $(\overline{\varphi}, \psi)$ is a soft isomorphism, and $([F : I^*], A) \simeq (G, B)$. ∎

Theorem 3.7. (Second Isomorphism Theorem) *Let I_1^m be hyperideals of R, and there exists $1 \leq j \leq m$ such that I_j is a normal hyperideal of R. If (F, A) is a soft Krasner (m,n)-hyperring over $f(I_1^{j-1}, 0, I_{j+1}^m)$, then*

$$([F : (f(I_1^{j-1}, 0, I_{j+1}^m) \cap I_j)^*], A) \simeq ([F_{I_j/0} : I_j^*], A),$$

where for all $x \in A$, $F_{I_j/0}(x) = f(\overline{I}_1^{j-1}, I_j, \overline{I}_{j+1}^m)$ if we assume that $F(x) = f(\overline{I}_1^{j-1}, 0, \overline{I}_{j+1}^m)$, $\overline{I}_k \subseteq I_k$, $k \in \{1, ..., j-1, j+1, ..., m\}$, and $f(I_1^{j-1}, 0, I_{j+1}^m) \cap I_j \subseteq F(x)$ for $x \in \mathrm{Supp}(F, A)$.

Proof. It is easy to deduce that $([F : (f(I_1^{j-1}, 0, I_{j+1}^m) \cap I_j)^*], A)$ and $([F_{I_j/0} : I_j^*], A)$ are soft hyperideals over $[f(I_1^{j-1}, 0, I_{j+1}^m) : (f(I_1^{j-1}, 0, I_{j+1}^m) \cap I_j)^*]$ and $[f(I_1^m) : I_j^*]$, respectively. We define $\varphi : f(I_1^{j-1}, 0, I_{j+1}^m) \to [f(I_1^m) : I_j^*]$ by $\varphi(x) = I_j^*[x]$ for all $x \in f(I_1^{j-1}, 0, I_{j+1}^m)$. Clearly, φ is a homomorphism. For any $I_j^*[y] \in [f(I_1^m) : I_j^*]$, where $y \in f(I_1^m)$, i.e., there exist $a_k \in I_k$, $k \in \{1, ..., m\}$ such that $y \in f(a_1^m)$, we have $I_j^*[y] = I_j^*[f(a_1^m)] = f(I_j, f(a_1^m), \overset{(m-2)}{0}) = f(a_1^{j-1}, f(I_j, a_j, \overset{(m-2)}{0}), a_{j+1}^m) = f(a_1^{j-1}, I_j, a_{j+1}^m) = f(I_j, f(a_1^{j-1}, 0, a_{j+1}^m), \overset{(m-2)}{0}) = I_j^*[x] = \varphi(x), x \in f(a_1^{j-1}, 0, a_{j+1}^m) \subseteq f(I_1^{j-1}, 0, I_{j+1}^m)$. It follows that φ is an epimorphism. Now, we define $\psi : A \to A$ by $\psi(x) = x$ for all $x \in A$. ψ is a bijective mapping. For all $x \in A$, we have $\varphi(F(x)) = \{I_j^*[a] \mid a \in F(x)\} = [F_{I_j/0} : I_j^*](x) = [F_{I_j/0} : I_j^*](\psi(x))$.

Next, we give the proof of $\{I_j^*[a] \mid a \in F(x)\} = [F_{I_j/0} : I_j^*](x)$. Clearly, $\{I_j^*[a] \mid a \in F(x)\} \subseteq [F_{I_j/0} : I_j^*](x)$. For any $I_j^*[b] \in [F_{I_j/0} : I_j^*](x)$, where $b \in F_{I_j/0}(x)$, which implies that there exist $a_k \in \overline{I}_k \subseteq I_k$, $k \in \{1, ..., j-1, j+1, ..., m\}$ and $a_j \in I_j$ such that $b \in f(a_1^m)$, we have $I_j^*[b] = I_j^*[f(a_1^m)] = f(I_j, f(a_1^m), \overset{(m-2)}{0}) = f(a_1^{j-1}, f(I_j, a_j, \overset{(m-2)}{0}), a_{j+1}^m) = f(a_1^{j-1}, I_j, a_{j+1}^m) = f(I_j, f(a_1^{j-1}, 0, a_{j+1}^m), \overset{(m-2)}{0}) = I_j^*[t] (t \in f(a_1^{j-1}, 0, a_{j+1}^m) \subseteq F(x)) \in \{I_j^*[a] \mid a \in F(x)\}$. Thus, (φ, ψ) is a soft epimorphism from (F, A) to $([F_{I_j/0} : I_j^*], A)$. Because $f(I_1^{j-1}, 0, I_{j+1}^m) \cap I_j$ is a normal hyperideal of $f(I_1^{j-1}, 0, I_{j+1}^m)$, if we have $ker\varphi = f(I_1^{j-1}, 0, I_{j+1}^m) \cap I_j$, then $([F : (f(I_1^{j-1}, 0, I_{j+1}^m) \cap I_j)^*], A) \simeq ([F_{I_j/0} : I_j^*], A)$. For any $x \in f(I_1^{j-1}, 0, I_{j+1}^m)$, $x \in ker\varphi \Leftrightarrow \varphi(x) = I_j^*[0] = I_j \Leftrightarrow I_j^*[x] = f(I_j, x, \overset{(m-2)}{0_{R_2}}) = I_j \Leftrightarrow x \in I_j$ (since $x \in f(I_1^{j-1}, 0, I_{j+1}^m)) \Leftrightarrow x \in I_j \cap f(I_1^{j-1}, 0, I_{j+1}^m)$. So $ker\varphi = f(I_1^{j-1}, 0, I_{j+1}^m) \cap I_j$. ∎

Theorem 3.8. (Third Isomorphism Theorem) *Let I and N be normal hyperideals of R such that $I \subseteq N$. If (F, A) is a soft Krasner (m, n)-hyperring over R, and $N \subseteq F(x)$ for all $x \in supp(F, A)$, then $([[F : I^*] : [N : I^*]^*], A) \simeq ([F : N^*], A)$.*

Proof. We can obtain that $[N : I^*]$ is a normal hyperideal of $[R : I^*]$. I and N are normal hyperideals of R, and $I \subseteq N$, then $[[R : I^*] : [N : I^*]^*]$ is defined. Also, it is easy to obtain that $([F : I^*], A)$, $([F : N^*], A)$ and $([[F : I^*] : [N : I^*]^*], A)$ are soft Krasner (m, n)-hyperrings over $[R : I^*]$, $[R : N^*]$ and $[[R : I^*] : [N : I^*]^*]$, respectively. $\varphi : [R : I^*] \to [R : N^*]$, defined by $\varphi(I^*[x]) = N^*[x]$, is an epimorphism, and $\psi : A \to A$, defined by $\psi(x) = x$ for all $x \in A$, is a bijective mapping. Also, we have $\varphi([F(x) : I^*]) = [F(x) : N^*] = [F(\psi(x)) : N^*]$, for all $x \in A$. Therefore, (φ, ψ) is a soft epimorphism from $([F : I^*], A)$ to $([F : N^*], A)$. According to the first isomorphism theorem, if $ker\varphi = [N : I^*]$, then $([[F : I^*] : [N : I^*]^*], A) \simeq ([F : N^*], A)$. For any $I^*[x] \in [R : I^*]$, $I^*[x] \in ker\varphi \Leftrightarrow \varphi(I^*[x]) = N^*[0] = N \Leftrightarrow N^*[x] = f(N, x, \overset{(m-2)}{0}) = N \Leftrightarrow x \in N \Leftrightarrow I^*[x] \in [N : I^*]$. It follows that $ker\varphi = [N : I^*]$. ∎

4. The relationships among idealistic soft Krasner (m, n)-hyperrings, fuzzy hyperideals and T-Fuzzy hyperideals

Let μ be a fuzzy subset of R, and $1 \leq i \leq n$. Then μ is called a fuzzy i-hyperideal of R if μ is a fuzzy subhypergroup of the canonical m-ary hypergroup (R, f), and $\mu(x_i) \leq \mu(g(x_1^n))$, for all $x_1^n \in R$. If for every $1 \leq i \leq n$, μ is a fuzzy i-hyperideal, then μ is called a fuzzy hyperideal of R [11]. Therefore, there is the following definition.

Definition 4.1([11]). A fuzzy subset μ of R is called a fuzzy hyperideal of R if the following conditions hold:

(1) $\min\{\mu(x_1), ..., \mu(x_m)\} \leq \inf\limits_{z \in f(x_1^m)} \{\mu(z)\}$, for all $x_1^m \in R$;

(2) $\mu(x) \leq \mu(-x)$, for all $x \in R$;

(3) $\max\{\mu(x_1), ..., \mu(x_n)\} \leq \mu(g(x_1^n))$, for all $x_1^n \in R$.

A triangular norm (t-norm) [8] is a function $T : [0, 1] \times [0, 1] \to [0, 1]$, such that for all $x, y, z \in [0, 1]$ the following axioms are satisfied:

(1) $T(x, T(y, z)) = T(T(x, y), z)$;

(2) $T(x, y) = T(y, x)$;

(3) $T(x, y) \leq T(x, z)$ if $y \leq z$;

(4) $T(x, 1) = x$.

Obviously, the function min, defined by $\min(a,b) = \min\{a,b\}$ for every $a,b \in [0,1]$, is a t-norm. There are many other t-norms. $T_{\text{Ł}uk}$ and T_{Prod} are two t-norms used frequently, defined by $T_{\text{Ł}uk}(a,b) = \max\{a+b-1,0\}$ and $T_{Prod}(a,b) = ab$, respectively, for all $a,b \in [0,1]$. Every t-norm T has the following property: $T(a,b) \leq \min\{a,b\}$ for all $a,b \in [0,1]$. Let T be a t-norm. Then the fuzzy set μ of a non-empty set X satisfies the imaginable property if $Im(\mu) \subseteq \Delta_T$, where $\Delta_T = \{\alpha \in [0,1] \mid T(\alpha,\alpha) = \alpha\}$ [30].

Definition 4.2. Let μ be a fuzzy subset of R. Then μ is called a T-fuzzy hyperideal of R if the following axioms hold:

(1) $T_m(\mu(x_1),...,\mu(x_m)) \leq \inf\limits_{z \in f(x_1^m)} \{\mu(z)\}$, for all $x_1^m \in R$;

(2) $\mu(x) \leq \mu(-x)$, for all $x \in R$;

(3) $\max\{\mu(x_1),...,\mu(x_m)\} \leq \mu(g(x_1^n))$, for all $x_1^n \in R$,

where T is a t-norm and for $a_1,...,a_m \in [0,1]$, if $m > 2$, $T_m(a_1,...,a_m) = T(a_i, T_{m-1}(a_1,...,a_{i-1},a_{i+1},...,a_m))$; $T_2 = T$; $T_1 = id$(identity)[1]. For convenience, we write T_m by T.

Example 4.3. Consider the Krasner $(2,n)$-hyperring described in Example 3.2. We define a fuzzy set μ in R by $\mu(0) = \mu(1) = 0.8$ and $\mu(2) = \mu(3) = 0.6$. Then μ is a $T_{\text{Ł}uk}$-fuzzy hyperideal of R.

For a fuzzy point x_t and a fuzzy set μ in a non-empty set X, we say $x_t \in \mu$, which means that $\mu(x) \geq t$. Let $A \subseteq [0,1]$. Consider the set-valued function $F : A \to \mathscr{P}(X)$, defined by $t \mapsto \mu_t = \{x \in X \mid x_t \in \mu\}$, then (F,A) is called a $\in$-soft set over X [49].

Theorem 4.4. *Let μ be a fuzzy set of R, and (F,A) be a $\in$-soft set over R with A=[0,1]. Then μ is a fuzzy hyperideal of R if and only if (F,A) is an idealistic soft Krasner (m,n)-hyperring over R.*

Proof. Assume that μ is a fuzzy hyperideal of R. Let $t \in \text{Supp}(F,A)$ and $x_1^m, x \in F(t)$. Then we have $(x_i)_t \in \mu$ for all $1 \leq i \leq m$ and $x_t \in \mu$. Since $t \leq \min\{\mu(x_1),...,\mu(x_m)\} \leq \inf\limits_{z \in f(x_1^m)} \{\mu(z)\}$ and $t \leq \mu(x) \leq \mu(-x)$, we obtain $f(x_1^m) \subseteq F(t)$ and $-x \in F(t)$. If $x_1^{i-1}, x_{i+1}^n \in R$ and $x_i \in F(t)$, then $(x_i)_t \in \mu$. So we have $t \leq \mu(x_i) \leq \max\{\mu(x_1),...,\mu(x_n)\} \leq \mu(g(x_1^n))$, which implies that $\mu(g(x_1^n)) \in F(t)$, i.e., $\mu(g(x_1,F(t),x^n)) \subseteq F(t)$. Hence (F,A) is an idealistic soft Krasner (m,n)-hyperring over R.

Conversely, let (F,A) be an idealistic soft Krasner (m,n)-hyperring over R. For all $x_1^m \in R$, if there exists $z \in f(x_1^m)$ such that $\mu(z) < \min\{\mu(x_1),...,\mu(x_m)\}$, then choose $t \in A$ such that $\mu(z) < t \leq \min\{\mu(x_1),...,\mu(x_m)\}$. It follows that $(x_i)_t \in \mu$ for all $1 \leq i \leq m$, but $z_t \overline{\in} \mu$. This contradicts to that $F(t)$ is a hyperideal

of R. So we have $\min\{\mu(x_1), ..., \mu(x_m)\} \leq \inf_{z \in f(x_1^m)} \{\mu(z)\}$. For all $x \in R$, assume that $\mu(-x) < \mu(x)$, then there is $r \in A$ such that $\mu(-x) < r \leq \mu(x)$. Thus, we have $(x)_r \in \mu$, but $(-x)_r \overline{\in} \mu$, which is a contradiction. So we have $\mu(x) \leq \mu(-x)$ for all $x \in R$. Let $x_1^n \in R$. Suppose that $s = \mu(x_i) = \max\{\mu(x_1), ..., \mu(x_n)\}$, then $x_i \in F(s)$. Since $F(s)$ is a hyperideal of R, $g(x_1^n) \in F(s)$, i.e., $\mu(g(x_1^n)) \geq s$, which implies that $\max\{\mu(x_1), ..., \mu(x_n\} \leq \mu(g(x_1^n))$. Therefore, μ is a fuzzy hyperideal of R. ∎

Theorem 4.5. *Let μ be an imaginable T-fuzzy hyperideal of R, then μ is a fuzzy hyperideal of R.*

Proof. Since μ is an imaginable T-fuzzy hyperideal of R, for all $x_1^m \in R$, we have $T(\mu(x_1), ..., \mu(x_m)) \leq \inf_{z \in f(x_1^m)} \mu(z)$. Furthermore, we obtain $\min\{\mu(x_1), ..., \mu(x_m)\} = T(\min\{\mu(x_1), ..., \mu(x_m)\}, ..., \min\{\mu(x_1), ..., \mu(x_m)\}) \leq T(\mu(x_1), ..., \mu(x_m)) \leq \min(\mu(x_1), ..., \mu(x_m))$. That is, $T(\mu(x_1), ..., \mu(x_m)) = \min(\mu(x_1), ..., \mu(x_m))$. It follows that $\min(\mu(x_1), ..., \mu(x_m)) \leq \inf_{z \in f(x_1^m)} \mu(z)$. Therefore, μ is a fuzzy hyperideal of R. ∎

Theorem 4.6. *Let μ be an imaginable fuzzy subset of R and (F, A) be a $\in$-soft set over R with A=[0,1]. Then μ is a T-fuzzy hyperideal of R if and only if (F, A) is an idealistic soft Krasner (m, n)-hyperring over R.*

Proof. Assume that μ is a T-fuzzy hyperideal of R. According to Theorem 4.5, we have that μ is a fuzzy hyperideal of R. Furthermore, by using Theorem 4.4, (F, A) is an idealistic soft Krasner (m, n)-hyperring over R. Conversely, if (F, A) is an idealistic soft Krasner (m, n)-hyperring over R, then μ is a fuzzy hyperideal of R by Therorem 4.4. It follows that $\inf_{z \in f(x_1^m)} \mu(z) \geq \min\{\mu(x_1), ..., \mu(x_m)\} \geq T(\mu(x_1), ..., \mu(x_m))$ for all $x_1^m \in R$. Therefore, we have μ is a T-fuzzy hyperideal of R. ∎

Theorem 4.7. *Let I be a hyperideal of R, and μ be a fuzzy subset of R given by for all $s, t \in [0, 1]$ with $s > t$, $\mu(x) = s$, if $x \in I$; otherwise, $\mu(x) = t$. Then μ is a $T_{\text{Ł}uk}$-fuzzy hyperideal of R. If $s = 1$ and $t = 0$ then μ is imaginable.*

Proof. Let $x_1^m \in R$. If $x_1^m \in I$ then $f(x_1^m) \subseteq I$ since I is a hyperideal of R. Thus, we have $T_{\text{Ł}uk}(\mu(x_1), ..., \mu(x_m)) = \max\{ms - (m-1), 0\} \leq s = \inf_{z \in f(x_1^m)} \mu(z)$. Otherwise, assume that $x_{\sigma(1)}, ..., x_{\sigma(i)} \in I$ and $x_{\sigma(i+1)}, ..., x_{\sigma(m)} \notin I$, where $\sigma \in \mathbb{S}_n$ is a permutation, then $T_{\text{Ł}uk}(\mu(x_1), ..., \mu(x_m)) = \max\{is + (m-i)t - (m-1), 0\} \leq t = \inf_{z \in f(x_1^m)} \mu(z)$. Let $x \in R$. If $x \in I$, then $-x \in I$. So we have $\mu(-x) = s = \mu(-x)$. If $x \notin I$, then $\mu(-x) \geq t = \mu(x)$. Let $x_1^n \in R$. If there exists $x_i \in I$, $i \in \{0, ..., n\}$, then $g(x_1^n) \in I$ since I is a hyperideal of R. Hence $\max\{\mu(x_1), ..., \mu(x_n)\} = s = \mu(g(x_1^n))$. Otherwise, $\max\{\mu(x_1), ..., \mu(x_n)\} = t \leq \mu(g(x_1^n))$. Consequently, μ is a $T_{\text{Ł}uk}$-fuzzy hyperideal of R. When $s = 1$ and $t = 0$, it is clear that μ is imaginable. ∎

Theorem 4.8. *Let μ be a fuzzy set of R with $Im(\mu) = \{\lambda_0, \lambda_1, ..., \lambda_t\}$, where $\lambda_i < \lambda_j$ whenever $i > j$. If there exists a chain of hyperideal of R: $I_0 \subseteq I_1 \subseteq ... \subseteq I_t = R$ such that $\mu(I'_k) = \lambda_k$, where $I'_k = I_k \backslash I_{k-1}, I_{-1} = \emptyset$ for $k = 0, 1, ..., t$, then μ is a T-fuzzy hyperideal of R.*

Proof. For all $x_1^m \in R$, assume that $x_1 \in I'_{\alpha_1}, ..., x_m \in I'_{\alpha_m}$, where $\alpha_1, ..., \alpha_m \in \{0, 1, ..., t\}$. Then $\mu(x_1) = \lambda_{\alpha_1}, ..., \mu(x_m) = \lambda_{\alpha_m}$. Set $\lambda_{\alpha_c} = \min\{\mu(x_1), ..., \mu(x_m)\}$, then we have $f(x_1^m) \subseteq I_{\alpha_c}$. It follows that

$$\inf_{z \in f(x_1^m)} \mu(z) \geq \lambda_{\alpha_c} = \min\{\mu(x_1), ..., \mu(x_m)\} \geq T(\mu(x_1), ..., \mu(x_m)).$$

For all $x \in R$, we have $x \in I'_k$ for some $k = \{0, 1, ..., t\}$. It is clear that $-x \in I'_k$. So we have $\mu(-x) = \lambda_k = \mu(x)$. Now, for all $x_1^n \in R$, suppose that $x_1 \in I'_{\beta_1}, ..., x_n \in I'_{\beta_n}$, where $\beta_1, ..., \beta_n \in \{0, 1, ..., t\}$. Then $\mu(x_1) = \lambda_{\beta_1}, ..., \mu(x_n) = \lambda_{\beta_n}$. Set $\lambda_{\beta_c} = \mu(x_c) = \max\{\mu(x_1), ..., \mu(x_n)\}$. Since I_{β_c} is a hyperideal of R, we have $g(x_1^n) \in I_{\beta_c}$. Therefore, we have $\mu(g(x_1^n)) \geq \lambda_{\beta_c} = \max\{\mu(x_1), ..., \mu(x_n)\}$. Consequently, μ is a T-fuzzy hyperideal of R. ∎

5. The relationships among idealistic soft Krasner (m, n)-hyperrings, T-Fuzzy hyperideals and falling fuzzy hyperideals

In probability theory, a probability space is a triple $(\Omega, \mathbb{A}, P)$, where Ω is the set of all possible outcomes, $\mathbb{A}$ is a σ-algebra of subsets of Ω called events, and P is a probability measure on measurable space $(\Omega, \mathbb{A})$, i.e., P is a countable additive and positive function and $P(\Omega) = 1$. The following definitions are given in [36].

Let U be a universal set, and $\mathscr{P}(U)$ be the power set of U. Then for each $u \in U$, define $\dot{u} = \{A \mid u \in A \text{ and } A \subseteq U\}$. Also, for each $A \in \mathscr{P}(U)$, let $\dot{A} = \{\dot{u} \mid u \in A\}$. Then an ordered pair $(\mathscr{P}(U), \mathbb{B})$ is said to be a hyper-measure structure on U if $\mathbb{B}$ is a σ-field in $\mathscr{P}(U)$ and $\dot{U} \subseteq \mathbb{B}$.

Given a probability space $(\Omega, \mathbb{A}, P)$ and a hyper-measurable structure $(\mathscr{P}(U), \mathbb{B})$ on U, a random set on U is defined to be a mapping $\xi : \Omega \to \mathscr{P}(U)$ that is $\mathbb{A} - \mathbb{B}$ measurable, i.e., $\forall C \in \mathbb{B}$, $\xi^{-1}(C) = \{\omega \mid \omega \in \Omega \text{ and } \xi(\omega) \in C\} \in \mathbb{A}$.

Definition 5.1. Let $(\Omega, \mathbb{A}, P)$ be a probability space, and $\xi : \Omega \to \mathscr{P}(R)$ be a random set, where $\mathscr{P}(R)$ denotes the power set of R. If for any $\omega \in \Omega$, $\xi(\omega)$ is a hyperideal of R, then the falling shadow S of the random set ξ, i.e., $S(x) = P(\{\omega \mid x \in \xi(\omega)\})$ for all $x \in R$, is called a falling fuzzy hyperideal of R.

Theorem 5.2. *Let μ be a fuzzy set of R, and (F, A) be a $\in$-soft set over R with A=[0,1]. If (F, A) is an idealistic soft Krasner (m, n)-hyperring over R, then μ is a falling fuzzy hyperideal of R.*

Proof. Consider the probability space $(\Omega, \mathbb{A}, P) = ([0, 1], \sigma, m)$, where σ is a Borel field on $[0, 1]$ and m is the usual Lebesgue measure. Define $\xi : [0, 1] \to \mathscr{P}(R)$ by $\xi(t) = F(t)$. Since $\xi^{-1}(\dot{x}) = [0, \mu(x)] \subseteq [0, 1]$, ξ is a random set on R.

Furthermore, we obtain that the falling shadow S of ξ is μ, i.e., $S(x) = m(t \mid x \in \xi(t)) = m([0, \mu(x)]) = \mu(x)$. Since (F, A) is an idealistic soft Krasner (m, n)-hyperring over R, $F(t)$ is a hyperideal of R, for $t \in \mathrm{Supp}(F, A)$. Therefore, μ is a falling fuzzy hyperideal of R. ∎

The following example demonstrated that the converse Theorem 5.2 is not true.

Example 5.3. Consider the Krasner $(2, n)$-hyperring R described in Example 3.2. Let $(\Omega, \mathbb{A}, P) = ([0, 1], \sigma, m)$ and $\xi : [0, 1] \to \mathscr{P}(R)$ be defined by

$$\xi(t) = \begin{cases} \{0, 1\} & \text{if } t \in [0, 0.45), \\ \{0, 2\} & \text{if } t \in [0.45, 0.75), \\ \{0, 1, 2, 3\} & \text{if } t \in [0.75, 1). \end{cases}$$

We can see that $\xi(t)$ is a hyperideal of R for all $t \in [0, 1]$, then $S(x) = m(\{t \mid x \in \xi(t)\})$ is a falling fuzzy hyperideal of R, which is denoted by $S(0) = 1$, $S(1) = 0.7$, $S(2) = 0.55$ and $S(3) = 0.25$. Let (F, A) be a $\in$-soft set over R with A=[0,1]. Then $F : A \to \mathscr{P}(R)$ is a set-valued function described by

$$F(t) = \begin{cases} \{0, 1, 2, 3\} & \text{if } t \in [0, 0.25], \\ \{0, 1, 2\} & \text{if } t \in (0.25, 0.55], \\ \{0, 1\} & \text{if } t \in (0.55, 0.7], \\ \{0\} & \text{if } t \in (0.7, 1], \end{cases}$$

Since $\{0, 1, 2\}$ is not a hyperideal of R, (F, A) is not an idealistic soft Krasner $(2, n)$-hyperring over R.

Theorem 5.4. *If S is a falling fuzzy hyperideal of R, then S is a T_{Luk}-fuzzy hyperideal of R.*

Proof. For all $x_1^m \in \xi(\omega)$, we have $f(x_1^m) \subseteq \xi(\omega)$, since $\xi(\omega)$ is a hyperideal of R for any $\omega \in \Omega$. So for every $z \in f(x_1^m)$, we have

$$\{\omega \mid z \in \xi(\omega)\} \supseteq \bigcap_{i=1}^{m} \{\omega \mid x_i \in \xi(\omega)\}.$$

It follows that

$$\begin{aligned} S(z) &= P(\{\omega \mid z \in \xi(\omega)\}) \geq P(\bigcap_{i=1}^{m} \{\omega \mid x_i \in \xi(\omega)\}) \\ &= P(\bigcap_{i=1}^{m-1} \{\omega \mid x_i \in \xi(\omega)\}) + P(\{\omega \mid x_m \in \xi(\omega)\}) \\ &\quad - P(\{\omega \mid x_1^{m-1} \in \xi(\omega) \text{ or } x_m \in \xi(\omega)\}) \\ &\geq P(\bigcap_{i=1}^{m-1} \{\omega \mid x_i \in \xi(\omega)\}) + S(x_m) - 1 \\ &\vdots \end{aligned}$$

$$\begin{aligned}
&\vdots\\
&\geq P(\{\omega \mid x_1 \in \xi(\omega)\} \cap \{\omega \mid x_2 \in \xi(\omega)\}) + \sum_{i=3}^{m} S(x_i) - (m-2)\\
&= P(\{\omega \mid x_1 \in \xi(\omega)\}) + P(\{\omega \mid x_2 \in \xi(\omega)\})\\
&\quad -P(\{\omega \mid x_1 \in \xi(\omega) \text{ or } x_2 \in \xi(\omega)\}) + \sum_{i=3}^{m} S(x_i) - (m-2)\\
&\geq \sum_{i=1}^{m} S(x_i) - (m-1).
\end{aligned}$$

Hence, we have

$$\inf_{z \in f(x_1^m)} \{S(z)\} \geq \max\{\sum_{i=1}^{m} S(x_i) - (m-1), 0\} = T_{\text{Ł}uk}(S(x_1), ..., S(x_m)).$$

For any $x \in \xi(\omega)$, we have $-x \in \xi(\omega)$. Hence $\{\omega \mid -x \in \xi(\omega)\} \supseteq \{\omega \mid x \in \xi(\omega)\}$. Furthermore, we have $P\{\omega \mid -x \in \xi(\omega)\} \geq P\{\omega \mid x \in \xi(\omega)\}$, i.e., $S(-x) \geq S(x)$.

Since $\xi(\omega)$ is a hyperideal of R for any $\omega \in \Omega$, it follows that

$$\{\omega \mid g(x_1^n) \in \xi(\omega)\} \supseteq \bigcup_{i=1}^{n} \{\omega \mid x_i \in \xi(\omega)\}.$$

So we have

$$\begin{aligned}
S(g(x_1^n)) &= P(\omega \mid g(x_1^n) \in \xi(\omega)) \geq P(\bigcup_{i=1}^{n} \{\omega \mid x_i \in \xi(\omega)\})\\
&= P(\bigcup_{i=1}^{n-1} \{\omega \mid x_i \in \xi(\omega)\}) + P(\{\omega \mid x_n \in \xi(\omega)\})\\
&-P(\bigcup_{i=1}^{n-1} \{\omega \mid x_i \in \xi(\omega)\} \cap \{\omega \mid x_m \in \xi(\omega)\})\\
&\geq \max\{P(\bigcup_{i=1}^{n-1} \{\omega \mid x_i \in \xi(\omega)\}), S(x_n)\}\\
&\vdots\\
&\geq \max\{P(\{\omega \mid x_1 \in \xi(\omega)\} \cup \{\omega \mid x_2 \in \xi(\omega)\}), S(x_3), ..., S(x_n)\}\\
&= \max\{P(\{\omega \mid x_1 \in \xi(\omega)\}) + P(\{\omega \mid x_2 \in \xi(\omega)\}) - P(\{\omega \mid x_1 \in \xi(\omega)\\
&\quad \text{and } x_2 \in \xi(\omega)\}), S(x_3), ..., S(x_n)\}\\
&\geq \max\{S(x_1), ..., S(x_n)\}..
\end{aligned}$$

Therefore, S is a $T_{\text{Ł}uk}$-fuzzy hyperideal of R. ∎

Let R be a Krasner (m,n)-ary hyperring, and Ω be a non-empty set. Define

$$\mathscr{P}(R)^{\Omega} = \{\eta \mid \eta : \Omega \to \mathscr{P}^*(R) \text{ is a mapping}\}$$

Let $\eta_i \in \mathscr{P}(R)^\Omega$, $i = 1, ..., m$. For any $\omega \in \Omega$, we put $\eta_i(\omega) = \{y^\omega_{i1}, ..., y^\omega_{i|\eta_i(\omega)|}\}$, $i = 1, ..., m$. Suppose

$$\begin{aligned}\overline{f}(\eta_1, ..., \eta_m) &= \{\zeta_{(t_1,...,t_m)} \mid \zeta_{(t_1,...,t_m)}(\omega) = f(y^\omega_{1t_1}, ..., y^\omega_{mt_m}), t_i \in \{1, ..., |\eta_i(\omega)|\}, \\ &\qquad\qquad i = 1, ..., m\}, \\ \overline{g}(\eta_1, ..., \eta_n)(\omega) &= \{g(y^\omega_{1t_1}, ..., y^\omega_{nt_n}) \mid t_i \in \{1, ..., |\eta_i(\omega)|\}, i = 1, ..., n\}.\end{aligned}$$

Then $(\mathscr{P}(R)^\Omega, \overline{f}, \overline{g})$ is a Krasner (m, n)-ary hyperring.

Theorem 5.5. Let $(\Omega, \mathbb{A}, P)$ be a probability space, I be a hyperideal of a Krasner (m, n)-ary hyperring R and B be a hyperideal of $\mathscr{P}(R)^\Omega$ such that $S_\eta = \{\omega \in \Omega \mid \eta(\omega) \subseteq I\} \in \mathbb{A}$ for all $\eta \in B$. Define $\mu : B \to [0, 1]$ by $\mu(\eta) = P(S_\eta)$ for all $\eta \in B$. Then μ is a $T_{\text{Ł}uk}$-fuzzy hyperideal of B.

Proof. Let $\eta_1, ..., \eta_m \in B$. For every $\omega \in \bigcap_{i=1}^m S_{\eta_i}$, we have $\eta_i(\omega) \subseteq I$, where $i = 1, ..., m$, that is, $y^\omega_{ij} \in I$, where $i = 1, ..., m$ and $j = 1, ..., |\eta_i(\omega)|$. Thus, for all $\zeta \in \overline{f}(\eta_1, ..., \eta_m)$, we have $\zeta(\omega) \subseteq I$. So $\omega \in S_\zeta$. Consequently, $S_\zeta \supseteq \bigcap_{i=1}^m S_{\eta_i}$. Furthermore,

$$\mu(\zeta) = P(S_\zeta) \geq P(\bigcap_{i=1}^m S_{\eta_i}) \geq \max\{\sum_{i=1}^m P(S_{\eta_i}) - (m-1), 0\} = T_{\text{Ł}uk}(\mu(\eta_1), ..., \mu(\eta_m)).$$

Hence, we have $\inf_{\zeta \in \overline{f}(\eta_1,...,\eta_m)} \{\mu(\zeta)\} \geq T_{\text{Ł}uk}(\mu(\eta_1), ..., \mu(\eta_m))$.

Let $\eta \in B$. For every $\omega \in S_\eta$, $\eta(\omega) = \{y_1, ..., y_{|\eta(\omega)|}\} \subseteq I$. Hence, $-\eta(\omega) = \{-y_1, ..., -y_{|\eta(\omega)|}\} \subseteq I$, and so $\omega \in S_{-\eta}$. Therefore, we have $S_{-\eta} \supseteq S_\eta$. It follows that $\mu(-\eta) = P(S_{-\eta}) \geq P(S_\eta) = \mu(\eta)$.

Let $\eta_1, ..., \eta_n \in B$. If $\omega \in \bigcup_{i=1}^n S_{\eta_i}$, then there exists $i \in \{1, ..., n\}$ such that $\eta_i(\omega) = \{y^\omega_{i1}, ..., y^\omega_{i|\eta_i(\omega)|}\} \subseteq I$. Hence, $\overline{g}(\eta_1, ..., \eta_n)(\omega) \subseteq I$ since I is a hyperideal of R. So we have $\omega \in S_{\overline{g}(\eta_1,...,\eta_n)}$. It follows that $S_{\overline{g}(\eta_1,...,\eta_n)} \supseteq \bigcup_{i=1}^n S_{\eta_i}$. Therefore,

$$\begin{aligned}\mu(\overline{g}(\eta_1, ..., \eta_n)) &= P(S_{\overline{g}(\eta_1,...,\eta_n)}) \geq P(\bigcup_{i=1}^n S_{\eta_i}) \geq \max\{P(S_{\eta_1}), ..., P(S_{\eta_n})\} \\ &= \max\{\mu(\eta_1), ..., \mu(\eta_n)\}.\end{aligned}$$

Therefore, μ is a $T_{\text{Ł}uk}$-fuzzy hyperideal of R. ∎

Based on the theory of falling shadows, Tan et al. [34] establish a theoretical approach to define a fuzzy inference relation. Let A and B be fuzzy sets in the universes U and V, respectively, ξ and η be cut-clouds of A and B, respectively. Then the fuzzy inference relation $I_{A\to B}$ of the implication $A \to B$ is defined to be

$$I_{A\to B}(u, v) = P((\lambda, \mu)|(u, v) \in I_{A_\lambda \to B_\mu}) = P((\lambda, \mu)|(u, v) \in (A_\lambda \times B_\mu) \cup (A^c_\lambda \times V)),$$

where P is a joint probability on $[0,1]^2$. So different probability distribution P will generate different formula for the fuzzy inference relation. The following three basic cases are considered.

(1) If the whole probability P of (λ,μ) on $[0,1]^2$ is concentrated and uniformly distributed on the main diagonal $\{(\lambda,\lambda)|\lambda\in[0,1]\}$ of the unit square $[0,1]^2$, then P is the diagonal distribution and $I_{A\to B}(u,v)=\min(1-A(u)+B(v),1)$.

(2) If the whole probability P of (λ,μ) on $[0,1]^2$ is concentrated and uniformly distributed on the anti-diagonal $\{(\lambda,1-\lambda)|\lambda\in[0,1]\}$ of the unit square $[0,1]^2$, then P is the anti-diagonal distribution and $I_{A\to B}(u,v)=\max(1-A(u),B(v))$.

(3) If the whole probability P of (λ,μ) on $[0,1]^2$ is uniformly distributed on the unit square $[0,1]^2$, then P is the independent distribution and $I_{A\to B}(u,v)=1-A(u)+A(u)B(v)$.

We call the three fuzzy inference relations falling implication operators on $[0,1]$.

Definition 5.6. Let μ be a fuzzy set of R, I_{FS} be a falling implication operator over $[0,1]$ and $\lambda\in(0,1]$. Then μ is called an I_{FS}-fuzzy hyperideal of R If the following conditions are satisfied:

(1) $I_{FS}(\min\{\mu(x_1),...,\mu(x_m)\},\inf\limits_{z\in f(x_1^m)}\{\mu(z)\})\geq\lambda$ for all $x_1^m\in R$;

(2) $I_{FS}(\mu(x),\mu(-x))\geq\lambda$ for all $x\in R$;

(3) $I_{FS}(\max\{\mu(x_1),...,\mu(x_n)\},\mu(g(x_1^n)))\geq\lambda$ for all $x_1^n\in R$.

Clearly, if $\lambda=1$ and P is the diagonal distribution, then Definition 5.6 is equivalent to Definition 4.1.

Theorem 5.7. *Let μ be a fuzzy set of R and $\lambda=0.5$, then*

(1) *if P is the diagonal distribution, then μ is an I_{FS}-fuzzy hyperideal of R, if and only if*

(a) $\min\{\mu(x_1),...,\mu(x_m)\}\leq\inf\limits_{z\in f(x_1^m)}\{\mu(z)\}$, *or*
$0<\min\{\mu(x_1),...,\mu(x_m)\}-\inf\limits_{z\in f(x_1^m)}\{\mu(z)\}\leq 0.5$ *for all* $x_1^m\in R$,

(b) $\mu(x)\leq\mu(-x)$ *or* $0\leq\mu(x)-\mu(-x)\leq 0.5$ *for all* $x\in R$,

(c) $\max\{\mu(x_1),...,\mu(x_n)\}\leq\mu(g(x_1^n))$, *or*
$0\leq\max\{\mu(x_1),...,\mu(x_n)\}-\mu(g(x_1^n))\leq 0.5$ *for all* $x_1^n\in R$;

(2) *if P is the anti-diagonal distribution, then μ is an I_{FS}-fuzzy hyperideal of R, if and only if*

(a) $\min\{\mu(x_1), ..., \mu(x_m)\} \leq \inf\limits_{z \in f(x_1^m)} \{\max\{\mu(z), 0.5\}\}$, *or*
$\min\{\mu(x_1), ..., \mu(x_m), 0.5\} \leq \inf\limits_{z \in f(x_1^m)} \{\mu(z)\}$ *for all* $x_1^m \in R$,

(b) $\mu(x) \leq \max\{\mu(-x), 0.5\}$ *or* $\min\{\mu(x), 0.5\} \leq \mu(-x)$ *for all* $x \in R$,

(c) $\max\{\mu(x_1), ..., \mu(x_n)\} \leq \max\{\mu(g(x_1^n)), 0.5\}$, *or*
$\max\{\min\{\mu(x_1), 0.5\}, ..., \min\{\mu(x_n), 0.5\}\} \leq \mu(g(x_1^n)$ *for all* $x_1^n \in R$;

(3) *if* P *is the independent distribution, then* μ *is an* I_{FS}*-fuzzy hyperideal of* R, *if and only if*

(a) $\min\{\mu(x_1), ..., \mu(x_m)\}(1 - \inf\limits_{z \in f(x_1^m)} \{\mu(z)\}) \leq 0.5$ *for all* $x_1^m \in R$,

(b) $\mu(x)(1 - \mu(-x)) \leq 0.5$ *for all* $x \in R$,

(c) $\max\{\mu(x_1), ..., \mu(x_n)\}(1 - \mu(g(x_1^n))) \leq 0.5$ *for all* $x_1^n \in R$.

Proof. We only prove (1), and the proofs of (2) and (3) are similar. P is the diagonal distribution, and μ is an I_{FS}-fuzzy hyperideal of R, then we have

(a) $\min(1 - \min\{\mu(x_1), ..., \mu(x_m)\} + \inf\limits_{z \in f(x_1^m)} \{\mu(z)\}, 1) \geq 0.5$ for all $x_1^m \in R$;

(b) $\min(1 - \mu(x) + \mu(-x), 1) \geq 0.5$ for all $x \in R$;

(c) $\min(1 - \max\{\mu(x_1), ..., \mu(x_n)\} + \mu(g(x_1^n)), 1) \geq 0.5$ for all $x_1^n \in R$.

It follows that

(a) $\min(1 - \min\{\mu(x_1), ..., \mu(x_m)\} + \inf\limits_{z \in f(x_1^m)} \{\mu(z)\}, 1) \geq 0.5$
$\Leftrightarrow 1 - \min\{\mu(x_1), ..., \mu(x_m)\} + \inf\limits_{z \in f(x_1^m)} \{\mu(z)\} \geq 1$, or
$1 \geq 1 - \min\{\mu(x_1), ..., \mu(x_m)\} + \inf\limits_{z \in f(x_1^m)} \{\mu(z)\} \geq 0.5$
$\Leftrightarrow \min\{\mu(x_1), ..., \mu(x_m)\} \leq \inf\limits_{z \in f(x_1^m)} \{\mu(z)\}$, or
$0 < \min\{\mu(x_1), ..., \mu(x_m)\} - \inf\limits_{z \in f(x_1^m)} \{\mu(z)\} \leq 0.5$;

(b) $\min(1 - \mu(x) + \mu(-x), 1) \geq 0.5 \Leftrightarrow 1 - \mu(x) + \mu(-x) \geq 1$, or
$1 \geq 1 - \mu(x) + \mu(-x) \geq 0.5 \Leftrightarrow \mu(x) \leq \mu(-x)$, or
$0 \leq \mu(x) - \mu(-x) \leq 0.5$;

(c) $\min(1 - \max\{\mu(x_1), ..., \mu(x_n)\} + \mu(g(x_1^n)), 1) \geq 0.5$
$\Leftrightarrow 1 - \max\{\mu(x_1), ..., \mu(x_n)\} + \mu(g(x_1^n)) \geq 1$, or
$1 \geq 1 - \max\{\mu(x_1), ..., \mu(x_n)\} + \mu(g(x_1^n)) \geq 0.5$
$\Leftrightarrow \max\{\mu(x_1), ..., \mu(x_n)\} \leq \mu(g(x_1^n)$, or
$0 \leq \max\{\mu(x_1), ..., \mu(x_n)\} - \mu(g(x_1^n)) \leq 0.5$. ∎

6. Conclusions

In this paper, soft Krasner (m,n)-hyperrings and idealistic soft Krasner (m,n)-hyperrings are introduced based on soft set theory, and three isomorphism theorems of soft Krasner (m,n)-hyperrings are derived. Furthermore, we consider T-fuzzy hyperideals of Krasner (m,n)-hyperrings by using triangular norms, and discuss the relationships among idealistic soft Krasner (m,n)-hyperrings, fuzzy hyperideals and T-fuzzy hyperideals. Also, we define falling fuzzy hyperideals of Krasner (m,n)-hyperrings by using the theory of falling shadows, and give the relationships among idealistic soft Krasner (m,n)-hyperrings, T-fuzzy hyperideals and falling fuzzy hyperideals. In the following work, we will consider the connection of fuzzy soft set theory and some algebraic hyperstructures.

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EXTENSIONS OF GENERALIZED LIE GROUPS IN TERMS OF COHOMOLOGY

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Abstract. In this paper we introduce cohomology of generalized Lie groups, explain the problem of generalized Lie group extensions and present them in terms of cohomology of generalized Lie groups. Moreover we find a cohomological obstruction to the existence of extensions in non-Abelian case.

1. Introduction

A Lie group is, roughly speaking, an analytic manifold with a group structure such that the group operations are analytic. Lie groups arise in a natural way as transformation groups of geometric objects. For example, the group of all affine transformations of a connected manifold with an affine connection and the group of all isometries of a pseudo-Riemannian manifold are known to be Lie groups in the compact open topology. However, the group of all diffeomorphisms of a manifold is too big to form a Lie group in any reasonable topology, [12]. The problem of extending a group in terms of cohomology can be found in [1], [10] and in the non-Abelian Case in [2]. This problem can be generalized to Lie groups and their generalizations. Since the case of Lie groupoids and Lie algebroids is spelled out in the literature [3], we had to choose another generalization of Lie groups. A special generalization of Lie groups is called generalized Lie groups or top spaces which was introduced by M.R. Molaei in 1998, [6]. In this generalized field, several authors (Araujo, Molaei, Mehrabi, Oloomi, Tahmoresi, Ebrahimi, etc.) have studied different aspects of generalized groups and top spaces, [8], [4], [9], [5]. We begin by the definition of a generalized group.

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Definition 1.1. [6] A generalized group is a non-empty set $\mathcal{G}$ admitting an operation called multiplication which satisfies the following conditions:

(i) $(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3)$, for all $g_1, g_2, g_3 \in \mathcal{G}$.

(ii) For each $g \in \mathcal{G}$, there exists a unique $e(g)$ in $\mathcal{G}$ such that

$$g \cdot e(g) = e(g) \cdot g = g.$$

(iii) For each $g \in \mathcal{G}$, there exists $h \in \mathcal{G}$ such that $g \cdot h = h \cdot g = e(g)$.

In this paper, by $e(\mathcal{G})$ we mean

$$\{e(g) : g \in \mathcal{G}\}.$$

For any generalized group $\mathcal{G}$ and any $g \in \mathcal{G}$,

$$e^{-1}(e(g)) = \{h \in \mathcal{G} | e(h) = e(g)\}$$

has a canonical group structure. If $e(g)e(h) = e(gh)$, for all $g, h \in \mathcal{G}$, then $e(\mathcal{G})$ is an idempotent semigroup with this product.

A top space is a smooth manifold which its points can be (smoothly) multiplied together by a generalized group operation and generally its identity is a semigroup morphism, i.e.,

Definition 1.2. [4] A top space T is a Hausdorff d-dimensional differentiable manifold which is endowed with a generalized group structure such that the generalized group operations:

(i) $\cdot : T \times T \to T$ by $(t_1, t_2) \mapsto t_1 \cdot t_2$ which is called the multiplication map;

(ii) $^{-1} : T \to T$ by $t \mapsto t^{-1}$ which is called the inverse map; are differentiable and it holds

(iii) $e(t_1 \cdot t_2) = e(t_1) \cdot e(t_2)$, for all $t_1, t_2 \in T$.

Example 1.3. [4] The d-dimensional Euclidean space $\mathbb{R}^d$ with the product:

$$((a_1, ..., a_d), (b_1, ..., b_d)) \mapsto \left(\frac{da_1 + \sum b_i}{d}, ..., \frac{da_d + \sum b_i}{d} \right),$$

is a top space which is not a Lie group.

Throughout this paper by T_a we mean $T \cap e^{-1}(e(a))$.

Definition 1.4. [8] If T and S are two top spaces, then a homomorphism $f : T \to S$ is called a morphism if it is also a C^∞ map.

By f_a we mean $f|_{e^{-1}(e(a))}$, where f is a morphism of top spaces.

For a smooth manifold M, the set of all smooth functions from M to M such that their restriction to a submanifold of M is a diffeomorphism, i.e., partial diffeomorphisms of M is denoted by $D_P(M)$.

Definition 1.5. [11] An action of a top space T on a smooth manifold M is a map

$$\phi : T \to D_P(M),$$

which satisfies the following conditions:

(i) The map $T \times M \to M$ which maps (t, m) to $\phi_t(m)$ is a smooth function;

(ii) $\phi_{ts} = \phi_t \circ \phi_s$, for all $t, s \in T$.

Example 1.6. [11] For a top space T,

$$Ad : T \to D_P(M),$$

where $Ad_t(s) = tst^{-1}$ is an action of top spaces on manifolds which is called adjoint action.

Definition 1.7. [11] If G is a Lie group and M a smooth manifold, a partial action of G on M is a map

$$\varphi : G \to D_P(M),$$

such that the map $G \times M \to M$ is a smooth function and $\varphi^{gh} = \varphi^g \circ \varphi^h$, for all $g, h \in G$.

Theorem 1.8. [11] *Let T be a top space. Then ϕ is an action of T on M if and only if there exists a family of partial actions $\{\varphi_i\}_{i \in e(T)}$ of $e^{-1}(i)$ on M, for any $i \in e(T)$, such that $\varphi^{ts}_{e(ts)} = \varphi^t_{e(t)} \circ \varphi^s_{e(s)}$ and $\phi_t = \varphi^t_{e(t)}$, for all $t, s \in T$.*

Proof. Let ϕ be an action of top space T on M and $\varphi_{e(t)} = \phi \mid_{e^{-1}(e(t))}$, for all $t \in T$. Then we know that $\varphi_{e(t)}$ is a partial action of $e^{-1}(e(t))$ on M. Since $\varphi^t_{e(t)} = \phi \mid_{e^{-1}(e(t))} (t) = \phi_t$, we have $\varphi^{ts}_{e(ts)} = \phi(st) = \phi_s \circ \phi_t = \varphi^t_{e(t)} \circ \varphi^s_{e(s)}$. Also $\phi_t = \varphi^t_{e(t)}$.

Conversely, assume that there exists a family of partial actions $\{\varphi_i\}_{i \in e(T)}$ of $e^{-1}(i)$ on M, such that $\varphi^{ts}_{e(ts)} = \varphi^t_{e(t)} \circ \varphi^s_{e(s)}$, for all $t, s \in T$. Define $\phi : T \to D_P(M)$ such that $\phi_t = \varphi^t_{e(t)}$ for all $t \in T$. Let $t, s \in T$, $\phi_{ts} = \varphi^{ts}_{e(ts)} = \varphi^t_{e(t)} \circ \varphi^s_{e(s)} = \phi_t \circ \phi_s$ and hence ϕ is an action of T on M. ∎

By Theorem (1.8) we have:

Example 1.9. [11] Let T be a top space. Then

$$L : T \to D_P(T),$$

where $L_t(s) = ts$ is an action of T on T, which is called left action of T.

A vector field X on T is called partially left invariant if $(L_{*t}(X))_s = X_{L_t(s)}$.

Example 1.10. [11] Any partially left invariant vector field on T is a left invariant vector field on $e^{-1}(e(t))$, for all $t \in T$.

Theorem 1.11. [11] *Let T be a generalized group such that $e(t)e(s) = e(ts)$, for any $t, s \in T$. Then there is a generalized group isomorphism between T and*

$$e(T) \ltimes \{G_i\}_{i \in e(T)},$$

where $G_i = e^{-1}(i)$, for all $i \in e(T)$ and

$$e(T) \ltimes \{G_i\}_{i \in e(T)} = \{(i, g) | g \in G_i\}.$$

Proof. Suppose $G_i = e^{-1}(i)$ for all $i \in e(T)$ and

$$f : T \to e(T) \ltimes \{G_i\}_{i \in S},$$

such that $f(t) = (e(t), t)$. It is easy to show that f is a generalized group isomorphism. ∎

2. Extensions of top spaces and cohomology

We begin this section by introducing the notation of an exact sequence. Let $\{f_i\}_{i \in \mathbb{Z}}$, where $f_i : T_i \to T_{i+1}$ be a collection of top spaces homomorphisms, i.e.

$$\cdots \longrightarrow T_i \xrightarrow{f_i} T_{i+1} \xrightarrow{f_{i+1}} T_{i+2} \longrightarrow \cdots .$$

The sequence f_i is called exact if

$$range(f_i)_a = ker(f_{i+1})_a, \ for \ all \ a \in e(T_{i+1}).$$

For an exact sequence $(f_i)_a \circ (f_{i+1})_a = e(a)$, i.e. the composition of two adjacent homomorphisms is the trivial homomorphism on each $e(T_{i+1})$. The canonical surjective homomorphism $f_a : T_a \to \frac{T_a}{S_a}$ maps $t \in T_a$ onto the element of T_a/S_a defined by the class to which t belongs in T_a. Clearly, $\ker f_a = S_a$ since the Lie subgroup S_a, viewed as an element of T_a/S_a, is the identity of T_a/S_a. Thus, if $S_a \prec T_a$, the sequence of homomorphisms

$$e(a) \longrightarrow S_a \longrightarrow T_a \xrightarrow{f_a} T_a/S_a \longrightarrow e(a),$$

where f_a is the mentioned canonical homomorphism is exact, since $S_a \to T_a$ is an injection and $T_a/S_a \to e(a)$ is the trivial homomorphism. Moreover, the exact sequence

$$e(a) \longrightarrow S_a \longrightarrow T_a \longrightarrow B_a \longrightarrow e(a)$$

implies that $B_a \approx T_a/S_a$.

Definition 2.1. Let T, K be top spaces. A top space $\tilde{T}$ is said to be an extension of T by K if K is a top generalized normal subgroup of $\tilde{T}$, i.e $K \prec \tilde{T}$, and $\tilde{T}/K = T$.

In terms of exact sequences, according to [8] and Theorem 1.11, Definition 2.1 is equivalent to saying that

$$e(a) \longrightarrow K_a \longrightarrow \tilde{T}_a \longrightarrow T_a \longrightarrow e(a)$$

is exact for all $a \in e(T)$; thus K_a is injected into $\tilde{T}_a$ and $\tilde{T}_a$ projected onto T_a by the canonical homomorphism so that $T_a = \tilde{T}_a/K_a$. Since K is a top generalized normal subgroup of $\tilde{T}$, the elements of $\tilde{T}_a$ act on K_a by conjugation. Let Aut$K$ be the group of all automorphisms of K. Then there exist functions $f_a : \tilde{T}_a \to AutK_a$, such that $t \mapsto [\tilde{t}]$, where $[\tilde{t}]$ is defined by

$$[\tilde{t}] : k \in K_a \mapsto \tilde{t}k\tilde{t}^{-1} \in K_a \ .$$

The kernel of f_a is the centeralizer $C_{\tilde{T}_a}(K_a)$ of K_a in $\tilde{T}_a$. Thus, we have the following exact sequence of top space homomorphisms:

$$e(a) \longrightarrow C_{\tilde{T}_a}(K_a) \longrightarrow \tilde{T}_a \xrightarrow{f_a} AutK_a \ .$$

The center C_{K_a} of K_a is top generalized normal subgroup in $C_{\tilde{T}}(K)$; if H_a denotes the quotient group $H_a = C_{\tilde{T}_a}(K_a)/C_{K_a}$,

$$e(a) \longrightarrow C_{K_a} \longrightarrow C_{\tilde{T}_a}(K_a) \longrightarrow H_a \longrightarrow e(a)$$

is exact.

Let T be a top space, and A an Abelian top space, on which T operates through the homomorphism $\sigma : T \to$Aut$A$ = Out A (since A is Abelian, Int A is reduced to the trivial automorphism). As a consequence of Theorem 1.8 there exist partial actions

$$\sigma_a : T_a \to AutA_a,$$

for all $a \in e(T)$.

To construct a cohomology for top spaces, we need to introduce the n-cochain maps. A mapping $\alpha_n : T \times \ldots \times T \to A$ is a n-cochain, i.e.,

$$\alpha_n : (t_1, \ldots, t_n) \mapsto \alpha_n(t_1, \ldots, t_n) \in A \ .$$

The n-cochains form an Abelian top space, i.e. a Lie group, which will be denoted $C^n(T, A)$, the addition being defined by the pointwise addition of functions and obtaining the usual topology of functions. $C^0(H, A)$ is defined by the zero-dimensional cochains, i.e., the constant mappings α_0; thus $C^0(T, A) = A$. Note that α_a for all $a \in e(T)$ is defined on $T_a \times \ldots \times T_a$ with values in A_a. So we may consider $C^n(T_a, A_a)$. The operator $\delta_a : C^n :\to C^{n+1}$ (the coboundary operator) can be defined according to the way that the action $\sigma(t) \in$ Aut$A$ of the elements t of T is defined on A.

Definition 2.2. For the left action σ, the operator δ_a is given by

$$\begin{aligned}(\delta_a\alpha_n)(t_1,\dots,t_{n+1}) :=&\sigma_a(t_1)\alpha_n(t_2,\dots,t_{n+1})\\ &+\sum_{i=1}^{n}(-1)^i\alpha_n(t_1,\dots,t_{i-1},t_it_{i+1},t_{i+2},\dots,t_{n+1})\\ &+(-1)^{n+1}\alpha_n(t_1,\dots,t_n).\end{aligned}$$

And when the action σ is a right action, δ_a is given by

$$\begin{aligned}(\delta_a\alpha_n)(t_1,\dots,t_{n+1}) :=&(-1)^{n+1}\alpha_n(t_2,\dots,t_{n+1})\\ &+\sum_{i=1}^{n}(-1)^{i+n+1}\alpha_n(t_1,\dots,t_{i-1},t_it_{i+1},t_{i+2},\dots,t_{n+1})\\ &+\alpha_n(t_1,\dots,t_n)\sigma_a(t_{n+1}).\end{aligned}$$

Theorem 2.3. *The operator δ_a satisfies $\delta_a \circ \delta_a = 0$.*

Proof. It's a straightforward but cumbersome calculations from the above definitions. ∎

Consider the following sequence of Abelian top spaces:

$$C_a^0 \xrightarrow{\delta_a^0} C_a^1 \xrightarrow{\delta_a^1} C_a^2 \longrightarrow \cdots \xrightarrow{\delta_a^n} C_a^{n+1} \longrightarrow \cdots .$$

Since $\delta_a^2 = 0$,

$$range(\delta_a^{n+1} \circ \delta_a^n) = 0,\ range\delta_a^n \subset ker\delta_a^{n+1},$$

for every $a \in e(T)$, which motivates the definitions

$$Z_{\sigma_a}^n := ker\delta_a^n \equiv \{cocycles\},$$

$$B_{\sigma_a}^n := range\delta_a^{n-1} \equiv \{coboundaries\}.$$

Both $Z_{\sigma_a}^n$ and $B_{\sigma_a}^n$ are Lie subgroups of $C^n(T_a, A_a)$.

The quotient group

$$H_{\sigma_a}^n(T_a, A_a) := Z_{\sigma_a}^n(T_a, A_a)/B_{\sigma_a}^n(T_a, A_a)$$

is called the n-th cohomology of T_a with values on A_a for every $a \in e(T)$.

As in the case of abstract groups, the elements of the second cohomology group, characterize the extensions $\tilde{T}$ of the top space T by the Abelian top space A for the given action σ of T on A.

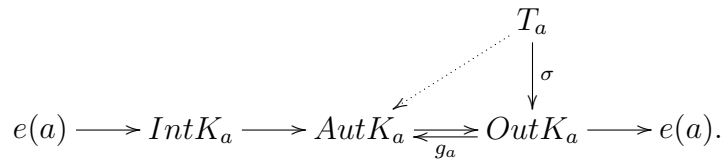
We are here concerning about extensions of the top space T by K in the case where K is not Abelian. The main difference from the Abelian case is that

not every top space is associated with one or more extensions, i.e., not every top space is extendible. In fact, one of the aims of this chapter is to show that the top space K determines an obstruction to the extension in the form of a certain three-cocycle; the top space T is extendible if this cocycle is, by an abuse of language, trivial.

For every $a \in e(T)$, consider the following exact sequence

$$e(a) \longrightarrow IntK_a \longrightarrow AutK_a \longrightarrow OutK_a \longrightarrow e(a),$$

which makes $AutK_a$ as an extension of $OutK_a$ by $IntK_a$. Let g_a be a trivializing section and let $\alpha_a = g_a \circ \sigma_a$ be defined by



It is clear that there exists an element $h_a(t', t) \in K_a$ such that

$$\alpha_a(t')\alpha_a(t) = [h_a(t', t)]\alpha_a(t't), \tag{2.1}$$

since $\alpha_a(t')\alpha_a(t)$ and $\alpha_a(t', t)$ differ by an internal automorphism. Consequently, (2.1) defines a mapping

$$[h]_a : T_a \times T_a \to IntK_a,\ [h]_a : (t', t) \mapsto [h_a(t', t)],$$

$$[h_a(t', t)]k := h_a(t', t)kh_a^{-1}(t', t).$$

The associative property in $AutK_a$,

$$\alpha_a(t'')(\alpha_a(t')\alpha_a(t)) = (\alpha_a(t'')\alpha_a(t'))\alpha_a(t),$$

leads to the two-cocycle property for $[h_a(t', t)] \in Z^2_{\alpha_a}(T_a, IntK_a)$, where

$$[\alpha_a(t'')h_a(t', t)] = \alpha_a(t'')[h_a(t', t)]\alpha_a(t'')^{-1},$$

or

$$[(\alpha_a(t'')h_a(t', t))h_a(t'', t't)] = [h_a(t'', t')h_a(t''t', t)]. \tag{2.2}$$

The above equation implies that the elements

$$(\alpha_a(t'')h_a(t', t))h_a(t'', t', t),\ h_a(t'', t')h_a(t''t', t)$$

of K_a determine the same element of $IntK_a$. Thus they differ by an element of the center C_{K_a}. Therefore the equality (2.2) in $IntK_a$ leads to an equality in K_a,

$$(\alpha_a(t'')h_a(t', t))h_a(t'', t't) = f_a(t'', t', t)h_a(t'', t')h_a(t''t', t); \tag{2.3}$$

note that $h_a(t', t)$ would itself be a two-cocycle for $f_a = e(a)$. Equation (2.3) determines a mapping $f_a : T_a \times T_a \times T_a \to C_{K_a}$, i.e. a three-cochain on T_a with values in the Abelian top space C_{K_a}.

Theorem 2.4. *The map $f_a \in Z^3_{(\sigma_0)_a}(T_a, C_{K_a})$ for $(\sigma_0)_a(t) = \sigma_a(t)$ acting on C_{K_a}, where it coincides with $\alpha_a(t)$ for all $a \in e(T)$.*

Proof. To prove the theorem it is sufficient to write f_a for every $a \in e(T)$ as

$$f_a(t'', t', t) = (\alpha_a(t'')h_a(t', t))h_a^{-1}(t''t', t)h_a^{-1}(t'', t') \tag{2.4}$$

and use this expression to check that f_a satisfies the cocycle condition,

$$\begin{aligned}(\delta_a f_a)&(t'', t', t, t_1)\\ &= (\alpha_a(t'')f_a(t', t, t_1))f_a^{-1}(t''t', t, t_1) \cdot f_a(t'', t't, t_1)f_a^{-1}(t'', t', tt_1) \cdot f_a(t'', t', t)\\ &= e(a),\end{aligned}$$

which is obtained from the definition (2.2) written in multiplicative notation. ∎

Our next step is to prove the following theorem.

Theorem 2.5. *Non-Abelian top space K together with the action σ characterize an element of the third cohomology group $H^3_{(\sigma_0)_a}(T_a, C_{K_a})$ for every $a \in e(T)$.*

Proof. We begin the proof by noting that for each action σ on K there exist partial actions σ_a on K_a for each $a \in e(T)$. We have to show that, if the choice of g_a (which modifies α_a) or that of h_a in (2.3) ($[h_a(t', t)]$ is uniquely determined by (2.1), but this fixes h_a only up to an element of C_{K_a}) is modified, the new f_a differ from the previous one by a three-coboundary. The proof is done in two steps.

a) Let h'_a be another element of the class $h_a * C_{K_a}$. Then $[h'_a(t', t)] = [h_a(t', t)]$ and it is related to h_a by

$$h'_a(t', t) = c_a(t', t)h_a(t', t), c_a(t', t) \in C_{K_a}. \tag{2.5}$$

Let us now consider (2.4) written for f'_a and h'_a and replace h'_a in it by its expression (2.5),

$$\begin{aligned}f'_a(t'', t', t) =&(\alpha_a(t'')(c_a(t', t)h_a(t', t)))c_a(t'', t't)h_a(t'', t't)\\ &h_a^{-1}(t''t', t)c_a^{-1}(t''t', t)h_a^{-1}(t'', t')c_a^{-1}(t'', t').\end{aligned} \tag{2.6}$$

Since h_a takes values in K_a and c_a in C_{K_a}, the c_a's locations are irrelevant, and (2.6) may be written as

$$\begin{aligned}f'_a(t'', t', t) =&(\alpha_a(t'')c_a(t', t))c_a(t'', t't)c_a^{-1}(t''t', t)c_a^{-1}(t'', t')\\ &(\alpha_a(t'')h_a(t', t))h_a(t'', t't)h_a^{-1}(t''t', t)h_a^{-1}(t'', t')\\ &\equiv (\delta_a c_a)(t'', t', t)f_a(t'', t', t).\end{aligned} \tag{2.7}$$

b) Now let α'_a be the mapping associated to a different trivializing section g'_a, and let $[k(t)] \in IntK_a$ be the element which satisfies

$$\alpha'_a(t) = [k(t)]\alpha_a(t), t \in T. \tag{2.8}$$

A way of proving that the change of α_a is unimportant is to take advantage of the freedom in h_a to choose an h'_a in such a way that f_a remains unaltered under the combined changes $\alpha_a \to \alpha'_a$, $h_a \to h'_a$. First we note that

$$\begin{aligned}(2.9)\qquad \alpha'_a(t')\alpha_a(t) =& [k(t')]\alpha_a(t')[k(t)]\alpha_a(t)\\ &= [k(t')][\alpha_a(t')k(t)]\alpha_a(t')\alpha_a(t)\\ &= [k(t')][\alpha_a(t')k(t)][h_a(t',t)]\alpha_a(t't)\\ &= [k(t')][\alpha_a(t')k(t)][h_a(t',t)][k^{-1}(t't)]\alpha'_a(t't)\\ &\quad [k(t')(\alpha_a(t')k(t))h_a(t',t)k^{-1}(t't)]\alpha'_a(t't).\end{aligned}$$

Now, looking at (2.9), we select an $h'_a(t',t)$ such that

$$h'_a(t',t) = k(t')(\alpha_a(t')k(t))h_a(t',t)k^{-1}(t't).$$

Then we write (2.4) for for α'_a and h'_a above,

$$\begin{aligned}(2.10)\qquad f'_a(t",t',t) =& \alpha'_a(t'')(k(t')(\alpha_a(t')k(t))h_a(t',t)k^{-1}(t't))\\ & k(t'')(\alpha_a(t'')k(t't))h_a(t'',t't)k^{-1}(t''t't)\\ & k(t''t't)h_a^{-1}(t''t',t)(\alpha_a(t''t')k^{-1}(t))k^{-1}(t''t')\\ & k(t''t')h_a^{-1}(t'',t')(\alpha_a(t'')k^{-1}(t'))k^{-1}(t'').\end{aligned}$$

Using the relation implied by (2.8),

$$(2.11)\qquad k(t'')\cdot\alpha_a(t'')() = \alpha'_a(t'')()\cdot k(t''),$$

equation (2.10) becomes equal to

$$\begin{aligned}(2.12)\qquad & \alpha'_a(t'')(k(t')(\alpha_a(t')k(t))h_a(t,t)k^{-1}(t't))\\ & (\alpha_a(t'')k(t't))k(t'')h_a(t'',t't)\\ & h_a^{-1}(t''t',t)(\alpha_a(t''t')k^{-1}(t))\\ & h_a^{-1}(t",t')(\alpha_a(t")k^{-1}(t'))k^{-1}(t").\end{aligned}$$

Using (2.11) again, (2.12) is seen to be equal to

$$\begin{aligned}(2.13)\qquad & \alpha'_a(t'')(k(t')\alpha_a(t')k(t))k(t'')\\ & \{(\alpha_a(t'')h_a(t',t))h_a(t'',t't)h_a^{-1}(t''t',t)h_a^{-1} - 1(t'',t')\}\\ & (\alpha_a(t'')\alpha_a(t')k^{-1}(t))(\alpha_a(t'')k^{-1}(t'))k^{-1}(t'')\\ & = f_a(t'',t',t)k(t'')\alpha_a(t'')(k(t')\alpha_a(t')k(t))\\ & \alpha_a(t'')((a(t')k^{-1}(t))k^{-1}(t'))k^{-1}(t'') = f_a(t'',t',t).\end{aligned}$$

■

If a top space T_a is extendible, then we have the diagram

$$\begin{array}{ccccccccc} e(a) & \longrightarrow & K_a & \xrightarrow{i} & \tilde{T}_a & \xrightarrow{\pi} & T_a & \longrightarrow & e(a) \\ & & & & \downarrow{\scriptstyle \psi_a} & & \downarrow{\scriptstyle \sigma_a} & & \\ e(a) & \longrightarrow & IntK_a & \longrightarrow & AutK_a & \longrightarrow & OutK_a & \longrightarrow & e(a), \end{array}$$

where $\psi_a : T_a \to AutK_a$ is the natural homomorphism defined by the action of the elements $t \in T_a$ on its top generalized normal subgroup K_a,

$$\psi_a : \tilde{t} \mapsto [\tilde{t}], [\tilde{t}]k = tkt^{-1}. \tag{2.14}$$

If $\tilde{T}_a$ is an extension, it is clear that the homomorphisms ψ_a and σ_a are compatible.

Theorem 2.6. *A top space T is extendible if and only if the cocycle f_a which it determines for every $a \in e(T)$ is a three-coboundary for such a.*

Proof. Let s_a be a trivializing section of the principal bundle $\tilde{T}_a(K_a, T_a)$. Now define

$$\alpha_a : T_a \to AutK_a,$$

by

$$\alpha_a = \psi_a \circ s_a, \alpha_a(t) = \psi_a(s_a(t)) = [s_a(t)].$$

We have

$$\begin{array}{ccccccccc} e(a) & \longrightarrow & K_a & \longrightarrow & \tilde{T}_a & \underset{s_a}{\overset{\pi}{\rightleftarrows}} & T_a & \longrightarrow & e(a) \\ & & & & \downarrow{\scriptstyle \psi_a} & \searrow^{\alpha_a} & \downarrow{\scriptstyle \sigma_a} & & \\ e(a) & \longrightarrow & IntK_a & \longrightarrow & AutK_a & \underset{g}{\overset{}{\rightleftarrows}} & OutK_a & \longrightarrow & e(a), \end{array}$$

(T_a is supposed here to be known, and so is ψ_a through (2.14)). Then

$$\begin{aligned} \alpha_a(t')\alpha_a(t) &\equiv (\psi_a \circ s_a)(t')(\psi_a o s_a)(t) \\ &\equiv [s_a(t')][s_a(t)] \equiv [\omega_a(t', t)][s_a(t't)] \\ &\equiv [\omega_a(t', t)](\psi_a \circ s_a)(t't) = [\omega_a(t', t)]\alpha_a(t't). \end{aligned} \tag{2.15}$$

Then from (2.1) we obtain

$$[\omega_a(t', t)] = [h_a(t', t)]$$

and an $h_a(t', t)$ in its class by C_{K_a} may be chosen such that

$$\omega_a(t', t) = h_a(t', t). \tag{2.16}$$

It is now easy to check that $f_a = e(a)$; from its definition we get

$$f_a(t'', t', t) = ([s_a(t'')]\omega_a(t', t))\omega_a(t'', t't)\omega_a^{-1}(t''t', t)\omega_a^{-1}(t'', t') = e(a),$$

where the last equality follows from the two-cocycle condition for ω_a.

Conversely, we follow the lower part of the diagram to define

$$\alpha_a = g \circ \sigma_a, \tag{2.17}$$

and choose h_a (by theorem (2.5)) so that $f_a = e(a)$. Let us now label the elements of $\tilde{T}_a$ by $\tilde{t} = (k, t)$ and define the multiplication law by

$$\tilde{t''} = (k', t')(k, t) = (k'(\alpha_a(t)k)h_a(t', t), t't). \tag{2.18}$$

Since $f_a = e(a)$, $h_a(t', t)$ is a two-cocycle. Then the two equations

$$\alpha_a(t')\alpha_a(t) = [h_a(t', t)]\alpha_a(t't), \tag{2.19}$$

$$(\alpha_a(t'')h_a(t', t))h_a(t'', t't) = h_a(t'', t')h_a(t''t', t), \tag{2.20}$$

show immediately that the group law is associative, that $\tilde{e} = (1, e)$ is the identity element and that

$$\tilde{t}^{-1} \equiv (k, t)^{-1} = (\alpha(t)^{-1}(k^{-1}h^{-1}(t, t^{-1})), t^{-1}).$$

Thus, $g \circ \sigma_a = \psi_a \circ s_a$, and our diagram is well defined. Also $\tilde{T}_a$ has a topology which makes it a top space according to [8]. ∎

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THE WIGNER-POISSON-FOKKER-PLANCK SYSTEM WITH EXCHANGE POTENTIAL IN WEIGHTED L^2 SPACE

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Abstract. This paper is concerned with the Wigner-Poisson-Fokker-Planck(WPFP) system subject to Coulomb and exchange potential. In this work, existence and uniqueness of the local mild solution are established on an appropriately weighted-L^2 space in one dimension. The main difficulties in establishing mild solution are to derive a-priori estimates on the appropriate potential term. The proof is based on contraction mapping principle and parabolic regularization of the quantum Fokker-Planck term.

Keywords: Wigner-Poisson-Fokker-Planck system; exchange potential; contraction mapping principle; parabolic regularization.

AMS 2010 Subject Classification: 35Q40; 47B44; 35K55.

1. Introduction and main result

In quantum mechanics, there is a widespread concern over the following version of the nonlinear Schrödinger-Poisson-Slater (Schrödinger-Poisson-X^α) system, such as [14], [13], [24], namely,

$$(1.1)\quad i\hbar\psi_t(t,x) = -\frac{\hbar^2}{2}\Delta\psi(t,x) + V_{\text{cou}}\psi(t,x) - V_{\text{ext}}\psi(t,x) - \alpha V_{\text{exc}}\psi(t,x),\ x \in R^n,$$

which has been studied by many researchers in recent years. The wave function $\psi(t,x)$ is complex-valued, α denotes a given constant and V_{ext} stands for a given external potential, for example a harmonic potential

$$V_{\text{ext}} = a|x|^2 + b\cdot x + c,\ a\in R^+,\ b\in R^n,\ c\in R. \tag{1.2}$$

The macroscopic density $\rho=\rho(t,x)$ is now given by the zeroth order moment in the kinetic variable v, i.e., by the physical observables from both the wave function ψ and the Winger function w, namely,

$$\rho(t,x) = |\psi|^2 = \int_{R^n} w(t,x,v)dv. \tag{1.3}$$

The Coulomb potential V_{cou} is a solution of the following Poisson equation

$$\Delta V_{\text{cou}} = -(\rho-\varrho), \tag{1.4}$$

where $\varrho(x)\geq 0$ denotes the fixed positively charged background, i.e., the doping profile in semiconductor modeling (see [25], [15] for more details), which is assumed to be given. The exchange potential V_{exc} in (1.1) is due to Slater [17], [28] who replaces it by

$$V_{\text{exc}} = \rho^{\frac{1}{n}},\ x\in R^n. \tag{1.5}$$

The Wigner transform of the wave function $\psi(t,x)$ is

$$w(t,x,v) = \frac{1}{(2\pi)^n}\int_{R^n}\psi\left(t,x-\frac{\hbar}{2}y\right)\overline{\psi}\left(t,x+\frac{\hbar}{2}y\right)\exp(iv\cdot y)dy, \tag{1.6}$$

where $\overline{\psi}$ denotes the complex conjugate of the wave function ψ. A direct calculation by applying the transform (1.6) to the Schrödinger-Poisson-Slater system (1.1) shows that $w(t,x,v)$ satisfies the so-called collisionless Wigner-Poisson-X^α system, see e.g. [28], [23],

$$\begin{aligned} w_t + (v\cdot\nabla_x)w - \Theta_\hbar[V_{\text{ext}}]w - \alpha\Theta_\hbar[\rho^{\frac{1}{n}}]w + \Theta_\hbar[V_{\text{cou}}]w &= 0,\\ w(0,x,v) &= w_0(x,v), \end{aligned} \tag{1.7}$$

where the Wigner function $w=w(t,x,v)$ is a probabilistic quasi-distribution function of particles at time $t\geq 0$, located at $x\in R^n$ with velocity $v\in R^n$. The operator $\Theta_\hbar[\varphi]$ in (1.7) is a pseudo-differential operator, as in [4], [6], [9], formally defined by

$$\begin{aligned} \Theta_\hbar[\varphi]w(t,x,v) &= \frac{i}{(2\pi)^n\hbar}\int_{R^n}\int_{R^n}\delta[\varphi](t,x,\eta)w(t,x,v')e^{i(v-v')\eta}dv'd\eta,\\ \delta[\varphi](t,x,\eta) &= \varphi\left(t,x+\frac{\hbar\eta}{2}\right)-\varphi\left(t,x-\frac{\hbar\eta}{2}\right). \end{aligned} \tag{1.8}$$

The system (1.7)–(1.8) is a Hamiltonian system (i.e., the collisionless Wigner-Poisson-X^α equation), which is (almost) equivalent to the Schrödinger-Poisson-X^α

system. This allows for a much simpler analysis, see [14], [13] and therein. In this work, we shall now consider an open quantum system that includes a collision operator $Q[w]$, for example a Fokker-Planck operator, such as [4], [6],

$$Q[w] = \beta \mathrm{div}_v(vw) + \sigma \Delta_v w + 2\gamma \mathrm{div}_x(\nabla_v w) + \kappa \Delta_x w, \tag{1.9}$$

which models diffusive effects (e.g. the electronphonon-interaction), see [20, 22, 29]. A combination of the Wigner-Poisson-X^α system (1.7) and the Fokker-Planck operator (1.9) gives us the Wigner-Poisson-Fokker-Planck system coupled with exchange potential:

$$\begin{aligned} w_t + (v \cdot \nabla_x)w - \Theta_\hbar[V_{\mathrm{ext}}]w - \alpha\Theta_\hbar[\rho^{\frac{1}{n}}]w + \Theta_\hbar[V_{\mathrm{cou}}]w &= Q[w], \\ w(0,x,v) &= w_0(x,v). \end{aligned} \tag{1.10}$$

Such a model is not any more equivalent to a system of Schrödinger-Poisson-X^α system. Therefore, the mathematical analysis must be done on the level of the Wigner function. Moreover, the methods are introduced in our analysis, which refer to [4], [2], [10], [16].

In the literature, two problems linked to the Wigner(-Poisson)-Fokker-Planck system without exchange potential (i.e., $\alpha = 0$) are mainly studied: the long time behavior of the solutions and the existence of stationary solutions of Wigner-Fokker-Planck system (i.e., $V_{cou} = 0$) have studied in [6], [11]; the existence and uniqueness of the solution of the Wigner-Poisson-Fokker-Planck system have established in [4], [2]. Some of these results (as well as considerations about other quantum physical problems) have been reviewed in [5].

The present paper, we aim to establish the existence and uniqueness of the solution for the WPFP system (1.10) coupled Poisson equations (1.4) and exchange potential (1.5) on $\Omega = I \times R_v = [0,1] \times R_v$ in a weighted L^2 space (in symbols X, see (2.1) for detail), with the following boundary conditions:

$$V_{\mathrm{cou}}(0,t) = V_{\mathrm{cou}}(1,t),\ w(0,v,t) = w(1,v,t),\ v \in R_v,\ t \geq 0. \tag{1.11}$$

The analysis will proceed as follows. The natural choice of the functional setting for the study of the Wigner-Poisson or Wigner-Poisson-Fokker-Planck problem is the Hilbert space $L^2(R_x^n \times R_v^n)$, see [18], [26]. However, it can be immediately observed that the density $\rho(t,x)$, given by (1.3), is not well-defined for any $w(t,x,v)$ belonging to this space. In other words, the nonlinear terms $\Theta_\hbar[\rho^{\frac{1}{n}}]w$ and $\Theta_\hbar[V_{\mathrm{cou}}]w$ are not defined pointwise in t on the state space of the Wigner function. Therefore, inspired by [4], [9], [27], we introduce a Hilbert space X in Section 2, such that, the existence of the density $\rho(t,x)$ is granted for any $w \in X$.

With the above notations, the main result of this paper can be described as the following theorem.

Theorem 1. *Let $\kappa\sigma \geq \gamma^2$, for every $w_0 \in X$, the WPFP problem (1.10) − (1.11), coupled the equations (1.4) − (1.5), has a unique mild solution $w \in C([0, t_{\max}), X)$.*

Remark 1. When considering the WPFP problem (1.10) without external potential and exchange potential, i.e., $V_{\text{ext}} = 0$ and $\alpha = 0$, Arnold et al.[2], [4] proved that $t_{\max} = +\infty$ by recovering a priori estimates for $\|w(t)\|_{L^2}, \||v|w\|_{L^2}, \||v|^2 w\|_{L^2}$ for all times in one or three dimensions.

In the present case $\alpha \neq 0$, we do not succeed in repeating the analogous strategy because the operator $\Theta_\hbar[\rho]w$ can be only controlled by $\|w\|_{H^1}$. Precisely speaking, we must use the regularity of the linear WFP equation to obtain existence of a solution, see Lemma 4. Thus, we are still working on results assuring the existence of the solution for t on the whole R^+.

Remark 2. Nevertheless, due to the regularity of the exchange potential, this result does not hold in higher dimensions. Precisely speaking, the symbols $\nabla(\rho^{\frac{1}{n}}), \Delta(\rho^{\frac{1}{n}})$ do not make sense at $\rho = 0$ when $n > 1$, see Lemma 4.

Our paper is structured as follows. In Section 2, we introduce a weighted space for the Wigner function w that allows to define the nonlinear terms $\Theta_\hbar[\rho]w$ and $\Theta_\hbar[V_{\text{cou}}]w$. And we also show that the quantum Fokker-Planck term is a semigroup-generator on the weighted L^2 space via the the Lumer-Phillips theorem [7]. In Section 3, we obtain a local-in-time, mild solution for WPFP with exchange potential in one dimensional using a fixed point argument and the parabolic regularization of the quantum Fokker-Planck term.

2. The functional setting and preliminaries

In this section, we shall discuss the functional analytic preliminaries for studying the nonlinear WPFP problem. First, we shall introduce an appropriate state space for the Wigner function w which allows to control the nonlinear terms $\Theta_\hbar[\rho]w$ and $\Theta_\hbar[V_{\text{cou}}]w$. Next, we shall discuss the linear WFP equation and the dissipativity of its (evolution) generator A defined in (2.15).

2.1. The State space and pseudo-differential operator

In this subsection, we shall analyze the properties of the pseudo-differential operator $\Theta_\hbar[\varphi]w$, which will be considered as a perturbation of the generator A defined in (2.15). This is one of the key ingredients for proving the Theorem 1.

We would show that the pseudo-differential operators $\Theta_\hbar[\rho]w$ and $\Theta_\hbar[V_{\text{cou}}]w$ are (local) bounded in the weighted L^2 space, in symbols:

$$X := L^2(I_x \times R_v, (1+v^2)dxdv),\ I = [0,1], \tag{2.1}$$

endowed with the following scalar product

$$\langle f, g\rangle_X := \int_0^1 \int_{R_v} f(x,v) \cdot \overline{g(x,v)}(1+v^2)dvdx,\ \forall f, g \in X. \tag{2.2}$$

In our calculations, we shall use the following norm:

$$||f||_X^2 := ||f||_{L^2}^2 + ||vf||_{L^2}^2, \tag{2.3}$$

The following proposition motivates our choice of the space X for the analysis.

Lemma 1. *Let $w \in X$ and $\rho(x,t)$ defined in* (1.3)*, for all $x \in I$, then ρ belongs to $L^2(I)$ and satisfies*

$$||\rho||_{L^2(I)} \leq \sqrt{\pi}||w||_X, \tag{2.4}$$

And ρ also belongs to $L^1(I)$ and satisfies

$$||\rho||_{L^1(I)} \leq \sqrt{\pi}||w||_X. \tag{2.5}$$

Moreover, ρ belongs to $L^p(I)$ with $1 \leq p \leq 2$ and $\|\rho\|_{L^p(I)} \leq \sqrt{\pi}\|w\|_X$.

Proof. Similar to [4], [2], [9], [10], the first assertion follows directly by using Cauchy–Schwartz inequality in v-integral. On the other hand, by Hölder inequality, we have

$$\begin{aligned}
\|\rho\|_{L^1(I)} &\leq \int_0^1 \left| \int_R w(x,v,t)dv \right| dx \\
&\leq \left[\int_0^1 1^2 dx \right]^{\frac{1}{2}} \left[\int_0^1 \left| \int_R w(x,v,t)dv \right|^2 dx \right]^{\frac{1}{2}} \\
&\leq \sqrt{\pi}\|w\|_X.
\end{aligned}$$

Using the interpolation inequality, we get

$$\|\rho\|_{L^p(I)} \leq \|\rho\|^{\theta}_{L^2(I)}\|\rho\|^{1-\theta}_{L^1(I)} \leq \sqrt{\pi}\|w\|_X. \qquad \blacksquare$$

Now, we introduce the weak version of the Poisson equation (1.4)

$$\int_0^1 \partial_x V_{\text{cou}} \partial_x \phi dx = \int_0^1 \phi(\rho - \varrho)dx, \ \forall \phi \in W_0^{1,2}(I). \tag{2.6}$$

We can summarize the results relevant to the solution of the equation (1.4) in the following lemma and remarks.

Lemma 2. *For all $\rho, \varrho \in L^2(I)$, an unique $V_{cou} \in W_0^{1,2}(I)$ exists which satisfies* (2.7) *and*

$$\|V_{cou}\|_{W_0^{1,2}(I)} \leq C(\|\rho\|_{L^2(I)} + \|\varrho\|_{L^2(I)}).$$

Moreover, $\|V_{cou}\|_{W^{2,2}(I)} \leq C(\|\rho\|_{L^2(I)} + \|\varrho\|_{L^2(I)})$.

Remark 3. In fact, because of the following Sobolev's Theorem

$$W^{2,2}(I) \hookrightarrow \mathcal{C}^1_B(I),$$

we have

$$\|V_{\text{cou}}\|_{\mathcal{C}^1_B(I)} \leq C\|V_{\text{cou}}\|_{W^{2,2}(I)}.$$

So, if we call $V_{\rm cou}$ the extension of the solution with value zero out of the I, we obtain

$$\begin{aligned}\|V_{\rm cou}\|_{W^{1,\infty}(R)} &\leq \max\left\{\sup_{x\in I}|V_{\rm cou}|+\sup_{x\in I}|\partial_x V_{\rm cou}|\right\}\\ &\leq C\|V_{\rm cou}\|_{W^{2,2}(I)}\\ &\leq C(\|\rho\|_{L^2(I)}+\|\varrho\|_{L^2(I)}).\end{aligned}$$

Actually, an extension of the Lemma 2 states that the weak solution $V_{\rm cou}$ is in $W^{3,2}(I)$ if $\rho,\varrho\in W^{1,2}(I)$. Furthermore, we can show that the following estimate is true by extending Lemma 1

$$\|\partial_x\rho\|_{L^2(I)}\leq C\|\partial_x w\|_X. \tag{2.7}$$

Remark 4. Let $T: X\mapsto W^{1,\infty}(R)$ be the map

$$(Tw)(x)=\begin{cases} V_{\rm cou} & : \quad x\in I,\\ 0 & : \quad x\in R\setminus I,\end{cases}$$

where $V_{\rm cou}$ is the continuous solution of the equation (1.4) with boundary conditions (1.11). Then $T\in\mathcal{B}(X;W^{1,\infty}(R))$, moreover, see [19], Theorem 5.4.1,

$$\|Tw\|_{W^{1,\infty}(R)}\leq C\|V_{\rm cou}\|_{W^{1,\infty}(I)}\leq C(\|w\|_X+\|\varrho\|_{L^2(I)}). \tag{2.8}$$

Next, we consider the Lipschitz properties of the pseudo-differential operator $\Theta_\hbar[\varphi]w$ defined by (1.8). But by the definition of it, the $V_{\rm cou}$ and $V_{\rm exc}$ have to be appropriately extended to R_x. However, the Remarks 3-4 enable us to give a less formal definition of the pseudo-differential operator $\Theta_\hbar[\varphi]w$; we show indeed that the operator $\Theta_\hbar[\varphi]w$ is well defined from the space X to itself. On the other hand, the operator $\Theta_\hbar[\varphi]w$ can be rewritten in a more compact form as, see [4], [3],

$$\begin{aligned}\Theta_\hbar[\varphi]\,w &= \frac{i}{\hbar}\mathcal{F}^{-1}_{\eta\to v}\left[\delta\varphi\mathcal{F}_{v\to\eta}[w]\right]=\frac{i}{\hbar}(\mathcal{F}^{-1}_{\eta\to v}\left[\delta\varphi\right]*_v w),\\ \mathcal{F}_{v\to\eta}(\Theta_\hbar[\varphi]\,w)(x,\eta) &= \frac{i}{\hbar}\delta\varphi(x,\eta)\mathcal{F}^{-1}_{\eta\to v}w(x,\eta),\end{aligned} \tag{2.9}$$

where the symbol $*_v$ is the partial convolution with respect to the variable v, $\mathcal{F}_{v\to\eta}$ is the Fourier transformation with respect to the variable v and $\mathcal{F}^{-1}_{\eta\to v}$ its inverse:

$$\mathcal{F}_{v\to\eta}[f(x,\cdot)](\eta)=\int_R f(x,v)e^{iv\cdot\eta}dv,\ \mathcal{F}^{-1}_{\eta\to v}[g(x,\cdot)](v)=\frac{1}{2\pi}\int_R g(x,v)e^{-iv\cdot\eta}d\eta$$

for suitable functions f and g. Moreover, we can state the following result:

Lemma 3. *The operator $\Theta_\hbar\left[V_{cou}\right]w$ maps X into itself and there exists $C>0$ such that*

$$||\Theta_\hbar\left[V_{cou}\right]w||_X\leq C(||w||_X+||\varrho||_{L^2(I)})||w||_X. \tag{2.10}$$

Moreover, the operator $\Theta_\hbar\left[V_{cou}\right]w$ is of class C^∞ in X, and satisfies

$$\begin{aligned}&\|\Theta_\hbar\left[V^1_{cou}\right]w_1-\Theta_\hbar\left[V^2_{cou}\right]w_2\|_X\\ &\qquad\leq C(\|w_1\|_X+\|w_2\|_X+\|\varrho\|_{L^2(I)})\|w_1-w_2\|_X.\end{aligned} \tag{2.11}$$

Proof. We refer the reader to [1], [4], [2], [9], [10]. ∎

Let $\widetilde{\rho}$ is an extension of ρ, i.e.,

$$\widetilde{\rho}(x) = \begin{cases} \rho & : \quad x \in I, \\ 0 & : \quad x \in R \setminus I. \end{cases}$$

It is obvious that $\|\widetilde{\rho}(x)\|_{L^p(R)} \leq C\|\rho\|_{L^p(I)}$ holds by Theorem 5.4.1 of [19]. Similarly, we can state the following result:

Lemma 4. *Let $\partial_x w \in X$, then one has*

$$||\Theta_\hbar[\rho]\, w||_X \leq C\|w\|_X(\|w\|_X + \|\partial_x w\|_X). \tag{2.12}$$

Moreover,

$$\begin{aligned} &\|\Theta_\hbar[\rho_1]\, w_1 - \Theta_\hbar[\rho_2]\, w_2\|_X \\ &\quad \leq C(\|w_1 - w_2\|_X + \|\partial_x(w_1 - w_2)\|_X)(\|w_1\|_X + \|\partial_x w_1\|_X + \|w_2\|_X). \end{aligned} \tag{2.13}$$

Proof. Using Plancherel's formula and Hölder's inequality, this implies

$$||\Theta_\hbar[\rho]\, w||_{L^2(\Omega)} \leq 2\|\rho\|_{L^\infty(I)}\|w\|_{L^2(\Omega)}.$$

From (2.9), we see that

$$\|v\Theta_\hbar[\rho]\, w\|_{L^2(\Omega)} \leq C\|\partial_\eta(\delta\rho\mathcal{F}_v w)\|_{L^2(\Omega)}.$$

where ∂_η denotes the derivation operator with respect to the variables η. Similar to Lemma 3, we have

$$\begin{aligned} \|\partial_\eta(\delta\rho\cdot(\mathcal{F}_v w))\|_{L^2(\Omega)} &= \|\partial_\eta\delta\rho\cdot\mathcal{F}_v w + \delta\rho\cdot\partial_\eta(\mathcal{F}_v w)||_{L^2(\Omega)} \\ &\leq ||\partial_\eta\delta\rho\cdot\mathcal{F}_v w||_{L^2(\Omega)} + ||\delta\rho\cdot\partial_\eta(\mathcal{F}_v w)||_{L^2(\Omega)} \\ &\leq ||\partial_\eta\delta\rho\cdot\mathcal{F}_v w||_{L^2(\Omega)} + ||\delta\rho||_{L^\infty(I)}||\partial_\eta(\mathcal{F}_v w)||_{L^2(\Omega)}. \end{aligned}$$

On the other hand, for all functions $f \in H^1(R)$, there exists a constant C, independent of f, such that

$$\|f\|_{L^\infty(R)} \leq C\|f\|_{H^1(R)}, \tag{2.14}$$

by Sobolev embedding theorem. Then, if we apply estimate (2.14) to the function $\rho, \mathcal{F}_v w$, we get

$$\|\rho\|_{L^\infty(I)}\|w\|_{L^2(\Omega)} \leq C\|\rho\|_{H^1}\|w\|_X$$

and

$$\begin{aligned} \|\partial_\eta(\delta\rho\cdot(\mathcal{F}_v w))\|_{L^2(\Omega)} &\leq C[\|\partial_x\rho\|_{L^2}\|\mathcal{F}_v w\|_{L^2(R;H^1)} + \|\rho\|_{H^1}\|vw\|_{L^2}] \\ &\leq C\|\rho\|_{H^1}\|w\|_X \\ &\leq C(\|w\|_X + \|\partial_x w\|_X)\|w\|_X. \end{aligned}$$

Moreover, the first assertion follows directly by combining these estimates. Subsequently, we study the Lipschitz property of the operator $\Theta_\hbar[\rho]w$. In fact,

$$\begin{aligned}
&\|\Theta_\hbar[\rho_1]w_1 - \Theta_\hbar[\rho_2]w_2\|_X \\
&\quad\le \|\Theta_\hbar[\rho_1]w_1 - \Theta_\hbar[\rho_1]w_2\|_X + \|\Theta_\hbar[\rho_1]w_2 - \Theta_\hbar[\rho_2]w_2\|_X \\
&\quad\le \|\Theta_\hbar[\rho_1](w_1 - w_2)\|_X + \|(\Theta_\hbar[\rho_1] - \Theta_\hbar[\rho_2])w_2\|_X = \Pi_1 + \Pi_2.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\Pi_1 &\le \|\Theta_\hbar[\rho_1](w_1 - w_2)\|_{L^2} + \|v\Theta_\hbar[\rho_1](w_1 - w_2)\|_{L^2} \\
&\le \|\delta\rho_1\mathcal{F}_{v\to\eta}(w_1 - w_2)\|_{L^2} + \|\partial_\eta[\delta\rho_1]\mathcal{F}_{v\to\eta}(w_1 - w_2)\|_{L^2} \\
&\quad + \|\delta\rho_1\partial_\eta\mathcal{F}_{v\to\eta}(w_1 - w_2)\|_{L^2} \\
&\le C(\|\rho_1\|_{L^\infty}\|w_1 - w_2\|_{L^2} + \|\partial_x\rho_1\|_{L^2}\|\mathcal{F}_{v\to\eta}(w_1 - w_2)\|_{L^2(R;L^\infty)} \\
&\quad + \|\rho_1\|_{L^\infty}\|\partial_\eta\mathcal{F}_{v\to\eta}(w_1 - w_2)\|_{L^2}) \\
&\le C(\|\rho_1\|_{H^1}\|w_1 - w_2\|_{L^2} + \|\partial_x\rho_1\|_{L^2}\|\mathcal{F}_{v\to\eta}(w_1 - w_2)\|_{L^2(R;H^1)}) \\
&\le C(\|w_1\|_X + \|\partial_x w_1\|_X)\|w_1 - w_2\|_X.
\end{aligned}$$

By an analogous way, we get

$$\begin{aligned}
\Pi_2 &\le \|\Theta_\hbar[\rho_1 - \rho_2]w_2\|_{L^2} + \|v\Theta_\hbar[\rho_1 - \rho_2]w_2\|_{L^2} \\
&\le C(\|w_1 - w_2\|_X + \|\partial_x(w_1 - w_2)\|_X)\|w_2\|_X.
\end{aligned}$$

This concludes the proof of result. ∎

2.2. Basic properties of the quantum Fokker-Planck

In this subsection, we will show that the quantum Fokker-Planck term is a semigroup-generator in the weighted L^2 space (2.1). An easy calculation shows that

$$\Theta_\hbar[V_{\text{ext}}]w(t,x,v) \to (2ax + b)\cdot\nabla_v w(t,x,v), \tag{2.15}$$

in $L^2(R^n_v \times R^n_x)$ with $\hbar \to 0$, see [16] and therein. Without loss of generality, we can shift this problem with respect to x and then set $b = 0$.

We will consider the linear operator $A : D(A) \to X$

$$Aw = 2axw_v + Q[w] - vw_x = 2axw_v - vw_x + \beta(vw)_v + \sigma w_{vv} + 2\gamma w_{xv} + \kappa w_{xx} \tag{2.16}$$

defined on

$$D(A) = \left\{ w \in X \;\middle|\; vw_x, vw_v, w_{vv}, w_{xv}, w_{xx} \in X, w(0,v,t) = w(1,v,t) \right\}$$

We will study the dissipation property of the operator A, which is defined on X space, i.e., whether $\langle Au, u\rangle_X \le 0, \forall u \in D(A)$ holds.

Lemma 5. *Let the coefficients of the operator A satisfy $\kappa\sigma \geq \gamma^2$, then $A - \left(\sigma + \frac{\beta}{2} + a\right) I$ and its closure are dissipative.*

Proof. Indeed, by [2], if the coefficients of the operator $Q[w] - vw_x$ satisfy $\kappa\sigma \geq \gamma^2$, then $Q[w] - vw_x - \left(\sigma + \frac{\beta}{2}\right) I$ and its closure are dissipative in X. On the other hand, since $\int_0^1 \int_R xw_v w dv dx = 0$ and

$$\int_0^1 \int_R xv^2 w_v w dv dx = -\int_0^1 \int_R (2xvw^2 + xv^2 w w_v) dv dx,$$

then we have

$$\int_0^1 \int_R xv^2 w_v w dv dx \leq \frac{1}{2}\int_0^1 \int_R (x^2 + v^2) w^2 dv dx \leq \frac{1}{2}\int_R (1 + v^2) w^2 dv dx \leq \frac{1}{2}\|w\|_X^2.$$

Moreover, $A - \left(a + \sigma + \frac{\beta}{2}\right) I$ and its closure are dissipative in X. ∎

Furthermore, we have the following theorem:

Theorem 2. *For $w_0 \in X$, the linear WFP system*

$$w_t = \overline{A} w, \;\; w(0, x, v) = w_0(x, v) \tag{2.17}$$

has a unique solution $w \in C((0, +\infty); X)$. If furthermore $w_0 \in \mathcal{D}(\overline{A})$, then the solution is a classical solution, i.e., it belongs to $w \in C^1((0, +\infty); X)$. Moreover,

$$||w(t)||_X =\leq e^{(a+\sigma+\frac{\beta}{2})t} ||w_0||_X. \tag{2.18}$$

Proof. By the Lumer-Phillips theorem (Corollary 1.4.4 of [7]), we should show that $(\overline{A})^* - (a + \sigma + \frac{\beta}{2})I$ is also dissipative. Indeed, A^* is densely defined on $D(A^*) \supseteq D(A)$, and hence A is a closable operator (see Theorem VIII.1.b of [21]). Its closure $\overline{A}$ satisfies $(\overline{A})^* = A^*$ (see [21], Theorem VIII.1.c). Thus, by the analogous arguments in [4, 2] (see Lemma 2.2 of [2] and Lemma 2.10 of [4] for detail), we can obtain that $A^* - (a + \sigma + \frac{\beta}{2})I$ is dissipative on all of $D(A^*)$. Applying Corollary 1.4.4 of [7] to $\overline{A} - (a + \sigma + \frac{\beta}{2})I$ with $(\overline{A})^* = A^*$ then implies that $\overline{A} - (a + \sigma + \frac{\beta}{2})$ generates a C_0 semigroup of contractions on X. Moreover, the semigroup C_0 generated by the operator $\overline{A}$ satisfies

$$||w(t)||_X = ||e^{\overline{A}t} w_0||_X \leq e^{lt} ||w_0||_X,$$

where $l = a + \sigma + \frac{\beta}{2}$. This concludes the proof of Theorem 2. ∎

Remark 5. In order to obtain the dissipative properties of $A^* - \tau I$ (τ is a positive constant) on the corresponding $D(A^*)$, see [2], [4] for detail, the arguments are introduced on the weighted $L^2(R_x^n \times R_v^n, (1 + |v|^2)^m dx dv, m > \frac{n}{2})$ space for the WPFP equation in R and R^3, respectively. It would be very interesting to study an analogous arguments for the Fokker-Planck type operators in [8], [12].

On the other hand, The unique solution of linear equation (2.17) can be expressed explicitly with the help of a fundamental solution, i.e., a distributional solution of the linear equation

$$\begin{aligned} \partial_t G(t,x,v,x_0,v_0) &= \overline{A}G(t,x,v,x_0,v_0), \\ \lim_{t\to 0} G(t,x,v,x_0,v_0) &= \delta(x-x_0,v-v_0), \end{aligned} \tag{2.19}$$

where δ is the Dirac distribution, see also [4], [16]. The Green's function reads

$$G(t,x,v,x_0,v_0) = e^{\beta t} g(t, X(-t,x,v)-x_0, \dot{X}(-t,x,v)-v_0),$$

where $g(t,x,v) = \frac{F_1(t,x,v)}{F_2(t,x,v)}$ with

$$\begin{aligned} F_1(t,x,v) &= \exp\Big(-\frac{\nu(t)|x|^2+\mu(t)(x\cdot v)+\lambda(t)|v|^2}{4\lambda(t)\nu(t)-\mu^2(t)}\Big), \\ F_2(t,x,v) &= 2\pi(4\lambda(t)\nu(t)-\mu^2(t))^{\frac{1}{2}}. \end{aligned}$$

The coefficients $\lambda(t), \nu(t), \mu(t)$ are defined by

$$\begin{aligned} \lambda(t) &= \int_0^t (\alpha(s)\ \beta(s)) D \begin{pmatrix} \alpha(s) \\ \beta(s) \end{pmatrix} ds, \\ \nu(t) &= \int_0^t (\dot{\alpha}(s)\ \dot{\beta}(s)) D \begin{pmatrix} \dot{\alpha}(s) \\ \dot{\beta}(s) \end{pmatrix} ds, \\ \mu(t) &= -2\int_0^t (\dot{\alpha}(s)\ \dot{\beta}(s)) D \begin{pmatrix} \alpha(s) \\ \beta(s) \end{pmatrix} ds \end{aligned}$$

where $\alpha(\cdot)$ and $\beta(\cdot)$ are such that

$$X(-t,x,v) = \alpha(t)x + \beta(t)v.$$

In our framework,

$$D = \begin{pmatrix} \kappa & \gamma \\ \gamma & \sigma \end{pmatrix}.$$

With these preliminaries, the following parabolic regularization result can be deduced, see Proposition 3.1 of [4] for $a = 0$.

Lemma 6. *For all $T > 0$, the derivatives of the solution f of the equation* (2.17) *can be estimated by:*

$$\|\partial_x w\|_X \le C_T t^{-\frac{1}{2}} \|w_0\|_X,\ \|\partial_v w\|_X \le C_T t^{-\frac{1}{2}} \|w_0\|_X,\ \forall t \in (0,T], \tag{2.20}$$

for all $w_0 \in X$.

3. Proof of Theorem 1

In this section, we shall use a contractive fixed point map to establish a local solution of the Wigner-Poisson-Fokker-Plank system with exchange potential in X. Without loss of generality, we can set $\varrho = 0$. Lemmas 4 and 6 motivate the definition of the Banach space

$$
\begin{aligned}
Y_T = \{ & z \in C([0,T];X) | \partial_x z \in C([0,T];X) \\
& \text{with } \|\partial_x z\|_X \leq C t^{-\frac{1}{2}} \text{ for } t \in (0,T),\ z(t=0) = z_0 \}
\end{aligned} \tag{3.1}
$$

endowed the norm

$$
||z||_{Y_T} = \sup_{t\in[0,T]} ||z(t)||_X + \sup_{t\in[0,T]} ||t^{\frac{1}{2}} \partial_x z(t)||_X, \tag{3.2}
$$

for every fixed $T > 0$. We shall obtain the (local-in-time) well-posedness result for the Wigner-Poisson–Fokker-Plank system with exchange potential by introducing a non-linear iteration in the space Y_T, with an appropriate (small enough) T. First, for a given $w \in Y_T$ we shall now consider the linear Cauchy problem for the function z,

$$
\begin{aligned}
z_t = \overline{A} z(t) + \Theta_\hbar[V_{\text{cou}}[z(t)]] w(t) + Q_\hbar[\rho[w(t)]] z(t), \\
\forall t \in [0,T],\ z(t=0) = w_0 \in X,
\end{aligned} \tag{3.3}
$$

where, $T > 0$ is a fixed time.

Lemma 7. *For all $w_0 \in X$ and $w \in Y_T$, the initial value problem* (3.3) *has a unique mild solution $z \in C([0,T];X)$, which satisfies*

$$
z(t) = e^{t\overline{A}} w_0 + \int_0^t e^{(t-s)\overline{A}} (\Theta_\hbar[V_{cou}[z(s)]] w(s) + Q_\hbar[\rho[w(s)]] z(s)),\ \forall t \in [0,T]. \tag{3.4}
$$

Moreover, the solution z belongs to the space Y_T.

Proof. The first assertion follows directly by applying Theorem 6.1.2 in [7]. Moreover, the following inequalities hold

$$
\begin{aligned}
||z(t)||_X &\leq e^{kt} ||w_0||_X + C \int_0^t e^{k(t-s)} ||z(s)||_X (||w(s)||_X + ||\partial_x w(s)||_X) ds \\
&\leq e^{kt} ||w_0||_X + C e^{kT} ||w||_{Y_T} \int_0^t ||z(s)||_X \left(1 + s^{-\frac{1}{2}}\right) ds,
\end{aligned}
$$

for all $t \in [0,T]$, $k = a + \sigma + \frac{\beta}{2}$. Then, by Gronwall′s inequality,

$$
||z(t)||_X \leq C e^{kT} ||w_0||_X \left[1 + \|w\|_{Y_T} (t + 2t^{\frac{1}{2}}) \exp[C e^{kT} \|w\|_{Y_T} (T + 2T^{\frac{1}{2}})]\right]. \tag{3.5}
$$

By differentiating equation (3.4) in the x-direction, we obtain

$$
\partial_x z(t) = \partial_x e^{t\overline{A}} w_0 + \int_0^t \partial_x e^{(t-s)\overline{A}} (\Theta_\hbar[V_{\text{cou}}[z(s)]] w(s) + Q_\hbar[\rho[w(s)]] z(s)),\ \forall t \in [0,T].
$$

Using Lemmas 3, 4 and 6, we can get

$$\|\partial_x z(t)\|_X \le C\left[e^{kT}t^{-\frac{1}{2}}\|w_0\|_X+\|w\|_{Y_T}\int_0^t \|z(s)\|_X e^{k(t-s)}(t-s)^{-\frac{1}{2}}(1+s^{-\frac{1}{2}})ds\right]. \tag{3.6}$$

Hence, substituting (3.5) into (3.6), by [4],

$$\int_0^t e^{k(t-s)}(t-s)^{-\frac{1}{2}}(1+s^{-\frac{1}{2}})ds \le e^{kt}(2t^{\frac{1}{2}}+\pi)$$

and

$$\int_0^t e^{k(t-s)}(t-s)^{-\frac{1}{2}}(1+s^{-\frac{1}{2}})(s+2s^{\frac{1}{2}})ds \le 2e^{kt}(2t^{\frac{1}{2}}+3t+t^{\frac{3}{2}}),$$

we can get that the function z belongs to the space Y_T. ∎

We now define the linear map τ on Y_T (for any fixed $0 < T \le T_0$):

$$w \mapsto \tau w = z,$$

where z is the unique mild solution of the (3.3). Next, we shall show that τ is a strict contraction on a closed subset of Y_T, for T sufficiently small. This will yield the local-in-time solution of the non-linear problem (1.10)–(1.11).

Lemma 8. *For any fixed $w_0 \in X$ and $w \in Y_T$, then there exists a $l > 0$ such that the map τ,*

$$\tau w(t) = e^{\overline{A}t}w_0 + \int_0^t e^{(t-s)\overline{A}}(\Theta_\hbar[V_{cou}[\tau w(s)]]w(s) + Q_\hbar[\rho[w(s)]]\tau w(s)),\ \forall t \in [0,\tau],$$

is a strict contraction from the ball of radius $R(>0)$ of Y_l into itself.

Proof. Similar to the proof of Lemma 7 the function $z = \tau w \in Y_l$ satisfies

$$||\tau w(t)||_X \ \le\ Ce^{kT}||w_0||_X\left[1+Ce^{kT}\|w\|_{Y_l}(t+2t^{\frac{1}{2}})\exp[Ce^{kT}\|w\|_{Y_l}(T+2T^{\frac{1}{2}})]\right].$$

Under the assumption $||w||_{Y_l} \le R$, this estimate reads

$$||\tau w(t)||_X \ \le\ Ce^{kT}||w_0||_X\left[1+Ce^{kT}R(t+2t^{\frac{1}{2}})\exp[Ce^{kT}R(T+2T^{\frac{1}{2}})]\right].$$

If we assume

$$Ce^{kT}||w_0||_X\left[1+Ce^{kT}R(t+2t^{\frac{1}{2}})\exp[ce^{kT}R(T+2T^{\frac{1}{2}})]\right] \le \frac{1}{3}R, \tag{3.7}$$

then $||\tau w(t)||_X \le \frac{1}{3}R$. Similar to (3.6), for enough small l and large R, we have $t^{\frac{1}{2}}\|\partial_x \tau w(t)\|_X \le \frac{1}{3}R$. Then, the estimate $\|\tau w(t)||_{Y_l} \le \frac{2}{3}\|w(t)\|_{Y_l}$ holds, and hence the operator τ maps the ball of radius R of Y_l into itself.

To prove contractivity we shall estimate $\tau u - \tau w$ for all $w, u \in Y_l$ with $||w||_{Y_l}, ||u||_{Y_l} \leq R$. Since

$$\tau u - \tau w = \int_0^t e^{\overline{A}(t-s)} \left(\Theta_\hbar[V_{\rm cou}[\tau u(s) - \tau w(s)]]u(s) + \Theta_\hbar[V_{\rm cou}[\tau w(s)]](u-w)(s)\right) ds$$
$$+ \int_0^t e^{\overline{A}(t-s)} \left(\Theta_\hbar[\rho[\tau u(s) - \tau w(s)]]u(s) + \Theta_\hbar[\rho[u(s)]](\tau u - \tau w)(s)\right) ds,$$

by analogous estimates,

$$\begin{aligned} ||\tau u - \tau w||_X &\leq Ce^{kl} \int_0^t \|u\|_X \|\tau u - \tau w\|_X + (\|u\|_X + \|\partial_x u\|_X)\|\tau u - \tau w\|_X ds \\ &+ Ce^{k\iota} \int_0^t \|u-w\|_X \|\tau w\|_X + \|\tau w\|_X (\|u-w\|_X + \|\partial_x u - \partial_x w\|_X) ds, \end{aligned}$$

moreover,

$$\begin{aligned} ||\tau u - \tau w||_X &\leq 2CRe^{kl} \int_0^t (1 + s^{-\frac{1}{2}})(\|\tau u - \tau w\|_X + \|u - w\|_{Y_l}) ds \\ &\leq 2CRe^{kl} \|u - w\|_{Y_l}(t + 2t^{\frac{1}{2}}) + \int_0^t (1 + s^{-\frac{1}{2}})\|\tau u - \tau w\|_X ds), \end{aligned}$$

hence, by applying Gronwall's Lemma,

$$||\tau u - \tau w||_X \leq 2CRe^{k\iota} \|u - w\|_{Y_l} \left(t + 2t^{\frac{1}{2}} + \left(\frac{3}{2}t^2 + \frac{4}{3}t^{\frac{3}{2}} + \frac{2}{5}t^{\frac{5}{2}}\right) e^{l + l^{\frac{3}{2}}}\right). \tag{3.8}$$

The estimate

$$t^{\frac{1}{2}} \|\partial_x(\tau u) - \partial_x(\tau w)\|_X \leq C(R, k, \iota) \|u - w\|_{Y_l} \tag{3.9}$$

can be proved in an analogous way as those of (3.8) by Lemmas 4, 6 and [4]. When choosing $l > 0$ small enough, estimates (3.8)–(3.9) imply that

$$||\tau u - \tau w||_{Y_l} \leq C(R, k, \iota) ||u - w||_{Y_l}, \tag{3.10}$$

for some $C(R, k, \iota) < 1$, and the assertion is proved. ∎

Proof of Theorem 1. By [7], there exists a $t_{\max} \leq \infty$ such that the initial boundary value problem (1.10)-(1.11) has a unique mild solution w in Y_T, for all $T < t_{\max}$, which satisfies

$$w(t) = e^{t\overline{A}} w_0 + \int_0^t e^{(t-s)\overline{A}} (\Theta_\hbar[V_{\rm cou}[w(s)]]w(s) + Q_\hbar[\rho[w(s)]]w(s)), \ \forall t \in [0, T].$$

Moreover, if $t_{\max} < \infty$, then

$$\lim_{t \to t_{\max}} ||w(t)||_{Y_T} = \infty.$$

Precisely speaking, the solution of the problem (1.10)–(1.11) is a fixed point of the map τ previously introduced. Using Lemma 8, this solution exists for a time interval of length l, depending only on $\|w_0\|_X$, and it belongs to the space Y_l. If the second assertion would not hold, there would be a sequence of times $t_n \to t_{\max}$ such that $\|w(t_n)\| \leq C$ for all n. Then, by solving a problem with the initial value $w(t_n)$, with t_n sufficiently close to $t_{\max}$, we would extend the solution up to a certain time $t'_n > t_{\max}$. This construction would contradict our definition of $t_{\max}$.

The uniqueness of the mild solution follows by arguments analogous to those in the proof of Theorem 6.1.4 in [7]. ∎

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ON CONVEXITY OF FUZZY MAPPINGS AND FUZZY OPTIMIZATIONS

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Abstract. In this paper, based on a new order and a new metric on the set of fuzzy numbers, we present the concepts of convexity of the fuzzy mappings and give characterization theorems for the convex fuzzy mappings and quasi-convex fuzzy mappings. Finally, we discuss the properties of convex fuzzy optimizations.

Keywords: convexity, fuzzy mapping, fuzzy optimization.

1. Introduction

In many scientific and engineering applications the fuzzy set concept plays an important role. The fuzziness appears when we need to perform, on manifold, calculations with imprecision variables. The fuzzy set theory was introduced initially by Zadeh [25] in 1965. In the theory and applications of fuzzy sets convexity is a most useful concept. In fact, in the basic and classical paper [25], Zadeh paid special attention to the investigation of the convex fuzzy sets which covers nearly the second half of the space of the paper.

Following the seminal work of Zadeh, a lot of scholars have discussed various aspects of the theory and applications of fuzzy convex analysis. Nanda and Kar [9] proposed a concept of convex fuzzy mapping and proved that a fuzzy mapping is convex if and only if its epigraph is a convex set. Yan and Xu [22] discussed the convexity and quasi-convexity of fuzzy mappings by considering the concept of ordering proposed by Goetschel and Voxman [6]. Noor [11] introduced the concept of fuzzy preinvex functions over the field of real numbers $\mathbb{R}$, and obtained some properties of fuzzy preinvex functions. In [16], Syau introduced the concepts of pseudo-convexity, invexity and pseudo-invexity for fuzzy mappings of one

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variable by using notion of differentiability and the results proposed by Goetschel and Voxman [6]. Syau [17] introduced and investigated a new kind of generalized convex fuzzy mapping known as a B-vex fuzzy mapping. Wang and Wu [19] by establishing the fuzzy subdifferential of a fuzzy mapping, considered the application to convex fuzzy programming. Panigrahi [12] extended and generalized these concepts to fuzzy mappings of several variables using Buckley and Feurings [2] approach for fuzzy differentiation and derived a K-K-T condition for a constrained fuzzy minimization problem. In [20], [21], Wu and Xu introduced the concepts of fuzzy pseudo-convex, fuzzy invex, fuzzy pseudo-invex and fuzzy preinvex mapping from $\mathbb{R}^n$ to the set of fuzzy numbers based on the concept of differentiability of fuzzy mapping due to Wang and Wu [19]. In order to solve the open problem in fuzzy analysis that we proposed in [13], we introduced some new and more general definitions in the area of fuzzy starshapedness, and developed several theorems on the shadows of starshaped fuzzy sets [14], which generalize the important results obtained by Liu [8].

In this paper, after defining a new order on the set of fuzzy numbers, which is finer than the usually used "fuzzy-max" order [1], we will present the concepts of convexity of the fuzzy mappings and give characterization theorems for the convex fuzzy mappings and quasi-convex fuzzy mappings in Section 3. Finally, we discuss the properties of convex fuzzy optimizations in Section 4.

2. Preliminaries

We now quote some concepts and results which will be needed in the sequel.

A fuzzy set $\tilde{u}$ of $\mathbb{R}$ is a function $\tilde{u} : \mathbb{R} \to [0, 1]$. For any fuzzy set $\tilde{u}$, we denote by $[\tilde{u}]^\alpha = \{x \in \mathbb{R} : \tilde{u}(x) \geq \alpha\}$ for any $\alpha \in (0, 1]$, its a-level set. We define the set $[\tilde{u}]^0$ by $[\tilde{u}]^0 = \overline{\{x \in \mathbb{R} : \tilde{u}(x) > 0\}}$, where $\overline{A}$ denotes the closure of a crisp set A. A fuzzy set $\tilde{u}$ is said to be a fuzzy number if it satisfies the following conditions [3]:

(i) $\tilde{u}$ is normal, i.e., there exists an $x_0 \in \mathbb{R}$ such that $\tilde{u}(x_0) = 1$;

(ii) $\tilde{u}$ is convex, i.e., $\tilde{u}(\lambda x_1 + (1 - \lambda)x_2) \geq \min\{\tilde{u}(x_1), \tilde{u}(x_2)\}$, for all $x_1, x_2 \in \mathbb{R}$ and $\lambda \in (0, 1)$;

(iii) $\tilde{u}$ is upper semi-continuous;

(iv) $[\tilde{u}]^0$ is compact.

Equivalently, a fuzzy number $\tilde{u}$ is a fuzzy set with non-empty bounded closed level sets $[\tilde{u}]^\alpha = [\tilde{u}_L(\alpha), \tilde{u}_R(\alpha)]$ for all $\alpha \in [0, 1]$, where $[\tilde{u}_L(\alpha), \tilde{u}_R(\alpha)]$ denotes a closed interval with the left end point $\tilde{u}_L(\alpha)$ and the right end point $\tilde{u}_R(\alpha)$. We denote the class of fuzzy numbers by $\mathscr{F}$. Notice that the real numbers $\mathbb{R}$ can be embedded in $\mathscr{F}$ by defining a fuzzy number $\tilde{a}$ as

$$\tilde{a}(x) = \begin{cases} 1, & \text{if } x = a, \\ 0, & \text{otherwise,} \end{cases}$$

for each $a \in \mathbb{R}$.

For any $\tilde{u}, \tilde{v} \in \mathscr{F}$ and $a \in \mathbb{R}$, owing to Zadeh's extension principle [26], scalar multiplication and addition are defined for any $x \in \mathbb{R}$ by

$$a \times \tilde{u}(x) = a\tilde{u}(x) = \begin{cases} \tilde{u}\left(\dfrac{x}{a}\right), & \text{if } a \neq 0, \\ \tilde{0}, & \text{if } a = 0, \end{cases}$$

and

$$(\tilde{u} + \tilde{v})(x) = \sup_{x_1, x_2 : x_1 + x_2 = x} \min\{\tilde{u}(x_1), \tilde{v}(x_2)\}.$$

For any $\tilde{u} \in \mathscr{F}$, we define the fuzzy number $-\tilde{u} \in \mathscr{F}$ by $-\tilde{u} = (-1) \times \tilde{u}$, i.e., $-\tilde{u}(x) = \tilde{u}(-x)$, for all $x \in \mathbb{R}$. By the level set representations of the fuzzy numbers $\tilde{u}, \tilde{v}, \tilde{w}$, we can get that

$$\begin{aligned} ([\tilde{u}]^\alpha + [\tilde{v}]^\alpha) + [\tilde{w}]^\alpha &= [\tilde{u}]^\alpha + ([\tilde{v}]^\alpha + [\tilde{w}]^\alpha) \\ &= [\tilde{u}_L(\alpha) + \tilde{v}_L(\alpha) + \tilde{w}_L(\alpha), \tilde{u}_R(\alpha) + \tilde{v}_R(\alpha) + \tilde{w}_R(\alpha)], \\ [\tilde{u}]^\alpha + [\tilde{v}]^\alpha &= [\tilde{v}]^\alpha + [\tilde{u}]^\alpha = [\tilde{u}_L(\alpha) + \tilde{v}_L(\alpha), \tilde{u}_R(\alpha) + \tilde{v}_R(\alpha)], \\ [-\tilde{u}]^\alpha &= -[\tilde{u}_L(\alpha), \tilde{u}_R(\alpha)] = [-\tilde{u}_R(\alpha), -\tilde{u}_L(\alpha)], \end{aligned}$$

which implies $\mathscr{F}$ is a commutative semigroup under addition.

Definition 2.1 [7] For a fuzzy number $\tilde{u}$, we define a function $M_{\tilde{u}} : [0,1] \to \mathbb{R}$ by assigning the midpoint of each α-level set to $M_{\tilde{u}}(\alpha)$ for all $\alpha \in [0,1]$, i.e.,

$$M_{\tilde{u}}(\alpha) = \frac{\tilde{u}_L(\alpha) + \tilde{u}_R(\alpha)}{2}.$$

It is interesting to note that the midpoint function $M_{\tilde{u}}$ is actually a gradual number and all of the gradual numbers form a group structure for addition [4, 5].

Lemma 2.1 [15] *For any $\tilde{u} \in \mathscr{F}$, the midpoint function $M_{\tilde{u}}$ is continuous from the right at 0 and continuous from the left on $[0,1]$. Furthermore it is a function of bounded variation on $[0,1]$.*

Definition 2.2 For any $\tilde{u}, \tilde{v} \in \mathscr{F}$, we say that $\tilde{u} \preceq_m \tilde{v}$ if for all $\alpha \in [0,1]$, $M_{\tilde{u}}(\alpha) \leq M_{\tilde{v}}(\alpha)$; we say that $\tilde{u} \prec_m \tilde{v}$ if $\tilde{u} \preceq_m \tilde{v}$ and there exists $\alpha_0 \in [0,1]$ such that $M_{\tilde{u}}(\alpha_0) < M_{\tilde{v}}(\alpha_0)$; we say that $\tilde{u} =_m \tilde{v}$ if $\tilde{u} \preceq_m \tilde{v}$ and $\tilde{v} \preceq_m \tilde{u}$.

It is often convenient to write $\tilde{v} \succeq_m \tilde{u}$ (resp. $\tilde{v} \succ_m \tilde{u}$) in place of $\tilde{u} \preceq_m \tilde{v}$ (resp. $\tilde{u} \preceq_m \tilde{v}$). A subset A of $\mathscr{F}$ is said to be bounded above if there exists a fuzzy number $\tilde{u} \in \mathscr{F}$, called an upper bound of A, such that $\tilde{v} \preceq_m \tilde{u}$ for every $\tilde{v} \in \mathscr{F}$. Further, a fuzzy number $\tilde{u}_0 \in \mathscr{F}$ is called the least upper bound (sup, in short) for A if

(i) $\tilde{u}_0$ is an upper bound of A,

(ii) $\tilde{u}_0 \preceq_m \tilde{u}$ for every upper bound $\tilde{u}$ of A.

A lower bound and the greatest lower bound (inf, in short) are defined similarly.

Remark 2.1 In the field of fuzzy optimizations, ranking of fuzzy numbers is an important and prerequisite procedure. In general, we rank two fuzzy numbers by considering their left-hand functions and right-hand functions [10], [18], which is called the "fuzzy-max" order. We will show that two fuzzy numbers are not comparable by using the general method but they are comparable in the sense of Definition 2.2.

Example 2.1 Define fuzzy numbers $\widetilde{u}, \widetilde{v}$, respectively, by their level sets as

$$[\widetilde{u}]^{\alpha} = \left[\frac{1}{2}, \frac{3}{2} - \alpha\right] \quad \text{and} \quad [\widetilde{v}]^{\alpha} = [0, 2 - 2\alpha], \alpha \in [0, 1].$$

Then we can get that the corresponding midpoint functions are respectively,

$$M_{\tilde{u}}(\alpha) = 1 - \frac{\alpha}{2} \quad \text{and} \quad M_{\tilde{v}}(\alpha) = 1 - \alpha,\ \alpha \in [0, 1].$$

It is obvious that $\widetilde{u} \succ \widetilde{v}$ in the sense of Definition 2.2. However, the fuzzy numbers $\widetilde{u}$ and $\widetilde{v}$ are not comparable with respect to the "fuzzy-max" order [10], [18].

We define a metric d_m on $\mathscr{F}$ by

$$d_m(\tilde{u}, \tilde{v}) = \sup_{0 \le \alpha \le 1} |M_{\tilde{u}}(\alpha) - M_{\tilde{v}}(\alpha)|$$

for any $u, v \in \mathscr{F}$. In general, this metric is not greater than the Hausdorff metric [15].

Throughout this paper, we suppose that V is a real vector space, and K is a convex subset of V. For any $x \in V$ and $\delta > 0$, let

$$B_{\delta}(x) = \{y \in V : \|x - y\| < \delta\},$$

where $\| \cdot \|$ is the norm on V. We present the definition of upper and lower semicontinuous fuzzy mappings, which is a variation of the one in [1] with respect to the order $\preceq_m$.

Definition 2.3 A fuzzy mapping $F : K \to \mathscr{F}$ is said to be:

(1) upper semicontinuous at $x_0 \in K$ if, for any $\varepsilon > 0$, there exists a $\delta = \delta(x_0, \varepsilon) > 0$ such that

$$F(x) \preceq_m F(x_0) + \tilde{\varepsilon} \text{ whenever } x \in K \cap B_{\delta}(x_0).$$

F is upper semicontinuous if it is upper semicontinuous at each point of K.

(2) lower semicontinuous at $x_0 \in K$ if, for any $\varepsilon > 0$, there exists a $\delta = \delta(x_0, \varepsilon) > 0$ such that

$$F(x_0) - \tilde{\varepsilon} \preceq_m F(x) \text{ whenever } x \in K \cap B_{\delta}(x_0).$$

F is lower semicontinuous if it is lower semicontinuous at each point of K.

3. Main results

Definition 3.1 A fuzzy mapping $F : K \to \mathscr{F}$ is said to be:

(1) convex if, for every $\lambda \in [0,1]$ and $x, y \in K$,

$$F(\lambda x + (1-\lambda)y) \preceq_m \lambda F(x) + (1-\lambda)F(y),$$

(2) concave if, for every $\lambda \in [0,1]$ and $x, y \in K$,

$$F(\lambda x + (1-\lambda)y) \succeq_m \lambda F(x) + (1-\lambda)F(y).$$

Theorem 3.1 *A fuzzy mapping $F : K \to \mathscr{F}$ is convex if and only if for all $x, y \in K$, $\lambda \in (0,1)$ and all $\tilde{u}, \tilde{v} \in \mathscr{F}$ such that $F(x) \preceq_m \tilde{u}, F(y) \preceq_m \tilde{v}$,*

$$F(\lambda x + (1-\lambda)y) \preceq_m \lambda\tilde{u} + (1-\lambda)\tilde{v}.$$

Proof. Suppose $F : K \to \mathscr{F}$ is a convex fuzzy mapping. For any $\lambda \in (0,1)$ and $\tilde{u}, \tilde{v} \in \mathscr{F}$, we have that

$$M_{(\lambda\tilde{u}+(1-\lambda)\tilde{v})}(\alpha) = \frac{\lambda(\tilde{u}_L(\alpha) + \tilde{u}_R(\alpha)) + (1-\lambda)(\tilde{v}_L(\alpha) + \tilde{v}_R(\alpha))}{2},$$

for all $\alpha \in [0,1]$. Now for any $x, y \in K$, $\lambda \in (0,1)$ and $\tilde{u}, \tilde{v} \in \mathscr{F}$ such that $F(x) \preceq_m \tilde{u}, F(y) \preceq_m \tilde{v}$, we have

$$\begin{aligned} M_{F(\lambda x+(1-\lambda)y)}(\alpha) &\leq M_{\lambda F(x)+(1-\lambda)F(y)}(\alpha) \\ &= \frac{\lambda(F(x)_L(\alpha) + F(x)_R(\alpha)) + (1-\lambda)(F(y)_L(\alpha) + F(y)_R(\alpha))}{2} \\ &\leq \frac{\lambda(\tilde{u}_L(\alpha) + \tilde{u}_R(\alpha)) + (1-\lambda)(\tilde{v}_L(\alpha) + \tilde{v}_R(\alpha))}{2} \\ &= M_{(\lambda\tilde{u}+(1-\lambda)\tilde{v})}(\alpha), \end{aligned}$$

for all $\alpha \in [0,1]$, i.e.,

$$F(\lambda x + (1-\lambda)y) \preceq_m \lambda\tilde{u} + (1-\lambda)\tilde{v}.$$

Conversely, suppose the conditions hold. Then for any $x, y \in K$, let $\tilde{u_0} =_m F(x)$ and $\tilde{v_0} =_m F(y)$. Thus we get that

$$F(\lambda x + (1-\lambda)y) \preceq_m \lambda\tilde{u_0} + (1-\lambda)\tilde{v_0} =_m \lambda F(x) + (1-\lambda)F(y). \qquad \blacksquare$$

Theorem 3.2 *A fuzzy mapping $F : K \to \mathscr{F}$ is convex if and only if the set*

$$A = \{(x, \tilde{u}) : x \in K, \tilde{u} \in \mathscr{F}, F(x) \preceq_m \tilde{u}\}$$

is convex.

Proof. Suppose F is a convex fuzzy mapping. For any $(x,\tilde{u}),(y,\tilde{v})\in A$, we have $F(x)\preceq_m \tilde{u}$ and $F(y)\preceq_m \tilde{v}$. By Theorem 3.1, we have that

$$F(\lambda x+(1-\lambda)y)\preceq_m \lambda\tilde{u}+(1-\lambda)\tilde{v},$$

which implies that

$$(\lambda x+(1-\lambda)y,\lambda\tilde{u}+(1-\lambda)\tilde{v})=\lambda(x,\tilde{u})+(1-\lambda)(y,\tilde{v})\in A,$$

for all $\lambda\in(0,1)$.

Conversely, suppose A is a convex set. For any $x,y\in K$, let $\tilde{u_0}=F(x)$ and $\tilde{v_0}=F(y)$. For all $\lambda\in(0,1)$, since $(x,\tilde{u_0}),(y,\tilde{v_0})\in A$, we have

$$\lambda(x,\tilde{u_0})+(1-\lambda)(y,\tilde{v_0})=(\lambda x+(1-\lambda)y,\lambda\tilde{u_0}+(1-\lambda)\tilde{v_0})\in A,$$

which implies that

$$F(\lambda x+(1-\lambda)y)\preceq_m \lambda\tilde{u_0}+(1-\lambda)\tilde{v_0}=\lambda F(x)+(1-\lambda)F(y).$$ ∎

Theorem 3.3 *A fuzzy mapping $F:K\to\mathscr{F}$ is convex if and only if for every $\alpha\in[0,1]$ the induced function $M_{F(x)}(\alpha)$ is convex with respect to x.*

Proof. By Definition 3.1, we have that $F:K\to\mathscr{F}$ is convex if and only for any $\lambda\in[0,1]$ and $x,y\in K$,

$$F(\lambda x+(1-\lambda)y)\preceq_m \lambda F(x)+(1-\lambda)F(y),$$

which is true if and only if for every $\alpha\in[0,1]$,

$$\begin{aligned}
M_{F(\lambda x+(1-\lambda)y)}(\alpha) &= \frac{F(\lambda x+(1-\lambda)y)_L(\alpha)+F(\lambda x+(1-\lambda)y)_R(\alpha)}{2}\\
&\le M_{\lambda F(x)+(1-\lambda)F(y)}(\alpha)\\
&= \frac{\lambda(F(x)_L(\alpha)+F(x)_R(\alpha))+(1-\lambda)(F(y)_L(\alpha)+F(y)_R(\alpha))}{2}\\
&= \lambda\frac{(F(x)_L(\alpha)+F(x)_R(\alpha))}{2}+(1-\lambda)\frac{(F(y)_L(\alpha)+F(y)_R(\alpha))}{2}\\
&= \lambda M_{F(x)}(\alpha)+(1-\lambda)M_{F(y)}(\alpha),
\end{aligned}$$

i.e., $M_{F(x)}(\alpha)$ is convex with respect to x. ∎

Since concave fuzzy mappings satisfy the opposite inequalities, by the similar proofs we can give the following three theorems for concave fuzzy mappings.

Theorem 3.4 *A fuzzy mapping $F:K\to\mathscr{F}$ is concave if and only if for all $x,y\in K$, $\lambda\in(0,1)$ and all $\tilde{u},\tilde{v}\in\mathscr{F}$ such that $F(x)\succeq_m\tilde{u}, F(y)\succeq_m\tilde{v}$,*

$$F(\lambda x+(1-\lambda)y)\succeq_m\lambda\tilde{u}+(1-\lambda)\tilde{v}.$$

Theorem 3.5 *A fuzzy mapping $F : K \to \mathscr{F}$ is concave if and only if the set*

$$A = \{(x, \tilde{u}) : x \in K, \tilde{u} \in \mathscr{F}, F(x) \succeq_m \tilde{u}\}$$

is convex.

Theorem 3.6 *A fuzzy mapping $F : K \to \mathscr{F}$ is concave if and only if for every $\alpha \in [0,1]$ the induced function $M_{F(x)}(\alpha)$ is concave with respect to x.*

Theorem 3.7 *Let $F : K \to \mathscr{F}$ be a fuzzy mapping. Then F is upper semicontinuous at $x_0 \in K$ if and only if $M_{F(x)}(\alpha)$ are upper semicontinuous at x_0 uniformly in $\alpha \in [0,1]$; F is lower semicontinuous at $x_0 \in K$ if and only if $M_{F(x)}(\alpha)$ are lower semicontinuous at x_0 uniformly in $\alpha \in [0,1]$.*

Proof. By Definition 2.1, 2.2 and 2.3, F is upper semicontinuous at $x_0 \in K$ if and only if, for any $\varepsilon > 0$, there exists a $\delta = \delta(x_0, \varepsilon) > 0$ such that

$$M_{F(x)}(\alpha) \leq M_{F(x_0)}(\alpha) + \varepsilon \text{ whenever } x \in K \cap B_\delta(x_0),$$

for all $\alpha \in [0,1]$; F is lower semicontinuous at $x_0 \in K$ if and only if, for any $\varepsilon > 0$, there exists a $\delta = \delta(x_0, \varepsilon) > 0$ such that

$$M_{F(x_0)}(\alpha) - \varepsilon \leq M_{F(x)}(\alpha) \text{ whenever } x \in K \cap B_\delta(x_0),$$

for all $\alpha \in [0,1]$. ∎

Theorem 3.8 *Let $F : K \to \mathscr{F}$ be a fuzzy mapping. Then F is upper semicontinuous and lower semicontinuous at $x_0 \in K$ if and only if F is continuous with respect to the metric d_m.*

Proof. By Theorem 3.7, F is upper semicontinuous and lower semicontinuous at $x_0 \in K$ if and only if for any $\varepsilon > 0$, there exists a $\delta = \delta(x_0, \varepsilon) > 0$ such that

$$M_{F(x_0)}(\alpha) - \varepsilon \leq M_{F(x)}(\alpha) \leq M_{F(x_0)}(\alpha) + \varepsilon \text{ whenever } x \in K \cap B_\delta(x_0),$$

for all $\alpha \in [0,1]$, which implies that

$$d_m(F(x_0), F(x)) = \sup_{0 \leq \alpha \leq 1} |M_{F(x_0)}(\alpha) - M_{F(x)}(\alpha)| \leq \varepsilon \text{ whenever } x \in K \cap B_\delta(x_0). \text{ ∎}$$

Theorem 3.9 *Let $F : K \to \mathscr{F}$ be an upper semicontinuous fuzzy mapping. If there exists a $\lambda \in (0,1)$ such that*

$$F(\lambda x + (1-\lambda)y) \preceq_m \lambda F(x) + (1-\lambda)F(y),$$

for all $x, y \in K$. Then F is a convex fuzzy mapping on K.

Proof. For any fixed $\alpha \in [0,1]$, by Theorem 3.7, $M_{F(x)}(\alpha)$ is upper semicontinuous with respect to x. Since there exists a $\lambda \in (0,1)$ such that

$$F(\lambda x + (1-\lambda)y) \preceq_m \lambda F(x) + (1-\lambda)F(y),$$

for all $x, y \in K$, by Definition 2.1, we have that

$$M_{F(\lambda x+(1-\lambda)y)}(\alpha) \leq M_{\lambda F(x)+(1-\lambda)F(y)}(\alpha) = \lambda M_{F(x)}(\alpha) + (1-\lambda)M_{F(y)}(\alpha),$$

for all $x, y \in K$. Thus by Theorem 2.3 in [23], we get $M_{F(x)}(\alpha)$ is convex with respect to x on K. From Theorem 3.3, it follows that F is a convex fuzzy mapping on K. ∎

Theorem 3.10 *Let $F : K \to \mathscr{F}$ be a lower semicontinuous fuzzy mapping. If, for any $x, y \in K$, there exists a $\lambda = \lambda(x,y) \in (0,1)$ such that*

$$F(\lambda x + (1-\lambda)y) \preceq_m \lambda F(x) + (1-\lambda)F(y).$$

Then F is a convex fuzzy mapping on K.

Proof. From the hypothesis, for any $x, y \in K$, by Definition 2.1, there exists a $\lambda = \lambda(x,y) \in (0,1)$ such that

$$M_{F(\lambda x+(1-\lambda)y)}(\alpha) \leq M_{\lambda F(x)+(1-\lambda)F(y)}(\alpha) = \lambda M_{F(x)}(\alpha) + (1-\lambda)M_{F(y)}(\alpha).$$

Thus by by Theorem 4 in [24], we get $M_{F(x)}(\alpha)$ is convex with respect to x on K for all $\alpha \in [0,1]$. From Theorem 3.3, it follows that F is a convex fuzzy mapping on K. ∎

Theorem 3.11 *For any $\tilde{u}, \tilde{v} \in \mathscr{F}$, the set $\{\tilde{u}, \tilde{v}\}$ has the least upper bound and the greatest lower bound.*

Proof. For any $\tilde{u}, \tilde{v} \in \mathscr{F}$, By Lemma 2.1 we have that the midpoint functions $M_{\tilde{u}}$ and $M_{\tilde{v}}$ are continuous from the right at 0, continuous from the left on $[0,1]$, and are functions of bounded variation on $[0,1]$. Thus we define two functions $M_{\sup} : [0,1] \to \mathbb{R}$ and $M_{\inf} : [0,1] \to \mathbb{R}$ by

$$M_{\sup}(\alpha) = \max\{M_{\tilde{u}}(\alpha), M_{\tilde{v}}(\alpha)\}$$

and

$$M_{\inf}(\alpha) = \min\{M_{\tilde{u}}(\alpha), M_{\tilde{v}}(\alpha)\},$$

respectively. It is easy to see that the functions $M_{\sup}$ and $M_{\inf}$ are continuous from the right at 0 and continuous from the left on $[0,1]$. Furthermore, we have that

$$V_0^1(M_{\sup}) \leq V_0^1(M_{\tilde{u}}) + V_0^1(M_{\tilde{v}})$$

and

$$V_0^1(M_{\inf}) \leq V_0^1(M_{\tilde{u}}) + V_0^1(M_{\tilde{v}}),$$

where $V_0^1(f)$ represents the total variation of the function f. Thus $M_{\sup}$ and $M_{\inf}$ are functions of bounded variation on $[0,1]$. By Theorem 3.10 in [15], the functions $M_{\sup}$ and $M_{\inf}$ can determine two fuzzy numbers $\tilde{m}$ and $\tilde{n}$ such that

$$M_{\tilde{m}} = M_{\sup} \text{ and } M_{\tilde{n}} = M_{\inf}.$$

It is easy to see that $\tilde{m}$ and $\tilde{n}$ are $\sup\{\tilde{u}, \tilde{v}\}$ and $\inf\{\tilde{u}, \tilde{v}\}$, respectively. ∎

Definition 3.2 A fuzzy mapping $F : K \to \mathscr{F}$ is said to be:

(1) quasi-convex if for every $\lambda \in [0,1]$ and $x, y \in K$,

$$F(\lambda x + (1-\lambda)y) \preceq_m \sup\{F(x), F(y)\},$$

(2) quasi-concave if for every $\lambda \in [0,1]$ and $x, y \in K$,

$$F(\lambda x + (1-\lambda)y) \succeq_m \inf\{F(x), F(y)\}.$$

Theorem 3.12 *Let $F : K \to \mathscr{F}$ be a convex (resp. concave) fuzzy mapping. Then it is quasi-convex (resp. quasi-concave) fuzzy mapping on K.*

Proof. Let $\lambda \in [0,1]$ and $x, y \in K$. By Definition 2.1 and Theorem 3. 11, we have that

$$\inf\{F(x), F(y)\} \preceq_m \lambda F(x) + (1-\lambda)F(y) \preceq_m \sup\{F(x), F(y)\}.$$

If F is convex, it follows that

$$F(\lambda x + (1-\lambda)y) \preceq_m \lambda F(x) + (1-\lambda)F(y) \preceq_m \sup\{F(x), F(y)\};$$

If F is concave, it follows that

$$\inf\{F(x), F(y)\} \preceq_m \lambda F(x) + (1-\lambda)F(y) \preceq_m F(\lambda x + (1-\lambda)y).$$ ∎

Theorem 3.13 *A fuzzy mapping $F : K \to \mathscr{F}$ is quasi-convex (resp. quasi-concave) if and only if for every $\alpha \in [0,1]$ the induced function $M_{F(x)}(\alpha)$ is quasi-convex (resp. quasi-concave) with respect to x.*

Proof. By Definition 3.2, we have that $F : K \to \mathscr{F}$ is quasi-convex if and only for any $\lambda \in [0,1]$ and $x, y \in K$,

$$F(\lambda x + (1-\lambda)y) \preceq_m \sup\{F(x), F(y)\},$$

which is true if and only if for every $\alpha \in [0,1]$,

$$\begin{aligned} M_{F(\lambda x+(1-\lambda)y)}(\alpha) &\leq M_{\sup\{F(x),F(y)\}}(\alpha) \\ &= \max\{M_{F(x)}(\alpha), M_{F(y)}(\alpha)\}, \end{aligned}$$

i.e., $M_{F(x)}(\alpha)$ is quasi-convex with respect to x.

Similarly, we can prove that $F : K \to \mathscr{F}$ is quasi-concave if and only if for every $\alpha \in [0,1]$ the induced function $M_{F(x)}(\alpha)$ is quasi-concave with respect to x. ∎

4. Applications to fuzzy optimization

Now we give some applications of the main results to fuzzy optimization.

Definition 4.1 For a fuzzy mapping $F : K \to \mathscr{F}$,

(1) an element $x_0 \in K$ is called a local minimizer of $F : K \to \mathscr{F}$ if there exists a $\delta > O$, such that

$$F(x_0) \preceq_m F(x) \text{ for all } x \in K \cap B_\delta(x_0);$$

(2) an element $x_0 \in K$ is called a strict local minimizer of $F : K \to \mathscr{F}$ if there exists a $\delta > O$, such that

$$F(x_0) \prec_m F(x) \text{ for all } x \neq x_0, x \in K \cap B_\delta(x_0);$$

(3) an element $x_0 \in K$ is called a global minimizer of $F : K \to \mathscr{F}$ if

$$F(x_0) \preceq_m F(x) \text{ for all } x \in K.$$

Theorem 4.1 *Let $F : K \to \mathscr{F}$ be a convex fuzzy mapping and $x_0 \in K$ is a local minimizer of F. Then x_0 is also a global minimizer of F.*

Proof. If there exists an $x' \in K$ such that $F(x') \prec_m F(x_0)$. Then since $F : K \to \mathscr{F}$ is convex, we have that

$$M_{F(\lambda x'+(1-\lambda)x_0)} \leq M_{\lambda F(x')+(1-\lambda)F(x_0)} = \lambda M_{F(x')} + (1-\lambda)M_{F(x_0)} < M_{F(x_0)},$$

which implies that

$$F(\lambda x' + (1-\lambda)x_0) \preceq_m \lambda F(x') + (1-\lambda)F(x_0) \prec_m F(x_0),$$

for all $\lambda \in (0,1)$. Thus for arbitrary small positive number λ, we have

$$F(\lambda x' + (1-\lambda)x_0) \prec_m F(x_0),$$

which contradicts with the definition of the local minimizer at x_0. ∎

Theorem 4.2 *Let $F : K \to \mathscr{F}$ be a quasi-convex fuzzy mapping and $x_0 \in K$ is a global minimizer of F. Then the set*

$$A = \{x \in K : F(x) =_m F(x_0)\}$$

is the set of all global minimizers of F and it is convex.

Proof. Let $x' \in K$ be a global minimizer of F. Then we have that $F(x') \preceq_m F(x_0)$ and $F(x_0) \preceq_m F(x')$, i.e., $F(x) =_m F(x_0)$. Thus $x' \in A$.

Conversely, let $x' \in A$. Since x_0 is a global minimizer of F, we have $F(x') =_m F(x_0) \preceq_m F(x)$ for all $x \in K$, which implies x' is also a global minimizer of F.

Let $x, y \in A$ and $\lambda \in [0,1]$, since F is a quasi-convex fuzzy mapping, we have that

$$F(\lambda x + (1-\lambda)y) \preceq_m \sup\{F(x), F(y)\} =_m F(x),$$

which implies $F(\lambda x + (1-\lambda)y) =_m F(x)$ because x is a global minimizer of F. Thus $\lambda x + (1-\lambda)y \in A$. ∎

5. Conclusions

In this present investigation, based on a new order and a new metric on the set of fuzzy numbers, we present the concepts of convexity of the fuzzy mappings and give characterization theorems for the convex fuzzy mappings and quasi-convex fuzzy mappings. Then, we discuss the properties of convex fuzzy optimizations.

The results of modern convex analysis research can be traced back to the work of Minkowski and Caratheodory. They investigated polyhedron and related convex problems, and funded the basic theories of convex analysis. During the last decades, the development of optimization theory has brought the fuzzy convexity into many theoretical and application problems. The fuzzy convex analysis has thus attracted more and more attention. Thus we hope our results would provide a background to ongoing work in the problems of those related fields.

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AN EFFICIENT ALGORITHM FOR WIDTHS OF CHANNEL ROUTING WITH GIVEN HORIZONTAL CONSTRAINT GRAPH

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Abstract. In VLSI design, One of the most important detailed routings is the channel routing. Channel routing in the 2-layer Manhattan model is one of the most investigated problem in VLSI design. In this paper, we consider the channel with horizontal constraint graph is a star. An efficient graph theoretic algorithm is presented, compared with the latest results, our algorithm yields a better bound on the width of the channel.

Keywords: channel routing, Manhattan model, VLSI.
AMS Subject Classification: 05C50, 15A18.

1. Introduction

In VLSI design the problem of completing the necessary interconnections among different modules is known as the routing problem. Typically, the routing problem can be divided into two steps due to the problem complexity: global routing and detailed routing. One of the most important detailed routing is channel routing [9,10,12]. The channel routing problem (CRP) is the problem of interconnecting all the nets in a channel using minimum possible routing area.

We use the expression of graph theory to describe the channel routing problem, a channel is defined by a rectangular grid G of size $(w+2) \times n$ consisting of horizontal tracks (numbered from 0 to $w+1$) and vertical columns (numbered from 1 to n), where w is the *width* and n is the *length* of the channel. Top and bottom sides points of G are called *terminals*.

A *channel routing problem* is a set $\mathcal{N} = \{N_1, \ldots, N_t\}$ of pairwise disjoint nets. A channel routing problem is called *bipartite* if each net contains exactly two terminals, one on the top, and one on the bottom side. A channel routing problem is *dense* if each terminal on the top and bottom sides belongs to some net. A net is called *trivial* if it consists of two terminals which are situated in the same column. In this paper we always assume that each net contains at least two terminals.

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There are two constraints for the net in a channel: horizontal constraint and vertical constraint.

The constraint that two nets cannot overlap on the horizontal layer is called the *horizontal constraint*. Let l_i be the leftmost and r_i be the rightmost column of net i. A net i is said to *span* the c-th column if $l_i \leq c \leq r_i$. The set of columns $[l_i, r_i]$ is called the span of net i.

There is a horizontal constraint between net i and net j if and only if their spans overlap. The horizontal constraints are often represented by an undirected graph, the horizontal constraint graph (HCG) (see Figure. 1), where vertices represent the nets and edges represent the horizontal constraints.

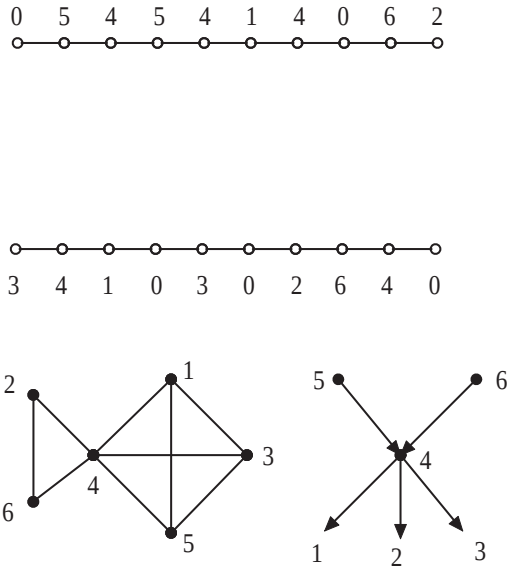


Figure 1.

Let Z_i be the set of nets that span the $i-$th column, $d_{\max} = \max\{|Z_i| : i \text{ is a column}\}$ is called the density of the CRP. Clearly, $d_{\max}$ is a lower bound on S^* because nets spanning the same column cannot be assigned to the same track.

The constraint that two nets cannot overlap on the vertical layer is called the *vertical constraint*. Note that if net i connects to the c-th column in the top row and net j connects to the c-th column in the bottom row, $i \neq j$, then net i must be assigned to a track *higher* than net j. In this case, we say that net i must precede net j and there is a vertical constraint from i to j . The vertical constraints define a partial ordering between nets. The vertical constraints are often represented by a directed graph, the vertical constraint graph (VCG) (see Figure.1), where vertices represent the nets and arcs represent the vertical constraints.

A solution of a channel routing problem is said to belong to the Manhattan model if consecutive layers contain wire segments of different directions only. That is, layers with horizontal(east-west) and vertical (north-south) wire segments alternate. In this paper we restrict that the length of the channel cannot be extended by introducing extra columns.

2-layer Manhattan routing has always been one of the most popular and most investigated problems in VLSI routing. The first classic result in the topic of VLSI routing is probably Gallai's linear time algorithm that solves the single row routing problem – a special case of channel routing – with optimal width in the 2-layer Manhattan model.

Theorem 1.1. (T. Gallai [2]) *The minimum width of a solution of a single row routing problem in the* 2*-layer Manhattan model is equal to the density of the problem. Moreover, such a minimum width routing can be found in linear time.*

The proof makes use of the fact that interval graphs are perfect. A horizontal interval is associated with every net, stretching from its leftmost terminal to its rightmost terminal. The density is equal to the clique number of the corresponding interval graph. A coloring with an equal number of colors can easily be transformed into an optimal width routing.

In light of the above theorem it might be surprising that channel routing turns out to be much more complicated than single row routing.

Theorem 1.2. (T.G. Szymanski, 1985 [10]) *It is NP-complete to decide whether a channel routing problem is solvable in the* 2*-layer Manhattan model with width at most k* (*where k is part of the input*).

In [9], the authors restrict the problem of 2-layer Manhattan channel routing, they do not allow extra empty columns to be added to the grid. They give a complete characterization of all specifications that are solvable if the width can be arbitrarily large, but the length of the channel cannot be extended by introducing extra columns. Furthermore, the author present a linear time algorithm to solve these specifications with a width at most constant times the length of the problem.

Theorem 1.3. (Dávid Szeszlér [8]) *A channel routing problem is not solvable in the* 2*-layer Manhattan model* (*with an arbitrary width*) *if and only if it is bipartite, dense and has at least one non-trivial net. Moreover, if a specification is solvable then it can be solved with width at most* $\frac{3}{2}n$ *in the bipartite, and* $\frac{7}{4}n$ *in the general case* (*where n is the length of the channel*).

2. Main results

Recall that a channel routing problem is called bipartite if each net contains exactly two terminals, one on the top, and one on the bottom side. Now we consider horizontal constraint graph of the net in this problem is a star.

Theorem 2.1. *If a channel with horizontal constraint graph be a star, then the vertical constraint graph has directed path with length at most* 1.

Proof. Let the horizontal constraint graph G has k vertices, denoted by $v_1, v_2, ..., v_k$. Without loss of generality, we let the vertex v_k has degree $k-1$. Since the graph G is a star, each vertex of vertices $v_1, v_2, ..., v_{k-1}$ has an edge joined with vertex v_k. Then is to say, each net of $v_1, v_2, ..., v_{k-1}$ has horizontal constraint with net v_k and each two nets belong to $v_1, v_2, ..., v_{k-1}$ have not horizontal constraint.

Now, we proof by contradiction. If these is an direct path belongs to the vertical constraint graph has length $s(s \geq 2)$, denoted by $u_1, u_2, ..., u_s$. Recall that if two nets have vertical constraint, then they also have horizontal constraint. Then the net u_i and $u_{i+1}(1 \leq i \leq s-1)$ have horizontal constraint, the vertex u_i and $u_{i+1}(1 \leq i \leq s-1)$ have edge in horizontal constraint graph, a contradiction.

So if a channel with horizontal constraint graph $v_1, v_2, \ldots, v_k$ be a star, then the vertical constraint graph has directed path with length at most 1, and if the vertical constraint graph has directed path with length 1, one of the vertex belong to this path has the largest degree. Since we consider channel routing problem is bipartite, that is all the net has two terminals, the directed path with length 1 of vertical constraint graph have at most two. ∎

2.1. Our Algorithm

An algorithm for 2-layer Manhattan routing problem with horizontal constraint graph be a star, based on the notions of horizontal constraint graph and vertical constraint graph, is presented in this section. The algorithm proceeds in three cases outlined below.

Route all trivial nets straight down in the obvious fashion. Henceforth we do not include these columns and nets below.

2.1.1. Algorithm

Case 1. Vertical constraint graph of the net have no directed path, that is to say, vertical constraint graph have not edge, each two nets have no vertical constraint.

In this case, we only consider horizontal constraint. Since horizontal constraint graph is a star, we only need to consider the corresponding net with maximum degree in the horizontal constraint graph. We assign the first track on the bottom layer to this net, then the rest nets can not use this track. The rest nets can route in the second track since each two nets belong to the rest nets have not horizontal constraint and vertical constraint. Then we route them in the most straightforward way: in the corresponding track of the bottom layer we introduce a horizontal wire segment connecting the columns of the two terminals, we switch to the top layer at both ends of this segment and connect to the two terminals.

Case 1. Vertical constraint graph of the net have no directed path, that is to say, vertical constraint graph have not edge, each two nets have no vertical constraint.

In this case, we only consider horizontal constraint. Since horizontal constraint graph is a star, we only need to consider the corresponding net with maximum degree in the horizontal constraint graph. We assign the first track on the bottom layer to this net, then the rest nets can not use this track. The rest nets can route in the second track since each two nets belong to the rest nets have not horizontal constraint and vertical constraint. Then we route them in the most straightforward way: in the corresponding track of the bottom layer we introduce a horizontal wire segment connecting the columns of the two terminals, we switch to the top layer at both ends of this segment and connect to the two terminals.

Case 2. Vertical constraint graph of the net have a directed path with length 1.

By Theorem 2.1, one of the vertex of this directed path is the center of star. If this vertex is the starting point, denoted by v_k, and the terminal vertex of this directed path denoted by v_{k+1}. By vertical constraint, the track assigned to the net corresponding to vertex v_k is above to the net corresponding to vertex v_{k+1}. In this case, we assign the first track on the bottom layer to the net corresponding to vertex v_k and assign the second track on the bottom layer to the net corresponding to vertex v_{k+1}. The rest nets can route in the second track since each nets belong to the rest nets have horizontal constraint with v_k but have not horizontal constraint with v_{k+1}. Then using the same method in Case 1, we can assign a track to each of these nets and route these nets in the above straightforward way.

If this vertex is the terminal point, denoted by v_k, and the starting vertex of this directed path denoted by v_{k-1}. By vertical constraint, the track assigned to the net corresponding to vertex v_k is below to the net corresponding to vertex v_{k-1}. In this case, We assign the first track on the bottom layer to the net corresponding to vertex v_{k-1} and assign the second track on the bottom layer to the net corresponding to vertex v_k. The rest nets must route in the third track since each nets belong to the rest nets have horizontal constraint with v_k. Then using the same method in case 1, we can assign a track to each of these nets and route these nets in the above straightforward way.

Case 3. Vertical constraint graph of the net have two directed paths with length 1.

We denote the center of star by v_k. By Theorem 2.1, one of the vertex of directed path is the center of star. Since vertical constraint graph of the net have two directed paths with length 1, then v_k is terminal point of one directed path and is starting point of another directed path. When v_k is the terminal point of directed path, then the starting vertex of this directed path denoted by v_{k-1}. When vertex v_k is the starting point of directed path, the terminal vertex of this directed path denoted by v_{k+1}. By vertical constraint, the track assigned to the net corresponding to vertex v_k is below to the net corresponding to vertex v_{k-1}, the track assigned to the net corresponding to vertex v_k is above to the net corresponding to vertex v_{k+1}. In this case, we assign the first track on the bottom layer to the net corresponding to vertex v_{k-1}, assign the second track on the bottom layer to the net corresponding to vertex v_k, and assign the third track on the bottom layer to the net corresponding to vertex v_{k+1}. The rest nets can route in the third track since each nets belong to the rest nets have not horizontal constraint with v_{k+1}. Then using the same method in Case 1, we can assign a track to each of these nets and route these nets in the above straightforward way.

2.1.2. Running time analysis

When we choose the directed path in algorithm, we need constant time to route the nets of this path. When we route the second directed path in step 2, we need 2 compares. Since every compare need constant time, So we give a constant time algorithm to solve this routing problems.

2.1.3. The upper bound

In the above algorithm, we give a solution for this particular routing problem. Now, we analyze the upper bound of our algorithm and compare to others algorithms. In [9], the author route this problem with n tracks, he assign a separate track on the bottom layer to each net. In our algorithm, we consider the horizontal constraint of the nets, we can route this problem with no more than 3 tracks. So we give a better polynomial time algorithm to solve the 2-layer Manhattan channel routing problem.

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COMPOSITE TRAPEZOID RULE FOR THE RIEMANN-STIELTJES INTEGRAL AND ITS RICHARDSON EXTRAPOLATION FORMULA

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Abstract. In this paper, the composite trapezoid rule for the Riemann-Stieltjes integralis presented and its error is investigated. And then, the rationality of the generalization of composite trapezoid rule for Riemann-Stieltjes integral is demonstrated. At last, Richardson extrapolation is applied to the composite trapezoid rule for the Riemann-Stieltjes integral to obtain high accuracy approximations with little computational cost.

Keywords: composite rule; trapezoid rules; Riemann-Stieltjes integral; error term.

2000 Mathematics Subject Classification: 65D30; 65D32.

1. Introduction

In mathematics, the Riemann-Stieltjes integral is a kind of generalization of the Riemann integral, named after Bernhard Riemann and Thomas Stieltjes. It is Stieltjes [1] that first give the definition of this integral in 1894. It serves as an instructive and useful precursor of the Lebesgue integral. It is known that the Riemann-Stieltjes integral has wide applications in the field of probability theory [2]-[3], stochastic process [4] and functional analysis [5], especially in original formulation of F. Riesz's theorem [2], [5] and the spectral theorem for self-adjoint operators in a Hilbert space [2], [5].

Definite integration is one of the most important and basic concepts in mathematics. And it has numerous applications in fields such as physics and engineering.

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In several practical problems, we need to calculate integrals. As is known to all, as for $I = \int_a^b f(x)dx$, once the primitive function $F(x)$ of integrand $f(x)$ is known, the definite integral of $f(x)$ over the interval $[a, b]$ is given by Newton-Leibniz formula, that is,

$$\int_a^b f(x)dx = F(b) - F(a). \tag{1.1}$$

However, the explicit primitive function $F(x)$ is not available or its primitive function is not easy to obtain, such as $e^{\pm x^2}$, $\sin x^2$, $\frac{\sin x}{x}$, etc. Moreover, the integrand $f(x)$ is only available at certain points x_i, $i = 1, 2, ..., n$. How to get high-precision numerical integration formulas becomes one of the challenges in fields of mathematics [6].

In numerical analysis, the trapezoidal rule is the most well known numerical integration rules for approximating the definite integral. Trapezoidal rule with error term for classical Riemann integral is

$$\int_a^b f(x)dx = \frac{b-a}{2}\left(f(a) + f(b)\right) - \frac{(b-a)^3}{12}f''(\xi), \tag{1.2}$$

where $\xi \in (a, b)$.

In spite of the many accurate and efficient methods for numerical integration being available in [7-9], recently Mercer has obtained trapezoid rule for Riemann-Stielsjes integral which engender a generalization of Hadamard's integral inequality [10]. Trapezoidal rule with error term for Riemann-Stieltjes integral is

$$\int_a^b f(t)dg = \left[G - g(a)\right] f(a) + \left[g(b) - G\right] f(b) - \frac{(b-a)^3}{12}f''(\xi)g'(\eta), \tag{1.3}$$

where $G = \dfrac{1}{b-a}\displaystyle\int_a^b g(t)dt$, $\xi,\ \eta \in (a, b)$.

Then Mercer develops Midpoint and Simpson's rules for Riemann-Stielsjes integral [11] by using the concept of relative convexity. Burg [12] has proposed derivative-based closed Newton-Cotes numerical quadrature which uses both the function value and the derivative value on uniformly spaced intervals. Zhao and Li have proposed midpoint derivative-based closed Newton-Cotes quadrature [13] and numerical superiority has been shown. Then, the derivative-based trapezoid rule for the Riemann-Stieltjes integral is presented by Zhao and Zhang [14], which uses derivative values at the endpoints. Recently, the midpoint derivative-based trapezoid rule for the Riemann-Stieltjes integral is presented by Zhao, Zhang and Ye [15], which only uses derivative values at the midpoint.

The trapezoid rule or the Riemann-Stieltjes integral uses only two points to approximate an integral, certainly not unsuitable for use over large integration intervals. A different direction is to subdivide the interval into smaller intervals and use lower-order schemes, like the trapezoid rule, on these smaller intervals. In numerical analysis, Richardson extrapolation is a sequence acceleration method,

used to improve the rate of convergence of a sequence. It is used to generate high-accuracy results while using low-order formulas. It is named after Lewis Fry Richardson, who introduced the technique in 1927 [16].

One motivation for the research presented here lies in construction of composite trapezoid rule for the Riemann-Stieltjes integral, which is generalization of the results in [10], [11], [14]. The other is apply Richardson extrapolation to the composite trapezoid rule for the Riemann-Stieltjes integral to obtain high accuracy approximations with little computational cost.

In this paper, we divide the interval into subintervals and apply an integration rule to each subinterval, so the composite trapezoid rule for the Riemann-Stieltjes integral is presented. These new scheme is investigated in Section 2. In Section 3, the error term is presented. In Section 4, Richardson extrapolation is applied to the composite trapezoid rule for the Riemann-Stieltjes integral to obtain high accuracy approximations. Finally, conclusions are drawn in Section 5.

2. Composite trapezoid rule for the Riemann-Stieltjes integral

In this section, by dividing the interval into subintervals and applying an integration rule to each subinterval, composite trapezoid rule for the Riemann-Stieltjes integral is presented.

Theorem 2.1 *Suppose that $f(t)$ and $g(t)$ are continuous on $[a,b]$ and $g(t)$ is increasing there. Suppose that the interval $[a,b]$ is subdivided into n subintervals $[x_k, x_{k+1}]$ of width $h = \frac{b-a}{n}$ by using the equally spaced nodes $x_k = a + kh$, for $k = 0, 1, ..., n$. The composite trapezoidal rule for n subintervals can be expressed as follows:*

$$\begin{aligned}\int_a^b f(t)dg \approx T_n = &\left[\frac{n}{b-a}\int_a^{x_1} g(t)dt - g(a)\right] f(a) \\ &+\frac{n}{b-a}\sum_{k=1}^{n-1}\left[\int_{x_k}^{x_{k+1}} g(t)dt - \int_{x_{k-1}}^{x_k} g(t)dt\right] f(x_k) \\ &+\left[g(b) - \frac{n}{b-a}\int_{x_{n-1}}^{b} g(t)dt\right] f(b).\end{aligned} \tag{2.1}$$

Proof. By (1.3), the trapezoidal rule for Riemann-Stieltjes integral is

$$\int_a^b f(t)dg \approx \left[\frac{1}{b-a}\int_a^b g(t)dt - g(a)\right] f(a) + \left[g(b) - \frac{1}{b-a}\int_a^b g(t)dt\right] f(b). \tag{2.2}$$

Applying formula (2.2) over each subinterval, we obtain

$$\begin{aligned}\int_a^b f(t)dg \approx &\left[\frac{1}{\frac{b-a}{n}}\int_a^{x_1} g(t)dt - g(a)\right] f(a) + \left[g(x_1) - \frac{1}{\frac{b-a}{n}}\int_a^{x_1} g(t)dt\right] f(x_1) \\ &+\left[\frac{1}{\frac{b-a}{n}}\int_{x_1}^{x_2} g(t)dt - g(x_1)\right] f(x_1) + \left[g(x_1) - \frac{1}{\frac{b-a}{n}}\int_{x_1}^{x_2} g(t)dt\right] f(x_2) + \cdots\end{aligned}$$

$$
\begin{aligned}
&+\left[\frac{1}{\frac{b-a}{n}}\int_{x_{k-1}}^{x_k} g(t)dt - g(x_k)\right] f(x_{k-1}) + \left[g(x_k) - \frac{1}{\frac{b-a}{n}}\int_{x_k}^{x_{k+1}} g(t)dt\right] f(x_k) + \cdots \\
&+\left[\frac{1}{\frac{b-a}{n}}\int_{x_{n-1}}^{b} g(t)dt - g(x_{n-1})\right] f(x_{n-1}) + \left[g(b) - \frac{1}{b-a}\int_{x_{n-1}}^{b} g(t)dt\right] f(b) \\
&= \left[\frac{n}{b-a}\int_a^{x_1} g(t)\,dt - g(a)\right] f(a) + \frac{n}{b-a}\left[\int_{x_1}^{x_2} g(t)\,dt - \int_a^{x_1} g(t)\,dt\right] f(x_1) \\
&+\frac{n}{b-a}\left[\int_{x_2}^{x_3} g(t)dt - \int_{x_1}^{x_2} g(t)dt\right] f(x_2) + \cdots \\
&+\frac{n}{b-a}\left[\int_{x_{n-1}}^{x_n} g(t)\,dt - \int_{x_{n-2}}^{x_{n-1}} g(t)\,dt\right] f(x_{n-1}) + \left[g(b) - \frac{n}{b-a}\int_{x_{n-1}}^{b} g(t)dt\right] f(b) \\
&= \left[\frac{n}{b-a}\int_a^{x_1} g(t)dt - g(a)\right] f(a) + \frac{n}{b-a}\sum_{k=1}^{n-1}\left[\int_{x_k}^{x_{k+1}} g(t)dt - \int_{x_{k-1}}^{x_k} g(t)dt\right] f(x_k) \\
&+\left[g(b) - \frac{n}{b-a}\int_{x_{n-1}}^{b} g(t)\,dt\right] f(b). \qquad \blacksquare
\end{aligned}
$$

So, we have the composite trapezoid rule for the Riemann-Stieltjes integral as desired.

3. The error term for Riemann-Stieltjes composite trapezoid rule

In this section, the error term of the composite trapezoid rule for the Riemann-Stieltjes is investigated.

Theorem 3.1 *Suppose that $f''(t)$ and $g'(t)$ are continuous on $[a,b]$ and $g(t)$ is increasing there. Other conditions are the same as Theorem* 2.1. *The composite trapezoid rule for the Riemann-Stieltjes integral with the error term is*

$$
\begin{aligned}
\int_a^b f(t)dg = &\left[\frac{n}{b-a}\int_a^{x_1} g(t)\,dt - g(a)\right] f(a) \\
&+\frac{n}{b-a}\sum_{k=1}^{n-1}\left[\int_{x_k}^{x_{k+1}} g(t)\,dt - \int_{x_{k-1}}^{x_k} g(t)\,dt\right] f(x_k) \\
&+\left[g(b) - \frac{n}{b-a}\int_{x_{n-1}}^{b} g(t)\,dt\right] f(b) - \frac{(b-a)^3}{12n^2} f''(\xi)g'(\eta),
\end{aligned} \tag{3.1}
$$

where $\xi, \eta \in (a,b)$. And the error term $R[f]$ of this method is

$$
-\frac{(b-a)^3}{12n^2} f''(\xi)g'(\eta). \tag{3.2}
$$

Proof. We know from (1.3) that in every subinterval the quadrature error is

$$
-\frac{h^3}{12} f''(\xi_k)g'(\eta_k), \tag{3.3}
$$

where $\xi_k,\ \eta_k \in (x_{k-1},\, x_k)$, $k = 1, ..., n$.

Hence, the overall error is obtained by summing over n such terms:

$$\sum_{k=1}^{n} -\frac{h^3}{12} f''(\xi_k) g'(\eta_k) = -\frac{nh^3}{12} \left[\frac{1}{n} \sum_{k=1}^{n} f''(\xi_k) g'(\eta_k) \right]. \tag{3.4}$$

Let $M = \dfrac{1}{n} \displaystyle\sum_{k=1}^{n} f''(\xi_k) g'(\eta_k)$.

Clearly, $\min\limits_{x\in[a,b]} \{f''(x)g'(x)\} \le M \le \max\limits_{x\in[a,b]} \{f''(x)g'(x)\}$. Since $f''(t)$ and $g'(t)$ are continuous on $[a, b]$, then there exists two points ξ and η such that $M = f''(\xi)g'(\eta)$. This implies that the error term is $R[f] = -\dfrac{nh^3}{12} f''(\xi)g'(\eta)$.

Recalling that $h = \dfrac{b-a}{n}$, we obtain $R[f] = -\dfrac{(b-a)^3}{12n^2} f''(\xi)g'(\eta)$. ∎

Corollary 3.2 *Conditions are the same as Theorem* 3.1. *When* $g(t) = t$, *equation* (2.2) *reduces to the composite trapezoid rule for the classical Riemann integral.*

Proof. It is easy to obtain $\dfrac{n}{b-a} \displaystyle\int_a^{x_1} t\,dt - a = \frac{b-a}{2n}$, $\displaystyle\int_{x_k}^{x_{k+1}} t\,dt - \int_{x_{k-1}}^{x_k} t\,dt = \left(\frac{b-a}{n}\right)^2$, $b - \dfrac{n}{b-a} \displaystyle\int_{x_{n-1}}^{b} t\,dt = \frac{b-a}{2n}$ and $g'(t) \equiv 1$. By Theorem 3.1,

$$\begin{aligned}
\int_a^b f(t)dg &= \int_a^b f(t)dt \\
&= \frac{b-a}{2n} f(a) + \frac{n}{b-a} \left(\frac{b-a}{n}\right)^2 \sum_{k=1}^{n-1} f(x_k) + \frac{b-a}{2n} f(b) - \frac{(b-a)^3}{12n^2} f''(\xi) \\
&= \frac{b-a}{2n} \left(f(a) + 2\sum_{k=1}^{n-1} f(x_k) + f(b) \right) - \frac{(b-a)^3}{12n^2} f''(\xi).
\end{aligned}$$

Remark 3.3 From Corollary 3.2, we know that the results in Theorem 3.1 possess the reducibility. When $g(t) = t$, formula (3.1) reduces to the composite trapezoid rule for the classical Riemann integral. So it is a reasonable generalization of composite trapezoid rule for Riemann-Stieltjes integral.

4. Richardson extrapolation formula for the composite trapezoid rule for the Riemann-Stieltjes integral

Extrapolation can be applied whenever it is known that an approximation technique has an error term with a predictable form, one that depends on a parameter, usually the step size h [8].

In this section, we will illustrate how Richardson extrapolation applied to results from the composite trapezoid rule for the Riemann-Stieltjes integral can be used to obtain high accuracy approximations with little computational cost.

Theorem 4.1 *Suppose that $f''(t)$ and $g'(t)$ are continuous on $[a,b]$ and $g(t)$ is increasing there. Other conditions are the same as Theorem* 2.1. *The Richardson extrapolation formula for the composite trapezoid rule for the Riemann-Stieltjes integral is*

$$\int_a^b f(t)dg \approx \bar{T}_n = \frac{4}{3}T_{2n} - \frac{1}{3}T_n. \tag{4.1}$$

Proof. By equation (3.4), it is easy to obtain the error term $R[f]$ of composite trapezoid rule for the Riemann-Stieltjes integral is

$$R[f] = \sum_{k=1}^{n} -\frac{h^3}{12} f''(\xi_k)g'(\eta_k) = -\frac{nh^3}{12}\left[\frac{1}{n}\sum_{k=1}^{n} f''(\xi_k)g'(\eta_k)\right],$$

where $\xi_k,\ \eta_k \in (x_{k-1},\, x_k)\,,\ \ k = 1, ..., n.$

By the definition of Riemann integral, when n is big enough, we obtain

$$-\frac{nh^3}{12}\left[\frac{1}{n}\sum_{k=1}^{n} f''(\xi_k)g'(\eta_k)\right] = -\frac{h^2}{12}\left[h\sum_{k=1}^{n} f''(\xi_k)g'(\eta_k)\right] \approx -\frac{h^2}{12}\int_a^b f''(x)g'(x)\,dx.$$

It means

$$\int_a^b f(t)dg - T_n \approx -\frac{h^2}{12}\int_a^b f''(x)g'(x)\,dx. \tag{4.2}$$

Similarly,

$$\int_a^b f(t)dg - T_{2n} = -\frac{\left(\frac{h}{2}\right)^2}{12}\int_a^b f''(x)g'(x)\,dx. \tag{4.3}$$

Therefore, $\frac{\int_a^b f(t)dg - T_{2n}}{\int_a^b f(t)dg - T_n} \approx \frac{1}{4}$. Thus,

$$\int_a^b f(t)dg - T_{2n} \approx \frac{1}{4}\left(\int_a^b f(t)dg - T_n\right), \tag{4.4}$$

that is,

$$\int_a^b f(t)dg \approx \frac{4}{3}T_{2n} - \frac{1}{3}T_n.$$

Denote $\bar{T}_n = \frac{4}{3}T_{2n} - \frac{1}{3}T_n$, so we have the Richardson extrapolation formula as desired. ∎

Corollary 4.2 *Conditions are the same as Theorem* 4.1. *The Richardson extrapolation formula for the composite trapezoid rule for the Riemann-Stieltjes integral is 4th-order accurate.*

Proof. We know that the composite trapezoid rule for the Riemann-Stieltjes integral is second-order accurate (see equation (3.2)).

A more detailed study of the quadrature error reveals that the difference between $\int_a^b f(t)dg$ and T_n can be written as

$$\int_a^b f(t)dg = T_n + c_1 h^2 + c_2 h^4 + \cdots c_k h^{2k} + o\left(h^{2k+2}\right). \tag{4.5}$$

The exact values of the coefficients, c_k, are of no interest to us as long as they do not depend on h.

We can now write a similar quadrature that is based on double the number of points, i.e., T_{2n}.

Therefore,

$$\int_a^b f(t)dg = T_{2n} + c_1 \left(\tfrac{h}{2}\right)^2 + c_2 \left(\tfrac{h}{2}\right)^4 + \cdots + c_k \left(\tfrac{h}{2}\right)^{2k} + o\left(\left(\tfrac{h}{2}\right)^{2k+2}\right). \tag{4.6}$$

4*(4.6)-(4.5), this enables us to eliminate the h^2 error term,

$$3\int_a^b f(t)dg = 4T_{2n} - T_n + \hat{c}_2 h^4 + \cdots + \hat{c}_k h^k + o\left(h^{2k+2}\right),$$

where $\hat{c}_k = \left(2^{2-2k} - 1\right) c_k$, $k = 2, 3, ...$ That is,

$$\begin{aligned}\int_a^b f(t)dg &= \frac{4}{3}T_{2n} - \frac{1}{3}T_n + \bar{c}_2 h^4 + \cdots + \bar{c}_k h^k + o\left(h^{2k+2}\right) \\ &= \bar{T}_n + \bar{c}_2 h^4 + \cdots + \bar{c}_k h^k + o\left(h^{2k+2}\right).\end{aligned} \tag{4.7}$$

where $\bar{c}_k = \frac{1}{3}\left(2^{2-2k} - 1\right) c_k$, $k = 2, 3, ...$

By equation (4.7), it is easy to obtain that the Richardson extrapolation formula for the composite trapezoid rule for the Riemann-Stieltjes integral is 4th-order accurate. ∎

5. Conclusions

We briefly summarize our main conclusions in this paper as follows.

1) The composite trapezoid rule for the Riemann-Stieltjes integral is presented and its error term is investigated.
2) The rationality of the generalization of composite trapezoid rule for Riemann-Stieltjes integral is demonstrated.
3) Richardson extrapolation formula for the composite trapezoid rule for the Riemann-Stieltjes integral is obtained.

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RELATIVE ENTROPY AND RELATIVE CONDITIONAL ENTROPY WITH INFINITE PARTITIONS

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Abstract. In this paper, we introduce the notions of relative entropy and relative conditional entropy for infinite partitions on a relative probability measure space. We present some examples and prove some theorems about relative conditional entropy.

Keywords: observer; entropy; relative entropy; relative conditional entropy.

AMS Subject Classification: 28D20, 37A35, 54C70.

1. Introduction

The study of concept entropy is very important in the current sciences. Entropy plays an important role in a variety of problem areas, including physics, computer science, general systems theory, information theory, statistics, biology, chemistry, sociology and others. In 1958, Kolmogorov introduced the concept of entropy in the ergodic theory. Entropy is defined on several structures [2]-[4]. Molaie in [9] has studied the notion of one dimensional observer. This notion has been applied in dynamical systems [7], [12], topology [4], [6], [8], geometry [10], and mathematical physics [10]. Let X be a non-empty set, then the function $\mu : X \to [0, 1]$ is called a one-dimensional observer of X. In this paper we assume that $f : X \to X$ is a mapping and $\mu : X \to [0, 1]$ is an observer of X and E is an arbitrary subset of X. The relative probability measure of E with respect to an

observer μ is the function $m_\mu^f(E) : X \to [0,1]$ is defined in [5] by

$$m_\mu^f(E)(x) = \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_E(f^i(x))\mu(f^i(x)).$$

The notation $m_\mu^f(E)(x)$ is the measure of E according to an observer viewpoint when it look at x.

Example 1.1. Suppose $X = \Re\backslash\{0,-1\}$ and also let $f : X \to X$ be defined by $x \mapsto \frac{1}{x}$. Also let $E = [0,1]$ and $\mu : X \to [0,1]$ be defined by $x \mapsto \frac{x}{x+1}$. Then

$$\begin{aligned} m_\mu^f(E)(3) &= \limsup \frac{1}{n} \sum_{i=0}^{n-1} \chi_E(f^i(3)) \cdot \mu(f^i(3)) \\ &= \limsup \frac{1}{n} \left(\chi_E(3) \cdot \frac{3}{3+1} + \chi_E(\frac{1}{3}) \cdot \frac{\frac{1}{3}}{\frac{1}{3}+1} + \chi_E(3) \cdot \frac{3}{3+1} + ... \right) \\ &= \limsup \frac{1}{n} \left(0 + \frac{1}{4} + 0 + \frac{1}{4} + ... \right) \\ &= \limsup \frac{1}{n} \left(\frac{n}{8} \right) \\ &= \frac{1}{8}. \end{aligned}$$

In Section 2, we introduce the notion of relative entropy for infinite partitions.

2. Relative entropy for partitions

We assume that (X, m_μ^f) is a relative probability space.

A partition of X is a disjoin collection of elements of $P(X)$ whose union is X, where $P(X)$ is the power set of X.

If $\xi = \{A_i : i \in \mathbb{N}\}$, and $\eta = \{C_j : j \in \mathbb{N}\}$ are two countable partitions of X, then their join is the partition :

$$\xi \vee \eta = \{A_i \cap C_j : m_\mu^f(A_i \cap C_j)(x) \geq m_\mu^f(A_i)(x) m_\mu^f(C_j)(x), \forall x \in X\}.$$

If $T : X \to X$ is a mapping, then T is called relative probability measure-preserving if $m_\mu^f(T^{-1}E)(x) = m_\mu^f(E)(x)$, for all $x \in X$.

Suppose $\mathcal{C}, \mathcal{D}$ are two subsets of $P(X)$ then we say $\mathcal{C} \subset_\mu^f \mathcal{D}$ if for each $C \in \mathcal{C}$ there exist $D \in \mathcal{D}$ such that $m_\mu^f(D\Delta C)(x) = 0$.

We say $\mathcal{C} =_\mu^f \mathcal{D}$ if $\mathcal{C} \subset_\mu^f \mathcal{D}$ and $\mathcal{D} \subset_\mu^f \mathcal{C}$. Also we say $\mathcal{D}$ is a refinement of $\mathcal{C}$, and write $\mathcal{C} \preccurlyeq_\mu^f \mathcal{D}$, when we can write each element of $\mathcal{C}$, as union some elements of $\mathcal{D}$.

Definition 2.1. Let $\mathcal{A} = \{A_i : i \in \mathbb{N}\}$ be a countable partition of X, then the relative entropy of $\mathcal{A}$ is the mapping $H_\mu^f(\mathcal{A}) : X \to \mathbb{R}$ defined by

$$H_\mu^f(\mathcal{A})(x) = -\log \sup_{i\in\mathbb{N}} m_\mu^f(A_i)(x).$$

Example 2.2. Let $X = (0, 1]$. If $A = \{A_i : i \in \mathbf{N}\}$ such that for each $i \in \mathbf{N}$,

$$A_i = \left(\frac{1}{i+1}, \frac{1}{i}\right].$$

A is a countable partition of X, because $\bigcup_{i=1}^{\infty} A_i = X$ and elements of $\{A_i : i \in \mathbf{N}\}$ are disjoint.

Now, let $f : X \to X$ be defined by $x \mapsto \frac{1}{k},\ k \geq 1$. Let $\mu : X \to [0, 1]$ be defined by $x \mapsto x$. Then

$$\begin{aligned} m_\mu^f(A_j)(x) &= \limsup \frac{1}{n} \sum_{i=0}^{n-1} \chi_{A_j}(f^i(x))\mu_(f^i(x)) \\ &= \limsup \frac{1}{n} \left[\chi_{A_j}(x) \cdot x + \sum_{i=1}^{n-1} \chi_{A_j}\left(\frac{1}{k}\right) \cdot \left(\frac{1}{k}\right)\right]. \end{aligned}$$

So $m_\mu^f(A_j)(x) = 0$ for $j \neq k$ and $m_\mu^f(A_j)(x) = \frac{1}{k}$ for $j = k$. Thus

$$H_\mu^f(\mathcal{A})(x) = -\log \sup_{i \in \mathbb{N}} m_\mu^f(A_i)(x) = -\log\left(\frac{1}{k}\right) = \log(k).$$

Remark 2.3. If $T : X \to X$ is a relative probability measure preserving map then $H_\mu^f(T^{-1}\mathcal{A})(x) = H_\mu^f(\mathcal{A})(x)$, for all $x \in X$.

Lemma 2.4. *If $\mathcal{A} =_\mu^f \mathcal{C}$, then $H_\mu^f(\mathcal{A})(x) = H_\mu^f(\mathcal{C})(x)$ for all $x \in X$.*

Proof. If $\mathcal{A} \subseteq_\mu^f \mathcal{C}$, then for each $A_i \in \mathcal{A}$ there exists $C_j \in \mathcal{C}$ such that $m_\mu^f(A_i \Delta C_j)(x) = 0$ for all $x \in X$.

We have $A_i = (A_i \bigcap C_j) \bigcup (A_i - C_j)$. So

$$\begin{aligned} A_i \cup (C_j - A_i) &= (A_i \cap C_j) \cup (C_j - A_i) \cup (A_i - C_j) \\ &= (A_i \cap C_j) \cup (A_i \Delta C_j). \end{aligned}$$

Hence

$$m_\mu^f(A_i)(x) + m_\mu^f(C_j - A_i)(x) = m_\mu^f(A_i \cap C_j)(x).$$

Thus

$$m_\mu^f(A_i)(x) = m_\mu^f(A_i \cap C_j)(x) \leq m_\mu^f(C_j)(x),$$

because $0 \leq m_\mu^f(C_j - A_i)(x) \leq m_\mu^f(A_i \Delta C_j)(x) = 0$. So for each $x \in X$,

$$\{m_\mu^f(A_i)(x) : i \in \mathbb{N}\} \subseteq \{m_\mu^f(C_j)(x) : j \in \mathbb{N}\}.$$

Hence

$$\sup_{i \in \mathbb{N}} m_\mu^f(A_i)(x) \leq \sup_{j \in \mathbb{N}} m_\mu^f(C_j)(x).$$

Thus $H_\mu^f(\mathcal{A})(x) \geq H_\mu^f(\mathcal{C})(x)$. Now, if $\mathcal{C} \subseteq_\mu^f \mathcal{A}$, then by changing the role of A and C we imply $H_\mu^f(\mathcal{A})(x) \leq H_\mu^f(\mathcal{C})(x)$. So $H_\mu^f(\mathcal{A})(x) = H_\mu^f(\mathcal{C})(x)$. ∎

3. Relative conditional entropy

In this section, the relative conditional entropy for infinite partitions and its properties are studied. Suppose that $\mathcal{A}$ and $\mathcal{C}$ are two infinite partitions of X.

Definition 3.1. The relative entropy of $\mathcal{A}$ given $\mathcal{C}$ is the number:

$$H_\mu^f(\mathcal{A}/\mathcal{C})(x) = -\log \frac{\sup\limits_{i,j\in\mathbb{N}} m_\mu^f(A_i \cap C_j)(x)}{\sup\limits_{j\in\mathbb{N}} m_\mu^f(C_j)(x)}.$$

$H_\mu^f(\mathcal{A}/\mathcal{C})(x)$ measure the uncertainty of the outcome of $\mathcal{A}$ when the observer who looks to the orbit of x, knows the outcome of $\mathcal{C}$.

Theorem 3.2. *Let $\mathcal{D} = \{\emptyset, X\}$. Then*

$$H_\mu^f(\mathcal{A}/\mathcal{D})(x) = H_\mu^f(\mathcal{A})(x) - H_\mu^f(X)(x).$$

Proof.

$$\begin{aligned} H_\mu^f(\mathcal{A}/\mathcal{D})(x) &= -\log \frac{\sup\limits_{i\in\mathbb{N}} m_\mu^f(A_i \cap X)(x)}{m_\mu^f(X)(x)} \\ &= -\log \frac{\sup\limits_{i\in\mathbb{N}} m_\mu^f(A_i)(x)}{m_\mu^f(X)(x)} - \log \sup_{i\in\mathbb{N}} m_\mu^f(A_i)(x) + \log m_\mu^f(X)(x) \\ &= H_\mu^f(\mathcal{A})(x) - H_\mu^f(X)(x). \end{aligned}$$

∎

Corollary 3.3. *If $\mathcal{D} = \{\emptyset, X\}$ and $m_\mu^f(X)(x) = 1$, then*

$$H_\mu^f(\mathcal{A}/\mathcal{D})(x) = H_\mu^f(\mathcal{A})(x).$$

Theorem 3.4.

1. *If $\mathcal{A} =_\mu^f \mathcal{D}$ then $H_\mu^f(\mathcal{A}/\mathcal{C})(x) = H_\mu^f(\mathcal{D}/\mathcal{C})(x)$.*
2. *If $\mathcal{C} =_\mu^f \mathcal{D}$ then $H_\mu^f(\mathcal{A}/\mathcal{C})(x) = H_\mu^f(\mathcal{A}/\mathcal{D})(x)$.*

Proof. 1. $\mathcal{A} =_\mu^f \mathcal{D}$ implies that $\mathcal{A} \vee \mathcal{C} =_\mu^f \mathcal{D} \vee \mathcal{C}$. So

$$\begin{aligned} H_\mu^f(\mathcal{A}/\mathcal{C})(x) &= H_\mu^f(\mathcal{A} \vee \mathcal{C})(x) - H_\mu^f(\mathcal{C})(x) \\ &= H_\mu^f(\mathcal{D} \vee \mathcal{C})(x) - H_\mu^f(\mathcal{C})(x) \\ &= H_\mu^f(\mathcal{D}/\mathcal{C})(x). \end{aligned}$$

2. $\mathcal{C} =_\mu^f \mathcal{D}$ implies that $\mathcal{A} \vee \mathcal{C} =_\mu^f \mathcal{A} \vee \mathcal{D}$. So

$$\begin{aligned} H_\mu^f(\mathcal{A}/\mathcal{C})(x) &= H_\mu^f(\mathcal{A} \vee \mathcal{C})(x) - H_\mu^f(\mathcal{C})(x) \\ &= H_\mu^f(\mathcal{A} \vee \mathcal{D})(x) - H_\mu^f(\mathcal{D})(x) \\ &= H_\mu^f(\mathcal{A}/\mathcal{D})(x). \end{aligned}$$

∎

Theorem 3.5. *If (X, m_μ^f) is a relative probability space, and $\mathcal{A}, \mathcal{C}, \mathcal{D}$ are countable partitions of X, then*

$$H_\mu^f(\mathcal{A} \vee \mathcal{C}/\mathcal{D})(x) = H_\mu^f(\mathcal{A}/\mathcal{D})(x) + H_\mu^f(\mathcal{C}/\mathcal{A} \vee \mathcal{D})(x).$$

Proof. Let $\mathcal{A} = \{A_i\}, \mathcal{C} = \{C_j\}$,and $\mathcal{D} = \{D_k\}$, then

$$\begin{aligned} H_\mu^f(\mathcal{A} \vee \mathcal{C}/\mathcal{D})(x) &= -\log \frac{\sup\limits_{i,j,k\in\mathbb{N}} m_\mu^f(A_i \cap C_j \cap D_k)(x)}{\sup\limits_{k\in\mathbb{N}} m_\mu^f(D_k)(x)} \\ &= -\log \left(\frac{\sup\limits_{i,k\in\mathbb{N}} m_\mu^f(A_i \cap D_k)(x)}{\sup\limits_{k\in\mathbb{N}} m_\mu^f(D_k)(x)} \times \frac{\sup\limits_{i,j,k\in\mathbb{N}} m_\mu^f(A_i \cap C_j \cap D_k)(x)}{\sup\limits_{i,k\in\mathbb{N}} m_\mu^f(A_i \cap D_k)(x)} \right) \\ &= H_\mu^f(\mathcal{A}/\mathcal{D})(x) + H_\mu^f(\mathcal{C}/\mathcal{A} \vee \mathcal{D})(x). \end{aligned}$$

∎

Corollary 3.6. *Suppose (X, m_μ^f) is a relative probability space. If $\mathcal{A}, \mathcal{C}$ are infinite partitions of X then*

$$H_\mu^f(\mathcal{A} \vee \mathcal{C})(x) = H_\mu^f(\mathcal{A})(x) + H_\mu^f(\mathcal{C}/\mathcal{A})(x).$$

Proof. Put $\mathcal{D} = \{\emptyset, X\}$ in Theorem 3.5 and apply Theorem 3.2. ∎

Theorem 3.7. *If $\mathcal{A} \preccurlyeq_\mu^f \mathcal{C}$ then $H_\mu^f(\mathcal{A})(x) \leq H_\mu^f(\mathcal{C})(x)$.*

Proof. Let $\mathcal{A} = \{A_i\}$,and $\mathcal{C} = \{C_j\}$. Since $\mathcal{A} \preccurlyeq_\mu^f \mathcal{C}$, we can write each element of $\mathcal{A}$, as union some elements of $\mathcal{C}$. So $\sup\limits_{i\in\mathbb{N}} m_\mu^f(A_i)(x) \leq \sup\limits_{j\in\mathbb{N}} m_\mu^f(C_j)(x)$. Thus $H_\mu^f(\mathcal{A})(x) \leq H_\mu^f(\mathcal{C})(x)$. ∎

Corollary 3.8. *Let (X, m_μ^f) be a relative probability space,and let $\mathcal{A}, \mathcal{C}, \mathcal{D}$ be countable partitions of X. Moreover let $\mathcal{A} \preccurlyeq_\mu^f \mathcal{C}$. Then*

$$H_\mu^f(\mathcal{A}/\mathcal{D})(x) \leq H_\mu^f(\mathcal{C}/\mathcal{D})(x).$$

Proof. Since $\mathcal{A} \preccurlyeq_\mu^f \mathcal{C}$, we have $\mathcal{A} \vee \mathcal{D} \preccurlyeq_\mu^f \mathcal{C} \vee \mathcal{D}$. By corollary 3.6, we can write

$$\begin{aligned} H_\mu^f(\mathcal{C}/\mathcal{D})(x) &= H_\mu^f(\mathcal{C} \vee \mathcal{D})(x) - H_\mu^f(\mathcal{D})(x) \\ &\geq H_\mu^f(\mathcal{A} \vee \mathcal{D})(x) - H_\mu^f(\mathcal{D})(x) \\ &= H_\mu^f(\mathcal{A}/\mathcal{D})(x). \end{aligned}$$

∎

Corollary 3.9. *If (X, m_μ^f) is a relative probability space, and $\mathcal{A}, \mathcal{C}$ are countable partitions of X, then $H_\mu^f(\mathcal{A} \vee \mathcal{C})(x) \leq H_\mu^f(\mathcal{A})(x) + H_\mu^f(\mathcal{C})(x)$.*

Proof. By definition, we have

$$\begin{aligned} H_\mu^f(\mathcal{A} \vee \mathcal{C})(x) &= -\log \sup_{i,j\in\mathbb{N}} m_\mu^f(A_i \cap C_j)(x) \\ &\leq -\log \sup_{i\in\mathbb{N}} m_\mu^f(A_i)(x) - \log \sup_{j\in\mathbb{N}} m_\mu^f(C_j)(x) \\ &= H_\mu^f(\mathcal{A})(x) + H_\mu^f(\mathcal{C})(x). \end{aligned}$$

∎

Corollary 3.10. *If (X, m_μ^f) is a relative probability space, T is a relative probability measure-preserving map, and $\mathcal{A}, \mathcal{C}$ are countable partitions of X then*

$$H_\mu^f(T^{-1}(\mathcal{A})/T^{-1}(\mathcal{C}))(x) = H_\mu^f(\mathcal{A}/\mathcal{D})(x).$$

Theorem 3.11. *Suppose $\mathcal{A}, \mathcal{C}$ are countable partitions of X and $\mathcal{C} \subset_\mu^f \mathcal{A}$, then for all $x \in X, H_\mu^f(\mathcal{A}/\mathcal{C})(x) = 0$.*

Proof. Since $\mathcal{C} \subset_\mu^f \mathcal{A}$, for each $j \in \mathbb{N}$ there exist $i \in \mathbb{N}$ such that

$$m_\mu^f(A_i \cap C_j)(x) = m_\mu^f(C_j)(x).$$

Hence

$$\{m_\mu^f(C_j)(x) : j \in \mathbb{N}\} \subseteq \{m_\mu^f(A_i \cap C_j)(x) : i, j \in \mathbb{N}\}.$$

So for each $x \in X$,

$$\sup_{j\in\mathbb{N}} m_\mu^f(C_j)(x) \leq \sup_{i,j\in\mathbb{N}} m_\mu^f(A_i \cap C_j)(x).$$

On the other hand,

$$\sup_{j\in\mathbb{N}} m_\mu^f(C_j)(x) \geq \sup_{i,j\in\mathbb{N}} m_\mu^f(A_i \cap C_j)(x).$$

Thus

$$H_\mu^f(\mathcal{A}/\mathcal{C})(x) = -\log \frac{\sup_{i,j\in\mathbb{N}} m_\mu^f(A_i \cap C_j)(x)}{\sup_{j\in\mathbb{N}} m_\mu^f(C_j)(x)} = 0. \qquad \blacksquare$$

Two countable partitions $\mathcal{A}$ and $\mathcal{C}$ are called independent if

$$m_\mu^f(A \cap C)(x) = m_\mu^f(A)(x) m_\mu^f(C)(x)$$

for all $A \in \mathcal{A}, C \in \mathcal{C}$, and $x \in X$.

Theorem 3.12. *Let $\mathcal{A}, \mathcal{C}$ be two countable partitions of (X, m_μ^f), and let $\mathcal{A}, \mathcal{C}$ be independent. Then*

$$H_\mu^f(\mathcal{A}/\mathcal{C})(x) = H_\mu^f(\mathcal{A})(x).$$

Proof. If $\mathcal{A}$ and $\mathcal{C}$ are independent then

$$\begin{aligned} H_\mu^f(\mathcal{A}/\mathcal{C})(x) &= -\log \frac{\sup_{i,j\in\mathbb{N}} m_\mu^f(A_i \bigcap C_j)(x)}{\sup_{j\in\mathbb{N}} m_\mu^f(C_j)(x)} \\ &= -\log \frac{\sup_{i,j\in\mathbb{N}} m_\mu^f(A_i)(x) m_\mu^f(C_j)(x)}{\sup_{j\in\mathbb{N}} m_\mu^f(C_j)(x)} \\ &= H_\mu^f(\mathcal{A})(x). \end{aligned}$$

$\blacksquare$

4. Conclusion

We extended the notions of relative entropy and relative conditional entropy to infinite partitions. Some properties about the notions with infinite partitions studied. Relative entropy of dynamical systems with infinite partitions can be a topic for further research.

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A GENERALIZATION OF SIMPSON TYPE INEQUALITY VIA DIFFERENTIABLE FUNCTIONS USING (s,m)-CONVEX FUNCTIONS

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Abstract. We derive a differentiable mapping integral identity, which is involved with two parameters. By using this result, we establish new inequalities of Simpson type based on (s,m)-convexity for differentiable mappings. This contributes to new better estimates than the earlier results. Finally, some applications to special means of positive real numbers have also been presented.

Keywords: Simpson's inequality; Höder inequality; (s,m)-convex function.

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1. Introduction

Throughout this paper, we consider a real number interval $I \subseteq \mathbb{R}$ and I_0 is the interior of I. We also use the notation: $\mathbb{R} = (-\infty, \infty), \mathbb{R}_0 = [0, \infty)$. We first recite some definitions concerning various kinds of generalized convex functions.

Definition 1.1 ([6]) Let $s \in (0,1]$ be a real number. A function $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ is said to be s-convex(in the second sense)if

$$f\big(\lambda x + (1-\lambda)y\big) \leq \lambda^s f(x) + (1-\lambda)^s f(y)$$

holds for all $x, y \in I$ and $\lambda \in [0,1]$.

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Definition 1.2 ([15]) For $f : [0, b] \to \mathbb{R}$ and $m \in (0, 1]$, if

$$f\big(\lambda x + m(1-\lambda)y\big) \le \lambda f(x) + m(1-\lambda)f(y)$$

is valid for all $x, y \in [0, b]$ and $\lambda \in [0, 1]$, then we say that $f(x)$ is m-convex function on $[0, b]$.

Definition 1.3 ([11]) Let $(s, m) \in (0, 1]^2$ be a real number. A function $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ is said to be (s, m)-convex (in the second sense) if

$$f\big(\lambda x + m(1-\lambda)y\big) \le \lambda^s f(x) + m(1-\lambda)^s f(y)$$

holds for all $x, y \in I$ and $\lambda \in [0, 1]$.

It can be easily seen that $(s, m) \in \big\{(1, 1), (s, 1), (1, m)\big\}$. One receives the following classes of functions respectively: convex, s-convex and m-convex.

The following inequality is very remarkable and well known in the literature as Simpson type inequality, which plays important roles in analysis. Particularly, it is well applied in numerical integration.

Theorem 1.1 ([3]) *Let $f : [a, b] \to \mathbb{R}$ be a four times continuously differentiable mapping on (a, b) and $||f^{(4)}||_\infty = \sup\limits_{x\in(a,b)} |f^{(4)}(x)| < \infty$. Then the following inequality holds:*

$$\left|\frac{1}{3}\Big[\frac{f(a)+f(b)}{2} + 2f\big(\frac{a+b}{2}\big)\Big] - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{1}{2880}||f^{(4)}||_\infty (b-a)^4. \tag{1.1}$$

In recent decades, a lot of inequalities of Simpson type and Hadamard type for various kinds of convex functions have been established and developed by many scholars, some of them may be reformulated as follows.

Theorem 1.2 ([4]) *Suppose that $f : \mathbb{R}_0 \to \mathbb{R}_0$ is an s-convex function in the second sense, where $s \in (0, 1]$ and let $a, b \in \mathbb{R}_0, a < b$. If $f \in L^1[a, b]$, then*

$$2^{s-1} f\big(\frac{a+b}{2}\big) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{s+1}. \tag{1.2}$$

Theorem 1.3 ([5]) *Let $f : [a, b] \to \mathbb{R}$ be an L-Lipschitzian mapping on $[a, b]$. Then*

$$\left|\int_a^b f(x)dx - \frac{b-a}{3}\Big[\frac{f(a)+f(b)}{2} + 2f\big(\frac{a+b}{2}\big)\Big]\right| \le \frac{5}{36}L(b-a)^2. \tag{1.3}$$

Theorem 1.4 ([7]) *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° and $a, b \in I$ with $a < b$. If $|f'(x)|$ is convex on $[a, b]$, then*

$$\left|f\big(\frac{a+b}{2}\big) - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{(b-a)(|f'(a)| + |f'(b)|)}{8}. \tag{1.4}$$

For some generalizations of (1.4), please refer to [8], [10], [12].

Theorem 1.5 *([13]) Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable on I°, where $a, b \in I$ such that $a < b$. If the mapping $|f'|^{p/(p-1)}$ is $(1, m)$-convex on $[a, b]$, for $m \in (0, 1]$ with $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\begin{aligned}
&\left|\frac{1}{6}\Big[f(ma) + 4f\big(\frac{b+ma}{2}\big) + f(b)\Big] - \frac{1}{b-ma}\int_{ma}^{b} f(x)dx\right| \\
&\leq (b-ma)\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{1}{q}} \\
&\times \left[\Big(|f'(\frac{b+ma}{2})|^q + |f'(ma)|^q\Big)^{\frac{1}{q}} + \Big(|f'(\frac{b+ma}{2})|^q + |f'(b)|^q\Big)^{\frac{1}{q}}\right].
\end{aligned} \tag{1.5}$$

For recent results and generalizations involving Simpson type inequalities and Hadamard's inequalities, we refer to [1], [2], [9], [14].

In the present paper, we explore new inequalities of Simpsons type based on (s, m)-convexity for differentiable mappings. This contributes to new better estimates than the earlier results. Finally, we apply these inequalities to deduce some inequalities of special means.

2. Main results

Before proceeding toward our main theorems regarding of Simpson type inequality using (s, m)-convex function, we need the following integral identity, which will be used in the sequel.

Lemma 2.1 *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable on I°, where $a, b \in I$ such that $a < b$. If $f' \in L[a, b]$ and $k, t \in \mathbb{R}$, then the following equality holds:*

$$\begin{aligned}
&tf(ma) + (1-k)f(b) + (k-t)f\big(\frac{b+ma}{2}\big) - \frac{1}{b-ma}\int_{ma}^{b} f(x)dx \\
&= (b-ma)\left[\int_0^{\frac{1}{2}} (\lambda - t)f'\big(\lambda b + m(1-\lambda)a\big)d\lambda + \int_{\frac{1}{2}}^{1} (\lambda - k)f'\big(\lambda b + m(1-\lambda)a\big)d\lambda\right]
\end{aligned}$$

for each $x \in [a, b]$.

Proof. We note that

$$J = \int_0^{\frac{1}{2}} (\lambda - t)f'\big(\lambda b + m(1-\lambda)a\big)d\lambda + \int_{\frac{1}{2}}^{1} (\lambda - k)f'\big(\lambda b + m(1-\lambda)a\big)d\lambda.$$

Integrating by parts, it yields that

$$\begin{aligned}J &= \frac{1}{b-ma}\left[(\lambda-t)f\big(\lambda b+m(1-\lambda)a\big)\Big|_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} f\big(\lambda b+m(1-\lambda)a\big)d\lambda\right]\\ &+ \frac{1}{b-ma}\left[(\lambda-k)f\big(\lambda b+m(1-\lambda)a\big)\Big|_{\frac{1}{2}}^1 - \int_{\frac{1}{2}}^1 f\big(\lambda b+m(1-\lambda)a\big)d\lambda\right]\\ &= \frac{1}{b-ma}\left[tf(ma)+(1-k)f(b)+(k-t)f(\frac{b+ma}{2}) - \int_0^1 f\big(\lambda b+m(1-\lambda)a\big)d\lambda\right]\end{aligned}$$

Let we substitute $x=\lambda b+m(1-\lambda)a$ and $dx=(b-ma)d\lambda$, which gives

$$\int_0^1 f\big(\lambda b+m(1-\lambda)a\big)d\lambda = \frac{1}{b-ma}\int_{ma}^b f(x)dx.$$

Therefore, we can show that

$$J = \frac{1}{b-ma}\left[tf(ma)+(1-k)f(b)+(k-t)f(\frac{b+ma}{2}) - \frac{1}{b-ma}\int_{ma}^b f(x)dx\right],$$

which is required.

Remark 2.1. Applying Lemma 2.1 for $t=\frac{1}{6}$ and $k=\frac{5}{6}$, we obtain Lemma 2.1 in [13].

In what follows, we establish another refinement of the Simpson's inequality for (s,m)-convex functions in the second sense.

Theorem 2.1 *Let f be defined as in Lemma 2.1. If the mapping $|f'|$ is (s,m)-convex on $[a,b]$, for $(s,m)\in(0,1]^2$ and $0\le k,t\le 1$, then the following inequality holds:*

$$\begin{aligned}(2.1)\qquad &\left|tf(ma)+(1-k)f(b)+(k-t)f\left(\frac{b+ma}{2}\right) - \frac{1}{b-ma}\int_{ma}^b f(x)dx\right|\\ &\le (b-ma)\big[v_1|f'(b)|+mv_2|f'(a)|\big],\end{aligned}$$

where

$$(2.2)\qquad v_1=\frac{2t^{s+2}+2k^{s+2}+\big[2(s+1)-2(s+2)(k+t)\big]\frac{1}{2^{s+2}}+(s+1-ks-2k)}{(s+1)(s+2)}$$

$$(2.3)\qquad v_2=\frac{2(1-t)^{s+2}+2(1-k)^{s+2}+\big[2(k+t)(s+2)-2(s+3)\big]\frac{1}{2^{s+2}}+(ts+2t-1)}{(s+1)(s+2)}.$$

Proof. From Lemma 2.1 and using the (s,m)-convexity of $|f'|$, we have

$$\begin{aligned}
&\left|tf(ma)+(1-k)f(b)+(k-t)f\big(\frac{b+ma}{2}\big)-\frac{1}{b-ma}\int_{ma}^{b}f(x)dx\right| \\
&\leq (b-ma)\bigg[\int_0^{\frac{1}{2}}|\lambda-t||f'\big(\lambda b+m(1-\lambda)a\big)|d\lambda+\int_{\frac{1}{2}}^{1}|\lambda-k||f'\big(\lambda b+m(1-\lambda)a\big)|d\lambda\bigg] \\
&\leq (b-ma)\bigg[\int_0^{\frac{1}{2}}|\lambda-t|\Big(\lambda^s|f'(b)|+m(1-\lambda)^s|f'(a)|\Big)d\lambda\bigg] \\
&+(b-ma)\bigg[\int_{\frac{1}{2}}^{1}|\lambda-k|\Big(\lambda^s|f'(b)|+m(1-\lambda)^s|f'(a)|\Big)d\lambda\bigg] \\
&=(b-ma)\bigg[\int_0^{\frac{1}{2}}|\lambda-t|\lambda^s d\lambda+\int_{\frac{1}{2}}^{1}|\lambda-k|\lambda^s d\lambda\bigg]\big|f'(b)\big| \\
&+(b-ma)\bigg[\int_0^{\frac{1}{2}}m|\lambda-t|(1-\lambda)^s d\lambda+\int_{\frac{1}{2}}^{1}m|\lambda-k|(1-\lambda)^s d\lambda\bigg]\big|f'(a)\big|
\end{aligned}$$

By simple calculations, we have

$$\begin{aligned}
&\int_0^{\frac{1}{2}}|\lambda-t|\lambda^s d\lambda+\int_{\frac{1}{2}}^{1}|\lambda-k|\lambda^s d\lambda \\
&=\frac{2t^{s+2}+2k^{s+2}+\big[2(s+1)-2(s+2)(k+t)\big]\frac{1}{2^{s+2}}+(s+1-ks-2k)}{(s+1)(s+2)}
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^{\frac{1}{2}}|\lambda-t|(1-\lambda)^s d\lambda+\int_{\frac{1}{2}}^{1}|\lambda-k|(1-\lambda)^s d\lambda \\
&=\frac{2(1-t)^{s+2}+2(1-k)^{s+2}+\big[2(k+t)(s+2)-2(s+3)\big]\frac{1}{2^{s+2}}+(ts+2t-1)}{(s+1)(s+2)}.
\end{aligned}$$

This proves as required.

Corollary 2.1 *Let f be defined as in Theorem 2.1. If the mapping $|f'|$ is (s,m)-convex on $[a,b]$, for $(s,m)\in(0,1]^2$, $t=\frac{1}{6}$ and $k=\frac{5}{6}$. Then we have the following inequality:*

$$\begin{aligned}
&\left|\frac{1}{6}\Big[f(ma)+f(b)+4f\big(\frac{b+ma}{2}\big)\Big]-\frac{1}{b-ma}\int_{ma}^{b}f(x)dx\right| \\
&\leq (b-ma)\big[w_1|f'(b)|+mw_2|f'(a)|\big],
\end{aligned}\tag{2.4}$$

where

$$w_1=w_2=\frac{6^{-s}-9(2)^{-s}+(5)^{s+2}(6)^{-s}+3s-12}{18(s+1)(s+2)}.\tag{2.5}$$

If $s = 1$, we can get:

$$\begin{aligned}(2.6)\qquad &\left|\frac{1}{6}\Big[f(ma)+4f\Big(\frac{b+ma}{2}\Big)+f(b)\Big]-\frac{1}{b-ma}\int_{ma}^{b}f(x)dx\right| \\ &\leq \frac{5(b-ma)}{72}\big[m|f'(a)|+|f'(b)|\big].\end{aligned}$$

The upper bound of the midpoint inequality for the first derivative is developed as follows:

Corollary 2.2 *By putting $f(ma) = f\Big(\dfrac{ma+b}{2}\Big) = f(b)$ in inequality* (2.1)*, we have:*

$$(2.7)\qquad \left|f\big(\frac{b+ma}{2}\big)-\frac{1}{b-ma}\int_{ma}^{b}f(x)dx\right| \leq (b-ma)\big[v_1|f'(b)|+mv_2|f'(a)|\big],$$

where v_1 and v_2 are defined in Theorem 2.1.

Corollary 2.3 *Putting $s = 1$, $m = 1$, $t = \dfrac{1}{6}$ and $k = \dfrac{5}{6}$ in the above inequality* (2.7)*, it yields that*

$$(2.8)\qquad \left|f\big(\frac{b+a}{2}\big)-\frac{1}{b-a}\int_{a}^{b}f(x)dx\right| \leq \frac{5(b-a)}{72}\big[|f'(b)|+|f'(a)|\big].$$

Remark 2.2. It is noted that the above midpoint inequality (2.8) is better than the inequality (1.4) presented by Kiramic in [7].

Theorem 2.2 *Let f be defined as in Theorem 2.1 with $\dfrac{1}{p}+\dfrac{1}{q}=1$. If the mapping $|f'|^{p/(p-1)}$ is (s,m)-convex on $[a,b]$, for $(s,m)\in(0,1]^2$ and $p>1$. Then the following inequality holds:*

$$\begin{aligned}&\left|tf(ma)+(1-k)f(b)+(k-t)f\big(\frac{b+ma}{2}\big)-\frac{1}{b-ma}\int_{ma}^{b}f(x)dx\right| \\ (2.9)\quad &\leq (b-ma)\Big(\frac{1}{p+1}\Big)^{\frac{1}{p}}\Big(\frac{1}{s+1}\Big)^{\frac{1}{q}}\Bigg\{\Big[t^{p+1}+(\frac{1}{2}-t)^{p+1}\Big]^{\frac{1}{p}}\Big[|f'(\frac{b+ma}{2})|^q+|f'(ma)|^q\Big]^{\frac{1}{q}} \\ &+\Big[(k-\frac{1}{2})^{p+1}+(1-k)^{p+1}\Big]^{\frac{1}{p}}\Big[|f'(\frac{b+ma}{2})|^q+|f'(b)|^q\Big]^{\frac{1}{q}}\Bigg\}.\end{aligned}$$

Proof. Using the the famous Höder integral inequality and by Lemma 2.1, we have

$$\Big|tf(ma)+(1-k)f(b)+(k-t)f\big(\frac{b+ma}{2}\big)-\frac{1}{b-ma}\int_{ma}^{b}f(x)dx\Big|$$
$$\leq (b-ma)\Big[\int_0^{\frac{1}{2}}|\lambda-t||f'\big(\lambda b+m(1-\lambda)a\big)|d\lambda+\int_{\frac{1}{2}}^{1}|\lambda-k||f'\big(\lambda b+m(1-\lambda)a\big)|d\lambda\Big]$$
$$\leq (b-ma)\Bigg\{\Big(\int_0^{\frac{1}{2}}|\lambda-t|^p d\lambda\Big)^{\frac{1}{p}}\Big[\int_0^{\frac{1}{2}}|f'\big(\lambda b+m(1-\lambda)a\big)|^q d\lambda\Big]^{\frac{1}{q}}$$
$$+\Big(\int_{\frac{1}{2}}^{1}|\lambda-k|^p d\lambda\Big)^{\frac{1}{p}}\Big[\int_{\frac{1}{2}}^{1}f'\big(\lambda b+m(1-\lambda)a\big)|^q d\lambda\Big]^{\frac{1}{q}}\Bigg\}.$$

By simple calculations, we acquire

$$\int_0^{\frac{1}{2}}|\lambda-t|^p d\lambda=\frac{t^{p+1}+(\frac{1}{2}-t)^{p+1}}{p+1} \tag{2.10}$$

and

$$\int_{\frac{1}{2}}^{1}|\lambda-k|^p d\lambda=\frac{(k-\frac{1}{2})^{p+1}+(1-k)^{p+1}}{p+1}. \tag{2.11}$$

Also the (s,m)-convexity of $|f'|^{p/(p-1)}$ implies that

$$\int_0^{\frac{1}{2}}|f'\big(\lambda b+m(1-\lambda)a\big)|^q d\lambda\leq\frac{|f'(\frac{b+ma}{2})|^q+|f'(ma)|^q}{s+1} \tag{2.12}$$

and

$$\int_{\frac{1}{2}}^{1}|f'\big(\lambda b+m(1-\lambda)a\big)|^q d\lambda\leq\frac{|f'(\frac{b+ma}{2})|^q+|f'(b)|^q}{s+1}. \tag{2.13}$$

Therefore, by combining (2.10), (2.11), (2.12) and (2.13), these lead to the desired result. The statement in Theorem 2.2 is proved.

Corollary 2.4 *Let f be defined as in Theorem 2.2. If the mapping $|f'|^{p/(p-1)}$ is (s,m)-convex on $[a,b]$, for $(s,m)\in(0,1]^2$ with $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$. Then we have the following inequalities:*

(1) *when $s=1$, we have*

$$\Big|tf(ma)+(1-k)f(b)+(k-t)f\big(\frac{b+ma}{2}\big)-\frac{1}{b-ma}\int_{ma}^{b}f(x)dx\Big|$$
$$\leq (b-ma)\Big(\frac{1}{p+1}\Big)^{\frac{1}{p}}\Big(\frac{1}{2}\Big)^{\frac{1}{q}}\Bigg\{\Big[t^{p+1}+(\frac{1}{2}-t)^{p+1}\Big]^{\frac{1}{p}}\Big[|f'(\frac{b+ma}{2})|^q+|f'(ma)|^q\Big]^{\frac{1}{q}} \tag{2.14}$$
$$+\Big[(k-\frac{1}{2})^{p+1}+(1-k)^{p+1}\Big]^{\frac{1}{p}}\Big[|f'(\frac{b+ma}{2})|^q+|f'(b)|^q\Big]^{\frac{1}{q}}\Bigg\};$$

(2) *when* $t=\frac{1}{6}$, $k=\frac{5}{6}$ *and* $s=1$, *we can deduce*

$$\begin{aligned}&\left|\frac{1}{6}\Big[f(ma)+4f\big(\frac{b+ma}{2}\big)+f(b)\Big]-\frac{1}{b-ma}\int_{ma}^{b}f(x)dx\right|\\&\leq(b-ma)\left[\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right]^{\frac{1}{p}}\Big(\frac{1}{2}\Big)^{\frac{1}{q}}\\&\times\left[\left(|f'(\frac{b+ma}{2})|^q+|f'(ma)|^q\right)^{\frac{1}{q}}+\left(|f'(\frac{b+ma}{2})|^q+|f'(b)|^q\right)^{\frac{1}{q}}\right],\end{aligned}\tag{2.15}$$

which is the same as inequality (1.5) *established by Qaisar et al. in* [13].

Corollary 2.5 *By putting* $|f'(ma)|=|f'(b)|=0$, $t=\frac{1}{6}$ *and* $k=\frac{5}{6}$ *in Theorem* 2.2, *we have*

$$\begin{aligned}&\left|\frac{1}{6}\Big[f(ma)+4f\big(\frac{b+ma}{2}\big)+f(b)\Big]-\frac{1}{b-ma}\int_{ma}^{b}f(x)dx\right|\\&\leq 2\frac{(b-ma)}{(s+1)^{\frac{1}{q}}}\left[\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right]^{\frac{1}{p}}\Big|f'(\frac{b+ma}{2})\Big|.\end{aligned}\tag{2.16}$$

In the following corollary, we have the midpoint inequality for powers in terms of the first derivative.

Corollary 2.6 *By substituting* $f(ma)=f(b)=f(\frac{ma+b}{2})$, $t=\frac{1}{6}$ *and* $k=\frac{5}{6}$ *in Theorem* 2.2, *we have*

$$\begin{aligned}&\left|f\big(\frac{b+ma}{2}\big)-\frac{1}{b-ma}\int_{ma}^{b}f(x)dx\right|\\&\leq(b-ma)\left[\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right]^{\frac{1}{p}}\Big(\frac{1}{s+1}\Big)^{\frac{1}{q}}\\&\times\left[\left(|f'(\frac{b+ma}{2})|^q+|f'(ma)|^q\right)^{\frac{1}{q}}+\left(|f'(\frac{b+ma}{2})|^q+|f'(b)|^q\right)^{\frac{1}{q}}\right].\end{aligned}\tag{2.17}$$

In the following theorem, we obtain another form of Simpson type inequality for powers in term of the first derivative.

Theorem 2.3 *Let* f *be defined as in Theorem* 2.2. *If the mapping* $|f'|^q$ *is* (s,m)*-convex on* $[a,b]$, *for* $(s,m)\in(0,1]^2$ *and* $q\geq 1$. *Then we have the following inequality:*

$$\begin{aligned}&\left|tf(ma)+(1-k)f(b)+(k-t)f\big(\frac{b+ma}{2}\big)-\frac{1}{b-ma}\int_{ma}^{b}f(x)dx\right|\\&\leq(b-ma)\Bigg\{\left(t^2-\frac{1}{2}t+\frac{1}{8}\right)^{1-\frac{1}{q}}\left[u_1|f'(b)|^q+mu_2|f'(a)|^q\right]^{\frac{1}{q}}\\&+\left(k^2-\frac{3}{2}k+\frac{5}{8}\right)^{1-\frac{1}{q}}\left[u_3|f'(b)|^q+mu_4|f'(a)|^q\right]^{\frac{1}{q}}\Bigg\},\end{aligned}\tag{2.18}$$

where

$$u_1 = \frac{2t^{s+2} + (s+1-2ts-4t)\frac{1}{2^{s+2}}}{(s+1)(s+2)},$$
$$u_2 = \frac{t(s+2) - 1 + 2(1-t)^{s+2} + (2ts+4t-s-3)\frac{1}{2^{s+2}}}{(s+1)(s+2)},$$
$$u_3 = \frac{2k^{s+2} + (s+1-2ks-4k)\frac{1}{2^{s+2}} + (s+1-ks-2k)}{(s+1)(s+2)}$$

and

$$u_4 = \frac{2(1-k)^{s+2} + (2ks+4k-s-3)\frac{1}{2^{s+2}}}{(s+1)(s+2)}.$$

Proof. From Lemma 2.1, and using power-mean inequality, it follows that

$$\begin{aligned}
&\Big|tf(ma) + (1-k)f(b) + (k-t)f\big(\frac{b+ma}{2}\big) - \frac{1}{b-ma}\int_{ma}^{b} f(x)dx\Big| \\
&\leq (b-ma)\Big[\int_0^{\frac{1}{2}} |\lambda - t||f'\big(\lambda b + m(1-\lambda)a\big)|d\lambda + \int_{\frac{1}{2}}^{1} |\lambda - k||f'\big(\lambda b + m(1-\lambda)a\big)|d\lambda\Big] \\
&\leq (b-ma)\Big[\Big(\int_0^{\frac{1}{2}} |\lambda - t|d\lambda\Big)^{1-\frac{1}{q}}\Big(\int_0^{\frac{1}{2}} |\lambda - t||f'\big(\lambda b + m(1-\lambda)a\big)|^q d\lambda\Big)^{\frac{1}{q}} \\
&+ \Big(\int_{\frac{1}{2}}^{1} |\lambda - k|d\lambda\Big)^{1-\frac{1}{q}}\Big(\int_{\frac{1}{2}}^{1} |\lambda - k||f'\big(\lambda b + m(1-\lambda)a\big)|^q d\lambda\Big)^{\frac{1}{q}}\Big].
\end{aligned}$$

By simple calculations, we can get

$$\int_0^{\frac{1}{2}} |\lambda - t|d\lambda = t^2 - \frac{1}{2}t + \frac{1}{8} \tag{2.19}$$

and

$$\int_{\frac{1}{2}}^{1} |\lambda - k|d\lambda = k^2 - \frac{3}{2}k + \frac{5}{8}. \tag{2.20}$$

The (s,m)-convexity of $|f'|^q$ implies that

$$\int_0^{\frac{1}{2}} |\lambda - t||f'\big(\lambda b + m(1-\lambda)a\big)|^q d\lambda \leq u_1|f'(b)|^q + mu_2|f'(a)|^q \tag{2.21}$$

and

$$\int_{\frac{1}{2}}^{1} |\lambda - k||f'\big(\lambda b + m(1-\lambda)a\big)|^q d\lambda \leq u_3|f'(b)|^q + mu_4|f'(a)|^q. \tag{2.22}$$

Thus, our desired result can be obtained by combining (2.19), (2.20), (2.21) and (2.22), the proof is completed.

Corollary 2.7 *Let f be defined as in Theorem 2.3, if $s = 1$, $t = \frac{1}{6}$ and $k = \frac{5}{6}$, the inequality holds for m-convex functions:*

$$\begin{aligned}&\left|\frac{1}{6}\left[f(ma) + 4f\big(\frac{b+ma}{2}\big) + f(b)\right] - \frac{1}{b-ma}\int_{ma}^{b} f(x)dx\right| \\ &\leq (b-ma)\Big(\frac{5}{72}\Big)^{1-\frac{1}{q}}\left[\Big(\frac{29}{1296}|f'(b)|^q + \frac{61}{1296}m|f'(a)|^q\Big)^{\frac{1}{q}}\right. \\ &\left. + \Big(\frac{61}{1296}|f'(b)|^q + \frac{29}{1296}m|f'(a)|^q\Big)^{\frac{1}{q}}\right].\end{aligned} \tag{2.23}$$

In particular, if $s = m = 1$, the inequality holds for convex function. If $|f'(x)| \leq L$, $\forall x \in I$, then we have

$$\left|\frac{1}{6}\left[f(a) + 4f\big(\frac{b+a}{2}\big) + f(b)\right] - \frac{1}{b-a}\int_{a}^{b} f(x)dx\right| \leq \frac{5(b-a)}{72}L. \tag{2.24}$$

Remark 2.3. It is observed that inequality (2.24) with $m = 1$ gives an improvement for inequality (1.3).

3. Applications to special means

For positive numbers, $\beta > \alpha > 0$ and $n \in \mathbb{Z} \setminus \{0, -1\}$, define $A(\alpha,\ \beta) = \frac{\alpha+\beta}{2}$ and $L_n(\alpha,\ \beta) = \frac{\beta^{n+1} - \alpha^{n+1}}{(n+1)(\beta-\alpha)}$. These quantities are respectively called the arithmetic and generalized logarithmic means of two positive number α and β.

Now, considering $f : [a, b] \to \mathbb{R}$, $f(x) = x^s$, $s \in (0, 1]$. Then we have the following means:

$$\frac{1}{b-a}\int_{a}^{b} f(x)dx = L_s^s(a,b), \quad \frac{f(a)+f(b)}{2} = A(a^s, b^s) \ \text{ and } \ f\big(\frac{a+b}{2}\big) = A^s(a,b).$$

Now, using the results of the Section 2, some new inequalities are derived for the above means.

Proposition 3.1 *Let $f : [a, b] \to R$, $0 < a < b$ and $s \in (0, 1]$. Then we have*

$$\begin{aligned}&\left|\frac{1}{3}A(a^s, b^s) + \frac{2}{3}A^s(a,b) - L_s^s(a,b)\right| \\ &\leq s(b-a)\frac{6^{-s} - 9(2)^{-s} + (5)^{s+2}(6)^{-s} + 3s - 12}{18(s+1)(s+2)}\left[|a|^{s-1} + |b|^{s-1}\right].\end{aligned} \tag{3.1}$$

Proof. The assertion follows by taking $m = 1$ and from inequality (2.4) applies to mapping $f(x) = x^s$, $x \in [a, b]$ with $s \in (0, 1]$. Moreover, by setting $s = 1$ in inequality (3.1), we obtain

$$\big|A(a,b) - L(a,b)\big| \leq \frac{5(b-a)}{36}. \tag{3.2}$$

Proposition 3.2 *Let $f : [a,b] \to \mathbb{R}$, $0 < a < b$ and $s \in (0,1]$. Then we can conclude*

$$\begin{aligned}
&\left|\frac{1}{3}A(a^s,b^s)+\frac{2}{3}A^s(a,b)-L_s^s(a,b)\right| \\
&\leq s(b-a)\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}\frac{s}{(s+1)^{1/q}} \\
&\quad\times\left[\left(|a^{s-1}|^q+|A^{s-1}(a,b)|^q\right)^{\frac{1}{q}}+\left(|A^{s-1}(a,b)|^q+|b^{s-1}|^q\right)^{\frac{1}{q}}\right].
\end{aligned} \tag{3.3}$$

Proof. The assertion follows by taking $m = 1$ and from inequality (2.17) applies to mapping $f(x) = x^s$, $x \in [a,b]$ with $s \in (0,1]$. Moreover, by setting $s = 1$ in inequality(3.3), we derive

$$\left|A(a,b)-L(a,b)\right| \leq 2(b-a)\left[\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right]^{\frac{1}{p}}. \tag{3.4}$$

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T_1 CONCEPTS IN FUZZY BITOPOLOGICAL SPACES

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Abstract. In this paper, we introduce some notions of fuzzy pairwise-T_1 bitopological spaces and find relations among them. We also study some other properties of these concepts.

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1. Introduction

The notion of bitopological spaces was initially introduced by Kelly [7] in 1963. Later in 1991, fuzzy pairwise-T_i (in short $FPTi$, $i = 0, 1, 2$) bitopological spaces have been introduced earlier by Kandil and El-Shafee [5]. Since then several other authors are investigating such concepts. Fuzzy pairwise-T_1 separation axioms have also been introduced by Abu Sufiya et al. [1] and Nouh [9]. Here we introduce a definition of fuzzy pairwise-T_1 bitopological space and obtain its several properties. Moreover, it will be seen that these two concepts are good extensions in the sense of Lowen [8].

2. Priliminaries

For the purpose of the main results, we need to introduce some definitions and notations.

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Definition 2.1 [13] A fuzzy set μ in a set X is a function from X into the closed unit interval $I = [0, 1]$. For every $x \in X$, $\mu(x) \in I$ is called the grade of membership of x. Throughout this paper, I^X will denote the set of all fuzzy sets from X into the closed unit interval I. A member of I^X may also be a called fuzzy subset of X.

Definition 2.2 [3] Let f be a mapping from a set X into a set Y and u be a fuzzy set in X. Then the image of u, written as $f(u)$, is a fuzzy set in Y whose membership function is given by

$$f(u)(y) = \begin{cases} \sup\{u(x)\}, & \text{if } f^{-1}[\{y\}] \neq \Phi,\ x \in X; \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.3 [4] Let f be a mapping from a set X into a set Y and v be a fuzzy set in Y. Then the inverse of v written as $f^{-1}(v)$ is a fuzzy set in X which is defined by $f^{-1}(v)(x) = v(f(x))$, for $x \in X$.

Definition 2.4 [12] A fuzzy set μ in X is called a fuzzy singleton iff $\mu(x) = r$,$(0 < r \leq 1)$ for a certain $x \in X$ and $\mu(y) = 0$ for all points y of X except x. The fuzzy singleton is denoted by x_r and x is its support. We call x_r is a fuzzy point if $0 < r < 1$. The class of all fuzzy singletons in X will be denoted by $S(X)$.

Definition 2.5 [3] A fuzzy topology t on X is a collection of members of I^X which is closed under arbitrary suprema and finite infima and which contains constant fuzzy sets 1 and 0. The pair (X, t) is called a fuzzy topological space (fts, in short) and members of t are called t-open (or simply open) fuzzy sets. A fuzzy set μ is called a t-closed (or simply closed) fuzzy set if $1 - \mu \in t$.

Definition 2.6 [10] Let λ be a fuzzy set in an fts (X, t). Then the closure of λ is denoted by $t - cl(\lambda)$ and defined by $t - cl(\lambda) = \cap\{\mu : \lambda \subseteq \mu \text{ and } \mu^c \in t\}$.

Definition 2.7 [3] Let (X, t) and (Y, s) be two fuzzy topological spaces. A mapping $f : (X, t) \to (Y, s)$ is called an fuzzy continuous iff for every $v \in s$, $f^{-1}(v) \in t$.

Definition 2.8 [11] Let f be a real valued function on a topological space (X, T). If $\{x : f(x) > \alpha\}$ is open for every real $\alpha \in I_1$, then f is called lower semi continuous function.

Definition 2.9 [2] Let X be a non-empty set and T be a topology on X. Let $t = w(T)$ be the set of all lower semi continuous (lsc, in short) functions from (X, T) into I (with usual topology) such that $w(T) = \{\mu \in I^X : \mu^{-1}(\alpha, 1] \in T\}$ for each $\alpha \in I_1$. Then $w(T)$ is a fuzzy topology on X.

Let P be a property of topological spaces and FP be its fuzzy topology analogue. Then FP is called a good extension of P iff the statement (X, T) has P iff $(X, w(T))$ has FP holds good for every topological space (X, T).

Definition 2.10 [6] *A fuzzy singleton x_r is said to be quasi-coincident with a fuzzy set μ, denoted by $x_r q \mu$ iff $r + \mu(x) > 1$. If x_r is not quasi-coincident with μ, we write $x_r \bar{q} \mu$.*

Definition 2.11 [9] A fuzzy set u of (X,t) is called quasi-neighborhood (Q-nbd, in short) of x_r iff there exists $v \in t$ such that $x_r q v$ and $v \subset u$. If x_r is a fuzzy point or a fuzzy singletone, then $N(x_r,t) = \{\mu \in t : x_r \in \mu\}$ is the family of all fuzzy t-open neighborhoods (t-nbds, in short) of x_r and $N_Q(x_r,t) = \{\mu \in t : x_r q \mu\}$ is the family of all Q-neighborhoods (Q-nbd, in short) of x_r.

Definition 2.12 [6] A fuzzy bitopological space (fbts, in short) is a triple (X,s,t), where s and t are arbitrary fuzzy topologies on X.

Definition 2.13 [7] A bitopological space (X,S,T) is called pairwise-T_1 (in short, PT_1) if for all $x,y \in X$, $x \neq y$, there exist $U \in S, V \in T$ such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$.

3. The main results

Definition 3.14 An fbts (X,s,t) is called

(*a*) $FPT_1(i)$-space iff for every pair of fuzzy singletons x_p, y_r in X with $x \neq y$, there exist fuzzy sets $u \in s$, $v \in t$ such that $(x_p q u, y_r \cap u = 0)$ and $(y_r q v, x_p \cap v = 0)$.

(*b*) [9] $FPT_1(ii)$-space iff for every pair of fuzzy singletons x_p, y_r in X such that $x \neq y$, there exist fuzzy sets $u, v \in s \cup t$ such that $(x_p q u, y_r \bar{q} u)$ and $(y_r q v, x_p \bar{q} v)$.

(*c*) [1] $FPT_1(iii)$-space iff for any two distinct fuzzy points x_p, y_r in X, there exist fuzzy sets $u \in s$, $v \in t$ such that $(x_p \in u, y_r \subseteq u^c)$ and $(y_r \in v, x_p \subseteq v^c)$.

Theorem 3.1 *Let (X,s,t) be an fbts. Then we have the following implications:*

(i) $(a) \Rightarrow (b)$, *and*

(ii) $(a) \Rightarrow (c)$.

The arrows are not reversible in general.

Proof. (*i*) $(a) \Rightarrow (b)$ Let x_r, $y_s \in S(X)$ with $x \neq y$. Since (X,s,t) be $FPT_1(i)$-space, there exist fuzzy sets $\mu \in s$, $\eta \in t$ such that $(x_r q \mu, y_s \cap \mu = 0)$ and $(y_s q \eta, x_r \cap \eta = 0)$. Now, $y_s \cap \mu = 0$ implies $\mu(y) = 0$. Then $\mu(y) + s \leq 1$. So, $y_s \bar{q} \mu$. Similarly, $x_r \cap \eta = 0$ implies $x_r \bar{q} \eta$. Again since $\mu \in s$, $\eta \in t$, then $\mu, \eta \in s \cup t$. Hence (X,s,t) be $FPT_1(ii)$-space.

(ii) $(a) \Rightarrow (c)$ Let x_r, y_s be two distinct fuzzy points in X. Choose $r^*, s^* \in (0,1)$ such that $r^* < 1 - r$ and $s^* < 1 - s$. Since x_{r^*}, y_s are two distinct fuzzy points in X and (X,s,t) is $FPT_1(i)$-space, there exist fuzzy sets $u \in s, v \in t$ such that $(x_{r^*} q u,\ y_s \cap u = 0)$ and $(y_s q v, x_{r^*} \cap v = 0)$. Now, $x_{r^*} q u, y_s \cap u = 0$ implies that $u(x) + r^* > 1$ and $u(y) = 0$. For $u(x) + r^* > 1$ and $r^* < 1 - r$, we have $u(x) > r$. So, $x_r \in u$. For $u(y) = 0$, we have $u(y) + s \leq 1$. That is, $s \leq 1 - u(y)$. So, $y_s \subseteq u^c$. Again, since x_r, y_{s^*} are two distinct fuzzy points in X and (X,s,t) is $FPT_1(i)$, there exist fuzzy sets $\mu \in s$, $\lambda \in t$ such that $(y_{s^*} q \mu,\ x_r \cap \mu = 0)$ and $(x_r q \lambda, y_{s^*} \cap \lambda = 0)$. Similarly, $y_{s^*} q \mu, x_r \cap \mu = 0$ implies that $y_s \in \mu$, $x_r \subseteq \mu^c$. Hence, (X,s,t) is $FPT_1(iii)$-space. ∎

The following counter example shows that $(c) \not\Rightarrow (b)$ as well as $(c) \not\Rightarrow (a)$ in general.

Example. Let $X = I, t_1 = t_2 = \{0, \lambda : \lambda(x) > 0, \forall x \in X\}$. Let x_r and y_s be distinct fuzzy singletons in X and $\gamma = \min\{1-r, 1-s\}$. Now, if $\gamma \neq 0$, we define μ, η as follows

$$\begin{array}{llll} \mu(x) = 1 & \text{and} & \mu(y) = \frac{\gamma}{8}, & \text{if } y \neq x; \\ \eta(y) = 1 & \text{and} & \eta(x) = \frac{\gamma}{4}, & \text{if } x \neq y. \end{array}$$

Again, if $\gamma = 0$, we define μ_1, η_1 as follows

$$\begin{array}{llll} \mu_1(x) = 1 & \text{and} & \mu_1(y) = 0.1, & \text{if } y \neq x; \\ \eta_1(y) = 1 & \text{and} & \eta_1(x) = 0.3, & \text{if } x \neq y. \end{array}$$

Then $\mu, \eta, \mu_1, \eta_1 \in t_1 = t_2$. For any pair of distinct fuzzy points x_r, y_s in X, we see that ($x_r \in \mu$, $y_s \subseteq \mu^c$) and ($y_s \in \eta$, $x_r \subseteq \eta^c$). Therefore, (X, t_1, t_2) is $FPT_1(iii)$-space. But if we take x_1, $y_1 \in S(X)$, then we see that $(x_1 q \mu_1, y_1 q \mu_1)$ and ($x_1 q \eta_1$, $y_1 q \eta_1$). Hence (X, t_1, t_2) is not $FPT_1(ii)$-space. Since $(a) \Rightarrow (b)$, then we see that $(c) \not\Rightarrow (a)$. ∎

The following counter example will show that (b) does not imply (c) as well as (b) does not imply (a) in general.

Example. Let $X = \{x, y, z\}$ and let $s = \{0, 1, x_1, y_1, x_1 \cup y_1\}$ and $t = \{0, 1, z_1\}$. Then (X, s, t) is $FPT_1(ii)$-space but not $FPT_1(iii)$-space, since for distinct fuzzy points x_r, y_s in X, we have $x_1 \in s$ such that $x_r \in x_1$, $y_s \subseteq x_1^c$ and there does not $v \in t$ such that $y_s \in v$, $x_r \subseteq v^c$. Since $(a) \Rightarrow (c)$, then we see that $(b) \not\Rightarrow (a)$.

Theorem 3.2 *An fbts (X, s, t) is $FPT_1(j)$-space iff for all $x_r \in S(X)$ $x_r = cl(x_r)$, where $j = i, ii, iii$.*

Proof. Suppose (X, s, t) is $FPT_1(i)$-space and suppose that x_r, $y_s \in S(X)$ with $x \neq y$. Then there exists $\mu \in s$ such that $y_s q \mu$ and $x_r \cap \mu = 0$. Now $y_s q \mu$ implies that $\mu(y) + s > 1$, that is, $1 - \mu(y) < s$. So, $y_s \notin \mu^c$. Again, $x_r \cap \mu = 0$ implies that $\mu(x) = 0$, that is, $\mu(x) + r \leq 1 \quad \Rightarrow 1 - \mu(x) \geq r$. So $x_r \in \mu^c$. Therefore we see that $y_s \notin s - cl(x_r)$. Hence $s - cl(x_r) \subseteq x_r$. But $x_r \subseteq s - cl(x_r)$. Hence $x_r = s - cl(x_r)$. Similarly, we can show that y_s is t-closed.

Conversely, suppose that $x_r = s - cl(x_r)$ and $y_s = t - cl(y_s)$ for all x_r, $y_s \in S(X)$ with $x \neq y$. Then $y_s \notin s - cl(x_1)$ and $x_r \notin t - cl(y_1)$. So, there exist fuzzy sets $\mu \in s$, $\eta \in t$ such that ($y_s \notin \mu^c$, $x_1 \in \mu^c$) and ($x_r \notin \eta^c$, $y_1 \in \eta^c$). Now, $y_s \notin \mu^c$, $x_1 \in \mu^c$ implies that $1 - \mu(y) < s$ and $1 - \mu(x) \geq 1$. That is, $\mu(y) + s > 1$ and $\mu(x) = 0$. That is, $y_s q \mu$ and $\mu(x) = 0$. So, for any $x_r \in S(X)$, we have $x_r \cap \mu = 0$ as $\mu(x) = 0$. Similarly, $x_r \notin \eta^c$, $y_1 \in \eta^c$ implies that $x_r q \eta$, $y_s \cap \eta = 0$. Hence (X, s, t) is $FPT_1(i)$-space.

Next, suppose (X, s, t) is $FPT_1(iii)$ and x_r, y_s are two distinct fuzzy points in X. Since x_r, y_{1-s} are two distinct fuzzy points in X and (X, s, t) is $FPT_1(iii)$-space, there exists $u \in s$ such that $y_{1-s} \in u$ and $x_r \subseteq u^c$.

Now $y_{1-s} \in u$ implies that $u(y) > 1 - s$, that is, $1 - u(y) < s$. So, $y_s \notin u^c$. Again, $x_r \subseteq u^c$ implies that $r \leq 1 - u(x)$. So $x_r \in u^c$. Therefore we see that $y_s \notin s - cl(x_r)$. Hence $s - cl(x_r) \subseteq x_r$. But $x_r \subseteq s - cl(x_r)$. Hence $x_r = s - cl(x_r)$. Similarly, we can show that y_s is t-closed.

Conversely, suppose that $x_r = s - cl(x_r)$ and $y_s = t - cl(y_s)$ for all distinct fuzzy points x_r, y_s in X. Then $y_{1-s} \notin s - cl(x_r)$ and $x_{1-r} \notin t - cl(y_s)$. So, there exist fuzzy sets $u \in s$, $v \in t$ such that ($y_{1-s} \notin u^c$, $x_r \in u^c$) and ($x_{1-r} \notin v^c$, $y_s \in v^c$). Now, $y_{1-s} \notin u^c$, $x_r \in u^c$ implies that $1 - u(y) < 1 - s$ and $1 - u(x) > r$. That is, $u(y) > s$ and $x_r \subseteq u^c$. That is, $y_s \in u$ and $x_r \subseteq u^c$. Similarly, $x_{1-r} \notin v^c$, $y_s \in v^c$ implies that $x_r \in v$ and $y_s \subseteq v^c$. Hence (X, s, t) is $FPT_1(iii)$. For the proof of $j = ii$, see [9].

Theorem 3.3 *Let (X, s, t) be a fuzzy bitopological space, $A \subset X$ and*

$$s_A = \{u/A : u \in s\}, \quad t_A = \{v/A : v \in t\}.$$

Then

(a) *(X, s, t) is $FPT_1(i)$ $\Rightarrow$ (A, s_A, t_A) is $FPT_1(i)$.*

(b) *(X, s, t) is $FPT_1(ii)$ $\Rightarrow$ (A, s_A, t_A) is $FPT_1(ii)$.*

(c) *(X, s, t) is $FPT_1(iii)$ $\Rightarrow$ (A, s_A, t_A) is $FPT_1(iii)$.*

Proof. (a) Suppose (X, s, t) is $FPT_1(i)$. We have to show that (A, s_A, t_A) is $FPT_1(i)$. Let a_r, $b_p \in S(A)$ with $a \neq b$. Then a_r, $b_p \in S(X)$ with $a \neq b$. Since (X, s, t) is $FPT_1(i)$, there exist fuzzy sets $u \in s$, $v \in t$ such that ($a_r q u$, $b_p \cap u = 0$) and ($b_p q v$, $a_r \cap v = 0$). Now, $a_r q u$, $b_p \cap u = 0$ implies that $u(a) + r > 1$ and $(b_p \cap u)(x) = 0$ for all $x \in X$. But, $u/A \in s_A$ and $(u/A)(a) = u(a)$. Then $(u/A)(a) + r > 1$. So, $a_r q (u/A)$. Also, for all $x \in X$, we have $(b_p \cap (u/A))(x) = b_p(x) \cap (u/A)(x) = b_p(x) \cap u(x) = (b_p \cap u)(x) = 0$, since $(u/A)(b) = u(b)$. So $b_p \cap (u/A) = 0$. Similarly, $b_p q v$, $a_r \cap v = 0$ implies that $b_p q (v/A)$ and $a_r \cap (v/A) = 0$, where $v/A \in t_A$. Hence (A, s_A, t_A) is $FPT_1(i)$-space. Similarly, (b) and (c) can be proved.

Theorem 3.4 *Let (X, T_1, T_2) be a bitopological space. Then*

(a) *(X, T_1, T_2) is PT_1 $\Leftrightarrow$ $(X, w(T_1), w(T_2))$ is $FPT_1(i)$.*

(b) *(X, T_1, T_2) is PT_1 $\Leftrightarrow$ $(X, w(T_1), w(T_2))$ is $FPT_1(i)$.*

(c) *(X, T_1, T_2) is PT_1 $\Leftrightarrow$ $(X, w(T_1), w(T_2))$ is $FPT_1(iii)$.*

Proof. (a) Suppose that (X, T_1, T_2) is PT_1. We have to show that $(X, w(T_1), w(T_2))$ is $FPT_1(i)$-space. Let x_p, $y_r \in S(X)$ with $x \neq y$. Since (X, T_1, T_2) is PT_1, there exist $U \in T_1$, $V \in T_2$ such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$. This implies ($1_U \in N_Q(x_p, w(T_1))$, $(1_U \cap y_r = 0)$) and ($1_V \in N_Q(y_r, w(T_2))$, $(1_V \cap x_p = 0)$). Hence $(X, w(T_1), w(T_2))$ is $FPT_1(i)$-space.

Conversely, suppose that $(X, w(T_1), w(T_2))$ is $FPT_1(i)$-space. We have to show that (X, T_1, T_2) is PT_1. Let x, $y \in X$ such that $x \neq y$. Since $(X, w(T_1), w(T_2))$ is $FPT_1(i)$, then $(\exists \mu \in N_Q(x_1, w(T_1))$, $y_1 \cap \mu = 0)$ and $(\exists \eta \in N_Q(y_1, w(T_2))$, $x_1 \cap \eta = 0)$. Now, $\mu \in N_Q(x_1, w(T_1))$, $y_1 \cap \mu = 0$ implies that $\mu(x) + 1 > 1$ and $\mu(y) = 0$ that is, $\mu(x) > 0$ and $\mu(y) = 0$. Hence $x \in \mu^{-1}(0, 1] \in T_1$ and $y \notin \mu^{-1}(0, 1] \in T_1$. Similarly, we can show that $\eta \in N_Q(y_1, w(T_2)), x_1 \cap \eta = 0$ implies that $y \in \eta^{-1}(0, 1] \in T_2$ and $x \notin \eta^{-1}(0, 1] \in T_2$. Hence (X, T_1, T_2) is PT_1.

Proof of (c) is similar, and for the proof (b), see [9].

Theorem 3.5 *Product of any two $FPT_1(j)$-spaces is $FPT_1(j)$-space, where $j = i, ii, iii$.*

Proof. Suppose (X_1, s_1, t_1) and (X_2, s_2, t_2) are $FPT_1(i)$-spaces. We have to show that $(X_1 \times X_2, s_1 \times s_2, t_1 \times t_2)$ is $FPT_1(i)$-space. Let $(x, y)_p$, $(x_1,\ y_1)_q$ be two distinct fuzzy singletons in $X_1 \times X_2$. We can assume without loss of generality that $x \neq x_1$. Since $x_p, (x_1)_q$ are two distinct fuzzy singletons in X_1 and (X_1, s_1, t_1) is $FPT_1(i)$-space, then $(\exists \mu \in N_Q(x_p, s_1),\ ((x_1)_q \cap \mu = 0))$ and $(\exists \eta \in N_Q((x_1)_q, t_1),\ (x_p \cap \eta = 0))$. Now, $\mu \in N_Q(x_p, s_1)$, $((x_1)_q \cap \mu = 0)$ implies that $\mu(x) + p > 1$ and

$$((x_1)_q \cap \mu)(x) = 0 \text{ for all } x \in X. \tag{A}$$

Since $\mu \times X_2 \in s_1 \times s_2$ and $(\mu \times X_2)(x, y) = \min\{\mu(x), X_2(y)\} = \mu(x)$, we have $(\mu \times X_2)(x, y) + p = \mu(x) + p > 1$. So, $(x, y)_p q(\ \times X_2)$. We know that

$$(x_1, y_1)_q(x, y) = \begin{cases} q & \text{if } (x, y) = (x_1, y_1); \\ 0, & \text{otherwise.} \end{cases} \tag{B}$$

When $(x, y) = (x_1, y_1)$, we have $\mu(x) = 0$, by (A), and

$$\begin{aligned} ((x_1, y_1)_q \cap (\mu \times X_2))(x, y) &= \min\{(x_1, y_1)_q(x, y),\ (\mu \times X_2)(x, y)\} \\ &= \min\{q, \mu(x)\} = \min\{q, 0\} = 0. \end{aligned}$$

Again when $(x, y) \neq (x_1, y_1)$, we have $(x_1, y_1)_q(x, y) = 0$, by (B), and

$$\begin{aligned} ((x_1, y_1)_q \cap (\mu \times X_2))(x, y) &= \min\{(x_1, y_1)_q(x, y), (\mu \times X_2)(x, y)\} \\ &= \min\{0, \mu(x)\} = \min\{0, \mu(x)\} = 0. \end{aligned}$$

Hence, for all $(x, y) \in X_1 \times X_2$, we have $((x_1, y_1)_q \cap (\mu \times X_2))(x, y) = 0$.

Similarly, we can show that $\eta \in N_Q((x_1)_q, t_1)$, $(x_p \cap \eta = 0)$ implies that $(x_1, y_1)_q q(\eta \times X_2)$ and $(x, y)_p \cap (\eta \times X_2) = 0$, where $\eta \times X_2 \in t_1 \times t_2$. Hence $(X_1 \times X_2, s_1 \times s_2, t_1 \times t_2)$ is $FPT_1(i)$.

Other proofs are similar.

Theorem 3.6 *A bijective mapping from an fts (X, t) to an fts (Y, s) preserves the value of a fuzzy singleton (fuzzy point).*

Proof. Let c_r be a fuzzy singleton in X. So, there exist a point $a \in Y$ such that $f(c) = a$. Now $f(c_r)(a) = f(c_r)(f(c)) = \sup c_r(c) = c_r(c) = r$, since f is bijective. Hence a_r has same value as c_r.

Note. Preimage of any fuzzy singleton (fuzzy point) under bijective mapping preserves its value.

The ideas of the following two theorems are taken from [4].

Theorem 3.7 *Let (X, s, t) and (Y, s_1, t_1) be two fuzzy bitopological spaces and let $f : X \to Y$ be bijective and FP-open. Then*

(a) (X, s, t) *is* $FPT_1(i) \Rightarrow (Y, s_1, t_1)$ *is* $FPT_1(i)$.
(b) (X, s, t) *is* $FPT_1(ii) \Rightarrow (Y, s_1, t_1)$ *is* $FPT_1(ii)$.
(c) (X, s, t) *is* $FPT_1(iii) \Rightarrow (Y, s_1, t_1)$ *is* $FPT_1(ii)$.

Proof. (a) Suppose (X, s, t) is $FPT_1(i)$. We have to show that (Y, s_1, t_1) is $FPT_1(i)$. Let a_r, b_q be two distinct fuzzy singletons in Y. Since f is bijective, there exist distinct fuzzy singletons c_r, d_q in X such that $f(c) = a$, $f(d) = b$ and $c \neq d$. Again since (X, s, t) is $FPT_1(i)$, there exist fuzzy sets $\mu \in s$, $\eta \in t$ such that $(c_r q\mu,\ d_q \cap \mu = 0)$ and $(d_q q\eta,\ c_r \cap \eta = 0)$.

Now, $c_r q\mu,\ d_q \cap \mu = 0$ implies that $\mu(c) + r > 1$ and $(d_q \cap \mu)(x) = 0$ for all $x \in X$. But $f(\mu)(a) = f(\mu)(f(c)) = \sup \mu(c) = \mu(c)$, since f is bijective. So $f(\mu)(a) + r > 1$, since $\mu(c) + r > 1$. Hence $a_r q f(\mu)$. Also,

$$\begin{aligned}(b_q \cap f(\mu))(y) &= (f(d_q) \cap f(\mu))(f(x)) = f(d_q)(f(x)) \cap f(\mu)(f(x)) \\ &= (d_q \cap \mu)(x) = 0.\end{aligned}$$

Similarly, we can prove that $d_q q\eta, c_r \cap \eta = 0$ implies that $b_q q f(\eta)$ and $a_r \cap f(\eta) = 0$.

Since f is FP-open, then $f(\mu) \in s_1$, $f(\eta) \in t_1$. Hence (Y, s_1, t_1) is $FPT_1(i)$.

Similarly, (b) and (c) can be proved.

Theorem 3.8 *Let (X, s, t) and (Y, s_1, t_1) be two fuzzy bitopological spaces and $f : X \to Y$ be FP-continuous and bijective. Then*

(a) (Y, s_1, t_1) *is* $FPT_1(i) \Rightarrow (X, s, t)$ *is* $FPT_1(i)$.
(b) (Y, s_1, t_1) *is* $FPT_1(ii) \Rightarrow (X, s, t)$ *is* $FPT_1(ii)$.
(c) (Y, s_1, t_1) *is* $FPT_1(iii) \Rightarrow (X, s, t)$ *is* $FPT_1(iii)$.

Proof. We now prove (a) only. Suppose (Y, s_1, t_1) is $FPT_1(i)$. We have to show that (X, s, t) is $FPT_1(i)$. Let c_r, d_q be two distinct fuzzy singletons in X. Then there exist distinct fuzzy singletons a_r, b_q in Y such that $f(c) = a$, $f(d) = b$ and $a \neq b$, since f is one-one. Again since (Y, s_1, t_1) is $FPT_1(i)$, there exist fuzzy sets $\mu \in s_1$, $\eta \in t_1$ such that $(a_r q\mu,\ b_q \cap \mu = 0)$ and $(b_q q\eta,\ a_r \cap \eta = 0)$. Now, $a_r q\mu,\ b_q \cap \mu = 0$ implies that $\mu(a) + r > 1$ and $(b_q \cap \mu)(f(x)) = 0$ for all $x \in X$. Since $f^{-1}(\mu)(c) = \mu(f(c)) = \mu(a)$, then we have

$$f^{-1}(\mu)(c) + r > 1.$$

So, $c_r q f^{-1}(\mu)$. Also,

$$\begin{aligned}(d_q \cap f^{-1}(\mu))(x) &= (f^{-1}(b_q) \cap f^{-1}(\mu))(x) = f^{-1}(b_q)(x) \cap f^{-1}(\mu)(x) \\ &= (b_q \cap \mu)(f(x)) = 0.\end{aligned}$$

Similarly, we can show that $b_q q\eta, a_r \cap \eta = 0$ implies that $d_q q f^{-1}(\eta)$ and $c_r \cap f^{-1}(\eta) = 0$. Since f is FP-continuous, then $f^{-1}(\mu) \in s$, $f^{-1}(\eta) \in t$. Hence (X, s, t) is $FPT_1(i)$.

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A NOTE ON SINGULAR VALUE INEQUALITIES FOR COMPACT OPERATORS

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Abstract. In this short note, we present some singular value inequalities for compact operators. Our results are generalizations of ones obtained by Audeh and Kittaneh [Linear Algebra Appl., 437 (2012), 2516-2522].

Keywords: singular values; compact operators; self-adjoint operators; positive operators.

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1. Introduction

We mainly adopt the notations and terminologies in [1]. For convenience, recall that, as usual, Let $B(H)$ denote the space of all bounded linear operators on a complex separable Hilbert space H, and let $K(H)$ denote the two-sided ideal of compact operators in $B(H)$. For $T \in K(H)$, the singular values of T , denoted by $s_1(T), s_2(T), \cdots$ are the eigenvalues of the positive operator $|T| = (TT^*)^{1/2}$ enumerated as $s_1(T) \geq s_2(T) \cdots$ and repeated according to multiplicity. For $T \in B(H)$, let $T^+ = \dfrac{|T|+T}{2}, T^- = \dfrac{|T|-T}{2}$. Let $S, T \in B(H)$ be self-adjoint, the order relation $S \geq T$ means, as usual, that $S - T$ is positive . We use the direct sum notation $S \oplus T$ for the block-diagonal operator $\begin{bmatrix} S & 0 \\ 0 & T \end{bmatrix}$ defined on $H \oplus H$.

Let $A, B \in K(H)$, where A is self-adjoint and $B \geq 0$. Audeh and Kittaneh have proved in [1] that if $\pm A \leq B$, then

$$s_j(A) \leq \frac{1}{2} s_j((B+A) \oplus (B-A)) \tag{1.1}$$

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for $j = 1, 2, \ldots$. Inequality (1.1) is equivalent to three inequalities considered by Tao [2].

As an application of inequality (1.1), Audeh and Kittaneh have proved in [1] that if $A, B \in K(H)$ and $X \in B(H)$, then

$$s_j(AX^*B^* + BXA^*) \leq \frac{1}{2} s_j \begin{pmatrix} (A|X|A^* + B|X^*|B^* + AX^*B^* + BXA^*) \\ \oplus (A|X|A^* + B|X^*|B^* - AX^*B^* - BXA^*) \end{pmatrix} \tag{1.2}$$

for $j = 1, 2, \ldots$. Bhatia and Kittaneh have proved in [2] that, if $A, B \in K(H)$, then

$$s_j(AB^* + BA^*) \leq s_j((AA^* + BB^*) \oplus (AA^* + BB^*)) \tag{1.3}$$

for $j = 1, 2, \ldots$. Inequality (1.2) is sharper and more general than inequality (1.3). As another application of inequality (1.1), these authors also have proved in [1] that if $A, B \in K(H)$ are self-adjoint operators, then

$$s_j(A + B) \leq s_j\left(\left(A^+ + B^+\right) \oplus \left(A^- + B^-\right)\right) \tag{1.4}$$

for $j = 1, 2, \ldots$.

In this note, we present generalizations of inequalities (1.1), (1.2), and (1.4).

2. Main results

To generalize inequalities (1.1), (1.2), and (1.4), we need the following lemma [4, Theorem 1].

Lemma 2.1. *Let $A, X, B \in B(H)$ such that A and B are compact and positive. Then*

$$s_j(AX - XB) \leq \|X\| s_j(A \oplus B)$$

for $j = 1, 2, \ldots$, where $\|\cdot\|$ denotes the usual operator norm on $B(H)$.

Theorem 2.1. *Let $A, B \in K(H)$ and $X \in B(H)$, where A, B are self-adjoint and $\pm A \leq B$. Then*

$$s_j((B + A)X - X(B - A)) \leq \|X\| s_j((B + A) \oplus (B - A)) \tag{2.1}$$

for $j = 1, 2, \ldots$.

Proof. Since $\pm A \leq B$, we have $B - A \geq 0$ and $B + A \geq 0$. Now, applying Lemma 2.1 to the operators $B + A$ and $B - A$, we get

$$s_j((B + A)X - X(B - A)) \leq \|X\| s_j((B + A) \oplus (B - A))$$

for $j = 1, 2, \ldots$. This completes the proof. ∎

Remark 2.1. Putting $X = I$ in Theorem 2.1, we obtain inequality (1.1). Note that, $B \geq 0$ is no longer the condition of Theorem 2.1.

Theorem 2.2. *Let $A, B \in K(H)$ and $X, Y \in B(H)$. Then*

$$s_j(Y_1Y + YY_1 + Y_2Y - YY_2) \leq \|Y\| s_j((Y_2 + Y_1) \oplus (Y_2 - Y_1))$$

for $j = 1, 2, \ldots$, where

$$Y_1 = AX^*B^* + BXA^*,\ Y_2 = A|X|A^* + B|X|B^*.$$

Proof. For $X \in B(H)$, it is known [5, p.15] that

$$\begin{bmatrix} |X| & X^* \\ X & |X^*| \end{bmatrix} \geq 0$$

and so

$$\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \begin{bmatrix} |X| & X^* \\ X & |X^*| \end{bmatrix} \begin{bmatrix} A^* & 0 \\ B^* & 0 \end{bmatrix} \geq 0$$

That is

$$A|X|A^* + B|X|B^* + AX^*B^* + BXA^* \geq 0.$$

Similarly, we also have

$$A|X|A^* + B|X|B^* - AX^*B^* - BXA^* \geq 0.$$

Now, applying Lemma 2.1 to the operators $Y_2 + Y_1$ and $Y_2 - Y_1$, we get

$$s_j((Y_2 + Y_1)Y - Y(Y_2 - Y_1)) \leq \|Y\| s_j((Y_2 + Y_1) \oplus (Y_2 - Y_1))$$

for $j = 1, 2, \ldots$, which is equivalent to

$$s_j(Y_1Y + YY_1 + Y_2Y - YY_2) \leq \|Y\| s_j((Y_2 + Y_1) \oplus (Y_2 - Y_1))$$

for $j = 1, 2, \ldots$. This completes the proof. ∎

Remark 2.2. Putting $Y = I$ in Theorem 2.2, we obtain inequality (1.2).

Theorem 2.3. *Let $A, B \in K(H)$ be self-adjoint operators and $X \in B(H)$. Then*

$$\begin{aligned} s_j((A + B)X + X(A + B) + (|A| + |B|)X - X(|A| + |B|)) \\ \leq 2\|X\| s_j((A^+ + B^+) \oplus (A^- + B^-)) \end{aligned}$$

for $j = 1, 2, \ldots$.

Proof. Since A and B are Hermitian, it follows that

$$\pm(A + B) \leq |A| + |B|.$$

Let

$$Y_1 = A + B,\ Y_2 = |A| + |B|.$$

Now, applying Lemma 2.1 to the operators $Y_2 + Y_1$ and $Y_2 - Y_1$, we have

$$s_j((Y_2 + Y_1)X - X(Y_2 - Y_1)) \leq \|X\| s_j((Y_2 + Y_1) \oplus (Y_2 - Y_1)),$$

which is equivalent to

$$s_j\left((A+B)X+X(A+B)+(|A|+|B|)X-X(|A|+|B|)\right) \leq 2\|X\|\, s_j\left((A^+ + B^+)\oplus(A^- + B^-)\right)$$

for $j = 1, 2, \ldots$. This completes the proof. ∎

Remark 2.3. Putting $X = I$ in Theorem 2.3, we obtain inequality (1.3). Let $g(t)$ be a polynomial. If $X = g(|A| + |B|)$, then, by Theorem 2.3, we have

$$s_j\left((A+B)X+X(A+B)\right) \leq 2\|X\|\, s_j\left(\left(A^+ + B^+\right)\oplus\left(A^- + B^-\right)\right)$$

for $j = 1, 2, \ldots$.

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ON SOME PROPERTIES OF M-HYPERCYCLIC C_0-SEMIGROUP

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Abstract. In this article, we justify that every separable infinite dimensional complex Banach space admits an M-hypercyclic C_0-semigroups. Also, we prove that if $(T_t)_{t\geq 0}$ is an M-hypercyclic C_0- semigroup of a generator A acting on such X then the point spectrum $\sigma_p(T_t^*)$ may be empty or not. On other hand we introduce the concept of diskcyclic, M-diskcyclic and M-disk transitive C_0-semigroup; we obtain some proprieties as well as an important characterization for these notions.

Keywords: hypercyclicity, M-hypercyclicity, C_0-semigroup, point spectrum, topologically transitive, diskcyclic operator, M-Diskcyclic operator.

1. Introduction

Let $B(X)$ be the complex Banach algebra of all bounded linear operators on a infinite dimensional complex separable Banach space X. Recall that an element $T \in B(X)$ is called hypercyclic if there exists a vector $x \in X$ such that the orbit $Orb(T, x) = \{T^n x : n \geq 0\}$ is dense in X (in this case x is called a hypercyclic vector). For more details about hypercyclic operators and their proprieties see [5] and [7].

Analogously, a C_0- semigroup $\mathcal{T} = (T_t)_{t\geq 0} \subset B(X)$ is called hypercyclic if there exists a vector $x \in X$ such that the orbit of $\mathcal{T}$ $Orb(\mathcal{T}, x) = \{T_t x : t \geq 0\}$ is dense in X (such a vector is called a hypercyclic vector for $\mathcal{T}$). The set of all hypercyclic vectors of $\mathcal{T}$ is denoted by $HC(\mathcal{T})$.

Desch, Schappacher and Webb [5] have shown that if $\mathcal{T}$ is a hypercyclic C_0-semigroup of a generator A, then the generator A^* (the adjoint of A) of the dual semigroup C_0-semigroup $\mathcal{T}^* = (T_t^*)_{t\geq 0}$ verifies the following conditions

1. If $\phi \in X^* \backslash \{0\}$, then $Orb(\mathcal{T}^*, \phi) = \{T_t^* \phi : t \geq 0\}$ is unbounded.
2. the point spectrum $\sigma_p(A^*) = \emptyset$.

T. Kalmes in [6] Showed that if $\mathcal{T}$ is a C_0-semigroup of a generator A such that the resolvent set $\rho(A) \neq \emptyset$ and if the injection $i : (D(A), \|.\|_A) \hookrightarrow (X, \|.\|)$ is compact then $\mathcal{T}$ is not hypercyclic.

At 1941, Oxtoby and Ulam [10] showed that, if $\mathcal{T}$ is a hypercyclic C_0-semigroup and $x \in HC(\mathcal{T})$, then there exists a G_δ-dense set I of $\mathbb{R}^+$ such that $x \in HC(T_t)$ for all $t \in I$ (by Baire's terminology a G_δ set is an intersection of countable collection of open subsets).

Recently, Conejero, Peris and Müller in [4] established a more general result. They proved that if $\mathcal{T}$ is a C_0-semigroup then we have the following equivalences:

1. $\mathcal{T}$ is hypercyclic.
2. There exists $t > 0$, such that T_t is hypercyclic.
3. For all $t > 0$, T_t is hypercyclic.

Recall that Peris (see [7]) proved that if $\mathcal{T}$ is a hypercyclic C_0-semigroup of a generator A then the range $R(T_t - \lambda I)$ is dense in X for every $t > 0$, $\lambda \in \mathbb{C}$, $\sigma_p(T_t^*) = \emptyset$ and $\sigma_p(A^*) = \emptyset$.

At 2011, Madore and Martinez-Avendano [9] introduced a new concept related of hypercyclicity it called subspace-hypercyclicity. We say that an operator $T \in B(X)$ is M-hypercyclic if there is a nonzero vector subspace M and a vector x of X such that $Orb(T, x) \cap M$ is dense in M, the same authors also gave the concept of M-transitivity and they showed that each M-transitive operator is M-hypercyclic and that the converse is false. Le in [8] and Talebi and Asadipour in [13] gave an example of operator which is M-hypercyclic but not M-transitive, for more details about relationship between M-hypercyclicity and M-transitivity and others properties see [8], [9], [11], [13].

Next, Recall that if $T \in B(X)$ and $S \in B(X)$ then the operator $T \oplus S$ is defined on the Banach spaces

$$X \oplus X := \{(x, y) : x \in X, y \in X\} = \{x \oplus y : x \in X, y \in X\}$$

endowing by the product topology by

$$(T \oplus S)(x \oplus y) = Tx \oplus Sy = (Tx, Sy).$$

We have the following properties [7]:

1. For all $n \in \mathbb{N}, (T \oplus S)^n = T^n \oplus S^n$.
2. $(T \oplus S)^* = T^* \oplus S^*$.
3. If $T \oplus S$ is hypercyclic, then T and S are hypercyclic.

On other hand, Zeana in [15] introduced and studied the notion of a diskcyclicity. An operator $T \in B(X)$ is called diskcyclic if there is a vector $x \in X$ such that the disc orbit $\mathbb{D}Orb(T, x) = \{\alpha T^n x : \alpha \in \mathbb{C}, |\alpha| \leq 1, n \geq 0\}$ is dense in X, in this case the vector x is called diskcyclic for T. For a systematic investigation, in [3] Bamerni and al introduced the concept of M-diskcyclicity and M-disk transitivity. An operator $T \in B(X)$ is called M-diskcyclic if there is non-trivial subspace M of X and a vector $x \in X$ such that $\mathbb{D}Orb(T, x) \cap M$ is dense in M, Such a vector x is called a M-diskcyclic vector for T.

More recently, Azimi in [2] studied the concept of diskcyclicity and its relevant criteria for a sequence $(T_n)_{n\geq 0}$ of bounded operators between separable Banach spaces and M-diskcyclicity for $(T_n)_{n\geq 0} \subset B(X)$ where X is a separable infinite dimensional Hilbert space.

In this article, we continue [12] to study some proprieties for the M-hypercyclicity of C_0-semigroup $\mathcal{T} = (T_t)_{t\geq 0}$. In section two, we know that if $(T_t)_{t\geq 0}$ is a hypercyclic C_0- semigroup of a generator A then T_t is hypercyclic for every $t > 0$, $\sigma_p(T_t^*) = \emptyset$ and $\sigma_p(A^*) = \emptyset$. We will show that it is not the case when $(T_t)_{t\geq 0}$ is a M-hypercyclic C_0-semigroup, precisely we give some examples proving that if $\mathcal{T}$ is M-hypercyclic then $\sigma_p(T_t^*)$ may be empty or not. Again, we justify that every infinite dimensional separable Banach space admits a M -hypercyclic C_0-semigroup. In section 3 we introduce the notion of diskcyclicity, M-diskcyclicity and M-disk transitivity for C_0-semigroup $\mathcal{T} = (T_t)_{t\geq 0}$. We shows that if $\mathcal{T}$ is M-disk transitive then $\mathcal{T}$ is M-diskcyclic and we prove that the set $\mathcal{DC}(\mathcal{T}, M)$ of all M-diskcyclic vectors of $\mathcal{T}$ is dense in X. Finally some characterizations of M-disk transitivity for $\mathcal{T}$ are given.

2. Point spectrum of M-hypercyclic C_0-semigroup

Let us recall some definitions and theorem from [12].

Definition 2.1. [12] Let $\mathcal{T} = (T_t)_{t\geq 0}$ be a C_0-semigroup and M is a nonzero subspace of X. $\mathcal{T}$ is called *M-hypercyclic* if there is a vector x of X such that $Orb(\mathcal{T}, x) \cap M$ is dense in M with $Orb(\mathcal{T}, x) = \{T_t x : t \geq 0\}$.

Definition 2.2. [12] Let $\mathcal{T} = (T_t)_{t\geq 0}$ be a C_0-semigroup on X and M is a nonzero subspace of X. We say that $\mathcal{T}$ is *M-transitive* if for every non-empty open sets U and V of M there is $t \geq 0$ such that $T_t^{-1}(U) \cap V$ is a non-empty open set of M.

Theorem 2.1. [12] *Let $\mathcal{T} = (T_t)_{t\geq 0}$ be a C_0-semigroup and $M \neq \{0\}$ a subspace of X. If $\mathcal{T}$ is M-transitive, then it is M-hypercyclic.*

In the sequel, we will need the following important propositions.

Proposition 2.1. *Let $(A_t)_{t\geq 0}$ be a C_0-semigroup of a generator A on X. Let $T_t = A_t \oplus I$ for all $t \geq 0$. Then we have*

1. *$(T_t)_{t\geq 0}$ is a C_0-semigroup on $X \oplus X$.*
2. *The generator of $(T_t)_{t\geq 0}$ is the operator T defined on $D(T) = D(A) \oplus X$ such that $T(x \oplus y) = Ax \oplus 0$, for all $x \in D(A)$, $y \in X$.*

Proof. 1. Since $(A_t)_{t\geq 0}$ is a C_0-semigroup, then

$$T_0 = A_0 \oplus I = I \oplus I = I_{X\oplus X}.$$

Let $x \oplus y \in X \oplus X$ and $t, s \geq 0$, we have

$$T_{t+s}(x \oplus y) = A_{t+s}x \oplus y = A_tA_s(x) \oplus y = T_t(A_sx \oplus y) = T_tT_s(x \oplus y).$$

On the other hand,

$$\lim_{t\to s} T_t(x \oplus y) = \lim_{t\to s}(A_tx \oplus y) = A_sx \oplus y = T_s(x \oplus y).$$

Therefore, $(T_t)_{t\geq 0}$ is a C_0-semigroup on $X \oplus X$.

2. Let $x \in D(A)$ and $y \in X$ we have:

$$\begin{aligned}\lim_{t\to 0} \frac{T_t(x \oplus y) - (x \oplus y)}{t} &= \lim_{t\to 0} \frac{A_tx \oplus y - x \oplus y}{t} \\ &= \lim_{t\to 0} \frac{(A_tx - x) \oplus 0}{t} \\ &= Ax \oplus 0.\end{aligned}$$

Then $D(T) = D(A) \oplus X$ and $T(x \oplus y) = A(x) \oplus 0$, for all $x \in D(A)$. ∎

Proposition 2.2. *Let $(A_t)_{t\geq 0}$ and $(B_t)_{t\geq 0}$ be two C_0-semigroups on X of generators respectively A and B. We set $T_t = A_t \oplus B_t$ for all $t \geq 0$. Then we have*

1. *$(T_t)_{t\geq 0}$ is a C_0-semigroup on $X \oplus X$.*
2. *The generator of $(T_t)_{t\geq 0}$ is the the operator T defined on $D(T)$=$D(A)\oplus D(B)$ by $T(x \oplus y) = Ax \oplus By$.*

Proof. 1. Let $x \oplus y \in X \oplus X$ and $t, s \geq 0$. Since $(A_t)_{t\geq 0}$ and $(B_t)_{\geq 0}$ are a C_0-semigroups then we have:

$$T_0(x \oplus y) = A_0x \oplus B_0y = Ix \oplus Iy = x \oplus y,$$

thus $T_0 = I \oplus I = I_{X\oplus X}$. Next,

$$\begin{aligned}T_{t+s}(x \oplus y) &= A_{t+s}x \oplus B_{t+s}y \\ &= A_tA_sx \oplus B_tB_sy \\ &= (A_t \oplus B_t)(A_sx \oplus B_sy) \\ &= T_tT_s(x \oplus y).\end{aligned}$$

Consequently $T_{t+s} = T_t T_s$. On the other hand,

$$\begin{aligned} \lim_{t\to s} T_t(x \oplus y) &= \lim_{t\to s}(A_t x \oplus B_t y) \\ &= A_s x \oplus B_s y \\ &= T_s(x \oplus y). \end{aligned}$$

This implies that $(T_t)_{t\geq 0}$ is a C_0-semigroup on $X \oplus X$.

2. Let $x \in D(A)$ and $y \in D(B)$. We have

$$\begin{aligned} \lim_{t\to 0} \frac{T_t(x \oplus y) - (x \oplus y)}{t} &= \lim_{t\to 0} \frac{(A_t x - x) \oplus (B_t y - y)}{t} \\ &= A(x) \oplus B(y). \end{aligned}$$

Then

$$D(T) = D(A) \oplus D(B) \text{ and } T(x \oplus y) = Ax \oplus By. \qquad \blacksquare$$

Remark 1. It is clear that $D(T)$ is dense in $X \oplus X$ either in Proposition 2.1 or in Proposition 2.2 since $D(A)$ and $D(B)$ are dense in X.

Remark 2. If $\mathcal{T} = (T_t)_{t\geq 0}$ is a hypercyclic C_0-semigroup then the point spectrum $\sigma_p(T_t^*) = \{\lambda \in \mathbb{C} : T_t^* - \lambda I \text{ is not injective } \} = \emptyset$ for every $t > 0$. We will show that it is not always the case when $\mathcal{T}$ is M-hypercyclic.

Remark 3. It is clear that Proposition 2.1 is a particular case for Proposition 2.2 by taking $B_t = t$ for all t and its generator $B = 0$.

Example 1. Let $(A_t)_{t\geq 0}$ be a hypercyclic C_0-semigroup on X of a hypercyclic vector x (such a hypercyclic C_0-semigroup exists according to [1] and [14]). Then the C_0-semigroup $(T_t)_{t\geq 0} = (A_t \oplus I)_{t\geq 0}$ is M-hypercyclic with $M = X \oplus \{0\}$, in addition $\sigma_p(T_t^*) \neq \emptyset$ for every $t > 0$.

Indeed, according to Proposition 2.1, $(T_t)_{t\geq 0}$ is a C_0-semigroup on $X \oplus X$ and since x is a hypercyclic vector of $(A_t)_{t\geq 0}$ then $Orb((A_t), x)$ is dense in X, hence $Orb((A_t), x) \oplus \{0\}$ is dense in $X \oplus \{0\}$.

Since $Orb((T_t), x \oplus 0) \cap M = Orb((A_t), x) \oplus \{0\} \cap X \oplus \{0\} = Orb((A_t), x) \oplus \{0\}$ is dense in $X \oplus \{0\}$. Therefore, $(T_t)_{t\geq 0}$ is $(X \oplus \{0\})$-hypercyclic.

On the other hand, for all $t > 0$ and every $x^* \in X^* \backslash \{0\}$, we have

$$T_t^*(0 \oplus x^*) = A^*(0 \oplus x^*) = 0 \oplus x^*.$$

Consequently, $1 \in \sigma_p(T_t^*)$ for every $t > 0$ and we conclude that $\sigma_p(T_t^*) \neq \emptyset$. $\blacksquare$

Example 2. Let $(A_t)_{t\geq 0}$ and $(B_t)_{t\geq 0}$ be two hypercyclic C_0-semigroup on X and x be a hypercyclic vector of $(A_t)_{t\geq 0}$. Then the C_0-semigroup $(T_t)_{t\geq 0} = (A_t \oplus B_t)_{t\geq 0}$ is M-hypercyclic with $M = X \oplus \{0\}$. In addition, $\sigma_p(T_t^*) = \emptyset$ for all $t > 0$.

Indeed, by Proposition 2.2, $(T_t)_{t\geq 0}$ is a C_0-semigroup on $X \oplus X$ and we have $Orb((T_t), x \oplus 0) \cap M = Orb((T_t), x) \oplus \{0\} \cap X \oplus \{0\} = Orb((A_t), x) \oplus \{0\}$ which is dense in $X \oplus \{0\}$ because $Orb((A_t), x)$ is dense in X.

On the other hand, we have $\sigma_p(T_t^*) = \emptyset$ for all $t > 0$, otherwise:

Let $\lambda \in \sigma_p(T_t^*)$, then there exists $x^* \oplus y^* \in (X^* \oplus X^*)\backslash\{0\}$ such that $\mathrm{T}_t^*(x^* \oplus y^*) = \lambda(x^* \oplus y^*) = \lambda x^* \oplus \lambda y^*$. Then we have $A_t^* x^* \oplus B_t^* y^* = \lambda x^* \oplus \lambda y^*$, on other word $A_t^* x^* = \lambda x^*$ and $B_t^* x^* = \lambda y^*$, therefore $\lambda \in \sigma_p(A_t^*)$ and $\lambda \in \sigma_p(B_t^*)$. Thus $\sigma_p(A_t^*) \neq \emptyset$ and $\sigma_p(B_t^*) \neq \emptyset$. This is a contradiction, because $(A_t)_{t\geq 0}$ and $(B_t)_{t\geq 0}$ are hypercyclic, consequently $\sigma_p(T_t^*) = \emptyset$. ∎

Corollairy 2.1. *If $\mathcal{T} = (T_t)_{t\geq 0}$ is a M-hypercyclic C_0-semigroup, then $\sigma_p(T_t^*)$ can be empty or not.*

Remark 4. In the two previous examples we have $\sigma_p(A^*) = \emptyset$ where A is the generator of $(A_t \oplus I)_{t\geq 0}$ in first example and $\sigma_p(A^* + B^*) = \emptyset$ where $A + B$ is the generator of $(A_t \oplus B_t)_{t\geq 0}$ in the second example.

Problem. *Let $\mathcal{T} = (T_t)_{t\geq 0}$ be a M-hypercyclic C_0-semigroup with $M \neq \{0\}$ a subspace of X. Do we have $\sigma_p(A^*) = \emptyset$ where A is the generator of $\mathcal{T}$?*

Theorem 2.2. [3] *If E is a dense subset of X, then there is a non-trivial closed subspace M of X such that $E \cap M$ is dense in M.*

Corollairy 2.2. *If $\mathcal{T}$ is hypercyclic C_0-semigroup on X, then there is a closed subspace M of X such that $\mathcal{T}$ is M-hypercyclic C_0-semigroup.*

Proof. Since $\mathcal{T}$ is hypercyclic, then there is a vecteur $x \in X$ such that $E = Orb(\mathcal{T}, x)$ is dense in X, by Theorem 2.2 there is a non trivial subspace M of X such that $E \cap M = Orb(\mathcal{T}, x) \cap M$ is dense in M. Then $\mathcal{T}$ is M-hypercyclic. ∎

Theorem 2.3. [14] *Every separable infinite dimensional Banach space admits a hypercyclic uniformly continuous semigroup.*

Corollairy 2.3. *Every infinite dimensional separable Banach space admits a subspace-hypercyclic C_0-semigroup.*

Proof. Theorem 2.2 gives in every infinite dimensional separable Banach admits a C_0-semigroup hypercyclic, applying Corollary 2.2 this C_0-semigroup is M-hypercyclic for some M. ∎

3. The diskcyclicity and subspace-diskcyclicity of C_0-semigroup

Definition 3.1. Let $\mathcal{T} = (T_t)_{t\geq 0}$ be a C_0-semigroup. We say that $\mathcal{T}$ *is diskcyclic* if there exists a vecteur $x \in X$, such that

$$\mathcal{D}Orb(\mathcal{T}, x) = \{\lambda T_t x : t \geq 0, \lambda \in \mathbb{C}, |\lambda| \leq 1\}$$

is dense in X. The vector x is called a *diskcyclic vector* for $\mathcal{T}$.

We denote the set of all diskcyclic vectors of $\mathcal{T}$ by $\mathcal{DC}(\mathcal{T})$.

Remark 5. Let $\mathcal{T} = (T_t)_{t\geq 0}$ be a C_0-semigroup.

1. If there is a $t_0 \geq 0$ such that T_{t_0} is diskcyclic, then $\mathcal{T}$ is diskcyclic.
2. If $\mathcal{T}$ is hypercyclic then $\mathcal{T}$ is diskcyclic.

Proposition 3.1. *Let $\mathcal{T} = (T_t)_{t\geq 0}$ be a C_0-semigroup on X. Then*

$$\mathcal{DC}(\mathcal{T}) = \bigcap_{m\geq 0} \bigcup_{t\geq 0} \bigcup_{|\lambda|\geq 1} T_t^{-1}(\lambda B_m),$$

where $(B_m)_{m\geq 0}$ is a countable open basis for X.

Proof. Of course, if $(B_m)_{m\geq 0}$ is a countable open basis for X then we have

$$\begin{aligned} x \in \mathcal{DC}(\mathcal{T}) &\iff \mathcal{DO}rb(\mathcal{T}, x) \text{ is dense in} X \\ &\iff \mathcal{DO}rb(\mathcal{T}, x) \cap B_m \neq \phi \text{ for all } \ m \geq 0 \\ &\iff \forall m \geq 0, \exists\, t \geq 0,\ \exists \lambda \in \mathbb{C}\ \ |\lambda| \leq 1 \text{ such that } \lambda T_t x \in B_m \\ &\iff \forall m \geq 0, \exists\, t \geq 0,\ \exists \lambda \in \mathbb{C}\ \ |\lambda| \geq 1 \text{ such that } x \in T_t^{-1}(\lambda B_m) \\ &\iff x \in \bigcap_{m\geq 0} \bigcup_{t\geq 0} \bigcup_{|\lambda|\geq 1} T_t^{-1}(\lambda B_m). \end{aligned}$$

∎

Proposition 3.2. *Let $\mathcal{T} = (T_t)_{t\geq 0}$ be a C_0-semigroup, and let $(\beta_t)_{t\geq 0}$ such that $\beta_t \geq 0$. If $(\beta_t T_t)_{t\geq 0}$ is diskcyclic then $(\alpha_t T_t)_{t\geq 0}$ is diskcyclic for all $\beta_t \leq \alpha_t$.*

Proof. Indeed, since

$$\{\lambda \beta_t T_t x : t \geq 0, \lambda \in \mathbb{C}; |\lambda| \leq 1\} \subseteq \{\lambda \alpha_t T_t x : t \geq 0,\ \lambda \in \mathbb{C}, |\lambda| \leq 1\}. \qquad ∎$$

Definition 3.2. Let $\mathcal{T} = (T_t)_{t\geq 0}$ be a C_0-semigroup. We say that $\mathcal{T}$ is disk topologically transitive if for any pair U, V of nonempty open subset of X there are $t \geq 0$ and $\lambda \in \mathbb{C}\ \ |\lambda| \leq 1$ such that $T_t(\lambda U) \cap V \neq \phi$.

Theorem 3.1. [3] *Let $\mathcal{T} = (T_t)_{t\geq 0}$ be a C_0-semigroup. $\mathcal{T}$ is diskcyclic if and only if $\mathcal{T}$ is disk topologically transitive.*

Definition 3.3. Let $\mathcal{T} = (T_t)_{t\geq 0}$ be a C_0-semigroup, and M be a nonzero subspace of X. We say that $\mathcal{T}$ is M-diskcyclic if there is a vector $x \in X$ such that $\mathcal{DO}rb(\mathcal{T}, x) \cap M$ is dense in M. The vector x is called the M-diskcyclic vector of $\mathcal{T}$.

The set of all M-dsikcyclic vectors is denote by $\mathcal{DC}(\mathcal{T}, M)$.

Remark 6. If $M = X$ then $\mathcal{T}$ is M-dsikcyclic and $\mathcal{T}$ is diskcyclic.

Example 3. Let $(A_t)_{t\geq 0}$ be a C_0-semigroup diskcyclic on X, with diskcyclic vector x, then the C_0-semigroup $T_t = A_t \oplus I$ is M-diskcyclic with $M = X \oplus \{0\}$. But $(T_t)_{t\geq 0}$ is not diskcyclic.

Proposition 3.3. *Let $\mathcal{T} = (T_t)_{t\geq 0}$ be a C_0-semigroup on X and M is a nonzero subspace of X, then*

$$\mathcal{DC}(\mathcal{T}, M) = \bigcap_{m\geq 0} \bigcup_{|\alpha|\geq 1} \bigcup_{t\geq 0} T_t^{-1}(\alpha B_m),$$

where $(B_m)_{m\geq 0}$ is a countable open basis for the relative topology of M.

Proof. If we fix $(B_m)_{m\geq 0}$ a countable open basis for the relative topology of M then

$$\begin{aligned}
x \in \mathcal{DC}(\mathcal{T}, M) &\iff \{\alpha T_t x : |\alpha| \leq 1, t \geq 0\} \cap M \text{ is dense in } M \\
&\iff \forall\, m \geq 0; \{\alpha T_t x : |\alpha| \leq 1, t \geq 0\} \cap B_m \neq \phi \\
&\iff \forall\, m \geq 0; \exists \alpha, |\alpha| \leq 1 \text{ and } t \geq 0 \text{ such that } \alpha T_t x \in B_m \\
&\iff \forall\, m \geq 0; \exists \alpha, 0 < |\alpha| \leq 1 \text{ and } t \geq 0 \text{ such that } x \in T_t^{-1}(\frac{1}{\alpha} B_m) \\
&\iff \forall\, m \geq 0; \exists \alpha, |\alpha| \geq 1 \text{ and } t \geq 0 \text{ such that } x \in T_t^{-1}(\alpha B_m) \\
&\iff x \in \bigcap_{m\geq 0} \bigcup_{|\alpha|\geq 1} \bigcup_{t\geq 0} T_t^{-1}(\alpha B_m)
\end{aligned}$$

The proof is complete. ∎

Definition 3.4. Let $\mathcal{T} = (T_t)_{t\geq 0}$ be a C_0-semigroup and M is a nonzero subspace of X. $\mathcal{T}$ is called M-disk transitive if for any nonempty open sets $U, V \subset M$ there exists $\alpha \in \mathbb{C}, |\alpha| \geq 1$ and $t \geq 0$ such that $T_t^{-1}(\alpha U) \cap V$ contains a nonempty open relatively subset of M.

Theorem 3.2. *Let $\mathcal{T} = (T_t)_{t\geq 0}$ be a C_0-semigroup on X and M is a nonzero subspace of X. Then the following conditions are equivalent*

1. *$\mathcal{T}$ is M-disk transitive.*
2. *For any nonempty open sets $U, V \subset M$ there exists $\alpha \in \mathbb{C}$, $|\alpha| \geq 1$ and $t \geq 0$ such that $T_t^{-1}(\alpha U) \cap V$ is nonempty and $T_t(M) \subset M$.*
3. *For any nonempty open sets $U, V \subset M$ there exists $\alpha \in \mathbb{C}$, $|\alpha| \geq 1$ and $t \geq 0$ such that $T_t^{-1}(\alpha U) \cap V$ is nonempty open subset of M.*

Proof. (1) ⇒ (2). Let U, V to be nonempty subset of M. From (1) there exists $\alpha \in \mathbb{C}, |\alpha| \geq 1$ and $t \geq 0$ such that $T_t^{-1}(\alpha U) \cap V$ contains a relatively non empty open subset of M. Let $W \subset M$ a non empty open subset such that $W \subset T_t^{-1}(\alpha U) \bigcap V$. Then $T_t^{-1}(\alpha U) \cap V$ is non empty.

Now, we prove that $T_t(M) \subset M$. Indeed, since $W \subset T_t^{-1}(\alpha U) \cap V$ this implies that $T_t(W) \subset \alpha U$ and $W \subset V$, consequently $\frac{1}{\alpha} T_t(W) \subset U \subset M$ and $W \subset V \subset M$.

Next, let $x \in M$. We justify that a fortiori $T_t(x) \in M$. Since $W \neq \emptyset$, put $a \in W$. For $r > 0$ small enough we have $a + rx \in W$, it follows that

$$\frac{1}{\alpha} T_t(a + rx) = \frac{1}{\alpha} T_t(a) + \frac{r}{\alpha} T_t(x) \in \frac{1}{\alpha} T_t(W) \subset M.$$

From $a \in W$ we obtain that

$$\frac{1}{\alpha}T_t(a) \in M \text{ and } \frac{r}{\alpha}T_t(x) \in M,$$

therefore $T_t(x) \in M$, we obtain $T_t(M) \subset M$.

(2) $\Rightarrow$ (3). Let U, V be a non empty open subset of M. By (2) there exists $\alpha \in \mathbb{C}, |\alpha| \geq 1$ and $t \geq 0$ such that $T_t^{-1}(\alpha U) \cap V \neq \emptyset$ and $T_t(M) \subset M$. But $T_t : M \to M$ is continuous and since U is open in M then $T_t^{-1}(\alpha U)$ is open subset in M, thus $T_t^{-1}(\alpha U) \cap V$ is a non empty open subset of M.

(3) $\Rightarrow$ (1) Clear. ∎

Theorem 3.3. *Let $\mathcal{T} = (T_t)_{t\geq 0}$ be a C_0-semigroup on X and M is a nonzero subspace of X. If $\mathcal{T}$ is M-disk transitive, then $\mathcal{DC}(\mathcal{T}, M)$ is a dense subset of M.*

Proof. Let $(B_m)_{m\geq 0}$ be a countable open basis for the relative topology of M, since $\mathcal{T}$ is M-disk transitive then from Theorem 3.2 for each i, j, there exists $t_{i,j} \geq 0$ and $\alpha_{i,j} \in \mathbb{C}$, $|\alpha_{i,j}| \geq 1$ such that the set $T_{t_{i,j}}^{-1}(\alpha_{i,j}B_i) \cap B_j \neq \emptyset$. Write $G_i = \bigcup_{j\geq 0} T_{t_{i,j}}^{-1}(\alpha_{i,j}B_i) \cap B_j$. G_i is a nonempty open set of M, by Baire's category theorem $\bigcap_{i\geq 0} G_i$ is dense in M. From Proposition 3.3, it follows that

$$\bigcap_{i\geq 0} G_i = \bigcap_i \bigcup_{j\geq 0} (T_{t_{i,j}}^{-1}(\alpha_{i,j}B_i) \bigcap B_j) \subseteq \bigcap_{i\geq 0} \bigcup_{t\geq 0} \bigcup_{|\alpha|\geq 1} T_t^{-1}(\alpha B_i) = \mathcal{DC}(\mathcal{T}, M).$$

We conclude that $\mathcal{DC}(\mathcal{T}, M)$ is dense in M. ∎

Corollairy 3.1. *If $\mathcal{T}$ is M-disk transitive, then $\mathcal{T}$ is M-diskcyclic.*

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TWO RESULTS ON THE ABELIAN IDEALS OF A BOREL SUBALGEBRA

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Abstract. Let $\mathfrak{b}$ a fixed Borel subalgebra of a finite-dimensional complex simple Lie algebra $\mathfrak{g}$. The Shi bijection associates to every ad-nilpotent ideal $\mathfrak{i}$ of $\mathfrak{b}$ a region $V_{\mathfrak{i}}$. In this paper, we show that $\mathfrak{i}$ is abelian if and only if $V_{\mathfrak{i}} \cap 2A$ is nonempty, if and only if the volume of $V_{\mathfrak{i}} \cap 2A$ equals to that of A, where A is the fundamental alcove of the affine Weyl group. We also determine the maximal eigenvalue m_{r-1} of the Casimir operator on $\Lambda^{r-1}\mathfrak{g}$ and the corresponding eigenspace M_{r-1}, where r is the number of positive roots.

Keywords: abelian ideal, ad-nilpotent ideal, Borel subalgebra, Casimir operator, Shi bijection.
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1. Introduction

Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra of rank l. Let $\mathfrak{b}$ a fixed Borel subalgebra of $\mathfrak{g}$. Two results of the abelian ideals of $\mathfrak{b}$ will be given in this paper. On one hand, we will describe the regions in the Shi arrangement corresponding to the abelian ideals of $\mathfrak{b}$. On the other hand, we will determine the maximal eigenvalue m_{r-1} of the Casimir operator on $\Lambda^{r-1}\mathfrak{g}$ and the corresponding eigenspace M_{r-1}, where r is the number of positive roots.

Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Then we have the root system $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$. Let $V = \mathfrak{h}^*_{\mathbb{R}}$ be the real vector space spanned by Δ. Let $(\,|\,)$ be the canonical inner product induced on V by the Killing form. Let $\Pi = \{\alpha_1, ..., \alpha_l\} \subseteq \Delta^+$ be a fixed choice of simple and positive root systems of Δ, respectively. Let $\mathfrak{n} = \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha$ is the root space relative to α. Then $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ is a Borel subalgebra of $\mathfrak{g}$. Let $W = W(\mathfrak{g}, \mathfrak{h})$ be the affine Weyl group.

Here, recall that an ideal $\mathfrak{i}$ of $\mathfrak{b}$ is called *ad-nilpotent* if $\mathfrak{i} \subseteq \mathfrak{n}$. Let us denote by $\mathcal{I}$ the set of all the ad-nilpotent ideals of $\mathfrak{b}$. Shi [10] associated a dominant

region $V_{\mathfrak{i}}$ of the now called Shi arrangement $\mathtt{Shi}$ to any $\mathfrak{i} \in \mathcal{I}$. To state his results, let us recall some notations concerning hyperplane arrangements.

A *hyperplane arrangement* is a finite collection of affine hyperplanes in a Euclidean space. For example, the *Shi arrangement* associated with Δ^+ is the arrangement in V defined by

$$\mathtt{Shi} := \{H_{\alpha,0} \mid \alpha \in \Delta^+\} \cup \{H_{\alpha,1} \mid \alpha \in \Delta^+\}. \tag{1.1}$$

Here for $\alpha \in \Delta^+$ and $k \in \mathbb{Z}$, we define an affine hyperplane

$$H_{\alpha,k} := \{v \in V \mid (v|\alpha) = k\}. \tag{1.2}$$

This arrangement was defined by Shi in the study of the Kazhdan-Lusztig cellular structure of the affine Weyl group of type A, see Chapter 7 of [9]. In general, if $\mathcal{A}$ is a hyperplane arrangement in V, the connected components of $V - \bigcup_{H\in\mathcal{A}} H$ are called *regions*. We refer the reader to Section 3.11 of [11] for basics about hyperplane arrangements. For $\mathtt{Shi}$, a region is called *dominant* if it is contained in V_∞, where

$$V_\infty := \{v \in V \mid (v|\alpha) > 0, \forall \alpha \in \Delta^+\}. \tag{1.3}$$

Now let us recall the *Shi bijection* from Theorem 1.4 of [10] as follows:

Theorem 1.1 (**Shi**) *There exists a natural bijective map from the set of all the ad-nilpotent ideals of* $\mathfrak{b}$ *to the set of all the dominant regions of the hyperplane arrangement* $\mathtt{Shi}$. *The map sends* $\mathfrak{i}$ *to* $V_{\mathfrak{i}} := \{v \in V \mid (v|\beta) > 1, \forall \beta \in \Phi(\mathfrak{i}); 0 < (v|\beta) < 1, \forall \beta \in \Delta^+ \setminus \Phi(\mathfrak{i})\}$. *Here* $\Phi(\mathfrak{i})$ *denotes the set of roots in* $\mathfrak{i}$.

It is easy to see that the dominant region of $\mathtt{Shi}$ corresponding to the zero ideal coincides with the fundamental alcove of the affine Weyl group $\widehat{W}$, which is defined as

$$A := \{v \in V \mid (v|\alpha_i) > 0 \text{ for } i = 1, \ldots, l \text{ and } (v|\theta) < 1\}, \tag{1.4}$$

where θ is the maximal positive root. Note that the Shi bijection is recently generalized to parabolic subalgebras in [3].

Theorem 1.1 gives us one way to visualize the set $\mathcal{I}$. It is then natural to ask for an according characterization of the abelian ideals among all the ad-nilpotent ones. A difficulty lies in that the region $V_{\mathfrak{i}}$ may be unbounded, even if $\mathfrak{i}$ is abelian. Indeed, we have

Proposition 1.2 *For any* $\mathfrak{i} \in \mathcal{I}$, *the following are equivalent:*

(a) $\mathfrak{i}$ *is abelian;*

(b) $V_{\mathfrak{i}} \cap 2A$ *is nonempty;*

(c) $\mathrm{vol}(V_{\mathfrak{i}} \cap 2A) = \mathrm{vol}(A)$. *Here "vol" means taking the volume.*

Another theme of the current paper comes from [6], where Kostant found a connection between the commutative Lie subalgebras of $\mathfrak{g}$ and the maximal eigenspaces of the Casimir operator $Cas \in U(\mathfrak{g})$ on certain degrees of $\wedge\mathfrak{g}$. To be more precise, let p denote the maximal dimension of all the abelian subalgebras of $\mathfrak{g}$. Let m_k be the maximal eigenvalue of Cas on $\wedge^k\mathfrak{g}$ and M_k be the corresponding eigenspace. Then

$$m_k \leq k,$$

and equality holds if and only if $k \leq p$. Moreover, in such a case, M_k is a multiplicity-free $\mathfrak{g}$-module whose highest weight vectors correspond to k-dimensional abelian subalgebras of $\mathfrak{g}$ (or equivalently, k-dimensional abelian ideals of $\mathfrak{b}$). In (1.2) of [7], Kostant mentioned that

$$m = \frac{1}{3}\dim \mathfrak{g}, \tag{1.5}$$

where m denotes the maximal eigenvalue of Cas on $\wedge\mathfrak{g}$, i.e., m is the maximum of m_k over all k. Recently, among other things, Han [4] determined the eigenspaces relative to m. To state the result, let r denote the number of positive roots and put $l = \operatorname{rank}\mathfrak{g}$. For any dominant integral linear form λ on $\mathfrak{h}$, let V_λ be the irreducible $\mathfrak{g}$-module with highest weight λ. Then, it is proved in [4] that

(i) one has $m_k \leq \frac{1}{3}\dim\mathfrak{g}$, and equality holds precisely for $k = r, r+1, \cdots, r+l$. Moreover, for $0 \leq s \leq l$, $M_{r+s} = C_l^s V_{2\rho}$;

(ii) For $0 \leq k < r$, one has $m_k < m_{k+1}$ and $\oplus_{k=0}^{r} M_k$ is a multiplicity-free $\mathfrak{g}$-module.

We find that m_{r-1} and the corresponding eigenspace M_{r-1} can be explicitly determined. Let Π_s be the set of all *short* simple roots. We view every root as a short one if $\mathfrak{g}$ is simply-laced, i.e., $\mathfrak{g}$ is of type A, D, E. Recall that the important integer

$$h^\vee = \frac{1}{(\theta|\theta)} \tag{1.6}$$

is the *dual Coxeter number*. Let α be any short root, then

$$k = \frac{(\theta|\theta)}{(\alpha|\alpha)} \tag{1.7}$$

is a positive integer. Now we are able to state

Theorem 1.3 *Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra. With notations as above, one has*

$$m_{r-1} = \frac{1}{3}\dim\mathfrak{g} - \frac{2}{kh^\vee}, \tag{1.8}$$

and

$$M_{r-1} = \oplus_{\alpha\in\Pi_s} V_{2\rho-\alpha}. \tag{1.9}$$

Moreover, for any $\alpha \in \Pi$ which is not necessarily short, the irreducible $\mathfrak{g}$-module $V_{2\rho-\alpha}$ occurs 2^l times in $\wedge\mathfrak{g}$, which are distributed as follows:

$r-1$	r	$r+1$	$r+2$	$\cdots$	$r+l-2$	$r+l-1$	$r+l$	$r+l+1$
C_{l-1}^0	C_{l-1}^1	C_{l-1}^2	$C_{l-1}^3+C_{l-1}^{l-1}$	$\cdots$	$C_{l-1}^{l-1}+C_{l-1}^3$	C_{l-1}^2	C_{l-1}^1	C_{l-1}^0

Here the first row denotes dimension, and the second row denotes the according times of occurrence of $V_{2\rho-\alpha}$.

The paper is organized as follows: after collecting necessary preliminaries in Section 2, we will prove Proposition 1.2 in Section 3, and Theorem 1.3 in Section 4.

2. Preliminaries

We extend V and its canonical inner product by setting $\widehat{V} = V \oplus \mathbb{R}\delta \oplus \mathbb{R}\lambda$, $(\delta|\delta) = (\delta|V) = (\lambda|\lambda) = (\lambda|V) = 0$, and $(\delta|\lambda) = 1$. The affine root system associated to Δ is $\widehat{\Delta} = \Delta + \mathbb{Z}\delta$. The set of positive affine roots is $\widehat{\Delta}^+ = (\Delta^+ + \mathbb{N}\delta) \cup (\Delta^- + \mathbb{N}^+\delta)$, where $\Delta^- = -\Delta^+$. We denote by $\alpha_0 = -\theta + \delta$ and set $\widehat{\Pi} = \{\alpha_0, \alpha_1, \cdots, \alpha_l\}$. For each $\alpha \in \widehat{\Delta}^+$, let s_α be the corresponding reflection of $\widehat{V}$. The affine Weyl group $\widehat{W}$ associated to Δ is the group generated by $\{s_\alpha | \alpha \in \widehat{\Delta}^+\}$. Now consider the $\widehat{W}$-invariant affine subspace $E := V \oplus \mathbb{R}\delta + \lambda$. Let $\pi : E \to V$ be the natural projection. For $w \in \widehat{W}$, set $\overline{w} = \pi \circ w|_E$. Then the map $w \mapsto \overline{w}$ gives an isomorphism of $\widehat{W}$ to a group of affine translations W_{af} of V. Thus we may and we will omit the bar, and use the same notation for the elements in $\widehat{W}$ and the corresponding elements in W_{af}. We refer the reader to [1] and the references therein for more about affine Weyl groups.

For any $\alpha \in \Delta^+$, $k \in \mathbb{N}^+$, $h \in \mathbb{N}$, by (1.1) of [1], we have

(2.1) $w^{-1}(-\alpha + k\delta) < 0$ if and only if $H_{\alpha,k}$ separates A and $w(A)$,

(2.2) $w^{-1}(\alpha + h\delta) < 0$ if and only if $H_{\alpha,-h}$ separates A and $w(A)$.

3. Proof of Proposition 1.2

This section is devoted to the proof of Proposition 1.2. Firstly, let us show that for any $\mathfrak{i} \in \mathcal{I}$, the following are equivalent:

(a') the index of nilpotence of $\mathfrak{i}$ equals to k;

(b') $V_\mathfrak{i} \cap kA$ is empty and $V_\mathfrak{i} \cap (k+1)A$ is non-empty.

Recall that the *index of nilpotence* of $\mathfrak{i}$, denoted by $n(\mathfrak{i})$, is the number of nonzero terms of the descending central series of $\mathfrak{i}$:

$$\mathfrak{i}^1 = \mathfrak{i}, \mathfrak{i}^2 = [\mathfrak{i}, \mathfrak{i}], \ldots, \mathfrak{i}^k = [\mathfrak{i}^{k-1}, \mathfrak{i}], \ldots.$$

For example, $n(0) = 0$ and $n(\mathfrak{i}) = 1$ for any nonzero abelian ideal $i \in \mathcal{I}$. For any $\mathfrak{i} \in \mathcal{I}$, let $\Phi(\mathfrak{i}^k)$ be the roots of $\mathfrak{i}^k$, $k \in \mathbb{N}^+$. Put

$$L_\mathfrak{i} = \bigcup_{k \geq 1} (-\Phi(\mathfrak{i}^k) + k\delta).$$

Recall that by Theorem 2.6 of [1], for any ad-nilpotent ideal $\mathfrak{i}$ of $\mathfrak{b}$, there exists a unique $w_\mathfrak{i} \in \widehat{W}$ such that $L_\mathfrak{i} = N(w_\mathfrak{i})$. Here, $N(w) := \{\alpha \in \widehat{\Delta}^+ | w^{-1}(\alpha) < 0\}$ for any $w \in \widehat{W}$. The equivalence of (a') and (b') will follow from the following two lemmas.

Lemma 3.1 *Let $\mathfrak{i} \in \mathcal{I}$ with $n(\mathfrak{i}) = k$. Then $V_\mathfrak{i} \cap kA$ is empty.*

Proof. Take any $x \in V_\mathfrak{i}$, it suffices to show that $(x|\theta) > k$. Indeed, since $n(\mathfrak{i}) = k$, we can find k roots $\varphi_1, \ldots, \varphi_k$ of $\mathfrak{i}$ such that $\varphi_1 + \cdots + \varphi_k$ is still a root of $\mathfrak{i}$. Then

$$(x|\theta) \geq (x|\varphi_1 + \cdots + \varphi_k) = (x|\varphi_1) + \cdots + (x|\varphi_k) > k. \qquad \blacksquare$$

Lemma 3.2 *Let $\mathfrak{i} \in \mathcal{I}$ with $n(\mathfrak{i}) = k$. Then*

$$w_\mathfrak{i}(A) \subseteq V_\mathfrak{i} \cap (k+1)A. \tag{3.1}$$

In particular, $V_\mathfrak{i} \cap (k+1)A$ is nonempty.

Proof. Take an arbitrary $\alpha \in \Delta^+$. Since $n(\mathfrak{i}) = k$, we have

$$-\alpha, -\alpha + (k+1)\delta \notin L(\mathfrak{i}) = N(w_\mathfrak{i}).$$

Thus, neither $H_{\alpha,0}$ nor $H_{\alpha,k+1}$ separates A and $w_\mathfrak{i}(A)$ by (2.2). Therefore, $w_\mathfrak{i}(A) \subseteq (k+1)A$. On the other hand,

$$\alpha \in \Phi(\mathfrak{i}) \iff -\alpha + \delta \in L_\mathfrak{i} = N(w_\mathfrak{i}) \iff H_{\alpha,1} \text{ separates } A \text{ and } w_\mathfrak{i}(A).$$

Therefore, $w_\mathfrak{i}(A) \subseteq V_\mathfrak{i}$. $\blacksquare$

Specializing k to be 0 and 1, the equivalence of (a') and (b') implies that of (a) and (b) in Proposition 1.2. The equivalence of (a) and (c) in Proposition 1.2 follows from the following

Lemma 3.3 *Let $\mathfrak{a} \in \mathcal{I}$ be abelian. Then* $\mathrm{vol}(V_\mathfrak{a} \cap 2A) = \mathrm{vol}(A)$.

Proof. By (3.1), we have

$$\sum_{\mathfrak{a} \text{ abelian}} \mathrm{vol}(w_\mathfrak{a}(A)) \leq \sum_{\mathfrak{a} \text{ abelian}} \mathrm{vol}(V_\mathfrak{a} \cap 2A).$$

We will show that both sides equal to $\mathrm{vol}(2A)$. Then for each abelian $\mathfrak{a}$, we would have $\mathrm{vol}(V_\mathfrak{a} \cap 2A) = \mathrm{vol}(w_\mathfrak{a}(A)) = \mathrm{vol}(A)$, as desired.

For the RHS, since $V_\mathfrak{i} \cap 2A$ is empty when $n(\mathfrak{i}) > 1$, we have

$$\sum_{\mathfrak{a} \text{ abelian}} \mathrm{vol}(V_\mathfrak{a} \cap 2A) = \sum_{\mathfrak{i} \in \mathcal{I}} \mathrm{vol}(V_\mathfrak{i} \cap 2A) = \mathrm{vol}(V_\infty \cap 2A) = \mathrm{vol}(2A).$$

For the LHS, the proof of Theorem 2.9 [1] shows that for any $w \in \widehat{W}$, $w(A) \subseteq 2A$ if and only if $w = w_\mathfrak{a}$ for some abelian ideal $\mathfrak{a}$ of $\mathfrak{b}$. Thus, we have

$$\sum_{\mathfrak{a} \text{ abelian}} \mathrm{vol}(w_\mathfrak{a}(A)) = \mathrm{vol}(2A). \qquad \blacksquare$$

4. The maximal eigenvalue of Cas on $\wedge^{r-1}\mathfrak{g}$

This section is devoted to determining the maximal eigenvalue of the Casimir operator on $\wedge^{r-1}\mathfrak{g}$, where r is the number of positive roots. We begin with some preliminaries.

Let $\wedge\mathfrak{g}$ be the exterior algebra. Then the adjoint action of $\mathfrak{g}$ extends naturally to $\wedge\mathfrak{g}$. One knows that, as $\mathfrak{g}$-modules, $\wedge^k\mathfrak{g}$ is isomorphic to $\wedge^{n-k}\mathfrak{g}$ for each k. Therefore,

$$m_k = m_{n-k} \tag{4.1}$$

and

$$M_k \cong M_{n-k}. \tag{4.2}$$

Recall that m_k is the maximal eigenvalue of Cas on $\wedge^k\mathfrak{g}$ and M_k is the corresponding eigenspace. Note also that, as $\mathfrak{g}$-modules one has (see [8])

$$\wedge\mathfrak{g} = 2^l\, V_\rho \otimes V_\rho. \tag{4.3}$$

Lemma 4.1 *For any simple root α, $V_{2\rho-\alpha}$ occurs exactly once in $V_\rho \otimes V_\rho$.*

Proof. For any simple root α, let s_α be the simple reflection relative to α. Then $s_\alpha\rho = \rho - \alpha$ is an extreme weight of V_ρ and it is easy to check that $2\rho - \alpha$ is dominant for Δ^+. Thus by the Steinberg formula (cf. Theorem 24.4 of [5]), the multiplicity of $V_{2\rho-\alpha}$ in $V_\rho \otimes V_\rho$ is equal to

$$\sum_{\sigma\in W}\sum_{\tau\in W} \operatorname{sign}(\sigma\tau)p(2\sigma\rho + 2\tau\rho - 4\rho + \alpha),$$

where p is the Kostant partition function. Note that

$$\sigma\rho = \rho - \langle\Phi(\sigma)\rangle,$$

where $\Phi(\sigma) = \sigma(-\Delta^+) \cap \Delta^+$ and $\langle\Phi(\sigma)\rangle$ denotes the sum of roots in $\Phi(\sigma)$. Similarly, we have $\tau\rho = \rho - \langle\Phi(\tau)\rangle$. Thus the multiplicity equals to

$$\sum_{\sigma\in W}\sum_{\tau\in W} \operatorname{sign}(\sigma\tau)p(\alpha - 2\langle\Phi(\sigma)\rangle - 2\langle\Phi(\tau)\rangle).$$

One sees easily that in the above sum the unique nonzero term occurs only when $\sigma = \tau = 1$. Thus the concerned multiplicity equals to one. ∎

Now let us recall some notation and results from [6]. An element $w \in \wedge^k\mathfrak{g}$ is called *decomposable* if $w = z_1 \wedge \cdots \wedge z_k$, where $z_i \in \mathfrak{g}$. In such a case, we let $\mathfrak{a}(w)$

be the subspace spanned by $z_1, \dots, z_k$. Now a $\mathfrak{g}$-submodule $V_1 \subseteq \wedge^k \mathfrak{g}$ is called *decomposably-generated* if it is spanned by decomposable elements.

A subspace $\mathfrak{a} \subseteq \mathfrak{g}$ is called $\mathfrak{b}$-*normal* if $[\mathfrak{b}, \mathfrak{a}] \subseteq \mathfrak{a}$. In such a case, $\mathfrak{a}$ is stable under $ad\,\mathfrak{h}$ and there exists a unique set of roots $\Delta(\mathfrak{a}) \subseteq \Delta$ such that

$$\mathfrak{a} = \mathfrak{a} \cap \mathfrak{h} + \sum_{\alpha \in \Delta(\mathfrak{a})} \mathfrak{g}_\alpha.$$

Define $\langle \mathfrak{a} \rangle = \sum_{\alpha \in \Delta(\mathfrak{a})} \alpha$.

Proposition 4 of [6] asserts that each M_k is a direct sum of decomposably-generated simple $\mathfrak{g}$-submodules of $\wedge^k \mathfrak{g}$. Proposition 6 of [6] establishes a one-one correspondence between the set of all k-dimensional $\mathfrak{b}$-normal subspaces of $\mathfrak{g}$ and all decomposably-generated simple $\mathfrak{g}$-submodules of $\wedge^k \mathfrak{g}$. The correspondence can be described as follows: starting with a k-dimensional $\mathfrak{b}$-normal subspace $\mathfrak{a}$ of $\mathfrak{g}$, then the cyclic submodule $V_\mathfrak{a}$ of $\wedge^k \mathfrak{g}$ generated by $\wedge^k \mathfrak{a}$ is irreducible, decomposably-generated. Actually, $V_\mathfrak{a} \cong V_{\langle \mathfrak{a} \rangle}$ and $\wedge^k \mathfrak{a}$ is the highest weight vector. Thus, to determine m_k and M_k, it suffices to consider all the k-dimensional $\mathfrak{b}$-normal subspaces of $\mathfrak{g}$.

Proof of Theorem 1.3. As mentioned above, to investigate m_{r-1} and M_{r-1}, it suffices to focus on the $(r-1)$-dimensional $\mathfrak{b}$-normal subspaces $\mathfrak{a}$ of $\mathfrak{g}$. As shown in the proof of Theorem 3.2(3) of [4], for $V_\mathfrak{a}$ to lie in M_{r-1}, we must have $\Delta(\mathfrak{a}) \subseteq \Delta^+$. Since $\dim \mathfrak{a} = r - 1$ and $\mathfrak{a}$ is $\mathfrak{b}$-normal, we must have $\Delta(\mathfrak{a}) = \Delta^+ \setminus \{\alpha\}$, where α is a simple root. Since

$$Cas(2\rho - \alpha) = \|3\rho - \alpha\|^2 - \|\rho\|^2 = 8\|\rho\|^2 - 2\|\alpha\|^2 = \frac{1}{3}\dim \mathfrak{g} - 2\|\alpha\|^2$$

for any simple root α, we see that

$$m_{r-1} = \frac{1}{3}\dim \mathfrak{g} - 2\|\alpha\|^2$$

for any *short* simple root $\alpha \in \Pi_s$. Now (1.8) follows from (1.6) and (1.7).

Note also that $V_{2\rho-\alpha}, \alpha \in \Pi_s$, are precisely the irreducible components of M_{r-1}. Now for any $\alpha \in \Pi_s$, it remains to figure out the multiplicity n_α of $V_{2\rho-\alpha}$ in M_{r-1}, which is also the multiplicity of $V_{2\rho-\alpha}$ in $\wedge^{r-1}\mathfrak{g}$. One way to do this is to cite Han's result [4], which says that M_{r-1} is multiplicity free. Thus $n_\alpha = 1$. We give another approach. By (4.3) and Lemma 4.1, $V_{2\rho-\alpha}$ occurs 2^l times in $\wedge\mathfrak{g}$. Let us determine these occurrences explicitly. Let $h_1, \dots, h_{l-1}$ be a basis of $\mathfrak{h}^\alpha := \{x \in \mathfrak{h} \mid \alpha(x) = 0\}$. Let u_α be the wedge product of all the r positive root vectors but e_α. Let $v_\alpha = u_\alpha \wedge [e_\alpha, e_{-\alpha}] \wedge e_\alpha \wedge e_{-\alpha}$. Then it is apparent that, for $0 \le s \le l-1$,

$$u_\alpha \wedge h_{j_1} \wedge \cdots \wedge h_{j_s}, v_\alpha \wedge h_{j_1} \wedge \cdots \wedge h_{j_s}, 1 \le j_1 < \cdots < j_s \le l-1$$

generate 2^l copies of $V_{2\rho-\alpha}$ in $\wedge\mathfrak{g}$. Thus they exhaust all the occurrences of $V_{2\rho-\alpha}$ in $\wedge\mathfrak{g}$. In particular $n_\alpha = 1$, we have (1.9), and the desired distribution of $V_{2\rho-\alpha}$

in $\wedge\mathfrak{g}$ follows. Finally, it is easy to see that the same distribution also holds for any simple root which is not necessarily short. ∎

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OPTIMAL INVESTMENT DECISION USING NEURAL REDUCT FUZZY SET OF IVFS

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Abstract. Taking optimal investment decision is very challenging in the current scenario. Traditionally, the investors were followed fundamental analysis or technical analysis in the investment decision. But, today the market and technology are growing very faster than the investors' ideas. Every new method is coming for the investment analysis. All the decisions are future oriented which relies uncertainty. It is inevitable to apply a tool which is linked with the uncertainty. Hence, in this study, 'IVFS' technique has been used which measures the uncertainty. Finding out the exact membership in Interval Valued Fuzzy is a crucial part in the investment analysis. To reduce this problem, here the notion of neural reduct fuzzy set of interval valued fuzzy set has been applied. In a clear sense, this paper finds the membership using interval value and converge into one single point that act as member for the fuzzy set.

Keywords: interval valued fuzzy set, soft set, ivfs-set, neural reduct fuzzy set, investment decision.

AMS Classification 2010: 03E72, 03E75, 62C86.

1. Introduction

Zadeh [6], [7] and Molodtsov [2] initiated interval valued fuzzy set and soft set respectively in their studies. Yang et al. [5] presented the concept of interval-valued fuzzy soft set by combining the interval-valued fuzzy set and soft set models. Later many researchers like [1], [4] using these tools for decision making for various needs. In this paper Interval valued Fuzzy Soft Decision Set is introduced for decision making situation and Neural Reduct fuzzy set [3] is applied for optimal investment decision.

2. Interval valued fuzzy soft set and interval valued fuzzy soft decision set

In this section, the basic definitions of interval valued fuzzy set and soft set have been presented. Definition of interval valued fuzzy soft set (IVFS-set) and interval valued fuzzy decision set of an IVFS-set are also used to construct a decision method using neural reduct fuzzy set of Inter valued fuzzy set.

Definition 2.1 [6] An *Interval-Valued Fuzzy Set* $\tilde{\mu}$ on a universe U is a mapping such that $\tilde{\mu} : U \to D([0,1])$, where $D([0,1])$ is the set of all closed sub intervals of $[0,1]$. The set of all interval-valued fuzzy sets on U is denoted by $\tilde{\mu}(U)$. Also, $\forall x \in U, \tilde{\mu}_x(x) = [\mu_x^L(x), \mu_x^U(x)]$ is called the degree of membership of an element x to $\tilde{\mu}$ where $\mu_x^L(x)$ and $\mu_x^U(x)]$ are the lower and upper degrees of membership of x to $\tilde{\mu}$ respectively such that $0 \leq \mu_x^L(x) \leq \mu_x^U(x) \leq 1$.

Example 2.2 Let $U = \{u_1, u_2, u_3, u_4\}$ be the Universe. Then $\tilde{\mu}(U)$ defined following is an Interval-Valued Fuzzy Set.

$$\begin{aligned}\tilde{\mu}(u_1) &= [0.4, 0.7]\\ \tilde{\mu}(u_2) &= [0.3, 0.5]\\ \tilde{\mu}(u_3) &= [0.1, 0.6]\\ \tilde{\mu}(u_4) &= [0.2, 0.8]\end{aligned}$$

Definition 2.3 [3] An *Neural Reduct fuzzy set* $\mu_N(U)$ of Interval-Valued Fuzzy Set $\tilde{\mu}(U)$ is defined by $\mu_N(U) = \left\{ \left\langle x \, , \frac{(\mu_x^L(x) + \mu_x^U(x))}{2} \right\rangle \; x \in U \right\}$.

Example 2.4 For the Universe defined in Example, 2.2, the Neural Reduct fuzzy set $\mu_N(U)$ is defined as follows: $\mu_N(U) = \{\langle u_1, 0.55\rangle \langle u_2, 0.4\rangle \langle u_3, 0.35\rangle \langle u_4, 0.5\rangle\}$.

Definition 2.5 [2] Let U refers to an initial universe, E is a set of all parameters, $P(U)$ is the power set of U, and $A \subseteq E$. A *Soft set* F_A over U is a set defined by a function f_A representing a mapping $f_A : E \to P(U)$ such that $f_A = \phi$ if $x \notin A$. Thus a soft set over U can be represented by the set of ordered pairs $F_A = \{(x, f_A x) : x \in E, f_A x \in P(U)\}$.

Example 2.6 Consider the Universe $U = \{u_1, u_2, u_3, u_4\}$ defined in Example. 2.2 Suppose $E = \{x_1, x_2, x_3, x_4, x_5\}$ is the set of parameters. Then the following is a Soft set F_A over U for the subset $A = \{x_1, x_2, x_3\}$ of E: $F_A = \{(x_1, \{u_1, u_2\}), (x_2, \{u_2, u_3\}), (x_3, \{u_1, u_2, u_3\})\}$.

Definition 2.7 [4] *Interval Valued Fuzzy Soft* [IVFS] *set.* Let U refers to an initial universe, E is a set of all parameters, $P(U)$ is the power set of U, and X be a fuzzy set over E. An IVFS-set on U is defined by the pair $F_X = \{(\tilde{\mu}_X(x)/x, f_X x) : x \in E, \; f_X x \in P(U), \tilde{\mu}_X(x) \in D([0,1])\}$, where the function $f_X : E \to P(U)$ is called the *approximate function* such that $f_X(x) = \phi$ if $\tilde{\mu}_X(x) = [0,0]$, and $\tilde{\mu}_X : E \to D([0,1])$ is called the *membership function* of IVFS-set F_X. The value of $\tilde{\mu}_X(x)$ is the degree of importance of the parameter x, and depends on the decision maker's requirements. The sets of all IVFS-sets over U will be denoted by IVFS(U).

Example 2.8 Consider the Universe $U=\{u_1, u_2, u_3, u_4\}$ and the set of parameters E as $E=\{x_1, x_2, x_3\}$. Then $F_X=\{([0.3, 0.6]/x_1, \{u_1, u_2\}), ([0.5, 0.8]/x_2, \{u_2, u_3\}), ([0.1, 0.4]/x_3, \{u_1, u_2, u_3\})\}$ is an Interval Valued Fuzzy Soft set.

Definition 2.9 *Interval Valued Fuzzy Soft Decision* (IVFSD) *set.* Let $F_X \in IVFS(U)$. Then, an interval valued fuzzy soft decision (IVFSD) set of F_X denoted by F_X^D and is defined by $F_X^D = \left\{\tilde{\mu}_{F_X^D}(u)/u : u \in U\right\}$, which is an interval valued fuzzy set over U and its membership function $\tilde{\mu}_{F_X^D}$ is defined by $\tilde{\mu}_{F_X^D} : U \to D([0,1])$ and $\tilde{\mu}_{F_X^D}(u)= \left[\frac{1}{|E|}\sum_{x\in E}\mu_X^L(x), \frac{1}{|E|}\sum_{x\in E}\mu_X^U(x)\right]$.

The following section gives an illustration for the IVFSD set.

3. Analysis of optimal investment decision

In this section, an algorithm has been introduced using Neural Reduct Fuzzy set of IVFSD-set to choose the optimal investment avenue from the alternatives.

1. Design a Soft set F_A over U.
2. Design a IVFS-set F_X over U.
3. Construct the IVFSD-set F_X^D.
4. Obtain Neural Reduct Fuzzy set of F_X^D.
5. Choose the maximum membership value from Neural Reduct Fuzzy set of F_X^D.

The study has been carried out by considering 50 investors in a basket of Middle Income Group having the annual income of Rs.3 Lacs to Rs.5 Lacs. A structured questionnaire has been circulated by asking their preference towards the selected investments along with corresponding constraints. Four major varieties of investment avenues namely Fixed Deposits in Bank, Gold, Real estate and Shares have been identified. In symbolic form, it is noted as I_1, I_2, I_3 and I_4. The criteria for the preferences of investments are indicated as Return, Growth, Risk less security, Tax benefit. These are denoted as C_1, C_2, C_3 and C_4 respectively.

Step 1. To apply IVFS Decision Set in Investment decision, a universal set $U = \{I_1, I_2, I_3, I_4\}$ has been framed and the parameters used for analyzing the investment characteristics by the investors are considered as the set of Constraints $C = \{C_1, C_2, C_3, C_4\}$. Based on the responses, the I1 (FD) having the characteristic of Return, Risk less security and Tax benefit, the I2 (Gold) is having the characteristic of Risk less security and Growth, I3(Real Estate) is having the characteristic of Growth and I4 (Shares) is having the characteristic of Return and Growth. Hence the Soft set F_A is designed as $F_A=\left\{ ((C_1), (I_3, I_4)); ((C_2), (I_2, I_3, I_4)); ((C_3), (I_1, I_2)); ((C_4), (I_1)) \right\}$.

Step 2. The points are given by the investors from 0 to 1 based on their preferences towards the selected investment avenues. From these points the lowest point and the highest point are taken as lower membership and upper membership grades for the IVF set respectively. By following the above procedure, the IVFS set has been framed.

The IVFS-set F_X is designed as

$$\left\{ \begin{array}{c} (([0.4,0.6]/C_1),(I_3,I_4));(([0.28,0.72]/C_2),(I_1,I_3,I_4)); \\ (([0.36,0.64]/C_3),(I_1,I_2));(([0.44,0.56]/C_4),(I_4)) \end{array} \right\}$$

Step 3. The IVFSD-set F_X^D has been attained using Definition 2.5.

$$F_X^D = \{[0.30,0.45]/I_1,\ [0.23,0.34]/I_2,\ [0.07,0.18]/I_3,\ [0.17,0.33]/I_4\}$$

Step 4. Neural Reduct Fuzzy set of the IVFSD-set F_X^D of given by

$$\{0.375/I_1,\ 0.285/I_2,\ 0.125/I_3,\ 0.250/I_4\}$$

Step 5. It is clear that the maximum membership value 0.375 occurs on I_1. i.e, Fixed Deposit is decided as the optimal investment.

4. Conclusion

The responses by the respondents are favor to the Fixed Deposit. It is a Risk less security having a reasonable return with tax benefit which impress the middle income group. Having all these factors, the FD is preferable investment with the selected criteria. Hence it is concluded that FD is the optimal investment decision for the middle income group. One can apply the notion of IVFSD in medical diagnosis, neural networks, Managerial and Engineering decision making situation to make the decision optimally.

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ON THE EXCHANGE PROPERTY FOR THE HARTLEY TRANSFORM

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Abstract. In this paper we investigate the exchange property for the Hartley transform by using the relation between the Fourier transform and the Hartley transform. Simplified construction of tempered Boehmians is also presented.

Keywords: distribution spaces, Boehmians, Fourier transform, Hartley transform.
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1. Introduction

The concept of Boehmians is motivated by the regular operator introduced by Boehme [2], which form a subalgebra of the field of Mikusiński operators and thus they include only such functions whose support is bounded from the left. The theory of Boehmians (quotient of sequences), its properties and different classes of Boehmian spaces can be studied in [1], [8], [9]. Tempered Boehmians is a natural extension of tempered distribution which, therefore, makes it possible to define an extension of the Fourier transform for this class of Boehmians. The Fourier transform of a tempered Boehmian is a distribution. A continuous function $f : \mathbb{R}^N \to \mathbb{C}$ is called slowly increasing if there is a polynomial p on $\mathbb{R}^N$ such that $|f(x)| \leq p(x)$ for all $x \in \mathbb{R}^N$. The space of slowly increasing function will be denoted by $J(\mathbb{R}^N)$ or simply by J. An infinitely differentiable function is called rapidly decreasing if

$$\sup_{|\alpha|\leq m} \sup_{x\in\mathbb{R}^N} \left(1 + x_1^2 + x_2^2 + \cdots + x_N^2\right)^m |D^\alpha f(x)| < \infty, \tag{1}$$

for every non-negative integer m, where $x = (x_1, x_2, ..., x_N)$, $\alpha = (\alpha_1, ...\alpha_N)$, α_N's are non negative integer, $|\alpha| = |\alpha_1| + \cdots + |\alpha_N|$, and

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}} \ . \tag{2}$$

The space of all rapidly decreasing functions is denoted by $S(\mathbb{R}^N)$ or simply by S. If $f \in J$ and $\varphi \in S$, then the convolution

$$(f * \varphi)(x) = \int_{\mathbb{R}^N} f(u)\varphi(x-u)du \tag{3}$$

is well defined and $f * \varphi \in J$. A sequence $\varphi_n \in S$ is called a delta sequence if it satisfies the following conditions

(i) $\int_{\mathbb{R}^N} \varphi_n dx = 1\ ,\ \forall n \in \mathbb{N}$

(ii) $\int_{\mathbb{R}^N} |\varphi_n|\, dx \leq C$, for some constant M and for all $n \in \mathbb{N}$

(iii) $\lim_{n\to\infty} \int_{\|x\|\geq\epsilon} \|x\|^k |\varphi_n(x)|\, dx = 0$, for every $k \in \mathbb{N}$ and $\epsilon > 0$.

If $\varphi \in S$ and $\int \varphi = 1$, then the sequence of functions φ_n is a delta sequence. A complex-valued function f on $\mathbb{R}^N$ is called slowly increasing if there exists a polynomial p on $\mathbb{R}^N$ such that $f(x)/p(x)$ is bounded. The space of all increasing continuous functions on $\mathbb{R}^N$ is denoted by $\mathcal{I}$. If $f_n \in \mathcal{I}$, $\{\varphi_n\}$ is a delta sequence under usual notion, then the space of equivalence classes of quotients of sequence will be denoted by $\mathcal{B}_\mathcal{I}$, elements of which will be called tempered Boehmians. For $F = [f_n/\varphi_n] \in \mathcal{B}_\mathcal{I}$, define $D^\alpha F = [(f_n * D^\alpha\varphi_n)/(\varphi_n * \varphi_n)]$. If F is a Boehmian corresponding to differentiable function, then $D^\alpha F \in \mathcal{B}_\mathcal{I}$.

If $F = [f_n/\varphi_n] \in \mathcal{B}_\mathcal{I}$ and $f_n \in S$, for all $n \in \mathbb{N}$, then F is called a rapidly decreasing Boehmian. The space of all rapidly decreasing Boehmian is denoted by $\mathcal{B}_S$. If $F = [f_n/\varphi_n] \in \mathcal{B}_\mathcal{I}$ and $G = [g_n/\psi_n] \in \mathcal{B}_S$, then the convolution is

$$F * G = [(f_n * g_n)/(\varphi_n * \psi_n)] \in \mathcal{B}_\mathcal{I}.$$

In what follows, we will denote by S' the space of tempered distributions, that is, the space of continuous linear functional on S. If $f \in S'$ and $\varphi \in S$, then the convolution $f * \varphi$ is defined as $(f * \varphi)(x) = f(\varphi_x)$, where $\varphi_x(z) = \varphi(x - z)$. The Hartley transform of a tempered distribution f, denoted by $\mathcal{H}f$, is defined by $\mathcal{H}f(\varphi) = f(\mathcal{H}\varphi)$, where $\mathcal{H}\varphi$ is the Hartley transform of φ defined by Nait *et al.* [11]. Further, Hartley transform on generalized functions and spaces of Boehmians is studied in [6], [7], [13], [14].

The Hartley transform is an integral transformation that maps a real-valued temporal or spacial function into a real-valued frequency function via the kernel, $cas(\nu x) \equiv cos(\nu x) + sin(\nu x)$. Hartley [5] studied a more symmetrical Fourier integral, which leads to a parallelism that exists between the function of the original variable and that of its transform.

The Hartley transform of a function $f(x)$ is defined by Olejniczak [12]:

$$\mathcal{H}(\nu) = \mathcal{H}[f](\nu) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)cas(\nu x)dx, \tag{4}$$

$$\text{or} \quad \mathcal{H}(f) = \int_{\mathbb{R}} f(x)cas(2\pi f x)dx, \tag{5}$$

where the angular frequency variable ν is related to the frequency variable f by $\nu = 2\pi f$, and

$$\mathcal{H}(f) = \sqrt{2\pi}\mathcal{H}(2\pi f) = \sqrt{2\pi}\mathcal{H}(\nu). \tag{6}$$

The integral kernel, known as the cosine and sine function, is define by

$$cas(\nu x) \equiv cos(\nu x) + sin(\nu x) = \sqrt{2}\sin\left(\nu x + \frac{\pi}{4}\right) = \sqrt{2}\cos\left(\nu x - \frac{\pi}{4}\right). \tag{7}$$

The inverse Hartley transform is defined by

$$\mathcal{H}^{-1}(\mathcal{H}(f)) = f(x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathcal{H}(\nu)cas(\nu x)d\nu = \int_{\mathbb{R}} \mathcal{H}(f)cas(2\pi f x)df. \tag{8}$$

The relation between the Hartley and Fourier transform is given by Olejniczak [12]

$$\mathcal{H}(\nu) = [Re\{F(\omega)\} - Im\{F(\omega)\}]_{\omega=\nu}, \tag{9}$$

$$\text{where} \quad Re\{F(\omega)\} = Re(\omega) = \mathcal{H}^e(\nu) = \mathcal{H}^e(-\nu), \tag{10}$$

$$Im\{F(\omega)\} = Im(\omega) = -\mathcal{H}^o(\nu) = \mathcal{H}^o(-\nu), \tag{11}$$

$\mathcal{H}$ is the Hartley transform and $\mathcal{H}^e(\nu)$ and $\mathcal{H}^o(\nu)$ are, respectively, even and odd parts of the Hartley transform. $Re\{F(\omega)\}$ is the real part and $Im\{F(\omega)\}$ is the imaginary part of the Fourier transform. We mention elementary properties of the Hartley transform as follows [7]:

1. Linearity: If $f_1(x)$ and $f_2(x)$ have the Hartley transforms $\mathcal{H}_1(f)$ and $\mathcal{H}_2(f)$, respectively, α and β are positive constants, $\alpha, \beta > 0$, then

$$\int_{\mathbb{R}} [\alpha f_1(x) + \beta f_2(x)]cas(2\pi f x)dx = \alpha\mathcal{H}_1(f) + \beta\mathcal{H}_2(f). \tag{12}$$

2. Scaling/Similarity: If $f(x)$ is the Hartley transform (4), then

$$\int_{\mathbb{R}} cas(2\pi f x)dx = \frac{1}{k}\mathcal{H}\left(\frac{f}{k}\right). \tag{13}$$

3. *nth* **Derivative of a Function** $f^n(x)$

$$f^n(x) = cas'\frac{n\pi}{2}(2\pi f)^n\mathcal{H}[(-1)^n f], \tag{14}$$

which is derived by recursive application of equation (9) to the Fourier transform of the function $\frac{df}{dx}$ and its higher order. Here $cas'(\xi) = cos(\xi) - sin(\xi)$ (complementary *cas* function) and $\frac{d}{d\tau}cas\tau = cas(-\tau) = cas'(\tau)$ (derivative relation), respectively.

4. Function Shift/Delay: If $f(x)$ is shifted in time by a constant T, then by substituting $x' = x - T$, the Hartley transform is

$$\mathcal{H}(f) = \int_{\mathbb{R}} f(x')cas[2\pi f(x'+T)]dx' = cos(2\pi fT)\mathcal{H}(f) + sin(2\pi fT)\mathcal{H}(-f). \tag{15}$$

5. Convolution: If $f_1(x)$ has the Hartley transform $\mathcal{H}_1(f)$ and $f_2(x)$ has the Hartley transform $\mathcal{H}_2(f)$, then

$$\begin{aligned} &f_1(x) * f_2(x) \\ &= \frac{1}{2}[\mathcal{H}_1(f)\mathcal{H}_2(f) + \mathcal{H}_1(-f)\mathcal{H}_2(f) + \mathcal{H}_1(f)\mathcal{H}_2(-f) - \mathcal{H}_1(-f)\mathcal{H}_2(-f)]. \end{aligned} \tag{16}$$

To obtain the result (16), simply substitute the convolution integral $f_1(x)*f_2(x) = \int_{\mathbb{R}} f_1(\lambda)f_1(x-\lambda)d\lambda$ in (5) and using shift /delay property, we get

$$\mathcal{H}(f) = \int_{\mathbb{R}} (f_1(x) * f_2(x))cas(2\pi fx)dx = \int_{\mathbb{R}} f_1(\lambda)[\int_{\mathbb{R}} f_2(x-\lambda)cas(2\pi fx)dx]d\lambda$$
$$= \int_{\mathbb{R}} f_1(\lambda)[cas(2\pi f\lambda)\mathcal{H}_2(f) + sin(2\pi f\lambda)\mathcal{H}_2(-f)]d\lambda, \quad \text{(by (12))}.$$

Factorizing the $\mathcal{H}_2(\cdot)$ of the right hand side, and using other properties of Hartley transform, equation (16) is defined. Simplification of equation (16) results into following symmetries in different forms, e.g.

(i) If $f_1(x)$ and (or) $f_2(x)$ is even, or if $f_1(x)$ is even and $f_2(x)$ is odd, or if $f_1(x)$ is odd and $f_2(x)$ is even, then $f_1(x) * f_2(x) = \mathcal{H}_1(f)\mathcal{H}_2(f)$.

(ii) If $f_1(x)$ is odd, then $f_1(x) * f_2(x) = \mathcal{H}_1(f)\mathcal{H}_2(-f)$.

(iii) If $f_2(x)$ is odd, then $f_1(x) * f_2(x) = \mathcal{H}_1(-f)\mathcal{H}_2(f)$.

(iv) (iv) If both functions are odd, then $f_1(x) * f_2(x) = -\mathcal{H}_1(f)\mathcal{H}_2(f)$.

In general, equation (16) can be written in the following form given by Loonker [6]:

$$\mathcal{H}[f * g] = \mathcal{H}[f]\mathcal{H}[f]. \tag{17}$$

6. Parseval relation: If $f_1(x)$ has the Hartley transform $\mathcal{H}_1(f)$ and $f_2(x)$ has the Hartley transform $\mathcal{H}_2(f)$, then

$$\int_{\mathbb{R}} f_1(x)f_2(x)dx = \int_{\mathbb{R}} \mathcal{H}_1(f)\mathcal{H}_2(f)df. \tag{18}$$

The Hartley transform of distributions is defined by Chaudhary and Thorat [4]. For the space of slow growth S', inversion theorems are proved by Nair and Banerji [11], using the Parseval equation for the Hartley transform

$$< \mathcal{H}(f), \varphi > = < f, \mathcal{H}(\varphi) >, \tag{19}$$

where function f is absolutely integrable and φ is a testing function of rapid descent, and for tempered distribution of Hartley transform, we have

$$|\nu^m \mathcal{H}^{(k)}(\nu)| \leq \Upsilon_{m,k}, \ \nu \in \mathbb{R} \tag{20}$$

where m and k are non-negative integers and Υ is a constant.

2. The exchange property

For a family $\{\varphi_i\}_{i\in I}$=$\{\varphi_i\}_I$, where I is an index set and φ_i∈S, $\forall i$∈I, we have [1]:

$$\Psi(\{\varphi_i\}_I) = \{x \in \mathbb{R}^N : \hat{\varphi}_i(x) = 0 \ \ \forall i \in I\}. \tag{21}$$

A family of pairs $\{(f_i, \varphi_i)\}_I$, where $f_i \in S'$ and $\varphi_i \in S$, is said to have the exchange property if

$$f_i * \varphi_k = f_k * \varphi_i, \qquad \forall i, k \in I. \tag{22}$$

We will denote by A the collection of all families of pairs $\{(f_i, \varphi_i)\}_I$, where I is an index set, $f_i \in S'$ and $\varphi_i \in S$ for all $i \in I$, satisfying the exchange property such that $\Psi(\{\varphi_i\}_I) = \phi$. If (φ_i) is a delta sequence, then $\Psi(\{\varphi_i\}_N) = \phi$.

Definition 1 If $\{(f_i, \varphi_i)\}_I \in A$, then the unique $F \in \mathcal{D}'(\mathbb{R}^N)$ such that $\mathcal{H}f_i = \mathcal{H}\varphi_i F$, $\forall i \in I$ will be denoted by $\mathcal{H}\{(f_i, \varphi_i)\}_I$.

Let $\{(f_i, \varphi_i)\}_I, \{(g_k, \psi_k)\}_K \in A$. If $f_i * \psi_k = g_k * \varphi_i$ for all $i \in I$ and $k \in K$, then we write $\{(f_i, \varphi_i)\}_I \sim \{(g_k, \psi_k)\}_K$. This relation is clearly symmetric and reflexive. We will show that it is also transitive.

Let $\{(f_i, \varphi_i)\}_I, \{(g_k, \psi_k)\}_K, \{(h_l, \gamma_l)\}_L \in A$. If $\{(f_i, \varphi_i)\}_I \sim \{(g_k, \psi_k)\}_K$ and $\{(g_k, \psi_k)\}_K \sim \{(h_l, \gamma_l)\}_L$, then

$$f_i * \psi_k = g_k * \varphi_i, \quad g_k * \gamma_l = h_l * \psi_k, \quad \forall i \in I, k \in K, l \in L. \tag{23}$$

Therefore,

$$f_i * \psi_k * \gamma_l = g_k * \varphi_i * \gamma_l, \quad g_k * \gamma_l * \varphi_i = h_l * \psi_k * \varphi_i, \quad \forall i \in I, k \in K, l \in L. \tag{24}$$

Since $*$ is commutative, we have

$$f_i * \gamma_l * \psi_k = h_l * \varphi_i * \psi_k. \tag{25}$$

Now, for $i \in I$ and $l \in L$. Since $\Psi(\{\psi_k\}_K) = \phi$ and (25) holds for every $k \in K$, we conclude that $f_i * \gamma_l = h_l * \varphi_i$ $\forall i \in I$, $l \in L$, which means $\{(f_i, \varphi_i)\}_I \sim \{(h_l, \gamma_l)\}_L$.

Theorem 2 *If a family of pair $\{(f_i, \varphi_i)\}_I$ has the exchange property and $\Omega = \Psi(\{\varphi_i\}_I)^c$ (the complement of $\Psi(\{\varphi_i\}_I)$ in $\mathbb{R}^N$), then there exists a unique $F \in \mathcal{D}'(\Omega)$ such that*

$$\mathcal{H}[f_i] = F\mathcal{H}[\varphi_i], \quad \forall i \in I. \tag{26}$$

Proof. For every $x \in \Omega$ there exists $i \in I$ and $\varepsilon > 0$ such that $|\mathcal{H}\varphi_i(x)| > \varepsilon$ in an open neighbourhood of x. Then we can define $F = \mathcal{H}f_i / \mathcal{H}\varphi_i$ in that neighbourhood. Let for some $\varepsilon > 0$, we have $|\mathcal{H}\varphi_i(x)| > \varepsilon$ for all $x \in U$ and $|\mathcal{H}\varphi_k(x)| > \varepsilon$ for all $x \in V$, where U and V are open sets. Since $f_i * \varphi_k = f_k * \varphi_i$, we have

$$\mathcal{H}f_i \mathcal{H}\varphi_k = \mathcal{H}f_k \mathcal{H}\varphi_i \text{ and } \frac{\mathcal{H}f_i}{\mathcal{H}\varphi_i} = \frac{\mathcal{H}f_k}{\mathcal{H}\varphi_k}, \tag{27}$$

on $U \cap V$. This shows that F is a unique function. ■

Theorem 3 *$\{(f_i, \varphi_i)\}_I \in A$ if and only if there exists a unique $F \in \mathcal{D}'(\mathbb{R}^N)$ such that $\mathcal{H}[f_i] = \mathcal{H}[\varphi_i]F$ for all $i \in I$.*

Proof. For any $i, k \in I$, we have

$$\mathcal{H}f_i\mathcal{H}\varphi_k = F\mathcal{H}\varphi_i\mathcal{H}\varphi_k = F\mathcal{H}\varphi_k\mathcal{H}\varphi_i = \mathcal{H}f_k\mathcal{H}\varphi_i \text{ [by Theorem 1].} \tag{28}$$

This proves that the existence of such a function $F \in \mathcal{D}'(\mathbb{R}^N)$ implies and justifies the exchange property. This completes the proof of the theorem. ■

Theorem 4 *There exists $\{(f_i, \varphi_i)\}_I \in A$ for every $F \in \mathcal{D}'(\mathbb{R}^N)$ and such that $F = \mathcal{H}(\{(f_i, \varphi_i)\}_I)$.*

Proof. Since $\mathcal{D}(\mathbb{R}^N)$ denotes the space of smooth function with compact support, there exists a total sequence $\{\varphi_i\}_N$ such that $\mathcal{H}\varphi_i \in \mathcal{D}(\mathbb{R}^N)$, $\forall I \in N$. Then for every $I \in N$, there is $f_i \in S'$ such that $\mathcal{H}f_i = \mathcal{H}\varphi_i F$. Clearly $\{(f_i, \varphi_i)\}_I \in A$ and $F = \mathcal{H}(\{(f_i, \varphi_i)\}_N)$. This completes the proof of theorem. ■

Definition 5 [1] *Let $\{U_i\}_I$ be an open covering of $\mathbb{R}^N$ and let $\{\varphi_i\}_I$ be such that $|\mathcal{H}\varphi_i(x)| > 0$ for $x \in U_i$. A family $\{\varphi_i\}_I$ such that $\Psi(\{\varphi_i\}_I)=\phi$ will be called total.*

Lemma 6 [1] *If $(\{\varphi_i\}_I)$ and $(\{\psi_k\}_K)$ are total, then $\{\varphi_i * \psi_k\}_{I\times K}$ is total.*

Theorem 7 *Let $\{(f_i, \varphi_i)\}_I, \{(g_k, \psi_k)\}_K \in A$. Then $\{(f_i, \varphi_i)\}_I \sim \{(g_k, \psi_k)\}_K$ if and only if $\mathcal{H}(\{(f_i, \varphi_i)\}_I) = \mathcal{H}(\{(g_k, \psi_k)\}_K)$.*

Proof. $F = \mathcal{H}(\{(f_i, \varphi_i)\}_I)$ and $G = \mathcal{H}(\{(g_k, \psi_k)\}_K)$. If $\{(f_i, \varphi_i)\}_I \sim \{(g_k, \psi_k)\}_K$, then $F\mathcal{H}\varphi_i\mathcal{H}\psi_k = \mathcal{H}f_i\mathcal{H}\psi_k = \mathcal{H}g_k\mathcal{H}\varphi_i$

$$\Rightarrow F\mathcal{H}\varphi_i\mathcal{H}\psi_k = G\mathcal{H}\psi_k\mathcal{H}\varphi_i, \ \ \forall i \in I, k \in K. \tag{29}$$

Hence $F = G$, by Lemma 6. Now assume $F = G$. Then

$$\mathcal{H}f_i\mathcal{H}\psi_k = F\mathcal{H}\varphi_i\mathcal{H}\psi_k = G\mathcal{H}\psi_k\mathcal{H}\varphi_i = \mathcal{H}g_k\mathcal{H}\varphi_i, \ \ \forall i \in I, k \in K. \tag{30}$$

Hence $\{(f_i, \varphi_i)\}_I \sim \{(g_k, \psi_k)\}_K$. This completes the proof of the theorem. ■

Theorem 8 *There exists a delta sequence φ_n such that, for every $T \in \mathcal{B}_j$, $T = (\{(f_n, \varphi_n)\}_N)$ for some $f_n \in J$.*

Proof. Let (ψ_n) be a delta sequence such that $\mathcal{H}\psi_n \in \mathcal{D}$. Then for any $T \in \mathcal{B}_j$, we have $\mathcal{H}T\mathcal{H}\psi_k \in S'$, since $\mathcal{H}T \in \mathcal{D}'$. Consequently, $\mathcal{H}T\mathcal{H}\psi_n = \mathcal{H}g_n$ for some $g_n \in S'$. It is easy to check that $T = [\{(g_n * \psi_n, \psi_n * \psi_n)\}_N]$. Since $f_n = g_n * \psi_n \in J$ and $\varphi_n = \psi_n * \psi_n$ is a delta sequence, where φ_n does not depend on T, hence the theorem is proved. ■

3. Algebric properties and convergence

$\mathcal{B}_j$ becomes a vector space with the addition operation, defined by

$$[\{(f_i, \varphi_i)\}_I] + [\{(g_k, \psi_k)\}_K] = [\{(f_i * \psi_k + g_k * \varphi_i, \varphi_i * \psi_k)\}_{I\times K}]. \tag{31}$$

Moreover, multiplication by a scalar is defined by

$$\lambda[\{(f_i,\varphi_i)\}_I] = [\{(\lambda f_i,\varphi_i)\}_I], \quad \lambda \in \mathbb{C}. \tag{32}$$

If $[\{(f_i,\varphi_i)\}_I]+[\{(g_k,\psi_k)\}_K] \in \mathcal{B}_j$ and $g_k \in S$ for all $k \in K$, then for the operation $*$ we can define

$$[\{(f_i,\varphi_i)\}_I] * [\{(g_k,\psi_k)\}_K] = [\{(f_i * g_k, \varphi_i * \psi_k)\}_{I\times K}]. \tag{33}$$

Definition 9 [1] Let $T_0, T_1, T_n, ... \in \mathcal{B}_j$. Then the sequence T_n is said to converge to T_0, which is written as $T_n \to T_0$ if there exists a total family $(\varphi_i)_I$ such that

(a) there exists tempered distribution $f_{i,n}$ where $i \in I$ and $n \in N$ such that $T_n = [\{(f_{i,n},\varphi_i)\}_I]$ for all $n = 0, 1, 2, ...,$

(b) $f_{i,n} \to f_{i,0}$ in S' as $n \to \infty$ for every $i \in I$.

Theorem 10 *The Hartley transform is an isomorphism from $\mathcal{B}_j$ to $\mathcal{D}'$.*

Proof. Since $T_n \to T_0$ in $\mathcal{B}_j$ if and only if $T_n - T_0 \to 0$, it suffices to prove the continuity at 0. Let $T_n \to 0$ is in $\mathcal{B}_j$. Then there exists tempered distribution $f_{i,n}$ where $i \in I$ and $n \in N$ such that $T_n = [\{(f_{i,n},\varphi_i)\}_I]$ for all $n = 1, 2, ...$ and $f_{i,n} \to 0$ in S' as $n \to \infty$ for every $i \in I$. If $\psi \in \mathcal{D}$, then there are $i_1, ...i_k$ such that $\text{supp}\psi \subset \bigcup_{m=1}^{k} \text{supp}\mathcal{H}\varphi_{i,m}$. Then

$$\begin{aligned} \lim_{n\to\infty} \mathcal{H}T_n\psi &= \lim_{n\to\infty} \sum_{m=1}^{k} (\mathcal{H}T_n\mathcal{H}\varphi_{i,m}) \frac{\overline{\mathcal{H}\varphi_{i,m}\psi}}{\sum_{m=1}^{k} |\mathcal{H}\varphi_{i,m}|^2} \\ &= \sum_{m=1}^{k} (\lim_{n\to\infty} \mathcal{H}f_{im,n}) \frac{\overline{\mathcal{H}\varphi_{i,m}\psi}}{\sum_{m=1}^{k} |\mathcal{H}\varphi_{i,m}|^2} = 0, \end{aligned} \tag{34}$$

because $\lim_{n\to\infty} \mathcal{H}f_{i,n} = 0, \forall i \in I$, due to the continuity of the Hartley transform in S'. This proves the continuity of $\mathcal{H} : \mathcal{B}_j \to \mathcal{D}'$, because $\lim_{n\to\infty} \mathcal{H}T_n\psi = 0$ in S' for every $\psi \in \mathcal{D}$, implies $\lim_{n\to\infty} \mathcal{H}T_n = 0$ in $\mathcal{D}'$.

Now, assume $\lim_{n\to\infty} \mathcal{H}T_n = 0$ in $\mathcal{D}'$. By Theorem 8, there exists a delta sequence $(\varphi_i), i \in N$ such that for every $n \in N$, we have $T_n = [\{(f_{i,n},\varphi_i)\}_N]$ for some $f_{i,n} \in J$. Let $(\psi_k), k \in N$ be a delta sequence such that $\mathcal{H}\psi_k \in \mathcal{D}$ for every $k \in N$. Then

$$\lim_{n\to\infty} \mathcal{H}T_n\mathcal{H}\varphi_i.\mathcal{H}\psi_k = 0 \ \in S' \quad \text{for every} \ \ i, k \in N.$$

Since $\mathcal{H}T_n\mathcal{H}\varphi_i = f_{i,n} \quad \forall i, k \in N$ and $\lim_{n\to\infty} \mathcal{H}f_{i,n}\mathcal{H}\psi_k = 0 \in S'$, which implies $\lim_{n\to\infty} f_{i,n} * \psi_k = 0 \in S'$. But

$$T_n = [\{(f_{i,n},\varphi_i)\}_I] = [\{(f_{i,n} * \psi_k, \varphi_k * \psi_k)\}_{I\times K}]$$

for all $n = 0, 1, 2, ...$ Thus we have $T_n \to 0$ in $\mathcal{B}_j$. This proves the theorem. ■

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MULTIPLIERS ON SPACES OF VECTOR VALUED ENTIRE DIRICHLET SERIES OF TWO COMPLEX VARIABLES

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Abstract. In this paper, we study a class of sequence spaces defined by using the type of an entire function represented by vector valued Dirichlet series of two complex variables. The main results concern with obtaining the nature of the dual spaces of this sequence space and coefficient multipliers for some classes of vector valued Dirichlet series.

Keywords: vector valued Dirichlet series, analytic function, entire function, type, dual space, norm.

Mathematics Subject Classification: 30B50, 32A15, 46E99.

1. Introduction

Let

$$f(s_1, s_2) = \sum_{m,n=1}^{\infty} a_{m,n} \exp(\lambda_m s_1 + \mu_n s_2), \ (s_j = \sigma_j + it_j \, , \, j = 1, 2) \tag{1.1}$$

be a Dirichlet series of two complex variables s_1, s_2; $a_{m,n}{}'s$ belong to a commutative complex Banach algebra $(E, ||.||)$ with the unit element ω and

$$0 < \lambda_1 < ... < \lambda_m \to \infty \ \text{ as } m \to \infty \, ; 0 < \mu_1 < ... < \mu_n \to \infty \ \text{as} \, n \to \infty. \tag{1.2}$$

Further, let

$$\limsup_{m,n\to\infty} \frac{\ln(m+n)}{\lambda_m + \mu_n} = D < +\infty \, , \tag{1.3}$$

and

$$\limsup_{m,n\to\infty} \frac{\ln(||a_{m,n}||)}{\lambda_m + \mu_n} = -\infty \ . \tag{1.4}$$

Then $f(s_1, s_2)$ represented by the vector valued Dirichlet series (VVDS) in (1.1) is an entire function (see [2]). We define the maximum modulus of $f(s_1, s_2)$ as

$$M(\sigma_1, \sigma_2) = \text{lub}\,||f(\sigma_1 + it_1, \sigma_2 + it_2)||; -\infty < t_j < \infty \ (j = 1, 2).$$

The entire function $f(s_1, s_2)$ is said to be of order ρ where ρ is defined as

$$\rho = \limsup_{\sigma_1,\sigma_2\to\infty} \frac{\ln \ln M(\sigma_1, \sigma_2)}{\ln(e^{\sigma_1} + e^{\sigma_2})}, \quad (0 \leqslant \rho \leqslant \infty)\,. \tag{1.5}$$

When $0 < \rho < \infty$,in order to further classify the growth, we define the type T of $f(s_1, s_2)$ as

$$T = \limsup_{\sigma_1,\sigma_2\to\infty} \frac{\ln M(\sigma_1, \sigma_2)}{e^{\rho\sigma_1} + e^{\rho\sigma_2}}\ , \qquad (0 \leqslant T \leqslant \infty)\ . \tag{1.6}$$

The coefficient characterizations of order and type of generalized vector valued Dirichlet series were obtained by Srivastava and Sharma [2]. Thus, if $f(s_1, s_2)$ is an entire function of order ρ, then

$$\rho = \limsup_{m,n\to\infty} \frac{\ln\left(\lambda_m^{\lambda_m}\mu_n^{\mu_n}\right)}{\ln ||a_{m,n}||^{-1}} \tag{1.7}$$

and, if $f(s_1, s_2)$ is entire function of order ρ $(0 < \rho < \infty)$, then it is of type T if and only if

$$e\rho T = \limsup_{m,n\to\infty} \left[{\lambda_m}^{\lambda_m}{\mu_n}^{\mu_n}||a_{m,n}||^{\rho}\right]^{1/(\lambda_m+\mu_n)}. \tag{1.8}$$

Let E_T denote the space of all entire functions $f(s_1, s_2)$ defined by VVDS (1.1) and satisfying

$$\limsup_{m,n\to\infty} \left[{\lambda_m}^{\lambda_m}{\mu_n}^{\mu_n}||a_{m,n}||^{\rho}\right]^{1/(\lambda_m+\mu_n)} \leqslant e\rho T \tag{1.9}$$

Further, the sequences $\{\lambda_m\}$ and $\{\mu_n\}$ satisfy the stronger condition

$$\limsup_{m,n\to\infty} \frac{\ln(m + n)}{\lambda_m + \mu_n} = 0 \tag{1.10}$$

From equation (1.9), for a given $\varepsilon > 0$ there exists a positive integer n_0 such that for $m, n > n_0$,

$$\left[{\lambda_m}^{\lambda_m}{\mu_n}^{\mu_n}||a_{m,n}||^{\rho}\right]^{1/(\lambda_m+\mu_n)} < e\rho(T + \varepsilon)\,.$$

In this paper, we have obtained various properties of the space E_T. In analogy with Khoi [1], we give some definitions regarding dual spaces in reference to double sequences.

A sequence $(u_{m,n})$ is said to be multiplier from a sequence space A into a sequence space B if $(u_{m,n}a_{m,n}) \in B$ whenever $(a_{m,n}) \in A$. The space of multipliers

from a sequence space A into a sequence space B is denoted by (A, B). If D is a fixed sequence space then the D-dual of a sequence space A is defined to be (A, D), the multipliers from A to D and denoted by A^D. Some duals are defined with some conditions such as Kothe dual, Abel dual. The Kothe dual is obtained when $D = l^1$,and will be denoted by A^α (it is also denoted by A^K).

In what follows, we shall always consider E to be a complex Banach algebra and the sequences $\{\lambda_m\}$ and$\{\mu_n\}$ satisfy the condition (1.10). We denote by E_T the sequence space

$$E_T = \{(a_{m,n}) \in E;\ (a_{m,n}) \text{ satisfies (1.9)}\}.$$

The Kothe dual of the space E_T is defined as

$$E_T^\alpha = \left\{(u_{m,n});\ \sum_{m,n=1}^{\infty} ||u_{m,n}a_{m,n}|| \text{ converges } \forall\, (a_{m,n}) \in E_T\right\}.$$

Now, we introduce another sequence space E_T^β defined as

$$E_T^\beta = \left\{(u_{m,n});\ \sum_{m,n=1}^{\infty} u_{m,n}a_{m,n} \text{ converges } \forall\, (a_{m,n}) \in E_T\right\}.$$

2. Main Results

We first study properties of some dual spaces of the space E_T. Later, we characterize the multipliers on E_T. It can be easily verified that, for the spaces defined above, $E_T^\alpha \subseteq E_T^\beta$. Now, we find the criteria for the reverse inclusion relation to be true.

We prove

Theorem 1. *For every T, $0 < T < \infty$, we have $E_T^\alpha = E_T^\beta$. Moreover, $(u_{m,n}) \in E_T^\beta$, if and only if*

$$\liminf_{m,n\to\infty} \left[{\lambda_m}^{\lambda_m}{\mu_n}^{\mu_n}||u_{m,n}||^{-\rho}\right]^{1/(\lambda_m+\mu_n)} > e\rho T. \tag{2.1}$$

Proof. Let us assume that $(u_{m,n}) \in E_T^\beta$, but (2.1) is not satisfied, i.e.,

$$\liminf_{m,n\to\infty} \left[{\lambda_m}^{\lambda_m}{\mu_n}^{\mu_n}||u_{m,n}||^{-\rho}\right]^{1/(\lambda_m+\mu_n)} \leqslant e\rho T.$$

For a given $\varepsilon > 0$, there exist increasing sequences (m_k) and (n_l) of positive integers such that

$$\left[{\lambda_m}^{\lambda_m}{\mu_n}^{\mu_n}||u_{m,n}||^{-\rho}\right]^{1/(\lambda_m+\mu_n)} \leqslant e\rho(T+\varepsilon),\ \forall\, m = m_k, n = n_l,\ k, l = 1, 2, ...$$

Let $(a_{m,n})$ be a sequence defined as

$$a_{m,n} = \begin{cases} \omega/||u_{m,n}||, & \text{if } m = m_k \text{ and } n = n_l;\ k, l = 1, 2, ..., \\ 0, & \text{for other values of } m \text{ and } n. \end{cases}$$

Then, we have

$$\begin{aligned} \limsup_{m,n\to\infty} \left[{\lambda_m}^{\lambda_m}{\mu_n}^{\mu_n}||a_{m,n}||^{\rho}\right]^{1/(\lambda_m+\mu_n)} &= \lim_{k,l\to\infty} \left[{\lambda_{m_k}}^{\lambda_{m_k}}{\mu_{n_l}}^{\mu_{n_l}}||a_{m_k,n_l}||^{\rho}\right]^{1/(\lambda_{m_k}+\mu_{n_l})} \\ &= \lim_{k,l\to\infty} \left[{\lambda_{m_k}}^{\lambda_{m_k}}{\mu_{n_l}}^{\mu_{n_l}}||u_{m_k,n_l}||^{-\rho}\right]^{1/(\lambda_{m_k}+\mu_{n_l})} \\ &\leqslant e\rho T. \end{aligned}$$

It follows that $(a_{m,n}) \in E_T$. But $||a_{m_k,n_l}u_{m_k,n_l}|| = 1,\ (k, l = 1, 2, ...)$, that is, $\lim_{m,n\to\infty} ||a_{m,n}u_{m,n}|| \neq 0$. So, the series $\sum_{m,n=1}^{\infty} u_{m,n}a_{m,n}$ does not converge, therefore our assumption is not valid. Hence, if $(u_{m,n}) \in E_T^{\beta}$, then (2.1) will always be satisfied.

Conversely, suppose that (2.1) holds, i.e.,

$$\liminf_{m,n\to\infty} \frac{\left[{\lambda_m}^{\lambda_m}{\mu_n}^{\mu_n}||u_{m,n}||^{-\rho}\right]^{1/\lambda_m+\mu_n}}{\rho e} = M > T.$$

Then, for a given $\delta > 0$, there exist positive integers M_1 and N_1 and such that, $\forall\, m \geq M_1$ and $\forall\, n \geq N_1$, we have

$$\left[{\lambda_m}^{\lambda_m}{\mu_n}^{\mu_n}||u_{m,n}||^{-\rho}\right]^{1/(\lambda_m+\mu_n)} > e\rho(M - \varepsilon),$$

or

$$||u_{m,n}||^{\rho} < \frac{{\lambda_m}^{\lambda_m}{\mu_n}^{\mu_n}}{[\rho e(M - \varepsilon)]^{(\lambda_m+\mu_n)}}.$$

Also, for every sequence $(a_{m,n}) \in E_T$, there exist M_2 and N_2 such that, $\forall\, m \geq M_2$, $n \geq N_2$,

$$||a_{m,n}||^{\rho} < \frac{[\rho e(T + \varepsilon)]^{(\lambda_m+\mu_n)}}{{\lambda_m}^{\lambda_m}{\mu_n}^{\mu_n}}.$$

Therefore, for all $m \geq \max\{M_1, M_2\}$ and $n \geq \max\{N_1, N_2\}$,

$$||a_{m,n}u_{m,n}|| < \left(\frac{T + \varepsilon}{M - \varepsilon}\right)^{(\lambda_m+\mu_n)/\rho}.$$

Since $M > T$, we can choose $\varepsilon > 0$ such that $M - \varepsilon > T + \varepsilon$. Then, from the above inequality, we can see that the series $\sum_{m,n=1}^{\infty} ||u_{m,n}a_{m,n}||$ converges. Hence $(u_{m,n}) \in E_T^{\alpha}$ and, therefore, $E_T^{\beta} \subseteq E_T^{\alpha}$. This completes the proof of Theorem 1. ∎

Next, we prove

Theorem 2. *The space E_T is perfect, i.e., $E_T^{\alpha\alpha} = E_T$.*

Proof. Let the sequence $(a_{m,n}) \notin E_T$. Then we have

$$\limsup_{m,n\to\infty} \left[{\lambda_m}^{\lambda_m} {\mu_n}^{\mu_n} ||a_{m,n}||^{\rho}\right]^{1/(\lambda_m+\mu_n)} \geqslant e\rho T.$$

We denote by $e\rho T^*$ the left hand side limit if it is finite, and a number $> e\rho T$ if the limit is infinite. Then, for arbitrary small $\delta > 0$, there exist infinitely increasing sequences (m_k) and (n_l) of positive integers such that

$$||a_{m,n}||^{\rho} \geqslant \frac{[\rho e(T^* - \delta)]^{(\lambda_m+\mu_n)}}{\lambda_m^{\lambda_m}\, \mu_n^{\mu_n}},\ m = m_k,\ n = n_l.$$

Let us define a sequence

$$u_{m,n} = \begin{cases} \omega\,/||a_{m_k,n_l}|| & \text{if } m = m_k,\ n = n_l, \text{ where } k, l = 1, 2, ... \\ 0 & \text{for other values of } m \text{ and } n. \end{cases}$$

Then, we have

$$\begin{aligned} \liminf_{m,n\to\infty} \left[\lambda_m^{\lambda_m}\, \mu_n^{\mu_n} ||u_{m,n}||^{-\rho}\right]^{1/(\lambda_m+\mu_n)} &= \lim_{k,l\to\infty} \left[\lambda_{m_k}^{\lambda_{m_k}}\, \mu_{n_l}^{\mu_{n_l}} ||u_{m_k,n_l}||^{-\rho}\right]^{1/(\lambda_{m_k}+\mu_{n_l})} \\ &= \lim_{k,l\to\infty} \left[\lambda_{m_k}^{\lambda_{m_k}}\, \mu_{n_l}^{\mu_{n_l}} ||a_{m_k n_l}||^{\rho}\right]^{1/(\lambda_{m_k}+\mu_{n_l})} \\ &\geqslant e\rho T^* > e\rho T. \end{aligned}$$

Hence, from Theorem 1, $(u_{m,n}) \in E_T^{\alpha}$. But $||a_{m,n}u_{m,n}|| = 1$, $\forall\, m = m_k$, $n = n_l$, i.e., $\sum a_{m,n}u_{m,n}$ does not converge. Therefore, $(a_{m,n}) \notin E_T^{\alpha\alpha}$. Hence $E_T^{\alpha\alpha} \subseteq E_T$. The reverse inclusion relation $E_T^{\alpha\alpha} \supseteq E_T$ always holds. Hence the space E_T is perfect.

Theorem 3. *For the sequence space E_T defined as above, we have*

$$(E_T, l^p) = E_T^{\alpha},\ \forall\, 0 < p \leq \infty.$$

Proof. Suppose that a sequence$(u_{m,n}) \notin E_T^{\alpha}$. Then, from Theorem 1, we have

$$\liminf_{m,n\to\infty} \left[\lambda_m^{\lambda_m}\, \mu_n^{\mu_n} ||u_{m,n}||^{-\rho}\right]^{1/(\lambda_m+\mu_n)} \leqslant e\rho T.$$

Then, for an arbitrarily small $\varepsilon > 0$, there exist monotonically increasing sequences (m_k) and (n_l) of positive integers such that

$$\left[\lambda_m^{\lambda_m} \mu_n^{\mu_n} ||u_{m,n}||^{-\rho}\right]^{1/(\lambda_m+\mu_n)} < e\rho(T + \varepsilon),\ m = m_k,\ n = n_l.$$

Let $0 < p < \infty$. We consider the sequence

$$a_{m,n} = \begin{cases} \omega\,/||u_{m_k,n_l}|| & \text{if } m = m_k,\ n = n_l \text{ and } k, l = 1, 2, ... \\ 0 & \text{for other values of } m \text{ and } n. \end{cases}$$

Then we have

$$\begin{aligned}\limsup_{m,n\to\infty}\left[\lambda_m^{\lambda_m}\,\mu_n^{\mu_n}||a_{m,n}||^{\rho}\right]^{1/(\lambda_m+\mu_n)} &= \lim_{k,l\to\infty}\left[\lambda_{m_k}^{\lambda_{m_k}}\,\mu_{n_l}^{\mu_{n_l}}||a_{m_k,n_l}||^{\rho}\right]^{1/(\lambda_{m_k}+\mu_{n_l})}\\ &= \lim_{k,l\to\infty}\left[\lambda_{m_k}^{\lambda_{m_k}}\,\mu_{n_l}^{\mu_{n_l}}||u_{m_k,n_l}||^{-\rho}\right]^{1/(\lambda_{m_k}+\mu_{n_l})}\\ &\leqslant e\rho T.\end{aligned}$$

Hence we get $(a_{m,n})\in E_T$. By the definition of (E_T,l^p) , $\sum_{m,n=1}^{\infty}||a_{m,n}u_{m,n}||^p$ should be convergent . But $||a_{m_k,n_l}u_{m_k,n_l}||=1; k,l=1,2....$ This implies $(a_{m,n}u_{m,n})\notin l^p$.

For the case $p=\infty$, we consider a sequence

$$a_{m,n}=\begin{cases}\omega(m+n)^{1/\rho}||u_{m,n}||^{-1} & \text{if } m=m_k,\ n=n_l \text{ and } k,l=1,2,....,\\ 0 & \text{for other values of } m \text{ and } n.\end{cases}$$

Then we have

$$\begin{aligned}\limsup_{m,n\to\infty}\left[\lambda_m^{\lambda_m}\,\mu_n^{\mu_n}||a_{m,n}||^{\rho}\right]^{1/(\lambda_m+\mu_n)} &= \lim_{k,l\to\infty}\left[\lambda_{m_k}^{\lambda_{m_k}}\,\mu_{n_l}^{\mu_{n_l}}||a_{m_k,n_l}||^{\rho}\right]^{1/(\lambda_{m_k}+\mu_{n_l})}\\ &= \lim_{k,l\to\infty}\left[\lambda_{m_k}^{\lambda_{m_k}}\,\mu_{n_l}^{\mu_{n_l}}||m_kn_l||^{\rho}||u_{m_k,n_l}||^{-\rho}\right]^{1/(\lambda_{m_k}+\mu_{n_l})}\\ &\leqslant e\rho T\end{aligned}$$

using (1.10) and the inequality above. This shows that $(a_{m,n})\in E_\rho$. Since $\lim_{k,l\to\infty}||a_{m_k,n_l}u_{m_k,n_l}||=+\infty$, this implies that $(a_{m,n},u_{m,n})\notin l^\infty$. Hence we conclude that, for $0<p\leq\infty$, $(u_{m,n})\notin E_T^\alpha\Rightarrow(u_{m,n})\notin(E_T,l^p)$. Thus $(E_T,l^p)\subseteq E_T^\alpha$, $0<p\leq\infty$.

Conversely, assume that $(u_{m,n})\in E_T^\alpha$. Then for a given $M>T$, there exist integers M_1 andN_1 such that $\forall\, m\geq M_1, n\geq N_1$,

$$||u_{m,n}||\leq\frac{{\lambda_m}^{\lambda_m/\rho}{\mu_n}^{\mu_n/\rho}}{[\rho eM]^{(\lambda_m+\mu_n)/\rho}}.$$

Suppose that $(a_{m,n})\in E_T$, then for $\delta\in(0,(M-T))$ there exist positive integers M_2 andN_2 such that$\forall\, m\geq M_2,\ n\geq N_2$,

$$||a_{m,n}||\leq\frac{[\rho e(T+\delta)]^{(\lambda_m+\mu_n)/\rho}}{{\lambda_m}^{\lambda_m/\rho}{\mu_n}^{\mu_n/\rho}}.$$

Consequently, for all $m\geq m_0=\max\{M_1,M_2\}, n\geq n_0=\max\{N_1,N_2\}$ we have

$$||a_{m,n}u_{m,n}||\leq||a_{m,n}||\,||u_{m,n}||<((T+\delta)/(M))^{(\lambda_m+\mu_n)/\rho}.$$

If $0<p<\infty$, then we have

$$\sum_{m=M,n=N}^{\infty}||a_{m,n}u_{m,n}||^p\leq\sum_{m=M,n=N}^{\infty}((T+\delta)/M)^{p(\lambda_m+\mu_n)/\rho}<\infty\,;$$

as $(T+\delta)/M < 1$, which implies that$(a_{m,n}u_{m,n}) \in l^p$.

Now, let us take $p = \infty$. Then we have

$$||a_{m,n}u_{m,n}|| \leq ((T+\delta)/M)^{(\lambda_m+\mu_n)/\rho} < 1,\ \forall m \geq m_0,\ n \geq n_0,$$

which shows that $(a_{m,n}u_{m,n}) \in l^\infty$. Thus, in both the cases $(u_{m,n}) \in (E_T, l^p)$ and consequently $E_T^\alpha \subset (E_T, l^p)$, $0 < p \leq \infty$. This completes the proof of Theorem 3. ∎

In the next theorem, we obtain the sequence space of multipliers from l^p to E_T. We prove

Theorem 4. *For the sequence space E_T defined as above, we have*

$$(l^p, E_T) = E_T \ , \ 0 < p \leq \infty.$$

Proof. First, we prove that $(l^p, E_T) \subseteq E_T$. Hence, for $0 < p < \infty$, let $(a_{m,n}) \in l^p$. Then $\sum\limits_{m,n=1}^{\infty} |a_{m,n}|^p < \infty$ and, therefore,

$$\lim_{m,n\to\infty} |a_{m,n}|^p = 0 \tag{2.2}$$

Let $(u_{m,n}) \in (l^p, E_T)$. Then, $(a_{m,n}\, u_{m,n}) \in E_T$ and, using (1.9), we have

$$\limsup_{m,n\to\infty} \left[{\lambda_m}^{\lambda_m}\mu_n{}^{\mu_n}||a_{m,n}\, u_{m,n}||^\rho\right]^{1/(\lambda_m+\mu_n)} \leqslant e\rho T. \tag{2.3}$$

Hence from (2.2) and (2.3), we get

$$\limsup_{m,n\to\infty} \left[{\lambda_m}^{\lambda_m}\mu_n{}^{\mu_n}||\, u_{m,n}||^\rho\right]^{1/(\lambda_m+\mu_n)} \leqslant e\rho T$$

and hence $(u_{m,n}) \in E_T$. If $p = \infty$, then $(a_{m,n})$ is a bounded sequence and from (2.3) we have the above inequality and $(u_{m,n}) \in E_T$. Hence we get $(l^p, E_T) \subseteq E_T$, $0 < p \leqslant \infty$.

To prove the converse, assume that $(u_{m,n}) \in E_T$. Then we have

$$\limsup_{m,n\to\infty} \left[\lambda_m^{\lambda_m}\,\mu_n^{\mu_n}||u_{m,n}||^\rho\right]^{1/(\lambda_m+\mu_n)} \leqslant e\rho T.$$

Let $(d_{m,n})$ be an arbitrary sequence such that $(d_{m,n}) \in l^p, 0 < p \leq \infty$. In both cases, i,e., $0 < p < \infty$ or $p = \infty$, there exists a constant P such that $|d_{m,n}| \leq P$, $\forall\, m, n \geq 1$. Hence we have

$$\begin{aligned}
\limsup_{m,n\to\infty} \left[\lambda_m^{\lambda_m}\,\mu_n^{\mu_n}||d_{m,n}u_{m,n}||^\rho\right]^{1/(\lambda_m+\mu_n)} &= \limsup_{m,n\to\infty} \left[\lambda_m^{\lambda_m}\,\mu_n^{\mu_n}|d_{m,n}|^\rho||u_{m,n}||^\rho\right]^{1/(\lambda_m+\mu_n)} \\
&\leqslant \limsup_{m,n\to\infty} \left[\lambda_m^{\lambda_m}\,\mu_n^{\mu_n}P^\rho||u_{m,n}||^\rho\right]^{1/(\lambda_m+\mu_n)} \\
&\leqslant e\rho T,
\end{aligned}$$

which shows that $(d_{m,n}u_{m,n}) \in E_T$. Thus, $(u_{m,n}) \in (l^p, E_T)$ and, consequently, $E_T \subseteq (l^p, E_T)\,,\ \forall\, 0 < p \leq \infty$. Hence the result follows. ∎

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MULTIPLICATION COMPONENTS OF GRADED MODULES

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Abstract. Let G be a group and $g \in G$. Let R be a commutative G-graded ring and M be a graded R-module. In this paper, we study some cases when R is strongly graded ring and the component M_e of M is multiplication R_e-module. Also, we prove that if R is strongly graded, then the components M_g of M are multiplication R_e-modules if and only if the component M_e is P-torsion or P-cyclic where P is a prime ideal of the component R_e of R.
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1. Introduction

A ring R with unity 1 graded by a group G will means that $R = \bigoplus_{g\in G} R_g$ where R_g is an additive subgroup of R and $R_gR_h \subseteq R_{gh}$ for all $g, h \in G$. If the inclusion is an equality, then the ring is called strongly graded. Clearly, R_e is a subring of R with $1 \in R_e$. An R-module is said to be graded if $M = \bigoplus_{g\in G} M_g$ for a family of subgroups $\{M_g\}_{g\in G}$ of M such that $R_gM_h \subseteq M_{gh}$ for all $g, h \in G$. Clearly, M_g is an R_e-module for all $g \in G$. In a similar way, we define a strongly graded module. The ring R is strongly graded if and only if every graded R-module is strongly graded. For more details, we refer the readers to [4], as well as [6], and the references therein.

If M is an R-module and N is an R-submodule of M, then the ideal $\{r \in R : rM \subseteq N\}$ of R will be denoted by $(N : M)$. An R-module M is said to be multiplication module if for every R-submodule N of M, there exists an ideal of R such that $N = IM$. Moreover, if $N = IM$ for some ideal of R, then $N = (N : M)M$. Given a ring R and a multiplicative subset S of R, the ring of fractions $S^{-1}R$ is $\{\frac{r}{s} : r \in R, s \in S\}$. For more details, we refer the readers to [1], [2], [3], as well as [5], and the references therein.

Throughout this paper, unless stated otherwise, R is commutative nontrivially graded ring.

2. Results

In this section, we introduce and prove the main results of the paper.

Theorem 2.1 *Let R be a strongly G-graded ring and M be a torsion free graded R-module. If M_e is a multiplication R_e-module and A is a proper ideal of R_e such that $AM_e = M_e$, then $M = \{0\}$.*

Proof. Let $g \in G$, $x \in M_g$. Then $R_{g^{-1}}x$ is an R_e-submodule of M_e. Since M_e is multiplication R_e-module, $R_{g^{-1}}x = BM_e$ for some ideal B of R_e and then $R_{g^{-1}}x = BM_e = BAM_e = ABM_e = AR_{g^{-1}}x = R_{g^{-1}}Ax$ and so $R_ex = Ax$. Now, $x = 1.x \in R_ex = Ax$ and then there exists $a \in A$ such that $(1-a)x = 0$. Since M is torsion free, either $1 = a$ or $x = 0$. If $1 = a \in A$, then $A = R_e$ a contradiction. So, $x = 0$, i.e., $M_g = \{0\}$ for all $g \in G$ and hence $M = \{0\}$. ∎

Theorem 2.2 *Let R be a strongly G-graded ring and M be a graded R-module. If M_e is a multiplication R_e-module and S is a multiplicative subset of R_e, then $S^{-1}M_g$ is multiplication as an $S^{-1}R_e$-module for all $g \in G$.*

Proof. Let $g \in G$ and X be an $S^{-1}R_e$-submodule of $S^{-1}M_g$. Then $X = S^{-1}N$ for some R_e-submodule N of M_g and then $R_{g^{-1}}N$ is an R_e-submodule of M_e and it follows that $R_{g^{-1}}N = AM_e$ for some ideal A of R_e. So, $X = R_eX = R_gR_{g^{-1}}S^{-1}N = R_gS^{-1}R_{g^{-1}}N = R_gS^{-1}AM_e = S^{-1}AR_gM_e = S^{-1}AM_g = S^{-1}AS^{-1}M_g$ where $S^{-1}A$ is an ideal of $S^{-1}R_e$ and this completes the proof. ∎

Given a prime ideal P of R_e, we consider the set

$$T_P(M_e) = \{m \in M_e : cm = 0 \text{ for some } c \in R_e - P\}.$$

It is easy to check that $T_P(M_e)$ is an R_e-submodule of M_e. If $T_P(M_e) = M_e$, then we will say that M_e is P-torsion. If there exists $x \in M_e$ and $c \in R_e - P$ such that $cM_e \subseteq R_ex$, we will say that M_e is P-cyclic. Now, we introduce the main result of our paper:

Theorem 2.3 *Let R be a strongly G-graded ring, and M be a graded R-module. Then for $g \in G$, M_g is multiplication R_e-module if and only if for every prime ideal P of R_e, either M_e is P-torsion or P-cyclic.*

Proof. Let $g \in G$. Suppose that M_g is multiplication R_e-module and P is a prime ideal of R_e. Firstly, we consider the case in which $PM_e = M_e$. Let $m \in M_e$. Then R_gm is an R_e-submodule of M_g and then there exists an ideal A of R_e such that $R_gm = AM_g$. So, $m = 1.m \in R_em = R_{g^{-1}}R_gm = R_{g^{-1}}AM_g = AR_{g^{-1}}M_g = AM_e = APM_e = APR_{g^{-1}}M_g = PR_{g^{-1}}AM_g = PR_{g^{-1}}R_gm = PR_em = R_ePm = Pm$ and then there exists $p \in P$ such that $(1-p)m = 0$ and it follows that $c = 1 - p \in R_e - P$ such that $cm = 0$ and therefore, $M_e = T_P(M_e)$, i.e., M_e is P-torsion. Now, we consider that $PM_e \neq M_e$. Then there exists $x \in M_e - PM_e$ and since R_gx is an R_e-submodule of M_g, there exists an ideal B of R_e such that $R_gx = BM_g$. If $B \subseteq P$, then $x = 1.x \in R_ex = R_{g^{-1}}R_gx = R_{g^{-1}}BM_g = BM_e \subseteq PM_e$ a contradiction. Therefore, $B \nsubseteq P$, then there exists $c \in B - P$ such that $cM_e = R_{g^{-1}}cM_g \subseteq R_{g^{-1}}BM_g = R_{g^{-1}}R_gx = R_ex$, i.e., M_e is P-cyclic. Conversely, let $g \in G$ and N be an R_e-submodule of M_g. Suppose that $A = (R_{g^{-1}}N : M_e)$, $n \in R_{g^{-1}}N$ and $K = (AM_e : R_en)$. Assume that $K \neq R_e$. Then there exists a maximal ideal P of R_e containing K. If M_e is P-torsion, then there exists $c \in R_e - P$ such that $cn = 0$ and it follows that $c \in K - P$ a contradiction. So, M_e is P-cyclic, i.e., there exists $x \in M_e$ and $c \in R_e - P$ such that $cM_e \subseteq R_ex$. Thus, $R_{g^{-1}}cN$ is an R_e-submodule of R_ex, and then cN is R_e-submodule of R_gx

but R_gx is multiplication because it is cyclic, hence there exists $J = (cN : R_gx)$ such that $cN = Jx$. It holds that

$$\begin{aligned} cJM_e = JcM_e \subseteq JR_ex = JR_{g^{-1}}R_gx = R_{g^{-1}}JR_gx \subseteq R_{g^{-1}}cN \\ = cR_{g^{-1}}N \subseteq R_eR_{g^{-1}}N = R_{g^{-1}}N \end{aligned}$$

and hence, $cJ \subseteq A$. Now, the element $c^2n \in c^2N = cJx \subseteq Ax \subseteq AM_e$. As a result, $c^2 \in K \subseteq P$ which is a contradiction. It follows that $K = R_e$ and then $R_e = (AM_e : R_{g^{-1}}N)$ and therefore, $R_{g^{-1}}N \subseteq (R_{g^{-1}}N : M_e)M_e$ and then $N \subseteq (N : M_g)M_g$. Since the other inclusion is always true, the proof ends. ∎

Corollary 2.4 *Let R be a strongly G-graded ring, and M be a graded R-module. Then for $g \in G$, M_g is multiplication R_e-module if and only if for every prime ideal P of R_e, either M_e is P-torsion or there exists an R_e-submodule N of M_e and $c \in R_e - P$ such that $cM_e \subseteq N$.*

Proof. To prove the sufficiency, let P be a prime ideal of R_e and suppose that M_e is not P-torsion. Then by hypothesis, there exists $c \in R_e - P$ such that $cM_e \subseteq N$ where N is an R_e-submodule of M_e. Since M_e is not P-torsion, N is not P-torsion. By Theorem 2.3, there exists $x \in N$ and $r \in R_e - P$ such that $rN \subseteq R_ex$. Thus, $crM_e = rcM_e \subseteq rN \subseteq R_ex$ and so M_e is P-cyclic and therefore, by Theorem 2.3, M_g is multiplication R_e-module for any $g \in G$. The necessity is obvious by Theorem 2.3. ∎

Corollary 2.5 *Let R be a strongly G-graded ring, and M be a graded R-module. If M_e is multiplication R_e-module and $Ann(M_e) = \{0\}$, then for $g \in G$,*

1. *$\bigcap_{k\in K}(I_kM_g) = (\bigcap_{k\in K} I_k)M_g$ for every family $I_k(k \in K)$ of ideals of R_e.*
2. *if N is an R_e-submodule of M_g and A is an ideal of R_e such that $N \subset AM_g$, then there exists an ideal B of R_e such that $B \subset A$ and $N \subseteq BM_g$.*

Proof. 1. Let $I_k(k \in K)$ be a family of ideals of R_e. We call $I = \bigcap_{k\in K} I_k$. Then it is always true that $IM_e \subseteq \bigcap_{k\in K}(I_kM_e)$. Let $x \in \bigcap_{k\in K}(I_kM_e)$ and $H = (IM_e : R_ex)$. Suppose that $H \neq R_e$. Then there exists a prime ideal P of R_e containing H. If $x \in T_P(M_e)$, then we find an element in $H - P$, so $x \notin T_P(M_e)$. By Theorem 2.3, M_e is P-cyclic and then there exists $m \in M_e$ and $c \in R_e - P$ such that $cM_e \subseteq R_em$ and so $cx \in \bigcap_{k\in K} I_km$. It follows that, for every $k \in K$, there exists $a_k \in I_k$ such that $cx = a_km$. Now, choose $k_0 \in K$ such that $cx \in I_{k_0}m$ and then $cx = a_{k_0}m$. Hence $a_{k_0}m = a_km$, i.e., $(a_{k_0} - a_k)m = 0$ for every $k \in K$. We have, $c(a_{k_0} - a_k)M_e = (a_{k_0} - a_k)cM_e \subseteq (a_{k_0} - a_k)R_em = R_e(a_{k_0} - a_k)m = \{0\}$. Since $Ann(M_e) = \{0\}$, $c(a_{k_0} - a_k) = 0$. Hence, $ca_{k_0} = ca_k \in I_k$ for every $k \in K$. As a consequence, $ca_{k_0} \in I$ and then $c^2x = ca_{k_0}m \in IM_e$ and so $c^2 \in H \subseteq P$ a contradiction. So, $H = R_e$ and hence $x \in IM_e$. Now, let $g \in G$. Then $\bigcap_{k\in K}(I_kM_g) = \bigcap_{k\in K}(I_kR_gM_e) = R_g\bigcap_{k\in K}(I_kM_e) = R_g(\bigcap_{k\in K} I_k)M_e = (\bigcap_{k\in K} I_k)R_gM_e = (\bigcap_{k\in K} I_k)M_g$.

2. Let $g \in G$, N be an R_e-submodule of M_g and A be an ideal of R_e such that $N \subset AM_g$. Then $R_{g^{-1}}N$ is an R_e-submodule of M_e such that $R_{g^{-1}}N \subset AM_e$.

Since M_e is multiplication, $R_{g^{-1}}N = CM_e$ for some ideal C of R_e and then $N = CM_g$. So, $N = AM_g \bigcap CM_g = (A \bigcap C)M_g$ by using (1). Hence, choose $B = A \bigcap C$. ∎

An R-module M is said to be finitely cogenerated if for every non-empty family of R-submodules $N_k (k \in K)$ of M such that $\bigcap_{k\in K} N_k = \{0\}$, there exists a finite subset F of K such that $\bigcap_{k\in F} N_k = \{0\}$. A ring R is said to be finitely cogenerated if it is finitely cogenerated as an R-module. We close the paper with the following result:

Theorem 2.6 *Let R be a strongly G-graded ring, and M be a graded R-module. If M_e is multiplication R_e-module and $Ann(M_e) = \{0\}$, then M_e is finitely cogenerated R_e-module if and only if M_g is finitely cogenerated R_e-module for all $g \in G$.*

Proof. Suppose that M_e is finitely cogenerated R_e-module. Firstly, we prove that R_e is finitely cogenerated. Let $I_k (k \in K)$ be a non-empty family of ideals of R_e such that $\bigcap_{k\in K} I_k = \{0\}$. Then by Corollary 2.5, $\bigcap_{k\in K}(I_k M_e) = \{0\}$. Since M_e is finitely cogenerated, there exists a finite subset F of K such that $\bigcap_{k\in F} I_k M_e = \{0\}$ and then by Corollary 2.5, $(\bigcap_{k\in F} I_k)M_e = \{0\}$. Since $Ann(M_e) = \{0\}$, $\bigcap_{k\in F} I_k = \{0\}$. Therefore, R_e is finitely cogenerated. Now, let $g \in G$ and $N_k (k \in K)$ be a non-empty family of R_e-submodules of M_g such that $\bigcap_{k\in K} N_k = \{0\}$. Then $R_{g^{-1}}N_k (k \in K)$ are R_e-submodules of M_e and then for $k \in K$, there exists an ideal A_k of R_e such that $R_{g^{-1}}N_k = A_k M_e$, it is clear that $(\bigcap_{k\in K} A_k)M_e = \bigcap_{k\in K}(A_k M_e) = \bigcap_{k\in K} R_{g^{-1}}N_k = R_{g^{-1}} \bigcap_{k\in K} N_k = \{0\}$. Since $Ann(M_e) = \{0\}$, $\bigcap_{k\in K} A_k = \{0\}$ and since R_e is finitely cogenerated, there exists a finite subset F of K such that $\bigcap_{k\in F} A_k = \{0\}$ and then

$$\begin{aligned}\bigcap_{k\in F} N_k &= R_e(\bigcap_{k\in F} N_k) = R_g R_{g^{-1}}(\bigcap_{k\in F} N_k)\\ &= R_g(\bigcap_{k\in F} R_{g^{-1}}N_k) = R_g(\bigcap_{k\in F} A_k M_e) = R_g(\bigcap_{k\in F} A_k)M_e = \{0\}.\end{aligned}$$

Hence, M_g is finitely cogenerated. The converse is obvious. ∎

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GENERALIZED CUBIC SOFT SETS AND THEIR APPLICATIONS TO ALGEBRAIC STRUCTURES

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Abstract. In this paper, we introduce the concepts of generalized cubic soft sets, generalized cubic soft $\mathcal{AG}$-subgroupoids and generalized cubic soft left (resp., right) ideals to study the algebraic structures and properties of $\mathcal{AG}$-groupoids. We also give some examples of generalized cubic soft $\mathcal{AG}$-subgroupoids and generalized cubic soft

left (resp., right) ideals. Moreover, we characterize intra-regular $\mathcal{AG}$-groupoids using the properties of generalized cubic soft sets and generalized cubic soft right ideals.

Keywords: cubic soft set, $\mathcal{AG}$-groupoid, left invertive law, medial law, paramedical law and $(\in_{(\tilde{\gamma}_1,\gamma_2)},\in_{(\tilde{\gamma}_1,\gamma_2)} \vee q_{(\tilde{\delta}_1,\delta_2)})$-cubic soft right ideal.

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1. Introduction

The concept of a fuzzy set was introduced by Zadeh in 1965 [14], which is now a days used in almost all branches of science. In fact, a fuzzy set is a suitable tool for modeling because there exist a lot of ambiguities in crisp models while handling problems in different fields of science like artificial intelligence, computer science, control engineering, decision theory, expert system, logic management science, operations research, robotics and many others. In other words the concept of a fuzzy set is used to control uncertainty problems arising in models representing real life phenomenon. This concept is also used in business, medical and related health sciences. Maiers and Sherif [12], reviewed the literature on fuzzy industrial controllers and provided an index of applications of fuzzy set theory to twelve subject areas including decision making, economics, engineering and operations research.

Molodtsov [11], in 1999 introduced the fundamental concept of a soft set which is now used in several basic notions of algebra. Ali et al. [6], introduced several new algebraic operations on soft sets. Cagman and Enginoglu [2], developed the uni-int decision making method in virtue of soft sets. Feng et al. [4], investigated soft semirings by using soft set theory. Aktas and Cagman [1], defined the notion of soft groups and derived some related properties. This initiated an important research direction concerning algebraic properties of soft sets in miscellaneous kinds of algebras such as BCK/BCI-algebras, d-algebras, semirings, rings, Lie algebras and K-algebras. Feng and Li [5], ascertained the relationship among five different types of soft subsets and considered the free soft algebras associated with soft product operations. It has been shown that soft sets have some nonclassical algebraic properties which are distinct from those of crisp sets and fuzzy sets. Further, Jun discussed the applications of soft sets in ideal theory of BCK/BCI-algebras and in d-algebras respectively. Recently, combining cubic sets and soft sets, Muhiuddin and Al-roqi [9], introduced the notions of (external, internal) cubic soft sets, P-cubic (resp., R-cubic) soft subsets, R-union (resp., R-intersection, P-union and P-intersection) of cubic soft sets and the complement of a cubic soft set. They investigated several related properties and applied the notion of cubic soft sets to BCK/BCI-algebras.

In [10], Khan et al. introduced the generalized version of Jun's cubic set and applied it to the ideal theory of semigroups. Here the purpose of this paper is to deal with the algebraic structure of $\mathcal{AG}$-groupoids by applying the generalized version of a soft set.

The idea of generalization of a commutative semigroup (which we call the left

almost semigroup) was introduced by M. A. Kazim and M. Naseeruddin in 1972 [8]. It is also known as an Abel-Grassmann groupoid ($\mathcal{AG}$-groupoid).

A groupoid $\mathcal{S}$ is called an $\mathcal{AG}$-groupoid if it satisfies the left invertive law, that is $(ab)c = (cb)a$, $\forall\ a, b, c \in \mathcal{S}$. In an $\mathcal{AG}$-groupoid medial law: $(ab)(cd) = (ac)(bd)$, holds $\forall\ a, b, c, d \in \mathcal{S}$ [8]. If an $\mathcal{AG}$-groupoid $\mathcal{S}$ contains left identity then it satisfies the paramedial law: $(ab)(cd) = (db)(ca)$, $\forall\ a, b, c, d \in \mathcal{S}$. Moreover, if an $\mathcal{AG}$-groupoid $\mathcal{S}$ contains the left identity, then the following law holds.

$$a(bc) = b(ac), \forall a, b, c \in \mathcal{S}.$$

Our interest in this paper is to introduce the concept of $(\in_{(\widetilde{\gamma}_1,\gamma_2)}, \in_{(\widetilde{\gamma}_1,\gamma_2)} \vee q_{(\widetilde{\delta}_1,\delta_2)})$-cubic soft sets, then we use this concept and introduced $(\in_{(\widetilde{\gamma}_1,\gamma_2)}, \in_{(\widetilde{\gamma}_1,\gamma_2)} \vee q_{(\widetilde{\delta}_1,\delta_2)})$-cubic soft $\mathcal{AG}$-subgroupoid, $(\in_{(\widetilde{\gamma}_1,\gamma_2)}, \in_{(\widetilde{\gamma}_1,\gamma_2)} \vee q_{(\widetilde{\delta}_1,\delta_2)})$-cubic soft left (resp., right, two-sided) ideals in an $\mathcal{AG}$-groupoid. Moreover, we discuss some characterizations of intra-regular $\mathcal{AG}$-groupoids using the properties of $(\in_{(\widetilde{\gamma}_1,\gamma_2)}, \in_{(\widetilde{\gamma}_1,\gamma_2)} \vee q_{(\widetilde{\delta}_1,\delta_2)})$-cubic soft sets and $(\in_{(\widetilde{\gamma}_1,\gamma_2)}, \in_{(\widetilde{\gamma}_1,\gamma_2)} \vee q_{(\widetilde{\delta}_1,\delta_2)})$-cubic soft right ideals.

2. Preliminaries notes

In this section, we define some basic definitions that are required for next sections.

Let $\mathcal{G}$ be an $\mathcal{AG}$-groupoid. By an $\mathcal{AG}$-subgroupoid of $\mathcal{G}$ we mean a non-empty subset A of $\mathcal{G}$ such that $A^2 \subseteq A$. A non-empty subset I of $\mathcal{G}$ is called left (resp., right) ideal of $\mathcal{G}$ such that $SI \subseteq I$ (resp., $IS \subseteq I$). An ideal I of $\mathcal{G}$ is called two sided ideal of $\mathcal{G}$ if it is both left and right ideal of $\mathcal{G}$. An $\mathcal{AG}$-subgroupoid B of $\mathcal{G}$ is called bi-ideal if $B(\mathcal{G}B) \subseteq B$. A non-empty subset Q of $\mathcal{G}$ is called quasi-ideal of $\mathcal{G}$ if $Q\mathcal{G} \cap \mathcal{G}Q \subseteq Q$.

A fuzzy subset f of an $\mathcal{AG}$-groupoid $\mathcal{G}$ is defined as a mapping from $\mathcal{G}$ into $[0, 1]$, where $[0, 1]$ is the usual closed interval of real numbers. Let f and g be any two fuzzy subsets of $\mathcal{G}$. Then, its product is denoted by $f \circ g$ and is defined as

$$(f \circ g)(x) = \begin{cases} \bigvee\limits_{x=yz} \{f(y) \wedge g(z)\} & \text{if there exist } x, y \in \mathcal{G}, \text{ such that } x = yz, \\ 0 & \text{otherwise.} \end{cases}$$

Next, we recall the concept of interval valued fuzzy sets. By an interval number, we mean a closed subinterval $\widetilde{a} = [a^-, a^+]$ of the closed interval $I = [0, 1]$, where $0 \leqslant a^- \leqslant a^+ \leqslant 1$. Let $D[0, 1]$ denote the family of all closed subintervals of $[0, 1]$, i.e.,

$$D[0, 1] = \{\widetilde{a} = [a^-, a^+] : a^- \leqslant a^+, \text{ for all } a^-, a^+ \in [0, 1]\},$$

where the elements in $D[0, 1]$ are called the interval numbers on $[0, 1]$, $\widetilde{0} = [0, 0]$ and $\widetilde{1} = [1, 1]$. For any two elements in $D[0, 1]$ the redefined minimum and redefined maximum, respectively, denoted by $r\min$ and $r\max$, and the symbols "$\preceq$","$\succeq$","$=$" are defined. We consider two elements $\widetilde{a} = [a^-, a^+]$ and $\widetilde{b} = [b^-, b^+]$ in $D[0, 1]$. Then,

$$\begin{aligned}
\widetilde{a} &\succeq \widetilde{b} \text{ if and only if } a^- \geq b^- \text{ and } a^+ \geq b^+ \\
\widetilde{a} &\preceq \widetilde{b} \text{ if and only if } a^- \leqslant b^- \text{ and } a^+ \leqslant b^+ \\
\widetilde{a} &= \widetilde{b} \text{ if and only if } a^- = b^- \text{ and } a^+ = b^+ \\
r\min\{\widetilde{a},\widetilde{b}\} &= \min\{a^-,b^-\},\min\{a^+,b^+\} \\
r\max\{\widetilde{a},\widetilde{b}\} &= \max\{a^-,b^-\},\max\{a^+,b^+\}.
\end{aligned}$$

Let $\widetilde{a}_i \in D[0,1]$, where $i \in \Pi$, we define

$$r\inf_{i\in\Pi}\widetilde{a}_i = \left[\inf a_i^-, \inf a_i^+\right]_{i\in\Pi} \text{ and } r\sup_{i\in\Pi}\widetilde{a}_i = \left[\sup a_i^-, \sup a_i^+\right]_{i\in\Pi}.$$

Let X be a non-empty set. A cubic set $\widetilde{\mu}_A$ on X is defined as

$$\widetilde{\mu}_A = \left\{\left\langle x, [\mu_A^-(x), \widetilde{\mu}_A^+(x)]\right\rangle : x \in X\right\},$$

where $\mu_A^-(x) \leq \mu_A^+(x)$, for all $x \in X$. Then the ordinary fuzzy sets $\mu_A^-(x) : X \to [0,1]$ and $\mu_A^+(x) : X \to [0,1]$ are called a lower fuzzy sets and upper fuzzy sets of $\widetilde{\mu}_A$ respectively. Let $\widetilde{\mu}_A(x) = [\mu_A^-(x), \widetilde{\mu}_A^+(x)]$ then

$$A = \{\langle x, \widetilde{\mu}_A(x)\rangle : x \in X\}, \text{ where } \widetilde{\mu}_A : X \to D[0,1].$$

To avoid symbols complications, we use the symbols $\Gamma = (\widetilde{\gamma}_1, \gamma_2)$ and $\Delta = (\widetilde{\delta}_1, \delta_2)$ for rest of the study.

3. $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-Cubic sets

Jun et al. [7], introduced the concept of cubic sets defined on a non-empty set X as objects having the form

$$\Xi = \left\{\left\langle x, \widetilde{\Psi}_\Xi(x), \eta_\Xi(x)\right\rangle : x \in X\right\},$$

which is briefly denoted by $\Xi = \left\langle \widetilde{\Psi}_\Xi, \eta_\Xi\right\rangle$, where the functions $\widetilde{\Psi}_\Xi : X \to D[0,1]$ and $\eta_\Xi : X \to [0,1]$.

Let $\Xi = \left\langle \widetilde{\Psi}_\Xi, \eta_\Xi\right\rangle$ and $\digamma = \left\langle \widetilde{\Psi}_\digamma, \eta_\digamma\right\rangle$ be two cubic sets of $\mathcal{S}$. Then,

$$\Xi \cap \digamma = \left\{\left\langle x, r\min\left\{\widetilde{\Psi}_\Xi(x), \widetilde{\Psi}_\digamma(x)\right\}, \max\{\eta_\Xi(x), \eta_\digamma(x)\}\right\rangle : x \in \mathcal{S}\right\}$$

and

$$\Xi \odot \digamma = \left\{\left\langle \left(\widetilde{\Psi}_\Xi \circ \widetilde{\Psi}_\eta\right)(x), \left(\eta_\Xi \circ \eta_\digamma\right)(x)\right\rangle : x \in \mathcal{S}\right\},$$

where,

$$\left(\widetilde{\Psi}_\Xi \circ \widetilde{\Psi}_\digamma\right)(x) = \begin{cases} r\sup\limits_{x=yz}\{r\min\left\{\widetilde{\Psi}_\Xi(y), \widetilde{\Psi}_\digamma(z)\right\} & \text{if } x = yz, \\ [0,0] & \text{otherwise.} \end{cases}$$

and

$$(\eta_{\Xi} \circ \eta_{\digamma})(x) = \begin{cases} \inf\limits_{x=yz} \{\max\{\eta_{\Xi}(y), \eta_{\digamma}(z)\}\} & \text{if } x = yz, \\ 1 & \text{otherwise.} \end{cases}$$

Let $\mathcal{C}(\mathcal{S})$ denote the family of all cubic sets in $\mathcal{S}$. Then, it becomes an $\mathcal{AG}$-groupoid.

Let $\widetilde{\alpha} \in D(0,1]$ and $\beta \in [0,1)$ be such that $\widetilde{0} \prec \widetilde{\alpha}$ and $\beta < 1$. Then, by cubic point (CP) we mean $x_{(\widetilde{\alpha},\beta)}(y) = \langle x_{\widetilde{\alpha}}(y), x_{\beta}(y)\rangle$, where

$$x_{\widetilde{\alpha}}(y) = \begin{cases} \widetilde{\alpha} & \text{if } x = y, \\ \widetilde{0} & \text{otherwise.} \end{cases}$$

and

$$x_{\beta}(y) = \begin{cases} \beta & \text{if } x = y, \\ 1 & \text{otherwise.} \end{cases}$$

Here we give the generalized version of a Jun's cubic set [10]. For any cubic set $\Xi = \left\langle \widetilde{\Psi}_{\Xi}, \eta_{\Xi} \right\rangle$ and for a cubic point $x_{(\widetilde{\alpha},\beta)}$, with the condition that $[\alpha,\beta]+[\alpha,\beta] = [2\alpha, 2\beta]$ such that $0 \leq 2\alpha < 1$ and $2\beta \leq 1$, we have

(i) $x_{(\widetilde{\alpha},\beta)} \in_{\Gamma} \Xi$ if $\widetilde{\Psi}_{\Xi}(x) \succeq \widetilde{\alpha} \succ \widetilde{\gamma}_1$ and $\eta_{\Xi}(x) \leq \beta < \gamma_2$.

(ii) $x_{(\widetilde{\alpha},\beta)} q_{\Delta} \Xi$ if $\widetilde{\Psi}_{\Xi}(x) + \widetilde{\alpha} \succ 2\widetilde{\delta}_1$ and $\eta_{\Xi}(x) + \beta < 2\delta_2$.

(iii) $x_{(\widetilde{\alpha},\beta)} \in_{\Gamma} \vee q_{\Delta} \Xi$ if $x_{(\widetilde{\alpha},\beta)} \in_{\Gamma} \Xi$ or $x_{(\widetilde{\alpha},\beta)} q_{\Delta} \Xi$.

(iv) $x_{(\widetilde{\alpha},\beta)} \in_{\Gamma} \wedge q_{\Delta} \Xi$ if $x_{(\widetilde{\alpha},\beta)} \in_{\Gamma} \Xi$ and $x_{(\widetilde{\alpha},\beta)} q_{\Delta} \Xi$.

Next, we define the generalized characteristic function.

Definition 1 Let $\mathcal{S}$ be an $\mathcal{AG}$-groupoid. Then, the cubic characteristic function $\mathcal{X}_{\Xi}^{(\Gamma,\Delta)} = \left\langle \widetilde{\Psi}_{\mathcal{X}_{\Xi}^{(\Gamma,\Delta)}}, \eta_{\mathcal{X}_{\Xi}^{(\Gamma,\Delta)}} \right\rangle$ of $\Xi = \left\langle \widetilde{\Psi}_{\Xi}, \eta_{\Xi} \right\rangle$ is defined as

$$\widetilde{\Psi}_{\mathcal{X}_{\Xi}^{(\Gamma,\Delta)}}(x) \succeq \begin{cases} \widetilde{\delta}_1 = [1,1] & \text{if } x \in \Xi, \\ \widetilde{\gamma}_1 = [0,0] & \text{if } x \notin \Xi, \end{cases}$$

and

$$\eta_{\mathcal{X}_{\Xi}^{(\Gamma,\Delta)}}(x) \leq \begin{cases} \delta_2 = 0 & \text{if } x \in \Xi, \\ \gamma_2 = 1 & \text{if } x \notin \Xi, \end{cases}$$

where $\widetilde{\delta}_1, \widetilde{\gamma}_1 \in D(0,1]$ such that $\widetilde{\gamma}_1 \prec \widetilde{\delta}_1$ and $\delta_2, \gamma_2 \in [0,1)$ such that $\delta_2 < \gamma_2$.

Now, we introduce a new relation on $\mathcal{C}(\mathcal{S})$ denoted by "$\subseteq \vee q_{(\Gamma,\Delta)}$" as follows.

Let $\Xi = \left\langle \widetilde{\Psi}_{\Xi}, \eta_{\Xi} \right\rangle, \digamma = \left\langle \widetilde{\Psi}_{\digamma}, \eta_{\digamma} \right\rangle \in \mathcal{C}(\mathcal{S})$, by $\Xi \subseteq \vee q_{(\Gamma,\Delta)} \digamma$ we mean that $x_{(\widetilde{\alpha},\beta)} \in_{\Gamma} \Xi$ implies that $x_{(\widetilde{\alpha},\beta)} \in_{\Gamma} \vee q_{\Delta} \digamma$ for all $x \in \mathcal{S}$. Moreover, Ξ and $\digamma$ are said to be (Γ,Δ)-equal if $\Xi \subseteq \vee q_{(\Gamma,\Delta)} \digamma$ and $\digamma \subseteq \vee q_{(\Gamma,\Delta)} \Xi$. The above definitions can be found in [10].

Lemma 1 *[10] Let* $\Xi = \left\langle \widetilde{\Psi}_{\Xi}, \eta_{\Xi} \right\rangle, \digamma = \left\langle \widetilde{\Psi}_{\digamma}, \eta_{\digamma} \right\rangle \in \mathcal{C}(\mathcal{S})$. *Then,* $\Xi \subseteq \vee q_{(\Gamma,\Delta)} \digamma$ *if and only if*

$$r \max\left\{\widetilde{\Psi}_{\digamma}(a), \widetilde{\gamma}_1\right\} \succeq r \min\left\{\widetilde{\Psi}_{\Xi}(a), \widetilde{\delta}_1\right\} \text{ and } \min\{\eta_{\digamma}(a), \gamma_2\} \leq \max\{\eta_{\Xi}(a), \delta_2\}.$$

Lemma 2 *[10] Let* $\Xi = \left\langle \widetilde{\Psi}_{\Xi}, \eta_{\Xi} \right\rangle, \digamma = \left\langle \widetilde{\Psi}_{\digamma}, \eta_{\digamma} \right\rangle, \Omega = \left\langle \widetilde{\Psi}_{\Omega}, \eta_{\Omega} \right\rangle \in \mathcal{C}(\mathcal{S})$. *If* $\Xi \subseteq \vee q_{(\Gamma,\Delta)} \digamma$ *and* $\digamma \subseteq \vee q_{(\Gamma,\Delta)} \Omega$. *Then,* $\Xi \subseteq \vee q_{(\Gamma,\Delta)} \Omega$.

From Lemmas 1 and 2 we say that " $=_{(\Gamma,\Delta)}$ " is an equivalence relation on $\mathcal{C}(\mathcal{S})$.

Lemma 3 *Let* $\mathcal{S}$ *be an* $\mathcal{AG}$*-groupoid and* $\Xi = \left\langle \widetilde{\Psi}_{\Xi}, \eta_{\Xi} \right\rangle, \digamma = \left\langle \widetilde{\Psi}_{\digamma}, \eta_{\digamma} \right\rangle$ *be the cubic sets of* $\mathcal{S}$. *Then, we have*

(i) $\Xi \subseteq \digamma$ *if and only if* $\mathcal{X}_{\Xi}^{(\Gamma,\Delta)} \subseteq \vee q_{(\Gamma,\Delta)} \mathcal{X}_{\digamma}^{(\Gamma,\Delta)}$.

(ii) $\mathcal{X}_{\Xi}^{(\Gamma,\Delta)} \cap \mathcal{X}_{\digamma}^{(\Gamma,\Delta)} =_{(\Gamma,\Delta)} \mathcal{X}_{\Xi\cap\digamma}^{(\Gamma,\Delta)}$.

(iii) $\mathcal{X}_{\Xi}^{(\Gamma,\Delta)} \odot \mathcal{X}_{\digamma}^{(\Gamma,\Delta)} =_{(\Gamma,\Delta)} \mathcal{X}_{\Xi\digamma}^{(\Gamma,\Delta)}$.

4. Cubic soft sets

In this section, we recall some fundamental concepts of soft sets. For further details and background, see [11], [13], [9].

We introduced the concept of an $(\in_{\Gamma}, \in_{\Gamma} \vee q_{\Delta})$-cubic soft set which is actually the generalization of a soft set.

Definition 2 [9] Let U be an initial universal set and E be set of parameters under consideration. Let $\mathcal{C}^U$ denotes the set of all cubic subsets of U. Let $A \subseteq E$. A pair (F, A) is called cubic soft set over U, where F is a mapping given by $F : A \to \mathcal{C}^U$. Note that the the pair (F, A) can be expressed as the following set:

$$(F, A) := \{F(\epsilon) : \epsilon \in A\}, \text{ where } F(\epsilon) = \left\langle \widetilde{\Psi}_{F(\epsilon)}, \eta_{F(\epsilon)} \right\rangle.$$

In general, for every $\epsilon \in A$, $F(\epsilon)$ is a cubic set of U and it is called cubic value set of parameter ϵ. The set of all cubic soft sets over U with parameters from E is called cubic soft class and is denoted by $\mathcal{F}_{\rho}(U, E)$.

Definition 3 [13] Let (F, A) and (G, B) be two cubic soft sets over U. Then (F, A) is called cubic soft set of (G, B) and write $(F, A) \subset (G, B)$ if

(i) $A \subseteq B$.

(ii) For any $\epsilon \in A$, $F(\epsilon) \subseteq G(\epsilon)$.

(F, A) and (G, B) are said to be cubic soft equal and write $(F, A) = (G, B)$ if $(F, A) \subset (G, B)$ and $(G, B) \subset (F, A)$.

Definition 4 [13] The union of two cubic soft sets (F, A) and (G, B) over U is called cubic soft set and is denoted by (H, C), where $C = A \cup B$ and

$$H(\epsilon) = \begin{cases} F(\epsilon), & \text{if } \epsilon \in A - B, \\ G(\epsilon), & \text{if } \epsilon \in B - A, \\ F(\epsilon) \cup G(\epsilon), & \text{if } \epsilon \in A \cap B, \end{cases}$$

for all $\epsilon \in C$. This is denoted by $(H, C) = (F, A)\widetilde{\cup}(G, B)$.

Definition 5 [13] The intersection of two cubic soft sets (F, A) and (G, B) over U is called cubic soft set and is denoted by (H, C), where $C = A \cup B$ and

$$H(\epsilon) = \begin{cases} F(\epsilon), & \text{if } \epsilon \in A - B, \\ G(\epsilon), & \text{if } \epsilon \in B - A, \\ F(\epsilon) \cap G(\epsilon), & \text{if } \epsilon \in A \cap B, \end{cases}$$

for all $\epsilon \in C$. This is denoted by $(H, C) = (F, A)\widetilde{\cap}(G, B)$.

Here, we introduced the concept of generalized version of cubic soft set.

Definition 6 Let $V \subseteq U$. A cubic soft set (F, A) over U is said to be relative whole (Γ, Δ)-cubic soft set (with respect to universe set V and parameter set A), denoted by $\Sigma(V, A)$, if

$$F(\epsilon) = \mathcal{X}_V^{(\Gamma,\Delta)} \text{ for all } \epsilon \in A.$$

Definition 7 Let (F, A) and (G, B) be two cubic soft sets over U. We say that (F, A) is an (Γ, Δ)-cubic soft subset of (G, B) and write $(F, A) \subset_{(\Gamma,\Delta)} (G, B)$ if

(i) $A \subseteq B$.

(ii) For any $\epsilon \in A$, $F(\epsilon) \subseteq \vee q_{(\Gamma,\Delta)} G(\epsilon)$.

(F, A) and (G, B) are said to be (Γ, Δ)-cubic soft equal and write $(F, A) \asymp_{(\Gamma,\Delta)} (G, B)$ if $(F, A) \subset_{(\Gamma,\Delta)} (G, B)$ and $(G, B) \asymp_{(\Gamma,\Delta)} (F, A)$.

The product of two cubic soft sets (F, A) and (G, B) over an $\mathcal{AG}$-groupoid $\mathcal{S}$, denoted by $(F \circ G, C)$, where $C = A \cup B$ and

$$(F \circ G)(\epsilon) = \begin{cases} F(\epsilon), & \text{if } \epsilon \in A - B, \\ G(\epsilon), & \text{if } \epsilon \in B - A, \\ F(\epsilon) \circ G(\epsilon), & \text{if } \epsilon \in A \cap B, \end{cases}$$

for all $\epsilon \in C$. This is denoted by $(F \circ G, C) = (F, A) \odot (G, B)$.

5. $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-Cubic soft ideals over an $\mathcal{AG}$-groupoid

Here, we will introduce the concepts of $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft $\mathcal{AG}$-subgroupoids and $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft left (resp., right,two-sided) ideals over an $\mathcal{AG}$-groupoid $\mathcal{S}$ and investigate the fundamental properties and relationships of $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft sets and $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft right ideals.

Definition 8 A cubic soft set (F, A) over an $\mathcal{AG}$-groupoid $\mathcal{S}$ is called an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft $\mathcal{AG}$-subgroupoid over $\mathcal{S}$ if it satisfies:

$$(F, A) \odot (F, A) \subset_{(\Gamma,\Delta)} (F, A).$$

Definition 9 A cubic soft set (F, A) over an $\mathcal{AG}$-groupoid $\mathcal{S}$ is called an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft left (resp., right) ideal over $\mathcal{S}$ if it satisfies:

$$\Sigma(\mathcal{S}, A) \odot (F, A) \subset_{(\Gamma,\Delta)} (F, A)(resp., (F, A) \odot \Sigma(\mathcal{S}, A) \subset_{(\Gamma,\Delta)} (F, A)).$$

A cubic soft set (F, A) over an $\mathcal{AG}$-groupoid $\mathcal{S}$ is called an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft ideal over $\mathcal{S}$ if it is both an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft left ideal and an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft right ideal over $\mathcal{S}$.

Theorem 1 *Let (F, A) be a cubic soft set over an $\mathcal{AG}$-groupoid $\mathcal{S}$ with left identity. Then, (F, A) is an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft $\mathcal{AG}$-subgroupoid over $\mathcal{S}$ if and only if*

$$r \max\left\{\widetilde{\Psi}_{F(\epsilon)}(xy), \widetilde{\gamma}_1\right\} \succeq r \min\left\{\left\{\widetilde{\Psi}_{F(\epsilon)}(x), \widetilde{\Psi}_{F(\epsilon)}(y)\right\}, \widetilde{\delta}_1\right\}$$

and

$$\min\{\eta_{F(\epsilon)}(xy), \gamma_2\} \leq \max\{\{\eta_{F(\epsilon)}(x), \eta_{F(\epsilon)}(y)\}, \delta_2\},$$

for all $x, y \in \mathcal{S}$, where $\epsilon \in A, \widetilde{\delta}_1, \widetilde{\gamma}_1 \in D(0, 1]$ such that $\widetilde{\gamma}_1 \prec \widetilde{\delta}_1$, and $\delta_2, \gamma_2 \in [0, 1)$ such that $\delta_2 < \gamma_2$.

Proof. Let (F, A) be a cubic soft set over an $\mathcal{AG}$-groupoid $\mathcal{S}$. Assume that (F, A) is an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft $\mathcal{AG}$-subgroupoid over $\mathcal{S}$. Let $x, y \in \mathcal{S}$, $\epsilon \in A$, $\widetilde{t}_1, \widetilde{\delta}_1, \widetilde{\gamma}_1 \in D(0, 1], t_2, \delta_2, \gamma_2 \in [0, 1)$ such that

$$r \max\left\{\widetilde{\Psi}_{F(\epsilon)}(xy), \widetilde{\gamma}_1\right\} \prec \widetilde{t}_1 \preceq r \min\left\{\left\{\widetilde{\Psi}_{F(\epsilon)}(x), \widetilde{\Psi}_{F(\epsilon)}(y)\right\}, \widetilde{\delta}_1\right\}$$

and

$$\min\{\eta_{F(\epsilon)}(xy), \gamma_2\} > t_2 \geq \max\{\{\eta_{F(\epsilon)}(x), \eta_{F(\epsilon)}(y)\}, \delta_2\}.$$

Then,

$$r \max\left\{\widetilde{\Psi}_{F(\epsilon)}(xy), \widetilde{\gamma}_1\right\} \prec \widetilde{t}_1 \text{ implies that } \widetilde{\Psi}_{F(\epsilon)}(xy) \prec \widetilde{t}_1 \prec \widetilde{\gamma}_1$$

and

$$\min\{\eta_{F(\epsilon)}(xy), \gamma_2\} > t_2 \text{ implies that } \eta_{F(\epsilon)}(xy) > t_2 > \gamma_2.$$

Thus, $(xy)_{(\widetilde{t}_1, t_2)} \overline{\in_\Gamma \vee q_\Delta} F(\epsilon)$. On the other hand, if

$$\widetilde{t}_1 \preceq r \min\left\{\left\{\widetilde{\Psi}_{F(\epsilon)}(x), \widetilde{\Psi}_{F(\epsilon)}(y)\right\}, \widetilde{\delta}_1\right\}$$

and

$$t_2 \geq \max\{\{\eta_{F(\epsilon)}(x), \eta_{F(\epsilon)}(y)\}, \delta_2\}$$

we have $\widetilde{\Psi}_{F(\epsilon)}(x) \succeq \widetilde{t}_1 \succ \widetilde{\gamma}_1$, $\widetilde{\Psi}_{F(\epsilon)}(y) \succeq \widetilde{t}_1 \succ \widetilde{\gamma}_1$ and $\eta_{F(\epsilon)}(x) \leq t_2 < \gamma_2$, $\eta_{F(\epsilon)}(y) \leq t_2 < \gamma_2$, then $(x)_{(\widetilde{t}_1,t_2)} \in_\Gamma F(\epsilon)$ and $(y)_{(\widetilde{t}_1,t_2)} \in_\Gamma F(\epsilon)$ but $(xy)_{(\widetilde{t}_1,t_2)} \overline{\in_\Gamma \vee q_\Delta} F(\epsilon)$. This is contradiction to the hypothesis. Hence,

$$r\max\left\{\widetilde{\Psi}_{F(\epsilon)}(xy), \widetilde{\gamma}_1\right\} \succeq r\min\left\{\left\{\widetilde{\Psi}_{F(\epsilon)}(x), \widetilde{\Psi}_{F(\epsilon)}(y)\right\}, \widetilde{\delta}_1\right\}$$

and

$$\min\{\eta_{F(\epsilon)}(xy), \gamma_2\} \leq \max\{\{\eta_{F(\epsilon)}(x), \eta_{F(\epsilon)}(y)\}, \delta_2\},$$

Conversely, let there exist $x \in \mathcal{S}$, $\epsilon \in A, \widetilde{t}, \widetilde{t}_1 \in D(0,1], s, s_1 \in [0,1)$ such that $(x)_{(\widetilde{t},s)} \in_\Gamma F(\epsilon), y \in \mathcal{S}$ such that $(y)_{(\widetilde{t}_1,s_1)} \in_\Gamma F(\epsilon)$. This shows that, $\widetilde{\Psi}_{F(\epsilon)}(x) \succeq \widetilde{t} \succ \widetilde{\gamma}_1, \eta_{F(\epsilon)}(x) \leq s < \gamma_2$ and $\widetilde{\Psi}_{F(\epsilon)}(y) \succeq \widetilde{t}_1 \succ \widetilde{\gamma}_1, \eta_{F(\epsilon)}(y) \leq s_1 < \gamma_2$. So

$$r\max\left\{\widetilde{\Psi}_{F(\epsilon)}(xy), \widetilde{\gamma}_1\right\} \succeq r\min\left\{\left\{\widetilde{\Psi}_{F(\epsilon)}(x), \widetilde{\Psi}_{F(\epsilon)}(y)\right\}, \widetilde{\delta}_1\right\} \succeq r\min\{\{\widetilde{t}, \widetilde{t}_1\}, \widetilde{\delta}_1\}$$

and

$$\min\{\eta_{F(\epsilon)}(xy), \gamma_2\} \leq \max\{\{\eta_{F(\epsilon)}(x), \eta_{F(\epsilon)}(y)\}, \delta_2\} \leq \max\{\{s, s_1\}, \delta_2\}.$$

Now we discuss the following cases:

(1) If $r\min\{\widetilde{t}, \widetilde{t}_1\} \preceq \widetilde{\delta}_1$ and $\max\{s, s_1\} \geq \delta_2$, then $\widetilde{\Psi}_{F(\epsilon)}(xy) \succeq r\min\{\widetilde{t}, \widetilde{t}_1\} \succ \widetilde{\gamma}_1$ and $\eta_{F(\epsilon)}(xy) \leq \max\{s, s_1\} < \gamma_2$. This shows that $(xy)_{\langle r\min\{\widetilde{t},\widetilde{t}_1\}, \max\{\{s,s_1\}\rangle} \in_\Gamma F(\epsilon)$.

(2) If $r\min\{\widetilde{t}, \widetilde{t}_1\} \succ \widetilde{\delta}_1$ and $\max\{s, s_1\} < \delta_2$, then $\widetilde{\Psi}_{F(\epsilon)}(xy) + r\min\{\widetilde{t}, \widetilde{t}_1\} \succ 2\widetilde{\delta}_1$ and $\eta_{F(\epsilon)}(xy) + \max\{\{s, s_1\} < 2\delta_2$. This shows that $(xy)_{(r\min\{\widetilde{t},\widetilde{t}_1\}, \max\{s,s_1\})} q_\Delta F(\epsilon)$.

From both cases, we get $(xy)_{(r\min\{\widetilde{t}_1,\widetilde{t}_2\}, \max\{s_1,s_2\})} \in_\Gamma \vee q_\Delta F(\epsilon)$. Hence, (F, A) is an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft $\mathcal{AG}$-subgroupoid over $\mathcal{S}$. ∎

Definition 10 A cubic soft set (F, A) over an $\mathcal{AG}$-groupoid $\mathcal{S}$ with left identity is called an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft left (resp., right) ideal over $\mathcal{S}$ if for all $x, y \in \mathcal{S}$, $y_{(\widetilde{t},s)} \in_\Gamma F(\epsilon)$ implies $(xy)_{(\widetilde{t},s)} \in_\Gamma \vee q_\Delta F(\epsilon)$ (resp., $x_{(\widetilde{t},s)} \in_\Gamma F(\epsilon)$ implies $(xy)_{(\widetilde{t},s)} \in_\Gamma \vee q_\Delta F(\epsilon)$), where $\epsilon \in A, \widetilde{\delta}_1, \widetilde{\gamma}_1 \in D(0,1]$ such that $\widetilde{\gamma}_1 \prec \widetilde{\delta}_1$ and $\delta_2, \gamma_2 \in [0,1)$ such that $\delta_2 < \gamma_2$.

Theorem 2 *Let A be a non-empty subset of an $\mathcal{AG}$-groupoid $\mathcal{S}$, then for an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft left (resp., right) ideal the following are equivalent.*

(1) $\Sigma(\mathcal{S}, A) \odot (F, A) \subset_{(\Gamma,\Delta)} (F, A)$

(2) $x_{(\widetilde{\alpha},\ \beta)} \in_\Gamma F(\epsilon) \Longrightarrow (yx)_{(\widetilde{\alpha},\ \beta)} \in_\Gamma F(\epsilon)$.

(3) $r\max\left\{(\widetilde{\Psi}_{F(\epsilon)}(yx), \widetilde{\gamma}_1\right\} \succeq r\min\{\widetilde{\Psi}_{F(\epsilon)}(x), \widetilde{\delta}_1\}$ *and* $\min\{\eta_{F(\epsilon)}(yx), \gamma_2\} \leq \max\{\eta_{F(\epsilon)}(x), \delta_2\}$.

Proof. (1) $\Longrightarrow$ (2): Let $x_{(\widetilde{\alpha},\,\beta)} \in_{(\widetilde{\gamma}_1,\gamma_2)} F(\epsilon)$, then $\widetilde{\Psi}_{F(\epsilon)}(x) \succeq \widetilde{\alpha} \succeq \widetilde{\gamma}_1$ and $\eta_{F(\epsilon)}(x) \leq \beta < \gamma_2$. Then,

$$\begin{aligned} r\max\left\{\widetilde{\Psi}_{F(\epsilon)}(yx), \widetilde{\gamma}_1\right\} &\succeq r\min\{(\widetilde{\Psi}_{\mathcal{X}_S^{(\Gamma,\Delta)}} \circ \widetilde{\Psi}_{F(\epsilon)})(yx), \}, \widetilde{\delta}_1\} \\ &= r\min\{r \sup_{yx=ab}\{r\min\{\widetilde{\Psi}_{\mathcal{X}_S^{(\Gamma,\Delta)}}(a), \widetilde{\Psi}_{F(\epsilon)}(b)\}\}, \widetilde{\delta}_1\} \\ &= r \sup_{yx=ab}[r\min\{\widetilde{\Psi}_{\mathcal{X}_S^{(\Gamma,\Delta)}}(a), \widetilde{\Psi}_{F(\epsilon)}(b), \widetilde{\delta}_1\} \\ &\succeq r\min\{\widetilde{\Psi}_{F(\epsilon)}(x), \widetilde{\delta}_1\} \\ &\succeq r\min\{\widetilde{\alpha}, \widetilde{\delta}_1\} \succeq \tilde{\gamma}_1. \end{aligned}$$

Thus, $\widetilde{\Psi}_{F(\epsilon)}(yx) \succeq r\min\{\widetilde{\alpha}, \widetilde{\delta}_1\} \succeq \tilde{\gamma}_1$ and $\widetilde{\alpha} \preceq \tilde{\gamma}_1$.

Case 1. $\widetilde{\alpha} \preceq \widetilde{\delta}_1$, this gives that $\widetilde{\Psi}_{F(\epsilon)}(yx) \succeq \widetilde{\alpha} \succ \tilde{\gamma}_1$.

Case 2. $\widetilde{\alpha} \succ \widetilde{\delta}_1$, thus $\widetilde{\Psi}_{F(\epsilon)}(yx) + \widetilde{\alpha} \succ 2\widetilde{\delta}_1$. Moreover,

$$\begin{aligned} \min\left\{(\eta_{F(\epsilon)}(yx), \gamma_2\right\} &\leq \max\{(\widetilde{\Psi}_{\mathcal{X}_S^{(\Gamma,\Delta)}} \circ \eta_{F(\epsilon)})(yx), \}, \delta_2\} \\ &= \max\{\min\{\max_{yx=ab}\{\widetilde{\Psi}_{\mathcal{X}_S^{(\Gamma,\Delta)}}(a), \eta_{F(\epsilon)}(b)\}\}, \delta_2\} \\ &= \max_{yx=ab}[\min\{\eta_{F(\epsilon)}(b), \delta_2\} \\ &\leq \max[\min\{\eta_{F(\epsilon)}(x), \delta_2\}] \\ &\leq \max\{\beta, \delta_2\} < \gamma_2. \end{aligned}$$

Thus, $\eta_{F(\epsilon)}(yx) \leq \max\{\beta, \delta_2\}$.

Case 3. If $\beta \leq \delta_2$, then $\eta_{F(\epsilon)}(yx) \leq \delta_2$.

Case 4. $\beta > \delta_2$, then $\eta_{F(\epsilon)}(yx) + \beta < 2\delta_2$.

Hence $(yx)_{(\widetilde{\alpha},\,\beta)} \in_{(\widetilde{\gamma}_1,\gamma_2)} \vee q_{(\widetilde{\delta}_1,\delta_2)} F(\epsilon)$.

(2) $\Longrightarrow$ (3) : is similar as in [10].

(3) $\Longrightarrow$ (1): Assume that on contrary that $\Sigma(\mathcal{S}, A) \odot (F, A)\overline{\subset_{(\Gamma,\Delta)}}(F, A)$. Then there exist $\epsilon \in A$ and $x_{(\widetilde{\alpha},\,\beta)} \in_{(\widetilde{\gamma}_1,\gamma_2)} (\widetilde{\Psi}_{\mathcal{X}_S^{(\Gamma,\Delta)}} \circ F)(\epsilon)$ such that

$$(x)_{(\widetilde{\alpha},\,\beta)}\overline{\in_{(\widetilde{\gamma}_1,\gamma_2)} \vee q_{(\widetilde{\delta}_1,\delta_2)}} F(\epsilon),$$

where $F(\epsilon) = \langle \widetilde{\Psi}_{F(\epsilon)}, \eta_{F(\epsilon)} \rangle$. Thus,

$$\begin{aligned} &(x)_{(\widetilde{\alpha},\,\beta)}\overline{\in_{(\widetilde{\gamma}_1,\gamma_2)}} F(\epsilon) \text{ and } (x)_{(\widetilde{\alpha},\,\beta)}\overline{q_{(\widetilde{\delta}_1,\delta_2)}} F(\epsilon), \text{ therefore} \\ \widetilde{\Psi}_{F(\epsilon)} \preceq\ & \widetilde{\alpha} \succ \widetilde{\gamma}_1 \ \&\ \eta_{F(\epsilon)} \geq \beta \text{ and } \widetilde{\Psi}_{F(\epsilon)} + \widetilde{\alpha} \preceq 2\widetilde{\delta}_1 \ \&\ \eta_{F(\epsilon)} + \beta > 2\delta_1. \end{aligned}$$

Therefore,

$$\widetilde{\Psi}_{F(\epsilon)} + \widetilde{\alpha} \preceq 2\widetilde{\delta}_1 \text{ and } \widetilde{\Psi}_{F(\epsilon)} - \widetilde{\alpha} \preceq \widetilde{\mathbf{0}}, \text{ also } \eta_{F(\epsilon)} - \beta \geq 1 \text{ and } \eta_{F(\epsilon)} + \beta > 2\delta_1$$

Thus we get,

$$\widetilde{\Psi}_{F(\epsilon)} \preceq \widetilde{\delta}_1 \text{ and } \eta_{F(\epsilon)} > \delta_1.$$

Therefore,

$$\begin{aligned}\widetilde{\delta}_1 &\succeq r\max\{\widetilde{\Psi}_{F(\epsilon)}(x),\widetilde{\gamma}_1\} = r\max\{r\min\{\widetilde{\Psi}_{F(\epsilon)}(ab),\widetilde{\gamma}_1\},\widetilde{\delta}_1\}\\ &= r\min\{r\max\{\widetilde{\Psi}_{F(\epsilon)}(ab),\widetilde{\gamma}_1\},\widetilde{\delta}_1\}\\ &\succeq r\min\{r\min\{\widetilde{\Psi}_{F(\epsilon)}(b),\widetilde{\delta}_1\},\widetilde{\delta}_1\}\\ &= r\min\{\widetilde{\Psi}_{F(\epsilon)}(b),\widetilde{\delta}_1\}.\end{aligned}$$

Thus, $r\max\{\widetilde{\Psi}_{F(\epsilon)}(x),\widetilde{\gamma}_1\} \succeq \widetilde{\Psi}_{F(\epsilon)}(b)$. Therefore

$$\begin{aligned}\widetilde{\alpha} &\preceq (\widetilde{\Psi}_{\mathcal{X}_{\mathcal{S}}^{(\Gamma,\Delta)}} \circ \widetilde{\Psi}_{F(\epsilon)})(x) = r\max\{r\min\{\widetilde{\Psi}_{\mathcal{X}_{\mathcal{S}}^{(\Gamma,\Delta)}}(a),\widetilde{\Psi}_{F(\epsilon)}(b)\}\}\\ &= r\max\{\widetilde{\Psi}_{F(\epsilon)}(b)\} \preceq r\max\{\widetilde{\Psi}_{F(\epsilon)}(b),\widetilde{\gamma}_1\}.\end{aligned}$$

This is a contradiction.

Hence, $\Sigma(\mathcal{S},A)\odot(F,A) \subset_{(\Gamma,\Delta)} (F,A)$. ∎

Lemma 4 *Let $\emptyset \neq A \subseteq \mathcal{S}$. Then, A is a left (right) ideal of $\mathcal{S}$ if and only if cubic characteristic function $\mathcal{X}_A^{(\Gamma,\Delta)} = \left\langle \widetilde{\Psi}_{\mathcal{X}_A^{(\Gamma,\Delta)}}, \eta_{\mathcal{X}_A^{(\Gamma,\Delta)}} \right\rangle$ of $A = \left\langle \widetilde{\Psi}_A, \eta_A \right\rangle$ is an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic left (right) ideal of $\mathcal{S}$, where $\widetilde{\delta}_1, \widetilde{\gamma}_1 \in D(0,1]$ such that $\widetilde{\gamma}_1 \prec \widetilde{\delta}_1$, and $\delta_2, \gamma_2 \in [0,1]$ such that $\delta_2 < \gamma_2$.*

Proof. It is same as in [10]. ∎

Let us now define the $\in_\Gamma \vee q_\Delta$-cubic level set for the cubic set $\Xi = \left(\widetilde{\psi}_\Xi, \eta_\Xi\right)$ as

$$[\Xi]_{\left(\tilde{t},\delta\right)} = \{x \in \mathcal{S} : x_{\left(\tilde{t},\delta\right)} \in_\Gamma \vee q_\Delta \Xi\}.$$

Theorem 3 *A cubic set $\Xi = \left(\widetilde{\Psi}_\Xi, \eta_\Xi\right)$ is said to be an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic $\mathcal{AG}$-subgroupoid (left ideal, right ideal) of $\mathcal{S}$ if and only if $\emptyset \neq [\Xi]_{\left(\tilde{t},\delta\right)}$ is a $\mathcal{AG}$-subgroupoid (left ideal, right ideal) of $\mathcal{S}$.*

Proof. It is same as in [10]. ∎

Theorem 4 *Let $\mathcal{S}$ be an $\mathcal{AG}$-groupoid and $P \subseteq \mathcal{S}$. Then, P is a left (resp., right) ideal of $\mathcal{S}$ if and only if $\Sigma(P,A)$ is an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft left (resp., right) ideal over $\mathcal{S}$ for any $P \subseteq E$.*

Proof. It follows from Theorem 3. ∎

Example 1 Let $\mathcal{S}$= $\{a,b,c\}$ and the binary operation " $\cdot$ " on $\mathcal{S}$ be defined as follows:

$\cdot$	a	b	c
a	a	a	a
b	a	a	c
c	a	a	a

Then, $\mathcal{S}$ is an $\mathcal{AG}$-groupoid. For $A = \{e_1, e_2, e_3\} \subseteq E$, the cubic soft set $(F, A) = \{F(e_1), F(e_2), F(e_3)\}$ over $\mathcal{S}$ is defined as follows:

$$\begin{aligned} F(e_1) &= \{\langle a, [0.2, 0.3), 0.1\rangle, \langle b, [0.3, 0.4), 0.2\rangle, \langle c, [0.4, 0.5), 0.3\rangle\}, \\ F(e_2) &= \{\langle a, [0.15, 0.16), 0.2\rangle, \langle b, [0.16, 0.18), 0.3\rangle, \langle c, [0.18, 0.2), 0.3\rangle\}, \\ F(e_3) &= \{\langle a, [0.1, 0.12), 0.1\rangle, \langle b, [0.12, 0.14), 0.2\rangle, \langle c, [0.14, 0.16), 0.2\rangle\}, \end{aligned}$$

such that $\widetilde{\gamma}_1 = [0.1, .18) \prec \widetilde{\delta}_1 = [0.3, 0.4)$ and $\delta_2 = 0.3 < \gamma_2 = 0.4$. Then, it is easy to see that (F, A) is an $(\in_{([0.1,.18),0.4)}, \in_{([0.1,.18),0.4)} \vee q_{([0.3,0.4),0.3)})$-cubic soft $\mathcal{AG}$-subgroupoid over $\mathcal{S}$.

Example 2 Let $\mathcal{S}$= $\{a, b, c\}$ and the binary operation " $\cdot$ " on $\mathcal{S}$ be defined as follow:

$\cdot$	a	b	c
a	a	a	a
b	a	a	c
c	a	a	a

Then, $\mathcal{S}$ is an $\mathcal{AG}$-groupoid. For $A = \{e_1, e_2, e_3\} \subseteq E$, the cubic soft set $(F, A) = \{F(e_1), F(e_2), F(e_3)\}$ over $\mathcal{S}$ is defined as follows.

$$\begin{aligned} F(e_1) &= \{\langle a, [0.3, 0.4), 0.5\rangle, \langle b, [0.3, 0.4), 0.5\rangle, \langle c, [0.4, 0.5), 0.4\rangle\}, \\ F(e_2) &= \{\langle a, [0.2, 0.15), 0.4\rangle, \langle b, [0.15, 0.18), 0.3\rangle, \langle c, [0.18, 0.2), 0.1\rangle\}, \\ F(e_3) &= \{\langle a, [0.1, 0.15), 0.3\rangle, \langle b, [0.15, 0.20), 0.3\rangle, \langle c, [0.20, 0.25), 0.2\rangle\}, \end{aligned}$$

such that $\widetilde{\gamma}_1 = [0.1, .18) \prec \widetilde{\delta}_1 = [0.19, 0.2)$ and $\delta_2 = 0.53 < \gamma_2 = 0.54$. Then, it is easy to see that (F, A) is an $(\in_{([0.1,.18),0.54)}, \in_{([0.1,.18),0.54)} \vee q_{([0.19,0.2),0.53)})$-cubic soft right ideal over $\mathcal{S}$.

Definition 11 A cubic soft set (F, A) over an $\mathcal{AG}$-groupoid $\mathcal{S}$ is called $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft semiprime over $\mathcal{S}$ if for all $x \in \mathcal{S}$, $\epsilon \in A, \widetilde{t}_1 \in D(0, 1]$ and $s_1 \in [0, 1)$ we have $x^2_{(\widetilde{t}_1, s_1)} \in_\Gamma F(\epsilon)$ implies that $x_{(\widetilde{t}_1, s_1)} \in_\Gamma F(\epsilon)$, where $\widetilde{\delta}_1, \widetilde{\gamma}_1, \in D(0, 1]$ such that $\widetilde{\gamma}_1 \prec \widetilde{\delta}_1$, and $\delta_2, \gamma_2 \in [0, 1)$ such that $\delta_2 < \gamma_2$.

Theorem 5 *A cubic soft set (F, A) over an $\mathcal{AG}$-groupoid $\mathcal{S}$ is an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft semiprime if and only if for all $a \in \mathcal{S}$*

$$r \max\left\{\widetilde{\Psi}_{F(\epsilon)}(a), \widetilde{\gamma}_1\right\} \succeq r \min\left\{\widetilde{\Psi}_{F(\epsilon)}(a^2), \widetilde{\delta}_1\right\}$$

and

$$\min\{\eta_{F(\epsilon)}(a), \gamma_2\} \leq \min\{\eta_{F(\epsilon)}(a^2), \delta_2\},$$

where $\widetilde{\delta}_1, \widetilde{\gamma}_1, \in D(0, 1]$ such that $\widetilde{\gamma}_1 \prec \widetilde{\delta}_1$, and $\delta_2, \gamma_2 \in [0, 1)$ such that $\delta_2 < \gamma_2$.

Proof. Let (F, A) be an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft semiprime. Assume that there exist $a \in \mathcal{S}$,$\epsilon \in A, \widetilde{t}_1, \widetilde{\delta}_1 \in D(0, 1], t_2, \delta_2 \in [0, 1)$, such that

$$r \max\left\{\widetilde{\Psi}_{F(\epsilon)}(a), \widetilde{\gamma}_1\right\} \prec \widetilde{t}_1 \preceq r \min\left\{\widetilde{\Psi}_{F(\epsilon)}(a^2), \widetilde{\delta}_1\right\}$$

and

$$\min\{\eta_{F(\epsilon)}(a), \gamma_2\} > t_2 \geq \min\{\eta_{F(\epsilon)}(a^2), \delta_2\},$$

Then,

$$r\max\left\{\widetilde{\Psi}_{F(\epsilon)}(a), \widetilde{\gamma}_1\right\} \prec \widetilde{t}_1 \text{ implies that } \widetilde{\Psi}_{F(\epsilon)}(a) \prec \widetilde{t}_1 \prec \widetilde{\gamma}_1$$

and

$$\min\{\eta_{F(\epsilon)}(a), \gamma_2\} > t_2 \text{ implies that } \eta_{F(\epsilon)}(a) > t_2 > \gamma_2.$$

Thus, $(a)_{(\widetilde{t}_1,t_2)}\overline{\in_\Gamma \vee q_\Delta} F(\epsilon)$. On the other hand if

$$\widetilde{t}_1 \preceq r\min\left\{\widetilde{\Psi}_{F(\epsilon)}(a^2), \widetilde{\delta}_1\right\}$$

and

$$t_2 \geq \max\{\eta_{F(\epsilon)}(a^2), \delta_2\}$$

we have, $\widetilde{\Psi}_{F(\epsilon)}(a^2) \succeq \widetilde{t}_1 \succ \widetilde{\gamma}_1$ and $\eta_{F(\epsilon)}(a^2) \leq t_2 < \gamma_2$ this implies that, $(a^2)_{(\widetilde{t}_1,t_2)} \in_\Gamma \vee q_\Delta F(\epsilon)$ but $(a)_{(\widetilde{t}_1,t_2)}\overline{\in_\Gamma \vee q_\Delta} F(\epsilon)$. This is contradiction to the hypothesis. Hence,

$$r\max\left\{\widetilde{\Psi}_{F(\epsilon)}(a), \widetilde{\gamma}_1\right\} \succeq r\min\left\{\widetilde{\Psi}_{F(\epsilon)}(a^2), \widetilde{\delta}_1\right\}$$

and

$$\min\{\eta_{F(\epsilon)}(a), \gamma_2\} \leq \max\{\{\eta_{F(\epsilon)}(a^2), \delta_2\}.$$

Conversely, let there exist $x \in \mathcal{S}$, $\epsilon \in A, \widetilde{t}_1 \in D(0,1], t_2 \in [0,1)$ such that $(a^2)_{(\widetilde{t}_1,t_2)} \in_\Gamma F(\epsilon)$. This implies that, $\widetilde{\Psi}_{F(\epsilon)}(a^2) \succeq \widetilde{t} \succ \widetilde{\gamma}_1$and $\eta_{F(\epsilon)}(a^2) \leq t_2 < \gamma_2$. So

$$r\max\left\{\widetilde{\Psi}_{F(\epsilon)}(a), \widetilde{\gamma}_1\right\} \succeq r\min\left\{\widetilde{\Psi}_{F(\epsilon)}(a^2), \widetilde{\delta}_1\right\} \succeq r\min\{\widetilde{t}_1, \widetilde{\delta}_1\}$$

and

$$\min\{\eta_{F(\epsilon)}(a), \gamma_2\} \leq \max\{\eta_{F(\epsilon)}(a^2), \delta_2\} \leq \max\{t_2, \delta_2\}.$$

Now we have the following cases:

(1) If $\widetilde{t}_1 \preceq \widetilde{\delta}_1$ and $t_2 \geq \delta_2$, then $\widetilde{\Psi}_{F(\epsilon)}(a) \succeq \widetilde{t}_1 \succ \widetilde{\gamma}_1$ and $\eta_{F(\epsilon)}(a) \leq t_2 < \gamma_2$. This implies that, $(a)_{(\widetilde{t}_1,t_2)} \in_\Gamma F(\epsilon)$.

(2) If $\widetilde{t}_1 \succ \widetilde{\delta}_1$ and $t_2 < \delta_2$, then $\widetilde{\Psi}_{F(\epsilon)}(a) + \widetilde{t}_1 \succ 2\widetilde{\delta}_1$ and $\eta_{F(\epsilon)}(a) + t_2 < 2\delta_2$. This implies that, $(a)_{(\widetilde{t}_1,t_2)} q_\Delta F(\epsilon)$.

From both cases we get, $(a)_{(\widetilde{t}_1,t_2)} \in_\Gamma \vee q_\Delta F(\epsilon)$.

Hence, (F, A) is an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft semiprime. ∎

Example 3 Let $\mathcal{S}= \{1, 2, 3\}$ and the binary operation " $\cdot$ " on $\mathcal{S}$ be defined as follows:

$\cdot$	1	2	3
1	2	2	2
2	2	2	2
3	1	2	2

Then, $\mathcal{S}$ is an $\mathcal{AG}$-groupoid. For $A = \{e_1, e_2\} \subseteq E$, the cubic soft set $(F, A) = \{F(e_1), F(e_2)\}$ over $\mathcal{S}$ is defined as follows:

$$\begin{aligned} F(e_1) &= \{\langle 1, [0.5, 0.8), 0.6\rangle, \langle 2, [0.1, 0.7), 0.8\rangle, \langle 3, [0.2, 0.6), 0.9\rangle\}, \\ F(e_2) &= \{\langle 1, [0.2, 0.5), 0.5\rangle, \langle 2, [0.5, 0.6), 0.3\rangle, \langle 3, [0.6, 0.7), 0.1\rangle\}, \end{aligned}$$

such that $\widetilde{\gamma}_1 = [0.1, .18) \prec \widetilde{\delta}_1 = [0.19, 0.2)$ and $\delta_2 = 0.53 < \gamma_2 = 0.54$. Then, it is easy to verify that (F, A) is an $(\in_{([0.1,.18),0.54)}, \in_{([0.1,.18),0.54)} \vee q_{([0.19,0.2),0.53)})$-cubic soft semiprime ideal over $\mathcal{S}$.

Theorem 6 *For a right ideal R of an $\mathcal{AG}$-groupoid $\mathcal{S}$ with left identity, the following conditions are equivalent:*

(i) *R is semiprime.*

(ii) *$\mathcal{X}_R^{(\Gamma,\Delta)}$ is an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft semiprime, where $\widetilde{\delta}_1, \widetilde{\gamma}_1, \in D(0, 1]$ such that $\widetilde{\gamma}_1 \prec \widetilde{\delta}_1$, and $\delta_2, \gamma_2 \in [0, 1)$ such that $\delta_2 < \gamma_2$.*

Proof. (i) $\Rightarrow$ (ii) : Let R be a semiprime ideal of an $\mathcal{AG}$-groupoid $\mathcal{S}$. Let a be an arbitrary element of $\mathcal{S}$ such that $a \in R$. Then $a^2 \in R$. Hence, $\widetilde{\Psi}_{\mathcal{X}_R^{(\Gamma,\Delta)}}(a) \succeq \widetilde{\delta}_1$, $\eta_{\mathcal{X}_R^{(\Gamma,\Delta)}}(a) \leq \delta_2$ and $\widetilde{\Psi}_{\mathcal{X}_R^{(\Gamma,\Delta)}}(a^2) \succeq \widetilde{\delta}_1, \eta_{\mathcal{X}_R^{(\Gamma,\Delta)}}(a^2) \leq \delta_2$. This implies that,

$$r \max\left\{\widetilde{\Psi}_{\mathcal{X}_R^{(\Gamma,\Delta)}}(a), \widetilde{\gamma}_1\right\} \succeq r \min\left\{\widetilde{\Psi}_{\mathcal{X}_R^{(\Gamma,\Delta)}}(a^2), \widetilde{\delta}_1\right\}$$

and

$$\min\left\{\eta_{\mathcal{X}_R^{(\Gamma,\Delta)}}(a), \gamma_2\right\} \leq \max\left\{\eta_{\mathcal{X}_R^{(\Gamma,\Delta)}}(a^2), \delta_2\right\}.$$

Now let $a \notin R$. Since R is semiprime we have $a^2 \notin R$. This implies that,

$$\widetilde{\Psi}_{\mathcal{X}_R^{(\Gamma,\Delta)}}(a) \preceq \widetilde{\gamma}_1, \widetilde{\Psi}_{\mathcal{X}_R^{(\Gamma,\Delta)}}(a^2) \preceq \widetilde{\gamma}_1$$

and

$$\eta_{\mathcal{X}_R^{(\Gamma,\Delta)}}(a) \geq \gamma_2, \eta_{\mathcal{X}_R^{(\Gamma,\Delta)}}(a^2) \geq \gamma_2.$$

Hence,

$$\begin{aligned} r \max\left\{\widetilde{\Psi}_{\mathcal{X}_R^{(\Gamma,\Delta)}}(a), \widetilde{\gamma}_1\right\} &\succeq r \min\left\{\widetilde{\Psi}_{\mathcal{X}_R^{(\Gamma,\Delta)}}(a^2), \widetilde{\delta}_1\right\} \\ \min\left\{\eta_{\mathcal{X}_R^{(\Gamma,\Delta)}}(a), \gamma_2\right\} &\leq \max\left\{\eta_{\mathcal{X}_R^{(\Gamma,\Delta)}}(a^2), \delta_2\right\}, \end{aligned}$$

for all $a \in \mathcal{S}, \widetilde{\delta}_1, \widetilde{\gamma}_1, \in D(0, 1]$ such that $\widetilde{\gamma}_1 \prec \widetilde{\delta}_1$, and $\delta_2, \gamma_2 \in [0, 1)$ such that $\delta_2 < \gamma_2$.

(ii) $\Rightarrow$ (i) : Let $\mathcal{X}_R^{(\Gamma,\Delta)}$ be cubic soft semiprime. If $a^2 \in R$, for some $a \in \mathcal{S}$, then $\widetilde{\Psi}_{\mathcal{X}_R^{(\Gamma,\Delta)}}(a^2) \succeq \widetilde{\delta}_1$ and $\eta_{\mathcal{X}_R^{(\Gamma,\Delta)}}(a^2) \leq \delta_2$. Since $\mathcal{X}_R^{(\Gamma,\Delta)}$ is an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft semiprime, we have

$$r \max\left\{\widetilde{\Psi}_{\mathcal{X}_R^{(\Gamma,\Delta)}}(a), \widetilde{\gamma}_1\right\} \succeq r \min\left\{\widetilde{\Psi}_{\mathcal{X}_R^{(\Gamma,\Delta)}}(a^2), \widetilde{\delta}_1\right\}$$

and

$$\min\left\{\eta_{\mathcal{X}_R^{(\Gamma,\Delta)}}(a), \gamma_2\right\} \leq \max\left\{\eta_{\mathcal{X}_R^{(\Gamma,\Delta)}}(a^2), \delta_2\right\}.$$

Therefore, $r \max\left\{\widetilde{\Psi}_{\mathcal{X}_R^{(\Gamma,\Delta)}}(a), \widetilde{\gamma}_1\right\} \succeq \widetilde{\delta}_1$ and $\min\left\{\eta_{\mathcal{X}_R^{(\Gamma,\Delta)}}(a), \gamma_2\right\} \leq \delta_2$. But $\widetilde{\gamma}_1 \prec \widetilde{\delta}_1$ and $\delta_2 < \gamma_2$, so $\widetilde{\Psi}_{\mathcal{X}_R^{(\Gamma,\Delta)}}(a) \succeq \widetilde{\delta}_1$ and $\eta_{\mathcal{X}_R^{(\Gamma,\Delta)}}(a) \leq \delta_2$. Thus $a \in R$. Hence, R is semiprime. ∎

6. Characterizations of intra-regular $\mathcal{AG}$-groupoids

An element r of an $\mathcal{AG}$-groupoid $\mathcal{S}$ is called **intra-regular** if there exist elements $s, t \in \mathcal{S}$ such that $r = (sr^2)t$ and $\mathcal{S}$ is called **intra-regular**, if every element of $\mathcal{S}$ is intra-regular.

Theorem 7 *Let $\mathcal{S}$ be an $\mathcal{AG}$-groupoid with left identity. Then, the following conditions are equivalent.*

(i) *$\mathcal{S}$ is intra-regular.*

(ii) *For a right ideal R of an $\mathcal{AG}$-groupoid $\mathcal{S}$, $R \subseteq R^2$ and R is semiprime.*

(iii) *For an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft right ideal (F, A) over $\mathcal{S}$, $(F, A) \subseteq \vee q_{(\Gamma,\Delta)}(F, A) \odot (F, A)$ and (F, A) is an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft semiprime.*

Proof. (i) ⇒ (iii) : Let (F, A) be an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft right ideal of intra-regular $\mathcal{AG}$-groupoid $\mathcal{S}$ with left identity. Since $\mathcal{S}$ is intra-regular so for any $r \in \mathcal{S}$ there exist $s, t \in \mathcal{S}$ such that $r = (sr^2)t$. Since $\mathcal{S} = \mathcal{S}^2$, so for each t in $\mathcal{S}$ there exists t_1, t_2 in $\mathcal{S}$ such that $t = t_1 t_2$, then

$$r = (sr^2)t = (sr^2)(t_1t_2) = (t_2t_1)(r^2s) = r^2[(t_2t_1)s] = [r(t_2t_1)](rs).$$

If $r = bc$. Then,

$$\begin{aligned}
&r \max\left\{(\widetilde{\Psi}_{F(\epsilon)} \circ \widetilde{\Psi}_{F(\epsilon)})(r), \widetilde{\gamma}_1\right\} \\
&\quad = r \max\left[r \sup_{r=bc}\left\{r \min\left\{\widetilde{\Psi}_{F(\epsilon)}(b), \widetilde{\Psi}_{F(\epsilon)}(c)\right\}\right\}, \widetilde{\gamma}_1\right] \\
&\quad \succeq r \max\left[r \min\left\{\widetilde{\Psi}_{F(\epsilon)}(r(t_2t_1)), \widetilde{\Psi}_{F(\epsilon)}(rs)\right\}, \widetilde{\gamma}_1\right] \\
&\quad = r \min\left[r \max\left\{\widetilde{\Psi}_{F(\epsilon)}(r(t_2t_1)), \widetilde{\gamma}_1\right\}, r \max\left\{\widetilde{\Psi}_{F(\epsilon)}(rs), \widetilde{\gamma}_1\right\}\right] \\
&\quad \succeq r \min\left[r \min\left\{\widetilde{\Psi}_{F(\epsilon)}(r), \widetilde{\delta}_1\right\}, r \min\left\{\widetilde{\Psi}_{F(\epsilon)}(r), \widetilde{\delta}_1\right\}\right] \\
&\quad = r \min\left\{\widetilde{\Psi}_{F(\epsilon)}(r), \widetilde{\delta}_1\right\}
\end{aligned}$$

and

$$\begin{aligned}
\min\left\{(\eta_{F(\epsilon)} \circ \eta_{F(\epsilon)})(r), \gamma_2\right\} &= \min\left[r \min_{r=bc}\left\{\max\left\{\eta_{F(\epsilon)}(b), \eta_{F(\epsilon)}(c)\right\}\right\}, \gamma_2\right] \\
&\leq \min\left[\max\left\{\eta_{F(\epsilon)}(r(t_2t_1)), \eta_{F(\epsilon)}(rs)\right\}, \gamma_2\right] \\
&= \max\left[\min\left\{\eta_{F(\epsilon)}(r(t_2t_1)), \gamma_2\right\}, \min\left\{\eta_{F(\epsilon)}(rs), \gamma_2\right\}\right] \\
&\leq \max\left[\max\left\{\eta_{F(\epsilon)}(r), \delta_2\right\}, \max\left\{\eta_{F(\epsilon)}(r), \delta_2\right\}\right] \\
&= \max\left\{\eta_{F(\epsilon)}(r), \delta_2\right\}.
\end{aligned}$$

Thus, by Lemma 1, $(F,A) \subseteq \vee q_{(\Gamma,\Delta)}(F,A) \odot (F,A)$. Next we show that (F,A) is an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft semiprime. Since

$$\begin{aligned} r \max\left\{\widetilde{\Psi}_{F(\epsilon)}(r), \widetilde{\gamma}_1\right\} &= r \max\left\{\widetilde{\Psi}_{F(\epsilon)}(r^2[(t_2t_1)s]), \widetilde{\gamma}_1\right\} \\ &\succeq r \min\left\{\widetilde{\Psi}_{F(\epsilon)}(r^2), \widetilde{\delta}_1\right\}. \end{aligned}$$

and

$$\begin{aligned} \min\{\eta_{F(\epsilon)}(r), \gamma_2\} &= \min\{\eta_{F(\epsilon)}(r^2[(t_2t_1)s]), \gamma_2\} \\ &\leq \max\{\eta_{F(\epsilon)}(r^2), \delta_2\}. \end{aligned}$$

Hence, (F,A) is an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft semiprime.

(iii) $\Rightarrow$ (ii) : Let R be a right ideal of an $\mathcal{AG}$-groupoid $\mathcal{S}$, then by Theorem 4, $\Sigma(R,E)$ is an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft right ideal over $\mathcal{S}$ by (iii) $\mathcal{X}_R^{(\Gamma,\Delta)}$ is cubic soft semiprime. Now by using assumption we have $\Sigma(R,E) \subset_{(\Gamma,\Delta)} \Sigma(R,E) \odot \Sigma(R,E)$. Now by using Lemma 3, we have

$$\mathcal{X}_R^{(\Gamma,\Delta)} = \mathcal{X}_{R\cap R}^{(\Gamma,\Delta)} =_{(\Gamma,\Delta)} \mathcal{X}_R^{(\Gamma,\Delta)} \cap \mathcal{X}_R^{(\Gamma,\Delta)} \subseteq \vee q_{(\Gamma,\Delta)} \mathcal{X}_R^{(\Gamma,\Delta)} \odot \mathcal{X}_R^{(\Gamma,\Delta)} =_{(\Gamma,\Delta)} \mathcal{X}_{R^2}^{(\Gamma,\Delta)}$$

Hence, by Lemma 3, $R \subseteq R^2$.

(ii) $\Rightarrow$ (i) : It is easy. ∎

Theorem 8 *Let $\mathcal{S}$ be an $\mathcal{AG}$-groupoid with left identity. Then, the following conditions are equivalent.*

(i) *$\mathcal{S}$ is intra-regular.*

(ii) *For any right ideal R and for any subset A of an $\mathcal{AG}$-groupoid $\mathcal{S}$, $R \cap A \subseteq RA$ and R is semiprime.*

(iii) *For any $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft right ideal (F,A) and for any $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft set (G,B) over $\mathcal{S}$, $(F,A)\widetilde{\cap}(G,B) \subseteq \vee q_{(\Gamma,\Delta)}(F,A) \odot (G,B)$ and (F,A) is an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft semiprime.*

Proof. (i) $\Rightarrow$ (iii) : Let $\mathcal{S}$ be intra-regular, (F,A) be an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft right ideal and (G,B) be an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft set over $\mathcal{S}$, respectively. Now let a be any element of $\mathcal{S}$, $\epsilon \in A \cup B$ and $(F,A)\widetilde{\cap}(G,B) = (H, A \cup B)$. We consider the following cases.

Case 1: $\epsilon \in A - B$. Then $H(\epsilon) = F(\epsilon) = (F \circ G)(\epsilon)$.

Case 2: $\epsilon \in B - A$. Then $H(\epsilon) = G(\epsilon) = (F \circ G)(\epsilon)$.

Case 3: $\epsilon \in A \cap B$. Then $H(\epsilon) = F(\epsilon) \cap G(\epsilon)$ and $(F \circ G)(\epsilon) = F(\epsilon) \circ G(\epsilon)$. Now we show that $F(\epsilon) \cap G(\epsilon) \subseteq \vee q_{((\Gamma,\Delta))} F(\epsilon) \circ G(\epsilon)$. Since $\mathcal{S}$ is intra-regular, then for any $r \in \mathcal{S}$ there exist $s, t \in \mathcal{S}$ such that $r = (sr^2)t$. Then,

$$r = (sr^2)t = [(s(rr))t] = [(r(sr))t] = [(t(sr))]r.$$

$$t(sr) = [t\{s((sr^2)t)\}] = [t\{(sr^2)(st)\}] = [(sr^2)(st^2)]$$
$$= [(t^2s)(r^2s)] = r^2(t^2s^2).$$

Thus, $r = [r^2(t^2s^2)]r$. If $r = pq$. Then

$$\begin{aligned} r \max &\left\{(\widetilde{\Psi}_{F(\epsilon)} \circ \widetilde{\Psi}_{G(\epsilon)})(r), \widetilde{\gamma}_1\right\} \\ &= r \max \left[r \sup_{r=pq} \left\{r \min \left\{\widetilde{\Psi}_{F(\epsilon)}(p), \widetilde{\Psi}_{G(\epsilon)}(q)\right\}\right\}, \widetilde{\gamma}_1\right] \\ &\succeq r \max \left[r \min \left\{\widetilde{\Psi}_{F(\epsilon)}(r^2(t^2s^2)), \widetilde{\Psi}_{G(\epsilon)}(r)\right\}, \widetilde{\gamma}_1\right] \\ &= r \min \left[r \max \left\{\widetilde{\Psi}_{F(\epsilon)}(r^2(t^2s^2)), \widetilde{\gamma}_1\right\}, r \max \left\{\widetilde{\Psi}_{G(\epsilon)}(r), \widetilde{\gamma}_1\right\}\right] \\ &\succeq r \min \left[r \min \left\{\widetilde{\Psi}_{F(\epsilon)}(r), \widetilde{\delta}_1\right\}, r \min \left\{\widetilde{\Psi}_{G(\epsilon)}(r), \widetilde{\delta}_1\right\}\right] \\ &= r \min \left\{(\widetilde{\Psi}_{F(\epsilon)} \cap \widetilde{\Psi}_{G(\epsilon)})(r), \widetilde{\delta}_1\right\} \end{aligned}$$

and

$$\begin{aligned} \min \left\{(\eta_{F(\epsilon)} \circ \eta_{G(\epsilon)})(r), \gamma_2\right\} &= \min \left[r \min_{r=pq}\{\max \left\{\eta_{F(\epsilon)}(p), \eta_{G(\epsilon)}(q)\right\}\}, \gamma_2\right] \\ &\leq \min \left[\max \left\{\eta_{F(\epsilon)}(r^2(t^2s^2)), \eta_{G(\epsilon)}(r)\right\}, \gamma_2\right] \\ &= \max \left[\min \left\{\eta_{F(\epsilon)}(r^2(t^2s^2)), \gamma_2\right\}, \min \left\{\eta_{G(\epsilon)}(r), \gamma_2\right\}\right] \\ &\leq \max \left[\max \left\{\eta_{F(\epsilon)}(r), \delta_2\right\}, \max \left\{\eta_{G(\epsilon)}(r), \delta_2\right\}\right] \\ &= \max \left\{(\eta_{F(\epsilon)} \cap \eta_{G(\epsilon)})(r), \delta_2\right\}. \end{aligned}$$

Thus, by Lemma 1, $(F, A)\widetilde{\cap}(G, B) \subseteq \vee q_{(\Gamma,\Delta)}(F, A) \odot (G, B)$. The rest of the proof is similar as in Theorem 7.

(iii) $\Rightarrow$ (ii) : Let R be a right ideal and A be any subset of an $\mathcal{AG}$-groupoid $\mathcal{S}$, then by Theorem 4, $\Sigma(R, E)$ and $\Sigma(A, E)$ is an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft right ideal and $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft set over $\mathcal{S}$, respectively. Now by using assumption we have $\Sigma(R, E) \cap \Sigma(A, E) \subset_{(\Gamma,\Delta)} \Sigma(R, E) \odot \Sigma(A, E)$. By using Lemma 3, we have

$$\mathcal{X}_{R\cap A}^{(\Gamma,\Delta)} = {}_{(\Gamma,\Delta)}\mathcal{X}_R^{(\Gamma,\Delta)} \cap \mathcal{X}_A^{(\Gamma,\Delta)} \subseteq \vee q_{(\Gamma,\Delta)} \mathcal{X}_R^{(\Gamma,\Delta)} \odot \mathcal{X}_A^{(\Gamma,\Delta)} = {}_{(\Gamma,\Delta)}\mathcal{X}_{RA}^{(\Gamma,\Delta)}.$$

By Lemma 3, $R \cap A \subseteq RA$.

(ii) $\Rightarrow$ (i) : It is easy. ∎

Theorem 9 *Let $\mathcal{S}$ be an $\mathcal{AG}$-groupoid with left identity. Then, the following conditions are equivalent.*

(i) *$\mathcal{S}$ is intra-regular.*

(ii) *For any right ideal R and for any subset A of an $\mathcal{AG}$-groupoid $\mathcal{S}$, $R \cap A \subseteq AR$ and R is semiprime.*

(iii) *For any $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft right ideal (F, A) and for any $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft set (G, B) over $\mathcal{S}$, $(F, A)\widetilde{\cap}(G, B) \subseteq \vee q_{(\Gamma,\Delta)}(G, B) \odot (F, A)$ and (F, A) is an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft semiprime.*

Proof. (i) $\Rightarrow$ (iii) : Let (F, A) be an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft right ideal and (G, B) be any $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft set over an intra-regular $\mathcal{AG}$-groupoid $\mathcal{S}$. Now, let a be any element of $\mathcal{S}$, $\epsilon \in A \cup B$ and $(F, A)\widetilde{\cap}(G, B) = (H, A \cup B)$. We consider the following cases.

Case 1: $\epsilon \in A - B$. Then $H(\epsilon) = F(\epsilon) = (G \circ F)(\epsilon)$.

Case 2: $\epsilon \in B - A$. Then $H(\epsilon) = G(\epsilon) = (G \circ F)(\epsilon)$.

Case 3: $\epsilon \in A \cap B$. Then $H(\epsilon) = G(\epsilon) \cap F(\epsilon)$ and $(G \circ F)(\epsilon) = G(\epsilon) \cap F(\epsilon)$.

Now we show that $F(\epsilon) \cap G(\epsilon) \subseteq \vee_{q_{(\Gamma,\Delta)}} G(\epsilon) \circ F(\epsilon)$. Since $\mathcal{S}$ is intra-regular, then for any $r \in \mathcal{S}$ there exist $s, t \in \mathcal{S}$ such that $r = (sr^2)t$. Now we obtain

$$\begin{aligned}
r &= (sr^2)t = (sr^2)(t_1t_2) = (t_2t_1)(r^2s) = r^2[(t_2t_1)s] \\
&= [s((t_2t_1)]r^2 = r[\{s(t_2t_1)\}r] = r[s\{(t_2t_1)\}\{(sr^2)t\}] \\
&= r[(sr^2)[\{s(t_2t_1)\}t]] = r[[t\{s(t_2t_1)\}](r^2s)] \\
&= r[r^2([t\{s(t_2t_1)\}]s] = r[(s[t\{s(t_2t_1)\}]r^2] \\
&= r[(s[t\{s(t_2t_1)\}])(rr)] \\
&= r[r((s[t\{s(t_2t_1)]\})r)] = r(ru), \text{ where } u = (s[t\{s(t_2t_1)\}])r.
\end{aligned}$$

If $r = pq$, then

$$\begin{aligned}
r \max &\left\{(\widetilde{\Psi}_{G(\epsilon)} \circ \widetilde{\Psi}_{F(\epsilon)})(r), \widetilde{\gamma}_1\right\} \\
&= r \max\left[r \sup_{r=pq}\left\{r \min\left\{\widetilde{\Psi}_{G(\epsilon)}(p), \widetilde{\Psi}_{F(\epsilon)}(q)\right\}\right\}, \widetilde{\gamma}_1\right] \\
&\succeq r \max\left[r \min\left\{\widetilde{\Psi}_{G(\epsilon)}(r), \widetilde{\Psi}_{F(\epsilon)}(ru)\right\}, \widetilde{\gamma}_1\right] \\
&= r \min\left[r \max\left\{\widetilde{\Psi}_{G(\epsilon)}(r), \widetilde{\gamma}_1\right\}, r \max\left\{\widetilde{\Psi}_{F(\epsilon)}(ru), \widetilde{\gamma}_1\right\}\right] \\
&\succeq r \min\left[r \min\left\{\widetilde{\Psi}_{G(\epsilon)}(r), \widetilde{\delta}_1\right\}, r \min\left\{\widetilde{\Psi}_{F(\epsilon)}(r), \widetilde{\delta}_1\right\}\right] \\
&= r \min\left\{(\widetilde{\Psi}_{F(\epsilon)} \cap \widetilde{\Psi}_{G(\epsilon)})(r), \widetilde{\delta}_1\right\},
\end{aligned}$$

and

$$\begin{aligned}
\min\left\{(\eta_{G(\epsilon)} \circ \eta_{F(\epsilon)})(r), \gamma_2\right\} &= \min\left[r \min_{r=pq} \max\left\{\eta_{G(\epsilon)}(p), \eta_{F(\epsilon)}(q)\right\}\}, \gamma_2\right] \\
&\leq \min\left[\max\left\{\eta_{G(\epsilon)}(r), \eta_{F(\epsilon)}(u)\right\}, \gamma_2\right] \\
&= \max\left[\min\left\{\eta_{G(\epsilon)}(r), \gamma_2\right\}, \min\left\{\eta_{F(\epsilon)}(ru), \gamma_2\right\}\right] \\
&\leq \max\left[\max\left\{\eta_{G(\epsilon)}(r), \delta_2\right\}, \max\left\{\eta_{F(\epsilon)}(r), \delta_2\right\}\right] \\
&= \max\left\{(\eta_{F(\epsilon)} \cap \eta_{G(\epsilon)})(r), \delta_2\right\}.
\end{aligned}$$

Hence, by Lemma 1, $(F, A)\widetilde{\cap}(G, B) \subseteq \vee q_{(\Gamma,\Delta)}(G, B) \odot (F, A)$. For the remaining proof see Theorem 7.

(iii) $\Rightarrow$ (ii) : Let R be a right ideal and A be any subset of an $\mathcal{AG}$-groupoid $\mathcal{S}$, then by Theorem 4, $\Sigma(R, E)$ and $\Sigma(A, E)$ is an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft right ideal and $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft set over $\mathcal{S}$, respectively. Now, by using assumption, we have $\Sigma(R, E) \cap \Sigma(A, E) \subset_{(\Gamma,\Delta)} \Sigma(R, E) \odot \Sigma(A, E)$. By using Lemma 3, we have

$$\begin{aligned}\mathcal{X}_{R\cap A}^{(\Gamma,\Delta)} &= \mathcal{X}_{A\cap R}^{(\Gamma,\Delta)} =_{(\Gamma,\Delta)} \mathcal{X}_{A}^{(\Gamma,\Delta)} \cap \mathcal{X}_{R}^{(\Gamma,\Delta)} \\ &\subseteq \vee q_{(\Gamma,\Delta)} \mathcal{X}_{A}^{(\Gamma,\Delta)} \odot \mathcal{X}_{R}^{(\Gamma,\Delta)} =_{(\Gamma,\Delta)} \mathcal{X}_{AR}^{(\Gamma,\Delta)}.\end{aligned}$$

By Lemma 3, $R \cap A \subseteq AR$. The rest of the proof is similar to Theorem 7.

(ii) $\Rightarrow$ (i) : Sine $\mathcal{S}p^2$ is a right ideal of $\mathcal{S}$ containing p^2. By (ii), it is semiprime. Therefore,

$$\begin{aligned}p \in \mathcal{S}p^2 \cap \mathcal{S}p \subseteq (\mathcal{S}p)(\mathcal{S}p^2) &= (p^2\mathcal{S})(p\mathcal{S}) = [(pp)(\mathcal{S}\mathcal{S})](p\mathcal{S}) \\ &= [(\mathcal{S}\mathcal{S})(pp)](p\mathcal{S}) \subseteq (\mathcal{S}p^2)\mathcal{S}.\end{aligned}$$

Hence, $\mathcal{S}$ is intra-regular. ∎

Theorem 10 *The following conditions are equivalent for an $\mathcal{AG}$-groupoid $\mathcal{S}$ with left identity:*

(i) *$\mathcal{S}$ is intra-regular.*

(ii) *For any subsets A, B and any right ideal R of $\mathcal{S}$, $A \cap B \cap R \subseteq (AB)R$ and R is semiprime ideal.*

(iii) *For any $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft sets (F, A), (G, B) and for any $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft right ideal (H, R) over $\mathcal{S}$, we have $(F, A)\widetilde{\cap}(G, B)\widetilde{\cap}(H, R) \subseteq \vee q_{(\Gamma,\Delta)}[(F, A) \odot (G, B)] \odot (H, R)$ and (H, R) is an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft semiprime ideal over $\mathcal{S}$.*

Proof. (i) $\Rightarrow$ (iii) Let a be any element of an intra-regular $\mathcal{AG}$-groupoid $\mathcal{S}$ with left identity, $[(F, A)\widetilde{\cap}(G, B)]\widetilde{\cap}(H, R) = (K, A \cup B \cup R)$. For any $\epsilon \in A \cup B \cup R$. We consider the following cases.

Case 1: $\epsilon \in A\backslash(B \cap R)$, then $F(\epsilon) = [(F \circ G) \circ H](\epsilon)$.
Case 2: $\epsilon \in B\backslash(A \cap R)$, then $G(\epsilon) = [(F \circ G) \circ H](\epsilon)$.
Case 3: $\epsilon \in R\backslash(A \cap B)$, then $H(\epsilon) = [(F \circ G) \circ H](\epsilon)$
Case 4: $\epsilon \in (A \cap B) \cap R$, then $[(F \circ G) \circ H](\epsilon) = [F(\epsilon) \circ G(\epsilon)] \circ H(\epsilon)$.

Since $\mathcal{S}$ is intra-regular, then for any $r \in \mathcal{S}$ there exist $s, t \in \mathcal{S}$ such that $r = (sr^2)t$. Then,

$$\begin{aligned}r &= (sr^2)t = (t_2t_1)(r^2s) = r^2[(t_2t_1)s] = [s(t_2t_1)]r^2 \\ &= r[\{s(t_2t_1)\}r] = r(pr), \text{ where } s(t_2t_1) = p, \text{ and} \\ pr &= p[(sr^2)t] = (sr^2)(pt) = [(tp)(r^2s)] \\ &= r^2[(tp)s] = [s(tp)](rr) = r[\{(tp)\}r] \\ &= [r(qr)], \text{ where } s(tp) = q, \text{ and} \\ qr &= q[(sr^2)t] = (sr^2)(qt) = (tq)(r^2s) \\ &= r^2[(tq)s].\end{aligned}$$

Thus,

$r = r[r(r^2c)] = r[r^2(rc)] = r^2[r(rc)]$, where $(tq)s = c$ and $s(tp) = q$ and $(t_2t_1) = p$

For any $r \in \mathcal{S}$ there exist $x, y \in \mathcal{S}$ such that $r = xy$, then

$$
\begin{aligned}
& r \max\left\{(\widetilde{\Psi}_{F(\epsilon)} \circ \widetilde{\Psi}_{G(\epsilon)}) \circ \widetilde{\Psi}_{H(\epsilon)})(r), \widetilde{r}_1\right\} \\
&\quad = r \max\left[r \sup_{r=xy}\left\{r \min\left\{\left(\widetilde{\Psi}_{F(\epsilon)} \circ \widetilde{\Psi}_{G(\epsilon)}\right)(x), \widetilde{\Psi}_{H(\epsilon)}(y)\right\}\right\}, \widetilde{\gamma}_1\right] \\
&\quad \succeq r \max\left[r \min\left\{\left(\widetilde{\Psi}_{F(\epsilon)} \circ \widetilde{\Psi}_{G(\epsilon)}\right)(rr), \widetilde{\Psi}_{H(\epsilon)}(r(rc))\right\}, \widetilde{\gamma}_1\right] \\
&\quad = r \max\left[r \sup_{rr=pq}\left\{r \min\left\{\begin{array}{c}\left\{\widetilde{\Psi}_{F(\epsilon)}(p), \widetilde{\Psi}_{G(\epsilon)}(q)\right\}, \\ \widetilde{\Psi}_{H(\epsilon)}(r(rc))\end{array}\right\}\right\}, \widetilde{\gamma}_1\right] \\
&\quad \succeq r \max\left[r \min\left\{\widetilde{\Psi}_{F(\epsilon)}(r), \widetilde{\Psi}_{G(\epsilon)}(r), \widetilde{\Psi}_{H(\epsilon)}(r(rc))\right\}, \widetilde{\gamma}_1\right] \\
&\quad = r \min\left[\begin{array}{c} r \max\left\{\widetilde{\Psi}_{F(\epsilon)}(r), \widetilde{\gamma}_1\right\}, r \max\left\{\widetilde{\Psi}_{G(\epsilon)}(r), \widetilde{\gamma}_1\right\}, \\ r \max\left\{\widetilde{\Psi}_{R(\epsilon)}(r(rc)), \widetilde{\gamma}_1\right\}\end{array}\right] \\
&\quad \succeq r \min\left[\begin{array}{c} r \min\left\{\widetilde{\Psi}_{F(\epsilon)}(r), \widetilde{\delta}_1\right\}, r \min\left\{\widetilde{\Psi}_{G(\epsilon)}(r), \widetilde{\delta}_1\right\}, \\ r \min\left\{\widetilde{\Psi}_{H(\epsilon)}(r), \widetilde{\delta}_1\right\}\end{array}\right] \\
&\quad = r \min\left[r \min\left\{\widetilde{\Psi}_{F(\epsilon)}(r), \widetilde{\Psi}_{G(\epsilon)}(r), \widetilde{\Psi}_{H(\epsilon)}(r)\right\}, \widetilde{\delta}_1\right] \\
&\quad = r \min\left\{\left(\widetilde{\Psi}_{F(\epsilon)} \cap \widetilde{\Psi}_{G(\epsilon)}(r) \cap \widetilde{\Psi}_{H(\epsilon)}\right)(r), \widetilde{\delta}_1\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
& \min\left\{(\eta_{F(\epsilon)} \circ \eta_{G(\epsilon)}) \circ \eta_{H(\epsilon)})(r), r_2\right\} \\
&\quad = \min\left[r \min_{r=xy} \max\left\{\left(\eta_{F(\epsilon)} \circ \eta_{G(\epsilon)}\right)(x), \eta_{H(\epsilon)}(y)\right\}\}, \gamma_2\right] \\
&\quad \leq \min\left[\max\left\{\left(\eta_{F(\epsilon)} \circ \eta_{G(\epsilon)}\right)(rr), \eta_{H(\epsilon)}(r(rc))\right\}, \gamma_2\right] \\
&\quad = \min\left[r \min_{rr=pq}\left\{\max\left\{\left\{\eta_{F(\epsilon)}(p), \eta_{G(\epsilon)}(q)\right\}, \eta_{H(\epsilon)}(r(rc))\right\}\right\}, \gamma_2\right] \\
&\quad \leq \min\left[\max\left\{\eta_{F(\epsilon)}(r), \eta_{G(\epsilon)}(r), \eta_{H(\epsilon)}(r(rc))\right\}, \gamma_2\right] \\
&\quad = \max\left[\begin{array}{c}\min\left\{\eta_{F(\epsilon)}(r), \gamma_2\right\}, \min\left\{\eta_{G(\epsilon)}(r), \gamma_2\right\}, \\ \min\left\{\eta_{H(\epsilon)}(r(rc)), \gamma_2\right\}\end{array}\right] \\
&\quad \leq \max\left[\begin{array}{c}\max\left\{\eta_{F(\epsilon)}(r), \delta_2\right\}, \max\left\{\eta_{G(\epsilon)}(r), \delta_2\right\}, \\ \max\left\{\eta_{H(\epsilon)}(r), \delta_2\right\}\end{array}\right] \\
&\quad = \max\left[\max\left\{\eta_{F(\epsilon)}(r), \eta_{G(\epsilon)}(r), \eta_{H(\epsilon)}(r)\right\}, \delta_2\right] \\
&\quad = \max\left\{\left(\eta_{F(\epsilon)} \cap \eta_{G(\epsilon)}(r) \cap \eta_{H(\epsilon)}\right)(r), \delta_2\right\}
\end{aligned}
$$

Thus, by Lemma 1,

$$(F, A)\widetilde{\cap}(G, B)\widetilde{\cap}(H, R) \subseteq \vee q_{(\Gamma,\Delta)}((F, A) \odot (G, B)) \odot (H, R).$$

The rest of the proof is similar as in Theorem 7.

(iii) $\Rightarrow$ (ii) : Let R be a right ideal and A, B are any subset of an $\mathcal{AG}$-groupoid $\mathcal{S}$, then by Theorem 4, $\Sigma(R,E)$, $\Sigma(A,E), \Sigma(B,E)$ are an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft right ideal and $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft set over $\mathcal{S}$, respectively. Now by using assumption we have $[\Sigma(A,E)\widetilde{\cap}\Sigma(B,E)]\widetilde{\cap}\Sigma(R,E) \subset_{(\Gamma,\Delta)} [\Sigma(A,E) \odot \Sigma(B,E)] \odot \Sigma(R,E)$. By using Lemma 3, we have

$$\begin{aligned} \mathcal{X}^{(\Gamma,\Delta)}_{(A\cap B)\cap R} &= {}_{(\Gamma,\Delta)}[\mathcal{X}^{(\Gamma,\Delta)}_A \cap \mathcal{X}^{(\Gamma,\Delta)}_B] \cap \mathcal{X}^{(\Gamma,\Delta)}_R \\ &\subseteq \vee q_{(\Gamma,\Delta)} \left(\mathcal{X}^{(\Gamma,\Delta)}_A \odot \mathcal{X}^{(\Gamma,\Delta)}_B\right) \odot \mathcal{X}^{(\Gamma,\Delta)}_R \\ &= {}_{(\Gamma,\Delta)}\mathcal{X}^{(\Gamma,\Delta)}_{(AB)R}. \end{aligned}$$

By Lemma 3, $(A \cap B) \cap R \subseteq (AB)R$. The rest of the proof is similar as in Theorem 7.

(ii) $\Rightarrow$ (i) : Since $\mathcal{S}p^2$ is a right ideal of an $\mathcal{AG}$-groupoid $\mathcal{S}$ containing p^2, so by (ii) it is semiprime. Therefore, by (ii) we have

$$\mathcal{S}p \cap \mathcal{S}p \cap \mathcal{S}p \subseteq [(\mathcal{S}p)(\mathcal{S}p)](\mathcal{S}p^2) = [(\mathcal{S}\mathcal{S})(pp)](\mathcal{S}p^2) \subseteq (\mathcal{S}p^2)\mathcal{S}.$$

Hence, $\mathcal{S}$ is intra-regular. ∎

Conclusion

In this paper, we introduce a new concept of generalized cubic soft sets and then apply it to the ideals theory of $\mathcal{AG}$-groupoids. We characterized intra-regular $\mathcal{AG}$-groupoids via cubic soft sets. In our future work we will be focusing on other generalized cubic soft ideals for characterizations of regular $\mathcal{AG}$-groupoids.

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CHARACTERIZATIONS OF ORDERED $\mathcal{AG}$-GROUPOIDS IN TERMS OF SOFT SETS

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Abstract. We introduced the concepts of soft intersection left (right, two-sided) ideals, (generalized) bi-ideals, interior ideals, quasi-ideals in ordered $\mathcal{AG}$-groupoids and obtained significant characterizations of intra-regular ordered $\mathcal{AG}$-groupoid via soft intersection left (right, two-sided) ideals, (generalized) bi-ideals, interior ideals, quasi-ideals of ordered $\mathcal{AG}$-groupoids.

Keywords: soft set, soft intersection left (right, two-sided, interior, quasi-) ideal, soft intersection (generalized) bi-ideal, interior ideal, intra-regular ordered $\mathcal{AG}$-groupoid.

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Introduction

Molodtsov in 1999 [1], introduced the fundamental concept of soft set which provides a natural frame work for generalization of several basic notions of algebra such as groups [2], [3], semirings [4], rings [5], BCK/BCI-algebras [6], [7], [8], BL-algebras [9], near-rings [10], [28], and soft substructures and union soft substructures [12], [13], and some other structures such as [14], [15], [16], [17].

Many related concepts with soft sets, especially soft sets operations, have also undergone tremendous studies. Maji et al. [18], presented some definitions on soft sets and based on the analysis of several operations on soft sets Ali et al. [19], introduced several operations on soft sets and Sezgin and Atagun [20], and Ali et al. [21], studied on soft sets as well.

Nowadays the theory of soft sets and its applications has been rapidly growing. These applications are used in various fields such as computer science and in soft decision making as in the following studies [24], [25], [26], [27], [28].

The concept of Abel-Grassmann's Groupoids ($\mathcal{AG}$-groupoids) [32], was first given by M.A. Kazim and M. Naseeruddin in 1972.

An $\mathcal{AG}$-groupoid is a groupoid having the left invertive law

$$(ab)c = (cb)a, \text{ for all } a,\, b,\, c, \in S. \tag{1}$$

An $\mathcal{AG}$-groupoid is a non-associative and non-commutative algebraic structure mid way between a groupoid and a commutative semigroup, nevertheless, it posses many interesting properties which we usually find in associative and commutative algebraic structures. The left identity in an $\mathcal{AG}$-groupoid if exists is unique [33].

In an $\mathcal{AG}$-groupoid, the medial law [32], holds

$$(ab)(cd) = (ac)(bd), \text{ for all } a,\, b,\, c,\, d \in S. \tag{2}$$

In an $\mathcal{AG}$-groupoid S with left identity, the paramedial law [33] holds

$$(ab)(cd) = (dc)(ba), \text{ for all } a, b, c, d \in S. \tag{3}$$

If an $\mathcal{AG}$-groupoid contains a left identity, then by using medial law, the following law [33] holds

$$a(bc) = b(ac), \text{ for all } a,\, b,\, c \in S. \tag{4}$$

The purpose of this paper, we have a new approach to the ordered $\mathcal{AG}$-groupoid theory via soft set theory with the concept of soft intersection $\mathcal{AG}$-groupoids and soft intersection $\mathcal{AG}$-ideals. The paper is organized as follows: First we remind some basic definitions about ordered $\mathcal{AG}$-groupoids, soft sets, soft intersection product and soft characteristic function, soft intersection ordered $\mathcal{AG}$-groupoid left (right, two-sided) ideal, (generalized) bi-ideals, interior ideal, quasi-ideal and soft semiprime ideals in ordered $\mathcal{AG}$-groupoids. Also we have shown that all these ideals coincide in intra-regular ordered $\mathcal{AG}$-groupoids with left identity. In the following sections intra-regular ordered $\mathcal{AG}$-groupoids are characterized by the properties of these soft intersection ideals.

Preliminary notes

In this section, we recall some basic notions and results on ordered $\mathcal{AG}$-groupoids.

Definition 1 [35] An ordered $\mathcal{AG}$-groupoid S (po-$\mathcal{AG}$-groupoid) is a structure $(S, \cdot, \leq)$ in which the following conditions hold:

(i) $(S, \cdot)$ is an $\mathcal{AG}$-groupoid.

(ii) $(S, \leq)$ is a poset. (Reflexive, anti symmetric and transitive)

(iii) $a \leq b \Longrightarrow ax \leq bx$ and $xa \leq xb$, $\forall\ a, b, x \in S$.

Example 1 Let $S = \{a, b, c\}$ be an $\mathcal{AG}$-groupoid with the following multiplication table.

$\cdot$	a	b	c
a	a	a	a
b	a	a	c
c	a	a	a

Then, S is an ordered $\mathcal{AG}$-groupoid with the following two orders:

$$\leq := \{(a,a),(b,b),(c,c),(c,a),(c,b)\}$$

$$\leq := \{(a,a),(b,b),(c,c),(a,c),(a,b)\}$$

Let $(S,\cdot,\leq)$ be an ordered $\mathcal{AG}$-groupoid. A fuzzy subset or a fuzzy set f of a non-empty set S is an arbitrary mapping $f : S \to [0,1]$, where $[0,1]$ is the unit segment of real line. A fuzzy subset f is a class of objects with a grades of membership having the form $f = \{(s,\ f(s))\ /s \in S\}$.

Let f and g be any two fuzzy subsets of $(S,\cdot,\leq)$. Then, the product of f and g is defined as:

$$(f \circ g)(x) = \begin{cases} \bigvee\limits_{x \leq yz} \{f(y) \wedge g(z)\} & \text{if } \exists\ y,z \in S \text{ such that } x \leq yz, \\ 0 & \text{otherwise.} \end{cases}$$

The order relation $\subseteq$ between any two fuzzy subsets f and g of $(S,\cdot,\leq)$ is defined as:

$$f \subseteq g \text{ if and only if } f(a) \leq g(a)\ \forall\ a \in S.$$

We define the symbols $f \wedge g$ and $f \vee g$, which will means the following fuzzy subsets of an ordered $\mathcal{AG}$-groupoid $(S,\cdot,\leq)$

$$(f \wedge g)(a) = \min\{f(a), g(a)\} \text{ for all } a \in S.$$
$$(f \vee g)(a) = \max\{f(a), g(a)\} \text{ for all } a \in S.$$

Let A be a non-empty subset of an ordered $\mathcal{AG}$-groupoid $(S,\cdot,\leq)$, then

$$(A] = \{t \in S \mid t \leq a,\ \text{for some } a \in A\}.$$

For $A = \{a\}$, we usually written as $(a]$.

For further information about $\mathcal{AG}$-groupoids ordered $\mathcal{AG}$-groupoids and fuzzy $\mathcal{AG}$-groupoids, we invite the reader to [35], [36], [37], [38]. A semilattice is a structure where "$\cdot$" is an infix binary operation, called semilattice operation, such that "$\cdot$" is commutative, associative and idempotent. From now on, U refers to an initial universe, E is a set of parameters, $P(U)$ is the power set of U and A, B are the subsets of E.

Definition 2 ([23, 1)] A soft set f_A over U is defined by

$$f_A : E \longrightarrow P(U) \text{ such that } f_A(x) = \phi \text{ if } x \notin A.$$

Here, f_A is also called approximate function. A soft set over U can be represented by the set of ordered pair

$$f_A = \{(x, f_A) : x \in E,\ f_A(x) \in P(U)\}.$$

Clearly, we see that a soft set is a parametrized family of subsets of set U. Note that throughout this paper $S(U)$ will denote the set of all soft sets over U.

Definition 3 [23] Let f_A and $f_B \in S(U)$. Then, f_A is called a soft subset of f_B and denoted by $f_A \subseteq f_B$ if and only if $f_A \leq f_B\ \forall\ x \in E$.

Definition 4 [23] Let f_A and $f_B \in S(U)$. Then, the union of f_A and f_B, denoted by $f_A \cup f_B$, is defined as $f_A \cup f_B = f_{A\cup B}$, where $f_{A\cup B}(x) = f_A(x) \vee f_B(x)\ \forall\ x \in E$.

Definition 5 [23] Let f_A and $f_B \in S(U)$. Then, the intersection of f_A and f_B, denoted by $f_A \cap f_B$, is defined as $f_A \cap f_B = f_{A\cap B}$, where $f_{A\cap B}(x) = f_A(x) \wedge f_B(x)\ \forall\ x \in E$.

Definition 6 Let f_A and $f_B \in S(U)$. Then, the product of f_A and f_B, denoted by $f_A \Theta f_B$, is defined as $f_A \Theta f_B = f_{A\Theta B}$, where $f_{A\Theta B}(x, y) = f_A(x) \wedge f_B(y)\ \forall\ (x, y) \in E \times E$.

Definition 7 Let S be an ordered $\mathcal{AG}$-groupoid and f_A and f_B be soft sets over the common universe set U. Then, the soft intersection product $f_A \circ f_B$ is defined by

$$(f_A \circ f_B)(x) = \begin{cases} \bigvee\limits_{x \le yz} \{f_A(y) \wedge f_B(z)\} & \text{if } \exists\ y, z \in S \text{ such that } x \le yz, \\ \phi & \text{otherwise,} \end{cases}$$

for all $x \in S$.

Theorem 1 [39] *Let f_A, f_B, $f_C \in S(U)$. Then,*

(i) $(f_A \circ f_B) \circ f_C = f_A \circ (f_B \circ f_C)$.

(ii) $f_A \circ (f_B \cup f_C) = (f_A \circ f_B) \cup (f_A \circ f_C)$ *and* $(f_A \cup f_B) \circ f_C = (f_A \circ f_C) \cup (f_B \circ f_C)$.

(iii) $f_A \circ (f_B \cap f_C) = (f_A \circ f_B) \cap (f_A \circ f_C)$ *and* $(f_A \cap f_B) \circ f_C = (f_A \circ f_C) \cap (f_B \circ f_C)$.

(iv) *if* $f_A \subseteq f_B$, *then* $f_A \circ f_C \subseteq f_B \circ f_C$ *and* $f_C \circ f_A \subseteq f_C \circ f_B$.

(v) *If* $t_A, t_B \in S(U)$ *such that* $t_A \subseteq f_A$ *and* $t_B \subseteq f_B$, *then* $t_A \circ t_B \subseteq f_A \circ f_B$.

Proposition 1 [39] *Let S be an ordered $\mathcal{AG}$-groupoid. Then, the medial law holds in $S(S)$.*

Theorem 2 [39] *Let S be an ordered $\mathcal{AG}$-groupoid with left identity and $f_A, f_B, f_C \in S(S)$. Then, the following properties hold in $S(S)$:*

(i) $f_A \circ (f_B \circ f_C) = f_B \circ (f_A \circ f_C)$.

(ii) $(f_A \circ f_B) \circ (f_C \circ f_D) = (f_D \circ f_C) \circ (f_B \circ f_A)$.

Definition 8 [39] Let X be a subst of S. We denote by $\mathcal{S}_X$, the soft characteristic function of X and define as

$$\mathcal{S}_X(x) = \begin{cases} U, & \text{if } x \in X, \\ \phi, & \text{if } x \notin X. \end{cases}$$

Lemma 1 [35] *In an ordered $\mathcal{AG}$-groupoid S, the following are true.*

(i) $A \subseteq (A]\,, \forall\ A \subseteq S$.

(ii) $A \subseteq B \subseteq G \Longrightarrow (A] \subseteq (B]\,,\ \forall\ A, B \subseteq S$.

(iii) $(A]\,(B] \subseteq (AB]\,,\ \forall\ A, B \subseteq S$.

(iv) $(A] = ((A]]\,,\ \forall\ A \subseteq S$.

(vi) $((A]\,(B]] = (AB]\,,\ \forall\ A, B \subseteq S$.

Theorem 3 *Let S be an ordered $\mathcal{AG}$-groupoid and X, Y be non-empty subsets of X. Then, the following properties hold:*

(i) *If $X \subseteq Y$, then $\mathcal{S}_X \subseteq \mathcal{S}_Y$.*

(ii) $\mathcal{S}_X \cap \mathcal{S}_Y = \mathcal{S}_{X\cap Y}$, $\mathcal{S}_X \cup \mathcal{S}_Y = \mathcal{S}_{XUY}$.

(iii) $\mathcal{S}_X \circ \mathcal{S}_Y = \mathcal{S}_{(XY]}$.

Proof. (i) It is obvious.

(ii) Let $x \in \mathcal{S}$. If $x \in X \cap Y$, then $x \in X$ and $x \in Y$. Thus, we have

$$(\mathcal{S}_X \cap \mathcal{S}_Y)(x) = \mathcal{S}_X(x) \cap \mathcal{S}_Y(x) = U = \mathcal{S}_{X\cap Y}(x).$$

If $x \notin X \cap Y$, then $x \notin X$ or $x \notin Y$. Hence, we have

$$(\mathcal{S}_X \cap \mathcal{S}_Y)(x) = \mathcal{S}_X(x) \cap \mathcal{S}_Y(x) = \phi = \mathcal{S}_{X\cap Y}(x).$$

Therefore, $\mathcal{S}_X \cap \mathcal{S}_Y = \mathcal{S}_{X\cap Y}$.

Similarly, we can show that $\mathcal{S}_X \cup \mathcal{S}_Y = \mathcal{S}_{XUY}$.

(iii) For any $x \in \mathcal{S}$. Let $x \in XY$. Then, there exist $y \in X$ and $z \in Y$ such that $x \leq yz$. Thus, we have

$$(\mathcal{S}_X \circ \mathcal{S}_Y)(x) = \bigvee_{x\leq yz} \{\mathcal{S}_X(y) \cap \mathcal{S}_Y(z)\} \supseteq \mathcal{S}_X(y) \cap \mathcal{S}_Y(z) = U,$$

and so $(\mathcal{S}_X \circ \mathcal{S}_Y)(x) = U$. Since $x \in XY$, we get $\mathcal{S}_{(XY]}(x) = U$. Suppose $x \notin XY$. Then, $x \nleqslant yz$ for any $y \in X$ and $z \in Y$. If $x \leq yz$ for some $y, z \in \mathcal{S}$, then $y \notin X$ or $z \notin Y$. Hence

$$(\mathcal{S}_X \circ \mathcal{S}_Y)(x) = \bigvee_{x\leq yz} \{\mathcal{S}_X(y) \cap \mathcal{S}_Y(z)\} = \phi = \mathcal{S}_{(XY]}(x).$$

If $x \nleqslant yz$ for all $x, y \in \mathcal{S}$, then

$$(\mathcal{S}_X \circ \mathcal{S}_Y)(x) = \phi = \mathcal{S}_{(XY]}(x).$$

In any case, we have $\mathcal{S}_X \circ \mathcal{S}_Y = \mathcal{S}_{(XY]}$. ∎

Definition 9 Let S be an ordered $\mathcal{AG}$-groupoid and f_A be a soft set over U. Then, f_A is called soft intersection ordered $\mathcal{AG}$-groupoid of S, if

$$f_A(xy) \geq f_A(x) \wedge f_A(y) \text{ for all } x, y \in S.$$

Definition 10 Let S be an ordered $\mathcal{AG}$-groupoid and f_A be a soft set over U. Then, f_A is called soft intersection left (*right*) ideal of S,if

(i) $x \leq y \Rightarrow f_A(x) \geq f_A(y)\ \forall\ x, y \in S$.

(ii) $f_A(xy) \geq f_A(y)$ $(f_A(xy) \geq f_A(x))\ \forall\ x, y \in S$.

A soft set over U is called a soft intersection two-sided ideal (soft intersection ideal) of S, if it is both soft intersection left and soft intersection right ideal of S over U.

Definition 11 A soft intersection ordered $\mathcal{AG}$-groupoid f_A over U is called a soft intersection bi-ideal of S over U if

(i) $x \leq y \Rightarrow f_A(x) \geq f_A(y) \ \forall \ x, y \in S$.

(ii) $f_A((xy)z) \geq f_A(x) \wedge f_A(z) \ \forall \ x, y, z \in S$.

Definition 12 A soft set f_A over U is called a soft intersection interior ideal of S over U, if

(i) $x \leq y \Rightarrow f_A(x) \geq f_A(y) \ \forall \ x, y \in S$.

(ii) $f_A((xy)z) \geq f_A(y) \ \forall \ x, y, z \in S$.

Definition 13 A soft set f_A over U is called a soft intersection generalized bi-ideal of S over U, if

(i) $x \leq y \Rightarrow f_A(x) \geq f_A(y) \ \forall \ x, y \in S$.

(ii) $f_A((xy)z) \geq f_A(x) \wedge f_A(z) \ \forall \ x, y, z \in S$.

Now, throughout this paper, soft intersection ordered $\mathcal{AG}$-groupoid soft intersection left (right, two-sided, interior, quasi-, generalized bi-) ideal are abbreviated by SI-ordered $\mathcal{AG}$-groupoid, SI-right (left, two-sided, interior, quasi-, generalized bi-) ideal respectively.

It is easy to show that $f_A(x) = U \ \forall \ x \in S$, then f_A is an SI-ordered $\mathcal{AG}$-groupoid (right ideal, left ideal, interior ideal, quasi-ideal, generalized bi-ideal) of S over U. We denote such type of SI-ordered $\mathcal{AG}$-groupoid (right ideal, left ideal, interior ideal, quasi-ideal, generalized bi-ideal) by $\widetilde{\mathcal{S}}$ [39].

Definition 14 A soft set f_A over U is called soft intersection quasi-ideal of S over U if

$$f_A(f_A \circ \widetilde{\mathcal{S}}) \wedge (\widetilde{\mathcal{S}} \circ f_A) \leq f_A$$

Lemma 2 [39] *Let f_A be any SI-ordered $\mathcal{AG}$-groupoid over U. Then, we have the following:*

(i) $\widetilde{\mathcal{S}} \circ \widetilde{\mathcal{S}} \subseteq \widetilde{\mathcal{S}}$.

(ii) $f_A \circ \widetilde{\mathcal{S}} \subseteq \widetilde{\mathcal{S}}$ *and* $\widetilde{\mathcal{S}} \circ f_A \subseteq \widetilde{\mathcal{S}}$.

(iii) $f_A \cup \widetilde{\mathcal{S}} = \widetilde{\mathcal{S}}$ *and* $f_A \cap \widetilde{\mathcal{S}} = \widetilde{\mathcal{S}}$.

Theorem 4 [39] *Let X be a nonempty subset of an ordered $\mathcal{AG}$-groupoid S. Then, X is left (right, two-sided, bi-, interior, quasi-, generalized bi-) ideal of S if and only if S_X is an SI-left (left, right, two-sided, bi-, interior, quasi-, generalized bi-) ideal of S.*

Proposition 2 [39] *Let f_A be a soft set over U. Then,*

(i) *f_A is an SI-ordered $\mathcal{AG}$-groupoid over U if $f_A \circ f_A \subseteq f_A$*

(ii) *f_A is an SI-left (right) ideal of S over U if and only if $\widetilde{\mathcal{S}} \circ f_A \subseteq f_A (f_A \circ \widetilde{\mathcal{S}}) \subseteq f_A$.*

(iii) *f_A is an SI-bi-ideal of S over U if and only if $f_A \circ f_A \subseteq f_A$ and $f_A \circ \widetilde{\mathcal{S}} \circ f_A \subseteq f_A$.*

(iv) *f_A is an SI-interior ideal of S over U if and only if $\widetilde{\mathcal{S}} \circ f_A \circ \widetilde{\mathcal{S}} \subseteq f_A$.*

(v) *f_A is an SI-generalized bi-ideal of S over U if and only if $f_A \circ \widetilde{\mathcal{S}} \circ f_A \subseteq f_A$.*

Theorem 5 [39] *Every SI-left (right, two-sided) ideal of an ordered $\mathcal{AG}$-groupoid S over U is an SI-bi-ideal of S.*

Proposition 3 [39] *For an ordered $\mathcal{AG}$-groupoid S, the following conditions are equivalent:*

(i) *Every SI-ideal of an ordered $\mathcal{AG}$-groupoid S over U is an SI-interior ideal of S over U.*

(ii) *Every SI-quasi-ideal of an ordered $\mathcal{AG}$-groupoid S is an SI-ordered $\mathcal{AG}$-groupoid of S.*

(iii) *Every one sided SI-ideal of S is an SI-quasi-ideal of S.*

(iv) *Every SI-quasi-ideal of S is an SI-bi-ideal of S.*

Characterizations of intra-regular ordered AG-groupoids

In this section, we characterize an intra-regular ordered $\mathcal{AG}$-groupoids in terms of SI-ideals. An ordered $\mathcal{AG}$-groupoid S is called intra-regular if for every $a \in S$ there exist elements $x, y \in S$ such that

$$a \leq (xa^2)y$$

Example 2 Let us consider an $\mathcal{AG}$-groupoid S with left identity e defined in the following multiplication table.

$\cdot$	a	b	c	d	e
a	e	a	b	c	d
b	d	e	a	b	c
c	c	d	e	a	b
d	b	c	e	d	a
e	a	b	c	d	e

Then, S is an ordered $\mathcal{AG}$-groupoid with the following order:

$$\leq := \{(a,a),(a,b),(c,c),(a,c),(d,d),(a,e),(e,e),(b,b)\}.$$

Note that S is an intra-regular ordered $\mathcal{AG}$-groupoid. Indeed for every $a \in S$ there exist elements $x, y \in S$ such that $a \leq (xa^2)y$.

Proposition 4 *Let S be an intra-regular ordered $\mathcal{AG}$-groupoid and f_A be a soft set of S. Then, f_A is an SI-right ideal if and only if it is an SI-left ideal.*

Proof. Suppose that f_A is an SI-right ideal of S. Since S is intra-regular, so for each $a \in S$ there exist elements $x, y \in S$ such that $a \leq (xa^2)y$. So, by using (1),

$$f_A(ab) \geq f_A((xa^2)y)b) = f_A((by)(xa^2)) \geq f_A(by) \geq f_A(b)$$

Thus, f_A is an SI-left ideal of S.

Conversely, suppose that f_A is an SI-left ideal of S. Then, by using (1), we have

$$f_A(ab) \geq f_A((xa^2)y)b) = f_A((by)(xa^2)) \geq f_A(xa^2) \geq f_A(a^2) \geq f_A(a).$$

Thus, f_A is an SI-right ideal of S. ∎

Proposition 5 *Every SI-two-sided ideal of an intra-regular ordered $\mathcal{AG}$-groupoid with left identity is idempotent.*

Proof. Let f_A be an SI-two sided ideal of S. Then,

$$f_A \circ f_A \subseteq f_A \circ \widetilde{\mathcal{S}} \subseteq f_A.$$

Since S is intra-regular, so for each $a \in S$ there exist elements $x, y \in S$ such that $a \leq (xa^2)y$. So,

$$a \leq (ax^2)y = (x(aa))y = (y(xa))a.$$

Thus, we have

$$\begin{aligned}(f_A \circ f_A)(a) &= \bigvee_{a \leq (y(xa))a} \{f_A(y(xa)) \wedge f_A(a)\} \\ &\geq f_A(y(xa)) \wedge f_A(a) \\ &\geq f_A(a) \wedge f_A(a) \\ &= f_A(a).\end{aligned}$$

Hence, $f_A \circ f_A = f_A$. ∎

Corollary 1 *Every SI-left ideal of an ordered $\mathcal{AG}$-groupoid S with left identity is idempotent.*

Proposition 6 *Let S be an intra-regular ordered $\mathcal{AG}$-groupoid with left identity. Then,*

$$f_A = (\widetilde{\mathcal{S}} \circ f_A)^2 \text{ for all } SI\text{-left ideals } f_A \text{ of } S.$$

Proof. Let S be an intra-regular ordered $\mathcal{AG}$-groupoid with left identity and f_A be any SI-left ideal of S. Then, $\widetilde{\mathcal{S}} \circ f_A \subseteq f_A$ and since $\widetilde{\mathcal{S}} \circ f_A$ is an SI-left ideal of S, it is idempotent. Thus,

$$(\widetilde{\mathcal{S}} \circ f_A)^2 = \widetilde{\mathcal{S}} \circ f_A \subseteq f_A$$

Moreover,

$$f_A = f_A \circ f_A \subseteq \widetilde{\mathcal{S}} \circ f_A = (\widetilde{\mathcal{S}} \circ f_A)^2.$$

This implies that, $f_A = (\widetilde{\mathcal{S}} \circ f_A)^2$. ∎

Theorem 6 *The following conditions are equivalent for an ordered $\mathcal{AG}$-groupoid S with left identity:*

(i) *f_A is an SI-ideal of S.*

(ii) *f_A is an SI-bi-ideal of S.*

Proof. (i) $\Rightarrow$ (ii) : Follows from Theorem 5.

(ii) $\Rightarrow$ (i) : Let f_A be an SI-bi-ideal of S. Since S is intra-regular, so for each $a, b \in S$ there exist $u, v \in S$ such that $a \leq (xa^2)y$ and $b \leq (ub^2)v$. So,

$$\begin{aligned} f_A(ab) &\geq f_A(((xa^2)y)b) = f_A((by)(xa^2)) = f_A((a^2x)(yb)) \\ &= f_A(((yb)x)a^2) = f_A(((yb)x)(aa)) \\ &= f_A((aa)(x(yb))) = f_A(((x(yb))a)a) \\ &= f_A(((x(yb)((xa^2)y))a) = f_A(((xa^2)((x(yb))y))a) \\ &= f_A(((y(x(yb)))(a^2x))a) = f_A((a^2((y(x(yb)))x))a) \\ &= f_A(((aa)((y(x(yb)))x))a) = f_A(((x(y(x(yb))))(aa))a) \\ &= f_A((a((x(y(x(yb))))a))a) \geq f_A(a) \wedge f_A(a) = f_A(a) \end{aligned}$$

and

$$\begin{aligned} f_A(ab) &\geq f_A(a((ub^2)v)) = f_A((ub^2)(av)) = f_A((va)(b^2u)) \\ &= f_A(b^2((va)u)) = f_A((bb)((va)u)) \\ &= f_A((((va)u)b)b) = f_A([((va)u)((ub^2)v)]b) \\ &= f_A([(ub^2)(((va)u)v)]b) = f_A([(v(va)u))(b^2u)]b) \\ &= f_A([b^2((v((va)u))u)]b) = f_A(((bb)((v((va)u))u))b) \\ &= f_A(((u(v((va)u)))(bb))b) = f_A((b((u(v((va)u)))b))b) \\ &\geq f_A(b) \wedge f_A(b) = f_A(b). \end{aligned}$$

Hence, f_A is an SI-ideal of S. ∎

Corollary 2 *Let S be an ordered $\mathcal{AG}$-groupoid with left identity. Then, an SI-right ideal of S is an SI-bi-ideal of S.*

Proposition 7 *For a soft set f_A of an ordered $\mathcal{AG}$-groupoid with left identity, the following conditions are equivalent:*

(i) *f_A is an SI-ideal of S.*

(ii) *f_A is an SI-interior ideal of S.*

Proof. (i) $\Rightarrow$ (ii) : It is easy.

(ii) $\Rightarrow$ (i) : Suppose that (ii) holds. Let S be an intra-regular ordered $\mathcal{AG}$-groupoid with left identity. Then, for any $a, b \in S$ there exist $x, y, u, v \in S$ such that $a \leq (xa^2)y$ and $b \leq (ub^2)v$. Since f_A is an SI-interior ideal of S, we have

$$\begin{aligned} f_A(ab) &\geq f_A(((xa^2)y)b) = f_A(((by)(xa^2)) = f_A((by)(x(aa))) \\ &= f_A((by)(a(xa))) = f_A((ba)(y(xa))) \geq f_A(a) \end{aligned}$$

and

$$\begin{aligned} f_A(ab) &\geq (a((ub^2)v)) = f_A((ub^2)(av)) = f_A((b(ub))(av)) \\ &= f_A((va)((ub)b)) = f_A((ub)((va)b)) \geq f_A(b). \end{aligned}$$

Hence, f_A is an SI-ideal of S. ∎

Proposition 8 *Let f_A be a soft set of an intra-regular ordered $\mathcal{AG}$-groupoid S with left identity. Then, the following conditions are equivalent:*

(i) *f_A is an SI-bi-ideal of S.*

(ii) *f_A is an SI-generalized bi-ideal of S.*

Proof. (i) $\Rightarrow$ (ii) : It is easy.

(ii) $\Rightarrow$ (i) : Suppose that (ii) holds. Let a be any element of an intra-regular ordered $\mathcal{AG}$-groupoid S. Then, there exist $x, y \in S$ such that $a \leq (xa^2)y$. Thus, we have

$$\begin{aligned} f_A(ab) &\geq f_A(((xa^2)y)b) = f_A(((xa^2)(ey)b) = f_A(((ye)(a^2x))b) \\ &= f_A((a^2((ye)x))b) = f_A(((aa)((ye)x))b) = f_A(((x(ye)(aa))b) \\ &= f_A((a((ye))a))b) \geq f_A(a) \wedge f_A(b). \end{aligned}$$

Hence, f_A is an SI-bi-ideal of S. ∎

Proposition 9 *Let S be an intra-regular ordered $\mathcal{AG}$-groupoid S with left identity and f_A be a soft set of S. Then, the following conditions are equivalent:*

(i) *f_A is an SI-ideal of S.*

(ii) *f_A is an SI-quasi-ideal of S.*

Proof. (i) $\Rightarrow$ (ii) : It is easy.

(ii) $\Rightarrow$ (i) : Suppose that (ii) holds. Since S is intra-regular so for any element $a \in S$ there exist $x, y \in S$ such that $a \leq (xa^2)y$. Thus, we have

$$\begin{aligned} a &\leq (xa^2)y = (xa^2)(ey) = (xe)(a^2y) = a^2((xe)y) \\ &= ((aa)((xe)y) = (ya)((xe)a) = (y(xe))(aa) = a((y((xe))a). \end{aligned}$$

Also,

$$\widetilde{\mathcal{S}} \circ f_A = (\widetilde{\mathcal{S}} \circ \widetilde{\mathcal{S}}) \circ f_A = (f_A \circ \widetilde{\mathcal{S}}) \circ \widetilde{\mathcal{S}}.$$

Therefore,

$$\begin{aligned} (\widetilde{\mathcal{S}} \circ f_A)(a) = ((f_A \circ \widetilde{\mathcal{S}}) \circ \widetilde{\mathcal{S}})(a) &= \bigvee_{a \leq a((y((xe))a)} \{(f_A \circ \widetilde{\mathcal{S}})(a) \wedge \widetilde{\mathcal{S}}((y((xe))a)\} \\ &\geq (f_A \circ \widetilde{\mathcal{S}})(a). \end{aligned}$$

Therefore,

$$f_A \circ \widetilde{\mathcal{S}} \subseteq (f_A \circ \widetilde{\mathcal{S}}) \cap (\widetilde{\mathcal{S}} \circ f_A) \subseteq f_A$$

Thus, f_A is an SI-right ideal of S. Now by proposition 4, f_A is an SI-left ideal of S. Hence f_A is an SI-ideal of S. ∎

Theorem 7 *For a soft set f_A of an intra-regular ordered $\mathcal{AG}$-groupoid S with left identity, the following conditions are equivalent:*

(i) *f_A is an SI-right ideal of S.*

(ii) *f_A is an SI-left ideal of S.*

(iii) f_A *is an SI-ideal of S.*

(iv) f_A *is an SI-bi-ideal of S.*

(v) f_A *is an SI-generalized bi-ideal of S.*

(vi) f_A *is an SI-interior ideal of S.*

(vii) f_A *is an SI-quasi-ideal of S.*

Definition 15 A soft set f_A over U is called soft semiprime* if

$$f_A(a) \geq f_A(a^2) \text{ for all } a \in S.$$

Theorem 8 *For a non-empty set A of an intra-regular ordered $\mathcal{AG}$-groupoid S, the following conditions are equivalent:*

(i) *A is semiprime.*

(ii) *The soft characteristic function $\mathcal{S}_A$ of A is soft semiprime*.*

Proof. Suppose that (i) holds. Let a be any arbitrary element of S. We need to show that $\mathcal{S}_A(a) \geq \mathcal{S}_A(a^2)$ for all $a \in S$. Now if $a^2 \in A$, then since A is semiprime, so $a \in A$. Thus,

$$\mathcal{S}_A(a) = U = \mathcal{S}_A(a^2).$$

If $a^2 \notin A$, then

$$\mathcal{S}_A(a) \geq \varphi = \mathcal{S}_A(a^2).$$

In both cases we see that, $\mathcal{S}_A(a) \geq \mathcal{S}_A(a^2)$ for all $a \in S$. Thus $\mathcal{S}_A$ is soft semiprime*. Hence, $(i) \Rightarrow (ii)$.

Conversely, suppose that (ii) holds. Let $a^2 \in A$. Since $\mathcal{S}_A$ is soft semiprime*, we have

$$\mathcal{S}_A(a) \geq \mathcal{S}_A(a^2) = U.$$

This implies that, $\mathcal{S}_A(a) = U$ and $a \in A$. Thus, A is semiprime. Hence, $(ii) \Rightarrow (i)$. ∎

Theorem 9 *For any SI-ordered $\mathcal{AG}$-groupoid f_A, the following conditions are equivalent:*

(i) *f_A is soft semiprime*.*

(ii) *$f_A(a) = f_A(a^2)$ for all $a \in S$.*

Proof. (ii) $\Rightarrow$ (i) : It is easy.

(i) $\Rightarrow$ (ii) : Let $a \in S$ be any element of S. Since f_A is an SI-ordered $\mathcal{AG}$-groupoid, we have

$$f_A(a) \geq f_A(a^2) = f_A(aa) \geq f_A(a) \wedge f_A(a) = f_A(a)$$

So, $f_A(a^2) = f_A(a)$. Hence,$(i) \Rightarrow (ii)$. ∎

Proposition 10 *In an intra-regular ordered $\mathcal{AG}$-groupoid S, every SI-interior ideal of S is a soft semiprime*.*

Proof. Let f_A be any SI-interior ideal of S and $a \in S$. Since S is intra-regular, there exist elements $x, y \in S$ such that $a \leq (xa^2)y$. Thus, we have

$$f_A(ab) \geq f_A((xa^2)y) \geq f_A(a^2).$$

Hence, f_A is soft semiprime*. ∎

Theorem 10 *Let S be an ordered $\mathcal{AG}$-groupoid with left identity. Then, the following conditions are equivalent:*

(i) *S is intra-regular.*

(ii) *$L \cap R \subseteq (LR]$ for every left ideal L and for every right ideal R of S and R is semiprime.*

(iii) *$f_A \cap f_B \subseteq f_A \circ f_B$ for every SI-left ideal f_A and for every SI-right ideal f_B of S and SI-right ideal f_B is soft semiprime*.*

Proof. (i) ⇒ (iii) : Let f_A and f_B be any SI-left ideal and SI-right ideal of S, respectively and a be any element of S. Since S is intra-regular, there exist $x, y \in S$ such that $a \leq (xa^2)y$. Thus,

$$\begin{aligned} a \leq (xa^2)y &= (a(xa))y = (y(xa))a = (y(xa))(ea) \\ &= (ye)((xa)a) = (xa)((ye)a) = (xa)((ye)a) = (xa)((ae)y). \end{aligned}$$

Thus, we have

$$\begin{aligned} (f_A \circ f_B)(a) &= \bigvee_{a \leq (xa)((ae)y)} \{f_A(xa) \wedge f_B((ae)y)\} \\ &\geq f_A(a) \wedge f_B \\ &= (f_A \cap f_B)(a). \end{aligned}$$

Thus, $f_A \circ f_B \supseteq f_A \cap f_B$, and

$$f_B(a) \geq f_B((xa^2)y) = f_B((xa^2)(ey)) = f_B((ye)(a^2x)) = f_B(a^2((ye)x))) \geq f_B(a^2).$$

Hence, f_B is soft semiprime*.

(iii) ⇒ (ii) : Let L and R be any left and right ideal of S. Then $\mathcal{S}_L$ and $\mathcal{S}_R$ are SI-left and SI-right ideal of S, respectively. Let $a \in L \cap R$. Then, $a \in L$ and $a \in R$, so by assumption, we have

$$U = \mathcal{S}_{L \cap R}(a) = (\mathcal{S}_L \cap \mathcal{S}_R)(a) \supseteq (\mathcal{S}_L \circ \mathcal{S}_R)(a) = \mathcal{S}_{(LR]}(a).$$

Thus, $L \cap R \subseteq (LR]$. By our assumption, the soft characteristic function $\mathcal{S}_R$ is soft semiprime*, and so, by Theorem 8, R is semiprime.

(ii) ⇒ (i) : Clearly, $(Sa]$ and $(a^2S]$ are left and right ideals of S such that $a \in (Sa]$ and $a^2 \in (a^2S]$. Since by assumption, $(a^2S]$ is semiprime, therefore, $a \in (a^2S]$. Now by using Lemma 1 and (1), (2), (3) and (4), we have

$$\begin{aligned} a \in (a^2S] \cap (Sa] &= ((a^2S](Sa]] \subseteq ((a^2S)(Sa)] = ((aS)(Sa^2)] \\ &= (((Sa^2)S)a] = (((Sa^2)(eS))a] \subseteq (((Sa^2)(SS))a] \\ &= (((SS)(a^2S))a] = ((a^2((SS)S))a] \subseteq ((a^2S)S] \\ &= ((SS)(aa)] = ((aa)(SS)] \subseteq ((aa)S] = ((Sa)a] \\ &\subseteq ((Sa)(a^2S)] = (((a^2S)a)S] = (((aS)a^2)S] \subseteq ((Sa^2)S]. \end{aligned}$$

This shows that there exist $x, y \in S$, such that $a \leq (xa^2)y$. Hence, S is intra-regular. ∎

Theorem 11 *Let S be an ordered $\mathcal{AG}$-groupoid with left identity. Then, the following conditions are equivalent:*

(i) *S is intra-regular.*

(ii) *$R \cap L = (RL]$ for every left ideal L and for every right ideal R of S and R is semiprime.*

Theorem 12 *The following conditions are equivalent for an ordered $\mathcal{AG}$-groupoid S with left identity.*

(i) *S is intra-regular.*

(ii) *$f_A \cap f_B \subseteq f_A \circ f_B$, for every SI-right ideal f_A and for every SI-left ideal f_B of S and f_A is soft semiprime*.*

Proof. (i) $\Longrightarrow$ (ii) : Let f_A be any SI-right ideal and f_A be any SI-left ideal of S with left identity. Then, we have

$$f_A \circ f_B \subseteq f_A \cap f_B.$$

Since S is right-regular, for each $a \in S$ there exist $x, y \in G$ such that $a \leq (xa^2)y$. Thus,

$$a \leq (xa^2)y = (x(aa))y = (a(xa))y = (y(xa))a = ((ey)(xa))a = ((ax)(ye))a.$$

Hence,

$$\begin{aligned}(f_A \circ f_B)(a) &= \bigvee_{a \leq ((ax)(ye))a} \{f_A((ax)(ye)) \wedge f_B(a)\} \\ &\geq \{f_A((ax)(ye)) \wedge f_B(a)\} \\ &\geq f_A(a) \wedge f_B(a) \\ &= (f_A \cap f_B)(a).\end{aligned}$$

Thus, $f_A \cap f_B \subseteq f_A \circ f_B$.

Moreover,

$$f_A(a) \geq f_A((xa^2)y) = f_A((xa^2)(ey)) = f_A((ye)(a^2x)) = f_A(a^2((ye)x)) \geq f_A(a^2).$$

Hence, f_A is a soft semiprime.

(ii) $\Longrightarrow$ (i) : Let R be any right ideal and L be any left ideal of S. Then, $\mathcal{S}_L$ and $\mathcal{S}_R$ are SI-left ideal and SI-right ideal of S.Let $a \in R \cap L$. Then, $a \in R$ and $a \in L$. So, by assumption, we have

$$U = \mathcal{S}_{R \cap L}(a) = (\mathcal{S}_R \cap \mathcal{S}_L)(a) = (\mathcal{S}_R \circ \mathcal{S}_L)(a) = (\mathcal{S}_{(Rl]}(a).$$

Thus, $R \cap L \subseteq (RL]$ and clearly $(RL] \subseteq R \cap L$. Thus $(RL] = R \cap L$. Hence, S is intra-regular by Theorem 11. Moreover, by assumption, the soft characteristic function $\mathcal{S}_R$ is soft semiprime*, and so R is semiprime by Theorem 8. This complete the proof. ∎

Corollary 3 *For an ordered $\mathcal{AG}$-groupoid S with left identity, $f_A \cap f_B = f_A \circ f_B$ for every SI-Ideals f_A and f_B of S.*

Proposition 11 *Let f_A and f_B be SI-left (right, two-sided) ideal of an ordered $\mathcal{AG}$-groupoid S with left identity. Then, $f_A \circ f_B$ is an SI-left (right, two-sided) ideal of S over U.*

Theorem 13 *Let S be an intra–regular ordered $\mathcal{AG}$-groupoid with left identity. Then, the set of all SI-ideals of S forms a samilattice structure with identity $\widetilde{\mathcal{S}}$.*

Proof. Let I_S be the set of all SI-ideals of an ordered $\mathcal{AG}$-groupoid S and f_A, f_B and $f_C \in I_S$. Clearly I_S is closed by Proposition 11. Moreover, by Proposition 5, we have $f_A = (f_A)^2$ and by Corollary 3, $f_A \circ f_B = f_A \cap f_B$, where f_A and f_B are $SI-$ideals. Obviously, $f_A \circ f_B = f_B \circ f_A$. Moreover by using (1),

$$(f_A \circ f_B) \circ f_C = f_C \circ (f_B \circ f_A).$$

Also by using (1) and Proposition 11,

$$f_A \circ \widetilde{\mathcal{S}} = (f_A \circ f_A) \circ \widetilde{\mathcal{S}} = (\widetilde{\mathcal{S}} \circ f_A) \circ f_A = f_A \circ f_A = f_A.$$

This complete the proof. ∎

Theorem 14 *Let S be an ordered $\mathcal{AG}$-groupoid with left identity. Then, the following conditions are equivalent:*

(i) *S is intra-regular.*

(ii) *$f_A \cap f_B \subseteq f_A \circ f_B$ for every SI-right ideal f_A and every SI-left ideal f_B of S and SI-right ideal f_A is soft semiprime*.*

(iii) *$f_A \cap f_B \subseteq f_A \circ f_B$ for every SI-right ideal f_A and every SI-bi- ideal (SI-quasi-ideal) f_B of S and SI-right ideal f_A is soft semiprime*.*

Proof. It is obvious that (i) ⇒ (iii) and (iii) ⇒ (ii).
(ii) ⇒ (i) follows from Theorem 12. ∎

Theorem 15 *Let S be an ordered $\mathcal{AG}$-groupoid with left identity. Then, the following conditions are equivalent:*

(i) *S is intra-regular.*

(ii) *$f_A \cap f_B \subseteq (f_A \circ f_B) \circ f_A$ for every SI-right ideal f_A and every SI-left ideal f_B of S and SI-right ideal f_A is soft semiprime*.*

(iii) *$f_A \cap f_B \subseteq (f_A \circ f_B) \circ f_A$ for every SI-right ideal f_A and every SI-bi- ideal f_B of S and SI-right ideal f_A is soft semiprime*.*

Proof. (i) ⇒ (iii) : Suppose S is intra-regular. Let f_A and f_B be any SI-right and SI-bi-ideal of S, respectively. Since S is intra-regular, for each $a \in S$, there exist $x, y \in S$ such that $a \leq (xa^2)y$. Thus,

$$\begin{aligned} a &\leq (xa^2)y = (a(xa))y = (y(xa))a = (y(x(xa^2)y))a \\ &= (y((xa^2)(xy)))a = ((xa^2)(y(xy)))a \\ &= ((x(aa))(y(xy)))a = ((a(xa))(y(xy)))a \\ &= (((y(xy)(xa))a)a = (((ax((xy)y))a)a. \end{aligned}$$

Thus, we have

$$\begin{aligned}((f_A \circ f_B) \circ f_A(a)) &= \bigvee_{a \leq (((ax((xy)y))a)a} \{f_A((ax((xy)y)) \wedge f_B(a) \wedge f_A(a)\} \\ &\geq f_A((ax((xy)y)) \wedge f_B(a) \wedge f_A(a) \\ &\geq f_A(a) \wedge f_B(a) \wedge f_A(a) \\ &= (f_A \cap f_B)(a).\end{aligned}$$

Thus, $f_A \cap f_B \subseteq (f_A \circ f_B) \circ f_A$.

Moreover,

$$f_A(a) \geq f_A((xa^2)y) = f_A((xa^2)(ey)) = f_A((ye)(a^2x)) = f_A(a^2((ye)x)) \geq f_A(a^2).$$

Hence, f_A is a soft semiprime*.

(iii) $\Rightarrow$ (ii) : It is obvious.

(ii) $\Rightarrow$ (i) : Suppose f_A and f_B be any SI-right and SI-left ideal of S, respectively. Then, by our assumption, we have

$$f_B \cap f_A = f_A \cap f_B \subseteq (f_A \circ f_B) \circ f_A \subseteq (\widetilde{\mathcal{S}} \circ f_B) \circ f_A \subseteq f_B \circ f_A.$$

Hence, by Theorem 10, S is intra-regular. ∎

Conclusions

Order theory is a branch of Mathematics which investigates our intuitive notion of order using binary relations. It provides a formal framework for describing statements such as "this is less than that" or "this precedes that". The study of an algebraic structure using the order theory plays a prominent role in Mathematics with wide ranging applications in many disciplines such as control engineering, computer arithmetics, coding theory, sequential machines and formal languages.

In this paper, we first remind the concept of soft intersection left (right, two-sided) ideals, generalized bi-ideals, interior ideals quasi-ideals of an ordered $\mathcal{AG}$-groupoid. We then show that all these ideals coincide in intra-regular ordered $\mathcal{AG}$-groupoid with left identity. Moreover, we characterize intra-regular ordered $\mathcal{AG}$-groupoid by the properties of soft intersection ideals. We are of the opinion that some results in this paper have already constituted a foundation for further investigation related with further development of ordered $\mathcal{AG}$-groupoids. In the following study of soft intersection ordered $\mathcal{AG}$-groupoids, applying soft intersection ordered $\mathcal{AG}$-groupoid to some applied fields, such as decision making, data analysis and forecasting and so on are worth to be considered.

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SOME GENERAL NUMERICAL RADIUS INEQUALITIES FOR THE OFF-DIAGONAL PARTS OF 2×2 OPERATOR MATRICES

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Abstract. We give some sharp inequalities involving powers of the numerical radii for the off-diagonal parts of 2×2 operator matrices. These inequalities, which are based on some classical convexity inequalities for the nonnegative real numbers, generalize earlier numerical radius inequalities.

Keywords: numerical radius, operator matrix, off-diagonal part, Cartesian decomposition, Jensen's inequality, mixed Schwarz inequality.

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1. Introduction

Let $B(H)$ denote the C^*- algebra of all bounded linear operators on a complex Hilbert space H with inner product $\langle \cdot, \cdot \rangle$. For $A \in B(H)$, let $\omega(A)$ and $\|A\|$ denote the numerical radius and the usual operator norm of A, respectively. It is well known that $\omega(\cdot)$ defines a norm on $B(H)$, which is equivalent to the usual operator norm $\| \cdot \|$. In fact, for every $A \in B(H)$,

$$\frac{1}{2}\|A\| \leq \omega(A) \leq \|A\|. \tag{1.1}$$

The inequalities in (1.1) are sharp. The first inequality becomes an equality if $A^2 = 0$. The second inequality becomes an equality if A is normal. For basic properties of the numerical radius, we refer to [4] and [6]. The inequalities in (1.1) have been improved considerably by Kittaneh. It has been shown in [10] and [11], respectively, that if $A \in B(H)$, then

$$\omega(A) \leq \frac{1}{2}\||A| + |A^*|\| \leq \frac{1}{2}\left(\|A\| + \|A^2\|^{\frac{1}{2}}\right), \tag{1.2}$$

where $|A| = (A^*A)^{\frac{1}{2}}$ is the absolute value of A, and

$$\frac{1}{4}\|A^*A + AA^*\| \leq \omega^2(A) \leq \frac{1}{2}\|A^*A + AA^*\|. \tag{1.3}$$

The inequalities in (1.2), which refine the second inequality in (1.1), have been utilized in [10] to derive an estimate for the numerical radius of the Frobenius companion matrix. Such an estimate can be employed to give new bounds for the zeros of polynomials (see, e.g., [9],[10], and references therein).

If $A = B + iC$ is the Cartesian decomposition of A, then B and C are self-adjoint, and $A^*A + AA^* = 2(B^2 + C^2)$. Thus, the inequalities in (1.3) can be written as

$$\frac{1}{2} \parallel B^2 + C^2 \parallel \leq \omega^2(A) \leq \parallel B^2 + C^2 \parallel . \tag{1.4}$$

The purpose of this paper is to establish a general inequalities involving powers of the numerical radii for the off-diagonal parts of 2×2 operator matrices that are based on the classical convexity inequalities for nonnegative real numbers and some operator inequalities.

Other recent numerical radius inequalities have been obtained by Dragomir [3], El-Haddad [5], and Yamazaki [12]. The inequalities in [3] are related to the Euclidean radius of two Hilbert space operators, the inequalities in [5] involving powers of the numerical radii for Hilbert space operators, and those in [12] involve the Aluthge transform.

2. Main results

To prove our generalized numerical radius inequalities for the off-diagonal parts of 2×2 operator matrices, we need several well known lemmas. The first lemma is a simple consequence of the classical Jensen's inequality concerning the convexity or the concavity of certain power functions. It is a special case of Schlömilch's inequality for weighted means of nonnegative real numbers (see, e.g., [7, p. 26]).

Lemma 2.1 *For $a, b \geq 0$, $0<\alpha<1$, and $r \neq 0$, let $M_r(a,b,\alpha)=(\alpha a^r+(1-\alpha)b^r)^{\frac{1}{r}}$ and let $M_0(a,b,\alpha) = a^\alpha b^{1-\alpha}$. Then*

$$M_r(a,b,\alpha) \leq M_s(a,b,\alpha) \quad for \quad r \leq s.$$

The second lemma is another application of Jensen's inequality (see. e.g., [7, p. 28]).

Lemma 2.2 *For $a, b \geq 0$, and $r > 0$, let $N_r(a,b) = (a^r + b^r)^{\frac{1}{r}}$. Then*

$$N_s(a,b) \leq N_r(a,b) \quad for \quad s \geq r > 0.$$

The third lemma follows from the spectral theorem for positive operators and Jensen's inequality (see, e.g., [8]).

Lemma 2.3 *Let $A \in B(H)$ be positive, and let $x \in H$ be any unit vector. Then*

(a) $\langle Ax, x\rangle^r \leq \langle A^r x, x\rangle \qquad for \quad r \geq 1.$

(b) $\langle A^r x, x\rangle \leq \langle Ax, x\rangle^r \qquad for \quad 0 < r \leq 1.$

The fourth lemma is an immediate consequence of the spectral theorem for self-adjoint operators. For generalizations of this lemma, we refer to [8].

Lemma 2.4 *Let $A \in B(H)$ be self-adjoint, and let $x \in H$ be any vector. Then*

$$| \langle Ax, x\rangle | \leq \langle | A | x, x\rangle.$$

The fifth lemma is a generalized for the mixed Schwarz inequality which has been proved by Kittaneh [8].

Lemma 2.5 *Let T be an operator in $B(H)$ and let f and g be nonnegative functions on $[0, \infty)$ which are continuous and satisfying the relation $f(t)g(t) = t$ for all $t \in [0, \infty)$. Then*

$$| \langle Tx, y\rangle | \leq \| f(|T|)x \| \| g(|T^*|)x \| \qquad for\, all \quad x, y \in H.$$

The sixth lemma contains two parts. Part (a) is well known and can be found in [2, p. 10]. Part (b) is also known and can be found in [1].

Lemma 2.6 *Let $X, Y \in B(H)$. Then*

(a) $\omega\left(\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}\right) = \max\{\omega(X), \omega(Y)\}.$

(b) $\omega\left(\begin{bmatrix} X & Y \\ Y & X \end{bmatrix}\right) = \max\{\omega(X+Y), \omega(X-Y)\}.$

In particular,

$$\omega\left(\begin{bmatrix} 0 & Y \\ Y & 0 \end{bmatrix}\right) = \omega(Y).$$

Our first result is a generalization of the first inequality in (1.2).

Theorem 2.7 *Let $S = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$ be a 2×2 operator matrix in $B(H_1 \oplus H_2)$, and let f and g be nonnegative functions on $[0, \infty)$ which are continuous and satisfying the relation $f(t)g(t) = t$ for all $t \in [0, \infty)$, and $r \geq 1$. Then*

$$\omega^r(S) \leq \frac{1}{2}\max\{\|f^{2r}(|C|) + g^{2r}(|B^*|)\|, \|f^{2r}(|B|) + g^{2r}(|C^*|)\|\}. \tag{2.1}$$

Proof. For every unit vector $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in (H_1 \bigoplus H_2)$, by using Lemma 2.5, Lemma 2.1, and Lemma 2.3(a) we have

$$
\begin{aligned}
&| \langle SX, X\rangle | \leq \| f(|S|)X \| \| g(|S^*|)X \| \\
&= \langle f^2(|S|)X, X\rangle^{\frac{1}{2}} \langle g^2(|S^*|)X, X\rangle^{\frac{1}{2}} \\
&= \left\langle f^2\left(\begin{bmatrix} |C| & 0 \\ 0 & |B| \end{bmatrix}\right)X, X\right\rangle^{\frac{1}{2}} \left\langle g^2\left(\begin{bmatrix} |B^*| & 0 \\ 0 & |C^*| \end{bmatrix}\right)X, X\right\rangle^{\frac{1}{2}} \\
&= \left\langle \begin{bmatrix} f^2(|C|) & 0 \\ 0 & f^2(|B|) \end{bmatrix} X, X\right\rangle^{\frac{1}{2}} \left\langle \begin{bmatrix} g^2(|B^*|) & 0 \\ 0 & g^2(|C^*|) \end{bmatrix} X, X\right\rangle^{\frac{1}{2}} \\
&\leq \frac{1}{2}\left(\left\langle \begin{bmatrix} f^2(|C|) & 0 \\ 0 & f^2(|B|) \end{bmatrix} X, X\right\rangle + \left\langle \begin{bmatrix} g^2(|B^*|) & 0 \\ 0 & g^2(|C^*|) \end{bmatrix} X, X\right\rangle\right) \\
&\leq \left(\frac{1}{2}\left(\left\langle \begin{bmatrix} f^2(|C|) & 0 \\ 0 & f^2(|B|) \end{bmatrix} X, X\right\rangle^r + \left\langle \begin{bmatrix} g^2(|B^*|) & 0 \\ 0 & g^2(|C^*|) \end{bmatrix} X, X\right\rangle^r\right)\right)^{\frac{1}{r}} \\
&\leq \left(\frac{1}{2}\left(\left\langle \begin{bmatrix} f^{2r}(|C|) & 0 \\ 0 & f^{2r}(|B|) \end{bmatrix} X, X\right\rangle + \left\langle \begin{bmatrix} g^{2r}(|B^*|) & 0 \\ 0 & g^{2r}(|C^*|) \end{bmatrix} X, X\right\rangle\right)\right)^{\frac{1}{r}} \\
&\leq \left(\frac{1}{2}\left\langle \begin{bmatrix} f^{2r}(|C|) + g^{2r}(|B^*|) & 0 \\ 0 & f^{2r}(|B|) + g^{2r}(|C^*|) \end{bmatrix} X, X\right\rangle\right)^{\frac{1}{r}}.
\end{aligned}
$$

Thus,

$$
| \langle SX, X\rangle |^r \leq \frac{1}{2}\left\langle \begin{bmatrix} f^{2r}(|C|) + g^{2r}(|B^*|) & 0 \\ 0 & f^{2r}(|B|) + g^{2r}(|C^*|) \end{bmatrix} X, X\right\rangle,
$$

and so

$$
\begin{aligned}
\omega^r(S) &= \sup\{| \langle SX, X\rangle |^r : X \in (H_1 \oplus H_2), \| X \| = 1\} \\
&\leq \frac{1}{2}\sup\left\{\left\langle \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix} X, X\right\rangle : X \in (H_1 \oplus H_2), \| X \| = 1\right\} \\
&= \frac{1}{2}\max\{\|\lambda\|, \|\mu\|\},
\end{aligned}
$$

where

$$
\lambda = f^{2r}(|C|) + g^{2r}(|B^*|)
$$

and

$$
\mu = f^{2r}(|B|) + g^{2r}(|C^*|),
$$

as required. ∎

Inequality (2.1) includes several numerical radius inequalities for operator matrices. Samples of inequalities are demonstrated in the following remarks.

Remark 2.8 For $f(t) = t^{\alpha}$ and $g(t) = t^{1-\alpha}$, $\alpha \in (0,1)$, in inequality (2.1), we get the following inequality

$$\omega^r(S) \leq \frac{1}{2}\max\{\| \mid C \mid^{2r\alpha} + \mid B^* \mid^{2r(1-\alpha)} \|, \| \mid B \mid^{2r\alpha} + \mid C^* \mid^{2r(1-\alpha)} \|\}.$$

Remark 2.9 If $B = C$ in the above Remark, and by using Lemma 2.6(b), then

$$\omega^r(B) = \omega^r(S) \leq \frac{1}{2}\| \mid B \mid^{2r\alpha} + \mid B^* \mid^{2r(1-\alpha)} \|,$$

and this inequality is given in Theorem 1 in [5].

Now, the second result is a generalization of the second inequality in (1.3).

Theorem 2.10 *Let* $S = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$ *be a* 2×2 *operator matrix in* $B(H_1 \bigoplus H_2)$, *and let* f *and* g *be nonnegative functions on* $[0,\infty)$ *which are continuous and satisfying the relation* $f(t)g(t) = t$ *for all* $t \in [0,\infty)$, *and* $r \geq 1$ *and* $0 < k < 1$. *Then*

$$\omega^{2r}(S) \leq \max\{\|kf^{\frac{2r}{k}}(|C|)+(1-k)g^{\frac{2r}{1-k}}(|B^*|)\|, \|kf^{\frac{2r}{k}}(|B|)+(1-k)g^{\frac{2r}{1-k}}(|C^*|)\|\}.$$

Proof. For every unit vector $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in (H_1 \bigoplus H_2)$, by using Lemma 2.5, Lemma 2.3(b), Lemma 2.1, and Lemma 2.3(a), we have

$$\begin{aligned}
&\mid \langle SX, X\rangle \mid^2 \leq \langle f^2(\mid S \mid)X, X\rangle\langle g^2(\mid S^* \mid)X, X\rangle \\
&= \left\langle \begin{bmatrix} f^2(\mid C \mid) & 0 \\ 0 & f^2(\mid B \mid) \end{bmatrix} X, X \right\rangle \left\langle \begin{bmatrix} g^2(\mid B^* \mid) & 0 \\ 0 & g^2(\mid C^* \mid) \end{bmatrix} X, X \right\rangle \\
&\leq \left\langle \begin{bmatrix} f^{\frac{2}{k}}(\mid C \mid) & 0 \\ 0 & f^{\frac{2}{k}}(\mid B \mid) \end{bmatrix} X, X \right\rangle^k \left\langle \begin{bmatrix} g^{\frac{2}{1-k}}(\mid B^* \mid) & 0 \\ 0 & g^{\frac{2}{1-k}}(\mid C^* \mid) \end{bmatrix} X, X \right\rangle^{1-k} \\
&\leq \left(k \left\langle \begin{bmatrix} f^{\frac{2}{k}}(\mid C \mid) & 0 \\ 0 & f^{\frac{2}{k}}(\mid B \mid) \end{bmatrix} X, X \right\rangle^r \right. \\
&\qquad \left. +(1-k)\left\langle \begin{bmatrix} g^{\frac{2}{1-k}}(\mid B^* \mid) & 0 \\ 0 & g^{\frac{2}{1-k}}(\mid C^* \mid) \end{bmatrix} X, X \right\rangle^r \right)^{\frac{1}{r}} \\
&\leq \left(k \left\langle \begin{bmatrix} f^{\frac{2r}{k}}(\mid C \mid) & 0 \\ 0 & f^{\frac{2r}{k}}(\mid B \mid) \end{bmatrix} X, X \right\rangle \right. \\
&\qquad \left. +(1-k)\left\langle \begin{bmatrix} g^{\frac{2r}{1-k}}(\mid B^* \mid) & 0 \\ 0 & g^{\frac{2r}{1-k}}(\mid C^* \mid) \end{bmatrix} X, X \right\rangle \right)^{\frac{1}{r}}.
\end{aligned}$$

Thus,

$$
\begin{aligned}
&| \langle SX, X\rangle |^{2r} \\
&\leq \left\langle \left(k \begin{bmatrix} f^{\frac{2r}{k}}(| C |) & 0 \\ 0 & f^{\frac{2r}{k}}(| B |) \end{bmatrix} + (1-k) \begin{bmatrix} g^{\frac{2r}{1-k}}(| B^* |) & 0 \\ 0 & g^{\frac{2r}{1-k}}(| C^* |) \end{bmatrix} \right) X, X \right\rangle \\
&= \left\langle \begin{bmatrix} kf^{\frac{2r}{k}}(| C |) + (1-k)g^{\frac{2r}{1-k}}(| B^* |) & 0 \\ 0 & kf^{\frac{2r}{k}}(| B |) + (1-k)g^{\frac{2r}{1-k}}(| C^* |) \end{bmatrix} X, X \right\rangle,
\end{aligned}
$$

and so

$$
\begin{aligned}
\omega^{2r}(S) &= \sup\{| \langle SX, X\rangle |^{2r} : X \in (H_1 \oplus H_2), \| X \| = 1\} \\
&\leq \sup\left\{ \left\langle \begin{bmatrix} \beta & 0 \\ 0 & \gamma \end{bmatrix} X, X \right\rangle : X \in (H_1 \oplus H_2), \| X \| = 1 \right\} \\
&= \max\{\|\beta\|, \|\gamma\|\},
\end{aligned}
$$

where

$$\beta = kf^{\frac{2r}{k}}(| C |) + (1-k)g^{\frac{2r}{1-k}}(| B^* |)$$

and

$$\gamma = kf^{\frac{2r}{k}}(| B |) + (1-k)g^{\frac{2r}{1-k}}(| C^* |),$$

as required. ∎

Now, Theorem 2.10 includes several numerical radius inequalities for operator matrices, and so we give some inequalities in the following remarks.

Remark 2.11 If $f(t) = t^k$ and $g(t) = t^{1-k}$, $k \in (0,1)$, in Theorem 2.10, then we get the following inequality

$$\omega^{2r}(S) \leq \max\{\| k \mid C \mid^{2r} + (1-k) \mid B^* \mid^{2r} \|, \| k \mid B \mid^{2r} + (1-k) \mid C^* \mid^{2r} \|\}.$$

Remark 2.12 If $B = C$ in the above Remark, and by using Lemma (2.6)b, then we have

$$\omega^{2r}(B) = \omega^{2r}(S) \leq \| k \mid B \mid^{2r} + (1-k) \mid B^* \mid^{2r} \|,$$

and this inequality can be found in Theorem 2 in [5].

Remark 2.13 If we take $r = 1$ and $k = \frac{1}{2}$ in the last Remark, we find

$$\omega^2(B) \leq \frac{1}{2} \| | B |^2 + | B^* |^2 \|,$$

which is the second inequality in (1.3).

Our next results are generalizations of the second inequality in (1.4).

Theorem 2.14 *Let* $R = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$ *be a* 2×2 *operator matrix in* $B(H_1 \oplus H_2)$, *with the Cartesian decomposition* $R = S + iT$ *and* $1 \leq r \leq 2$. *Then*

$$\omega^r(R) \leq \frac{1}{2^r} \max\{\| \mid C + B^* \mid^r + \mid C - B^* \mid^r \|, \| \mid B + C^* \mid^r + \mid B - C^* \mid^r \|\}.$$

Proof. For every unit vector $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in (H_1 \oplus H_2)$, and for $1 \leq r \leq 2$, we have

$$\begin{aligned}
\mid \langle RX, X \rangle \mid &= \mid \langle (S + iT)X, X \rangle \mid \\
&= \mid \langle SX, X \rangle + i \langle TX, X \rangle \mid \\
&= \sqrt{\langle SX, X \rangle^2 + \langle TX, X \rangle^2} \\
&\leq \sqrt[r]{\mid \langle SX, X \rangle \mid^r + \mid \langle TX, X \rangle \mid^r} \quad \text{(by Lemma 2.2)} \\
&\leq \sqrt[r]{\langle \mid S \mid X, X \rangle^r + \langle \mid T \mid X, X \rangle^r} \quad \text{(by Lemma 2.4)} \\
&\leq \sqrt[r]{\langle \mid S \mid^r X, X \rangle + \langle \mid T \mid^r X, X \rangle} \quad \text{(by Lemma (2.3)a)} \\
&= \sqrt[r]{\langle (\mid S \mid^r + \mid T \mid^r) X, X \rangle}.
\end{aligned}$$

Thus,

$$\mid \langle RX, X \rangle \mid^r \leq \langle (\mid S \mid^r + \mid T \mid^r) X, X \rangle$$

and so,

$$\begin{aligned}
\omega^r(R) &= \sup \{ \mid \langle RX, X \rangle \mid^r : X \in (H_1 \oplus H_2), \| X \| = 1 \} \\
&\leq \sup \{ \langle (\mid S \mid^r + \mid T \mid^r) X, X \rangle : X \in (H_1 \oplus H_2), \| X \| = 1 \} \\
&= \frac{1}{2^r} \max\{\| \mid C + B^* \mid^r + \mid C - B^* \mid^r \|, \| \mid B + C^* \mid^r + \mid B - C^* \mid^r \|\},
\end{aligned}$$

as required. ∎

Remark 2.15 Let $B = C$ and $r = 2$ in Theorem (2.14). Then we get

$$\begin{aligned}
\omega^2 \left(\begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix} \right) &= \omega^2(B) \qquad \text{(by Lemma (2.6)b)} \\
&\leq \frac{1}{4} \max\{\| \mid B + B^* \mid^2 + \mid B - B^* \mid^2 \|, \| \mid B + B^* \mid^2 + \mid B - B^* \mid^2 \|\} \\
&= \frac{1}{4} \| \mid B + B^* \mid^2 + \mid B - B^* \mid^2 \| \\
&= \frac{1}{2} \| B^*B + BB^* \|,
\end{aligned}$$

which is the second inequality in (1.3).

Theorem 2.16 *Let* $R = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$ *be a* 2×2 *operator matrix in* $B(H_1 \oplus H_2)$, *with the Cartesian decomposition* $R = S + iT$ *and* $r \geq 2$. *Then*

$$\omega^r(R) \leq 2^{\frac{-r}{2}-1} \max\{|||\ C + B^*\ |^r + |\ C - B^*\ |^r||, |||\ B + C^*\ |^r + |\ B - C^*\ |^r||\}.$$

Proof. For every unit vector $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in (H_1 \oplus H_2)$, we have

$$\begin{aligned}
\frac{1}{\sqrt{2}} |\ \langle RX, X\rangle\ | &= \frac{1}{\sqrt{2}} |\ \langle (S+iT)X, X\rangle\ | \\
&= \frac{1}{\sqrt{2}} |\ \langle SX, X\rangle + i\langle TX, X\rangle\ | \\
&= \sqrt{\frac{\langle SX, X\rangle^2 + \langle TX, X\rangle^2}{2}} \\
&\leq \sqrt[r]{\frac{|\ \langle SX, X\rangle\ |^r + |\ \langle TX, X\rangle\ |^r}{2}} \qquad \text{(by Lemma 2.1)} \\
&\leq 2^{\frac{-1}{r}} (\langle |\ S\ |\ X, X\rangle^r + \langle |\ T\ |\ X, X\rangle^r)^{\frac{1}{r}} \quad \text{(by Lemma 2.4)} \\
&\leq 2^{\frac{-1}{r}} (\langle |\ S\ |^r\ X, X\rangle + \langle |\ T\ |^r\ X, X\rangle)^{\frac{1}{r}} \quad \text{(by Lemma (2.3)a)} \\
&= 2^{\frac{-1}{r}} \langle (|\ S\ |^r + |\ T\ |^r)\, X, X\rangle^{\frac{1}{r}} \\
&= 2^{\frac{-1}{r}-1} \left\langle \begin{bmatrix} \eta & 0 \\ 0 & \theta \end{bmatrix} X, X \right\rangle^{\frac{1}{r}}.
\end{aligned}$$

Thus,

$$|\ \langle RX, X\rangle\ |^r \leq 2^{\frac{-r}{2}-1} \left\langle \begin{bmatrix} \eta & 0 \\ 0 & \theta \end{bmatrix} X, X \right\rangle.$$

and so,

$$\begin{aligned}
\omega^r(R) &= \sup\{|\ \langle RX, X\rangle\ |^r : X \in (H_1 \oplus H_2), \|\ X\ \| = 1\} \\
&\leq 2^{\frac{-r}{2}-1} \sup\left\{ \left\langle \begin{bmatrix} \eta & 0 \\ 0 & \theta \end{bmatrix} X, X \right\rangle : X \in (H_1 \oplus H_2), \|\ X\ \| = 1 \right\} \\
&= 2^{\frac{-r}{2}-1} \max\{\|\ \eta\ \|, \|\ \theta\ \|\},
\end{aligned}$$

where

$$\eta = |\ C + B^*\ |^r + |\ C - B^*\ |^r$$

and

$$\theta = |\ B + C^*\ |^r + |\ B - C^*\ |^r,$$

as required. ∎

Remark 2.17 Let $B = C$ and $r = 2$ in Theorem (2.16). Then we get

$$\begin{aligned}\omega^2\left(\begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix}\right) &= \omega^2(B) \qquad (\text{by Lemma (2.6)b}) \\ &\leq \frac{1}{4}\max\{|||\, B+B^*|^2+|B-B^*|^2 \,||, |||\, B+B^*|^2+|B-B^*|^2 \,||\} \\ &= \frac{1}{4}\,|||\, B+B^* \,|^2 + |\, B-B^* \,|^2|| \\ &= \frac{1}{2}\,\|\, B^*B+BB^* \,\|,\end{aligned}$$

which is the second inequality in (1.3).

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ON ROUGH APPROXIMATIONS OF SUBGROUPS VIA CONJUGACY

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Abstract. Conjugacy is a very significant equivalence relation in the theory of groups and it has several important applications as well. In this present paper, we use this equivalence relation to generate Rough sets. A few results are presented in this context by assigning a group structure to the universe set. Some interesting properties of Lower and Upper approximations of subgroups are investigated.

Keywords: group, subgroup, conjugacy, rough set, upper approximation, lower approximation, information system, equivalence relation.

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1. Introduction

The problem of imperfect knowledge has been tackled for a long time by philosophers, logicians and mathematicians. Recently it became a crucial issue for computer scientists, particularly in the area of Artificial Intelligence. There are many approaches to the problem of how to understand and manipulate imperfect knowledge. The most successful approaches to tackle this problem are the Fuzzy set theory and the Rough set theory. Theories of Fuzzy sets and Rough sets are powerful mathematical tools for modeling various types of uncertainties. Fuzzy set theory was introduced by *L. A. Zadeh* in his classical paper [6] of 1965.

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A polish applied mathematician and computer scientist *Zdzislaw Pawlak* introduced Rough set theory in his classical paper [3] of 1982. Rough set theory is a new mathematical approach to imperfect knowledge. This theory presents still another attempt to deal with uncertainty or vagueness. The Rough set theory has attracted the attention of many researchers and practitioners who contributed essentially to its development and application. Rough sets have been proposed for a very wide variety of applications. In particular, the Rough set approach seems to be important for Artificial Intelligence and cognitive sciences, especially for machine learning, knowledge discovery, data mining, pattern recognition and approximate reasoning.

In this present work, we construct Rough sets by considering the equivalence relation Conjugacy on a group. We investigate a few results on lower and upper approximations of subgroups.

2. Preliminaries

Definition 2.1 A relation R on a non-empty set S is said to be an *equivalence relation* on S if

(a) xRx for all $x \in S$ (*reflexivity*)

(b) $xRy \Leftrightarrow yRx$ (*symmetry*)

(c) xRy and $yRz \Rightarrow xRz$ (*transitivity*)

We denote the equivalence class of an element $x \in S$ with respect to the equivalence relation R by the symbol $R[x]$ and $R[x] = \{y \in S : yRx\}$.

Definition 2.2 Let U be a universe set and $X \subseteq U$. Let R be an equivalence relation on U . Then we define the following.

(a) The *lower approximation* of X with respect to R is the set

$$R_*(X) = \{x : R[x] \subseteq X\}$$

(b) The *upper approximation* of X with respect to R is the set

$$R^*(X) = \{x : R[x] \cap X \neq \phi\}$$

(c) The *boundary region* of X with respect to R is the set

$$B_R(X) = R^*(X) - R_*(X)$$

It is clear that $R_*(X) \subseteq X \subseteq R^*(X)$.

Definition 2.3 A set $X \subseteq U$ is said to be a *Rough set* with respect to an equivalence relation R on U, if the boundary region $\mathcal{B}_{\mathcal{R}}(X) = R^*(X) - R_*(X)$ is non-empty.

Definition 2.4 A non-empty set G of elements is said to form a *group* if there is defined a binary operation, called the product and denoted by $*$, such that

(a) $a, b \in G \quad \Rightarrow a * b \in G$ (*Closure*)

(b) $a * (b * c) = (a * b) * c$ for all a, b, c in G (*Associative law*)

(c) there exists an element $e \in G$ such that $a * e = e * a = a \ \forall \ a \in G$. The element e is called the identity element in G. (*Existence of identity*)

(d) for every $a \in G$ there exists an element $a^{-1} \in G$ such that $a * a^{-1} = a^{-1} * a = e$. The element a^{-1} is called the inverse element of a in G. (*Existence of inverse*)

Definition 2.5 A group G with respect to a binary operation $*$ is said to be an *abelian group* if $a * b = b * a$ for all a, b in G.

Definition 2.6 A non-empty subset H of a group G with respect to a binary operation $*$ is said to be a *subgroup* of G if H itself forms a group with respect to $*$.

Remark 2.7 A non-empty subset H of a group G is a subgroup of G if and only if (i) $a, b \in H \quad \Rightarrow a * b \in H$ and (ii) $a \in H \quad \Rightarrow a^{-1} \in H$

Definition 2.8 A subgroup H of a group G is said to be a *normal subgroup* of G if $x \in G$ and $h \in H$ then $xhx^{-1} \in H$.

3. Construction of rough sets

In the Literature of Rough set theory, information systems are considered. An *information system* is a pair $(U, \mathcal{A})$ where $\mathcal{A}$ is a set of attributes. Each attribute $a \in \mathcal{A}$ is a mapping $a : U \to V_a$ where V_a is the range set of the attribute $a \in \mathcal{A}$. Corresponding to each attribute $a \in \mathcal{A}$, an equivalence relation R_a is defined on U such that $xR_ay \Leftrightarrow a(x) = a(y)$. Rough sets are constructed through this relation as usual.

In this section, we slightly deviate from the above traditional setting to construct Rough sets. We consider the relation *conjugacy* on a group to construct Rough sets and present a few results in this context. In what follows G stands for a group and we take the universe set U to be G.

Definition 3.1 If $x, y \in G$ then y is said to be a conjugate of x in G if there exists an element $c \in G$ such that $y = c^{-1}xc$. We write, for this, $x \sim y$ and refer to this relation as *conjugacy*.

Theorem 3.2 Conjugacy *is an equivalence relation on* G.

Remark 3.3 Since conjugacy is an equivalence relation on G, the group G is partitioned into disjoint equivalence classes. If $x \in G$ then the equivalence class of x under the relation conjugacy is the set $C[x] = \{c^{-1}xc : c \in G\}$ and it is called the *conjugate class* of x in G.

Definition 3.4 The lower approximation, upper approximation and boundary region of a subset X of G under the equivalence relation conjugacy on G are as follows.

(a) $C_*(X) = \{x \in G : C[x] \subseteq X\}$,

(b) $C^*(X) = \{x \in G : C[x] \cap X \neq \phi\}$,

(c) $\mathcal{B}_\sim(X) = C^*(X) - C_*(X)$.

4. Rough approximations of subgroups

In this section, we present a few results on lower and upper approximations of subgroups of G under the relation conjugacy.

Proposition 4.1 *A group G is abelian $\Leftrightarrow C_*(X) = X = C^*(X)$ for any subset X of G.*

Proof. Let G be a group. Suppose that G is abelian. Then the conjugate class of an element $x \in G$ is $C[x] = \{x\}$. Let X be any subset of G. If $X = \phi$ or $X = G$, then $C_*(X) = X = C^*(X)$. Now suppose that $X \neq \phi$ and $X \neq G$. Let $x \in X$. Since $C[x] = \{x\} \subset X, \quad x \in C_*(X) \Rightarrow X \subseteq C_*(X) \Rightarrow X = C_*(X)$.

Let $y \in C^*(X)$. Then $\{y\} \cap X \neq \phi \Rightarrow y \in X$. This shows that $C^*(X) \subseteq X$. Hence $C_*(X) = X = C^*(X)$ for any subset X of G.

Conversely, suppose that $C_*(X) = X = C^*(X)$ for any subset X of G. Let $x, y \in G$. Since $C_*(\{y\}) = \{y\} = C^*(\{y\})$, we have $C[y] \subseteq \{y\}$ and $C[y] \cap \{y\} \neq \phi \Rightarrow C[y] = \{y\}$. Since $x^{-1}yx \in C[y]$, $x^{-1}yx = y \Rightarrow xy = yx$. Hence G is abelian. ∎

Remark 4.2 By the above Proposition 4.1, it follows that there does not exist Rough sets in an abelian group under the equivalence relation conjugacy.

Example 4.3 Consider the symmetric group $S_3 = \{e, x, x^2, y, xy, x^2y\}$ of order $|S_3| = 6$, where e is the identity permutation, x is the cyclic permutation (1 2 3) and y is the transposition (1 2). It can be observed that $x^3 = e$ and $y^2 = e$. This group S_3 is a finite non-abelian group under the product of permutations. In this group S_3, the conjugate classes are as follows.

(a) $C[e] = \{e\}$

(b) $C[x] = C[x^2] = \{x, x^2\}$

(c) $C[y] = C[xy] = C[x^2y] = \{y, xy, x^2y\}$.

Let $A = \{e, x\}$. Then $C_*(A) = \{e\}$ and $C^*(A) = \{e, x, x^2\}$. Then $\mathcal{B}_\sim(A) = C^*(A) - C_*(A) = \{x, x^2\}$. Hence A is a Rough set in S_3 under the relation conjugacy. From this example it is clear that Rough sets do exist in non-abelian groups.

Proposition 4.4 *If H is a subgroup of G then the lower approximation $C_*(H)$ of H is a subgroup of G.*

Proof. Let H be a subgroup of G. Let $x, y \in C_*(H)$. Then $C[x] \subseteq H$ and $C[y] \subseteq H$. Let $z \in C[xy]$.

$\Rightarrow \quad z = c^{-1}(xy)c$ for some $c \in G$

$\Rightarrow \quad z = (c^{-1}xc)\,(c^{-1}yc)$.

Since $c^{-1}xc \in C[x]$ and $c^{-1}yc \in C[y]$, we have $c^{-1}xc \in H$ and $c^{-1}yc \in H$

$\Rightarrow \quad (c^{-1}xc)\,(c^{-1}yc) \in H$

$\Rightarrow \quad z \in H$

Hence $C[xy] \subseteq H$. This shows that $xy \in C_*(H)$.

Let $w \in C[y^{-1}]$. Then $w = d^{-1}y^{-1}d$ for some $d \in G$.

Since $d^{-1}y^{-1}d \in C[y]$, $\; d^{-1}y^{-1}d \in H$

$\Rightarrow \quad d^{-1}y^{-1}d \in H$

$\Rightarrow \quad w \in H$

Hence $C[y^{-1}] \subseteq H$. This proves that $y^{-1} \in C_*(H)$. Hence $C_*(H)$ is a subgroup of G. ∎

Remark 4.5 The converse of the above Proposition 4.4 is not true. That is, a subset H of G need not be a subgroup of G even though $C_*(H)$ of H is a subgroup of G. It can be seen from the following example.

Example 4.6 In S_3, take $H = \{e, x^2\}$. Then $C_*(H) = \{e\}$. Now it can be observed that H is not a subgroup of G but $C_*(H)$ is a subgroup of G.

Proposition 4.7 *If H is a cyclic subgroup of G then the lower approximation $C_*(H)$ of H is a cyclic subgroup of G.*

Proposition 4.8 *A subgroup H of a group G is a normal subgroup of G if and only if $C_*(H) = H$.*

Proof. Suppose that H is a subgroup of G. Let H be a normal subgroup of G.

Let $x \in H$ and let $z \in C[x]$.

$\Rightarrow \quad z = c^{-1}xc$ for some $c \in G$.

Since $c \in G$, $\; c^{-1} \in G$.

Since $c^{-1} \in G$ and $x \in H$, $\; c^{-1}xc \in H \Rightarrow z \in H$.

This shows that $C[x] \subseteq H \Rightarrow x \in C_*(H)$. Hence $C_*(H) = H$.

Conversely, suppose that $C_*(H) = H$. Let $c \in G$ and $x \in H$

$\Rightarrow \quad c^{-1} \in G$ and $x \in C_*(H)$

$\Rightarrow \quad c^{-1} \in G$ and $C[x] \subseteq H$

$\Rightarrow \quad cxc^{-1} \in C[x]$

$\Rightarrow \quad cxc^{-1} \in H$.

Hence H is a normal subgroup of G. ∎

Proposition 4.9 *For any subset A of G, the following conditions are equivalent to one another.*

(a) $C_*(A) = A$,

(b) $C^*(A) = (A)$,

(c) $C_*(A) = C^*(A)$.

Proof. For any subset A of G, we always have $C_*(A) \subseteq A \subseteq C^*(A)$. Suppose $C_*(A) = A$ and let $x \in C^*(A)$. Then $C[x] \cap A \neq \phi$

$\Rightarrow$ $C[x] \cap C_*(A) \neq \phi$

$\Rightarrow$ there exists a point y in G such that $y \in C[x] \cap C_*(A)$

$\Rightarrow$ $y \sim x$ and $y \in C_*(A)$

$\Rightarrow$ $C[x] = C[y]$ and $C[y] \subseteq A$

$\Rightarrow$ $C[x] \subseteq A$

$\Rightarrow$ $x \in A$.

This shows that $C^*(A) \subseteq A$. Hence $C^*(A) = A$.

This completes the proof of (a) $\Rightarrow$ (b).

Now, consider $C^*(A) = A$. Let $x \in A$. Then $x \in C^*(A) \Rightarrow$ $C[x] \cap A \neq \phi$.

Let $z \in C[x]$. Then $x \sim z \quad \Rightarrow \quad z \in A$.

This proves that $C[x] \subseteq A$ and hence $x \in C_*(A)$.

Hence $C_*(A) = A$. This completes the proof of (b) $\Rightarrow$ (a).

Obviously, (c) is equivalent to both (a) and (b). ∎

Proposition 4.10 *A subgroup H of G is a normal subgroup of G if and only if $C^*(H) = H$.*

Remark 4.11 The lower and upper approximations of a subset A of G can be equal even though A is not a subgroup of G. It can be observed from the following example.

Example 4.12 In S_3, take $B = \{e, y, xy, x^2y\}$. Then B is not a subgroup of G, but $C^*(B) = B = C_*(B)$.

Example 4.13 *Let $K = \{e, y\}$. Then K is a cyclic subgroup of S_3 generated by y but K is not a normal subgroup of S_3. It can be observed that $C^*(K) = \{e, y, xy, x^2y\}$ is not a cyclic subgroup of S_3. From this example, it follows that the upper approximation of a cyclic subgroup of G is not necessarily a cyclic subgroup of G.*

Example 4.14 In S_3, $C^*(W) = \{e, y, xy, x^2\}$ for $W = \{e, x\}$. Here $C^*(W)$ is a normal subgroup of S_3 eventhough W is not a subgroup of S_3. Also $C^*(W)$ is a cyclic subgroup of S_3, generated by x and $C^*(W) = A_3$, the Alternating group in S_3. From this example, it can be observed that the upper approximation of a subset can be a normal subgroup of G.

5. Homomorphisms and rough approximations

In this section, we establish some interesting results on the Rough approximations of homomorphic images of subsets of G. In what follows G and G' stand for two groups.

Definition 5.1 A mapping $f : G \to G'$ is said to be a *homomorphism* of G into G' if $f(xy) = f(x)\ f(y)$ for all x and y in G.

Definition 5.2 A homomorphism $f : G \to G'$ is said to be an *isomorphism* of G into G' if f is one-one.

Definition 5.3 A homomorphism $f : G \to G$ is said to be an *automorphism* of G onto itself if f is a bijection.

Proposition 5.4 *If $f : G \to G'$ is an isomorphism of G onto G', then*

$$C_*(f(A)) = f(C_*(A)) \text{ for any subset } A \text{ of } G.$$

Proof. Let $f : G \to G'$ be an isomorphism of G onto G' and let A be any subset of G. If $A = \phi$ or $A = G$ then $C_*(f(A)) = f(C_*(A))$. Now suppose that $A \neq \phi$ and $A \neq G$. Let $y \in f(C_*(A))$

$\Rightarrow \quad y = f(x)$ for some $x \in C_*(A)$.

Since $x \in C_*(A),\ C[x] \subseteq A$. Let $z \in C[y]$

$\Rightarrow \quad z = c^{-1}yc$ for some $c \in G'$.

Since $c \in G'$, there exists $d \in G$ such that $c = f(d)$.

Hence $z = c^{-1}yc = f(d^{-1})f(x)f(d) = f(d^{-1}xd)$.

Since $d^{-1}xd \in C[x], d^{-1}xd \in A$

$\Rightarrow \quad f(d^{-1}xd) \in f(A)$

$\Rightarrow \quad z \in f(A)$.

Hence $C[y] \subseteq f(A)$.

This proves that

$$f(C_*(A)) \subseteq C_*(f(A)) \tag{1}$$

Let $y \in C_*(f(A))$. Then $C[y] \subset f(A)$.

$\Rightarrow \quad y = f(x)$ for some $x \in A$.

Let $z \in C[x]$. Then $z = c^{-1}xc$ for some $c \in G$

$\Rightarrow \quad f(z) = f(c^{-1}xc) = f(c^{-1})y\ f(c)$

$\Rightarrow \quad f(z) \in C[y]$

$\Rightarrow \quad f(z) \in f(A)$

$\Rightarrow \quad f(z) \in f(a)$ for some $a \in A$

$\Rightarrow \quad z = a$ and hence $z \in A$.

Hence $C[x] \subseteq A$

$\Rightarrow x \in C_*(A)$

$\Rightarrow y \in f(C_*(A))$.

This shows that

$$C_*(f(A)) \subseteq f(C_*(f(A)) \tag{2}$$

From (1) and (2), we have

$$C_*(f(A)) = f(C_*(A)).$$ ∎

Proposition 5.5 *If $f : G \to G'$ is a homomorphism of G onto G' then*

$$C^*(f(A)) = f(C^*(A)) \text{ for any subset } A \text{ of } G.$$

Proof. Let $f : G \to G'$ be a homomorphism of G onto G' and let A be any subset of G. If $A = \phi$ or $A = G$, then $C^*(f(A)) = f(C^*(A))$.

Now, suppose that $A \neq \phi$ and $A \neq G$. Let $y \in f(C^*(A))$.

$\Rightarrow$ $y = f(x)$ for some $x \in C^*(A)$.

Since $x \in C^*(A)$, $C[x] \cap A \neq \phi$

$\Rightarrow$ there exists z such that $z \in C[x] \cap A$

$\Rightarrow$ $z = c^{-1}xc$ for some $c \in G$ and $z \in A$

$\Rightarrow$ $f(z) = f(c^{-1})y\ f(c)$

$\Rightarrow$ $y = f(czc^{-1}) = f(d^{-1}zd)$ where $d = c^{-1}$

$\Rightarrow$ $y = f(d^{-1})f(z)f(d)$

$\Rightarrow$ $f(z) \in C[y]$ and $f(z) \in f(A)$.

$\Rightarrow$ $f(z) \in C[y] \cap f(A)$.

Hence $C[y] \cap f(A) \neq \phi$ and $y \in C^*(f(A))$. This proves that

$$f(C^*(A)) \subseteq C^*(f(A)) \tag{1}$$

Let $y \in C^*(f(A))$. Then $C[y] \cap f(A) \neq \phi$

$\Rightarrow$ there exists $z \in G$ such that $f(z) \in C[y] \cap f(A)$

$\Rightarrow$ $f(z) = c^{-1}yc$ for some $c \in G'$ and $z \in A$.

Since $c \in G'$, there exists $d \in G$ such that $c = f(d)$. Hence $f(z) = f(d^{-1})y\ f(d)$

$\Rightarrow$ $y = f(d)f(z)f(d^{-1}) = f(dzd^{-1})$.

Since $z = d^{-1}(dzd^{-1})d$, $z \in C[dzd^{-1}] \cap A$

$\Rightarrow$ $C[dzd^{-1}] \cap A \neq \phi$

$\Rightarrow dzd^{-1} \in C^*(A)$

$\Rightarrow f(dzd^{-1}) \in f(C^*(A))$

$\Rightarrow$ $y \in f(C^*(A))$.

This shows that

$$C^*(f(A)) \subseteq f(C^*(f(A))) \tag{2}$$

From (1) and (2), we have

$$C^*(f(A)) = f(C^*(A)).$$ ∎

Proposition 5.6 *If $f : G \to G'$ is an isomorphism of G into G' then*

$$C_*(f(A)) \subseteq f(C_*(A)) \text{ for any subset } A \text{ of } G.$$

Proposition 5.7 *If $f : G \to G'$ is a homomorphism of G into G' then $C^*(f(A)) \subseteq f(C^*(A))$ for any subset A of G.*

Proposition 5.8 *If $f : G \to G'$ is an isomorphism of G into G' then $f(\mathcal{B}_\sim(A)) = \mathcal{B}_\sim(f(A))$ for any subset A of G.*

6. Topological aspects

In this section, we assign a topological structure to G and investigate some topological aspects in this context.

Definition 6.1 A *topological group* is a group G together with a topology on G that satisfies the following two properties.

(a) The map $\varphi : G \times G \to G$ defined by $\varphi(g, h) = gh$ is continuous when $G \times G$ is endowed with the product topology.

(b) The map $G \to G$ defined by $\psi(g) = g^{-1}$ is continuous.

Proposition 6.2 *A collection $\mathcal{T} = \{A : C_*(A) = A\}$ of subsets of G forms a topology on G and hence $(G, \mathcal{T})$ is a topological space.*

Proposition 6.3 *The map $\psi : (G, \mathcal{T}) \to (G, \mathcal{T})$ defined by $\psi(g) = g^{-1}$ for every $g \in G$ is continuous.*

Proof. Let U be an open set in $(G, \mathcal{T})$ and let $U^{-1} = \{a^{-1} : a \in U\}$.
If U is empty then so is U^{-1}. Now, suppose that U is non-empty.
Since U is open, $C_*(U) = U$.
If $y \in U^{-1}$, then $y = x^{-1}$ for some $x \in U$.
Since $x \in U$, $\quad x \in C_*(U) \quad \Rightarrow \quad C[x] \subseteq U$.
Let $z \in C[y]$. Then $z = c^{-1}yc$ for some $c \in G$
$\Rightarrow \quad z^{-1} = c^{-1}xc$
$\Rightarrow \quad z^{-1} \in C[x]$
$\Rightarrow \quad z^{-1} \in U$
$\Rightarrow \quad z \in U^{-1}$.
Hence $C[y] \subseteq U^{-1}$
$\Rightarrow \quad y \in C_*(U^{-1})$.
This shows that $C_*(U^{-1}) = U^{-1}$. Hence U^{-1} is open in $(G, \mathcal{T})$. ∎

Example 6.4 The topology $\mathcal{T} = \{A : C_*(A) = A\}$ on S_3 is given by

$$\mathcal{T} = \{\phi, S_3, \{e\}, \{x, x^2\}, \{e, x, x^2\}, \{y, xy, x^2y\}, \{e, y, xy, x^2y\}, \{x, x^2, y, xy, x^2y\}\}$$

The map $\varphi : S_3 \times S_3 \to S_3$ defined by $\varphi(g, h) = gh$ is not continuous when $S_3 \times S_3$ is endowed with the product topology. Because $x.x^2 = x^3 = e \in \{e\}$ and $\{e\} \in \mathcal{T}$, but there does not exist two open neighborhoods V_x and V_{x^2} of x and x^2 respectively in $(S_3, \mathcal{T})$ such that $V_x\, V_{x^2} \subseteq \{e\}$.

Now, it is so natural to ask the following Question-5.5. The investigation of the following problem is under progress. The results that are under investigation in this context may appear in later Research Articles.

Question 6.5 *Under what conditions* $(G, \mathcal{T})$ *forms a topological group ?*

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EXISTENCE AND UNIQUENESS OF Ψ-BOUNDED SOLUTIONS FOR NONLINEAR MATRIX DIFFERENCE EQUATIONS

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Abstract. Sufficient conditions are established for the existence and uniqueness of Ψ-bounded solutions for nonlinear vector difference equation on $\mathbb{Z}$, using Banach contraction principle. Further, we obtain sufficient conditions for the existence and uniqueness of Ψ-bounded solutions for nonlinear matrix difference equation on $\mathbb{Z}$, using Kronecker product of matrices.

Keywords: Ψ-bounded solutions, Banach contraction principle, nonlinear difference equations.

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1. Introduction

The aim of this paper is to give sufficient conditions for the nonlinear matrix difference equation

$$X(n+1) = A(n)X(n)B(n) + F(n, X(n)) \tag{1.1}$$

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has a unique Ψ-bounded solution on $\mathbb{Z}$, where $A \in \mathbb{R}^{m\times m}$, $F : \mathbb{R}\times\mathbb{R}^{m\times m} \to \mathbb{R}^{m\times m}$ and $F(n, O) = O$. Here Ψ is an invertible matrix function on $\mathbb{Z}$. The basic problem under consideration is to determine sufficient conditions for the existence of a solution with some specified boundedness conditions. Classical results of this type, for linear and nonlinear differential equations were given by Coppel [2] and for linear and nonlinear difference equations were given by Agarwal [1]. The problem of Ψ-bounded solutions for the system of linear difference equations have been studied by many authors [3], [4], [6]. Recently, Suresh kumar et al. [7], [8], [9] studied Ψ-bounded solutions for linear matrix difference equations. In [3], Diamandescu proved a necessary and sufficient condition for the existence of Ψ-bounded solutions for the nonhomogeneous linear difference equation on $\mathbb{Z}$. Suresh kumar et al. [7] extended these results to matrix difference equations, using technique of Kronecker product of matrices.

In this paper, we present sufficient conditions for the existence and uniqueness of Ψ-bounded solutions for the nonlinear difference equation

$$x(n+1) = A(n)x(n) + f(n, x(n)) \tag{1.2}$$

on $\mathbb{Z}$. Further, we established sufficient conditions for the existence and uniqueness of Ψ-bounded solutions for nonlinear matrix difference equation (1.1) on $\mathbb{Z}$ with the help of Kronecker product of matrices.

2. Preliminaries

Denote $\mathbb{R}^n$ the Euclidean n-space. $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}$ denote the set of all real, integers and nonnegative integers respectively, denote $N(n_0) = \{n_0, n_0+1, n_0+2, \dots\}$. I_m and O_m denote the unit matrix and zero matrix of order m respectively. For $x = \{x_1, x_2, \dots, x_m\} \in \mathbb{R}^m$ and $A = [a_{ij}] \in \mathbb{R}^{m\times m}$, we use the following vector and matrix norms

$$\|x\| = \max\{|x_1|, |x_2|, \dots |x_m|\} \quad \text{and} \quad |A| = \sup_{\|x\|\le 1} \|Ax\|.$$

Let $\Psi_i : \mathbb{Z} \to (0, \infty)$, $i = 1, 2, \dots, m$ be functions, and define a matrix function

$$\Psi = \operatorname{diag}[\Psi_1, \Psi_2, \dots \Psi_m].$$

Then $\Psi(n)$ is an invertible matrix function on $\mathbb{Z}$.

Definition 2.1 [5] Let $A \in \mathbb{R}^{p\times q}$ and $B \in \mathbb{R}^{r\times s}$ then the Kronecker product of A and B written $A \otimes B$ is defined to be the partitioned matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1q}B \\ a_{21}B & a_{22}B & \dots & a_{2q}B \\ \dots & \dots & \dots & \dots \\ a_{p1}B & a_{p2}B & \dots & a_{pq}B \end{bmatrix}$$

is an $pr \times qs$ matrix and is in $\mathbb{R}^{pr\times qs}$.

Definition 2.2 [5] Let $A = [a_{ij}] \in \mathbb{R}^{p\times q}$, then the vectorization operator $Vec : \mathbb{R}^{p\times q} \to \mathbb{R}^{pq}$, defined and denote by

$$\hat{A} = VecA = \begin{bmatrix} A_{.1} \\ A_{.2} \\ \vdots \\ A_{.q} \end{bmatrix}, \text{ where } A_{.j} = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{pj} \end{bmatrix} (1 \leq \ j \leq \ q).$$

Lemma 2.1 [7] *The vectorization operator $Vec : \mathbb{R}^{m\times m} \to \mathbb{R}^{m^2}$, is a linear and one-to-one operator. In addition, Vec and Vec^{-1} are continuous operators.*

Regarding properties and rules for Kronecker product of matrices we refer to [5].

Now, by applying the Vec operator to the nonlinear matrix difference equation (1.1) and using Kronecker product properties, we have

$$\hat{X}(n+1) = G(n)\hat{X}(n) + \hat{F}(n, \hat{X}(n)), \tag{2.1}$$

where $G(n) = B^T(n)\otimes A(n)$ is a $m^2\times m^2$ matrix and $\hat{F}(n, \hat{X}(n)) = VecF(n, X(n))$ is a column matrix of order m^2. The equation (2.1) is called the Kronecker product difference equation associated with (1.1). It is clear that, if $X(n)$ is a solution of (1.1) if and only if $\hat{X}(n) = \text{Vec } X(n)$ is a solution of (2.1).

The corresponding homogeneous difference equation of (2.1) is

$$\hat{X}(n+1) = G(n)\hat{X}(n). \tag{2.2}$$

Definition 2.3 A function $\varphi : \mathbb{Z} \to \mathbb{R}^m$ is said to be Ψ-bounded on $\mathbb{Z}$ if $\Psi\varphi$ is bounded on $\mathbb{Z}$ (i.e., there exists $M > 0$ such that $\|\Psi(n)\varphi(n)\| < M$, for all $n \in \mathbb{Z}$).

Extend this definition to matrix functions.

Definition 2.4 A matrix sequence $F : \mathbb{Z} \to \mathbb{R}^{m\times m}$ is said to be Ψ-bounded on $\mathbb{Z}$ if ΨF is bounded on $\mathbb{Z}$ (i.e., there exists $L > 0$ such that $|\Psi(n)F(n)| \leq L$, for all $n \in \mathbb{Z}$).

The following lemmas play a vital role in the proof of main results.

Lemma 2.2 *The matrix function $F(n)$ is Ψ-bounded on $\mathbb{Z}$ if and only if the vector function $VecF(n)$ is $I_m \otimes \Psi$-bounded on $\mathbb{Z}$.*

Proof. From the proof of Lemma 2.1, it follows that

$$\frac{1}{m}|A| \leq \|VecA\|_{\mathbb{R}^{m^2}} \leq |A|,$$

for every $A \in \mathbb{R}^{m\times m}$.

Put $A = \Psi(n)F(n)$ in the above inequality, we have

$$\frac{1}{m}|\Psi(n)F(n)| \leq \|(I_m \otimes \Psi(n)).VecF(n)\|_{\mathbb{R}^{m^2}} \leq |\Psi(n)F(n)|, \tag{2.3}$$

for all matrix functions $F(n)$, $n \in \mathbb{Z}$. The proof easily follows from inequality (2.3). ∎

Consider the linear difference equation

$$x(n+1) = A(n)x(n), \tag{2.4}$$

where $A(n)$ is an invertible square matrix of order m on $\mathbb{Z}$. The following lemma is well-known and is given in [3].

Lemma 2.3 *Let $Y(n)$ be the fundamental matrix and satisfies $Y(0) = I_m$, then*

(i) $$Y(n) = \begin{cases} A(n-1)A(n-2)\dots A(1)A(0), & n > 0 \\ I_m, & n = 0 \\ [A(-1)A(-2)\dots A(n-1)A(n)]^{-1}, & n < 0 \end{cases}$$

(ii) *$Y(n+1) = A(n)Y(n)$, for all $n \in \mathbb{Z}$.*

(iii) *the solution of (2.4) with the initial condition $x(0) = x_0$ is*

$$x(n) = Y(n)x_0, \ \ n \in \mathbb{Z}.$$

(iv) *$Y(n)$ is invertible for each $n \in \mathbb{Z}$ and*

$$Y^{-1}(n) = \begin{cases} A^{-1}(0)A^{-1}(1)\dots A^{-1}(n-2)A^{-1}(n-1), & n > 0 \\ I_m, & n = 0 \\ A(-1)A(-2)\dots A(n-1)A(n), & n < 0. \end{cases}$$

Lemma 2.4 [7] *Let $Y(n)$ and $Z(n)$ be the fundamental matrices for the matrix difference equations*

$$X(n+1) = A(n)X(n) \tag{2.5}$$

and

$$X(n+1) = B^T(n)X(n), \tag{2.6}$$

$n \in \mathbb{Z}$ respectively. Then the matrix $Z(n) \otimes Y(n)$ is fundamental matrix of (2.2).

Lemma 2.5 *Let $Y(n)$ be an invertible matrix function on $\mathbb{N}$ and let P be a projection. If there exists a positive constant $L > 1$ such that*

$$\sum_{k=n_0}^{n-1} |\Psi(n)Y(n)PY^{-1}(k+1)\Psi^{-1}(k+1)| \leq L, \ \text{for all } n \in \mathbb{N}(n_0), \tag{2.7}$$

then there exists a constant $L_1 > 0$ such that

$$|\Psi(n)Y(n)P| \leq L_1 \left(\frac{L-1}{L}\right)^{n-n_0}, \ \text{for all } n \in \mathbb{N}(n_0). \tag{2.8}$$

Proof. Let $a(n) = |\Psi(n+1)Y(n+1)P|^{-1}$. From the identity

$$\Psi(n)Y(n)P\left(\sum_{k=n_0}^{n-1} a(k)\right) = \sum_{k=n_0}^{n-1} \Psi(n)Y(n)PY^{-1}(k+1)\Psi^{-1}(k+1)\Psi(k+1)Y(k+1)Pa(k),$$

it follows that

$$\begin{aligned} &|\Psi(n)Y(n)P|\left(\sum_{k=n_0}^{n-1} a(k)\right) \\ &\leq \sum_{k=n_0}^{n-1} |\Psi(n)Y(n)PY^{-1}(k+1)\Psi^{-1}(k+1)|\ |\Psi(k+1)Y(k+1)P|a(k) \leq L. \end{aligned} \tag{2.9}$$

Setting $b(n) = \sum_{k=n_0}^{n-1} a(k)$, we obtain $b(n) - b(n-1) = |\Psi(n)Y(n)P|^{-1}$, for $n \in \mathbb{N}(n_0+1)$. After substituting in (2.9), we have

$$b(n) - b(n-1) \geq \frac{b(n)}{L} \quad \text{and} \quad b(n) \geq \frac{L}{L-1} b(n-1),$$

which implies that

$$b(n) \geq \left(\frac{L}{L-1}\right)^{n-1-n_0} b(n_0+1), \ \text{ for all } \ n \in \mathbb{N}(n_0+1).$$

From (2.9), we get

$$|\Psi(n)Y(n)P|b(n) \leq L,$$

which implies that

$$\begin{aligned} |\Psi(n)Y(n)P| &\leq L(b(n))^{-1} \leq L\left(\frac{L-1}{L}\right)^{n-n_0-1} b^{-1}(n_0+1) \\ &= L\left(\frac{L-1}{L}\right)^{n-n_0-1} |\Psi(n_0+1)Y(n_0+1)P|. \end{aligned}$$

If we choose $L_1 = \max\left\{|\Psi(n_0)Y(n_0)P|, \frac{L^2}{L-1}|\Psi(n_0+1)Y(n_0+1)P|\right\}$, then (2.8) follows. ∎

Lemma 2.6 *Let $Y(n)$ be an invertible matrix which is defined on $\mathbb{N}$ and let P be a projection. If there exists a constant $L > 0$ such that*

$$\sum_{k=n}^{\infty} |\Psi(n)Y(n)PY^{-1}(k+1)\Psi^{-1}(k+1)\| \leq L, \ \textit{for all } n \in \mathbb{N}, \tag{2.10}$$

then for any vector $\xi \in \mathbb{R}^m$ such that $P\xi \neq 0$,

$$\limsup_{n\to\infty} \|\Psi(n)Y(n)P\xi\| = \infty. \tag{2.11}$$

Proof. For any $n \in \mathbb{N}(n_0)$, we have $\|\Psi(n+1)Y(n+1)P\xi\| > 0$. Then, from

$$\sum_{k=n}^{n_1} \|\Psi(k+1)Y(k+1)P\xi\|^{-1}\Psi(n)Y(n)P\xi$$
$$=\sum_{k=n}^{n_1} \|\Psi(k+1)Y(k+1)P\xi\|^{-1}\Psi(n)Y(n)PY^{-1}(k+1)\Psi^{-1}(k+1)\Psi(k+1)Y(k+1)P\xi$$

and (2.10), we get

$$\|\Psi(k+1)Y(k+1)P\xi\| \sum_{k=n}^{n_1} \|\Psi(k+1)Y(k+1)P\xi\|^{-1} \leq L, \text{ for } n_1 \geq n, n, n_1 \in \mathbb{N}(n).$$

Therefore, $\sum_{k=n}^{\infty} \|\Psi(k+1)Y(k+1)P\xi\|^{-1}$ exists and so

$$\limsup_{n\to\infty} \|\Psi(n+1)Y(n+1)P\xi\|^{-1} = 0, \text{ or } \limsup_{n\to\infty} \|\Psi(n+1)Y(n+1)P\xi\| = \infty. \blacksquare$$

3. Main result

In this section, first we obtain sufficient conditions for the existence and uniqueness of Ψ-bounded solution of the nonlinear difference equation (1.2), using Banach contraction principle.

Let the vector space $\mathbb{R}^m$ be represented as a direct sum of three subspaces X_-, X_0, X_+ such that a solution $y(n)$ of (2.4) is Ψ-bounded on $\mathbb{Z}$ if and only if $y(0) \in X_0$ and Ψ-bounded on $\mathbb{Z}$ if and only if $y(0) \in X_- \oplus X_0$. Also let P_{-1}, P_0, P_1 denote the corresponding projections of $\mathbb{R}^m$ onto X_-, X_0, X_+ respectively.

In the general case where ($P_0 \neq 0$), the solution for (1.2) is as follows

$$\begin{aligned} x(n) =& \sum_{k=-\infty}^{n-1} Y(n)P_{-1}Y^{-1}(k+1)f(k,x(k)) \\ &+ \sum_{k=0}^{n-1} Y(n)P_0Y^{-1}(k+1)f(k,x(k)) - \sum_{k=n}^{\infty} Y(n)P_1Y^{-1}(k+1)f(k,x(k)). \end{aligned} \tag{3.1}$$

For simplicity, assume that the linear equation (2.4) has no nontrivial Ψ-bounded solution ($P_0 = 0$).

Theorem 3.1 (Existence and Uniqueness) *Suppose that there exist supplementary projections P_{-1},P_1 and a positive constant K such that*

$$\begin{aligned} &\sum_{k=-\infty}^{n-1} |\Psi(n)Y(n)P_{-1}Y^{-1}(k+1)\Psi^{-1}(k)| \\ &\quad + \sum_{k=n}^{\infty} |\Psi(n)Y(n)P_1Y^{-1}(k+1)\Psi^{-1}(k)| \leq K. \end{aligned} \tag{3.2}$$

Let $f(n,x)$ be a vector function such that

$$||\Psi(n)[f(n,x)-f(n,y)]|| \le \alpha||\Psi(n)(x-y)||, \tag{3.3}$$

for $n \in \mathbb{Z}$, $\|\Psi x\| \le \rho$, $\|\Psi y\| \le \rho$, where $\alpha K < 1$, then the equation (1.2) has a unique Ψ-bounded solution $x(n)$ for which $\|\Psi x\| \le \rho$.

Proof. From Lemmas 2.5 and 2.6, the condition (3.2) implies that $|\Psi(n)Y(n)P_{-1}\xi|$ is unbounded for $n \le 0$ if $P_{-1}\xi \ne 0$ and bounded for $n \ge 0$, and that $|\Psi(n)Y(n)P_1\xi|$ is unbounded for $n \ge 0$ if $P_1\xi \ne 0$ and bounded for $n \le 0$. Hence the linear equation (2.4) has no nontrivial Ψ-bounded solution.

Let $x(n)$ be the solution of (1.2), then from (3.2) and (3.3) the function

$$\begin{aligned} y(n) = x(n) &- \sum_{k=-\infty}^{n-1} Y(n)P_{-1}Y^{-1}(k+1)f(k,x(k)) \\ &+ \sum_{k=n}^{\infty} Y(n)P_1Y^{-1}(k+1)f(k,x(k)), \end{aligned} \tag{3.4}$$

exists and is Ψ-bounded for all $n \in \mathbb{Z}$. Moreover, it follows that

$$\begin{aligned} y(n+1) &= x(n+1) - \sum_{k=-\infty}^{n} Y(n+1)P_{-1}Y^{-1}(k+1)f(k,x(k)) \\ &\quad + \sum_{k=n+1}^{\infty} Y(n+1)P_1Y^{-1}(k+1)f(k,x(k)) \\ &= Ax(n) + f(n,x(n)) - Y(n+1)P_{-1}Y^{-1}(n+1)f(n,x(n)) \\ &\quad - \sum_{k=-\infty}^{n-1} A(n)Y(n)P_{-1}Y^{-1}(k+1)f(k,x(k)) \\ &\quad - Y(n+1)P_1Y^{-1}(n+1)f(n,x(n)) \\ &\quad + \sum_{k=n}^{\infty} A(n)Y(n)P_1Y^{-1}(k+1)f(k,x(k)) \\ &= A(n)\left[x(n) - \sum_{k=-\infty}^{n-1} Y(n)P_{-1}Y^{-1}(k+1)f(k,x(k))\right. \\ &\quad \left. - \sum_{k=n}^{\infty} Y(n)P_1Y^{-1}(k+1)f(k,x(k))\right] + f(n,x(n)) - f(n,x(n)) \\ &= A(n)y(n). \end{aligned}$$

Therefore, $y(n)$ is a Ψ-bounded solution of (1.2). Thus $y(n) = 0$ that is

$$x(n) = \sum_{k=-\infty}^{n-1} Y(n)P_{-1}Y^{-1}(k+1)f(k,x(k)) - \sum_{k=n}^{\infty} Y(n)P_1Y^{-1}(k+1)f(k,x(k)). \tag{3.5}$$

Define $C_\Psi=\{x:\mathbb{R}\to\mathbb{R}^m : x \text{ is } \Psi\text{-bounded functions on } \mathbb{Z} \text{ such that } \|\Psi x\|\le\rho\}$ and $\|x\|_\Psi=\sup_{n\in\mathbb{Z}}\|\Psi(n)x(n)\|$. Clearly, this defines a norm on C_Ψ and $(C_\Psi,||\cdot\|_\Psi)$ is a Banach space. Let T be a mapping defined by

$$Tx(n)=\sum_{k=-\infty}^{n-1}Y(n)P_{-1}Y^{-1}(k+1)f(k,x(k))-\sum_{k=n}^{\infty}Y(n)P_1Y^{-1}(k+1)f(k,x(k)), \tag{3.6}$$

for all $x\in C_\Psi$. Consider

$$\begin{aligned}
\|\Psi(n)Tx(n)\| &= |\sum_{k=-\infty}^{n-1}\Psi(n)Y(n)P_{-1}Y^{-1}(k+1)\Psi^{-1}(k)\Psi(k)f(k,x(k))\\
&\quad-\sum_{k=n}^{\infty}\Psi(n)Y(n)P_1Y^{-1}(k+1)\Psi^{-1}(k)\Psi(k)f(k,x(k))|\\
&\le\Big[\sum_{k=-\infty}^{n-1}|\Psi(n)Y(n)P_{-1}Y^{-1}(k+1)\Psi^{-1}(k)|\\
&\quad+\sum_{k=n}^{\infty}|\Psi(n)Y(n)P_1Y^{-1}(k+1)\Psi^{-1}(k)|\Big]\|\|\Psi(k)f(k,x(k))\|\\
&\le K\alpha\|\Psi(k)x(k)\|\\
&\le K\alpha\rho<\rho.
\end{aligned}$$

which implies $Tx(n)\in C_\Psi$ and hence $T:C_\Psi\to C_\Psi$.

Now, we show that T is a contraction mapping on C_Ψ. Consider

$$\begin{aligned}
\|\Psi(n)(Tx-Ty)(n)\| &= \Big\|\sum_{k=-\infty}^{n-1}\Psi(n)Y(n)P_{-1}Y^{-1}(k+1)\Psi^{-1}(k)\Psi(k)f(k,x(k))\\
&\quad-\sum_{k=n}^{\infty}\Psi(n)Y(n)P_1Y^{-1}(k+1)\Psi^{-1}(k)\Psi(k)f(k,x(k))\\
&\quad-\Big[\sum_{k=-\infty}^{n}\Psi(n)Y(n)P_{-1}Y^{-1}(k+1)\Psi^{-1}(k)\Psi(k)f(k,y(k))\\
&\quad-\sum_{k=n}^{\infty}\Psi(n)Y(n)P_1Y^{-1}(k+1)\Psi^{-1}(k)\Psi(k)f(k,y(k))\Big]\Big\|\\
&\le\Big[\sum_{k=-\infty}^{n}|\Psi(n)Y(n)P_{-1}Y^{-1}(k+1)\Psi^{-1}(k)|\\
&\quad+\sum_{k=n}^{\infty}|\Psi(n)Y(n)P_1Y^{-1}(k+1)\Psi^{-1}(k)|\Big]\\
&\qquad\|\Psi(k)[f(k,x(k))-f(k,y(k))]\|\\
&\le K\alpha\|\Psi(k)(x(k)-y(k))\|.
\end{aligned}$$

Thus $\|Tx-Ty\|_\Psi\le K\alpha\|x-y\|_\Psi$.

Therefore T is a contraction mapping on C_Ψ. Hence by Banach contraction principle, T has a unique fixed point $x(n)$ on C_Ψ. Thus, the nonlinear difference equation (1.2) has a unique fixed point for which $\|\Psi x\| \leq \rho$.

Conversely, if $x(n)$ is a solution of (1.2) such that $\|\Psi x\| \leq \rho$, then $y = x - Tx$ is a Ψ-bounded solution of the linear equation (2.4), therefore $y = 0$. ∎

Example 3.1 Consider the nonlinear difference equation (1.2) with

$$A(n) = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \text{ and } f(n, x(n)) = \frac{1}{10} \begin{bmatrix} 3^{-|n|} \tan^{-1}(x_1(n)) \\ 5^{-|n|} \frac{x_2(n)}{1+n^2} \end{bmatrix}.$$

The fundamental matrix of (2.4) is

$$Y(n) = \begin{bmatrix} 2^n & 0 \\ 0 & 3^{-n} \end{bmatrix}.$$

Consider

$$\Psi(n) = \begin{bmatrix} 3^{-n} & 0 \\ 0 & 5^n \end{bmatrix}, \text{ for all } n \in \mathbb{Z}.$$

Clearly, the linear equation (2.4) has no nontrivial Ψ-bounded solution on $\mathbb{Z}$. There exist supplementary projections

$$P_{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } P_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

such that

$$\sum_{k=-\infty}^{n-1} |\Psi(n)Y(n)P_{-1}Y^{-1}(k+1)\Psi^{-1}(k)| = \left(\frac{1}{3}\right) \sum_{k=-\infty}^{n-1} \left(\frac{2}{3}\right)^{k-(n+1)} = 1,$$

$$\sum_{k=n}^{\infty} |\Psi(n)Y(n)P_1Y^{-1}(k+1)\Psi^{-1}(k)| = (3) \sum_{k=n}^{\infty} \left(\frac{5}{3}\right)^{n-k} = 7.5,$$

which implies

$$\sum_{k=-\infty}^{n-1} |\Psi(n)Y(n)P_{-1}Y^{-1}(k+1)\Psi^{-1}(k)| + \sum_{k=n}^{\infty} |\Psi(n)Y(n)P_1Y^{-1}(k+1)\Psi^{-1}(k)| = 8.5.$$

And also

$$\|\Psi(n)[f(n,x) - f(n,y)]\| \leq \frac{1}{10} \|\Psi(n)[x-y]\|.$$

Therefore, all the conditions of Theorem 3.1 are satisfied with $\alpha = 1/10$ and $K = 8.5$. Hence the nonlinear difference equation (1.2) has a unique Ψ-bounded solution

$$x(n) = \frac{1}{10} \begin{bmatrix} \sum_{k=-\infty}^{n-1} (2)^{n-(k+1)} 3^{-|k|} \tan^{-1}(x_1(k)) \\ \sum_{k=n}^{\infty} 3^{(k+1)-n} 5^{-|k|} \frac{x_2(k)}{1+k^2} \end{bmatrix}$$

on $\mathbb{Z}$.

Now, we obtain sufficient conditions for the existence and uniqueness of the nonlinear matrix difference equation (1.1), using Theorem 3.1 and the technique of Kronecker product of matrices.

Let the matrix space $\mathbb{R}^{m\times m}$ be represented as a direct sum of three subspaces Y_-, Y_0, Y_+ such that a solution $V(n)$ of

$$X(n+1) = A(n)X(n)B(n) \tag{3.7}$$

is a Ψ-bounded solution on $\mathbb{Z}$ if and only if $V(0) \in Y_0$ and Ψ-bounded on $\mathbb{N}$ if and only if $V(0) \in Y_- \oplus Y_0$. Also, let R_{-1}, R_0, R_1 denote the corresponding projections of $\mathbb{R}^{m\times m}$ onto Y_-, Y_0, Y_+ respectively.

Then, the vector space $\mathbb{R}^{m^2}$ represents direct sum of three subspaces S_-, S_0, S_+ such that a solution $\hat{V}(n) = VecV(n)$ of (2.2) is $(I_m \otimes \Psi)$-bounded on $\mathbb{Z}$ if and only if $\hat{V}(0) \in S_0$ and $(I_m \otimes \Psi)$-bounded on $\mathbb{N}$ if and only if $\hat{V}(0) \in S_- \oplus S_0$. Also, let Q_{-1}, Q_0, Q_1 denote the corresponding projections of $\mathbb{R}^{m^2}$ onto S_-, S_0, S_+ respectively.

In the general case where $(Q_0 \neq 0)$, the solution for (2.1) is as follows

$$\begin{aligned}\hat{X}(n) = &\sum_{k=-\infty}^{n-1} (Z(n)\otimes Y(n))Q_{-1}(Z^{-1}(k+1)\otimes Y^{-1}(k+1))\hat{F}(k,\hat{X}(k))\\ &+\sum_{k=0}^{n-1}(Z(n)\otimes Y(n))Q_0(Z^{-1}(k+1)\otimes Y^{-1}(k+1))\hat{F}(k,\hat{X}(k))\\ &-\sum_{k=n}^{\infty}(Z(n)\otimes Y(n))Q_1(Z^{-1}(k+1)\otimes Y^{-1}(k+1))\hat{F}(k,\hat{X}(k)).\end{aligned} \tag{3.8}$$

For simplicity, assume that the linear equation (2.2) has no nontrivial $(I_m \otimes \Psi)$-bounded solution $(Q_0 = 0)$.

Theorem 3.2 *Suppose that there exist supplementary projections Q_{-1}, Q_1 and a positive constant M such that*

$$\begin{aligned}&\sum_{k=-\infty}^{n-1} |(Z(n)\otimes\Psi(n)Y(n))Q_{-1}(Z^{-1}(k+1)\otimes(Y^{-1}(k+1)\Psi^{-1}(k)))|\\ &+\sum_{k=n}^{\infty}|(Z(n)\otimes\Psi(n)Y(n))Q_1(Z^{-1}(k+1)\otimes(Y^{-1}(k+1)\Psi^{-1}(k)))| \le M.\end{aligned} \tag{3.9}$$

Let $F(n,X)$ be a matrix function such that

$$|\Psi(n)(F(n,U)-F(n,V))| \le \beta|\Psi(n)(U-V)| \tag{3.10}$$

for $n \in \mathbb{Z}$, $|\Psi U| \le \gamma$, $|\Psi V| \le \gamma$, where $m\beta M < 1$, then the equation (1.1) has a unique Ψ-bounded solution $X(n)$ from which $|\Psi X| \le \gamma$.

Proof. Let $F(n, X)$ be a matrix function satisfies (3.10). From inequality (2.3), we have

$$\begin{aligned}|(I_m \otimes \Psi(n+1))(\hat{F}(n,\hat{U}) - \hat{F}(n,\hat{V}))| &\leq |\Psi(n+1)(F(n,U) - F(n,V))| \\ &\leq \beta|\Psi(n)(U-V)| \\ &\leq m\beta|(I_m \otimes \Psi(n))(\hat{U}-\hat{V})|,\end{aligned}$$

for $\hat{U}, \hat{V} \in \mathbb{R}^{m^2}$. Also

$$\begin{aligned}|(I_m \otimes \Psi(n))\hat{U}| &\leq |\Psi(n)U(n)| \leq \gamma, \\ |(I(n) \otimes \Psi(n))\hat{V}| &\leq |\Psi(n)V(n)| \leq \gamma.\end{aligned}$$

From Kronecker product properties, equations (3.9) and (3.10), we have that the fundamental matrix of (2.2) satisfies condition (3.2) of Theorem 3.1, and the vectorization function $\hat{F}(n, \hat{X})$ satisfies condition (3.3). Therefore, from Theorem 3.1, the Kronecker product difference equation (2.1) has a unique $(I_m \otimes \Psi)$-bounded solution on $\mathbb{Z}$. From Lemma 2.2, the matrix difference equation (1.1) has a unique Ψ-bounded solution on $\mathbb{Z}$. ∎

Example 3.2 Consider the nonlinear matrix difference equation (1.1) with

$$A(n) = \begin{bmatrix} 5 & 0 \\ 0 & \frac{1}{3} \end{bmatrix},\ B(n) = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 2 \end{bmatrix},\ F(n, X(n)) = \frac{1}{30}\begin{bmatrix} \frac{5^{-|n|}}{1+|n|}x_1(n) & 3^{-|n|}\tan^{-1}x_3(n) \\ \frac{5^{-|n|}}{1+n^2}x_2(n) & \frac{3^{-|n|}}{1+|n|}\tan^{-1}x_4(n) \end{bmatrix}$$

Then $Y(n) = \begin{bmatrix} 5^n & 0 \\ 0 & 3^{-n} \end{bmatrix}$ and $Z(n) = \begin{bmatrix} 3^{-n} & 0 \\ 0 & 2^n \end{bmatrix}$ are fundamental matrices for (2.5) and (2.6) respectively.

Let $\Psi(n) = \begin{bmatrix} 5^{-n} & 0 \\ 0 & 3^n \end{bmatrix}$, for all $n \in \mathbb{Z}$. Then, their exist supplementary projections

$$Q_{-1} = \begin{bmatrix} I_2 & O_2 \\ O_2 & O_2 \end{bmatrix} \text{ and } Q_1 = \begin{bmatrix} O_2 & O_2 \\ O_2 & I_2 \end{bmatrix},$$

such that condition (3.9) and (3.10) satisfied with $M = 15/2$ and $\beta = 1/30$. Moreover, $m\beta M = 1/2 < 1$. Therefore all the conditions of Theorem 3.2 are satisfied. Hence nonlinear difference equation (1.1) has a unique Ψ-bounded solution on $\mathbb{Z}$.

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ON RADIAL DISTRIBUTION OF JULIA SETS OF SOLUTIONS TO COMPLEX DIFFERENTIAL EQUATIONS WITH MEROMORPHIC COEFFICIENTS

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Abstract. In this paper, we mainly investigate the radial distribution of Julia set of solutions to some second order complex linear differential equations with meromorphic coefficients, and find out the lower bound of it. The radial distributions of Julia sets of derivatives of these solutions are also obtained.

Keywords: radial distribution, Julia set, complex differential equation, derivative, deficient value.

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1. Introduction and main results

In this article, we assume the reader is familiar with standard notations and basic results of Nevanlinna's value distribution theory; see [6], [7], [10], [18], [19]. Let f be a meromorphic function in the whole complex plane. We use $\sigma(f)$ and $\mu(f)$ to denote the order and lower order of f respectively; see [19, p.10] for the definitions. Some basic knowledge of complex dynamics of meromorphic functions is also needed; see [3], [21]. We define $f^n, n \in \mathbb{N}$ denote the nth iterate of f. The Fatou set $F(f)$ of transcendental meromorphic function f is the subset of the plane $\mathbb{C}$ where the iterates f^n of f form a normal family. The complement of $F(f)$ in $\mathbb{C}$ is called the Julia set $J(f)$ of f. It is well known that $F(f)$ is open and completely invariant under f, $J(f)$ is closed and non-empty.

We denote $\Omega(\alpha, \beta) = \{z \in \mathbb{C} | \arg z \in (\alpha, \beta)\}$, where $0 < \alpha < \beta < 2\pi$. Given $\theta \in [0, 2\pi)$, if $\Omega(\theta-\varepsilon, \theta+\varepsilon) \cap J(f)$ is unbounded for any $\varepsilon > 0$, then we call the ray $\arg z = \theta$ the radial distribution of $J(f)$. Define $\Delta(f) = \{\theta \in [0, 2\pi) | \arg z = \theta$ is the radial distribution of $J(f)\}$. Obviously, $\Delta(f)$ is closed and so measurable. We use the $mes\Delta(f)$ to denote the linear measure of $\Delta(f)$. Many important results of radial distribution of transcendental meromorphic functions have been obtained, for example [1], [12], [13], [14], [15], [16], [22]. Recently, Huang and Wang [8], [9]

considered the radial distribution of Julia sets of entire solutions of some special linear complex differential equations and obtained the lower bound of them.

In the present paper, we devote to investigate the radial distributions of Julia sets of solutions of some kind of second order complex differential equations which was studied in [17]. The coefficients of this equation are meromorphic and have some special properties. In order to introduce these properties, we give the definition of the so called $\mathscr{EF}$ Class firstly. In what follows we use the notations $\Omega(\alpha,\beta,r)=\{z:\arg z\in\Omega(\alpha,\beta),|z|<r\}$, $\Omega(r,\alpha,\beta)=\{z:\arg z\in\Omega(\alpha,\beta),|z|\geq r\}$ and denote by $\overline{\Omega}(r,\alpha,\beta)$ the closure of $\Omega(r,\alpha,\beta)$.

Definition. Let f be a meromorphic function in the finite complex plane $\mathbb{C}$ of order $0<\sigma(f)<\infty$. A ray $\arg z=\theta$ starting from the origin is called a zero-pole accumulation ray of $f(z)$, if for any given real number $\varepsilon>0$, the following equality holds

$$\limsup_{r\to\infty}\frac{\log n\{\overline{\Omega}(\theta-\varepsilon,\theta+\varepsilon),f=0\}+\log n\{\overline{\Omega}(\theta-\varepsilon,\theta+\varepsilon),f=\infty\}}{\log r}=\sigma(f). \tag{1.1}$$

Edrei and Fuchs [4], [5] proved that the number of deficient values of a meromorphic function is finite when its zeros and poles are located near some curves. The weaker form of their result can be stated as follows.

Theorem A. [20, Theorem 3.10] *Let f be a meromorphic function in the complex plane $\mathbb{C}$ of order $0<\sigma(f)<\infty$. Assume that $f(z)$ has q zero-poles accumulation rays and p deficient values other than 0 and ∞, then $p\leq q$.*

We shall say that a meromorphic function $f\in\mathscr{EF}$, called it Edrei-Fuchs Class if $f(z)$ satisfies the conditions of Theorem A with $p=q\geq 1$, that is, $f(z)$ is of finite and positive order, and has p zero-pole accumulation rays and p non-zero finite deficient values. In [17], assume that one of coefficients of the second order linear complex differential equations belongs to Edrei-Fuchs Class, the authors proved every nontrivial solutions of this equation is of infinite order. Actually, they obtained

Theorem B. [17] *Let $A(z)\in\mathscr{EF}$ be a meromorphic function and let $B(z)$ be a transcendental meromorphic function having a deficient value ∞. Then every nontrivial solution f of equation*

$$f''+A(z)f'+B(z)f=0 \tag{1.2}$$

is of infinite order.

Our main aim of this article is to estimate the lower bound of radial distribution of Julia set to solutions of equation (1.2).

Theorem 1.1 *Let $A(z) \in \mathscr{EF}$ be a meromorphic function and let $B(z)$ be a transcendental meromorphic function with finite order having a deficient value ∞, let f be a nontrivial solution of equation* (1.2) *and $J(f)$ has an unbounded component, then* $\operatorname{mes}\Delta(f) \geq M_2 > 0$, *where M_2 is a constant defined in Remark* 1, *Section* 2.

Furthermore, we study the radial distribution of Julia set of the derivatives of the nontrivial solutions of equations (1.2). Indeed, we obtain the following results.

Theorem 1.2 *Let $A(z) \in \mathscr{EF}$ be a meromorphic function and let $B(z)$ be a transcendental meromorphic function with finite order having a deficient value ∞, let f be a nontrivial solution of equation* (1.2). *Moreover, if $J(f)$ and $J(f^{(k)})$ both have an unbounded component, then* $\operatorname{mes}(\Delta(f) \cap \Delta(f^{(k)})) \geq M_2$, *where k is a positive integer.*

By Theorem 1.2, we immediately have

Corollary 1.1 *Under the hypothesis of Theorem* 1.2 *we have $mes\Delta(f^{(k)}) \geq M_2$, where k is a positive integer.*

2. Preliminary lemmas

In this article, for a measurable set $E \subset (0, \infty)$, we define the logarithmic measure of E by $m_l(E) = \int_E \frac{dt}{t}$. We also define the upper and lower logarithmic densities of $E \subset [1, \infty)$ respectively, by

$$\begin{aligned} \overline{\log \operatorname{dens}} E &= \limsup_{r\to\infty} \frac{m_l(E \cap [0, r])}{\log r}, \\ \underline{\log \operatorname{dens}} E &= \liminf_{r\to\infty} \frac{m_l(E \cap [0, r])}{\log r}. \end{aligned} \tag{2.1}$$

In the following, we recall the Nevanlinna characteristic in an angle; see [6]. Let $g(z)$ be meromorphic on the angle $\overline{\Omega}(\alpha, \beta)$, where $\beta - \alpha \in (0, 2\pi]$. Following [6], we define

$$\begin{aligned} A_{\alpha,\beta}(r, g) &= \frac{w}{\pi} \int_1^r \left(\frac{1}{t^w} - \frac{t^w}{r^{2w}}\right) \{\log^+ |g(te^{i\alpha})| + \log^+ |g(te^{i\beta})|\} \frac{dt}{t}; \\ B_{\alpha,\beta}(r, g) &= \frac{2w}{\pi r^w} \int_\alpha^\beta \log^+ |g(re^{i\theta})| \sin w(\theta - \alpha) d\theta; \\ C_{\alpha,\beta}(r, g) &= 2 \sum_{1<|b_n|<r} \left(\frac{1}{|b_n|^w} - \frac{|b_n|^w}{r^{2w}}\right) \sin w(\beta_n - \alpha), \end{aligned}$$

where $w = \pi/(\beta - \alpha)$, and $b_n = |b_n|e^{i\beta_n}$ are poles of $g(z)$ in $\overline{\Omega}(\alpha, \beta)$ appearing according to their multiplicities. The Nevanlinna angular characteristic is defined as

$$S_{\alpha,\beta}(r, g) = A_{\alpha,\beta}(r, g) + B_{\alpha,\beta}(r, g) + C_{\alpha,\beta}(r, g).$$

In particular, we denote the order of $S_{\alpha,\beta}(r,g)$ by

$$\sigma_{\alpha,\beta}(g)=\limsup_{r\to\infty}\frac{\log S_{\alpha,\beta}(r,g)}{\log r}.$$

We call W is a hyperbolic domain if $\overline{\mathbb{C}}\backslash W$ contains at least three points, where $\overline{\mathbb{C}}$ is the extended complex plane. For an $a\in\mathbb{C}\backslash W$, define $C_W(a)=\inf\{\lambda_W(z)|z-a|:\forall z\in W\}$, where $\lambda_W(z)$ is the hyperbolic density on W. It is well known that if every component of W is simply connected, then $C_W(a)\geq 1/2$; see [22]. For a finite number $a\in J(f)$, if there is a component U in $F(f)$ such that $C_U(a)>0$, then we call $C_{F(f)}(a)>0$, where $f(z)$ is a transcendental meromorphic function in $\mathbb{C}$.

Lemma 2.1. ([22, Lemma 2.2]) *Let $f(z)$ be analytic in $\Omega(r_0,\theta_1,\theta_2)$, U be a hyperbolic domain, and $f:\Omega(r_0,\theta_1,\theta_2)\to U$. If there exists a point $a\in\partial U\backslash\{\infty\}$ such that $C_U(a)>0$, then there exists a constant $d>0$ such that, for sufficiently small $\varepsilon>0$, we have*

$$|f(z)|=O(|z|^d),\quad z\to\infty,\ z\in\Omega(r_0,\theta_1+\varepsilon,\theta_2-\varepsilon).$$

The next lemma shows some estimates for the logarithmic derivative of functions being analytic in an angle. Before this, we recall the definition of an R-set; for reference, see [10]. Set $B(z_n,r_n)=\{z:|z-z_n|<r_n\}$. If $\sum\limits_{n=1}^{\infty}r_n<\infty$ and $z_n\to\infty$, then $\bigcup\limits_{n=1}^{\infty}B(z_n,r_n)$ is called an R-set. Clearly, the set $\left\{|z|:z\in\bigcup\limits_{n=1}^{\infty}B(z_n,r_n)\right\}$ is of finite linear measure.

Lemma 2.2. ([9, Lemma 2.2]) *Let $z=re^{i\psi}, r_0+1<r$ and $\alpha\leq\psi\leq\beta$, where $0<\beta-\alpha\leq 2\pi$. Suppose that n (≥ 2) is an integer, and that $g(z)$ is analytic in $\Omega(r_0,\alpha,\beta)$ with $\sigma_{\alpha,\beta}(g)<\infty$. Choose $\alpha<\alpha_1<\beta_1<\beta$. Then, for every $\varepsilon_j\in(0,(\beta_j-\alpha_j)/2)$ $(j=1,2,\ldots,n-1)$ outside a set of linear measure zero with*

$$\alpha_j=\alpha+\sum_{s=1}^{j-1}\varepsilon_s,\quad \beta_j=\beta-\sum_{s=1}^{j-1}\varepsilon_s,\quad j=2,3,\ldots,n-1.$$

there exists $K>0$ and $M>0$ only depending on $g,\varepsilon_1,\ldots,\varepsilon_{n-1}$ and $\Omega(\alpha_{n-1},\beta_{n-1})$, and not depending on z, such that

$$\left|\frac{g'(z)}{g(z)}\right|\leq Kr^M(\sin k(\psi-\alpha))^{-2}$$

and

$$\left|\frac{g^{(n)}(z)}{g(z)}\right|\leq Kr^M\left(\sin k(\psi-\alpha)\prod_{j=1}^{n-1}\sin k_{\varepsilon_j}(\psi-\alpha_j)\right)^{-2}$$

for all $z\in\Omega(\alpha_{n-1},\beta_{n-1})$ outside an R-set H, where $k=\pi/(\beta-\alpha)$ and $k_{\varepsilon_j}=\pi/(\beta_j-\alpha_j)$ $(j=1,2,\ldots,n-1)$.

Lemma 2.3. ([11]) *Let $T(r) > 1$ be a nonconstant increasing function in $(0, +\infty)$ of finite order σ, i.e.,*

$$\limsup_{r\to\infty} \frac{\log T(r)}{\log r} = \sigma < \infty. \tag{2.2}$$

For any η such that $0 \le \eta < \sigma$, if $\sigma > 0$, and $\eta = 0$, if $\sigma = 0$, define

$$E(\eta) = \{r \ge 1 : r^\eta < T(r)\}. \tag{2.3}$$

Then $\overline{\log \text{dens}}\, E(\eta) > 0$.

Lemma 2.4. ([17]) *Let $A(z)$ be a meromorphic function of order $0 < \sigma(A) < +\infty$ having p finite deficient values $a_1, a_2, \ldots, a_p, (p \ge 1)$ and let $B(z)$ be a meromorphic function with finite order having a deficient value ∞. Suppose that $\beta > 1$ and $0 < \eta < \sigma(A)$ are two constants. Then there exists a sequence $\{t_n\}$ such that*

$$\lim_{n\to\infty} \frac{t_n^\eta}{T(t_n, A)} = 0. \tag{2.4}$$

Moreover, for every sufficiently large n, there is a set $F_n \subset [t_n, (\beta+1)t_n]$ with $mes(F_n) \le (\beta-1)t_n/4$ such that, for all $R \in [t_n, \beta t_n]\backslash F_n$, the arguments θ sets $E_\nu(R)$, $(\nu = 1, 2, \ldots, p)$ and $E_\infty(R)$ satisfying the following inequalities

$$\text{mes}(E_\nu(R)) := \text{mes}\left(\left\{\theta \in [0, 2\pi) | \log \frac{1}{|A(Re^{i\theta}) - a_\nu|} \ge \frac{\delta_0}{4} T(R, A)\right\}\right) \ge M_1 > 0; \tag{2.5}$$

and

$$\text{mes}(E_\infty(R)) := \text{mes}\left(\left\{\theta \in [0, 2\pi) | \log |B(Re^{i\theta})| \ge \frac{\delta_1}{4} T(R, B)\right\}\right) \ge M_2 > 0, \tag{2.6}$$

where M_1, M_2 are two positive constants depending only on $\sigma(A), \sigma(B)$, $\delta_0 = \min\{\delta(a_\nu, A),\ \nu = 1, 2, \ldots, p\}$, $\delta_1 = \delta(\infty, B), \beta$ and η.

Remark 1. From the proof of Lemma 2.4 in [17], we know that

$$M_1 := \frac{\delta_0}{4}\left\{\frac{(2\beta+4)^{H_1}}{2\pi}\left[(2\beta+1) + \frac{2\log\frac{16e(2\beta+1)}{\beta-1}}{\log\frac{\beta+2}{\beta+1}}\right]\right\}^{-1}; \tag{2.7}$$

$$M_2 := \frac{\delta_1}{4}\left\{\frac{(2\beta+4)^{H_2}}{2\pi}\left[(2\beta+1) + \frac{2\log\frac{16e(2\beta+1)}{\beta-1}}{\log\frac{\beta+2}{\beta+1}}\right]\right\}^{-1} \tag{2.8}$$

where $H_1 = \frac{\sigma(A)}{h_1} + 1$ and h_1 is defined by $h_1 := \overline{\log \text{dens}}\, E(\eta_1) := \overline{\log \text{dens}}\,\{r \ge 1 : r^{\eta_1} < T(r, A)\} > 0$, $\eta < \eta_1 < \sigma(A)$; $H_2 = \frac{\sigma(B)}{h_2} + 1$ and h_2 is defined by $h_2 := \overline{\log \text{dens}}\, E(\eta_2) := \overline{\log \text{dens}}\,\{r \ge 1 : r^{\eta_2} < T(r, B)\} > 0$, $\eta < \eta_2 < \sigma(B)$ according to Lemma 2.3.

Lemma 2.5. ([17, Lemma 2.6]) *Let $A(z) \in \mathscr{EF}$, then for any given sufficiently small $\varepsilon > 0$ and $\beta > 1$, when n is sufficiently large, there exists a sequence of angular regions $\overline{\Omega}(\theta_{k_\nu} + 2\varepsilon, \theta_{k_\nu+1} - 2\varepsilon, t_n, \beta t_n) := \{z : t_n \le |z| \le \beta t_n, \theta_{k_\nu} + 2\varepsilon \le \arg z \le \theta_{k_\nu+1} - 2\varepsilon\}$, $n = 1, 2, \ldots, \nu = 1, 2, \ldots, p$ such that for every $1 \le \nu \le p$, the following inequality*

$$\log \frac{1}{|A(z) - a_\nu|} > \log \frac{4}{d} \tag{2.9}$$

holds for $z \in \overline{\Omega}(\theta_{k_\nu} + 2\varepsilon, \theta_{k_\nu+1} - 2\varepsilon, t_n, \beta t_n) \backslash \bigcup_{\nu=1}^{p} (\gamma_\nu)_n$, where $\bigcup_{\nu=1}^{p} (\gamma_\nu)_n$ are some disks with the sum of total radius not exceeding $p\varepsilon t_n/8$ and $t_n, \beta t_n$ are defined by Lemma 2.4 and $d = \min_{1 \le \nu \ne \nu' \le p} \{|a_\nu - a_{\nu'}|\}$ and a_ν are deficient values of $A(z)$.

3. Proof of theorems

Proof of Theorem 1.1. By Theorem B, we have already known that every nontrivial meromorphic solution f of (1.2) is of infinite order. We shall obtain the assertion by reduction to contradiction. At first, we suppose that $\operatorname{mes} \Delta(f) < M_2$, so $\zeta = M_2 - \operatorname{mes} \Delta(f) > 0$. Since $\Delta(f)$ is closed, obviously $S = [0, 2\pi) \backslash \Delta(f)$ is open, so it consists of at most countably many open intervals. We can choose finitely many open intervals $I_i = (\alpha_i, \beta_i)$, $i = 1, 2, \ldots, m$ satisfying $[\alpha_i, \beta_i] \subset S$ and $\operatorname{mes}\left(S \backslash \bigcup_{i=1}^{m} I_i\right) < \zeta/4$. For the angular domain $\Omega(\alpha_i, \beta_i)$, it is easy to see that $(\alpha_i, \beta_i) \cap \Delta(f) = \emptyset$ and $\Omega(r, \alpha_i, \beta_i) \cap J(f) = \emptyset$ for sufficiently large r. This implies that, for each $i = 1, 2, \ldots, m$, there exist the corresponding r_i and unbounded Fatou component U_i of $F(f)$ such that $\Omega(r_i, \alpha_i, \beta_i) \subset U_i$; see [2]. Because the poles of f are in the set $J(f)$; see [3, Section 2.1], then f does not have poles in $\Omega(r, \alpha_i, \beta_i)$. By the assumption $J(f)$ has an unbounded component, say Γ, then the mapping $f : \Omega(r_i, \alpha_i, \beta_i) \to \mathbb{C} \backslash \Gamma$ is analytic. Since we have chosen Γ such that $\mathbb{C} \backslash \Gamma$ is simply connected, for any $a \in \Gamma \backslash \{\infty\}$, we have $C_{\mathbb{C}\backslash\Gamma}(a) \ge 1/2$. Thus, applying Lemma 2.1 to f in every $\Omega(r_i, \alpha_i, \beta_i)$, there exist a positive constant d_1 such that, for $z \in \bigcup_{i=1}^{m} \Omega(r_i, \alpha_i + \varepsilon, \beta_i - \varepsilon)$,

$$|f(z)| = O(|z|^{d_1}), \qquad as |z| \to \infty, \tag{3.1}$$

where $0 < \varepsilon < \min\{\zeta/(16m), (\beta_i - \alpha_i)/8\}$, $i = 1, 2, \ldots, m$; see [16], [22]. Thus, recalling the definition of $S_{\alpha,\beta}(r, f)$, we immediately have that

$$S_{\alpha_i+\varepsilon, \beta_i-\varepsilon}(r, f) = O(1), \qquad (i = 1, 2, \ldots, m) \tag{3.2}$$

So $\sigma_{\alpha_i+\varepsilon, \beta_i-\varepsilon}(r, f)$ is finite. Therefore, by Lemma 2.2, there exist two constants $M > 0$ and $K > 0$ such that

$$\left| \frac{f^{(s)}(z)}{f(z)} \right| \le K r^M, \qquad (s = 1, 2) \tag{3.3}$$

for all $z \in \bigcup_{i=1}^{m} \Omega(r_i, \alpha_i + 2\varepsilon, \beta_i - 2\varepsilon)$, outside a R-set H.

By Lemma 2.4, there exists a sequence $\{t_n\}$ satisfying (2.4), such that for all $R_n \in [t_n, \beta t_n]\backslash F_n$, where $F_n \subset [t_n, (\beta+1)t_n]$ is a set with $\mathrm{mes}(F_n) \leq (\beta-1)t_n/4$, and for sufficiently large n, we have

$$\mathrm{mes}(E_\infty(R_n)) > M_2 - \zeta/4. \tag{3.4}$$

Therefore, we have

$$\begin{aligned} \mathrm{mes}(E_\infty(R_n) \cap S) &= \mathrm{mes}(E_\infty(R_n)\backslash(\Delta(f) \cap E_\infty(R_n))) \\ &\geq \mathrm{mes}(E_\infty(R_n)) - \mathrm{mes}(\Delta(f)) > \frac{3\zeta}{4} > 0. \end{aligned} \tag{3.5}$$

Then, for each n, we have

$$\begin{aligned} \mathrm{mes}\left(\left(\bigcup_{i=1}^{m} I_i\right) \cap E_\infty(R_n)\right) &= \mathrm{mes}(S \cap E_\infty(R_n)) \\ -\mathrm{mes}\left(\left(S\backslash\bigcup_{i=1}^{m} I_i\right) \cap E_\infty(R_n)\right) &> \frac{3\zeta}{4} - \frac{\zeta}{4} = \frac{\zeta}{2}. \end{aligned} \tag{3.6}$$

Thus, there exists an open interval $I_{i_0} = (\alpha, \beta) \subset \bigcup_{i=1}^{m} I_i \subset S$ such that, for infinitely many n,

$$\mathrm{mes}(E_\infty(R_n) \cap (\alpha, \beta)) > \frac{\zeta}{2m} > 0. \tag{3.7}$$

Without loss of generality, we can assume that (3.7) holds for all n.

Suppose that $A(z)$ has p non-zero finite deficient values, $a_1, a_2, \ldots, a_p$ with deficiency $\delta(a_\nu, A) > 0, 1 \leq \nu \leq p$ and has p zero-pole accumulation rays, $0 \leq \theta_1 < \theta_2 < \ldots < \theta_p < \theta_1 + 2\pi$. From equation (1.2), we have the following inequality

$$|B(z)| \leq \left|\frac{f''(z)}{f(z)}\right| + \left|\frac{f'(z)}{f(z)}\right| (|A(z) - a_\nu| + |a_\nu|). \tag{3.8}$$

Let $\omega = \min\limits_{1\leq k\leq \nu} (\theta_{k+1} - \theta_k)$ and $0 < \varepsilon_0 < \min\left\{\frac{\zeta}{8pm}, \frac{\omega}{2}, \frac{\beta-1}{2p}\right\}$. According to Lemma 2.5, we choose $R_n^* \in [t_n, \beta t_n]\backslash F_n$ such that for every $n \geq n_0$

$$\{z : |z| = R_n^*\} \cap \left(\bigcup_{\nu=1}^{p} (\gamma_\nu)_n\right) \cap H = \emptyset \tag{3.9}$$

where $\bigcup\limits_{\nu=1}^{p} (\gamma_\nu)_n$ are some disks with the sum of total radius not exceeding $p\varepsilon_0 t_n/8 < \frac{(\beta-1)t_n}{16}$ and H is the R-set mentioned above. Hence, by Lemma 2.5 the following inequalities

$$\log \frac{1}{|A(R_n^* e^{i\varphi}) - a_\nu|} > \log \frac{4}{d}, \qquad \nu = 1, 2, \ldots, p \tag{3.10}$$

holds for sufficiently large n and $R_n^* e^{i\varphi} \in \bigcup_{\nu=1}^{p} \overline{\Omega}(\theta_{k_\nu} + 2\varepsilon_0, \theta_{k_\nu+1} - 2\varepsilon_0, t_n, \beta t_n)$.

On the other hand, from Lemma 2.4, for the sequence R_n^*, the following inequality

$$\text{(3.11)} \quad \operatorname{mes}(E_\infty(R_n^*)) := \operatorname{mes}\left(\left\{\theta \in [0,2\pi) | \log|B(R_n^* e^{i\theta})| \geq \frac{\delta_1}{4} T(R_n^*, B)\right\}\right) \geq M_2 > 0$$

also holds for sufficiently large n.

Combining with (3.7), there exists a set $E_\infty(R_n^*) \cap (\alpha,\beta) \cap [\theta_{k_{\nu_0}} + 2\varepsilon_0, \theta_{k_{\nu_0}+1} - 2\varepsilon_0]$, $(1 \leq k_{\nu_0} \leq p)$ such that

$$\text{(3.12)} \qquad \operatorname{mes}(E_\infty(R_n^*) \cap (\alpha,\beta) \cap [\theta_{k_{\nu_0}} + 2\varepsilon_0, \theta_{k_{\nu_0}+1} - 2\varepsilon_0]) \geq \frac{\zeta}{4mp}.$$

Thus, for sufficiently large n, we choose $\varphi_n \in E_\infty(R_n^*) \cap (\alpha,\beta) \cap [\theta_{k_{\nu_0}} + 2\varepsilon_0, \theta_{k_{\nu_0}+1} - 2\varepsilon_0]$ such that (3.3), (3.10) and (3.11) hold. Combining (3.3), (3.8) with (3.10), we get

$$\text{(3.13)} \qquad |B(R_n^* e^{i\varphi_n})| \leq K (R_n^*)^M \left(1 + \frac{d}{4} + |a_\nu|\right).$$

Then, by (3.11) we obtain

$$\text{(3.14)} \qquad \frac{\delta_1}{4} T(R_n^*, B) \leq \log K + M \log R_n^* + \log\left(1 + \frac{d}{4} + |a_\nu|\right).$$

This implies that $B(z)$ is a rational function, which is a contradiction. Thus, we complete the proof. ∎

Proof of Theorem 1.2. We know that every nontrivial solution f of equation (1.2) is a transcendental meromorphic function with infinite order. We also obtain the assertion by reduction to contradiction. Assume that

$$\text{(3.15)} \qquad \operatorname{mes}(\Delta(f) \cap \Delta(f^{(k)})) < M_2$$

and so

$$\text{(3.16)} \qquad \xi := M_2 - \operatorname{mes}(\Delta(f) \cap \Delta(f^{(k)})) > 0.$$

We shall show that there must exist an open interval

$$\text{(3.17)} \qquad I = (\alpha,\beta) \subset \Delta(f^{(k)})^c, \quad 0 < \beta - \alpha < M_2$$

such that

$$\text{(3.18)} \qquad \lim_{n\to\infty} \operatorname{mes}(\Delta(f) \cap E_\infty(R_n) \cap I) > 0,$$

where $\Delta(f^{(k)})^c := [0,2\pi) \backslash \Delta(f^{(k)})$ and $E_\infty(R_n)$ is as defined in (2.6). In order to achieve this goal, we shall prove the following firstly.

$$\text{(3.19)} \qquad \lim_{n\to\infty} \operatorname{mes}(E_\infty(R_n) \backslash \Delta(f)) = 0.$$

Otherwise, suppose that there is a subseries $\{R_{n_k}\}$ such that

$$\lim_{k\to\infty} \mathrm{mes}(E_\infty(R_{n_k})\backslash\Delta(f)) > 0, \tag{3.20}$$

then there exists $\theta_0 \in \Delta(f)^c$ and $\eta > 0$ satisfying

$$\lim_{k\to\infty} \mathrm{mes}((\theta_0-\eta,\theta_0+\eta)\cap(E_\infty(R_{n_k})\backslash\Delta(f))) > 0. \tag{3.21}$$

Since $\arg z = \theta_0$ is not a radial distribution of $J(f)$, there exists $r_0 > 0$ such that

$$\Omega(r_0,\theta_0-\eta,\theta_0+\eta)\cap J(f) = \emptyset. \tag{3.22}$$

This implies that there exists an unbounded component U of Fatou set $F(f)$, such that $\Omega(r_0,\theta_0-\eta,\theta_0+\eta)\subset U$ and $f(\Omega(r_0,\theta_0-\eta,\theta_0+\eta))\subset F(f)$ is analytic. Note that $J(f)$ has an unbounded component, applying Lemma 2.1 to f in $\Omega(r_0,\theta_0-\eta,\theta_0+\eta)$, for any $\zeta>0,\zeta<\eta$, we have

$$|f(z)| = O(|z|^{d_1}),\quad z\in\Omega(r_0,\theta_0-\eta+\zeta,\theta_0+\eta-\zeta),\ |z|\to\infty, \tag{3.23}$$

where d_1 is a positive constant. Recalling the definition of $S_{\alpha,\beta}(r,f)$, we immediately get that

$$S_{\theta_0-\eta+\zeta,\theta_0+\eta-\zeta}(r,f) = O(1). \tag{3.24}$$

Thus, $\sigma_{\theta_0-\eta+\zeta,\theta_0+\eta-\zeta}(r,f)$ is finite. Therefore, by Lemma 2.2, there exists constants $M>0$ and $K>0$ such that (3.3) holds for all $z\in\Omega(r_0,\theta_0-\eta+\zeta,\theta_0+\eta-\zeta)$, outside a R-set H.

Since ζ can be chosen sufficiently small, from (3.21) we have

$$\lim_{k\to\infty} \mathrm{mes}((\theta_0-\eta+\zeta,\theta_0+\eta-\zeta)\cap E_\infty(R_{n_k})) > 0. \tag{3.25}$$

Thus, similarly as (3.9) we can find an infinite series $\{R^*_{n_k}e^{i\theta_{n_k}}\}$ such that, for all sufficiently large k and sufficiently small ε_0, (3.3) (3.10) and (3.11) hold when $\theta_{n_k}\in(\theta_0-\eta+\zeta,\theta_0+\eta-\zeta)\cap E_\infty(R_{n_k})\cap[\theta_{k_{\nu_0}}+2\varepsilon_0,\theta_{k_{\nu_0}+1}-2\varepsilon_0]$. From (3.3), (3.8), (3.10), (3.11), as the same argument of (3.12) and (3.13) in the proof of Theorem 1.1, we can obtain contradiction. These implies (3.19) is valid.

By Theorem 1.1, we know that

$$\mathrm{mes}\Delta(f) \geq M_2. \tag{3.26}$$

From Lemma 2.4, we have, for all sufficiently large n and any positive ε,

$$\mathrm{mes}(E_\infty(R_n)) > M_2-\varepsilon. \tag{3.27}$$

Combining (3.19), (3.26) and (3.27) follows that, for all sufficiently large n,

$$\mathrm{mes}(\Delta(f)\cap E_\infty(R_n)) \geq M_2-\xi/4, \tag{3.28}$$

where ξ is defined in (3.16). Since $\Delta(f^{(k)})$ is closed, clearly $\Delta(f^{(k)})^c$ is open, so it consists of at most countably open intervals. We can choose finitely many open intervals I_j $(j = 1, 2, \ldots, m)$, satisfying

$$I_j \subset \Delta(f^{(k)})^c, \quad \operatorname{mes}\left(\Delta(f^{(k)})^c \backslash \bigcup_{i=1}^{m} I_i\right) < \xi/4. \tag{3.29}$$

Since, for sufficiently large n,

$$\begin{aligned} &\operatorname{mes}\left(\Delta(f) \cap E_\infty(R_n) \cap \left(\bigcup_{i=1}^{m} I_i\right)\right) + \operatorname{mes}(\Delta(f) \cap E_\infty(R_n) \cap \Delta(f^{(k)})) \\ &= \operatorname{mes}\left(\Delta(f) \cap E_\infty(R_n) \cap \left(\Delta(f^{(k)}) \cup \left(\bigcup_{i=1}^{m} I_i\right)\right)\right) \geq M_2 - \xi/2, \end{aligned} \tag{3.30}$$

we have

$$\begin{aligned} &\operatorname{mes}(\Delta(f) \cap E_\infty(R_n) \cap \left(\bigcup_{i=1}^{m} I_i\right)) \\ &\quad \geq M_2 - \xi/2 - \operatorname{mes}(\Delta(f) \cap E_\infty(R_n) \cap \Delta(f^{(k)})) \\ &\quad \geq M_2 - \xi/2 - \operatorname{mes}(\Delta(f) \cap \Delta(f^{(k)})) = \xi/2. \end{aligned} \tag{3.31}$$

Thus, there exists an open interval $I_{i_0} = (\alpha, \beta) \subset \bigcup_{i=1}^{m} I_i \subset \Delta(f^{(k)})^c$ such that, for infinitely many sufficiently large n,

$$\operatorname{mes}(\Delta(f) \cap E_\infty(R_n) \cap I_{i_0}) \geq \frac{\xi}{2m} > 0. \tag{3.32}$$

Then, we prove (3.18) holds.

From (3.18), we know that there are $\widetilde{\theta_0}$ and $\widetilde{\eta} > 0$ such that

$$(\widetilde{\theta_0} - \widetilde{\eta}, \widetilde{\theta_0} + \widetilde{\eta}) \subset I \tag{3.33}$$

and

$$\lim_{n\to\infty} \operatorname{mes}(\Delta(f) \cap E_\infty(R_n) \cap (\widetilde{\theta_0} - \widetilde{\eta}, \widetilde{\theta_0} + \widetilde{\eta})) > 0. \tag{3.34}$$

Then, there exists $\widetilde{r_0}$ such that $\Omega(\widetilde{r_0}, \widetilde{\theta_0} - \widetilde{\eta}, \widetilde{\theta_0} + \widetilde{\eta}) \cap J(f^{(k)}(z)) = \emptyset$. Noting that $J(f^{(k)})$ has an unbounded component, by Lemma 2.1 and the similar argument between (3.22) and (3.23), for any $\widetilde{\zeta} > 0, \widetilde{\zeta} < \widetilde{\eta}$, we have

$$|f^{(k)}(z)| = O(|z|^{d_2}), \quad z \in \Omega(\widetilde{r_0}, \widetilde{\theta_0} - \widetilde{\eta} + \widetilde{\zeta}, \widetilde{\theta_0} + \widetilde{\eta} - \widetilde{\zeta}), \ |z| \to \infty, \tag{3.35}$$

where d_2 is a positive constant.

Fixed $r_N e^{i\theta_N} \in \{r_n e^{i\theta_n}\}$, and take a $r_n e^{i\theta_n} \in \{r_n e^{i\theta_n}\}, n > N$. Take a simple Jordan arc γ in $\Omega(\widetilde{r_0}, \widetilde{\theta_0} - \widetilde{\eta}, \widetilde{\theta_0} + \widetilde{\eta})$ which connecting $r_N e^{i\theta_N}$ to $r_N e^{i\theta_n}$ along

$|z| = r_N$, and connecting $r_N e^{i\theta_n}$ to $r_n e^{i\theta_n}$ along $\arg z = \theta_n$. For any $z \in \gamma$, γ_z denotes a part of γ, which connecting $r_N e^{i\theta_N}$ to z. Let $L(\gamma)$ be the length of γ. Clearly, $L(\gamma) = O(r_n), \quad n \to \infty$. By (3.35), it follows

$$|f^{(k-1)}(z)| \leq \int_{\gamma_z} |f^{(k)}(z)||dz| + c_k \leq O(|z|^{d_2} L(\gamma)) + c_k \leq O(r_n^{d_2+1}), \ n \to \infty. \tag{3.36}$$

Similarly, we have

$$\begin{aligned} |f^{(k-2)}(z)| &\leq \int_{\gamma_z} |f^{(k-1)}(z)||dz| + c_{k-1} \leq O(r_n^{d_2+2}), \quad n \to \infty \\ &\vdots \\ |f(z)| &\leq \int_{\gamma_z} |f'(z)||dz| + c_1 \leq O(r_n^{d_2+k}), \quad n \to \infty. \end{aligned} \tag{3.37}$$

where $c_1, c_2, \ldots, c_k$ are constants, which are independent of n. Therefore,

$$S_{\widetilde{\theta}_0 - \widetilde{\eta} + \widetilde{\zeta}, \widetilde{\theta}_0 + \widetilde{\eta} - \widetilde{\zeta}}(r, f) = O(1). \tag{3.38}$$

By Lemma 2.2, we know (3.3) also holds for all $z \in \Omega(\widetilde{r_0}, \widetilde{\theta_0} - \widetilde{\eta} + \widetilde{\zeta}, \widetilde{\theta_0} + \widetilde{\eta} - \widetilde{\zeta})$, outside a R-set H. By the similar argument between (3.3) and (3.14) in the proof of Theorem 1.1, we can deduce contradictions. Therefore, it follows

$$\operatorname{mes}(\Delta(f) \cap \Delta(f^{(k)})) \geq M_2. \tag{3.39}$$

The proof is complete. ∎

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EXTENSION OF MITTAG–LEFFLER DENSITY AND PROCESSES

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Abstract. An attempt is made to investigate some properties of Mittag-Leffler density. In this paper, structural representation of the Mittag–Leffler variable and extension of Mittag–Leffler stochastic process have also been discussed.

Keywords: Mittag–Leffler density, moment, Laplace transformation, positive levy random variable, Levy process, infinite divisibility, stochastic process.

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1. Introduction

In 1903, the Swedish mathematician Gosta Mittag–Leffler [11] introduced the function $E_\alpha(z)$, is defined by

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad \alpha \in \mathbf{C},\ \Re(\alpha) > 0,\ z \in \mathbf{C}, \tag{1}$$

where $\Gamma(z)$ is the familiar Gamma function. The Mittag–Leffler function reduces immediately to the exponential function $e^z = E_1(z)$ when $\alpha = 1$ for $0 < \alpha < 1$, it interpolates between the pure exponential e^z and a geometric function $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, (|z| < 1)$. Its importance has been realized during the last two decades due to its involvement in the problems of applied sciences such as physics, chemistry, biology and engineering. Mittag–Leffler function occurs naturally in the solution of fractional order differential or integral equations.

In 1905, a generalization of $E_\alpha(z)$ was studied by Wiman [11] who defined the function $E_{\alpha,\beta}(z)$ as follows:

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad \alpha, \beta \in \mathbf{C},\ \Re(\alpha) > 0,\ \Re(\beta) > 0,\ z \in \mathbf{C} \tag{2}$$

The function $E_{\alpha,\beta}(z)$ is now known as Wiman function.

In 1971, Prabhakar [8] introduced the function $E^{\gamma}_{\alpha,\beta}(z)$ defined by,

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n + \beta)} \cdot \frac{z^n}{n!}, \quad \alpha, \beta, \gamma \in \mathbf{C}, \Re(\alpha) > 0, \Re(\beta) > 0, \tag{3}$$

where

$$(\gamma)_n = \gamma(\gamma + 1) \dots (\gamma + n - 1) = \frac{\Gamma(\gamma + n)}{\Gamma(\gamma)},$$

whenever $\Gamma(\gamma)$ is defined, $(\gamma)_0 = 1$, $\gamma \neq 0$.

In the sequel to this study, Shukla and Prajapati [10] investigated the function $E^{\gamma,q}_{\alpha,\beta}(z)$ defined for $\alpha, \beta, \gamma \in \mathbf{C},\ \Re(\alpha) > 0,\ \Re(\beta) > 0,\ q \in (0,1) \cup \mathbf{N}$ as,

$$E^{\gamma,q}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\alpha n + \beta)} \cdot \frac{z^n}{n!} \tag{4}$$

The function $E^{\gamma,q}_{\alpha,\beta}(z)$ converges absolutely for all $z \in \mathbf{C}$ if $q < \Re(\alpha) + 1$ (entire function of order $\Re(\alpha)^{-1}$) and for $|z| < 1$ if $q = \Re(\alpha) + 1$.

It is easily seen that (4) is an obvious generalization of (1),(2),(3) and the exponential function e^z as follows:

$$E^{1,1}_{1,1} = e^z, E^{1,1}_{\alpha,1} = E_\alpha(z), E^{1,1}_{\alpha,\beta}(z) = E_{\alpha,\beta}(z), E^{\gamma,1}_{\alpha,\beta}(z) = E^{\gamma}_{\alpha,\beta}(z)$$

Many properties of the function follow from Mittag–Leffler integral representation,

$$E_\alpha(z) = \frac{1}{2\pi i} \int_{\mathbf{C}} \frac{y^{\alpha-1} e^y}{y^\alpha - z} dz,$$

where the path of integration C is a loop which starts and ends at $-\infty$ and encircles the circular disc $|y| \leq z^{\frac{1}{\alpha}}$.

Mittag–Leffler functions and distributions have received the attention from mathematicians, statisticians, and scientists in physical and chemical sciences. Pillai and Sandhya [7] introduced the Mittag–Leffler distribution in terms of Mittag-Leffler functions.

We will use following terminologies in the analysis of the problem undertaken.

Mellin transform

Let a function $f(t)$ be defined for positive t and let it satisfy the conditions

$$\int_0^1 |f(t)|t^{\sigma_1-1}dt < +\infty, \quad \int_1^\infty |f(t)|t^{\sigma_2-1}dt < +\infty$$

for a proper choice of the numbers σ_1 and σ_2. The function

$$F(s) = \int_0^\infty f(t)t^{s-1}dt, \quad s = \sigma + i\tau, \sigma_1 < \sigma < \sigma_2 \tag{5}$$

is the Mellin transform [1] of the function $f(t)$. The inversion formula of the Mellin transform is,

$$f(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} F(s)t^{-s}ds, \quad t > 0, \quad \sigma_1 < \sigma < \sigma_2,$$

where the integral is taken along the straight line l, $\Re(s) = \sigma$ parallel to the imaginary axis of the s plane and is understood to be the principal value.

Laplace transform

The function $F(s)$ of the complex variable defined by,

$$F(s) = \mathcal{L}\{f(t); s\} = \int_0^\infty e^{-st}f(t)dt, \tag{6}$$

is called the Laplace transform [4] of the function $f(t)$.

The original $f(t)$ can be restored from the Laplace transform $F(s)$ with the help of the inverse Laplace transform,

$$f(t) = \mathcal{L}^{-1}\{F(s); t\} = \int_{c-i\infty}^{c+i\infty} e^{-st}F(s)ds,$$

where, c lies in the right half plane of the absolute convergence of the Laplace integral.

Levy random variable

A positive Levy random variable [5] $u > 0$, with parameter α is such that the Laplace transform of the density of $u > 0$ is given by e^{-t^α}.

$$E[e^{-tu}] = e^{-t^\alpha},$$

where $E(\cdot)$ denotes the expected value of $(\cdot)$ or the statistical expectation of $(\cdot)$. When $\alpha = 1$ the random variable is degenerate with the density function

$$f_1(x) = \begin{cases} 1; & \text{for} \quad x = 1 \\ 0; & \text{elsewhere} \end{cases}$$

Consider an exponential random variable with density function

$$f_1(x) = \begin{cases} e^{-x}; & \text{for} \quad 0 \leq x < \infty \\ 0; & \text{elsewhere} \end{cases}$$

and with the Laplace transform $L_{f_1}(t) = (1+t)^{-1}$.

Levy processes and infinite divisibility [2]

• Brownian motion

A real-valued process $B = \{B_t : t \geq 0\}$, defined on a probability space (Ω, F, P), is said to be a Brownian motion if the following hold:

(i) The paths of B are P-almost surely continuous.

(ii) $P(B_0 = 0) = 1$.

(iii) For $0 \leq s \leq t, B_t - B_s$ is equal in distribution to B_{t-s}.

(iv) For $0 \leq s \leq t, B_t - B_s$ is independent of $\{B_u : u \leq s\}$.

(v) For each $t > 0$, B_t is equal in distribution to a normal random variable with variance t.

• Poisson process

A process valued on the non-negative integers $N = \{N_t : t \geq 0\}$, defined on a probability space (Ω, F, P), is said to be a Poisson process with intensity $\lambda > 0$ if the following hold:

(i) the paths of N are P-almost surely right continuous with left limits.

(ii) $P(N_0 = 0) = 1$.

(iii) For $0 \leq s \leq t, N_t - N_s$ is equal in distribution to N_{t-s}.

(iv) For $0 \leq s \leq t, N_t - N_s$ is independent of $\{N_u : u \leq s\}$.

(v) For each $t > 0$, N_t is equal in distribution to a Poisson random variable with variance λt.

• Levy process

A process $X = \{X_t : t \geq 0\}$, defined on a probability space(Ω, F, P), is said to be a Levy process if it possesses the following properties.

(i) the paths of X are P-almost surely right continuous with left limits.

(ii) $P(X_0 = 0) = 1$.

(iii) For $0 \leq s \leq t, X_t - X_s$ is equal in distribution to X_{t-s}.

(iv) For $0 \leq s \leq t, X_t - X_s$ is independent of $\{X_u : u \leq s\}$.

Conditional expectation [5]

For two random variables x and y having a joint distribution,

$$e(x) = E[E(x|y)]$$

whenever all the expected values exist, where the inside expectation is taken in the conditional space of x given y and the outside expectation is taken in the marginal space of y.

Moments

The r^{th} moment of a variable x about any point $x = A$, usually denoted by $\mu_r{}'$, and defined as,

$$\mu_r{}' = \frac{1}{N}\sum_i f_i(x_i - A)^r, \qquad \sum_i f_i = N = \frac{1}{N}\sum_i f_i(d_i)^r,$$

where $d_i = x_i - A$. The r^{th} moment of a variable about the mean $\bar{x}$, usually denoted by μ_r is given by,

$$\mu_r = \frac{1}{N}\sum_i f_i\,(x_i - \bar{x})^r = \frac{1}{N}\sum_i f_i {z_i}^r,$$

where, $z_i = x_i - \bar{x}$.

2. Theorem on generalized Mittag–Leffler and related results

Theorem 1. *If*

$$f(x) = \frac{x^{\alpha\beta-1}}{\delta^\beta} E^{\beta,\gamma}_{\alpha,\alpha\beta}\left(-\frac{x^\alpha}{\delta}\right), \tag{7}$$

then

$$L_f(t) = \frac{1}{(\delta t^\alpha)^\beta}\left(1 + \frac{1}{\delta t^\alpha}\right)^{-\beta,\gamma}, \tag{8}$$

where $|\frac{1}{(\delta t^\alpha)^\beta}(1+\delta t^\alpha)^{-\beta,\gamma}| \le k$, *$k$ is an arbitrary constant. α and β are non-negative and nonzero values, $E^{\beta,\gamma}_{\alpha,\alpha\beta}\left(-\frac{x^\alpha}{\delta}\right)$ is the generalized Miitag–Leffler function defined by* (4) *and* $(1+\delta t^\alpha)^{-\beta,\gamma}$ *is defined by Prajapati and Shukla* [9].

Proof. Since

$$f(x) = \frac{x^{\alpha\beta-1}}{\delta^\beta} E^{\beta,\gamma}_{\alpha,\alpha\beta}\left(-\frac{x^\alpha}{\delta}\right),$$

by simplification, this can be easily written as

$$= \frac{x^{\alpha\beta-1}}{\delta^{\beta}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{(\beta)_{\gamma k}}{\Gamma(\alpha k+\alpha\beta)} \left(\frac{x^{\alpha}}{\delta}\right)^k$$

$$f(x) = \frac{1}{\delta^{\beta}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{(\beta)_{\gamma k}}{\delta^k \Gamma(\alpha k+\alpha\beta)} x^{\alpha k+\alpha\beta-1}.$$

By taking the Laplace transform, we get

$$\begin{aligned} L_f(t) &= \frac{1}{\delta^{\beta}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{(\beta)_{\gamma k}}{\delta^k \Gamma(\alpha k+\alpha\beta)} \int_0^{\infty} e^{-tx} x^{\alpha k+\alpha\beta-1} dx \\ &= \frac{1}{\delta^{\beta}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{(\beta)_{\gamma k}}{\delta^k \Gamma(\alpha k+\alpha\beta)} \frac{\Gamma(\alpha k+\alpha\beta)}{t^{\alpha k+\alpha\beta}} \\ &= \frac{1}{\delta^{\beta}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{(\beta)_{\gamma k}}{\delta^k} t^{-\alpha k-\alpha\beta} \\ L_f(t) &= \frac{1}{(\delta t^{\alpha})^{\beta}} \left(1+\delta t^{\alpha}\right)^{-\beta,\gamma}. \end{aligned}$$

∎

By writing $f(x)$ in the Mellin–Barnes integral form as

$$\begin{aligned} f(x) &= \frac{x^{\alpha\beta-1}}{\delta^{\beta}\Gamma(\beta)} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s)\Gamma(\beta-\gamma s)}{\Gamma(\alpha\beta-\alpha s)} \left(\frac{x^{\alpha}}{\delta}\right)^{-s} ds,\ 0<c<\beta, \\ &= \frac{1}{\Gamma(\beta)} \frac{1}{2\pi i} \int_{c_1-i\infty}^{c_1+i\infty} \frac{\Gamma\left(\frac{1}{\alpha}-\frac{s}{\alpha}\right)\Gamma\left(\beta-\gamma\left(\frac{1}{\alpha}-\frac{s}{\alpha}\right)\right)}{\delta^{\frac{1}{\alpha}}\alpha\Gamma(1-s)} \left(\frac{x}{\delta^{\frac{1}{\alpha}}}\right)^{-s} ds,\ 1-\alpha\beta<c_1<1, \\ &= \frac{1}{\Gamma(\beta)} \frac{1}{2\pi i} \int_{c_1-i\infty}^{c_1+i\infty} \frac{\Gamma\left(1+\frac{1}{\alpha}-\frac{s}{\alpha}\right)\Gamma\left(\beta-\left(\frac{\gamma}{\alpha}-\frac{\gamma s}{\alpha}\right)\right)}{\delta^{\frac{1}{\alpha}}\Gamma(2-s)} \left(\frac{x}{\delta^{\frac{1}{\alpha}}}\right)^{-s} ds. \end{aligned}$$

If $f(x)$ is a density, then the kernel in the Mellin–Barnes integral as the $(s-1)^{\text{th}}$ moment $E(s^{s-1})$

$$E(s^{s-1}) = \frac{1}{\Gamma(\beta)\delta^{\frac{\gamma}{\alpha}}} \frac{\Gamma\left(1+\frac{1}{\alpha}-\frac{s}{\alpha}\right)\Gamma\left(\beta-\left(\frac{\gamma}{\alpha}-\frac{\gamma s}{\alpha}\right)\right)}{\delta^{-\frac{\gamma s}{\alpha}}\Gamma(2-s)} \tag{9}$$

$$= \frac{1}{\Gamma(\beta)} \frac{\Gamma\left(\frac{1}{\alpha}-\frac{s}{\alpha}\right)\Gamma\left(\beta-\frac{\gamma}{\alpha}+\frac{\gamma s}{\alpha}\right)\delta^{\frac{\gamma(s-1)}{\alpha}}}{\alpha\Gamma(1-s)}. \tag{10}$$

If we subsitute $s=1$ in the equation (9) then R.H.S. will be equal to 1. Thus $f(x)$ is a density function, which is known as the Extension of Generalized Mittag–Leffler density. On setting $\gamma = 1$, this reduces to Generalized Mittag–Leffler density as suggested by Mathai [4].

The $\rho-$ moment of the Mittag–Leffler density $f(x)$ (Mathai [4]) is,

$$\begin{aligned}
E(x^{\rho}) &= \frac{1}{\Gamma(\beta)}\,\frac{1}{\delta^{-\frac{\gamma\rho}{\alpha}}}\,\frac{\Gamma\left(1-\frac{\rho}{\alpha}\right)\Gamma\left(\beta+\frac{\gamma\rho}{\alpha}\right)}{\Gamma(1-\rho)}; -\alpha < \mathrm{R}(\rho) < \alpha < 1 \\
E(x^{s-1}) &= \frac{1}{\Gamma(\beta)}\,\frac{1}{\delta^{\frac{\gamma}{\alpha}}}\,\frac{\Gamma\left(1+\frac{1}{\alpha}-\frac{s}{\alpha}\right)\Gamma\left(\beta-\frac{\gamma}{\alpha}+\frac{\gamma s}{\alpha}\right)}{\Gamma(2-s)\delta^{-\frac{\gamma s}{\alpha}}} \\
&= \frac{1}{\Gamma(\beta)}\,\frac{1}{\delta^{\frac{\gamma}{\alpha}}}\,\frac{\left(\frac{1}{\alpha}-\frac{s}{\alpha}\right)\Gamma\left(\frac{1}{\alpha}-\frac{s}{\alpha}\right)\Gamma\left(\beta-\frac{\gamma}{\alpha}+\frac{\gamma s}{\alpha}\right)}{(1-s)\Gamma(1-s)\delta^{-\frac{\gamma s}{\alpha}}} \\
&= \frac{1}{\alpha}\,\frac{1}{\Gamma(\beta)}\,\frac{1}{\delta^{\frac{\gamma}{\alpha}}}\,\frac{\Gamma\left(\frac{1}{\alpha}-\frac{s}{\alpha}\right)\Gamma\left(\beta-\frac{\gamma}{\alpha}+\frac{\gamma s}{\alpha}\right)}{\Gamma(1-s)\delta^{-\frac{\gamma s}{\alpha}}},
\end{aligned}$$

where, $(1-s)>0, \beta-\frac{\gamma}{\alpha}+\frac{\gamma s}{\alpha}>0$, which is same as (10).

Case I. ([5]) If $y=ax,\ a>0$ and if x has a Mittag–Leffler distribution then the density of y can also be represented as a Mittag–Leffler function with the Laplace transform,

$$\begin{aligned}
L_x(t) &= (1+t^{\alpha})^{-1} \\
\therefore L_y(t) &= (1+(at)^{\alpha})^{-1}, \quad a>0,\ |(at)^{\alpha}|<1.
\end{aligned}$$

Case II. Let

$$E(x^h)=\frac{1}{\Gamma(\beta)}\frac{1}{\delta^{-\frac{\gamma h}{\alpha}}}\frac{\Gamma\left(1-\frac{h}{\alpha}\right)\Gamma\left(\beta+\frac{\gamma h}{\alpha}\right)}{\Gamma(1-h)}; \quad -\alpha < R(h) < \alpha < 1$$

This can easily reduce to,

$$E(x^0)=\frac{1}{\Gamma(\beta)}\frac{\Gamma(1)\Gamma(\beta)}{\Gamma(1)}=1$$

from (5), $f(x)$ is non-negative function for all x,

$$E(x^h)=\int_0^{\infty} x^h g(x)dx \qquad \text{for } h=0.$$

Hence, $f(x)$ is a density function for a positive random variable x. However this is interesting to see that, it is not possible to show that $\int_0^{\infty} x^h g(x)dx=1$ by using the series of Mittag–Leffler function.

3. Structural representation of the Mittag–Leffler variable

Let u be a positive Levy variable with the Laplace transform $e^{-t^{\alpha}}; 0<\alpha\le 1$ and let v be a gamma random variable with parameters η and δ or with the Laplace transform $\frac{1}{(\delta t^{\alpha})^{\beta}}\left(1+\frac{1}{\delta t^{\alpha}}\right)^{-\beta,\gamma}$, where u and v are statistically independently distributed.

Theorem 2. *If u and v are defined as above, then*

$$w \sim uv^{\frac{\gamma}{\alpha}}, \tag{11}$$

where w is a generalized Mittag–Leffler variable with the Laplace transform $\frac{1}{(\delta t^\alpha)^\beta}\left(1+\frac{1}{\delta t^\alpha}\right)^{-\beta,\gamma}$, provided $\frac{1}{(\delta t^\alpha)^\beta}\left(1+\frac{1}{\delta t^\alpha}\right)^{-\beta,\gamma}$ is convergent and β is non-negative value, and $\sim$ means 'distributed as' or both sides have the same distribution.

Proof. By denoting the Laplace transform of the density of w by $L_w(t)$ and treating it as an expected value,

$$L_w(t) = E\left(e^{-tv^{\frac{\gamma}{\alpha}}u}\right) = E\left(E\left(e^{-\left(tv^{\frac{\gamma}{\alpha}}u\right)}|v\right)\right)$$

By simplification, we obtain

$$= E\left[e^{-\left(tv^{\frac{\gamma}{\alpha}}\right)^\alpha}\right] = E\left[-t^\alpha v^\gamma\right] = \frac{1}{(\delta t^\alpha)^\beta}\left(1+\frac{1}{\delta t^\alpha}\right)^{-\beta,\gamma}$$

From the structural representation (11) and the Mellin transform (5), we have,

$$E(w^{s-1}) = E(u^{s-1})E\left(v^{\frac{\gamma}{\alpha}}\right)^{s-1} \tag{12}$$

Since, u and v are statistically independent, the left hand side of (12) is written as,

$$E(w^{s-1}) = \frac{\Gamma\left(\beta-\frac{\gamma}{\alpha}+\frac{\gamma s}{\alpha}\right)\Gamma\left(\frac{1}{\alpha}-\frac{s}{\alpha}\right)\delta^{\frac{\gamma(s-1)}{\alpha}}}{\alpha\Gamma(\beta)\Gamma(1-s)} \tag{13}$$

One can easily compute $E\left(v^{\frac{\gamma}{\alpha}}\right)^{s-1}$ by using Gamma density, this gives

$$\begin{aligned}
E(v^{\frac{\gamma}{\alpha}})^{s-1} &= \frac{1}{\Gamma(\beta)}\,\frac{1}{\delta^\beta}\int_0^\infty \left(v^{\frac{\gamma}{\alpha}}\right)^{s-1} v^{\beta-1}e^{-\frac{v}{\delta}}dv \\
&= \frac{1}{\Gamma(\beta)}\frac{1}{\delta^\beta}\int_0^\infty v^{\left(\beta-\frac{\gamma}{\alpha}+\frac{\gamma s}{\alpha}\right)-1}e^{-\frac{v}{\delta}}dv \\
&= \frac{1}{\Gamma(\beta)}\,\frac{1}{\delta^\beta}\Gamma\left(\beta-\frac{\gamma}{\alpha}+\frac{\gamma s}{\alpha}\right)\delta^{\beta-\frac{\gamma}{\alpha}+\frac{\gamma s}{\alpha}} \\
&= \frac{1}{\Gamma(\beta)}\Gamma\left(\beta-\frac{\gamma}{\alpha}+\frac{\gamma s}{\alpha}\right)\delta^{\frac{\gamma(s-1)}{\alpha}}
\end{aligned}$$

Finally, we arrive at

$$E(v^{\frac{\gamma}{\alpha}})^{s-1} = \frac{1}{\Gamma(\beta)}\Gamma\left(\beta-\frac{\gamma}{\alpha}+\frac{\gamma s}{\alpha}\right)\delta^{\frac{\gamma(s-1)}{\alpha}} \tag{14}$$

For $R(s) > 1-\alpha\beta,\ 0<\alpha\le 1,\ \beta>0$ (13) and (14), we have $(s-1)^{th}$ moment of Levy variable.

$$E[u^{s-1}] = \frac{\Gamma\left(\frac{1}{\alpha}-\frac{s}{\alpha}\right)}{\alpha\Gamma(1-s)} = \frac{\Gamma\left(1+\frac{1}{\alpha}-\frac{s}{\alpha}\right)}{\Gamma(2-s)};\ 0<\alpha\le 1.$$

This leads us to the proof of the theorem. ∎

4. Extension of Mittag–Leffler stochastic process

The stochastic process $\{x(t), t > 0\}$ having stationary independent increment with $x(0) = 0$ & $x(1)$ having the Laplace transform,

$$L_{x(1)}(\xi) = (1 + \xi^{\alpha})^{-1}, 0 < \alpha \leq 1, \xi > 0$$

which is the Laplace transform of a Mittag–Leffler random variable, is called the Mittag–Leffler stochastic process. Then the Laplace transform of $x(t)$, denoted by $L_{x(t)}(\xi)$, is given by,

$$L_{x(1)}(\xi) = \frac{1}{(\xi^{\alpha})^{\beta}} \left(1 + \frac{1}{\xi^{\alpha}}\right)^{-\beta,\gamma} \tag{15}$$

The density corresponding to the Laplace transform of (15) or the density of $x(t)$ is then available as the following:

$$\begin{aligned} f_{x(t)}(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{(\beta)_{\gamma k}}{\Gamma(\alpha k + \alpha\beta)} x^{\alpha k + \alpha\beta - 1} \\ &= x^{\alpha\beta-1} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{(\beta)_{\gamma k}}{\Gamma(\alpha k + \alpha\beta)} x^{\alpha k} \\ &= x^{\alpha\beta-1} E^{\beta,\gamma}_{\alpha,\alpha\beta}(-x^{\alpha}),\ 0 < \alpha \leq 1,\ x \geq 0, t > 0. \end{aligned} \tag{16}$$

The distribution function of $x(t)$ is given by,

$$\begin{aligned} F_{x(t)}(x) &= \int_0^x f_{x(t)}(y)dy \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{(\beta)_{\gamma k}}{k!} \frac{x^{\alpha k + \alpha\beta}}{(\alpha k + \alpha\beta)\Gamma(\alpha k + \alpha\beta)} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\Gamma(\beta + \gamma k)}{\Gamma(\beta)} \frac{x^{\alpha k + \alpha\beta}}{\Gamma(\alpha k + \alpha\beta + 1)};\ 0 < \alpha \leq 1. \end{aligned} \tag{17}$$

Special case of distribution function

On replacing $\alpha\beta$ by αt, β by t & γ by 1 in equation (16), it reduces to

$$F_x(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\Gamma(t + k)}{\Gamma(t)} \frac{x^{\alpha k + \alpha t}}{\Gamma(\alpha k + \alpha t + 1)}; 0 < \alpha \leq 1, t > 0 \tag{18}$$

This result was given by Pillai [6].

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SOME RESULTS ON HYPERCYCLICITY OF TUPLE OF OPERATORS

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Abstract. In this paper we extend some results of hypercyclicity from a single operator to a tuple of commuting operators. In particular we extend theorem of Nathan S. Feldman to a tuple of commuting operators.

Keywords: tuple of operators, hypercyclic vector, hypercyclic operator for tuple, topologically transitive, $d-$dense orbit.

1. Introduction

For a infinite-dimensional separable complex Banach space $X, B(X)$ will denote the algebra of all bounded linear operators on X. For $x \in X$, the orbit of x under T is the set $Orb(T, x) = \{T^n x : n \in \mathbb{N}\}$. A vector x is called a hypercyclic vector for T if $Orb(T, x)$ is dense in X and the operator T is said to be hypercyclic if there is some vector $x \in X$ is hypercyclic see [2]. Let T be a bounded linear operator and U a non-empty open subset of X. Suppose that $x \in X$, and $d > 0$. The orbit $Orb(T, x)$ is called $d-$dense in U if for any $y \in U, B(y, d) \cap Orb(T, x) \neq \emptyset$. In 2002 Nathan S. Feldman studied the perturbations of hypercyclic vectors and show that if T has a $d-$dense orbit, then T is hypercyclic see [4, Theorem 2.1] and see also [6, Theorem 8, p. 344]. In 2007 Enhui Shi , Yuwu Yao, Lizhen Zhou, Youcheng Zhou give a local version of Theorem of Nathan S. Feldman see [8, Theorem 3.7]. Recently, in 2013 B. Yousefi and K. Jahedi extends Theorem 3.7 in [8] for a tuples see [10, Theorem 2.2]. In 2010, Vladimir M*ü*ller proved for $T \in B(X)$ be mixing, $\lambda \in \mathbb{C}, |\lambda| = 1$. Then λT is mixing see [7, Proposition 6].

By an n-tuple of operators we mean a finite sequence of length n of commuting continuous linear operators on a Banach space X.

Let $\mathcal{T} = (T_1, T_2, ..., T_n)$ be an n-tuple of operators acting on an infinite dimensional separable Banach space X. If $d > 0$, we say a n-tuple $\mathcal{T} = (T_1, T_2, ..., T_n)$ of

bounded linear operators acting on a separable infinite dimensional Banach space X has a $d-$dense orbit in a nonempty open subset U of X, if there exists $x \in X$ such that for any $y \in U, B(y,d) \cap Orb(T,x) \neq \emptyset$.

In this paper, we extend theorem of Nathan S. Feldman see [4, Theorem 2.1] and see also [6, Theorem 8, p. 344] from a single operator to a tuple of commuting operators. Also, we show that if $\mathcal{T} = (T_1, T_2, ..., T_n)$ is topologically mixing, with $|\lambda| = 1$ for all $\lambda \in \mathbb{C}$, then there exist $M(i) \in \mathbb{N}$ such that

$$||\lambda^{m(i)}(T_1^{m(1)}T_2^{m(2)}...T_n^{m(n)})u - y|| \leq \varepsilon, \ \forall\, m(i) \geq M(i), \text{ for } i = 1, 2, ..., n.$$

Definition 1.1. [5], [9] Let $\mathcal{T} = (T_1, T_2, ..., T_n)$ be an n-tuple of operators acting on an infinite dimensional separable Banach space X. We will let

$$\mathcal{F} = \{T_1^{k_1}T_2^{k_2}...T_n^{k_n} : \ k_i \in \mathbb{Z}_+, \quad i = 1, ..., n\}$$

be the semigroup generated by $\mathcal{T}$. For $x \in X$, the orbit of x under the tuple $\mathcal{T}$ is the set

$$Orb(\mathcal{T}, x) = \{Sx : S \in \mathcal{F}\}.$$

A vector x is called a hypercyclic vector for $\mathcal{T}$ if $Orb(\mathcal{T}, x)$ is dense in X and in this case the tuple $\mathcal{T}$ is called hypercyclic.

Definition 1.2. [11] An n-tuple $\mathcal{T} = (T_1, T_2, ..., T_n)$ is called topologically mixing if for any given open sets U and V, there exist positive integers $M(1), M(2), ..., M(n)$ such that

$$T_1^{m(1)}...T_n^{m(n)}(U) \cap V \neq \emptyset, \ \forall m(i) \geq M(i), \ i = 1, ..., n.$$

Definition 1.3. [9] We say that the pair $\mathcal{T} = (T_1, T_2)$ is topologically transitive if for every nonempty open subsets U and V of X there exists $S \in \mathcal{F}$ such that $S(U) \cap V \neq \emptyset$.

Definition 1.4. [3, Definition 3.68] Let A be a subset of a metric space (X, d). A subset A_ε of A is an ε-net for A if for every point x of A there exists a point y in A_ε such that $d(x, y) < \varepsilon$. A subset A of X is totally bounded in (X, d) if for every real number $\varepsilon > 0$ there exists a finite ε-net for A.

We have the equivalent the hypercyclic and topologically transitive for $\mathcal{T}$ (see [9, Lemma 2.2] and see also [5, Proposition 2.3]).

2. Main results

For simplicity, we state and prove our results for a pair that is a tuple with $n = 2$, and the general case follows by a similar method. Note that if T_1, T_2 are commutative bounded linear operators on a Banach space X, and $\{m_j\}, \{n_j\}$ are

two sequences of natural numbers, then we say $\{T_1^{m_j}T_2^{n_j} : j \in \mathbb{N}\}$ is hypercyclic if there exists $x \in X$ such that $\{T_1^{m_j}T_2^{n_j}x : j \in \mathbb{N}\}$ is dense in X.

The following theorem extends the theorem of Nathan S. Feldman (see [4, Theorem 2.1] and also [6, Theorem 8, p. 344]) from a single operator to a tuple of commuting operators.

Theorem 2.1. *Let $\mathcal{T} = (T_1, T_2, ..., T_n)$ be an n-uplet of commuting continuous linear operators acting on an infinite dimensional separable Banach space X, $d > 0$, and let $x \in X$ satisfying that for each $y \in X$ there is the sequences $\{m_j^i\}_j$, for $i = 1, ..., n$, of natural numbers with $||T_1^{m_j^1}T_2^{m_j^2}...T_n^{m_j^n}x - y|| < d$. Then, $\mathcal{T} = (T_1, T_2, ..., T_n)$ is hypercyclic.*

Proof. For simplicity, we take $n = 2$. Let U, V be non-empty open subsets of X. We show that

$$T_1^{m_j}T_2^{n_j}U \cap V \neq \emptyset$$

for some $m_j, n_j \in \mathbb{N}$ for each $j = 0, 1, ...$ ∎

Choose $u \in U, v \in V$, and $\varepsilon > 0$ such that $\{y \in X : ||y - u|| < \varepsilon\} \subset U$ and $\{y \in X : ||y - v|| < \varepsilon\} \subset V$.

Let $x' = \frac{\varepsilon x}{3d}$. We show first that the set $Orb(\mathcal{T}, x') = \{T_1^{m_j}T_2^{n_j}x', j \in \mathbb{N}\}$ intersects each open ball with radius $\frac{\varepsilon}{3}$. If $y \in X$, then there is an $m_j, n_j \in \mathbb{N}$ such that

$$\left\| T_1^{m_j}T_2^{n_j}x - \frac{3dy}{\varepsilon} \right\| < d.$$

Therefore,

$$\|T_1^{m_j}T_2^{n_j}x' - y\| < \frac{\varepsilon}{3d}\left\| T_1^{m_j}T_2^{n_j}x - \frac{3dy}{\varepsilon} \right\| < \frac{\varepsilon}{3}.$$

Next, we show that $Orb(\mathcal{T}, x')$ intersects each open ball with radius ε in infinite set. Suppose on the contrary that there is a $y \in X$ such that the set

$$\{m_j, n_j, j \in \mathbb{N} : ||T_1^{m_j}T_2^{n_j}x' - y|| < \varepsilon\}$$

is finite. Since the ball $B(y, \frac{2\varepsilon}{3})$ cannot be covered by a finite number of balls of radii $\frac{\varepsilon}{3}$, hence there is a

$$y_1 \in B\left(y, \frac{2\varepsilon}{3}\right)$$

such that

$$dist\{y_1, Orb(\mathcal{T}, x')\} \geq \frac{\varepsilon}{3}.$$

Thus

$$Orb(\mathcal{T}, x') \cap B\left(y_1, \frac{\varepsilon}{3}\right) = \emptyset,$$

a contradiction. Hence

$$B(v, \varepsilon) \cap \{T_1^{m_j}T_2^{n_j}x' : j \in \mathbb{N}\}$$

has infinite elements. In particular, there exist $m_j, n_j \in \mathbb{N}$ satisfying $m_j^{'} > m_j$ and $n_j^{'} > n_j$ such that

$$T_1^{m_j} T_2^{n_j} x^{'} \in B(u,\varepsilon) \subset U \quad \text{and} \quad T_1^{m_j^{'}} T_2^{n_j^{'}} x^{'} \in B(v,\varepsilon) \subset V.$$

Hence

$$T_1^{m_j^{'}-m_j} T_2^{n_j^{'}-n_j} T_1^{m_j} T_2^{n_j} x^{'} \in V,$$

and so

$$T_1^{m_j^{'}-m_j} T_2^{n_j^{'}-n_j} U \cap V \neq \emptyset.$$

Thus there exist $S \in \mathcal{F}$ such that $SU \cap V \neq \emptyset$. By [9, Lemma 2.2] and see also [5, Proposition 2.3], $\mathcal{T} = (T_1, T_2)$ is hypercyclic. ∎

In the proof of the following proposition, we use a method of the proof of Proposition 6 [7]. Denote by $\mathbb{T}$ the unit circle in the complex plane, $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

Proposition 2.1. *Let $\mathcal{T} = (T_1, T_2, ...T_n)$ be a n-uplet of commuting continuous linear operators acting on an infinite dimensional separable Banach space X, Let $\mathcal{T}$ is topologically mixing, with $|\lambda| = 1$ for all $\lambda \in \mathbb{C}$. Then there exist $M(i) \in \mathbb{N}$ such that*

$$||\lambda^{m(i)}(T_1^{m(1)} T_2^{m(2)} ... T_n^{m(n)})u - y|| \leq \varepsilon, \quad \forall\, m(i) \geq M(i), \quad \text{for } i = 1, 2, ..., n.$$

Proof. For simplicity, we take $n = 2$. Let $U, V \subset X$ be non-empty open subsets of X. Choose $y \in V, y \neq 0$. Let $\varepsilon > 0$ satisfy $B(y,\varepsilon) \subset V$, where $B(y,\varepsilon)$ denotes the open ball centered at y with radius ε.

Let $\mu_1, ..., \mu_k \in \mathbb{T}$ be a finite $\frac{\varepsilon}{2||y||}$-net in $\mathbb{T}$. Since $\mathcal{T} = (T_1, T_2)$ is topologically mixing, for each $j = 1, ..., k$, there exists $M_j(i) \in \mathbb{N}$ where $i = 1, 2$ such that

$$T_1^{m(1)} T_2^{m(2)} U \cap B(\mu_j y, \frac{\varepsilon}{2}) \neq \emptyset \ \text{ for all } m(i) \geq M_j(i).$$

Let $m(i) \geq \max\{M_j(i),\ j = 1, ..., k\}$, for all $i = 1, 2$. Find j such that $|\lambda^{-m(i)} - \mu_j| \leq \frac{\varepsilon}{2||y||}$ for all $i = 1, 2$. Find $u \in U$ such that $||T_1^{m(1)} T_2^{m(2)} u - \mu_j y|| < \frac{\varepsilon}{2}$. Then

$$\begin{aligned} ||\lambda^{m(i)}(T_1^{m(1)} T_2^{m(2)})u - y|| &= ||(T_1^{m(i)} T_2^{m(2)})u - \lambda^{-m(i)} y|| \\ &\leq ||T_1^{m(1)} T_2^{m(2)} u - \mu_j y|| + ||\mu_j y - \lambda^{-m(i)} y|| \\ &\leq \frac{\varepsilon}{2} + ||y||\,|\mu_j - \lambda^{-m(i)}| \leq \varepsilon. \end{aligned}$$

Hence

$$||\lambda^{m(i)}(T_1^{m(1)} T_2^{m(2)})u - y|| \leq \varepsilon.$$

Then, there exist $M(i) = \max\{M_j(i),\ j = 1, ..., k\}$, for all $i = 1, 2$, such that

$$||\lambda^{m(i)}(T_1^{m(1)} T_2^{m(2)})u - y|| \leq \varepsilon \qquad \forall\, m(i) \geq M(i), \text{ for } \ i = 1, 2.$$ ∎

We will finish with a corollary of the following theorem.

Theorem 2.2. [1, Theorem 1.1]. *Let X be a complex topological vector space, let $T \in \mathcal{L}(X)$ and let $x \in X$. The following are equivalent:*

(i) *x is hypercyclic for T;*

(ii) $\overline{\{e^{iP(n)}T^n x;\, n \in \mathbb{N}^*\}} = X$ *for any polynomial* $P \in \mathbb{R}[t]$;

(iii) $\overline{\{e^{iP(n)}T^n x;\, n \in \mathbb{N}^*\}}^{\circ} \neq \emptyset$ *for some polynomial* $P \in \mathbb{R}[t]$.

Corollary 2.1. *Let $T \in \mathcal{L}(X)$, $k \in \mathbb{N}^*$, and $P \in \mathbb{R}[t]$. Suppose there are vectors $x_1, x_2, ..., x_k \in X$ such that*

$$\overline{\bigcup_{j=1}^{k} \{e^{iP(n)}T^n x_j;\, n \in \mathbb{N}^*\}} = X.$$

Then T is hypercyclic.

Proof. We have

$$X = \overline{\bigcup_{j=1}^{k} \{e^{iP(n)}T^n x_j;\, n \in \mathbb{N}^*\}} = \bigcup_{j=1}^{k} \overline{\{e^{iP(n)}T^n x_j;\, n \in \mathbb{N}^*\}}.$$

By [3, Theorem 3.58, p. 147], there is $1 \leq j_0 \leq k$ such that

$$\overline{\{e^{iP(n)}T^n x_{j_0};\, n \in \mathbb{N}^*\}}^{\circ} \neq \emptyset.$$

By Theorem 1.2, x_{j_0} is hypercyclic for T. ∎

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BI-HYPERIDEALS AND QUASI-HYPERIDEALS IN ORDERED SEMIHYPERGROUPS

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Abstract. In this paper, we introduce concepts of bi-hyperideals and quasi-hyperideals of an ordered semihypergroup and present several examples of them. Some properties and relationships between bi-hyperideals and quasi-hyperideals are investigated. In particular, we introduce the concept of intra-regular ordered semihypergroups and give their characterizations in terms of bi-hyperideals and quasi-hyperideals.

Keywords: algebraic hyperstructure, ordered semihypergroup, subsemihypergroup, hyperideal, bi-hyperideal, quasi-hyperideal, quasi-simple, intra-regular.

AMS Mathematics Subject Classification: 20N20, 06F05.

1. Introduction and basic definitions

Good and Hughes [12] introduced the notion of bi-ideals. The notion of quasi-ideals was first introduced by Steinfeld [29] for rings and semigroups. Then, many authors studied these concepts, for example see [15], [17], [18], [26]. Ordered semigroups have been studied by several authors, for example, Alimov [1] and Clifford [6], also see [19], [20], [21], [27], [28]. The concept of algebraic hyperstructures was introduced by Marty [22]. Semihypergroups are studied by many authors, for example, Anvariyeh et al. [2], [3], Bonansinga and Corsini [4], Davvaz [7, 8], De Salvo et al. [10], Freni [11], Heidari et al. [14], Hila et al. [15], [16], Leoreanu [25], Yaqoob et al. [30], and many others. The concept of ordering hypergroups

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investigated by Chvalina [5] as a special class of hypergroups and studied by him and many others. In [13], Heidari and Davvaz studied a semihypergroup $(S, \circ)$ besides a binary relation $\leq$, where $\leq$ is a partial order relation such that satisfies the monotone condition.

Let S be a nonempty set. A mapping $\circ : S \times S \to \mathcal{P}^*(S)$, where $\mathcal{P}^*(S)$ denotes the family of all nonempty subsets of S, is called a *hyperoperation* on S. The couple $(S, \circ)$ is called a *hypergroupoid*. In the above definition, if A and B are two nonempty subsets of S and $x \in S$, then we denote

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b, \; x \circ A = \{x\} \circ A \;\text{ and }\; A \circ x = A \circ \{x\}.$$

A hypergroupoid $(S, \circ)$ is called a *semihypergroup* if for every x, y, z in S,

$$x \circ (y \circ z) = (x \circ y) \circ z.$$

That is,

$$\bigcup_{u \in y \circ z} x \circ u = \bigcup_{v \in x \circ y} v \circ z.$$

In [13], Heidari and Davvaz studied a semihypergroup $(S, \circ)$ besides a binary relation $\leq$, where $\leq$ is a partial order relation such that satisfies the monotone condition. Indeed, an *ordered semihypergroup* $(S, \circ, \leq)$ is a semihypergroup $(S, \circ)$ together with a partial order $\leq$ that is *compatible* with the hyperoperation, meaning that for any x, y, z in S,

$$x \leq y \Rightarrow z \circ x \leq z \circ y \text{ and } x \circ z \leq y \circ z.$$

Here, $z \circ x \leq z \circ y$ means for any $a \in z \circ x$ there exists $b \in z \circ y$ such that $a \leq b$. The case $x \circ z \leq y \circ z$ is defined similarly.

A nonempty subset A of an ordered semihypergroup $(S, \circ, \leq)$ is called a *subsemihypergroup* of S if $A \circ A \subseteq A$.

Example 1. Suppose that $S = [0, 1]$, the unit real interval numbers. We define the following hyperoperation on S:

$$\circ : S \times S \to \mathcal{P}^*(S) \text{ by } x \circ y = [0, xy] \text{ for all } x, y \text{ in } S$$

and we consider usual inequality relation $\leq$ of real numbers as a partial order relation on S. Then, it is easy to see that $(S, \circ, \leq)$ is an ordered semihypergroup. If $A = [0, a]$ for some a in S, then A is a subsemihypergroup of S.

Definition 1.1. A nonempty subset A of an ordered semihypergroup $(S, \circ, \leq)$ is called a *left* (respectively, *right*) *hyperideal* of S if it satisfies the following conditions:

(i) $S \circ A \subseteq A$ (respectively, $A \circ S \subseteq A$);

(ii) if $x \in A$ and $y \in S$ such that $y \leq x$, then $y \in A$.

If A is both a left and a right hyperideal of S, then it is called a *two-sided hyperideal* of S, or simply a *hyperideal* of S.

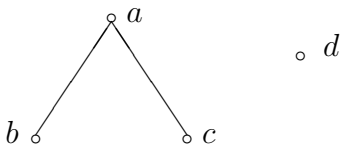
Example 2. We have $(S, \circ, \leq)$ is an ordered semihypergroup where the hyperoperation and the order relation are defined by:

$\circ$	a	b	c	d
a	a	$\{a, b\}$	$\{a, c\}$	a
b	a	$\{a, b\}$	$\{a, c\}$	a
c	a	$\{a, b\}$	$\{a, c\}$	a
d	a	$\{a, b\}$	$\{a, c\}$	a

$$\leq = \{(a, a), (b, b), (c, c), (d, d), (b, a), (c, a)\}.$$

The detailed proof that $(S, \circ, \leq)$ is an ordered semihypergroup is straightforward. The covering relation and the figure of S are given by:

$$\prec = \{(b, a), (c, a)\}$$



Now, it is easy to see that $\{a, b, c\}$ is a hyperideal of S.

Let A be a nonempty subset of an ordered semihypergroup $(S, \circ, \leq)$. Define

$$(A] = \{x \in S \mid x \leq a \text{ for some } a \in A\}.$$

Note that condition (ii) in Definition 1.1 is equivalent to $A = (A]$. If A and B are nonempty subsets of S, then

(1) $A \subseteq (A]$;

(2) $(A \cup B] = (A] \cup (B]$;

(3) $((A] \circ (B]] = (A \circ B]$;

(4) $(A] \circ (B] \subseteq (A \circ B]$.

Let a be an element of an ordered semihypergroup $(S, \circ, \leq)$. We define

$$\begin{aligned} L(a) &: \text{the left hyperideal of } S \text{ generated by } a; \\ R(a) &: \text{the right hyperideal of } S \text{ generated by } a; \\ I(a) &: \text{the hyperideal of } S \text{ generated by } a. \end{aligned}$$

Lemma 1.2. *Let a be an element of an ordered semihypergroup $(S, \circ, \leq)$.*

(1) $L(a) = (a \cup S \circ a]$.

(2) $R(a) = (a \cup a \circ S]$.

(3) $I(a) = (a \cup S \circ a \cup a \circ S \cup S \circ a \circ S]$.

Proof. We will prove (1). The statements (2) and (3) are proved similarly. Clearly, $(a \cup S \circ a] \neq \emptyset$. We have

$$S \circ (a \cup S \circ a] \subseteq (S \circ (a \cup S \circ a)] = (S \circ a \cup S \circ (S \circ a)] \subseteq (S \circ a] \subseteq (a \cup S \circ a].$$

Then, $(a \cup S \circ a]$ is a left hyperideal of S containing a; hence $L(a) \subseteq (a \cup S \circ a]$. The reverse inclusion is clear. ∎

2. Bi-hyperideals

In this section, we introduce the class of bi-hyperideals of an ordered semihypergroup and we present some examples of them. In particular, we show that an ordered semihypergroup is left and right simple if and only if it does not contains proper bi-hyperideals.

Definition 2.1. A subsemihypergroup A of an ordered semihypergroup $(S, \circ, \leq)$ is called a *bi-hyperideal* of S if it satisfies the following conditions:

(i) $A \circ S \circ A \subseteq A$;

(ii) if $x \in A$ and $y \in S$ such that $y \leq x$, then $y \in A$.

Note that every left and right hyperideals are bi-hyperideals.

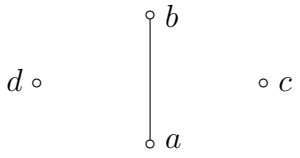
Example 3. We have $(S, \circ, \leq)$ is an ordered semihypergroup where the hyperoperation and the order relation are defined by:

$\circ$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	$\{a, b\}$	a
d	a	a	$\{a, b\}$	$\{a, b\}$

$$\leq = \{(a, a), (b, b), (c, c), (d, d), (a, b)\}.$$

The detailed proof that $(S, \circ, \leq)$ is an ordered semihypergroup is straightforward. The covering relation and the figure of S are given by:

$$\prec = \{(a, b)\}$$



It is easy to see that $\{a, b\}$ is a bi-hyperideal of S.

Definition 2.2. An ordered semihypergroup $(S, \circ, \leq)$ is said to be *left* (respectively, *right*) *simple* if it does not contain proper left (respectively, right) hyperideals.

Lemma 2.3. *An ordered semihypergroup $(S, \circ, \leq)$ is left simple if and only if $(S \circ x] = S$ for all x in S.*

Proof. Assume that $(S \circ x] = S$ for all x in S. Let A be a left hyperideal of S, and let $a \in A$. Then, $(S \circ a] = S$. If $y \in S$, then $y \in (S \circ a]$; hence $y \in (w \circ a]$ for some w in S. Since $(w \circ a] \subseteq A$, we have $y \in A$, and so $A = S$. The opposite direction follows by $(S \circ x]$ is a left hyperideal of S for all x in S. ∎

Similarly, we have the following lemma:

Lemma 2.4. *An ordered semihypergroup $(S, \circ, \leq)$ is right simple if and only if $(x \circ S] = S$ for all x in S.*

Definition 2.5. An ordered semihypergroup $(S, \circ, \leq)$ is said to be *regular* if for any $a \in S$, $a \in (a \circ S \circ a]$.

Lemma 2.6. *If an ordered semihypergroup $(S, \circ, \leq)$ is left and right simple, then it is regular.*

Proof. Assume that S is left and right simple. If $a \in S$, then by Lemma 2.3 and 2.4 we have $S = (S \circ a]$ and $S = (a \circ S]$; hence

$$a \in S = (a \circ S] = (a \circ (S \circ a]] \subseteq (a \circ S \circ a].$$ ∎

Now, we prove the main result of this section:

Theorem 2.7. *Let $(S, \circ, \leq)$ be an ordered semihypergroup. Then, $(S, \circ, \leq)$ is left and right simple if and only if it does not contain proper bi-hyperideals.*

Proof. Assume that $(S, \circ, \leq)$ is left and right simple. Let A be a bi-hyperideal of S. We will show that $S \subseteq A$. Let $x \in S$ and $y \in A$. Since S is left simple, we have $S = (y \cup S \circ y]$. Then, $x \leq y$ or $x \in (w \circ y]$ for some w in S. If $x \leq y$, then $x \in A$. Assume that $x \in (w \circ y]$ for some w in S. Since S is right simple, we have $S = (y \cup y \circ S]$. Since $w \in S$, so $w \leq y$ or $w \in (y \circ u]$ for some u in S. By Lemma 2.6, $(S, \circ, \leq)$ is regular. Then, there exists $b \in S$ such that $y \in (y \circ b \circ y]$. Now, if $w \leq y$, then

$$(w \circ y] \subseteq (y \circ y] \subseteq (y \circ y \circ b \circ y] \subseteq (A \circ S \circ A] \subseteq A;$$

hence $x \in A$. If $w \in (y \circ u]$, then

$$(w \circ y] \subseteq (y \circ u \circ y)] \subseteq (A \circ S \circ A)] \subseteq A;$$

hence $x \in A$. Thus, $S \subseteq A$.

Conversely, assume that $(S, \circ, \leq)$ does not contain proper bi-hyperideals. If A is a left hyperideal of S, then A is a bi-hyperideal of S. By assumption, $S = A$. Similarly, if A is a right hyperideal of S, then A is a bi-hyperideal of S, so $S = A$. ∎

3. Quasi-hyperideals

In this section, we introduce the class of quasi-hyperideals of an ordered semihypergroup and we give some of their properties and relationships between them and bi-hyperideals. We try to use sets instead of elements in the proof of our results similar to [17, 25].

Definition 3.1. A nonempty subset Q of an ordered semihypergroup $(S, \circ, \leq)$ is called a *quasi-hyperideal* of S if it satisfies the following conditions:

(i) $(S \circ Q) \cap (Q \circ S) \subseteq Q$;

(ii) if $x \in Q$ and $y \in S$ such that $y \leq x$, then $y \in Q$.

Note that every quasi-hyperideal is a subsemihypergroup. Indeed, if Q is a quasi-hyperideal of an ordered semihypergroup $(S, \circ, \leq)$, then $Q \circ Q \subseteq S \circ Q$ and $Q \circ Q \subseteq Q \circ S$; hence

$$Q \circ Q \subseteq S \circ Q \cap Q \circ S \subseteq Q.$$

Thus, Q is a subsemihypergroup of S.

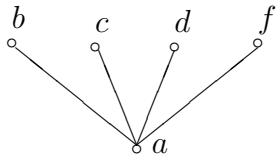
Example 4. We have $(S, \circ, \leq)$ is an ordered semihypergroup where the hyperoperation and the order relation are defined by:

$\circ$	a	b	c	d	f
a	a	a	a	a	a
b	a	$\{a, b\}$	a	$\{a, d\}$	a
c	a	$\{a, f\}$	$\{a, c\}$	$\{a, c\}$	$\{a, f\}$
d	a	$\{a, b\}$	$\{a, d\}$	$\{a, d\}$	$\{a, b\}$
f	a	$\{a, f\}$	a	$\{a, c\}$	a

$$\leq = \{(a, a), (b, b), (c, c), (d, d), (f, f), (a, b), (a, c), (a, d), (a, f)\}.$$

The covering relation and the figure of S are given by:

$$\prec = \{(a, b), (a, c), (a, d), (a, f)\}$$



It is easy to see that the quasi-hyperideals of S are $\{a\}$, $\{a, b\}$, $\{a, d\}$, $\{a, f\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{a, b, f\}$, $\{a, c, f\}$.

Theorem 3.2. *Let Q be a quasi-hyperideal of an ordered semihypergroup $(S, \circ, \leq)$. If T is a subsemihypergroup of S, then $Q \cap T = \emptyset$ or $Q \cap T$ is a quasi-hyperideal of T.*

Proof. Let $Q_1 = Q \cap T$. Assume that $Q_1 \neq \emptyset$. Since $Q_1 \subseteq Q$, it follows that

$$(T \circ Q_1) \cap (Q_1 \circ T) \subseteq (S \circ Q) \cap (Q \circ S) \subseteq Q.$$

Since $Q_1 \subseteq T$ and T is a subsemihypergroup of S, we have

$$(T \circ Q_1) \cap (Q_1 \circ T) \subseteq T.$$

Then,

$$(T \circ Q_1) \cap (Q_1 \circ T) \subseteq Q_1.$$

If $x \in Q_1$ and $y \in T$ such that $y \leq x$, then since $x \in Q$ we have $y \in Q$; hence $y \in Q_1$. Therefore, Q_1 is a quasi-hyperideal of T. ∎

Theorem 3.3. *Let $\{Q_i \mid i \in I\}$ be an indexed family of quasi-hyperideals of an ordered semihypergroup $(S, \circ, \leq)$. If $\bigcap_{i \in I} Q_i \neq \emptyset$, then it is a quasi-hyperideal of S.*

Proof. Let $Q = \bigcap_{i \in I} Q_i$. Since, for each $i \in I$,

$$(S \circ Q) \cap (Q \circ S) \subseteq (S \circ Q_i) \cap (Q_i \circ S) \subseteq Q_i,$$

we have

$$(S \circ Q) \cap (Q \circ S) \subseteq Q.$$

If $x \in Q$ and $y \in S$ such that $y \leq x$, then for each $i \in I$ we have $y \in Q_i$; hence $y \in Q$. Therefore, Q is a quasi-hyperideal of S. ∎

Let A be a nonempty subset of an ordered semihypergroup $(S, \circ, \leq)$. Then, the intersection of all quasi-hyperideals of S containing A, denoted by $(A)_q$, is a quasi-hyperideal of S containing A. This is called the *quasi-hyperideal* of S generated by A.

Theorem 3.4. *The intersection of a left and a right hyperideals of an ordered semihypergroup $(S, \circ, \leq)$ is a quasi-hyperideal of S.*

Proof. Let L and R be a left hyperideal and a right hyperideal of S. Let $Q = L \cap R$. We choose $l \in L$ and $r \in R$. Since $l \circ r \subseteq L \cap R$, so $Q \neq \emptyset$. Since

$$S \circ Q \subseteq L \text{ and } Q \circ S \subseteq R,$$

it follows that

$$S \circ Q \cap Q \circ S \subseteq L \cap R = Q.$$

If $x \in Q$ and $y \in S$ such that $y \leq x$, then by $x \in L \cap R$ we have $y \in L \cap R$. Hence, Q is a quasi-hyperideal of S. ∎

Lemma 3.5. *If Q is a quasi-hyperideal of an ordered semihypergroup $(S, \circ, \leq)$, then*

$$Q = (Q \cup S \circ Q] \cap (Q \cup Q \circ S].$$

Proof. Let Q be a quasi-hyperideal of an ordered semihypergroup $(S, \circ, \leq)$. Clearly,

$$Q \subseteq (Q \cup S \circ Q] \cap (Q \cup Q \circ S].$$

Let $x \in (Q \cup S \circ Q] \cap (Q \cup Q \circ S]$. Then,

$$x \in Q \text{ or } x \in (S \circ Q \cap Q \circ S].$$

By

$$S \circ Q \cap Q \circ S \subseteq Q$$

it follows that $x \in Q$. Hence, $Q = (Q \cup S \circ Q] \cap (Q \cup Q \circ S]$. ∎

Theorem 3.6. *If Q is a quasi-hyperideal of an ordered semihypergroup $(S, \circ, \leq)$, then there exist a left hyperideal L and a right hyperideal R of S such that $Q = L \cap R$.*

Proof. Assume that Q is a quasi-hyperideal of S. Let

$$L = (Q \cup Q \circ S] \text{ and } R = (Q \cup S \circ Q].$$

We have $L \neq \emptyset$. Let $x \in L \circ S$, then $x \in l \circ s$ for some l in L and s in S. Let $l \leq p$ for some p in $Q \cup Q \circ S$. Then, $l \circ s \leq p \circ s$. There are two cases to consider:

Case 1: $p \in Q$. By

$$p \circ s \subseteq Q \circ S \subseteq Q \cup Q \circ S,$$

it follows $x \in L$.

Case 2: $p \in Q \circ S$. Let $p \in q \circ s_1$ for some q in Q and s_1 in S. Since

$$p \circ s \subseteq q \circ s_1 \circ s \subseteq Q \circ S \circ S \subseteq Q \cup Q \circ S \circ S,$$

we have $x \in L$. Then, $L \circ S \subseteq L$. Let $x \in L$ and $y \in S$ be such that $y \leq x$. Then, $x \leq z$ for some z in $Q \cup Q \circ S$. Since $y \leq z$, $y \in L$. Hence, L is a left hyperideal of S. Similarly, R is a right hyperideal of S.

By Lemma 3.5, we obtain

$$L \cap R = (Q \cup Q \circ S] \cap (Q \cup S \circ Q] = Q.$$

This completes the proof. ∎

Theorem 3.7. *Let A be a nonempty subset of an ordered semihypergroup $(S, \circ, \leq)$. Then, the intersection of all quasi-hyperideals of S containing A is of the form*

$$(A)_q = (A \cup S \circ A] \cap (A \cup A \circ S].$$

Proof. Let Q be the intersection of all quasi-hyperideals $Q_i, i \in I$ of S containing A. By Theorem 3.3, $(A)_q$ is a quasi-hyperideal of S containing A. We have

$$(A \cup S \circ A] \text{ and } (A \cup A \circ S]$$

are left and right hyperideals of S, respectively. By Theorem 3.4,

$$(A \cup S \circ A] \cap (A \cup A \circ S]$$

is a quasi-hyperideal of S containing A. Thus,

$$(A)_q \subseteq (A \cup S \circ A] \cap (A \cup A \circ S].$$

For each $i \in I$, since $A \subseteq Q_i$ and Lemma 3.5, we have

$$(A \cup S \circ A] \cap (A \cup A \circ S] \subseteq (Q_i \cup S \circ Q_i] \cap (Q_i \cup Q_i \circ S] = Q_i.$$

Then,

$$(A)_q \supseteq (A \cup S \circ A] \cap (A \cup A \circ S].$$

Hence,

$$(A)_q = (A \cup S \circ A] \cap (A \cup A \circ S]. \quad \blacksquare$$

Definition 3.8. An ordered semihypergroup $(S, \circ, \leq)$ is said to be *quasi-simple* if it is the unique quasi-hyperideal.

Definition 3.9. An ordered semihypergroup $(S, \circ, \leq)$ is called an ordered hypergroup if $(S, \circ)$ is a hypergroup. In fact, for all a in S, we have $a \circ S = S \circ a = S$.

Corollary 3.10. *Every ordered hypergroup is quasi-simple.*

Theorem 3.11. *An ordered semihypergroup $(S, \circ, \leq)$ is quasi-simple if and only if $S = (x \circ S] \cap (S \circ x]$ for all x in S.*

Proof. Assume that S is quasi-simple and let $x \in S$. We will show that

$$R = (x \circ S]$$

is a right hyperideal of S. Let $y \in R \circ S$. Then, $y \in r \circ s$ for some r in R and s in s. Since $r \in R$, so $r \in x \circ s_1$ for some s_1 in S. Thus,

$$y \in r \circ s \subseteq x \circ s_1 \circ s \subseteq x \circ S \subseteq R.$$

Let $y \in R$ and $z \in S$ be such that $z \leq y$. Since $y \in R$, so $y \leq w$ for some w in $x \circ S$. Since $z \leq w$, so $z \in x \circ S \subseteq R$. Therefore, R is a right hyperideal of S. Similarly, $(S \circ x]$ is a left hyperideal of S. By Theorem 3.3,

$$S = (x \circ S] \cap (S \circ x].$$

Conversely, assume that $S = (x \circ S] \cap (S \circ x]$ for all x in S. Let Q be a quasi-hyperideal of S . If $q \in Q$, then by assumption and Lemma 3.5 we get

$$S = (q \circ S] \cap (S \circ q] \subseteq (Q \circ S] \cap (S \circ Q] = Q$$

Thus, $S = Q$. This completes the proof. $\blacksquare$

Definition 3.12. A quasi-hyperideal Q of an ordered semihypergroup $(S, \circ, \leq)$ is said to be *minimal* if it contains no proper quasi-hyperideal.

Theorem 3.13. *Every quasi-simple hyperideal of an ordered semihypergroup $(H, f, \leq)$ is minimal.*

Proof. Let Q be a quasi-simple hyperideal of H. Let Q' be a quasi-hyperideal of H such that $Q' \subseteq Q$. Since Q' is a quasi-hyperideal of Q, $Q' = Q$. Then, Q is a minimal quasi-hyperideal of H. ∎

The notion of intra-regular ordered semihypergroups in introduced by Kehayupulu et al. in [17] and Lee and Lee [25] gave some characterizations of the intra-regular ordered semihypergroups. Note that Definition 3.14 is a generalization of the concept of intra-regular ordered semigroups, since every ordered semigroup is an ordered semihypergroup.

Definition 3.14. An ordered semihypergroup $(S, \circ, \leq)$ is called *intra-regular* if for all s in S there exist x, y in S such that $s \leq x \circ s^2 \circ y$, where we means $s^2 = s \circ s$.

Example 5. Suppose that S is a nonempty set and $\leq$ be a partial order relation on S. We define the following hyperoperation on S:

$$\circ : S \times S \to \mathcal{P}^*(S), \text{ by } x \circ y = \{x, y\} \text{ for all } x, y \text{ in } S.$$

Then, $(S, \circ, \leq)$ is an ordered semihypergroup. Since for every s in S, $s \leq s \circ s^2 \circ s$, we conclude that S is intra-regular.

Example 6. The order semihypergroup $(S, \circ, \leq)$ defined in Example 3 is not intra-regular, since if we consider the element f, then $f^2 = a$ and for all x, y in S, we have $x \circ f^2 \circ y = x \circ a \circ y = a$ and $f \leq a$ is false.

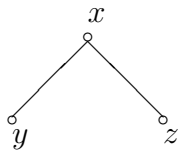
Example 7. We have $(S, \circ, \leq)$ is an ordered semihypergroup where the hyperoperation and the order relation are defined by:

$\circ$	x	y	z
x	x	$\{x, y\}$	$\{x, z\}$
y	x	$\{x, y\}$	$\{x, y\}$
z	x	$\{x, y\}$	z

$$\leq = \{(x, x), (y, y), (z, z), (y, x), (z, x)\}.$$

The covering relation and the figure of S are given by:

$$\prec = \{(y, x), (z, x)\}$$



Now, we have

$$x \leq x \circ x^2 \circ x = x \circ x \circ x = x,$$
$$y \leq x \circ y^2 \circ x = x \circ \{x, y\} \circ x = \{x, y\} \circ x = x,$$
$$z \leq z \circ z^2 \circ z = z \circ z \circ z = z.$$

Therefore, $(S, \circ, \leq)$ is intra-regular.

Let a be an element of an ordered semihypergroup $(S, \circ, \leq)$. We define

$$\begin{aligned} B(a) &: \text{the bi-hyperideal of } S \text{ generated by } a; \\ Q(a) &: \text{the quasi-hyperideal of } S \text{ generated by } a. \end{aligned}$$

Lemma 3.15. *Let a be an element of an ordered semihypergroup $(S, \circ, \leq)$.*

(1) $B(a) = (a \cup a^2 \cup a \circ S \circ a]$.

(2) $Q(a) = (a \cup ((a \circ S] \cap (S \circ a])]$.

Proof. (1) Clearly, $(a \cup a^2 \cup a \circ S \circ a] \neq \emptyset$. We have

$$\begin{aligned} &(a \cup a^2 \cup a \circ S \circ a] \circ S \circ (a \cup a^2 \cup a \circ S \circ a] \\ &\quad \subseteq ((a \cup a^2 \cup a \circ S \circ a) \circ S \circ (a \cup a^2 \cup a \circ S \circ a)] \\ &\quad \subseteq (a^2 \cup a \circ S \circ a] \\ &\quad \subseteq (a \cup a^2 \cup a \circ S \circ a]. \end{aligned}$$

Then, $(a \cup a^2 \cup a \circ S \circ a]$ is a subsemihypergroup of S. Now,

$$\begin{aligned} &(a \cup a^2 \cup a \circ S \circ a](a \cup a^2 \cup a \circ S \circ a] \\ &\quad \subseteq \quad ((a \cup a^2 \cup a \circ S \circ a) \circ (a \cup a^2 \cup a \circ S \circ a)] \\ &\quad \subseteq \quad (a \circ S \circ a] \\ &\quad \subseteq \quad (a \cup a^2 \cup a \circ S \circ a]. \end{aligned}$$

Hence, $(a \cup a^2 \cup a \circ S \circ a]$ is a bi-hyperideal of S, and thus $B(a) \subseteq (a \cup a^2 \cup a \circ S \circ a]$. The reverse inclusion is clear.

(2) Clearly, $(a \cup ((a \circ S] \cap (S \circ a])] \neq \emptyset$. We have

$$\begin{aligned} (a \cup ((a \circ S] \cap (S \circ a])] \circ S &\subseteq (a \cup (a \circ S]] \circ S \\ &\subseteq (a \circ S \cup (a \circ S] \circ S] \\ &\subseteq (a \circ S \cup (a \circ S]] \\ &= ((a \circ S]] \\ &= (a \circ S]. \end{aligned}$$

Similarly,

$$\begin{aligned} S \circ (a \cup ((a \circ S] \cap (S \circ a])] &\subseteq S \circ (a \cup (S \circ a]] \\ &\subseteq (S \circ a \cup S \circ (S \circ a]] \\ &\subseteq (S \circ a \cup (S \circ a]] \\ &= ((S \circ a]] \\ &= (S \circ a]. \end{aligned}$$

Hence,

$$\begin{aligned} (a \cup ((a \circ S] \cap (S \circ a])] \circ S \cap S \circ (a \cup ((a \circ S] \cap (S \circ a])] &\subseteq (a \circ S] \cap (S \circ a] \\ &\subseteq ((a \circ S] \cap (S \circ a]] \\ &\subseteq (a \cup ((a \circ S] \cap (S \circ a])]. \end{aligned}$$

Therefore, $(a \cup ((a \circ S] \cap (S \circ a])]$ is a quasi-hyperideal of S, and thus

$$Q(a) \subseteq (a \cup ((a \circ S] \cap (S \circ a])].$$

The reverse inclusion is clear. ∎

Theorem 3.16. *Let $(S, \circ, \leq)$ be an ordered semihypergroup. Then, the following statements are true:*

(1) *S is intra-regular if and only if for a bi-hyperideal B and a quasi-hyperideal Q of S, we have $B \cap Q \subseteq (S \circ B \circ Q \circ S]$;*

(2) *S is intra-regular if and only if for a bi-hyperideal B and a quasi-hyperideal Q of S, we have $B \cap Q \subseteq (S \circ Q \circ B \circ S]$.*

Proof. (1) Assume that S is intra-regular. Let $a \in B \cap Q$. Since S is intra-regular, there exist x, y in S such that $a \leq x \circ a^2 \circ y$. We have

$$\begin{aligned} a \leq x \circ a^2 \circ y &\leq x \circ a \circ (x \circ a^2 \circ y) \circ y \\ &= x \circ (a \circ x \circ a) \circ a \circ y^2 \\ &\subseteq S \circ (B \circ S \circ B) \circ Q \circ S \\ &\subseteq S \circ B \circ Q \circ S. \end{aligned}$$

Then, $B \cap Q \subseteq (S \circ B \circ Q \circ S]$.

Conversely, let $a \in S$. We consider a bi-hyperideal $B(a)$ and a quasi-hyperideal $Q(a)$. Then,

$$\begin{aligned} a \in B(a) \cap Q(a) &\subseteq (S \circ B(a) \circ Q(a) \circ S] \\ &= (S \circ (a \cup a^2 \cup a \circ S \circ a] \circ (a \cup ((a \circ S] \cap (S \circ a])] \circ S] \\ &\subseteq ((S \circ a \cup S \circ a^2 \cup S \circ a] \circ (a \cup (a \circ S]] \circ S] \\ &\subseteq ((S \circ a] \circ (a \cup (a \circ S]] \circ S] \\ &\subseteq ((S \circ a] \circ (a \circ S \cup (a \circ S^2]] \\ &\subseteq ((S \circ a^2 \circ S] \cup (S \circ a^2 \circ S^2]] \\ &\subseteq ((S \circ a^2 \circ S]] \\ &= (S \circ a^2 \circ S]. \end{aligned}$$

Thus, S is intra-regular.

(2) Assume that S is intra-regular. Let $a \in B \cap Q$. Since S is intra-regular, there exist x, y in S such that $a \leq x \circ a^2 \circ y$. By

$$\begin{aligned} a \leq x \circ a^2 \circ y &\leq x \circ (x \circ a^2 \circ y) \circ a \circ y \\ &= x^2 \circ a \circ (a \circ y \circ a) \circ y \\ &\subseteq S \circ Q \circ (B \circ S \circ B) \circ S \\ &\subseteq S \circ Q \circ B \circ S, \end{aligned}$$

it follows that $B \cap Q \subseteq (S \circ Q \circ B \circ S]$.

Conversely, let $a \in S$. We consider a bi-hyperideal $B(a)$ and a quasi-hyperideal $Q(a)$ of S; then

$$\begin{aligned} a \in B(a) \cap Q(a) &\subseteq (S \circ Q(a) \circ B(a) \circ S] \\ &= (S \circ (a \cup ((a \circ S] \cap (S \circ a])] \circ (a \cup a^2 \cup a \circ S \circ a] \circ S] \\ &\subseteq (S \circ (a \cup (S \circ a]] \circ (a \circ S \cup a^2 \circ S \cup a \circ S \circ a \circ S]] \\ &\subseteq ((Sa \cup (S^2 \circ a]] \circ (a \circ S]] \\ &\subseteq ((S \circ a](a \circ S]] \\ &\subseteq ((S \circ a^2 \circ S]] \\ &= (S \circ a^2 \circ S]. \end{aligned}$$

Thus, S is intra-regular. ∎

Theorem 3.17. *The following statements hold for an ordered semihypergroup $(S, \cdot, \leq)$:*

(1) *S is intra-regular if and only if for a left hyperideal L and a bi-hyperideal B of S, we have $L \cap B \subseteq (L \circ B \circ S]$;*

(2) *S is intra-regular if and only if for a right hyperideal R and a bi-hyperideal B of S, we have $B \cap R \subseteq (S \circ B \circ R]$.*

Proof. (1) Assume that S is intra-regular. Let L be a left hyperideal of S, and B a bi-hyperideal of S. If $a \in L \cap B$, then by S is intra-regular there exists x, y in S such that $a \leq x \circ a^2 \circ y$. We have

$$\begin{aligned} x \circ a^2 \circ y &\leq x \circ (x \circ a^2 \circ y) \circ a \circ y \\ &\subseteq x^2 \circ a \circ (a \circ y \circ a) \circ y \\ &\subseteq S \circ L \circ (B \circ S \circ B) \circ S \\ &\subseteq L \circ B \circ S. \end{aligned}$$

Then, $a \in (L \circ B \circ S]$. This proves that $L \cap B \subseteq (L \circ B \circ S]$.

Conversely, let $a \in S$. We consider a bi-hyperideal $B(a)$ and a left hyperideal $L(a)$ of S; then

$$\begin{aligned} L(a) \cap B(a) &\subseteq (L(a) \circ B(a) \circ S] \\ &\subseteq ((a \cup S \circ a] \circ (a \cup a^2 \cup a \circ S \circ a] \circ S] \\ &\subseteq ((a \cup S \circ a] \circ (a \circ S \cup a^2 \circ S \cup a \circ S \circ a \circ S]] \\ &\subseteq ((a \cup S \circ a] \circ (a \circ S]] \\ &\subseteq (a^2 \circ S \cup S \circ a^2 \circ S]. \end{aligned}$$

Thus, $a \leq t$ for some t in $a^2 \circ S \cup S \circ a^2 \circ S$. If $t \in a^2 \circ S$, then $a \leq a^2 \circ x$ for some x in S. Thus, we have

$$a \leq a^2 \circ x \leq a \circ (a^2 \circ x) \circ x \subseteq a \circ a^2 \circ x^2 \subseteq S \circ a^2 \circ S.$$

If $t \in S \circ a^2 \circ S$, it is obvious. Hence, S is intra-regular.

(2) Let R be a right hyperideal of S, and let B a bi-hyperideal of S. Let $a \in B \cap R$. Since S is intra-regular, there exist x, y in S such that $a \leq x \circ a^2 \circ y$, and so

$$\begin{aligned} a \leq x \circ a^2 \circ y &\leq x \circ a \circ (x \circ a^2 \circ y) \circ y \\ &= x \circ (a \circ x \circ a) \circ a \circ y^2 \\ &\subseteq S \circ (B \circ S \circ B) \circ R \circ S \\ &\subseteq S \circ B \circ R. \end{aligned}$$

Thus, $B \cap R \subseteq (S \circ B \circ R]$.

Conversely, to show that S is intra-regular, let $a \in S$. Consider a bi-hyperideal $B(a)$ and a right hyperideal $R(a)$ of S we have

$$\begin{aligned} B(a) \cap R(a) &\subseteq (S \circ B(a) \circ R(a)] \\ &= (S \circ (a \cup a^2 \cup a \circ S \circ a] \circ (a \cup a \circ S]] \\ &\subseteq ((S \circ a \cup S \circ a^2 \cup S \circ a \circ S \circ a] \circ (a \cup a \circ S]] \\ &\subseteq ((S \circ a] \circ (a \cup a \circ S]] \\ &\subseteq ((S \circ a^2 \cup S \circ a^2 \circ S]] \\ &= (S \circ a^2 \cup S \circ a^2 \circ S]. \end{aligned}$$

Then, $a \leq t$ for some t in $S \circ a^2 \cup S \circ a^2 \circ S$. If $t \in S \circ a^2$, then $a \leq x \circ a^2$ for some x in S; hence

$$a \leq x \circ a^2 \leq x \circ (x \circ a^2) \circ a = x^2 \circ a^2 \circ a \subseteq S \circ a^2 \circ S.$$

If $t \in S \circ a^2 \circ S$, it is obvious. Hence, S is intra-regular. ∎

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FILTER THEORY ON HYPER BE-ALGEBRAS

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Abstract. In this paper, we focus on investigating some types of hyper filters on hyper BE-algebras and discuss the relations among them. Also we construct quotient hyper BE-algebras and deliver some related results.

Keywords: hyper BE-algebra; hyper filter; upper sets; (positive) implicative hyper filter; quotient hyper BE-algebra.

1. Introduction

The hyper structure theory was introduced by Marty [6], at the 8th Congress of Scandinavian Mathematicians. Since then hyper structure theory has been intensively researched [1],[3],[4],[10],[11]. Recently, Radfar applied the hyper theory to the BE-algebras [5],[8],[9], and introduced the notion of a hyper BE-algebras [7], which is a generalization of a dual hyper K-algebra [2],[11].

The other part of this paper is organized as follows: in Section 2, we recall notions of hyper BE-algebras. In Section 3, we mainly introduce the notion of upper sets in hyper BE-algebras and give some important theorems about hyper filters. In Section 4, we introduce the notion of (positive) implicative hyper filters on hyper BE-algebras and discuss some relations among various hyper filters.

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Also, we give some characterizations of (positive) implicative hyper filters. In Section 5, we construct the quotient hyper BE-algebras and give out some related results.

2. Preliminaries

Definition 2.1 [5] An algebra $(X; *, 1)$ of type $(2, 0)$ is called a BE-algebra if it satisfies the following conditions: for any $x, y, z \in X$,

(BE1) $x * x = 1$;
(BE2) $x * 1 = 1$;
(BE3) $1 * x = x$;
(BE4) $x * (y * z) = y * (x * z)$.

In any BE algebra, the relation $\leq$ can be defined as follows: $x \leq y$ iff $x * y = 1$ for all $x, y \in X$.

Definition 2.2 [7] Let H be a nonempty set and $\circ : H \times H \to P^*(H)$ be a hyperoperation. Then $(H; \circ, 1)$ is called a hyper BE-algebra provided it satisfies the following axioms:

(HBE1) $x < 1$ and $x < x$;
(HBE2) $x \circ (y \circ z) = y \circ (x \circ z)$;
(HBE3) $x \in 1 \circ x$;
(HBE4) $1 < x$ implies $x = 1$,

for all $x, y \in H$, where the relation $<$ is defined by $x < y \Leftrightarrow 1 \in x \circ y$. For any two nonempty subsets A and B of H, $A < B$ means that there exist $a \in A$ and $b \in B$ such that $a < b$ and $A \circ B := \bigcup_{a \in A, b \in B} a \circ b$.

In addition, we write $A \ll B$ if for any $a \in A$, there exists $b \in B$ such that $a < b$.

Example 2.3 [7] Define the hyperoperation " $\circ$ " on R as follows:

$$x \circ y = \begin{cases} \{\mathrm{y}\} & \text{if } x = 1 \\ R & \text{otherwise.} \end{cases}$$

Then $(R; \circ, 1)$ is a hyper BE-algebra.

Example 2.4 [7] Let $H = \{1, a, b\}$. Define two hyperoperations " $\circ_1$ " and " $\circ_2$ " on H as follows:

$\circ_1$	1	a	b
1	$\{1\}$	$\{a\}$	$\{b\}$
a	$\{1\}$	$\{1, a, b\}$	$\{b\}$
b	$\{1\}$	$\{a, b\}$	$\{1, b\}$

$\circ_2$	1	a	b
1	$\{1\}$	$\{a, b\}$	$\{b\}$
a	$\{1\}$	$\{1, a, b\}$	$\{b\}$
b	$\{1, b\}$	$\{1, a, b\}$	$\{1, a, b\}$

Then $(H; \circ_1, 1)$ and $(H; \circ_2, 1)$ are hyper BE-algebras.

Example 2.5 [7] Let $H = \{1, a, b, c\}$. Define the hyperoperation " $\circ$ " on H as follows:

$\circ$	1	a	b	c
1	$\{1\}$	$\{a\}$	$\{b\}$	$\{c\}$
a	$\{1\}$	$\{1\}$	$\{a\}$	$\{b, c\}$
b	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
c	$\{1\}$	$\{1\}$	$\{a\}$	$\{1, b, c\}$

Then $(H; \circ, 1)$ is a hyper BE-algebra.

Proposition 2.6 [7] *Let $(H; \circ, 1)$ be a hyper BE-algebra. Then*
(1) $A \circ (B \circ C) = B \circ (A \circ C)$;
(2) $A < A$;
(3) $1 < A$ *implies* $1 \in A$;
(4) $x < y \circ x$, ;
(5) $x < y \circ z$ *implies* $y < x \circ z$;
(6) $x < (x \circ y) \circ y$;
(7) $z \in x \circ y$ *implies* $x < z \circ y$;
(8) $y \in 1 \circ x$ *implies* $y < x$, *for all* $x, y, z \in H$ *and* $A, B, C \subseteq H$.

Proposition 2.7 *Let $(H; \circ, 1)$ be a hyper BE-algebra. Then*
(1) $A \ll B \circ A$;
(2) $A < B$ *iff* $1 \in A \circ B$;
(3) $A \subseteq 1 \circ A$;
(4) $A \subseteq B$ *implies* $A < B$ *and* $A \ll B$;
(5) $A \ll B$ *and* $1 \in A$ *imply* $1 \in B$, *for all* $A, B, C \subseteq H$.

Proof. (3) Since $1 \circ A = \bigcup_{a \in A} 1 \circ a$ and $a \in 1 \circ a$, we have $A \subseteq 1 \circ A$.

(4) Let $x \in A$. Then $x \in B$. Hence $1 \in x \circ x$, which implies $1 \in A \circ B$. Using (4), the proof is finished.

(5) Since $A \ll B$ and $1 \in A$, then there exists $b \in B$ such that $1 < b$. By (HBE4), $b = 1$ and so $1 \in B$.

Definition 2.8 [7] A hyper BE-algebra $(H; \circ, 1)$ is called
(1) row hyper BE-algebra (for briefly, R-hyper BE-algebra), if $1 \circ x = \{x\}$, for all $x \in H$;
(2) column hyper BE-algebra (for briefly, C-hyper BE-algebra), if $x \circ 1 = \{1\}$, for all $x \in H$;
(3) diagonal hyper BE-algebra (for briefly, D-hyper BE-algebra), if $x \circ x = \{1\}$, for all $x \in H$;
(4) RC-hyper BE-algebra, if H is both R-hyper BE-algebra and C-hyper BE-algebra;
(5) RCD-hyper BE-algebra, if H is R-hyper BE-algebra,C-hyper BE-algebra and D-hyper BE-algebra.

3. Hyper subalgebras and (weak) hyper filters

Definition 3.1 [7] A nonempty subset S of a hyper BE-algebra H is said to be a hyper subalgebra of H, if $x \circ y \subseteq S$, for all $x, y \in S$.

Theorem 3.2 *Let $(H_1; \circ_1, 1_1)$ and $(H_2; \circ_2, 1_2)$ be two hyper BE-algebras and $H = H_1 \times H_2$. We define a hyperoperation $\circ$ on H: $(x_1, y_1) \circ (x_2, y_2) = (x_1 \circ x_2, y_1 \circ y_2)$ for all $(x_1, y_1), (x_2, y_2) \in H$, where for $A \subseteq H_1$ and $B \subseteq H_2$, we define $(A, B) := \{(x, y) : x \in A, y \in B\}$ and $1 := (1_1, 1_2)$. Then $(H; \circ, 1)$ is a hyper BE-algebra and we call it the product of hyper BE-algebras H_1 and H_2.*

Proof. Let $(x, y) \in H$. Since $(x, y) \circ (1_1, 1_2) = (x \circ_1 1_1, y \circ_2 1_2)$ and $1_1 \in x \circ_1 1_1$, $1_2 \in y \circ_2 1_2$, we have $1 \in (x, y) \circ 1$. That is, $(x, y) < 1$. The proof of $(x, y) < (x, y)$ is obtained by $x < x$ and $y < y$. Therefore (HBE1) holds.

Let $x = (x_1, y_1), y = (x_2, y_2), z = (x_3, y_3) \in H$. Then

$$\begin{aligned} x \circ (y \circ z) &= (x_1, y_1) \circ (x_2 \circ x_3, y_2 \circ y_3) \\ &= \bigcup\{(x_1, y_1) \circ (a, b) : a \in x_2 \circ_1 x_3, b \in y_2 \circ_2 y_3\} \\ &= \bigcup\{(x_1 \circ_1 a, y_1 \circ_2 b) : a \in x_2 \circ_1 x_3, b \in y_2 \circ_2 y_3\} \\ &= (x_1 \circ_1 (x_2 \circ x_3), y_1 \circ_2 (y_2 \circ y_3)) \\ &= \bigcup\{(c, d) : c \in x_1 \circ_1 (x_2 \circ x_3), d \in y_1 \circ_2 (y_2 \circ y_3)\} \\ &= \bigcup\{(c, d) : c \in x_2 \circ_1 (x_1 \circ x_3), d \in y_2 \circ_2 (y_1 \circ y_3)\} \\ &= (x_2 \circ_1 (x_1 \circ x_3), y_2 \circ_2 (y_1 \circ y_3)) \\ &= y \circ (x \circ z). \end{aligned}$$

So (HBE2) holds.

Let $(x, y) \in H$. Then $x \in 1_1 \circ_1 x, y \in 1_2 \circ_2 y$.

Note that $(1_1, 1_2) \circ (x, y) = (1_1 \circ_1 x, 1_2 \circ y)$, so that $(x, y) \in (1_1 \circ_1 x, 1_2 \circ y)$. That is, $(x, y) \in 1 \circ (x, y)$ and thus (HBE3) holds.

Let $(x, y) \in H$ and $(1_1, 1_2) < (x, y)$. Then $1_1 < x, 1_2 < y$ and this shows $x = 1_1, y = 1_2$. So we have $(x, y) = (1_1, 1_2) = 1$. It implies that (HBE4) holds and the proof is completed.

Definition 3.3 [7] Let F be a nonempty subset of hyper BE-algebra H and $1 \in F$. Then F is called

(1) a weak hyper filter of H if $x \circ y \subseteq F$ and $x \in F$ imply $y \in F$, for all $x, y \in H$;
(2) a hyper filter of H if $x \circ y \approx F$ and $x \in F$ imply $y \in F$, where $x \circ y \approx F$ means that $x \circ y \cap F \neq \emptyset$, for all $x, y \in H$.

Remark 3.4

(1) Every hyper filter of a hyper BE-algebra H is a weak hyper filter of H, but the converse needn't hold in general (see [7]).
(2) $\{1\}$ is not a hyper subalgebra of H. If H satisfies $1 \circ 1 = \{1\}$, then $\{1\}$ is a hyper subalgebra of H.
(3) If H is a R-hyper (C-hyper, D-hyper) BE-algebra, then $\{1\}$ is a hyper subalgebra of H.

(4) $\{1\}$ is a hyper filter. Furthermore, $\{1\}$ is a weak hyper filter by (1). In fact, Let $x \in \{1\}$ and $x \circ y \approx \{1\}$ i.e. $x \circ y \cap \{1\} \neq \emptyset$ for any $x, y \in H$. Then $x = 1$ and so $1 \circ y \cap 1 \neq \emptyset$. This implies $1 \in 1 \circ y$. Hence $1 < y$, which shows $y = 1$ by (HBE4). It follows that $\{1\}$ is a hyper filter.

(5) A hyper filter F is called proper if $F \neq H$; A hyper filter F is called non-trival if $F \neq \{1\}$ and $F \neq H$.

Theorem 3.5 [7] *Let F be a hyper filter of a hyper BE-algebra H. For all $x, y \in F$, if $x \in F$ and $x < y$, then $y \in F$.*

Theorem 3.6 *Let F be a nonempty subset of a hyper BE-algebra H. Then F is a hyper filter of H if and only if*

(1) $1 \in F$;

(2) $F < x \circ y$ *and* $x \in F$ *imply* $y \in F$, *for all* $x, y \in H$.

Proof. Assume that F is a hyper filter of H. Clearly, (1) holds. To prove (2), we let $x \in F$ and $F < x \circ y$. Then there exist $a \in F$ and $b \in x \circ y$ such that $a < b$. Again, $a \in F$, we get $b \in F$ by Theorem 3.5. This implies that $x \circ y \cap F \neq \emptyset$ and so $y \in F$. Conversely, assume that (1) and (2) hold. Now let $x \in F$ and $x \circ y \approx F$, for any $y \in F$. Then there exists $a \in H$ such that $a \in F$ and $a \in x \circ y$, which imply that $F < x \circ y$. Combing $x \in F$, by the assumption, we conclude $y \in F$. Therefore F is a hyper filter of H.

Theorem 3.7 *Let $\{F_i : i \in I\}$ be a family of nonempty subsets of a hyper BE-algebra H.*

(1) *If F_i is a (weak) hyper filter of H for all $i \in I$, then $\bigcap F_i$ is a (weak) hyper filter of H.*

(2) *If $\{F_i : i \in I\}$ is a chain of hyper filters of H for all $i \in I$, then $\bigcup F_i$ is a hyper filter of H.*

Proof. (1) Suppose that F_i is a hyper filter of H, for all $i \in I$. Clearly, $1 \in \bigcap F_i$. Let $\bigcap F_i < x \circ y$ and $x \in \bigcap F_i$ for any $x, y \in H$. From $\bigcap F_i \subseteq F_i$, it follows that $F_i < x \circ y$ for each $i \in I$. Note that $x \in F_i$ and F_i is a hyper filter, we get $y \in F_i$ for each $i \in I$. This implies that $y \in \bigcap F_i$ and so $\bigcap F_i$ is a hyper filter of H by Theorem 3.6. The proof the other case is similar.

(2) Suppose that $\{F_i : i \in I\}$ is a chain of hyper filters of H for all $i \in I$, and $x \in \bigcup F_i$ and $x \circ y \approx \bigcup F_i$, i.e., $x \circ y \cap \bigcup F_i \neq \emptyset$. Then there exist $m, n \in I$ such that $x \in F_m$ and $x \circ y \cap F_n \neq \emptyset$. The rest proof is similar to the above case. Therefore $\bigcup F_i$ is a hyper filter of H.

Theorem 3.8 *Let $(H_1 \times H_2; \circ, (1_1, 1_2))$ be the product of hyper BE-algebras $(H_1; \circ_1, 1_1)$ and $(H_2; \circ_2, 1_2)$. If F_1 and F_2 are (weak) hyper filters of H_1 and H_2, respectively, then $F_1 \times F_2$ is a (weak) hyper filter of $H_1 \times H_2$.*

Proof. Assume that F_1 and F_2 are hyper filters of H_1 and H_2, respectively. Since $1_1 \in F_1, 1_2 \in F_2$, then $(1_1, 1_2) \in F_1 \times F_2$. For any $(x_1, y_1), (x_2, y_2) \in H_1 \times H_2$, let $(x_1, y_1) \in F_1 \times F_2$ and $F_1 \times F_2 < (x_1, y_1) \circ (x_2, y_2)$. Then $F_1 \times F_2 < (x_1 \circ_1 x_2, y_1 \circ_2 y_2)$.

It follows that $F_1 < x_1 \circ_1 x_2$, $F_2 < y_1 \circ_2 y_2$. Since $x_1 \in F_1$ and $y_1 \in F_2$, we get that $x_2 \in F_1$ and $y_2 \in F_2$, which implies $(x_2, y_2) \in F_1 \times F_2$. This shows that $F_1 \times F_2$ is a hyper filter by Theorem 3.6. The proof of the other case is similar.

In any hyper BE-algebra H, let us denote by $[S)$ the least hyper filter of H containing S, called the hyper filter generated by S, where S is a nonempty subset of H. In particular, if $S = \{a\}$, we write $[\{a\}) = [a)$, called the principal hyper filter generated by the element a in H. In addition , we use $[F \cup \{x\})$ to denote the hyper filter generated by F and x, where $x \in H - F$. The following are some results about the generated hyper filters.

Theorem 3.9 *Let S be a nonempty subset of a hyper BE-algebra H. Then*

$$[S) \supseteq \{x \in H : 1 \in a_n \circ (\cdots (a_2 \circ (a_1 \circ x)) \cdots) \text{ for some } a_1, a_2, ..., a_n \in S\}.$$

Proof. Denote the right side of the result by B and let $x \in B$. Then there exist $a_1, a_2, ..., a_n \in S$ such that $1 \in a_n \circ (\cdots (a_2 \circ (a_1 \circ x)) \cdots)$. Hence $a_n \circ (\cdots (a_2 \circ (a_1 \circ x)) \cdots) \cap [S) \neq \emptyset$. Since $a_n \in S \subseteq [S)$ and $[S)$ is a hyper filter, we get $a_{n-1} \circ (\cdots (a_2 \circ (a_1 \circ x)) \cdots) \cap [S) \neq \emptyset$. Repeating the above argument, we verify $x \in [S)$ and thus $[S) \supseteq B$. This proof is finished.

Theorem 3.10 *Let F be a hyper filter of a hyper BE-algebra H and $a \in H - F$. Then*

$$[F \cup \{a\}) \supseteq \{x \in H : a \circ x \cap F \neq \emptyset\}.$$

Proof. Denote $B = \{x \in H : a \circ x \cap F \neq \emptyset\}$ and let $x \in B$. Then $a \circ x \cap F \neq \emptyset$, $a \in H - F$. Since $F \subseteq [F \cup \{a\})$, we have $a \circ x \cap [F \cup \{a\}) \neq \emptyset$. Considering that $a \in [F \cup \{a\})$ and $[F \cup \{a\})$ is a hyper filter, it follows that $x \in [F \cup \{a\})$, proving this theorem.

Corollary 3.11 *Let H be a hyper BE-algebra and $a \in H$. Then*

$$[a) \supseteq \{x \in H : 1 \in a \circ x\}.$$

In the following, we introduce the notion of a upper set of a hyper BE-algebra H, and get some results about the upper set of H. Moreover, by means of the upper set the hyper filter of H will be characterized well.

Definition 3.12 Let $(H; \circ, 1)$ be a hyper BE-algebra and let $x, y \in H$. We define an upper set of x and y by $\{z \in H : 1 \in x \circ (y \circ z)\}$, which is denoted by $U(x, y)$.

Proposition 3.13 *Let $(H; \circ, 1)$ be a hyper BE-algebra. Then $U(x, y)$ has the following properties:*
(1) $1, x, y \in U(x, y)$;
(2) $U(1, 1) = \{1\}$;
(3) $U(x, y) = U(y, x)$.

Proof. (1) Note that $1 \in y \circ 1$ and $x \circ (y \circ 1) = \bigcup\{x \circ b : b \in y \circ 1\}$, so that $1 \in x \circ (y \circ 1)$ and thus $1 \in U(x, y)$. By Proposition 2.3 (4), $x \in U(x, y)$ holds. By (HBE1), $1 \in y \circ y$ and $1 \in x \circ 1$. Again by $x \circ (y \circ y) = \bigcup\{x \circ b : b \in y \circ y\}$, it implies that $1 \in x \circ (y \circ y)$. This shows that $y \in U(x, y)$.

(2) Clearly, $\{1\} \subseteq U(1,1)$ by (1). Let $z \in U(1,1)$. Then $1 \in 1 \circ (1 \circ z)$. This means that there exists $b \in 1 \circ z$ such that $1 \in 1 \circ b$. By (HBE4), $b = 1$. This shows that there is unique element 1 such that $1 \in 1 \circ z$. Again using (HBE2), we get $z = 1$.

(3) By (HBE2).

From Remark 3.4 we see $U(1,1) = \{1\}$ is a hyper filter, but the following examples shows that $U(x,y)$ need not be a (weak) hyper filter of H in general.

Example 3.14 In Example 2.4, $(H; \circ_1, 1)$ is a hyper BE-algebra. One can calculate that the upper set $U(1,a) = \{1,a\}$ is a (weak) hyper filter; The upper set $U(1,b) = \{1,b\}$ is a weak hyper filter, but it is not a hyper filter, since $b \circ a \approx U(1,b)$, $b \in U(1,b)$, but $a \notin U(1,b)$.

Example 3.15 Let $(H; \circ, 1)$ be a hyper BE-algebra from the Example 2.5. It is easy to check that the upper set $U(1,c) = \{1,a,c\}$ is not a weak hyper filter, since $c \circ b = \{a\} \subseteq U(1,c)$, $c \in U(1,c)$, but $b \notin U(1,c)$. Of course, $U(1,c)$ is not also a hyper filter.

Theorem 3.16 *Let H be a hyper BE-algebra with a bottom element 0 and $x \in H$. Then*

$$U(x,0) = H = U(0,x).$$

Proof. Clearly, $U(x,y) \subseteq H$. Now Let $z \in H$. Since $1 \in 0 \circ z$, then by (HBE1) $1 \in x \circ (0 \circ z)$, which implies $z \in U(x,0)$. It follows that $U(x,0) = H$. Again using Proposition 3.13 (3), this proof is finished.

Theorem 3.17 *Let F be a nonempty subset of a hyper BE-algebra H. Then F is a hyper filter of H if and only if $U(x,y) \subseteq F$ for all $x, y \in F$.*

Proof. Assume that F is a hyper filter and $x, y \in F$. If $z \in U(x,y)$, then $1 \in x \circ (y \circ z)$. Since $1 \in F$, we have $F < x \circ (y \circ z)$. Hence there exists $a \in y \circ z$ such that $F < x \circ a$ and so $a \in F$. It shows that $F < y \circ z$ and $z \in F$. Therefore $U(x,y) \subseteq F$. Conversely, Let $U(x,y) \subseteq F$ for all $x, y \in F$. First, $1 \in U(x,y)$ by Proposition 3.13 (1), we have $1 \in F$. If $F < a \circ b$ and $a \in F$, then there exist $u \in F$ and $v \in a \circ b$ such that $1 \in u \circ v \subseteq u \circ (a \circ b)$. Hence $b \in U(u,a) \subseteq F$. This shows that F is a hyper filter by applying Theorem 3.6.

Theorem 3.18 *Let F be a hyper filter of a hyper BE-algebra H. Then*

$$F = \bigcup \{U(x,y) : x, y \in F\}.$$

Proof. Assume that F be a hyper filter of H. If $z \in \bigcup \{U(x,y) : x, y \in F\}$, then there exist $u, v \in F$ such that $z \in U(u,v)$. By Theorem 3.17, we get $z \in F$ and so $\bigcup \{U(x,y) : x, y \in F\} \subseteq F$. Now let $z \in F$. Then $1 \in z \circ z \subseteq z \circ (1 \circ z)$ by (HBE1) and (HBE3), which implies $z \in U(z,1)$. Thus $F \subseteq \bigcup \{U(z,1) : z \in F\} \subseteq \bigcup \{U(x,y) : x, y \in F\}$. We complete this proof.

Corollary 3.19 *Let F be a hyper filter of a hyper BE-algebra H. Then*
$$F = \bigcup\{U(z,1) : z \in F\}.$$

Proof. By the proof of Theorem 3.18, $F \subseteq \bigcup\{U(z,1) : z \in F\}$. Now we prove the converse holds. Let $z \in \bigcup\{U(z,1) : z \in F\}$. Then there exists $a \in F$ such that $z \in U(a,1)$, i.e., $1 \in a \circ (1 \circ z)$. Thus there exists $b \in 1 \circ z$ such that $a < b$. Since $a \in F$ and F is a hyper filter, we have $b \in F$ by Theorem 3.5. It follows that $1 \circ z \cap F \neq \emptyset$. Again applying that F is a hyper filter and $1 \in F$, we conclude that $z \in F$ and therefore this proof is finished.

4. (Positive) Implicative hyper filters

Definition 4.1 Let $(H; \circ, 1)$ be a hyper BE-algebra and F be a nonempty subset of H containing 1. Then for all $x, y, z \in H$, F is called:

(1) an implicative hyper filter of H if $x \circ (y \circ z) \cap F \neq \emptyset$ and $x \circ y \cap F \neq \emptyset$ imply $x \circ z \cap F \neq \emptyset$.

(2) a positive implicative hyper filter of H if $x \circ ((y \circ z) \circ y) \cap F \neq \emptyset$ and $x \in F$ imply $y \in F$.

Example 4.2 Let $H = \{1, a, b\}$. Define the hyperoperation " $\circ$" on H as follows:

$\circ$	1	a	b
1	$\{1\}$	$\{a\}$	$\{b\}$
a	$\{1\}$	$\{1\}$	$\{b\}$
b	$\{1\}$	$\{1, a\}$	$\{1\}$

Then $(H; \circ, 1)$ is a hyper BE-algebra[7]. We can check that $F = \{1, a\}$ is a hyper filter. Moreover, F is both positive implicative and implicative.

Example 4.3 Let $(H; \circ_1, 1)$ be a hyper BE-algebra given by Example 2.4. One can easily check that $F_1 = \{1, a\}$ is not a positive implicative hyper filter, since $1 \circ_1 ((b \circ_1 a) \circ_1 b) \cap F \neq \emptyset$ and $1 \in F_1$, but $b \notin F_1$; $F_2 = \{1, b\}$ is not an implicative hyper filter, since $1 \circ_1 (b \circ_1 a) \cap F_2 \neq \emptyset$, $1 \circ_1 b \cap F_2 \neq \emptyset$, but $1 \circ_1 a \cap F_2 = \emptyset$.

In the following, we discuss about the relations among various hyper filters.

Theorem 4.4 *Let $(H; \circ, 1)$ be a hyper BE-algebra and F is a nonempty subset of H.*

(1) *If F is a positive implicative hyper filter of H, then F is a hyper filter of H;*

(2) *If H is a R-hyper BE-algebra and F is an implicative hyper filter of H, then F is a hyper filter of H.*

Proof. (1) Clearly, $1 \in F$. Assume that $x \circ y \approx F$ and $x \in F$. This implies that $x \circ y \cap F \neq \emptyset$. Then there exists $a \in H$ such that $a \in x \circ y$ and $a \in F$. Hence by (HBE1), (HBE2) and Proposition 2.3 (1), we get that $a \in (y \circ y) \circ (x \circ y) = x \circ ((y \circ y) \circ y)$. Thus $x \circ ((y \circ y) \circ y) \cap F \neq \emptyset$. Since $x \in F$ and F is a positive implicative hyper filter of H, then $y \in F$ and so F is a hyper filter of H.

(2) Clearly, $1 \in F$. Let $x \circ y \approx F$ and $x \in F$. Then $x \circ y \subseteq 1 \circ (x \circ y)$ by Proposition 2.4 (3) and $x \circ y \cap F \neq \emptyset$. Hence $1 \circ (x \circ y) \cap F \neq \emptyset$. In addition, since H is a R-hyper BE-algebra, then $1 \circ x = x \in F$. This shows $1 \circ x \cap F \neq \emptyset$. Since F is an implicative hyper filter, we conclude $1 \circ y \cap F \neq \emptyset$. Therefore $y = 1 \circ y \in F$ and so this proof is completed.

In the following, we give out some remarks about Theorem 4.4.

Remark 4.5

(1) About the example of Theorem 4.4, one can see Example 4.2.

(2) The converse of Theorem 4.4 (1) is not true in general. Let $(H; \circ_1, 1)$ be a hyper BE-algebra given by Example 2.4. we can easily check that $F_1 = \{1, a\}$ is a hyper filter, but F_1 is not a positive implicative hyper filter, since $1 \circ_1 ((b \circ_1 a) \circ_1 b) \cap F \neq \emptyset$ and $1 \in F$, but $b \notin F_1$.

(3) The converse of Theorem 4.4 (2) is not true in general. Let $(H; \circ, 1)$ be a hyper BE-algebra given by Example 2.5. we know $\{1\}$ is a hyper filter by Remark 3.4, but $\{1\}$ is not an implicative hyper filter, since $a \circ (a \circ b) \cap \{1\} \neq \emptyset$ and $a \circ a \cap \{1\} \neq \emptyset$, but $a \circ b \cap \{1\} = \emptyset$.

(4) The condition that H is R-hyper BE-algebra is not necessary. Let $(H; \circ_2, 1)$ be a hyper BE-algebra given by Example 2.4. Clearly, $(H; \circ_2, 1)$ is not a R hyper BE-algebra. But $F = \{1, a\}$ is a hyper filter. Moreover, F is an implicative hyper filter.

(5) In general, An implicative hyper filter of a hyper BE-algebra is not a hyper filter. Let $H = \{1, a, b\}$ and define the hyperoperation " $\circ$ " on H as follows:

$\circ$	1	a	b
1	$\{1\}$	$\{a, b\}$	$\{b\}$
a	$\{1\}$	$\{1, a\}$	$\{1, b\}$
b	$\{1\}$	$\{1, a, b\}$	$\{1\}$

Then $(H; \circ, 1)$ is a C-hyper BE-algebra[7]. One can verify that $F = \{1, b\}$ is an implicative hyper filter. But F is not a hyper filter by Theorem 3.17, since the upper set $U(1, b) = \{1, a, b\} \nsubseteq F$.

Definition 4.6 Let $(H; \circ, 1)$ be a hyper BE-algebra and S be a nonempty subset of H. Then S is said to be $S_\circ-$reflexive if $x \circ y \cap S \neq \emptyset$ implies $x \circ y \subseteq S$ for all $x, y \in H$.

Definition 4.7 Let $(H; \circ, 1)$ be a hyper BE-algebra. Then H is said to be transitive if the following holds:

(HT) $y \circ z \ll (x \circ y) \circ (x \circ z)$ and $x \circ y \ll (y \circ z) \circ (x \circ z)$ for all $x, y, z \in H$.

Example 4.8 Let $H = \{1, a, b, c\}$ be a hyper BE-algebra defined by Example 2.5. Then H is not transitive. In fact, neither $b \circ c \ll (a \circ b) \circ (a \circ c)$ nor $a \circ b \ll (b \circ c) \circ (a \circ c)$ holds.

Example 4.9 Let $(H; \circ_1, 1)$ be a hyper BE-algebra given by Example 2.4. Then we can check that $H = \{1, a, b\}$ is transitive.

Proposition 4.10 *Let $(H; \circ, 1)$ be a transitive hyper BE-algebra. Then $y \circ A \ll (x \circ y) \circ (x \circ A)$ and $x \circ A \ll (A \circ z) \circ (x \circ z)$ for all $x, y, z \in H$.*

Proof. Straightforward by (HT).

Lemma 4.11 *Let $(H; \circ, 1)$ be a hyper BE-algebra and F be a hyper filter of H. Then for any nonempty subsets A, B of H, if $A \cap F \neq \emptyset$ and $A \ll B$, then $B \cap F \neq \emptyset$.*

Proof. Since $A \cap F \neq \emptyset$, then there exists $c \in H$ such that $c \in A$ and $c \in F$. Again as $A \ll B$, for the above $c \in F$ it follows that there exists $b \in B$ such that $c < b$. Since F is a hyper filter, we have $b \in F$ by Theorem 3.5 and so $B \cap F \neq \emptyset$.

Theorem 4.12 *Let $(H; \circ, 1)$ be a transitive hyper BE-algebra and F be a $F_\circ$-reflexive nonempty subset of H. Then F is a positive implicative hyper filter implies F is an implicative hyper filter.*

Proof. Let $x \circ (y \circ z) \cap F \neq \emptyset$ and $x \circ y \cap F \neq \emptyset$. First, we have $x \circ (y \circ z) = y \circ (x \circ z) \ll (x \circ y) \circ (x \circ (x \circ z))$ by (HBE2) and Proposition 4.10. Since $x \circ (y \circ z) \cap F \neq \emptyset$, then $(x \circ y) \circ (x \circ (x \circ z)) \cap F \neq \emptyset$ by Lemma 4.11. Hence there exist $a \in x \circ y \subseteq F$ by the $S_\circ-$reflexivity of F and $b \in x \circ (x \circ z)$ such that $a \circ b \cap F \neq \emptyset$. Since $a \in F$ and F is a hyper filter by Theorem 4.4, we get $b \in F$. This means $x \circ (x \circ z) \cap F \neq \emptyset$. Since H is transitive, we get $x \circ (x \circ z) \ll ((x \circ z) \circ z) \circ (x \circ z)$. Hence by Lemma 4.11 $((x \circ z) \circ z) \circ (x \circ z) \cap F \neq \emptyset$. As $((x \circ z) \circ z) \circ (x \circ z) \subseteq 1 \circ (((x \circ z) \circ z) \circ (x \circ z))$ by Proposition 2.4 (3), it follows that $1 \circ (((x \circ z) \circ z) \circ (x \circ z)) \cap F \neq \emptyset$. Then there exists $c \in x \circ z$ such that $1 \circ ((c \circ z) \circ c) \cap F \neq \emptyset$, combining $1 \in F$ and F is a positive implicative filter we obtain $c \in F$. This shows $x \circ z \cap F \neq \emptyset$. This finishes the proof.

Example 4.13 Let $H = \{1, a, b\}$ be a hyper BE-algebra given by Example 4.2. Then it is not difficult to check that H is transitive and $F = \{1, a\}$ is $F_\circ$-reflexive. Moreover, F is both implicative and positive implicative.

Note that In Theorem 4.12, the condition $F_\circ$-reflexivity is not necessary. In Remark 4.5 (5), we can easily check that $H = \{1, a, b\}$ is transitive, and $\{1, a\}$ is neither implicative nor positive implicative. However, $\{1, a\}$ is not $F_\circ$-reflexive, since $a \circ b \cap \{1, a\} \neq \emptyset$ doesn't imply $a \circ b \subseteq \{1, a\}$.

Now, we verify some results and give equivalent characterizations about the implicative and positive implicative hyper filter in H.

Theorem 4.14 *Let F be a hyper filter of a hyper BE-algebra H. Then the following conditions are equivalent:*
(1) *F is a positive implicative hyper filter;*
(2) *for all $x, y \in H$, $(x \circ y) \circ x \cap F \neq \emptyset$ implies $x \in F$;*
(3) *for all $x, y, u \in H$, $(x \circ y) \circ (u \circ x) \cap F \neq \emptyset$ and $u \in F$ imply $x \in F$.*

Proof. (1) $\Rightarrow$ (2) Assume that F is a positive implicative hyper filter of H. Let $(x\circ y)\circ x\cap F\neq\emptyset$ for any $x,y\in H$. Then there exists $a\in H$ such that $a\in(x\circ y)\circ x$ and $a\in F$. Since $a\in 1\circ a\subseteq 1\circ((x\circ y)\circ x)$, combing $1\in F$ and the definition of a positive implicative hyper filter, it follows that $x\in F$.

(2) $\Rightarrow$ (3) Assume that (2) holds. Let $(x\circ y)\circ(u\circ x)\cap F\neq\emptyset$ and $u\in F$ for any $x,y,u\in H$. Then there exists $a\in H$ such that $a\in(x\circ y)\circ(u\circ x)$ and $a\in F$. Hence $u\circ((x\circ y)\circ x)=(x\circ y)\circ(u\circ x)\cap F\neq\emptyset$. Combining $u\in F$ and F is a hyper filter, we get $(x\circ y)\circ x\cap F\neq\emptyset$ and thus $x\in F$ by (2).

(3) $\Rightarrow$ (1) Assume that (3) holds. Let $x\circ((y\circ z)\circ y)\cap F\neq\emptyset$ and $x\in F$ for all $x,y,z\in H$. Since $(y\circ z)\circ(x\circ y)=x\circ((y\circ z)\circ y)$, then $(y\circ z)\circ(x\circ y)\cap F\neq\emptyset$. Considering $x\in F$ and (3) holds, it follows that $y\in F$. This complete the proof.

Theorem 4.15 *Let F be a hyper filter of a hyper BE-algebra H. Then the following are equivalent:*
(1) *for all $x,y\in H$, $x\circ(x\circ y)\cap F\neq\emptyset$ implies $x\circ y\cap F\neq\emptyset$;*
(2) *for all $x,y,z\in H$, $z\circ(x\circ(x\circ y))\cap F\neq\emptyset$ and $z\in F$ imply $x\circ y\cap F\neq\emptyset$.*

Proof. (1) $\Rightarrow$ (2) Assume that (1) holds. Let $z\circ(x\circ(x\circ y))\cap F\neq\emptyset$ and $z\in F$ for any $x,y,z\in H$. Since F is a hyper filter, then $x\circ(x\circ y)\cap F\neq\emptyset$. Therefore $x\circ y\cap F\neq\emptyset$ by (2).

(2) $\Rightarrow$ (1) Assume that (2) holds. Let $x\circ(x\circ y)\cap F\neq\emptyset$ for any $x,y\in H$. Since $x\circ(x\circ y)\subseteq 1\circ(x\circ(x\circ y))$ by Proposition 2.4 (3), then $1\circ(x\circ(x\circ y))\cap F\neq\emptyset$. Combining $1\in F$, it implies that $x\circ y\cap F\neq\emptyset$ by (2). This proof is completed.

Theorem 4.16 *Let F be a hyper filter of a hyper BE-algebra H. If F is implicative, then*
(1) *for all $x,y\in H$, $x\circ(x\circ y)\cap F\neq\emptyset$ implies $x\circ y\cap F\neq\emptyset$;*
(2) *for all $x,y,z\in H$, $z\circ(x\circ(x\circ y))\cap F\neq\emptyset$ and $z\in F$ imply $x\circ y\cap F\neq\emptyset$.*

Proof. By Theorem 4.15, we only need to prove (1) holds. Assume that F is an implicative hyper filter. Let $x\circ(x\circ y)\cap F\neq\emptyset$ for any $x,y\in H$. Since $1\in x\circ x$ by (HBE1) and so $x\circ x\cap F\neq\emptyset$. Combing the definition of an implicative hyper filter, we obtain $x\circ y\cap F\neq\emptyset$ and the condition (1) follows.

Theorem 4.17 *Let $(H;\circ,1)$ be a transitive hyper BE-algebra, and F be a $F_\circ$-reflexive non-empty subset containing 1 of H. If for all $x,y,z\in H$, $z\circ(x\circ(x\circ y))\cap F\neq\emptyset$ and $z\in F$ imply $x\circ y\cap F\neq\emptyset$, then F is an implicative hyper filter of H.*

Proof. Let $x\circ(y\circ z)\cap F\neq\emptyset$ and $x\circ y\cap F\neq\emptyset$ for any $x,y,z\in H$. Since H is transitive and so by Proposition 4.10 we have $x\circ(y\circ z)=y\circ(x\circ z)\ll(x\circ y)\circ(x\circ(x\circ z))$. Since $x\circ(y\circ z)\cap F\neq\emptyset$, From Lemma 4.11 we obtain $(x\circ y)\circ(x\circ(x\circ z))\cap F\neq\emptyset$. As F is $S_\circ-$reflexive, this shows $x\circ y\subseteq F$. Hence there exists $a\in x\circ y$, i.e., $a\in F$ such that $a\circ(x\circ(x\circ z))\cap F\neq\emptyset$. Therefore, by the hypothesis, $x\circ z\cap F\neq\emptyset$ holds, which implies F is an implicative hyper filter.

From Theorem 4.15, 4.16 and 4.17 we can obtain the following important corollary.

Corollary 4.18 *Let $(H;\circ,1)$ be a transitive hyper BE-algebra, and F be a $F_\circ$-reflexive nonempty subset containing 1 of H. Then the following are equivalent:*
(1) *F is an implicative hyper filter;*
(2) *for all $x,y\in H$, $x\circ(x\circ y)\cap F\neq\emptyset$ implies $x\circ y\cap F\neq\emptyset$;*
(3) *for all $x,y,z\in H$, $z\circ(x\circ(x\circ y))\cap F\neq\emptyset$ and $z\in F$ imply $x\circ y\cap F\neq\emptyset$.*

Definition 4.19 A hyper BE-algebra $(H;\circ,1)$ is called RD-hyper BE-algebra, if H is both R-hyper BE-algebra and D-hyper BE-algebra.

Clearly, in Example 4.2, $(H;\circ,1)$ is a RD-hyper BE-algebra.

Lemma 4.20 *Let $(H;\circ,1)$ be a RD-hyper BE-algebra, and F be a hyper filter of H. For all $x,y,z\in H$, If $x\circ(y\circ z)\cap F\neq\emptyset$ implies $(x\circ y)\circ(x\circ z)\cap F\neq\emptyset$, then $z\circ(x\circ(x\circ y))\cap F\neq\emptyset$ and $z\in F$ imply $x\circ y\cap F\neq\emptyset$.*

Proof. Let $z\circ(x\circ(x\circ y))\cap F\neq\emptyset$ and $z\in F$ for any $x,y,z\in H$. Since $x\circ(x\circ(z\circ y))=z\circ(x\circ(x\circ y))$ by (HBE2), then $x\circ(x\circ(z\circ y))\cap F\neq\emptyset$. By the hypothesis, it follows that $(x\circ x)\circ(x\circ(z\circ y))\cap F\neq\emptyset$. Since $z\circ(x\circ y)=x\circ(z\circ y)=(x\circ x)\circ(x\circ(z\circ y))$. Thus $z\circ(x\circ y)\cap F\neq\emptyset$. Again as F be a hyper filter, it follows that $x\circ y\cap F\neq\emptyset$ and so this proof is finished.

By Theorem 4.18 and Lemma 4.20, the following theorem can be obtained immediately.

Theorem 4.21 *Let $(H;\circ,1)$ be a transitive RD-hyper BE-algebra, and F be a $F_\circ$-reflexive hyper filter of H. If for all $x,y,z\in H$, $x\circ(y\circ z)\cap F\neq\emptyset$ implies $(x\circ y)\circ(x\circ z)\cap F\neq\emptyset$, then F is an implicative hyper filter of H.*

In the following, we give an equivalent characterization of an implicative hyper filter of a hyper BE-algebra.

Theorem 4.22 *Let F be a nonempty subset of a hyper BE-algebra H. Then F is an implicative hyper filter of H if and only if $\{x\in H: a\circ x\cap F\neq\emptyset\}$ is a hyper filter of H for all $a\in H$.*

Proof. Suppose F is an implicative hyper filter and $F_a=\{x\in H: a\circ x\cap F\neq\emptyset\}$. Clearly, $1\in F_a$. Let $u\circ v\cap F_a\neq\emptyset$ and $u\in F_a$ for any $u,v\in F$. Then $a\circ(u\circ v)\cap F\neq\emptyset$ and $a\circ u\cap F\neq\emptyset$. Since F is an implicative hyper filter, we have $a\circ v\cap F\neq\emptyset$. This implies that $v\in F_a$. Thus F_a is a hyper filter. Conversely, assume that for all $a\in H$, $F_a=\{x\in H: a\circ x\cap F\neq\emptyset\}$ is a hyper filter of H. Let $m\circ(n\circ p)\cap F\neq\emptyset$ and $m\circ n\cap F\neq\emptyset$, for any $m,n,p\in H$. Then $n\circ p\cap F_m\neq\emptyset$ and $n\in F_m$. Since F_m is a hyper filter, then $p\in F_m$. That is, $m\circ n\cap F\neq\emptyset$. This shows that F is an implicative hyper filter.

Corollary 4.23 *Let F be an implicative hyper filter of a hyper BE-algebra H and $a\in H-F$. Then the hyper filter generated by F and a can be represented as $[F\cup\{a\})=\{x\in H: a\circ x\cap F\neq\emptyset\}$.*

Proof. According to Theorem 3.10, we have $[F \cup \{a\}) \supseteq \{x \in H : a \circ x \cap F \neq \emptyset\}$. Hence we need to prove $[F \cup \{a\}) \subseteq \{x \in H : a \circ x \cap F \neq \emptyset\}$. In fact, we only need to verify that $\{x \in H : a \circ x \cap F \neq \emptyset\}$ is a hyper filter containing $F \cup \{a\}$. Since $1 \in a \circ a$, then $a \circ a \cap F \neq \emptyset$, i.e., $a \in \{x \in H : a \circ x \cap F \neq \emptyset\}$. For each $x \in F$, we have $x \cap F \neq \emptyset\}$. Again as $x \ll a \circ x$ by Proposition 2.3 (4), it follows that $a \circ x \cap F \neq \emptyset$ from Lemma 4.11, which shows $F \subseteq \{x \in H : a \circ x \cap F \neq \emptyset\}$. By Theorem 4.22, we can know that $\{x \in H : a \circ x \cap F \neq \emptyset\}$ is a hyper filter. This proof is completed.

Corollary 4.24 *Let $(H; \circ, 1)$ be a hyper BE-algebra and let $a \in H$. Then $\{1\}$ is an implicative hyper filter of H if and only if $[a) = \{x \in H : 1 \in a \circ x\}$.*

In fact, if $H = \{1, a, b\}$ is a hyper BE-algebra given by Example 4.2, then $\{1\}$ is an implicative hyper filter and $[a) = \{a, 1\}$, $[b) = H$.

5. Quotient hyper BE-algebras

Definition 5.1 Let H be a hyper BE-algebra and θ be an equivalence relation on H.

(1) for any $A, B \subseteq H$, $A\theta B$ means for all $a \in A$ there exists $b \in B$ such that $a\theta b$ and for all $b \in B$ there exists $a \in A$ such that $a\theta b$;

(2) θ is called a congruence relation if for all $x, y, u, v \in H$, $x\theta y$ and $u\theta v$ imply $x \circ u\theta y \circ v$.

Let θ be a congruence relation on a hyper BE-algebra H. Denote $H/\theta = \{[x] : x \in H\}$, where $[x] = \{y \in H : y\theta x\}$. $\overline{\circ}$ and $<$ on H/F are defined by $[x]\overline{\circ}[y] = \{[a] : a \in x \circ y\}$ and $[x] < [y]$ iff $[1] \in [x]\overline{\circ}[y]$ for any $[x], [y] \in H/\theta$, respectively. Clearly, $x < y$ implies that $[x] < [y]$.

Theorem 5.2 *Let θ be a congruence relation on a hyper BE-algebra H. Then $(H/\theta; \overline{\circ}, [1])$ is a hyper BE-algebra, which is called as a quotient hyper BE-algebra with respect to θ.*

Proof First, we prove that $\overline{\circ}$ is well defined. Assume that $[x_1] = [x_2]$ and $[y_1] = [y_2]$, $x_1, x_2, y_1, y_2 \in H$. Then $x_1\theta x_2$ and $y_1\theta y_2$. Thus $x_1 \circ y_1\theta x_2 \circ y_2$. Let $[a] \in [x_1]\overline{\circ}[y_1]$ for some $a \in x_1 \circ y_1$. Then $[a] = [b]$, i.e., $a\theta b$ for some $b \in x_1 \circ y_1$. In addition, since $x_1 \circ y_1\theta x_2 \circ y_2$, then there exists $c \in x_2 \circ y_2$ such that $b\theta c$. Hence $a\theta c$ and so $[a] = [c] \in [x_2]\overline{\circ}[y_2]$. This shows that $[x_1]\overline{\circ}[y_1] \subseteq [x_2]\overline{\circ}[y_2]$. Similarly, we have $[x_2]\overline{\circ}[y_2] \subseteq [x_1]\overline{\circ}[y_1]$. It follows that $[x_1]\overline{\circ}[y_1] = [x_2]\overline{\circ}[y_2]$. Therefore $\overline{\circ}$ is well defined. Next, let $x, y, z \in H$.

(HBE1) Since $x < 1$ and $x < x$, we have $[x] < [1]$ and $[x] < [x]$.

(HBE2) Let $[a] \in [x]\overline{\circ}([y]\overline{\circ}[z])$. Then there exists $b \in y \circ z$, $[b] \in [y]\bar{[}z]$ such that $[a]_F = [x]\bar{[}b]$. Further, there exists $c \in x \circ b$, $[c] \in [x]\bar{[}b]$ such that $[a] = [c]$. Since $c \in x \circ b \subseteq x \circ (y \circ z) = y \circ (x \circ z)$, we have $[c] \in [y]\overline{\circ}([x]\overline{\circ}[z])$. Thus $[a] \in [y]\overline{\circ}([x]\overline{\circ}[z])$. This implies that $[x]\overline{\circ}([y]\overline{\circ}[z]) \subseteq [y]\overline{\circ}([x]\overline{\circ}[z])$. Similarly, we have $[y]\overline{\circ}([x]\overline{\circ}[z]) \subseteq [y]\overline{\circ}([x]\overline{\circ}[z])$. Therefore $[x]\overline{\circ}([y]\overline{\circ}[z]) = [y]\overline{\circ}([x]\overline{\circ}[z])$.

(HBE3) Since $x \in 1 \circ x$, we have $[x] \in [1]\overline{\circ}[x]$.

(HBE4) Let $[1] < [x]$. Then $[1] \in [1]\overline{\circ}[x]$, where $1 \in 1 \circ x$ and so $x = 1$. Hence $[x] = [1]$.

Definition 5.3 Let H be a hyper BE-algebra and F be a hyper filter of H. Define a binary relation θ_F as follows: $x\theta_F y$ iff $x \circ y \cap F \neq \emptyset$ and $y \circ x \cap F \neq \emptyset$.

Lemma 5.4 *Let H be a hyper BE-algebra and F be an implicative hyper filter of H. Then θ_F is an equivalence relation on H.*

Proof. Clearly, θ_F satisfies the reflectivity and the symmetry. Let $x, y, z \in H$ such that $x\theta_F y$ and $y\theta_F z$. Then $x \circ y \cap F \neq \emptyset$ and $y \circ z \cap F \neq \emptyset$. Hence by Proposition 2.7, $y \circ z \ll x \circ (y \circ z)$ and Lemma 4.11, we have $x \circ (y \circ z) \cap F \neq \emptyset$. Since F is an implicative hyper filter and $x \circ y \cap F \neq \emptyset$, we get that $x \circ z \cap F \neq \emptyset$. Similarly, $z \circ x \cap F \neq \emptyset$. Thus, the transitivity holds, which shows that θ_F is an equivalence relation.

Proposition 5.5 *Let H be a hyper BE-algebra and F be an implicative hyper filter of H. Then $[1]_F$ is a hyper filter of H and $F = [1]_F$.*

Proof. Let $x \in [1]_F$ and $x \circ y \cap [1]_F \neq \emptyset$ for any $x, y \in H$. Then $1 \circ x \cap F \neq \emptyset$ and $1 \circ (x \circ y) \cap F \neq \emptyset$. Since F is implicative, then $1 \circ y \cap F \neq \emptyset$. Again, $y \circ 1 \cap F \neq \emptyset$. This implies that $y\theta_F 1$, that is, $y \in [1]_F$. It follows that $[1]_F$ is a hyper filter. The following proves $F = [1]_F$. Let $x \in F$. Since $x \in 1 \circ x$ and $1 \in x \circ 1$, we have $1 \circ x \cap F \neq \emptyset$ and $x \circ 1 \cap F \neq \emptyset$. Thus $x\theta_F 1$, i.e., $x \in [1]_F$ and so $F \subseteq [1]_F$. On the other hand, let $x \in [1]_F$. Then $x\theta_F 1$ and so $1 \circ x \cap F \neq \emptyset$. Considering F is a hyper filter, we get $x \in F$. Hence $[1]_F \subseteq F$. This shows that $F = [1]_F$.

Definition 5.6 Let H be a hyper BE-algebra and F be an implicative hyper filter of H. Then F is called normal if the relation θ_F preserves the operation $\circ$, that is, $x\theta_F y$ and $u\theta_F v$ implies that $(x \circ u)\theta_F(y \circ v)$ for all $x, y, u, v \in H$.

Note that in the above definition F is a normal hyper filter if and only if the relation θ_F is a congruence relation. In this condition, we denote H/θ_F by $H/F = \{[x]_F : x \in H\}$.

Example 5.7 Let $H = \{1, a, b\}$ be a hyper BE-algebra given by Example 4.2. Then $F = \{1, a\}$ is a hyper filter. Moreover, F is an implicative hyper filter of H. One can check easily that $\theta_F = \{(1,1), (1,a), (a,1), (a,a), (b,b)\}$ is a congruence relation, so F is a normal hyper filter.

Theorem 5.8 *Let H be a hyper BE-algebra and F be a normal hyper filter of H. Then $(H/F; \overline{\circ}, [1]_F)$ is a hyper BE-algebra and $\{[1]_F\}$ is a hyper filter of H/F.*

Proof. By Theorem 5.2, we only prove that $\{[1]_F\}$ is a hyper filter of H/F. Assume that $A = [1]_F$ and $A \circ B \bigcap \{[1]_F\} \neq \emptyset$ for any $B \in H/F$. The following proves that $B = [1]_F$. Let $B \neq [1]_F$. Then $B = [x]_F$, where $x \in H - F$ and so $[1]_F \circ [x]_F \bigcap \{[1]_F\} \neq \emptyset$, i.e., $[1 \circ x]_F \bigcap \{[1]_F\} \neq \emptyset$. Hence there exists $a \in 1 \circ x$ such that $[a]_F = [1]_F$. This implies that $a \in F$. It follows that $1 \circ x \cap F \neq \emptyset$ and thus $x \in F$, which is a contraction. Therefore $B = [1]_F$. This proof is completed.

Example 5.9 Let $H = \{1, a, b\}$ be a hyper BE-algebra defined by Example 4.2. According to Example 5.7, $F = \{1, a\}$ is a normal hyper filter of H. The following constructs the quotient hyper BE-algebra H/F via F. Since $\theta_F = \{(1,1),(1,a),(a,1),(a,a),(b,b)\}$, then $[1]_F = F = \{1, a\} = [a]_F$, $[b]_F = \{b\}$, $H/F = \{[1]_F, [b]_F\}$ and

$\overline{\circ}$	$[b]_F$	$[1]_F$
$[b]_F$	$[1]_F$	$[1]_F$
$[1]_F$	$[b]_F$	$[1]_F$

Then $(H/F; \overline{\circ}, [1]_F)$ is a hyper BE-algebra.

Definition 5.10 A proper hyper filter F of a hyper BE-algebra H is said to be maximal if it is not a proper subset of any proper hyper filter of H.

Example 5.11 Let $(H; \circ_2, 1)$ be a hyper BE-algebra which is defined in Example 2.4, we can check that $\{1, a\}$ is a maximal hyper filter.

Clearly, a proper hyper filter F of H is maximal if and only if $[F \cup \{a\}) = H$ for each $a \in H - F$.

Definition 5.12 A hyper BE-algebra H is called to be

(1) bounded if H has a bottom element 0;
(2) simple if H has no non-trivial hyper filter.

Example 5.13 (1) Let $H = \{1, a, b\}$ be a hyper BE-algebra as in Example 4.2. Then H is a bounded hyper BE-algebra, where the bottom element $b := 0$.
(2) Let $H = \{1, a, b\}$. Define the hyperoperation " $\circ$ " on H as follows:

$\circ$	1	a	b
1	$\{1\}$	$\{a\}$	$\{b\}$
a	$\{1\}$	$\{1\}$	$\{1, a\}$
b	$\{1\}$	$\{1\}$	$\{1, a\}$

Then $(H; \circ, 1)$ is a simple hyper BE-algebra.

Remark 5.14 A hyper BE-algebra $(H; \circ, 1)$ is simple if and only if $\{1\}$ is a maximal hyper filter of H.

Theorem 5.15 *Let H be a bounded hyper BE-algebra and F be a proper hyper filter of H. Then there is a maximal hyper filter of H containing F.*

Proof. Denote $\Gamma = \{J \in F(L) : F \neq H \text{ and } F \subseteq J\}$. Then $\Gamma \neq \emptyset$ and $(\Gamma, \subseteq)$ is a poset, where $\subseteq$ is the inclusion relation. Let $\{J_i : i \in I\}$ is a chain in Γ. Then $\bigcup\{J_i : i \in I\}$ is a filter of L by Theorem 3.7 and it is the upper bound of this chain. Since $0 \notin J_i$, we have $0 \notin \bigcup\{J_i : i \in I\}$. Hence $\bigcup\{J_i : i \in I\} \in \Gamma$. By Zorn Lemma, there exists a maximal element M in Γ. Therefore M is indeed a maximal hyper filter.

By Corollary 4.23, the following conclusion holds.

Theorem 5.16 *Let $(H; \circ, 1)$ be a hyper BE-algebra and F be an implicative hyper filter of H. Then F is maximal if and only if $\{x \in H : a \circ x \cap F \neq \emptyset\} = H$ for $a \in H - F$.*

Theorem 5.17 *Let H be a hyper BE-algebra and F be an implicative hyper filter. Then F is maximal if and only if $x \circ y \cap F \neq \emptyset$ and $y \circ x \cap F \neq \emptyset$ for all $x, y \in H - F$.*

Proof. Assume that F is maximal. Then by Corollary 4.26, $[F \cup \{x\}) = \{x \in H : a \circ x \cap F \neq \emptyset\} = H$ for $a \in H - F$. We prove that $x \circ y \cap F \neq \emptyset$ and $y \circ x \cap F \neq \emptyset$ for all $x, y \in H - F$. Or else, without loss of any generality, there exist $x, y \in H - F$ such that $x \circ y \cap F = \emptyset$. Hence $y \notin [F \cup \{x\})$, which implies $y \notin H$. This is a contradiction. Conversely, let $x \in H - F$. Then $x \circ y \cap F \neq \emptyset$. Hence $y \in [F \cup \{x\})$. This shows that $H - F \subseteq [F \cup \{x\})$. Again, $F \subseteq [F \cup \{x\})$. It follows that $H = [F \cup \{x\})$, which implies that F is maximal.

Lemma 5.18 *Let H be a hyper BE-algebra and F be a normal hyper filter of H. If J is a hyper filter containing F, then J/F is also a hyper filter of H/F.*

Proof. Assume that $[x]_F \in J/F$ and $[x]_F \circ [y]_F \bigcap J/F \neq \emptyset$ for any $[x]_F, [y]_F \in H/F$. Then $x \in J$ and there exists $t \in x \circ y$ such that $t \in J$. Hence $x \circ y \cap J \neq \emptyset$. Since J is a hyper filter, we have $y \in J$ and thus $[y]_F \in J/F$. Therefore J/F is a hyper filter of H/F.

Theorem 5.19 *Let H be a bounded hyper BE-algebra and F be a non-trivial normal hyper filter of H. Then the following are equivalent:*

(1) *F is maximal;*
(2) *$H/F = \{H - F, F\}$, where $F = [1]_F$ and $H - F = [x]_F = \{y \in H - F : y \theta_F x, x \in H - F\}$;*
(3) *H/F is simple.*

Proof. (1) $\Leftrightarrow$ (2) Obvious by Theorem 5.17.

(2) $\Rightarrow$ (3) Assume that (2) holds and H/F is not simple. Then $\{[1]_F\}$ is not a maximal hyper filter of H/F by Remark 5.16. Hence according to Theorem 5.8 and Theorem 5.17, there exists a maximal hyper filter of H/F containing $\{[1]_F\}$, which contradicts to $H/F = \{H - F, F\}$. This shows that H/F is simple.

(3) $\Rightarrow$ (1) Assume that H/F is simple. Then $\{[1]_F\}$ is a maximal hyper filter of H/F and $\{[1]_F\} \neq H/F$. Hence there exists $[x]_F \in H/F$ such that $[x]_F \neq [1]_F$. This shows that $x \notin F$ and $F \neq H$. Let F be not a maximal hyper filter of H. Then by Theorem 5.8 and Theorem 5.17, there exists a maximal hyper filter J

of H satisfying $F \subset J \subset H$. Since J is a hyper filter of H, then J/F is a hyper filter of H/F by Lemma 5.18 and $\{[1]_F\} \subseteq J/F \subseteq H/F$. The following proves that $\{[1]_F\} \neq J/F$ and $J/F \neq H/F$. Let $\{[1]_F\} = J/F$. Then for all $x \in J$, we have $[x]_F = [1]_F$ and so $x \in F$, which implies that $J \subseteq F$. This is a contraction. Again, let $J/F = H/F$. Since $J \neq H$, then there exists $x \in H$ such that $x \notin J$. Hence $[x]_F \in H/F = J/F$ and so there exists $a \in J$ such that $[x]_J = [a]_J$. This shows that $x\theta_F a$ and thus $a \circ x \cap F \neq \emptyset$. Therefore $a \circ x \cap J \neq \emptyset$. Considering J is a hyper filter of H, we get $x \in J$, which is a contraction. It follows that $\{[1]_F\} \subset J/F \subset H/F$. This contradicts to the simplicity of H/F. The proof is complete.

6. Conclusions

In this paper, we investigate some types of hyper filters on hyper BE-algebras. we first give some results about hyper filters, then use upper sets to get some representations of hyper filter; Next, we introduce two types of hyper filters: (positive) implicative hyper filters, and focus on discussing the relations of various hyper filters. Also, we give some characterizations of (positive) implicative hyper filters. At last, we consider quotient hyper BE-algebras and get some results.

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A CLASS OF HIGH ACCURACY EXPLICIT DIFFERENCE SCHEMES FOR SOLVING THREE-DIMENSIONAL PARABOLIC EQUATIONS

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Abstract. A class of explicit difference schemes with high accuracy for solving three-dimensional parabolic equations are given. First, a difference approximation expression of $\frac{\partial u}{\partial t}$ is deduced at a special node (x_j, y_k, z_l, t_{n+1}); a class of explicit difference schemes are constructed by the method of undetermined coefficients, and appropriate parameters are chosen to endow the truncation error of schemes is $O\left(\Delta t^3 + \Delta x^4\right)$. In turn, the new difference schemes are proved to be stable if $r < \frac{1}{4}$ with the Fourier analysis method. Finally, the numerical experiment shows the numerical solutions of difference schemes and the exact solutions are matched and the difference schemes are effective.

Keywords: three-dimensional parabolic equations, explicit difference scheme, truncation error.

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1. Introduction

Partial differential equations (PDE) can be seen in many fields, such as physics, chemistry, biology, economics and financial projections, so PDE is an indispensable mathematical tool in the application of production and life. As we known, it is difficult to solve the PDE to gain its exact analytical solutions, so when solving

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the PDE problems, it is especially important to get its numerical solutions. By doing this, we can understand the mathematical meaning of these PDE problems.

This paper consider the following three-dimensional parabolic differential equation with initial and boundary conditions:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2},\ 0 < x, y, z < 1, t > 0, \tag{1}$$

$$u(x, y, z, 0) = \varphi(x, y, z),\ 0 \le x, y, z \le 1, \tag{2}$$

$$u(0, y, z, t) = f_1(y, z, t), u(1, y, z, t) = f_2(y, z, t),\ 0 \le y, z \le 1, 0 \le t \le T, \tag{3}$$

$$u(x, 0, z, t) = g_1(x, z, t), u(x, 1, z, t) = g_2(x, z, t),\ 0 \le x, z \le 1, 0 \le t \le T, \tag{4}$$

$$u(x, y, 0, t) = h_1(x, y, t), u(x, y, 1, t) = h_2(x, y, t),\ 0 \le x, y \le 1, 0 \le t \le T, \tag{5}$$

where $\varphi, f_1, f_2, g_1, g_2, h_1$ and h_2 are sufficiently smooth functions.

Equation (1) can describe various physical quantities, and can be encountered in many fields of engineering and science, such as seepage, diffusion, heat conduction and so on.

Various numerical finite difference schemes have been proposed to solve parabolic problems approximately. For multidimensional problems, The explicit difference scheme and implicit difference scheme are the common finite difference schemes. The implicit difference scheme has the advantage of good stability to the explicit one, but it is needed to solve different linear equations on each time layer which will cost to big computation. The alternating-direction implicit (ADI) difference scheme can overcome these disadvantages.

As we known, the ADI scheme is unconditional stable and only need to solve a sequence of tridiagonal linear systems [1], [2]. In recent years, there are many new methods which use ADI scheme to solve the high dimensional parabolic equations [3]–[8], some of them have the accuracy of $O(\Delta t^2 + \Delta x^4)$ [3], [6], [7], [8].

The explicit difference scheme has worse stability than the implicit difference scheme, but has the advantage of smaller amount of calculation. The general explicit scheme is the classical explicit scheme with the stability condition of $r \le \frac{1}{6}$, where $r = \frac{\Delta t}{\Delta x^2}$ is the mesh spacing ratio. Its deficiency is that its accuracy is not high, and its truncation error is $O(\Delta t + \Delta x^2)$ [1]. Recently, there has been a interest in the development and application of explicit difference schemes for the numerical solution of equation (1) [9]–[12]. In reference [11], Zeng constructed a class of explicit schemes with the truncation error of $O(\Delta t^2 + \Delta x^4)$ and the stability condition of $r \le \frac{1}{6}$. In reference [12], Ma also constructed a class of explicit schemes with the truncation error of $O(\Delta t^2 + \Delta x^4)$ and the stability condition of $r \le \frac{1}{4}$. This paper presents a class of explicit schemes for solving equation (1), the stability condition is $r < \frac{1}{4}$, and the truncation error is $O(\Delta t^3 + \Delta x^4)$, the schemes have higher accuracy than the above schemes.

The remainder of this paper is organized as follows. In Section 2, we establish a approximate difference scheme of one-order partial derivative $\frac{\partial u}{\partial t}$ at the node

of (x_j, y_k, z_l, t_{n+1}). In Section 3, we construct a class of a three-layer explicit difference schemes with the accuracy of $O(\Delta t^3 + \Delta x^4)$. In Section 4, by using the Fourier analysis method, it is proved that the difference schemes are stable when $r < \frac{1}{4}$. In Section 5, we compare the difference of exact solution and the scheme constructed in this paper with that in the reference [12], and compare the computational efficiency of the two difference schemes and the classical explicit scheme. The results shows that the difference schemes in this paper are effective.

2. Difference approximation

Suppose the solution $u(x, y, z, t)$ of equation (1) is sufficiently smooth, firstly establish the approximate expression of the first order partial derivative of the $u(x, y, z, t)$ at node (x_j, y_k, z_l, t_{n+1}) at time t, where $x_j = j\Delta x$, $y_k = k\Delta y$, $z_l = l\Delta z, t_n = n\Delta t$, let $u^n_{jkl} = u(x_j, y_k, z_l, t_n)$, by the Taylor expansion, we have

$$u^n_{jkl} = u^{n+1}_{jkl} - \frac{\Delta t}{1!}\left(\frac{\partial u}{\partial t}\right)^{n+1}_{jkl} + \frac{\Delta t^2}{2!}\left(\frac{\partial^2 u}{\partial t^2}\right)^{n+1}_{jkl} - \frac{\Delta t^3}{3!}\left(\frac{\partial^3 u}{\partial t^3}\right)^{n+1}_{jkl} + \cdots$$

$$u^{n-1}_{jkl} = u^{n+1}_{jkl} - \frac{(2\Delta t)}{1!}\left(\frac{\partial u}{\partial t}\right)^{n+1}_{jkl} + \frac{(2\Delta t)^2}{2!}\left(\frac{\partial^2 u}{\partial t^2}\right)^{n+1}_{jkl} - \frac{(2\Delta t)^3}{3!}\left(\frac{\partial^3 u}{\partial t^3}\right)^{n+1}_{jkl} + \cdots$$

we have

$$\left(\frac{\partial u}{\partial t}\right)^{n+1}_{jkl} = \frac{\frac{3}{2}u^{n+1}_{jkl} - 2u^n_{jkl} + \frac{1}{2}u^{n-1}_{jkl}}{\Delta t} + \frac{1}{3}\Delta t^2\left(\frac{\partial^3 u}{\partial t^3}\right)^{n+1}_{jkl} + \cdots$$

$$= \frac{\frac{3}{2}u^{n+1}_{jkl} - 2u^n_{jkl} + \frac{1}{2}u^{n-1}_{jkl}}{\Delta t} + O\left(\Delta t^2\right)$$

For convenience, denote the difference quotient to be

$$\eta_t u^{n+1}_{jkl} = \frac{\frac{3}{2}u^{n+1}_{jkl} - 2u^n_{jkl} + \frac{1}{2}u^{n-1}_{jkl}}{\Delta t}.$$

It is a finite difference approximation of $\left(\frac{\partial u}{\partial t}\right)^{n+1}_{jkl}$.

3. Construction of the difference scheme

Let Δt denote the step length of time and $\Delta x = \Delta y = \Delta z$ be the step length of space in the direction of x, y, z, respectively. We approximate equation (1) with the following parametered difference equation

$$\begin{aligned}&\eta_t u^{n+1}_{jkl} + \theta_1\Delta_t u^{n-1}_{jkl} + \theta_2\Delta_t\left(\frac{1}{2}\diamondsuit\right)u^{n-1}_{jkl}\\ &\quad = \frac{1}{\Delta x^2}\left[\left(\frac{\theta_3}{4}\square + \frac{\theta_4}{2}\diamondsuit\right)u^n_{jkl} + \left(\frac{\theta_5}{4}\square + \frac{\theta_6}{2}\diamondsuit\right)u^{n-1}_{jkl}\right],\end{aligned} \tag{6}$$

where u_{jkl}^n denotes the value of u at node $(j\Delta x, k\Delta y, l\Delta z, n\Delta t)$, $\Delta_t u_{jkl}^n = \dfrac{u_{jkl}^{n+1} - u_{jkl}^n}{\Delta t}$, respectively, and

$$\begin{aligned}
\Box u_{jkl}^n &= \left({}_x\Box + {}_y\Box + {}_z\Box\right) u_{jkl}^n, \\
\diamondsuit u_{jkl}^n &= \left({}_x\diamondsuit + {}_y\diamondsuit + {}_z\diamondsuit\right) u_{jkl}^n, \\
{}_x\Box u_{jkl}^n &= u_{j,k+1,l+1}^n + u_{j,k-1,l+1}^n + u_{j,k+1,l-1}^n + u_{j,k-1,l-1}^n - 4u_{jkl}^n \\
{}_x\diamondsuit u_{jkl}^n &= u_{j,k+1,l}^n + u_{j,k-1,l}^n + u_{j,k,l+1}^n + u_{j,k,l-1}^n - 4u_{jkl}^n
\end{aligned}$$

the rest can be inferred by analogy. $\theta_1 - \theta_6$ are parameters to be determined. A proper choice of undetermined parameters $\theta_1 - \theta_6$ can make difference equation (6) approach equation (1), and not only has truncation error with order as high as possible, but also has higher stability.

When the solution of equation (1) is smooth enough, we can get the following relation:

$$\frac{\partial^n}{\partial t^n}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)^m u = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)^{m+2n} u \tag{7}$$

Substituting the Taylor series of u at node $(j\Delta x, k\Delta y, l\Delta z, n\Delta t)$ into (6) and using relation (7), we can obtain

$$\begin{aligned}
&(1+\theta_1)\frac{\partial u}{\partial t} + \Delta t\left(1 - \frac{\theta_1}{2}\right)\frac{\partial^2 u}{\partial t^2} + \theta_2 \Delta x^2 \frac{\partial^2 u}{\partial t^2} + \frac{\Delta t^2}{6}(1+\theta_1)\frac{\partial^3 u}{\partial t^3} - \frac{\theta_2 \Delta x^2 \Delta t}{2}\frac{\partial^3 u}{\partial t^3} \\
&\quad = (\theta_3 + \theta_4 + \theta_5 + \theta_6)\frac{\partial u}{\partial t} - \Delta t(\theta_5 + \theta_6)\frac{\partial^2 u}{\partial t^2} \\
&\quad + \frac{\Delta t^2}{2}(\theta_5 + \theta_6)\frac{\partial^3 u}{\partial t^3} + \frac{\Delta x^2}{12}(\theta_3 + \theta_4 + \theta_5 + \theta_6)\frac{\partial^2 u}{\partial t^2} \\
&\quad + \frac{\Delta x^2}{12}\left[(\theta_3 + \theta_5) - 2(\theta_4 + \theta_6)\right]\left(\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^2 \partial z^2} + \frac{\partial^4 u}{\partial x^2 \partial z^2}\right) \\
&\quad - \frac{\Delta x^2 \Delta t}{12}(\theta_5 + \theta_6)\frac{\partial^3 u}{\partial t^3} \\
&\quad + \frac{\Delta x^2 \Delta t}{12}(2\theta_6 - \theta_5)\left(\frac{\partial^5 u}{\partial x^2 \partial y^2 \partial t} + \frac{\partial^5 u}{\partial y^2 \partial z^2 \partial t} + \frac{\partial^5 u}{\partial x^2 \partial z^2 \partial t}\right) \\
&\quad + O\left(\Delta t^3 + \Delta x^4\right)
\end{aligned}$$

In order to make the truncation error of scheme (6) getting to $O(\Delta t^3 + \Delta x^4)$, the following equation system should be available.

$$\begin{cases}
1 + \theta_1 = \theta_3 + \theta_4 + \theta_5 + \theta_6 \\
1 - \dfrac{\theta_1}{2} + \dfrac{\theta_2}{r} = -\theta_5 - \theta_6 + \dfrac{1}{12r}(\theta_3 + \theta_4 + \theta_5 + \theta_6) \\
(\theta_3 + \theta_5) - 2(\theta_4 + \theta_6) = 0 \\
\dfrac{1}{6}(1 + \theta_1) - \dfrac{\theta_2}{2r} = \left(\dfrac{1}{2} - \dfrac{1}{12r}\right)(\theta_5 + \theta_6) \\
2\theta_6 - \theta_5 = 0
\end{cases} \tag{8}$$

where $r = \dfrac{\Delta t}{\Delta x^2}$. Let $\theta_6 = \theta$, the solution of the above equation system is:

$$\begin{aligned} \theta_1 &= \frac{16r + 6\theta - 1}{2r+1}; \qquad \theta_2 = \frac{\theta - 12r^2\theta + 12r^2}{2(2r+1)}; \\ \theta_3 &= \frac{2(6r + \theta - 2r\theta)}{2r+1}; \quad \theta_4 = \frac{6r + \theta - 2r\theta}{2r+1}; \theta_5 = 2\theta. \end{aligned}$$

Substituting the above values into (6), we obtain the following single parameter three level explicit difference schemes with its truncation error getting to $O(\Delta t^3 + \Delta x^4)$.

$$\begin{aligned} &6(2r+1)\,u_{jkl}^{n+1} \\ &= [12(1-2\theta-4r) + 2r(6r+\theta-2r\theta)\,\square + (8r^2\theta - \theta + 2r\theta)\,\diamondsuit]\,u_{jkl}^{n} \\ &+ [(60r + 24\theta - 6) + 2r(2r+1)\,\theta\square + (-8r^2\theta + 12r^2 + 2r\theta + \theta)\,\diamondsuit]\,u_{jkl}^{n-1}. \end{aligned} \tag{9}$$

4. Analysis of stability

According to the Fourier method for analyzing stability, the two-level equation system equivalent to (9) is

$$\begin{cases} u_{jkl}^{n+1} = \dfrac{12(1-2\theta-4r) + 2r(6r+\theta-2r\theta)\,\square + (8r^2\theta - \theta + 2r\theta)\,\diamondsuit}{6(2r+1)} u_{jkl}^{n} \\ \qquad + \dfrac{60r + 24\theta - 6 + 2r(2r+1)\,\theta\square + (-8r^2\theta + 12r^2 + 2r\theta + \theta)\,\diamondsuit}{6(2r+1)} v_{jkl}^{n} \\ v_{jkl}^{n+1} = u_{jkl}^{n} \end{cases} \tag{10}$$

Let

$$u_{jkl}^{n} = U^n e^{i(j\theta + k\varphi + l\psi)}, v_{jkl}^{n} = V^n e^{i(j\theta + k\varphi + l\psi)} \tag{11}$$

where $i = \sqrt{-1}$. And through a simple calculation, we know

$$\square u_{jkl}^{n} = -4s_2 u_{jkl}^{n}, \diamondsuit u_{jkl}^{n} = -8s_1 u_{jkl}^{n} \tag{12}$$

where

$$\begin{aligned} s_1 &= \sin^2\frac{\theta}{2} + \sin^2\frac{\varphi}{2} + \sin^2\frac{\psi}{2} \in [0,3] \\ s_2 &= \sin^2\frac{\theta+\varphi}{2} + \sin^2\frac{\theta-\varphi}{2} + \sin^2\frac{\varphi+\psi}{2} + \sin^2\frac{\varphi-\psi}{2} + \sin^2\frac{\theta+\psi}{2} \\ &\quad + \sin^2\frac{\theta-\psi}{2} \in [0,6] \end{aligned}$$

Substituting (11) into (10) and using (12), we obtain

$$\begin{bmatrix} U^{k+1} \\ V^{k+1} \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} U^{k} \\ V^{k} \end{bmatrix} = \mathbf{G}(\mathbf{s_1}, \mathbf{s_2}) \begin{bmatrix} U^{k} \\ V^{k} \end{bmatrix}$$

where

$$g_{11} = \frac{12\left(1-2\theta-4r\right)-8r\left(6r+\theta-2r\theta\right)s_2-8\left(8r^2\theta-\theta+2r\theta\right)s_1}{6\left(2r+1\right)}$$
$$g_{12} = \frac{60r+24\theta-6-8r\left(2r+1\right)\theta s_2-8\left(-8r^2\theta+12r^2+2r\theta+\theta\right)s_1}{6\left(2r+1\right)},$$
$$g_{21} = 1,$$
$$g_{22} = 0$$

The characteristic equation of propagation matrix $\mathbf{G}\left(\mathbf{s_1},\mathbf{s_2}\right)$ is

$$\lambda^2 - g_{11}\lambda - g_{12} = 0. \tag{13}$$

Lemma 1. [13] *The two roots of real coefficient quadratic equation* (13) *are less than or equal to* 1 *in norm if and only if*

$$|g_{11}| \leq 1 - g_{12} \leq 2. \tag{14}$$

Lemma 2. [13] *The difference scheme* (9) *is stable, i.e., the family of matrices* $\mathbf{G^n}\left(\mathbf{s_1},\mathbf{s_2}\right), ((s_1,s_2) \in [0,3]\times[0,6], n = 1,2,...)$ *is uniformly bounded if and only if*

(1) $|\lambda_{1,2}| \leq 1$ ($\lambda_{1,2}$ *are roots of* (13))*;*

(2) (s_1,s_2) *which assures* $1 - g_{11}^2/4 = g_{11}^2 + 4g_{12} = 0$ *is not existent or not in the region of* $[0,3]\times[0,6]$.

Theorem 1. *A sufficient condition for scheme* (9) *being stable is*

$$r < \frac{1}{4}, \quad -3r < \theta \leq \frac{12r^2}{8r^2-2r-1}\cdot$$

Proof. If $g_{12} \neq -1, 1 - g_{11}^2/4 = g_{11}^2 + 4g_{12} = 0$ do not hold for any (s_1,s_2). By Lemmas 1 and 2, the stability conditions of scheme (9) become

$$-1 + g_{12} \leq g_{11} \leq 1 - g_{12} < 2.$$

From $g_{11} \leq 1 - g_{12}$, we have

$$\frac{12r+6-16r\left(3r+\theta\right)s_2-32r\left(3r+\theta\right)s_1}{6\left(2r+1\right)} \leq 1. \tag{15}$$

Hence

$$3r + \theta \geq 0. \tag{16}$$

Because $1 - g_{12} < 2$, we have

$$24\left(3r+\theta\right)-8r\left(2r+1\right)\theta s_2-8\left(-8r^2\theta+12r^2+2r\theta+\theta\right)s_1 > 0. \tag{17}$$

A sufficient condition which assures the above inequality hold is

$$\begin{cases} 3r+\theta>0 & (18)\\ \theta\le 0 & (19)\\ \left(8r^2-2r-1\right)\theta\ge 12r^2. & (20)\end{cases}$$

It is equivalent to

$$\begin{cases} 8r^2-2r-1<0\\ -3r<\theta\le\dfrac{12r^2}{8r^2-2r-1}\end{cases}$$

i.e.,

$$\begin{cases} r<\dfrac{1}{4} & (21)\\ \theta>-3r & (22)\\ \theta\le\dfrac{12r^2}{8r^2-2r-1} & (23)\end{cases}$$

Using $-1+g_{12}\le g_{11}$, we obtain

$$12r+6\theta-3-2r^2\left(2\theta-3\right)s_2-2\left(-8r^2\theta+6r^2+\theta\right)s_1\le 0. \tag{24}$$

A sufficient condition which assures the above inequality hold is

$$\begin{cases} 2\theta-3\le 0 & (25)\\ -8r^2\theta+6r^2+\theta\le 0 & (26)\\ 12r+6\theta-3-12r^2\left(2\theta-3\right)-6\left(-8r^2\theta+6r^2+\theta\right)\le 0 & (27)\end{cases}$$

Considering the conditions (19) and (21), conditions (25) and (27) hold. From (26) we have

$$\theta\le\frac{6r^2}{8r^2-1}. \tag{28}$$

When $r<\dfrac{1}{4}$, condition (23) is superior to (28).

By combining inequalities (21)–(23), we complete the proof. ∎

In particular, taking $\theta=\dfrac{12r^2}{8r^2-2r-1}$, we obtain a three-level explicit scheme as follows:

$$\begin{aligned} &6\left(2r+1\right)u_{jkl}^{n+1}\\ &=\left[-\frac{288r^2}{8r^2-2r-1}-48r+12+\left(12r^2+\frac{24r^3-48r^4}{8r^2-2r-1}\right)\square+\frac{12r^2\left(8r^2+2r-1\right)}{8r^2-2r-1}\diamondsuit\right]u_{jkl}^{n}\\ &+\left[60r-6+\frac{288r^2}{8r^2-2r-1}+\frac{24r^3\left(2r+1\right)}{8r^2-2r-1}\square\right]u_{jkl}^{n-1}\end{aligned} \tag{29}$$

5. Numerical experiment

Consider initial and boundary value problem as follows:

$$
(30)\quad \begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}, & (0 < x, y, z < 1, t > 0) \\ u(x, y, z, 0) = \sin(x + y + z), & (0 \le x, y, z \le 1) \\ u(0, y, z, t) = e^{-3t} \sin(y + z), \ u(1, y, z, t) = e^{-3t} \sin(1 + y + z), & \\ \qquad (0 \le y, z \le 1, t \ge 0) & \\ u(x, 0, z, t) = e^{-3t} \sin(x + z), \ u(x, 1, z, t) = e^{-3t} \sin(x + 1 + z), & \\ \qquad (0 \le x, z \le 1, t \ge 0) & \\ u(x, y, 0, t) = e^{-3t} \sin(x + y), \ u(x, y, 1, t) = e^{-3t} \sin(x + y + 1), & \\ \qquad (0 \le x, y \le 1, t \ge 0). & \end{cases}
$$

Taking $\Delta x = \Delta y = \Delta z = 0.1, \Delta t = r\Delta x^2 = r/100, r = 1/10, 1/8, 1/6, 1/5$. For convenience, we use the exact solution of (30) $u(x, y, z, t) = e^{-3t} \sin(x + y + z)$ to calculate the value of the first level u^1_{jkl}.

Table 1 shows the comparison of the exact solutions and the scheme (29) constructed in this paper and that in the reference [12] at the time strata n=200. From Table 1, one can easily see that numerical results of schemes (29) are completely identical with theoretical analysis, it has higher accuracy than the scheme in reference [12]. Table 2 shows the comparison of the efficiency of the scheme (29) constructed in this paper and the classical scheme one and that in the reference [12]. The result is tested by the unit of second. From Table 2, we can see that the three schemes' computational efficiency are similar. Among these three schemes, the classical explicit scheme has the highest computational efficiency, the second one is the scheme in this paper, and the lowest is the one constructed in the reference [12]. From the result of these two tables, we can see that the schemes constructed in this paper is a class of high accuracy and efficiency difference schemes.

Table 1 Comparison of calculating results of difference schemes with exact solution

r	result	(x, y, z)				
		$(0.1, 0.1, 0.1)$	$(0.3, 0.3, 0.3)$	$(0.5, 0.5, 0.5)$	$(0.7, 0.7, 0.7)$	$(0.9, 0.9, 0.9)$
$\frac{1}{10}$	exact solution	0.162 184 928	0.429 898 922	0.547 436 855	0.473 739 344	0.234 551 051
	scheme (29)	0.162 184 912	0.429 898 757	0.547 436 580	0.473 739 170	0.234 551 032
	reference [12]	0.162 184 903	0.429 898 666	0.547 436 427	0.473 739 073	0.234 551 021
$\frac{1}{8}$	exact solution	0.139 593 861	0.370 017 431	0.471 183 268	0.407 751 232	0.201 879 960
	scheme (29)	0.139 593 848	0.370 017 301	0.471 183 051	0.407 751 095	0.201 879 945
	reference [12]	0.139 593 844	0.370 017 253	0.471 182 970	0.407 751 043	0.201 879 940
$\frac{1}{6}$	exact solution	0.108 715 808	0.288 169 865	0.366 957 898	0.317 556 979	0.157 224 271
	scheme (29)	0.108 715 802	0.288 169 798	0.366 957 786	0.317 556 908	0.157 224 263
	reference [12]	0.108 715 792	0.288 169 789	0.366 957 777	0.317 556 899	0.157 224 253
$\frac{1}{5}$	exact solution	0.089 008 975	0.235 933 531	0.300 439 716	0.259 993 664	0.128 724 346
	scheme (29)	0.089 008 976	0.235 933 539	0.300 439 730	0.259 993 673	0.128 724 347
	reference [12]	0.089 008 960	0.235 933 366	0.300 439 441	0.259 993 490	0.128 724 327

Table 2 Comparison of the calculation efficiency among three kinds of difference schemes (unit:s)

difference scheme	r=0.1,n=100	r=0.2,n=100	r=0.1,n=200	r=0.2,n=200
classical explicit scheme	21.868 497	23.463 492	46.445 442	46.507 839
scheme (29)	22.362 985	23.629 915	46.671 319	46.826 391
reference [12] scheme	22.398 267	23.784 045	46.732 014	46.874 218

6. Conclusions

In this paper, we proposed a class of explicit difference schemes for solving three-dimensional parabolic problems. The stable character of the schemes are which has been verified by a discrete Fourier analysis. The schemes which proposed in this paper is fourth-order accurate in space and third-order accurate in time and allows a considerable saving in computing time. Numerical examples are given to test its high accuracy and to show its superiority over some other schemes in terms of accuracy and computational costs.

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HESITANT FUZZY IDEALS IN ABEL-GRASSMANN'S GROUPOIDS

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Abstract. In this paper, we introduced the concepts of hesitant fuzzy sets, hesitant fuzzy product, characteristic hesitant fuzzy set, hesitant fuzzy $\mathcal{AG}$-groupoids, hesitant fuzzy left (resp., right, two-sided) ideal, hesitant fuzzy bi-ideal, hesitant fuzzy interior ideal and hesitant fuzzy quasi-ideal in $\mathcal{AG}$-groupoids, their examples and basic properties are given. We also characterized regular, completely regular, weakly regular and quasi-regular $\mathcal{AG}$-groupoid by the properties of their hesitant fuzzy ideals.

Keywords: hesitant fuzzy sets, hesitant fuzzy $\mathcal{AG}$-groupoid, hesitant fuzzy left (resp., right, two-sided, quasi-, bi-, interior) ideal.

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Introduction

A fuzzy subset f of a given set S (or a fuzzy set in S) is, described as an arbitrary function $f : S \to [0, 1]$, where $[0, 1]$ is the usual closed interval of real numbers. This fundamental concept of fuzzy set, was introduced by Zadeh in 1965 [17], it has been widely studied, developed and successfully applied in various fields, such as multicriteria decision making (see [19], [20]), fuzzy logic and approximate reasoning [18], and pattern recognition [21]. In real cases, due to the fuzziness and uncertainty of decision making problems, the criteria's weights and evaluation values of alternatives can be inaccurate, uncertain, or incomplete. For the problems like those, fuzzy sets, especially fuzzy numbers, can provide good solutions. However, in fuzzy sets the membership degree of the element is represented by

a single value between zero and one, and a major drawback of fuzzy sets is that single values cannot convey information precisely.

In practice, the information regarding alternatives, when referring to a fuzzy set concept, may be incomplete, that is, the sum of membership and non-membership degree of an element in the universe can be less then one. The fuzzy set fails when it comes to managing the insufficient understanding of membership degrees. Thus Atanassov's intuitionistic fuzzy sets, interval valued fuzzy sets, and trapezoidal or triangular intuitionistic fuzzy sets, as the extensions of Zadeh's fuzzy sets, were introduced ([4], [5], [6]). Interval valued fuzzy sets, intuitionistic fuzzy sets has been widely applied in solving multicriteria decision making problems ([6], [7]).

Hesitant fuzzy set is a novel and recent extension of fuzzy sets that aims to model the uncertainty originated by the hesitation that might arise in the assignment of membership degrees of the elements to a fuzzy set. Despite the previous extensions overcome in different ways the managing of simultaneous sources of vagueness, Torra [10], introduced a useful generalization of the fuzzy set that is designed for situations in which it is difficult to determine the membership of an element to a set owing to ambiguity between a few different values. The hesitant fuzzy set permits the membership degree of an element to a set to be represented by a set of possible values between 0 and 1 (see [10], [11]). The hesitant fuzzy set therefore provides a more accurate representation of peoples hesitancy in stating their preferences over objects than the fuzzy set or its classical extensions. Hesitant fuzzy sets are very useful to deal with group decision making problems when experts have hesitation among several possible memberships for an element to a set. During the evaluating process in practice, however, these possible memberships may not be only crisp values in $[0, 1]$, but also interval values. Hesitant fuzzy sets have attracted the attention of many researchers in a short period of time because hesitant situations are very common in different real world problems and this new approach facilitates the management of uncertainty provoked by hesitation. A deep revision of the specialized literature shows the quick growth and applicability of hesitant fuzzy sets which have been extended from different points of view, (see [9], [11], [12], [13], [14], [15], [16]). Jun and Song applied the notion of hesitant fuzzy sets to MTL-algebras and EQ-algebras (see [1], [2]). Additionally, many operators for hesitant fuzzy sets and their extensions have been introduced to deal with such type of information in different applications where decision making has been the most remarkable one. Here the purpose of this paper is to deal with the algebraic structure of $\mathcal{AG}$-groupoids by applying the concept of hesitant fuzzy sets.

In this study, we applied the notion of hesitant fuzzy sets to $\mathcal{AG}$-groupoids. We introduced the notions of hesitant fuzzy product, characteristic hesitant fuzzy set, hesitant fuzzy $\mathcal{AG}$-groupoids, hesitant fuzzy left (resp., right, two-sided) ideal, hesitant fuzzy bi-ideal, hesitant fuzzy interior ideal and hesitant fuzzy quasi-ideal on $\mathcal{AG}$-groupoids and investigate several properties. Moreover, we consider characterizations of regular, completely regular, weakly regular and quasi-regular $\mathcal{AG}$-groupoids in terms of hesitant fuzzy ideals.

Preliminaries notes

The concept of an $\mathcal{AG}$-groupoid (Abel-Grassmann's groupoid) [3] was first given by M. A. Kazim and M. Naseeruddin in 1972.

A groupoid $(\mathcal{S}, .)$ is called an $\mathcal{AG}$-groupoid ($\mathcal{LA}$-semigroup in some articles [8]), if its elements satisfy left invertive law: $(ab)c = (cb)a$ for all $a, b, c \in \mathcal{S}$. In an $\mathcal{AG}$-groupoid medial law [3], $(ab)(cd) = (ac)(bd)$, holds for all a, b, c, $d \in \mathcal{S}$. It is also known that in an $\mathcal{AG}$-groupoid $\mathcal{S}$ with left identity, the paramedial law: $(ab)(cd) = (db)(ca)$, [8] holds for all a, b, c, $d \in \mathcal{S}$. If an $\mathcal{AG}$-groupoid contains left identity, then the following law [8] holds,

$$a(bc) = b(ac), \text{for all } a, b, c \in \mathcal{S}. \tag{1}$$

Let $\mathcal{S}$ be an $\mathcal{AG}$-groupoid and X, Y be subsets of $\mathcal{S}$. Then, the multiplication of X and Y is defined as follows:

$$XY = \{xy \in \mathcal{S} \setminus x \in X \text{ and } y \in Y\}.$$

Let $\mathcal{S}$ be an $\mathcal{AG}$-groupoid. By $\mathcal{AG}$-subgroupoid of $\mathcal{S}$ we means a non-empty subset A of $\mathcal{S}$ such that $A^2 \subseteq A$, and by a left (resp., right) ideal of $\mathcal{S}$ we mean a non-empty subset I of $\mathcal{S}$ such that $\mathcal{S}I \subseteq I$ ($I\mathcal{S} \subseteq I$). By two-sided ideals or simply ideal, we mean a non-empty subset of $\mathcal{S}$ which is both a left and a right ideal of $\mathcal{S}$. An $\mathcal{AG}$-subgroupoid B of $\mathcal{S}$ is called a bi-ideal of $\mathcal{S}$ if $(B\mathcal{S})B \subseteq B$. A non-empty subset B of $\mathcal{S}$ is called a generalized bi-ideal of $\mathcal{S}$ if $(B\mathcal{S})B \subseteq B$. A non-empty subset Q of $\mathcal{S}$ is called quasi-ideal of $\mathcal{S}$ if $Q\mathcal{S} \cap \mathcal{S}Q \subseteq Q$.

Obviously, every one sided ideal of an $\mathcal{AG}$-groupoid $\mathcal{S}$ is a quasi-ideal, every quasi-ideal is a bi-ideal and every bi-ideal is a generalized bi-ideal.

Sometimes, it is difficult to determine the membership of an element into a fixed set and which may be caused by a doubt among the set of different values. For the sake of better description of this situation, Torra [10], introduced the concept of hesitant fuzzy sets as a generalization of fuzzy sets. Torra defined hesitant fuzzy sets in terms of a function that returns a set of membership values for each element in the domain. This is formally defined in the next definition.

Definition 1 [10] Let $\mathcal{S}$ be a reference set. Then, we define hesitant fuzzy set on $\mathcal{S}$ in terms of a function h that when applied to $\mathcal{S}$ returns a finite subset of $[0, 1]$. A hesitant fuzzy set $\mathcal{H}$ can also be viewed as the following mathematical representation:

$$\mathcal{H} := \{(x, h(x)) \setminus \forall\, x \in \mathcal{S}\},$$

where $h(x)$ is the set of some different values in $[0, 1]$, denoting the possible membership degrees of the elements of $x \in \mathcal{S}$ to the set $\mathcal{H}$. The set of all hesitant fuzzy sets on $\mathcal{S}$ is denoted by

$$H(\mathcal{S}) = \left\{ \begin{array}{c} \{(x, h(x)) \setminus \forall x \in \mathcal{S}\} \setminus \text{ for any } x \in \mathcal{S} \text{ and any} \\ \text{function } h \text{ is the set of some different valuesin } [0, 1] \end{array} \right\}$$

Note that, throughout this paper, the set of all subsets of $[0, 1]$ will be denoted by $\mathcal{P}([0, 1]$.

Let g and h be two hesitant fuzzy sets on $\mathcal{S}$. Then, hesitant union $g \sqcup h$ and hesitant intersection $g \sqcap h$ of g and h are defined to be hesitant fuzzy sets on $\mathcal{S}$ as follows:

$$g \sqcup h :\to \mathcal{P}([0,1], x \mapsto g(x) \sqcup h(x) \text{ and } g \sqcap h :\to \mathcal{P}([0,1], x \mapsto g(x) \sqcap h(x),$$

respectively.

Definition 2 The hesitant fuzzy product of two hesitant fuzzy sets f and g on an $\mathcal{AG}$-groupoid $\mathcal{S}$ is defined to be a hesitant fuzzy set $f\widetilde{\circ}g$ on $\mathcal{S}$ which is given by

$$(f\widetilde{\circ}g)(x) = \begin{cases} \bigcup\limits_{x=yz} \{f(y) \cap g(z)\} & \text{if there exist } x,\, z \in \mathcal{S} \text{ such that } x = yz, \\ \phi & \text{otherwise.} \end{cases}$$

We denote by $H(\mathcal{S})$ the set of all hesitant fuzzy sets on $\mathcal{S}$.

Definition 3 Let f and g be the elements of $H(\mathcal{S})$. Then, f is called the subset of g, denoted by $f \sqsubseteq g$, if $f(x) \subseteq g(x)$ for all $x \in \mathcal{S}$.

Theorem 1 *For any hesitant fuzzy sets $f, g, h \in H(\mathcal{S})$, the following conditions holds.*

(i) $(f\widetilde{\circ}g)\widetilde{\circ}h = f\widetilde{\circ}(g\widetilde{\circ}h)$.

(ii) $f\widetilde{\circ}g \neq g\widetilde{\circ}f$, *in general.*

(iii) $f\widetilde{\circ}(g \sqcup h) = (f\widetilde{\circ}g) \sqcup (f\widetilde{\circ}h)$ *and* $(f \sqcup g)\widetilde{\circ}h = (f\widetilde{\circ}h) \sqcup (g\widetilde{\circ}h)$.

(iv) $f\widetilde{\circ}(g \sqcap h) = (f\widetilde{\circ}g) \sqcap (f\widetilde{\circ}h)$ *and* $(f \sqcap g)\widetilde{\circ}h = (f\widetilde{\circ}h) \sqcap (g\widetilde{\circ}h)$.

(v) *If* $f \sqsubseteq g$, *then* $f\widetilde{\circ}h \sqsubseteq g\widetilde{\circ}h$ *and* $h\widetilde{\circ}f \sqsubseteq h\widetilde{\circ}g$.

(vi) *If* $t, l \in H(\mathcal{S})$ *such that* $t \sqsubseteq f$ *and* $l \sqsubseteq g$, *then* $t\widetilde{\circ}l \sqsubseteq f\widetilde{\circ}g$.

Proposition 1 *Let $\mathcal{S}$ be an $\mathcal{AG}$-groupoid. Then, the set $(H(\mathcal{S}), \widetilde{\circ})$ forms an $\mathcal{AG}$-groupoid.*

Proof. It is obvious that $H(\mathcal{S})$ is closed. Let $f, g, h \in H(\mathcal{S})$. Let $a \in \mathcal{S}$. If a is not expressible as $a = bc$ for some $b, c \in \mathcal{S}$. Then, $((f\widetilde{\circ}g)\widetilde{\circ}h)(a) = ((h\widetilde{\circ}g)\widetilde{\circ}f)(a) = \phi$. Let for $a \in \mathcal{S}$ there exist $b, c \in \mathcal{S}$ such that $a = bc$. Then,

$$\begin{aligned}((f\widetilde{\circ}g)\widetilde{\circ}h)(a) &= \bigcup_{a=bc} \{(f\widetilde{\circ}g)(b) \cap h(c)\} = \bigcup_{a=bc} \left\{ \bigcup_{b=pq} \{f(p) \cap g(q)\} \cap h(c) \right\} \\ &= \bigcup_{a=(pq)c} \{f(p) \cap g(q) \cap h(c)\} = \bigcup_{a=(cq)p} \{h(c) \cap g(q) \cap f(p)\} \\ &= \bigcup_{a=wp} \left\{ \bigcup_{w=cq} \{h(c) \cap g(q)\} \cap f(p) \right\} = \bigcup_{a=wp} \{(h\widetilde{\circ}g)(w) \cap f(p)\} \\ &= ((h\widetilde{\circ}g)\widetilde{\circ}f)(a).\end{aligned}$$

Thus, $H(\mathcal{S})$ satisfies left invertive law. Hence, $(H(\mathcal{S}), \widetilde{\circ})$ is an $\mathcal{AG}$-groupoid. ∎

Proposition 2 *Let $\mathcal{S}$ be an $\mathcal{AG}$-groupoid, then $H(\mathcal{S})$ satisfies the medial law.*

Proof. Let f, g, h and k be any hesitant fuzzy sets of $H(\mathcal{S})$. Then, by using left invertive law,

$$(f\widetilde{\circ}g)\widetilde{\circ}(h\widetilde{\circ}k) = ((h\widetilde{\circ}k)\widetilde{\circ}g)\widetilde{\circ}f = ((g\widetilde{\circ}k)\widetilde{\circ}h)\widetilde{\circ}f = (f\widetilde{\circ}h)\widetilde{\circ}(g\widetilde{\circ}k). \qquad \blacksquare$$

Theorem 2 *Let $\mathcal{S}$ be an $\mathcal{AG}$-groupoid with left identity. Then, the following properties holds in $H(\mathcal{S})$.*

(i) $f\widetilde{\circ}(g\widetilde{\circ}h) = g\widetilde{\circ}(f\widetilde{\circ}h)$, *for all hesitant fuzzy sets* $f, g, h \in H(\mathcal{S})$.

(ii) $(f\widetilde{\circ}g)\widetilde{\circ}(h\widetilde{\circ}k) = (k\widetilde{\circ}h)\widetilde{\circ}(g\widetilde{\circ}f)$, *for all hesitant fuzzy sets* $f, g, h, k \in H(\mathcal{S})$.

Proof. (i) Let $a \in \mathcal{S}$. If $a \neq bc$ for some $b, c \in \mathcal{S}$, then $(f\widetilde{\circ}(g\widetilde{\circ}h))(a) = (g\widetilde{\circ}(f\widetilde{\circ}h))(a) = \phi$. If a is expressible as $a = bc$ for some $b, c \in \mathcal{S}$, then

$$\begin{aligned}
(f\widetilde{\circ}(g\widetilde{\circ}h))(a) &= \bigcup_{a=bc}\{f(b) \cap (g\widetilde{\circ}h)(c)\} = \bigcup_{a=bc}\left\{f(b) \cap \bigcup_{c=pq}\{g(p) \cap h(q)\}\right\} \\
&= \bigcup_{a=b(pq)}\{f(b) \cap g(p) \cap h(q)\} = \bigcup_{a=p(bq)}\{g(p) \cap f(b) \cap h(q)\} \\
&= \bigcup_{a=pw}\left\{g(p) \cap \bigcup_{w=bq}\{f(b) \cap h(q)\}\right\} = \bigcup_{a=pw}\{g(p) \cap (f\widetilde{\circ}h)(w)\} \\
&= (g\widetilde{\circ}(f\widetilde{\circ}h))(a).
\end{aligned}$$

Hence, $f\widetilde{\circ}(g\widetilde{\circ}h) = g\widetilde{\circ}(f\widetilde{\circ}h)$ for all $a \in \mathcal{S}$.

(ii) Let $a \in \mathcal{S}$ and $a \neq bc$ for some $b, c \in \mathcal{S}$, then $(f\widetilde{\circ}g)\widetilde{\circ}(h\widetilde{\circ}k) = (k\widetilde{\circ}h)\widetilde{\circ}(g\widetilde{\circ}f) = \phi$. Assume that $a \in \mathcal{S}$ is expressible as $a = bc$ for some $b, c \in \mathcal{S}$. Then,

$$\begin{aligned}
((f\widetilde{\circ}g)\widetilde{\circ}(h\widetilde{\circ}k))(a) &= \bigcup_{a=bc}\{(f\widetilde{\circ}g)(b) \cap (h\widetilde{\circ}k)(c)\} \\
&= \bigcup_{a=bc}\left\{\bigcup_{b=pq}\{f(p) \cap g(q)\} \cap \bigcup_{c=uv}\{h(u) \cap k(v)\}\right\} \\
&= \bigcup_{a=(pq)(uv)}\{f(p) \cap g(q) \cap h(u) \cap k(v)\} \\
&= \bigcup_{a=(vu)(qp)}\{k(p) \cap h(q) \cap g(u) \cap f(v)\} \\
&= \bigcup_{a=mn}\left\{\bigcup_{m=vu}\{k(p) \cap h(u)\} \cap \bigcup_{n=qp}\{g(q) \cap f(p)\}\right\} \\
&= \bigcup_{a=mn}\{(k\widetilde{\circ}h)(m) \cap (g\widetilde{\circ}f)(n)\} \\
&= ((k\widetilde{\circ}h)\widetilde{\circ}(g\widetilde{\circ}f))(a)
\end{aligned}$$

Hence, $(f\widetilde{\circ}g)\widetilde{\circ}(h\widetilde{\circ}k) = (k\widetilde{\circ}h)\widetilde{\circ}(g\widetilde{\circ}f)$ for all $a \in \mathcal{S}$. $\blacksquare$

Definition 4 For a non-empty subset A of an $\mathcal{AG}$-groupoid $\mathcal{S}$, define a map

$$\mathcal{H}_A : \mathcal{S} \to \mathcal{P}([0,1],\ x \mapsto \begin{cases} [0,1] & \text{if } x \in A, \\ \phi & \text{otherwise.} \end{cases}$$

Then, $\mathcal{H}_A$ is a hesitant fuzzy set on $\mathcal{S}$ and is called characteristic hesitant fuzzy set on $\mathcal{S}$. The hesitant fuzzy set $\mathcal{H}_\mathcal{S}$ is called identity hesitant fuzzy set on $\mathcal{S}$.

Theorem 3 *Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be hesitant fuzzy sets on $\mathcal{AG}$-groupoid $\mathcal{S}$, where A and B are non-empty subsets of $\mathcal{S}$. Then, the following properties hold.*

(i) *If $A \subseteq B$, then $\mathcal{H}_A \sqsubseteq \mathcal{H}_B$.*

(ii) $\mathcal{H}_A \sqcap \mathcal{H}_B = \mathcal{H}_{A\cap B}$.

(iii) $\mathcal{H}_A \widetilde{\circ} \mathcal{H}_B = \mathcal{H}_{AB}$.

Proof. (i) It is obvious.

(ii) Let $x \in \mathcal{S}$. If $x \in A \cap B$, then $x \in A$ and $x \in B$. Thus, we have

$$(\mathcal{H}_A \sqcap \mathcal{H}_B)(x) = \mathcal{H}_A(x) \cap \mathcal{H}_B(x) = [0,1] = \mathcal{H}_{A\cap B}(x).$$

If $x \notin A \cap B$, then $x \notin A$ or $x \notin B$. Hence, we have

$$(\mathcal{H}_A \sqcap \mathcal{H}_B)(x) = \mathcal{H}_A(x) \cap \mathcal{H}_B(x) = \phi = \mathcal{H}_{A\cap B}(x).$$

Therefore, $\mathcal{H}_A \sqcap \mathcal{H}_B = \mathcal{H}_{A\cap B}$.

(iii) For any $x \in \mathcal{S}$. Let $x \in AB$. Then, there exist $y \in A$ and $z \in B$ such that $x = yz$. Thus, we have

$$(\mathcal{H}_A \widetilde{\circ} \mathcal{H}_B)(x) = \bigcup_{x=yz} \{\mathcal{H}_A(y) \cap \mathcal{H}_B(z)\} \supseteq \mathcal{H}_A(y) \cap \mathcal{H}_B(z) = [0,1],$$

and so $(\mathcal{H}_A \widetilde{\circ} \mathcal{H}_B)(x) = [0,1]$. Since $x \in AB$, we get $\mathcal{H}_{AB}(x) = [0,1]$. Suppose $x \notin AB$. Then $x \neq yz$ for any $y \in A$ and $z \in B$. If $x = yz$ for some $y, z \in \mathcal{S}$, then $y \notin A$ or $z \notin B$. Hence,

$$(\mathcal{H}_A \widetilde{\circ} \mathcal{H}_B)(x) = \bigcup_{x=yz} \{\mathcal{H}_A(y) \cap \mathcal{H}_B(z)\} = \phi = \mathcal{H}_{AB}(x).$$

If $x \neq yz$ for all $x, y \in \mathcal{S}$, then

$$(\mathcal{H}_A \widetilde{\circ} \mathcal{H}_B)(x) = \phi = \mathcal{H}_{AB}(x).$$

In any case, we have $\mathcal{H}_A \widetilde{\circ} \mathcal{H}_B = \mathcal{H}_{AB}$. ∎

Definition 5 A hesitant fuzzy set h on $\mathcal{AG}$-groupoid $\mathcal{S}$ is called a hesitant fuzzy $\mathcal{AG}$-groupoid on $\mathcal{S}$ if it satisfies:

$$(\forall x, y \in \mathcal{S})\ (h(xy) \supseteq h(x) \cap h(y).$$

Example 1 Let $\mathcal{S} = \{1, 2, 3\}$ be an $\mathcal{AG}$-groupoid with the following Cayley multiplication table.

$\cdot$	1	2	3
1	1	1	1
2	1	1	3
3	1	1	1

Let h be a hesitant fuzzy set on $\mathcal{S}$ defined as follows:

$$h : \mathcal{S} \to \mathcal{P}([0,1],\ x \mapsto \begin{cases} (0.2, 0.8] & \text{if } x = 1, \\ [0.3, 0.7] & \text{if } x = 2, \\ [0.5, 0.7) & \text{if } x = 3. \end{cases}$$

It is easy to observe that h is a hesitant fuzzy $\mathcal{AG}$-groupoid on $\mathcal{S}$.

If $h(a) = [0,1]\ \forall\ a \in \mathcal{S}$, then it is easy to see that h is a hesitant fuzzy $\mathcal{AG}$-groupoid on $\mathcal{S}$. We denote such type of hesitant fuzzy $\mathcal{AG}$-groupoids by $\widetilde{\mathcal{H}}$. It is obvious that $\widetilde{\mathcal{H}} = \mathcal{H}_{\mathcal{S}}$ that is $\widetilde{\mathcal{H}}(a) = [0,1]\ \forall\ a \in \mathcal{S}$.

Lemma 1 *Let $\mathcal{S}$ be an $\mathcal{AG}$-groupoid and f be a hesitant fuzzy $\mathcal{AG}$-groupoid on $\mathcal{S}$. Then, $\widetilde{\mathcal{H}} \tilde{\circ} \widetilde{\mathcal{H}} \sqsubseteq \widetilde{\mathcal{H}}, f \tilde{\circ} \widetilde{\mathcal{H}} \sqsubseteq \widetilde{\mathcal{H}}, \widetilde{\mathcal{H}} \tilde{\circ} f \sqsubseteq \widetilde{\mathcal{H}}, f \sqcup \widetilde{\mathcal{H}} = \widetilde{\mathcal{H}}$ and $f \sqcap \widetilde{\mathcal{H}} \sqsubseteq f$.*

Proposition 3 *Let $\mathcal{S}$ be an $\mathcal{AG}$-groupoid with left identity. Then, $\widetilde{\mathcal{H}} \tilde{\circ} \widetilde{\mathcal{H}} = \widetilde{\mathcal{H}}$.*

Proof. Let $a \in \mathcal{S}$. Since every element $a \in \mathcal{S}$ can be expressed as $a = ea$, where e is the left identity in $\mathcal{S}$, so we have

$$(\widetilde{\mathcal{H}} \tilde{\circ} \widetilde{\mathcal{H}})(a) = \bigcup_{a=bc} \left\{ \widetilde{\mathcal{H}}(b) \cap \widetilde{\mathcal{H}}(c) \right\} \supseteq \left\{ \widetilde{\mathcal{H}}(e) \cap \widetilde{\mathcal{H}}(a) \right\} = [0,1].$$

Hence, $(\widetilde{\mathcal{H}} \tilde{\circ} \widetilde{\mathcal{H}})(a) = [0,1] = \widetilde{\mathcal{H}}(a)\ \forall\ a \in \mathcal{S}$. ∎

Theorem 4 *Let f be a hesitant fuzzy set on $\mathcal{AG}$-groupoid $\mathcal{S}$. Then, f is a hesitant fuzzy $\mathcal{AG}$-groupoid on $\mathcal{S}$ if and only if*

$$f \tilde{\circ} f \sqsubseteq f.$$

Proof. Let f be a hesitant fuzzy $\mathcal{AG}$-groupoid on $\mathcal{S}$. Let $a \in \mathcal{S}$.

If $a \neq bc$ for some $b, c \in \mathcal{S}$. Then, $(f \tilde{\circ} f)(a) = \phi \sqsubseteq f(a)$, implies that $(f \tilde{\circ} f)(a) \sqsubseteq f(a)$. Hence, $f \tilde{\circ} f \sqsubseteq f$.

If $a = bc$ for some $b, c \in \mathcal{S}$. Then, since f is a hesitant fuzzy $\mathcal{AG}$-groupoid on $\mathcal{S}$, so we have $(f \tilde{\circ} f)(a) = \bigcup\limits_{a=bc} \{f(b) \cap f(c)\} \subseteq \bigcup\limits_{a=bc} f(bc) = \bigcup\limits_{a=bc} f(a) = f(a)$. Hence, $f \tilde{\circ} f \sqsubseteq f$.

Conversely, suppose that $f \tilde{\circ} f \sqsubseteq f$. Let $a, b, c \in \mathcal{S}$ and $a = bc$, then we have $f(bc) = f(a) \supseteq (f \tilde{\circ} f)(a) = \bigcup\limits_{a=bc} \{f(b) \cap f(c)\} \supseteq f(b) \cap f(c)$. Hence, f is a hesitant fuzzy $\mathcal{AG}$-groupoid on $\mathcal{S}$. ∎

Theorem 5 *Let $\mathcal{S}$ be an $\mathcal{AG}$-groupoid and A be a non-empty subset of $\mathcal{S}$. Then, A is an $\mathcal{AG}$-subgroupoid of $\mathcal{S}$ if and only if $\mathcal{H}_A$ is a hesitant fuzzy $\mathcal{AG}$-groupoid on $\mathcal{S}$.*

Proof. Suppose that A is an $\mathcal{AG}$-subgroupoid of $\mathcal{S}$, then $AA \subseteq A$. Thus, we have $\mathcal{H}_A \widetilde{\circ} \mathcal{H}_A = \mathcal{H}_{AA} \sqsubseteq \mathcal{H}_A$. By Theorem 4, $\mathcal{H}_A$ is a hesitant fuzzy $\mathcal{AG}$-groupoid on $\mathcal{S}$.

Conversely, assume that $\mathcal{H}_A$ is a hesitant fuzzy $\mathcal{AG}$-groupoid on $\mathcal{S}$ and let $a \in AA$. Then, by using Theorem 4, we have $\mathcal{H}_A(a) \supseteq (\mathcal{H}_A \widetilde{\circ} \mathcal{H}_A)(a) = \mathcal{H}_{AA}(a) = [0,1] \Rightarrow \mathcal{H}_A(a) = [0,1] \Rightarrow a \in A$. Thus, $AA \subseteq A$. Hence, A is an $\mathcal{AG}$-subgroupoid of $\mathcal{S}$. ∎

Definition 6 A hesitant fuzzy set h on an $\mathcal{AG}$-groupoid $\mathcal{S}$ is called a hesitant fuzzy left (resp., right) ideal on $\mathcal{S}$ if it satisfies:

$$(\forall x, y \in \mathcal{S})\ (h(xy) \supseteq h(y)\ (\text{resp. }, h(xy) \supseteq h(x)).$$

A hesitant fuzzy set h on $\mathcal{S}$ is called a hesitant fuzzy two-side ideal on $\mathcal{S}$ if it is both hesitant fuzzy left ideal and hesitant fuzzy right ideal on $\mathcal{S}$.

Example 2 Let $\mathcal{S} = \{a, b, c, d\}$ be an $\mathcal{AG}$-groupoid with the following Cayley multiplication table.

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	a	b
d	a	a	b	c

Let h be a hesitant fuzzy set on $\mathcal{S}$ defined as follows:

$$h : \mathcal{S} \to \mathcal{P}([0,1],\ x \mapsto \begin{cases} (0.2, 0.8] & \text{if } x = a, \\ [0.3, 0.7] & \text{if } x = b, \\ [0.5, 0.7] & \text{if } x = c, \\ (0.5, 0.6) \cup (0.6, 0.7) & \text{if } x = d. \end{cases}$$

Then, it is easy to verify that h is a hesitant fuzzy two-sided ideal on $\mathcal{S}$.

Obviously, every hesitant fuzzy left (resp., right) ideal on $\mathcal{S}$ is a hesitant fuzzy $\mathcal{AG}$-groupoid on $\mathcal{S}$. But the converse is not true as seen in the following example.

Example 3 Let $\mathcal{S} = \{0, 1, 2, 3, 4, 5\}$ be an $\mathcal{AG}$-groupoid with the following Cayley multiplication table.

.	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	1	1	1	1
2	1	2	2	3	1	1
3	0	1	1	1	3	3
4	0	3	4	5	1	1
5	0	1	1	1	4	5

Let h be a hesitant fuzzy set on $\mathcal{S}$ defined as follows:

$$h : \mathcal{S} \to \mathcal{P}([0,1],\ x \mapsto \begin{cases} [0,1) & \text{if } x = 0, \\ \{0\} \cup [0.4,1) & \text{if } x = 1, \\ [0.5,0.8] \cup \{0.9\} & \text{if } x \in \{2,4\}, \\ [0.5,0.9] & \text{if } x = 3, \\ \{0,0.2\} \cup [0.5,1) & \text{if } x = 5. \end{cases}$$

Then, h is a hesitant fuzzy $\mathcal{AG}$-groupoid on $\mathcal{S}$. But it is not a hesitant fuzzy left ideal on $\mathcal{S}$, since $h(3.5) = h(3) = [0.5,0.9] \not\supseteq \{0,0.2\} \cup [0.5,1) = h(5)$.

Example 4 Let $\mathcal{S} = \{0,1,2,3,4,5\}$ be an $\mathcal{AG}$-groupoid with the Cayley multiplication table which is appeared in Example 3. Let g be a hesitant fuzzy set on $\mathcal{S}$ defined as follows:

$$g : \mathcal{S} \to \mathcal{P}([0,1],\ x \mapsto \begin{cases} [0.3,1] & \text{if } x \in \{0,1\}, \\ [0.4,0.9] & \text{if } x = 2, \\ (0.3,1) & \text{if } x = 3, \\ (0.4,0.9) & \text{if } x \in \{4,5\}. \end{cases}$$

Then, g is a hesitant fuzzy $\mathcal{AG}$-groupoid on $\mathcal{S}$. But it is not a hesitant fuzzy right ideal on $\mathcal{S}$, since $g(3.4) = g(2) = [0.4,0.9] \not\supseteq (0.3,1) = g(3)$.

Theorem 6 *Let f be a hesitant fuzzy set on $\mathcal{AG}$-groupoid $\mathcal{S}$. Then, f is a hesitant fuzzy left (resp., right) ideal on $\mathcal{S}$ if and only if $(\widetilde{\mathcal{H}} \widetilde{\circ} f \sqsubseteq f)$ (resp., $f \widetilde{\circ} \widetilde{\mathcal{H}} \sqsubseteq f$).*

Proof. Assume that f be a hesitant fuzzy left ideal on $\mathcal{S}$. Let $a \in \mathcal{S}$. If $a \neq bc$ for some $b, c \in \mathcal{S}$, then $(\widetilde{\mathcal{H}} \widetilde{\circ} f)(a) = \phi \sqsubseteq f(a)$, that is $\widetilde{\mathcal{H}} \widetilde{\circ} f \sqsubseteq f(a)$. If a is expressible as $a = bc$ for some $b, c \in \mathcal{S}$. Then, since f is a hesitant fuzzy left ideal on $\mathcal{S}$, we have $(\widetilde{\mathcal{H}} \widetilde{\circ} f)(a) = \bigcup\limits_{a=bc} \left\{ \widetilde{\mathcal{H}}(b) \cap f(c) \right\} \subseteq \{[0,1] \cup f(bc)\} = \bigcup\limits_{a=bc} \{[0,1] \cap f(a)\} = f(a)$. Hence, $\widetilde{\mathcal{H}} \widetilde{\circ} f \sqsubseteq f$.

Conversely, let $\widetilde{\mathcal{H}} \widetilde{\circ} f \sqsubseteq f$. Let there exist $b, c \in \mathcal{S}$ such that $a = bc$. Then, we have $f(bc) = f(a) \supseteq (\widetilde{\mathcal{H}} \widetilde{\circ} f)(a) = \bigcup\limits_{a=bc} \left\{ \widetilde{\mathcal{H}}(b) \cap f(c) \right\} \supseteq \widetilde{\mathcal{H}}(b) \cap f(c) = [0,1] \cap f(c) = f(c)$. Hence, f is a hesitant fuzzy left ideal on $\mathcal{S}$.

The other case can be seen in a similar way. ∎

Theorem 7 *For any non-empty subset A of an $\mathcal{AG}$-groupoid $\mathcal{S}$, the following are equivalent.*

(i) *A is left (resp., right, two-sided) ideal of $\mathcal{S}$.*

(ii) *The characteristic hesitant fuzzy set $\mathcal{H}_A$ on $\mathcal{S}$ is a left (resp., right, two-sided) ideal on $\mathcal{S}$.*

Proof. (i) $\Rightarrow$ (ii) : Let A be left ideal of an $\mathcal{AG}$-groupoid $\mathcal{S}$. For any $x, y \in \mathcal{S}$, if $y \notin A$ then $\mathcal{H}_A(y) = \phi \subseteq \mathcal{H}_A(xy)$. If $y \in A$, then $x, y \in A$, since A is left ideal of $\mathcal{S}$. Hence, $\mathcal{H}_A(xy) = [0,1] = \mathcal{H}_A(y)$. Therefore, $\mathcal{H}_A$ is a hesitant fuzzy left ideal on $\mathcal{S}$. Similarly, $\mathcal{H}_A$ is a hesitant fuzzy right and hesitant fuzzy two-sided ideal on $\mathcal{S}$ when A is a right ideal and two-sided ideal of $\mathcal{S}$.

(ii) $\Rightarrow$ (i) : Let $\mathcal{H}_A$ be a hesitant fuzzy left ideal on $\mathcal{S}$. Let $x \in \mathcal{S}$ and $y \in A$. Then $\mathcal{H}_A(y) = [0,1]$, and so $\mathcal{H}_A(xy) \supseteq \mathcal{H}_A(y) = [0,1]$, that is, $\mathcal{H}_A(xy) = [0,1]$. Thus $x, y \in A$ and therefore A is a left ideal of $\mathcal{S}$. Similarly, we can show that A is a right ideal and two-sided ideal of $\mathcal{S}$, when $\mathcal{H}_A$ is a hesitant fuzzy right ideal and hesitant fuzzy two-sided ideal on $\mathcal{S}$. ∎

Theorem 8 *Let $\mathcal{S}$ be an $\mathcal{AG}$-groupoid and f be a hesitant fuzzy set on $\mathcal{S}$. If f is a hesitant fuzzy left (resp., right, two-sided) ideal on $\mathcal{S}$, then f is a hesitant fuzzy $\mathcal{AG}$-groupoid on $\mathcal{S}$.*

Proof. Let f be a hesitant fuzzy left ideal on $\mathcal{AG}$-groupoid $\mathcal{S}$. Then, $f(ab) \supseteq f(b)$ $\forall\ a, b \in \mathcal{S}$ implies that $f(ab) \supseteq f(b) \supseteq f(a) \cap f(b)$. Hence, f is a hesitant fuzzy $\mathcal{AG}$-groupoid on $\mathcal{S}$.

Similarly, we can show that if f is a hesitant fuzzy (right, two-sided) ideal on $\mathcal{S}$, then f is a hesitant fuzzy $\mathcal{AG}$-groupoid on $\mathcal{S}$. ∎

Definition 7 Let $\mathcal{S}$ be an $\mathcal{AG}$-groupoid. A hesitant fuzzy $\mathcal{AG}$-groupoid f on $\mathcal{S}$ is called a hesitant fuzzy bi-ideal on $\mathcal{S}$ if it satisfies:

$$(\forall\ a, b, c \in \mathcal{S})\ (h((ab)c) \supseteq h(a) \cap h(c)).$$

Example 5 Let $\mathcal{S} = \{1, 2, 3\}$ be an $\mathcal{AG}$-groupoid with the following Cayley multiplication table.

$\cdot$	1	2	3
1	1	1	1
2	1	1	3
3	1	1	1

Let f be a hesitant fuzzy set on $\mathcal{S}$ defined as follows:

$$f : \mathcal{S} \to \mathcal{P}([0,1],\ x \mapsto \begin{cases} (0.3, 0.9] & \text{if } x = 1, \\ [0.4, 0.8] & \text{if } x = 2, \\ [0.6, 0.8] & \text{if } x = 3. \end{cases}$$

Then, it is easy to verify that f is a hesitant fuzzy bi-ideal on $\mathcal{S}$.

Theorem 9 *Let $\mathcal{S}$ be an $\mathcal{AG}$-groupoid and f be a hesitant fuzzy set on $\mathcal{S}$. Then, f is a hesitant fuzzy bi-ideal on $\mathcal{S}$ if and only if*

$$f \widetilde{\circ} f \sqsubseteq f \text{ and } (f \widetilde{\circ} \widetilde{\mathcal{H}}) \widetilde{\circ} f \sqsubseteq f.$$

Proof. Let f be a hesitant fuzzy bi-ideal on $\mathcal{AG}$-groupoid $\mathcal{S}$. Then, f is a hesitant fuzzy $\mathcal{AG}$-groupoid on $\mathcal{S}$ and by Theorem 4, we have $f \widetilde{\circ} f \sqsubseteq f$. Next let $a \in \mathcal{S}$. If a is not expressible as $a = bc$ for some $b, c \in \mathcal{S}$, then $((f \widetilde{\circ} \widetilde{\mathcal{H}}) \widetilde{\circ} f)(a) = \phi \sqsubseteq f$. If a is expressible as $a = bc$ for some $b, c \in \mathcal{S}$, then we have

$$\begin{aligned}((f\widetilde{\circ}\widetilde{\mathcal{H}})\widetilde{\circ}f)(a) &= \bigcup_{a=bc}\left\{(f\widetilde{\circ}\widetilde{\mathcal{H}})(b)\cap f(c)\right\} = \bigcup_{a=bc}\left\{\bigcup_{b=mn}\left\{f(m)\cap\widetilde{\mathcal{H}}(n)\right\}\cap f(c)\right\}\\ &= \bigcup_{a=bc}\left\{\bigcup_{b=mn}\{f(m)\cap[0,1]\}\cap f(c)\right\} = \bigcup_{a=(mn)c}\{f(m)\cap f(c)\}\\ &\subseteq \bigcup_{a=(mn)c} f((mn)c) = f_a.\end{aligned}$$

Hence, $(f\widetilde{\circ}\widetilde{\mathcal{H}})\widetilde{\circ}f \sqsubseteq f$.

Conversely, let $f\widetilde{\circ}f \sqsubseteq f$. Then, by Theorem 4, f is a hesitant fuzzy $\mathcal{AG}$-groupoid on $\mathcal{S}$. Now, since $(f\widetilde{\circ}\widetilde{\mathcal{H}})\widetilde{\circ}f \sqsubseteq f$, let $a,b,c\in\mathcal{S}$, then we have

$$\begin{aligned}f((ab)c) \;\supseteq\; ((f\widetilde{\circ}\widetilde{\mathcal{H}})\widetilde{\circ}f)((ab)c) &= \bigcup_{((ab)c)=pq}\left\{(f\widetilde{\circ}\widetilde{\mathcal{H}})(p)\cap f(q)\right\}\\ &= \bigcup_{((ab)c)=pq}\left\{\bigcup_{p=mn}\left\{f(m)\cap\widetilde{\mathcal{H}}(n)\right\}\cap f(q)\right\}\\ &= \bigcup_{((ab)c)=(mn)q}\{f(m)\cap f(q)\}\supseteq f(a)\cap f(c).\end{aligned}$$

This shows that, $f((ab)c)\supseteq f(a)\cap f(c)\ \forall\ a,b,c\in\mathcal{S}$. Hence, f is a hesitant fuzzy bi-ideal on $\mathcal{S}$. ∎

Theorem 10 *Let $\mathcal{S}$ be an $\mathcal{AG}$-groupoid. Then, every hesitant fuzzy left (resp., right, two-sided) ideal on $\mathcal{S}$ is a hesitant fuzzy bi-ideal on $\mathcal{S}$.*

Proof. Let f be a hesitant fuzzy left (resp., right, two-sided) ideal on $\mathcal{S}$. Then, by Theorem 8, f is a hesitant fuzzy left (resp., right, two-sided) ideal on $\mathcal{S}$. Also $f((ab)c)\supseteq f(c)\supseteq f(a)\cap f(c)\ \forall\ a,b,c\in\mathcal{S}$. Hence, f is a hesitant fuzzy bi-ideal on $\mathcal{S}$. ∎

Theorem 11 *Let A be a non-empty subset of an $\mathcal{AG}$-groupoid $\mathcal{S}$. Then, A is a bi-ideal of $\mathcal{S}$ if and only if the characteristic hesitant fuzzy set $\mathcal{H}_A$ is a hesitant fuzzy bi-ideal on $\mathcal{S}$.*

Definition 8 Let $\mathcal{S}$ be an $\mathcal{AG}$-groupoid and f be a hesitant fuzzy set on $\mathcal{S}$. Then, f is called a hesitant fuzzy interior ideal on $\mathcal{S}$, if

$$(\forall\ a,b,c\in\mathcal{S})\ (f((ab)c)\supseteq f(b)).$$

Example 6 Let $\mathcal{S}=\{p,q,r\}$ be an $\mathcal{AG}$-groupoid with the following Cayley multiplication table.

$\cdot$	p	q	r
p	r	r	q
q	q	q	q
r	q	q	q

Let f be a hesitant fuzzy set on $\mathcal{S}$ defined as follows:

$$f : \mathcal{S} \to \mathcal{P}([0,1],\ x \mapsto \begin{cases} [0.5, 0.7) & \text{if } x = p, \\ [0.3, 0.7] & \text{if } x = q, \\ (0.3, 0.6] & \text{if } x = r. \end{cases}$$

Then, it is easy to verify that f is a hesitant fuzzy interior on $\mathcal{S}$.

Theorem 12 *A heitant fuzzy set f on $\mathcal{AG}$-groupoid $\mathcal{S}$ is a hesitant fuzzy interior ideal on $\mathcal{S}$ if and only if*

$$(\widetilde{\mathcal{H}} \widetilde{\circ} f) \widetilde{\circ} \widetilde{\mathcal{H}} \sqsubseteq f.$$

Proof. Let $a \in \mathcal{S}$ and let f be a hesitant fuzzy interior ideal on $\mathcal{S}$. If a is not expressible as $a = bc$ for some $b, c \in \mathcal{S}$, then $((\widetilde{\mathcal{H}} \widetilde{\circ} f) \widetilde{\circ} \widetilde{\mathcal{H}})(a) = \phi \subseteq f(a)$. Thus, $(\widetilde{\mathcal{H}} \widetilde{\circ} f) \widetilde{\circ} \widetilde{\mathcal{H}} \sqsubseteq f$. If a is expressible as $a = bc$ for some $b, c \in \mathcal{S}$, then we have

$$\begin{aligned} ((\widetilde{\mathcal{H}} \widetilde{\circ} f) \widetilde{\circ} \widetilde{\mathcal{H}})(a) &= \bigcup_{a=bc} \left\{ (\widetilde{\mathcal{H}} \widetilde{\circ} f)(b) \cap \widetilde{\mathcal{H}}(c) \right\} = \bigcup_{a=bc} \left\{ \bigcup_{b=pq} \left\{ \widetilde{\mathcal{H}}(p) \cap f(q) \right\} \cap \widetilde{\mathcal{H}}(c) \right\} \\ &= \bigcup_{a=bc} \left\{ \bigcup_{b=pq} \{[0,1], f(q)\} \cap [0,1] \right\} \subseteq \bigcup_{a=bc} \left\{ \bigcup_{b=pq} \{[0,1], f((pq)c)\} \cap [0,1] \right\} \\ &= \bigcup_{a=(pq)c} \{[0,1], f((pq)c)\} \cap [0,1] = f(a). \end{aligned}$$

Also if $b \neq pq$, then $(\widetilde{\mathcal{H}} \widetilde{\circ} f)(b) = \phi$, and so $((\widetilde{\mathcal{H}} \widetilde{\circ} f) \widetilde{\circ} \widetilde{\mathcal{H}})(a) = \phi \subseteq f(a)$.

In any case, we have $(\widetilde{\mathcal{H}} \widetilde{\circ} f) \widetilde{\circ} \widetilde{\mathcal{H}} \sqsubseteq f$.

Conversely, let $(\widetilde{\mathcal{H}} \widetilde{\circ} f) \widetilde{\circ} \widetilde{\mathcal{H}} \sqsubseteq f$. Let $a, b, c \in \mathcal{S}$, then

$$\begin{aligned} f((ab)c) &\supseteq ((\widetilde{\mathcal{H}} \widetilde{\circ} f) \widetilde{\circ} \widetilde{\mathcal{H}})((ab)c) = \bigcup_{(ab)c=mn} \left\{ (\widetilde{\mathcal{H}} \widetilde{\circ} f)(m) \cap \widetilde{\mathcal{H}}(n) \right\} \\ &\supseteq (\widetilde{\mathcal{H}} \widetilde{\circ} f)(ab) \cap \widetilde{\mathcal{H}}(c) = (\widetilde{\mathcal{H}} \widetilde{\circ} f)(ab) \cap [0,1] \\ &= \bigcup_{ab=pq} \left\{ \widetilde{\mathcal{H}}(p) \cap f(q) \right\} \supseteq \widetilde{\mathcal{H}}(a) \cap f(b) = [0,1] \cap f(b) = f(b). \end{aligned}$$

This shows that, f is a hesitant fuzzy interior ideal on $\mathcal{S}$. ∎

Theorem 13 *Let $\mathcal{S}$ be an $\mathcal{AG}$-groupoid. Then, a hesitant fuzzy left (resp., right, two-sided) ideal f on $\mathcal{S}$ is a hesitant fuzzy interior ideal on $\mathcal{S}$ but the converse is not true in general.*

Proof. Assume that f be a hesitant fuzzy ideal on $\mathcal{S}$. Let $a, b, c \in \mathcal{S}$, then $f((ab)c) \supseteq f(ab) \supseteq f(b)$. This implies that f is a hesitant fuzzy interior ideal on $\mathcal{S}$. Conversely, since in Example 6, f is a hesitant fuzzy interior ideal on $\mathcal{S}$ and it is not a hesitant fuzzy ideal on $\mathcal{S}$ because $f(pq) = f(r) = (0.3, 0.6] \not\supseteq [0.3, 0.7] = f(q)$. ∎

Theorem 14 *Let A be a non-empty subset of an $\mathcal{AG}$-groupoid $\mathcal{S}$. Then, A is an interior ideal of $\mathcal{S}$ if and only if the characteristic hesitant fuzzy set $\mathcal{H}_A$ is a hesitant fuzzy interior ideal on $\mathcal{S}$.*

If $h(a) = [0,1]\ \forall\ a \in \mathcal{S}$, then it is easy to see that h is a hesitant fuzzy $\mathcal{AG}$-groupoid, left (resp., right, bi-ideal, interior ideal, generalized bi-ideal) on $\mathcal{S}$. We denote such type of hesitant fuzzy $\mathcal{AG}$-groupoid, hesitant fuzzy left (resp., right, bi-ideal, interior ideal, generalized bi-ideal) by $\widetilde{\mathcal{H}}$. It is obvious that $\widetilde{\mathcal{H}} = \mathcal{H}_{\mathcal{S}}$ that is $\widetilde{\mathcal{H}}(a) = [0,1]\ \forall\ a \in \mathcal{S}$.

Definition 9 A hesitant fuzzy set f on $\mathcal{AG}$-groupoid $\mathcal{S}$ is called a hesitant fuzzy quasi-ideal on $\mathcal{S}$ if the following condition is valid.

$$(f\widetilde{\circ}\widetilde{\mathcal{H}}) \sqcap (\widetilde{\mathcal{H}}\widetilde{\circ}f) \sqsubseteq f.$$

Example 7 Let $\mathcal{S} = \{0, a, b, c\}$ be an $\mathcal{AG}$-groupoid with the following Caley multiplication table.

$\cdot$	0	a	b	c
0	0	0	0	0
a	c	a	b	0
b	0	0	0	0
c	0	c	b	0

Let f be a hesitant fuzzy set defined on $\mathcal{S}$ as follows:

$$f : \mathcal{S} \to \mathcal{P}([0,1],\ x \mapsto \begin{cases} \epsilon & \text{if } x \in \{0, a\} \\ \phi & \text{if } x \in \{b, c\} \end{cases}$$

where ϵ is a non-empty subset of $[0,1]$. Then, f is a hesitant fuzzy quasi-ideal on $\mathcal{S}$.

Proposition 4 *Let $\mathcal{S}$ be an $\mathcal{AG}$-groupoid. Then, every hesitant fuzzy quasi-ideal on $\mathcal{S}$ is a hesitant fuzzy $\mathcal{AG}$-groupoid on $\mathcal{S}$.*

Proof. Assume that f be a hesitant fuzzy quasi-ideal on $\mathcal{S}$. Since $f \sqsubseteq \widetilde{\mathcal{H}}$, $f\widetilde{\circ}f \sqsubseteq \widetilde{\mathcal{H}}\widetilde{\circ}f$ and $f\widetilde{\circ}f \sqsubseteq f\widetilde{\circ}\widetilde{\mathcal{H}}$. So we have, $f\widetilde{\circ}f \sqsubseteq (\widetilde{\mathcal{H}}\widetilde{\circ}f) \sqcap (f\widetilde{\circ}\widetilde{\mathcal{H}}) \sqsubseteq f$ implies that $f\widetilde{\circ}f \sqsubseteq f$. Hence, f is a hesitant fuzzy $\mathcal{AG}$-groupoid on $\mathcal{S}$. ∎

Proposition 5 *Every hesitant fuzzy left (resp., right) ideal on $\mathcal{AG}$-groupoid $\mathcal{S}$ is a hesitant fuzzy quasi-ideal on $\mathcal{S}$.*

Proof. Assume that f is a hesitant fuzzy quasi-ideal on $\mathcal{S}$. Now, since $\widetilde{\mathcal{H}}\widetilde{\circ}f \sqsubseteq f$, so we have $(\widetilde{\mathcal{H}}\widetilde{\circ}f)\sqcap(f\widetilde{\circ}\widetilde{\mathcal{H}}) \sqsubseteq \widetilde{\mathcal{H}}\widetilde{\circ}f \sqsubseteq f$. Hence, f is a hesitant fuzzy left ideal on $\mathcal{S}$.∎

Proposition 6 *Let $\mathcal{S}$ be an $\mathcal{AG}$-groupoid. Then, every idempotent hesitant fuzzy quasi-ideal on $\mathcal{S}$ is a hesitant fuzzy bi-ideal on $\mathcal{S}$.*

Proof. Let f be a hesitant fuzzy quasi-ideal on $\mathcal{AG}$-groupoid$\mathcal{S}$. Then, by Proposition 4, f is a hesitant fuzzy $\mathcal{AG}$-groupoid on $\mathcal{S}$. Next, since f is idempotent hesitant fuzzy quasi-ideal on $\mathcal{S}$, so we have

$$(f\widetilde{\circ}\widetilde{\mathcal{H}})\widetilde{\circ}f \sqsubseteq (\widetilde{\mathcal{H}}\widetilde{\circ}\widetilde{\mathcal{H}})\widetilde{\circ}f \sqsubseteq \widetilde{\mathcal{H}}\widetilde{\circ}f \text{ and } (f\widetilde{\circ}\widetilde{\mathcal{H}})\widetilde{\circ}f = (f\widetilde{\circ}\widetilde{\mathcal{H}})\widetilde{\circ}(f\widetilde{\circ}f) = (f\widetilde{\circ}f)\widetilde{\circ}(\widetilde{\mathcal{H}}\widetilde{\circ}f) \sqsubseteq f\widetilde{\circ}\widetilde{\mathcal{H}}$$

This implies that, $(f\widetilde{\circ}\widetilde{\mathcal{H}})\widetilde{\circ}f \sqsubseteq (\widetilde{\mathcal{H}}\widetilde{\circ}f) \sqcap (f\widetilde{\circ}\widetilde{\mathcal{H}}) \sqsubseteq f$ implies that $(f\widetilde{\circ}\widetilde{\mathcal{H}})\widetilde{\circ}f \sqsubseteq f$. Hence, by using Theorem 9, f is a hesitant fuzzy bi-ideal on $\mathcal{S}$. ∎

Theorem 15 *Let Q be a non-empty subset of an $\mathcal{AG}$-groupoid $\mathcal{S}$. Then, Q is a quasi-ideal of $\mathcal{S}$ if and only if the characteristic hesitant fuzzy set $\mathcal{H}_Q$ is a hesitant fuzzy quasi-ideal on $\mathcal{S}$.*

Proof. We first assume that Q is a quasi-ideal of $\mathcal{S}$. Let a be any element of $\mathcal{S}$. If $a \in Q$, then

$$((\mathcal{H}_Q \widetilde{\circ} \widetilde{\mathcal{H}}) \sqcap (\widetilde{\mathcal{H}} \widetilde{\circ} \mathcal{H}_Q))(a) \sqsubseteq [0,1] = \mathcal{H}_Q(a).$$

If $a \notin Q$, then $[\mathcal{H}_Q](a) = \phi$. On the other hand assume that

$$((\mathcal{H}_Q \widetilde{\circ} \widetilde{\mathcal{H}}) \sqcap (\widetilde{\mathcal{H}} \widetilde{\circ} \mathcal{H}_Q))(a) = [0,1].$$

Then

$$\bigcup_{a=xy} \left\{ \mathcal{H}_Q(x) \cap \widetilde{\mathcal{H}}(y) \right\} = (\mathcal{H}_Q \widetilde{\circ} \widetilde{\mathcal{H}})(a) = [0,1]$$

and

$$\bigcup_{a=xy} \left\{ \widetilde{\mathcal{H}}(x) \cap \mathcal{H}_Q(y) \right\} = (\widetilde{\mathcal{H}} \widetilde{\circ} \mathcal{H}_Q)(a) = [0,1].$$

This implies that, there exist b, c, d and e of $\mathcal{S}$ with $a = bc = de$ such that $\mathcal{H}_Q(b) = [0,1]$ and $\mathcal{H}_Q(e) = [0,1]$. Hence $a = bc = de \in Q\mathcal{S} \cap \mathcal{S}Q \subseteq Q$, which contradicts that $a \notin Q$. Thus, we have $(\mathcal{H}_Q \widetilde{\circ} \widetilde{\mathcal{H}}) \sqcap (\widetilde{\mathcal{H}} \widetilde{\circ} \mathcal{H}_Q) \sqsubseteq \mathcal{H}_Q$ and so $\mathcal{H}_Q$ is a hesitant fuzzy quasi-ideal on $\mathcal{S}$.

Conversely, suppose that $\mathcal{H}_Q$ is a hesitant fuzzy quasi-ideal on $\mathcal{S}$. Let a be any element of $Q\mathcal{S} \cap \mathcal{S}Q$. Then, $bx = a = yc$ for some $x, y \in \mathcal{S}$. It follows from Example 2, that

$$\begin{aligned} \mathcal{H}_Q(a) &\supseteq ((\mathcal{H}_Q \widetilde{\circ} \widetilde{\mathcal{H}}) \sqcap (\widetilde{\mathcal{H}} \widetilde{\circ} \mathcal{H}_Q))(a) = (\mathcal{H}_Q \widetilde{\circ} \widetilde{\mathcal{H}})(a) \cap (\widetilde{\mathcal{H}} \widetilde{\circ} \mathcal{H}_Q)(a) \\ &= \left(\bigcup_{a=uv} \left\{ \mathcal{H}_Q(u) \cap \widetilde{\mathcal{H}}(v) \right\} \right) \cap \left(\bigcup_{a=uv} \{ \widetilde{\mathcal{H}}(u) \cap \mathcal{H}_Q(v) \} \right) \\ &= \left(\bigcup_{a=uv} \{ \mathcal{H}_Q(u) \} \right) \cap \left(\bigcup_{a=uv} \{ \mathcal{H}_Q(v) \} \right) = [0,1] \end{aligned}$$

and so $a \in Q$. Thus, $Q\mathcal{S} \cap \mathcal{S}Q \subseteq Q$. Hence, Q is a quasi-ideal of $\mathcal{S}$. ∎

Characterizations of regular $\mathcal{AG}$-groupoids

In this section, we characterize a regular $\mathcal{AG}$-groupoid in terms of hesitant fuzzy ideals.

Definition 10 An $\mathcal{AG}$-groupoid $\mathcal{S}$ is called regular $\mathcal{AG}$-groupoid if for every element $a \in \mathcal{S}$ there exist an element $x \in \mathcal{S}$ such that

$$a = (ax)a.$$

Theorem 16 *Let $\mathcal{S}$ be a regular $\mathcal{AG}$-groupoid. Then, for every hesitant fuzzy right ideal f and for every hesitant fuzzy left ideal g on $\mathcal{S}$, $f\widetilde{\circ}g = f \sqcap g$.*

Proof. Let f be a hesitant fuzzy right ideal and g be a hesitant fuzzy left ideal on a regular $\mathcal{AG}$-groupoid $\mathcal{S}$. Then, $f\widetilde{\circ}g \sqsubseteq f\widetilde{\circ}\widetilde{\mathcal{H}} \sqsubseteq f$ and $f\widetilde{\circ}g \sqsubseteq \widetilde{\mathcal{H}}\widetilde{\circ}g \sqsubseteq g$. This implies that, $f\widetilde{\circ}g \sqsubseteq f \sqcap g$. Now let a be any element of $\mathcal{S}$. Then, since $\mathcal{S}$ is regular $\mathcal{AG}$-groupoid, so there exist an element $x \in \mathcal{S}$ such that $a = (ax)a$. Thus, we have $(f\widetilde{\circ}g)(a) = \bigcup_{a=xy} \{f(x) \cap g(y)\} \supseteq f(ax) \cap g(a) \supseteq f(a) \cap g(a) = (f \cap g)(a)$. This implies that, $f\widetilde{\circ}g \sqsupseteq f \sqcap g$. Hence, $f\widetilde{\circ}g = f \sqcap g$. ∎

Corollary 1 *Let $\mathcal{S}$ be a regular $\mathcal{AG}$-groupoid. Then, for every hesitant fuzzy ideal f and for every hesitant fuzzy ideal g on $\mathcal{S}$, $f\widetilde{\circ}g = f \sqcap g$.*

Proposition 7 *Let $\mathcal{S}$ be a regular $\mathcal{AG}$-groupoid. Then, every hesitant fuzzy right ideal on $\mathcal{S}$ is idempotent.*

Proof. Let $\mathcal{S}$ be a regular $\mathcal{AG}$-groupoid and g be a hesitant fuzzy right ideal on $\mathcal{S}$. Then, $g\widetilde{\circ}g \sqsubseteq g\widetilde{\circ}\widetilde{\mathcal{H}} \sqsubseteq g$. Next, since $\mathcal{S}$ is regular, so for any $a \in \mathcal{S}$ there exist $x \in \mathcal{S}$ such that $a = (ax)a$. Thus, we have $(g\widetilde{\circ}g)(a) = \bigcup_{a=(ax)a} \{g(ax) \cap g(a)\} \supseteq g(a) \cap g(a) = g(a)$. Hence, $g \sqsubseteq (g\widetilde{\circ}g)$ and so $(g)^2 = g\widetilde{\circ}g = g$. ∎

Corollary 2 *Let $\mathcal{S}$ be a regular $\mathcal{AG}$-groupoid. Then, every hesitant fuzzy ideal on $\mathcal{S}$ is idempotent.*

Corollary 3 *Let $\mathcal{S}$ be a regular $\mathcal{AG}$-groupoid. Then, the set of all hesitant fuzzy ideals on $\mathcal{S}$ forms a samilattice structure under the hesitant fuzzy product.*

Proposition 8 *Every hesitant fuzzy right ideal on a regular $\mathcal{AG}$-groupoid $\mathcal{S}$ is a hesitant fuzzy left ideal on $\mathcal{S}$.*

Proof. Let $\mathcal{S}$ be a regular $\mathcal{AG}$-groupoid and f be a hesitant fuzzy right ideal on $\mathcal{S}$. Then, since $\mathcal{S}$ is regular, so for any $a \in \mathcal{S}$ there exist $x \in \mathcal{S}$ such that $a = (ax)a$. Thus, we have $f(ab) = f(((ax)a)b) = f((ba)(ax)) \supseteq f(ba) \supseteq f(b)$. Hence, f is a hesitant fuzzy left ideal on $\mathcal{S}$. ∎

Proposition 9 *Let $\mathcal{S}$ be an $\mathcal{AG}$-groupoid and let the set of all hesitant fuzzy ideals on $\mathcal{S}$ forms a regular $\mathcal{AG}$-groupoid on $\mathcal{S}$ under the hesitant fuzzy product. Then, every hesitant fuzzy ideal on $\mathcal{S}$ has the form $f = (f\widetilde{\circ}\widetilde{\mathcal{H}})\widetilde{\circ}f$.*

Proof. Let f be a hesitant fuzzy ideal on $\mathcal{S}$. Then, by assumption there exist a hesitant fuzzy ideal on $\mathcal{S}$ such that $f = (f\widetilde{\circ}g)\widetilde{\circ}f$ Thus,

$$f = (f\widetilde{\circ}g)\widetilde{\circ}f \sqsubseteq (f\widetilde{\circ}\widetilde{\mathcal{H}})\widetilde{\circ}f \sqsubseteq (f\widetilde{\circ}\widetilde{\mathcal{H}}) \sqcap (\widetilde{\mathcal{H}}\widetilde{\circ}f) \sqsubseteq f \sqcap f = f.$$

Now since $(f\widetilde{\circ}\widetilde{\mathcal{H}})\widetilde{\circ}f \sqsubseteq (f\widetilde{\circ}\widetilde{\mathcal{H}})\widetilde{\circ}\widetilde{\mathcal{H}} \sqsubseteq f\widetilde{\circ}\widetilde{\mathcal{H}}$ and $(f\widetilde{\circ}\widetilde{\mathcal{H}})\widetilde{\circ}f \sqsubseteq (\widetilde{\mathcal{H}}\widetilde{\circ}\widetilde{\mathcal{H}})\widetilde{\circ}f \sqsubseteq \widetilde{\mathcal{H}}\widetilde{\circ}f$, hence, $f = (f\widetilde{\circ}\widetilde{\mathcal{H}})\widetilde{\circ}f$. ∎

Proposition 10 *Let $\mathcal{S}$ be a regular $\mathcal{AG}$-groupoid and let f be a hesitant fuzzy interior ideal on $\mathcal{S}$. Then, f is a hesitant fuzzy ideal on $\mathcal{S}$.*

Proof. Let f be a hesitant fuzzy interior ideal on $\mathcal{S}$ and a be any element of $\mathcal{S}$. Then, since $\mathcal{S}$ is regular so there exist $x \in \mathcal{S}$ such that $a = (ax)a$. Thus, we have$f(ab) = f(((ax)a)b) \supseteq f(a)$. Hence, f is a hesitant fuzzy right ideal on $\mathcal{S}$. ∎

Theorem 17 *Let $\mathcal{S}$ be a regular $\mathcal{AG}$-groupoid. Then, for every hesitant fuzzy bi-ideal f on $\mathcal{S}$, $f = (f\widetilde{\circ}\widetilde{\mathcal{H}})\widetilde{\circ}f$.*

Proof. Let f be a hesitant fuzzy bi-ideal on $\mathcal{S}$ and let a be an arbitrary element of $\mathcal{S}$. Then, since $\mathcal{S}$ is regular, so there exist $x \in \mathcal{S}$ such that $a = (ax)a$. Thus, we have

$$\begin{aligned}((f\widetilde{\circ}\widetilde{\mathcal{H}})\widetilde{\circ}f)(a) &= \bigcup_{a=bc}\Big\{(f\widetilde{\circ}\widetilde{\mathcal{H}})(b)\cap f(c)\Big\} \supseteq (f\widetilde{\circ}\widetilde{\mathcal{H}})(xa)\cap f(a)\\ &= \bigcup_{ax=pq}\Big\{f(p)\cap\widetilde{\mathcal{H}}(q)\Big\}\cap f(a) \supseteq f(a)\cap\widetilde{\mathcal{H}}(x)\cap f(a)\\ &= f(a)\cap[0,1]\cap f(a) = f(a)\end{aligned}$$

This shows that, $(f\widetilde{\circ}\widetilde{\mathcal{H}})\widetilde{\circ}f \sqsupseteq f$. By assumption, since f is a hesitant fuzzy bi-ideal on $\mathcal{S}$, so $(f\widetilde{\circ}\widetilde{\mathcal{H}})\widetilde{\circ}f \sqsubseteq f$. Hence, $f = (f\widetilde{\circ}\widetilde{\mathcal{H}})\widetilde{\circ}f$. ∎

Proposition 11 *Let $\mathcal{S}$ be a regular $\mathcal{AG}$-groupoid. Let f be a hesitant fuzzy right ideal and g be a hesitant fuzzy left ideal on $\mathcal{S}$. Then, $f\widetilde{\circ}g$ is a hesitant fuzzy quasi-ideal on $\mathcal{S}$.*

Proof. Let f be a hesitant fuzzy right ideal and g be a hesitant fuzzy left ideal on $\mathcal{S}$. It follows that $f \sqcap g$ is a hesitant fuzzy quasi-ideal on $\mathcal{S}$. Now, since $\mathcal{S}$ is regular, so by Theorem 16, $f\widetilde{\circ}g{=}f \sqcap g$. Hence, $f\widetilde{\circ}g$ is a hesitant fuzzy quasi-ideal on $\mathcal{S}$. ∎

Characterizations of completely regular $\mathcal{AG}$-groupoids

In this section, we characterize a completely regular $\mathcal{AG}$-groupoid in terms of hesitant fuzzy ideals. An $\mathcal{AG}$-groupoid $\mathcal{S}$ is called left regular if for all $a \in \mathcal{S}$ there exist $x \in \mathcal{S}$ such that

$$a = xa^2 = x(aa).$$

An $\mathcal{AG}$-groupoid $\mathcal{S}$ is called right regular if for all $a \in \mathcal{S}$ there exist $x \in \mathcal{S}$ such that

$$a = a^2x = (aa)x.$$

Definition 11 An $\mathcal{AG}$-groupoid $\mathcal{S}$ is called completely regular $\mathcal{AG}$-groupoid if $\mathcal{S}$ is regular, left regular and right regular.

Theorem 18 *Let $\mathcal{S}$ be an $\mathcal{AG}$-groupoid. Then, the following coditions are equivalent.*

(i) $\mathcal{S}$ *is left regular.*

(ii) $f(a) = f(a^2)$ *for all* $a \in \mathcal{S}$*, where* f *is any hesitant fuzzy left ideal on* $\mathcal{S}$*.*

Proof. (i) $\Rightarrow$ (ii) : Let $\mathcal{S}$ be a left regular $\mathcal{AG}$-groupoid and f be any hesitant fuzzy left ideal on $\mathcal{S}$. Then, since $\mathcal{S}$ is a left regular so for any $p \in \mathcal{S}$ there exist $q \in \mathcal{S}$ such that $p = q(pp)$. Thus,

$$f(p) = f(q(pp)) \supseteq f(pp) \supseteq f(p).$$

This shows that, $f(p) = f(p^2)$.

(ii) $\Rightarrow$ (i) : Let $f(p) = f(p^2)$ for all $p \in \mathcal{S}$. Since $L(p^2)$ is a left ideal of $\mathcal{S}$, the characteristic hesitant fuzzy set $\mathcal{H}_{L(p^2)}$ is a hesitant fuzzy left ideal on $\mathcal{S}$. Since $p^2 \in L(p^2)$, we have

$$\mathcal{H}_{L(p^2)}(p) = \mathcal{H}_{L(p^2)}(p^2) = [0, 1].$$

This implies that, $p \in L(p^2) = \mathcal{S}(pp)$. This obviously means that $\mathcal{S}$ is a left regular. ∎

Theorem 19 *Let* $\mathcal{S}$ *be an* $\mathcal{AG}$*-groupoid. Then, the following coditions are equivalent.*

(i) $\mathcal{S}$ *is left regular.*

(ii) $f(a) = f(a^2)$ *for all* $a \in \mathcal{S}$*, where* f *is any hesitant fuzzy right ideal on* $\mathcal{S}$*.*

Characterizations of weakly regular $\mathcal{AG}$-groupoids

In this section, we characterize a weakly regular $\mathcal{AG}$-groupoid in terms of hesitant fuzzy ideals.

Definition 12 An $\mathcal{AG}$-groupoid $\mathcal{S}$ is called weakly regular $\mathcal{AG}$-groupoid if for all $a \in \mathcal{S}$ there exist $b, c \in \mathcal{S}$ such that

$$a = (ab)(ac)$$

Theorem 20 *Let* $\mathcal{S}$ *be a weakly regular* $\mathcal{AG}$*-groupoid. Let* f *be any hesitant fuzzy right ideal and* g *be any hesitant fuzzy ideal on* $\mathcal{S}$*. Then,* $f \sqcap g = f \widetilde{\circ} g$*.*

Proof. Let f be any hesitant fuzzy right ideal and g be any hesitant fuzzy ideal on weakly regular $\mathcal{AG}$-groupoid $\mathcal{S}$. Let a be any element of $\mathcal{S}$. Then, since $\mathcal{S}$ is weakly regular, so there exist $b, c \in \mathcal{S}$ such that $a = (ab)(ac)$. Thus, we have

$$(f \widetilde{\circ} g)(a) = \bigcup_{a=(ab)(ac)} \{f(ab) \cap g(ac)\} \supseteq f(a) \cap f(a) = (f \cap g)(a).$$

This shows that, $f \sqcap g \sqsubseteq f \widetilde{\circ} g$. Now since $f \sqcap g \sqsupseteq f \widetilde{\circ} g$ holds for all hesitant fuzzy right ideal and for all hesitant fuzzy left ideal on $\mathcal{S}$. Hence, $f \sqcap g = f \widetilde{\circ} g$. ∎

Characterizations of quasi-regular $\mathcal{AG}$-groupoids

In this section, we characterize quasi-regular $\mathcal{AG}$-groupoid in terms of hesitant fuzzy ideals. An $\mathcal{AG}$-groupoid $\mathcal{S}$ is called left quasi-regular if for all $a \in \mathcal{S}$ there exist $b, c \in \mathcal{S}$ such that $a = (ba)(ca)$ and $\mathcal{S}$ is called right quasi-regular if for all $a \in \mathcal{S}$ there exist $b, c \in \mathcal{S}$ such that $a = (ab)(ac)$.

Definition 13 An $\mathcal{AG}$-groupoid $\mathcal{S}$ is called quasi-regular, if it is both left quasi-regular and right quasi-regular.

Theorem 21 *Let $\mathcal{S}$ be an $\mathcal{AG}$-groupoid. Then, $\mathcal{S}$ is left (resp., right) quasi-regular if and only if every hesitant fuzzy left (resp., right) ideal on $\mathcal{S}$ is idempotent.*

Proof. Let $\mathcal{S}$ be left quasi-regualr and f be a hesitant fuzzy left ideal on $\mathcal{S}$. Then, for any $x \in \mathcal{S}$ there exist $b, c \in \mathcal{S}$ such that $x = (bx)(cx)$. Thus, we have $(f\widetilde{\circ}f)(x) = \bigcup_{x=(bx)(cx)} \{f(bx) \cap f(cx)\} \supseteq f(bx) \cap f(cx) \supseteq f(x) \cap f(x) = f(x)$. This shows that, $f\widetilde{\circ}f \supseteq f$. Also it is obvious that $f\widetilde{\circ}f \subseteq f$. Thus, $f\widetilde{\circ}f = f$. Hence, f is idempotent.

Conversely, let every hesitant fuzzy left ideal on $\mathcal{S}$ is idempotent. Let $x \in \mathcal{S}$. Then, since $L[x]$ is a principal left ideal of $\mathcal{S}$, so the characteristic hesitant fuzzy set is a hesitant fuzzy left ideal on $\mathcal{S}$. Thus, by using assumption we have

$$\mathcal{H}_{L[x]L[x]}(x) = (\mathcal{H}_{L[x]}\widetilde{\circ}\mathcal{H}_{L[x]})(x) = \mathcal{H}_{L[x]}(x) = [0, 1].$$

This implies that, $a \in L[x]L[x] = (\mathcal{S}x)(\mathcal{S}x)$. Hence, $\mathcal{S}$ is left quasi-regular.

The other case can be shown in a similar way. ∎

Proposition 12 *Let $\mathcal{S}$ be left (resp., right) quasi-regular $\mathcal{AG}$-groupoid and f be a hesitant fuzzy right ideal on $\mathcal{S}$. Then, f is a hesitant fuzzy ideal on $\mathcal{S}$.*

Proof. Let $\mathcal{S}$ be left (resp., right) quasi-regular $\mathcal{AG}$-groupoid and f be a hesitant fuzzy right ideal on $\mathcal{S}$. Then, since $\widetilde{\mathcal{H}}$ is a hesitant fuzzy right ideal of itself on $\mathcal{S}$, and by using assumption $\widetilde{\mathcal{H}}$ is idempotent, so we have$\widetilde{\mathcal{H}}\widetilde{\circ}f = (\widetilde{\mathcal{H}}\widetilde{\circ}\widetilde{\mathcal{H}})\widetilde{\circ}f = (f\widetilde{\circ}\widetilde{\mathcal{H}})\widetilde{\circ}\widetilde{\mathcal{H}} \sqsubseteq f\widetilde{\circ}\widetilde{\mathcal{H}} \sqsubseteq f$. Hence, f is a hesitant fuzzy ideal on $\mathcal{S}$. ∎

Theorem 22 *Let $\mathcal{S}$ be an $\mathcal{AG}$-groupoid. Let for every hesitant fuzzy ideal f on $\mathcal{S}$, $f = (f\widetilde{\circ}\widetilde{\mathcal{H}})^2 \sqcap (\widetilde{\mathcal{H}}\widetilde{\circ}f)^2$, then $\mathcal{S}$ is quasi-regular.*

Proof. Let f be any hesitant fuzzy right ideal on $\mathcal{S}$. Then, by assumption we have $f = (f\widetilde{\circ}\widetilde{\mathcal{H}})^2 \sqcap (\widetilde{\mathcal{H}}\widetilde{\circ}f)^2 \sqsubseteq (f\widetilde{\circ}\widetilde{\mathcal{H}})^2 \sqsubseteq f\widetilde{\circ}f \sqsubseteq f\widetilde{\circ}\widetilde{\mathcal{H}} \sqsubseteq f$. Thus, $f = f\widetilde{\circ}f = (f)^2$ and from Theorem 21, it follows that $\mathcal{S}$ is right quasi-regular. In similar way we can show that $\mathcal{S}$ is left quasi-regular. ∎

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FIXED POINTS IN INTUITIONISTIC MENGER SPACE

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Abstract. The purpose of this paper is to use the new concept of absorbing mappings in intuitionistic Menger space and prove fixed point theorems, without appeal to continuity. The results thus obtained, generalizes and extends the result of Rashwan and Hedar [5] in intuitionistic Menger space. We also cited an example in support of our result.

Keywords and phrases: intuitionistic Menger space; common fixed points; compatible mappings; absorbing Mappings.

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1. Introduction

There have been a number of generalizations of metric space. One such generalization is Menger space initiated by Menger [3]. It is a probabilistic generalization in which we assign to any two points x and y, a distribution function $F_{x,y}$. Schweizer and Sklar [6], [7] studied this concept and gave some fundamental results on this space. Sehgal and Bharucha-Reid [8] obtained a generalization of Banach Contraction Principle on a complete Menger space which is a milestone in developing fixed point theory in Menger space.

Kutukcu et al. [2] introduced the notion of intuitionistic Menger spaces with the help of t-norms and t-conorms as a generalization of Menger space due to Menger [3]. Using the concept of weakly compatible maps in intuitionistic Menger space, Pant et. al. [4] proved a common fixed point theorem for six self maps without appeal to continuity.

In this paper, we prove some common fixed point theorems for absorbing mappings in Intuitionistic Menger spaces without using the condition of continuity. We extend and generalize the result Rashwan and Hedar [5]. We support our theorem by providing a suitable example.

2. Preliminaries

Definition 2.1. [4] A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is called a t-norm if it satisfies the following conditions:

(1) $*$ is commutative and associative,
(2) $*$ is continuous,
(3) $a * 1 = a$, for all $a \in [0,1]$,
(4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for all $a, b, c, d \in [0,1]$.

Definition 2.2. [1] A binary operation $\Diamond : [0,1] \times [0,1] \to [0,1]$ is called a t-conorm if it satisfies the following conditions:

(1) $\Diamond$ is commutative and associative.
(2) $\Diamond$ is continuous,
(3) $a \Diamond 0 = a$, for all $a \in [0,1]$
(4) $a \Diamond b \leq c \Diamond d$ whenever $a \leq c$ and $b \leq d$, for all $a, b, c, d \in [0,1]$.

Remark 2.1. [4] The concept of triangular norms (t-norms) and triangular conorms (t-conorms) are known as the axiomatic skeletons that we use for characterizing fuzzy intersection and union respectively. These concepts were originally introduced by Menger [3] in his study of statistical metric spaces.

Definition 2.3. [4] A distance distribution function is a function $\mathbf{F} : R \to R^+$ which is left continuous on R, non-decreasing and $\inf_{t \in R} F(t) = 0$, $\sup_{t \in R} F(t) = 1$. We will denote by D the family of all distance distribution function and by H a special of D defined by

$$H(t) = \begin{cases} 0, & \text{if } t \leq 0 \\ 1, & \text{if } t > 0. \end{cases}$$

If X is a non-empty set, $\boldsymbol{F} : X \times X \to D$ is called a probabilistic distance on X and $F(x,y)$ is usually denoted by $F_{x,y}$.

Definition 2.4. [4] A non-distance distribution function is a function $\boldsymbol{L} : R \to R^+$ which is right continuous on R, non-increasing and $\inf_{t \in R} L(t) = 1$, $\sup_{t \in R} L(t) = 0$. We

will denote by E the family of all distance distribution function and by G a special of E defined by

$$G(t) = \begin{cases} 1, & \text{if } t \leq 0 \\ 0, & \text{if } t > 0 \end{cases}.$$

If X is a non-empty set, $\boldsymbol{L} : X \times X \to E$ is called a probabilistic non-distance on X and $L(x, y)$ is usually denoted by $L_{x,y}$.

Definition 2.5. [4] A 5-tuple $(X, \boldsymbol{F}, \boldsymbol{L}, *, \lozenge)$ is said to be an intuitionistic Menger space if X is an arbitrary set, $*$ is a continuous t-norm, $\lozenge$ is continuous t-conorm, $\boldsymbol{F}$ is a probabilistic distance and $\boldsymbol{L}$ is a probabilistic non-distance on X satisfying the following conditions :
for all $x, y, z \in X$ and $t, s \geq 0$,

(1) $F_{x,y}(t) + L_{x,y}(t) \leq 1$,
(2) $F_{x,y}(0) = 0$,
(3) $F_{x,y}(t) = H(t)$ if and only if $x = y$,
(4) $F_{x,y}(t) = F_{y,x}(t)$,
(5) if $F_{x,y}(t) = 1$ and $F_{y,z}(s) = 1$, then $F_{x,z}(t + s) = 1$,
(6) $F_{x,z}(t + s) \geq F_{x,y}(t) * F_{y,z}(s)$,
(7) $L_{x,y}(0) = 1$,
(8) $L_{x,y}(t) = G(t)$ if and only if $x = y$,
(9) $L_{x,y}(t) = L_{y,x}(t)$,
(10) if $L_{x,y}(t) = 0$ and $L_{y,z}(s) = 0$, then $L_{x,z}(t + s) = 0$,
(11) $L_{x,z}(t + s) \leq L_{x,y}(t) \lozenge L_{y,z}(s)$.

The functions $F_{x,y}(t)$ and $L_{x,y}(t)$ denote the degree of nearness and degree of non-nearness between x and y with respect to t, respectively.

Example 2.1. [4] Let (X, d) be a usual metric space. Then the metric d induces a distance distribution function $\boldsymbol{F}$ defined by $F_{x,y}(t) = H(t - d(x, y))$ and non-distance distribution function $\boldsymbol{L}$ defined by $L_{x,y}(t) = G(t - d(x, y))$ for all $x, y \in X$ and $t \geq 0$. Then $(X, \boldsymbol{F}, \boldsymbol{L})$ is an intuitionistic probabilistic metric space. We call this intuitionistic probabilistic metric space induced by a metric d, the induced intuitionistic probabilistic metric space. If t-norm $*$ is $a * b = \min\{a, b\}$ and t-conorm $\lozenge$ is $a \lozenge b = \max\{a, b\}$ for all $a, b \in [0, 1]$ then $(X, \boldsymbol{F}, \boldsymbol{L}, *, \lozenge)$ is an intuitionistic Menger space.

Definition 2.6. Self mappings A and S of an intuitionistic Menger space $(X, \boldsymbol{F}, \boldsymbol{L}, *, \lozenge)$ are said to be compatible if

$$\lim_{n\to\infty} F_{ASx_n, SAx_n}(t) = 1 \text{ and } \lim_{n\to\infty} L_{ASx_n, SAx_n}(t) = 0$$

whenever there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$, $z \in X$.

Definition 2.7. Self mappings A and S of an intuitionistic Menger space $(X, \boldsymbol{F}, \boldsymbol{L}, *, \lozenge)$ are said to be reciprocal continuous if

$$\lim_{n\to\infty} ASx_n = At \text{ and } \lim_{n\to\infty} SAx_n = St$$

for some $t \in X$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$, $t \in X$.

Lemma 2.1. [4] Let $(X, \boldsymbol{F}, \boldsymbol{L}, *, \lozenge)$ be an intuitionistic Menger space with $t*t \geq t$ and $(1-t) \lozenge (1-t) \leq (1-t)$ and for all $x, y \in X$, $t > 0$ and if for a number $k \in (0,1)$

$$F_{x,y}(kt) \geq F_{x,y}(t) \text{ and } L_{x,y}(kt) \leq L_{x,y}(t).$$

Then x = y.

Definition 2.8. Suppose A and S be two self mappings of an intuitionistic Menger space $(X, \boldsymbol{F}, \boldsymbol{L}, *, \lozenge)$, then A is called S-absorbing if there exists a positive integer $R > 0$ such that $F_{Sx,SAx}(t) \geq F_{Sx,Ax}(t/R)$ and $L_{Sx,SAx}(t) \leq L_{Sx,Ax}(t/R)$ for all $x \in X$.

Similarly, S is called A-absorbing if there exists a positive integer $R > 0$ such that $F_{Ax,ASx}(t) \geq F_{Ax,Sx}(t/R)$ and $L_{Sx,SAx}(t) \leq L_{Sx,Ax}(t/R)$ for all $x \in X$.

Map A is called pointwise S-absorbing if for given $x \in X$ there exists a positive integer $R > 0$ such that $F_{Sx,SAx}(t) \geq F_{Sx,Ax}(t/R)$ and $L_{Sx,SAx}(t) \leq L_{Sx,Ax}(t/R)$ for all $x \in X$.

Similarly, we can define pointwise A-absorbing map.

Example 2.2. Let $X = [2, 20]$ for each $t \in (0, \infty)$ and $x, y \in X$. Define $(\boldsymbol{F}, \boldsymbol{L})$ by

$$F_{x,y}(t) = \begin{cases} \frac{t}{t+|x-y|}, & \text{if } t > 0 \\ 0, & \text{if } t = 0 \end{cases}$$

$$L_{x,y}(t) = \begin{cases} \frac{|x-y|}{t+|x-y|}, & \text{if } t > 0 \\ 1, & \text{if } t = 0 \end{cases}$$

Define self maps A and B as follows:

$$A(x) = \begin{cases} 6, & \text{if } 2 \leq x \leq 5 \text{ and } x = 6 \\ 10, & \text{if } x > 6 \\ \frac{x-1}{2}, & \text{if } 5 < x < 6 \end{cases}$$

$$B(x) = \begin{cases} 2, & \text{if } 2 \leq x \leq 5 \\ \frac{x+1}{3}, & \text{if } x > 5 \end{cases}$$

Now, consider a sequence $x_n = 5 + \frac{1}{2n}$ for $n = 1, 2, 3, ...$ then it is easy to see that both pairs (A, B) and (B, A) are not-compatible but A is B-absorbing and B is A-absorbing.

Remark 2.2. In view of above example, it is clear that absorbing maps are more general than that of compatible maps.

Lemma 2.2. [4] Let $(X, \boldsymbol{F}, \boldsymbol{L}, *, \lozenge)$ be an intuitionistic Menger space with $t*t \geq t$ and $(1-t) \lozenge (1-t) \leq (1-t)$ and for all $x, y \in X,\ t > 0$. If there exists a constant $k \in (0,1)$ such that for $n = 1, 2, ...$

$$F_{y_{n+2},y_{n+1}}(kt) \geq F_{y_{n+1},y_n}(t) \text{ and } L_{y_{n+2},y_{n+1}}(kt) \leq L_{y_{n+1},y_n}(t).$$

Then $\{y_n\}$ is a Cauchy sequence in X.

Lemma 2.3. [4] In an intuitionistic Menger space $(X, \boldsymbol{F}, \boldsymbol{L}, *, \lozenge)$, $F_{x,y}(.)$ is non-decreasing and $L_{x,y}(.)$ is non-increasing for all $x, y \in X$.

3. Main result

Theorem 3.1. *Let P be pointwise S-absorbing and Q be pointwise T-absorbing self maps on a complete intuitionistic Menger space $(X, \mathbf{F}, \mathbf{L}, *, \lozenge)$ with continuous t-norm $*$ and continuous t-conorm $\lozenge$ defined by $a * b = \min\{a, b\}$ and $a \lozenge b = \max\{a, b\}$ for all $a, b \in [0, 1]$ satisfying the following conditions:*

(i) $P(X) \subseteq T(X),\ \ Q(X) \subseteq S(X)$;

(ii) *there exists $k \in (0,1)$ such that for every $x, y \in X$ and $t > 0$*

$$F_{Px,Qy}(kt) \geq \min\Big\{F_{Sx,Ty}(t), F_{Px,Sx}(t), F_{Qy,Ty}(t), F_{Px,Ty}(t)\Big\}$$
$$L_{Px,Qy}(kt) \leq \max\Big\{L_{Sx,Ty}(t), L_{Px,Sx}(t), L_{Qy,Ty}(t), L_{Px,Ty}(t)\Big\};$$

(iii) *for all $x, y \in X$, $F_{x,y}(t) = 1$ and $L_{x,y}(t) = 0$.*

If the pair (P, S) is reciprocally continuous compatible maps then P, Q, S and T have a unique common fixed point in X.

Proof. Let x_0 be any arbitrary point in X, construct a sequence $\{y_n\} \in X$ such that $y_{2n-1} = Tx_{2n-1} = Px_{2n-2}$ and $y_{2n} = Sx_{2n} = Qx_{2n-1}, n = 1, 2, 3, ...$ This can be done by virtue of (i).
By using contractive condition, we obtain

$$\begin{aligned} F_{y_{2n+1},y_{2n+2}}(kt) &= F_{Px_{2n},Qx_{2n+1}}(kt) \\ &\geq \min\{F_{Sx_{2n},Tx_{2n+1}}(t), F_{Px_{2n},Sx_{2n}}(t), F_{Qx_{2n+1},Tx_{2n+1}}(t), F_{Px_{2n},Tx_{2n+1}}(t)\} \\ &\geq \min\{F_{y_{2n},y_{2n+1}}(t), F_{y_{2n+1},y_{2n}}(t), F_{y_{2n+2},y_{2n+1}}(t), 1\} \end{aligned}$$

$$\begin{aligned} L_{y_{2n+1},y_{2n+2}}(kt) &= L_{Px_{2n},Qx_{2n+1}}(kt) \\ &\leq \max\{L_{Sx_{2n},Tx_{2n+1}}(t), L_{Px_{2n},Sx_{2n}}(t), L_{Qx_{2n+1},Tx_{2n+1}}(t), L_{Px_{2n},Tx_{2n+1}}(t)\} \\ &\leq \max\{L_{y_{2n},y_{2n+1}}(t), L_{y_{2n+1},y_{2n}}(t), L_{y_{2n+2},y_{2n+1}}(t), 1\} \end{aligned}$$

which implies,

$$F_{y_{2n+1},y_{2n+2}}(kt) \geq F_{y_{2n},y_{2n+1}}(t)$$
$$L_{y_{2n+1},y_{2n+2}}(kt) \leq L_{y_{2n},y_{2n+1}}(t).$$

In general,

$$\begin{aligned} F_{y_n,y_{n+1}}(kt) &\geq F_{y_{n-1},y_n}(t) \\ L_{y_n,y_{n+1}}(kt) &\leq L_{y_{n-1},y_n}(t). \end{aligned} \tag{1}$$

To prove $\{y_n\}$ is a Cauchy sequence, we have to show

$$F_{y_n,y_{n+1}}(t) \to 1 \text{ and } L_{y_n,y_{n+1}}(t) \to 0.$$

For this, from (iii) we have

$$F_{y_n,y_{n+1}}(t) \geq F_{y_{n-1},y_n}(t/k) \geq F_{y_{n-2},y_{n-1}}(t/k^2) \geq ... \geq F_{y_0,y_1}(t/k^n) \to 1 \text{ as } n \to \infty,$$
$$L_{y_n,y_{n+1}}(t) \leq L_{y_{n-1},y_n}(t/k) \leq L_{y_{n-2},y_{n-1}}(t/k^2) \leq ... \leq L_{y_0,y_1}(t/k^n) \to 0 \text{ as } n \to \infty.$$

For $p \in N$, by (1) we have

$$\begin{aligned} F_{y_n,y_{n+p}}(t) &\geq F_{y_n,y_{n+1}}((1-k)t) * F_{y_{n+1},y_{n+p}}(kt) \\ &\geq F_{y_0,y_1}\left(\frac{(1-k)t}{k^n}\right) * F_{y_{n+1},y_{n+2}}(t) * F_{y_{n+2},y_{n+p}}((k-1)t) \\ &\geq F_{y_0,y_1}\left(\frac{(1-k)t}{k^n}\right) * F_{y_0,y_1}\left(\frac{t}{k^n}\right) * F_{y_{n+2},y_{n+3}}(t) * F_{y_{n+3},y_{n+p}}((k-2)t) \\ &\geq F_{y_0,y_1}\left(\frac{(1-k)t}{k^n}\right) * F_{y_0,y_1}\left(\frac{t}{k^n}\right) * F_{y_0,y_1}\left(\frac{(1-k)t}{k^{n+2}}\right) * \cdots * F_{y_0,y_1}\left(\frac{(k-p)t}{k^{n+p+1}}\right) \end{aligned}$$

and

$$\begin{aligned} L_{y_n,y_{n+p}}(t) &\leq L_{y_n,y_{n+1}}((1-k)t) \lozenge L_{y_{n+1},y_{n+p}}(kt) \\ &\leq L_{y_0,y_1}\left(\frac{(1-k)t}{k^n}\right) \lozenge L_{y_{n+1},y_{n+2}}(t) \lozenge L_{y_{n+2},y_{n+p}}((k-1)t) \\ &\leq L_{y_0,y_1}\left(\frac{(1-k)t}{k^n}\right) \lozenge L_{y_0,y_1}\left(\frac{t}{k^n}\right) \lozenge L_{y_{n+2},y_{n+3}}(t) \lozenge L_{y_{n+3},y_{n+p}}((k-2)t) \\ &\leq L_{y_0,y_1}\left(\frac{(1-k)t}{k^n}\right) \lozenge L_{y_0,y_1}\left(\frac{t}{k^n}\right) \lozenge L_{y_0,y_1}\left(\frac{(1-k)t}{k^{n+2}}\right) \\ &\qquad \lozenge \ldots \lozenge L_{y_0,y_1}\left(\frac{(k-p)t}{k^{n+p+1}}\right). \end{aligned}$$

Thus, $F_{y_n,y_{n+p}}(t) \to 1$ and $L_{y_n,y_{n+p}}(t) \to 0$ for all $t > 0$.

Therefore, $\{y_n\}$ is a Cauchy sequence in X.

But $(X, \boldsymbol{F}, \boldsymbol{L}, *, \lozenge)$ is complete, so there exists a point z (say) in X such that $\{y_n\} \to z$.

Also, using (i), we have

$$\{Px_{2n-2}\},\ \{Tx_{2n-1}\},\ \{Sx_{2n}\},\ \{Qx_{2n-1}\} \to z.$$

Since the pair (P, S) is reciprocally continuous, then we have

$$\lim_{n\to\infty} PSx_{2n} = Pz \ \text{ and } \ \lim_{n\to\infty} SPx_{2n} = Sz.$$

Compatibility of P and S yields

$$\begin{aligned}&\lim_{n\to\infty} F_{PSx_{2n},SPx_{2n}}(t) = 1 \ \text{ and}\\ &\lim_{n\to\infty} L_{PSx_{2n},SPx_{2n}}(t) = 0,\end{aligned}$$

i.e.,

$$\begin{aligned}&F_{Pz,Sz}(t) = 1 \ \text{ and}\\ &L_{Pz,Sz}(t) = 0.\end{aligned}$$

Hence $Pz = Sz$.

Since $P(X) \subseteq T(X)$ then there exists a point u in X such that $Pz = Tu$.

Now, by the contractive condition (ii), we get

$$\begin{aligned}F_{Pz,Qu}(kt) &\geq \min\Big\{F_{Sz,Tu}(t), F_{Pz,Sz}(t), F_{Qu,Tu}(t), F_{Pz,Tu}(t)\Big\}\\ &\geq \min\Big\{F_{Pz,Pz}(t), F_{Pz,Pz}(t), F_{Qu,Pz}(t), F_{Pz,Pz}(t)\Big\}\\ &> F_{Pz,Qu}(t)\end{aligned}$$

and

$$\begin{aligned}L_{Pz,Qu}(kt) &\leq \max\Big\{L_{Sz,Tu}(t), L_{Pz,Sz}(t), L_{Qu,Tu}(t), L_{Pz,Tu}(t)\Big\}\\ &\leq \max\Big\{L_{Pz,Pz}(t), L_{Pz,Pz}(t), L_{Qu,Pz}(t), L_{Pz,Pz}(t)\Big\}\\ &< L_{Pz,Qu}(t)\end{aligned}$$

i.e., $Pz = Qu$.

Thus, $Pz = Sz = Qu = Tu$. Since P is pointwise S-absorbing then for $R > 0$, we have

$$F_{Sz,SPz}(t) \geq F_{Sz,Pz}(t/R) = 1 \ \text{ and } \ L_{Sz,SPz}(t) \leq L_{Sz,Pz}(t/R) = 0,$$

i.e., $Pz = SPz = Sz$.

Now, by the contractive condition, we have

$$\begin{aligned} F_{Pz,PPz}(t) &= F_{PPz,Qu}(t) \\ &\geq \min\left\{F_{SPz,Tu}(t), F_{PPz,Su}(t), F_{Qu,Tu}(t), F_{PPz,Tu}(t)\right\} \\ &= \min\left\{F_{Pz,Pz}(t), F_{PPz,Pz}(t), F_{Qu,Qu}(t), F_{PPz,Pz}(t)\right\} \\ &= F_{PPz,Pz}(t) \end{aligned}$$

$$\begin{aligned} L_{Pz,PPz}(t) &= L_{PPz,Qu}(t) \\ &\leq \max\left\{L_{SPz,Tu}(t), L_{PPz,Su}(t), L_{Qu,Tu}(t), L_{PPz,Tu}(t)\right\} \\ &= \max\left\{L_{Pz,Pz}(t), L_{PPz,Pz}(t), L_{Qu,Qu}(t), L_{PPz,Pz}(t)\right\} \\ &= L_{PPz,Pz}(t) \end{aligned}$$

i.e., $PPz = Pz = SPz$.

Therefore, Pz is a common fixed point of P and S.
Similarly, Q is pointwise T-absorbing, therefore, we have

$$\begin{aligned} F_{Tu,TQu}(t) &\geq F_{Tu,Qu}(t/R) = 1 \\ \text{and } L_{Tu,TQu}(t) &\leq F_{Tu,Qu}(t/R) = 0 \end{aligned}$$

i.e., $Tu = TQu = Qu$.

Now, by the contractive condition, we have

$$\begin{aligned} F_{QQu,Qu}(t) &= F_{Pz,QQu}(t) \\ &\geq \min\left\{F_{Sz,TQu}(t), F_{Pz,Su}(t), F_{QQu,TQu}(t), F_{Pz,TQu}(t)\right\} \\ &= \min\left\{F_{Sz,Qu}(t), F_{Pz,Pz}(t), F_{QQu,Qu}(t), F_{Pz,Qu}(t)\right\} \\ &= F_{QQu,Qu}(t) \end{aligned}$$

$$\begin{aligned} L_{QQu,Qu}(t) &= L_{Pz,QQu}(t) \\ &\leq \max\left\{L_{Sz,TQu}(t), L_{Pz,Su}(t), L_{QQu,TQu}(t), L_{Pz,TQu}(t)\right\} \\ &= \max\left\{L_{Sz,Qu}(t), L_{Pz,Pz}(t), L_{QQu,Qu}(t), L_{Pz,Qu}(t)\right\} \\ &= L_{QQu,Qu}(t) \end{aligned}$$

i.e., $QQu = Qu = TQu$.

Hence $Qu = Pz$ is a common fixed point of P, Q, S and T.

Uniqueness of Pz can easily follow from the contractive condition.

The proof is similar when Q and P are assumed compatible and reciprocally continuous.

Now, we give an example which supports our result.

Example 3.1. Let $X = [2, 20]$ with metric defined by $d(x, y) = |x - y|$ and for each $t \in [0, 1]$ defined

$$F_{x,y}(t) = \begin{cases} \frac{t}{t+|x-y|}, & \text{if } t > 0 \\ 0, & \text{if } t = 0 \end{cases}$$

$$L_{x,y}(t) = \begin{cases} \frac{|x-y|}{t+|x-y|}, & \text{if } t > 0 \\ 1, & \text{if } t = 0 \end{cases}$$

for all $x, y \in X$. Clearly $(X, \boldsymbol{F}, \boldsymbol{L}, \boldsymbol{*}, \lozenge)$ is an intuitionistic Menger space where $*$ is defined by $t * t \geq t$ and is defined by $(1 - t) \lozenge (1 - t) \leq (1 - t)$.

Define mappings $P, Q, R, S : X \to X$ by

$$P(X) = \begin{cases} 2 & \text{if } x = 2 \\ 3 & \text{if } x > 2 \end{cases}$$

$$S(X) = \begin{cases} 2 & \text{if } x = 2 \\ 6 & \text{if } x > 2 \end{cases}$$

$$Q(X) = \begin{cases} 2 & \text{if } x = 2 \text{ or } x > 5 \\ 6 & \text{if } 2 < x \leq 5 \end{cases}$$

$$T(X) = \begin{cases} 2 & \text{if } 2 \leq x \leq 5 \\ x - 3 & \text{if } x > 5. \end{cases}$$

Then P, Q, S and T satisfy all the conditions of above theorem with $k \in (0, 1)$ and have a unique common fixed point $x = 2$. Here P and S are reciprocally continuous compatible maps. But neither P nor S is continuous even at fixed point $x = 2$.

The mappings Q and T are non-compatible but Q is pointwise T-absorbing. To see Q and T are non-compatible, let us consider the sequence $\{x_n\}$ in X defined by

$$x_n = 5 + (1/n), \quad n \geq 1.$$

Then $\{Tx_n\} \to 2, \{Qx_n\} \to 2, \{TQx_n\} \to 2$ and $\{QTx_n\} \to 6$.

Hence Q and T are non-compatible.

Now, we prove the result by assuming the range of one of the mappings P, Q, S or T to be a complete subspace of X.

Theorem 3.2. *Let P be pointwise S-absorbing and Q be pointwise T-absorbing self maps on a complete intuitionistic Menger space $(X, \mathbf{F}, \mathbf{L}, \boldsymbol{*}, \lozenge)$ with continuous t-norm $*$ and continuous t-conorm $\lozenge$ defined by $a * b = \min\{a, b\}$ and $a \lozenge b = \max\{a, b\}$ for all $a, b \in [0, 1]$ satisfying the following conditions:*

(a) $P(X) \subseteq T(X),\ \ Q(X) \subseteq S(X)$;
(b) *there exists* $k \in (0,1)$ *such that for every* $x, y \in X$ *and* $t > 0$

$$F_{Px,Qy}(kt) \geq \min\left\{F_{Sx,Ty}(t), F_{Px,Sx}(t), F_{Qy,Ty}(t), F_{Px,Ty}(t)\right\}$$
$$L_{Px,Qy}(kt) \leq \max\left\{L_{Sx,Ty}(t), L_{Px,Sx}(t), L_{Qy,Ty}(t), L_{Px,Ty}(t)\right\} \text{ and}$$

(c) *for all* $x, y \in X$, $\lim_{n\to\infty} F_{x,y}(t) = 1$ *and* $\lim_{n\to\infty} L_{x,y}(t) = 0$.

If the range of one of the mapping P, Q, S *or* T *be a complete subspace of* X *then* P, Q, S *and* T *have a unique common fixed point in* X.

Proof. Let x_0 be any arbitrary point in X, construct a sequence $\{y_n\} \in X$ such that

$$y_{2n-1} = Tx_{2n-1} = Px_{2n-2} \text{ and } y_{2n} = Sx_{2n} = Qx_{2n+1},\ n = 1, 2, 3, ... \tag{3.2.1}$$

This can be done by virtue of (a) and by using the same technique of above theorem, we can show that $\{y_n\}$ is a Cauchy sequence.

Let $S(X)$ the range of X be a complete metric subspace then there exists a point Su such that $\lim_{n\to\infty} Sx_{2n} = Su$.

By (3.2.1), we get

$$Qx_{2n+1} \to Su,\ Px_{2n-2} \to Su,\ Tx_{2n-1} \to Su \text{ and } \{y_n\} \to Su \text{ as } n \to \infty.$$

By using contractive condition, we obtain

$$F_{Pu,Qx_{2n+1}}(kt) \geq \min\left\{F_{Su,Tx_{2n+1}}(t), F_{Pu,Su}(t), F_{Qx_{2n+1},Tx_{2n+1}}(t), F_{Pu,Tx_{2n+1}}(t)\right\}$$
$$L_{Pu,Qx_{2n+1}}(kt) \leq \max\left\{L_{Su,Tx_{2n+1}}(t), L_{Pu,Su}(t), L_{Qx_{2n+1},Tx_{2n+1}}(t), L_{Pu,Tx_{2n+1}}(t)\right\}.$$

Letting $n \to \infty$, we get

$$F_{Pu,Su}(kt) \geq \min\left\{F_{Su,Su}(t), F_{Pu,Su}(t), F_{Su,Su}(t), F_{Pu,Su}(t)\right\}$$
$$L_{Pu,Su}(kt) \leq \max\left\{L_{Su,Su}(t), L_{Pu,Su}(t), L_{Su,Su}(t), L_{Pu,Su}(t)\right\}$$

i.e., $Pu = Su$.

Since $P(X) \subseteq T(X)$ then there exists $w \in X$ such that $Su = Tw$.

Again, by using the contractive condition, we get

$$F_{Pu,Qw}(kt) \geq \min\left\{F_{Su,Tw}(t), F_{Pu,Su}(t), F_{Qw,Tw}(t), F_{Pu,Tw}(t)\right\}$$
$$L_{Pu,Qw}(kt) \leq \max\left\{L_{Su,Tw}(t), L_{Pu,Su}(t), L_{Qw,Tw}(t), L_{Pu,Tw}(t)\right\}$$

i.e., $Pu = Su = Qw = Tw$.

Since P is pointwise S-absorbing, then we have

$$F_{Su,SPu}(t) \geq F_{Su,Qu}(t/R)$$
$$L_{Su,SPu}(t) \leq L_{Su,Qu}(t/R)$$

i.e., $Su = SPu = SSu$ and, similarly, Q is pointwise T-absorbing then we have

$$F_{Tw,TQw}(t) \geq F_{Tw,Qw}(t/R)$$
$$L_{Tw,TQw}(t) \leq L_{Tw,Qw}(t/R)$$

i.e., $Tw = TQw = QQw$.

Thus, $Su(= Tw)$ is a common fixed point of P, Q, S and T.

Uniqueness of common fixed point follows from the contractive condition. The proof is similar when $T(X)$, the range of T, is assumed to be a complete subspace of X. Moreover, since $P(X) \subseteq T(X)$ and $Q(X) \subseteq S(X)$.

The proof follows on similar lines when either the range of P or range of Q is assumed complete. This completes the proof of the theorem.

Conclusion. As compatible mappings are absorbing mappings but reverse implication not follows. So, our Theorem 3.1 generalizes and extends the result of Rashwan and Hedar [5] in the sense that the mappings involved are absorbing mappings and we establish the theorems without appeal to continuity.

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THE INDEX OF A SPECIAL BIPARTITE GRAPH[1]

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Abstract. The Harary index of a graph is defined as the sum of reciprocals of distances between all pairs of vertices of the graph. In this paper we provide an upper bound of the Harary index in the class of all connected n-vertex bipartite graphs with a given matching number q. We characterize the unique graph with the maximum Harary index in the class of all connected n-vertex bipartite graphs with a given matching number q.
Keywords: Harary index; matching number; bipartite graphs.

1. Introduction

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. The distance between two vertices u and v of G, denoted by $d_G(u, v)$, is defined as the minimum length of the paths between u and v in G. The Harary index of a graph G, denoted by $H(G)$, has been introduced independently by Plavšić et al. [21] and by Ivanciuc et al. [19] in 1993 for the characterization of molecular graphs. It has been named in honor of Professor Frank Harary on the occasion of his 70th birthday. The Harary index $H(G)$ is defined as the sum of reciprocals of distances between all pairs of vertices of the graph G, i.e.,

$$H(G) = \sum_{u,v \in V(G)} \frac{1}{d_G(u, v)} \cdot$$

Mathematical properties and applications of the Harary index are reported in [15], [17], [20]. Note that in any disconnected graph G, the distance is infinite between any two vertices from two distinct components. Therefore, its reciprocal can be viewed as 0. Thus, we can define validly the Harary index of disconnected graph G as follows:

$$H(G) = \sum_{i=1}^{k} H(G_i),$$

where $G_1, G_2, ..., G_k$ are the components of G.

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Another distance-based topological index of a graph G is the Wiener index, denoted by $W(G)$. As an oldest topological index, the Wiener index of a graph G, first introduced by Wiener [22] in 1947, was defined as

$$W(G) = \sum_{u,v \in V(G)} d_G(u,v).$$

The motivation for introduction of the Harary index was pragmatic – the aim was to design a distance index differing from the Wiener index in that the contributions to it from the distant atoms in a molecule should be much smaller than from near atoms, since in many instances the distant atoms influence each other much less than near atoms.

Let $\gamma(G,k)$ be the number of vertex pairs of the graph G that are at distance k. Then

$$H(G) = \sum_{k \geq 1} \frac{1}{k} \gamma(G,k). \tag{1.1}$$

It will be convenient to determine the exact value by equation (1.1) for some graphs with the simple structure (e.g., the graphs with small diameter), but in general it is very difficult to give the exact value of $\gamma(G,k)$. So it is very useful to provide upper or lower bounds for the Harary index; see, e.g., [1], [5], [16]. In addition, the extremal Harary index of a given class of graphs has also been studied extensively; see, e.g., [4], [6], [11]–[15].

In this paper, we provide an upper bound of the Harary index in the class of all connected n-vertex bipartite graphs with a given matching number q. We characterize the unique graph with the maximum Harary index in the class of all connected n-vertex bipartite graphs with a given matching number q.

In this paper, we only consider connected, simple and undirected graphs. Let $G = (V_G, E_G)$ be a graph with $u, v \in V_G$. Then $G - v$, $G - uv$ denote the graph obtained from G by deleting vertex $v \in V_G$, or edge $uv \in E_G$, respectively (this notation is naturally extended if more than one vertex or edge is deleted). Similarly, $G + uv$ is obtained from G by adding an edge $uv \notin E_G$. For $v \in V_G$, let $N_G(v)$ denote the set of all the adjacent vertices of v in G and $d(v) = |N_G(v)|$, the degree of v in G. In particular, let $\Delta(G) = \max\{d(x) \mid x \in V_G\}$ and $\delta(G) = \min\{d(x) \mid x \in V_G\}$. For convenience, let $N_G[u] = N_G(u) \cup \{u\}$. The distance $d(u,v)$ between vertices u and v in G is defined as the length of a shortest path between them. The diameter of G is the maximal distance between any two vertices of G. $D_G(u)$ denotes the sum of all distances from u in G.

Recall that G is called k-connected if $|G| > k$ and $G - X$ is connected for every set $X \in V_G$ with $|X| < k$. The greatest integer k such that G is k-connected is the connectivity $\kappa(G)$ of G. Thus, $\kappa(G) = 0$ if and only if G is disconnected or K_1, and $\kappa(K_n) = n - 1$ for all $n \geq 1$.

A bipartite graph G is a simple graph, whose vertex set V_G can be partitioned into two disjoint subsets V_1 and V_2 such that every edge of G joins a vertex of V_1 with a vertex of V_2. A bipartite graph in which every two vertices from different

partition classes are adjacent is called complete, which is denoted by $K_{m,n}$, where $m = |V_1|$, $n = |V_2|$.

A vertex (edge) independent set of a graph G is a set of vertices (edges) such that any two distinct vertices (edges) of the set are not adjacent (incident on a common vertex). The vertex (edge) independence number of G, denoted by $\alpha(G)$ ($\alpha'(G)$), is the maximum of the cardinalities of all vertex (edge) independent sets. A vertex (edge) cover of a graph G is a set of vertices (edges) such that each edge (vertex) of G is incident with at least one vertex (edge) of the set. The vertex (edge) cover number of G, denoted by $\beta(G)$ ($\beta'(G)$), is the minimum of the cardinalities of all vertex (edge) covers. For a connected graph G of order n, its matching number $\alpha'(G)$ satisfies $1 \leq \alpha'(G) \leq [\frac{n}{2}]$. When we consider an edge cover of a graph, we always assume that the graph contains no isolated vertex. It is known that for a graph G of order n, $\alpha(G) + \beta(G) = n$; and if in addition G has no isolated vertex, then $\alpha'(G) + \beta'(G) = n$. For a bipartite graph G, one has $\alpha;(G) = \beta(G)$, and $\alpha(G) = \beta'(G)$.

Let B_n^q be the class of all bipartite graphs of order n with matching number q. In this paper we study the quantity $H(G)$ in the case of n-vertex bipartite graphs, which is an important class of graphs in graph theory. Based on the structure of bipartite graphs, sharp bounds on $H(G)$ among B_n^q are determined. The corresponding extremal graphs are identified, respectively.

2. Maximum index among B_n^q

In this section, we determine the sharp bound on the transmission of all n-vertex bipartite graphs with matching q. The unique corresponding extremal graph is identified.

Lemma 2.1. ([12]). *Let G be a graph with $u, v \in V(G)$. If $uv \notin E(G)$, then $H(G) < H(G + uv)$. If $uv \in E(G)$, then $H(G) < H(G - uv)$.*

Theorem 2.2. *Let G be in B_n^q . Then $H(G) \leq \frac{1}{4}n^2 - \frac{1}{2}q^2 + \frac{1}{2}nq - \frac{1}{4}n$ with equality if and only if $G \cong K_{q,n-q}$.*

Proof. It is routine to check that

$$\begin{aligned} H(K_{q,n-q}) &= \frac{1}{2}\left[(n-q)q + \frac{1}{2}(q-1)q + (n-q)q + \frac{1}{2}(n-q-1)(n-q)\right] \\ &= \frac{1}{2}\left[2nq - \frac{3}{2}q^2 - \frac{1}{2}q + \frac{1}{2}n^2 + \frac{1}{2}q^2 - nq - \frac{1}{2}n + \frac{1}{2}q\right] \\ &= \frac{1}{2}\left[\frac{1}{2}n^2 - q^2 + nq - \frac{1}{2}n\right] = \frac{1}{4}n^2 - \frac{1}{2}q^2 + \frac{1}{2}nq - \frac{1}{4}n \end{aligned}$$

So, in what follows, we show that $K_{q,n-q}$ is the unique graph in B_n^q with the maximum transmission.

Choose G in B_n^q such that its transmission is as large as possible. If $q = \left[\frac{n}{2}\right]$, by Lemma 2.1 the extremal graph is just $K_{[\frac{n}{2}],[\frac{n}{2}]}$, as desired. So, in what follows, we consider $q < \left[\frac{n}{2}\right]$ ·



Figure 1

Let (U, W) be the bipartition of the vertex set of G such that $|W| \geq |U| \geq q$, and let M be a maximal matching of G. By Lemma 2.1, the sum of all distances of a graph decreases with addition of edges, so if $|U| = q$, then the extremal graph is $G \cong K_{q,n-q}$. So, we assume that $|U| > q$ in what follows. Let U_M, W_M be the sets of vertices of U, W which are incident to the edges of M, respectively. Therefore, $|U_M| = |W_M| = q$. Note that G contains no edges between the vertices of $U \setminus U_M$ and the vertices of $W \setminus W_M$, otherwise any such edge may be united with M to produce a matching of cardinality greater than that of M, violating the maximality of M.

Adding all possible edges between the vertices of U_M and W_M, U_M and $W \setminus W_M$, $U \setminus U_M$ and W_M, we get a graph G' with $H(G') > H(G)$. Note that the matching number of G' is at least $k+1$. Hence, $G' \notin B_n^k$ and $G \cong G'$. Based on G', we construct a new graph, say G'', which is obtained from G' by deleting all the edges between $U \setminus U_M$ and W_M, and adding all the edges between $U \setminus U_M$ and W_M. G'' is depicted in Fig. 1. It is routine to check that $G'' \cong K_{k,n-k}$.

Let $|U \setminus U_M| = n_1$, $|W \setminus W_M| = n_2$. Suppose $n_1 \leq n_2$. We partition $V_{G'} = V_{G''}$ into $U_M \cup W_M \cup (U \setminus U_{<M}) \cup (W \setminus W_M)$ as shown in Fig. 1. By direct calculation, for all $x \in W \setminus W_M$ (resp. $y \in U_M$, $z \in W_M$, $w \in U \setminus U_M$), one has

$$\begin{aligned}
D_G(x) &= q + \frac{1}{2}q + \frac{1}{3}n_1 + \frac{1}{2}(n_2 - 1) = \frac{3}{2}q + \frac{1}{3}n_1 + \frac{1}{2}n_2 - \frac{1}{2} \\
D_{G'}(x) &= q + \frac{1}{2}(q + n_1) + \frac{1}{2}(n_2 - 1) = \frac{3}{2}q + \frac{1}{2}n_1 + \frac{1}{2}n_2 - \frac{1}{2} \\
D_G(y) &= n_2 + q + \frac{1}{2}n_1 + \frac{1}{2}(q - 1) = \frac{3}{2}q + \frac{1}{2}n_1 + n_2 - \frac{1}{2} \\
D_{G'}(y) &= n_1 + n_2 + q + \frac{1}{2}(q - 1) = \frac{3}{2}q + n_1 + n_2 - \frac{1}{2} \\
D_G(z) &= q + \frac{1}{2}n_2 + n_1 + \frac{1}{2}(q - 1) = \frac{3}{2}q + \frac{1}{2}n_2 + n_1 - \frac{1}{2} \\
D_{G'}(z) &= q + \frac{1}{2}(n_2 + n_1) + \frac{1}{2}(q - 1) = \frac{3}{2}q + \frac{1}{2}n_2 + \frac{1}{2}n_1 - \frac{1}{2} \\
D_G(w) &= q + \frac{1}{2}q + \frac{1}{3}n_2 + \frac{1}{2}(n_1 - 1) = \frac{3}{2}q + \frac{1}{2}n_1 + \frac{1}{3}n_2 - \frac{1}{2} \\
D_{G'}(w) &= q + \frac{1}{2}(q + n_2) + \frac{1}{2}(n_1 - 1) = \frac{3}{2}q + \frac{1}{2}n_1 + \frac{1}{2}n_2 - \frac{1}{2}
\end{aligned}$$

$$
\begin{aligned}
H(G')-H(G) &= \frac{1}{2}\left(\sum D_{G'}(x)-\sum D_G(x)\right)\\
&= \frac{1}{2}\left(\sum_{x\in W\setminus W_m} D_{G'}(x)-\sum_{x\in W\setminus W_m} D_G(x)+\sum_{y\in U_m} D_{G'}(y)-\sum_{y\in U_m} D_G(y)\right.\\
&\quad \left.+\sum_{z\in W_m} D_{G'}(z)-\sum_{z\in W_m} D_G(z)+\sum_{w\in U\setminus U_m} D_{G'}(w)-\sum_{w\in U\setminus U_m} D_G(w)\right)\\
&= \frac{1}{2}\left[n_2\left(\frac{3}{2}q+\frac{1}{2}n_1+\frac{1}{2}n_2-\frac{1}{2}\right)-n_2\left(\frac{3}{2}q+\frac{1}{3}n_1+\frac{1}{2}n_2-\frac{1}{2}\right)\right.\\
&\quad +q\left(\frac{3}{2}q+n_1+n_2-\frac{1}{2}\right)-q\left(\frac{3}{2}q+\frac{1}{2}n_1+n_2-\frac{1}{2}\right)\\
&\quad +q\left(\frac{3}{2}q+\frac{1}{2}n_2+\frac{1}{2}n_1-\frac{1}{2}\right)-q\left(\frac{3}{2}q+\frac{1}{2}n_2+n_1-\frac{1}{2}\right)\\
&\quad \left.-n_1\left(\frac{3}{2}q+\frac{1}{2}n_1+\frac{1}{2}n_2-\frac{1}{2}\right)-n_1\left(\frac{3}{2}q+\frac{1}{2}n_1+\frac{1}{3}n_2-\frac{1}{2}\right)\right]\\
&= \frac{1}{6}n_1n_2>0
\end{aligned}
$$

This completes the proof. ∎

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CENTRALIZERS ON PRIME AND SEMIPRIME GAMMA RINGS

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Abstract. Let M be a noncommutative 2-torsion free semiprime Γ-ring satisfying a certain assumption and let S and T be left centralizers on M. We prove the following results:

(i) If $[S(x), T(x)]_\alpha \beta S(x) + S(x)\beta[S(x), T(x)]_\alpha = 0$ holds for all $x \in M$ and $\alpha, \beta \in \Gamma$, then $[S(x), T(x)]_\alpha = 0$.

(ii) If $S \neq 0 (T \neq 0)$, then there exists $\lambda \in C$,(the extended centroid of M) such that $T = \lambda\alpha S$ $(S = \lambda\alpha T)$ for all $\alpha \in \Gamma$.

(iii) Suppose that $[[S(x), T(x)]_\alpha, S(x)]_\beta = 0$ holds for all $x \in M$ and $\alpha, \beta \in \Gamma$. Then $[S(x), T(x)]_\alpha = 0$ for all $x \in M$ and $\alpha \in \Gamma$.

(iv) If M is a prime Γ-ring satisfying a certain assumption and $S \neq 0 (T \neq 0)$, then there exists $\lambda \in C$, the extended centroid, such that $T = \lambda\alpha S (S = \lambda\alpha T)$.

Keywords: prime Γ-ring, semiprime Γ-ring, central closure, extended centroid, left(right) centralizer.

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1. Introduction

The notion of a Γ-ring was first introduced as an extensive generalization of the concept of a classical ring. From its first appearance, the extensions and generalizations of various important results in the theory of classical rings to the theory of Γ-rings have been attracted a wider attentions as an emerging area of research

to the modern algebraists to enrich the world of algebra. All over the world, many prominent mathematicians have worked out on this interesting area of research to determine many basic properties of Γ-rings and have executed more productive and creative results of Γ-rings in the last few decades. We begin with the definition.

Let M and Γ be additive abelian groups. If there exists an additive mapping $(x, \alpha, y) \longrightarrow x\alpha y$ of $M \times \Gamma \times M \longrightarrow M$, which satisfies the conditions $(x\alpha y)\beta z = x\alpha(y\beta z)$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$, then M is called a Γ-ring.

Every ring M is a Γ-ring with M=Γ. However a Γ-ring need not be a ring. Gamma rings, more general than rings, were introduced by Nobusawa [8]. Barnes [1] weakened slightly the conditions in the definition of Γ-ring in the sense of Nobusawa.

Let M be a Γ-ring. Then an additive subgroup U of M is called a left (right) ideal of M if $M\Gamma U \subset U (U\Gamma M \subset U)$. If U is both a left and a right ideal, then we say U is an ideal of M. Suppose again that M is a Γ-ring. Then M is said to be a 2-torsion free if $2x$=0 implies x=0 for all $x \in M$.

An ideal P_1 of a Γ-ring M is said to be prime if for any ideals A and B of M, $A\Gamma B \subseteq P_1$ implies $A \subseteq P_1$ or $B \subseteq P_1$. An ideal P_2 of a Γ-ring M is said to be semiprime if for any ideal U of M, $U\Gamma U \subseteq P_2$ implies $U \subseteq P_2$. A Γ-ring M is said to be prime if $a\Gamma M\Gamma b$=(0) with $a, b \in M$, implies a=0 or b=0 and semiprime if $a\Gamma M\Gamma a$=(0) with $a \in M$ implies a=0. Furthermore, M is said to be commutative Γ-ring if $x\alpha y$=$y\alpha x$ for all $x, y \in M$ and $\alpha \in \Gamma$.

Moreover, the set $Z(M) = \{x \in M : x\alpha y = y\alpha x$ for all $\alpha \in \Gamma, y \in M\}$ is called the centre of the Γ-ring M.

For the definitions of the quotent Γ-ring, the extended centroid and the central closure, we refer to [7].

If M is a Γ-ring, then $[x, y]_\alpha$=$x\alpha y - y\alpha x$ is known as the commutator of x and y with respect to α, where $x, y \in M$ and $\alpha \in \Gamma$. We make the basic commutator identities:

$$[x\alpha y, z]_\beta = [x, z]_\beta \alpha y + x[\alpha, \beta]_z y + x\alpha[y, z]_\beta \quad \text{and}$$
$$[x, y\alpha z]_\beta = [x, y]_\beta \alpha z + y[\alpha, \beta]_x z + y\alpha[x, z]_\beta \quad \text{for all } x, y, z \in M \text{ and } \alpha, \beta \in \Gamma.$$

We consider the following assumption:

(*A*) $x\alpha y\beta z$=$x\beta y\alpha z$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$.

According to assumption (*A*), the above two identities reduce to

$$[x\alpha y, z]_\beta = [x, z]_\beta \alpha y + x\alpha[y, z]_\beta \quad \text{and}$$
$$[x, y\alpha z]_\beta = [x, y]_\beta \alpha z + y\alpha[x, z]_\beta,$$

which we extensively used.

An additive mapping $T : M \to M$ is a left(right) centralizer if $T(x\alpha y)$=$T(x)\alpha y$ ($T(x\alpha y) = x\alpha T(y)$) holds for all $x, y \in M$ and $\alpha \in \Gamma$. A centralizer is an additive mapping which is both a left and a right centralizer. For any fixed $a \in M$ and $\alpha \in \Gamma$, the mapping $T(x) = a\alpha x$ is a left centralizer and $T(x) = x\alpha a$ is a right

centralizer. We shall restrict our attention on left centralizer since all results represented in this paper are true also for right centralizers because of left and right symmetry.

Barnes [1], Lue [6] and Kyuno [5] studied the structure of Γ-rings and obtained various generalizations of corresponding parts in ring theory.

Borut Zalar [14] worked on centralizers of semiprime rings and proved that Jordan centralizers and centralizers of these rings coincide. Joso Vukman[11], [12], [13] developed some remarkable results using centralizers on prime and semiprime rings.

In [2], Hoque and Paul have proved that every Jordan centralizer of a 2-torsion free semiprime Γ-ring satisfying a certain assumption is a centralizer. Also, they proved in [3], if T is an additive mapping on a 2-torsion free semiprime Γ-ring M with a certain assumption such that $T(x\alpha y\beta x) = x\alpha T(y)\beta x$ for all $x, y \in M$ and $\alpha, \beta \in \Gamma$, then T is a centralizer and in [4], if $2T(x\alpha b\beta a) = T(x)\alpha y\beta x + x\alpha y\beta T(x)$ holds for all $x, y \in M$ and $\alpha, \beta \in \Gamma$, then T is also a centralizer.

In this paper, we generalize some results of Joso Vukman [11] in Γ-rings.

2. Centralizers of prime and semiprime Γ-rings

To prove our main results, we need the following lammas:

Lemma 1.1 *Suppose that the elements a_i, b_i in the central closure of a prime Γ-ring M satisfy $\Sigma a_i\alpha_i x\beta_i b_i = 0$ for all $x \in M$ and $\alpha_i, \beta_i \in \Gamma$. If $b_i \neq 0$ for some i, then a_i's are C-dependent, where C is the extended centroid.*

Proof. Let M be a prime Γ-ring and let $C_\Gamma = C$ be the extended centroid of M. If a_i and b_i are non-zero elements of M such that $\Sigma a_i\alpha_i x\beta_i b_i = 0$ for all $x \in M$ and $\alpha_i, \beta_i \in \Gamma$, then a_i's (also b_i's) are linearly dependent over C. Moreover, if $a\alpha x\beta b = b\alpha x\beta a$ for all $x \in M$ and $\alpha, \beta \in \Gamma$, where $a(\neq 0)$, $b \in M$ are fixed, then there exists $\lambda \in C$ such that $a = \lambda\alpha b$ for $\alpha \in \Gamma$. Clearly, the lemma is proved. ∎

Lemma 1.2 *Suppose that M is a noncommutative prime Γ-ring satisfying the assumption (A) and $T : M \to M$ is a left centralizer. If $T(x) \in Z(M)$ for all $x \in M$, then $T = 0$.*

Proof. Since T is a left centralizer on M, we have $T(x\alpha y) = T(x)\alpha y$ holds for all $x, y \in M$ and $\alpha \in \Gamma$ and hence $[T(x), y]_\alpha = 0$ for all $x, y \in M$ and $\alpha \in \Gamma$. Putting $x = x\beta z$ in the above relation, we have

$$\begin{aligned} 0 &= [T(x\beta z), y]_\alpha \\ &= [T(x)\beta z, y]_\alpha \\ &= [T(x), y]_\alpha\beta z + T(x)\beta[z, y]_\alpha \\ &= T(x)\beta[z, y]_\alpha \end{aligned}$$

Hence $T(x)\beta[z, y]_\alpha = 0$, which gives $T(x)\beta w\gamma[z, y]_\alpha = 0$ for all $x, y.z, w \in M$ and $\alpha, \beta, \gamma \in \Gamma$, whence it follows that $T = 0$, otherwise $M = 0$. ∎

Lemma 1.3 *Suppose that M is a noncommutative prime Γ-ring satisfying the assumption (A) and $S, T : M \to M$ are left centralizers. If $[S(x), T(x)]_\alpha = 0$ holds for all $x \in M$ and $\alpha \in \Gamma$ and $T \neq 0$, then there exists $\lambda \in C$ such that $S = \lambda\alpha T$, where C is the extended centroid.*

Proof. First, we put $x = x + y$ in the relation $[S(x), T(x)]_\alpha = 0$ and linearize, we have

$$[S(x), T(y)]_\alpha + [S(y), T(x)]_\alpha = 0. \tag{1}$$

Replace y by $y\beta z$ in (1), we have

$$\begin{aligned} 0 &= [S(x), T(y)\beta z]_\alpha + [S(y)\beta z, T(x)]_\alpha \\ &= [S(x), T(y)]_\alpha\beta z + T(y)\beta[S(x), z]_\alpha + [S(y), T(x)]_\alpha\beta z + S(y)\beta[z, T(x)]_\alpha \\ &= T(y)\beta[S(x), z]_\alpha + S(y)\beta[z, T(x)]_\alpha \end{aligned}$$

Thus we have

$$T(y)\beta[S(x), z]_\alpha + S(y)\beta[z, T(x)]_\alpha = 0$$

Putting $y = y\gamma w$ in the above relation, we obtain,

$$T(y)\gamma w\beta[S(x), z]_\alpha + S(y)\gamma w\beta[z, T(x)]_\alpha = 0 \tag{2}$$

Our assumption $T \neq 0$ follows from Lemma 2.2 that there exist $x, z \in M$ and $\alpha \in \Gamma$ such that $[T(x), z]_\alpha \neq 0$. Now, relation (2) and Lemma 2.1 imply that $S(y) = \lambda(y)\alpha T(y)$, where $\lambda(y)$ is from C. If we put $S(y) = \lambda(y)\alpha T(y)$ and $S(x) = \lambda(x)\alpha T(x)$ in relation (2), we obtain $(\lambda(x) - \lambda(y))\alpha T(y)\gamma w\beta[T(x), z]_\alpha = 0$ for all pairs $y, w \in M$, whence it follows $(\lambda(x) - \lambda(y))\alpha T(y) = 0$, since $[T(x), z]_\alpha \neq 0$. Thus we have $\lambda(x)\alpha T(y) = \lambda(y)\alpha T(y)$ which completes the proof of the lemma. ∎

Theorem 1.1 *Suppose that M is a 2-torsion free noncommutative semiprime Γ-ring satisfying the assumption (A) and S, T are left centralizers on M. If $[S(x), T(x)]_\alpha\beta S(x) + S(x)\beta[S(x), T(x)]_\alpha = 0$ holds for all $x \in M$ and $\alpha, \beta \in \Gamma$. Then $[S(x), T(x)]_\alpha = 0$ for all $x \in M$ and $\alpha \in \Gamma$. Also, if M is prime Γ-ring satisfying the assumption (A) and $S \neq 0 (T \neq 0)$, then there exists $\lambda \in C$,(the extended centroid of M) such that $T = \lambda\alpha S (S = \lambda\alpha T)$.*

Proof. By the hypothesis, we have

$$[S(x), T(x)]_\alpha\beta S(x) + S(x)\beta[S(x), T(x)]_\alpha = 0 \tag{3}$$

The linearization of the above relation, we have

$$\begin{aligned} 0 = {} & [S(x), T(x)]_\alpha\beta S(y) + S(y)\beta[S(x), T(x)]_\alpha \\ & + [S(x), T(y)]_\alpha\beta S(x) + S(x)\beta[S(x), T(y)]_\alpha \\ & + [S(y), T(x)]_\alpha\beta S(x) + S(x)\beta[S(y), T(x)]_\alpha \\ & + [S(y), T(y)]_\alpha\beta S(x) + S(x)\beta[S(y), T(y)]_\alpha \\ & + [S(y), T(x)]_\alpha\beta S(y) + S(y)\beta[S(y), T(x)]_\alpha \\ & + [S(x), T(y)]_\alpha\beta S(y) + S(y)\beta[S(x), T(y)]_\alpha \end{aligned} \tag{4}$$

Replacing $-x$ for x in the above relation, we have

$$\begin{aligned}(5)\qquad 0 &= [S(x),T(x)]_\alpha\beta S(y)+S(y)\beta[S(x),T(x)]_\alpha\\ &\quad +[S(x),T(y)]_\alpha\beta S(x)+S(x)\beta[S(x),T(y)]_\alpha\\ &\quad +[S(y),T(x)]_\alpha\beta S(x)+S(x)\beta[S(y),T(x)]_\alpha\\ &\quad -[S(y),T(y)]_\alpha\beta S(x)-S(x)\beta[S(y),T(y)]_\alpha\\ &\quad -[S(y),T(x)]_\alpha\beta S(y)-S(y)\beta[S(y),T(x)]_\alpha\\ &\quad -[S(x),T(y)]_\alpha\beta S(y)-S(y)\beta[S(x),T(y)]_\alpha\end{aligned}$$

Adding (4) and (5), we have

$$\begin{aligned}0 &= 2[S(x),T(x)]_\alpha\beta S(y)+2S(y)\beta[S(x),T(x)]_\alpha\\ &\quad +2[S(x),T(y)]_\alpha\beta S(x)+2S(x)\beta[S(x),T(y)]_\alpha\\ &\quad +2[S(y),T(x)]_\alpha\beta S(x)+2S(x)\beta[S(y),T(x)]_\alpha\end{aligned}$$

Hence by 2-torsion freeness of M, it follows that

$$\begin{aligned}(6)\qquad 0 &= [S(x),T(x)]_\alpha\beta S(y)+S(y)\beta[S(x),T(x)]_\alpha\\ &\quad +[S(x),T(y)]_\alpha\beta S(x)+S(x)\beta[S(x),T(y)]_\alpha\\ &\quad +[S(y),T(x)]_\alpha\beta S(x)+S(x)\beta[S(y),T(x)]_\alpha\end{aligned}$$

Replacing y by $x\gamma y$ in the above relation, we have

$$\begin{aligned}0 &= [S(x),T(x)]_\alpha\beta S(x)\gamma y+S(x)\gamma y\beta[S(x),T(x)]_\alpha\\ &\quad +[S(x),T(x)\gamma y]_\alpha\beta S(x)+S(x)\beta[S(x),T(x)\gamma y]_\alpha\\ &\quad +[S(x)\gamma y,T(x)]_\alpha\beta S(x)+S(x)\beta[S(x)\gamma y,T(x)]_\alpha\\ &= [S(x),T(x)]_\alpha\beta S(x)\gamma y+S(x)\gamma y\beta[S(x),T(x)]_\alpha\\ &\quad +[S(x),T(x)]_\alpha\gamma y\beta S(x)+T(x)\gamma[S(x),y]_\alpha\beta S(x)\\ &\quad +S(x)\beta[S(x),T(x)]_\alpha\gamma y+S(x)\beta T(x)\gamma[S(x),y]_\alpha\\ &\quad +[S(x),T(x)]_\alpha\gamma y\beta S(x)+S(x)\gamma[y,T(x)]_\alpha\beta S(x)\\ &\quad +S(x)\beta[S(x),T(x)]_\alpha\gamma y+S(x)\beta S(x)\gamma[y,T(x)]_\alpha\end{aligned}$$

According to (6), the above relation reduces to

$$\begin{aligned}(7)\qquad 0 &= S(x)\gamma y\beta[S(x),T(x)]_\alpha+2[S(x),T(x)]_\alpha\gamma y\beta S(x)\\ &\quad +T(x)\gamma[S(x),y]_\alpha\beta S(x)+S(x)\beta T(x)\gamma[S(x),y]_\alpha\\ &\quad +S(x)\gamma[y,T(x)]_\alpha\beta S(x)+S(x)\beta[S(x),T(x)]_\alpha\gamma y\\ &\quad +S(x)\beta S(x)\gamma[y,T(x)]_\alpha\end{aligned}$$

Putting $y=y\delta S(x)$ in (7), we obtain

$$\begin{aligned}0 &= S(x)\gamma y\delta S(x)\beta[S(x),T(x)]_\alpha+2[S(x),T(x)]_\alpha\gamma y\delta S(x)\beta S(x)\\ &\quad +T(x)\gamma[S(x),y]_\alpha\beta S(x)\delta S(x)+S(x)\beta T(x)\gamma[S(x),y]_\alpha\delta S(x)\\ &\quad +S(x)\gamma[y,T(x)]_\alpha\delta S(x)\beta S(x)+S(x)\gamma y\delta[S(x),T(x)]_\alpha\beta S(x)\\ &\quad +S(x)\beta[S(x),T(x)]_\alpha\gamma y\delta S(x)+S(x)\beta S(x)\gamma y\delta[S(x),T(x)]_\alpha\\ &\quad +S(x)\beta S(x)\gamma[y,T(x)]_\alpha\delta S(x)\end{aligned}$$

which gives, according to (7), to

$$0 = S(x)\gamma y\delta S(x)\beta[S(x),T(x)]_\alpha + S(x)\beta S(x)\gamma y\delta[S(x),T(x)]_\alpha \tag{8}$$

Putting $y = T(x)\omega y$ in (8), we have

$$\begin{aligned} 0 &= S(x)\gamma T(x)\omega y\delta S(x)\beta[S(x),T(x)]_\alpha \\ &\quad +S(x)\beta S(x)\gamma T(x)\omega y\delta[S(x),T(x)]_\alpha \end{aligned} \tag{9}$$

Also left multiplication of (8) by $T(x)\omega$ gives

$$\begin{aligned} 0 &= T(x)\omega S(x)\gamma y\delta S(x)\beta[S(x),T(x)]_\alpha \\ &\quad +T(x)\omega S(x)\beta S(x)\gamma y\delta[S(x),T(x)]_\alpha \end{aligned} \tag{10}$$

From (9) and (10), we obtain,

$$\begin{aligned} 0 &= [S(x),T(x)]_\gamma\omega y\delta S(x)\beta[S(x),T(x)]_\alpha \\ &\quad +[S(x)\beta S(x),T(x)]_\gamma\omega y\delta[S(x),T(x)]_\alpha \\ &= [S(x),T(x)]_\gamma\omega y\delta S(x)\beta[S(x),T(x)]_\alpha \\ &\quad +([S(x),T(x)]_\gamma\beta S(x) + S(x)\beta[S(x),T(x)]_\gamma)\omega y\delta[S(x),T(x)]_\alpha \\ &= [S(x),T(x)]_\gamma\omega y\delta S(x)\beta[S(x),T(x)]_\alpha \end{aligned}$$

Thus we have

$$0 = [S(x),T(x)]_\gamma\omega y\delta S(x)\beta[S(x),T(x)]_\alpha$$

Left multiplication of the above relation by $S(x)\beta$ gives

$$0 = S(x)\beta[S(x),T(x)]_\gamma\omega y\delta S(x)\beta[S(x),T(x)]_\alpha \tag{11}$$

for all $x, y \in M$ and $\alpha, \beta, \gamma, \delta, \omega \in \Gamma$. Hence, from (11), it follows

$$S(x)\beta[S(x),T(x)]_\alpha = 0 \tag{12}$$

From (3) and (12), we have also

$$[S(x),T(x)]_\alpha\beta S(x) = 0 \tag{13}$$

From (12) one obtains the relation

$$\begin{aligned} 0 &= S(y)\beta[S(x),T(x)]_\alpha + S(x)\beta[S(y),T(x)]_\alpha \\ &\quad +S(x)\beta[S(x),T(y)]_\alpha \end{aligned} \tag{14}$$

(see the proof of (6)). Putting $y = x\gamma y$ in (14), we have

$$\begin{aligned} 0 &= S(x)\gamma y\beta[S(x),T(x)]_\alpha + S(x)\beta[S(x)\gamma y,T(x)]_\alpha \\ &\quad +S(x)\beta[S(x),T(x)\gamma y]_\alpha \\ &= S(x)\gamma y\beta[S(x),T(x)]_\alpha + S(x)\beta[S(x),T(x)]_\alpha\gamma y \\ &\quad +S(x)\beta S(x)\gamma[y,T(x)]_\alpha + S(x)\beta[S(x),T(x)]_\alpha\gamma y \\ &\quad +S(x)\beta T(x)\gamma[S(x),y]_\alpha \\ &= S(x)\gamma y\beta[S(x),T(x)]_\alpha + S(x)\beta S(x)\gamma[y,T(x)]_\alpha \\ &\quad +S(x)\beta T(x)\gamma[S(x),y]_\alpha \end{aligned}$$

Thus we have the above relation

$$\begin{aligned} 0 &= S(x)\gamma y\beta[S(x),T(x)]_\alpha + S(x)\beta S(x)\gamma[y,T(x)]_\alpha \\ &\quad +S(x)\beta T(x)\gamma[S(x),y]_\alpha \end{aligned}$$

which can be written in the form

$$\begin{aligned} 0 = \; & S(x)\gamma y\beta[S(x),T(x)]_\alpha + S(x)\beta S(x)\gamma y\alpha T(x) \\ & -S(x)\beta T(x)\gamma y\alpha S(x) + S(x)\beta[T(x),S(x)]_\gamma \alpha y \end{aligned}$$

whence it follows

$$0 = S(x)\gamma y\beta[S(x),T(x)]_\alpha + S(x)\beta S(x)\gamma y\alpha T(x) - S(x)\beta T(x)\gamma y\alpha S(x) \tag{15}$$

according to (12). Taking $T(x)\delta$ of (15) on the left side, we have

$$\begin{aligned} 0 = \; & T(x)\delta S(x)\gamma y\beta[S(x),T(x)]_\alpha + T(x)\delta S(x)\beta S(x)\gamma y\alpha T(x) \\ & -T(x)\delta S(x)\beta T(x)\gamma y\alpha S(x) \end{aligned} \tag{16}$$

Putting $y = T(x)\delta y$ in (15) gives

$$\begin{aligned} 0 = \; & S(x)\gamma T(x)\delta y\beta[S(x),T(x)]_\alpha + S(x)\beta S(x)\gamma T(x)\delta y\alpha T(x) \\ & -S(x)\beta T(x)\gamma T(x)\delta y\alpha S(x) \end{aligned} \tag{17}$$

From (16) and (17), we have

$$\begin{aligned} 0 = \; & [S(x),T(x)]_\gamma \delta y\beta[S(x),T(x)]_\alpha + [S(x)\beta S(x),T(x)]_\gamma \delta y\alpha T(x) \\ & +[T(x),S(x)]_\delta \beta T(x)\gamma y\alpha S(x) \\ = \; & [S(x),T(x)]_\gamma \delta y\beta[S(x),T(x)]_\alpha + ([S(x),T(x)]_\gamma \beta S(x) \\ & +S(x)\beta[S(x),T(x)]_\gamma)\delta y\alpha T(x) + [T(x),S(x)]_\delta \beta T(x)\gamma y\alpha S(x) \end{aligned}$$

which reduces to

$$0 = [S(x),T(x)]_\gamma \delta y\beta[S(x),T(x)]_\alpha + [T(x),S(x)]_\delta \beta T(x)\gamma y\alpha S(x) \tag{18}$$

The substitution $y\omega S(x)\rho z$ for y in (18) gives

$$\begin{aligned} 0 &= [S(x),T(x)]_\gamma \delta y\omega S(x)\rho z\beta[S(x),T(x)]_\alpha \\ &\quad +[T(x),S(x)]_\delta \beta T(x)\gamma y\omega S(x)\rho z\alpha S(x) \end{aligned} \tag{19}$$

Again, right multiplication of (18) by $\omega z\rho S(x)$, we have

$$\begin{aligned} 0 &= [S(x),T(x)]_\gamma \delta y\beta[S(x),T(x)]_\alpha \omega z\rho S(x) \\ &\quad +[T(x),S(x)]_\delta \beta T(x)\gamma y\alpha S(x)\omega z\rho S(x) \end{aligned} \tag{20}$$

From (19) and (20), we obtain

$$0 = [S(x),T(x)]_\gamma \delta y\beta A(x,z) \tag{21}$$

where $A(x,z) = [S(x),T(x)]_\alpha \omega z\rho S(x) - S(x)\rho z\omega[S(x),T(x)]_\alpha$.

Replacing y by $z\rho S(x)\omega y$ in (21) gives

$$0 = [S(x),T(x)]_\gamma \delta z\rho S(x)\omega y\beta A(x,z) \tag{22}$$

Left multiplication of (21) by $S(x)\rho z\omega$ gives

$$0 = S(x)\rho z\omega[S(x),T(x)]_\gamma \delta y\beta A(x,z) \tag{23}$$

Combining (22) and (23), we arrive at

$$0 = A(x,z)\delta y\beta A(x,z)$$

for all $x,y,z \in M$ and $\delta,\beta \in \Gamma$. Hence by semiprimeness of M, it follows $A(x,z)=0$ and hence

$$[S(x),T(x)]_\alpha \omega z\rho S(x) = S(x)\rho z\omega[S(x),T(x)]_\alpha \tag{24}$$

The substitution of z by $T(x)\gamma y$ in (24) gives

$$[S(x),T(x)]_\alpha \omega T(x)\gamma y\rho S(x) = S(x)\rho T(x)\gamma y\omega[S(x),T(x)]_\alpha \tag{25}$$

The relation (25) makes it possible to replace in (18), $[S(x),T(x)]_\delta \beta T(x)\gamma y\alpha S(x)$ by $S(x)\alpha T(x)\gamma y\beta[S(x),T(x)]_\delta$. Thus we have

$$0 = [S(x),T(x)]_\gamma \delta y\beta[S(x),T(x)]_\alpha - S(x)\alpha T(x)\gamma y\beta[S(x),T(x)]_\delta$$

which reduces to

$$0 = T(x)\gamma S(x)\delta y\beta[S(x),T(x)]_\alpha \tag{26}$$

Putting $y = T(x)\omega y$ in (26), we have

$$0 = T(x)\gamma S(x)\delta T(x)\omega y\beta[S(x),T(x)]_\alpha \tag{27}$$

Multiplying (26) from the left side by $T(x)\omega$, we have

$$0 = T(x)\omega T(x)\gamma S(x)\delta y\beta[S(x),T(x)]_\alpha \tag{28}$$

Subtracting (28) from (27), we have

$$0 = T(x)\omega[S(x),T(x)]_\gamma \delta y\beta[S(x),T(x)]_\alpha$$

which gives putting $y = y\omega T(x)$,

$$\begin{aligned} 0 &= T(x)\omega[S(x),T(x)]_\gamma \delta y\omega T(x)\beta[S(x),T(x)]_\alpha \\ &= T(x)\omega[S(x),T(x)]_\gamma \delta y\beta T(x)\omega[S(x),T(x)]_\alpha \end{aligned}$$

whence it follows

$$0 = T(x)\omega[S(x),T(x)]_\alpha \tag{29}$$

The substitution $y = y\beta T(x)$ in (25) gives because of (29)

$$0 = [S(x), T(x)]_\alpha \omega y \beta T(x) \rho S(x) \tag{30}$$

From (13), we obtain the relation

$$0 = [S(x), T(x)]_\alpha \beta S(y) + [S(x), T(y)]_\alpha \beta S(x) + [S(y), T(x)]_\alpha \beta S(x)$$

(see the proof of (6)). Putting in the above relation $y = x\gamma y$, we have

$$\begin{aligned} 0 &= [S(x), T(x)]_\alpha \beta S(x)\gamma y + [S(x), T(x)\gamma y]_\alpha \beta S(x) + [S(x)\gamma y, T(x)]_\alpha \beta S(x) \\ &= T(x)\gamma[S(x), y]_\alpha \beta S(x) + [S(x), T(x)]_\alpha \gamma y \beta S(x) \\ &\quad + S(x)\gamma[y, T(x)]_\alpha \beta S(x) + [S(x), T(x)]_\alpha \gamma y \beta S(x) \end{aligned}$$

Thus we have

$$0 = 2[S(x), T(x)]_\alpha \gamma y \beta S(x) + T(x)\gamma[S(x), y]_\alpha \beta S(x) + S(x)\gamma[y, T(x)]_\alpha \beta S(x)$$

which can be written after some calculation in the form

$$0 = [S(x), T(x)]_\alpha \gamma y \beta S(x) + S(x)\gamma y \alpha T(x) \beta S(x) - T(x)\gamma y \alpha S(x) \beta S(x) \tag{31}$$

The relation (24) makes it possible to replace in (31), $[S(x), T(x)]_\alpha \gamma y \beta S(x)$ by $S(x)\beta y \gamma [S(x), T(x)]_\alpha$. Thus we have

$$\begin{aligned} 0 &= S(x)\beta y \gamma [S(x), T(x)]_\alpha + S(x)\gamma y \alpha T(x) \beta S(x) - T(x)\gamma y \alpha S(x) \beta S(x) \\ &= S(x)\beta y \gamma S(x) \alpha T(x) - T(x)\gamma y \alpha S(x) \beta S(x) \end{aligned}$$

Therefore, we have

$$S(x)\beta y \gamma S(x) \alpha T(x) = T(x)\gamma y \alpha S(x) \beta S(x) \tag{32}$$

Putting in the above relation $y = T(x)\omega y$, we have

$$S(x)\beta T(x)\omega y \gamma S(x) \alpha T(x) = T(x)\gamma T(x)\omega y \alpha S(x) \beta S(x) \tag{33}$$

Left multiplication of (32) by $T(x)\omega$ gives

$$T(x)\omega S(x)\beta y \gamma S(x) \alpha T(x) = T(x)\omega T(x)\gamma y \alpha S(x) \beta S(x) \tag{34}$$

Combining (33) and (34), we have

$$0 = [S(x), T(x)]_\beta \omega y \gamma S(x) \alpha T(x)$$

which gives together with (30),

$$0 = [S(x), T(x)]_\alpha \omega y \beta [S(x), T(x)]_\alpha$$

Hence by semiprimeness of M, we have

$$[S(x), T(x)]_\alpha = 0 \tag{35}$$

If M is a prime Γ-ring, then relation (35) and Lemma 2.3 complete the proof of the theorem. ∎

Theorem 1.2 *Suppose that M is a 2-torsion free noncommutative semiprime Γ-ring satisfying the assumption (A) and S, T are left centralizers on M. If $[[S(x),T(x)]_\alpha,S(x)]_\beta=0$ holds for all $x\in M$ and $\alpha,\beta\in\Gamma$. Then $[S(x),T(x)]_\alpha=0$ for all $x\in M$ and $\alpha\in\Gamma$. Moreover, if M is prime Γ-ring satisfying the assumption (A) and $S\neq 0 (T\neq 0)$, then there exists $\lambda\in C$, (the extended centroid of M) such that $T=\lambda\alpha S (S=\lambda\alpha T)$.*

Proof. By the assumption

$$[[S(x),T(x)]_\alpha,S(x)]_\beta = 0 \tag{36}$$

the linearization of (36) gives

$$0=[[S(x),T(x)]_\alpha,S(y)]_\beta+[[S(x),T(y)]_\alpha,S(x)]_\beta+[[S(y),T(x)]_\alpha,S(x)]_\beta \tag{37}$$

Putting $y=x\gamma y$ in (37), we have

$$\begin{aligned}
0 &= [[S(x),T(x)]_\alpha,S(x)\gamma y]_\beta+[[S(x),T(x)\gamma y]_\alpha,S(x)]_\beta\\
&\quad +[[S(x)\gamma y,T(x)]_\alpha,S(x)]_\beta\\
&= [[S(x),T(x)]_\alpha,S(x)]_\beta\gamma y+S(x)\gamma[[S(x),T(x)]_\alpha,y]_\beta\\
&\quad +[[S(x),T(x)]_\alpha\gamma y+T(x)\gamma[S(x),y]_\alpha,S(x)]_\beta\\
&\quad +[[S(x),T(x)]_\alpha\gamma y+S(x)\gamma[y,T(x)]_\alpha,S(x)]_\beta\\
&= S(x)\gamma[[S(x),T(x)]_\alpha,y]_\beta+[[S(x),T(x)]_\alpha,S(x)]_\beta\gamma y\\
&\quad +[S(x),T(x)]_\alpha\gamma[y,S(x)]_\beta+T(x)\gamma[[S(x),y]_\alpha,S(x)]_\beta\\
&\quad +[T(x),S(x)]_\beta\gamma[S(x),y]_\alpha+[[S(x),T(x)]_\alpha,S(x)]_\beta\gamma y\\
&\quad +[S(x),T(x)]_\alpha\gamma[y,S(x)]_\beta+S(x)\gamma[[y,T(x)]_\alpha,S(x)]_\beta
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
0 &= S(x)\gamma[[S(x),T(x)]_\alpha,y]_\beta+3[S(x),T(x)]_\alpha\gamma[y,S(x)]_\beta\\
&\quad +T(x)\gamma[[S(x),y]_\alpha,S(x)]_\beta+S(x)\gamma[[y,T(x)]_\alpha,S(x)]_\beta
\end{aligned} \tag{38}$$

Replacing y by $y\delta S(x)$ in the above relation, we have

$$\begin{aligned}
0 &= S(x)\gamma[[S(x),T(x)]_\alpha,y\delta S(x)]_\beta+3[S(x),T(x)]_\alpha\gamma[y\delta S(x),S(x)]_\beta\\
&\quad +T(x)\gamma[[S(x),y\delta S(x)]_\alpha,S(x)]_\beta+S(x)\gamma[[y\delta S(x),T(x)]_\alpha,S(x)]_\beta\\
&= S(x)\gamma[[S(x),T(x)]_\alpha,y]_\beta\delta S(x)+S(x)\gamma y\delta[[S(x),T(x)]_\alpha,S(x)]_\beta\\
&\quad +3[S(x),T(x)]_\alpha\gamma[y,S(x)]_\beta\delta S(x)+T(x)\gamma[[S(x),y]_\alpha\delta S(x),S(x)]_\beta\\
&\quad +S(x)\gamma[[y,T(x)]_\alpha\delta S(x)+y\delta[S(x),T(x)]_\alpha,S(x)]_\beta\\
&= S(x)\gamma[[S(x),T(x)]_\alpha,y]_\beta\delta S(x)+3[S(x),T(x)]_\alpha\gamma[y,S(x)]_\beta\delta S(x)\\
&\quad +T(x)\gamma[[S(x),y]_\alpha,S(x)]_\beta\delta S(x)+S(x)\gamma[[y,T(x)]_\alpha,S(x)]_\beta\delta S(x)\\
&\quad +S(x)\gamma[y,S(x)]_\beta\delta[S(x),T(x)]_\alpha+S(x)\gamma y\delta[[S(x),T(x)]_\alpha,S(x)]_\beta
\end{aligned}$$

Thus we have, according to (36) and (38),

$$0 = S(x)\gamma[y,S(x)]_\beta\delta[S(x),T(x)]_\alpha$$

which can be written in the form

$$S(x)\gamma y\beta S(x)\delta[S(x),T(x)]_\alpha = S(x)\gamma S(x)\beta y\delta[S(x),T(x)]_\alpha \tag{39}$$

Putting in the above relation $y = T(x)\omega y$, we have

$$S(x)\gamma T(x)\omega y\beta S(x)\delta[S(x),T(x)]_\alpha = S(x)\gamma S(x)\beta T(x)\omega y\delta[S(x),T(x)]_\alpha \tag{40}$$

On the other hand, left multiplication of (39) by $T(x)\omega$, we have

$$T(x)\omega S(x)\gamma y\beta S(x)\delta[S(x),T(x)]_\alpha = T(x)\omega S(x)\gamma S(x)\beta y\delta[S(x),T(x)]_\alpha \tag{41}$$

Subtracting (41) from (40), we obtain

$$\begin{aligned} 0 &= [S(x),T(x)]_\gamma\omega y\beta S(x)\delta[S(x),T(x)]_\alpha - [S(x)\gamma S(x),T(x)]_\beta\omega y\delta[S(x),T(x)]_\alpha \\ &= [S(x),T(x)]_\gamma\omega y\beta S(x)\delta[S(x),T(x)]_\alpha \\ &\quad -([S(x),T(x)]_\beta\gamma S(x) + S(x)\gamma[S(x),T(x)]_\beta\omega y\delta[S(x),T(x)]_\alpha \end{aligned}$$

According to the requirement of the theorem one can replace in the above calculation $[S(x),T(x)]_\beta\gamma S(x)$ by $S(x)\gamma[S(x),T(x)]_\beta$ which gives

$$\begin{aligned} &[S(x),T(x)]_\gamma\omega y\beta S(x)\delta[S(x),T(x)]_\alpha \\ &\quad = 2S(x)\gamma[S(x),T(x)]_\beta\omega y\delta[S(x),T(x)]_\alpha \end{aligned}$$

Left multiplication of the above relation by $S(x)\rho$ gives

$$\begin{aligned} &S(x)\rho[S(x),T(x)]_\gamma\omega y\beta S(x)\delta[S(x),T(x)]_\alpha \\ &\quad = 2S(x)\rho S(x)\gamma[S(x),T(x)]_\beta\omega y\delta[S(x),T(x)]_\alpha \end{aligned} \tag{42}$$

On the other hand, putting $y = [S(x),T(x)]\rho y$ in (39), we have

$$\begin{aligned} &S(x)\gamma[S(x),T(x)]_\omega\rho y\beta S(x)\delta[S(x),T(x)]_\alpha \\ &\quad = S(x)\gamma S(x)\beta[S(x),T(x)]_\omega\rho y\delta[S(x),T(x)]_\alpha \end{aligned} \tag{43}$$

Combining (42) with (43), we obtain

$$0 = S(x)\rho[S(x),T(x)]_\gamma\omega y\beta S(x)\delta[S(x),T(x)]_\alpha$$

Hence by semiprimeness of M, we obtain

$$S(x)\delta[S(x),T(x)]_\alpha = 0 \tag{44}$$

From (44) and the assumption of the theorem, we have

$$[S(x),T(x)]_\alpha\delta S(x) = 0$$

The rest of the proof goes through in the same way as in the proof of Theorem 2.1. ∎

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A NOTE ON SPECIAL MATRICES

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Abstract. The word "matrix" comes from the Latin word for "womb" because of the way that the matrix acts as a womb for the data that it holds. The first known example of the use matrices was found in a Chinese text called Nine Chapters of the Mathematical Art, which is thought to have originated somewhere between 300 B.C. and 200 A.D. The modern method of matrix solution was developed by a German mathematician and scientist Carl Friedrich Gauss. There are many different types of matrices used in different modern career fields. We introduce and discuss the different types of matrices that play important roles in various fields.
Keywords: matrix, types of matrices, Linear Algebra, matrix theory

1. Introduction

The history of matrices goes back to ancient times! Around 4000 years ago, the Babylon knew how to solve a simple 2 2 system of linear equations with two unknowns. But the term "matrix" was not applied to the concept until 1850. The origins of mathematical matrices lie with the study of systems of simultaneous linear equations. An important Chinese text from between 300 BC and AD 200, Nine Chapters of the Mathematical Art [1], gives the first known example of the use of matrix methods to solve simultaneous equations [2]. In the seventh chapter, the concept of a determinant first appears, nearly two millennia before its supposed invention by the Japanese mathematician Seki Kowa [3] in

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1683 or his German contemporary Gottfried Leibnitz [4]. More uses of matrix-like arrangements of numbers appear in chapter eight, "Methods of rectangular arrays," in which a method is given for solving simultaneous equations using a counting board that is mathematically identical to the modern matrix method of solution outlined by Carl Friedrich Gauss (1777–1855) [5], also known as Gaussian elimination.

In the beginning of 19th century, Gauss introduced a procedure to be used for solving a system of linear equations. His work mainly dealt with the linear equations and had yet to bring the idea of matrices. He dealt with equations of different numbers and variables and also the traditional works of Euler, Leibnitz and Cramer. Gauss work is now termed as Gauss Elimination method. In 1848, James Joseph Sylvester [6] introduced the term "matrix", the Latin word for womb, as a name for an array of numbers. He used womb because he viewed a matrix as a generator of determinants [7]. On the other part, matrix multiplication or matrix algebra came from the work of Arthur Cayley in 1855. Cayley's efforts were published in two papers, one in 1850 and the other in 1858. His works introduced the idea of the identity matrix as well as the inverse of a square matrix [8]. The elevation of the matrix from mere tool to important mathematical theory owes a lot to the work of female mathematician Olga Taussky Todd (1906–1995) [9], who began by using matrices to analyze vibrations on airplanes during World War II. Later on, she became the "mother" of matrix theory.

2. Matrices

2.1 Definition. A rectangular arrangement of mn numbers, in m rows and n columns and enclosed within a bracket is called a matrix. We shall denote the matrices by capital letters like A, B, C etc.

$$A = \begin{pmatrix} A_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ A_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = (a_{ij})_{m\times n}$$

A is a matrix of order $m \times n$. A matrix is just not a collection of elements but every element has assigned a definite position in a particular row and column.

2.2 Fundamental matrices

- **Square matrix:** A matrix which has equal number of rows and columns is called a square matrix.
- **Diagonal matrix:** A square matrix $A = (a_{ij})_{n\times n}$ is called a diagonal matrix, if each of its non-diagonal elements is zero.
- **Identity matrix:** A diagonal matrix whose diagonal elements are equal to 1 is called identity matrix.

- **Upper Triangular matrix:** A square matrix is said to be an Upper triangular matrix, if $a_{ij} = 0$ for $i > j$.
- **Lower Triangular matrix:** A square matrix is said to be a Lower triangular matrix, if $a_{ij} = 0$ for $i < j$.
- **Symmetric matrix:** A square matrix $A = (a_{ij})_{n\times n}$ is said to be a Symmetric matrix, if $a_{ij} = a_{ji}$ for all i and j.
- **Skew - Symmetric matrix:** A square matrix $A = (a_{ij})_{n\times n}$ is said to be a Skew–Symmetric matrix, if $a_{ij} = -a_{ji}$ for all i and j.
- **Zero matrix:** A matrix whose all the elements are zero is called a Zero matrix, and a zero matrix of order $n \times m$ is denoted by $O_{n\times m}$.
- **Row matrix:** A matrix which has only one row is called a Row matrix or Row vector.
- **Column matrix:** A matrix which has only one column is called a Column matrix or Column vector.

3. Special matrices in Mathematics

Matrices have numerous applications, both in mathematics and other sciences. We will list the matrices which have applications in the field of mathematics. The matrices are given in alphabetical order with special matrices under each category, if applicable.

3.1 Alternate matrix

An alternate matrix is a matrix with a particular structure, in which successive columns have a particular function applied to their entries. Examples of alternate matrices include Vandermonde matrices and Moore matrices. Alternate matrices are used in coding theory in the construction of alternate codes.

3.1.1 Vandermonde matrix

A Vandermonde matrix [10], [11] of order n is of the form

$$V = \begin{bmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} \end{bmatrix}$$

A Vandermonde matrix is also called an alternate matrix. Sometimes, the transpose of an alternate matrix is known as the Vandermonde matrix. A Vandermonde

matrix is a matrix that arises in the Lagrange interpolating polynomials, polynomial least squares fitting, and the reconstruction of a statistical distribution from the distribution's moments.

3.1.2 Moore matrix

A Moore matrix [12] is a matrix of the form

$$M = \begin{bmatrix} x_1 & x_1^q & \cdots & x_1^{q^{n-1}} \\ x_2 & x_2^q & \cdots & x_2^{q^{n-1}} \\ x_3 & x_3^q & \cdots & x_3^{q^{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ x_m & x_m^q & \cdots & x_m^{q^{n-1}} \end{bmatrix}$$

It is a matrix defined over a finite field. This matrix was introduced by E.H. Moore in the year 1896. The Moore matrix has successive powers of the Frobenius automorphism applied to the first column. Therefore, it is an $m \times n$ matrix.

3.2 Anti-diagonal matrix

Anti-diagonal is the diagonal of a matrix starting from the lower left corner to the upper right corner of the matrix. An anti-diagonal matrix is a matrix where all the entries are zero except the anti-diagonal ($\nearrow$). Example of Anti-diagonal matrix is

$$A = \begin{bmatrix} 0 & 0 & 0 & a \\ 0 & 0 & b & 0 \\ 0 & c & 0 & 0 \\ d & 0 & 0 & 0 \end{bmatrix}$$

3.3 Arrowhead matrix

An arrowhead matrix [13, 14] is a square matrix which has zeros in all entries except the first row, first column, and the main diagonal. The general form of an Arrowhead matrix is

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & 0 & a_{33} & 0 \\ a_{41} & 0 & 0 & a_{44} \end{bmatrix}$$

Real symmetric arrowhead matrices are used in some algorithms for finding of eigenvalues and eigenvectors. Arrowhead matrices are important for the computation of the eigenvalues, via divide and conquer approaches.

3.4 Band matrix

A band matrix [15] is a sparse matrix whose non-zero entries are confined to a diagonal *band*, comprising the main diagonal and zero or more diagonals on either side. The general form of a Band matrix is

$$A = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 \\ 0 & a_{32} & a_{33} & a_{34} & 0 \\ 0 & 0 & a_{43} & a_{44} & a_{45} \\ 0 & 0 & 0 & a_{54} & a_{55} \end{bmatrix}$$

Let us consider an $n \times n$ matrix $A = (a_{ij})$. If all the elements of the matrix are zero outside a diagonally bordered band whose range is determined by the constants p and q as follows:

$$a_{ij} = 0 \text{ if } j < i - p \text{ or } j > i + q,\ p, q \geq 0,$$

then the quantities p and q are called the lower and upper bandwidth, respectively. The bandwidth of the matrix is the maximum of p and q, that is, it is the number r such that

$$a_{ij} = 0 \text{ if } |i - j| > r.$$

A matrix is called a band matrix or banded matrix if its bandwidth is reasonably small. A band matrix with $p = q = 0$ is a diagonal matrix. A band matrix with $p = q = 1$ is a tridiagonal matrix. When $p = q = 2$, it is a pentadiagonal matrix. If $p = 0$, $q = n - 1$, we get an upper triangular matrix. Similarly, for $p = n - 1$, $q = 0$, we obtain a lower triangular matrix. In numerical analysis, matrices from finite element or finite difference problems are often banded.

3.5 Binary matrix

A Binary matrix [16] or a Boolean matrix is a matrix with entries being either 0 or 1. Such a matrix can be used to represent a binary relation between a pair of finite sets. Such type of matrices are also called as Logical matrix, Relation matrix, or (0,1) matrix. Some of the (0,1) matrices are discussed below.

3.5.1 Adjacency matrix

An adjacency matrix [17], also called the connection matrix, is a matrix with rows and columns representing which vertices (or nodes) of a graph are adjacent to which other vertices. Another matrix representation for a graph is the incidence matrix. In the special case of a finite simple graph, an Adjacency matrix is a (0,1)-matrix with zeros on its diagonal. For an undirected graph, the adjacency matrix is symmetric. The adjacency matrix of a complete graph contains all ones except along the diagonal where there are only zeros. The adjacency matrix of an empty graph is a zero matrix. The *biadjacency matrix* of a simple, undirected bipartite graph is a (0,1)-matrix.

3.5.2 Design matrix

A design matrix [18], [19] is a matrix which contains explanatory variables, and is used in statistical models like the general linear model.

3.5.3 Incidence matrix

The incidence matrix [20] of a graph has one row for each vertex of the graph and one column for each edge of the graph. If an edge runs from node a to node b, the row corresponding to that edge has $-$ in column a and 1 in column b and all other entries in that row are 0. However, some authors define the incidence matrix to be the transpose of this, with a column for each vertex and a row for each edge. The physicist Kirchhoff (1847) was the first to define the incidence matrix. Incidence matrices are mostly used in graph theory. In graph theory an undirected graph G has two kinds of incidence matrices: unoriented and oriented.

3.5.4 Permutation matrix

A permutation matrix [15] is a square matrix obtained from the same size identity matrix by a permutation of rows. Every row and column has a single 1 with 0s everywhere else. There are $n!$ permutation matrices of size n.

The permutation matrices of order two are given by

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and of order three are given by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

A permutation matrix is nonsingular, and the determinant is always ± 1. In addition, a permutation matrix A satisfies

$$AA^{\top} = I,$$

where $A^{\top}$ is a transpose and I is the identity matrix.

3.5.5 Exchange matrix

The exchange matrix is an anti-diagonal matrix in which all the entries in the anti-diagonal are 1 and all other elements are zero. It is also a special case of permutation matrix. In other words, it is a 'row-reversed' or 'column-reversed' version of the identity matrix. The exchange matrix is also called the reversal matrix, backward identity, or standard involutory permutation.

3.6 Bi-Symmetric matrix

A bisymmetric matrix [21] is a square matrix which is symmetric about both of its main diagonals. More precisely, an $n \times n$ matrix B is bisymmetric if it satisfies both $B = B^\top$ and $BE = EB$ where E is the $n \times n$ exchange matrix. For example:

$$\begin{bmatrix} a & b & c & d & e \\ b^f & g & h & d & \\ c & g & i & g & c \\ d & h & g & f & b \\ e & d & c & b & a \end{bmatrix}$$

Bisymmetric matrices are both symmetric centrosymmetric and symmetric persymmetric.

3.6.1 Centrosymmetric matrix

A matrix which is symmetric about its center is called a centrosymmetric matrix. More precisely, an $m \times m$ matrix $A = [A_{ij}]$ is centrosymmetric when its entries satisfy

$$A_{i,j} = A_{m-i+1,m-j+1} \text{ for } 1 \leq i,\ j \leq m.$$

If E denotes the $m \times m$ matrix with all the counterdiagonal elements as 1 and 0 elsewhere, then a matrix A is centrosymmetric if and only if $AE = EA$. The matrix E is sometimes named as the exchange matrix.

3.6.2 Persymmetric matrix

A persymmetric matrix [21] is a square matrix which is symmetric in the northeast-to-southwest diagonal or a square matrix such that the values on each line perpendicular to the main diagonal are the same for a given line.

3.7 Block matrix

A block matrix [22] is a matrix which is obtained using smaller matrices, called blocks. For example,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where A, B, C and D are matrices, is a block matrix. In the above example

$$A = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix},\ B = \begin{bmatrix} 6 & 6 & 6 \\ 6 & 6 & 6 \end{bmatrix},\ C = \begin{bmatrix} 7 & 7 \\ 7 & 7 \\ 7 & 7 \end{bmatrix},\ D = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

therefore, it can be written as the matrix

$$\begin{bmatrix} 0 & 3 & 6 & 6 & 6 \\ 3 & 0 & 6 & 6 & 6 \\ 7 & 7 & 2 & 0 & 2 \\ 7 & 7 & 0 & 2 & 0 \\ 7 & 7 & 2 & 0 & 2 \end{bmatrix},$$

Block matrices are also called partitioned matrices.

3.8 Cauchy matrix

Cauchy matrix [23] is an $m \times n$ matrix if its entries a_{ij} are of the form

$$a_{ij} = \frac{1}{\alpha_i - \beta_j}, \text{ or } a_{ij} = \frac{1}{\alpha_i + \beta_j}, \ \alpha_i \mp \beta_j \neq 0, \ 1 \leq i \leq m, \ 1 \leq j \leq n,$$

where α_i and β_j are elements which are distinct and not repeated. Every submatrix of a Cauchy matrix is a Cauchy matrix. The Hilbert matrix is a special case of the Cauchy matrix, with $x_i - y_j = i + j - 1$.

3.9 Conference matrix

Conference matrix or a C-matrix [24] is a square matrix C with diagonal entries 0 and off-diagonal entries ± 1, such that the product of C and its transpose is the multiple of the identity matrix, that is,

$$C^\top C = (n-1)I, \text{ where } n \text{ is the order of } C.$$

Conference matrix is either symmetric or antisymmetric. Sometimes, the Conference matrix or a C-matrix is defined as the matrix, which requires a single 0 in each row and column but not necessarily on the diagonal [25], [26].

The two symmetric C- matrices of order 2 are

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and two antisymmetric C-matrices of order 2 are

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

3.10 Complex matrix

Complex matrix is a matrix whose entries are complex numbers.

3.11 Copositive matrix

A Copositive matrix [27] is a real $n \times n$ square matrix A, if

$$x^\top Ax \geq 0$$

for all non-negative n-vectors x. The collection of real positive-definite matrices is the subset of copositive matrix.

3.12 Diagonal matrix

A Diagonal matrix [28] is a square matrix D, in which all the entries other than the main diagonal are zero. The diagonal entries themselves may be zero or may not be. Thus, the matrix $D = (d_{ij}$ with n columns and n rows is diagonal if

$$d_{ij} = 0 \text{ if } i \neq j, \ \forall \ i, j \in \{1, 2, ..., n\}.$$

Sometimes, the term *diagonal matrix* refers to a rectangular diagonal matrix, which is an $m \times n$ matrix with only the entries of $d_{i,j}$ will be non-zero. For example,

$$\begin{bmatrix} 8 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -5 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 7 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -6 & 0 & 0 \end{bmatrix}.$$

Any square diagonal matrix is also a symmetric matrix. A diagonal matrix is also defined as a matrix which is both upper- and lower-triangular. The identity matrix and square zero matrix are diagonal matrices. A one-dimensional matrix is always a diagonal matrix. The different types of diagonal matrices are discussed below.

3.12.1 Bidiagonal matrix

A bidiagonal matrix [29] is a matrix with non-zero entries along the main diagonal and *either* the diagonal above or the diagonal below the main diagonal. A bidiagonal matrix has exactly two non- zero diagonals in the matrix.

When the entries above the main diagonal are non-zero, the matrix is called an upper bidiagonal matrix and when the entries below the main diagonal are non-zero, the matrix is called lower bidiagonal matrix.

3.12.2 Tridiagonal matrix

A tridiagonal matrix is a matrix which has non-zero elements along the main diagonal and the diagonals immediately above and below the main diagonal.

3.12.3 Pentadiagonal matrix

A pentadiagonal matrix is a matrix which has non-zero elements along the main diagonal and two diagonals immediately above and below the main diagonal.

3.12.4 Block diagonal matrix

A block diagonal matrix [30] or a diagonal block matrix, is a square diagonal matrix in which the diagonal elements are square matrices of any size (possibly even 1×1), and the off-diagonal elements are 0. Therefore, a block diagonal matrix is a block matrix in which the blocks off the diagonal are the zero matrices, and the diagonal matrices are square.

3.12.5 Block tridiagonal matrix

A block tridiagonal matrix [31] is another special block matrix, which is also a square matrix, having square matrices (blocks) in the main diagonal, lower diagonal and upper diagonal, with all other blocks being zero matrices.

3.13 Diagonally Dominant matrix

A matrix A of order $n \times n$ is said to be row diagonally dominant (rdd) [32] if

$$|a_{ii}| \geq \sum_{j \neq i} |a_{ij}| \text{ for all } i,$$

where a_{ij} denotes the entry in the i^{th} row and j^{th} column. If a strict inequality ($>$) is used, then this matrix is called strictly diagonally dominant. It is said to be column diagonally dominant (cdd), if $A^{\top}$ is row diagonally dominant. In other words, a matrix A of order $n \times n$ is said to be diagonally dominant for every row of the matrix, the magnitude of the diagonal entry in a row is larger than or equal to the sum of the magnitudes of all the other non-diagonal entries in that row.

3.14. Echelon matrix

A matrix that has undergone Gaussian elimination is said to be in row echelon form [33] or, more properly, "reduced echelon form" or "row-reduced echelon form." Such a matrix has the following characteristics:

- All zero rows are at the bottom of the matrix
- The leading entry of each nonzero row after the first occurs to the right of the leading entry of the previous row.
- The leading entry in any nonzero row is 1.
- All entries in the column above and below a leading 1 are zero.

Another common definition of echelon form only requires zeros below the leading ones, while the above definition also requires them above the leading ones.

3.15 Elementary matrix

An elementary matrix [34] is a matrix which can be obtained from the identity matrix by one single elementary row operation. Left multiplication (pre-multiplication) by an elementary matrix represents elementary row operations, while right multiplication (post-multiplication) represents elementary column operations. The acronym "ERO" is commonly used for "elementary row operations".

3.16 Equivalent matrix

An $m \times n$ matrix A is said to be equivalent [35] to a matrix $m \times n$ matrix B iff $B = PAQ$, for any non-singular matrices P and Q of order $m \times m$ and $n \times n$ respectively.

3.17 Frobenius matrix

A Frobenius matrix is a special kind of square matrix from numerical mathematics. A matrix is a Frobenius matrix if it has the following three properties:

- all entries on the main diagonal are ones,
- the entries below the main diagonal of at most one column are arbitrary,
- every other entry is zero.

Example for Frobenius matrix is

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & a_{32} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & a_{n2} & 0 & \cdots & 1 \end{bmatrix}$$

Frobenius matrices are named after Ferdinand Georg Frobenius. An alternative name for this class of matrices is Gauss transformation, after Carl Friedrich Gauss [15].

3.18 Hadamard matrix and Complex Hadamard matrix

A Hadamard matrix [36], named after the French mathematician Jacques Hadamard, is a square matrix whose entries are either $+1$ or -1 and whose rows are mutually orthogonal.

A complex Hadamard matrix is any complex $N \times N$ matrix H satisfying the following two conditions:

- unimodularity (the modulus of each entry is unity)
- orthogonality: $HH^{\theta} = NI$, where H^{θ} is the Hermitian transpose of H and I is the identity matrix.

3.19 Hankel matrix

A square matrix with constant skew diagonals is called a Hankel matrix [37]. In other words, a Hankel matrix is a matrix in which the $(i, j)^{\text{th}}$ entry depends only on the sum $i + j$. Such matrices are sometimes known as persymmetric matrices or, in older literature, orthosymmetric matrices.

3.20 Hermitian and skew–Hermitian matrix

A Hermitian matrix (or self-adjoint matrix) [38] is a square matrix with complex entries that is equal to its own conjugate transpose-that is, the element in the i^{th} row and j^{th} column is equal to the complex conjugate of the element in the j^{th} row and i^{th} column, for all indices i and j:

$$a_{ij} = \overline{a_{ji}} \quad \text{or} \quad A = \overline{A^{\top}} \text{ in matrix form.}$$

Hermitian matrices are named after Charles Hermite.

A square matrix with complex entries is said to be skew-Hermitian or anti-hermitian, if its conjugate transpose is equal to its negative. That is, the matrix A is skew-Hermitian if it satisfies the relation

$$a_{ij} = \overline{-a_{ji}} \quad \text{or} \quad A^{\top} = \overline{-A} \text{ in matrix form.}$$

3.21 Hessenberg matrix

A Hessenberg matrix [39] is a matrix of the form

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1(n-1)} & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2(n-1)} & a_{2n} \\ 0 & a_{32} & a_{33} & \cdots & a_{3(n-1)} & a_{3n} \\ 0 & 0 & a_{43} & \cdots & a_{4(n-1)} & a_{4n} \\ 0 & 0 & 0 & \cdots & a_{5(n-1)} & a_{5n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{(n-1)(n-1)} & a_{(n-1)n} \\ 0 & 0 & 0 & \cdots & a_{n(n-1)} & a_{nn} \end{bmatrix}$$

Hessenberg matrices were given by a German engineer Karl Hessenberg (1904–1959), whose dissertation investigated the computation of eigenvalues and eigenvectors of linear operators.

3.22 Hollow matrix

A hollow matrix [40] is a square matrix whose diagonal elements are all zero. The real skew-symmetric matrix is an example of a hollow matrix. That is, any square matrix which takes the form

$$\begin{bmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & 0 \end{bmatrix}$$

is a hollow matrix.

3.23 Integer matrix

A matrix whose all entries are integers is called an integer matrix. Binary matrices, the zero matrix, the unit matrix, and the adjacency matrices used in graph theory, are examples of integer matrices.

3.24 Metzler matrix

A matrix is called Metzler [41] or quasi-positive or essentially nonnegative if all of its elements are non-negative except those elements on the main diagonal, which

are unconstrained. That is, a Metzler matrix is any matrix A which satisfies the following condition:

$$a_{ij} \geq 0;\ i \neq j.$$

3.25 Monomial matrix

A generalized permutation matrix (or monomial matrix) is a matrix with the same nonzero pattern as a permutation matrix. That is, entry in each row and each column has exactly one nonzero element. In a permutation matrix, the nonzero entry is 1, whereas in a generalized permutation matrix the nonzero entry can be any nonzero value.

3.26 Non - negative matrix

A nonnegative matrix is a matrix in which all the elements are greater than or equal to zero. That is,

$$x_{ij} \geq 0,\ \forall i, j.$$

3.27 Polynomial matrix

A polynomial which has matrices as its coefficients is called a polynomial matrix [42], [43]. An n^{th} order matrix polynomial in a variable x is given by

$$Q(x) = A_0 + A_1 x + A_2 x^2 + \cdots + A_n x^n,$$

where each A is a square matrix.

3.28 Positive matrix

A positive matrix is a matrix in which all the elements are greater than zero. The set of positive matrices is a subset of all non-negative matrices.

3.29 Signature matrix

A matrix, whose diagonal elements are either $+1$ or -1, is called a signature matrix [44].

3.30 Sparse matrix

A matrix in which most of its elements are zero is called a sparse matrix. Also, if most of the elements are nonzero, then the matrix is said to be dense.

3.31 Stochastic matrix

A stochastic matrix [45] is a matrix used to describe the transitions of a Markov chain. Each entry of this matrix is a nonnegative real number and represents a probability. It is also termed as probability matrix, transition matrix, substitution matrix, or Markov matrix. The other types of stochastic matrices are:

- A doubly stochastic matrix is a square matrix of nonnegative real numbers with the sum of each row and column as 1.

- A left stochastic matrix is a real square matrix, in which the sum of each column is 1.
- A right stochastic matrix is a real square matrix, in which the sum of each row is 1.

3.32 Sylvester matrix

Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ and $Q(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0$ be two polynomials of degree n and m respectively. The Sylvester matrix [46] of order $(n + m) \times (n + m)$ corresponding to these polynomials is obtained by writing the coefficients of $P(x)$ from the leftmost corner of the first row followed by zeros, then shifting down one row and one column to the right and writing the coefficients starting from that position. This is continued until the coefficients of $P(x)$ touches the last position of the right hand side.The process is then repeated for the coefficients of $Q(x)$. The Sylvester matrix for the above polynomials $P(x) = a_2x^2 + a_1x + a_0$ and $Q(x) = b_3x^3 + b_2x^2 + b_1x + b_0$ is given as follows:

$$S = \begin{bmatrix} a_2 & a_1 & a_0 & 0 & 0 \\ 0 & a_2 & a_1 & a_0 & 0 \\ 0 & 0 & a_2 & a_1 & a_0 \\ b_3 & b_2 & b_1 & b_0 & 0 \\ 0 & b_3 & b_2 & b_1 & b_0 \end{bmatrix}$$

3.33 Toeplitz matrix

Any nn matrix in which the negative sloping diagonal elements are constants is called a Toeplitz matrix [47]. A toeplitz matrix is of the form

$$T = \begin{bmatrix} x_0 & x_{-1} & x_{-2} & \cdots & x_{-(n-1)} \\ x_1 & x_0 & x_{-1} & \ddots & \ddots \\ x_2 & x_1 & x_0 & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ x_{n-1} & \ddots & \ddots & \ddots & x_0 \end{bmatrix}$$

3.34 Unitary matrix

A matrix U is said to be unitary [48], if $U^*U = I$, where U^* is the conjugate transpose of U and I is the identity matrix.

3.35 Walsh matrix

A square matrix with entries being $+1$ or -1, with the condition that the dot product of any two distinct rows or columns being zero is called a Walsh matrix [49]. The dimension of the matrix will be the powers of 2.

4. Conclusion

In this paper, we have discussed different types of matrices, which have great applications in the field of Mathematics and sciences. There are also other matrices that have important role in science, Engineering and other fields.

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FUZZY ISOMORPHISM THEOREMS OF SOFT GROUPS

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Abstract. In this paper, we introduce the concept of fuzzy normal subgroups of soft groups. The first, second and third fuzzy isomorphism theorems of soft groups are established, respectively. In particular, some classes of quotient groups are characterized by their fuzzy normal subgroups.

Keywords: soft groups; fuzzy normal subgroups; quotient groups; fuzzy isomorphism theorems.

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1. Introduction

The fuzzy set theory, proposed by Zadeh [15], has been extensively applied to many scientific fields. This work opened a new direction, new exploration, new path of thinking to mathematicians, engineers, computer scientists and many other researchers, so the field grew enormously and applications were found in many areas. In 1971, Rosenfeld [13] applied the concept to the theory of groupoids and groups, this job was the first fuzzification of algebraic structures. Since then, various fuzzy algebraic structures have been appeared in the literature [4], [9], [10]. Fang [4] introduced the fuzzy homomorphism and fuzzy isomorphism between two fuzzy groups by a natural way, and studied some of their properties. Liu [9], [10] investigated the various constructions of fuzzy quotient groups, fuzzy quotient rings and fuzzy isomorphisms, respectively.

Following the discovery of fuzzy sets, much attention has been paid to generalize the concept of soft sets. Molodtsov [11] introduced the concept of soft sets as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches in 1999. At present, the research on the soft set theory is developing rapidly. Several directions

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for the applications of soft sets have been pointed out by several researchers (see, e.g., [5], [6], [17], [18], and they also studied some operations on the theory of soft sets. Aktas et al. [1] researched the basic concepts of soft set theory and compared soft sets to fuzzy and rough sets. Ali et al. [2] introduced a new concept of sets with some order among the parameters and some properties of lattice ordered soft sets were given. The soft algebraic structures have been investigated by some researchers. Aktas et al. [1] applied the notion of soft sets to the theory of groups. Liu et al. [7], [8] proposed the definition of soft rings, investigated some properties of soft rings, and established three isomorphism theorems and three fuzzy isomorphism theorems, respectively. Zhan et al. [17], [18] applied the concepts of soft union sets to the hemirings, h-hemiregular and h-intrahemiregular hemirings, and researched some of their properties. In particular, Zhan investigated the ideal theory on hemirings, most of relevant conclusions have been already demonstrated in Zhan's book, which is referred to [16].

In this paper, we introduce the fuzzy normal subgroups of soft groups. Then, we use fuzzy normal subgroups to describe some classes of quotient soft groups. Finally, the first, second and third fuzzy isomorphism theorems of soft groups are given.

2. Preliminaries

Definition 2.1 ([13]) A fuzzy subset μ of a group G is said to be a fuzzy subgroup of G if

(1) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$,
(2) $\mu(x^{-1}) \geq \mu(x)$,

holds for all $x, y \in G$.

Equivalently, $\mu(xy^{-1}) \geq \mu(x) \wedge \mu(y)$ for all $x, y \in G$.

Proposition 2.2 ([3]) *For a fuzzy subgroup μ of a group G, the following hold:*

(1) $\mu(e) \geq \mu(x)$ *for all* $x \in G$,
(2) $\mu(x^{-1}) = \mu(x)$ *for all* $x \in G$,
(3) $\mu(xy^{-1}) = \mu(e)$ *implies* $\mu(x) = \mu(y)$ *for all* $x, y \in G$.

Definition 2.3 ([3]) A fuzzy subset μ of a group G is said to be a fuzzy normal subgroup of G if the following axioms hold:

(1) $\mu(xyx^{-1}) \geq \mu(y)$ for all $x, y \in G$.

Equivalently, $\mu(xyx^{-1}) = \mu(y)$ for all $x, y \in G$, or $\mu(xy) = \mu(yx)$ for all $x, y \in G$.

Proposition 2.4 ([3]) *A fuzzy subset μ of a group G is a fuzzy normal subgroup of G if and only if for all $t \in [0, 1]$, $\mu_t = \{x \in G | \mu(x) \geq t\}$ is a normal subgroup of G.*

Definition 2.5 ([12]) Let $f : X \to Y$ be a mapping of sets, μ a fuzzy subset of X and ν a fuzzy subset of Y. Then, for any $x \in X$, the image $f(\mu)$ of μ and preimage $f^{-1}(\nu)$ of ν are both fuzzy sets defined respectively as follows:

$$\begin{aligned}(f(\mu))(y) &= \bigvee_{f(x)=y} \mu(x),\\ (f^{-1}(\nu))(x) &= \nu(f(x)).\end{aligned}$$

Molodtsov defined the soft set in the following way: let U be an initial universe and E a set of parameters. Denote the power set of U by $P(U)$ and consider $A \subset E$.

Definition 2.6 [11] A pair (F, A) is called a soft set over U, where F is a mapping given by $F : A \to P(U)$.

Definition 2.7 ([1],[14])

(1) Let (F, A) be a soft set over group G. Then (F, A) is said to be a soft group over G if and only if $F(x) < G$ for all $x \in A$.

(2) Let G be a group and (F, A) be a non-null soft set over G. Then (F, A) is called a normalistic soft group over G if $F(x)$ is a normal subgroup of G for all $x \in \mathrm{Supp}(F, A)$.

Let G be an algebra and A a non-empty set. Now, we use ρ to refer an arbitrary binary relation between an element of A and an element of G. Thus, a set-valued function $F : A \to P(G)$ can be defined by $F(x) = \{y \in G | (x, y) \in \rho, x \in A\}$.

Definition 2.8 ([7],[8])

(1) Let (F, A) be a soft set over an algebra G. Then (F, A) is said to be a soft algebra over G if and only if $F(x)$ is a subalgebra of G for all $x \in A$. For our convenience, the empty set $\emptyset$ is regarded as a subalgebra of G.

(2) Given soft groups (F, A) and (H, B) over G and K respectively, a soft homomorphism $(f, g) : (F, A) \to (H, B)$ comprises a map $g : A \to B$ and a group homomorphism $f : G \to K$. If f is an isomorphism from G to K and g is a bijective mapping, then (f, g) is called a soft isomorphism from (F, A) to (H, B), denoted by $(F, A) \simeq (H, B)$.

3. Fuzzy normal subgroups of soft groups

Definition 3.1 Let (F, A) be a soft group over a group G and μ a fuzzy subset in G. Then μ is said to be a fuzzy subgroup of (F, A) if, for any $t \in A$ and $x, y \in F(t)$, the following conditions hold:

(1) $\mu(xy) \geq \mu(x) \wedge \mu(y)$ for all x, y of G,

(2) $\mu(x^{-1}) \geq \mu(x)$ for all x of G.

Equivalently, $\mu(xy^{-1}) \geq \mu(x) \wedge \mu(y)$ for all $x, y \in G$.

Definition 3.2 Let (F, A) be a soft group over a group G and μ a fuzzy subset in G. Then μ is said to be a fuzzy normal subgroup of (F, A) if

(1) $\mu(xyx^{-1}) \geq \mu(y)$ for all $x, y \in G$.

Equivalently, $\mu(xyx^{-1}) = \mu(y)$ for all $x, y \in G$ or $\mu(xy) = \mu(yx)$ for all $x, y \in G$.

Remark 3.3

(1) Let (F, A) be a soft group over a group G and μ a fuzzy subset in G. Then μ is a fuzzy subgroup (normal subgroup) of (F, A) if and only if μ is a fuzzy subgroup (normal subgroup) of $F(t)$ for any $t \in A$;

(2) Let (F, A) be a soft group over a group G. If μ is a fuzzy subgroup (normal subgroup) of G, then μ is a fuzzy subgroup (normal subgroup) of (F, A), but the converse is not true.

Example 3.4 Suppose that $G = S_3 = \{e, (12), (13), (23), (123), (132)\}$ and (F, A) be a soft set over G, where $A = \{e, (12)\}$ and $F : A \to P(G)$ is defined by $F(x) = \{y \in G | x\rho y \Leftrightarrow y = x^n, n \in N\}$ for all $x \in A$. Then, $F(e) = \{e\}$, $F(12) = \{e, (12)\}$ and (F, A) is a soft group of G.

Let μ be a fuzzy subset in G, where μ is defined by

$$\mu(x) = \begin{cases} 0.8 & \text{if } t \in \{e, (123)\}, \\ 0.1 & \text{if } t \in \{(12), (13)\}, \\ 0.5 & \text{if } t \in \{(23), (132)\}. \end{cases}$$

We can verify that for any $t \in A$ and $x, y \in F(t)$, μ is a fuzzy normal subgroup of (F, A). But μ is not a fuzzy normal subgroup of G, because $\mu((123)(23)(123)^{-1}) = \mu(12) < \mu(23)$.

Proposition 3.5 *Let (F, A) be a soft group over a group G and μ a fuzzy subset in G. Then μ is a fuzzy subgroup(normal subgroup) of (F, A) if and only if, for any $x \in A$ and $t \in [0, 1]$, $U(t, x) = \{r \in F(t) | \mu(r) \geq t\}$ is a subgroup (normal subgroup) of $F(x)$.*

Proof. It is straightforward. ∎

Let μ be a fuzzy normal subgroup of a group G. For any $x, y \in G$, define a binary relation $\sim$ on G by $x \sim y$ if and only if $\mu(xy^{-1}) = \mu(e)$, where e is the unit element of G.

Lemma 3.6 ([10]) $\sim$ is a congruence relation of G.

The following lemma is similar to fuzzy ideals in [7]:

Lemma 3.7 *Let N be a group G and μ a fuzzy normal subgroup of G. We have*

(1) *if μ is restricted to N, written μ_N, then μ_N is fuzzy normal subgroup of N;*

(2) *N/μ_N is a normal subgroup of G/μ.*

For our convenience, we can denote μ instead of μ_N.

Let μ be a fuzzy normal subgroup of a group G and (F, A) a normalistic soft group over G. Now, we restrict μ to $F(x)$ for all $x \in A$; then, for all $x \in A$, μ is fuzzy normal subgroup of $F(x)$ and $F(x)/\mu$ is a normal subgroup of G/μ. Thus, we can define a set-valued function $F/\mu : A \to P(G/\mu)$ by $(F/\mu)(x) = F(x)/\mu$.

By Lemma 3.7, we can very easily deduce the following theorem.

Theorem 3.8 *If μ is a fuzzy normal subgroup of a group G and (F, A) is a normalistic soft group over G, then*

(1) *μ is a fuzzy normal subgroup of (F, A);*

(2) *$(F/\mu, A)$ is a normalistic soft group over G/μ.*

Notation 3.9 The $(F/\mu, A)$ is called the quotient soft group of (F, A) induced by fuzzy normal subgroup μ.

The following result is similar to fuzzy subgroups in [10]:

Lemma 3.10 *If μ, ν are two fuzzy normal subgroups of a group G, then $\mu \cap \nu$ is also a fuzzy normal subgroup of G.*

By Lemma 3.10, we can very easily deduce the following proposition.

Proposition 3.11 *If (F, A) is a soft group over a group G and μ, ν are two fuzzy normal subgroups of (F, A), then $\mu \cap \nu$ is also a fuzzy normal subgroup of (F, A).*

Let μ be be a fuzzy subgroup of a group G. We denote $G_\mu = \{x \in G | \mu(x) = \mu(e)\}$. Clearly, G_μ is a subgroup of G. Furthermore, if μ is a fuzzy normal subgroup of G, then G_μ is a normal subgroup of G.

Proposition 3.12 *If μ is a fuzzy subgroup of a group G and (F, A) is a soft group over G, let $F_\mu(t) = \{x \in F(t) | \mu(x) = \mu(e)\}$ for all $t \in A$. Then (F_μ, A) is a soft group over G_μ.*

Proof. The proof is easy and is omitted. ∎

In a similar way, we have the following proposition.

Proposition 3.13 *If μ is a fuzzy subgroup of a group G and (F, A) is a normalistic soft group over G, let $F_\mu(t) = \{x \in F(t) | \mu(x) = \mu(e)\}$ for all $t \in A$. Then (F_μ, A) is a normalistic soft group over G_μ.*

Lemma 3.14 *Let μ be a fuzzy normal subgroup of a group G. Then the fuzzy subset μ^* of G/G_μ, defined by $\mu^*(xG_\mu) = \mu(x)$ for all $x \in G$, is a fuzzy normal subgroup of G/G_μ.*

Theorem 3.15 *If μ is a fuzzy normal subgroup of a group G and (F, A) is a soft group over G with $F(t) \supset G_\mu$ for all $t \in A$, then*

(1) *$(F/G_\mu, A)$ is a soft group of G/G_μ, where $(F/G_\mu)(t) = F(t)/G_\mu$ for all $t \in A$,*

(2) *the fuzzy subset μ^* of G/G_μ, defined in Lemma 3.14, is a fuzzy normal subgroup of the soft group $(F/G_\mu, A)$.*

Proof. (1) Since μ is a fuzzy normal subgroup of a group G and (F, A) is a soft group over G with $F(t) \supset G_\mu$ for all $t \in A$, $F(t)$ is a subgroup of G and G_μ is a subgroup of $F(t)$ for all $t \in A$. Let $xG_\mu, yG_\mu \in F(t)/\mu$, where $x, y \in F(t)$, it follows that $y^{-1} \in F(t)$, then $(xG_\mu)(y^{-1}G_\mu) = (xy^{-1})G_\mu \in F(t)/\mu$. Hence, $(F/G_\mu, A)$ is a soft group of G/G_μ.

(2) By Lemma 3.14, μ^* is a fuzzy normal subgroup of G/G_μ, then μ^* is a fuzzy normal subgroup of the soft group $(F/G_\mu, A)$. ∎

4. Fuzzy isomorphism theorems of soft groups

In this section, we give the First, Second and Third Fuzzy Isomorphism Theorems of soft groups by means of fuzzy normal subgroups.

Lemma 4.1 ([10]) *Let $f : G \to K$ be an epimorphism of groups and μ a fuzzy normal subgroup of G, then $f(\mu)$ is a fuzzy normal subgroup of K.*

Lemma 4.2 ([10]) *Let $f : G \to K$ be a homomorphism of groups, μ a fuzzy normal subgroup of G and ν a fuzzy normal subgroup of K, then*

(1) *If f is an epimorphism, then $f(f^{-1}(\nu)) = \nu$,*

(2) *If μ is constant on $kerf$, then $f^{-1}(f(\mu)) = \mu$.*

Lemma 4.3 ([10]) *Let $f : G \to K$ be a homomorphism of groups and ν a fuzzy normal subgroup of K, then $f^{-1}(\nu)$ is a fuzzy normal subgroup of G.*

Theorem 4.4 (First fuzzy isomorphism theorem) *Let (F, A) and (H, B) be two normalistic soft groups over groups G and K, respectively. If (f, g) is soft homomorphism from (F, A) to (H, B) and μ is a fuzzy normal subgroup of G with $G_\mu \supset kerf$, then*

(1) *$(F/\mu, A) \simeq (f(F)/f(\mu), A)$,*

(2) *If g is a bijective mapping, then $(F/\mu, A) \simeq (H/f(\mu), B)$.*

Proof. (1) Since (F, A) is a normalistic soft group over G and μ is a fuzzy normal subgroup of G, $(F/\mu, A)$ is a normalistic soft group over G/μ. (f, g) is a soft homomorphism from (F, A) to (H, B), then for any $t \in A$, $f(F(t)) = H(g(t))$ is a normal subgroup of K, and then we can deduce that $(f(F)/f(\mu), A)$ is a normalistic soft group over $K/f(\mu)$. We define $\varphi : G/\mu \to K/f(\mu)$ by $\varphi(\mu_x) = (f(\mu))_{f(x)}$.

(a) If $\mu_x = \mu_y$, then $\mu(xy^{-1}) = \mu(e)$. Since $G_\mu \supset kerf$, then μ is a constant on $kerf$, by Lemma 4.2, $f^{-1}(f(\mu)) = \mu$. Thus we have $(f^{-1}(f(\mu)))(xy^{-1}) = (f^{-1}(f(\mu)))(e)$, i.e., $f(\mu)(f(xy^{-1})) = f(\mu)(f(e))$, then $f(\mu)(f(x)(f(y))^{-1}) = f(\mu)(e')$, and so $(f(\mu))_{f(x)} = (f(\mu))_{f(y)}$. Hence φ is well-defined.

(b) $\varphi(\mu_x\mu_y) = \varphi(\mu_{xy}) = (f(\mu))_{f(xy)} = (f(\mu))_{f(x)f(y)} = (f(\mu))_{f(x)}(f(\mu))_{f(y)} = \varphi(\mu_x)\varphi(\mu_y)$. Hence φ is a homomorphism.

(c) φ is an epimorphism: For any $(f(\mu))_y \in K/f(\mu)$, since f is epimorphic, there exists $x \in G$ such that $f(x) = y$. So $\varphi(\mu_x) = (f(\mu))_{f(x)} = (f(\mu))_y$.

(d) φ is an monomorphism: $(f(\mu))_{f(x)} = (f(\mu))_{f(y)}$, thus $f(\mu)(f(x)(f(y))^{-1}) = f(\mu)(e')$, so $f(\mu)(f(xy^{-1})) = f(\mu)(f(e))$, i.e., $(f^{-1}(f(\mu)))(xy^{-1}) = (f^{-1}(f(\mu)))(e)$, then $\mu(xy^{-1}) = \mu(e)$, thus $\mu_x = \mu_y$.

By (a),(b),(c) and (d), we deduce that φ is an isomorphism from G/μ to $K/f(\mu)$.

Let $\phi : A \to A$ defined by $\phi(t) = t$. Then ϕ is bijective.

Moreover, we have $\varphi((F/\mu)(t)) = \{(f(\mu))_{f(x)} | x \in F(t)\} = f(F(t))/f(\mu) = f(F(\phi(t)))/f(\mu) = (f(F)/f(\mu))(\phi(t))$. Hence, (φ, ϕ) is an isomorphism and $(F/\mu, A) \simeq (f(F)/f(\mu), A)$.

(2) If g is a bijective, we define $\varphi : G/\mu \to K/f(\mu)$ as same as (1), and we have $\varphi((F/\mu)(t)) = \{(f(\mu))_{f(x)} | x \in F(t)\} = f(F(t))/f(\mu) = H(g(t))/f(\mu)$. Hence, (φ, g) is an isomorphism so that $(F/\mu, A) \simeq (H/f(\mu), B)$. ∎

Theorem 4.5 *Let (F, A) and (H, B) be two normalistic soft groups over groups G and K, respectively. If (f, g) is soft homomorphism from (F, A) to (H, B) and ν is a fuzzy normal subgroup of K, then*

(1) $(F/f^{-1}(\nu), A) \simeq (f(F)/\nu, A)$,

(2) *If g is a bijective mapping, then* $(F/f^{-1}(\nu), A) \simeq (H/\nu, B)$.

Proof. It follows from Lemma 4.3, that $(F/f^{-1}(\nu), A)$ and $(f(F)/\nu, A)$ are normalistic soft groups over groups $G/f^{-1}(\nu)$ and K/ν, respectively. Since f is epimorphic, by Lemma 4.2, we have $\nu = f(f^{-1}(\nu))$. Now, we show that $G_{f^{-1}(\nu)} \supset kerf$. For any $x \in kerf$, then $f(x) = e' = f(e)$, and so $\nu(f(x)) = \nu(f(e))$, i.e., $f^{-1}(\nu)(x) = f^{-1}(\nu)(e)$. Hence $x \in G_f^{-1}(\nu)$ and $G_f^{-1}(\nu) \supset kerf$. By Theorem 4.4, we have (1) $(F/f^{-1}(\nu), A) \simeq (f(F)/\nu, A)$ and (2) if g is a bijiective mapping, $(F/f^{-1}(\nu), A) \simeq (H/\nu)\, B)$. ∎

Lemma 4.6 *Let χ_S be a characteristic function of a subset S of a group G. Then χ_S is a fuzzy normal subgroup of G if and only if S is a normal subgroup of G.*

Theorem 4.7 *Let (F, A) and (H, B) be two normalistic soft groups over groups G and K, respectively. If (f, g) is soft homomorphism from (F, A) to (H, B) and N a is normal subgroup of G with $N \supset kerf$, then*

(1) $(F/\chi_N, A) \simeq (f(F)/\chi_{f(N)}, A)$,
(2) *If g is a bijective mapping, then* $(F/\chi_N, A) \simeq (H/\chi_{f(N)}, B)$.

Proof. By Lemma 4.6, χ_N and $\chi_{f(N)}$ are both fuzzy normal subgroups of G and K, respectively. So they are fuzzy normal subgroups of (F, A) and (H, B), respectively. Putting $\mu = \chi_N$ in Theorem 4.4, then $G_\mu = G_{\chi_N} = N \supset kerf$, and we have $f(\mu) = f(\chi_N) = \chi_{f(N)}$. Then (1) $(F/\chi_N, A) \simeq (f(F)/\chi_{f(N)}, A)$ and (2) if g is a bijective mapping, $(F/\chi_N, A) \simeq (H/\chi_{f(N)}, B)$. ∎

Theorem 4.8 *Let (F, A) and (H, B) be two normalistic soft groups over groups G and K, respectively. If (f, g) is soft homomorphism from (F, A) to (H, B) and J is a normal subgroup of K, then*

(1) $(F/\chi_{f^{-1}(J)}, A) \simeq (f(F)/\chi_J, A)$,
(2) *If g is a bijective mapping, then* $(F/\chi_{f^{-1}(J)}, A) \simeq (H/\chi_J, B)$.

Proof. We know that $\chi_{f^{-1}(J)}$ and χ_J are both fuzzy normal groups of (F, A) and (H, B), respectively. Setting $\nu = \chi_J$ in Theorem 4.5, we have $f^{-1}(\nu) = f^{-1}(\chi_J) = \chi_{f^{-1}(J)}$. In fact, for any $x \in G$, if $x \in f^{-1}(J)$, then $f(x) \in J$, $f^{-1}(\chi_J)(x) = \chi_J(f(x)) = 1 = \chi_{f^{-1}(J)}(x)$, if $x \notin f^{-1}(J)$, then $f(x) \notin J$, $f^{-1}(\chi_J)(x) = \chi_J(f(x)) = 0 = \chi_{f^{-1}(J)}(x)$. Hence,

(1) $(F/\chi_{f^{-1}(J)}, A) \simeq (f(F)/\chi_J, A)$ and
(2) if g is a bijective mapping, then $(F/\chi_{f^{-1}(J)}, A) \simeq (H/\chi_J, B)$. ∎

We can easy verify that $\chi_{\{e\}}f = \chi_{kerf}$ and $G/\chi_{\{e\}} \cong G$ for any group G. Putting $N = kerf$ in Theorem 4.7, we have the following.

Theorem 4.9 *Let (F, A) and (H, B) be two normalistic soft groups over groups G and K, respectively. If (f, g) is a soft homomorphism from (F, A) to (H, B), then*

(1) $(F/\chi_{kerf}, A) \simeq (f(F), A)$,
(2) *If g is a bijective mapping, then* $(F/\chi_{kerf}, A) \simeq (H, B)$.

Let N be a normal subgroup of a group G. Recall that a quotient group G/N induced by a normal subgroup N is determined by an equivalent relation $\sim$, where $x \sim y$ is defined by $xy^{-1} \in N$. For no confusion, we write $x \sim y(N)$ to show that x is equivalent to y with respect to the normal subgroup N, and $x \sim y(N)$ to mean that x is equivalent to y with respect to the fuzzy normal subgroup χ_N.

Lemma 4.10 *Suppose that N is a normal subgroup of a group G. Then $x \sim y(N)$ if and only if $x \sim y(\chi_N)$.*

Proof. $x \sim y(N) \Longleftrightarrow xy^{-1} \in N \Longleftrightarrow \chi_N(xy^{-1}) = 1 \Longleftrightarrow \chi_N(xy^{-1}) = \chi_N(e) \Longleftrightarrow x \sim y(\chi_N)$. The proof is complete. ∎

By Lemma 4.10, we have $G/N \cong G/\chi_N$, where N is the normal subgroup of group G. Applying Theorem 4.7, Theorem 4.8, Theorem 4.9, we have the following corollaries.

Corollary 4.11 *Let (F, A) and (H, B) be two normalistic soft groups over groups G and K, respectively. If (f, g) is a soft homomorphism from (F, A) to (H, B) and N is a normal subgroup of G with $F(t) \supset N \supset kerf$ for all $t \in A$, then*

(1) $(F/N, A) \simeq (f(F)/f(N), A)$,

(2) *If g is a bijective mapping, then $(F/N, A) \simeq (H/f(N), B)$.*

Corollary 4.12 *Let (F, A) and (H, B) be two normalistic soft groups over groups G and K, respectively. If (f, g) is a soft homomorphism from (F, A) to (H, B) and J is a normal subgroup of K with $F(t) \supset f^{-1}(J)$ for all $t \in A$, then*

(1) $(F/f^{-1}(J), A) \simeq (f(F)/J, A)$,

(2) *If g is a bijective mapping, then $(F/f^{-1}(J), A) \simeq (H/J, B)$.*

Corollary 4.13 *Let (F, A) and (H, B) be two normalistic soft groups over groups G and K, respectively. If (f, g) is a soft homomorphism from (F, A) to (H, B) with $F(t) \supset kerf$ for all $t \in A$, then*

(1) $(F/kerf, A) \simeq (f(F), A)$,

(2) *If g is a bijective mapping, then $(F/kerf, A) \simeq (H, B)$.*

Theorem 4.14 (Second fuzzy isomorphism theorem) *Let (F, A) be a normalistic soft group over group G and μ, ν two fuzzy normal subgroups of G with $\mu(e) = \nu(e)$, then*

$$(F_\mu F_\nu/\nu, A) \simeq (F_\mu/(\mu \cap \nu), A).$$

Proof. We know that $(F_\mu F_\nu/\nu, A)$ and $(F_\mu/(\mu \cap \nu), A)$ are normalistic soft groups over $G_\mu G_\nu/\nu$ and $G_\mu/(\mu \cap \nu)$, respectively. For any $x \in G_\mu G_\nu$, $x = ab$, where $a \in G_\mu$ and $b \in G_\nu$. Define $\varphi : G_\mu G_\nu/\nu \to G_\mu/(\mu \cap \nu)$ by

$$\varphi(\nu_x) = (\mu \cap \nu)_a.$$

(1) If $\nu_x = \nu_y$, where $y = a_1 b_1$, $a_1 \in G_\mu$ and $b_1 \in G_\nu$, then $\nu(ab(a_1 b_1)^{-1}) = \nu(abb_1^{-1}a_1^{-1}) = \nu(a_1^{-1}abb_1^{-1}) = \nu(a_1^{-1}a(b_1 b^{-1})^{-1}) = \nu(e)$. Hence $\nu(a_1^{-1}a) = \nu(b_1 b^{-1}) = \nu(e)$. Thus $(\mu \cap \nu)(aa_1^{-1}) = \mu(aa_1^{-1}) \wedge \nu(aa_1^{-1}) = \mu(e) \wedge \nu((a_1^{-1}a)^{-1}) = \mu(e) \wedge \nu(e) = (\mu \cap \nu)(e)$. That is $(\mu \cap \nu)_a = (\mu \cap \nu)_{a_1}$. Hence φ is well-defined.

(2) For any $\nu_x, \nu_y \in G_\mu G_\nu/\nu$, where $x = ab$, $y = a_1 b_1$, $a, a_1 \in G_\mu$, $b, b_1 \in G_\nu$. then $xy = aba_1 b_1$. Since G_μ is normal, $ba_1 b_1 \in G_\mu$. Hence $\varphi(\nu_x \nu_y) = \varphi(\nu_{xy}) = (\mu \cap \nu)_{a(ba_1 b_1)} = (\mu \cap \nu)_a (\mu \cap \nu)_{ba_1 b_1}$. Now we show $(\mu \cap \nu)_{ba_1 b_1} = (\mu \cap \nu)_{a_1}$. In fact, $(\mu \cap \nu)((ba_1 b_1)a_1^{-1}) = \mu(ba_1 b_1 a_1^{-1}) \wedge \nu(ba_1 b_1 a_1^{-1}) = \mu((ba_1 b_1)a_1^{-1}) \wedge \nu(b(a_1 b_1 a_1^{-1})) =$

$\mu(e) \wedge \nu(e) = (\mu \cap \nu)(e)$. Hence $\varphi(\nu_x\nu_y) = (\mu \cap \nu)_a(\mu \cap \nu)_{a_1} = \varphi(\nu_x)\varphi(\nu_y)$ and φ is a homomorphism.

(3) For any $(\mu\cap\nu)_a \in G_\mu/(\mu\cap\nu)$, taking $b \in G_\nu$, then $x = ab \in G_\mu G_\nu$. Hence $f(\nu_x) = (\mu \cap \nu)_a$ and φ is an epimorphism.

(4) For any $x, y \in G_\mu G_\nu$, where $x = ab$, $y = a_1b_1$, $a, a_1 \in G_\mu$, $b, b_1 \in G_\nu$, if $(\mu \cap \nu)_a$=$(\mu \cap \nu)_{a_1}$, then $(\mu \cap \nu)(aa_1^{-1})$=$(\mu \cap \nu)(e)$, i.e., $\mu(aa_1^{-1}) \wedge \nu(aa_1^{-1}) = \mu(e) \wedge \nu(e)$. Since $\mu(e) = \nu(e)$ and $\mu(aa_1^{-1}) = \mu(e)$, then $\nu(aa_1^{-1}) = \nu(e)$. Hence $\nu(xy^{-1}) = \nu(ab(a_1b_1)^{-1}) = \nu(abb_1^{-1}a_1^{-1}) = \nu(a_1^{-1}abb_1^{-1}) \geq \nu(a_1^{-1}a) \wedge \nu(bb_1^{-1}) = \nu((a_1^{-1}a)^{-1}) \wedge \nu(bb_1^{-1}) = \nu(e) \wedge \nu(e) = \nu(e)$. Thus $\nu_x = \nu_y$. Hence φ is a monomorphism.

By (1), (2), (3) and (4), we deduce that φ is an isomorphism from $G_\mu G_\nu/\nu$ to $G_\mu/(\mu \cap \nu)$.

Let $\phi : A \to A$ defined by $\phi(t) = t$. Then ϕ is bijective.

Also, we have $\varphi(((F/\nu)/(F_\mu/\nu))(t)) = F(t)/\mu = F(\varphi(t))/\mu = (F/\mu)(\phi(t))$. Hence $(\varphi, \phi) : (F_\mu F_\nu/\nu, A) \simeq (F_\mu/(\mu \cap \nu), A)$. ∎

Corollary 4.15 *Let (F, A) be a normalistic soft group over group G and N, J two normal subgroups of G, then*

$$(F_{\chi_N} F_{\chi_J}/\chi_J, A) \simeq (F_{\chi_N}/\chi_{N\cap J}, A).$$

Corollary 4.16 *Let (F, A) be a normalistic soft group over group G and N, J two normal subgroups of G with $F_{\chi_N}(t) \supset J$ for any $t \in A$, then*

$$(F_{\chi_N} F_{\chi_J}/J, A) \simeq (F_{\chi_N}/(N \cap J), A).$$

Theorem 4.17 (Third fuzzy isomorphism theorem) *Let (F, A) be a normalistic soft group over group G and μ, ν two fuzzy normal subgroups of G with $F_\mu(t) = G_\mu$ for all $t \in A$, then*

$$((F/\nu)/(F_\mu/\nu), A) \simeq (F/\mu, A).$$

Proof. $((F/\nu)/(F_\mu/\nu), A)$ is a normalistic soft group over $(G/\nu)/(G_\mu/\nu)$ and $(F/\mu, A)$ is a normalistic soft group over G/μ. Define $\varphi : (G/\nu)/(G_\mu/\nu) \to G/\mu$ by $\varphi(\nu_x(G_\mu/\nu)) = \mu_x$.

(1) If $\nu_x(G_\mu/\nu) = \nu_y(G_\mu/\nu)$, then $\nu_x(\nu_y)^{-1} = \nu_{xy^{-1}} \in G_\mu/\nu$, then $xy^{-1} \in G_\mu$, i.e., $\mu(xy^{-1}) = \mu(e)$ and $\mu(x) = \mu(y)$. Hence, $\mu_x = \mu_y$ and φ is well-defined.

(2) For any $\nu_x(G_\mu/\nu), \nu_y(G_\mu/\nu) \in (G/\nu)/(G_\mu/\nu)$, $\varphi[(\nu_x(G_\mu/\nu))(\nu_y(G_\mu/\nu))] = \varphi[(\nu_x\nu_y)(G_\mu/\nu)] = \varphi[\nu_{xy}(G_\mu/\nu)] = \mu_{xy} = \mu_x\mu_y = \varphi(\nu_x(G_\mu/\nu)\varphi(\nu_y(G_\mu/\nu)$. Thus, φ is a homomorphism.

(3) If $\mu_x = \mu_y$, then $\mu(xy^{-1}) = \mu(e)$ and $xy^{-1} \in G/\mu$, $\nu_{xy^{-1}} = \nu_x(\nu_y)^{-1} \in G_\mu/\nu$, we have $\nu_x(G_\mu/\nu) = \nu_y(G_\mu/\nu)$. Thus, φ is a monomorphism.

(4) For any $\mu_x \in G/\mu$, there exists $\nu_x(G_\mu/\nu) \in (G/\nu)/(G_\mu/\nu)$ such that $\varphi(\nu_x(G_\mu/\nu)) = \mu_x$, so φ is an epimorphism.

It follows that φ is an isomorphism from (1), (2), (3) and (4).

Let $\phi : A \to A$ defined by $\phi(t)$=t. Then ϕ is bijective. That is, $\varphi((G_\mu G_\nu)(t)/\nu)$ =$F_\mu(t)/(\mu\cap\nu)$=$F_\mu(\phi(t))/(\mu\cap\nu)$. Hence $(\varphi, \phi) : ((F/\nu)/(F_\mu/\nu), A) \simeq (F/\mu, A)$. ∎

Corollary 4.18

(1) *Let (F, A) be a normalistic soft group over group G and N, J two normal subgroups of G with $F_{\chi_J}(t) = J$ for all $t \in A$, then*

$$((F/\chi_N)/(J/\chi_N), A) \simeq (F/\chi_J, A).$$

(2) *Let (F, A) be a normalistic soft group over group G and N, J two normal subgroups of G with $N \subset J$ and $F_{\chi_J}(t) = J$ for all $t \in A$, then*

$$((F/N)/(J/N), A) \simeq (F/J, A).$$

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CERTAIN TRANSFORMATIONS OF BASIC AND POLY-BASIC HYPERGEOMETRIC SERIES

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Abstract. In the present work, certain new transformations of basic and poly-basic hypergeometric series have been established with the help of Bailey lemma.

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1. Introduction

In 1944, Bailey [5] established a powerful series identity which was later known as Bailey's lemma. The Bailey's lemma states that, if

$$\beta_n = \sum_{r=0}^{n} \alpha_r u_{n-r} v_{n+r} \tag{1.1}$$

and

$$\gamma_n = \sum_{r=n}^{\infty} \delta_r u_{r-n} v_{n+r} \tag{1.2}$$

then, under the suitable convergence conditions and if change in the order of summations is allowable

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n \tag{1.3}$$

where α_r, δ_r, u_r and v_r are functions of r, such that γ_n exists. The proof of the lemma is trivial.

The Bailey lemma has been a powerful tool in proving Rogers-Ramanujan type of identities and also a verity of transformations of basic and Poly-basic hypergeometric series. Slater [15], [16] used Bailey's lemma systematically to produced long list of 130 identities of Roger-Ramanujan type. The theory of

basic hypergeometric series has been extensively studied and developed during last decades due to its various application in mathematics as well as in other disciplines. For a complete account of the theory and its some of the applications one is referred to the notes of Gasper and Rahman [12], Exton [9], Agarwal [2], Fine[10], Andrew [3], [4] Ernst [8] and Tariboon et al [17].

In the present paper, we have made an attempt to establish some interesting transformations and summations of basic and polybasic hypergeometric series by making use of Bailey lemma. Some applications have also been discussed.

In what follows, the following notations and definitions [12] have been used.

For $|q| < 1$,

$$(a;q)_n = \begin{cases} a(1-a)(1-aq)(1-aq^2)...(1-aq^{n-1}); & n \in N \\ 1; & n = 0 \end{cases}$$

or, equivalently,

$$(a;q)_n = \prod_{j=0}^{\infty} \frac{(1-aq^j)}{(1-aq^{n+j})} \equiv \frac{(a;q)_\infty}{(aq^n;q)_\infty},$$

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n),$$

where a is real or complex.

A Basic Hypergeometric Series is defined as

$${}_r\Phi_s \left[\begin{matrix} a_1, a_2, ..., a_r; q; z \\ b_1, b_2, ..., b_s \end{matrix} \right] = \sum_{n=0}^{\infty} \frac{(a_1;q)_n \ (a_2;q)_n ... (a_r;q)_n}{(q;q)_n (b_1;q)_n \ (b_2;q)_n .... (b_s;q)_n} [(-1)^n q^{\frac{n(n-1)}{2}}]^{1+s-r} z^n$$

For $0 < |q| < 1$, the series converges absolutely for all z if $r \leq s$ and for $|z| < 1$ if $r = s+1$.

This series also converges absolutely if $|q| > 1$ and $|z| < |b_1 b_2 ... b_s| / |a_1 a_2 ... a_r|$.

We define the Poly-Basic Hypergeometric Series as

$$\Phi \left[\begin{matrix} a_1, a_2, ...a_r & : c_{1,1}, ...c_{1,r_1} & ; ...; c_{m,1}; ...c_{m,r_m} & ; q, q_1 q_2 ... q_m, z \\ b_1, b_2, ...b_s & : d_{1,1}, ...d_{1,s_1} & ; ...; d_{m,1}; ...d_{m,s_m} & \end{matrix} \right]$$

$$= \sum_{t=0}^{n-1} \frac{(a_1, a_2 .... a_r; q)_t}{(q, b_1, b_2 .... b_s; q)_t} z^t \prod_{j=1}^{m} \frac{(c_{j,1} .... c_{j,r_j}; q_j)_t}{(d_{j,1} .... d_{j,s_j}; q_j)_t}$$

which converges for $\max(|q|, |q_1|, ..., |q_m|) < 1$.

We shall also require the following known results in our work

$$(1.4) \qquad {}_2\Phi_1 \left[\begin{matrix} a, & y; & q; & q \\ & ayq & & \end{matrix} \right]_n = \frac{(aq, yq; q)_n}{(q, ayq; q)_n} \qquad ([1], \text{App.II}(8))$$

$$(1.5) \quad {}_4\Phi_3 \left[\begin{matrix} \alpha, & q\sqrt{\alpha}, & -q\sqrt{\alpha}, & e; & q; 1/e \\ & \sqrt{\alpha}, & -\sqrt{\alpha}, & \alpha q/e & & \end{matrix} \right]_n = \frac{(\alpha q, eq; q)_n}{(q, \alpha q/e; q)_n e^n} \quad ([1], \text{App.II}(8))$$

$$
\begin{aligned}
&{}_6\Phi_5\left[\begin{array}{ccccccc} \alpha, & q\sqrt{\alpha}, & -q\sqrt{\alpha}, & \beta, & \gamma, & \delta; & q;\ q \\ & \sqrt{\alpha}, & -\sqrt{\alpha}, & \alpha q/\beta, & \alpha q/\gamma, & \alpha q/\delta & & \end{array}\right]_n \\
&\qquad = \frac{(\alpha q, \beta q, \gamma q, \delta q; q)_n}{(q, \alpha q/\beta, \alpha q/\gamma, \alpha q/\delta; q)_n} \quad ([1],\ \text{App.II(25)})
\end{aligned} \tag{1.6}
$$

where $\alpha = \beta\gamma\delta$

$$
\sum_{i=0}^{n} \frac{(1-\alpha p^i q^i)(\alpha;p)_i(\beta;q)_i\beta^{-i}}{(1-\alpha)(q;q)_i(\alpha p/\beta;p)_i} = \frac{(\alpha p;p)_n(\beta q;q)_n\beta^{-n}}{(q;q)_n(\alpha p/\beta;p)_n} \quad ([12],\text{App.(II.34)}) \tag{1.7}
$$

$$
\begin{aligned}
&\sum_{i=0}^{n} \frac{(1-\alpha p^i q^i)(1-\beta p^i q^{-i})(\alpha;p)_i(\beta;p)_i(\gamma;q)_i(\alpha/\beta\gamma;q)_i q^i}{(1-\alpha)(1-\beta)(q;q)_i(\alpha q/\beta;q)_i(\alpha p/\gamma;p)_i(\beta\gamma p;p)_i} \\
&\qquad = \frac{(\alpha p, \beta p;p)_n(\gamma q;q)_n(\alpha q/\beta\gamma;q)_n}{(q, \alpha q/\beta;q)_n(\alpha p/\gamma;p)_n(\beta\gamma p;p)_n} \quad ([12]\ ,\ \text{App.(II.35)})
\end{aligned} \tag{1.8}
$$

$$
\begin{aligned}
&\sum_{r=0}^{n} \frac{(1-\alpha\delta p^r q^r)(1-\beta p^r/\delta q^r)(\alpha,\beta;p)_r(\gamma, \alpha\delta^2/\beta\gamma;q)_r}{(1-\alpha\delta)(1-\beta/\delta)(\delta q, \alpha\delta q/\beta;q)_r(\alpha\delta p/\gamma, \beta\gamma p/\delta;p)_r} q^r \\
&\qquad \frac{(1-\alpha)(1-\beta)(1-\gamma)(1-\alpha\delta^2/\beta\gamma)}{\delta(1-\alpha\delta)(1-\beta/\delta)(1-\gamma/\delta)(1-\alpha\delta/\beta\gamma)} \\
&\times \left(\frac{(\alpha p, \beta p;p)_n(\gamma q, \alpha\delta^2 q/\beta\gamma;q)_n}{(\delta q, \alpha\delta q/\beta;q)_n(\alpha\delta p/\gamma, \beta\gamma p/\delta;p)_n} - \frac{(\gamma/\alpha\delta, \delta/\beta\gamma;p)_1(1/\delta, \beta/\alpha\delta;q)_1}{(1/\gamma, \beta\gamma/\alpha\delta^2;q)_1(1/\alpha, 1/\beta;p)_1} \right) \\
&\qquad ([12],\text{App.(II.36) for } m=0)
\end{aligned} \tag{1.9}
$$

2. Main results

The following results have been established in this section.

$$
\begin{aligned}
&{}_4\Phi_3\left[\begin{array}{cccccc} a, & y, & \alpha q, & eq; & q; & q/e \\ & q, & ayq, & \alpha q/e & & \end{array}\right] \\
&\quad + {}_6\Phi_5\left[\begin{array}{ccccccc} aq, & yq, & \alpha, & q\sqrt{\alpha}, & -q\sqrt{\alpha}, & e; & q;\ 1/e \\ & q, & ayq, & \sqrt{\alpha}, & -\sqrt{\alpha}, & \alpha q/e & & \end{array}\right] \\
&\quad = {}_6\Phi_5\left[\begin{array}{ccccccc} a, & y, & \alpha, & q\sqrt{\alpha}, & -q\sqrt{\alpha}, & e; & q;\ q/e \\ & q, & ayq, & \sqrt{\alpha}, & -\sqrt{\alpha}, & \alpha q/e & & \end{array}\right]
\end{aligned} \tag{2.1}
$$

where $|q/e| < 1$ and $|1/e| < 1$. Choose $a = 0$ in (2.1), we have

$$
\begin{aligned}
&{}_3\Phi_2\left[\begin{array}{ccccc} y, & \alpha q, & eq; & q; q/e \\ & q, & \alpha q/e & & \end{array}\right] \\
&\quad + {}_5\Phi_4\left[\begin{array}{ccccccc} yq, & \alpha, & q\sqrt{\alpha}, & -q\sqrt{\alpha}, & e; & q, 1/e \\ & q, & \sqrt{\alpha}, & -\sqrt{\alpha}, & \alpha q/e & & \end{array}\right] \\
&\quad = {}_5\Phi_4\left[\begin{array}{ccccccc} y, & \alpha, & q\sqrt{\alpha}, & -q\sqrt{\alpha}, & e; & q; & q/e \\ & q, & \sqrt{\alpha}, & -\sqrt{\alpha}, & \alpha q/e & & \end{array}\right]
\end{aligned} \tag{2.2}
$$

where $|q/e|<1$ and $|1/e|<1$

$$
\begin{aligned}
&{}_6\Phi_5\left[\begin{matrix} a, & y, & \alpha q, & \beta q, & \gamma q, & \delta q; & q; & q \\ & q, & ayq, & \alpha q/\beta, & \alpha q/\gamma, & \alpha q/\delta & & \end{matrix}\right] \\
&\quad + {}_8\Phi_7\left[\begin{matrix} aq, & yq, & \alpha, & q\sqrt{\alpha}, & -q\sqrt{\alpha}, & \beta, & \gamma, & \delta; & q; & q \\ & q, & ayq, & \sqrt{\alpha}, & -\sqrt{\alpha}, & \alpha q/\beta, & \alpha q/\gamma, & \alpha q/\delta & & \end{matrix}\right] \\
&\quad = \frac{(aq, yq, \alpha q, \beta q, \gamma q, \delta q; q)_\infty}{(q, qyq, q, \alpha q/\beta, \alpha q/\gamma, \alpha q/\delta; q)_\infty} \\
&\quad + {}_8\Phi_7\left[\begin{matrix} a, & y, & \alpha, & q\sqrt{\alpha}, & -q\sqrt{\alpha}, & \beta, & \gamma, & \delta; & q; & q^2 \\ & q, & ayq, & \sqrt{\alpha}, & -\sqrt{\alpha}, & \alpha q/\beta, & \alpha q/\gamma, & \alpha q/\delta & & \end{matrix}\right]
\end{aligned}
\tag{2.3}
$$

$$
\begin{aligned}
&\Phi\left[\begin{matrix} a, & y, & \beta q: & \alpha p; & q, & p; & q/\beta \\ q, & ayq: & \alpha p/\beta & & & & \end{matrix}\right] \\
&\quad + \Phi\left[\begin{matrix} aq, & yq, & \beta: & \alpha: & \alpha pq; & q, & p, & pq; & 1/\beta \\ q, & ayq: & \alpha p/\beta: & \alpha & & & & & \end{matrix}\right] \\
&\quad = \Phi\left[\begin{matrix} a, & y, & \beta: & \alpha: & \alpha pq; & q, & p, & pq; & q/\beta \\ q, & ayq: & \alpha p/\beta: & \alpha & & & & & \end{matrix}\right]
\end{aligned}
\tag{2.4}
$$

Taking $p=q$ in (2.4), we get

$$
\begin{aligned}
&{}_4\Phi_3\left[\begin{matrix} a, & y, & \beta q, & \alpha q; & q; & q/\beta \\ & q, & ayq, & \alpha q/\beta & & \end{matrix}\right] \\
&\quad + \Phi\left[\begin{matrix} aq, & yq, & \beta, & \alpha: & \alpha q^2; & q, & q^2; & 1/\beta \\ q, & ayq, & \alpha q/\beta: & \alpha & & & & \end{matrix}\right] \\
&\quad = \Phi\left[\begin{matrix} a, & y, & \beta, & \alpha: & \alpha q^2; & q, & q^2; & q/\beta \\ q, & ayq, & \alpha q/\beta: & \alpha & & & & \end{matrix}\right]
\end{aligned}
\tag{2.5}
$$

$$
\begin{aligned}
&\Phi\left[\begin{matrix} aq, & yq, \ \gamma, \ \alpha/\beta\gamma: & \alpha, & \beta: & \alpha pq: & \beta p/q; & q, \ p, \ pq, \ p/q; \ q \\ q, & ayq, \ \alpha q/\beta: & \alpha p/\gamma, & \beta\gamma p: & \alpha: & \beta & & \end{matrix}\right] \\
&\quad + \Phi\left[\begin{matrix} a, & y, & \gamma q, & \alpha q/\beta\gamma: & \alpha p, & \beta p; & q, & p; & q \\ q, & ayq, & \alpha q/\beta: & \alpha p/\gamma, & \beta\gamma p & & & & \end{matrix}\right] \\
&\quad = \frac{(aq, yq, \gamma q, \alpha q/\beta\gamma; q)_\infty (\alpha p, \beta p; p)_\infty}{(q, q, ayq, \alpha q/\beta; q)_\infty (\alpha p/\gamma, \beta\gamma p; p)_\infty} \\
&\quad + \Phi\left[\begin{matrix} a, \ y, \ \gamma \ \alpha/\beta\gamma: & \alpha \ , \beta: & \alpha pq: & \beta p/q; & q \ , p \ , pq, \ p/q; \ q^2 \\ q, \ ayq, \ \alpha q/\beta: & \alpha p/\gamma, \ \beta\gamma p: & \alpha: & \beta & & \end{matrix}\right]
\end{aligned}
\tag{2.6}
$$

$$
\begin{aligned}
&\frac{(1-\alpha)(1-\beta)(1-\gamma)(1-\alpha\delta^2/\beta\gamma)}{\delta(1-\alpha\delta)(1-\beta/\delta)(1-\gamma/\delta)(1-\alpha\delta/\beta\gamma)} \\
&\quad\Bigg(\Phi\left[\begin{matrix} a, & y, & \gamma q, & \alpha\delta^2 q/\beta\gamma: & \alpha p, & \beta p; & q, & p; & q \\ ayq, & \delta q, & \alpha\delta q/\beta: & \alpha\delta p/\gamma, & \beta\gamma p/\delta & & & & \end{matrix}\right] \\
&\quad -\frac{(aq,yq;q)_\infty}{(q,ayq;q)_\infty}\times\frac{(1-\gamma/\alpha\delta)(1-\delta/\beta\gamma)(1-1/\delta)(1-\beta/\alpha\delta)}{(1-1/\gamma)(1-\beta\gamma/\alpha\delta^2)(1-1/\alpha)(1-1/\beta)}\Bigg) \\
&\quad =\frac{(aq,yq,\gamma q,\alpha\delta^2 q/\gamma;q)_\infty(\alpha p,\beta p;p)_\infty}{(q,ayq,\delta q,\alpha\delta q/\beta;q)_\infty(\alpha\delta p/\gamma,\beta\gamma p/\delta;p)_\infty}\frac{(1-\alpha)(1-\beta)(1-\gamma)}{\delta(1-\alpha\delta)(1-\beta/\delta)} \\
&\quad \times\frac{(1-\alpha\delta^2/\beta\gamma)}{(1-\gamma/\delta)(1-\alpha\delta/\beta\gamma)}-\frac{(aq,yq;q)_\infty}{(q,ayq;q)_\infty}\frac{(1-\gamma/\alpha\delta)(1-\delta/\beta\gamma)}{(1-1/\alpha)(1-1/\beta)} \\
&\quad \times\frac{(1-1/\delta)(1-\beta/\alpha\delta)}{(1-1/\gamma)(1-\beta\gamma/\alpha\delta^2)}\frac{(1-\alpha)(1-\beta)(1-\gamma)(1-\alpha\delta^2/\beta\gamma)}{\delta(1-\alpha\delta)(1-\beta/\delta)(1-\gamma/\delta)(1-\alpha\delta/\beta\gamma)} \\
&\quad -\Phi\left[\begin{matrix} aq,yq,\gamma,\alpha\delta^2/\beta\gamma:\alpha,\beta:\alpha\delta pq:\beta p/\delta q; & q, & p, & pq, & p/q; & q \\ ayq, & \delta q, & \alpha\delta q/\beta:\alpha\delta p/\gamma, & \beta\gamma p/\delta:\alpha\delta:\beta/\delta & & \end{matrix}\right] \\
&\quad +\Phi\left[\begin{matrix} a, & y, & \gamma, & \alpha\delta^2/\beta\gamma:\alpha, & \beta:\alpha\delta pq:\beta p/\delta q; & q, & p, & pq, & p/q; & q^2 \\ ayq, & \delta q, & \alpha\delta q/\beta:\alpha\delta p/\gamma, & \beta\gamma p/\delta:\alpha\delta:\beta/\delta & & & & & & \end{matrix}\right]
\end{aligned}
\tag{2.7}
$$

Proof of 2.1. Let us choose $u_r = v_r = 1$ and $\delta_r = \dfrac{(a,y;q)_r}{(q,ayq;q)_r}q^r$ in (1.1) and (1.2), we get

$$
\begin{aligned}
\beta_n &= \sum_{r=0}^{n}\alpha_r \\
\gamma_n &= \sum_{r=n}^{\infty}\delta_r
\end{aligned}
\tag{2.8}
$$

and γ_r can be written as,

$$\gamma_r = \sum_{r=0}^{\infty}\delta_r - \sum_{r=0}^{n}\delta_r + \delta_n$$

By choosing $\delta_r = \dfrac{(a,y;q)_n}{(q,ayq;q)_n}q^n$ in the above relation, gives

$$\gamma_r = \sum_{r=0}^{\infty}\frac{(a,y;q)_r}{(q,ayq;q)_r}q^r - \sum_{r=0}^{n}\frac{(a,y;q)_r}{(q,ayq;q)_r}q^r + \frac{(a,y;q)_n}{(q,ayq;q)_n}q^n$$

By using (1.4), we have,

$$\gamma_r = \frac{(aq,yq;q)_\infty}{(q,ayq;q)_\infty} - \frac{(aq,yq;q)_n}{(q,ayq;q)_n} + \frac{(a,y;q)_n}{(q,ayq;q)_n}q^n \tag{2.9}$$

Substituting (2.8) and (2.9) in (1.3), we obtain the following relation

$$\sum_{n=0}^{\infty}\alpha_n\left[\frac{(aq,yq;q)_\infty}{(q,ayq;q)_\infty}-\frac{(aq,yq;q)_n}{(q,ayq;q)_n}+\frac{(a,y;q)_n}{(q,ayq;q)_n}q^n\right]$$
$$=\sum_{n=0}^{\infty}\beta_n\frac{(a,y;q)_n}{(q,ayq;q)_n}q^n \tag{2.10}$$

By choosing $\alpha_r=\dfrac{(\alpha,q\sqrt{\alpha},-q\sqrt{\alpha},e;q)_r}{(q,\sqrt{\alpha},-\sqrt{\alpha},\alpha q/e;q)_r e^r}$ in (2.8), we get

$$\beta_n=\sum_{r=0}^{n}\frac{(\alpha,q\sqrt{\alpha},-q\sqrt{\alpha},e;q)_r}{(q,\sqrt{\alpha},-\sqrt{\alpha},\alpha q/e;q)_r e^r}$$

By using (1.5), we get

$$\beta_n=\frac{(\alpha q,eq;q)_n}{(q,\alpha q/e;q)_n e^n}$$

Now, put α_n and β_n in (2.10). We have (2.1).

Proof of 2.3. Let $\alpha_r=\dfrac{(\alpha,q\sqrt{\alpha},-q\sqrt{\alpha},\beta,\gamma,\delta;q)_r}{(q,\sqrt{\alpha},-\sqrt{\alpha},\alpha q/\beta,\alpha q/\gamma,\alpha q/\delta;q)_r}q^r$ in (2.8). We get

$$\beta_r=\sum_{r=0}^{n}\frac{(\alpha,q\sqrt{\alpha},-q\sqrt{\alpha},\beta,\gamma,\delta;q)_r}{(q,\sqrt{\alpha},-\sqrt{\alpha},\alpha q/\beta,\alpha q/\gamma,\alpha q/\delta;q)_r}q^r$$

By employing (1.6), we have

$$\beta_n=\frac{(\alpha q,\beta q,\gamma q,\delta q;q)_n}{(q,\alpha q/\beta,\alpha q/\gamma,\alpha q/\delta;q)_n}$$

By substituting α_n and β_n in (2.10), we get (2.3).

Proof of 2.4. By taking $\alpha_r=\dfrac{(\alpha pq;pq)_r(\alpha;p)_r(\beta;q)_r\beta^{-r}}{(\alpha;pq)_r(\alpha p/\beta;p)_r(q;q)_r}$ in (2.8), we have

$$\beta_n=\sum_{r=0}^{n}\frac{(\alpha pq;pq)_r(\alpha;p)_r(\beta;q)_r}{(\alpha;pq)_r(\alpha p/\beta;p)_r(q;q)_r}\beta^{-r}$$

By using (1.7), we obtain

$$\beta_n=\frac{(\alpha p;p)_n(\beta q;q)_n}{(q;q)_n(\alpha p/\beta;p)_n}\beta^{-n}$$

Now, put α_n and β_n in (2.10). We obtain (2.4).

Proof of 2.6. Let us choose

$$\alpha_r=\frac{(\alpha pq;pq)_r(\beta p/q;p/q)_r(\alpha,\beta;p)_r(\gamma,\alpha/\beta\gamma;q)_r q^r}{(\alpha;pq)_r(\beta;p/q)_r(q,\alpha q/\beta;q)_r(\alpha p/\gamma,\beta\gamma p;p)_r}$$

in (2.8) and we have

$$\beta_n = \sum_{r=0}^{n} \frac{(\alpha pq; pq)_r (\beta p/q; p/q)_r (\alpha, \beta; p)_r (\gamma, \alpha/\beta\gamma; q)_r}{(\alpha; pq)_r (\beta; p/q)_r (q, \alpha q/\beta; q)_r (\alpha p/\gamma, \beta\gamma p; p)_r} q^r$$

By employing (1.8), we get

$$\beta_n = \frac{(\alpha p, \beta p; p)_n (\gamma q, \alpha q/\beta\gamma; q)_n}{(q, \alpha q/\beta; q)_n (\alpha p/\gamma, \beta\gamma p; p)_n}$$

By substituting α_n and β_n in (2.10), we get (2.6).

Proof of 2.7. Let us choose

$$\alpha_r = \frac{(\alpha\delta pq; pq)_r (\beta p/\delta q; p/q)_r (\alpha, \beta; p)_r (\gamma, \alpha\delta^2/\beta\gamma; q)_r q^r}{(\alpha\delta; pq)_r (\beta/\delta; p/q)_r (\alpha\delta p/\gamma, \beta\gamma p/\delta; p)_r (\delta q, \alpha\delta q/\beta; q)_r}$$

in (2.8) and, by employing (1.9), we have

$$\beta_n = \frac{(1-\alpha)(1-\beta)(1-\gamma)(1-\alpha\delta^2/\beta\gamma)}{\delta(1-\alpha\delta)(1-\beta/\delta)(1-\gamma/\delta)(1-\alpha\delta/\beta\gamma)}$$
$$\times \left(\frac{(\alpha p, \beta p; p)_n (\gamma q, \alpha\delta^2 q/\beta\gamma; q)_n}{(\delta q, \alpha\delta q/\beta; q)_n (\alpha\delta p/\gamma, \beta\gamma p/\delta; p)_n} - \frac{(\gamma/\alpha\delta, \delta/\beta\gamma; p)_1 (1/\delta, \beta/\alpha\delta; q)_1}{(1/\gamma, \beta\gamma/\alpha\delta^2; q)_1 (1/\alpha, 1/\beta; p)_1} \right)$$

By substituting α_n and β_n in (2.10), we get (2.7).

Concluding remark

In the above section, we have demonstrated the power of Bailey lemma as a tool for discovering new transformations of basic hypergeometric series from the known summations and transformations. Some of the transformations in the previous section generalize the known transformation formulae. The study may be continued to further develop the theory of summation and transformation of basic hypergeometric series

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CLEAN MULTIPLICATIVE HYPERRINGS

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Abstract. In this work, we introduce the notation of clean multiplicative hyperrings and we will obtain some properties of them. In particular, we study some topological concepts to realize clean elements of a multiplicative hyperring by clopen subsets of its Zariski topology.

Keywords: multiplicative hyperring, hyperideal, clean element, clean multiplicative hyperring, Zariski topology.
AMS Mathematics Subject Classification: 20N20, 16Y99.

1. Introduction

The theory of hyperstructures has been introduced by Marty in 1934 during the 8^{th} Congress of the Scandinavian Mathematicians [22]. Marty introduced hypergroups as a generalization of groups. He published some notes on hypergroups, using them in different contexts as algebraic functions, rational fractions, non commutative groups and then many researchers have been worked on this new field of modern algebra and developed it. It was later observed that the theory of hyperstructures has many applications in both pure and applied sciences; for example, semi-hypergroups are the simplest algebraic hyperstructures that possess the properties of closure and associativity. The theory of hyperstructures has been widely reviewed [10], [11], [16], [39].

In [11], Corsini and Leoreanu-Fotea have collected numerous applications of algebraic hyperstructures, especially those from the last fifteen years to the following subjects: geometry, hypergraphs, binary relations, lattices, fuzzy sets and rough sets, automata, cryptography, codes, median algebras, relation algebras, artificial intelligence, and probabilities. The hyperrings were introduced and studied by Krasner [21], Nakasis [25], Massouros [23] and especially studied by Davvaz and Leoreanu-Fotea [16], Zahedi and Ameri [40], Ameri and Norouzi [3], [4]. The study

on hyperrings in [38] ends with an outline of applications in chemistry and physics, analyzing several special kinds of hyperstructures: e-hyperstructures and transposition hypergroups. The theory of suitable modified hyperstructures can serve as a mathematical background in the field of quantum communication systems. A well-known type of a hyperring, called the Krasner hyperring [21]. Krasner hyperrings are essentially rings, with approximately modified axioms in which addition is a hyperoperation, while the multiplication is an operation. Then, this concept has been studied by a variety of authors. Some principal notions of hyperring theory can be found in [15], [16], [24], [37], [39]. The another type of hyperrings was introduced by Rota in 1982 which the multiplication is a hyperoperation, while the addition is an operation, and it is called it a multiplicative hyperring (for more details see [32], [33], [34], [35]) which was subsequently investigated by Olson and Ward [26] and many others. De Salvo [17] introduced hyperrings in which the additions and the multiplications are hyperoperations. Moreover, there exists another types of hyperrings that both the addition and multiplication are hyperoperations and instead associativity, commutativity and distributivity satisfy in weak associativity, weak commutativity and weak distributivity, which is called H_V-hyperrings, this type of hyperrings can be seen in [38], [39]. Also, there are other types of hyperrings which were completely studied in [15]. These hyperrings are studied by Rahnamai Barghi [31]. Procesi and Rota in [29] have studied ring of fractions in Krasner hyperrings and also they conceptualized in [30] the notion of primeness of hyperideal in a multiplicative hyperring, and in [12], Dasgupta extended the prime and primary hyperideals in multiplicative hyperrings.

A special equivalence relations which is called fundamental relations play important roles in the the theory of algebraic hyperstructures. The fundamental relations are one of the most important and interesting concepts in algebraic hyperstructures that ordinary algebraic structures are derived from algebraic hyperstructures by them. The fundamental relation β^* on hypergroups was defined by Koskas [20], mainly studied by Corsini [23], Freni [18], [19], Vougiouklis [39] (for more details about hyperrings and fundamental relations on hyperrings see [3], [4], [13], [15], [37], [39]). In [8], the authors studied the notation of regular multiplicative hyperrings and in [7], we studied Zariski topology on multiplicative hyperrings. In this paper, we introduce the notation of clean multiplicative hyperrings and we will obtain some properties of them. Also we study some topological concepts to realize clean elements of a multiplicative hyperring by clopen subsets of its Zariski topology.

2. Preliminaries

A hyperoperation $''.''$ on H is a mapping of $H \times H$ into the family of non-empty subsets of H. Let $''.''$ be a hyperoperation on H. Then, $(H, .)$ is called a hypergroupoid. A hypergroup is a hypergroupoid $(H, .)$, that satisfies:

(1.) $\forall a, b, c \in H : a(bc) = (ab)c$;

(2.) $\forall a \in H, aH = H = Ha$.

In the above definition, if $A, B \subseteq H$ and $h \in H$, then

$$AB = \bigcup_{a\in A, b\in B} ab, \quad Ah = A\{h\}, hB = \{h\}B.$$

A non-empty set R with two hyperoperations $+$ and $.$ is said to be a hyperring if $(R, +)$ is a canonical hypergroup, $(R, .)$ is a semihypergroup with $r.0 = 0.r = 0$ for all $r \in R$ (0 as a bilaterally absorbing element) and the hyperoperation $''.''$ is distributive over $+$, i.e., for every $a, b, c \in R; a(b+c) = ab+ac$ and $(a+b)c = ac+bc$.

A multiplicative hyperring is an additive commutative group $(R, +)$ endowed with a hyperoperation $''.''$ which satisfies the following conditions:

(1) $\forall a, b, c \in R : a(bc) = (ab)c$;

(2) $\forall a, b, c \in R : (a + b)c \subseteq ac + bc, a(b + c) \subseteq ab + ac$;

(3) $\forall a, b \in R : (-a)b = a(-b) = -(ab)$.

If, in (2), we have equalities instead of inclusions, then we say that the multiplicative hyperring is strongly distributive.

Let $(R, +, .)$ be a hyperring. We define the relation γ as follows:

$a\gamma b$ if and only if $\{a, b\} \subseteq U$ where U is a finite sum of finite products of elements of R, i.e.,

$$a\gamma b \Longleftrightarrow \exists z_1, ..., z_n \in R \text{ such that } \{a, b\} \subseteq \sum_{j\in J} \prod_{i\in I_j} z_i; \quad I_j, J \subseteq \{1, ..., n\}.$$

We denote the transitive closure of γ by γ^*. The relation γ^* as the smallest equivalence relation on a multiplicative hyperring $(R, +, .)$ such that the quotient R/γ^*, the set of all equivalence classes, is a fundamental ring. Let $\mathcal{U}$ be the set of all finite sums of products of elements of R we can rewrite the definition of γ^* on R as follows:

> $a\gamma^* b \Longleftrightarrow \exists z_1, ..., z_{n+1} \in R$ with $z_1 = a, z_{n+1} = b$ and $u_1, ..., u_n \in \mathcal{U}$ such that $\{z_i, z_{i+1}\} \subseteq u_i$ for $i \in \{1, ..., n\}$.

Suppose that $\gamma^*(a)$ is the equivalence class containing $a \in R$. Then, both the sum $\oplus$ and the product $\odot$ in R/γ^* are defined as follows:$\gamma^*(a) \oplus \gamma^*(b) = \gamma^*(c)$ for all $c \in \gamma^*(a) + \gamma^*(b)$ and $\gamma^*(a) \odot \gamma^*(b) = \gamma^*(d)$ for all $d \in \gamma^*(a).\gamma^*(b)$. Then R/γ^* is a ring, which is called fundamental ring of R (for more details see [37] and [38]).

Definition 2.1. Let R be a multiplicative hyperring. Then

(i) an element $e \in R$ is said to be a left (resp. right) identity if $a \in e.a$ (resp. $a \in a.e$) for $a \in R$. An element e is called an *identity element* if it is both left and right identity element.

(ii) an element $e \in R$ is said to be a left (resp. right) scalar identity if $a = e.a$ (resp., $a = a.e$) for $a \in R$. An element e is called an *scalar identity element* if it is both left and right scalar identity element.

(iii) Let R be a a multiplicative hyperring with an identity e. An element A is called a left (right) invertible (with respect to e), if there exists $x \in R$, such that $e \in xa (e \in ax)$ and a is called invertible if it is both a left and a right invertible.

A multiplicative hyperring R is called a *left (right) invertible* if every element of R has a left (right) invertible and R is called *invertible* if it is both a left and a right invertible. Denote the set of all invertible elements in R by $U(R)$ (with respect to the identity e by $U_e(R)$).

Definition 2.2. Let R be a multiplicative hyperring. We say that $a \in R$ is regular, if there exists $x \in R$ such that $0 \in a(1 - xa)$. Also, we say that R is regular multiplicative hyperring, if all of elements in R are regular elements. Denote the set of all regular elements in R by $V(R)$.

Definition 2.3. Let R be a multiplicative hyperring. An element $e \in R$ is an idempotent, if there exists $e' \in R$ such that $0 \in ee'$ and $e + e' = 1$. Also, we say that a set A of R is idempotent if every element in A is an idempotent. Denote the set of all idempotent elements of R by $Idem(R)$.

Remark 2.4. In throughout of the paper we define $A^2 = \bigcup_{a \in A} a^2$.

Definition 2.5. We say that I is a hyperideal of multiplicative hyperring $(R, +, .)$ if it satisfies the following conditions:

(1) $I - I \subseteq I$,

(2) $\forall x \in I$, $r \in R$, $xr \cup rx \subseteq I$.

Definition 2.6. An element a of R is nilpotent if there exists an n such that $0 \in a^n$ and denote the set of all nilpotent elements of R by $nil_\omega(R)$. Also, we define $rad(I) = \{r \in R | r^n \subseteq I, n \in \mathbb{N}\}$.

Definition 2.7. ([12]) A hyperideal $I(\neq R)$ of a multiplicative hyperring $(R, +, \circ)$ is a prime hyperideal of R, if for all $a, b \in R$, $a \circ b \subseteq I$ then $a \in I$ or $b \in I$.

A hyperideal $I(\neq R)$ of a multiplicative hyperring R is maximal if for any hyperideal J of R, $I \subsetneq J \subseteq R$ then $J = R$.

Definition 2.8. ([12])Let $\mathcal{C}$ be the class of all finite products of elements of a multiplicative hyperring $(R, +, \circ)$, i.e.,

$$\mathcal{C} = \left\{ \prod_{i=1}^{n} r_i | r_i \in R, i \in \mathbb{N} \right\} \subseteq P^*(R).$$

A hyperideal I of R is said to be a $\mathcal{C}$-ideal of R if, for any $A \in \mathcal{C}$, $A \cap I \neq \emptyset$, then $A \subseteq I$.

Definition 2.9. ([16]) A homomorphism (resp. good homomorphism) between two multiplicative hyperrings $(R, +, \circ)$ and $(R', +', \circ')$ is a map ϕ from R into R' such that for all $a, b \in R$,

$$f(a+b) = f(a) +' f(b) \quad \text{and} \quad \phi(a \circ b) \subseteq \phi(a) \circ' \phi(b),$$

(resp. $\phi(a \circ b) = \phi(a) \circ' \phi(b)$).

Therefore, let $(R, +, \circ)$ be a multiplicative hyperring and consider the canonical projection $\phi : R \to R/\gamma^*$. The heart of R is the set $\omega_R = \{r \in R | \phi(r) = \bar{0}\}$.

Definition 2.10. Let R be a multiplicative hyperring. An element $a \in R$ is ω-regular, if there exists $x \in R$ such that $a - axa \subseteq \omega_R$. Also, we say that R is ω-regular multiplicative hyperring, if all of elements in R are regular elements. Denote the set of all regular elements in R by $V_\omega(R)$.

Definition 2.11. Let R be a multiplicative hyperring. An element $e \in R$ is said to be ω-idempotent, if there exists $e - e^2 \subseteq \omega_R$. Denote the set of all ω-idempotent elements of R by $Idem_\omega(R)$.

Definition 2.12. An element a of R is ω-nilpotent if there exists a positive integer n such that $a^n \subseteq \omega_R$ and denote the set of all ω-nilpotent elements of R by $nil_\omega(R)$.

3. Clean Multiplicative Hyperrings

Definition 3.1. Let R be a commutative multiplicative hyperring with a unity. Then we say that an element $a \in R$ is clean if there exist $e \in Idem(R)$ and $u \in U(R)$ such that $a = e + u$ and we denote R is clean, if every element in R is clean. Also, we say that an element $a \in R$ is ω-clean if there exist $e \in Idem(R)$ and $u \in U(R)$ such that $a - (e + u) \in \omega_R$ and we denote R is ω-clean, if every element in R is ω-clean.

Example 3.2. (see [8]) Let $(R, +, .)$ be clean commutative ring with a unity 1. Let $A \in P^*(R) = P(R) - \{\emptyset\}, |A| \geq 2$, and $1 \in A$, define a multiplicative hyperring $(R_A, +, \circ)$, where $R_A = R$ and for all $x, y \in R_A$, $x \circ y = \{xay \mid a \in A\}$. Then $(R_A, +, \circ)$ is a clean multiplicative hyperring.

Example 3.3. (see [8]) Let $(R, +, .)$ be a non-zero clean ring and for all $a, b \in R$ define a hyper-operation $a \circ b = \{a.b, 2a.b, 3a.b, \dots\}$. Then $(R, +, \circ)$ is a clean multiplicative hyperring, which is not strongly distributive.

Theorem 3.4. *Let R be a commutative multiplicative hyperring with a scalar identity. Then we have the following statements:*

(i) *If $e \in Idem(R)$, then $1 - e \in Idem(R)$,*

(ii) *If $e \in Idem(R)$, then $e \in Idem_\omega(R)$,*

(iii) *If* $e \in Idem(R)$, *then* $\gamma^*(e) \in Idem(R/\gamma^*)$,

(iv) *If* $e \in Idem_\omega(R)$, *then* $\gamma^*(e) \in Idem(R/\gamma^*)$,

(v) *If* $\gamma^*(e) \in Idem(R/\gamma^*)$, *then* $e \in Idem_\omega(R)$,

(vi) *If* $u \in U(R)$, *then* $\gamma^*(u) \in U(R/\gamma^*)$.

(vii) $a \in R$ *is clean* (ω-*clean*) *if and only if* $1 - a$ *is clean* (ω-*clean*).

Proof. Straightforward. ∎

Theorem 3.5. *Let* R *be a commutative multiplicative hyperring with a scalar identity. Then the following statements hold:*

(i) *If* R *is clean, then* R/γ^* *is clean,*

(ii) *If* R/γ^* *is clean then* R *is* ω-*clean.*

Proof. (i) Assume that R is clean, then for all $\gamma^*(a) \in R/\gamma^*$, we have $a \in R$. Thus there exist $e \in Idem(R)$ and $u \in U(R)$ such that $a = e + u$. Therefore $\gamma^*(a) = \gamma^*(e) + \gamma^*(u)$. Now, by 3.4(iii)(vi), we have $\gamma^*(a)$ is clean. Hence R/γ^* is clean.

(ii) It is straightforward. ∎

Lemma 3.6. *Let* R *be a commutative multiplicative hyperring with a scalar identity. If* $u + nil(R)$ *is a unit in* $R/nil(R)$ *(resp.,* $R/nil_\omega(R)$*) and* $nil(R)$ *is* $\mathcal{C}$-*ideal, then we have the following statements:*

(i) $u + \omega_R \in U(R/\omega_R)$.

(ii) *if* $|\omega_R| = 1$, *then* $u \in U(R)$.

Proof. (i) Assume that $u + nil(R)$ is an unit in $R/nil(R)$, then there exists $\ell + nil(R)$ in $R/nil(R)$ such that $1 + nil(R) \in (u + nil(R))(\ell + nil(R))$. Thus $(1 - u\ell) \cap nil(R) \neq \emptyset$, and since $nil(R)$ is $\mathcal{C}$-ideal, then $1 - u\ell \subseteq nil(R)$. So, there exists $n \in \mathbb{N}$ such that $0 \in (1 - u\ell)^n$. Therefore, $\gamma^*(0) = \gamma^*((1 - u\ell)^n) = \gamma^*(1 - uv)$, where $v = \sum_{i=1}^{n} \binom{n}{i}(-1)^i u^{i-1}\ell^i$. Now, we have $1 - uv \subseteq \omega_R$, then $1 + \omega_R = (u + \omega_R)(v + \omega_R)$. Hence $u + \omega_R \in U(R/\omega_R)$.

(ii) It follows immediately from part (i). ∎

Theorem 3.7. *Let* R *be a commutative multiplicative hyperring with a scalar identity* 1. *If* $a + nil_\omega(R)$ *is clean element in* $R/nil_\omega(R)$, *then the following statements hold:*

(i) a *is* ω-*clean.*

(ii) *If* $|\omega_R| = 1$, *then* a *is clean in* R.

Proof. (i) Since $a+nil_\omega(R)$ is clean, then there exist $u+nil_\omega(R) \in U(R/nil_\omega(R))$ and $s+nil_\omega(R) \in Idem(R/nil_\omega(R))$ such that $a+nil_\omega(R) = (u+s)+nil_\omega(R)$. Thus, by Lemma 3.6(i) and Lemma 4.13 of [7], there exists $u+\omega_R \in U(R/\omega_R)$ and $e \in Idem_\omega(R)$ such that $u+nil_\omega(R) = (u+\omega_R)+nil_\omega(R)$ and $s+nil_\omega(R) = (e+\omega_R)+nil_\omega(R)$. Therefore it's enough to set $a+\omega_R = (u+e)+\omega_R$, then $a-(u+e) \subseteq \omega_R$, i.e., a is ω-clean.

(ii) Obvious from part (i). ∎

Corollary 3.8. *Let R be a commutative multiplicative hyperring with a scalar identity. Then we have the following statements:*

(i) *If $R/nil_\omega(R)$ is clean, then R/ω_R is so.*

(ii) *If $|\omega_R| = 1$ and $R/nil_\omega(R)$ is clean, then R is so.*

Proof. It follows immediately from Theorem 3.7. ∎

Theorem 3.9. *Let R be a regular commutative multiplicative hyperring with a scalar identity 1. Then there exist $u+\omega_R \in U(R/\omega_R)$ and $A \subseteq A^2$ such that $a-uA \subseteq \omega_R$.*

Proof. Since for all $a \in R$, there exists $y \in R$ such that, $0 \in a(1-ya)$, so we set $A = ya$, then $A \subseteq A^2$. Now, we can put $U = u+\omega_R$, where $u = aA+(1-A)$. Thus by getting $V = \nu+\omega_R$, where $\nu = yA+(1-A)$, we have $1-u\nu \subseteq \omega_R$. Hence $a-uA \subseteq \omega_R$, where $u+\omega_R \in U(R/\omega_R)$ and $A \subseteq A^2$. ∎

Theorem 3.10. *Let R be a regular commutative multiplicative hyperring with a scalar identity 1. Then there exist $u+\omega_R \in U(R/\omega_R)$ and $A \subseteq A^2$ such that $a-(u+A) \subseteq \omega_R$.*

Proof. By Theorem 3.9, there exist $u+\omega_R \in U(R/\omega_R)$ and $A \subseteq A^2$ such that $a-uA \subseteq \omega_R$. Now, we take $\ell = uA+(1-A)$ and $\mathfrak{m} = 1-A$. Thus $\ell+\omega_R \in U(R/\omega_R)$ and $\mathfrak{m} \in Idem_\omega(R)$ and $a-(\ell+\mathfrak{m}) \subseteq \omega_R$. ∎

Let R be a multiplicative hyperring with identity 1. Define

$$\mathfrak{M} = \{(a_{ij})_{n\times n} \mid 0 \in a_{ij}a_{jj}a_{jk};\ 1 \le i,j,k \le n,\ i \ne j \ne k\}.$$

Now, we have the following properties:

Lemma 3.11. *Let for all $(a_{ij})_{n\times n} \in \mathfrak{M}$, $a_{jj} \in U(R)$. Then $\mathfrak{M} \subseteq U(M_n(R))$.*

Proof. Assume that $A = (a_{ij})_{n\times n} \in \mathfrak{M}$ and $u_{jj} = a_{jj} \in U(R)$ and u'_{jj} is inverse of u_{jj}. Then by some calculation we obtain that

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -a_{21}u'_{11} & 1 & 0 & \cdots & 0 \\ -a_{31}u'_{11} & -a_{32}u'_{22} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{n1}u'_{11} & -a_{n2}u'_{22} & -a_{n3}u'_{33} & \cdots & 1 \end{pmatrix} A \begin{pmatrix} 1 & -u'_{11}a_{12} & -u'_{11}a_{13} & \cdots & -u'_{11}a_{1n} \\ 0 & 1 & -u'_{22}a_{23} & \cdots & -u'_{22}a_{2n} \\ 0 & 0 & 1 & \cdots & -u'_{33}a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix},$$

is containing the diagonal matrix B such that its diagonal entries are u_{jj}. Therefore, since B is in $U(M_n(R))$, then A is in $U(M_n(R))$. ∎

Theorem 3.12. *Let R be a multiplicative hyperring with identity 1. If R is clean, then $\mathfrak{M}$ is clean, so.*

Proof. Assume that R is clean. For all $A = (a_{ij})_{n\times n} \in \mathfrak{M}$ we have $a_{jj} = e_{jj} + u_{jj}$, where $e_{jj} \in Idem(R)$ and $u_{jj} \in U(R)$. Now we have $A = E + B$ where:

$$E = \begin{pmatrix} e_{11} & 0 & \dots & 0 \\ 0 & e_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e_{nn} \end{pmatrix} \quad and \quad B = \begin{pmatrix} u_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & u_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & u_{nn} \end{pmatrix}.$$

Therefore $E \in Idem(M_n(R))$ and by Lemma 3.11, $B \in U(M_n(R))$. Hence $\mathfrak{M}$ is clean. ∎

4. Characterizations of clean multiplicative hyperrings

By $Spec(R)$ and $Max(R)$ we mean the spectrum of prime ideals and maximal ideals of R, respectively. For any hyperideal I of R and $a \in R$, we set

$$V(a) = \{P \in Spec(R) | a \in P\} \text{ and } V(I) = \{P \in Spec(R) | I \subseteq P\}.$$

Then, the set $V(I) = \bigcap\limits_{a\in I} V(a)$, where I is an hyperideal of R, satisfies the axioms for the closed sets of a topology on $Spec(R)$, called the Zariski topology (see [7]). For any subset S of $Spec(R)$, we consider the topology of S as subspace topology of $Spec(R)$. Also we define, $V_S(a) = S \bigcap V(a)$ and $V_S(I) = S \bigcap V(I)$.

Theorem 4.1. *Let R be a commutative multiplicative hyperring with an identity 1 such that $nil(R) = \{0\}$ and $Max(R) \subseteq S$. Then R is a clopen subset of S if and only if $A = V(e)$ for an idempotent $e \in R$.*

Proof. Assume that A is a clopen subset of S and $I = \bigcap A$ and $J = \bigcap A^c$. Clearly, $A = \bar{A} = V_S(\bigcap A) = V_S(I)$ and $A^c = V_S(J)$ and $V_S(I) \bigcap V_S(J) = \emptyset$. Therefore $I + J = R$ by the assumption. Since $1 \in R$, there exist $e \in I$ and $e' \in J$ such that $e + e' = 1$. Also, since $V_S(e) \bigcup V_S(e') = S$, we have $V(ee') = S$ and by [[7], Theorem 3.6(ii)] and it follows that $ee' \subseteq nil(R) = \{0\}$, i.e., $ee' = \{0\}$. Hence e is an idempotent and $A = V_S(I) = V_S(e)$. The converse is straightforward. ∎

Theorem 4.2. *Let R be a commutative multiplicative hyperring with an identity 1 such that $nil(R) = \{0\}$. Then $a \in R$ is clean if and only if there exists U in $Spec(R)$ such that $V(a-1) \subseteq U \subseteq Spec(R) \setminus V(a)$.*

Proof. Assume that $a \in R$ is clean, then there exist $e \in U(R)$ and $e \in Idem(R)$ such that $a = u + e$. Thus for all $P \in V(a-1)$ we have $P \notin V(1-e)$. So $P \in V(e)$ and we have $V(a-1) \subseteq V(e) \subseteq Spec(R) \setminus V(a)$. For the converse, let there exists U in $Spec(R)$ such that $V(a-1) \subseteq U \subseteq Spec(R) \setminus V(a)$, then for some $e \in Idem(R)$ we have $U = V(e)$, by Theorem 4.1. Set $u = a - e$ and we

prove that u is in $U(R)$. Assume that $u \notin U(R)$. Then there exists $P \in Spec(R)$ such that $u \in P$. But we know that either $P \in V(e)$ or $P \in V(1-e)$, because in otherwise if $P \in V(e)$ and $P \in V(1-e)$ we have $1 \in P$ and if $P \notin V(e)$ and $P \notin V(1-e)$ we have $0 \notin P$ which both cases are incorrect. Therefore either $P \in V(a) \bigcap V(e) = \emptyset$ or $P \in V(a-1) \bigcap V(e-1) = \emptyset$, which is a contradiction. Hence $u \in U(R)$ and $a \in R$ is clean. ∎

Corollary 4.3. *Let R be a commutative multiplicative hyperring with an identity 1 such that $nil(R) = \{0\}$. Then for all $a \in R$, we have the following statements:*

(i) *a^2 is clean if and only if a and $-a$ are clean.*

(ii) *for a positive integer n, a^n is clean, then a is clean.*

(iii) *If I is a clean hyperideal of R, then $rad(I)$ is so.*

Proof. (i) Suppose that a^2 is clean. Then there exists a clopen subset U of R such that $V(a-1) \bigcup V(a+1) = V(a^2-1) \subseteq U \subseteq Spec(R) \setminus V(a^2) = Spec(R) \setminus V(a) = Spec(R) \setminus V(-a)$. Therefore a and $-a$ are clean. The converse is similarly.

(ii) Assume that a^n is clean. Then, there exists U if $Spec(R)$ such that $V(a-1) \subseteq V(a^n-1) \subseteq U \subseteq Spec(R) \setminus V(a^n) = Spec(R) \setminus V(a)$. Hence, by Theorem 4.2, a is clean.

(iii) It follows immediately from part (ii). ∎

Lemma 4.4. *Let R be a commutative multiplicative hyperring with identity 1. If every prime hyperideal of R is $\mathcal{C}$-ideal, then for $a \in R$ and $e \in Idem(R)$, $V(ae-1) = V(a-1) \bigcap V(e-1)$.*

Proof. Let $P \in V(ae-1)$, then $e \notin P$ and $e-1 \in P$. Since $(a-1)e \subseteq (ae-1)+(e-1) \subseteq P$, then $a-1 \in P$. Therefore, $P \in V(a-1) \bigcap V(e-1)$. Conversely, suppose $P \in V(a-1) \bigcap V(e-1)$. Then $a-1 \in P$ and $e-1 \in P$. Since $(ae-1) \supseteq a(e-1)+(a-1)$ and $a(e-1)+(a-1) \subseteq P$ then $(ae-1) \bigcap P \neq \emptyset$ and since P is $\mathcal{C}$-ideal, $(ae-1) \subseteq P$, i.e., $P \in V(ae-1)$. ∎

Theorem 4.5. *Let R be a commutative multiplicative hyperring with identity 1 such that $nil(R) = \{0\}$ and every prime hyperideal of R is $\mathcal{C}$-ideal. If $a \in R$ is clean and $e \in Idem(R)$, then ae is clean.*

Proof. Since a is clean then by Theorem 4.1, there exists $e' \in Idem(R)$ such that $V(a-1) \subseteq V(e') \subseteq Spec(R) \setminus V(a)$. Now $U = V(e') \bigcap V(e-1)$ is a clopen and by Lemma 4.4 we have $V(ae-1) = V(a-1) \bigcap V(e-1) \subseteq U \subseteq (Spec(R) \setminus V(a)) \bigcap (Spec(R) \setminus V(e)) = Spec(R) \setminus V(ae)$. Hence ae is a clean. ∎

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A FAMILY OF NONBINARY SEQUENCES WITH OPTIMAL CORRELATION PROPERTY

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Abstract. In this paper, we first present a family of p-ary sequences with ideal two-level autocorrelation property, which could be regarded as a generalization of the well-known nonbinary sequences introduced by Helleseth and Gong. Utilizing the proposed sequences and m-sequences, we later construct a family of p-ary sequences of which the correlation property is optimal in terms of the Welch lower bound.

Keywords: p-ary sequences, correlations, autocorrelations, Welch bound.

1. Introduction

Pseudorandom sequences with good correlation properties have many applications in modern communication systems and cryptography, such as radar, CDMA communication systems, and stream cipher cryptosystems [2], [3], [9]. The design of sequences with two-level ideal autocorrelation, which play important roles in synchronization applications and also have close connection to different sets, has been an interesting research topic for decades in [3] and [6]. In recent years, there have been numerous researches on binary sequences with two-level autocorrelation property, see [3] for details. However, for p-ary sequences with two-level autocorrelation, without using a subfield constructions, there is only one general construction for any arbitrary odd prime p, which is the m-sequences. Helleseth and Gong [5] presented a construction of p-ary sequences with ideal two-level autocorrelation for any odd p, which generalized the ternary family by Helleseth, Kumar and Martinsen [4]. This is another general construction for p-ary sequences with ideal autocorrelation property and we refer it as HG sequence in the sequel.

In the present paper, we generalize certain parameters of the HG sequences and deduce a family of p-ary sequences with two-level ideal autocorrelation. With the proposed sequences and p-ary m-sequences, a family of p-ary sequences with period $p^n - 1$, size p^n and the maximal nontrivial correlation value $R_{\max}$ not exceeding $p^{n/2} + 1$ are given.

2. Prelimilaries

First, we fix some notations throughout the paper.

- n is a positive integer with $n = (2m+1)e$;
- k is a positive integer such that $\gcd(k, n) = e$;
- p is an odd prime and $q = p^e$;
- $\mathbb{F}_q$ is the finite field with $q = p^e$ elements and $\mathbb{F}_{p^n} = \mathbb{F}_{q^{2m+1}}$;
- $\mathrm{Tr}_n(x)$ (resp. $\mathrm{Tr}_e(x)$) is the absolute trace function from $\mathbb{F}_{p^n}$ (resp. $\mathbb{F}_{p^e}$) to $\mathbb{F}_p$, and $\mathrm{Tr}_e^n(x)$ is the trace function from $\mathbb{F}_{p^n}$ to the subfield $\mathbb{F}_{p^e}$;

2.1. Quadratic forms over finite fields

Let $x = \sum_{i=0}^{2m} x_i\alpha_i$ where $x_i \in \mathbb{F}_q$ and α_i, $i = 0, 1, ..., 2m$ is a basis for $\mathbb{F}_{q^{2m+1}}$ over $\mathbb{F}_q$. Then the function $Q(x)$ in $\mathbb{F}_{q^{2m+1}}$ is a quadratic form over $\mathbb{F}_q$ if it can be expressed as

$$Q(x) = Q\left(\sum_{i=0}^{2m} x_i\alpha_i\right) = \sum_{i=0}^{2m}\sum_{j=0}^{2m} b_{i,j}x_ix_j$$

where $b_{i,j} \in \mathbb{F}_q$. The quadratic form in odd characteristic has been well analyzed in [8]. The rank of a quadratic form is the minimum number of variables required to represent the function under the nonsingular coordinate transformations, which is related to the dimension of the vector space $\mathcal{W}$ in $\mathbb{F}_{q^{2m+1}}$, i.e.,

$$\mathcal{W} = \{y \in \mathbb{F}_{q^{2m+1}} \mid Q(x+y) = Q(x) \text{ for all } x \in \mathbb{F}_{q^{2m+1}}\}.$$

More precisely, $\rho = 2m + 1 - \dim(\mathcal{W})$.

The main result depends on the following lemma which is an extension and consequence of results from Trachtenberg [10] and Helleseth and Gong [5].

Lemma 1 *Let $Q(x)$ be a quadratic form over $\mathbb{F}_q$ in $2m+1$ variables of rank ρ. Let r be a non-square in $\mathbb{F}_q$ and define*

$$S = \frac{1}{2}\left(\sum_{x\in\mathbb{F}_{q^{2m+1}}} \omega^{Tr_e(Q(x))} + \sum_{x\in\mathbb{F}_{q^{2m+1}}} \omega^{Tr_e(rQ(x))}\right).$$

Then

$$S = \begin{cases} 0, & \text{if } \rho \text{ is odd,} \\ \pm q^{(2m+1)-\rho/2}, & \text{if } \rho \text{ is even.} \end{cases}$$

2.2. Sequences with two-level ideal autocorrelation function

Given two sequences $a = \{a_0, a_1, ..., a_{N-1}\}$ and $b = \{b_0, b_1, ..., b_{N-1}\}$ of period N, we define the periodic cross correlation between a and b at shift τ as

$$C_{a,b}(\tau) = \sum_{t=0}^{N-1} \omega^{a_t - b_{t+\tau}},$$

where ω is a p-th primitive root of unity given by

$$\omega = e^{\frac{2\pi\sqrt{-1}}{p}}.$$

Sequences with ideal autocorrelation properties are of considerable interest because of their applications in spread spectrum communication systems, cryptography and their close connections with difference sets. A number of ideal two-level autocorrelation sequences of period $p^n - 1$ have been discovered during the past few decades [1]. For p-ary sequences with two-level autocorrelation, without using a subfield constructions, there are only two general constructions for any arbitrary odd prime p, which are the well-known m-sequences and the sequences introduced by Helleseth and Gong [5].

Theorem 1 ([5]) *Let s, $1 \le s \le 2m$ be an integer such that $\gcd(s, 2m+1) = 1$. Define $b_0 = 1$, $b_i = b_{2m+1-i}$ and $b_{is} = (-1)^i$ for $i = 1, 2, ..., m$, where indices of b_{is} are taken modulo $2m+1$. Let $u_0 = b_0/2 = (p+1)/2$ and $u_i = b_{2i}$ for $i = 1, 2, ..., m$. Define*

$$f_0(x) = \sum_{i=0}^{m} u_i x^{(p^{2ie}+1)/2}. \tag{1}$$

Then the sequence over $\mathbb{F}_p$ defined by $a(t) = Tr_n(f_0(\alpha^t))$ has an ideal two-level autocorrelation.

Let $\mathcal{F}$ be a family of sequences, $C_{i,j}(\tau)$ denote the crosscorrelation between the i-th and j-th sequences at shift τ, i.e.,

$$C_{i,j}(\tau) = \sum_{t=0}^{p^n-2} \omega^{s_i(t+\tau) - s_j(t)}, \quad 0 \le \tau \le p^n - 2.$$

The maximum correlation value $C_{\max}$ of $\mathcal{F}$ is defined by

$$C_{\max} = \max\{|C_{i,j}(\tau)| : \text{either } i \ne j \text{ or } \tau \ne 0\ \}. \tag{2}$$

The crosscorrelation function is important in code-division multiple-access (CDMA) communication systems. Here each user is assigned a distinct signature sequence. To minimize interference due to the other users, it is desirable that the signature sequences have pairwise low values of crosscorrelation function. The classical goal in sequences design for CDMA systems has been minimization of the parameter $C_{\max}$ and maximization of the family size M for a given period, which are conflicting requirements.

3. Nonbinary sequences from difference-balanced functions

In this section we firstly generalize a parameter of the Helleseth-Gong (HG) sequences. Then we will employ the proposed sequences and m-sequences to construct a sequence family $\mathcal{F}$, which is optimal in the sense that it attains the Welch lower bound [11].

Theorem 2 *Let k be an positive integer such that $\gcd(n,k)=e$. Let $s, 1 \le s \le 2m$ be an integer such that $\gcd(s, 2m+1)=1$. Define $b_0=1$, $b_i=b_{2m+1-i}$ and $b_{is}=(-1)^i$ for $i=1,2,...,m$, where indices of b_{is} are taken modulo $2m+1$. Let $u_0=b_0/2=(p+1)/2$ and $u_i=b_{2i}$ for $i=1,2,...,m$. Define*

$$f(x)=\sum_{i=0}^{m} u_i x^{(p^{2ik}+1)/2}. \tag{3}$$

Then the sequence over $\mathbb{F}_p$ defined by $b(t)=Tr_n(f(\alpha^t))$ has an ideal two-level autocorrelation.

Remark 1 In the case of $k=e$, the function $f(x)$ defined in Theorem 2 reduces to the function f_0 in Theorem 1. Then the sequences $\{a(t)\}$ and $\{b(t)\}$ are the same. Hence Theorem 2 generalizes the result of Theorem 1. Moreover, it is easy to verify that both $f(x)$ and $f_0(x)$ defined in Theorems 1 and 2 are difference-balanced functions.

First, we give an interesting result which will be later used in the proof of Theorem 2.

Define a vector

$$\varepsilon=(\varepsilon_0,\varepsilon_1,...,\varepsilon_{2m}),\ \varepsilon_i=(-1)^i \text{ for } 0\le i\le 2m$$

and the right circular shift operator σ over ε as

$$\sigma(\varepsilon)=(\varepsilon_{2m},\varepsilon_0,...,\varepsilon_{2m-1}).$$

By iterating the shift operator σ over ε, we can obtain a set $\Gamma=\{\varepsilon,\sigma(\varepsilon),...,\sigma^{2m}(\varepsilon)\}$.

Define a $(2m+1)\times(2m+1)$ matrix M as

$$M_{i,j}=\beta_{j-i}(tz_iz_j-1), \tag{4}$$

where

$$\beta=\sigma^r(\varepsilon)=(\varepsilon_{2m+1-r},...,\varepsilon_{2m},\varepsilon_0,...,\varepsilon_{2m-r}) \tag{5}$$

is an element of Γ and the indices of β_i's and z_i's are taken modulo $2m+1$.

For example, when $m=2$ and $r=1$, denoting $t_{i,j}=tz_iz_j-1$ for $0\le i,j\le 4$ for simplicity, we obtain the matrix M as follows.

$$M = \begin{pmatrix} t_{0,0} & t_{0,1} & -t_{0,2} & t_{0,3} & -t_{0,4} \\ -t_{1,0} & t_{1,1} & t_{1,2} & -t_{1,3} & t_{1,4} \\ t_{2,0} & -t_{2,1} & t_{2,2} & t_{2,3} & -t_{2,4} \\ -t_{3,0} & t_{3,1} & -t_{3,2} & t_{3,3} & t_{3,4} \\ t_{4,0} & -t_{4,1} & t_{4,2} & -t_{4,3} & t_{4,4} \end{pmatrix}.$$

The determinant of the matrix M as defined in (4) is characterized in Proposition 1.

Proposition 1 *Let $\beta = \sigma^r(\varepsilon)$ with $0 \le r \le 2m$. The determinant $det(M)$ is given by*

$$det(M) = 2^{2m-1}\left(\prod_{i=0}^{2m}(tz_iz_{i+r} - 1) + \prod_{i=0}^{2m}(tz_iz_{i+r-1} - 1)\right).$$

Proof. Note that for $i = 0, 1, ..., 2m$,

$$\beta_i + \beta_{i-1} = \begin{cases} 2, & \text{if } i = r, \\ 0, & \text{otherwise.} \end{cases}$$

For $i = 0, 1, ..., 2m$, by replacing the i-th row by the sum of the i-th and $(i+1)$-th row (where indices are taken modulo $2m+1$), we obtain a matrix N with the entity

$$\begin{aligned} N_{i,j} &= \beta_{j-i}(tz_iz_j - 1) + \beta_{j-(i+1)}(tz_{i+1}z_j - 1) \\ &= \begin{cases} tz_j(z_i + z_{i+1}) - 2, & \text{if } j - i = r, \\ \beta_{j-i}tz_j(z_i - z_{i+1}), & \text{otherwise.} \end{cases} \end{aligned}$$

Then dividing the elements in the i-th row by $t(z_i - z_{i+1})$ and the elements in the j-th column by z_j, the determinant of the matrix N becomes

$$det(N) = \prod_{i=0}^{2m}(t(z_i - z_{i+1}))\prod_{j=0}^{2m} z_j \cdot det(R),$$

where R is a $(2m+1) \times (2m+1)$ matrix such that

$$R_{i,j} = \begin{cases} \dfrac{(tz_j(z_i + z_{i+1}) - 2)}{tz_j(z_i - z_{i+1})}, & \text{if } j - i = r, \\ \beta_{j-i}, & \text{otherwise.} \end{cases}$$

We repeat the process above for the matrix R, i.e., for $i = 0, 1, ..., 2m$ we replace the i-th row with the sum of the i-th and $(i+1)$-th rows (where indices are taken modulo $2m+1$). Performing these row operations on R leads to a matrix where all elements are zeros except for only two nonzero elements in each row. The only two nonzero elements in the resulting matrix S are

$$\begin{aligned} S_{i,i+r} &= \beta_{r-1} + (tz_{i+r}(z_i + z_{i+1}) - 2)/tz_{i+r}(z_i - z_{i+1}) \\ &= 2(tz_iz_{i+r} - 1)/tz_{i+r}(z_i - z_{i+1}) \end{aligned}$$

and for $j = i + r + 1$,

$$\begin{aligned} S_{i,j} &= \beta_{r+1} + (tz_j(z_{i+1} + z_{i+2}) - 2)/tz_j(z_{i+1} - z_{i+2}) \\ &= 2(tz_{i+2}z_j - 1)/tz_j(z_{i+1} - z_{i+2}). \end{aligned}$$

Thus, the determinant of S is a product of two terms along two "diagonals" corresponding to indices $(i, i + r)$ and $(i, i + r + 1)$ respectively for $i = 0, 1, ..., 2m$. That is to say, the determinant

$$\begin{aligned} det(S) =& \prod_{i=0}^{2m} \frac{2(tz_iz_{i+r} - 1)}{tz_{i+r}(z_i - z_{i+1})} + \prod_{i=0}^{2m} \frac{2(tz_{i+2}z_{i+r+1} - 1)}{tz_{i+r+1}(z_{i+1} - z_{i+2})} \\ =& \prod_{i=0}^{2m} \frac{2(tz_iz_{i+r} - 1)}{tz_{i+r}(z_i - z_{i+1})} + \prod_{i=0}^{2m} \frac{2(tz_{i+1}z_{i+r} - 1)}{tz_{i+r}(z_i - z_{i+1})}. \end{aligned}$$

Note that the determinant of the matrix with rows r_i for $i = 0, 1, ..., 2m$ is one half of the matrix with row $r_i + r_{i+1}$ for $i = 0, 1, ..., 2m$. This implies,

$$\begin{aligned} det(M) &= det(N)/2 \\ &= \prod_{i=0}^{2m}(t(z_i - z_{i+1})) \prod_{j=0}^{2m} z_j \cdot det(R)/2 \\ &= \prod_{i=0}^{2m}(t(z_i - z_{i+1})) \prod_{j=0}^{2m} z_j \cdot det(S)/4 \\ &= 2^{2m-1} \left(\prod_{i=0}^{2m}(tz_iz_{i+r} - 1) + \prod_{i=0}^{2m}(tz_{i+1}z_{i+r} - 1) \right) \\ &= 2^{2m-1} \left(\prod_{i=0}^{2m}(tz_iz_{i+r} - 1) + \prod_{i=0}^{2m}(tz_iz_{i+r-1} - 1) \right). \end{aligned}$$

The proof is ended. ∎

We are now ready for the proof of Theorem 2.

Proof of Theorem 2. The autocorrelation of $\{b(t)\}$ at shift τ is given by

$$\begin{aligned} A_b(\tau) &= \sum_{t=0}^{p^n-2} \omega^{b(t+\tau)-b(t)} \\ &= \sum_{t=0}^{p^n-2} \omega^{\mathrm{Tr}_n(f(\alpha^{t+\tau})-f(\alpha^t))} \\ &= -1 + \sum_{x \in \mathbb{F}_{p^n}} \omega^{\mathrm{Tr}_n(f(\alpha^\tau x)-f(x))}. \end{aligned}$$

In the following, we will investigate the value of $A_b(\tau)$ when τ is nonzero. From the definition, one has $f(\lambda x) = \lambda f(x)$ for any $\lambda \in \mathbb{F}_{p^e}$ since $(p^{2ik}+1)/2 \equiv 1 \pmod{p^e-1}$ for $i = 0, 1, ..., m$. Denote $p(x) = f(\alpha^\tau x) - f(x)$. It follows from Lemma 1 that

$$A_b(\tau) + 1 = \frac{1}{2}\left(\sum_{x\in\mathbb{F}_{p^n}} \omega^{\mathrm{Tr}_e(Q(x))} + \sum_{x\in\mathbb{F}_{p^n}} \omega^{\mathrm{Tr}_e(rQ(x))}\right)$$

$$= \begin{cases} 0, & \text{if } \rho \text{ is odd,} \\ \pm q^{2m+1-\rho/2}, & \text{otherwise,} \end{cases}$$

where

$$Q(x) = \mathrm{Tr}_e^n(p(x^2)) = \mathrm{Tr}_e^n(f(\alpha^\tau x^2) - f(x^2)) \tag{6}$$

is a quadratic form over $\mathbb{F}_q$ and ρ is the rank of $Q(x)$. To determine the rank ρ, as stated in Section 2.1, we need to consider the number of solutions $y \in \mathbb{F}_{p^n} (= \mathbb{F}_{q^{2m+1}})$, such that $Q(x+y) = Q(x)$ for all $x \in \mathbb{F}_{p^n}$. In the following we represent each element $c = \alpha^\tau \in \mathbb{F}_{p^n}$ in the form $c = t\gamma^2$ where $t = 1$ or t is a non-square in the subfield $\mathbb{F}_q$. For simplicity, we denote $v_i = u_i(c^{(p^{2ik}+1)/2} - 1) = u_i(t\gamma^{p^{2ik}+1} - 1)$, then

$$Q(x) = \mathrm{Tr}_e^n(f(\alpha^\tau x^2) - f(x^2)) = \mathrm{Tr}_e^n\left(\sum_{i=0}^{m} v_i x^{p^{2ik}+1}\right).$$

Thus, $Q(x+y) = Q(x)$ is equivalent to

$$\mathrm{Tr}_e^n\left(\sum_{i=0}^{m} v_i((x+y)^{p^{2ik}+1}\right) = \mathrm{Tr}_e^n\left(\sum_{i=0}^{m} v_i x^{p^{2ik}+1}\right),$$

i.e.,

$$\mathrm{Tr}_e^n\left(\sum_{i=0}^{m} v_i(x^{p^{2ik}}y + xy^{p^{2ik}}) + \sum_{i=0}^{m} v_i y^{p^{2ik}+1}\right) = 0.$$

If this holds for all $x \in \mathbb{F}_{q^{2m+1}}$, we must have

$$\mathrm{Tr}_e^n\left(\sum_{i=0}^{m} v_i(x^{p^{2ik}}y + xy^{p^{2ik}})\right) = \mathrm{Tr}_e^n\left(x\left(\sum_{i=0}^{m}(v_i^{p^{-2ik}}y^{p^{-2ik}} + v_i y^{p^{2ik}})\right)\right) = 0$$

and

$$\mathrm{Tr}_e^n\left(\sum_{i=0}^{m} v_i y^{p^{2ik}+1}\right) = 0.$$

The first equation holds for all $x \in \mathbb{F}_{q^{2m+1}}$ if and only if

$$L(y) = \sum_{i=0}^{m}\left(v_i^{p^{-2ik}}y^{p^{-2ik}} + v_i y^{p^{2ik}}\right) = 0.$$

The second equation follows directly as a consequence of $L(y) = 0$ by considering $\mathrm{Tr}_e^n(yL(y)) = 0$. Hence, $Q(x+y) = Q(x)$ holds for all $x \in \mathbb{F}_{q^{2m+1}}$ if and only if

$L(y) = 0$. That is to say, in order to show the rank of $Q(x)$ is $2m+1$, it suffices to show that the equation $L(y) = 0$ has only one solution $y = 0$.

From the definition of v_i, we have

$$L(y) = \sum_{i=0}^{m} u_i \left((t\gamma^{p^{-2ik}+1} - 1)y^{p^{-2ik}} + (t\gamma^{p^{2ik}+1} - 1)y^{p^{2ik}} \right).$$

Further, since $u_i = b_{2i}$, $b_i = b_{2m+1-i}$ for $i = 0, 1, ..., m$ and $\gcd(s, 2m + 1) = 1$,

$$\begin{aligned} L(y) &= \sum_{i=0}^{2m} b_i (t\gamma^{p^{ik}+1} - 1)y^{p^{ik}} \\ &= \sum_{i=0}^{2m} b_{is} (t\gamma^{q^{isf}+1} - 1)y^{q^{isf}}, \end{aligned}$$

where $f = k/e$.

Raising the linearized equations $L(y) = 0$ to the q^{isf} power for $i = 0, 1, ..., 2m$, we can obtain a linear equation system with $2m + 1$ equations in the $2m + 1$ unknowns $y^{q^{jsf}}$ for $j = 0, 1, ..., 2m$. The coefficient matrix $M = (m_{i,j})$ of this system is given by

$$m_{i,j} = b_{(j-i)s}(t\gamma^{q^{isf}+q^{jsf}} - 1)$$

where the indices are taken modulo $2m + 1$ and $m_{i,j}$ is the coefficient of $y^{q^{jsf}}$ in the equation $(L(y))^{q^{isf}} = 0$.

Note that $b_{is} = (-1)^i$ and $b_i = b_{2m+1-i}$, for $i = 1, ..., m$. Thus, the vector $b = (b_0, b_s, ..., b_{2ms})$ becomes

$$\begin{aligned} b &= (1, -1, ..., (-1)^m, (-1)^m, ..., -1) \\ &= (-1)^m((-1)^m, ..., (-1)^{2m}, (-1)^0, ..., (-1)^{m-1}) \\ &= (-1)^m \sigma^{m+1}(\varepsilon) \end{aligned}$$

where $\sigma^{m+1}(\varepsilon)$ is as given in (5). Denote the variables $z_i = \gamma^{q^{isf}}$ for $i = 0, 1, ..., 2m$. Then it follows from Proposition 1 that the determinant of the coefficient matrix M is

$$\begin{aligned} \Delta &= (-1)^m 2^{2m-1} \left(\prod_{i=0}^{2m} (tz_i z_{i+m+1} - 1) + \prod_{i=0}^{2m} (tz_i z_{i+m} - 1) \right) \\ &= (-1)^m 2^{2m} \prod_{i=0}^{2m} (tz_i z_{i+m} - 1). \end{aligned}$$

Note that $z_i z_{i+m} = \gamma^{q^{isf}+q^{(i+m)sf}}$ is a square. Thus, if t is a non-square, $tz_i z_{i+m} - 1 \neq 0$ for $i = 0, 1, ..., 2m$, which implies $\Delta \neq 0$. When $t = 1$, suppose the determinant $\Delta = 0$, then we have $\gamma^{q^{isf}+q^{(i+m)sf}} = 1$ for some integer i, which is equivalent to $\gamma^{\gcd(q^{msf}+1, q^{2m+1}-1)} = 1$. Since $(q^{msf} + 1, q^{2m+1} - 1) = 2$, we have $\gamma^2 = 1$. This leads to a contradiction $c = t\gamma^2 = 1$. Thus, the linear equation system with $(L(y))^{q^{isf}} = 0$ for $i = 0, 1, ..., 2m$ has $y = 0$ as its only solution. This implies the quadratic form

$Q(x) = \mathrm{Tr}_e^n(f(\alpha^\tau x^2) - f(x^2))$ has rank $2m+1$ when $\tau \neq 0$. Thus, $A_b(\tau) + 1 = 0$ when τ is nonzero. ∎

The following proposition characterizes the sufficient and necessary condition for cyclic equivalence of the sequences given in Theorems 1 and 2.

Proposition 2 *Let $k_1 = k/e$. Let $\{a(t)\}$ and $\{b(t)\}$ be the sequences as defined in Theorems 1 and 2. Then $C_{a,b}(\tau) = p^n - 1$ if and only if $\tau = 0$ and for any $0 \leq i \leq 2m$, $b_i = b_{\sigma(i)}$, where $\sigma(i) \equiv i \cdot k_1^{-1} \pmod{2m+1}$.*

Proof. As discussed in the proof of Theorem 2,

$$C_{a,b}(\tau) + 1 = \frac{1}{2}\left(\sum_{x\in\mathbb{F}_{p^n}} \omega^{\mathrm{Tr}_e(Q_1(x))} + \sum_{x\in\mathbb{F}_{p^n}} \omega^{\mathrm{Tr}_e(rQ_1(x))}\right),$$

where $Q_1(x) = \mathrm{Tr}_e^n(f_0(x^2) - f(cx^2))$ is a quadratic form in $\mathbb{F}_{q^{2m+1}}$, $c = \alpha^\tau$ and r is a nonsquare of $\mathbb{F}_q$. Note that $C_{a,b}(\tau) = p^n - 1$ if and only if the rank of the quadratic form $Q_1(x)$ equals to zero. That is to say, $C_{a,b}(\tau) = p^n - 1$ if and only if $Q_1(x+y) = Q_1(x)$ holds for any $x, y \in \mathbb{F}_{q^{2m+1}}$. Similar to the method adopted in Theorem 2, $Q_1(x+y) = Q_1(x)$ holds for any $x, y \in \mathbb{F}_{q^{2m+1}}$ is equivalent to the linearized equation

$$\sum_{i=0}^{m}\left(u_i(y^{q^{2i}} + y^{q^{2m+1-2i}})\right) = \sum_{i=0}^{m}\left(u_i c^{\frac{q^{2if}+1}{2}} y^{q^{2if}} + u_i(c^{\frac{q^{-2if}+1}{2}} y^{q^{-2if}})^{q^{(2m+1)f}}\right)$$

holds for any $y \in \mathbb{F}_{q^{2m+1}}$. From the definition of $u_i = b_{2i}$ for $i = 0, 1, ..., m$, this equation is rewritten as

$$\begin{aligned}\sum_{i=0}^{2m} b_i y^{q^i} &= \sum_{i=0}^{2m} b_i c^{\frac{q^{if}+1}{2}} y^{q^{if}} \\ &= \sum_{i=0}^{2m} b_{if^{-1}} c^{\frac{q^i+1}{2}} y^{q^i}.\end{aligned}$$

Therefore, $Q_1(x+y) = Q_1(x)$ holds for any $x, y \in \mathbb{F}_{q^{2m+1}}$ if and only if $b_i = b_{if^{-1}} c^{\frac{q^i+1}{2}}$ for $i = 0, 1, ..., 2m$. It is easily obtained that $c = \alpha^\tau = 1$ for $i = 0$ and $b_i = b_{ik_1^{-1}}$ for $i = 0, 1, ..., 2m$. ∎

Due to Proposition 2, the task of finding cyclically inequivalent sequences in Theorems 1 and 2 can be reduced to find parameters m, s, f such that the condition $b_i = b_{\sigma(i)}$ for $i = 0, 1, ..., 2m$ is not fulfilled, which is independent of the value of p and e.

By employing the HG sequences, a family of p-ary sequences of period $p^n - 1$ with size p^n [7]. The maximum nontrivial correlation value $C_{\max}$ of all pairs of distinct sequences in the family does not exceed $p^{n/2} + 1$, which means the family has optimal correlation with respect to Welch's lower bound [11]. Similar to the idea adopted in [7], we define

$$\mathcal{F} = \left\{s_i(t) \,|\, 0 \leq i \leq p^n - 1, 0 \leq t \leq p^n - 2\right\} \tag{7}$$

with

$$s_i(t) = \mathrm{Tr}_n(\alpha^t) + \mathrm{Tr}_n(f(\delta_i \alpha^{2t})), \tag{8}$$

where $f(x)$ is defined as in Theorem 2 and $\{\delta_i \,|\, 0 \le i \le p^n - 1\}$ is an enumeration of elements in the field $\mathbb{F}_{p^n}$. Specially, let $\delta_{p^n-1} = 0$.

The correlation property can be investigated in a similar way as in [7]. We give a brief proof here. By the definition of the cross-correlation function, we have

$$\begin{aligned} C_{i,j}(\tau) &= \sum_{t=0}^{p^n-2} \omega^{s_i(t+\tau)-s_j(t)} \\ &= \sum_{t=0}^{p^n-2} \omega^{\mathrm{Tr}_n(\alpha^{t+\tau}+f(\delta_i\alpha^{2(t+\tau)})-\alpha^t-f(\delta_j\alpha^{2t})} \\ &= -1 + \sum_{x\in\mathbb{F}_{p^n}} \omega^{\mathrm{Tr}_n(f(\delta_i\varepsilon^2x^2)-f(\delta_jx^2)+(\varepsilon-1)x)}, \end{aligned} \tag{9}$$

where $\varepsilon = \alpha^\tau \in \mathbb{F}_{p^n}^*$. We can divide the computation of the cross-correlation $C_{i,j}(\tau)$ into the following cases depending on different values of t, i and j.

1) $\tau = 0$, $i = j$. In this trivial case, $\varepsilon = 1$ and $\delta_i = \delta_j$. Thus, $C_{i,j}(\tau) = p^n - 1$.

2) $\tau \neq 0$ and $\delta_i\varepsilon^2 = \delta_j$. In this case,

$$C_{i,j}(\tau) = -1 + \sum_{x\in\mathbb{F}_{p^n}} \omega^{\mathrm{Tr}_n((\varepsilon-1)x)} = -1.$$

3) $\delta_i\varepsilon^2 \neq \delta_j$. Denote $\delta_i = \theta_i\eta_i^2$ for $0 \le i \le p^n - 1$, where θ_i is a non-square in $\mathbb{F}_{p^e}$ if δ is a non-square in $\mathbb{F}_{p^n}$ and $\theta_i = 1$ otherwise. Then,

$$\begin{aligned} C_{i,j}(\tau) &= -1 + \sum_{x\in\mathbb{F}_{p^n}} \omega^{\mathrm{Tr}_n(f(\theta_i\eta_i^2\varepsilon^2x^2)-f(\theta_j\eta_j^2x^2)+(\varepsilon-1)x)} \\ &= -1 + \sum_{x\in\mathbb{F}_{p^n}} \omega^{\mathrm{Tr}_n\left(f(\theta_i\frac{\eta_i^2\varepsilon^2}{\eta_j^2}x^2)-f(\theta_jx^2)+\frac{(\varepsilon-1)}{\eta_j}x\right)} \end{aligned} \tag{10}$$

Let $\zeta = \theta_i\frac{\eta_i^2\varepsilon^2}{\eta_j^2}$ and $\lambda = \frac{(\varepsilon-1)}{\eta_j}$. Define

$$Q_2(x) = \mathrm{Tr}_e^n(f(\zeta x^2) - f(\delta_j x^2)). \tag{11}$$

Then we have

$$C_{i,j}(\tau) = -1 + \sum_{x\in\mathbb{F}_{p^n}} \omega^{\mathrm{Tr}_e(Q_2(x))+\mathrm{Tr}_n(\lambda x)}.$$

It is shown in Theorem 2 that the quadratic form $Q(x) = \mathrm{Tr}_e^n(f(cx^2) - f(x^2)$ has full rank for any nonzero element $c \in \mathbb{F}_{p^n}^*$. By the definition of $f(x)$, it follows that the quadratic form

$$Q_2(x) = \delta_j\mathrm{Tr}_e^n(f(\tfrac{\zeta}{\delta_j}x^2) - f(x^2)) = \delta_jQ(x)$$

with $c = \frac{\zeta}{\delta_j}$ has the full rank. Then

$$\Big|\sum_{x\in\mathbb{F}_{p^n}} \omega^{\mathrm{Tr}_e(Q_2(x))+\mathrm{Tr}_n(\lambda x)}\Big| = p^{n/2} \text{ for any } \lambda \in \mathbb{F}_{p^n}.$$

Furthermore, one deduces $|C_{i,j}(\tau) + 1| = p^{n/2}$ in this case.

By combining cases 1), 2) and 3), we have the following theorem.

Theorem 3 *The family $\mathcal{F}$ defined in* (7) *has the optimal correlation property with* $C_{max} = p^{n/2} + 1$.

4. Conclusion

In this paper, we generalized the function introduced by Helleseth and Gong [5], which generates sequences over $\mathbb{F}_p$ with two-level ideal autocorrelation. Moreover, by properly choosing the parameter k we can deduce sequences among the proposed sequences that are cyclically inequivalent to the HG sequences. By employing the proposed sequences and m-sequences, a nonbinary sequence family with optimal correlation property was constructed.

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CONNECTEDNESS IN INTUITIONISTIC FUZZY TOPOLOGICAL SPACES IN ŠOSTAK'S SENSE

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Abstract. In this paper, we introduce various types of fuzzy connectedness in intuitionistic fuzzy topological spaces in view of Šostak's sense. The interrelationship between different notions of intuitionistic fuzzy connectedness are investigate. Also, we inspect some interrelations between these types of intuitionistic fuzzy connectedness together with the preservation properties under intuitionistic fuzzy continuous maps.

Keywords: intuitionistic fuzzy topology; intuitionistic fuzzy $(c_i^{\alpha,\beta}, c_S^{\alpha,\beta}, c_M^{\alpha,\beta}, O^{\alpha,\beta}, O_q^{\alpha,\beta})$-connectedness; (α,β)-intuitionistic fuzzy super connectedness.

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1. Introduction and preliminaries

Zadeh[34] introduced the fundamental concept of a fuzzy set. Later Chang [7] defined fuzzy topological spaces. Šostak [29] introduced the fundamental concept of a fuzzy topological structure, as an extension of both crisp topology and Chang's fuzzy topology. The fuzzy topology in Šostak's sense were rediscovered by Chattopadhyay et al. [8]. In the same year, Ramadan [22] gave a similar definition of a fuzzy topology under the name "smooth topology".

On the other hand, Atanassove and his colleagues [2]–[6] introduced the fundamental concept of an intuitionistic fuzzy set. Çoker [12], [14] used this type of generalized fuzzy set to define "intuitionistic fuzzy topological spaces". Also,

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Çoker and Demirci [13] introduced the basic definition and properties of "intuitionistic fuzzy topological spaces in Šostak's sense" which is a generalized form of "fuzzy topological spaces" developed by Šostak [29], [30]. In this sense many works have been launched [15], [17]–[20], [25], [32].

Connectedness of fuzzy sets is an important subject in fuzzy topology, it won the attention of many researchers [1], [9], [16], [21], [24], [26]–[28], [33].

In this paper, many different notions of connectedness of fuzzy sets are extended to intuitionistic fuzzy topological spaces in Šostak's sense and the interrelationship between them are studied. Also, we inspect some interrelations between these types of intuitionistic fuzzy connectedness together with the preservation properties under intuitionistic fuzzy continuous maps.

Definition 1.1. ([2]) Let X be a nonempty fixed set. An intuitionistic fuzzy set (briefly, IFS) A is an object having the form

$$A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\},$$

where the map $\mu_A : X \longrightarrow I$ and $\gamma_A : X \longrightarrow I$ denote the degree of membership (namely, $\mu_A(x)$) and the degree of nonmembership (namely, $\gamma_A(x)$) of each element $x \in X$ to the set A, respectively, and $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for each $x \in X$. Obviously, every fuzzy set A on a nonempty set X is an IFS having the form

$$A = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle : x \in X\}.$$

For the sake of simplicity, we shall use the symbol $A = \langle x, \mu_A(x), \gamma_A(x)\rangle$ for the IFS $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$. For a given nonempty set X, let us denote the family of all IFSs in X by the symbol ζ^X.

Definition 1.2. ([2],[14]) Let X be a nonempty set, and the IFSs A and B in X be in the form $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$, $B = \{\langle x, \mu_B(x), \gamma_B(x)\rangle : x \in X\}$. Furthermore, let $\{A_i : i \in J\}$ be an arbitrary family of IFSs in X. Then,

(i) $A \subseteq B$ iff $\mu_A(x) \leq \mu_B(x)$ and $\gamma_A(x) \geq \gamma_B(x)$, for all $x \in X$;

(ii) $A = B$ iff $A \subseteq B$ and $B \subseteq A$;

(iii) $\overline{A} = \{\langle x, \gamma_A(x), \mu_A(x)\rangle : x \in X\}$;

(iv) $A - B = A \cap \overline{B}$;

(v) $\cap A_i = \{\langle x, \bigwedge \mu_{A_i}(x), \bigvee \gamma_{A_i}(x)\rangle : x \in X\}$;

(vi) $\cup A_i = \{\langle x, \bigvee \mu_{A_i}(x), \bigwedge \gamma_{A_i}(x)\rangle : x \in X\}$;

(vii) $0_\sim = \{\langle x, 0, 1\rangle : x \in X\}$ and $1_\sim = \{\langle x, 1, 0\rangle : x \in X\}$.

Definition 1.3. ([11]) Let a and b be two real numbers in [0,1] satisfying the inequality $a + b \leq 1$. Then the pair $\langle a, b\rangle$ is called an intuitionistic fuzzy pair.

Let $\langle a_1, b_1\rangle, \langle a_2, b_2\rangle$ be two intuitionistic fuzzy pairs. Then we define

(i) $\langle a_1, b_1\rangle \leq \langle a_2, b_2\rangle \Leftrightarrow a_1 \leq a_2$ and $b_1 \geq b_2$;

(ii) $\langle a_1, b_1\rangle = \langle a_2, b_2\rangle \Leftrightarrow a_1 = a_2$ and $b_1 = b_2$;

(iii) If $\{\langle a_i, b_i\rangle : i \in J\}$ is a family of intuitionistic fuzzy pairs, then $\bigvee\langle a_i, b_i\rangle = \langle \bigvee a_i, \bigwedge b_i\rangle$ and $\bigwedge\langle a_i, b_i\rangle = \langle \bigwedge a_i, \bigvee b_i\rangle$;

(iv) The complement of an intuitionistic fuzzy pair $\langle a, b\rangle$ is the intuitionistic fuzzy pair defined by $\overline{\langle a, b\rangle} = \langle b, a\rangle$;

(v) $1^{\sim} = \langle 1, 0\rangle$ and $0^{\sim} = \langle 0, 1\rangle$.

Definition 1.4. ([14]) Let X and Y be two nonempty sets and $f : X \to Y$ be a map.

(i) If $B = \{\langle y, \mu_B(y), \gamma_B(y)\rangle : y \in Y\}$ is an IFS in Y, then the preimage of B under f, denoted by $f^{-1}(B)$, is the IFS in X defined by
$f^{-1}(B) = \{\langle x, f^{-1}(\mu_B)(x), f^{-1}(\gamma_B)(x)\rangle : x \in X\}$.

(ii) If $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$ is an IFS in X, then the image of A under f , denoted by $f(A)$ is the IFS in Y defined by
$f(A) = \{\langle y, f(\mu_A)(y), \underline{f}(\gamma_A)(y)\rangle : y \in Y\}$, where $\underline{f}(\gamma_A) = 1 - f(1 - \gamma_A)$.

Definition 1.5. ([10]) Let $A, B \in \zeta^X$. Then, A and B are said to be quasi-coincident, denoted by AqB iff there exists an element $x \in X$ such that $\mu_A(x) > \gamma_B(x)$ or $\gamma_A(x) < \mu_B(x)$, otherwise $A \not{q} B$.

Theorem 1.6. ([10],[31]) *Let $A, B \in \zeta^X$. Then,*

(i) $A \not{q} \overline{B}$ *iff* $A \subseteq B$,

(ii) AqB *iff* $A \not\subseteq \overline{B}$,

(iii) *if* $A \cap B = 0_{\sim}$, *then* $A \subseteq \overline{B}$,

(iv) *if* $A \not\subseteq \overline{B}$, *then* $A \cap B \neq 0_{\sim}$.

Definition 1.7. ([11]) An IFS ξ on the set ζ^X is called an intuitionistic fuzzy family (IFF for short) on X. In symbols, denote such an IFF in form $\xi = \langle \mu_\xi, \gamma_\xi\rangle$.

Let ξ be an IFF on X. Then the complemented IFF of ξ on X is defined by $\xi^* = \langle \mu_{\xi^*}, \gamma_{\xi^*}\rangle$, where $\mu_{\xi^*}(A) = \mu_\xi(\overline{A})$ and $\gamma_{\xi^*}(A) = \gamma_\xi(\overline{A})$, for each $A \in \zeta^X$.

If τ is an IFF on X, then for any $A \in \zeta^X$, construct the intuitionistic fuzzy pair $\langle \mu_\tau(A), \gamma_\tau(A)\rangle$ and use the symbol $\tau(A) = \langle \mu_\tau(A), \gamma_\tau(A)\rangle$.

Definition 1.8. ([13]) An intuitionistic fuzzy topology in Šostak's sense (IFT for short) on a nonempty set X is an IFF τ on X satisfying the following axioms:

(T_1) $\tau(0_{\sim}) = \tau(1_{\sim}) = 1^{\sim}$;

(T_2) $\tau(A \cap B) \geq \tau(A) \wedge \tau(B)$, for any $A, B \in \zeta^X$;

(T_3) $\tau(\cup A_i) \geq \bigwedge \tau(A_i)$ for any $\{A_i : i \in J\} \subseteq \zeta^X$.

In this case, the pair (X, τ) is called an intuitionistic fuzzy topological space in Šostak's sence (IFTS for short). For any $A \in \zeta^X$, the number $\mu_\tau(A)$ is called the openness degree of A, while $\gamma_\tau(A)$ is called the nonopenness degree of A.

Definition 1.9. ([13]) Let (X, τ) be an IFTS on X. Then, the IFF τ^* of complemented IFSs on X is defined by: $\tau^*(A) = \tau(\overline{A})$. The number $\mu_{\tau^*}(A) = \mu_\tau(\overline{A})$ is called the closedness degree of A, while $\gamma_{\tau^*}(A) = \gamma_\tau(\overline{A})$ is called the nonclosedness degree of A.

Theorem 1.10. ([13]) *The IFF τ^* on X satisfies the following properties:*

(C_1) $\tau^*(0_\sim) = \tau^*(1_\sim) = 1^\sim$;

(C_2) $\tau^*(A \cup B) \geq \tau^*(A) \wedge \tau^*(B)$, *for any* $A, B \in \zeta^X$;

(C_3) $\tau^*(\cap A_i) \geq \bigwedge \tau^*(A_i)$, *for any* $\{A_i : i \in J\} \subseteq \zeta^X$.

Definition 1.11. ([13]) Let (X, τ) be an IFTS and A be an IFS in X. Then the fuzzy closure and fuzzy interior of A are defined by

$$\begin{aligned} cl_{\alpha,\beta}(A) &= \cap\{K \in \zeta^X : A \subseteq K, \tau^*(K) \geq \langle \alpha, \beta \rangle\} \\ int_{\alpha,\beta}(A) &= \cup\{G \in \zeta^X : G \subseteq A, \tau(G) \geq \langle \alpha, \beta \rangle\}, \end{aligned}$$

where $\alpha \in I_0 = (0, 1], \beta \in I_1 = [0, 1)$ with $\alpha + \beta \leq 1$.

Theorem 1.12. ([13]) *The closure and interior operators satisfy the following properties:*

(i) $A \subseteq cl_{\alpha,\beta}(A)$;

(ii) $int_{\alpha,\beta}(A) \subseteq A$;

(iii) $cl_{\alpha,\beta}(cl_{\alpha,\beta}(A)) = cl_{\alpha,\beta}(A)$;

(iv) $int_{\alpha,\beta}(int_{\alpha,\beta}(A)) = int_{\alpha,\beta}(A)$;

(v) $cl_{\alpha,\beta}(A \cup B) = cl_{\alpha,\beta}(A) \cup cl_{\alpha,\beta}(B)$;

(vi) $int_{\alpha,\beta}(A \cap B) = int_{\alpha,\beta}(A) \cap int_{\alpha,\beta}(B)$;

(vii) $\overline{cl_{\alpha,\beta}(A)} = int_{\alpha,\beta}(\overline{A})$;

(viii) $\overline{int_{\alpha,\beta}(A)} = cl_{\alpha,\beta}(\overline{A})$.

Definition 1.13. ([13]) Let (X, τ_1) and (Y, τ_2) be two IFTSs and $f : X \to Y$ be a map. Then, f is said to be intuitionistic fuzzy continuous iff

$$\tau_1(f^{-1}(B)) \geq \tau_2(B),$$

for each $B \in \zeta^Y$.

Theorem 1.14. ([13]) *The following properties are equivalent:*

(i) $f : (X, \tau_1) \to (Y, \tau_2)$ *is an intuitionistic fuzzy continuous.*

(ii) $\tau_1^*(f^{-1}(B)) \geq \tau_2^*(B)$*, for each* $B \in \zeta^Y$.

Definition 1.15. ([23]) Let A be an IFS in an IFTS (X, τ). For $\alpha \in I_0, \beta \in I_1$ with $\alpha + \beta \leq 1$, A is called:

(i) an (α, β)-intuitionistic fuzzy regular open (briefly, (α, β)-ifro) set of X, if $int_{\alpha,\beta}(cl_{\alpha,\beta}A) = A$,

(ii) an (α, β)-intuitionistic fuzzy regular closed (briefly, (α, β)-ifrc) set of X, if $cl_{\alpha,\beta}(int_{\alpha,\beta}A) = A$.

Theorem 1.16. ([23]) *Let A be an IFS in an IFTS (X, τ). Then, for $\alpha \in I_0$, $\beta \in I_1$ with $\alpha + \beta \leq 1$.*

(i) *If A is an (α, β)-ifro (resp. (α, β)-ifrc) set then, $\tau(A) \geq \langle \alpha, \beta \rangle$ (resp. $\tau^*(A) \geq \langle \alpha, \beta \rangle$).*

(ii) *A is an (α, β)-ifro set iff $\overline{A}$ is an (α, β)-ifrc set.*

2. Different notions of connectedness of intuitionistic fuzzy sets

Definition 2.1. Let (X, τ) be an IFTS and $N \in \zeta^X$. For $\alpha \in I_0, \beta \in I_1$ with $\alpha + \beta \leq 1$,

(i) If there exist the IFSs $U, V \in \zeta^X$ with $\tau(U) \geq \langle \alpha, \beta \rangle$, $\tau(V) \geq \langle \alpha, \beta \rangle$ satisfying the following properties, then N is called an intuitionistic fuzzy $c_i^{\alpha,\beta}$-disconnected(briefly, IF$c_i^{\alpha,\beta}$-disconnected), $(i = 1, 2, 3, 4)$:
IF$c_1^{\alpha,\beta}$: $N \subseteq U \cup V$, $U \cap V \subseteq \overline{N}$, $N \cap U \neq 0_\sim$, $N \cap V \neq 0_\sim$.
IF$c_2^{\alpha,\beta}$: $N \subseteq U \cup V$, $N \cap U \cap V = 0_\sim$, $N \cap U \neq 0_\sim$, $N \cap V \neq 0_\sim$.
IF$c_3^{\alpha,\beta}$: $N \subseteq U \cup V$, $U \cap V \subseteq \overline{N}$, $U \not\subseteq \overline{N}$, $V \not\subseteq \overline{N}$.
IF$c_4^{\alpha,\beta}$: $N \subseteq U \cup V$, $N \cap U \cap V = 0_\sim$, $U \not\subseteq \overline{N}$, $V \not\subseteq \overline{N}$.

(ii) N is said to be intuitionistic fuzzy $c_i^{\alpha,\beta}$-connected(briefly, IF$c_i^{\alpha,\beta}$-connected) if N is not an IF$c_i^{\alpha,\beta}$-disconnected, $(i = 1, 2, 3, 4)$.

Remark 2.2. From Definition 2.1, we have the following implication between IF$c_i^{\alpha,\beta}$-connectedness $(i = 1, 2, 3, 4)$.

$$\begin{array}{ccc} \text{IF}c_1^{\alpha,\beta}\text{-connectedness} & \Longrightarrow & \text{IF}c_2^{\alpha,\beta}\text{-connectedness} \\ \Downarrow & & \Downarrow \\ \text{IF}c_3^{\alpha,\beta}\text{-connectedness} & \Longrightarrow & \text{IF}c_4^{\alpha,\beta}\text{-connectedness} \end{array}$$

But the reciprocal implication are not true in general as shown by the following examples.

Example 2.3. Let $X = \{a, b, c\}$ and $N_1, N_2, G_i \in \zeta^X$ $(i = 1, 2, 3, 4)$ defined as follows:

$$N_1 = \langle x, (\frac{a}{0.2}, \frac{b}{0.3}, \frac{c}{0.2}), (\frac{a}{0.4}, \frac{b}{0.6}, \frac{c}{0.4})\rangle$$

$$N_2 = \langle x, (\frac{a}{0.5}, \frac{b}{0.4}, \frac{c}{0.4}), (\frac{a}{0.4}, \frac{b}{0.5}, \frac{c}{0.3})\rangle$$

$$G_1 = \langle x, (\frac{a}{0.5}, \frac{b}{0.5}, \frac{c}{0.4}), (\frac{a}{0.2}, \frac{b}{0.4}, \frac{c}{0.4})\rangle$$

$$G_2 = \langle x, (\frac{a}{0.4}, \frac{b}{0.6}, \frac{c}{0.2}), (\frac{a}{0.5}, \frac{b}{0.3}, \frac{c}{0.3})\rangle$$

$$G_3 = \langle x, (\frac{a}{0.5}, \frac{b}{0.6}, \frac{c}{0.4}), (\frac{a}{0.2}, \frac{b}{0.3}, \frac{c}{0.3})\rangle$$

$$G_4 = \langle x, (\frac{a}{0.4}, \frac{b}{0.5}, \frac{c}{0.2}), (\frac{a}{0.5}, \frac{b}{0.4}, \frac{c}{0.4})\rangle$$

Let $\tau : \zeta^X \longrightarrow I \times I$ defined as follows:

$$\tau(A) = \begin{cases} 1^{\sim}, & \text{if } A \in \{0_{\sim}, 1_{\sim}\} \\ \langle 0.6, 0.3\rangle, & \text{if } A = G_i (i = 1, 2, 3, 4) \\ 0^{\sim}, & \text{otherwise.} \end{cases}$$

Let $\alpha = 0.4, \beta = 0.5$. Then, N_1 is both an IF$c_2^{\alpha,\beta}$-connected and IF$c_3^{\alpha,\beta}$-connected but not an IF$c_1^{\alpha,\beta}$-connected (i.e., N_1 is an IF$c_2^{\alpha,\beta}$-connected since, for every $G_i \in \zeta^X$ with $\tau(G_i) \geq \langle \alpha, \beta\rangle$, $i = 1, 2, 3, 4$, and satisfies $N_1 \subseteq G_1 \cup G_2$, $N_1 \subseteq G_1 \cup G_3$, $N_1 \subseteq G_1 \cup G_4$, $N_1 \subseteq G_3 \cup G_2$, $N_1 \subseteq G_3 \cup G_4$, we have $N_1 \cap G_1 \cap G_2 \neq 0_{\sim}$, $N_1 \cap G_1 \cap G_3 \neq 0_{\sim}$, $N_1 \cap G_1 \cap G_4 \neq 0_{\sim}$, $N_1 \cap G_3 \cap G_2 \neq 0_{\sim}$, $N_1 \cap G_3 \cap G_4 \neq 0_{\sim}$. Similarly, N_1 is an IF$c_3^{\alpha,\beta}$-connected. N_1 is not an IF$c_1^{\alpha,\beta}$-connected since, there exist $G_1, G_2 \in \zeta^X$ with $\tau(G_i) \geq \langle \alpha, \beta\rangle$, $i = 1, 2$ and satisfies $N_1 \subseteq G_1 \cup G_2$, $G_1 \cap G_2 \subseteq \overline{N_1}$, $N_1 \cap G_1 \neq 0_{\sim}$ and $N_1 \cap G_2 \neq 0_{\sim}$). By the same technique we have, N_2 is an IF$c_4^{\alpha,\beta}$-connected but not an IF$c_3^{\alpha,\beta}$-connected.

Example 2.4. Let $X = \{a, b, c\}$ and $N, G_i \in \zeta^X$ $(i = 1, 2, 3, 4)$ defined as follows:

$$N = \langle x, (\frac{a}{0.5}, \frac{b}{0.3}, \frac{c}{0.0}), (\frac{a}{0.4}, \frac{b}{0.6}, \frac{c}{1.0})\rangle$$

$$G_1 = \langle x, (\frac{a}{0.0}, \frac{b}{0.4}, \frac{c}{0.3}), (\frac{a}{1.0}, \frac{b}{0.5}, \frac{c}{0.4})\rangle$$

$$G_2 = \langle x, (\frac{a}{0.6}, \frac{b}{0.0}, \frac{c}{0.2}), (\frac{a}{0.3}, \frac{b}{1.0}, \frac{c}{0.4})\rangle$$

$$G_3 = \langle x, (\frac{a}{0.6}, \frac{b}{0.4}, \frac{c}{0.3}), (\frac{a}{0.3}, \frac{b}{0.5}, \frac{c}{0.4})\rangle$$

$$G_4 = \langle x, (\frac{a}{0.0}, \frac{b}{0.0}, \frac{c}{0.2}), (\frac{a}{1.0}, \frac{b}{1.0}, \frac{c}{0.4})\rangle$$

Let $\tau : \zeta^X \longrightarrow I \times I$ defined as follows:

$$\tau(A) = \begin{cases} 1^{\sim}, & \text{if } A \in \{0_{\sim}, 1_{\sim}\} \\ \langle 0.5, 0.2\rangle, & \text{if } A \in \{G_1, G_1\} \\ \langle 0.7, 0.2\rangle, & \text{if } A \in \{G_3, G_4\} \\ 0^{\sim}, & \text{otherwise.} \end{cases}$$

Let $\alpha = 0.3, \beta = 0.6$. Then, N is an IF$c_4^{\alpha,\beta}$-connected but not an IF$c_2^{\alpha,\beta}$-connected.

Definition 2.5. Let X be a nonempty set and $A, B \in \zeta^X$. For $\alpha \in I_0, \beta \in I_1$ with $\alpha + \beta \leq 1$, A and B are said to be

(i) (α, β)-intuitionistic fuzzy weakly separated (briefly,(α, β)-IFWS) if there exist IFSs $U, V \in \zeta^X$ with $\tau(U) \geq \langle \alpha, \beta\rangle$, $\tau(V) \geq \langle \alpha, \beta\rangle$ such that $A \subseteq U$, $B \subseteq V$, $A \not{q} V$, $B \not{q} U$.

(ii) (α, β)-intuitionistic fuzzy q-separated (briefly, (α, β)-IFqS) if $Cl_{\alpha,\beta}(A) \cap B = 0_{\sim}$ and $A \cap Cl_{\alpha,\beta}(B) = 0_{\sim}$.

Theorem 2.6. *Let (X, τ) be an IFTS and $A, B \in \zeta^X$. For $\alpha \in I_0, \beta \in I_1$ with $\alpha + \beta \leq 1$, A and B are (α, β)-IFWS iff $cl_{\alpha,\beta}A \subseteq \overline{B}$ and $cl_{\alpha,\beta}B \subseteq \overline{A}$.*

Proof. Suppose that A, B are (α, β)-IFWS. Then, there exist $U, V \in \zeta^X$ with $\tau(U) \geq \langle \alpha, \beta\rangle$ such that $A \subseteq U$, $B \subseteq V$, $A \not{q} V$, $B \not{q} U$. By Theorem 1.6, $A \subseteq \overline{V}$, since $\tau^*(\overline{V}) = \tau(V) \geq \langle \alpha, \beta\rangle$ then, $cl_{\alpha,\beta}A \subseteq \overline{V} \subseteq \overline{B}$. Similarly, $cl_{\alpha,\beta}B \subseteq \overline{A}$.

Conversely, suppose that $cl_{\alpha,\beta}A \subseteq \overline{B}$ and $cl_{\alpha,\beta}B \subseteq \overline{A}$. Then, $B \subseteq \overline{cl_{\alpha,\beta}A} = V$, $A \subseteq \overline{cl_{\alpha,\beta}B} = U$ which implies that, $\tau(U) = \tau(\overline{cl_{\alpha,\beta}B}) = \tau^*(cl_{\alpha,\beta}B) \geq \langle \alpha, \beta\rangle$ and $\tau(V) = \tau(\overline{cl_{\alpha,\beta}A}) = \tau^*(cl_{\alpha,\beta}A) \geq \langle \alpha, \beta\rangle$. Also, $A \subseteq cl_{\alpha,\beta}A = \overline{V}$ and $B \subseteq cl_{\alpha,\beta}B = \overline{U}$, which implies that $A \not{q} V$, $B \not{q} U$. Hence, A, B are (α, β)-IFWS.

Definition 2.7. Let (X, τ) be an IFTS and $N \in \zeta^X$. For $\alpha \in I_0, \beta \in I_1$ with $\alpha + \beta \leq 1$,

(i) N is called an intuitionistic fuzzy $c_s^{\alpha,\beta}$-disconnected (briefly, IF$c_s^{\alpha,\beta}$-disconnected) if there exist an (α, β)-IFWS sets $A, B \in \zeta^X$ such that $A \cup B = N$ and $A \neq 0_{\sim}$, $B \neq 0_{\sim}$.

(ii) N is called an intuitionistic fuzzy $c_M^{\alpha,\beta}$-disconnected (briefly, IF$c_M^{\alpha,\beta}$-disconnected) if there exist (α, β)-IFqS sets $A, B \in \zeta^X$ such that $A \cup B = N$ and $A \neq 0_{\sim}$, $B \neq 0_{\sim}$.

(iii) N called an IF$c_s^{\alpha,\beta}$-connected if N is not an IF$c_s^{\alpha,\beta}$-disconnected.

(iv) N called an IF$c_M^{\alpha,\beta}$-connected if N is not an IF$c_M^{\alpha,\beta}$-disconnected.

Theorem 2.8. *Let (X, τ) be an IFTS and $N \in \zeta^X$. For $\alpha \in I_0, \beta \in I_1$ with $\alpha + \beta \leq 1$, if N is an IF$c_s^{\alpha,\beta}$-connected then, N is an IF$c_M^{\alpha,\beta}$-connected.*

Proof. Suppose for a contradiction that N is an IF$c_M^{\alpha,\beta}$-disconnected. Then, there exist $A, B \in \zeta^X$ such that $A \cup B = N$, $(cl_{\alpha,\beta}A) \cap B = 0_{\sim}$, $(cl_{\alpha,\beta}B) \cap A = 0_{\sim}$, $A \neq 0_{\sim}$, $B \neq 0_{\sim}$. By Theorem 1.6, we have $cl_{\alpha,\beta}A \subseteq \overline{B}$, $cl_{\alpha,\beta}B \subseteq \overline{A}$. Then by Theorem 2.6, N is an IF$c_s^{\alpha,\beta}$-disconnected which is a contradiction. Hence, N is an IF$c_M^{\alpha,\beta}$-connected.

Theorem 2.9. *Let (X,τ) be an IFTS and $N \in \zeta^X$. For $\alpha \in I_0, \beta \in I_1$ with $\alpha+\beta \leq 1$, if N is an IF$c_1^{\alpha,\beta}$-connected then, N is an IF$c_s^{\alpha,\beta}$-connected.*

Proof. Suppose that N is an IF$c_s^{\alpha,\beta}$-disconnected. Then, there exist $A, B \in \zeta^X$ such that, $A \cup B = N$, $cl_{\alpha,\beta}A \subseteq \overline{B}$, $cl_{\alpha,\beta}B \subseteq \overline{A}$, $A \neq 0_\sim$, $B \neq 0_\sim$. By Theorem 2.6, there exist $U, V \in \zeta^X$ with, $\tau(U) \geq \langle \alpha, \beta \rangle$, $\tau(V) \geq \langle \alpha, \beta \rangle$ such that, $A \subseteq U$, $B \subseteq V$, $A \not/qV$, $B \not/qU$. Then, $N = A \cup B \subseteq U \cup V$. Also, $N \cap U \neq 0_\sim$. For, if $N \cap U = 0_\sim$, then $N \cap A = 0_\sim$, so that $A = 0_\sim$(since, $A \subseteq N$) which a contradicts that $A \neq 0_\sim$. Similarly, $N \cap V \neq 0_\sim$. Also, $U \cap V \subseteq \overline{N}$. For, if $U \cap V \not\subseteq \overline{N}$ then, $(U \cap V)qN$ which implies that $(U \cap V)qA$ or $(U \cap V)qB$. Then (UqA and VqA) or (UqB and VqB), a contradiction with $A \not/qV$ and $B \not/qU$. Thus, N is an IF$c_1^{\alpha,\beta}$-disconnected, which is a contradiction. Hence, N is an IF$c_s^{\alpha,\beta}$-connected.

Theorem 2.10. *Let (X,τ) be an IFTS and $N \in \zeta^X$. For $\alpha \in I_0, \beta \in I_1$ with $\alpha+\beta \leq 1$, if N is an IF$c_s^{\alpha,\beta}$-connected then, N is an IF$c_2^{\alpha,\beta}$-connected.*

Proof. Suppose that N is an IF$c_2^{\alpha,\beta}$-disconnected. Then, there exist $U, V \in \zeta^X$ with, $\tau(U) \geq \langle \alpha, \beta \rangle$, $\tau(V) \geq \langle \alpha, \beta \rangle$ such that, $N \subseteq U \cup V$, $N \cap U \cap V = 0_\sim$, $N \cap U \neq 0_\sim$, $N \cap V \neq 0_\sim$. put $A = N \cap U \subseteq U$ and $B = N \cap V \subseteq V$. Then, $A \cup B = (N \cap U) \cup (N \cap V) = N \cap (U \cup V) = N$. Now, if AqV then, there exists $x \in X$ such that $\mu_A(x) > \gamma_V(x)$ or $\gamma_A(x) < \mu_V(x)$. First if $\mu_A(x) > \gamma_V(x)$ then, $\mu_A(x) > 0$. Since $N = A \vee B$ and $A \leq U$ then, $\mu_N(x) > 0$ and $\mu_U(x) > 0$. $\mu_V(x) \neq 0_\sim$ (since, if $\mu_V(x) = 0_\sim$, then $\gamma_V(x) = 1$, a contradiction with $\mu_A(x) > \gamma_V(x)$). Thus, $(\mu_N(x) \wedge \mu_U(x) \wedge \mu_V(x)) > 0$. So, $N \cap U \cap V \neq 0_\sim$, a contradiction with $N \cap U \cap V = 0_\sim$. Thus $A \not/qV$. Second: if $\gamma_A(x) < \mu_V(x)$ then, $\mu_V(x) > 0$. $\mu_N(x) > 0$ (for, if $\mu_N(x) = 0$ then, $N = A \cup B$ implies that, $\mu_A(x) = \mu_B(x) = 0$, so $\gamma_A(x) = 1$ a contradicts with $\gamma_A(x) < \mu_V(x)$). Since, $N \subseteq U \cup V$, $\mu_V(x) > 0$, $\mu_N(x) > 0$ then, $(\mu_N \wedge \mu_U \wedge \mu_V)(x) \neq 0$ and so, $N \cap U \cap V \neq 0_\sim$, a contradiction, then $A \not/qV$. Similarly, $B \not/qU$. Then, N is an IF$c_s^{\alpha,\beta}$-disconnected, a contradiction. Thus, N is an IF$c_2^{\alpha,\beta}$-connected.

Theorem 2.11. *Let (X,τ) be an IFTS and $N \in \zeta^X$. For $\alpha \in I_0, \beta \in I_1$ with $\alpha+\beta \leq 1$, if N is an IF$c_s^{\alpha,\beta}$-connected then, N is an IF$c_3^{\alpha,\beta}$-connected.*

Proof. Suppose that N is an IF$c_3^{\alpha,\beta}$-disconnected. Then, there exist $U, V \in \zeta^X$ with, $\tau(U) \geq \langle \alpha, \beta \rangle$, $\tau(V) \geq \langle \alpha, \beta \rangle$ such that, $N \subseteq U \cup V$, $U \cap V \subseteq \overline{N}$, $U \not\subseteq \overline{N}$, $V \not\subseteq \overline{N}$. Put $A = N \cap U \subseteq U$ and $B = N \cap V \subseteq V$. Then, $A \cup B = N$.
Let $R = \langle x, \mu_R(x), \gamma_R(x) \rangle$ and $S = \langle x, \mu_S(x), \gamma_S(x) \rangle$. Where,

$$\mu_R(x) = \begin{cases} \mu_A(x), & \text{if } \mu_U(x) \geq \mu_V(x) \\ 0, & \text{otherwise} \end{cases}$$

$$\gamma_R(x) = \begin{cases} \gamma_A(x), & \text{if } \gamma_U(x) \leq \gamma_V(x) \\ 1, & \text{otherwise} \end{cases}$$

$$\mu_S(x) = \begin{cases} \mu_B(x), & \text{if } \mu_U(x) < \mu_V(x) \\ 0, & \text{otherwise} \end{cases}$$

$$\gamma_S(x) = \begin{cases} \gamma_B(x), & \text{if } \gamma_U(x) > \gamma_V(x) \\ 1, & \text{otherwise} \end{cases}.$$

Then, $N = R \cup S$. Now, $R \neq 0_\sim$ (since, if $R = 0_\sim$, then $U \subset V$ which implies that, $U = U \cap V \subseteq \overline{N}$, a contradiction). Similarly, $S \neq 0_\sim$. Also, $R \subseteq A \subseteq U$ and $S \subseteq B \subseteq V$. Now, $R \not qV$. For, if RqV then, there exists, $x \in X$ such that, $\mu_R(x) > \gamma_V(x)$ or $\gamma_R(x) < \mu_V(x)$. First, if $\mu_R(x) > \gamma_V(x)$ then, $\mu_R(x) > 0$ which implies that, $\mu_U(x) \geq \mu_V(x)$. Since $N = R \cup S$ then, $\mu_N(x) \geq \mu_R(x) > \gamma_V(x)$. Since, $U \cap V \subseteq \overline{N}$ then, $\mu_N(x) \leq \gamma_U(x) \vee \gamma_V(x)$ but, $\mu_N(x) > \gamma_V(x)$ this implies that $\gamma_U(x) > \gamma_V(x)$ then, $\gamma_R(x) = 1$ this implies that $\mu_R(x) = 0$, a contradiction. Then, $R \not qV$. Second, if $\gamma_R(x) < \mu_V(x)$ then , $\gamma_R(x) < 1$ which implies that $\gamma_U(x) \leq \gamma_V(x)$. Since $N = R \cup S$ then, $\gamma_N(x) = (\gamma_R \wedge \gamma_S)(x)$ implies that $\gamma_N(x) \leq \gamma_R(x)$ then, $\gamma_N(x) < \mu_V(x)$. Since, $U \cap V \subseteq \overline{N}$ then, $\gamma_N(x) \geq (\mu_U \wedge \mu_V)(x)$ but, $\gamma_N(x) < \mu_V(x)$ this implies that $\mu_V(x) > \mu_U(x)$ then, $\mu_R(x) = 0$ implies $\gamma_R(x) = 1$ a contradiction. Then, $R \not qV$. Similarly, $S \not qU$. Then N is an IF$c_s^{\alpha,\beta}$-disconnected, which is a contradiction. Thus, N is an IF$c_3^{\alpha,\beta}$-connected.

Theorem 2.12. *Let (X, τ) be an IFTS and $N \in \zeta^X$. For $\alpha \in I_0, \beta \in I_1$ with $\alpha + \beta \leq 1$, if N is an IF$c_3^{\alpha,\beta}$-connected then, N is an IF$c_M^{\alpha,\beta}$-connected.*

Proof. Suppose that N is an IF$c_M^{\alpha,\beta}$-disconnected. Then, there exist $A, B \in \zeta^X$ such that, $A \cup B = N$, $(cl_{\alpha,\beta}A) \cap B = 0_\sim$, $(cl_{\alpha,\beta}B) \cap A = 0_\sim$, $A \neq 0_\sim$, $B \neq 0_\sim$. Let $U = \overline{cl_{\alpha,\beta}A}$ and $V = \overline{cl_{\alpha,\beta}B}$. Then, $\tau(U) = \tau^*(\overline{U}) = \tau^*(cl_{\alpha,\beta}A) \geq \langle \alpha, \beta \rangle$ and $\tau(V) = \tau^*(\overline{V}) = \tau^*(cl_{\alpha,\beta}B) \geq \langle \alpha, \beta \rangle$.

Now, $U \cap V = \overline{cl_{\alpha,\beta}A} \cap \overline{cl_{\alpha,\beta}B} = \overline{cl_{\alpha,\beta}A \cup cl_{\alpha,\beta}B} = \overline{cl_{\alpha,\beta}(A \cup B)} = \overline{cl_{\alpha,\beta}N} \subseteq \overline{N}$. Also, $U \cup V = \overline{cl_{\alpha,\beta}A} \cup \overline{cl_{\alpha,\beta}B} = \overline{cl_{\alpha,\beta}A \cap cl_{\alpha,\beta}B} = \overline{cl_{\alpha,\beta}(A \cap B)} = \overline{cl_{\alpha,\beta}0_\sim} = \overline{0_\sim} = 1_\sim$. Then, $N \subseteq U \cup V$. Also, $U \not\subseteq \overline{N}$. For if $U \subseteq \overline{N}$ then, $N \subseteq \overline{U} = cl_{\alpha,\beta}A$ this implies that, $cl_{\alpha,\beta}A \supseteq A \cup B$ implies $cl_{\alpha,\beta}A \supseteq B$ implies $B \cap cl_{\alpha,\beta}A \neq 0_\sim$, a contradiction. Similarly, $V \not\subseteq \overline{N}$. Therefore, N is an IF$c_3^{\alpha,\beta}$-disconnected which is a contradiction, then N is an IF$c_M^{\alpha,\beta}$-connected.

Definition 2.13. Let X be a nonempty set and $A, B \in \zeta^X$. For $\alpha \in I_0, \beta \in I_1$ with $\alpha + \beta \leq 1$, A and B are said to be

(i) (α, β)-intuitionistic fuzzy separated (briefly, (α, β)-IFS), if there exist IFSs $U, V \in \zeta^X$ with $\tau(U) \geq \langle \alpha, \beta \rangle$ and $\tau(V) \geq \langle \alpha, \beta \rangle$ such that $A \subseteq U$, $B \subseteq V$, $U \cap B = 0_\sim$ and $A \cap V = 0_\sim$.

(ii) (α, β)-intuitionistic fuzzy strongly separated (briefly, (α, β)-IFSS) if there exist IFSs $U, V \in \zeta^X$ with $\tau(U) \geq \langle \alpha, \beta \rangle$ and $\tau(V) \geq \langle \alpha, \beta \rangle$ such that $A \subseteq U$, $B \subseteq V$, $U \cap B = 0_\sim$, $A \cap V = 0_\sim$, UqA and VqB.

Definition 2.14. Let (X, τ) be an IFTS and $N \in \zeta^X$. For $\alpha \in I_0, \beta \in I_1$ with $\alpha + \beta \leq 1$,

(i) N is called an intuitionistic fuzzy $O^{\alpha,\beta}$-disconnected (briefly, IF$O^{\alpha,\beta}$-disconnected) if there exist an (α, β)-IFS sets $A, B \in \zeta^X$ such that $A \cup B = N$, $A \neq 0_\sim$ and $B \neq 0_\sim$.

(ii) N is called an intuitionistic fuzzy $O_q^{\alpha,\beta}$-disconnected (briefly,IF$O_q^{\alpha,\beta}$-disconnected) if there exist an (α, β)-IFSS sets $A, B \in \zeta^X$ such that $A \cup B = N$, $A \neq 0_\sim$ and $B \neq 0_\sim$.

(iii) N is called an IF$O^{\alpha,\beta}$-connected if N is not an IF$O^{\alpha,\beta}$-disconnected.

(iv)) N is called an IF$O_q^{\alpha,\beta}$-connected if N is not an IF$O_q^{\alpha,\beta}$-disconnected.

Theorem 2.15. *Let (X,τ) be an IFTS and $N \in \zeta^X$. For $\alpha \in I_0, \beta \in I_1$ with $\alpha+\beta \leq 1$, N is an IF$c_2^{\alpha,\beta}$-connected iff N is an IF$O^{\alpha,\beta}$-connected.*

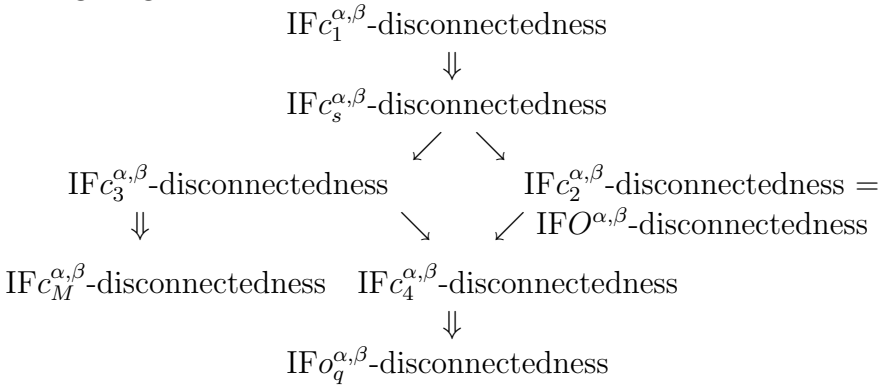
Proof. Suppose that N is an IF$O^{\alpha,\beta}$-disconnected. Then, there exist $A, B \in \zeta^X$ such that, A, B are (α,β)-IFS, $N = A \cup B$, $A \neq 0_\sim$, $B \neq 0_\sim$. Since A, B are (α,β)-IFS, there exist $U, V \in \zeta^X$ with $\tau(U) \geq \langle \alpha,\beta\rangle$ and $\tau(V) \geq \langle \alpha,\beta\rangle$ such that, $A \subseteq U$, $B \subseteq V$, $U \cap B = 0_\sim$, $V \cap A = 0_\sim$. $N = A \cup B \subseteq U \cup V$. Now, $N \cap U \cap V = (A \cup B) \cap (U \cap V) = (A \cap U \cap V) \cup (B \cap U \cap V) = 0_\sim$. Also, $N \cap U = (A \cup B) \cap U = (A \cap U) \cup (B \cap U) = A \cup 0_\sim = A \neq 0_\sim$. Similarly, $N \cap V \neq 0_\sim$. Then, N is an IF$c_2^{\alpha,\beta}$-disconnected, which is a contradiction. Hence, N is an IF$O^{\alpha,\beta}$-connected.

Conversely, suppose that N is an IF$c_2^{\alpha,\beta}$-disconnected. Then, there exist $U, V \in \zeta^X$ with $\tau(U) \geq \langle \alpha,\beta\rangle$ and $\tau(V) \geq \langle \alpha,\beta\rangle$ such that, $N \subseteq U \cup V$, $N \cap U \cap V = 0_\sim$, $N \cap U \neq 0_\sim$, $N \cap V \neq 0_\sim$. Let $A = N \cap U \subseteq U$ and $B = N \cap V \subseteq V$. Then, $A \cup B = (N \cap U) \cup (N \cap V) = N \cap (U \cup V) = N$. Also, $U \cap B = U \cap N \cap V = 0_\sim$. Similarly, $V \cap A = 0_\sim$. So, N is an IF$O^{\alpha,\beta}$-disconnected which is a contradiction. Then, N is an IF$c_2^{\alpha,\beta}$-connected.

Theorem 2.16. *Let (X,τ) be an IFTS and $N \in \zeta^X$. For $\alpha \in I_0, \beta \in I_1$ with $\alpha+\beta \leq 1$, if N is an IF$c_4^{\alpha,\beta}$-connected then, N is an IF$O_q^{\alpha,\beta}$-connected.*

Proof. Suppose that N is an IF$O_q^{\alpha,\beta}$-disconnected.Then, there exist $A, B \in \zeta^X$ such that, A, B are (α,β)-IFSS, $N = A \cup B$. Since A, B are (α,β)-IFSS, there exist $U, V \in \zeta^X$ with $\tau(U) \geq \langle \alpha,\beta\rangle$ and $\tau(V) \geq \langle \alpha,\beta\rangle$ such that, $A \subseteq U$, $B \subseteq V$, $U \cap B = 0_\sim$, $V \cap A = 0_\sim$, UqA, VqB. $N = A \cup B \subseteq U \cup V$. Now, $N \cap U \cap V = (A \cup B) \cap U \cap V = (A \cap U \cap V) \cup (B \cap U \cap V) = 0_\sim$. Also, since UqA and $A \subseteq N$, there exists $x \in X$ such that $\mu_U(x) > \gamma_A(x) \geq \gamma_N(x)$ or $\gamma_U(x) < \mu_A(x) \leq \mu_N(x)$ this implies that, $U \not\subseteq \overline{N}$. Similarly, $V \not\subseteq \overline{N}$. Therefore, N is an IF$c_4^{\alpha,\beta}$-disconnected which is a contradiction. Then, N is an IF$O_q^{\alpha,\beta}$-connected.

Remark 2.17. From Remark 2.2 and Theorems 2.8-2.12,2.15,2.16 we can build the following diagram



Examples 2.3, 2.4 and the next examples show that the reverse implications in Remark 2.17 are not true in general.

Example 2.18. Let $X = \{a, b\}$ and $N, G_i \in \zeta^X$ $(i = 1, 2)$ be defined as follows:

$$N = \langle x, (\frac{a}{0.4}, \frac{b}{0.4}), (\frac{a}{0.5}, \frac{b}{0.4})\rangle$$

$$G_1 = \langle x, (\frac{a}{0.7}, \frac{b}{0.8}), (\frac{a}{0.3}, \frac{b}{0.1})\rangle$$

$$G_2 = \langle x, (\frac{a}{0.2}, \frac{b}{0.3}), (\frac{a}{0.4}, \frac{b}{0.5})\rangle$$

Let $\tau : \zeta^X \longrightarrow I \times I$ defined as follows:

$$\tau(A) = \begin{cases} 1^{\sim}, & \text{if } A \in \{0_{\sim}, 1_{\sim}\} \\ \langle 0.5, 0.3\rangle, & \text{if } A = G_1 \\ \langle 0.7, 0.2\rangle, & \text{if } A = G_2 \\ 0^{\sim}, & \text{otherwise.} \end{cases}$$

Let $\alpha = 0.4,\ \beta = 0.5$. Then, N is an IF$c_s^{\alpha,\beta}$-connected but not an IF$c_1^{\alpha,\beta}$-connected.

Example 2.19. Let $X = \{a, b\}$ and $N, G_i \in \zeta^X$ $(i = 1, 2, 3, 4, 5, 6)$ defined as follows:

$$N = \langle x, (\frac{a}{0.3}, \frac{b}{0.3}), (\frac{a}{0.5}, \frac{b}{0.6})\rangle$$

$$G_1 = \langle x, (\frac{a}{0.3}, \frac{b}{0.4}), (\frac{a}{0.5}, \frac{b}{0.4})\rangle$$

$$G_2 = \langle x, (\frac{a}{0.4}, \frac{b}{0.3}), (\frac{a}{0.3}, \frac{b}{0.4})\rangle$$

$$G_3 = \langle x, (\frac{a}{0.3}, \frac{b}{0.3}), (\frac{a}{0.5}, \frac{b}{0.4})\rangle$$

$$G_4 = \langle x, (\frac{a}{0.4}, \frac{b}{0.4}), (\frac{a}{0.3}, \frac{b}{0.4})\rangle$$

$$G_5 = \langle x, (\frac{a}{0.0}, \frac{b}{0.3}), (\frac{a}{0.5}, \frac{b}{0.6})\rangle$$

$$G_6 = \langle x, (\frac{a}{0.3}, \frac{b}{0.0}), (\frac{a}{0.4}, \frac{b}{0.5})\rangle$$

Let $\tau : \zeta^X \longrightarrow I \times I$ defined as follows:

$$\tau(A) = \begin{cases} 1^{\sim}, & \text{if } A \in \{0_{\sim}, 1_{\sim}\} \\ \langle 0.4, 0.2\rangle, & \text{if } A \in \{G_1, G_2\} \\ \langle 0.5, 0.4\rangle, & \text{if } A \in \{G_3, G_4\} \\ 0^{\sim}, & \text{otherwise.} \end{cases}$$

Let $\alpha = 0.4, \beta = 0.5$. Then, N is both IF$c_2^{\alpha,\beta}$-connected and IF$c_3^{\alpha,\beta}$-connected but not IF$c_s^{\alpha,\beta}$-connected.

Example 2.20. Let $X = \{a, b, c\}$ and $N, G_i \in \zeta^X$ $(i = 1, 2, 3)$ defined as follows:

$$N = \langle x, (\frac{a}{0.6}, \frac{b}{0.4}, \frac{c}{0.2}), (\frac{a}{0.3}, \frac{b}{0.5}, \frac{c}{0.4})\rangle$$

$$G_1 = \langle x, (\frac{a}{0.6}, \frac{b}{0.0}, \frac{c}{0.0}), (\frac{a}{0.3}, \frac{b}{1.0}, \frac{c}{1.0})\rangle$$

$$G_2 = \langle x, (\frac{a}{0.0}, \frac{b}{0.5}, \frac{c}{0.3}), (\frac{a}{1.0}, \frac{b}{0.2}, \frac{c}{0.3})\rangle$$

$$G_3 = \langle x, (\frac{a}{0.6}, \frac{b}{0.5}, \frac{c}{0.3}), (\frac{a}{0.3}, \frac{b}{0.2}, \frac{c}{0.3})\rangle$$

Let $\tau : \zeta^X \longrightarrow I \times I$ defined as follows:

$$\tau(A) = \begin{cases} 1^{\sim}, & \text{if } A \in \{0_{\sim}, 1_{\sim}\} \\ < 0.3, 0.5 >, & \text{if } A \in \{G_1, G_2\} \\ < 0.7, 0.2 >, & \text{if } A = G_3 \\ 0^{\sim}, & \text{otherwise.} \end{cases}$$

Let $\alpha = 0.1, \beta = 0.8$. Then, N is an IF$O_q^{\alpha,\beta}$-connected but not an IF$c_4^{\alpha,\beta}$-connected.

Example 2.21. Let $X = \{a, b, c\}$ and $N, G_i \in \zeta^X$ $(i = 1, 2, 3, 4)$ defined as follows:

$$N = \langle x, (\frac{a}{0.4}, \frac{b}{0.4}, \frac{c}{0.3}), (\frac{a}{0.3}, \frac{b}{0.4}, \frac{c}{0.2})\rangle$$

$$G_1 = \langle x, (\frac{a}{0.4}, \frac{b}{0.4}, \frac{c}{0.3}), (\frac{a}{0.1}, \frac{b}{0.3}, \frac{c}{0.3})\rangle$$

$$G_2 = \langle x, (\frac{a}{0.3}, \frac{b}{0.5}, \frac{c}{0.1}), (\frac{a}{0.4}, \frac{b}{0.2}, \frac{c}{0.2})\rangle$$

$$G_3 = \langle x, (\frac{a}{0.4}, \frac{b}{0.5}, \frac{c}{0.3}), (\frac{a}{0.1}, \frac{b}{0.2}, \frac{c}{0.2})\rangle$$

$$G_4 = \langle x, (\frac{a}{0.3}, \frac{b}{0.4}, \frac{c}{0.1}), (\frac{a}{0.4}, \frac{b}{0.3}, \frac{c}{0.3})\rangle$$

Let $\tau : \zeta^X \longrightarrow I \times I$ defined as follows:

$$\tau(A) = \begin{cases} 1^{\sim}, & \text{if } A \in \{0_{\sim}, 1_{\sim}\} \\ \langle 0.3, 0.6\rangle, & \text{if } A \in \{G_1, G_2\} \\ \langle 0.5, 0.4\rangle, & \text{if } A \in \{G_3, G_4\} \\ 0^{\sim}, & \text{otherwise.} \end{cases}$$

Let $\alpha = 0.2, \beta = 0.7$. Then, N is an IF$C_M^{\alpha,\beta}$-connected but not an IF$c_3^{\alpha,\beta}$-connected.

3. Intuitionistic fuzzy $c_5^{\alpha,\beta}$-connectedness

Definition 3.1. Let (X,τ) be an IFTS. For $\alpha \in I_0, \beta \in I_1$ with $\alpha+\beta \leq 1$,

(i) X is called an intuitionistic fuzzy $c_5^{\alpha,\beta}$-disconnected (briefly, IF$c_5^{\alpha,\beta}$-disconnected) if there exist an IFS $A \in \zeta^X$ such that $\tau(A) \geq \langle \alpha, \beta \rangle$, $\tau^*(A) \geq \langle \alpha, \beta \rangle$, $A \neq 0_\sim$ and $A \neq 1_\sim$.

(ii) X is called an (α,β)-intuitionistic fuzzy disconnected (briefly, (α,β) IF-disconnected) if there exist an IFSs $A, B \in \zeta^X$ with $\tau(A) \geq \langle \alpha, \beta \rangle$, $\tau(B) \geq \langle \alpha, \beta \rangle$, such that $A \cup B = 1_\sim$, $A \cap B = 0_\sim$, $A \neq 0_\sim$ and $B \neq 0_\sim$.

(iii) X called an IF$c_5^{\alpha,\beta}$-connected if X is not an IF$c_5^{\alpha,\beta}$-disconnected.

(iv) X called an (α,β)IF-connected if X is not an (α,β)IF-disconnected.

Theorem 3.2. *Let (X,τ) be an IFTS. For $\alpha \in I_0, \beta \in I_1$ with $\alpha+\beta \leq 1$, if (X,τ) is an IF$c_5^{\alpha,\beta}$-connected then, (X,τ) is an (α,β)IF-connected.*

Proof. Suppose that (X,τ) is an (α,β)IF-disconnected.Then, there exist $A, B \in \zeta^X$ with, $\tau(A) \geq \langle \alpha, \beta \rangle$, $\tau(B) \geq \langle \alpha, \beta \rangle$ such that, $A \cup B = 1_\sim$, $A \cap B = 0_\sim$, $A \neq 0_\sim$, $B \neq 0_\sim$. This implies that, $\mu_A \vee \mu_B = 1_X$, $\gamma_A \wedge \gamma_B = 0_X$, $\mu_A \wedge \mu_B = 0_X$, $\gamma_A \vee \gamma_B = 1_X$. Let $C = \{x \in X : \mu_A(x) > 0\}$ and $D = \{x \in X : \mu_A(x) = 0\}$.

If $x \in C$ then, $\mu_A(x) > 0 \Rightarrow \mu_B(x) = 0 \Rightarrow \mu_A(x) = 1 \Rightarrow \gamma_A(x) = 0 \Rightarrow \gamma_B(x) = 1$.

If $x \in D$ then, $\mu_A(x) = 0 \Rightarrow \gamma_A(x) = 1 \Rightarrow \gamma_B(x) = 0 \Rightarrow \mu_B(x) = 1$. Then, $\mu_A = \gamma_B$ and $\gamma_A = \mu_B$; in other words, $B = \overline{A}$ then, $\tau^*(A) = \tau(\overline{A}) = \tau(B) \geq \langle \alpha, \beta \rangle$. and since $B \neq 0_\sim$, $A \neq 1_\sim$. Thus, (X,τ) is an IF$c_5^{\alpha,\beta}$-disconnected which is a contradiction. Hence, (X,τ) is an (α,β)IF-connected.

Theorem 3.3. *Let (X,τ_1), (Y,τ_2) be two IFTSs and $f : (X,\tau_1) \to (Y,\tau_2)$ be an intuitionistic fuzzy continuous and surjective map. For $\alpha \in I_0, \beta \in I_1$ with $\alpha+\beta \leq 1$, if (X,τ_1) is an (α,β)IF-connected then so is (Y,τ_2).*

Proof. Suppose that (Y,τ_2) is an (α,β)IF-disconnected. Then, there exist $U, V \in \zeta^Y$ with, $\tau_2(U) \geq \langle \alpha, \beta \rangle$, $\tau_2(V) \geq \langle \alpha, \beta \rangle$ such that, $U \cup V = 1_\sim$, $U \cap V = 0_\sim$, $U \neq 0_\sim$ and $V \neq 0_\sim$. Since f is an intuitionistic fuzzy continuous then,

$\tau_1(f^{-1}(U)) \geq \tau_2(U) \geq \langle \alpha, \beta \rangle$ and $\tau_1(f^{-1}(V)) \geq \tau_2(V) \geq \langle \alpha, \beta \rangle$. Let $A = f^{-1}(U)$, $B = f^{-1}(V)$, then $\tau_1(A) \geq \langle \alpha, \beta \rangle$ and $\tau_1(B) \geq \langle \alpha, \beta \rangle$. Since f is surjective and $U \neq 0_\sim$ then, $A = f^{-1}(U) \neq 0_\sim$ (For, if $f^{-1}(U) = 0_\sim$ then, $U = f(f^{-1}(U)) = f(0_\sim) = 0_\sim$ a contradiction). Similarly, $B = f^{-1}(V) \neq 0_\sim$. Now, $A \cup B = f^{-1}(U) \cup f^{-1}(V) = f^{-1}(U \cup V) = f^{-1}(1_\sim) = 1_\sim$. Similarly, $A \cup B = 0_\sim$. Thus, (X,τ_1) is an (α,β)IF-disconnected which is a contradiction. Hence, (Y,τ_2) is an (α,β)IF-connected.

Theorem 3.4. *Let (X,τ_1), (Y,τ_2) be two IFTSs and $f : (X,\tau_1) \to (Y,\tau_2)$ be an intuitionistic fuzzy continuous and surjective map. For $\alpha \in I_0, \beta \in I_1$ with $\alpha+\beta \leq 1$, if (X,τ_1) is an IF $c_5^{\alpha,\beta}$-connected then so is (Y,τ_2).*

Proof. It is similar to Theorem 3.3.

Theorem 3.5. *Let (X,τ) be an IFTS. For $\alpha \in I_0, \beta \in I_1$ with $\alpha+\beta \leq 1$, (X,τ) is an IF $c_5^{\alpha,\beta}$-connected iff there is no exist IFSs $A, B \in \zeta^X$ with $\tau(A) \geq \langle \alpha, \beta \rangle$, $\tau(B) \geq \langle \alpha, \beta \rangle$ such that $A = \overline{B}$, $A \neq 0_\sim$ and $B \neq 0_\sim$.*

Proof. Assume that there exist $A, B \in \zeta^X$ with $\tau(A) \geq \langle \alpha, \beta \rangle$, $\tau(B) \geq \langle \alpha, \beta \rangle$ such that $A = \overline{B}$, $A \neq 0_\sim$ and $B \neq 0_\sim$. Now, $\tau^*(B) = \tau(\overline{B}) = \tau(A) \geq \langle \alpha, \beta \rangle$ and $A \neq 0_\sim$ implies that $B \neq 1_\sim$. Then, (X, τ) is an IF $c_5^{\alpha,\beta}$-disconnected which is a contradiction. Conversely, assume that (X, τ) is an IF $c_5^{\alpha,\beta}$-disconnected. Then, there exists an IFS $A \in \zeta^X$ such that $\tau(A) \geq \langle \alpha, \beta \rangle$, $\tau^*(A) \geq \langle \alpha, \beta \rangle$, $A \neq 0_\sim$, $A \neq 1_\sim$. Now, take $B = \overline{A}$ then, $\tau(B) = \tau^*(\overline{B}) = \tau(A) \geq \langle \alpha, \beta \rangle$ and since $A \neq 1_\sim$ then $B \neq 0_\sim$, which is a contradiction. Hence, (X, τ) is an IF $c_5^{\alpha,\beta}$-connected.

Theorem 3.6. *Let (X, τ) be an IFTS. For $\alpha \in I_0, \beta \in I_1$ with $\alpha + \beta \leq 1$, (X, τ) is an IF $c_5^{\alpha,\beta}$-connected iff there is no exist IFSs $A, B \in \zeta^Y$ such that $B = \overline{A}$, $B = \overline{cl_{\alpha,\beta}A}$, $A = \overline{cl_{\alpha,\beta}B}$, $A \neq 0_\sim$ and $B \neq 0_\sim$.*

Proof. Assume that there exist $A, B \in \zeta^Y$ such that $B = \overline{A}$, $B = \overline{cl_{\alpha,\beta}A}$, $A = \overline{cl_{\alpha,\beta}B}$, $A \neq 0_\sim$, $B \neq 0_\sim$. Then, $\tau(A) = \tau(\overline{cl_{\alpha,\beta}B}) = \tau^*(cl_{\alpha,\beta}B) \geq \langle \alpha, \beta \rangle$ and $\tau(\overline{A}) = \tau(B) = \tau(\overline{cl_{\alpha,\beta}A}) = \tau^*(cl_{\alpha,\beta}A) \geq \langle \alpha, \beta \rangle$. Then, (X, τ) is an IF $c_5^{\alpha,\beta}$-disconnected, a contradiction. Conversely, suppose that, (X, τ) is an IF $c_5^{\alpha,\beta}$-disconnected. Then there exists an IFS $A \in \zeta^Y$ such that $\tau(A) \geq \langle \alpha, \beta \rangle$, $\tau^*(A) \geq \langle \alpha, \beta \rangle$, $A \neq 0_\sim$ and $A \neq 1_\sim$. Let $B = \overline{A}$. Then, $B \neq 1_\sim$, $B \neq 0_\sim$ and since $\tau^*(A) \geq \langle \alpha, \beta \rangle$ then, $A = cl_{\alpha,\beta}A$, this implies that $B = \overline{cl_{\alpha,\beta}A}$. Since $\tau^*(B) = \tau^*(\overline{A}) = \tau(A) \geq \langle \alpha, \beta \rangle$, then $A = \overline{B} = \overline{cl_{\alpha,\beta}B}$, a contradiction.

Definition 3.7. Let (X, τ) be an IFTS. For $\alpha \in I_0, \beta \in I_1$ with $\alpha + \beta \leq 1$, X is called an (α, β)-intuitionistic fuzzy strong connected(briefly, (α, β)IF-strong connected) if there exist IFSs $A, B \in \zeta^X$ with $\tau^*(A) \geq \langle \alpha, \beta \rangle$, $\tau^*(B) \geq \langle \alpha, \beta \rangle$, such that $\mu_A + \mu_B \leq 1$, $\gamma_A + \gamma_B \geq 1$, $A \neq 0_\sim$ and $B \neq 0_\sim$.

Remark 3.8. The notions of $c_5^{\alpha,\beta}$-connectedness and (α, β)IF-strong connectedness are independent as indicated by the following examples.

Example 3.9. Let $X = \{a, b, c\}$ and $G_i \in \zeta^X$ $(i = 1, 2, 3, 4)$ defined as follows:

$$G_1 = \langle x, (\frac{a}{0.3}, \frac{b}{0.4}, \frac{c}{0.2}), (\frac{a}{0.4}, \frac{b}{0.5}, \frac{c}{0.3}) \rangle$$

$$G_2 = \langle x, (\frac{a}{0.4}, \frac{b}{0.5}, \frac{c}{0.3}), (\frac{a}{0.3}, \frac{b}{0.4}, \frac{c}{0.2}) \rangle$$

$$G_3 = \langle x, (\frac{a}{0.3}, \frac{b}{0.4}, \frac{c}{0.2}), (\frac{a}{0.4}, \frac{b}{0.5}, \frac{c}{0.3}) \rangle$$

$$G_4 = \langle x, (\frac{a}{0.4}, \frac{b}{0.5}, \frac{c}{0.3}), (\frac{a}{0.3}, \frac{b}{0.4}, \frac{c}{0.2}) \rangle$$

Let $\tau : \zeta^X \longrightarrow I \times I$ defined as follows:

$$\tau(A) = \begin{cases} 1^\sim, & \text{if } , A \in \{0_\sim, 1_\sim\} \\ \langle 0.4, 0.5 \rangle, & \text{if } A \in \{G_1, G_2\} \\ \langle 0.5, 0.5 \rangle, & \text{if } A \in \{G_3, G_4\} \\ 0^\sim, & \text{otherwise} \end{cases}$$

Let $\alpha = 0.3, \beta = 0.5$. Then, X is an (α, β)-strong connected but not an IF$c_5^{\alpha,\beta}$-connected.

Example 3.10. Let $X = \{a, b, c\}$ and $G_1, G_2 \in \zeta^X$ defined as follows:

$$G_1 = \langle x, (\frac{a}{0.6}, \frac{b}{0.6}, \frac{c}{0.7}), (\frac{a}{0.4}, \frac{b}{0.3}, \frac{c}{0.2})\rangle$$

$$G_2 = \langle x, (\frac{a}{0.4}, \frac{b}{0.4}, \frac{c}{0.3}), (\frac{a}{0.5}, \frac{b}{0.6}, \frac{c}{0.7})\rangle$$

Let $\tau : \zeta^X \longrightarrow I \times I$ defined as follows:

$$\tau(A) = \begin{cases} 1^{\sim}, & \text{if } A \in \{0_{\sim}, 1_{\sim}\} \\ \langle 0.4, 0.3\rangle, & \text{if } A = G_1 \\ \langle 0.6, 0.2\rangle, & \text{if } A = G_2 \\ 0^{\sim}, & \text{otherwise} \end{cases}$$

Let $\alpha = 0.2, \beta = 0.5$. Then, X is an IF$c_5^{\alpha,\beta}$-connected but not an (α, β)-strong connected.

Theorem 3.11. *Let (X, τ) be an IFTS. For $\alpha \in I_0, \beta \in I_1$ with $\alpha + \beta \leq 1$, (X, τ) is an (α, β)-IF strong connected iff there is no exist IFSs $A, B \in \zeta^Y$ with $\tau(A) \geq \langle \alpha, \beta\rangle$, $\tau(B) \geq \langle \alpha, \beta\rangle$ such that $\mu_A + \mu_B \geq 1$, $\gamma_A + \gamma_B \leq 1$, $A \neq 1_{\sim}$ and $B \neq 1_{\sim}$.*

Proof. Let $A, B \in \zeta^X$ with $\tau(A) \geq \langle \alpha, \beta\rangle$, $\tau(B) \geq \langle \alpha, \beta\rangle$ such that $\mu_A + \mu_B \geq 1$, $\gamma_A + \gamma_B \leq 1$, $A \neq 1_{\sim}$ and $B \neq 1_{\sim}$. If we take $C = \overline{A}$ and $D = \overline{B}$, then $\tau^*(C) = \tau^*(\overline{A}) = \tau(A) \geq \langle \alpha, \beta\rangle$, $\tau^*(D) = \tau^*(\overline{B}) = \tau(B) \geq \langle \alpha, \beta\rangle$, $C \neq 0_{\sim}$ and $D \neq 0_{\sim}$. Moreover, $\mu_C + \mu_D = \gamma_A + \gamma_B \leq 1$, $\gamma_C + \gamma_D = \mu_A + \mu_B \geq 1$, a contradiction.

The converse of the proof is obtained by using a similar technique.

Theorem 3.12. *Let $f : (X, \tau_1) \to (Y, \tau_2)$ be an intuitionistic fuzzy continuous and surjective map from an IFTS (X, τ_1) to another IFTS (Y, τ_2). For $\alpha \in I_0, \beta \in I_1$ with $\alpha + \beta \leq 1$, if (X, τ_1) is an (α, β)IF-strong connected then so is (Y, τ_2).*

Proof. Suppose that (Y, τ_2) is not an (α, β)IF-strong connected. Then, there exist $C, D \in \zeta^Y$ with, $\tau_2^*(C) \geq \langle \alpha, \beta\rangle$, $\tau_2^*(D) \geq \langle \alpha, \beta\rangle$ such that, $\mu_C + \mu_D \leq 1$, $\gamma_C + \gamma_D \geq 1$, $C \neq 0_{\sim}$, $D \neq 0_{\sim}$. By using Theorem 1.14, we have, $\tau_1^*(f^{-1}(C)) \geq \tau_2^*(C) \geq \langle \alpha, \beta\rangle$ and $\tau_1^*(f^{-1}(D)) \geq \tau_2^*(D) \geq \langle \alpha, \beta\rangle$. Also, $\mu_{f^{-1}(C)} + \mu_{f^{-1}(D)} = f^{-1}(\mu_C) + f^{-1}(\mu_D) = \mu_C \circ f + \mu_D \circ f \leq 1_{\sim}$(since, $\mu_C + \mu_D \leq 1_{\sim}$). Similarly, $\gamma_{f^{-1}(C)} + \gamma_{f^{-1}(D)} = f^{-1}(\gamma_C) + f^{-1}(\gamma_D) = \gamma_C \circ f + \gamma_D \circ f \geq 1_{\sim}$(since, $\gamma_C + \gamma_D \geq 1_{\sim}$). Moreover, $f^{-1}(C) \neq 0_{\sim}$ (For, if $f^{-1}(C) = 0_{\sim}$ then, $C = f(f^{-1}(C)) = f(0_{\sim}) = 0_{\sim}$, a contradiction). Similarly, $f^{-1}(C) \neq 0_{\sim}$. This is a contradiction, thus (Y, τ_2) is (α, β)IF-strong connected.

4. (α, β)-intuitionistic fuzzy super connectedness

Definition 4.1. Let (X, τ) be an IFTS. For $\alpha \in I_0, \beta \in I_1$ with $\alpha + \beta \leq 1$,

(i) X is called an (α, β)-intuitionistic fuzzy super disconnected(briefly, (α, β)IF-super disconnected) if there exist an (α, β)-ifro set A in X such that, $A \neq 0_{\sim}$ and $A \neq 1_{\sim}$.

(ii) X called an (α,β)IF-super connected if X is not an (α,β)IFS-disconnected.

Theorem 4.2. *Let (X,τ) be an IFTS. For $\alpha \in I_0, \beta \in I_1$ with $\alpha+\beta \leq 1$, the following statements are equivalent:*

(i) *X is an (α,β)IF-super connected.*

(ii) *For each $A \in \zeta^X$, $A \neq 0_\sim$ such that $\tau(A) \geq \langle \alpha,\beta\rangle$ we have $cl_{\alpha,\beta}A = 1_\sim$.*

(iii) *For each $A \in \zeta^X$, $A \neq 1_\sim$ such that $\tau^*(A) \geq \langle \alpha,\beta\rangle$ we have $int_{\alpha,\beta}A = 0_\sim$.*

(iv) *There is no exist IFSs $A,B \in \zeta^X$ with, $\tau(A) \geq \langle \alpha,\beta\rangle$ and $\tau(B) \geq \langle \alpha,\beta\rangle$ such that $A \subseteq \overline{B}$, $A \neq 0_\sim$ and $B \neq 0_\sim$.*

(v) *There is no exist IFSs $A,B \in \zeta^X$ with, $\tau(A) \geq \langle \alpha,\beta\rangle$ and $\tau(B) \geq \langle \alpha,\beta\rangle$ such that $B = \overline{cl_{\alpha,\beta}A}$, $A = \overline{cl_{\alpha,\beta}B}$, $A \neq 0_\sim$ and $B \neq 0_\sim$.*

(vi) *There is no exist IFSs $A,B \in \zeta^X$ with, $\tau^*(A) \geq \langle \alpha,\beta\rangle$ and $\tau^*(B) \geq \langle \alpha,\beta\rangle$ such that $B = \overline{int_{\alpha,\beta}A}$, $A = \overline{int_{\alpha,\beta}B}$, $A \neq 1_\sim$ and $B \neq 1_\sim$.*

Proof. (i)⇒(ii) Assume that there exist $A \in \zeta^X$, $A \neq 0_\sim$ with $\tau(A) \geq \langle \alpha,\beta\rangle$ such that $cl_{\alpha,\beta}A \neq 1_\sim$. Then, $B = int_{\alpha,\beta}(cl_{\alpha,\beta}A) \neq 1_\sim$ is an (α,β)-ifro set in X and $0_\sim \neq A \subseteq int_{\alpha,\beta}(cl_{\alpha,\beta}A) = B$, which is a contradiction. Then, $cl_{\alpha,\beta}A = 1_\sim$.

(ii)⇒(iii) Let $A \neq 1_\sim$ be an IFS in X such that $\tau^*(A) \geq \langle \alpha,\beta\rangle$. Then, $\overline{A} \neq 0_\sim$ and $\tau(\overline{A}) = \tau^*(A) \geq \langle \alpha,\beta\rangle$. By (ii) we have, $cl_{\alpha,\beta}(\overline{A}) = 1_\sim$ implies that $\overline{cl_{\alpha,\beta}(\overline{A})} = 0_\sim$ and by Theorem 1.12, we have $int_{\alpha,\beta}A = 0_\sim$.

(iii)⇒(iv) Let $A,B \in \zeta^X$ with, $\tau(A) \geq \langle \alpha,\beta\rangle$ and $\tau(B) \geq \langle \alpha,\beta\rangle$ such that $A \subseteq \overline{B}$, $A \neq 0_\sim$ and $B \neq 0_\sim$. Then, $\overline{B} \neq 1_\sim$ and $\tau^*(\overline{B}) = \tau(B) \geq \langle \alpha,\beta\rangle$. By (iii) we have $int_{\alpha,\beta}\overline{B} \neq 0_\sim$, and since $A \leq \overline{B}$, then $0_\sim \neq A = int_{\alpha,\beta}A \subseteq int_{\alpha,\beta}\overline{B} = 0_\sim$ which is a contradiction.

(iv)⇒(i) Assume for a contradiction that X is an (α,β)IF-super disconnected. Then, there exists an (α,β)-ifro set A in X such that $A \neq 0_\sim$ and $A \neq 1_\sim$. By Theorem 1.16, $\tau(A) \geq \langle \alpha,\beta\rangle$. If we take $B = \overline{cl_{\alpha,\beta}A}$, then $\tau(B) \geq \langle \alpha,\beta\rangle$ and $B \neq 0_\sim$ (For, if $B = 0_\sim \Rightarrow \overline{cl_{\alpha,\beta}A} = 0_\sim \Rightarrow cl_{\alpha,\beta}A = 1_\sim \Rightarrow A = int_{\alpha,\beta}(cl_{\alpha,\beta}A) = 1_\sim$ which is a contradiction with the fact $A \neq 0_\sim$). We also , have $A \leq \overline{B}$ and this is a contradiction too.

(i)⇒(v) Suppose that there exist IFSs $A,B \in \zeta^X$ with, $\tau(A) \geq \langle \alpha,\beta\rangle$ and $\tau(B) \geq \langle \alpha,\beta\rangle$ such that $B = \overline{cl_{\alpha,\beta}A}$, $A = \overline{cl_{\alpha,\beta}B}$, $A \neq 0_\sim$ and $B \neq 0_\sim$. Then, $int_{\alpha,\beta}(cl_{\alpha,\beta}A) = int_{\alpha,\beta}\overline{B} = \overline{cl_{\alpha,\beta}B} = A$ and $A \neq 0_\sim$, $A \neq 1_\sim$ (For, if $A = 1_\sim$, then $1_\sim = \overline{cl_{\alpha,\beta}B}$ implies $0_\sim = cl_{\alpha,\beta}B$ implies $B = 0_\sim$). A contradiction with X is an (α,β)IF-super connected.

(v)⇒ (i) Suppose that X is an (α,β)IF-super disconnected. Then, there is an (α,β)-ifro set A in X such that, $A \neq 0_\sim$, $A \neq 1_\sim$. Now, take $B = \overline{cl_{\alpha,\beta}A}$. Then, $\tau(B) \geq \langle \alpha,\beta\rangle$, $B \neq 0_\sim$ and $\overline{cl_{\alpha,\beta}B} = \overline{cl_{\alpha,\beta}(\overline{cl_{\alpha,\beta}A})} = \overline{\overline{int_{\alpha,\beta}(cl_{\alpha,\beta}A)}} = int_{\alpha,\beta}(cl_{\alpha,\beta}A) = A$ which is a contradiction.

(v)⇒(vi) Let A,B IFSs in X with $\tau^*(A) \geq \langle \alpha,\beta\rangle$ and $\tau^*(B) \geq \langle \alpha,\beta\rangle$ such that $B = \overline{int_{\alpha,\beta}A}$, $A = \overline{int_{\alpha,\beta}B}$, $A \neq 1_\sim$ and $B \neq 1_\sim$. Take $C = \overline{A}$ and $D = \overline{B}$.

Then, $C \neq 0_\sim$, $D \neq 0_\sim$, $\tau(C) = \tau(\overline{A}) = \tau^*(A) \geq \langle \alpha, \beta \rangle$, $\tau(D) = \tau(\overline{B}) = \tau^*(B) \geq \langle \alpha, \beta \rangle$ and $\overline{cl_{\alpha,\beta}C} = \overline{cl_{\alpha,\beta}\overline{A}} = \overline{\overline{int_{\alpha,\beta}A}} = int_{\alpha,\beta}A = \overline{B} = D$. Similarly, $\overline{cl_{\alpha,\beta}D} = C$. This is a contradiction.

(vi)⇒(v) It is similarly to that (v)⇒(vi).

Theorem 4.3. *Let (X, τ) be an IFTS. For $\alpha \in I_0, \beta \in I_1$ with $\alpha + \beta \leq 1$, if X is an (α, β)IF-super connected then, X is an IF$c_5^{\alpha,\beta}$-connected.*

Proof. It is clear.

The converse of Theorem 4.3, is not true in general as shows in the following example:

Example 4.4. Let $X = \{a, b, c, d\}$ and $G_i \in \zeta^X$ $(i = 1, 2, 3, 4)$ defined as follows:

$$G_1 = \langle x, (\frac{a}{1.0}, \frac{b}{0.0}, \frac{c}{0.0}, \frac{d}{0.0}), (\frac{a}{0.0}, \frac{b}{0.0}, \frac{c}{1.0}, \frac{d}{1.0}) \rangle$$

$$G_2 = \langle x, (\frac{a}{0.0}, \frac{b}{0.0}, \frac{c}{0.0}, \frac{d}{1.0}), (\frac{a}{1.0}, \frac{b}{0.0}, \frac{c}{1.0}, \frac{d}{0.0}) \rangle$$

$$G_3 = \langle x, (\frac{a}{1.0}, \frac{b}{0.0}, \frac{c}{0.0}, \frac{d}{1.0}), (\frac{a}{0.0}, \frac{b}{0.0}, \frac{c}{1.0}, \frac{d}{0.0}) \rangle$$

$$G_4 = \langle x, (\frac{a}{0.0}, \frac{b}{0.0}, \frac{c}{0.0}, \frac{d}{0.0}), (\frac{a}{1.0}, \frac{b}{0.0}, \frac{c}{1.0}, \frac{d}{1.0}) \rangle$$

Let $\tau : \zeta^X \longrightarrow I \times I$ defined as follows:

$$\tau(A) = \begin{cases} 1^\sim, & \text{if } , A \in \{0_\sim, 1_\sim\} \\ \langle 0.5, 0.4 \rangle, & \text{if } A \in \{G_1, G_2\} \\ \langle 0.7, 0.3 \rangle, & \text{if } A \in \{G_3, G_4\} \\ 0^\sim, & \text{otherwise} \end{cases}$$

Let $\alpha = 0.4, \beta = 0.6$. Then, X is an IF$c_5^{\alpha,\beta}$-connected but not an (α, β)IF-super connected.

Theorem 4.5. *Let (X, τ_1), (Y, τ_2) be two IFTSs and $f : (X, \tau_1) \to (Y, \tau_2)$ be a surjective intuitionistic fuzzy continuous map. Then, for $\alpha \in I_0, \beta \in I_1$ with $\alpha + \beta \leq 1$, if X is an (α, β)IF-super connected, then so is Y.*

Assume that Y is an (α, β)IF-super disconnected. By Theorem 4.4(iv), there exist IFSs $C, D \in \zeta^Y$ with $\tau_2(C) \geq \langle \alpha, \beta \rangle$ and $\tau_2(D) \geq \langle \alpha, \beta \rangle$ such that $C \subseteq \overline{D}$, $C \neq 0_\sim$ and $D \neq 0_\sim$. Since f is intuitionistic fuzzy continuous, $\tau_1(f^{-1}(C)) \geq \tau_2(C) \geq \langle \alpha, \beta \rangle$ and $\tau_1(f^{-1}(D)) \geq \tau_2(D) \geq \langle \alpha, \beta \rangle$. $C \subseteq \overline{D}$ implies that $f^{-1}(C) \subseteq f^{-1}(\overline{D}) = \overline{f^{-1}(D)}$. Also, $f^{-1}(C) \neq 0_\sim$ and $f^{-1}(D) \neq 0_\sim$. By Theorem 4.4(iv), X is an (α, β)IF-super disconnected, a contradiction.

5. Conclusions

We defined and studied several types of fuzzy connectedness in intuitionistic fuzzy topological spaces in view of Šostak's sense. The relationships between different kinds of intuitionistic fuzzy connectedness were investigated. We Built a diagram to sum up the interrelationships between these types of intuitionistic fuzzy connectedness and illustrated that the converses are not true in general by giving several examples.

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ON SOME MORE q-SERIES IDENTITIES

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Abstract. In this paper we have discussed and also generalized some interesting identities found in the 'Lost' notebook of Ramanujan.

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1. Introduction, notations and definitions

The q-rising factorial $(a;q)_k$ is defined as,

$$(a,q)_k = \begin{cases} 1, & \text{if } k=0; \\ (1-a)(1-aq)(1-aq^2)\dots(1-aq^{k-1}), & \text{if } k \geq 1. \end{cases}$$

Similarly, the infinite q-rising factorial is defined by

$$(a;q)_\infty = \prod_{r=0}^{\infty}(1-aq^r), \quad \text{for } |q|<1.$$

Ramanujan's Notebooks, specially the 'Lost' notebook contains a large number of q-series identities. Some of the identities found in the 'Lost' notebook are very interesting and noteworthy. Our aim is to discuss and also generalize few identities found in the 'Lost' notebook of Ramanujan.

2. Main results

In this section, we shall discuss following identities. Making use of the q-binomial theorem Ramanujan established following transformation

$$\sum_{m=0}^{\infty}\frac{(a;q^h)_m(b;q)_{mh}t^m}{(q^h;q^h)_m(c;q)_{mh}} = \frac{(b;q)_\infty(at;q^h)_\infty}{(c;q)_\infty(t;q^h)_\infty}\sum_{n=0}^{\infty}\frac{(c/b;q)_m(t;q^h)_m b^m}{(q;q)_m(at;q^h)_m}, \tag{2.1}$$

where $|t|<1, |b|<1$. [2; Theorem (1.2.1) p.6]

Further, he deduced a large number of identities as special cases of (2.1). One of the identities is,

$$\sum_{n=0}^{\infty}\frac{(a;q^2)_n(b;q)_n t^n}{(q^2;q^2)_n(c;q)_n}=\frac{(b;q)_\infty(at;q^2)_\infty}{(c;q)_\infty(t;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(c/b;q)_{2n}(t;q^2)_n b^{2n}}{(q;q)_{2n}(at;q^2)_n}$$
$$+\frac{(b;q)_\infty(atq^2;q^2)_\infty}{(c;q)_\infty(tq;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(c/b;q)_{2n+1}(tq;q^2)_n}{(q;q)_{2n+1}(atq;q^2)_n}b^{2n+1}, \tag{2.2}$$

where $|t|<1, |b|<1$. [2; Theorem (1.2.2) p.7]

(a) The two variable generalization of (2.1) is

$$\sum_{m,n=0}^{\infty}\frac{(a;q^{h_1})_m(d;q^{h_2})_n}{(q^{h_1};q^{h_1})_m(q^{h_2};q^{h_2})_n}\frac{(b;q)_{mh_1+nh_2}}{(c;q)_{mh_1+nh_2}}x^m y^n$$
$$=\frac{(b;q)_\infty(ax;q^{h_1})_\infty(dy;q^{h_2})_\infty}{(c;q)_\infty(x;q^{h_1})_\infty(y;q^{h_2})_\infty}\sum_{r=0}^{\infty}\frac{(c/b;q)_r(x;q^{h_1})_r(y;q^{h_2})_r b^r}{(q;q)_r(ax;q^{h_1})_r(dy;q^{h_2})_r}, \tag{2.3}$$

where $|x|, |y|, |b|<1$.

Proof of (2.3). The left hand side of (2.3) is

$$=\frac{(b;q)_\infty}{(c;q)_\infty}\sum_{m,n=0}^{\infty}\frac{(a;q^{h_1})_m(d;q^{h_2})_n}{(q^{h_1};q^{h_1})_m(q^{h_2};q^{h_2})_n}\frac{(cq^{mh_1+nh_2};q)_\infty}{(bq^{mh_1+nh_2};q)_\infty}x^m y^n$$

Applying q-binomial theorem [2; (1.22) p. 6].

$$=\frac{(b;q)_\infty}{(c;q)_\infty}\sum_{m,n=0}^{\infty}\frac{(a;q^{h_1})_m(d;q^{h_2})_n x^m y^n}{(q^{h_1};q^{h_1})_m(q^{h_2};q^{h_2})_n}\sum_{r=0}^{\infty}\frac{(c/b;q)_r b^r q^{r(mh_1+nh_2)}}{(q;q)_r}.$$

Since $|x|, |y|, |b|<1$, both series are convergent. So changing the order of summations we have

$$=\frac{(b;q)_\infty}{(c;q)_\infty}\sum_{r=0}^{\infty}\frac{(c/b;q)_r b^r}{(q;q)_r}\sum_{m=0}^{\infty}\frac{(a;q^{h_1})_m(xq^{rh_1})^m}{(q^{h_1};q^{h_1})_m}\sum_{n=0}^{\infty}\frac{(d;q^{h_2})_n(yq^{rh_2})^n}{(q^{h_2};q^{h_2})_n}$$

Again, applying q-binomial theorem

$$=\frac{(b;q)_\infty}{(c;q)_\infty}\sum_{r=0}^{\infty}\frac{(c/b;q)_r b^r}{(q;q)_r}\frac{(axq^{h_1 r};q^{h_1})_\infty}{(xq^{h_1 r};q^{h_1})_\infty}\frac{(dyq^{h_2 r};q^{h_2})_\infty}{(yq^{h_2 r};q^{h_2})_\infty}$$

$$=\frac{(b;q)_\infty(ax;q^{h_1})_\infty(dy;q^{h_2})_\infty}{(c;q)_\infty(x;q^{h_1})_\infty(y;q^{h_2})_\infty}\sum_{r=0}^{\infty}\frac{(c/b;q)_r(x;q^{h_1})_r(y;q^{h_2})_r b^r}{(q;q)_r(ax;q^{h_1})_r(dy;q^{h_2})_r},$$

which is precisely the right hand side of (2.3).

Proceeding in the above manner we have the multi-variables generalization of (2.1) as

$$\sum_{m_1,m_2,\ldots m_r=0}^{\infty}\frac{(a_1;q^{h_1})_{m_1}(a_2;q^{h_2})_{m_2}\ldots(a_r;q^{h_r})_{m_r}(b;q)_{m_1h_1+m_2h_2+\ldots+m_rh_r}x_1^{m_1}x_2^{m_2}\ldots x_r^{m_r}}{(q^{h_1};q^{h_1})_{m_1}(q^{h_2};q^{h_2})_{m_2}\ldots(q^{h_r};q^{h_r})_{m_r}(c;q)_{m_1h_1+m_2h_2+\ldots+m_rh_r}}$$

$$=\frac{(b;q)_\infty(a_1x_1;q^{h_1})_\infty\ldots(a_rx_r;q^{h_r})_\infty}{(c;q)_\infty(x_1;q^{h_1})_\infty\ldots(x_r;q^{h_r})_\infty}\sum_{s=0}^{\infty}\frac{(c/b;q)_s(x_1;q^{h_1})_s\ldots(x_r;q^{h_r})_s b^s}{(q;q)_s(a_1x_1;q^{h_1})_s\ldots(a_rx_r;q^{h_r})_s}, \tag{2.4}$$

where $|x_1|, |x_2|, \ldots, |x_r|, |b|<1$.

(b) Two variables generalization of (2.2) is given by,

$$\sum_{m,n=0}^{\infty} \frac{(a;q^2)_m (d;q^2)_n (b;q)_{m+n}}{(q^2;q^2)_m (q^2;q^2)_n (c;q)_{m+n}} x^m y^n$$

$$= \frac{(b;q)_\infty (ax;q^2)_\infty (dy;q^2)_\infty}{(c;q)_\infty (x;q^2)_\infty (y;q^2)_\infty} \sum_{r=0}^{\infty} \frac{(c/b;q)_{2r} (x;q^2)_r (y;q^2)_r b^{2r}}{(q;q)_{2r} (ax;q^2)_r (dy;q^2)_r}$$

$$+ \frac{(b;q)_\infty (axq;q^2)_\infty (dyq;q^2)_\infty}{(c;q)_\infty (xq;q^2)_\infty (yq;q^2)_\infty} \sum_{r=0}^{\infty} \frac{(c/b;q)_{2r+1} (xq;q^2)_r (yq;q^2)_r b^{2r+1}}{(q;q)_{2r+1} (axq;q^2)_r (dyq;q^2)_r}, \tag{2.5}$$

where $|x|, |y|, |b| < 1$.

Proof of (2.5). The left hand side of (2.5) is

$$= \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{m,n=0}^{\infty} \frac{(a;q^2)_m (d;q^2)_n (cq^{m+n};q)_\infty}{(q^2;q^2)_m (q^2;q^2)_n (dq^{m+n};q)_\infty} x^m y^n$$

Applying q-binomial theorem

$$= \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{m,n=0}^{\infty} \frac{(a;q^2)_m (d;q^2)_n x^m y^n}{(q^2;q^2)_m (q^2;q^2)_n} \sum_{r=0}^{\infty} \frac{(c/b;q)_r b^r q^{mr+nr}}{(q;q)_r}$$

$$= \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{m,n=0}^{\infty} \frac{(a;q^2)_m (d;q^2)_n x^m y^n}{(q^2;q^2)_m (q^2;q^2)_n} \times$$

$$\times \left\{ \sum_{r=0}^{\infty} \frac{(c/b;q)_{2r} b^{2r} q^{2mr+2nr}}{(q;q)_{2r}} + \sum_{r=0}^{\infty} \frac{(c/b;q)_{2r+1} b^{2r+1} q^{m(2r+1)} q^{n(2r+1)}}{(q;q)_{2r+1}} \right\}$$

$$= \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{r=0}^{\infty} \frac{(c/b;q)_{2r} b^{2r}}{(q;q)_{2r}} \sum_{m=0}^{\infty} \frac{(a;q^2)_m (xq^{2r})^m}{(q^2;q^2)_m} \sum_{n=0}^{\infty} \frac{(d;q^2)_n (yq^{2r})^n}{(q^2;q^2)_n}$$

$$+ \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{r=0}^{\infty} \frac{(c/b;q)_{2r+1} b^{2r+1}}{(q;q)_{2r+1}} \sum_{m=0}^{\infty} \frac{(a;q^2)_m (xq^{2r+1})^m}{(q^2;q^2)_m} \sum_{n=0}^{\infty} \frac{(d;q^2)_n (yq^{2r+1})^n}{(q^2;q^2)_n}.$$

Applying again q-binomial theorem

$$= \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{r=0}^{\infty} \frac{(c/b;q)_{2r} b^{2r} (axq^{2r};q^2)_\infty (dyq^{2r};q^2)_\infty}{(q;q)_{2r} (xq^{2r};q^2)_\infty (yq^{2r};q^2)_\infty}$$

$$+ \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{r=0}^{\infty} \frac{(c/b;q)_{2r+1} b^{2r+1} (axq^{2r+1};q^2)_\infty (dyq^{2r+1};q^2)_\infty}{(q;q)_{2r+1} (xq^{2r+1};q^2)_\infty (yq^{2r+1};q^2)_\infty}.$$

$$= \frac{(b;q)_\infty (ax;q^2)_\infty (dy;q^2)_\infty}{(c;q)_\infty (x;q^2)_\infty (y;q^2)_\infty} \sum_{r=0}^{\infty} \frac{(c/b;q)_{2r} (x;q^2)_r (y;q^2)_r b^{2r}}{(q;q)_{2r} (ax;q^2)_r (dy;q^2)_r}$$

$$+ \frac{(b;q)_\infty (axq;q^2)_\infty (dyq;q^2)_\infty}{(c;q)_\infty (xq;q^2)_\infty (yq;q^2)_\infty} \sum_{r=0}^{\infty} \frac{(c/b;q)_{2r+1} (xq;q^2)_r (yq;q^2)_r b^{2r+1}}{(q;q)_{2r+1} (axq;q^2)_r (dyq;q^2)_r},$$

which is the right hand side of (2.5).

(c) One noteworthy identity given in the 'Lost' notebook of Ramanujan [2; (1.4.20) p. 19] is

$$(2.6)\quad (-bq^n;q^n)_\infty \sum_{m=0}^{\infty} \frac{a^m q^{m(m+1)/2}}{(q;q)_m(-bq^n;q^n)_m} = (-aq;q)_\infty \sum_{m=0}^{\infty} \frac{b^m q^{m(m+1)/2}}{(q^n;q^n)_m(-aq;q)_{nm}}.$$

In order to establish (2.6) let us consider (2.1).
Putting t/a for t and then taking $a \to \infty$ and b=0 in (2.1) we get,

$$\sum_{m=0}^{\infty} \frac{q^{hm(m-1)/2}(-)^m t^m}{(q^h;q^h)_m(c;q)_{mh}} = \frac{(t;q^h)_\infty}{(c;q)_\infty} \sum_{m=0}^{\infty} \frac{(-)^m c^m q^{m(m-1)/2}}{(q;q)_m(t;q^h)_m},$$

which yields (2.6) if we put h=n, $t = -bq^n$ and c=-aq.

Two variables generalization of (2.6) can be easily deduced from (2.3) in the following form

$$(2.7)\quad \begin{aligned}&(-aq;q)_\infty \sum_{m,n=0}^{\infty} \frac{q^{h_1 m(m+1)/2 + h_2 n(n+1)/2} \alpha^m \beta^n}{(q^{h_1};q^{h_1})_m (q^{h_2};q^{h_2})_n (-aq;q)_{mh_1+nh_2}} \\ &= (-\alpha q^{h_1};q^{h_1})_\infty (-\beta q^{h_2};q^{h_2})_\infty \sum_{r=0}^{\infty} \frac{a^r q^{r(r+1)/2}}{(q;q)_r(-\alpha q^{h_1};q^{h_1})_r(-\beta q^{h_2};q^{h_2})_r}.\end{aligned}$$

(d) Another interesting identity due to Ramanujan is

$$(2.8)\quad (-bq;q)_\infty \sum_{n=0}^{\infty} \frac{(-\lambda/a;q)_n a^n q^{n(n+1)/2}}{(q;q)_n(-bq;q)_n} = (-aq;q)_n \sum_{n=0}^{\infty} \frac{(-\lambda/b;q)_n b^n q^{n(n+1)/2}}{(q;q)_n(-aq;q)_n}.$$

[1; (6.2.9) p.146]

Identity (2.8) is noteworthy for several reasons.
It contains large number of identities as its special cases.
If we take $\lambda = 0$ in (2.8) we get the identity,

$$(2.9)\qquad (-bq;q)_\infty \sum_{n=0}^{\infty} \frac{a^n q^{n(n+1)/2}}{(q;q)_n(-bq;q)_n} = (-aq;q)_\infty \sum_{n=0}^{\infty} \frac{b^n q^{n(n+1)/2}}{(q;q)_n(-aq;q)_n},$$

which is n=1, case of (2.6).

Taking $\lambda = ab$ in (2.8), we get

$$(2.10)\qquad (-b;q)_\infty \sum_{n=0}^{\infty} \frac{a^n q^{n(n+1)/2}}{(q;q)_n(1+bq^n)} = (-a;q)_\infty \sum_{n=0}^{\infty} \frac{b^n q^{n(n+1)/2}}{(q;q)_n(1+aq^n)}.$$

If we put $\lambda = -ac$ in (2.8) we get the new form of the identity as,

$$(-bq;q)_\infty \sum_{n=0}^{\infty} \frac{(c;q)_n a^n q^{n(n+1)/2}}{(q;q)_n(-bq;q)_n} = (-aq;q)_\infty \sum_{n=0}^{\infty} \frac{(ac/b;q)_n b^n q^{n(n+1)/2}}{(q;q)_n(-aq;q)_n}. \qquad (2.11)$$

From (2.11) it is easy to have

$$\frac{\sum_{n=0}^{\infty}\frac{(c;q)_n a^n q^{n(n+1)/2}}{(q;q)_n(-bq;q)_n}}{\sum_{n=0}^{\infty}\frac{(c;q)_n a^n q^{n(n-1)/2}}{(q;q)_n(-bq;q)_n}} = \frac{\sum_{n=0}^{\infty}\frac{(ac/b;q)_n b^n q^{n(n+1)/2}}{(q;q)_n(-aq;q)_n}}{(1+a)\sum_{n=0}^{\infty}\frac{(ac/bq;q)_n b^n q^{n(n+1)/2}}{(q;q)_n(-a;q)_n}}$$

$$= \frac{1}{1+}\,\frac{a(1-c)/(1+bq)}{1-}\,\frac{aq(c+bq)/(1+bq)(1+bq^2)}{1+} \tag{2.12}$$

$$\frac{aq(1-cq)/(1+bq^2)(1+bq^3)}{1-}\,\frac{aq^3(c+bq^2)/(1+bq^3)(1+bq^4)}{1+}$$

$$\frac{aq^2(1-cq^2)/(1+bq^4)(1+bq^5)}{1-}\cdots\quad.$$

Proof of (2.12).

$$\frac{\sum_{n=0}^{\infty}\frac{(c;q)_n a^n q^{n(n+1)/2}}{(q;q)_n(-bq;q)_n}}{\sum_{n=0}^{\infty}\frac{(c;q)_n a^n q^{n(n-1)/2}}{(q;q)_n(-bq;q)_n}} = \frac{1}{1+\dfrac{\sum_{n=0}^{\infty}\frac{(c;q)_n a^n q^{n(n-1)/2}(1-q^n)}{(q;q)_n(-bq;q)_n}}{\sum_{n=0}^{\infty}\frac{(c;q)_n a^n q^{n(n+1)/2}}{(q;q)_n(-bq;q)_n}}}$$

$$= \frac{1}{1+\dfrac{a(1-c)/(1+bq)}{\dfrac{\sum_{n=0}^{\infty}\frac{(c;q)_n a^n q^{n(n+1)/2}}{(q;q)_n(-bq;q)_n}}{\sum_{n=0}^{\infty}\frac{(c;q)_n a^n q^{n(n+1)/2}}{(q;q)_n(-bq^2;q)_n}}}}$$

$$= \frac{1}{1+\dfrac{a(1-c)/(1+bq)}{1+\dfrac{\sum_{n=0}^{\infty}\frac{a^n q^{n(n+1)/2}}{(q;q)_n}\left\{\frac{(c;q)_n}{(-bq;q)_n}-\frac{(cq;q)_n}{(-bq^2;q)_n}\right\}}{\sum_{n=0}^{\infty}\frac{a^n(cq;q)_n q^{n(n+1)/2}}{(q;q)_n(-bq^2;q)_n}}}}$$

$$= \frac{1}{1+\dfrac{a(1-c)/(1+bq)}{1-\dfrac{aq(c+bq)/(1+bq)(1+bq^2)}{\dfrac{\sum_{n=0}^{\infty}\frac{(cq;q)_n a^n q^{n(n+1)/2}}{(q;q)_n(-bq^2;q)_n}}{\sum_{n=0}^{\infty}\frac{(cq;q)_n a^n q^{n(n+3)/2}}{(q;q)_n(-bq^3;q)_n}}}}}$$

Proceeding in the same way we get the continued fraction for the above ratio.

3. A theorem of Ramanujan

In this section, we shall give a simple proof of the following theorem of Ramanujan.

Theorem. *Let a and b be any complex numbers and* $|xy| < 1$.

If $\Phi(a,x,y) = \sum_{n=0}^{\infty} \frac{a^n x^{n(n+1)/2}}{(xy;xy)_n}$, *then*

$$\Phi(a,x,y)\Phi(b,y,x) = \sum_{n=0}^{\infty} \frac{(ax+by^n)(ax^2+by^{n-1})...(ax^n+by)}{(xy;xy)_n} \tag{3.1}$$

Proof. It is easy to show that right hand side of (3.1) is

$$= \sum_{n=0}^{\infty} \frac{b^n y^{n(n+1)/2}(-axy^{-n}/b;xy)_n}{(xy;xy)_n} = \sum_{n=0}^{\infty} \frac{b^n y^{n(n+1)/2}(-axy^{-n}/b;xy)_\infty}{(xy;xy)_n(-ax^{n+1}/b;xy)_\infty}$$

Applying q-binomial theorem

$$= \sum_{n=0}^{\infty} \frac{b^n y^{n(n+1)/2}}{(xy;xy)_n} \sum_{r=0}^{\infty} \frac{(x^{-n}y^{-n};xy)_r}{(xy;xy)_r}(-)^r \frac{a^r}{b^r} x^{r(r+1)}$$

Applying the identity [4; lemma 1(2) p.100] we get

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \frac{b^{n+r} y^{nr+n(n+1)/2+r(r+1)/2}(x^{-n-r}y^{-n-r};xy)_r}{(xy;xy)_{n+r}(xy;xy)_r} \left(-\frac{a}{b}\right)^r x^{nr+r(r+1)}$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \frac{b^{n+r} y^{nr+n(n+1)/2+r(r+1)/2}}{(xy;xy)_{n+r}(xy;xy)_r} \left(-\frac{a}{b}\right)^r x^{nr+r(r+1)} \frac{(x^{-n-r}y^{-n-r};xy)_\infty}{(x^{-n}y^{-n};xy)_\infty}$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \frac{(-)^r a^r b^r y^{nr+n(n+1)/2+r(r+1)/2} x^{nr+r(r+1)}}{(xy;xy)_{n+r}(xy;xy)_r (x^{-n}y^{-n};xy)_{-r}}$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \frac{(-)^r a^r b^r y^{nr+n(n+1)/2+r(r+1)/2} x^{nr+r(r+1)} (-x^{-n}y^{-n})^r}{(xy;xy)_{n+r}(xy;xy)_r (xy)^{r(r+1)/2}} (x^{n+1}y^{n+1};xy)_r$$

which on simplification gives

$$= \sum_{n=0}^{\infty} \frac{b^n y^{n(n+1)/2}}{(xy;xy)_n} \sum_{r=0}^{\infty} \frac{a^r x^{r(r+1)/2}}{(xy;xy)_r} = \Phi(b,y,x)\Phi(a,x,y).$$

This is precisely the left hand side of (3.1) if we put y=x and b=a in (3.1). We get

$$\left\{\sum_{n=0}^{\infty} \frac{a^n x^{n(n+1)/2}}{(x^2;x^2)_n}\right\}^2 = \sum_{n=0}^{\infty} \frac{a^n x^{n(n+1)/2}(-x^{-n+1};x^2)_n}{(x^2;x^2)_n} \tag{3.2}$$

4. Special cases

In this section, we shall deduce certain special cases of the results established in previous sections.

Taking $h_1 = h_2 = 1$ in (2.3), we get

$$\begin{aligned}\Phi_D[b : a, d; c : q; x, y] &= \sum_{m,n=0}^{\infty} \frac{(b;q)_{m+n}(a;q)_m(d;q)_n x^m y^n}{(c;q)_{m+n}(q;q)_m(q;q)_n} \\ &= \frac{(b;q)_\infty(ax;q)_\infty(dy;q)_\infty}{(c;q)_\infty(x;q)_\infty(y;q)_\infty}\ {}_3\Phi_2\left[\begin{matrix} c/b, x, y; q; b \\ ax, dy \end{matrix}\right],\end{aligned} \tag{4.1}$$

which is n=2 case of a known result [3; Theorem 5, p.207].

Putting x/a for x, y/d for y and then taking $a, d \to \infty$ in (2.3) we get

$$\begin{aligned}&\sum_{m,n=0}^{\infty} \frac{(-)^{m+n} q^{h_1 m(m-1)/2 + h_2 n(n-1)/2}(b;q)_{mh_1+nh_2} x^m y^n}{(q^{h_1};q^{h_1})_m(q^{h_2};q^{h_2})_n(c;q)_{mh_1+nh_2}} \\ &\quad = \frac{(b;q)_\infty(x;q^{h_1})_\infty(y;q^{h_2})_\infty}{(c;q)_\infty} \sum_{r=0}^{\infty} \frac{(c/b;q)_r b^r}{(q;q)_r(x;q^{h_1})_r(y;q^{h_2})_r},\end{aligned} \tag{4.2}$$

provided $|b| < 1$.

As $b \to 0$, (4.2) yields

$$\begin{aligned}&\sum_{m,n=0}^{\infty} \frac{(-)^{m+n} q^{h_1 m(m-1)/2 + h_2 n(n-1)/2} x^m y^n}{(q^{h_1};q^{h_1})_m(q^{h_2};q^{h_2})_n(c;q)_{mh_1+nh_2}} \\ &\quad = \frac{(x;q^{h_1})_\infty(y;q^{h_2})_\infty}{(c;q)_\infty} \sum_{r=0}^{\infty} \frac{(-)^r c^r q^{r(r-1)/2}}{(q;q)_r(x;q^{h_1})_r(y;q^{h_2})_r}\end{aligned} \tag{4.3}$$

Taking $c = -aq, x = -\alpha q^{h_1}$ and $y = \beta q^{h_2}$ in (4.3), we get

$$\begin{aligned}&\sum_{m,n=0}^{\infty} \frac{q^{h_1 m(m+1)/2 + h_2 n(n+1)/2} \alpha^m \beta^n}{(q^{h_1};q^{h_1})_m(q^{h_2};q^{h_2})_n(c;q)_{mh_1+nh_2}} \\ &\quad = \frac{(-\alpha q^{h_1};q^{h_1})_\infty(-\beta q^{h_2};q^{h_2})_\infty}{(-aq;q)_\infty} \sum_{r=0}^{\infty} \frac{a^r q^{r(r+1)/2}}{(q;q)_r(-\alpha q^{h_1};q^{h_1})_r(-\beta q^{h_2};q^{h_2})_r}.\end{aligned} \tag{4.4}$$

Putting c=bq in (2.5) we have,

$$\begin{aligned}\sum_{m,n=0}^{\infty} \frac{(a;q^2)_m(d;q^2)_n x^m y^n}{(q^2;q^2)_m(q^2;q^2)_n(1-bq^{m+n})} &= \frac{(ax;q^2)_\infty(dy;q^2)_\infty}{(x;q^2)_\infty(y;q^2)_\infty} \sum_{r=0}^{\infty} \frac{(x;q^2)_r(y;q^2)_r b^{2r}}{(ax;q^2)_r(dy;q^2)_r} \\ &+ \frac{(axq;q^2)_\infty(dyq;q^2)_\infty}{(xq;q^2)_\infty(yq;q^2)_\infty} \sum_{r=0}^{\infty} \frac{(xq;q^2)_r(yq;q^2)_r b^{2r+1}}{(axq;q^2)_r(dyq;q^2)_r}\end{aligned} \tag{4.5}$$

Taking b=1 and a=-1 in (2.9) we get,

$$\sum_{n=0}^{\infty} \frac{(-)^n q^{n(n+1)/2}}{(q^2;q^2)_n} = (q;q)_\infty(q;q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q;q)_n^2}. \tag{4.6}$$

Putting c=0 in (2.12), we get

$$\frac{\sum_{n=0}^{\infty}\frac{a^n q^{n(n+1)/2}}{(q;q)_n(-bq;q)_n}}{\sum_{n=0}^{\infty}\frac{a^n q^{n(n-1)/2}}{(q;q)_n(-bq;q)_n}} = \frac{\sum_{n=0}^{\infty}\frac{b^n q^{n(n+1)/2}}{(q;q)_n(-aq;q)_n}}{(1+a)\sum_{n=0}^{\infty}\frac{b^n q^{n(n+1)/2}}{(q;q)_n(-a;q)_n}}$$

$$= \frac{1}{1+}\frac{a/(1+bq)}{1-}\frac{abq^2/(1+bq)(1+bq^2)}{1+}$$

$$\frac{aq/(1+bq^2)(1+bq^3)}{1-}\frac{abq^5/(1+bq^3)(1+bq^4)}{1+}\cdots \quad . \tag{4.7}$$

Putting c=q in (2.12) we find,

$$\frac{\sum_{n=0}^{\infty}\frac{a^n q^{n(n+1)/2}}{(-bq;q)_n}}{\sum_{n=0}^{\infty}\frac{a^n q^{n(n-1)/2}}{(-bq;q)_n}} = \frac{\sum_{n=0}^{\infty}\frac{(aq/b;q)_n b^n q^{n(n+1)/2}}{(q;q)_n(-aq;q)_n}}{(1+a)\sum_{n=0}^{\infty}\frac{(a/b;q)_n b^n q^{n(n+1)/2}}{(q;q)_n(-a;q)_n}}$$

$$= \frac{1}{1+}\frac{a(1-q)/(1+bq)}{1-}\frac{aq^2(1+b)/(1+bq)(1+bq^2)}{1+}$$

$$\frac{aq(1-q^2)/(1+bq^2)(1+bq^3)}{1-}\frac{aq^4(1+bq)/(1+bq^3)(1+bq^4)}{1+}$$

$$\frac{aq^2(1-q^3)/(1+bq^4)(1+bq^5)}{1-}\cdots \quad . \tag{4.8}$$

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ENDOGENOUS CONTROL IN A TERNARY LOTKA-VOLTERRA MODEL AND ITS APPLICATIONS

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Abstract. This paper aims at highlighting the role of endogenous controls in Lotka-Volterra predator-prey models. Unlike other studies in which the core lies in expanding the number of conflicting species, here the stress is laid upon a control variable that does not need to be of the same nature as the species involved in the conflict. As in the case of the logistic equation Lotka-Volterra models have proved highly fruitful also outside the pure biological frame, clarifying many socio-economical and psychological evolution phenomena. A main problem lies in the change of the structural parameters, that can alter the evolution. Rather than an exterior action a model is most satisfactory in presence of an endogenous change, that should happen without any external intervention. The critical model is described in Section 2 under the name of tripartite antagonist model, where there are three actors, and three predator-prey equations involving, two at each time, the actors. The structure is that each actor once is predator and once prey in a circular scheme. In some natural cases the three variables are all biological, but in most meaningful cases the control has a different nature, usually social or behavioral. The paper highlights the particular case where the initial existence of a control, albeit at a very low level, is essential to allow the development towards equilibrium, while its absence leads to the destruction of the prey. A similar case arises when predator and prey are in temporary equilibrium, but a hidden control variable operates until a sudden change of equilibrium bursts out. Section 1 is devoted to recall some preliminary knowledge of system theory and of Lotka.Volterra classical equations. Section 3 is devoted to illustrate some actual cases where tripartite antagonist system can explain social, psychological, economic developments, even with reference to the academic world. The long lasting competition between Stoicism and Epicurism, between hedonism and Aristotle's eudaimonia fully enters in this scheme.

Keywords: evolution, tripartite antagonist model, predator-prey, ecology, eudaimonia, top-down and bottom-up controls.
AMS Mathematics Subject Classification: 34D23, 65P40, 92D15, 93C15, 93C40.

Introduction

The key word of this study is evolution, within a framework of endogenous dynamics. There are at least three cultural lines of thought: an evolution that tends towards a state of equilibrium in which transient phenomena gradually subside [1], an evolution that tends towards a final catastrophe, possibly followed by a regeneration, and a cyclic evolution with "twists and turns" in a perpetual coming and going of the story in both collective and individual terms. External phenomena cannot be ignored, but a good explanatory model aims at reducing the number of the exogenous variables that govern them. As soon as one steps away from the physical phenomena, the predictive value of the models becomes more limited (think of the econometric models), but still the ability to explain the trends and the possible structural transitions may survive. The tendency to make more realistic models adds new variables and new equations, making the models less and less controllable. On the other hand lies the tendency to seek the core of phenomena, with models that, beyond the odd accidental disturbance, make reason of their essential lines. This fact is important, since it allows the detection of situations of irreversibility and stability that may occur even in simple cases, so that the complexity of the system is not a necessary condition for their onset. The simpler models allow a mathematical treatment that is much more complete, since the mechanisms by which they operate can be discovered, whilst the more complex often end in a series of computer simulations that give no guarantee of being resolutive in situations of instability and bifurcation.

Lotka-Volterra models of the fight for life have become more and more popular because they can explain the kernel of many biological, social, economical, cultural evolution phenomena. From the first legendary papers of the 30's of last century, due to Lotka [21] and Volterra [48, 49], many developments have been performed. A complete structural analysis can be found in [37], while a beautiful experimental description is in [42]. A summary of those equations and their relations with stability forms part of chapter 5 of the book by Piccinini, Stampacchia, and Vidossich [34]. A wide book about the dynamics of population is [17]. An application to the themes of landscape is to be found in Piccinini-Taverna-Chang [35]. Up to date contributions are [25] and [16]. The present paper studies a different problem, namely not connected with the multiplication of conflicting species, but rather with the problem of controlling conflicts. The problem arose since the statement of Lotka-Volterra predator-prey model, and it was clear that it was enough to change the parameters in order to reach new equilibriums. But who could change the parameters? An exogenous change (*deus ex machina*) is obvious, and experimented since the ancient tragedy. What is required is an endogenous change, that should happen without any external intervention. Incidentally it must be observed that also the equation of exponential growth became Verhulst's (= logistic) equation transposing the control into the interior of the system. The

[1] The Hegelian thesis-antithesis-synthesis conceptual scheme enters this category when damping is moderately fast: not so fast as to prevent any antithesis, yet not so slow that multiple oscillations are required.

authors therefore considered a similar predator-prey system and inserted an endogenous control, according to a same predator-prey model, obtaining thus a tripartite antagonist system. The original theme was the evolution of touristic resorts and their limiting capabilities, and were dealt with in [31]. A similar model was applied to social sciences and fine arts in [30]. The authors decided that the model deserved to be analyzed from a mathematical point of view, highlighting the phenomena of hidden instability and checking some conditions under which they can arise.

The first section is devoted to Lotka-Volterra predator-prey system. In particular it recalls that also periodic linear systems may be interpreted as simplified Lotka-Volterra predator-prey models. The main objective is the construction of a tripartite antagonist system that glues together in a cycle three Lotka-Volterra linear systems. It will help understanding the main models of the second section since it represents a linear approximation for small non damped oscillations.

The second section is the original part where three Lotka-Volterra predator-prey systems are superimposed; formally all the variables have the same role, but it is shown that it is easy to consider two of them as conflicting analogous variables, while the third one may be considered as a control. It is no longer exogenous but becomes endogenous to the model and explains the possible outcomes according to the structural parameters and initial data. Many different situations may arise, but the most interesting are those in which in absence of control the prey vanishes, while if the control is present, albeit at a very low level, the equilibrium is finally achieved. Another interesting case is that where the prey survives (at a low level) even if the control is absent, but in presence of a control it finally jumps from an apparent equilibrium to a different final equilibrium.

The third section discusses the nature of the possible control in different real situations, distinguishing in particular the extreme cases of a top-down control (state, academy, school) or a bottom-up control (individual, public opinion, social nets), envisaging also meaningful mixed forms of control. Finally, Section 3 offers numerous examples of situations in which the tripartite antagonist model can be applied to life, behavior and well-being, both private and public.

1. From periodicity to Predator-Prey model

A first aim in this section is to show how periodic phenomena may arise and how damped oscillations lead to stability [2]. The section then recalls non linearity considering Verhulst's equation (logistic) and the two-variables predator-prey Lotka-Volterra equations.

1.1. From periodicity to stability

The simplest case is that of the non-forced harmonic oscillator, which satisfies the equation $y'' + \omega^2 y = 0$. It can be read as a model of the predator-prey type. In the case of biological balance, predator and prey numbers remain stable

[2]The experienced reader can skip the parts that he knows. He should just check the knowledge of system (10), that can form a "starred" exercise of system theory.

because the losses suffered by the prey are offset by its biological capability to reproduce, whilst the benefits gained by the predator are overshadowed by the limits of available prey.

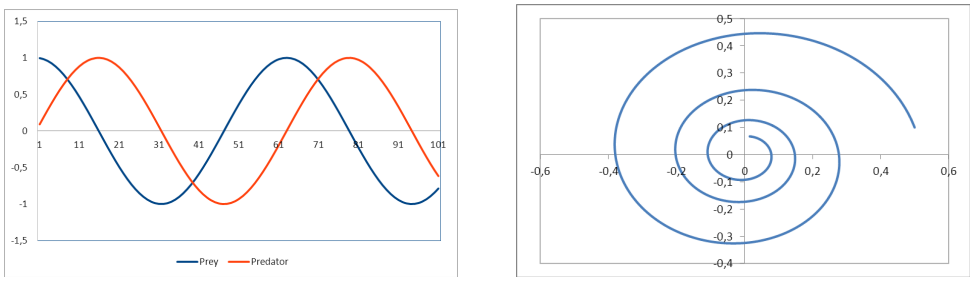
Let X be the abundance of prey, X° its equilibrium value, Y the abundance of the predator, Y° its equilibrium value. The easiest description is the following, which corresponds to the harmonic oscillator. The parameters m_{21} and m_{21} are assumed to be positive [3].

$$\begin{aligned} X' &= -\, m_{21}(Y - Y\circ) \\ Y' &= +\, m_{12}(X - X^\circ) \end{aligned} \tag{1}$$

The first equation states that prey decreases (negative derivative) when the abundance of the predator exceeds the equilibrium value ($Y > Y^\circ$) and then $Y - Y^\circ > 0$, whereas predator increases when the amount of prey exceeds the equilibrium value ($X > X^\circ$) and then $X - X^\circ > 0$. The general solution of the system is

$$\begin{aligned} X &= X^\circ + R\sqrt{k}\cos(\omega(t-\phi)) \\ Y &= Y^\circ + \tfrac{R}{\sqrt{k}}\sin(\omega(t-\phi)) \end{aligned}$$

where $k^2 = \frac{m_{21}}{m_{12}}$ and $\omega^2 = m_{21}m_{12}$ with $m_{21}, m_{12} > 0$. It represents an ellipse with semiaxes $R\sqrt{k}$, $\frac{R}{\sqrt{k}}$ running counterclockwise at a constant angular velocity, ϕ is the phase, which depends on the initial position; R also depends on the initial position, while the parameters of the equilibrium position are exogenous. The essential requisite is that the second variable generates a negative effect on the first, while the first variable generates a positive effect on the second. When each variable exerts an effect of the same sign on the other cycles can no longer happen, and there is a trend either to equilibrium or to instability. For a general theory of systems of linear differential equations, please refer to the specific texts, such as [38] and [6].



(a) The phase shift between two periodic variables

(b) Phase space for damped oscillations

Figure 1: Periodicity and stability

An important feature of system (1) is the shift between the two variables that can be observed in Figure 1(a) (centered in $(0, 0)$ and is to be compared with the symmetric behavior of the tripartite antagonist model of Section 1.2.

[3]Lotka-Volterra equations are more complicated than the elementary harmonic oscillator, in view of their nonlinearity, but approaching the equilibrium this is a good approximation allowing the substance of the phenomenon to be grasped.

The previous systems have periodic solutions, centered around an exogenous equilibrium point, but they do not converge to it. A stabilizing action may arise from the change of the structural parameters of the system of equations, in order to bring it back to the damped oscillations around the value of final equilibrium, regardless of the initial situation. Such systems are called asymptotically stable. In this case the trajectories become spirals converging towards equilibrium. A system that is asymptotically stable can have this form

$$\begin{aligned} X' =& -a(X - X^\circ) - m_{21}(Y - Y^\circ) \\ Y' =& m_{12}(X - X^\circ) - b(Y - Y^\circ) \end{aligned}$$

where the positive constants a and b are the intrinsic rates at which the variables X and Y tend to shrink to their reference values X° and Y°.

Fig. 1(b) shows the spiral trajectories that arise with this type of parameters (as long as $(a-b)^2 < 4m_{12}m_{21}$).

1.2. Linear tripartite antagonist model

The models with three actors provide a more complex picture, which will become very rich in the nonlinear case of the second section. Suppose that there are three variables X, Y, and C[4] that satisfy two by two a predator-prey system. For simplicity, we set $X^\circ = Y^\circ = C^\circ = 0$. A tripartite antagonist situation is one in which each variable is once predator and once prey; adding symmetry conditions, the following three systems of equations arise (we recall that $m_{ij} > 0$)

$$X' = -2m_{12}Y \qquad Y' = 2m_{21}X \tag{2a}$$

$$Y' = -2m_{23}C \qquad C' = 2m_{32}Y \tag{2b}$$

$$C' = -2m_{31}X \qquad X' = 2m_{13}C \tag{2c}$$

Each system separately would lead to the harmonic oscillator of equation (1). When the effects overlap we arrive at a system of three equations

$$\begin{aligned} X' &= -m_{12}Y + m_{13}C \\ Y' &= -m_{23}C + m_{21}X \\ C' &= -m_{31}X + m_{32}Y \end{aligned} \tag{3}$$

Even if each equation (2) entrains periodic solutions, (3) may have non periodic unstable solutions. Let M the matrix of coefficients of the system, i.e.

$$M = \begin{bmatrix} 0 & -m_{12} & m_{13} \\ m_{21} & 0 & -m_{23} \\ -m_{31} & m_{32} & 0 \end{bmatrix} \tag{4}$$

Let us remember that if $\boldsymbol{Z(t)} = (X(t), Y(t), C(t))$, $\boldsymbol{Z(t)} = exp(Mt)\boldsymbol{Z^\circ}$ is the solution of the system $\boldsymbol{Z'} = M\boldsymbol{Z}$, with $\boldsymbol{Z(0)} = \boldsymbol{Z^\circ}$ ([34] cap. II, 2 and II, 3, see also [20]). The characteristic equation becomes

$$\lambda^3 + (m_{12}m_{21} + m_{23}m_{32} + m_{31}m_{13})\lambda + m_{12}m_{23}m_{31} - m_{13}m_{12}m_{32} = 0 \tag{5}$$

[4] C is used for analogy to Section 2, where it denotes the control variable.

An interesting case is given by the uniformity condition, $m_{12} = m_{23} = m_{31}$ and $m_{13} = m_{21} = m_{32}$, since the uniformity condition is preserved in the exponential, for example starting from

$$M = \begin{bmatrix} 0 & -3 & 2 \\ 2 & 0 & -3 \\ -3 & 2 & 0 \end{bmatrix} \tag{6}$$

the characteristic equation (5) becomes

$$\lambda^3 + 18\lambda + 19 = 0$$

with eigenvalues -1, $\frac{1}{2} \pm i\,\frac{5\sqrt{3}}{2}$. Hence

$$exp(Mt) = \begin{bmatrix} f & g & h \\ h & f & g \\ g & h & f \end{bmatrix} \tag{7}$$

where

$$\begin{aligned} f &= \exp(-t)/3 + 2/3\exp(t/2)\cos(Ht) \\ g &= \exp(-t)/3 - 1/3\exp(t/2)\cos(Ht) - 5/2\exp(t/2)sen(Ht)/Ht \\ h &= \exp(-t)/3 - 1/3\exp(t/2)\cos(Ht) + 5/2\exp(t/2)sen(Ht)/Ht \end{aligned}$$

with $H = 5\sqrt{3}/2$. Compare Fig. 2

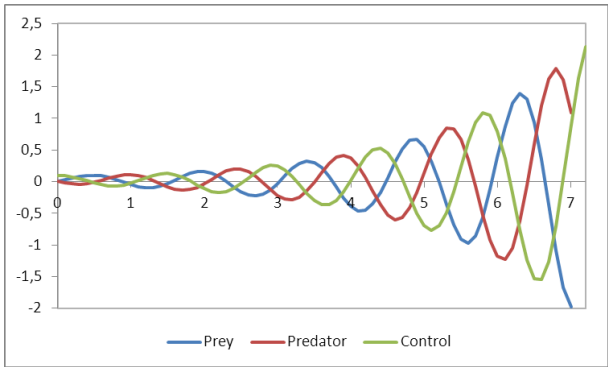


Figure 2: Time sheet of a uniform unstable system

In the symmetric situation when $m_{12}m_{23}m_{31} = m_{13}m_{12}m_{32}$, determinant is 0 and there is stability. For example, $m_{12} = m_{21} = 2$ and $m_{23} = m_{13} = m_{31} = m_{32} = 1$, lead to

$$M = \begin{bmatrix} 0 & -2 & 1 \\ 2 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \tag{8}$$

and $\exp(Mt)$ becomes

$$\begin{bmatrix} 1+5\cos(\sqrt{6}t) & 1-\cos(\sqrt{6}t)-2\sqrt{6}\sin(\sqrt{6}t) & 2-2\cos(\sqrt{6}t)+\sqrt{6}\sin(\sqrt{6}t) \\ 1-\cos(\sqrt{6}t)-2\sqrt{6}\sin(\sqrt{6}t) & 1+5\cos(\sqrt{6}t) & 2-2\cos(\sqrt{6}t)-\sqrt{6}\sin(\sqrt{6}t) \\ 2-2\cos(\sqrt{6}t)-\sqrt{6}\sin(\sqrt{6}t) & 2-2\cos(\sqrt{6}t)+\sqrt{6}\sin(\sqrt{6}t) & 4+2\cos(\sqrt{6}t) \end{bmatrix}$$

For the time sheet compare Fig. 3

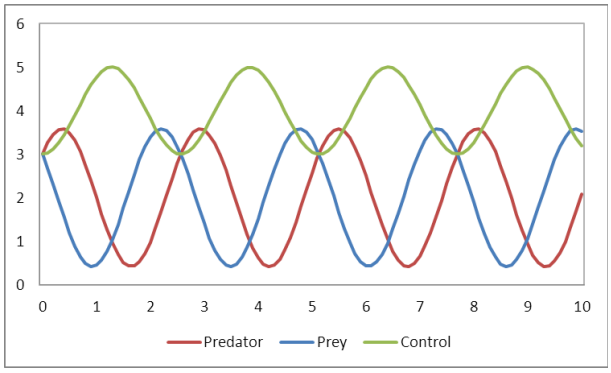


Figure 3: Time sheet of a symmetric non uniform system

A uniform symmetric system arises when $m_{12} = m_{21} = m_{23} = m_{13} = m_{31} = m_{32} = \omega$, the system is

$$\begin{aligned} X' &= \omega(-Y + C) \\ Y' &= \omega(-C + X) \\ C' &= \omega(-X + Y) \end{aligned} \tag{9}$$

The matrix M is

$$M = \begin{bmatrix} 0 & -\omega & \omega \\ \omega & 0 & -\omega \\ -\omega & \omega & 0 \end{bmatrix} \tag{10}$$

and the exponential matrix has the form (7) where

$$\begin{aligned} f &= 1 + 2/3\cos(\sqrt{3}\omega t) \\ g &= 1 - \cos(\sqrt{3}\omega t)/3 - \sqrt{3}sen(\sqrt{3}\omega t)/3 \\ h &= 1 - \cos(\sqrt{3}\omega t)/3 + \sqrt{3}sen(\sqrt{3}\omega t)/3 \end{aligned}$$

Observe the uniform shift of trajectories, as shown in Fig. 4, to be compared with Fig. 1(a).

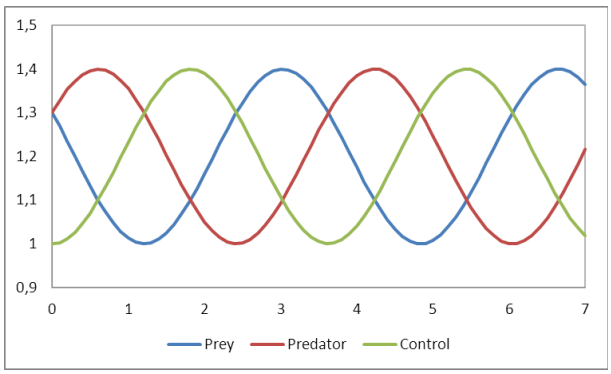


Figure 4: Time sheet of a symmetric uniform system

Some remarks for (9)

• There is a nonzero constant solution, given by any set $X = Y = C = A$, while in separate equations (2) only the constant solutions $X = Y = C = 0$ exists.

• By adding the three equations member by member it follows that

$$(X + Y + C)' = X' + Y' + C' = 0 \tag{11}$$

and multiplying the first for X, the second for Y and the third for C and summing up it follows that

$$(X^2 + Y^2 + C^2)' = 2(XX' + YY' + CC') = 0. \tag{12}$$

so that the trajectories of system (10) are circumferences that lie in a plane perpendicular to the vector $(1, 1, 1)$.

Therefore, a more geometrical representation is obtained using the three independent solutions

$$\begin{aligned}
(X_1, Y_1, C_1) =& (1, 1, 1)\\
(X_2, Y_2, C_2) =& (\cos(\sqrt{3}\omega t), \cos(\sqrt{3}\omega t - \frac{2\pi}{3}), \cos(\sqrt{3}\omega t - \frac{4\pi}{3}))\\
(X_3, Y_3, C_3) =& (\sin(\sqrt{3}\omega t), \sin(\sqrt{3}\omega t - \frac{2\pi}{3}), \sin(\sqrt{3}\omega t - \frac{4\pi}{3}))
\end{aligned}$$

by which the general solution, depending on the data A, R, ω, is generated:

$$\begin{aligned}
X =& A + R\cos(\sqrt{3}\omega t - \phi),\\
Y =& A + R\cos(\sqrt{3}\omega t - \frac{2\pi}{3} - \phi),\\
C =& A + R\cos(\sqrt{3}\omega t - \frac{4\pi}{3} - \phi).
\end{aligned}$$

1.3. From Verhulst to Lotka-Volterra equations

A significant example of a non-linear model is given by the equation of Verhulst, better known as logistic equation. Its origin is the following: the first hypothesis on the growth of a population in the absence of any constraint is that it evolves according to the exponential equation $P' = kP$, where k represents the coefficient of growth or decrease according to its sign. The solution is given by $P(t) = P(0)\exp(kt)$, which tends to infinity if $k > 0$.

The salient point of Verhulst's model is the belief that the growth rate of the population does not remain constant, but rather decreases with the increase of the population (in relation to the environment available), and may become negative in case of overcrowding. The simplest formulation implies a linear dependence descending around the equilibrium level C (called endogenous carrying capability), i.e., $k = v(C-P)$. With this modification, the exponential equation is transformed into the equation of Verhulst

$$P' = v(C - P)P \tag{13}$$

where all the parameters have already been lumped together in the constants v and C. The equation (13) has the important feature of possessing a solution of asymptotic equilibrium, given by $P = C$. The evolution governed by equation (13), if the initial data are below the threshold value C, presents a trend of sigmoid curves, increasing asymptotically to the value C, whereas if the initial data are higher than this value, the trend is decreasing asymptotically towards C_1 [5]. This equation is of considerable interest because it presents a level of stable equilibrium, that is to say that even if the system is disturbed (for example by an epidemic), it will tend to return to the equilibrium state provided that the structural parameters have not been changed. The logistic equation was originally referred to the evolution of biological species, and was then applied to many social and economic phenomena, in particular the spread of epidemics and of fashions.

The extension to multiple components present in the same area took place about a hundred years later, and is mainly due, independently, to Volterra [48, 49] and Lotka [21]. The predator-prey model in particular is the frame for the subsequent discussion. Its standard form may be written as

$$\begin{aligned} X' &= uX(C - X - hY) \\ Y' &= vY(D + kX - Y), \end{aligned} \tag{14}$$

where u and v are positive constants representing the speed of adjustment of the two components, h and k are positive constants. The logistic evolution of the X component is damaged (according to h) by the presence of the Y component (and therefore X is the "prey"), while the Y component is advantaged (according to k) by the presence of the species X (and then is the "predator"). The carrying

[5]The corresponding finite difference equation has a similar behavior when parameters are small enough, but as they increase it shows a set of periodic solutions, then it degenerates into a chaotic behavior (compare [33], ch 6.4)

capacities C and D are those that would separately hold for each component should the other component be absent [6]. The situation of equilibrium in presence of both species is given by the solution of the linear system

$$\begin{aligned} X + hY &= C \\ -kX + Y &= D \end{aligned} \tag{15}$$

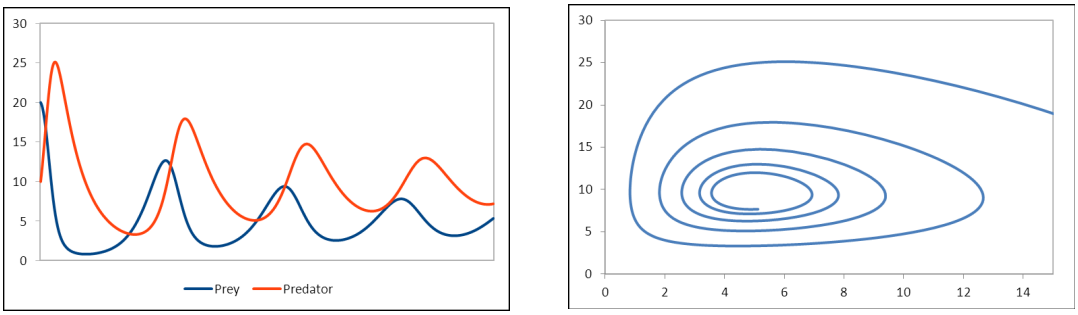
namely

$$\begin{aligned} X &= (C - hD)/(1 + hk) \\ Y &= (kC + D)/(1 + hk). \end{aligned} \tag{16}$$

The existence of equilibrium requires that the solutions (16) are both positive, otherwise the only possible solution is the disappearance of one of the two components. The evolution of the system involves oscillations around the equilibrium, damped according to the magnitude of the speed of adaptation. Also in this case, as already in the logistic equation, the values of equilibrium do not depend on the initial situation.

If one of the two variables is a candidate for disappearance, it is usually the prey, provided the survival of the predator does not depend on it alone. However, if the model predicts that D, the carrying capability of the predator, is negative then it follows that the predator component can exist only in the presence of the prey component, since the essential resource for the development of the predator is that very prey. In this case, the oscillations around equilibrium are showier and more persistent. Fig. 2 show precisely this case, and respectively represent the evolution in phase space and the laws of temporal evolution, when the values of the parameters are $u = v = 0.001$, $C = 100$, $D = -40$, $h = k = 10$. The balance is achieved when $X = 4.95...$, $Y = 9.505....$

Figure 5(a) shows in particular the shift between the maximum of the prey and that of the predator, similar to what was seen in Fig. 1 for the linear system (1).



(a) Phase space of predator-prey model (b) Time sheet of predator-prey model

Figure 5: Predator-prey model

[6]In other presentations there is a constant before X in the first equation and Y in the second, but so the role of C and D is no longer that of carrying capacity.

A very different case arises when the coefficient k becomes negative, because then the two components have a symmetrical role, so that each component damages the other. This situation of conflict no longer presents oscillatory phenomena, even if also in this case a rich choice of possible exits arises. Refer to [34] pp. 365-368 for further details, and to [35] for some special cases of multiple equilibria.

2. Lotka-Volterra tripartite antagonist models

2.1. Open and closed chains

The models with multiple components are very articulate, and have been widely proposed over the last fifty years. In many cases, the studies have examined how the main structural features change as the parameters vary, but usually all the components are biological species, and none of them have an essentially different role. Multiple models mostly represent open chains, where a lattice hierarchical sequence is available. A new articulated and far reaching model involving human and natured dynamics has recentrly been presented by Motesharrei, Rivas, and Kalnay in [25]. In what follows, on the contrary, a closed (i.e., circular) model will be presented.

Lotka-Volterra models deal with the competition between two or more species, but usually there is no endogenous control. In this section, we construct a model with three components, where two variables correspond to biological species (prey and predator) and the third corresponds to a control that is not necessarily biological, but endogenously arises from biological evolution. To the authors this seems to be the simplest model where the role of control is interior to the system. The model is apt to describe the competition between the prey) and the predator, in presence of a control.

2.2. Non linear tripartite antagonist model

There are three actors on the stage: the prey, the predator, and the possibility of a control. X is the perceived value of the prey, Y is the the level of predator, C the control capability. We will assume that each of these variables evolves according to logistics, i.e. self-sustaining until the respective level of equilibrium $(X^\circ, Y^\circ, C^\circ)$ is attained [7]. Suppose also that the evolution of each variable is conditioned by the level of the other two, dominant with respect to one of them, inferior to the other, in a cyclic structure depending on the m_{hk} positive coefficients. It corresponds and generalizes the linear model (3). The model will be called "tripartite antagonist model". We interpret the conceptual schema:

The prey is the fundamental resource, but is continually debased by the predator, while it is protected by the ability to control that can slow the degeneration caused by the thirst for profit.

[7]Some carrying capacities may be negative, but this does not seem particularly consistent with the meaning of the model.

The predator is self-powered ("appetite comes with eating") and generally tends to draw on the resources of the prey; it is moderated by control and critical consciousness.

The control tend to spread, but is actually deterred by the excess of the prey, which dulls and trivializes it. It feeds much more forcefully on the antagonism against the ugly caused by obtuse profit, and in any case the profit tends to donate a portion of its benefit for saving a part of prey

The tripartite antagonist model takes the following (general) form:

$$\begin{aligned} X' =& uX(X^\circ - X - m_{12}Y + m_{13}C) \\ Y' =& vY(Y^\circ + m_{21}X - Y - m_{23}C) \\ C' =& wC(C^\circ - m_{31}X + m_{32}Y - C) \end{aligned} \tag{17}$$

Many different equilibrium situations may arise, but the most interesting is achieved when the master system

$$\begin{aligned} X^\circ - X^* - m_{12}Y^* + m_{13}C^* = 0 \\ Y^\circ + m_{21}X^* - Y^* - m_{23}C^* = 0 \\ C^\circ - m_{31}X^* + m_{32}Y^* - C^* = 0 \end{aligned} \tag{18}$$

has positive solutions, what implies asymptotical stability with coexistence of all the variables. The reduced systems are obtained when a variable is absent, so that the master system shrinks to a pure predator-prey model

$$X^\circ - \hat{X} - m_{12}\hat{Y} = 0, \qquad Y^\circ + m_{21}\hat{X} - \hat{Y} = 0, \qquad \hat{C} = 0 \tag{19a}$$

$$Y^\circ - \hat{Y} - m_{23}\hat{C} = 0, \qquad C^\circ + m_{32}\hat{Y} - \hat{C} = 0, \qquad \hat{X} = 0 \tag{19b}$$

$$C^\circ - m_{31}\hat{X} - \hat{C} = 0, \qquad X^\circ - \hat{X} + m_{13}\hat{C} = 0, \qquad \hat{Y} = 0 \tag{19c}$$

Positive solutions are conditionally stable solutions, that is solutions that remain stable provided the lacking variable is not allowed to become positive. The most meaningful case of the tripartite antagonist model arises when there is no conditional stability for (19a), while asymptotic stability of the master system exists, what means that without any endogenous control the prey vanishes. Remark that in some unbalanced cases, when a positive solution of the master system does not exist, one of systems (19) may have an asymptotically stable solution, otherwise this is to be looked for in one of the one-variable logistics; in any case these solutions are at least conditionally stable if the two remaining variables are constrained to 0.

Adding conditions to the structural parameters in system (17), it becomes possible to find sufficient conditions for the solution of the master system (18) to be positive. A first hypothesis is the condition of symmetry $m_{23} = m_{32} = b, m_{31} = m_{13} = e, m_{12} = m_{21} = c$. If $b = e = c(= a)$. The system is uniform and becomes conservative, namely

$$X^* + Y^* + C^* = X^\circ + Y^\circ + C^\circ$$

since $X^* = [X^\circ(1+a^2)+Y^\circ(a^2-a)+C^\circ(a^2+a)]/D, Y^* = [Y^\circ(1+a^2)+C^\circ(a^2-a)+X^\circ(a^2+a)]/D, C^* = [C^\circ(1+a^2)+X^\circ(a^2-a)+Y^\circ(a^2+a)]/D$, where $D = 1+3a^2$. The condition does not imply the existence of positive solutions as it can be seen by the example $X^\circ = 1$, $Y^\circ = 8$, $C^\circ = 1$, $a = 0,5$, where $X^* = 0$, $Y^* = 6$, $C^* = 4$. This case arises from the coexistence of an unbalanced structure of carrying capacities joined with small interactions. The positivity condition is obviously unconditionally satisfied in case of strong interrelation, that is $a \geq 1$. An important structural remark is that, on the contrary, for reduced systems (19) the positivity condition is $a < 1$.

We say that the system is balanced if $X^\circ = Y^\circ = C^\circ (= A)$. If the system is balanced and symmetric it is sub-conservative, namely it holds

$$X^* + Y^* + C^* \leq X^\circ + Y^\circ + C^\circ$$

and equality holds only for conservative systems. Remark that a non positive solution could still arise when one of the interrelations is much greater than 1 and the others are much smaller than 1. For example $c = 2, b = 0,2, e = 0,2, A = 1$ leads to $X^* = -6.3.., Y^* = 64.6.., C^* = 114.1...$ On the contrary, a symmetric system that is balanced and uniform has always positive solutions since they coincide exactly with the carrying capacity A. Remark that the speeds of adjustment (u, v, w in (17)) do not modify the equilibrium position. This simplified situation allows to detect interesting delay phenomena that arise in the study of real models.

2.3. Critical tripartite antagonist models

The main model corresponds to the linear model (10). Here $A = 1, a = 1$, time is normalized so that $u = v = w = 1$.

$$\begin{aligned} X' &= X(1 - X - Y + C) \\ Y' &= Y(1 + X - Y - C) \\ C' &= C(1 - X + Y - C) \end{aligned} \tag{20}$$

The asymptotically stable equilibrium is $X^* = Y^* = C^* = 1$. However, as in all Lotka-Volterra systems, there are solutions that provide for the elimination of some components. With the parameter values of (20), there are no final solutions with two nonzero components, while there are those with only one nonzero component. This case occurs when a component is absent since the initial time (and then remains definitively null). The model then becomes a predator-prey model, with the disappearance of the prey:

- If $X(0) = 0$, $Y(0) > 0$, $C(0) > 0$, the system arrives at $C = 1$, $X = Y = 0$. The predator is destroyed by an excess of control, and is not compensated by the prey.
- If $C(0) = 0$, $X(0) > 0$, $Y(0) > 0$, the system arrives at $Y = 1$, $X = C = 0$. The predator destroys the prey because of the absence of control. The result is destructive and entrains insane development at the expense of the prey.

- If $Y(0) = 0$, $X(0) > 0$, $C(0) > 0$, the system arrives at $X = 1$, $Y = C = 0$. A dreamy prey eliminates the need for controls in a model that is seemingly idyllic, but is born in the absence of prey (golden Age): ataraxic in the contemplation of unproductive prey.

However, if each element is present, albeit at a very small, but positive level, it will be able to bear fruit and rebalance the system toward the global situation. The presence-absence of a component is then a bifurcation point.

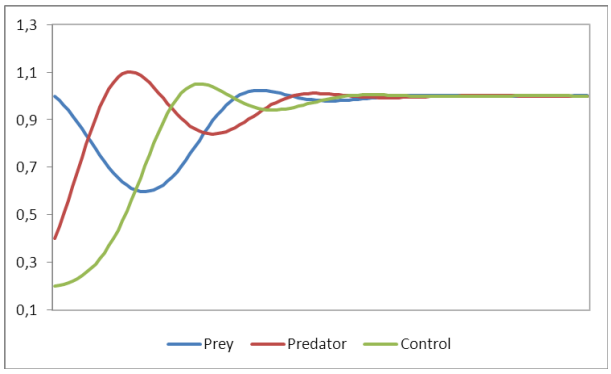
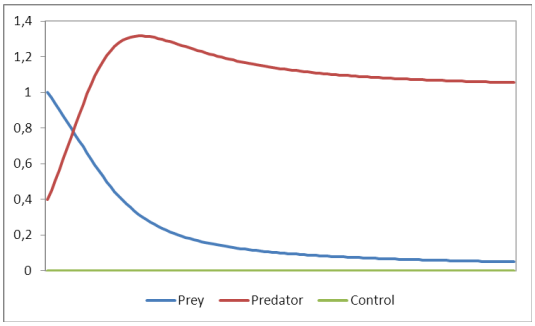
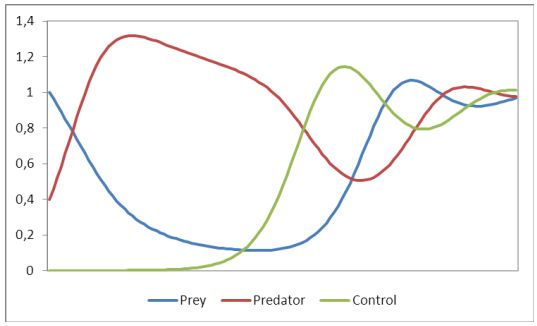


Figure 6: Evolution of the tripartite antagonist system (20)

Fig. 6 shows the evolution of a system in which the component C (Control) starts from a lower level than the other two and then rises to the equilibrium level. Fig. 7(a) instead shows the evolution in the case of total absence of the C component: X (Prey) tends to disappear, despite starting from a higher value than Y (Predator).



(a) Tripartite antagonist system with degenerate initial data (system (20), C (0) = 0)



(b) Tripartite antagonist delayed system ((20), C(0) = 0.001)

Figure 7: Degeneration or final equilibrium?

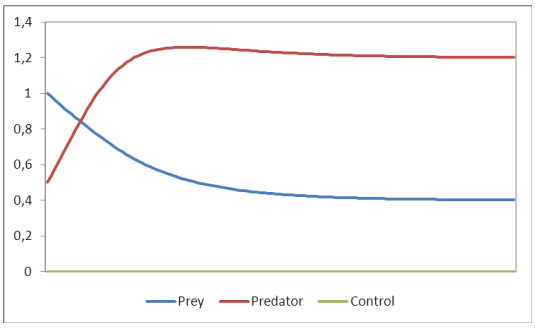
This fact seems paradoxical, since the initial presence of a minimum quantity of the variable C is sufficient in order to eventually reach the equilibrium, albeit after some fluctuations of no small importance. Fig. 7(b) shows a case in which

the initial level of control is 0.001. The initial trend is similar to Fig. 7(a), but later the presence of control becomes significant and leads to equilibrium.

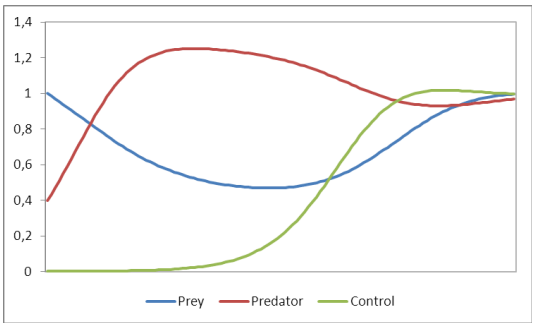
In the case of system (20) such extreme situations occur because of the high level of the interaction coefficients between the variables. Should coefficients be lower, the effects are not so obvious, and in particular there might be no bifurcation given by the presence-absence of a variable. It is in fact sufficient that the interaction coefficients fall below 1 to eliminate the phenomenon. An example is the following system (halved coefficients).

$$\begin{aligned} X' &= X(1 - X - 0.5Y + 0.5C) \\ Y' &= Y(1 + 0.5X - Y - 0.5C) \\ C' &= C(1 - 0.5X + 0.5Y - C) \end{aligned} \tag{21}$$

If $C(0) = 0$, the absence of control will be maintained all the time, however, the variables X and Y will continue to exist, even if the limit value $X(= 0.4)$ is much lower than that of $Y(= 1.2)$, compare Fig. 8(a). If the system proceeded from a positive value of C, though very close to 0, the limit value would be, as in the other cases, 1 for each of the variables. This apparent bifurcation at 0 should in fact be read as an extension of the time required to reach equilibrium, becoming more pronounced as the initial values are close to 0, and diverging to infinity at 0.



(a) Antagoniste stable tripartite system (21)



(b) Sudden change of equilibrium (system (21))

Figure 8: Which equilibrium?

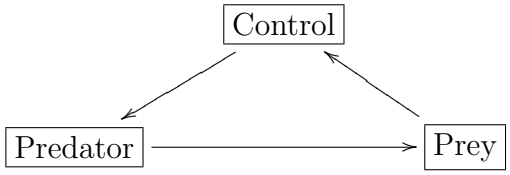
Even in model (21) the only stable equilibrium is $X = C = Y = 1$, for which balance disturbances lead back to the same final situation. All other equilibrium positions are not stable, since a small deviation from the value of unstable equilibrium may deflect towards a new equilibrium position. The situation $X = 0.4$, $C = 0$, $Y = 1.2$ is unstable, as a deviation of C towards a positive value leads to stable equilibrium, whilst the subsystem in which $C(0) = 0$ has a stable equilibrium position when it is restricted to X and Y components. This is very interesting because it means that there are cases of apparent stable equilibrium in which even a small perturbation of the exogenous control (fashion or awareness, or ecological consciousness) can lead to pronounced changes in the overall situation, albeit over the course of time.

Remark that if $C(0) > 0$, but very small, a long transition period can be expected, where the solutions initially tend to an unstable equilibrium and then suddenly pass to the stable equilibrium as shown in fig 8(b).

3. Examples of interpretation of the tripartite antagonist model

The abstract tripartite antagonist model stated in equations (17) can now be applied to more realistic situations. For brevity, we will use the symbol

(22)

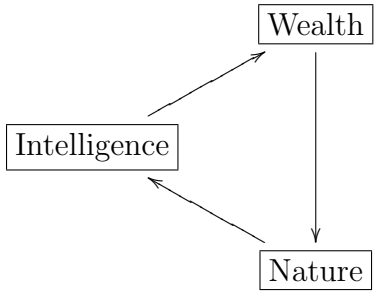


to indicate the circular chain where $X$is the predator of Y. C is the control and acts as a predator of X. Y is the predator of C, in the sense that the presence of C allows the increase of Y, while it is the lack of Y that generates the need for C, and on the contrary a strong presence of Y makes it useless. It can be summarized that the endogenous control responds positively or negatively to the unbalance between predator and prey. The examples are sorted according to the nature of the control: top-down (public control, legislation, school, rule), or bottom-up (social networks, market, ecology, individual freedom). There are cases of mixed control where the two forms coexist or succeed one another over time from private to public, such as political consensus, or from public to private through education campaigns and awareness raising. In the case of top-down control the prey is some form of social good and the predator is some form of private interest or benefit, while in the opposite case the prey is some form of individual liberty and the predator is the constriction of the rules and of the lack of privacy. We start with three mixed control systems

3.1. Mixed control

Many instances of the basic model (20) (more generally system (17)) are represented by the formula

(23)



Here nature is the fundamental resource that needs to be saved. Non reproducible resources end up in their vanishing, but also reproducible resources (like

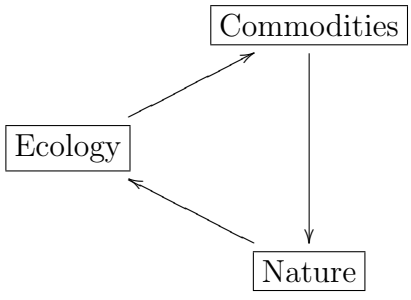
agriculture and cattle breeding) require some kind of balance otherwise they reach lower levels of equilibrium. Intelligence cannot mean short period maximization of revenues and must be connected with medium term temporal horizon. Intelligence is strengthened by wealth, both as a reaction and as a perfunctory good supported by the very wealth.

Remark that under the parameters stated in (20) without intelligence nature vanishes, hence wealth will be totally artificial. The result is destructive and entrains insane development at the expense of the environment and its beauty: this is a risk well recognized by ecologists and nowadays also by many stakeholders. What is somewhat paradoxical is that in the absence of nature also wealth will suffer or vanish while the "intelligent" control will become overwhelming. The economic sustainability is destroyed by an excess of control, and is not compensated by natural and artistic resources. In the absence of wealth intelligence becomes needless, hence a dreamy natural beauty eliminates the need for controls in a model that is seemingly idyllic, but is born in the absence of productive instances (golden Age): ataraxic in the contemplation of unproductive beauty.

Provided we replace nature by beauty, the model may easily be applied with reference to resorts and also to many forms of artistic production such as literature, figurative arts, music, movies. The art, for being readable by a wide audience, must humanize, otherwise it remains closed within the ivory tower of critics. So there is a broad fuzzy band of uncertainty between art and bad taste (compare Eco [11]) , and also this band, in turn, changes with time. Often the remains of bad taste become extremely demanded, as was the case with Art Nouveau or neo-medieval reconstructions. Here beauty synthesizes art, creativity, originality; wealth distribution and sale to many users (to be motivated and convinced); intelligence critical skills and reaction to mass production: the tripartite model recommends not indulging in the mannerism of uncritical reproduction of what people like and can easily be sold, and at the same time recognizes the need for a certain amount of bad taste to stimulate innovation and creativity, since the reaction to decadence fertilizes minds more than the mere imitation of the beautiful, when devoid of novelty.

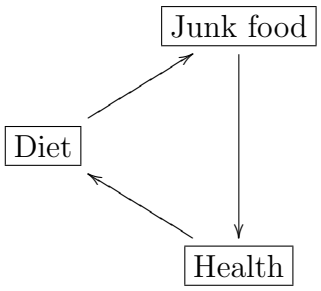
A scheme similar to (23) holds nowadays for the food. The role nature contains also biodiversity, tradition; that of wealth means production/distribution of commodities (also acquired by biotechnology), loss of biodiversity, slavery of multinational corporations; that of intelligence involves ecological private and public consciousness. The model is therefore

(24)



or its more personal variant

(25)



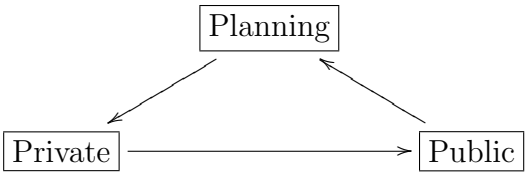
This model expounds that the difference in the speed with which different groups of consumers have reacted to the mass production of commodities is due to the dose of ecology present at the outset. The evolution towards equilibrium is endogenous, and is not only due to an imported mode. It should be noted, moreover, that the model affirms the necessity of commodities, otherwise the regression to a state of paleo-naturalness would lead to starvation as well as to the loss of ecological consciousness (as indeed it has for many thousands of years).

3.2. Top-down control

We now turn to models with top-down control. When systems are troubled by overly individualistic subsystems that tend to degenerate into self-interest and chaos [8], the key is to restore order. A decision maker (unit or team) is required. Other situations of conflict arise when decisions are taken with contrasting temporal horizons (often private prefers short period, while public can afford medium period) or when a solution with low return and high probability is preferred to a solution of high return and low probability (game theory with all its paradoxes).

The basic model is that of the town-planning scheme, where the public interest is undermined by the interests of the private sector. The control is exercised by the planning scheme.

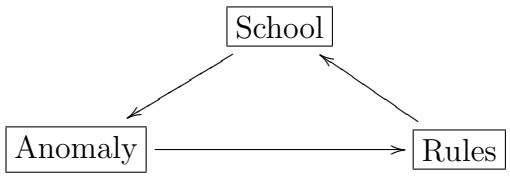
(26)



A variant of the model (26) may be found in semiotics and linguistics, where the anomaly of particularism tends to destroy the established order and the rules of analogy. Control instruments include the school, the academy, and the media: remark that nowadays Google has acquired an almost monopolistic position.

[8] A reference to Hirschmann's work ([15]) is essential, while the transitions between various states have been widely discussed in [29].

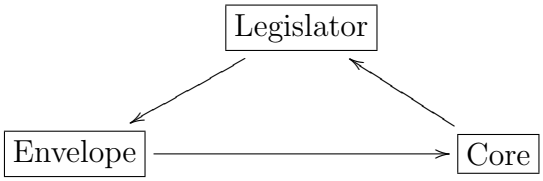
(27)



The organization of most forms of structured education obey this scheme. It can be remarked that also leisure, tourism [9], and environment discovery are covered by this scheme, where the role of the school is carried out by the various forms of guidance (book, network, organizer, personal trainer); a user's autonomous position could be envisaged in a generalized model (32). Also the alimentary education and its economic spin-off according to [9] can belong to scheme (27).

A rich model in this section is the Kernel model. The authors took the definition from the book by Lorch [20] where he explained that his is a kernel not a hull presentation, what made his book a lifelong companion for functional analysis. In any form of network, both physical and conceptual, there is the contrast between the core of the system and its developments and additions (envelope), so that some legislator [10] must govern it. The model is as follows

(28)



In particular, in the absence of a legislator the core becomes unreadable and unrecognizable due to the complexity of the accretions. In the teaching the over determination typical of most textbooks is overcome by good teachers that play the role of a legislator. In the case of networks a rearrangement was attempted, at least at the cognitive level, by Barabasi ([1], but had no consequences. In literature we can recall the wonderful (short) story of Calvino in Invisible Cities ([5]) where Ersilia, fourth among the "Town and exchanges", is characterized by a network of colored wires that expresses all the relationships in the city: when the network becomes incomprehensible the city is abandoned, and the new city begins with the core of a new network. Encyclopedic knowledge tends to destroy the structured knowledge and diagrams recall Calvino's cities. In [2] this defect is attributed to the absence of a clear conceptual scheme, according to the scientific statements of [24] and [26].

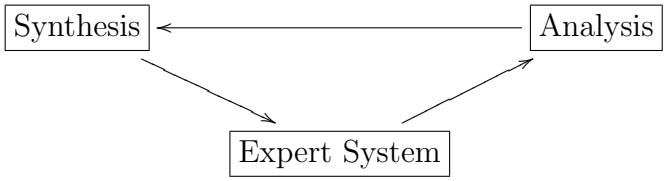
[9]The organization and falsification of touristic "authenticity" is considered in [22] where typical shows and trips *ad usum delphini* are described and their didactic necessity is explained. Ejarque in [12] follows a similar path, applying it to the artificial development of a tourist city.

[10]This is the model of Corpus Iuris due to Emperor Justinian, who, according to Dante's Divine Comedy (see [8], Par. 6, 12), remarks: "*I pruned the law of waste, excess, and sham*".

3.3. Bottom-up control

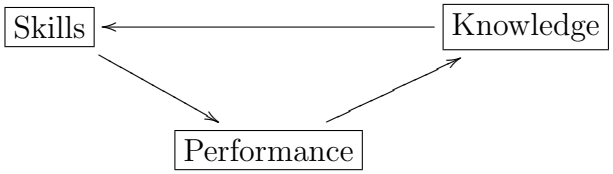
The core model (28) has some consequences as was pointed out by one of the fathers of artificial intelligence, namely the Nobel laureate H. Simon, as he recalls in his autobiography ([45]) with reference to [44]. Actually he says academy prefers teaching analysis, where it is possible to state sound basis and methods that can be learnt by the pupils, rather than proposing synthesis, where rules are fuzzy. Synthesis is left to hospital training, to projects, to auditing, in general to non structured science practical. The solution lies in artificial intelligence (today we prefer to say "expert systems"), where the need for explicit rules bursts out. Of course not every situation can be solved [11], but a good expert system that works at 95% nowadays is usually accepted, since the supervision requires much less time than starting from scratch. Typical cases are pattern and voice recognition and translation between different languages. Simon's scheme can thus be expressed as

(29)



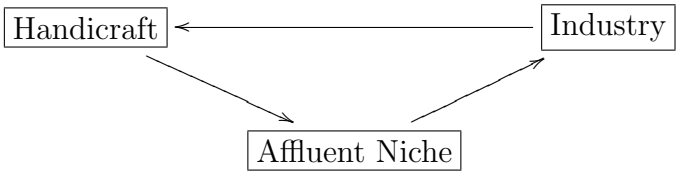
General purpose everyday activities require skills ready for use, so that theoretical knowledge is considered a hindrance rather than a help [12].

(30)



A somewhat similar effect is achieved in production, where coasty skills may be saved only in a market niche of affluent customers:

(31)

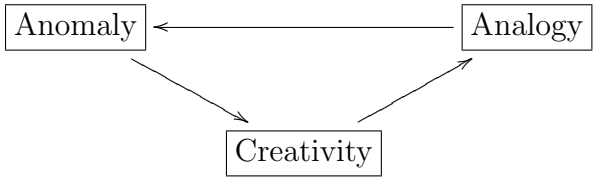


[11]This is in accordance with phenomenological psychology in comparison with cognitivism, as pointed by Vicario ([47])

[12]That is true also in mathematics. Vicario [46] reported of an experiment of Kanisza at the University of Trieste: a group of students were shown examples of solutions of some problems, a second group were taught the underlying theory. In the presence of a new similar problem the students of the first group easily succeeded while in the second group very few could understand what to do (and this was expected). But there was a third group that were shown the examples and then taught the theory. Instead of finding the best solution, the third group scored worse than the purely practical group, even if better than the theoretical group.

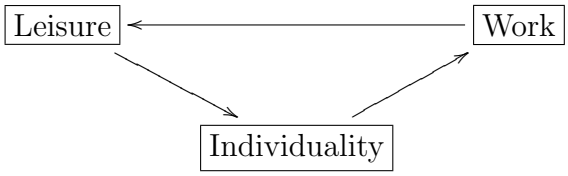
We come finally to the reaction of liberty against the rules that sterilize it. The model is that of subjective creation. Creativity enhances the anomaly in the conformist and well-ordered world of rule and analogy.

(32)



A model essential for a balanced life is the use of time and its economic assessment [13]. The productive working time destroys the time of leisure, if there is no instance of individuality to protect it.

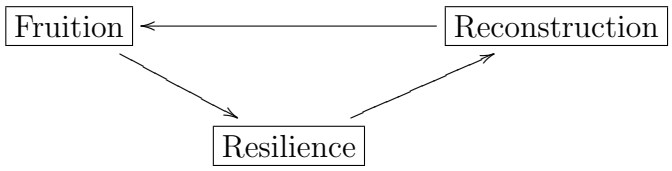
(33)



The model also says that individual freedom can manifest itself fully only in the balance between work and leisure, in agreement with the issues of vacation time and retirement [14].

It is also interesting to remark that in case of catastrophes the resilience operates in the opposite direction with respect to the long term planning of equation (26), since the prey in this case is the immediate survival (hence a private need) rather that the reconstruction from scratch. The contrast between (26) and (34) becomes evident in retrospect, where usually any solution is criticized.

(34)



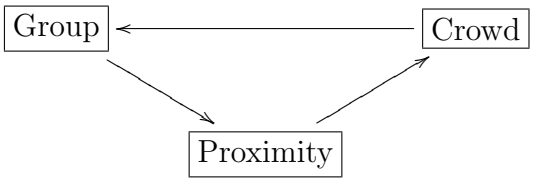
For researchers an important question concerns the optimal dimension of research groups. The tendency to huge research groups not always leads to success in pure science, while it can help in applied sciences where many different competences are required. But also in this case some clustering is required, in order to prepare interdisciplinar contacts. Proximity should avoid that groups get confused in crowds as it happens in world general congresses. In the case of scientific research proximity is not always physical, but usually it helps, just as it happens in territorial phenomena. The difficulty of defining structural proximity becomes

[13]Forerunners to this analysis were Becker and De Grazia (see [10] and [3], with important progress in [4])

[14]The characteristics of the tourist were fully analized in [23], while in [18] many economic implications were derived.

even greater in economic analysis, as it was shown in Section 3.1 of [7]. Positive effects of some special cases of clustering in random Bak Sneppen evolution models were discussed in Section 4 of [32]. The scheme is

(35)



This local frame explains in particular how some research and development structures work when they resort to brainstorming, official or informal: the members must be not too little, but also not too many, as explained in [41] and [36], and described in a glamorous example in [43]. The most striking case is perhaps the arising of Institute for Advanced Study of Princeton (compare Flexner [14]).

4. Conclusions

Paulo maiora canamus
Let us sing something nobler
Vergilius, Bucolics, 4,1

Models of Lotka-Volterra type present many practical problems of estimation of the parameters, beginning with the carrying capabilities of the single variables. In particular a tripartite model can find sharp application only in the case of biological defense of some natural species, where the three actors are homogeneous. An accurate discussion of the way of choosing and measuring parameters between non homogeneous data can be found in [25].

Simplified models represent somehow a synthesis according to platonic ideas of what a perfect abstract science should look for. Reality does not strictly obey to the abstract model, that expresses only a potentiality, but not actual reality. A summary of this interpretation of platonic ideas related to mathematics can be found in the Section 4 of the fundamental article by Enriques [13]. In many practical cases a model can suggest various forms of scenario analysis, that show a fuzzy panorama of possible realizations of an idea. Also in our paper the reader can expand the research to check what change of parameters (in particular dropping symmetry of equations) implies.

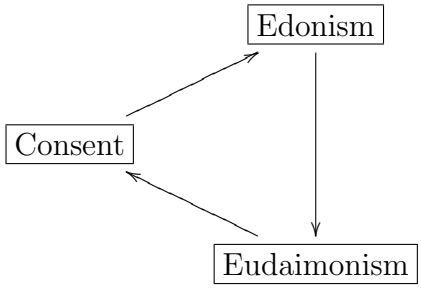
From a philosophical point of view it is also worth to observe that the tripartite antagonist frame can be easily referred to the long lasting competition between Epicurism and Stoicism, and their search for human happiness. While Stoicism searched it in the transformation beyond the self, also in everyday life, Epicurism followed the suggestion "lathe biosas" (live hidden, avoid public life). Both philosophical theories aim at man's well-being, although in the latter case this is through (moderate) hedonism, whereas in the former it is achieved through Aristotle's eudaimonia: the promotion of "self acceptance, positive relationships with others, personal growth, purpose in life, ability to realize everyday demands,

and autonomy", as summarized by Kalvand in [19]. Strangely enough, in the modern age, hedonism and its limits are well studied by psychologists, whilst the heavier eudaimonia (that entrains also prestige, self-realization, power) has started to be really recognized only in the last few years. Some essential references for it are the recent [40] and [39].

The approach to the psychological competition between eudaimonism and hedonism is essentially the same of the master model (23), where intelligence is replaced by the search for socio-cultural consent. This consent actually is a mixed control, since no longer stems only from authority or religion, but is founded on public opinion through information and nets. Recall that a first strong privatization of control arose with the diffusion of photography in the "roaring Twenties" when the classical pair "Private vices, public virtues" could no longer save the privacy of vices. More generally happiness is not confined to personal well-being alone, but relies also on the surrounding world, both at the macro and micro levels: city and landscape, nature, art, friends, hobbies, sport. The competition between personal and public happiness involves also this world: its solution, in terms of achieving a non-traumatic equilibrium, requires a control that can be top-down (legislator), or bottom-up (social network, market).

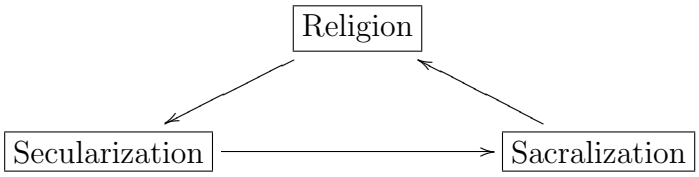
The model can be represented as

(36)



When consent arises from religion or authority the control may be considered top-down. In this case the role of hedonism is taken by a general, skeptical, secularization, while the role of eudaimonism is assumed by sharper forms of sacralization, so that the scheme becomes

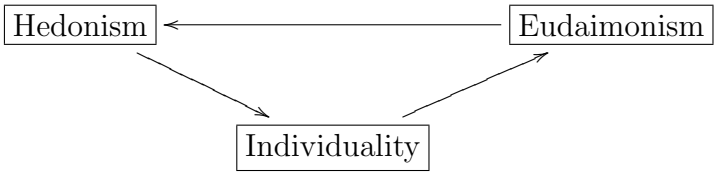
(37)



This may hold also in more profane forms of beliefs. An important feature of the model is that control may start from very small groups and then burst out, while in some cases the absence of any control leads to the vanishing of the prey (in this case sacralization). On the other side, the absence of secularization may lead to reduce the interior strength of the religion, confining it to a pure sequence of formal rules and rites.

Attention is required in using scheme (36) and specially its more severe form (37) since it should be used for lighthearted persons and pupils, whilst for severe persons the scheme ought to be turned into a bottom-up reverse control of type (33) so that it may become hedonism-preserving, otherwise private life could actually suffer exceeding restrictions. Also public evolution might be hindered, as in the case of European austerity, since the excess of constraints can lead to stagnation. The defense must start from the bottom, so we have the scheme

(38)



A form of this reaction is described in the book of Ordine [OR] where the schemes (32) and (33) are superimposed. The author proves the utility of the useless, both in fine arts/literature and in science, since unconditioned ideas are usually required at the basis of new creations.

In particular, there is a contrast between the freedom of artistic creation and the preservation of tradition or nature. On this basis, some renowned monuments such as the Sacra di San Michele, which crowns the mountains at the entrance of the Susa Valley near Turin, would never have been built. In the modern age it would not have been allowed to construct the Villa Malaparte in Capri, the stunning work of A. Libera that seems to timelessly complement the rocky seascape. The fishing and sailor villages of the Amalfi Coast or Cinque Terre would never have developed. To the natural landscape would be admitted, albeit reluctantly, only ports, dams, chair lifts or cable car stations, and only a few works of major artistic and cultural value.

A systemic view, with attention to all the participants, present and future, becomes essential although all too often there is a tendency of the stakeholders to shine a light only on the little stretch of the route that most moves them personally, and offers the greatest return of personal and political prestige.

The great risk of such a shorteyed view is that catastrophes can suddenly arise. The last example of Section 2 (Fig. 8(b)) shows clearly what can happen. At the beginning it seems that there exists only the predator-prey component, and that it is stable, apart some small deviations that can be attributed to random perturbations. Some hidden forces that act as a bottom-up controls can develop up to a non controllable level before the global predator-prey system reaches a perturbation, say of 10% of the equilibrium value; in the example 10% deviation is reached when the control has already reached about 50% of its capacity level, so that it becomes very costly to eradicate it. Remark also that its growth at the moment of the discovery is at the highest speed. Of course a faster detection is possible when the attention level is put at 5% of deviation from the norm or less. But random phenomena and a path that seems to approximate the equilibrium level may make the early discovery difficult. This happens in medicine, in territorial geology, in climate control, in social and political revolutions.

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GLOBAL ASYMPTOTIC STABILITY FOR A STOCHASTIC COMPETITION AND COOPERATION MODEL OF TWO ENTERPRISES

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Abstract. This paper deals with a stochastic competition and cooperation model of two enterprises. Some very verifiable criteria on the global stability of the positive equilibrium of the deterministic system are established. An example with its computer simulations is given to illustrate our main theoretical findings.

Keywords: competition and cooperation model; global stability; stochastic perturbation; enterprise.
2000 Mathematics Subject Classification: 60H30, 60H10, 92D25.

1. Introduction

Competition and cooperation of two enterprises has been the hot topic of international community, many governments and researchers. During the past several decades, numerous authors have widely investigated enterprises' inborn path, operating mechanism, management process, risk and efficiency, evolvement cycle, economic performance and so on [1]. For example, Tian and Nie [2] investigated the competition and cooperation model of two enterprises based on ecosystem

$$\begin{cases} \dfrac{dx_1(t)}{dt} = r_1(t)x_1(t)\left[1 - \dfrac{x_1(t)}{K} - \dfrac{\alpha(x_2(t)-c_2)^2}{K}\right], \\ \dfrac{dx_2(t)}{dt} = r_2(t)x_2(t)\left[1 - \dfrac{x_2(t)}{K} - \dfrac{\beta(x_1(t)-c_1)^2}{K}\right], \end{cases} \tag{1.1}$$

where $x_1(t), x_2(t)$ represent the output of interprises A and B, r_1, r_2 are the intrinsic growth rates, K denotes the carrying capacity of mark under nature unlimited conditions, α, β are the competitive coefficient of two enterprises, c_1, c_2 are the initial production of two enterprises. Let $a_1 = \frac{r_1}{K}, a_2 = \frac{r_2}{K}, b_1 = \frac{r_1\alpha}{K}, b_2 = \frac{r_2\beta}{K}$ in system (1.1). Then system (1.1) takes the form

$$\begin{cases} \dfrac{dx_1(t)}{dt} = x_1(t)[r_1 - a_1x_1(t) - b_1(x_2(t) - c_2)^2], \\ \dfrac{dx_2(t)}{dt} = x_2(t)[r_2 - a_2x_2(t) + b_2(x_1(t) - c_1)^2]. \end{cases} \tag{1.2}$$

In real world, two enterprises are constantly in the competition and cooperation, and when an enterprise suffers damage from another one by competition, another one could benefit, and two enterprises can simultaneously benefit by cooperation, meantime, the duration time of output for enterprises would also play an important role. Motivated by the viewpoint, Liao et al. [3] modified system (1.2) as follows

$$\begin{cases} \dfrac{dx_1(t)}{dt} = x_1(t)[r_1 - a_1x_1(t - \tau_1) - b_1(x_2(t - \tau_2) - c_2)^2], \\ \dfrac{dx_2(t)}{dt} = x_2(t)[r_2 - a_2x_2(t - \tau_1) + b_2(x_1(t - \tau_2) - c_1)^2], \end{cases} \tag{1.3}$$

where τ_1 is nonnegative constant which stands for the gestation periodic of production for two enterprises, τ_2 in the first equation of the system (1.3) represents the block delay of enterprise B to A, and τ_2 in the second equation of the system (1.3) represents the promoting delay of enterprise A to B. By choosing the two delays τ_1 and τ_2 as bifurcation parameters, Liao et al. [3] analyzed the effect of different delays on the dynamics of model (1.3). If $\tau_1 = \tau_2 = \tau$, then model (1.3) takes the form

$$\begin{cases} \dfrac{dx_1(t)}{dt} = x_1(t)[r_1 - a_1x_1(t - \tau) - b_1(x_2(t - \tau) - c_2)^2], \\ \dfrac{dx_2(t)}{dt} = x_2(t)[r_2 - a_2x_2(t - \tau) + b_2(x_1(t - \tau) - c_1)^2]. \end{cases} \tag{1.4}$$

By regarding the time delay τ as bifurcation parameter, Liao et al. [4] investigated the stability and Hopf bifurcation nature of model (1.4).

In 2012, Li and Zhang [1] studied the model (1.2) with nonconstant coefficients, which can be derived as a modified model (1.2) with variable coefficients

$$\begin{cases} \dfrac{dx_1(t)}{dt} = x_1(t)[r_1(t) - a_1(t)x_1(t) - b_1(t)(x_2(t) - c_2(t))^2], \\ \dfrac{dx_2(t)}{dt} = x_2(t)[r_2(t) - a_2(t)x_1(t) + b_2(t)(x_1(t) - c_1(t))^2]. \end{cases} \tag{1.5}$$

Using coincidence degree theory and differential inequality theory, Xu [5] established some sufficient conditions to ensure the existence of periodic solutions of model (1.5).

Considering that the realistic models require the inclusion of the effect of changing environment and the output of enterprises A and B are subjected to rapid change at certain instants in time, Xu and Shao [6] focused on the existence and global attractivity of periodic solution for the following enterprise clusters based on ecology theory with impulse and varying coefficients

$$(1.6)\quad \begin{cases} \dfrac{dx_1(t)}{dt} = x_1(t)[r_1(t) - a_1(t)x_1(t) - b_1(t)(x_2(t) - c_2(t))^2],\ t \neq t_k, \\ \dfrac{dx_2(t)}{dt} = x_2(t)[r_2(t) - a_2(t)x_1(t) + b_2(t)(x_1(t) - c_1(t))^2],\ t \neq t_k, \\ \Delta_i(t_k) = x_i(t_k^+) - x_i(t_k^-) = -\gamma_{ik}(t_k),\ i = 1, 2,\ k = 1, 2, ..., q, \end{cases}$$

where $\Delta_i(t_k) = x_i(t_k^+) - x_i(t_k^-)$ denotes the impulses at moments t_k and $t_1 < t_2 < \cdots$ is a strictly increasing sequence such that $\lim_{k\to+\infty} t_k = +\infty$ and q is a positive integer. Applying continuation theorem of coincidence degree theory, Xu and Shao [6] obtained a set of sufficient conditions that ensure the existence of at least a positive periodic solution, and by constructing a suitable Lyapunov functional, the uniqueness and global attractivity of the positive periodic solution are included.

Numerous scholars argue [7]-[18] that in the real word the enterprises usually live in a changing circumstances which have important effect on the intrinsic growth rates. So we can think that the intrinsic growth rate can be expressed as an average rate plus an error term. In view of the well known central limit theorem, we know that the error term follows a normal distribution [19]-[21]. Therefore the error term can be denoted by a white noise $\sigma_i(t)\dot{W}_i(t)(i = 1, 2)$, where $\sigma_i^2(t)$ denotes the intensity of the noise and $\dot{W}_i(t)$ denotes a standard white noise, where $W_i(t)$ represents a Brownian motion defined on a complete probability space (Ω, F, P). Motivated by the discussed above, we can let the growth rates r_1 and r_2 become $r_1 + \sigma_1\dot{W}_1(t)$ and $r_2 + \sigma_2\dot{W}_2(t)$, respectively. In view of model (1.1), we can establish the stochastic competition and cooperation model of two enterprises which takes the form:

$$(1.7)\quad \begin{cases} dx_1 = x_1(t)\left[1 - \dfrac{x_1(t)}{K} - \dfrac{\alpha(x_2(t) - c_2)^2}{K}\right][r_1dt + \sigma_1dW_1(t)], \\ dx_2 = x_2(t)\left[1 - \dfrac{x_2(t)}{K} - \dfrac{\beta(x_1(t) - c_1)^2}{K}\right][r_2dt + \sigma_2dW_2(t)]. \end{cases}$$

The initial conditions of system (1.7) are given by

$$(1.8)\qquad x_{10} = \varphi_1(0) > 0, x_{20} = \varphi_2(0) > 0.$$

The remainder of the paper is organized as follows: in Section 2, some sufficient conditions for the existence of global positive solution of system (1.7) with the initial conditions (1.8) are established. Moreover, the sufficient condition is obtained to ensure that the equilibrium of system (1.7) with the initial conditions (1.8) is globally asymptotically stable. In Section 3, an example with its computer simulations is given to illustrate the feasibility and effectiveness of our main results. A brief conclusion is drawn in Section 4.

2. Main results

In this section, we shall present our main results.

Theorem 2.1 *For any given initial value* $(x_{10}, x_{20}) \in R^2_+$, *system* (1.7) *has a unique global solution* $(x_1(t), x_2(t))$ *almost sure* (*a.s.*), *where* $R^2_+ = \{x_1 > 0,\ x_2 > 0\}$.

Proof. Consider the following system

$$
\left\{
\begin{aligned}
dh_1 = &\left\{ r_1 \left[1 - \frac{e^{h_1}}{K} - \frac{\alpha(e^{h_2} - c_2)^2}{K} \right] \right. \\
&\left. - \frac{\sigma_1^2}{2} \left[1 - \frac{e^{h_1}}{K} - \frac{\alpha(e^{h_2} - c_2)^2}{K} \right]^2 \right\} dt \\
&+ \sigma_1 \left[1 - \frac{e^{h_1}}{K} - \frac{\alpha(e^{h_2} - c_2)^2}{K} \right] dW_1(t), \\
dh_2 = &\left\{ r_2 \left[1 - \frac{e^{h_2}}{K} - \frac{\beta(e^{h_1} - c_1)^2}{K} \right] \right. \\
&\left. - \frac{\sigma_2^2}{2} \left[1 - \frac{e^{h_2}}{K} - \frac{\beta(e^{h_1} - c_1)^2}{K} \right]^2 \right\} dt \\
&+ \sigma_2 \left[1 - \frac{e^{h_2}}{K} - \frac{\beta(e^{h_1} - c_1)^2}{K} \right] dW_2(t)
\end{aligned}
\right. \tag{2.1}
$$

with initial value $h_{10} = \ln x_{10}, h_{20} = \ln x_{20}$.

Obviously, the coefficients of (2.1) satisfy the local Lipschitz condition, then there is unique local solution $(h_1(t), h_2(t))$ on $[0, \tau_e)$, where τ_e is the explosion time. Thus, in view of Itö's formula, $(x_1(t), x_2(t)) = (e^{h_1(t)}, e^{h_2(t)})$ is the unique positive local solution to system (1.7) with the initial value $x_{10} > 0, x_{20} > 0$.

Assume that $k_0 > 0$ is sufficiently large such that x_{10} and x_{20} lying within $[\frac{1}{k_0}, k_0]$. For integer $k > k_0$, define the stopping times

$$
\tau_k = \inf \left\{ t \in [0, \tau_e] : x_1(t) \notin \left(\frac{1}{k}, k \right) \ or\ x_2(t) \notin \left(\frac{1}{k}, k \right) \right\}. \tag{2.2}
$$

Let $\tau_\infty = \lim_{k \to \infty} \tau_k$. Then $\tau_\infty \leq \infty$ a.s. To complete the proof of Lemma 2.1 we need only to prove $\tau\infty = \infty$. Assume that the statement does not hold, then there exists a constant $T > 0$ and $\epsilon \in (0, 1)$ such that $P\{\tau_\infty \leq T\} > \epsilon$. Thus there is an integer $k_1 \geq k_0$ such that $P\{\tau_k \leq T\} \geq \epsilon$. We define

$$
V(x_1, x_2) = (\sqrt{x_1} - 1 - 0.5 \ln x_1) + (\sqrt{x_2} - 1 - 0.5 \ln x_2). \tag{2.3}
$$

If $(x_1(t), x_2(t)) \in R^2_+$, then

$$
\begin{aligned}
dV(x_1, x_2) \ = \ & 0.5 r_1 (\sqrt{x_1} - 1) \left[1 - \frac{x_1}{K} - \frac{\alpha(x_2 - c_2)^2}{K} \right] dt \\
& + \frac{\sigma_1^2 (2 - \sqrt{x_1})}{8} \left[1 - \frac{x_1}{K} - \frac{\alpha(x_2 - c_2)^2}{K} \right]^2 dt \\
& + 0.5 \sigma_1 (\sqrt{x_1} - 1) \left[1 - \frac{x_1}{K} - \frac{\alpha(x_2 - c_2)^2}{K} \right] dW_1(t) \\
& + 0.5 r_2 (\sqrt{x_2} - 1) \left[1 - \frac{x_2}{K} - \frac{\beta(x_1 - c_1)^2}{K} \right] dt
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\sigma_2^2(2-\sqrt{x_2})}{8}\left[1-\frac{x_2}{K}-\frac{\beta(x_1-c_1)^2}{K}\right]^2 dt\\
&+0.5\sigma_2(\sqrt{x_2}-1)\left[1-\frac{x_2}{K}-\frac{\beta(x_1-c_1)^2}{K}\right]dW_2(t)\\
=\;& 0.5r_1\left[\sqrt{x_1}-\frac{x_1^{\frac{3}{2}}}{K}-\frac{\sqrt{x_1}\alpha(x_2-c_2)^2}{K}-1+\frac{x_1}{K}+\frac{\alpha(x_2-c_2)^2}{K}\right]dt\\
&+0.5r_2\left[\sqrt{x_2}-\frac{x_2^{\frac{3}{2}}}{K}-\frac{\sqrt{x_2}\beta(x_1-c_1)^2}{K}-1+\frac{x_2}{K}+\frac{\beta(x_1-c_1)^2}{K}\right]dt\\
&+\frac{\sigma_1^2}{8}\left[2+\frac{2x_1^2}{K^2}+\frac{2\alpha^2(x_2-c_2)^4}{K^2}-\frac{4x_1+4\alpha(x_2-c_2)^2}{K}\right.\\
&\left.+\frac{4\alpha x_1(x_2-c_2)^2}{K^2}-\sqrt{x_1}-\frac{x_1^{\frac{3}{2}}}{K^2}-\frac{\alpha^2\sqrt{x_1}(x_2-c_2)^2}{K^2}+\frac{2x_1^{\frac{3}{2}}}{K}\right.\\
&\left.+\frac{2\alpha\sqrt{x_1}(x_2-c_2)^2}{K}-\frac{2x_1^{\frac{3}{2}}\alpha(x_2-c_2)^2}{K^2}\right]dt\\
&+\frac{\sigma_2^2}{8}\left[2+\frac{2x_2^2}{K^2}+\frac{2\beta^2(x_1-c_1)^4}{K^2}-\frac{4x_2+4\beta(x_1-c_1)^2}{K}\right.\\
&\left.+\frac{4\alpha x_2(x_1-c_1)^2}{K^2}-\sqrt{x_2}-\frac{x_2^{\frac{3}{2}}}{K^2}-\frac{\beta^2\sqrt{x_2}(x_1-c_1)^2}{K^2}+\frac{2x_2^{\frac{3}{2}}}{K}\right.\\
&\left.+\frac{2\beta\sqrt{x_2}(x_1-c_1)^2}{K}-\frac{2x_2^{\frac{3}{2}}\beta(x_1-c_1)^2}{K^2}\right]dt\\
&+0.5\sigma_1(\sqrt{x_1}-1)\left[1-\frac{x_1}{K}-\frac{\alpha(x_2-c_2)^2}{K}\right]dW_1(t)\\
&+0.5\sigma_2(\sqrt{x_2}-1)\left[1-\frac{x_2}{K}-\frac{\beta(x_1-c_1)^2}{K}\right]dW_2(t)\\
\leq\;& 0.5r_1\left[\sqrt{x_1}+\frac{x_1}{K}+\frac{\alpha(x_2-c_2)^2}{K}\right]dt\\
&+0.5r_2\left[\sqrt{x_2}+\frac{x_2}{K}+\frac{\beta(x_1-c_1)^2}{K}\right]dt\\
&+\frac{\sigma_1^2}{8}\left[\frac{2x_1^2}{K^2}+\frac{2\alpha^2(x_2-c_2)^4}{K^2}+\frac{4\alpha x_1(x_2-c_2)^2}{K^2}\right.\\
&\left.+\frac{2x_1^{\frac{3}{2}}}{K}+\frac{2\alpha\sqrt{x_1}(x_2-c_2)^2}{K}+2\right]dt\\
&+\frac{\sigma_2^2}{8}\left[\frac{2x_2^2}{K^2}+\frac{2\beta^2(x_1-c_1)^4}{K^2}+\frac{4\beta x_2(x_1-c_1)^2}{K^2}\right.\\
&\left.+\frac{2x_2^{\frac{3}{2}}}{K}+\frac{2\beta\sqrt{x_1}(x_1-c_1)^2}{K}+2\right]dt
\end{aligned}
$$

$$
\begin{aligned}
&+0.5\sigma_1(\sqrt{x_1}-1)\left[1-\frac{x_1}{K}-\frac{\alpha(x_2-c_2)^2}{K}\right]dW_1(t)\\
&+0.5\sigma_2(\sqrt{x_2}-1)\left[1-\frac{x_2}{K}-\frac{\beta(x_1-c_1)^2}{K}\right]dW_2(t)\\
\leq\ &\chi_1 dt+\chi_2 dt\\
&+0.5\sigma_1(\sqrt{x_1}-1)\left[1-\frac{x_1}{K}-\frac{\alpha(x_2-c_2)^2}{K}\right]dW_1(t)\\
&+0.5\sigma_2(\sqrt{x_2}-1)\left[1-\frac{x_2}{K}-\frac{\beta(x_1-c_1)^2}{K}\right]dW_2(t),
\end{aligned}
\tag{2.4}
$$

where χ_1 and χ_2 are positive numbers. Integrating both sides of (2.4) from 0 to $t_k \wedge T$, and taking the expectations, we have

$$
EV(x_1(\tau_k \wedge T), x_2(\tau_k \wedge T)) \leq V(x_{10}, x_{20}) + (\chi_1+\chi_2)T. \tag{2.5}
$$

Set $\Phi_k = \{\tau_k \leq T\}$, then it follows that $P(\Phi_k) \geq \epsilon$. Since for every $\varpi \in \Phi_k$, there exists some i such that $x_i(\tau_k, \varpi)$ is equal to either k or $\frac{1}{k}$ for $i = 1, 2$. Thus $V(x_1(\tau_k, \varpi), x_2(\tau_k, \varpi)$ is no less than

$$
\min\left\{\left(\sqrt{k}-1-0.5\ln k\right), \left(\sqrt{\frac{1}{k}}-1-0.5\ln\frac{1}{k}\right)\right\}. \tag{2.6}
$$

In view of (2.5), we have

$$
\begin{aligned}
&V(x_{10}, x_{20}) + (\chi_1+\chi_2)T \geq E[1_{\Phi_k}(\varpi)V(x_1(\tau_k), x_2(\tau_n))]\\
&\geq \epsilon \min\left\{\left(\sqrt{k}-1-0.5\ln k\right), \left(\sqrt{\frac{1}{k}}-1-0.5\ln\frac{1}{k}\right)\right\},
\end{aligned}
\tag{2.7}
$$

where 1_{Φ_k} stands for the indicator function of Φ_k. Let $k \to \infty$, then

$$
\infty > V(x_{10}, x_{20}) + (\chi_1+\chi_2)T = \infty, \tag{2.8}
$$

which is a contradiction. The proof of Theorem 2.1 is complete. ∎

Let $E(x_1^*, x_2^*)$ be the positive equilibrium of system (1.7), then x_1^* and x_2^* satisfy the following equations

$$
\begin{cases}
1-\dfrac{x_1^*}{K}-\dfrac{\alpha(x_2^*-c_2)^2}{K}=0,\\
1-\dfrac{x_2^*}{K}-\dfrac{\beta(x_1^*-c_1)^2}{K}=0.
\end{cases}
\tag{2.9}
$$

Now we will give the second main result of the paper.

Theorem 2.2 *Let*

$$
\begin{aligned}
\mu_1 &= -\left(\frac{r_1}{K}-\frac{\sigma_1^2 x_1^*}{2K}\right)+\frac{\sigma_2^2 x_2^* r_2^2 \beta^2 (x_1^*+K-2c_2)^2}{2K^2},\\
\mu_2 &= (1+\sigma_1^2 x_1^*)\frac{r_1\alpha|x_2^*+K-2c_2|}{K}+(1+\sigma_2^2 x_2^*)\frac{r_2\beta|x_1^*+K-2c_1|}{K},\\
\mu_3 &= -\left(\frac{r_2}{K}-\frac{\sigma_2^2 x_2^*}{2K}\right)+\frac{\sigma_1^2 x_1^* r_1^2 \alpha^2 (x_2^*+K-2c_2)^2}{2K^2}.
\end{aligned}
$$

If $\mu_1 < 0, 4\mu_1\mu_3 > \mu_2^2$, *then the equilibrium* (x_1^*, x_2^*) *of system* (1.7) *is globally asymptotically stable, that is, for any initial value* $x_{10}(0) > 0, x_{20}(0) > 0$, *the solution of system* (1.7) *satisfies*

$$\lim_{t\to+\infty} x_1(t) = x_1^*, \ \lim_{t\to+\infty} x_2(t) = x_2^*, a.s.$$

Proof. Define the following functions

$$V_1(x_1) = \int_0^{x_1-x_1^*} \frac{\theta}{\theta + x_1^*} d\theta, \tag{2.10}$$

$$V_2(x_2) = \int_0^{x_2-x_2^*} \frac{\theta}{\theta + x_2^*} d\theta. \tag{2.11}$$

By Itô's formula, we have

$$\begin{aligned}
LV_1(x_1) &= r_1(x_1 - x_1^*)\left[1 - \frac{x_1}{K} - \frac{\alpha(x_2 - c_2)^2}{K}\right] \\
&\quad + \frac{\sigma_1^2 x_1^*}{2}\left[1 - \frac{x_1}{K} - \frac{\alpha(x_2 - c_2)^2}{K}\right]^2 \\
&= r_1(x_1 - x_1^*)\left[\frac{1}{K}(x_1^* - x_1) + \frac{\alpha(x_2^* - c_2)^2}{K} - \frac{\alpha(x_2 - c_2)^2}{K}\right] \\
&\quad + \frac{\sigma_1^2 x_1^*}{2}\left[\frac{1}{K}(x_1^* - x_1) + \frac{\alpha(x_2^* - c_2)^2}{K} - \frac{\alpha(x_2 - c_2)^2}{K}\right]^2 \\
&= -\frac{r_1}{K}(x_1 - x_1^*)^2 - \frac{r_1\alpha(x_2^* + x_2 - 2c_2)}{K}(x_1 - x_1^*)(x_2 - x_2^*) \\
&\quad + \frac{\sigma_1^2 x_1^*}{2}\left[\frac{1}{K}(x_1 - x_1^*)^2 + \frac{r_1^2\alpha^2(x_2^* + x_2 - 2c_2)^2}{K^2}(x_2 - x_2^*)^2\right. \\
&\quad \left. - \frac{2r_1\alpha(x_2^* + x_2 - 2c_2)}{K^2}(x_1 - x_1^*)(x_2 - x_2^*)\right] \\
&\le -\left(\frac{r_1}{K} - \frac{\sigma_1^2 x_1^*}{2K}\right)(x_1 - x_1^*)^2 + (1 + \sigma_1^2 x_1^*)\frac{r_1\alpha|x_2^* + K - 2c_2|}{K} \\
&\quad \times |x_1 - x_1^*||x_2 - x_2^*| + \frac{\sigma_1^2 x_1^* r_1^2\alpha^2(x_2^* + K - 2c_2)^2}{2K^2}(x_2 - x_2^*)^2, \qquad (2.12) \\
LV_2(x_2) &= r_2(x_2 - x_2^*)\left[1 - \frac{x_2}{K} - \frac{\beta(x_1 - c_1)^2}{K}\right] \\
&\quad + \frac{\sigma_2^2 x_2^*}{2}\left[1 - \frac{x_2}{K} - \frac{\beta(x_1 - c_1)^2}{K}\right]^2 \\
&= r_2(x_2 - x_2^*)\left[\frac{1}{K}(x_2^* - x_2) + \frac{\beta(x_1^* - c_1)^2}{K} - \frac{\beta(x_1 - c_1)^2}{K}\right] \\
&\quad + \frac{\sigma_2^2 x_2^*}{2}\left[\frac{1}{K}(x_2^* - x_2) + \frac{\beta(x_1^* - c_1)^2}{K} - \frac{\beta(x_1 - c_1)^2}{K}\right]^2
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{r_2}{K}(x_2-x_2^*)^2 - \frac{r_2\beta(x_1^*+x_1-2c_1)}{K}(x_1-x_1^*)(x_2-x_2^*)\\
&\quad +\frac{\sigma_2^2x_2^*}{2}\left[\frac{1}{K}(x_2-x_2^*)^2+\frac{r_2^2\beta^2(x_1^*+x_1-2c_1)^2}{K^2}(x_1-x_1^*)^2\right.\\
&\quad \left.-\frac{2r_2\beta(x_1^*+x_1-2c_1)}{K^2}(x_1-x_1^*)(x_2-x_2^*)\right]\\
&\le -\left(\frac{r_2}{K}-\frac{\sigma_2^2x_2^*}{2K}\right)(x_2-x_2^*)^2+(1+\sigma_2^2x_2^*)\frac{r_2\beta|x_1^*+K-2c_1|}{K}\\
&\quad \times|x_1-x_1^*||x_2-x_2^*|+\frac{\sigma_2^2x_2^*r_2^2\beta^2(x_1^*+K-2c_2)^2}{2K^2}(x_1-x_1^*)^2. \qquad (2.13)
\end{aligned}
$$

Now we define

$$V(t)=V_1(x_1)+V_2(x_2). \qquad (2.14)$$

Then we get

$$
\begin{aligned}
LV(t) &= LV_1(x_1)+LV_2(x_2)\\
&\le -\left(\frac{r_1}{K}-\frac{\sigma_1^2x_1^*}{2K}\right)(x_1-x_1^*)^2+(1+\sigma_1^2x_1^*)\frac{r_1\alpha|x_2^*+K-2c_2|}{K}\\
&\quad \times|x_1-x_1^*||x_2-x_2^*|+\frac{\sigma_1^2x_1^*r_1^2\alpha^2(x_2^*+K-2c_2)^2}{2K^2}(x_2-x_2^*)^2\\
&\quad -\left(\frac{r_2}{K}-\frac{\sigma_2^2x_2^*}{2K}\right)(x_2-x_2^*)^2+(1+\sigma_2^2x_2^*)\frac{r_2\beta|x_1^*+K-2c_1|}{K}\\
&\quad \times|x_1-x_1^*||x_2-x_2^*|+\frac{\sigma_2^2x_2^*r_2^2\beta^2(x_1^*+K-2c_2)^2}{2K^2}(x_1-x_1^*)^2\\
&= \mu_1(x_1-x_1^*)^2+\mu_2|x_1-x_1^*||x_2-x_2^*|+\mu_3(x_2-x_2^*)^2. \qquad (2.15)
\end{aligned}
$$

Let $|Y-Y^*|=(|x_1-x_1^*|,|x_2-x_2^*|)^T$. By (2.15), we have

$$LV(t)\le\frac{1}{2}|Y-Y^*|^T\begin{bmatrix}2\varrho_1 & \varrho_2\\ \varrho_2 & 2\varrho_3\end{bmatrix}|Y-Y^*|. \qquad (2.16)$$

According to the conditions in Theorem 2.2, we know that $LV(t)<0$ along all trajectories in the first quadrant except (x_1^*,x_2^*). Then $\lim_{t\to+\infty}x_1(t)=x_1^*$ and $\lim_{t\to+\infty}x_2(t)=x_2^*$. The proof is complete. ∎

3. Computer simulations

In this section, we give an example to illustrate our main results obtained in previous sections by the Milstein method [22]. Consider the following stochastic competition and cooperation model of two enterprises

$$
(3.1)\qquad \begin{cases} x_{1,k+1} = x_{1,k} + r_1 x_{1,k}\left[1 - \dfrac{x_{1,k}}{K} - \dfrac{\alpha(x_{2,k}-c_2)^2}{K}\right]\Delta t \\ \qquad\qquad + \sigma_1 x_{1,k}\left[1 - \dfrac{x_{1,k}}{K} - \dfrac{\alpha(x_{2,k}-c_2)^2}{K}\right]\sqrt{\Delta t}\xi_k \\ \qquad\qquad + \dfrac{\sigma_1^2}{2} x_{1,k}\left[1 - \dfrac{x_{1,k}}{K} - \dfrac{\alpha(x_{2,k}-c_2)^2}{K}\right](\xi_k^2 - 1)\Delta t, \\ x_{2,k+1} = x_{2,k} + r_2 x_{2,k}\left[1 - \dfrac{x_{2,k}}{K} - \dfrac{\beta(x_{1,k}-c_1)^2}{K}\right]\Delta t \\ \qquad\qquad + \sigma_2 x_{2,k}\left[1 - \dfrac{x_{2,k}}{K} - \dfrac{\beta(x_{1,k}-c_1)^2}{K}\right]\sqrt{\Delta t}\eta_k \\ \qquad\qquad + \dfrac{\sigma_2^2}{2} x_{2,k}\left[1 - \dfrac{x_{2,k}}{K} - \dfrac{\beta(x_{1,k}-c_1)^2}{K}\right](\eta_k^2 - 1)\Delta t, \end{cases}
$$

where ξ_k and η_k are Gaussian random variables that follow $N(0,1)$. We choose $r_1 = 0.45, r_2 = 0.74, \alpha = 0.33, \beta = 0.63, K_1 = 2.4, k_2 = 1.7$. Let $\sigma_1^2 = 0.1, \sigma_2^2 = 0.3$. Then we can easily check that all the conditions of Theorem 2.2 are satisfied. Thus we can conclude that the equilibrium (x_1^*, x_2^*) is globally asymptotically stable which is illustrated in Figure 1 and Figure 2.

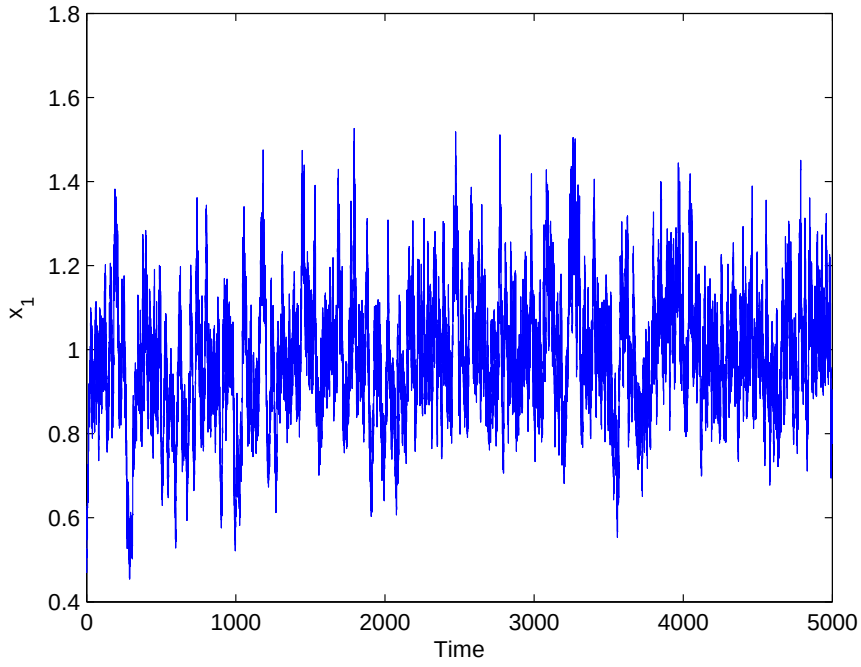


Figure 1: The solutions of system (3.1) with $\sigma_1^2 = 0.1$, $\sigma_2^2 = 0.3$.

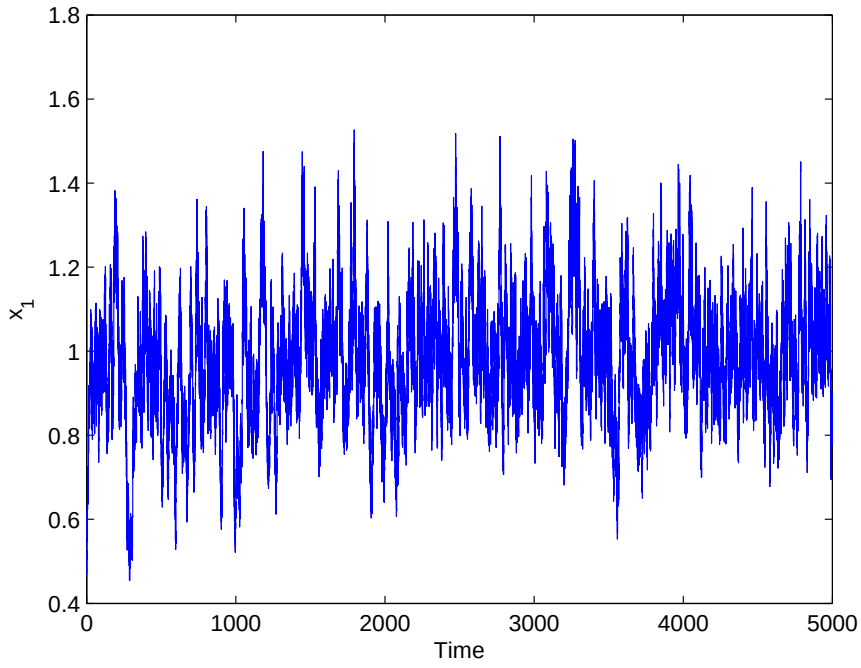


Figure 2: The solutions of system (3.1) with $\sigma_1^2 = 0.1$, $\sigma_2^2 = 0.3$.

4. Conclusions and further research

In this paper, we have investigated the global asymptotic stability of a stochastic competition and cooperation model of two enterprises. It is shown that under appropriate conditions, the stochastic competition and cooperation model has a unique global positive solutions for any initial values. Some sufficient conditions which guarantee the global stability of the stochastic model are derived. From the perspective of enterprise, the results play an prominent role in practical applications. The fact that a positive equilibrium is globally asymptotically stable implies that both enterprise could be co-exist. To the best of our knowledge, it is the first attempt to carry out such a investigation on competition and cooperation of a stochastic competition and cooperation model of two enterprises. In real society, we know that discontinuity often appears. Thus the investigation on the fractal competition and cooperation model of two enterprises has important value. This aspect will be our future work.

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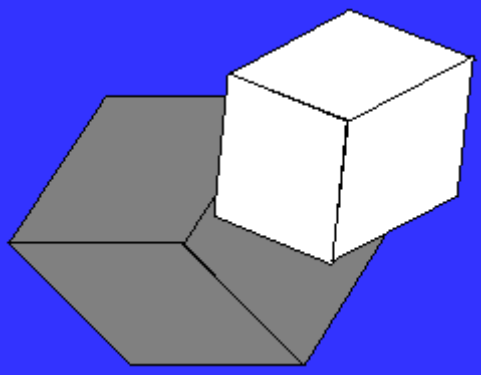
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