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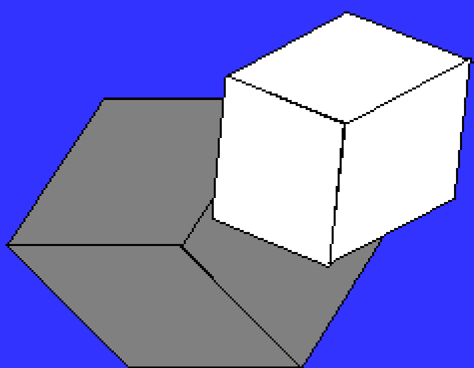

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A RECURSIVE FORMULA FOR POWER MOMENTS OF 2-DIMENSIONAL KLOOSTERMAN SUMS ASSOCIATED WITH GENERAL LINEAR GROUPS[1]

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Abstract. In this paper, we construct a binary linear code connected with the Kloosterman sum for $GL(2, q)$. Here q is a power of two. Then we obtain a recursive formula generating the power moments 2-dimensional Kloosterman sum, equivalently that generating the even power moments of Kloosterman sum in terms of the frequencies of weights in the code. This is done via Pless power moment identity and by utilizing the explicit expression of the Kloosterman sum for $GL(2, q)$.

Keywords: recursive formula, power moment, Kloosterman sum, 2-dimensional Kloosterman sum, general linear group, Pless power moment identity, weight distribution.

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1. Introduction

Let ψ be a nontrivial additive character of the finite field $\mathbb{F}_q$ with $q = p^r$ elements (p a prime), and let m be a positive integer. Then the m-dimensional Kloosterman sum $K_m(\psi; a)$([10]) is defined by

$$K_m(\psi; a) = \sum_{\alpha_1,\ldots,\alpha_m \in \mathbb{F}_q^*} \psi(\alpha_1 + \cdots + \alpha_m + a\alpha_1^{-1} \cdots \alpha_m^{-1}) \quad (a \in \mathbb{F}_q^*).$$

In particular, if $m = 1$, then $K_1(\psi; a)$ is simply denoted by $K(\psi; a)$, and is called the Kloosterman sum. For this, we have the Weil bound(cf. [10])

$$|K(\lambda; a)| \leq 2\sqrt{q}. \tag{1.1}$$

The Kloosterman sum was introduced in 1926([8]) to give an estimate for the Fourier coefficients of modular forms.

[1]The work was supported by National Foundation of Korea Grant funded by the Korean Government (2009-0072514).

For each nonnegative integer h, we denote by $MK_m(\psi)^h$ the h-th moment of the m-dimensional Kloosterman sum $K_m(\psi; a)$, i.e.,

$$MK_m(\psi)^h = \sum_{a \in \mathbb{F}_q^*} K_m(\psi; a)^h.$$

If $\psi = \lambda$ is the canonical additive character of $\mathbb{F}_q$, then $MK_m(\lambda)^h$ will be simply denoted by MK_m^h. If further $m = 1$, for brevity, MK_1^h will be indicated by MK^h.

Explicit computations on power moments of Kloosterman sums were initiated in the paper [13] of Salié in 1931, where it is shown that for any odd prime q,

$$MK^h = q^2 M_{h-1} - (q-1)^{h-1} + 2(-1)^{h-1} \ (h \geq 1).$$

Here $M_0 = 0$, and for $h \in \mathbb{Z}_{>0}$,

$$M_h = \left| \left\{ (\alpha_1, ..., \alpha_h) \in (\mathbb{F}_q^*)^h | \sum_{j=1}^{h} \alpha_j = 1 = \sum_{j=1}^{h} \alpha_j^{-1} \right\} \right|.$$

For $q = p$ odd prime, Salié obtained MK^1, MK^2, MK^3, MK^4 in that same paper by determining M_1, M_2, M_3.

From now on, let us assume that $q = 2^r$. Carlitz [1] evaluated MK^h for $h \leq 4$. Moisio was able to find explicit expressions of MK^h, for $h \leq 10$ (cf. [12]). This was done, via Pless power moment identity, by connecting moments of Kloosterman sums and the frequencies of weights in the binary Zetterberg code of length $q+1$, which were known by the work of Schoof and Vlugt in [14].

In [5], for both n, q powers of two, a binary linear code $C(SL(n, q))$ associated with the finite special linear group $SL(n, q)$ was constructed in order to produce a recursive formula for the power moments of multi-dimensional Kloosterman sums in terms of the frequencies of weights in that code. On the other hand, in [6], for q a power of three, two infinite families of ternary linear codes associated with double cosets in the symplectic group $Sp(2n, q)$ were constructed in order to generate infinite families of recursive formulas for the power moments of Kloosterman sums with square arguments and for the even power moments of those in terms of the frequencies of weights in those codes.

In this paper, we will utilize one simple identity connecting the Kloosterman sum for $GL(2, q)$ and the ordinary Kloosterman sum (cf. (2.3)). Then we will be able to produce a recursive formula generating the power moments of 2-dimensional Kloosterman sums, equivalently that generating the even power moments of Kloosterman sums. To do that, we construct a binary linear code connected with the Kloosterman sum for $GL(2, q)$.

Theorem 1.1 of the following (cf. (1.2)-(1.4)) is the main result of this paper. Henceforth, we agree that the binomial coefficient $\binom{b}{a} = 0$, if $a > b$ or $a < 0$.

Theorem 1.1 *Let $q = 2^r$. Then we have the following:*

(*a*) *For* $r \geq 2$, *and* $h = 1, 2, ...,$

$$\begin{aligned} MK_2^h = &\sum_{l=0}^{h-1}(-1)^{h+l+1}\binom{h}{l}(q^3-2q^2-q+1)^{h-l}MK_2^l \\ &+ q^{1-h}\sum_{j=0}^{min\{N,h\}}(-1)^{h+j}C_j\sum_{t=j}^{h}t!S(h,t)2^{h-t}\binom{N-j}{N-t}, \end{aligned} \tag{1.2}$$

(*b*) *For* $r \geq 2$, *and* $h = 1, 2, ...,$

$$\begin{aligned} MK^{2h} = &\sum_{l=0}^{h-1}(-1)^{h+l+1}\binom{h}{l}(q^3-2q^2+1)^{h-l}MK^{2l} \\ &+ q^{1-h}\sum_{j=0}^{min\{N,h\}}(-1)^{h+j}C_j\sum_{t=j}^{h}t!S(h,t)2^{h-t}\binom{N-j}{N-t}, \end{aligned} \tag{1.3}$$

where $N = |GL(2,q)| = q(q-1)(q^2-1)$, *and* $\{C_j\}_{j=0}^N$ *is the weight distribution of* $C(GL(2,q))$ *given by*

$$C_j = \sum\binom{m_0}{\nu_0}\prod_{|t|<2\sqrt{q},\ t\equiv -1(4)}\ \prod_{K(\lambda;\beta^{-1})=t}\binom{m_t}{\nu_\beta}\ (j=0,\cdots,N), \tag{1.4}$$

with the sum running over all the sets of nonnegative integers $\{\nu_\beta\}_{\beta\in\mathbb{F}_q}$ *satisfying*

$$\sum_{\beta\in\mathbb{F}_q}\nu_\beta = j \quad and \quad \sum_{\beta\in\mathbb{F}_q}\nu_\beta\beta = 0,$$

$$m_0 = q(2q^2-2q-1) \quad and \quad m_t = q(q^2-2q-1+t),$$

for all integers t *satisfying* $|t| < 2\sqrt{q}$ *and* $t \equiv -1(mod\ 4)$.

In addition, $S(h,t)$ *is the Stirling number of the second kind given by*

$$S(h,t) = \frac{1}{t!}\sum_{j=0}^{t}(-1)^{t-j}\binom{t}{j}j^h. \tag{1.5}$$

2. Preliminaries

Throughout this paper, the following notations will be used:

$$\begin{aligned} q &= 2^r\ (r \in \mathbb{Z}_{>0}), \\ \mathbb{F}_q &= \text{the finite field with } q \text{ elements}, \\ tr(x) &= x + x^2 + \cdots + x^{2^{r-1}} \text{ the trace function } \mathbb{F}_q \to \mathbb{F}_2, \\ \lambda(x) &= (-1)^{tr(x)} \text{ the canonical additive character of } \mathbb{F}_q. \end{aligned}$$

Then any nontrivial additive character ψ of $\mathbb{F}_q$ is given by $\psi(x) = \lambda(ax)$, for a unique $a \in \mathbb{F}_q^*$.

For any nontrivial additive character ψ of $\mathbb{F}_q$ and $a \in \mathbb{F}_q^*$, the Kloosterman sum $K_{GL(t,q)}(\psi; a)$ for $GL(t, q)$ is defined as

$$K_{GL(t,q)}(\psi; a) = \sum_{g \in GL(t,q)} \psi(Trg + aTrg^{-1}).$$

Observe that, for $t = 1$,$K_{GL(1,q)}(\psi; a)$ denotes the Kloosterman sum $K(\psi; a)$.

In [4], it is shown that $K_{GL(t,q)}(\psi; a)$ satisfies the following recursive relation: for integers $t \geq 2$, $a \in \mathbb{F}_q^*$,

$$K_{GL(t,q)}(\psi; a) = q^{t-1} K_{GL(t-1,q)}(\psi; a) K(\psi; a) + q^{2t-2}(q^{t-1} - 1) K_{GL(t-2,q)}(\psi; a), \tag{2.1}$$

where we understand that $K_{GL(0,q)}(\psi; a) = 1$.

Theorem 2.1 ([2]) *For the canonical additive character λ of $\mathbb{F}_q$, and $a \in \mathbb{F}_q^*$,*

$$K_2(\lambda; a) = K(\lambda; a)^2 - q. \tag{2.2}$$

Our paper will be based on the $t = 2$ case of the identity in (2.1).

Proposition 2.2 *For the canonical additive character λ of $\mathbb{F}_q$, we have:*

$$K_{GL(2,q)}(\lambda; a) = qK(\lambda; a)^2 + q^2(q - 1) = qK_2(\lambda; a) + q^3. \tag{2.3}$$

Proposition 2.3 ([7]) *For $n = 2^s$ $(s \in \mathbb{Z}_{\geq 0})$, λ the canonical additive character of $\mathbb{F}_q$, and $a \in \mathbb{F}_q^*$,*

$$K(\lambda; a^n) = K(\lambda; a). \tag{2.4}$$

Remark 2.4 In fact, (2.4) holds more generally for multi-dimensional Kloosterman sums. For $n = 2^s$ $(s \in \mathbb{Z}_{\geq 0})$, λ the canonical additive character of $\mathbb{F}_q$, $a \in \mathbb{F}_q^*$, and any positive integer m,

$$K_m(\lambda; a^n) = K_m(\lambda; a). \tag{2.5}$$

The order of the general linear group $GL(n, q)$ is given by

$$g_n = \prod_{j=0}^{n-1} (q^n - q^j) = q^{\binom{n}{2}} \prod_{j=1}^{n} (q^j - 1). \tag{2.6}$$

3. Construction of codes

Let

$$N = |GL(2, q)| = q(q - 1)(q^2 - 1). \tag{3.1}$$

Here we will construct a binary linear code C of length N connected with the Kloosterman sum for $GL(2, q)$.

Let $g_1, ..., g_N$ be a fixed ordering of the elements in $GL(2,q)$, and let $v = (Trg_1 + Trg_1{}^{-1}, ..., Trg_N + Trg_N{}^{-1}) \in \mathbb{F}_q^N$. The binary linear code $C = C(GL(2,q))$ is defined as

$$C = \{u \in \mathbb{F}_2^N \mid u \cdot v = 0\}. \tag{3.2}$$

The following Delsarte's theorem is well-known.

Theorem 3.1 ([11]) *Let B be a linear code over $\mathbb{F}_q$. Then $(B|_{\mathbb{F}_2})^{\perp} = tr(B^{\perp})$.*

In view of this theorem, the dual $C^{\perp}$ of C is given by

$$C^{\perp} = \{c(a) = (tr(a(Trg_1 + Trg_1{}^{-1})), ..., tr(a(Trg_N + Trg_N{}^{-1}))) \mid a \in \mathbb{F}_q\}. \tag{3.3}$$

The following estimate is very coarse but will serve for our purpose.

Lemma 3.2 *For any $a \in \mathbb{F}_q^*$, and ψ any nontrivial additive character of $\mathbb{F}_q$,*

$$\begin{aligned} &|K_{GL(n,q)}(\psi; a)| < |GL(n,q)|, \quad \textit{for } n \geq 2 \textit{ and } q \geq 4, \textit{ and} \\ &|K_{GL(1,q)}(\psi; a)| < |GL(1,q)|, \quad \textit{for } q \geq 8. \end{aligned} \tag{3.4}$$

Proof. For n=1, this is trivial, since $2\sqrt{q} < q-1$, for $q \geq 8$. For n=2, from (2.1)

$$K_{GL(2,q)}(\psi; a) = qK(\psi; a)^2 + q^2(q-1), \tag{3.5}$$

and hence from (1.1) and (3.5), for $q \geq 4$,

$$|K_{GL(2,q)}(\psi; a)| \leq q^3 + 3q^2 < q(q-1)(q^2-1) = |GL(2,q)|. \tag{3.6}$$

For $n = 3$, from (2.1),

$$K_{GL(3,q)}(\psi; a) = q^2 K_{GL(2,q)}(\psi; a)K(\psi; a) + q^4(q^2-1)K(\psi; a), \tag{3.7}$$

and hence from (1.1), (3.6), and (4.6), for $q \geq 4$,

$$|K_{GL(3,q)}(\psi; a)| < 2q^{\frac{7}{2}}(q^2-1)(2q-1) < q^3(q-1)(q^2-1)(q^3-1) = |GL(3,q)|.$$

Assume now that $n \geq 4$ and that (3.4) holds for all integers less than n and greater than and equal to 2, for $q \geq 4$. Then, from (1.1), (2.1), and (2.6), and for $q \geq 4$,

$$|K_{GL(n,q)}(\psi; a)| < q^{\binom{n}{2}}(q + 2\sqrt{q}) \prod_{j=1}^{n-1}(q^j - 1) < q^{\binom{n}{2}} \prod_{j=1}^{n}(q^j - 1) < |GL(n,q)|. \quad \blacksquare$$

Remark 3.3 It was shown in [3, *Theorem* 2] that, for any nontrivial additive character ψ of $\mathbb{F}_q$ and $a \in \mathbb{F}_q^*$,

$$K_{GL(n,q)}(\psi; a^2) = \sum_{g \in GL(n,q)} \psi(a(Trg + Trg^{-1})) = (-1)^n q^{\binom{n}{2}} \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q \omega^j \bar{\omega}^{n-j},$$

where ω, $\bar{\omega}$ are complex numbers, depending on ψ and a, with $|\omega|$=$|\bar{\omega}|$=$\sqrt{q}$. Thus

$$|K_{GL(n,q)}(\psi; a^2)| \leq q^{\frac{1}{2}n^2} \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q ,$$

and, in particular, we get

$$|K_{GL(2,q)}(\psi; a^2)| \leq q^2 \sum_{j=0}^{2} \begin{bmatrix} 2 \\ j \end{bmatrix}_q = q^2(q+3).$$

Proposition 3.4 *The map $\mathbb{F}_q \to C^{\perp}$ ($a \mapsto c(a)$) is an $\mathbb{F}_2$-linear isomorphism for $q \geq 4$.*

Proof. The map is clearly $\mathbb{F}_2$-linear and surjective. Let a be in the kernel of the map. Then $tr(a(Trg + Trg^{-1})) = 0$, for all $g \in GL(2,q)$. Suppose that $a \neq 0$. Then, on the one hand,

$$(3.8)\quad \begin{aligned} |GL(2,q)| &= \sum_{g \in GL(2,q)} (-1)^{tr(a(Trg+Trg^{-1}))} = \sum_{g \in GL(2,q)} \lambda(a(Trg + Trg^{-1})) \\ &= \sum_{g \in GL(2,q)} \lambda(Trg + a^2 Trg^{-1}) \ (g \to a^{-1}g) = K_{GL(2,q)}(\lambda; a^2). \end{aligned}$$

As $q \geq 4$, (3.8) is on the other hand strictly less than $|GL(2,q)|$ by Lemma 3.2. This is a contradiction. So we must have $a = 0$. ∎

Remark 3.5 (a) If $q = 2$, one checks easily that the kernel of the map $\mathbb{F}_2 \to C^{\perp}$ is $\mathbb{F}_2$.

(b) The fact that the map in Proposition 3.4 is injective follows also from (1.1) and (3.11), since they imply that $n(\beta) > 0$, for all β, provided that $q \geq 4$.

Proposition 3.6 ([7]) *Let λ be the canonical additive character of $\mathbb{F}_q$, $m \in \mathbb{Z}_{>0}$, $\beta \in \mathbb{F}_q$. Then*

$$(3.9)\quad \sum_{a \in \mathbb{F}_q^*} \lambda(-a\beta) K_m(\lambda; a) = \begin{cases} qK_{m-1}(\lambda; \beta^{-1}) + (-1)^{m+1}, & \text{if } \beta \neq 0, \\ (-1)^{m+1}, & \text{if } \beta = 0. \end{cases}$$

with the convention $K_0(\lambda; \beta^{-1}) = \lambda(\beta^{-1})$.

Let

$$(3.10)\quad n(\beta) = |\{g \in GL(2,q) | Trg + Trg^{-1} = \beta\}|.$$

Then, with N as in (3.1),

$$\begin{aligned} qn(\beta) &= N + \sum_{\alpha \in \mathbb{F}_q^*} \lambda(-\alpha\beta) \textstyle\sum_{g \in GL(2,q)} \lambda(\alpha(Trg + Trg^{-1})) \\ &= N + \sum_{\alpha \in \mathbb{F}_q^*} \lambda(-\alpha\beta) K_{GL(2,q)}(\lambda; \alpha^2) \\ &= N + \sum_{\alpha \in \mathbb{F}_q^*} \lambda(-\alpha\beta)(qK_2(\lambda; \alpha^2) + q^3)(cf.(2.3)) \\ &= N + q \sum_{\alpha \in \mathbb{F}_q^*} \lambda(-\alpha\beta) K_2(\lambda; \alpha^2) + q^3 \sum_{\alpha \in \mathbb{F}_q^*} \lambda(-\alpha\beta) \\ &= N + q \sum_{\alpha \in \mathbb{F}_q^*} \lambda(-\alpha\beta) K_2(\lambda; \alpha) + q^3 \sum_{\alpha \in \mathbb{F}_q^*} \lambda(-\alpha\beta) \ (cf.\ (2.5)). \end{aligned}$$

Now, from Proposition 3.6, we obtain the following.

Proposition 3.7 *Let* $n(\beta)$ *be as in* (3.10). *Then we have*

$$n(\beta) = \begin{cases} q(q^2-2q-1+K(\lambda;\beta^{-1})), & \text{if } \beta \neq 0, \\ q(2q^2-2q-1), & \text{if } \beta = 0. \end{cases} \tag{3.11}$$

4. Power moments of 2-dimensional Kloosterman sums

In this section, we will be able to find, via Pless power moment identity, a recursive formula for the power moments of 2-dimensional Kloosterman sums or equivalently for the even power moments of Kloosterman sums in terms of the frequencies of weights in $C = C(GL(2,q))$.

Theorem 4.1 (Pless power moment identity): Let B be a q-ary $[n,k]$ code, and let B_i (resp. $B_i^{\perp}$) denote the number of codewords of weight i in B (resp. in $B^{\perp}$). Then, for $h = 0,1,2,...,$

$$\sum_{j=0}^{n} j^h B_j = \sum_{j=0}^{min\{n,h\}} (-1)^j B_j^{\perp} \sum_{t=j}^{h} t! S(h,t) q^{k-t}(q-1)^{t-j}\binom{n-j}{n-t}, \tag{4.1}$$

where $S(h,t)$ is the Stirling number of the second kind defined in (1.5).

From now on, we will assume that $q \geq 4$ (i.e., $r \geq 2$), so that every codeword in $C(GL(2,q))^{\perp}$ can be written as $c(a)$, for a unique $a \in \mathbb{F}_q$ (cf. Proposition 3.4). This also allows one to use Theorem 4.5.

Lemma 4.2 *Let* $c(a)=(tr(a(Trg_1+Trg_1{}^{-1})),...,tr(a(Trg_N+Trg_N{}^{-1})))\in C(GL(2,q))^{\perp}$, *for* $a \in \mathbb{F}_q^*$. *Then the Hamming weight* $w(c(a))$ *can be expressed as follows:*

$$w(c(a)) = \frac{1}{2}q(q^3-2q^2+1-K(\lambda;a)^2) \tag{4.2}$$

$$= \frac{1}{2}q(q^3-2q^2-q+1-K_2(\lambda;a)). \tag{4.3}$$

Proof.

$$\begin{aligned} w(c(a)) &= \frac{1}{2}\sum_{i=1}^{N}(1-(-1)^{tr(a(Trg_i+Trg_i^{-1}))}) \\ &= \frac{1}{2}(N - \sum_{g\in GL(2,q)} \lambda(a(Trg+Trg^{-1}))) \\ &= \frac{1}{2}(N - \sum_{g\in GL(2,q)} \lambda(Trg+a^2Trg^{-1})) \\ &= \frac{1}{2}(N - K_{GL(2,q)}(\lambda;a^2)) \\ &= \frac{1}{2}(N - qK(\lambda;a)^2 - q^2(q-1)) \ (cf.(2.3),\ (2.4)) \\ &= \frac{1}{2}q(q^3-2q^2+1-K(\lambda;a)^2) \ (cf.(3.1)) \\ &= \frac{1}{2}q(q^3-2q^2-q+1-K_2(\lambda;a)) \ (cf.(2.2)). \end{aligned}$$

■

Let $u = (u_1, ..., u_N) \in \mathbb{F}_2^N$, with ν_β 1's in the coordinate places where $Trg_j + Trg_j^{-1} = \beta$, for each $\beta \in \mathbb{F}_q$. Then we see from the definition of the code $C(GL(2,q))$ (cf. (3.2)) that u is a codeword with weight j if and only if $\sum_{\beta \in \mathbb{F}_q} \nu_\beta = j$ and $\sum_{\beta \in \mathbb{F}_q} \nu_\beta \beta = 0$ (an identity in $\mathbb{F}_q$). As there are $\prod_{\beta \in \mathbb{F}_q} \binom{n(\beta)}{\nu_\beta}$ many such codewords with weight j, we obtain the following result.

Proposition 4.3 *Let $\{C_j\}_{j=0}^N$ be the weight distribution of $C(GL(2,q))$, where C_j denotes the frequency of the codewords with weight j in C. Then*

$$C_j = \sum \prod_{\beta \in \mathbb{F}_q} \binom{n(\beta)}{\nu \beta}, \tag{4.4}$$

where the sum runs over all the sets of nonnegative integers $\{\nu_\beta\}_{\beta \in \mathbb{F}_q}$ $(0 \leq \nu_\beta \leq n(\beta))$, satisfying

$$\sum_{\beta \in \mathbb{F}_q} \nu_\beta = j \text{ and } \sum_{\beta \in \mathbb{F}_q} \nu_\beta \beta = 0. \tag{4.5}$$

Corollary 4.4 *Let $\{C_j\}_{j=0}^N$ be the weight distribution of $C(GL(2,q))$. Then we have:*

$$C_j = C_{N-j},$$

for all j, with $0 \leq j \leq N$.

Proof. Under the replacements $\nu_\beta \to n(\beta) - \nu_\beta$, for each $\beta \in \mathbb{F}_q$, the first equation in (4.5) is changed to $N - j$, while the second one in (4.5) and the summands in (4.4) are left unchanged. Here the second sum in (4.5) is left unchanged, since $\sum_{\beta \in \mathbb{F}_q} n(\beta)\beta = 0$, as one can see by using the explicit expression of $n(\beta)$ in (3.11). ■

Theorem 4.5 ([9]) *Let $q = 2^r$, with $r \geq 2$. Then the range R of $K(\lambda; a)$, as a varies over $\mathbb{F}_q^*$, is given by*

$$R = \{t \in \mathbb{Z} \mid |t| < 2\sqrt{q},\ t \equiv -1 (mod\ 4)\}.$$

In addition, each value $t \in R$ is attained exactly $H(t^2 - q)$ times, where $H(d)$ is the Kronecker class number of d.

Now, we get the following formula in (4.6), by applying the formula in (4.4) to $C(GL(2,q))$, using the explicit values of $n(\beta)$ in (3.11) and taking Theorem 4.5 into consideration.

Theorem 4.6 *Let $\{C_j\}_{j=0}^N$ be the weight distribution of $C(GL(2,q))$. Then*

$$C_j = \sum \binom{m_0}{\nu_0} \prod_{|t|<2\sqrt{q},\ t \equiv -1(4)} \prod_{K(\lambda;\beta^{-1})=t} \binom{m_t}{\nu_\beta} \ (j = 0, ..., N), \tag{4.6}$$

where the sum is over all the sets of nonnegative integers $\{\nu_\beta\}_{\beta\in\mathbb{F}_q}$ *satisfying*

$$\sum_{\beta\in\mathbb{F}_q}\nu_\beta = j \ \textit{and} \ \sum_{\beta\in\mathbb{F}_q}\nu_\beta\beta = 0,$$

$$m_0 = q(2q^2-2q-1), \quad \textit{and} \quad m_t = q(q^2-2q-1+t),$$

for all integers t *satisfying* $|t| < 2\sqrt{q}$ *and* $t \equiv -1(mod\ 4)$.

We now apply the Pless power moment identity in (4.1) to $C(GL(2,q))^{\perp}$, in order to obtain the results in Theorem 1.1 (cf. (1.2)-(1.4)) about recursive formulas.

Then the left hand side of that identity in (4.1) is equal to

$$\sum_{a\in\mathbb{F}_q^*} w(c(a))^h, \tag{4.7}$$

with the $w(c(a))$ given either by (4.2) or by (4.3).

Using the expression of $w(c(a))$ in (4.3), (4.7) is

$$\begin{aligned}
&\left(\frac{q}{2}\right)^h \sum_{a\in\mathbb{F}_q^*}(q^3-2q^2-q+1-K_2(\lambda;a))^h \\
&= \left(\frac{q}{2}\right)^h \sum_{a\in\mathbb{F}_q^*}\sum_{l=0}^{h}(-1)^l\binom{h}{l}(q^3-2q^2-q+1)^{h-l}K_2(\lambda;a)^l \\
&= \left(\frac{q}{2}\right)^h \sum_{l=0}^{h}(-1)^l\binom{h}{l}(q^3-2q^2-q+1)^{h-l}MK_2^l.
\end{aligned} \tag{4.8}$$

Equivalently, using the expression of $w(c(a))$ in (4.2), (4.7) is

$$\left(\frac{q}{2}\right)^h \sum_{l=0}^{h}(-1)^l\binom{h}{l}(q^3-2q^2+1)^{h-l}MK^{2l}. \tag{4.9}$$

On the other hand, the right hand side of the identity in (4.1) is

$$q\sum_{j=0}^{min\{N,h\}}(-1)^jC_j\sum_{t=j}^{h}t!S(h,t)2^{-t}\binom{N-j}{N-t}. \tag{4.10}$$

Our main results in Theorem 1.1 (cf. (1.2)-(1.4)) now follow by equating (4.8) and (4.10), and (4.9) and (4.10). Also, one has to separate the term corresponding to $l = h$ in (4.8) and (4.9), and note $dim_{\mathbb{F}_2}C(GL(2,q)) = r$.

Note here that, in view of (2.2), obtaining power moments of 2-dimensional Kloosterman sums is equivalent to getting even power moments of Kloosterman sums.

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A NOTE ON THE CAFIERO CRITERION IN EFFECT ALGEBRAS

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Abstract. We give an alternative proof of a Cafiero type theorem for measures on effect algebras.

1. Introduction

In this note, we want to give an alternative proof of the Cafiero theorem valid for measures on effect algebras as contained in (cf. [1]). Avallone reduced the proof to the classical case using techniques elaborated in [11]; we here give a direct proof imitating de Lucia and Cavaliere's paper (see [7]). Effect algebras (alias D-posets) have been independently introduced in 1994 by D.J. Foulis and M. K. Bennett in [3] and by F. Chovanek and F. Kopka in [5] for modelling unsharp measurement in a quantum mechanical system. They are a generalization of many structures which arise in Quantum Physics [8] and in Mathematical Economics [6], [4], in particular they are a generalization of orthomodular posets and MV-algebras and therefore of Boolean algebras.

2. Preliminaries

Definition 2.1 Let $(L, \leq)$ be a poset with a smallest element 0 and a greatest element 1 and let $\ominus$ be a partial operation on L such that $b \ominus a$ is defined if and only if $a \leq b$ and for all $a, b, c \in L$:

If $a \leq b$, then $b \ominus a \leq b$ and $b \ominus (b \ominus a) = a$;
If a $\leq b \leq c$, then $c \ominus b \leq c \ominus a$ and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.

Then $(L, \leq, \ominus)$ is called a difference poset (D-poset for short), or a difference lattice (D-lattice for short) if L is a lattice.

One defines in L a partial operation $\oplus$ as follows:

$a \oplus b$ is defined and $a \oplus b = c$ if and only if $c \ominus b$ is defined and $c \ominus b = a$.

The operation $\oplus$ is well-defined by the cancellation law [8, page 13] ($a \leq b$, c and $b \ominus a = c \ominus a$ implies $b = c$), and $(L, \oplus, 0, 1)$ is an effect algebra (see [8, Theorem 1.3.4]), that is the following conditions are satisfied for all $a, b, c \in L$:

If $a \oplus b$ is defined, then $b \oplus a$ is defined and $a \oplus b = b \oplus a$;

If $b \oplus c$ is defined and $a \oplus (b \oplus c)$ is defined, then $a \oplus b$ and $(a \oplus b) \oplus c$ are defined, and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$;

There exists a unique $a' \in E$ such that $a \oplus a'$ is defined and $a \oplus a' = 1$;

If $a \oplus 1$ is defined, then $a = 0$.

We say that a and b are orthogonal if $a \leq b'$ and we write $a \perp b$. Therefore $a \oplus b$ is defined if and only if $a \perp b$, and in this case $a \oplus b = (a' \ominus b)'$ by [8, Lemma 1.2.5].

From now on, let L be a D-lattice.

In the sequel we deal with functions defined on L with values in a topological space (S, τ).

Definition 2.2 A map $\mu \colon L \to S$ is called a measure if $\mu(a \oplus b) = \mu(a) + \mu(b)$ whenever $a, b \in L$ are orthogonal.

Classical measures on Boolean algebras are example of measures on effect algebras. We employ the notation:

Notation 2.3 *Let e be a point of S. We denote by $\mathbb{M}$ the collection of all functions $\mu \colon L \to S$ such that $\mu(0) = e$ and by $\tau[e]$ a fundamental system of neighbourhoods of e. Moreover $M \in I_\infty(\mathbb{N})$ means that M is an infinite subset of $\mathbb{N}$.*

Definition 2.4 A function of $\mathbb{M}$ is said exhaustive whenever $\lim_k \mu(a_k) = e$ for every orthogonal sequence (a_k) as well as a sequence (μ_n) of elements of $\mathbb{M}$ is said to be uniformly exhaustive if $\lim_k \mu_n(a_k) = e$, uniformly with respect to $n \in \mathbb{N}$, for any orthogonal sequence (a_k) in L. For any function $\mu \in \mathbb{M}$ we put

$$\tilde{\mu}(a) := \{\mu(b) : \ b \in L,\, b \leq a\} \text{ for every } a \in L.$$

Lemma 2.5 *If $\mu \in \mathbb{M}$ is exhaustive and (a_k) is an orthogonal sequence in L, then for every $P \in I_\infty(\mathbb{N})$ and every $U \in \tau[0]$, there exists $M \in I_\infty(P)$ such that $\bigoplus_{k \in M} a_k$ exists in L and $\tilde{\mu}\left(\bigoplus_{k \in M} a_k\right) \subseteq U$.*

Proof. The proof is straightforward. ∎

In [1] Avallone introduced the following definition:

Definition 2.6 We say that L satisfies the D-subsequential completeness property (D-SCP, for short) if for every orthogonal sequence (a_n) in L there is $M \in I_\infty(\mathbb{N})$ such that $\bigoplus_{n \in M} a_n$ exists.

Lemma 2.7 *Let L be with the D-SCP property. If (μ_n) is a sequence of exhaustive elements of $\mathbb{M}$, then, for every $U \in \tau[0]$, any orthogonal sequence (a_k) in L admits a subsequence a_{k_i} such that the sum $\bigoplus_{i \in \mathbb{N}} a_{k_i}$ exists in L and $\tilde{\mu}_{k_j}\left(\bigoplus_{i>j} a_{k_i}\right) \subseteq U$ for every $j \in \mathbb{N}$.*

Proof. Let $U \in \tau[0]$ and let (a_k) be an orthogonal sequence in L.

Since μ_1 is exhaustive, by Lemma 2.5, there exists $M_0 \in I_\infty(\mathbb{N} \setminus \{1\})$ such that $\bigoplus_{k \in M_0} a_k$ exists in L and $\tilde{\mu}_1(\oplus_{k \in M_0} a_k) \subseteq U$. Let $k_1 := \min M_0$. By Lemma 2.5 again, there exists $M_1 \in I_\infty(M_0 \setminus \{k_1\})$ such that $\bigoplus_{k \in M_1} a_k$ exists in L and $\tilde{\mu}_{k_1}\left(\bigoplus_{k \in M_1} a_k\right) \subseteq U$. Going on by induction, one can determine an increasing sequence (k_m) in $\mathbb{N}$ and a decreasing sequence (M_m) in $I_\infty(\mathbb{N})$ such that for every $m \in \mathbb{N}$, $\bigoplus_{k \in M_m} a_k$ exists in L and $\tilde{\mu}_{k_m}\left(\bigoplus_{k \in M_m} a_k\right) \subseteq U$ with $k_m \notin M_m$.

By the D-SCP property, the orthogonal sequence (a_{k_m}) admits a subsequence $(a_{k_{m_i}})$ such that there exists in L the supremum $\bigoplus a_{k_{m_i}}$.

Since for every $j \in \mathbb{N}$ it holds that $\bigoplus_{i>j} a_{k_{m_i}} \leq \bigoplus_{k \in M_{m_j}} a_k$ one has

$$\tilde{\mu}_{k_{m_j}}\left(\bigoplus_{i>j} a_{k_{m_i}}\right) \subseteq U \ \forall j \in \mathbb{N}$$

which ends the proof. ∎

Definition 2.8 The function μ in $\mathbb{M}$ is called quasi-triangular whenever for every U in $\tau[0]$ there exists $V(U) \in \tau[0]$ such that it holds

$$a \perp b,\ \mu(a) \in V,\ \mu(b) \in V \Longrightarrow \mu(a \oplus b) \in U;$$

$$a \perp b,\ \mu(a) \in V, \mu(a \oplus b) \in V \Longrightarrow \mu(b) \in U.$$

The functions (μ_n) in $\mathbb{M}$ are called uniformly quasi-triangular whenever for every U in $\tau[0]$ there exists $V(U) \in \tau[0]$ such that, for all $n \in \mathbb{N}$, it holds

$$a \perp b,\ \mu_n(a) \in V,\ \mu_n(b) \in V \Longrightarrow \mu_n(a \oplus b) \in U;$$

$$a \perp b,\ \mu_n(a) \in V, \mu_n(a \oplus b) \in V \Longrightarrow \mu_n(b) \in U.$$

Quasi-triangular functions generalize functions $\mu \colon L \to [0, +\infty]$ satisfying

$$|\mu(a \oplus b) - \mu(a)| \leq \mu(b)$$

for orthogonal elements $a, b \in L$. Such functions were considered in the classical context and are called triangular by some authors.

Lemma 2.9 *Let L be with the D-SCP property. Given a sequence (μ_n) of exhaustive and uniformly quasi-triangular elements of M, if*

> *for every $U_0 \in \tau[0]$ and for every orthogonal sequence (b_k) in L there exists $k_0 \in \mathbb{N}$ such that $\{n \in \mathbb{N} : \mu_n(b_{k_0}) \notin U_0\}$ is finite,*

then for every $U \in \tau[0]$ and every orthogonal sequence (a_k) in L such that $\mu_k(a_k) \notin U$ for all $k \in \mathbb{N}$, there exist an increasing sequence (k_m) in $\mathbb{N}$ and $M \in I_\infty(\mathbb{N})$ such that there exists $\bigoplus_{j\in M} a_{k_j}$ and $\tilde{\mu}_{k_m}\left(\bigoplus_{j\in M} a_{k_j}\right) \notin V(U)$ for all $m \in \mathbb{N}$.

Proof. Let $U \in \tau[0]$ be given. Since the μ_n's are uniformly quasi-triangular, one can consider $V_0 := V(U)$ and $V_n = V_{n-1} \cap V(V_{n-1})$ like in Definition 2.8. By Lemma 2.7, taking subsequences if needed, one has

$$\tilde{\mu}_m(\oplus_{k>m} a_k) \subseteq V_1 \; \forall m \in \mathbb{N}. \tag{1}$$

Moreover, from assumptions there exist two natural numbers k_1 and n_1 such that

$$\mu_n(a_{k_1}) \in V_2 \; \forall n > n_1$$

as well as there exist n_2 and k_2 such that

$$k_2 > \max\{k_1, n_1\},\; n_2 > n_1,\; \mu_n(a_{k_2}) \in V_3 \; \forall n > n_2.$$

Thus, by induction, one can construct two strictly increasing sequence (k_j) and (n_j) such that

$$k_j > n_{j-1} \text{ and } \mu_{k_m}(a_{k_j}) \in V_{j+1}; \forall m > j. \tag{2}$$

Since L has the D-SCP property, there exists $M \in I_\infty(\mathbb{N})$ such that there exists $\oplus_{j\in M} a_{k_j}$.

Moreover, one infers from (1) that $\tilde{\mu}_{k_m}\left(\bigoplus_{j>m, j\in M} a_{k_j}\right) \subseteq V_1 \; \forall m \in \mathbb{N}$, and from (2) that

$$\mu_{k_m}\left(\bigoplus_{j<m, j\in M} a_{k_j}\right) \in V_1 \; \forall m \in \mathbb{N}.$$

Hence, by the uniform quasi-triangularity of the μ_n, it follows that

$$\mu_{k_m}\left(\bigoplus_{j\neq m, j\in M} a_{k_j}\right) \in V_0 \; \forall m \in \mathbb{N},$$

so by assumptions one can establish that $\tilde{\mu}_{k_m}\left(\bigoplus_{j\in M} a_{k_j}\right) \notin V_0$ for all $m \in \mathbb{N}$, as desired. ∎

3. Cafiero criterion

Now, we are able to proof our main result.

Theorem 3.1 *Let L be with the D-SCP property. Let (μ_n) be a sequence of exhaustive and uniformly quasi-triangular functions. Then (μ_n) is uniformly exhaustive if and only if the following condition holds for every $U \in \tau[0]$ and every orthogonal sequence (a_k) there exist k_0, $n_0 \in \mathbb{N}$ such that $\mu_n(a_{k_0}) \in U$ for all $n \geq n_0$.*

Proof. The necessity of the condition is trivial.

For the sufficiency, we argue by contradiction. Let us assume, by passing to a subsequence if necessary, that there exists an orthogonal sequence (a_k) such that $\mu_n(a_n) \notin U_0$ for all $n \in \mathbb{N}$. Let (P_k) be a disjoint sequence in $I_\infty(\mathbb{N})$ whose elements cover $\mathbb{N}$. By 2.9 for every $k \in \mathbb{N}$ there exists $M_k \in I_\infty(P_k)$ such that there exists $\oplus_{j\in M_k} a_j$ and the set $\left\{ n \in \mathbb{N} : \mu_n \left(\bigoplus_{j\in M_k} a_j \right) \notin V(U_0) \right\}$ is infinite.

The above construction guarantees that the sequence $(\oplus_{j\in M_k} a_j)_k$ is orthogonal and that for every $k \in \mathbb{N}$ the set $\left\{ n \in \mathbb{N} : \ \mu_n \left(\bigoplus_{j\in M_k} a_j \right) \notin V(U_0) \right\}$ is infinite, but this contradicts the hypothesis. ■

Theorem 3.2 *Let L be with the D-SCP property. Let (μ_n) be a sequence of exhaustive and uniformly quasi-triangular elements of M. If (μ_n) converges pointwise to a exhaustive element μ of $\mathbb{M}$, then (μ_n) is uniformly exhaustive.*

Proof. Let us consider an open element U of $\tau[0]$ and an orthogonal sequence (a_k). Since μ is exhaustive, there exists $k_0 \in \mathbb{N}$ such that $\mu(a_k) \in U$ for every $k \geq k_0$. Thus, the result comes applying 3.1. ■

Theorem 3.2 furnishes an alternative proof of [1, Theorem 4.3].

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ON INCLUSION BETWEEN $\Lambda BV^{(p)}$, CHANTURYIA AND H_w^p CLASSES

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Abstract. In this paper We prove inclusion relations between $\Lambda BV^{(p)}$ and V[v] and give a necessary condition for the inclusion of $\Lambda BV^{(p)}$ in classes H_ω^p.

Keywords: $\Lambda BV^{(p)}$, H_ω^p classes.

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1. Introduction

In 1972, Waterman [7] introduced the class of functions of ΛBV. In 1980, Shiba [5] generalized this class and introduced the class $\Lambda BV^{(p)}$ ($p \geq 1$). In 2006, Vyas [6] proved $\Lambda BV^{(p)}$ is a Banach space with suitable norm, the intersection of $\Lambda BV^{(p)}$, over all sequences Λ, is the class of functions $BV^{(p)}$ and the union of $\Lambda BV^{(p)}$, over all sequences Λ, is the class of functions having right- and left-hand limits at every point. In [1], Avdispahic gave inclusion relations between ΛBV and V[v]. For $p \geq 1$, we give inclusion relation between $\Lambda BV^{(p)}$ and V[v]. Goginava in [3] gave a necessary condition for inclusion ΛBV in H_ω^p. Here, we give a necessary condition for inclusion $\Lambda BV^{(p)}$ in H_ω^p. First, we define the classes $\Lambda BV^{(p)}$, Chanturia and H_ω^p.

Definition 1.1 Given an interval I, and a sequence of positive real numbers $\Lambda = \{\lambda_m\}$, (m=1,2,...) such that $\sum_{m=1}^{\infty}(1/\lambda_m)$ diverges and $1 \leq p < \infty$, we say that $f \in \Lambda BV^{(p)}$(I) (that is f is a function of $p-\Lambda-$bounded variation over I) if

$$V(f) = V_\Lambda(f,p,I) = \sup_{\{I_m\}} V_\Lambda(\{I_m\}, f, p, I) < \infty,$$

where $V_\Lambda(\{I_m\}, f, p, I) = \left(\sum_m \frac{|f(a_m)-f(b_m)|^p}{\lambda_m}\right)^{1/p}$, and $\{I_m\}$ is a sequence of non-overlapping subintervals $I_m = [a_m, b_m] \subset I = [a, b]$. For any $x \in I = [a, b]$, we define

$$v(x) = v_\Lambda(f; x) = v_\Lambda(f, p, [a, x]).$$

For $f \in \Lambda BV^{(p)}(I)$, we define $||f|| = |f(a)| + V(f)$ where $I = [a, b]$.

Definition 1.2 The modulus of variation of a Λ function f is the function $\nu_f(n)$ with domain the positive integers, defined by

$$\nu_f(n) = \sup_{\Pi_n} \sum_{k=1}^{n} |f(I_k)|$$

where Π_n is an arbitrary system of n disjoint intervals $I_k = [a_k, b_k] \subset (0, 1)$ and $f(I_k) = f(b_k) - f(a_k)$.

The modulus of variation of any function is nondecreasing and upwards convex. If the modulus of variation $\nu(n)$ is given, then $V[\nu]$ denotes the class of functions for which $\nu_f(n) = \mathrm{O}(\nu(n))$ when $n \to \infty$.

Definition 1.3 If $\omega(\delta)$ is a modulus of continuity , then $H_p^\omega, p \geq 1$, denotes the class of functions $f \in L^p([0, 1])$ for which $\omega(\delta, f)_p = O(\omega(\delta))$ as $\delta \to 0+$, where

$$\omega(\delta, f)_p = \sup_{0<h\leq\delta} \left(\int_0^1 |f(x + h) - f(x)|^p dx\right)^{1/p}.$$

2. On the classes $\Lambda BV^{(p)}$ and $V[\nu]$

In [1], Avdispahic gave inclusion relations between ΛBV and $V[v]$. Here, we give inclusion relations between $\Lambda BV^{(p)}$ and $V[v]$. Theorem 1 of [4] shows that

Theorem 2.1 $\Lambda BV^{(p)} \subset V[\left(n/\left(\sum\limits_{i=1}^{n}\right) 1/\lambda_i\right)]$.

Theorem 2.2 *$\Lambda BV^{(p)}$ contains every class $V[\nu]$ such that the condition*

$$\sum_{k=1}^{\infty} \Delta(1/\lambda_k)\nu^p(k) < \infty$$

is satisfied, where $\Delta a_k = a_k - a_{k+1}$.

Proof. Let $\{I_k\}$, $k = 1, \dots, n$, be an arbitrary collection of nonoverlapping intervals, $I_k \subset [0, 1]$. By partial summaion we obtain

$$\begin{aligned} \sum_{k=1}^{n} |f(I_k)|^p/\lambda_k &= \sum_{k=1}^{n-1} \Delta(1/\lambda_k) \sum_{i=1}^{k} |f(I_i)|^p + 1/\lambda_n \sum_{i=1}^{n} |f(I_i)|^p \\ &\leq \sum_{k=1}^{n-1} \Delta(1/\lambda_k)\nu^p(k) + \nu^p(n)/\lambda_n \end{aligned}$$

and $\nu^p(n)/\lambda_n \leq \sum_{k=n} \Delta(1/\lambda_k)\nu^p(k)$. ∎

Theorem 2.3 *If $p > 1$ and $k\Delta(\frac{1}{\lambda_k}) = O(1)$, then*

$$V\left[\frac{n}{(\sum_{i=1}^{n} 1/\lambda_i)^p}\right] \subset \Lambda BV^{(p)}.$$

Proof. Let us denote $u_k = k\Delta(1/\lambda_k)$, $S_n = \sum_{k=1}^{n} u_k$. By [2, Theorem 2], $S_n \to \infty$ as $n \to \infty$. Hence by [1, p. 905, Corollary to theorem 1] we have

$$\sum_{k=1}^{\infty} \Delta\left(\frac{1}{\lambda_k}\right)\left(\frac{k^{\frac{1}{p}}}{\sum_{i=1}^{k}(1/\lambda_i)}\right)^p \le \sum_{k=1}^{\infty} \frac{u_k}{S_k^p} < \infty$$

The conclusion follows by Theorem 2.2. ∎

It can be observed that the sequence $\nu_n := \frac{n^{1/p}}{\sum_{i=1}^{n}\frac{1}{\lambda_i}}$ is equivalent to a modulus of variation. Thus, the class appearing in the statement of theorem 2.4 is a Chanturia class, in fact.

Immediately, from Theorem 2.3 and Theorem 2.1 we see that

Theorem 2.4 $V[n^{\frac{1}{p}}/ln\ n] \subset HBV^{(p)} \subset V[n/\ln^{\frac{1}{p}} n]$.

3. On the imbedding of $\Lambda BV^{(p)}$ class in the class H^p_ω

In [3], Goginava gave a necessary condition for inclusion ΛBV in H^p_ω. Here, we give a necessary condition for inclusion $\Lambda BV^{(p)}$ in H^p_ω.

Theorem 3.1 *Let $\Lambda BV^{(p)} \subset H^\omega_p$ for some $p \in [1, \infty)$ then*

$$\limsup_{n\to\infty} \frac{1}{\omega(1/n)n^{1/p}} \max_{1\le m\le n} \frac{m^{1/p}}{\sum_{i=1}^{m} 1/\lambda_i} < \infty. \tag{1}$$

Proof. Assume condition (1) is not satisfied. As an example, we construct a function from $\Lambda BV^{(p)}$ that is not in H^ω_p. Since condition (1) is not satisfied, there exists a sequence of integers $\{\gamma_k, k \ge 1\}$ such that

$$\lim_{k\to\infty} \frac{1}{\omega(1/\gamma_k)\gamma_k^{1/p}} \max_{1\le m\le \gamma_k} \frac{m^{1/p}}{\sum_{i=1}^{m} 1/\lambda_i} = \infty\ .$$

Let $\{\gamma_k', k \ge 1\}$ be a sequence of integers for which $2^{\gamma_k'-1} \le \gamma_k < 2^{\gamma_k'}$. The fact that $\omega(\delta)$ is nondecreasing yields

$$\frac{2^{1/p}}{\omega(2^{-\gamma_k'})2^{\gamma_k'/p}} \max_{1\le m\le 2^{\gamma_k'}} \frac{m^{1/p}}{\sum_{i=1}^{m} 1/\lambda_i} \ge \frac{1}{\omega(1/\gamma_k)\gamma_k^{1/p}} \max_{1\le m\le \gamma_k} \frac{m^{1/p}}{\sum_{i=1}^{m} 1/\lambda_i}\ ,$$

where

$$\limsup_{k\to\infty} \frac{1}{\omega(2^{-\gamma_k'})2^{-\gamma_k'/p}} \max_{1\le m\le 2^{\gamma_k'}} \frac{m^{1/p}}{\sum_{i=1}^{m} 1/\lambda_i} < \infty.$$

Then, a sequence of integers $\{n_k^{'} : k \geq 1\} \subset \{\gamma_k^{'} : k \geq 1\}$ exist such that

$$\lim_{k\to\infty} \frac{1}{\omega(2^{-n_k^{'}})} \frac{1}{\sum_{i=1}^{m(n_k^{'})} 1/\lambda_i} \left(\frac{m(n_k^{'})}{2^{n_k^{'}}}\right)^{1/p} < \infty \ , \tag{2}$$

where

$$\max_{1\leq m\leq 2^{n_k^{'}}} \frac{m^{1/p}}{\sum_{i=1}^{m} 1/\lambda_i} = \frac{(m(n_k^{'}))^{1/p}}{\sum_{i=1}^{m(n_k^{'})} 1/\lambda_i}.$$

The following three cases are possible:

(a) (a) there exists a sequence of integers $\{s_k^{'} : k \geq 1\} \subset \{n_k^{'} : k \geq 1\}$ such that

$$m(s_k^{'}) < 2^{2s_{k-1}^{'}};$$

(b) there exists a sequence of integers $\{q_k^{'} : k \geq 1\} \subset \{n_k^{'} : k \geq 1\}$ such that

$$2^{2q_{k-1}^{'}} \leq m(q_k^{'}) < 2^{q_k^{'}-q_{k-1}^{'}};$$

(c) $2^{n_k^{'}-n_{k-1}^{'}} \leq m(n_k^{'}) < 2^{n_k^{'}}$ for all $k \geq k_0$.

First, consider case (a). We choose a sequence of integers $\{s_k : k \geq 1\}$ of $\{s_k^{'} : k \geq 1\}$ such that

$$\sum_{i=1}^{m(s_k)} \frac{1}{\lambda_i} \geq 2^{2s_{k-1}/p}.$$

Then, relation (2) yields

$$\lim_{k\to\infty} \omega\left(\frac{1}{2^{s_k}}\right) 2^{s_k/p} = 0.$$

Let $\{r_k : k \geq 1\} \subset \{s_k : k \geq 1\}$ be such that

$$\omega^p\left(\frac{1}{2^{r_k}}\right) 2^{r_k/p} \leq \omega\left(\frac{1}{2^{r_k}}\right) 2^{r_k/p} \leq 4^{-k}. \tag{3}$$

Consider the function f defined as follows:

$$f(x) = \begin{cases} 2c_j(2^{r_j}x - 1), & x \in [2^{-r_j}, 3(2^{-r_j-1})), \\ -2c_j(2^{r_j}x - 2), & x \in [3(2^{-r_j-1}),\ 2(2^{-r_j})) \text{ for } j = 1, 2, ..., \\ 0, & \text{otherwise} \ , \end{cases}$$

and

$$f(x + l) = f(x), \quad l = \pm 1, \pm 2, ...,$$

where $c_j = \sqrt{\omega\left(\frac{1}{2^{r_j}}\right) 2^{r_j/p}}$. From the construction of the function f and relation (3) it follows that $f \in \Lambda BV^{(p)}$.

Now, consider case (b). Let $\{q_k : k \geq 1\} \subset \{q_k^{'} : k \geq 1\}$ be such that

$$\frac{1}{\omega(2^{-q_k})} \frac{1}{\sum_{i=1}^{m(q_k)} 1/\lambda_i} \left(\frac{m(q_k)}{2^{q_k}}\right)^{1/p} \geq 4^k. \tag{4}$$

Consider the function g_k defined as follows:

$$\mathrm{g}_k(x) = \begin{cases} h_k(2^{q_k}x - 2j + 1), & x \in [(2j-1)/2^{q_k}, 2j/2^{q_k}), \\ -h_k(2^{q_k}x - 2j - 1), & x \in [2j/2^{q_k}, (2j+1)/2^{q_k}) \\ & \text{for } j = m(q_{k-1}), ..., m(q_k) - 1, \\ 0, & \text{otherwise }, \end{cases}$$

where $h_k = \frac{1}{2^k \sum_{j=1}^{m(q_k)} 1/\lambda_j}$.

Let

$$\mathrm{g}(x) = \sum_{k=2}^{\infty} \mathrm{g}_k(x), \qquad \mathrm{g}(x+l) = \mathrm{g}(x), \qquad l = \pm 1, \pm 2, ...$$

First, we prove that $\mathrm{g} \in \Lambda BV^{(p)}$. For each non overlapping intervals $\{I_n : n \geq 1\}$, we have

$$\sum_{j=1}^{\infty} \frac{|\mathrm{g}(I_j)|^p}{\lambda_j} \leq 2^p \sum_{i=1}^{\infty} h_i^p \sum_{j=1}^{m(q_i)} \frac{1}{\lambda_j} \leq 2^p \sum_{i=1}^{\infty} h_i \sum_{j=1}^{m(q_i)} \frac{1}{\lambda_j} = 2^p \sum_{i=1}^{\infty} \frac{1}{2^i} < \infty.$$

Hence, $\mathrm{g} \in \Lambda BV^{(p)}$.

Finally, consider case (c). Let $\{n_k : k \geq 1\} \subset \{n_k^{'} : k \geq k_0\}$ be such that

$$n_k \geq 2n_{k-1} + 1, \tag{5}$$

$$\frac{1}{\omega\left(2^{-n_k}\right)} \frac{1}{\sum_{i=1}^{m(n_k)} 1/\lambda_i} \left(\frac{m(n_k)}{2^{n_k}}\right)^{1/p} \geq 2^{2n_{k-1}/p+k}. \tag{6}$$

Consider the function ϕ_k defined as follows:

$$\phi_k(x) = \begin{cases} d_k(2^{n_k}x - 2j + 1), & x \in [(2j-1)/2^{n_k}, 2j/2^{n_k}), \\ -d_k(2^{n_k}x - 2j - 1), & x \in [2j/2^{n_k}, (2j+1)/2^{n_k}) \\ & \text{for } j = 2^{n_{k-1}-n_{k-2}}, ..., 2^{n_k - n_{k-1}-1} - 1, \\ 0, & \text{otherwise }, \end{cases}$$

where $d_k = \frac{1}{2^k \sum_{j=1}^{m(n_k)} 1/\lambda_j}$.

Let

$$\phi(x) = \sum_{k=3}^{\infty} \phi_k(x), \qquad \phi(x+l) = \phi(x), \qquad l = \pm 1, \pm 2, ...$$

For every choice of nonoverlapping intervals $\{I_n, n \geq 1\}$, we have

$$\begin{aligned}\sum_{j=1}^{\infty} \frac{|\phi(I_j)|^p}{\lambda_j} &\leq 2^p \sum_{i=2}^{\infty} d_i^p \sum_{j=1}^{2^{n_i-n_{i-1}-1}} \frac{1}{\lambda_j} \leq 2^p \sum_{i=2}^{\infty} d_i \sum_{j=1}^{2^{n_i-n_{i-1}-1}} \frac{1}{\lambda_j} \\ &\leq 2^p \sum_{i=2}^{\infty} d_i \sum_{j=1}^{m(n_i)} \frac{1}{\lambda_j} \leq 2^p \sum_{i=2}^{\infty} \frac{1}{2^i} < \infty.\end{aligned}$$

Hence $\phi \in \Lambda BV^{(p)}$, by [3, Theorem 1] we have f, g and ϕ do not belong to H_ω^p. Therefore the proof is completed. ∎

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INFLUENCE OF VARIABLE FLUID PROPERTIES, THERMAL RADIATION AND CHEMICAL REACTION ON MHD SLIP FLOW OVER A FLAT PLATE

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Abstract. In the present study the magneto hydrodynamic (MHD) slip flow and heat transfer over a flat plate with convective surface heat flux at the boundary and temperature dependent fluid properties has been presented in presence of chemical reactions, thermal radiation and non-uniform heat source/sink. The transverse magnetic field is assumed as a function of the distance from the origin. Also it is assumed that the fluid viscosity and the thermal conductivity vary as an inverse function and linear function of temperature respectively. Using the similarity transformation, the governing system of equations are transformed into similarity non-linear ordinary differential equations which are solved numerically using symbolic software MATHEMATICA. As a result, the dimensionless velocity, temperature, concentration, the skin friction coefficient, the Nusselt number and the local Sherwood number are presented through graphs and tables for several sets of values of the involved parameters of the problem and discussed in details from the physical point of view.

Keywords: slip flow, variable viscosity, variable thermal conductivity, chemical reaction.

2010 Mathematics Subject Classification: 76W05, 76V05.

1. Introduction

The boundary layer flow for an electrically conducting fluid have been discussed by many authors [1]-[9] and historically Rossow [1] was the first to study the hydrodynamic behavior of the boundary layer on a semi-infinite plate in the presence of a uniform transverse magnetic field. Varshney and Kumar [10] studied magnetohydrodynamic boundary layer flow of non-Newtonian fluid past a flat plate.

The similarity solution for the thermal boundary layer for the case of constant surface temperature at the plate is well established [11]. Kays and Crawford [12] proposed that similarity solution does not exist for the boundary condition of constant heat flux at the plate. Bejan [13] disproved their claim by suggesting a different similarity temperature variable which reduced the energy equation to an ordinary differential equation. Aziz [14] has studied thermal boundary layer flow over a flat plate considering convective surface heat flux at the lower surface of the plate and established a condition for similarity solution. Later on Ishak [15], Yao and Zhong [16] developed the problem under different conditions and in the presence of various physical effects.

In all the aforementioned papers the thermo physical properties of the ambient fluid were assumed to be constant. However, it is well known [17]-[21] that these physical properties may change with temperature, especially fluid viscosity and thermal conductivity. For lubricating fluids, heat generated by internal friction and the corresponding rise in the temperature affects the physical properties of the fluid and so the properties of the fluid are no longer assumed to be constant. The increase in temperature leads to increase in the transport phenomena by reducing the physical properties across the thermal boundary layer and so the heat transfer at the wall is also affected. Therefore to predict the flow and heat transfer rates, it is necessary to take into account the variable fluid properties.

Slip flow happens if the characteristic size of the flow system is small or the flow pressure is very low. In no-slip-flow, as a requirement of continuum physics, the fluid velocity is zero at a solid-fluid interface. When fluid flows in micro electro mechanical system (MEMS), the no slip condition at the solid-fluid interface is no longer applicable. Beavers and Joseph [22] were the first to investigate the fluid flow at the interface between a porous medium and fluid layer in an experimental study and proposed a slip boundary conditions at the porous interface. The slip flows under different flow configurations have been studied in recent years [23]-[27]. Recently, Das [28] have considered the slip effects on heat and mass transfer in MHD micropolar fluid flow over an inclined plate with thermal radiation and chemical reaction.

However, the effect of thermal radiation on the flow and heat transfer have not been taken into account in the most of the investigations. The effect of radiation on MHD flow and heat transfer problem have become more important industrially. At high operating temperature, radiation effect can be quite significant. Cogley et al. [29] showed that in the optically thin limit, the fluid does not absorb its own emitted radiation but the fluid does absorb radiation emitted by the boundaries. Raptis [30] investigated the steady flow of a viscous fluid through a porous medium bounded by a porous plate subject to a constant suction velocity in presence of thermal radiation. Makinde [31] examined the transient free convection interaction with thermal radiation of an absorbing emitting fluid along moving vertical permeable plate. Ibrahim et al. [32] discussed the case of mixed convection flow of a micropolar fluid past a semi infinite, steady moving porous plate with varying suction velocity normal to the plate in presence of thermal radiation and viscous dissipation. Recently, Das [33] investigated the impact of thermal radiation on MHD slip flow over a flat plate with variable fluid properties.

The present trend in the field of chemical reaction analysis is to give a mathematical model for the system to predict the reactor performance. In particular, the study of heat and mass transfer with chemical reaction is of considerable importance in chemical and hydro metallurgical industries. Chamkha [34] investigated the problem of heat and mass transfer by steady flow of an electrically conducting fluid past a moving vertical surface in presence of first order chemical reaction. The problems involving chemical reactions can be found in the studies of Damseh et al. [35], Magyari and Chamkha [36] and Das [37]. Yazdi et al. [38] discussed slip MHD liquid flow and heat transfer over non-linear permeable stretching surface with chemical reaction.

In this paper, the work of Das [33] has been extended to investigate the effect of chemical reaction on the hydro-magnetic flow and heat transfer over an impermeable flat plate with variable fluid properties in presence of thermal radiation. The resulting governing equations have been transformed into a system of non-linear ordinary differential equations by applying a suitable similarity transformation.

2. Mathematical formulation of the problem

Consider a steady two dimensional laminar flow of an electrically conducting incompressible fluid moving over an impermeable flat plate under the influence of a transverse magnetic field $\vec{B}$ in the presence of non-uniform heat source/sink, chemical reaction and thermal radiation. The magnetic Reynolds number of the flow is taken to be small enough so that induced magnetic field is assumed to be negligible in comparison with applied magnetic field so that $\vec{B}=[0, B(x)]$, where $B(x)$ is the applied magnetic field acting normal to the plate and varies in strength as a function of x. The flow is assumed to be in the x-direction which is taken along the plate and y-axis is normal to it. The viscosity and thermal conductivity of the fluid are assumed to be functions of temperature.

Under the foregoing assumptions, the governing boundary layer equations [20, 33] for the present problem can be written as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2.1}$$

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) - \sigma B^2(x)(u - U_\infty), \tag{2.2}$$

$$\rho c_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = \frac{\partial}{\partial y}\left(\kappa\frac{\partial T}{\partial y}\right) - \frac{\partial q_r}{\partial y} + q''', \tag{2.3}$$

$$u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = D_m\frac{\partial^2 T}{\partial y^2} - k_r C^m, \tag{2.4}$$

where u, v are velocity components along x, y-axis respectively, U_∞ is the free stream velocity, σ is the electrical conductivity of the fluid, T is the temperature of the fluid within the boundary layer, κ is the thermal conductivity of the fluid, c_p is the specific heat at constant pressure p, μ is the dynamic viscosity, ρ is the constant fluid density, C is the concentration of the fluid within the boundary layer, D_m is the chemical molecular diffusivity, k_r is the chemical reaction rate constant and m is order of chemical reaction.

The radiative heat flux term q_r by using the Rosseland approximation is given by

$$q_r = -\frac{4\sigma^*}{3k^*}\frac{\partial T^4}{\partial y} \tag{2.5}$$

where σ^* is the Stefan-Boltzmann constant and k^* is the mean absorption coefficient. Assuming that the differences in temperature within the flow are such that T^4 can be expressed as a linear combination of the temperature, we expand T^4 in Taylor's series about T_∞ and neglecting higher order terms, we get

$$T^4 = 4T_\infty^3 T - 3T_\infty^4 \tag{2.6}$$

Thus we have

$$\frac{\partial q_r}{\partial y} = -\frac{16T_\infty^3\sigma^*}{3k^*}\frac{\partial^2 T}{\partial y^2} \tag{2.7}$$

Using equation (2.7) in equation (2.3), we obtain

$$\rho c_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = \frac{\partial}{\partial y}\left[\left(\kappa + \frac{16T_\infty^3\sigma^*}{3k^*}\right)\frac{\partial T}{\partial y}\right] + q''', \tag{2.8}$$

The appropriate boundary conditions for the present problem are

$$\left.\begin{aligned} &u = L\frac{\partial u}{\partial y} \text{ (partial slip), v=0 (impermeable surface),}\\ &C = C_w - \kappa\frac{\partial T}{\partial y} = h_w(T_w - T) \text{ (convective surface heat flux) for } y = 0,\\ &u = U_\infty,\ T = T_\infty,\ C = C_\infty \text{ as } y \to \infty \end{aligned}\right\} \tag{2.9}$$

where L is the slip length and h_w is the convective heat transfer coefficient.

Now we transform the system of equations (2.2), (2.4), (2.8) and (2.9) into a dimensionless form. To this end, let the us introduce the following dimensionless variables:

$$\begin{aligned} &\eta = y\left(\frac{U_\infty}{v_\infty x}\right)^{1/2}, \quad f(\eta) = \frac{\psi}{(U_\infty v_\infty x)^{1/2}},\\ &\theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}, \quad \phi(\eta) = \frac{C - C_\infty}{C_w - C_\infty} \end{aligned} \tag{2.10}$$

where $\psi(x, y)$ is the stream function, $v_\infty = \mu_\infty/\rho$ is the kinematic viscosity of the ambient fluid. Since $u = \frac{\partial\psi}{\partial y}$ and $v = -\frac{\partial\psi}{\partial x}$, we have from (2.10)

$$u = U_\infty f' \ \textit{and}\ v = -\frac{1}{2}\left(\frac{v_\infty U_\infty}{x}\right)^{1/2}(f - \eta f') \tag{2.11}$$

where f is non-dimensional stream function and prime denotes differentiation with respect to η.

In order to predict the flow and heat transfer rates accurately, Ling and Dybbs [42] suggested a temperature dependent viscosity of the form

$$\frac{1}{\mu} = \frac{1}{\mu_\infty}[1 + \gamma(T - T_\infty)] \tag{2.12}$$

where γ is the thermal property of fluid, T_∞ is the temperature of the fluid outside the boundary layer and μ_∞ is the dynamic viscosity at ambient temperature.

Equation (2.12) can be written as

$$\frac{1}{\mu} = A(T - T_r) \tag{2.13}$$

where $A = \frac{\gamma}{\mu_\infty}$ and $T_r = T_\infty - \frac{1}{\gamma}$. In general, $A > 0$ corresponds to liquids and $A < 0$ to gases when the temperature at the plate is larger than that of the temperature at far away from the plate.

The non-uniform heat source/sink q''' is given by [33]

$$q''' = \frac{\kappa_\infty U_0}{2\nu_\infty x}[Q(T - T_\infty) + Q^*(T_w - T_\infty)e^{-\alpha' y}] \tag{2.14}$$

where κ_∞ is the thermal conductivity at ambient temperature, Q and Q^* are the coefficients of space and temperature dependent heat source/sink terms respectively and α' is the thermal property of fluid.

The dimensionless temperature θ can also be written as

$$\theta = \frac{T - T_r}{T_w - T_\infty} + \theta_r \tag{2.15}$$

where $\theta_r = T_r - T_\infty/(T_w - T_\infty) = -1/\delta(T_w - T_\infty)$. Using (2.15), equation (2.13) becomes

$$\mu = \mu_\infty \left(\frac{\theta_r}{\theta_r - \theta} \right) \tag{2.16}$$

Following Chiam [17], we consider the specific model for variable thermal conductivity as

$$\kappa = \kappa_\infty \left(1 + \varepsilon \frac{T - T_\infty}{\Delta T} \right) \tag{2.17}$$

where ε is the thermal conductivity parameter and $\Delta T = T_w - T_\infty$. This relation can be written as

$$\kappa = \kappa_\infty(1 + \varepsilon\theta) \tag{2.18}$$

Now, introducing equations (2.16) and (2.18) into equations (2.2), (2.4) and (2.8), we obtain,

$$\left(\frac{\theta_r}{\theta_r-\theta}\right)f''' + \frac{1}{2}ff'' + \frac{\theta_r}{(\theta_r-\theta)^2}f''\theta' - M(f'-1) = 0, \tag{2.19}$$

$$(1+\varepsilon\theta+Nr)\theta'' + \varepsilon\theta'^2 + \frac{1}{2}Pr_\infty f\theta' + Q\theta + Q^* e^{-\alpha\eta} = 0, \tag{2.20}$$

$$\phi'' + Scf\phi' - Kr\phi^m = 0 \tag{2.21}$$

where $M = \sigma' B^2(x)/\rho U_\infty$ is the magnetic field parameter, $Pr_\infty = \mu_\infty c_p/\kappa_\infty$ is the ambient Prandtl number, $Nr = 16T_\infty^3\sigma^*/3k^*\kappa_\infty$ is the thermal radiation parameter, $\alpha = \alpha'(\frac{\nu_\infty x}{U_\infty})$ is the thermal property of fluid, $Sc = \nu/D_m$ is the Schmidt number and $Kr = kr\nu^2/D_m U_\infty^2$ is the chemical reaction rate parameter.

The corresponding boundary conditions (2.9) become

$$\left.\begin{array}{l} f=0, f'=\delta f'', \theta' = -a\left(\dfrac{1-\theta(0)}{1+\varepsilon\theta(0)}\right), \phi = 1 \text{ for } \eta = 0, \\ f'=1, \theta=0, \phi=0 \text{ as } \eta\to\infty \end{array}\right\} \tag{2.22}$$

where $a = \frac{h_w}{\kappa_\infty}\left(\frac{\nu_\infty x}{U_\infty}\right)^{1/2}$ is the surface convection parameter and $\delta = L\left(\frac{U_\infty}{\nu_\infty x}\right)^{1/2}$ is the slip parameter.

In the present study, both viscosity and thermal conductivity vary across the boundary layer so it is reasonable to consider the Prandtl number as a variable and is defined as (see Rahman [20] and Rahman et al. [21])

$$Pr = \frac{\mu c_p}{\kappa} = \frac{(\frac{\theta_r}{\theta_r-\theta})\mu_\infty c_p}{\kappa_\infty(1+\varepsilon\theta)} = \frac{1}{(1-\frac{\theta}{\theta_r})(1+\varepsilon\theta)}Pr_\infty \tag{2.23}$$

Using equation (2.23), the non-dimensional energy equation (2.20) can be written as

$$(1+\varepsilon\theta+Nr)\theta''+\varepsilon\theta'^2+Pr\left(1-\frac{\theta}{\theta_r}\right)(1+\varepsilon\theta)(f\theta'-f'\theta)+Q\theta+Q^*e^{-\alpha\eta} = 0 \tag{2.24}$$

It should be noted that for large θ_r and small ε i.e $\theta_r \to \infty$ and $\varepsilon \to 0$, the variable Prandtl number Pr becomes the ambient Prandtl number Pr_∞ and in that case equation (2.24) reduces to the equation (2.20).

The quantities of main physical interest are the skin friction coefficient (rate of shear stress), the Nusselt number (rate of heat transfer) and the Sherwood number (rate of mass transfer). The equation defining the wall shear stress is

$$\tau_w = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} \tag{2.25}$$

The local skin friction coefficient is defined as

$$C_f = 2Re_x^{-1/2}\left[\left(\frac{\theta_r}{\theta_r-\theta(0)}\right)\right]f''(0) \tag{2.26}$$

or,

$$C_f^* = \left(\frac{\theta_r}{\theta_r - \theta(0)}\right) f''(0) \text{ where } C_f^* = \frac{1}{2} Re_x^{1/2} C_f \tag{2.27}$$

Knowing the temperature field, it is interesting to study the effect of the free convection and thermal radiation on the rate of heat transfer q_w, is given by

$$q_w = -\kappa \left(\frac{\partial T}{\partial y}\right)_{y=0} - \frac{4\sigma^*}{3k^*}\left(\frac{\partial T^4}{\partial y}\right)_{y=0} \tag{2.28}$$

So the rate of heat transfer in terms of the dimensionless Nusselt number is defined as follows:

$$Nu = -\frac{1}{2} Re_x^{1/2}(1 + \varepsilon\theta(0) + Nr)\theta'(0) \tag{2.29}$$

or,

$$Nu^* = -(1 + \varepsilon\theta(0) + Nr)\theta'(0) \text{ where } Nu^* = 2Re_x^{-1/2} Nu \tag{2.30}$$

Similarly, the rate of mass transfer in terms of local Sherwood number is given by

$$Sh^* = -\phi'(0) \tag{2.31}$$

3. Method of solution

The non-linear differential equations (2.19), (2.21) and (2.24) with boundary conditions (2.22) have been solved in the symbolic computation software MATHEMATICA using finite difference code that implements the 3-stage Lobatto IIIa formula for partitioned Runge-Kutta method. For numerical computation infinity condition has been taken at a large but finite value of η where no considerable variation in velocity, temperature etc. occur. To check the validity of the present code, the values of $-\theta'(0)$ have been calculated for different values of the surface convection parameter a and Prandtl number Pr in Table 1. From Table 1, it has been observed that the data produced by present code and those of Rahman [20] and Das [33] show excellent agreement and so justifies the use of the present numerical code.

4. Numerical results and discussions

In order to get a clear insight of the present problem, the numerical results for velocity, temperature, concentration etc. have been presented graphically in Figs. 1-10 and in Tables 1-3 for several sets of values of the pertinent parameters. In the simulation the default values of the parameters are considered as $\delta = 0.2$, $a = 0.2$, $\theta_r = 2.5$, $\varepsilon = 0.5$, $M = 0.5$, $Nr = 0.2$, $Pr = 0.71$, $\alpha = 1$, $Q = 0.2$, $Q^* = 0.3$, $Kr = 0.3$, $Sc = 0.4$ and $m = 1$ unless otherwise specified.

4.1. Computational results for velocity profiles

In Figs. 1-4 we presented the behavior of the fluid velocity for various material parameters. It can be easily seen from Fig. 1 that the fluid velocity within the boundary layer increases with the increase of δ and, as a result, thickness of momentum boundary layer increases. From Fig. 2 we see that $f'(\eta)$ is considerably increased with an increase in the surface convection parameter a but effect is not significant for higher values of a. The variations of the velocity profiles against transverse coordinate η are shown in Figs.3 for various values of viscosity parameter θ_r. The results indicate that with increase in the parameter θ_r, the velocity profiles increases within the boundary region. Fig. 4 illustrates the effect of thermal radiation parameter Nr on velocity profiles. From figure we see that the velocity increases as η increases for a fixed value of Nr. For a non-zero fixed value of η, the velocity distribution across the boundary layer increases with the increasing values of Nr. Table 2 shows that surface convection parameter a enhances the skin friction coefficient C_f^*. It is evident from the table 3 that the skin friction coefficient C_f^* decreases on increasing δ and θ_r.

4.2. Computational results for temperature profiles

The effect of various physical parameters on the fluid temperature are illustrated in Figs. 5-8. Fig. 5 shows that the fluid temperature is the maximum near the boundary layer region and it decreases on increasing boundary layer coordinate η to approach free stream value. Also fluid temperature decreases on increasing δ in the boundary layer region and, as a consequence, thickness of the thermal boundary layer decreases. Fig. 6 demonstrates the effects of a on fluid temperature in the presence of non-uniform heat source/sink. It is observed from the figure that temperature $\theta(\eta)$ decreases on increasing a in the boundary layer region and is maximum at the surface of the plate. The solution approaches to the solution for constant surface temperature for large values of a, i.e., $a \to \infty$. For a non-zero fixed value of η, temperature distribution across the boundary layer decreases with the increasing values of Nr and hence the thickness of thermal boundary layer decreases as shown in Fig. 7. The influence of viscosity parameter θ_r on temperature distribution are highlighted in Fig. 8. It is seen that as θ_r increases, the thickness of the thermal boundary layer decreases with a consequent reduction of the temperature in the boundary layer. From Table 2, we observed that Nu^* increases with increasing a and Nr. The influence of variable viscosity parameter θ_r on Nu^* is presented in Table 3. It is observed from this table that θ_r enhances the dimensionless Nusselt number.

4.3. Computational results for concentration profiles

Fig. 9 illustrates the variation of the concentration distribution across the boundary layer for various values of the chemical reaction parameter Kr. It is seen that the effect of increasing values of the chemical reaction parameter results in decreasing concentration distribution across the boundary layer. Fig. 10 shows the variation of concentration profiles for different values of reaction order parameter m. It is observed from this figure that the concentration profiles increase with increasing m but effect is not significant for higher order reaction. It is found from Table 2 that an increase in Kr leads to increase in the values of the dimensionless Sherwood number Sh^*. It is observed from Table 3 that as reaction-order parameter m increases, the dimensionless Sherwood number Sh^* decreases.

5. Conclusions

The effects of chemical reaction and thermal radiation on steady two dimensional boundary layer flow of an incompressible electrically conducting fluid over a flat plate with partial slip at the surface of the boundary with temperature dependent fluid viscosity as well as with variable thermal conductivity have been studied in the present paper. Numerical results are presented to illustrate the details of the flow, heat and mass transfer characteristics and their dependence on material parameters. Following conclusions can be drawn from the present investigation:

(i) The velocity distribution are increasing for increasing values of slip parameter δ, surface convection parameter a and variable viscosity parameter θ_r.

(ii) The temperature profile decreases with a increasing of slip parameter δ, surface convection parameter a and thermal radiation parameter Nr while the opposite effect is observed for variable viscosity parameter θ_r.

(iii) The chemical species concentration decreases with increase of Kr but reverse effect occurs for m.

(iv) The skin friction coefficient decreases with increase of thermal radiation parameter Nr, slip parameter δ and variable viscosity parameter θ_r but effect is reverse for surface convection parameter a.

(v) Nusselt number Nu^* increases for increasing of surface convection parameter a, thermal radiation parameter Nr and variable viscosity parameter θ_r while it decreases for increasing slip parameter δ.

(vi) Sherwood number Sh^* decreases with increase of reaction-order parameter m but effect is opposite for chemical reaction parameter Kr.

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Table 1. Comparison of the values of $-\theta'(0)$ for various values of a in the absence of mass transfer

	Rahman [20]		Das [33]		Present results	
a	$Pr = 0.1$	$Pr = 0.71$	$Pr = 0.1$	$Pr = 0.71$	$Pr = 0.1$	$Pr = 0.71$
0.05	0.036900	0.042781	0.036866	0.042767	0.036863	0.042762
0.2	0.082477	0.119358	0.082473	0.119295	0.082483	0.119288
0.6	0.113688	0.198155	0.113741	0.198051	0.113722	0.198051
1.0	0.122999	0.228303	0.123074	0.228178	0.123039	0.228178
5.0	0.136400	0.279283	0.136515	0.279131	0.136519	0.279135

Table 2. Effects of a, Kr and Q, Q^* on C_f^*, Nu^* and Sh^*.

a	Kr	$Q = Q^*$	C_f^*	Nu^*	Sh^*
0.0	0.3	0.5	0.583515	0.000000	
0.4			0.605034	0.122074	
1.0			0.612411	0.158639	
0.2	0.0	0.5			0.407903
	0.6				0.82648
	0.9				0.984198
0.2	0.3	0.0	0.576377	0.183116	
		0.5	0.788644	-0.890034	
		1.0	0.476148	0.876308	

Table 3. Effects of θ_r, δ and m on C_f^*, Nu^* and Sh^*

θ_r	δ	m	C_f^*	Nu^*	Sh^*
2.0	0.3	1.0	0.677068	0.0882478	
5.0			0.598672	0.0934037	
∞			0.559764	0.0964655	
2.0	0.0	1.0	0.732151		
	0.6		0.58307		
	1.2		0.476659		
2.0	0.3	1.0			0.640209
		2.0			0.577629
		3.0			0.543216

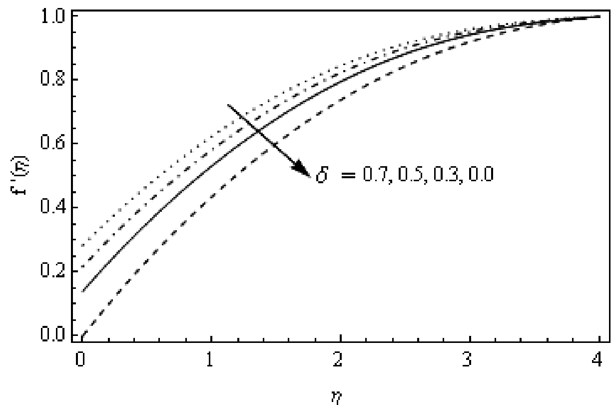

Figure 1: Velocity profiles for various values of δ

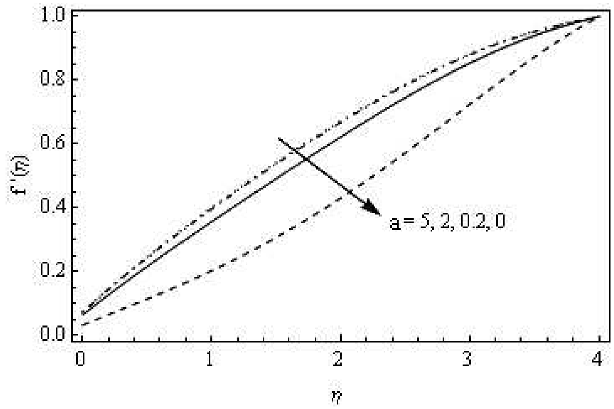

Figure 2: Velocity profiles for various values of a

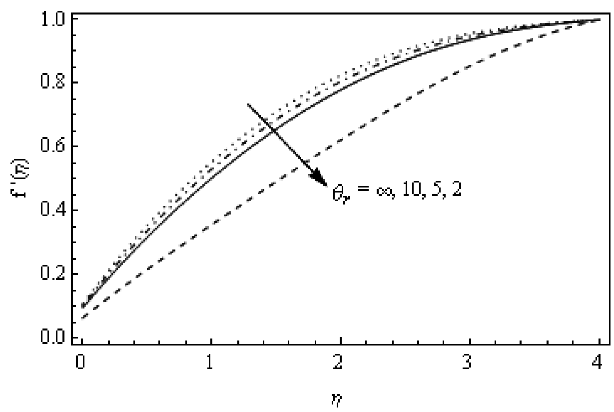

Figure 3: Velocity profiles for various values of θ_r.

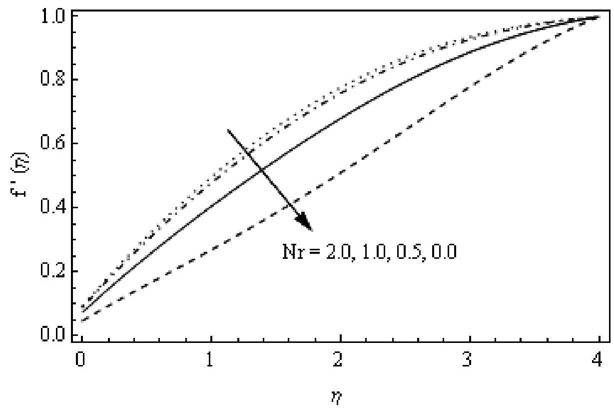

Figure 4: Velocity profiles for various values of Nr.

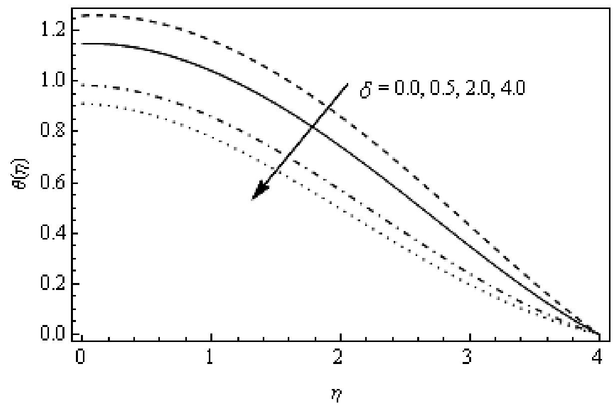

Figure 5: Temperature profiles for various values of δ.

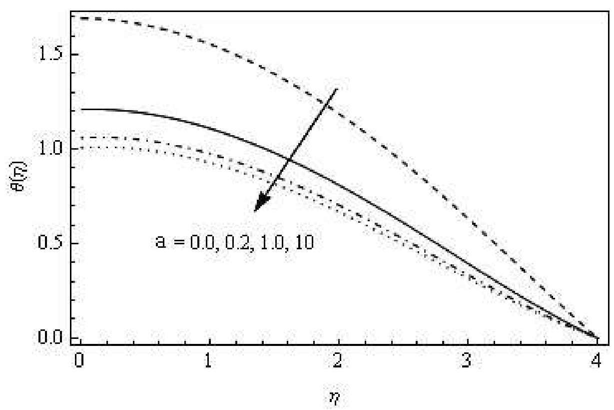

Figure 6: Temperature profiles for various values of a.

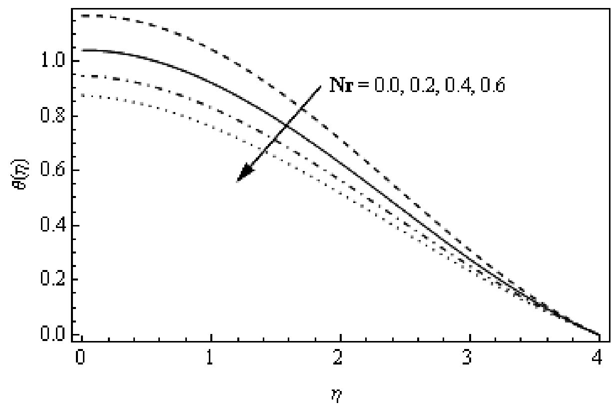

Figure 7: Temperature profiles for various values of Nr.

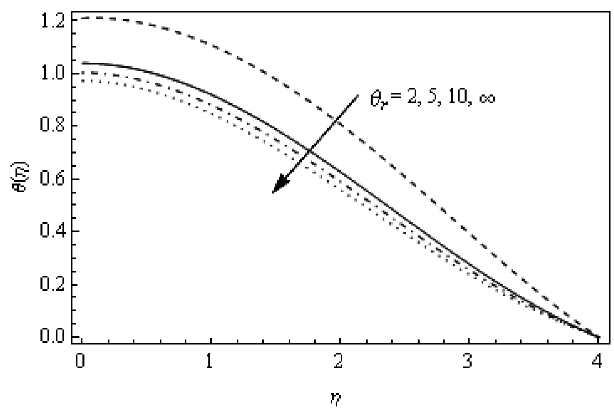

Figure 8: Temperature profiles for various values of θ_r.

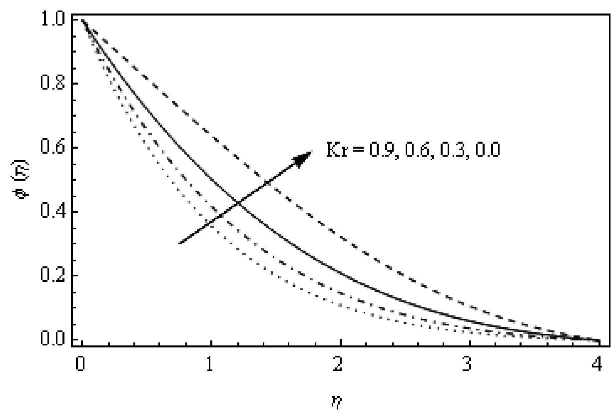

Figure 9: Concentration profiles for various values of Kr.

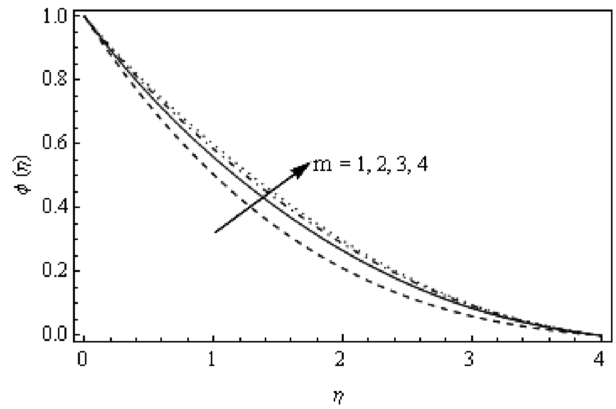

Figure 10: Concentration profiles for various values of m.

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NORMAL INDUCED FUZZY TOPOLOGICAL SPACES

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Abstract. The motto of the present treatise is to introduce and characterize the concept of n-infy induced fuzzy topological spaces generated by normal lower semi-continuous functions. Examples of n-infy induced fuzzy topological spaces are given and its properties are studied. Interrelationship between the newly defined induced spaces and their corresponding topological spaces are examined.

Keywords: Regular open and regular closed subsets, normal lower semi-continuous function, topology, induced fuzzy topological space, n-continuous mapping, r-continuous mapping.

AMS Classification: 54 A 40.

1. Introduction

In 1965, L.A. Zadeh introduced the fuzzy set in his classical paper [16]. Since then many researchers used this tool to generalize different concepts of Mathematics. One such successful generalization is fuzzy topology from general topology. With the help of fuzzy set, C.L. Chang [6] defined fuzzy topological space as a generalization of topological space as follows:

Let X be a non-empty set. A family $\mathscr{F}$ of fuzzy subsets of X is called fuzzy topology on X if

(i) $0(\equiv \mu_\phi)$, $1(\equiv \mu_X) \in \mathscr{F}$

(ii) arbitrary supremum of members of $\mathscr{F}$ is in $\mathscr{F}$ (iii) finite infimum of members of $\mathscr{F}$ is in $\mathscr{F}$.

Later, R. Lowen [11] suggested an alternative and more natural definition of fuzzy topology for achieving more results which are compatible to the general case in topology by incorporating all constant functions instead of only 0 and 1 in (i) of Chang's definition. It has been seen that fuzzy topology has numerous applications in science and technology including medical sciences.

After this, several researchers successfully generalized the concepts of fuzzy topology further. Fuzzy supra topology and fuzzy minimal structure are two examples of these kinds. Monsef and Ramadan [12] introduced the concept of fuzzy supra topology and Alimohammady and Roohi [1] introduced fuzzy minimal structure. Again it has been seen that these generalized fuzzy structures also have applications in various branches of science and technology.

Again Weiss [15] showed that there is a natural way to associate a fuzzy topology $\mathscr{F}$ on a set X with a given topology T on X by means of collections of lower semi continuous (LSC) functions from a topological space (X, T) to unit closed interval I and he called this fuzzy topology as induced fuzzy topology on X. In 1976, R. Lowen [11] noticed the natural association between a topological space and a fuzzy topological space on a set X and introduced the notion of topologically generated space which is same as the induced fuzzy topological space of Weiss and further studied category of the fuzzy topological space. The notion of induced fuzzy topological spaces was further studied in [4], [5], [13].

Defining all the concepts stated above, the LSC function and their stronger forms (viz. Completely LSC, δ-LSC) played the key role. But it is interesting to notice that all the stronger forms of LSC functions fails to satisfy the conditions to become a fuzzy topology. So, it would be interesting to study such structures formed by other forms of LSC functions which either

(i) fail to preserve point wise arbitrary suprema but is closed under countable point wise suprema and finite infima, or

(ii) is closed under finite infima only but not closed even under finite suprema.

In this paper, we study the structure formed by the functions that are of category (ii) only. The structure formed by the functions of the category (i) has been defined as countable fuzzy topology in [2] and with the help of regular lower semi continuous (RLSC) functions [10] a countably induced fuzzy topological space viz. r-countably induced fuzzy topological space has been introduced and studied in [2], [3].

The motivation of the present paper is to search for a function which is not closed even under finite suprema but is closed under finite infima and to form a generalized induced fuzzy topological structure viz. n-infy induced fuzzy topological space. This has further tempted us to define infy fuzzy topology in this paper and the properties of this generalized fuzzy structure will be studied elsewhere.

Let us first introduce the concept of fuzzy infy topological space in the following way.

Definition 1.1: A family $\mathscr{F}$ of fuzzy sets in X is said to form a fuzzy infy topology in X if i) $r1_X \in \mathscr{F}$, for r $\in$ I. ii) For any two fuzzy subset λ and μ of $\mathscr{F}$, $\lambda \wedge \mu \in \mathscr{F}$.

The space (X, $\mathscr{F}$) is called fuzzy infy topological space. Every member of $\mathscr{F}$ is called fuzzy i-open set and complement of an i-open set is called i-closed set.
In section 2, we recall the definition of NLSC function and then prove that the collection of all NLSC functions from a topological space X to I is closed under finite point wise infima but fails to be closed even under finite suprema. This led us to define a new structure viz. infy induced fuzzy topological space.
The infy induced fuzzy topological spaces generalize the usual notion of induced fuzzy topological spaces; for, every induced fuzzy topological space is infy induced fuzzy topological space but not conversely (Example 2.14). So, the properties possessed by infy induced fuzzy topological spaces will also valid for induced fuzzy topological spaces, which makes the study of such spaces worthy and meaningful.

2. n-infy induced fuzzy space

In this section, we introduce the concept of n-infy induced fuzzy topological space with the help of NLSC functions. Before proceeding further, we define the concepts which are relevant for our proposed study and the results will be used in the sequel without any specific reference.

Definition 2.1 [14] A subset A of X is said to be regular open if $A = \text{int}(\text{cl } A)$. Alternatively, A is regular open if A is the interior of some closed set.

Clearly, every regular open set is open but the converse is not true, e.g. let $X = [0, 1]$, the set $A = (1/2, 1)$ is open in X. But cl $A = [1/2, 1]$ and int (cl A) $= (1/2, 1] \neq A$. Thus A is open but not regular open in X.

Definition 2.2. [14] A subset A of X is said to be regular closed if $A = \text{cl}(\text{int } A)$. Alternatively, A is regular closed if A is the closure of some open set.

Clearly, every regular closed set is closed but the converse is not true, e.g., let $X = [0, 1]$, $A = [0, 1/2] \cup \{1\}$ which is closed in X. Again, int $A = (0, 1/2)$ and cl (int A) $= [0, 1/2] \neq A$. Thus, A is closed but not regular closed in X.

It may be remarked here that the intersection of two regular open sets is regular open and union of two regular closed sets is regular closed. The complement of a regular open set is regular closed and conversely.

Definition 2.3. [9] A function $f : X \to R$ is called upper semi-continuous (respectively, lower semi-continuous) if for each $r \in R$, the set $\{x : f(x) < r\}$ (respectively $\{x : f(x) > r\}$) is open in X, i.e. the set $\{x : f(x) \geq r\}$ (respectively, $\{x : f(x) \leq r\}$) is closed in X.

Definition 2.4. [7] An upper semi-continuous (USC) function ϕ on X is normal iff for each $x \in X$, $\Phi(x) > \lambda$ and an open set U containing x, there exists a non-void open set V, $\overline{V} \subset U$ with $\phi(v) < \lambda$ for each $v \in V$, here $\overline{V} = cl(V)$.

Definition 2.5. [7] An USC function Φ on X is normal iff for each $\lambda \in R$, the set $\{x : \Phi(x) > \lambda\}$ is a union of regular closed sets.

Dually, one can define and characterize normal lower semi-continuous function as follows:

Definition 2.6. [7] A lower semi-continuous (LSC) function ϕ on X is normal iff for each $x \in X$, $\phi(x) < \lambda$ and an open set U containing x, there exists a non-void open set V, $\overline{V} \subset U$ with $\phi(v) < \lambda$ for each $v \in V$.

Theorem 2.7. [7] *An LSC function ϕ on X is normal iff for each $\lambda \in R$, the set $\{x : \phi(x) < \lambda\}$ is a union of regular closed sets.*

Corollary 2.8. *The characteristic function of a regular open* (*resp. regular closed*) *set is NLSC* (*resp. NUSC*) [8].

It can be easily seen that every continuous function is NLSC but the converse is not true, which is shown in the following example.

Example 2.9. Let $X = [0, 1]$ and $A = (1/2, 1]$. Then $\mathrm{cl}(A) = [1/2, 1]$ and int $\mathrm{cl}(A) = (1/2, 1] = A$. Thus, A is regular open. Let us now define a function $f : X \to I$ such that

$$f(x) = \begin{cases} 0, & x \notin A \\ 1, & x \in A. \end{cases}$$

Clearly, f is NLSC. Let us now consider an open set B of I, given by $B = [0, 2/3)$. So $f^{-1}(B) = X - A$, which is not open in X. Hence, f is not continuous.

Next, we study some properties of NLSC functions.

Theorem 2.10. *If f is NLSC, then for a ¿ 0, af is also NLSC.*

Proof. Let f is NLSC, so for each $r \in R$, the set $\{x : f(x) < r\}$ is a union of regular closed sets. Again, let us consider the set $\{x : af(x) < t\}$, $t \in R$. Now, $\{x : af(x) < t\} = \{x : f(x) < t/a\} = \{x : f(x) < p\}, p = t/a$, which is a union of regular closed sets, since f is NLSC. Similarly, we can check that, since f is LSC, af $(a > 0)$ is also LSC. Hence, af is NLSC. ∎

That the finite suprema of NLSC functions may not be NLSC is shown in the following example.

Example 2.11. Let $X = [0, 1]$, then $A_1 = [0, 1/3)$ and $A_2 = (1/3, 1]$ be two regular open subset of X. Now, we define

$$\phi_1(x) = \begin{cases} 1/2, & x \in A_1 \\ 0, & x \notin X - A_1 \end{cases}$$

and

$$\phi_2(x) = \begin{cases} 1/3, & x \in A_2 \\ 0, & x \notin X - A_2 \end{cases}$$

Then, ϕ_1, ϕ_2 are two NLSC functions on X. Now, $\Phi = \text{Sup}\{\phi_1, \phi_2\}$ is given by

$$\Phi(x) = \begin{cases} 1/2, & x \in [0, 1/3) \\ 0, & x \in 1/3 \\ 1/3 & x \in (1/3, 1] \end{cases}.$$

Now, we see that Φ is not NLSC at $x = 1/3$. For this, we note that $\Phi(1/3) = 0 < 1/6$ $(= \lambda$, say) and $U = (1/6, 1/2)$ is an open set containing 1/3. It can be seen that there does not exist any non-void open subset V of U with $\overline{V} \subset U$ such that $\Phi(v) < \lambda$, for all $v \in V$. Thus suprema of two NLSC functions may not be NLSC.

Remark. The above example further shows that the sum of two NLSC functions may not be an NLSC function, because the sum of the two functions $\phi_1(x)$ and $\phi_2(x)$ is same as their supremum $\Phi(x)$, which is not NLSC.

That the finite pointwise infima of NLSC functions is NLSC is shown in the following theorem.

Theorem 2.12. *The finite infima of NLSC functions is NLSC.*

Proof. Let $\psi(x) = \inf\{\phi_i(x)\}$, $i = 1, 2, ..., n$ and $x \in X$, where each $\phi_i(x)$ is NLSC. We prove that $\psi(x)$ is NLSC. Let for $x \in X, \psi(x) < \lambda$ and U be an open set containing x. Now, $\psi(x) < \lambda$ i.e. $\inf(\phi_i(x)) < \lambda$ implies that, there exists at least one i, say j, for which $\phi_j(x) < \lambda$. Again, ϕ_j being NLSC, $\phi_j(x) < \lambda$ and U is an open set containing x, so there exists a non-void open set V_j such that $\overline{V_j} \subset U$ and $\phi_j(v_{\alpha,j}) < \lambda$ for all $v_{\alpha,j} \in V_j$. Now, $\phi_j(v_{\alpha,j)} < \lambda$ for all $v_{\alpha,j} \in V_j$ implies $inf\phi_i(v) < \lambda$ for all $v \in \cup_j V_j$, i.e. $\psi(v) < \lambda$ for all $v \in \cup_j V_j = V$ (say), where $\overline{V} = \overline{\cup_j V_j} \subseteq \cup_j \overline{V_j} \subset U$. Again, since the finite infimum of LSC functions is also LSC, so ψ is also LSC. Hence, $\psi = inf\phi_i$, $i = 1, 2, ..., n$ is NLSC.

Now, we cite examples of infy induced fuzzy topological space.

Example 2.13. Let A be a regular open set of a topological space X. We define a family $\mathscr{F} = \{f_a : a \in I\}$ of functions as follows:

$$f_a(x) = \begin{cases} a & \text{if } x \in A \\ 0 & \text{if } x \in X - A \end{cases}$$

Clearly, all the members of $\mathscr{F}$ are NLSC. Now, we construct a family $\mathscr{F}^*$ of functions as follows: $\mathscr{F}^* = \{rf_a : f_a \in \mathscr{F}, 0 \leq r \leq 1\}$

So, $\mathscr{F}^*$ is a collection of NLSC functions on X such that

(i) $t1_X \in \mathscr{F}^*$, for all $t \in I$.

(ii) Let, $r_1 f_{a^1}, r_2 f_{a^2}, ..., r_n f_{a^n} \in \mathscr{F}^*$.

Then, $\wedge_i r_i f_{a^i} = g$ (say) is given by

$$g(x) = \begin{cases} \wedge_i a^i & \text{if } x \in A \\ 0 & \text{if } x \in X - A \text{ where } 0 \leq a^i \leq 1, \end{cases}$$

which shows that $g(x)$ belongs to $\mathscr{F}^*$.

Thus, we see that $(X, \mathscr{F}^*)$ forms an infy induced fuzzy topological space.

Example 2.14. Let us consider the family of functions $\Sigma = \{0, 1, \phi_1, \phi_2\}$ defined on the topological space $X = [0, 1]$ with usual topology, where ϕ_1 and ϕ_2 are defined as in Example 2.11. Then Σ forms an infy induced fuzzy topological space on X but (X, Σ) is not a induced fuzzy topological space.

Now, let us consider the family $\mathscr{F} = \{f_\alpha : \alpha \in \wedge\}$ of all NLSC functions from a topological space X to the closed unit interval I. Thus, by Theorems 2.10 and 2.12, we see that $\mathscr{F}$ satisfies the following properties:

(i) $t1_X \in \mathscr{F}$, for all $t \in I$.

(ii) $f_i \in \mathscr{F} \Rightarrow \inf_i f_i \in \mathscr{F}, \qquad i = 1, 2, ..., n.$

Thus, by Definition 1.1, the family $\mathscr{F}$ of all NLSC functions from X to I (henceforth denoted by $nL(X)$) forms a fuzzy infy topological space.

Definition 2.15. The family $\mathscr{F} = \{f_\alpha : \alpha \in \wedge\}$ of all NLSC functions from a topological space X to the closed unit interval I forms a fuzzy infy topology on X, this induced fuzzy infy topology is called normal infy induced fuzzy topology on X and the corresponding space denoted by $(X, n(T))$ is called normal infy induced fuzzy topological space or in short n-infy induced fts. The members of $n(T)$ are called n-open sets.

Definition 2.16. A subset H of X is said to be G^* if it can be expressed as intersection of regular open sets, i.e. H is G^* if $H = \cap_a \operatorname{int} F_a$, where $a \in \wedge$ and each Fa is closed. Here, we note that every regular open subset is clearly open G^*. With the help of G^* subset we have the following characterization of NLSC functions.

Definition 2.17. An LSC function $f : X \to R$ is NLSC iff the set $\{x : f(x) \geq r\}$ is a G^* set.

Proof. Let f be NLSC. So the set $A = \{x \in X : f(x) < r\}$ is a union of regular closed sets. i.e., $A = \cup_a clG_a$, where each G_a is open.

Now, $B = \{x \in X : f(x) \geq r\} = X - \{x \in X : f(x) < r\} = X - \cup_a clG_a = \cap_a (X - clG_a) = \cap_a \operatorname{int}(X - G_a) = \cap_a \operatorname{int} F_a$, where each $F_a = X - G_a$ is closed. Thus, B is G^* set.

Conversely, let f be LSC and the set $B = \{x \in X : f(x) \geq r\}$ be a G^* set. So $B = \cap_a \operatorname{int} F_a$, where each F_a is closed. Let us consider the set $A = \{x \in X : f(x) < r\}$. Then, $A = X - \{x \in X : f(x) \geq r\} = X - \cap_a \operatorname{int} F_a = \cup_a (X - \operatorname{int} F_a) = \cup_a cl(X - F_a) = \cup_a clG_a$, where each $G_a = X - F_a$ is open. Thus A is a union of regular open subsets of X and hence NLSC. ∎

Theorem 2.18. *Characteristic function of an open G^* set is NLSC.*

Proof. It is straightforward. ∎

Again, since every regular open set is open G^*, as a corollary of this result it follows that the characteristic function of a regular open set is NLSC.

For further discussion of the newly introduced n-infy induced fuzzy space, we assume the following property to hold in its underlying topological space (X, T).

Definition 2.19. A topological space (X, T) is said to have the property $*$ if for each NLSC function f on X, $\{x \in X : f(x) \leq r\} = cl_X\{x \in X : f(x) < r\}$.

Since f is LSC, so the set $A = \{x \in X : f(x) > r\}$ is open and so no point of A is the limit point of $\{x \in X : f(x) < r\}$. i.e., $\mathrm{cl}_X\{x \in X : f(x) < r\} \subseteq \{x \in X : f(x) \leq r\}$.

Now, let $x_0 \in \{x \in X : f(x) \leq r\} - \mathrm{cl}_X\{x \in X : f(x) < r\}$ which implies that $f(x_0) = r$ and there exists a neighbourhood of x_0, say $N(x_0)$ such that $N(x_0) \cap \{f(x) < rl\} = \phi$.

Thus, we infer that if there exists a point $x_0 \in X, f(x_0) = r$ such that $N(x_0)$ is a subset of $\{x : f(x) > r\} \cup \{x_0\}$, then the reverse inclusion is not true. Thus the property $*$ follows from the condition that the set of the form $\{x : f(x) < r\}$, which is a union of regular closed sets, is not closed in X, i.e. the set$\{x : f(x) \geq r\}$ is not open.

In the sequel all the topological spaces considered are assumed to have the property $*$.

Theorem 2.20. *A fuzzy subset $\lambda \in I^X$ in an n-infy induced fts $(X, n(T))$ is n-open iff for each $r \in I$, the strong r-cut $\sigma_r(\lambda)$ is regular open in the topological space (X, T).*

Proof. Let $\lambda \in n(T)$, i.e., λ be an NLSC function. Now, as λ is NLSC, so λ is LSC and for each $r \in R$, the set $\{x \in X : \lambda(x) < r\}$ is a union of regular closed sets. i.e., $\{x \in X : \lambda(x) < r\} = \cup_\alpha(\mathrm{cl}_X G_\alpha)$, where each G_α is open. Again, the set $\{x \in X : \lambda(x) > r\}$ is open, i.e., the set $\{x \in X : \lambda(x) \leq r\}$ is closed. But, by property $*$, we have

$$\{x \in X : \lambda(x) \leq r\} = \mathrm{cl}_X\{x \in X : \lambda(x) < r\} = \mathrm{cl}_X(\cup_\alpha(\mathrm{cl}_X G_\alpha)),$$

which is regular closed [8]. Hence, $\sigma_r(\lambda) = \{x \in X : \lambda(x) > r\}$ is regular open.

Conversely, let us assume that the strong r-cut $\sigma_r(\lambda) = \{x \in X : \lambda(x) > r\}$ is regular open, i.e., the sets of the form $\{x \in X : \lambda(x) \leq r\}$ is regular closed. So λ is LSC (since the regular open sets are open). Now, let us consider the set $B = \{x \in X : \lambda(x) < r\}$. We see that

$$B = \{x \in X : \lambda(x) < r\} = \cup_{n \in N}\{x \in X : \lambda(x) \leq r - 1/n\}.$$

But $\{x \in X : \lambda(x) \leq r - 1/n\}$ is a regular closed set for each n. Thus B is a union of regular closed sets and hence λ is NLSC and thus belongs to $n(T)$.

Lemma 2.21. *A function $f : (X, T) \to (R, \sigma_1)$, where $\sigma_1 = \{(r, \infty) : r \in R\}$ is NLSC function iff the inverse image of an open subset of (R, σ_1) is regular open in (X, T).*

Proof. Let $f : (X, T) \to (R, \sigma_1)$ be NLSC. We consider the open subset (r, ∞) of (R, σ_1), where $r \in R$. We are to show that $f^{-1}(r, \infty) = \{x \in X : f(x) > r\}$ is regular open. Since f is NLSC, we have $\{x \in X : f(x) < r\} = \cup_\alpha(cl_X G_\alpha)$, where each G_α is open in X. Now, let us consider the set $A = \{x \in X : f(x) \leq r\}$. Then, $A = \{x \in X : f(x) \leq r\} = cl_X\{x \in X : f(x) < r\} = cl_X(\cup_\alpha(cl_X G_\alpha))$ [8], which is regular closed.

Again, $f^{-1}(r, \infty) = \{x \in X : f(x) > r\} = X - \{x \in X : f(x) \leq r\}$, which is regular open.

Conversely, let $f^{-1}(r, \infty) = \{x \in X : f(x) > r\}$ be regular open, i.e., $\{x \in X : f(x) \leq r\}$ is regular closed.

But, $\{x \in X : \lambda(x) < r\} = \cup_{n \in N}\{x \in X : \lambda(x) \leq r - 1/n\}$, which is a union of regular closed sets. Also, since $\{x \in X : f(x) > r\}$ is regular open and hence open; so f is LSC. Therefore, $f : (X, T) \to (R, \sigma_1)$ is NLSC.

Definition 2.22. Let $f : (X, n(T_1)) \to (Y, n(T_2))$ be a mapping between two n-infy induced fuzzy topological spaces. Then f is called fuzzy n-continuous if the inverse image of an n-open fuzzy subset of $n(T_2)$ is an n-open fuzzy subset of $n(T_1)$.

Definition 2.23. A mapping $f : X \to Y$ is said to be an r-continuous if the inverse image under f of any regular open subset of Y is regular open in X.

Theorem 2.24. *A mapping $f : (X, n(T_1)) \to (Y, n(T_2))$ is fuzzy n-continuous iff the mapping $f : (X, T_1) \to (Y, T_2)$ is r-continuous.*

Proof. Let $f : (X, n(T_1)) \to (Y, n(T_2))$ be a fuzzy n-continuous mapping and let B be a regular open subset of (Y, T_2). Now,

$$\begin{aligned} f^{-1}(B) &= \{x \in X : f(x) \in B\} \\ &= \{x \in X : \mu_B f(x) = 1\}, \\ &\quad \text{where } \mu_B \text{ is the characteristic function of the crisp set B.} \\ &= \{x \in X : \mu_B f(x) > r, \ \ 0 \leq r < 1\} \\ &= \{x \in X : (f^{-1}(\mu_B))(x) > r, \ \ 0 \leq r < 1\} \\ &= \sigma_r(f^{-1}(\mu_B)) \end{aligned}$$

Now μ_B, being the characteristic function of a regular open set of Y, is an NLSC function. So, $\mu_B \in n(T_2)$. Again, $f : (X, n(T_1)) \to (Y, n(T_2))$ being a fuzzy n-continuous mapping, so $f^{-1}(\mu_B) \in n(T_1)$.

Hence, by Theorem 2.20, $f^{-1}(B) = \sigma_r(f_{-1}(\mu_B))$ is a regular open subset of X, i.e. $f : (X, T_1) \to (Y, T_2)$ is r-continuous.

Conversely, let $f : (X, T_1) \to (Y, T_2)$ be an r-continuous mapping and β be an n-open fuzzy subset of $(Y, n(T_2))$. We are to show that $f^{-1}(\beta)$ is a member of $n(T_1)$. Now, for $0 < r < 1$,

$$\begin{aligned}\sigma_r(f^{-1}(\beta)) &= \{x \in X : (f^{-1}(\beta))(x) > r\}\\ &= \{x \in X : \beta(f(x)) > r\}\\ &= (\beta f)^{-1}(r, 1]\\ &= f^{-1}\beta^{-1}(r, 1]\\ &= f^{-1}(\beta^{-1}(r, 1])\\ &= f^{-1}(\sigma_r(\beta))\end{aligned}$$

Now, since $\beta \in n(T_2)$ i.e. β is NLSC in Y. So $\sigma_r(\beta) = \{y \in Y : \beta(y) > r\}$ is a regular open subset of Y. Finally, $f : (X, T_1) \to (Y, T_2)$, being an r-continuous mapping, $\sigma_r(f^{-1}(\beta)) = f^{-1}(\beta^{-1}(r, 1])$ is a regular open subset of X.

Hence, $f^{-1}(\beta) \in n(T_1)$ and thus f is fuzzy n-continuous.

To conclude, we investigate the conditions under which a fuzzy topological space $(X, \mathscr{F})$ on a given set X becomes an infy-induced fuzzy topological space of the topology formed by the crisp members of $\mathscr{F}$. We first recall a theorem from [5].

Theorem 2.25. *Let $(X, \mathscr{F})$ be a fuzzy topological space and $T = \mathscr{F} \cap 2^X$ be the crisp members of $\mathscr{F}$. Then, the following two statements are equivalent:*

(a) *For any fuzzy subset λ and $r \in I$,*
$\omega_r(cl_{\mathscr{F}}(\lambda)) = \cap\{cl_T(\omega_s(\lambda)) : s < r\}$ is regular closed in X.

(b) *For any fuzzy subset μ and $p \in I$,*
$\sigma_p(int_{\mathscr{F}}(\mu)) = \cup\{int_T(\sigma_t(\mu)) : p < t\}$ is regular open in X.

Theorem 2.26. *Let $(X, \mathscr{F})$ be a fuzzy topological space and let $T = \mathscr{F} \cap 2^X$ the crisp part of $\mathscr{F}$. Then, the following statements are equivalent:*

(a) *For any fuzzy subset λ of X,*
$\omega_r(\lambda) = \sigma_r(cl_{\mathscr{F}}(\lambda)) = \cup\{cl_T(\omega_s(\lambda)) : s > r\}$is regular closed in (X, T).

(b) *For any fuzzy subset μ of X,*
$\sigma_p(\mu) = \omega_p(int_{\mathscr{F}}(\mu)) = \cap\{(\sigma_t(\mu)) : p > t\}$ is regular open in (X, T).

Proof. Let λ be a closed subset of $(X, \mathscr{F})$ and $r \in I$. Then, by the given condition,

$$\omega_r(\lambda) = \omega_r(cl_{\mathscr{F}}(\lambda)) = \cap\{cl_T(\omega_t(\lambda)) : t < r\}$$

is a regular closed subset of (X, T). Taking complement, we get

$$X - \omega_r(\lambda) \text{ is a regular open subset of (X,T),}$$

i.e., $\sigma_{1-r}(1_X - \lambda)$ is a regular open subset of (X, T), i.e., $(1_X - \lambda)$ is n-open in $(X, n(T))$, i.e., λ is closed in $(X, n(T))$.

Again, let μ be a closed fuzzy subset in $(X, n(T))$ and $r \in I$. Then, by given condition,

$$\omega_r(cl_{\mathscr{F}}(\mu)) = \cap\{cl_T(\omega_t(\mu)) : t < r\} \text{ is a regular closed subset of } (X, T).$$

Since μ is closed in $n(T)$, it can be seen that $\omega_t(\mu)$ is regular closed in (X, T) and hence closed in (X, T).

Thus, $\mathrm{cl}_T(\omega_t(\mu)) = \omega_t(\mu)$.

Hence, $\omega_r(cl_{\mathscr{F}}(\mu)) = \cap\{(\omega_t(\mu)) : t < r\} = \omega_r(\mu)$, i.e., $\mu = cl_{\mathscr{F}}(\mu)$.

Hence, μ is closed in $(X, \mathscr{F})$, which implies that the above two spaces $(X, \mathscr{F})$ and $(X, n(T))$ are equivalent, i.e., $\mathscr{F} = n(T)$.

Theorem 2.27. *Let $(X, \mathscr{F})$ be a infy induced fuzzy topological space and $T = \mathscr{F} \cap 2^X$, the crisp part of $\mathscr{F}$. Then the following statements are equivalent:*

(a) *For any fuzzy subset λ of X,*
$\omega_r(\lambda) = \sigma_r(cl_{\mathscr{F}}(\lambda)) = \cup\{cl_T(\omega_s(\lambda)) : s > r\}$ *is regular closed in* (X, T).

(b) *For any fuzzy subset μ of X,*
$\sigma_p(\mu) = \omega_p(int_{\mathscr{F}}(\mu)) = \cap\{int_T(\sigma_t(\mu)) : p > t\}$ *is regular open in (X,T).*

Proof. (**a**) $\Rightarrow$ (**b**). $\omega_r(\lambda) = \sigma_r(cl\mathscr{F}(\lambda)) = \cup\{cl_T(\omega_s(\lambda)) : s > r\}$ is regular closed in (X, T). i.e., $X - \omega_r(\lambda) = X - \sigma_r(cl_{\mathscr{F}}(\lambda)) = X - \cup\{cl_T(\omega_s(\lambda)) : s > r\}$ is regular open in (X, T).

Now, $X - \omega_r(\lambda) = \sigma_{1-r}(1_X - \lambda)$. Thus, $X - \omega_r(cl_{\mathscr{F}}(\lambda)) = \omega_{1-r}(1_X - cl_{\mathscr{F}}(\lambda)) = \omega_{1-r}(\mathrm{int}_{\mathscr{F}}(1_X - \lambda))$. Again,

$$\begin{aligned} X - \cup\{\mathrm{cl}_T(\omega_s(\lambda)) : s > r\} &= \cap(X\{\mathrm{cl}_T(\omega_s(\lambda)) : s > r\}) \\ &= \cap\{\mathrm{int}_T(X(\omega_s(\lambda))) : s > r\} \\ &= \cap\{\mathrm{int}_T(\sigma_{1-s}(1_X - \lambda)) : s > r\} \end{aligned}$$

i.e., $\sigma_{1-r}(1_X - \lambda) = \omega_{1-r}(\mathrm{int}_{\mathscr{F}}(1_X\lambda)) = \cap(\omega_{1-s}(1_X - \lambda)) : s > r, \}$ is regular open in (X, T). Putting $1 - r = p$, $1 - s = t$ and $1_X - \lambda = \mu$, we get, $\sigma_p(\mu) = \omega_p(\mathrm{int}_{\mathscr{F}}(\mu)) = \cap \mathrm{int}_(\sigma_t(\mu)) : p > t,\ p, t \in I$ is regular open in (X, T).

(**b**) $\Rightarrow$ (**a**). $\sigma_p(\mu) = \omega_p(\mathrm{int}_{\mathscr{F}}(\mu)) = \{\mathrm{int}_T(\sigma_t(\mu)) : p > t,\ p, t \in I\}$ is regular open in (X,T). i.e., $X - \sigma_p(\mu) = X - \omega_p(\mathrm{int}_{\mathscr{F}}\mu)) = X - \cap\{\mathrm{int}_T(\sigma_t(\mu)) : p > t,$ $p, t \in I\}$ is regular closed in (X,T). Now, $X - \sigma_p(\mu) = \omega_{1-p}(1_X - \mu) = \omega_r(\lambda)$. Again,

$$\begin{aligned} X - \omega_p(\mathrm{int}_{\mathscr{F}}(\mu)) &= \sigma_{1-p}(1_X - int_{\mathscr{F}}(\mu)) \\ &= \sigma_{1-p}(\mathrm{cl}_{\mathscr{F}}(1_X - \mu)) \\ &= \sigma_r(cl_{\mathscr{F}}(\lambda)) \end{aligned}$$

and

$$\begin{aligned} X - \cap\{\mathrm{int}_T(\sigma_t(\mu)) : p > t\} &= \cup\{X - int_T(\sigma_t(\mu)) : p > t\} \\ &= \cup\{cl_T(X - \sigma_t(\mu)) : p > t\} \\ &= \cup\{cl_T(\omega_{1-t}(1_X - \mu)) : p > t\} \\ &= \{cl_T(\omega_s(\lambda)) : s > r\} \end{aligned}$$

i.e., $\omega_r(\lambda) = \sigma_r(cl_{\mathscr{F}}(\lambda)) = \cup\{cl_T(\omega_s(\lambda)) : s > r\}$ is regular closed in (X,T).

This completes the proof. ∎

Theorem 2.28. *Let $(X, \mathscr{F})$ be a fuzzy topological space and let $T = \mathscr{F} \cap 2^X$ be the crisp members of $\mathscr{F}$. Then the n-infy induced fuzzy topology $n(T)$ on X is equivalent to the fuzzy topology $\mathscr{F}$ if for any fuzzy subset λ and $r \in I$*

$$\omega_r(\lambda) = \sigma_r(\mathrm{cl}_{\mathscr{F}}(\lambda)) = \cup\{cl_T(\omega_s(\lambda)) : s > r \textit{ is regular closed in } (X, T).$$

Proof. We are to show that, under the given condition, $\mathscr{F} = n(T)$. Let the given condition holds and let λ be closed in $\mathscr{F}$. Then, $\sigma_r(\lambda) = \sigma_r(cl_{\mathscr{F}}(\lambda)) = \omega_r(\lambda)$ is regular closed in (X, T) and hence $\sigma_{1-r}(1_X - \lambda)$ is regular open in (X, T), i.e., $1_X - \lambda \in n(T)$, i.e., λ is closed in $n(T)$.

Again, let α be closed in $n(T)$ and $t \in I$. Then,

$$\sigma_t(\mathrm{cl}_{\mathscr{F}}(\alpha)) = \cup\{\mathrm{cl_T}(\omega_\mathrm{p}(\alpha)) : \mathrm{p} > \mathrm{t}\} = \cup\{\omega_\mathrm{p}(\alpha) : \mathrm{p} > \mathrm{t}\} = \sigma_\mathrm{t}(\alpha).$$

Thus, $\sigma_t(cl_{\mathscr{F}}(\alpha)) = \sigma_t(\alpha)$ for all $t \in I$, i.e., $\mathrm{cl}_{\mathscr{F}}(\alpha) = \alpha$, i.e., α is closed in $\mathscr{F}$. Hence, $\mathscr{F} = n(T)$. ∎

In this paper, a new fuzzy topological structure namely infy induced fuzzy topological space has been introduced with the help of NLSC functions and under the influence of the space various properties of fuzzy subsets has been studied. Also, the conditions under which a fuzzy topological space becomes an infy induced fuzzy topological space has been investigated. The newly introduced space actually generalizes the concept of fuzzy topology and so there is ample scope to investigate various topological properties on this generalized space.

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SOME RESULTS ON LAGUERRE TRANSFORM IN TWO VARIABLES

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Abstract. An attempt is made to investigate some results on Laguerre transform in two variables [8]. In this paper, Laguerre transform of some particular functions and integral formulas have been obtained.

Keywords: Laguerre transform, Laguerre polynomials, Laguerre transform in two variables.

AMS Classification: 44A15, 44A30, 33C45.

1. Introduction

Basic concepts and applications of the Laguerre transform and the generalized Laguerre transform can be found in Debnath et al. [2]. Recently Shukla et al. [8] introduced the Laguerre transform of $f(x,y)$ as

$$L\{f(x,y)\} = F_n(\alpha,\beta) = \int_0^\infty \int_0^\infty e^{-(x+y)} x^\alpha y^\beta K_n^{(\alpha,\beta)}(x,y)\, f(x,y)\, dxdy, \tag{1.1}$$

where $f(x,y)$ be a Riemann integrable function (see [7]) defined on the set $S = R^+ \times R^+$,$\alpha > -1$, $\beta > -1$, n is non-negative integer and $K_n^{(\alpha,\beta)}(x,y) = L_n^\alpha(x)\, L_n^\beta(y)$.

Shukla et al. [8] proved the following theorem:

Theorem 1.1. *If* $K_n^{(\alpha,\beta)}(x,y) = L_n^{\alpha}(x)\, L_n^{\beta}(y)$, *then*

$$\int_0^\infty \int_0^\infty e^{-(x+y)} x^\alpha y^\beta K_n^{(\alpha,\beta)}(x,y) K_m^{(\alpha,\beta)}(x,y)\, dx dy = \delta_n \delta_{m\,n}, \tag{1.2}$$

where $\delta_{m\,n}$ *(Kronecker delta symbol) is defined as*

$$\delta_{m\,n} = \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases},$$

$$\delta_n = \frac{\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}{(n!)^2},$$

$$\alpha > -1 \text{ and } \beta > -1.$$

Howell [4] proved the following results:

$$\begin{aligned} \int_0^\infty x^{\alpha+a-1} e^{-x} L_n^\alpha(x) dx &= \frac{\Gamma(a+\alpha)\Gamma(n-a+1)}{n!\Gamma(1-a)} \\ \int_0^\infty e^{-(1+a)x} y^\beta L_n^\beta(y) L_m^\beta(y) dy &= \frac{\Gamma(n+\alpha+1)\Gamma(m+\alpha+1)(a-1)^{n-m+\alpha+1}}{n!m!\Gamma(1+\alpha)a^{n+m+2\alpha+2}} \end{aligned} \tag{1.3}$$

$$\times {}_2F_1\left(n+\alpha+1, \frac{m+a+1}{a+1}, \frac{1}{a^2}\right) \tag{1.4}$$

$$\int_0^\infty e^{-st} t^\alpha L_n^\alpha(t)\, dt = \frac{\Gamma(n+\alpha+1)(s-1)^n}{n! s^{n+\alpha+1}}, \tag{1.5}$$

where $\mathrm{Re}(\alpha) > -1$,$\mathrm{Re}(s) > 0$, and

$$\begin{aligned} &e^{x+y}(xy)^{-\alpha}\Gamma(\alpha, \max(x,y))\,\gamma(\alpha, \min(x,y)) \\ &= \sum_{m=0}^\infty \frac{m!}{(m+1)(\alpha)_{m+1}} L_m^\alpha(x)\, L_m^\alpha(y), \end{aligned} \tag{1.6}$$

where $\gamma(\alpha, x)$ is incomplete gamma functions and $\Gamma(\alpha,x) = \Gamma(\alpha) - \gamma(\alpha,x)$.

$$\begin{aligned} &e^{x+y}(xy)^{-\alpha}\left\{\Gamma(\alpha, \max(x,y)) - \Gamma(\alpha,x)\,\Gamma(\alpha,y)/\Gamma(\alpha)\right\} \\ &= \sum_{m=0}^\infty \frac{m!}{(m+1)\Gamma(m+\alpha+1)} L_m^\alpha(x)\, L_m^\alpha(y) \end{aligned} \tag{1.7}$$

$$\begin{aligned} &e^{-\frac{1}{2}(x+y)}(xy)^{-\frac{1}{2}\alpha} e^{-\alpha\pi i}\gamma\left(\alpha, e^{i\pi}\min(x,y)\right) \\ &= \sum_{m=0}^\infty \frac{m!}{(m+\alpha)\Gamma(m+\alpha+1)} L_m^\alpha(x)\, L_m^\alpha(y). \end{aligned} \tag{1.8}$$

2. Laguerre Transforms of some particular functions

In this section, some properties of Laguerre transform in two variables [8] have been obtained, this work can be considered as an extension of [8] and [9].

By using the orthogonal property (1.2), we can prove:

$$L\left\{K_n^{(\alpha,\beta)}(x,y)\right\} = \begin{cases} 0, & m \neq n \\ \dfrac{\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}{(n!)^2}, & m = n \end{cases} \tag{2.1}$$

From definition (1.1) and result (1.3) of Howell [4], we get the following result as:

If $f(x,y) = x^{a-1}y^{b-1}$, where a and b are positive numbers, then

$$L\{f(x,y)\} = \frac{\Gamma(a+\alpha)\,\Gamma(b+\beta)\,\Gamma(n-a+1)\,\Gamma(n-b+1)}{(n!)^2\Gamma(1-a)\,\Gamma(1-b)} \tag{2.2}$$

From (1.5), we can prove the following result as:

If $f(x,y) = e^{-(a\,x\,+\,b\,y)}$, where $a > -1$ and $b > -1$ then,

$$L\{f(x,y)\} = \frac{(ab)^n\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}{(n!)^2(a+1)^{n+\alpha+1}(b+1)^{n+\beta+1}} \tag{2.3}$$

Applying (1.1) to equation (1.4) and further simplification gives the following result.

If $f(x,y) = e^{-(a\,x\,+\,b\,y)}K_m^{(\alpha,\beta)}(x,y)$, where $a > -1$ and $b > -1$ then,

$$L\{f(x,y)\} = \frac{\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}{(n!)^2(m!)^2\Gamma(1+\alpha)\,\Gamma(1+\beta)}$$

$$\times\frac{\Gamma(m+\alpha+1)\,\Gamma(m+\beta+1)\,(a-1)^{n-m+\alpha+1}(b-1)^{n-m+\beta+1}}{a^{n+m+2\alpha+2}b^{n+m+2\beta+2}}$$

$$\times {}_2F_1\left(n+\alpha+1, \frac{m+a+1}{a+1}, \frac{1}{a^2}\right) {}_2F_1\left(n+\beta+1, \frac{m+b+1}{b+1}, \frac{1}{b^2}\right). \tag{2.4}$$

From (1.2) and (1.6), we arrive at following result:

If $f(x,y) = e^{x+y}(xy)^{-\alpha}\Gamma(\alpha, \max(x,y))\,\gamma(\alpha, \min(x,y))$ and $F_n(\alpha,\beta) = L\{f(x,y), \alpha, \beta, n\}$ then,

$$F_n(\alpha,\alpha) = \frac{\Gamma(\alpha)\,\Gamma(n+\alpha+1)}{(n+1)!} \tag{2.5}$$

Also from (1.7) and (1.2) further proceeding as above, this yields:

If $f(x,y) = e^{x+y}(xy)^{-\alpha}\left\{\Gamma(\alpha, \max(x,y)) - \Gamma(\alpha, x)\Gamma(\alpha, y)/\Gamma(\alpha)\right\}$ and $F_n(\alpha,\beta) = L\{f(x,y), \alpha, \beta, n\}$ then,

$$F_n(\alpha,\alpha) = \frac{\Gamma(n+\alpha+1)}{(n+1)!}. \tag{2.6}$$

By using (1.8) and (1.2), we can say:

If $f(x,y) = e^{-\frac{1}{2}(x+y)}(xy)^{-\frac{1}{2}\alpha}e^{-\alpha\pi i}\gamma\left(\alpha, e^{i\pi}\min(x,y)\right)$ then,

$$L\{f(x,y), \alpha, \alpha, n\} = \frac{\Gamma(n+\alpha+1)}{(n+1)!} \tag{2.7}$$

3. Some integral formula

Laguerre polynomials occur in many fields of research in science, engineering and numerical mathematics such as, in quantum mechanics [5], communication theory [1] and numerical inverse Laplace transform [6]. Explicit evaluation of integrals involving Laguerre polynomials is very often required in these and other applied areas of research. In this section, we derive some integral formula.

This is interesting to write (2.5) in the following form as,

$$\int_0^\infty\int_0^\infty L_n^\alpha(x)L_n^\alpha(y)\Gamma(\alpha, \max\,(x,y))\gamma(\alpha, \min(x,y))dxdy = \frac{\Gamma(\alpha)\Gamma(n+\alpha+1)}{(n+1)!} \tag{3.1}$$

Proof of (3.1). To prove (3.1), first replace β by α in the definition (1.1), and we have

$$F_n(\alpha,\alpha) = \int_0^\infty\int_0^\infty e^{-(x+y)}(xy)^\alpha L_n^\alpha(x)\,L_n^\alpha(y)\,f(x,y)\,dxdy$$

By substituting the value of $f(x,y)$, we get

$$F_n(\alpha,\alpha) = \int_0^\infty\int_0^\infty e^{-(x+y)}(xy)^\alpha L_n^\alpha(x)\,L_n^\alpha(y)\sum_{m=0}^\infty \frac{m!}{(m+1)\,(\alpha)_{m+1}} L_m^\alpha(x)\,L_m^\alpha(y)dxdy$$

By applying the orthogonal property (1.2),

$$F_n(\alpha,\alpha) = \sum_{m=0}^\infty \frac{m!}{(m+1)\,(\alpha)_{m+1}}\delta_n\delta_{m\,n}$$

and using the definition of Kronecker delta $\delta_{m\,n}$ and δ_n , we have

$$F_n(\alpha,\alpha) = \frac{n!}{(n+1)\,(\alpha)_{n+1}}\frac{\Gamma(n+\alpha+1)\,\Gamma(n+\alpha+1)}{(n!)^2}$$

Further simplification yields (3.1).

By using (1.2),(1.7) and (2.6),we get

$$\int_0^\infty\int_0^\infty L_n^\alpha(x)L_n^\alpha(y)\{\Gamma(\alpha,\max(x,y))-\Gamma(\alpha,x)\Gamma(\alpha,y)/\Gamma(\alpha)\}dxdy=\frac{\Gamma(n+\alpha+1)}{(n+1)!} \tag{3.2}$$

$$\int_0^\infty\int_0^\infty e^{\frac{1}{2}(x+y)}(xy)^{\frac{1}{2}\alpha}e^{-\alpha\pi i}\gamma(\alpha,e^{i\pi}\min(x,y))L_n^\alpha(x)L_n^\alpha(y)dxdy=\frac{\Gamma(n+\alpha+1)}{(n+1)!} \tag{3.3}$$

Proof (3.3). By setting $\beta=\alpha$ in (1.1), this reduces to

$$F_n(\alpha,\alpha)=\int_0^\infty\int_0^\infty e^{-(x+y)}(xy)^\alpha L_n^\alpha(x)\,L_n^\alpha(y)\,f(x,y)\,dxdy$$

and substituting the value of $f(x,y)$, we arrive at

$$F_n(\alpha,\alpha)$$
$$=\int_0^\infty\int_0^\infty e^{-(x+y)}(xy)^\alpha L_n^\alpha(x)\,L_n^\alpha(y)\,e^{-\frac{1}{2}(x+y)}(xy)^{-\frac{1}{2}\alpha}e^{-\alpha\pi i}\gamma\left(\alpha,e^{i\pi}\min(x,y)\right)dxdy$$

Now, applying the same argument as in proof (3.1), gives

$$F_n(\alpha,\alpha)=\frac{n!}{(n+1)\,\Gamma(n+\alpha+1)}\frac{\Gamma(n+\alpha+1)\,\Gamma(n+\alpha+1)}{(n!)^2}$$

Further simplification yields (3.3).

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GROWTH ANALYSIS OF WRONSKIANS BASED ON RELATIVE L^*-ORDER AND RELATIVE L^*-TYPE

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Abstract. In this paper, we establish the relationship between the relative L-order (relative L^*-order), relative L-type (relative L^*-type) and relative L-weak type (relative L^*-weak type) of a transcendental meromorphic function f with respect to an entire function g and that of Wronskian generated by the meromorphic f and entire g.

Keywords and phrases: transcendental entire function, transcendental meromorphic function, relative order (relative lower order), relative type (relative lower type), relative weak type, wronskian, slowly changing function.

AMS Subject Classification (2010): 30D20,30D30,30D35.

1. Introduction, definitions and notations

Let $\mathbb{C}$ be the set of all finite complex numbers. For a meromorphic function f defined on $\mathbb{C}$, the Wronskian determinant $W(f) = W(a_1, a_2, ...a_k, f)$ is defined as

$$W(f) = \begin{vmatrix} a_1 & a_2 & . & . & . & a_k & f \\ a_1^{'} & a_2^{'} & . & . & . & a_k^{'} & f^{'} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ a_1^{(k)} & a_2^{(k)} & . & . & . & a_k^{(k)} & f^{(k)} \end{vmatrix}$$

where $a_1, a_2, ...a_k$ are linearly independent meromorphic functions and small with respect to f (i.e., $T_{a_i}(r) = S(r, f)$ or, in other words, $\frac{T_{a_i}(r)}{S(r,f)} \to 0$ as $r \to \infty$ for $i = 1, 2, 3...k$, $T_f(r)$ being the Nevanlinna Characteristic function of f).

We do not explain the standard notations and definitions in the theory of entire and meromorphic functions as those are available in [3] and [6]. From the Nevanlinna's second fundamental theorem, it follows that the set of values of $a \in \mathbb{C} \cup \{\infty\}$ for which $\delta(a; f) > 0$ is countable and $\sum\limits_{a \neq \infty} \delta(a; f) + \delta(\infty; f) \leq 2$ (cf. [3], p. 43), where $\delta(a; f) = 1 - \limsup\limits_{r\to\infty} \frac{N(r,a;f)}{T_f(r)} = \liminf\limits_{r\to\infty} \frac{m(r,a;f)}{T_f(r)}$. If, in particular, $\sum\limits_{a \neq \infty} \delta(a; f) + \delta(\infty; f) = 2$, we say that f has the maximum deficiency sum.

Somasundaram and Thamizharasi [5] introduced the notions of L-order and L-lower order for entire functions where $L \equiv L(r)$ is a positive continuous function increasing slowly i.e.,$L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant "a". Their definitions are as follows:

Definition 1 [5] The L-order ρ_f^L and the L-lower order λ_f^L of a meromorphic function f are defined as follows:

$$\rho_f^L = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log [rL(r)]} \quad \text{and} \quad \lambda_f^L = \liminf_{r\to\infty} \frac{\log T(r, f)}{\log [rL(r)]}.$$

For $L(r) \equiv 1$, the definitions of L-order and the L-lower order of a meromorphic function f respectively reduce to the classical definitions of order and lower order of the same.

The more generalized concept of L-order and L-lower order of meromorphic functions are L^*-order and L^*-lower order respectively which are as follows:

Definition 2 The L^*-order $\rho_f^{L^*}$ and the L^*-lower order $\lambda_f^{L^*}$ of a meromorphic function f are defined by

$$\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log [re^{L(r)}]} \quad \text{and} \quad \lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log T(r, f)}{\log [re^{L(r)}]}.$$

It is well known that the Nevanlinna's characteristic function $T_g(r)$ of an entire function g is defined as

$$T_g(r) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| g(re^{i\theta}) \right| d\theta,$$

where $\log^+ x = \max(0, \log x)$ for $x > 0$. If g is non-constant then $T_g(r)$ is strictly increasing and continuous and its inverse $T_g^{-1} : (T_g(0), \infty) \to (0, \infty)$ exists and is such that $\lim_{s\to\infty} T_g^{-1}(s) = \infty$.

Lahiri and Banerjee [4] introduced the definition of relative order of a meromorphic function with respect to an entire function in the following way:

Definition 3 [4]Let f be meromorphic and g be entire. The relative order of f with respect to g denoted by $\rho_g(f)$ is defined as

$$\rho_g(f) = \inf \left\{ \begin{array}{c} \mu > 0 : T_f(r) < T_g(r^\mu) \\ \text{for all sufficiently large } r \end{array} \right\} = \limsup_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log r}.$$

For $g(z) = \exp z$, the above definition coincides with the classical one [4].

Analogously, the relative lower order $\lambda_g(f)$ of a meromorphic function f with respect to an entire function g is defined.

Datta and Biswas [1] gave the definition of relative type and relative weak type of a meromorphic function with respect to an entire function g which are as follows:

Definition 4 [1] The relative type $\sigma_g(f)$ and lower relative type $\bar{\sigma}_g(f)$ of a meromorphic function f with respect to an entire function g are defined as

$$\sigma_g(f) = \limsup_{r\to\infty} \frac{T_g^{-1} T_f(r)}{r^{\rho_g(f)}} \quad \text{and} \quad \bar{\sigma}_g(f) = \liminf_{r\to\infty} \frac{T_g^{-1} T_f(r)}{r^{\rho_g(f)},}$$

where $0 < \rho_g(f) < \infty$.

Definition 5 [1] The relative weak type $\tau_g(f)$ of a meromorphic function f with respect to an entire function g with finite positive relative lower order $\lambda_g(f)$ is defined by

$$\tau_g(f) = \liminf_{r\to\infty} \frac{T_g^{-1} T_f(r)}{r^{\lambda_g(f)}}.$$

Similarly, one can define the growth indicator $\bar{\tau}_g(f)$ by replacing "lim inf" with "lim sup" in Definition 5.

In order to prove our results, we require the following definitions:

Definition 6 The relative L-order $\rho_g^L(f)$ (the relative L-lower order $\lambda_g^L(f)$), relative L-type $\sigma_g^L(f)$ (relative L-lower type $\bar{\sigma}_g^L(f)$) and relative L-weak type $\tau_g^L(f)$ (growth indicator $\bar{\tau}_g^L(f)$) of a meromorphic function f with respect to an entire function g are defined as follows:

$$\rho_g^L(f) = \limsup_{r\to\infty}\frac{\log T_g^{-1}T_f(r)}{\log[rL(r)]} \quad \left(\lambda_g^L(f) = \liminf_{r\to\infty}\frac{\log T_g^{-1}T_f(r)}{\log[rL(r)]}\right),$$

$$\sigma_g^L(f) = \limsup_{r\to\infty}\frac{T_g^{-1}T_f(r)}{[rL(r)]^{\rho_g^L(f)}} \quad \left(\bar{\sigma}_g^L(f) = \liminf_{r\to\infty}\frac{T_g^{-1}T_f(r)}{[rL(r)]^{\rho_g^L(f)}}\right),$$

where $0 < \rho_g^L(f) < \infty$, and

$$\tau_g^L(f) = \liminf_{r\to\infty}\frac{T_g^{-1}T_f(r)}{[rL(r)]^{\lambda_g^L(f)}} \quad \left(\bar{\tau}_g^L(f) = \limsup_{r\to\infty}\frac{T_g^{-1}T_f(r)}{[rL(r)]^{\lambda_g^L(f)}}\right),$$

where $0 < \lambda_g^L(f) < \infty$.

The more generalized concept of relative L-order (relative L-lower order), relative L-type (relative L-lower type) and relative L-weak type of a meromorphic function with respect to an entire function are relative L^*-order (relative L^*-lower order), relative L^*-type (relative L^*-lower type) and relative L^*-weak type respectively which may be defined as follows:

Definition 7 The relative L^*-order $\rho_f^{L^*}$(relative L^*-lower order $\lambda_f^{L^*}$), relative L^*-type $\sigma_g^{L^*}(f)$ (relative L^*-lower type $\bar{\sigma}_g^{L^*}(f)$) and relative L^*-weak type $\tau_g^{L^*}(f)$ (the growth indicator $\bar{\tau}_g^{L^*}(f)$) of a meromorphic function f with respect to entire g are respectively defined as follows:

$$\rho_g^{L^*}(f) = \limsup_{r\to\infty}\frac{\log T_g^{-1}T_f(r)}{\log[re^{L(r)}]} \quad \left(\lambda_g^{L^*}(f) = \liminf_{r\to\infty}\frac{\log T_g^{-1}T_f(r)}{\log[re^{L(r)}]}\right),$$

$$\sigma_g^{L^*}(f) = \limsup_{r\to\infty}\frac{T_g^{-1}T_f(r)}{[re^{L(r)}]^{\rho_g^{L^*}(f)}} \quad \left(\bar{\sigma}_g^{L^*}(f) = \liminf_{r\to\infty}\frac{T_g^{-1}T_f(r)}{[re^{L(r)}]^{\rho_g^{L^*}(f)}}\right)$$

where $0 < \rho_g^{L^*}(f) < \infty$, and

$$\tau_g^{L^*}(f) = \liminf_{r\to\infty}\frac{T_g^{-1}T_f(r)}{[re^{L(r)}]^{\lambda_g^{L^*}(f)}} \quad \left(\bar{\tau}_g^{L^*}(f) = \limsup_{r\to\infty}\frac{T_g^{-1}T_f(r)}{[re^{L(r)}]^{\lambda_g^{L^*}(f)}}\right),$$

where $0 < \lambda_g^{L^*}(f) < \infty$.

Since the natural extension of a derivative is a differential polynomial, in this paper we prove our results for a special type of linear differential polynomials viz. the Wronskians. In the paper, we establish the relationship between the relative L-order (relative L^*-order), relative L-type (relative L^*-type) and relative L-weak type (relative L^*-weak type) of a transcendental meromorphic function f with respect to an entire function g and that of Wronskian generated by the meromorphic f and entire g.

2. Lemmas

In this section, we present a lemma which will be needed in the sequel.

Lemma 1 [2] *Let f be a transcendental meromorphic function having the maximum deficiency sum and g be a transcendental entire function with regular growth and non zero finite order. Also, let $\sum\limits_{a\neq\infty}\delta(a;g)+\delta(\infty;g)=2$. Then*

$$\lim_{r\to\infty}\frac{\log T^{-1}_{W(g)}T_{W(f)}(r)}{\log T^{-1}_{g}T_{f}(r)}=1.$$

Lemma 2 [2] *Let f be a transcendental meromorphic function having the maximum deficiency sum and g be a transcendental entire function of regular growth and non zero finite type. Also let $\sum\limits_{a\neq\infty}\delta(a;g)+\delta(\infty;g)=2$. Then*

$$\lim_{r\to\infty}\frac{T^{-1}_{W(g)}T_{W(f)}(r)}{T^{-1}_{g}T_{f}(r)}=\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;g)}\right)^{\frac{1}{\rho_g}},$$

where $W(f)=W(a_1,a_2,....a_{k_1},f)$ and $W(g)=W(a_1,a_2,....a_{k_2},g)$.

3. Theorems

In this section, we present the main results of the paper.

Theorem 1 *Let f be a transcendental meromorphic function with the maximum deficiency sum and g be a transcendental entire function of regular growth having non zero finite order and $\sum\limits_{a\neq\infty}\delta(a;g)+\delta(\infty;g)=2$. Then the relative L-order and relative L-lower order of $W(f)$ with respect to $W(g)$ are same as those of f with respect to g.*

Proof. By Lemma 1 we obtain that

$$
\begin{aligned}
\rho^{L}_{W(g)}\left(W\left(f\right)\right) &= \limsup_{r\to\infty}\frac{\log T^{-1}_{W(g)}T_{W(f)}\left(r\right)}{\log\left[rL\left(r\right)\right]} \\
&= \limsup_{r\to\infty}\left\{\frac{\log T^{-1}_{g}T_{f}\left(r\right)}{\log\left[rL\left(r\right)\right]}\cdot\frac{\log T^{-1}_{W(g)}T_{W(f)}\left(r\right)}{\log T^{-1}_{g}T_{f}\left(r\right)}\right\} \\
&= \limsup_{r\to\infty}\frac{\log T^{-1}_{g}T_{f}\left(r\right)}{\log\left[rL\left(r\right)\right]}\cdot\lim_{r\to\infty}\frac{\log T^{-1}_{W(g)}T_{W(f)}\left(r\right)}{\log T^{-1}_{g}T_{f}\left(r\right)} \\
&= \rho^{L}_{g}\left(f\right)\cdot 1=\rho^{L}_{g}\left(f\right)\ .
\end{aligned}
$$

In a similar manner,

$$\lambda^{L}_{W(g)}\left(W\left(f\right)\right)=\lambda^{L}_{g}\left(f\right).$$

This proves the theorem. ∎

Theorem 2 *Let f be a transcendental meromorphic function with the maximum deficiency sum and g be a transcendental entire function of regular growth having non zero finite order and $\sum\limits_{a\neq\infty}\delta\left(a;g\right)+\delta\left(\infty;g\right)=2$. Then the relative L^{*}-order and relative L^{*}-lower order of $W(f)$ with respect to $W(g)$ are same as those of f with respect to g.*

We omit the proof of Theorem 2 because it can be carried out in the line of Theorem 1.

Theorem 3 *Let f be a transcendental meromorphic function with the maximum deficiency sum and g be a transcendental entire function of regular growth having non zero finite type and $\sum\limits_{a\neq\infty}\delta\left(a;g\right)+\delta\left(\infty;g\right)=2$. Then the relative L-type and relative L-lower type of $W(f)$ with respect to $W(g)$ are $\left(\frac{1+k_{1}-k_{1}\delta\left(\infty;f\right)}{1+k_{2}-k_{2}\delta\left(\infty;g\right)}\right)^{\frac{1}{\rho_{g}}}$ times that of f with respect to g if $\rho^{L}_{g}\left(f\right)$ is positive finite.*

Proof. From Lemma 2 and Theorem 1, we get that

$$
\begin{aligned}
\sigma^{L}_{W(g)}\left(W(f)\right) &= \limsup_{r\to\infty}\frac{T^{-1}_{W(g)}T_{W(f)}\left(r\right)}{\left[rL\left(r\right)\right]^{\rho_{w(g)}(w(f))}} \\
&= \lim_{r\to\infty}\frac{T^{-1}_{W(g)}T_{W(f)}\left(r\right)}{T^{-1}_{g}T_{f}\left(r\right)}\cdot\limsup_{r\to\infty}\frac{T^{-1}_{g}T_{f}\left(r\right)}{\left[rL\left(r\right)\right]^{\rho^{L}_{g}(f)}} \\
&= \left(\frac{1+k_{1}-k_{1}\delta\left(\infty;f\right)}{1+k_{2}-k_{2}\delta\left(\infty;g\right)}\right)^{\frac{1}{\rho_{g}}}\sigma^{L}_{g}\left(f\right)\ .
\end{aligned}
$$

Similarly,

$$\overline{\sigma}^{L}_{W(g)}\left(W(f)\right)=\left(\frac{1+k_{1}-k_{1}\delta\left(\infty;f\right)}{1+k_{2}-k_{2}\delta\left(\infty;g\right)}\right)^{\frac{1}{\rho_{g}}}\cdot\overline{\sigma}^{L}_{g}\left(f\right).$$

Thus, the theorem is established. ∎

Theorem 4 *If f be a transcendental meromorphic function with the maximum deficiency sum and g be a transcendental entire function of regular growth having non zero finite type and* $\sum\limits_{a\neq\infty}\delta(a;g)+\delta(\infty;g)=2$ *then the relative L^*-type and relative L^*-lower type of $W(f)$ with respect to $W(g)$ are* $\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;g)}\right)^{\frac{1}{\rho_g}}$ *times that of f with respect to g if $\rho_g^{L^*}(f)$ is positive finite.*

We omit the proof of Theorem 4 because it can be carried out in the line of Theorem 3.

Now, we state the following two theorems without their proofs because those can be carried out in the line of Theorem 3 and Theorem 4 respectively.

Theorem 5 *Let f be a transcendental meromorphic function with the maximum deficiency sum and g be a transcendental entire function of regular growth having non zero finite type and* $\sum\limits_{a\neq\infty}\delta(a;g)+\delta(\infty;g)=2$. *Then $\tau_{W(g)}^{L}(W(f))$ and $\overline{\tau}_{W(g)}^{L}(W(f))$ are* $\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;g)}\right)^{\frac{1}{\rho_g}}$ *times that of f with respect to g i.e.,*

$$\tau_{W(g)}^{L}(W(f))=\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;g)}\right)^{\frac{1}{\rho_g}}\cdot\tau_g^L(f)$$

and

$$\overline{\tau}_{W(g)}^{L}(W(f))=\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;g)}\right)^{\frac{1}{\rho_g}}\cdot\overline{\tau}_g^{L}(f)$$

when $\lambda_g^L(f)$ is positive finite.

Theorem 6 *If f be a transcendental meromorphic function with the maximum deficiency sum and g be a transcendental entire function of regular growth having non zero finite type and* $\sum\limits_{a\neq\infty}\delta(a;g)+\delta(\infty;g)=2$, *then $\tau_{W(g)}^{L^*}(W(f))$ and $\overline{\tau}_{W(g)}^{L^*}(W(f))$ are* $\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;g)}\right)^{\frac{1}{\rho_g}}$ *times that of f with respect to g, i.e.,*

$$\tau_{W(g)}^{L^*}(W(f))=\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;g)}\right)^{\frac{1}{\rho_g}}\cdot\tau_g^{L^*}(f)$$

and

$$\overline{\tau}_{W(g)}^{L^*}(W(f))=\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;g)}\right)^{\frac{1}{\rho_g}}\cdot\overline{\tau}_g^{L^*}(f)$$

when $\lambda_g^{L^*}(f)$ is positive finite.

4. Conclusion

The notions of order (type), L-order (L-type) and L^*-order (L^*-type) which are the main tools to study the composite growth properties of entire and meromorphic functions are very much classical in complex analysis. On the basis of the order (type), L-order (L-type) and L^*-order (L^*-type) of entire or meromorphic functions, several researchers have already explored their works in the area of comparative growth rates of composite entire and meromorphic functions in different directions. In fact, the main aim of this paper is actually to extend these notions to the relativeness of growth indicators in case of wronskians. Actually, the relative order, relative type etc. are the gradation of the growth indicators of entire and meromorphic functions. So keeping all these in mind, it is quite expected to explore and establish similar strong results using the existing literature and theorems of this paper in the field of growth analysis of complex valued, bi-complex valued and fuzzy complex valued functions.

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SOFT FUZZY DISCONNECTEDNESS IN DIMENSION THEORY

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Abstract. In this paper the concept of soft fuzzy B disconnected space is introduced and studied. In particular, soft fuzzy B disconnectedness via dimension theory is established.

Keywords: soft fuzzy B boundary, soft fuzzy B extremally disconnected, soft fuzzy B basically disconnected, soft fuzzy large inductive dimension function.

2000 Mathematics Subject Classification: 54A40, 03E72.

1. Introduction

The concept of fuzzy set was introduced by L.A. Zadeh [19]. Fuzzy sets have applications in many fields such as information [12] and control [13]. The theory of fuzzy topological spaces was introduced and developed by C.L. Chang [8], and since then various notions in classical topology have been extended to fuzzy topological spaces. G. Balasubramanian [6], [7] introduced the concepts of fuzzy extremally disconnectedness and fuzzy basically disconnectedness. The concept of soft fuzzy topological space was introduced by Ismail U. Triyaki [15]. J. Tong [16] introduced the concept of B-set in topological space. The concept of fuzzy B-set was introduced by M.K. Uma, E. Roja and G. Balasubramanian [17]. D. Vidhya, E. Roja and M.K. Uma [18] introduced the concept of soft fuzzy B-open set.

J.R. Munkers [10], A.R. Pears [11] and R. Engleking [9] discussed the concept of "dimension" in topological space. A.R. Pears [11] and R. Engleking [9] discussed the concepts of small inductive dimension function, *ind* and large inductive dimension function, *Ind*. D. Adnadjevic [1], [2] introduced the concepts of F-*ind* and F-*Ind* and discussed some of its basic properties in GF-spaces. D. Adnadjevic and A.P. Sostak [3] developed these concepts in Hutton fuzzy topological

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spaces. In 2007, S.S. Benchalli, B.M. Ittanagi and P.G. Patil [4] proved that, if X is a fuzzy topological space such that $IndfX = 0$ then X is a normal fuzzy topological space.

Based on the above concepts, the present, study is on "soft fuzzy disconnectedness in dimension theory". Some basic definitions are given in section 2. Section 3 is divided into two parts: soft fuzzy extremally disconnectedness and soft fuzzy basically disconnectedness. In these two sections, we have introduced the concepts of soft fuzzy extremally disconnectedness and soft fuzzy basically disconnectedness and their basic propositions are also discussed. The main purpose of section 4 is to introduce, a new concept called soft fuzzy large inductive dimension function, $SFInd$ on soft fuzzy topological space. This concept is applied, for proving the soft fuzzy large inductive zero dimension function of soft fuzzy topological space is disconnectedness.

2. Preliminaries

Definition 2.1. [14] Let A and B be any two sets. The *relative complement* of B in A (or of B with respect to A), written as $A - B$, is that the set consisting of all elements of A which are not elements of B, that is,

$$A - B = \{x|x \in A \wedge x \notin B\} = \{x|x \in A \wedge \neg(x \in B)\}.$$

The relative complement of B in A is also called the *difference* of A and B.

Definition 2.2. [14] Let A and B be any two sets. The *symmetric difference* (or *Boolean sum*) of A and B is the set $A+B$ is defined by $A+B = (A-B)\cup(B-A)$ or $x \in A + B \Longleftrightarrow x \in \{x \in A \bar{\vee} x \in B\}$ where $\bar{\vee}$ is the exclusive disjunction.

Definition 2.3. [6] Suppose $(X, T(X))$ be any fuzzy topological space. X is said to be *fuzzy extremally disconnected* if $\lambda \in T(X)$ implies $\bar{\lambda} \in T(X)$.

Definition 2.4. [7] Let (X, T) be a fuzzy topological space. A fuzzy set $\lambda : X \to [0, 1]$ is said to a fuzzy G_δ-set if $\lambda = \wedge_{i=1}^{\infty}\lambda_i$ where each $\lambda_i \in T$. The complement of a G_δ-set is F_σ-set.

Definition 2.5. [7] Let (X, T) be any fuzzy topological space. (X, T) is called *fuzzy basically disconnected* if the closure of every fuzzy open F_σ set is fuzzy open.

Definition 2.6. [15] Let X be a non-empty set. A *soft fuzzy set*(in short, SFS) A have the form $A = (\lambda, M)$ where the function $\lambda : X \to I$ denote the degree of membership and M is the subset of X. The set of all soft fuzzy set will be denoted by $SF(X)$.

Proposition 2.1. [15] If $(\mu_j, N_j) \in SF(X), j \in J$, then the family $\{(\mu_j, N_j)|j \in J\}$ has a meet, ie., g.l.b., in $(SF(X), \sqsubseteq)$ denoted by $\sqcap_{j\in J}(\mu_j, N_j)$ and given by $\sqcap_{j\in J}(\mu_j, N_j) = (\mu, N)$ where $\mu(x) = \wedge_{j\in J}\mu_j(x)\forall x \in X$ and $M = \cap M_j$ for $j \in J$.

Proposition 2.2. [15] *If $(\mu_j, N_j) \in SF(X)$, $j \in J$, then the family $\{(\mu_j, N_j)|j \in J\}$ has a join, ie., l.u.b., in $(SF(X), \sqsubseteq)$ denoted by $\sqcup_{j\in J}(\mu_j, N_j)$ and given by $\sqcup_{j\in J}(\mu_j, N_j) = (\mu, N)$ where $\mu(x) = \vee_{j\in J}\mu_j(x)\forall x \in X$ and $M = \cup M_j$ for $j \in J$.*

Definition 2.7. [15] Let X be a non-empty set and the soft fuzzy sets A and C be in the form $A = (\lambda, M)$ and $C = (\mu, N)$. Then

(i) $A \sqsubseteq C$ if and only if $\lambda(x) \leq \mu(x)$ for all $x \in X$ and $M \subseteq N$.

(ii) $A = C$ if and only if $\lambda(x) = \mu(x)$ for all $x \in X$ and $M = N$.

(iii) $A \sqcap C = (\lambda, M) \sqcap (\mu, N)$ if and only if $\lambda(x) \wedge \mu(x)$ for all $x \in X$, $M \cap N$.

(iv) $A \sqcup C = (\lambda, M) \sqcup (\mu, N)$ if and only if $\lambda(x) \vee \mu(x)$ for all $x \in X$, $M \cup N$.

Definition 2.8. [15] For $(\mu, N) \in SF(X)$ the soft fuzzy set $(\mu, N)' = (1-\mu, X\backslash N)$ is called the *complement* of (μ, N).

Remark 2.1. $(1 - \mu, X/N) = (1, X) - (\mu, N)$.

Proof. $(1, X)-(\mu, N) = (1, X)\sqcap(\mu, N)' = (1, X)\sqcap(1-\mu, X/N) = (1-\mu, X/N)$.

Definition 2.9. [15] Let X be a set. Let T be family of soft fuzzy subsets of X. Then T is called a soft fuzzy topology on X if T satisfies the following conditions:

(i) $(0, \emptyset)$ and $(1, X) \in T$.

(ii) If $(\mu_j, N_j) \in T, j = 1, 2, ..., n$ then $\sqcap_{j=1}^{n}(\mu_j, N_j) \in T$.

(iii) If $(\mu_j, N_j) \in T, j \in J$ then $\sqcup_{j\in J}(\mu_j, N_j) \in T$.

The pair (X, T) is called a *soft fuzzy topological space*(in short, *SFTS*). The members of T are *soft fuzzy open sets* and its complement is *soft fuzzy closed sets.*

Definition 2.10. [18] Let (X, T) be a soft fuzzy topological space. Let (λ, M) be any soft fuzzy set. Then (λ, M) is said to be a *soft fuzzy t open set* if $SFint(\lambda, M) = SFint(SFcl(\lambda, M))$

Definition 2.11. [18] Let (X, T) be a soft fuzzy topological space. Let (λ, M) be any soft fuzzy set. Then (λ, M) is said to be a *soft fuzzy B open set*(in short, *SFBoS*) if $(\lambda, M) = (\mu, N) \sqcap (\gamma, L))$ where (μ, N) is a soft fuzzy open set and (γ, L) is a soft fuzzy t open set. The complement of soft fuzzy B open set is a *soft fuzzy B closed set* (in short, *SFBcS*).

Definition 2.12. [18] Let (X, T) be a soft fuzzy topological space. Let (λ, M) be any soft fuzzy set. Then the *soft fuzzy B closure* (in short, *SFBcl*) of (λ, M) is defined as follows:

$SFBcl(\lambda, M) = \sqcap \{(\mu, N)|(\mu, N)$ is a soft fuzzy B closed set and $(\lambda, M) \sqsubseteq (\mu, N)\}$.

Definition 2.13. [18] Let (X,T) be a soft fuzzy topological space. Let (λ,M) be any soft fuzzy set. Then the *soft fuzzy B interior*(in short, $SFBint$) of (λ,M) is defined as follows:

$$SFBint(\lambda,M)=\sqcup\{(\mu,N)|(\mu,N) \text{ is a soft fuzzy } B \text{ open set and } (\mu,N)\sqsubseteq(\lambda,M)\}.$$

Property 2.1. [18] *Let (X,T) be a soft fuzzy topological space. For any two soft fuzzy sets (λ,M) and (μ,N) the following statements are valid.*

(i) $(\lambda,M)\sqsubseteq(\mu,N)$ *implies* $SFBcl(\lambda,M)\sqsubseteq SFBcl(\mu,N)$.

(ii) $SFBcl((\lambda,M)\sqcap(\mu,N))\sqsubseteq SFBcl(\lambda,M)\sqcap SFBcl(\mu,N)$.

(iii) $SFBcl((\lambda,M)\sqcup(\mu,N))=SFBcl(\lambda,M)\sqcup SFBcl(\mu,N)$.

Property 2.2. [18] *Let (X,T) be a soft fuzzy topological space. For any soft fuzzy set (λ,M) in X, the following statements are valid.*

(i) $SFBint(\lambda,M)\sqsubseteq(\lambda,M)\sqsubseteq SFBcl(\lambda,M)$.

(ii) $(SFBint(\lambda,M))'=SFBcl(\lambda,M)'$.

(iii) $(SFBcl(\lambda,M))'=SFBint(\lambda,M)'$.

Definition 2.14. [19] Let (X,T) be a soft fuzzy topological space. Let (λ,M) be a soft fuzzy set of X. Then (λ,M) is said to be a *soft fuzzy G_δ* set (in short, SFG_δ) if $(\lambda,M)=\sqcap_{i=1}^{\infty}(\mu_i,N_i)$, where each (μ_i,N_i) is a soft fuzzy open set. The complement of soft fuzzy G_δ set is a *soft fuzzy F_σ* set (in short, SFF_σ).

3. Soft fuzzy B extremally disconnected and soft fuzzy B basically disconnected spaces via soft fuzzy large inductive dimension function

3.1. Soft fuzzy B extremally disconnectedness

Definition 3.15. Let (λ,M) and (μ,N) be any two soft fuzzy sets. Then $(\lambda,M)+(\mu,N)$ is defined by $(\lambda,M)+(\mu,N)=((\lambda,M)\sqcap(\mu,N)')\sqcup((\lambda,M)'\sqcap(\mu,N))$.

Definition 3.16. Let (X,T) be a soft fuzzy topological space. Then (X,T) is said to be *soft fuzzy B extremally disconnected space* if the soft fuzzy B closure of every soft fuzzy B open set is a soft fuzzy B open set.

Property 3.3. *Let (X,T) be a soft fuzzy topological space. Then the following conditions are equivalent:*

(i) (X,T) *is a soft fuzzy B extremally disconnected space.*

(ii) *For each soft fuzzy B closed set (λ,M), $SFBint(\lambda,M)$ is soft fuzzy B closed.*

(iii) *For each soft fuzzy B open set* (λ, M)*,*

$$SFBcl(\lambda, M) + SFBcl(SFBcl(\lambda, M))' = (1, X).$$

(iv) *For every pair of soft fuzzy B open sets* (λ, M) *and* (μ, N) *with* $SFBcl(\lambda, M)+ (\mu, N) = (1, X)$*, we have* $SFBcl(\lambda, M) + SFBcl(\mu, N) = (1, X)$*.*

Proof. (i)⇒(ii). Let (λ, M) be any soft fuzzy B closed G_δ set in X. Then $(\lambda, M)'$ is a soft fuzzy B open set. Now,

$$SFBcl(\lambda, M)' = (SFBint(\lambda, M))'.$$

By (i), $SFBcl(\lambda, M)'$ is a soft fuzzy B open set. Then $SFBint(\lambda, M)$ is a soft fuzzy B closed set.

(ii)⇒(iii). Let (λ, M) be any soft fuzzy B open set. Then

$$\begin{aligned} &SFBcl(\lambda, M) + SFBcl(SFBcl(\lambda, M))' \\ &\quad = SFBcl(\lambda, M) + SFBcl(SFBint(\lambda, M)') \end{aligned} \tag{3.1}$$

Since (λ, M) is a soft fuzzy B open set, $(\lambda, M)'$ is a soft fuzzy B closed set. Hence by (ii), $SFBint(\lambda, M)'$ is soft L-fuzzy B closed. Therefore, by (3.1)

$$\begin{aligned} &SFBcl(\lambda, M) + SFBcl(SFBcl(\lambda, M))' \\ &\quad = SFBcl(\lambda, M) + SFBcl(SFBint(\lambda, M)') \\ &\quad = SFBcl(\lambda, M) + SFBint(\lambda, M)' \\ &\quad = SFBcl(\lambda, M) + (SFBcl(\lambda, M))' \\ &\quad = (1, X). \end{aligned}$$

Therefore, $SFBcl(\lambda, M) + SFBcl(SFBcl(\lambda, M))' = (1, X)$.

(iii)⇒(iv). Let (λ, M) and (μ, N) be any two soft fuzzy B open sets such that

$$SFBcl(\lambda, M) + (\mu, N) = (1, X). \tag{3.2}$$

Then by (iii),

$$\begin{aligned} (1, X) &= SFBcl(\lambda, M) + SFBcl(SFBcl(\lambda, M))' \\ &= SFBcl(\lambda, M) + SFBcl(\mu, N), \text{ by (3.2)}. \end{aligned}$$

Therefore, $SFBcl(\lambda, M) + SFBcl(\mu, N) = (1, X)$.

(iv)⇒(i). Let (λ, M) be any soft fuzzy B open F_σ set. Put $(\mu, N) = (SFB\,cl(\lambda, M))' = (1, X) - SFBcl(\lambda, M)$. Then $SFBcl(\lambda, M) + (\mu, N) = (1, X)$. Therefore by (iv), $SFBcl(\lambda, M) + SFBcl(\mu, N) = (1, X)$. This implies that $SFBcl(\lambda, M)$ is a soft fuzzy B open set and so (X, T) is a soft fuzzy B extremally disconnected space.

Property 3.4. *Let (X,T) be a soft fuzzy topological space. Then (X,T) is soft fuzzy B extremally disconnected if and only if for all soft fuzzy B open set (λ,M) and every soft fuzzy B closed set (μ,N) such that $(\lambda,M)\sqsubseteq(\mu,N)$, $SFBcl(\lambda,M)\sqsubseteq SFBint(\mu,N)$.*

Proof. Let (λ,M) be any soft fuzzy B open set and (μ,N) be any soft fuzzy B closed set with $(\lambda,M)\sqsubseteq(\mu,N)$. By (ii) of Property 3.1, $SFBint(\mu,N)$ is a soft fuzzy B closed set. Therefore, $SFBcl(SFBint(\mu,N)) = SFBint(\mu,N)$. Also, since (λ,M) is a soft fuzzy B open set and $(\lambda,M)\sqsubseteq(\mu,N)$, $SFBcl(\lambda,M)$ $\sqsubseteq SFBint(\mu,N)$. Therefore, $SFBcl(\lambda,M)\sqsubseteq SFBint(\mu,N)$.

Conversely, let (μ,N) be any soft fuzzy B closed set then $SFBint(\mu,N)$ is soft fuzzy B open and $SFBint(\mu,N)\sqsubseteq(\mu,N)$. Therefore by assumption, $SFBcl(SFBint(\mu,N))\sqsubseteq SFBint(\mu,N)$. This implies that, $SFBint(\mu,N)$ is a soft fuzzy B closed set. Hence by (ii) of Property 3.1, it follows that (X,T) is a soft fuzzy B extremally disconnected space.

Property 3.5. *Let (X,T) be a soft fuzzy B extremally disconnected space. Let $\{(\lambda_i,M_i),(\mu_i,N_i)'/i\in\mathbb{N}\}$ be collection such that every $(\lambda_i,M_i)'s$ be a soft fuzzy B open sets and every $(\mu_i,N_i)'s$ be a soft fuzzy B closed sets and let (λ,M) and (μ,N) be soft fuzzy B clopen sets. If*

$$(\lambda_i,M_i)\sqsubseteq(\lambda,M)\sqsubseteq(\mu_j,N_j) \text{ and } (\lambda_i,M_i)\sqsubseteq(\mu,N)\sqsubseteq(\mu_j,N_j)$$

for all $i,j\in\mathbb{N}$, then there exists a soft fuzzy B clopen set (γ,L) such that $SFBcl$ $(\lambda_i,M_i)\sqsubseteq(\gamma,L)\sqsubseteq SFBint(\mu_j,N_j)$ for all $i,j\in\mathbb{N}$.

Proof. By Property 3.2, $SFBcl(\lambda_i,M_i)\sqsubseteq SFBcl(\lambda,M)\sqcap SFBint(\mu,N)\sqsubseteq$ $SFBint\,(\mu_j,N_j)$ for all $i,j\in\mathbb{N}$. Letting, $(\gamma,L)=SFBcl(\lambda,M)\sqcap SFBint(\mu,N)$ is a soft fuzzy B clopen set satisfying the required conditions.

3.2. Soft fuzzy B basically disconnectedness

Definition 3.17. Let (X,T) be a soft fuzzy topological space. Let (λ,M) be any soft fuzzy set. Then (λ,M) is said to be a *soft fuzzy G_δ set* (in short, SFG_δ) if $(\lambda,M)=\sqcap_{i=1}^{\infty}(\mu_i,N_i)$, where each (μ_i,N_i) is a soft fuzzy open set. The complement of soft fuzzy G_δ set is a *soft fuzzy F_σ set* (in short, SFF_σ).

Definition 3.18. Let (X,T) be a soft fuzzy topological space. Let (λ,M) be any soft fuzzy set. Then (λ,M) is said to be a

(i) *soft fuzzy B open F_σ set* if (λ,M) is both soft fuzzy B open and soft fuzzy F_σ set.

(ii) *soft fuzzy B closed G_δ set* if (λ,M) is both soft fuzzy B closed and soft fuzzy G_δ set.

(iii) *soft fuzzy B clopen $G_\delta F_\sigma$ set* if (λ,M) is both soft fuzzy B open F_σ set and soft fuzzy B closed G_δ set.

Definition 3.19. Let (X, T) be a soft fuzzy topological space. Then (X, T) is said to be a *soft fuzzy B basically disconnected space* if the soft fuzzy B closure of every soft fuzzy B open F_σ set is a soft fuzzy B open set.

Property 3.6. *Let (X, T) be a soft fuzzy topological space, the following statements are equivalent:*

(i) *(X, T) is a soft fuzzy B basically disconnected space.*

(ii) *For each soft fuzzy B closed G_δ set (λ, M), $SFBint(\lambda, M)$ is soft fuzzy B closed G_δ set.*

(iii) *For each soft fuzzy B open F_σ set (λ, M),*

$$SFBcl(\lambda, M) + SFBcl(SFBcl(\lambda, M))' = (1, X).$$

(iv) *For every pair of soft fuzzy B open F_σ sets (λ, M) and (μ, N) with SFB $cl(\lambda, M) + (\mu, N) = (1, X)$, we have $SFBcl(\lambda, M) + SFBcl(\mu, N) = (1, X)$.*

Proof. The proof is similar to that of Property 3.1.

Property 3.7. *Let (X, T) be soft fuzzy topological space. Then (X, T) is a soft fuzzy B basically disconnected space if and only if for all soft fuzzy B clopen $G_\delta F_\sigma$ sets (λ, M) and (μ, N) such that $(\lambda, M) \sqsubseteq (\mu, N)$, $SFBcl(\lambda, M) \sqsubseteq SFBint(\mu, N)$.*

Proof. Let (λ, M) and (μ, N) be any two soft fuzzy B clopen $G_\delta F_\sigma$ sets with $(\lambda, M) \sqsubseteq (\mu, N)$. By (ii) of Property 3.4, $SFBint(\mu, N)$ is a soft fuzzy B closed G_δ set. Therefore, $SFBcl(SFBint(\mu, N)) = SFBint(\mu, N)$. Also, since (λ, M) is soft fuzzy B clopen $G_\delta F_\sigma$ set and $(\lambda, M) \sqsubseteq (\mu, N)$, $SFBcl(\lambda, M)$ $\sqsubseteq SFBint(\mu, N)$. Therefore, $SFBcl(\lambda, M) \sqsubseteq SFBint(\mu, N)$.

Conversely, let (μ, N) be any soft fuzzy B clopen $G_\delta F_\sigma$ set then $SFBint(\mu, N)$ is a soft fuzzy B open F_σ set and $SFBint(\mu, N) \sqsubseteq (\mu, N)$. Therefore by assumption, $SFBcl(SFBint(\mu, N)) \sqsubseteq SFBint(\mu, N)$. This implies that $SFBint(\mu, N)$ is a soft fuzzy B closed G_δ set. Hence by (ii) of Property 3.4, it follows that (X, T) is a soft fuzzy B basically disconnected space.

Property 3.8. *Let (X, T) be a soft fuzzy B basically disconnected space. Let $\{(\lambda_i, M_i), (\mu_i, N_i)'/i \in \mathbb{N}\}$ be a collection such that $(\lambda_i, M_i)'s$ and $(\mu_i, N_i)'s$ are soft fuzzy B clopen $G_\delta F_\sigma$ sets and let (λ, M) and (μ, N) be soft fuzzy B clopen $G_\delta F_\sigma$ sets. If*

$$(\lambda_i, M_i) \sqsubseteq (\lambda, M) \sqsubseteq (\mu_j, N_j) \text{ and } (\lambda_i, M_i) \sqsubseteq (\mu, N) \sqsubseteq (\mu_j, N_j)$$

for all $i, j \in \mathbb{N}$, then there exists a soft fuzzy B clopen $G_\delta F_\sigma$ set (γ, L) such that $SFBcl$ $(\lambda_i, M_i) \sqsubseteq (\gamma, L) \sqsubseteq SFBint(\mu_j, N_j)$ for all $i, j \in \mathbb{N}$.

Proof. By Property 3.5, $SFBcl(\lambda_i, M_i) \sqsubseteq SFBcl(\lambda, M) \sqcap SFBint(\mu, N) \sqsubseteq SFBint\ (\mu_j, N_j)$ for all $i, j \in \mathbb{N}$. Therefore, (γ, L)=$SFBcl(\lambda, M) \sqcap SFBint(\mu,\ N)$ is a soft fuzzy B clopen $G_\delta F_\sigma$ set satisfying the required conditions.

3.3. Soft fuzzy dimension theory

Definition 3.20. Let (X, T) be a soft fuzzy topological space. Let (λ, M) be any soft fuzzy set. Then the *soft fuzzy boundary* of (λ, M), is denoted and defined as

$$SFbd(\lambda, M) = SFcl(\lambda, M) - SFint(\lambda, M).$$

Definition 3.21. Let (X, T) be a soft fuzzy topological space. Let (λ, M) be any soft fuzzy set. Then the *soft fuzzy B boundary* of (λ, M), is denoted and defined as

$$SFBbd(\lambda, M) = SFBcl(\lambda, M) - SFBint(\lambda, M).$$

Property 3.9. *Let (X, T) be a soft fuzzy topological space. Let (λ, M) be any soft fuzzy set. Then $SFBbd(\lambda, M) = (0, \emptyset)$ iff (λ, M) is both soft fuzzy B open and soft fuzzy B closed.*

Proof. The proof follows from Definition 4.2.

Definition 3.22. Let (X, T) be a soft fuzzy topological space. Then the *soft fuzzy large inductive dimension* of (X, T), denoted by $SFIndX$, is defined as follows.

(i) $SFIndX = -1$ iff $X = \emptyset$.

(ii) For any positive integer n, $SFIndX \sqsubseteq n$ if for each soft fuzzy B closed set (λ, M) and each soft fuzzy B open set (μ, N) in (X, T) such that $(\lambda, M) \sqsubseteq (\mu, N)$ there exists a soft fuzzy B open set (γ, L) in (X, T) such that $(\lambda, M) \sqsubseteq (\gamma, L) \sqsubseteq (\mu, N)$ and $SFIndSFBbd(\gamma, L) \sqsubseteq n - 1$.

(iii) $SFIndX = n$ if $SFIndX \sqsubseteq n$ is true and $SFIndX \sqsubseteq n - 1$ is not true.

(iv) $SFIndX = \infty$ if $SFIndX \sqsubseteq n$ is not true for every n.

Property 3.10. *Let (X, T) be a soft fuzzy topological space. If $SFIndX = 0$ then (X, T) is a soft fuzzy B extremally disconnected space.*

Proof. Let (λ, M) be any soft fuzzy B open set and (μ, N) be any soft fuzzy B closed set such that $(\lambda, M) \sqsubseteq (\mu, N)$. By Property 3.2, $SFBcl(\lambda, M) \sqsubseteq SFBint(\mu, N)$. Since $SFIndX \sqsubseteq 0$ and by Definition 4.3, there exist a soft fuzzy B open set (γ, L) in (X, T) such that $SFBcl(\lambda, M) \sqsubseteq (\gamma, L) \sqsubseteq SFBint(\mu, N)$ and $SFIndSFBbd(\gamma, L) \sqsubseteq 0 - 1$. Therefore, $SFBbd(\gamma, L) = (0, \emptyset)$. Since $SFBbd(\gamma, L) = (0, \emptyset)$ from Property 4.1, (γ, L) is both soft fuzzy B open and soft fuzzy B closed. By Property 3.3, $SFBcl(\lambda, M) \sqsubseteq (\gamma, L) \sqsubseteq SFBint(\mu, N)$. Hence (X, T) is a soft fuzzy B extremally disconnected space.

Property 3.11. *Let (X, T) be a soft fuzzy topological space. If $SFIndX = 0$ then (X, T) is a soft fuzzy B basically disconnected space.*

Proof. The proof is similar to that of Property 4.2.

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FACTOR BISEMIRINGS

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Abstract. In this paper we define congruence relations on bisemirings and bisemiring homomorphisms. We show that each bisemiring homomorphism defines a congruence relation on a bisemiring and then we introduce factor bisemirings. In the last section, we prove analogue of the isomorphism theorems.

Keywords: congruences, homomorphisms, analogue of the isomorphism theorems.

1. Introduction

In 1934, Vandever introduced the notion of a semiring which was a common generalization of rings and distributive lattices. The following definition has been taken from [4].

A semiring $(R, +, \cdot)$ is a non-empty set in which $(R, +)$ and $(R, \cdot)$ are semigroups such that "$\cdot$" is distributive over "$+$".

Corresponding to semiring in 2001, M.K. Sen, Shamik Ghosh and Suma Ghosh introduced the concept of a bisemiring in [3]. A bisemiring $(R, +, \cdot, \times)$ is a non-empty set in which $(R, +, \cdot)$ and $(R, +, \times)$ are semirings. In other words $(R, +)$, $(R, \cdot)$ and $(R, \times)$ are semigroups such that for all $a, b, c \in R$,

$$a \cdot (b + c) = a \cdot b + a \cdot c$$
$$(b + c) \cdot a = b \cdot a + c \cdot a$$

and

$$a \times (b \cdot c) = (a \times b) \cdot (a \times c)$$
$$(b \cdot c) \times a = (b \times a) \cdot (c \times a)$$

where "$+$" is called addition "$\cdot$" is called multiplication and "$\times$" is called product. To understand the above concept we give some examples. The following examples have been taken from [3].

Example 1.1.

(i) Let $P(X)$ be the power set on a non-empty set X. Then $(P(X), \triangle, \cap, \cup)$ is a bisemiring.

(ii) Let N be the set of natural numbers. Then $(N, \min, \max, +)$, $(N, \min, \max, \cdot)$ are bisemirings.

(iii) Consider again the set N of natural numbers. Define $a + b = gcd(a, b)$ and $a \cdot b = lcm(a, b)$ and $a \times b = ab$. Then $(N, +, \cdot, \times)$ is a bisemiring.

2. Congruences on Bisemirings

In this section we define left compatible (left congruence), right compatible (right congruence) and compatible (congruence) relations on bisemirings. The idea comes from the book [2] in which the author has defined these terms for semigroups. At the end of this section we prove a result which gives equivalent conditions for congruence relations on bisemirings by using the same idea of [2].

Definition 2.1. Let $(R, +, \cdot)$ be a bisemiring. A relation ρ on R is said to be left compatible if for all s, t and $a \in R$ such that $(s, t) \in \rho$ implies that $(a + s, a + t)$, $(a \cdot s, a \cdot t)$ and $(a \times s, a \times t) \in \rho$.

The relation ρ is said to be right compatible if for all s, t and $b \in R$ such that $(s, t) \in \rho$ implies $(s + b, t + b)$, $(s \cdot b, s \cdot b)$ and $(s \times b, t \times b) \in \rho$.

It is called compatible if for all s, t, u and $v \in R$, (s, t) and $(u, v) \in \rho$ implies that $(s + u, t + v)$, $(s \cdot u, t \cdot v)$ and $(s \times u, t \times v) \in \rho$.

A left (right) compatible equivalence relation is called a left (right) congruence relation. A compatible equivalence relation is called a congruence relation.

To understand the above notion we give an example.

Example 2.2. Let $P(X)$ be the power set on a non-empty set X. Then $(P(X), \triangle, \cap, \cup)$ is a bisemiring as discussed in Section 1. Let ρ=$\{(A, B) : A$=$B\}$ be a relation on $P(X)$. Then one can easily verify that ρ is a congruence relation on $P(X)$.

We are now going to state and prove a result which gives equivalent conditions between left (right) congruence relations and congruence relations.

Proposition 2.3. *A relation ρ on a bisemiring R is a congruence relation if and only if it is both a left and a right congruence relation.*

Proof. Suppose that ρ is a congruence relation on R. Let s, t and $a \in R$ such that $(s, t) \in \rho$, then $(a + s, a + t)$, $(a \cdot s, a \cdot t)$ and $(a \times s, a \times t) \in \rho$, since $(a, a) \in \rho$. This shows that ρ is a left congruence relation. In the same way we can show that ρ is a right congruence relation. Conversely, assume that ρ is both right and left congruence relation. Let s, t, u and $v \in R$ such that (s, t) and $(u, v) \in \rho$. This implies that $(s+u, t+u)$, $(s\cdot u, t\cdot u)$ and $(s\times u, t\times u) \in \rho$, as ρ is a right compatible relation and $(t+u, t+v)$, $(t \cdot u, t \cdot v)$ and $(t \times u, t \times v) \in \rho$, as ρ is a left compatible realtion. This implies that $(s+u, t+v)$, $(s \cdot u, t \cdot v)$ and $(s \times u, t \times v) \in \rho$, as ρ is transitive. Thus ρ is a congruence relation. ∎

Like homomorphisms of other algebraic structures such as groups and rings, homomorphisms of bisemirings are maps which preserve binary operations. In the following section we give a proper definition of a bisemiring homomorphism.

3. Homomorphisms of bisemirings

Definition 3.1. Let $(R, +, \cdot, \times)$ and $(S, \oplus, \circ, \otimes)$ be two bisemirings. A function $f : R \to S$ is said to be a bisemiring homomorphism or simply a homomorphism if it satisfies the following conditions:

(i) $f(r+s) = f(r) \oplus f(s)$ for all $r, s \in R$;

(ii) $f(r \cdot s) = f(r) \circ f(s)$ for *all* $r, s \in R$;

(iii) $f(r \times s) = f(r) \otimes f(s)$ for all $r, s \in R$.

The terms monomorphism, epimorphism, isomorphism, endomorphism and automorphism can be defined in the same way. If there is an isomorphism from a bisemiring R to a bisemiring S, then we say that R is isomorphic to S and write $R \cong S$.

To understand the above concept we give an example.

Example 3.2. Let N be the set of natural numbers. Then $(N, \min, \max, +)$ and $(2N, \min, \max, +)$ are bisemirings as discussed in Section 1. Now define a map $\theta : N \to 2N$ by $\theta(n) = 2n$. Then it can be easily verified that θ is a bisemiring homomorphism.

We are now going to state a result in which we prove that corresponding to every homomorphism there is a congruence relation. The result is important because once we get this congruence relation we can get factor bisemiring.

Theorem 3.3. *If f is a homomorphism from a bisemiring R to a bisemiring S, then f defines a congruence relation ρ on R given by $r\rho s$ if and only if $f(r) = f(s)$.*

Proof. First, we show that this is an equivalence relation. Since $f(r) = f(r)$ for all $r \in R$, therefore $r\rho r$ and the relation is reflexive. If $r\rho s$, then $f(r) = f(s)$ and this implies that $f(s) = f(r)$. Thus $s\rho r$ and so the relation is symmetric. Now, if $r\rho s$ and $s\rho t$, then $f(r) = f(s)$ and $f(s) = f(t)$ and this gives us $f(r) = f(t)$. This shows that $r\rho t$ and so the relation is transitive. Now, let $r\rho s$ and $t\rho u$, then $f(r) = f(s)$ and $f(t) = f(u)$. As $f(r+t) = f(r) + f(t) = f(s) + f(u) = f(s+u)$. So we get $r+t \ \rho \ s+u$. Similarly $r \cdot t \ \rho \ s \cdot u$ and $r \times t \ \rho \ s \times u$. Thus the relation is compatible. This completes the proof. ∎

Let ρ be an equivalence relation on a set A. Then the equivalence class corresponding to an element a of A is denoted by the symbol $a\rho$ and is defined as:

$$a\rho = \{x \in A = (a, x) \in \rho\}.$$

If ρ is a congruence relation on a bisemiring R, then we say that $a\rho$ is a congruence class corresponding to the element a of R. Let R/ρ denote the set of all congruence classes, i.e., $R/\rho = \{a\rho : a \in R\}$. We are now going to state a result which has been taken from [1] and will be used later. The result is true for classes but since we know that every class is a set, so in particular it is true for sets as well.

Lemma 3.4. *Let ρ be an equivalence relation on a set A, then $a\rho = b\rho$ if and only if $(a, b) \in \rho$.*

Let $a, b \in R$ and $a\rho$, $b\rho$ represent the congruence classes corresponding to a and b, then we can define binary operations on the quotient set R/ρ as follows:

$$a\rho + b\rho = (a + b)\rho,$$
$$a\rho \cdot b\rho = (a \cdot b)\rho$$

and

$$a\rho \times b\rho = (a \times b)\rho.$$

These operations are well defined, since for all a, b, c and $d \in R$ if $a\rho = c\rho$ and $b\rho = d\rho$, then by the above lemma, $(a, c) \in \rho$ and $(b, d) \in \rho$. Thus $(a + b, c + d), (a \cdot b, c \cdot d), (a \times b, c \times d) \in \rho$, as ρ is a congruence relation. Thus again by the above lemma this implies that $(a + b)\rho = (c + d)\rho$, $(a \cdot b)\rho = (c \cdot d)\rho$ and $(a \times b)\rho = (c \times d)\rho$. Further,

Associative laws:

(i) With respect to addition:

Let a, b and $c \in R$ such that $a\rho, b\rho$ and $c\rho \in R/\rho$, then

$$\begin{aligned}(a\rho + b\rho) + c\rho &= (a + b)\rho + c\rho \\ &= ((a + b) + c))\rho \\ &= ((a + (b + c))\rho \\ &= a\rho + (b + c)\rho \\ &= a\rho + (b\rho + c\rho).\end{aligned}$$

(ii) With respect to multiplication:

$$\begin{aligned}(a\rho \cdot b\rho) \cdot c\rho &= (a \cdot b)\rho \cdot c\rho \\ &= ((a \cdot b) \cdot c))\rho \\ &= ((a \cdot (b \cdot c))\rho \\ &= a\rho \cdot (b \cdot c)\rho \\ &= a\rho \cdot (b\rho \cdot c\rho).\end{aligned}$$

(iii) With respect to product:

$$\begin{aligned}(a\rho \times b\rho)c\rho &= (a \times b)\rho \times c\rho\\ &= ((a \times b) \times c))\rho\\ &= ((a \times (b \times c))\rho\\ &= a\rho \times (b \times c)\rho\\ &= a\rho \times (b\rho \times c\rho).\end{aligned}$$

Thus $(R/\rho, +)$, $(R/\rho, \cdot)$ and $(R/\rho, \times)$ are semigroups.

Distributive laws:

(i) Multiplication is distributive over addition:

$$\begin{aligned}a\rho \cdot (b\rho + c\rho) &= a\rho \cdot \ (b + c)\rho\\ &= (a \cdot (b + c))\rho\\ &= (a \cdot b + a \cdot c)\rho\\ &= (a \cdot b)\rho + (a \cdot c)\rho\\ &= a\rho \cdot b\rho + a\rho \cdot c\rho\end{aligned}$$

Similarly,

$$(a\rho + b\rho) \cdot c\rho = a\rho \cdot c\rho + b\rho \cdot c\rho$$

(ii) Product is distributive over multiplication:

$$\begin{aligned}a\rho \times (b\rho \cdot c\rho) &= a\rho \times (b \cdot c)\rho\\ &= (a \times (b \cdot c))\rho\\ &= (a \times b \cdot a \times c)\rho\\ &= (a \times b)\rho \cdot (a \times c)\rho\\ &= a\rho \times b\rho \cdot a\rho \times c\rho.\end{aligned}$$

Similarly,

$$(a\rho \cdot b\rho) \times c\rho = a\rho \times c\rho \cdot b\rho \times c\rho.$$

Thus $(R/\rho, +, \cdot, \times)$ is a bisemiring which is called a factor bisemiring.

We are now going to state and prove the bisemiring analogues of the first, second and third isomorphism theorems. The semigroup equivalents can be found in [2].

Theorem 3.5. (First Isomorphism Theorem) *If ρ is a congruence on a bisemiring R. Then R/ρ is a bisemiring with respect to the operations*

$$\begin{aligned}a\rho + b\rho &= (a + b)\rho,\\ a\rho \cdot b\rho &= (a \cdot b)\rho\\ a\rho \times b\rho &= (a \times b)\rho.\end{aligned}$$

The mapping $\rho : R \to R/\rho$ *defined by* $\rho^{\#}(a) = a\rho$ *for all* $a \in R$ *is an epimorphism. If*

$$\phi : R \to S$$

is a homomorphism where R *and* S *are bisemirings, then the relation*

$$ker\phi = \{(a, b) \in R \times R : \phi(a) = \phi(b)\}$$

is a congruence relation on R *and there is a monomorphism* $\alpha : R/ker \to S$ *such that* $ran\alpha = ran\phi$ *and the diagram*

$$\begin{tikzcd} R \arrow[r, "\phi"] \arrow[d, "(\ker\phi)^{\#}"'] & S \\ R/\ker\phi \arrow[ur, "\alpha"'] & \end{tikzcd}$$

commutes.

Proof. We have already proved that R/ρ is a bisemiring. Now, let a and $b \in R$, then

$$\begin{aligned} \rho^{\#}(a+b) &= (a+b)\rho = a\rho + b\rho = \rho^{\#}(a) + \rho^{\#}(b), \\ \rho^{\#}(a \cdot b) &= (a \cdot b)\rho = a\rho \cdot b\rho = \rho^{\#}(a) \cdot \rho^{\#}(b) \\ \rho^{\#}(a \times b) &= (a \times b)\rho = a\rho \times b\rho = \rho^{\#}(a) \times \rho^{\#}(b) \end{aligned}$$

Thus ρ is a homomorphism. $ker\phi$ is a congruence relation on R by Theorem 3.3. Next we define $\alpha : R/ker\phi \to S$ by $\alpha(aker\phi) = \phi(a)$. Then α is both well defined and one-one, since for all $a, b \in R$

$$aker\phi = bker\phi \Leftrightarrow (a, b) \in ker\phi \Leftrightarrow \phi(a) = \phi(b) \Leftrightarrow \alpha(aker\phi) = \alpha(bker\phi).$$

It is a homomorphism, since for all $a, b \in R$

$$\begin{aligned} \alpha[(aker\phi) + (bker\phi)] &= \alpha[(a+b)ker\phi] \\ &= \phi(a+b) \\ &= \phi(a) + \phi(b) \\ &= \alpha(aker\phi) + \alpha(bker\phi), \\ \alpha[(aker\phi) \cdot (bker\phi)] &= \alpha[(a \cdot b)ker\phi] \\ &= \phi(a \cdot b) \\ &= \phi(a) \cdot \phi(b) \\ &= \alpha(aker\phi) \cdot \alpha(bker\phi) \end{aligned}$$

and

$$\begin{aligned} \alpha[(aker\phi) \times (bker\phi)] &= \alpha[(a \times b)ker\phi] \\ &= \phi(a \times b) \\ &= \phi(a) \times \phi(b) \\ &= \alpha(aker\phi) \times \alpha(bker\phi). \end{aligned}$$

Clearly, $ran\alpha = ran\phi$ and from the definition it is obvious that for all $a \in R$, $\alpha[(ker\phi)^{\#}(a)] = \alpha(aker\phi) = \phi(a)$. That is, the diagram commutes. ∎

Theorem 3.6. (Second Isomorphism Theorem) *Let ρ be congruence relation on a bisemiring R. If $\phi : R \longrightarrow S$ is a homomorphism where R and S are bisemirings such that $\rho \subseteq ker\phi$, then there is a unique homomorphism $\beta : R/\rho \longrightarrow S$ such that $ran\beta = ran\phi$ and the diagram*

$$\begin{tikzcd} R \arrow[r, "\phi"] \arrow[d, "\rho^{\#}"'] & S \\ R/\rho \arrow[ur, "\beta"'] & \end{tikzcd}$$

commutes.

Proof. Define $\beta : R/\rho \longrightarrow S$ by $\beta(a\rho) = \phi(a)$ where $a\rho \in R/\rho$. Then β is well defined, since for all $a, b \in R$,

$$a\rho = b\rho \Rightarrow (a, b) \in \rho \subseteq ker\phi \Rightarrow \phi(a) = \phi(b) \Rightarrow \beta(a\rho) = \beta(b\rho)\beta.$$

β is a homomorphism, because if $a\rho, b\rho \in R/\rho$, then

$$\begin{aligned} \beta(a\rho + b\rho) &= \beta[(a + b)\rho] = \phi(a + b) = \phi(a) + \phi(b) = \beta(a\rho) + \beta(a\rho), \\ \beta(a\rho \cdot b\rho) &= \beta[(a \cdot b)\rho] = \phi(a \cdot b) = \phi(a) \cdot \phi(b) = \beta(a\rho) \cdot \beta(a\rho) \\ \beta(a\rho \times b\rho) &= \beta[(a \times b)\rho] = \phi(a \times b) = \phi(a) \times \phi(b) = \beta(a\rho) \times \beta(a\rho). \end{aligned}$$

Now, $\beta[\rho^{\#}(a)] = \beta(a\rho) = \phi(a)$. That is the diagram commutes and it is obvious that $ran\beta = ran\phi$. Finally, let $\beta_1 : R/\rho \longrightarrow S$ be another homomorphism such that $\beta_1\rho^{\#} = \phi$. Let $a \in R$, then

$$\beta_1[\rho^{\#}(a)] = \phi(a) = \beta[\rho^{\#}(a)].$$

So

$$\beta_1(a\rho) = \beta(a\rho),$$

i.e.

$$\beta_1 = \beta.$$

■

Theorem 3.7. (Third Isomorphism Theorem) *Let ρ and σ be congruence relations on a bisemiring R such that $\rho \subseteq \sigma$. Then*

$$\sigma/\rho = \{(x\rho, y\sigma) \in R/\rho \times R/\rho : (x, y) \in \sigma\}$$

is a congruence relation on R/ρ and

$$R/\rho|\sigma/\rho \cong R/\sigma.$$

Proof. First, we show that σ/ρ is a congruence relation. Let $x \in R$, then $(x, x) \in \sigma$, as σ is reflexive. Then $(x\rho, x\rho) \in \sigma/\rho$, so σ/ρ is reflexive. Now, let $x, y \in R$ such that $(x\rho, y\rho) \in \sigma/\rho$, then $(x, y) \in \sigma$ and this implies $(y, x) \in \sigma$, as σ is symmetric. Then $(y\rho, x\rho) \in \sigma/\rho$, so σ/ρ is symmetric. Now, let x, y and $z \in R$ such that $(x\rho, y\rho)$ and $(y\rho, z\rho) \in \sigma/\rho$, then (x, y) and $(y, z) \in \sigma$ and this implies $(x,z) \in \sigma$, as σ is transitive. Then $(x\rho, z\rho) \in \sigma/\rho$, so σ/ρ is transitive. This shows that σ/ρ is an equivalence relation.

Now, let w, x, y and $z \in R$ such that $(w\rho, x\rho)$ and $(y\rho, z\rho) \in \sigma/\rho$, then (w,x) and $(y,z) \in \sigma$.This implies that $(w+y, x+z), (w \cdot y, x \cdot z)$ and $(w \times y, x \times z) \in \sigma$, as σ is compatible, and this implies

$$((w+y)\rho, (x+z)\rho), ((w \cdot y)\rho, (x \cdot z)\rho) \text{ and } ((w \times y)\rho, (x \times z)\rho) \in \sigma/\rho.$$

This shows that σ/ρ is compatible.

Now, define $\beta : R/\rho \longrightarrow R/\sigma$ by $\beta(a\rho) = a\sigma$. Let $a\rho$ and $b\rho \in R/\rho$, then

$$\begin{aligned} \beta(a\rho + b\rho) &= \beta((a+b)\rho) = (a+b)\sigma = a\sigma + b\sigma = \beta(a\rho) + \beta(b\rho) \\ \beta(a\rho \cdot b\rho) &= \beta((a \cdot b)\rho) = (a \cdot b)\sigma = a\sigma \cdot b\sigma = \beta(a\rho) \cdot \beta(b\rho). \end{aligned}$$

Similarly,

$$\beta(a\rho \times b\rho) = \beta((a \times b)\rho) = (a \times b)\sigma = a\sigma \times b\sigma = \beta(a\rho) \times \beta(b\rho)$$

Thus, β is homomorphism. So, by Theorem 3.5, there is a monomorphism

$$\alpha : R/\rho | ke\beta \longrightarrow R/\sigma$$

defined by $\alpha((a\rho)\ker\beta) = a\sigma$. Clearly, it is onto. Thus, $R/\rho|\ker\beta \cong R/\sigma$. Now,

$$\begin{aligned} \ker\beta &= \{(x\rho, y\rho) \in R/\rho \times R/\rho : \beta(x\rho) = \beta(y\rho)\} \\ &= \{(x\rho, y\rho) \in R/\rho \times R/\rho : x\sigma = y\sigma\} \\ &= \{(x\rho, y\rho) \in R/\rho \times R/\rho : (x,y) \in \sigma\} \\ &= \sigma/\rho. \end{aligned}$$

Thus, $R/\rho|\sigma/\rho \cong R/\sigma$, as required. ∎

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FUZZY PARAMETERIZED FUZZY SOFT RINGS AND APPLICATIONS

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Abstract. The concepts of FP-fuzzy soft rings, FP-equivalent fuzzy soft rings and FP-increasing(decreasing) fuzzy soft rings are introduced. Then some properties of them are given. Finally, aggregate fuzzy subrings are proposed by aggregate fuzzy sets of FP-fuzzy soft rings.

Keywords: FP-fuzzy soft rings; FP-equivalent fuzzy soft rings; FP-increasing (decreasing) fuzzy soft rings; FP-fuzzy soft homomorphism; aggregate fuzzy subrings.

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1. Introduction

In dealing with the complicated problems in economics, engineering and environmental sciences, we are usually unable to apply the classical methods because there are various uncertainties in these problems. Some kinds of theories were developed like theory of fuzzy sets, soft sets, i.e., which can be used as the fundamental tools for dealing with uncertainties.

The concept of fuzzy sets and fuzzy set operations, introduced by L.A. Zadeh [18], have been extensively applied to many scientific fields. In 1971, A. Rosenfeld [17] applied the concept to the theory of groupoids and groups. In 1982, W. Liu [8] defined and studied fuzzy subrings as well as fuzzy ideals. Since then many papers concerning various fuzzy algebraic structures have appeared in the literature.

The concept of soft sets was introduced by D. Molodtsov in 1999 [15], which was another mathematical tool for dealing with uncertainties. At present, the algebraic structure of set theories dealing with uncertainties has been studied by

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many authors. H. Aktaş et al. [1] applied the notion of soft sets to the theory of groups. Y.B. Jun [6] introduced the notions of soft BCK/BCI-algebras, and then investigated their basic properties [7]. We also noticed that F. Feng et al. [5] have already investigated the structure of soft semirings. In [9], we have proposed the definition of soft rings and established three isomorphism theorems. Furthermore, we gave three fuzzy isomorphism theorems of soft rings in [10].

In 2001, P.K. Maji et al. [13] presented the definition of fuzzy soft set, and Roy et al. presented some applications of this notion to decision-making problems in [14]. We notice that E. İnan et al. [4] have already introduced the definition of fuzzy soft rings and studied some of their basic properties.

Furthermore, N. Çağman introduced fuzzy parameterized soft sets [2] and fuzzy parameterized fuzzy soft sets [3], in short written FP-soft sets and FP-fuzzy soft sets, respectively, whose parameters sets are fuzzy sets and have improved several results. In [3], the authors also defined their operation and soft aggregation operator to form FP-fuzzy soft decision making method that allows constructing more efficient decision processes. FP-soft sets and FP-fuzzy soft sets have already been studied by some authors. We have studied FP-soft rings on FP-soft set theory in [11].

In this paper, we study FP-fuzzy soft rings on FP-fuzzy soft set theory. We first introduce FP-fuzzy soft rings generated by FP-fuzzy soft sets and some properties of FP-fuzzy soft rings will be given. Then FP-equivalent soft rings and FP-accelerating(decelerating) fuzzy soft rings will be studied. Moreover, the notions of FP-fuzzy homomorphisms of FP-fuzzy soft rings are proposed and some examples are given. Finally, aggregate fuzzy subrings will be proposed by aggregate operator and an example will be given to show that the methods can be successfully applied to many problems that contain uncertainties.

2. Preliminaries

Definition 2.1 [16]

(i) A fuzzy set μ in a ring R is said to be a fuzzy subring of R if the following conditions hold for all $x, y \in R$:

(1) $\mu(x-y) \geq \min\{\mu(x), \mu(y)\}$, and

(2) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$.

(ii) A fuzzy set μ in a ring R is said to be a fuzzy left (right) ideal of R if the following conditions hold for all $x, y \in R:$ (1) $\mu(x-y) \geq \min\{\mu(x), \mu(y)\}$, and (3) $\mu(xy) \geq \mu(y)$ $(\mu(xy) \geq \mu(x))$.

(iii) A fuzzy set μ is said to be a fuzzy ideal of R if it is both a fuzzy left ideal of R and a fuzzy right ideal of R.

Definition 2.2 [12] Let $f : X \to Y$ be a mapping of sets, μ a fuzzy set of X and ν a fuzzy set of Y. Then the image $f(\mu)$ of μ and preimage $f^{(-1)}(\nu)$ of ν are both fuzzy sets defined respectively as follows:

$$
\begin{aligned}
f(\mu)(y) &= \begin{cases} \sup\limits_{x\in f^{(-1)}(y)} \mu(x) & \text{if } f^{(-1)}(y) \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases} \\
f^{(-1)}(\nu)(x) &= \nu(f(x)), x \in X.
\end{aligned}
$$

Definition 2.3 [3] Let U be an initial universe, E be the set of all parameters and X be a fuzzy set over E with the membership function $\mu_X : E \longrightarrow [0,1]$ and $\gamma_X(x)$ be a fuzzy set over U for all $x \in E$, $F(U)$ be the set of all fuzzy set of U. Then an fuzzy parameterized fuzzy soft set Γ_X on U is defined by a function $\gamma_X(x)$ representing a mapping

$$\gamma_X : E \longrightarrow F(U)$$

such that $\gamma_X(x) = \emptyset$ if $\mu_X(x) = 0$. Here, γ_X is called the fuzzy approximate function of the fuzzy parameterized fuzzy soft set Γ_X, and the value $\gamma_X(x)$ is a fuzzy set called $x-$element of the fuzzy parameterized fuzzy soft set for all $x \in E$. Thus a fuzzy parameterized fuzzy soft set Γ_X over U can be represented by the set of ordered pairs

$$\Gamma_X = \{(\mu_X(x)/x, \gamma_X(x)) : x \in E, \gamma_X(x) \in F(U), \mu_X(x) \in [0,1]\}.$$

A fuzzy parameterized fuzzy soft set is briefly said to be an FP-fuzzy soft set. The set of all FP-fuzzy soft sets is denoted by $FPFS(U)$.

Definition 2.4 [3] Let $\Gamma_X \in FPFS(U)$.

(i) If $\gamma_X(x) = \emptyset$ for all $x \in E$, then Γ_X is called an X-empty FP-fuzzy soft set, denoted by $\Gamma_{\emptyset_X}$.

(ii) If $X = \emptyset$, then the Γ_X is called an empty FP-fuzzy soft set, denoted by $\Gamma_\emptyset$.

(iii) If $\mu_X(x) = 1$ and $\gamma_X(x) = U$ for all $x \in E$, then Γ_X is called an X-universal FP-fuzzy soft set, denoted by $(\Gamma_U)_X$.

(iv) If $X = E$, then the X-universal FP-fuzzy soft set is called an universal FP-fuzzy set, denoted by Γ_E.

Definition 2.5 [3] Let $\Gamma_X, \Gamma_Y \in FPFS(U)$. Then

(i) Γ_X is an FP-fuzzy soft subset of Γ_Y, denoted by $\Gamma_X \widetilde{\subseteq} \Gamma_Y$, if $\mu_X(x) \leq \mu_Y(x)$ and $\gamma_X(x) \subseteq \gamma_Y(x)$ for all $x \in E$.

(ii) Γ_X and Γ_Y are FP-equal, denoted by $\Gamma_X = \Gamma_Y$, if $\mu_X(x) = \mu_Y(x)$ and $\gamma_X(x) = \gamma_Y(x)$ for all $x \in E$.

Definition 2.6 [3] Let $\Gamma_X \in FPFS(U)$. Then the complement of Γ_X, denoted by $\Gamma_X^{\bar{c}}$, is an FP-fuzzy soft set defined by

$$\mu_X^{\bar{c}}(x) = 1 - \mu(x) \text{ and } \gamma_X^{\bar{c}}(x) = U \setminus \gamma_X(x).$$

Definition 2.7 [3] Let $\Gamma_X, \Gamma_Y \in FPFS(U)$.

(i) The union of Γ_X and Γ_Y, denoted by $\Gamma_X \widetilde{\cup} \Gamma_Y$, is defined by $\mu_{X\widetilde{\cup}Y}(x) = \max\{\mu_X(x), \mu_Y(x)\}$ and $\gamma_{X\widetilde{\cup}Y}(x) = \gamma_X(x) \cup \gamma_Y(x)$ for all $x \in E$.

(ii) The intersection of Γ_X and Γ_Y, denoted by $\Gamma_X \widetilde{\cap} \Gamma_Y$, is defined by $\mu_{X\widetilde{\cap}Y}(x) = \min\{\mu_X(x), \mu_Y(x)\}$ and $\gamma_{X\widetilde{\cap}Y}(x) = \gamma_X(x) \cap \gamma_Y(x)$ for all $x \in E$.

3. Fuzzy parameterized fuzzy soft rings

Definition 3.1 Let R be a ring, E be a set of parameters and X be a fuzzy set over E, $\Gamma_X = \{(\mu_X(x)/x, \gamma_X(x)) : x \in E, \gamma_X(x) \in F(R), \mu_X(x) \in [0,1]\} \in FPFS(R)$. Then Γ_X is said to be an FP-fuzzy soft ring over R if, for any $x \in E$, $\gamma_X(x)$ is a fuzzy subring of R.

Example 3.2 Let $R = Z_4 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}\}$ be a ring and $E = \{a, b\}$ be a set of parameters. If $X = \{0.2/a, 0.4/b\}$, $\gamma_X(a) = \{\overline{0}/0.5, \overline{1}/0.3, \overline{2}/0.4, \overline{3}/0.3\}$, $\gamma_X(b) = \{\overline{0}/0.4, \overline{1}/0.2, \overline{2}/0.3, \overline{3}/0.2\}$, then Γ_X is an FP-fuzzy soft ring over R.

Theorem 3.3 *Let R be a ring, E be a set of parameters. Then*

(1) $\Gamma_{\emptyset_X}$ *and* $\Gamma_\emptyset$ *are FP-fuzzy soft rings.*

(2) $(\Gamma_R)_X$ *and* $(\Gamma_R)_X^{\bar{c}}$ *are FP-fuzzy soft rings.*

(3) $\Gamma_\emptyset^{\bar{c}}$ *and* $\Gamma_E^{\bar{c}}$ *are FP-fuzzy soft rings.*

Proof. By Definitions 2.4 and 3.1, the proofs of (1) and (2) are straightforward. Since $\Gamma_\emptyset^{\bar{c}} = (\Gamma_R)_E$ and $\Gamma_E^{\bar{c}} = \Gamma_\emptyset$, then (3) is hold. ∎

Theorem 3.4 *Let R be a ring, E be a set of parameters and Γ_X and Γ_Y be FP-fuzzy soft rings over R. Then their intersection $\Gamma_X \widetilde{\cap} \Gamma_Y$ is still an FP-fuzzy soft ring over R.*

Proof. We can write $\Gamma_X \widetilde{\cap} \Gamma_Y = \Gamma_{X\widetilde{\cap}Y}$. For all $x \in E$, $\mu_{X\widetilde{\cap}Y}(x) = \min\{\mu_X(x), \mu_Y(x)\}$, $\gamma_X(x)$ and $\gamma_Y(x)$ are fuzzy subrings of R, then $\gamma_{X\widetilde{\cap}Y}(x) = \gamma_X(x) \cap \gamma_Y(x)$ is a fuzzy subring of R. Therefore, $\Gamma_X \widetilde{\bigcap} \Gamma_Y$ is an FP-fuzzy soft ring over R. ∎

Theorem 3.5 *Let R be a ring, E be a set of parameters and Γ_X and Γ_Y be FP-fuzzy soft rings over R with $X \cap Y = \emptyset$. Then their union $\Gamma_X \widetilde{\cup} \Gamma_Y$ is still an FP-fuzzy soft ring over R.*

Proof. We can write $\Gamma_X \widetilde{\cup} \Gamma_Y = \Gamma_{X\widetilde{\cup}Y}$. For all $x \in E$, $\mu_{X\widetilde{\cup}Y}(x) = \max\{\mu_X(x), \mu_Y(x)\}$, then $\mu_{X\widetilde{\cup}Y}(x) = \mu_X(x)$ or $\mu_{X\widetilde{\cup}Y}(x) = \mu_Y(x)$ since $X \cap Y = \emptyset$. Therefore, $\gamma_{X\widetilde{\cup}Y}(x) = \gamma_X(x)$ or $\gamma_{X\widetilde{\cup}Y}(x) = \gamma_Y(x)$, so $\gamma_{X\widetilde{\cup}Y}(x)$ is a fuzzy subring of R, then $\Gamma_X \widetilde{\bigcup} \Gamma_Y$ is an FP-fuzzy soft ring over R. ■

Definition 3.6 Let Γ_X, Γ_Y be FP-fuzzy soft rings over R. Then Γ_X is said to be an FP-fuzzy soft subring of Γ_Y, if $\mu_X(x) \leq \mu_Y(x)$ and $\gamma_X(x)$ is a fuzzy subset of $\gamma_Y(x)$ for all $x \in E$.

Example 3.7 Let $R = Z_4 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}\}$ be a ring and $E = \{a, b\}$ be a set of parameters. If $X = \{0.2/a, 0/b\}$, $\gamma_X(a) = \{\overline{0}/0.5, \overline{1}/0.3, \overline{2}/0.4, \overline{3}/0.3\}$, $\gamma_X(b) = \emptyset$, and Y=$\{0.4/a, 0.3/b\}$, $\gamma_Y(a)$=$\{\overline{0}/0.6, \overline{1}/0.4, \overline{2}/0.5, \overline{3}/0.4\}$, $\gamma_Y(b) = \{\overline{0}/0.4, \overline{1}/0.2, \overline{2}/0.3, \overline{3}/0.2\}$, then Γ_X and Γ_Y are FP-fuzzy soft rings over R, and Γ_X is an FP-fuzzy soft subring of Γ_Y.

Theorem 3.8 *Let R be a ring, E be a set of parameters, Γ_X and Γ_Y are FP-fuzzy soft subrings of Γ_Z.*

(1) $\Gamma_X \widetilde{\bigcap} \Gamma_Y$ *is an FP-fuzzy soft subring of Γ_Z.*

(2) If $X \cap Y = \emptyset$, then $\Gamma_X \widetilde{\bigcup} \Gamma_Y$ is an FP-fuzzy soft subring of Γ_Z.

Proof. The proofs are similar to the proofs of Theorems 3.4 and 3.5. ■

Definition 3.9 Let Γ_X=$\{(\mu_X(x)/x, \gamma_X(x)) : x{\in}A, \gamma_X(x){\in}F(R), \mu_X(x){\in}[0,1]\}$ and $\Gamma_Y = \{(\mu_Y(y)/y, \gamma_Y(y) : y \in B, f_Y(y) \in F(K), \mu_Y(y) \in [0,1]\}$ be FP-fuzzy soft rings over rings R and K, respectively. If $f : R \to K$ and $g : A \to B$ are two functions, then (f, g) is called an FP-fuzzy soft homomorphism such that (f, g) is an FP-fuzzy soft homomorphism from Γ_X to Γ_Y. The latter is written by $\Gamma_X \sim \Gamma_Y$ if the following conditions are satisfied:

(1) f is an epimorphism from R to K,

(2) g is a surjective mapping, and

(3) $f(\gamma_X(x)) = \gamma_Y(g(x))$ and $\mu_X(x) = \mu_Y(g(x))$ for all $x \in A$.

In the above definition, if f is an isomorphism from R to K and g is a bijective mapping, then (f, g) is called an FP-fuzzy soft isomorphism so that (f, g) is an FP-fuzzy soft isomorphism from Γ_X to Γ_Y, denoted by $\Gamma_X \simeq \Gamma_Y$.

Example 3.10 Let R=$(Z, +, \times)$ and K=$(4Z, +, \times)$, A=$\{1, 3\}$ and B=$\{2, 6\}$. Define a homomorphism f from R onto K by $f(r) = 4r$ for $r \in R$, and a mapping g from A onto B by $g(x) = 2x$, for $x \in A$.

Let X be a fuzzy set over A defined by $\mu_X = \{1/0.5, 3/0.8\}$,

Let Y be a fuzzy set over B defined by $\mu_Y = \{2/0.5, 6/0.8\}$.

Let $\gamma_X : A \to F(R)$ defined by

$$(\gamma_X(1))(r) = \begin{cases} 0.1, & r = 2k+1, k \in Z, \\ 0.3, & r = 2k, k \in Z. \end{cases}$$

$$(\gamma_X(3))(r) = \begin{cases} 0.2, & r = 2k+1, k \in Z, \\ 0.4, & r = 2k, k \in Z. \end{cases}$$

Let $\gamma_Y : B \to F(K)$ defined by

$$(\gamma_Y(2))(r) = \begin{cases} 0.1, & r = 8k+4, k \in Z, \\ 0.3, & r = 8k, k \in Z. \end{cases}$$

$$(\gamma_Y(6))(r) = \begin{cases} 0.2, & r = 8k+4, k \in Z, \\ 0.4, & r = 8k, k \in Z. \end{cases}$$

It is clear that Γ_X and Γ_Y are FP-fuzzy soft rings over R and K, respectively. We can immediately see that f is an isomorphism from R to K and g is a bijective mapping, $\mu_X(x) = \mu_Y(g(x))$ and we can deduce that $f(\gamma_X(x)) = \gamma_Y(g(x))$ for all $x \in A$. Hence (f, g) is an FP-fuzzy soft isomorphism from Γ_X to Γ_Y.

The following lemma is similar to fuzzy subgroups in [12], and we omit the proof.

Lemma 3.11 *If $f : R \to K$ is an epimorphism of rings and μ a fuzzy subring(ideal) of R, then $f(\mu)$ is a fuzzy subring(ideal) of K.*

Theorem 3.12 *Let $\Gamma_X = \{(\mu_X(x)/x, \gamma_X(x)) : x\in A, \gamma_X(x)\in F(R), \mu_X(x)\in[0,1]\}$ be an FP-fuzzy soft ring over R and $\Gamma_Y=\{(\mu_Y(y)/y, \gamma_Y(y)) : y\in B, \gamma_Y(y) \in F(K)$, $\mu_Y(y) \in [0,1]\}$ be an FP-fuzzy soft set over ring K. If Γ_X is FP-fuzzy soft homomorphic to Γ_Y, then Γ_Y is an FP-fuzzy soft ring over K.*

Proof. Let (f, g) be an FP-fuzzy soft homomorphism from Γ_X to Γ_Y. Since Γ_X is an FP-fuzzy soft ring over R, $f(R) = K$ and $\gamma_X(x)$ is a fuzzy subring of R for all $x \in A$. Now, for all $y \in B$, there exists $x \in A$ such that $g(x) = y$. Hence, $\gamma_Y(y) = \gamma_Y(g(x)) = f(\gamma_X(x))$ is a fuzzy subring of the ring K and $\mu_Y(y) = \mu_Y(g(x)) = \mu_X(x)$, so Γ_Y must be an FP-fuzzy soft ring over K as well. ∎

4. FP-equivalent fuzzy soft rings

Definition 4.1 Let $\Gamma_X=\{(\mu_X(x)/x, \gamma_X(x)) : x\in E, \gamma_X(x)\in F(R), \mu_X(x) \in [0,1]\}$ be an FP-fuzzy soft ring over R. Then Γ_X is said to be FP-equivalent fuzzy soft ring over R if, for any $x, y \in E$, $\mu_X(x) = \mu_X(y)$, we have $\gamma_X(x) = \gamma_X(y)$.

Example 4.2 Let $R = Z_4$, $E = \{x_1, x_2, x_3, x_4\}$ be and X be a fuzzy set over E defined by $X = \{0.1/x_1, 0.5/x_2, 0.5/x_3, 0.3/x_4\}$, $\gamma_X(x_1) = \{\overline{0}/1, \overline{1}/0.4, \overline{2}/0.5, \overline{3}/0.4\}$, $\gamma_X(x_2) = \{\overline{0}/0.8, \overline{1}/0.5, \overline{2}/0.6, \overline{3}/0.5\}$, $\gamma_X(x_3) = \{\overline{0}/0.8, \overline{1}/0.5, \overline{2}/0.6, \overline{3}/0.5\}$, $\gamma_X(x_4) = \{\overline{0}/0.7, \overline{1}/0.3, \overline{2}/0.5, \overline{3}/0.3\}$, It is clearly that Γ_X is an FP-equivalent fuzzy soft ring over R.

Theorem 4.3 *Let R be a ring, E be a set of parameters. Then*

(1) $\Gamma_{\emptyset_X}$ *and* $\Gamma_\emptyset$ *are FP-equivalent fuzzy soft rings.*

(2) $(\Gamma_R)_X$ *and* $(\Gamma_R)_X^{\bar{c}}$ *are FP-equivalent fuzzy soft rings.*

(3) $\Gamma_\emptyset^{\bar{c}}$ *and* $\Gamma_E^{\bar{c}}$ *are FP-equivalent fuzzy soft rings.*

Proof. By Definitions 2.4 and 4.1, the proofs of (1) and (2) are straightforward. Since $\Gamma_\emptyset^{\bar{c}} = (\Gamma_R)_E$ and $\Gamma_E^{\bar{c}} = \Gamma_\emptyset$, then (3) is hold. ∎

Notation 4.4 If $\Gamma_X = \{\mu_X(x)/x, \gamma_X(x)) : x \in E, \gamma_X(x) \in F(R), \mu_X(x) \in [0,1]\}$ and $\Gamma_Y = \{(\mu_Y(x)/x, \gamma_Y(x)) : x \in E, \gamma_Y(x) \in F(R), \mu_Y(x) \in [0,1]\}$ are FP-equivalent fuzzy soft rings over ring R, $\Gamma_X \widetilde{\cap} \Gamma_Y$ is not always an FP-equivalent fuzzy soft ring over R.

Example 4.5 Let $R = Z_4$, $E = \{x_1, x_2, x_3, x_4\}$. Let Γ_X be an FP-fuzzy soft set over R defined by

$X = \{0.1/x_1, 0.5/x_2, 0.5/x_3, 0.3/x_4\}$,
$\gamma_X(x_1) = \{\overline{0}/1, \overline{1}/0.4, \overline{2}/0.5, \overline{3}/0.4\}$, $\gamma_X(x_2) = \{\overline{0}/0.8, \overline{1}/0.5, \overline{2}/0.6, \overline{3}/0.5\}$,
$\gamma_X(x_3) = \{\overline{0}/0.8, \overline{1}/0.5, \overline{2}/0.6, \overline{3}/0.5\}$, $\gamma_X(x_4) = \{\overline{0}/0.7, \overline{1}/0.3, \overline{2}/0.5, \overline{3}/0.3\}$.

And let Γ_Y be an FP-fuzzy soft set over R defined by

$Y = \{0.3/x_1, 0.3/x_2, 0.6/x_3, 0.1/x_4\}$,
$\gamma_Y(x_1) = \{\overline{0}/0.5, \overline{1}/0.3, \overline{2}/0.4, \overline{3}/0.3\}$, $\gamma_Y(x_2) = \{\overline{0}/0.5, \overline{1}/0.3, \overline{2}/0.4, \overline{3}/0.3\}$,
$\gamma_Y(x_3) = \{\overline{0}/0.6, \overline{1}/0.4, \overline{2}/0.5, \overline{3}/0.4\}$, $\gamma_Y(x_4) = \{\overline{0}/0.8, \overline{1}/0.2, \overline{2}/0.7, \overline{3}/0.2\}$,

It is clearly that Γ_X and Γ_Y are FP-equivalent fuzzy soft rings over R.

We can see that
$X \cap Y = \{0.1/x_1, 0.3/x_2, 0.5/x_3, 0.1/x_4\}$, $\mu_{(X\cap Y)}(x_1) = \mu_{(X\cap Y)}(x_4)$, but
$\gamma_{(X\cap Y)}(x_1) = \{\overline{0}/0.5, \overline{1}/0.3, \overline{2}/0.4, \overline{3}/0.3\}$,
$\gamma_{(X\cap Y)}(x_4) = \{\overline{0}/0.7, \overline{1}/0.2, \overline{2}/0.5, \overline{3}/0.2\}$.

Then $\Gamma_X \widetilde{\cap} \Gamma_Y$ is not an FP-equivalent fuzzy soft ring over R.

Notation 4.6 If $\Gamma_X = \{\mu_X(x)/x, \gamma_X(x)) : x \in E, \gamma_X(x) \in F(R), \mu_X(x) \in [0,1]\}$ and $\Gamma_Y = \{(\mu_Y(x)/x, \gamma_Y(x)) : x \in E, \gamma_Y(x) \in F(R), \mu_Y(x) \in [0,1]\}$ are FP-equivalent fuzzy soft rings over ring R with $X \cap Y = \emptyset$, $\Gamma_X \widetilde{\cup} \Gamma_Y$ is not always an FP-equivalent fuzzy soft ring over R.

Example 4.7 Let $R = Z_4$, $E = \{x_1, x_2, x_3, x_4\}$. Let Γ_X be an FP-fuzzy soft set over R defined by

$X = \{0/x_1, 0.5/x_2, 0.5/x_3, 0/x_4\}$,
$\gamma_X(x_1) = \emptyset$, $\gamma_X(x_2) = \{\overline{0}/0.6, \overline{1}/0.4, \overline{2}/0.5, \overline{3}/0.4\}$
$\gamma_Y(x_3) = \{\overline{0}/0.6, \overline{1}/0.4, \overline{2}/0.5, \overline{3}/0.4\}$, $\gamma_Y(x_4) = \emptyset$.

Let Γ_Y be an FP-fuzzy soft set over R defined by

$Y = \{0.5/x_1, 0/x_2, 0/x_3, 0.5/x_4\}$,
$\gamma_Y(x_1) = \{\overline{0}/0.4, \overline{1}/0.2, \overline{2}/0.3, \overline{3}/0.2\}$, $\gamma_Y(x_2) = \emptyset$,
$\gamma_Y(x_3) = \emptyset$, $\gamma_Y(x_4) = \{\overline{0}/0.4, \overline{1}/0.2, \overline{2}/0.3, \overline{3}/0.2\}$.

It is clear that Γ_X and Γ_Y are FP-equivalent fuzzy soft rings over R and $X \cap Y = \emptyset$.

We can see that
$\mu_{(X\cup Y)}(x_1) = \mu_{(X\cup Y)}(x_2)$, but
$\gamma_{(X\cup Y)}(x_1) = \{\overline{0}/0.4, \overline{1}/0.2, \overline{2}/0.3, \overline{3}/0.2\} \neq \gamma_{(X\cup Y)}(x_2)$
$= \{\overline{0}/0.6, \overline{1}/0.4, \overline{2}/0.5, \overline{3}/0.4\}$.
Then $\Gamma_X \widetilde{\cup} \Gamma_Y$ is not an FP-equivalent fuzzy soft ring over R.

Theorem 4.8 *Let* $\Gamma_X = \{(\mu_X(x)/x, \gamma_X(x)) : x \in A, \gamma_X(x) \in F(R), \mu_X(x) \in [0,1]\}$ *be an FP-equivalent fuzzy soft ring over R and* $\Gamma_Y = \{(\mu_Y(x)/x, \gamma_Y(x)) : x \in B, \gamma_Y(x) \in F(K), \mu_Y(x) \in [0,1]\}$ *be an FP-fuzzy soft set over ring K. If Γ_X is FP-fuzzy soft homomorphic to Γ_Y, then Γ_Y is an FP-equivalent fuzzy soft ring over K.*

Proof. Let (f, g) be an FP-fuzzy soft homomorphism from Γ_X to Γ_Y. Since Γ_X is an FP-equivalent fuzzy soft ring over R, $\gamma_X(x_1) = \gamma_X(x_2)$ for all $x_1, x_2 \in A$, $\mu_X(x_1) = \mu_X(x_2)$. Now, for all $y_1, y_2 \in B$ and $\mu_Y(y_1) = \mu_Y(y_2)$, then there exist $x_1, x_2 \in A$ such that $g(x_1) = y_1$, $g(x_2) = y_2$. Since $\mu_Y(y_1) = \mu_Y(g(x_1)) = \mu_X(x_1)$ and $\mu_Y(y_2) = \mu_Y(g(x_2)) = \mu_X(x_2)$, then $\mu_X(x_1) = \mu_X(x_2)$. Hence, $\gamma_Y(y_1) = \gamma_Y(g(x_1)) = f(\gamma_X(x_1)) = f(\gamma_X(x_2)) = \gamma_Y(g(x_2)) = \gamma_Y(y_2)$ and Γ_Y must be an FP-soft fuzzy ring over K as well. ∎

5. FP-increasing(decreasing) fuzzy soft rings

Definition 5.1 Let $\Gamma_X = \{(\mu_X(x)/x, \gamma_X(x)) : x \in E, \gamma_X(x) \in F(R), \mu_X(x) \in [0,1]\}$ be an FP-fuzzy soft ring over R. Then Γ_X is said to be FP-increasing fuzzy soft ring over R if, for any $x, y \in E$, $\mu_X(x) \leq \mu_X(y)$, we have $\gamma_X(x) \subseteq \gamma_X(y)$, and Γ_X is said to be FP-decreasing fuzzy soft ring over R if, for any $x, y \in E$, $\mu_X(x) \leq \mu_X(y)$, we have $\gamma_X(x) \supseteq \gamma_X(y)$.

Example 5.2 Let $R = Z_4$, $E = \{x_1, x_2, x_3, x_4\}$ and X be a fuzzy set over E defined by
$X = \{0.6/x_1, 0.5/x_2, 0.3/x_3, 0.2/x_4\}$,
$\gamma_X(x_1) = \{\overline{0}/1, \overline{1}/0.6, \overline{2}/0.7, \overline{3}/0.6\}$, $\gamma_X(x_2) = \{\overline{0}/0.8, \overline{1}/0.5, \overline{2}/0.6, \overline{3}/0.5\}$,
$\gamma_X(x_3) = \{\overline{0}/0.8, \overline{1}/0.5, \overline{2}/0.6, \overline{3}/0.5\}$, $\gamma_X(x_4) = \{\overline{0}/0.7, \overline{1}/0.3, \overline{2}/0.5, \overline{3}/0.3\}$,
It is clearly that F_X is an FP-increasing fuzzy soft ring over R.

Notation 5.3 If $\Gamma_X = \{\mu_X(x)/x, \gamma_X(x)) : x \in E, \gamma_X(x) \in F(R), \mu_X(x) \in [0,1]\}$ and $\Gamma_Y = \{(\mu_Y(x)/x, \gamma_Y(x)) : x \in E, \gamma_Y(x) \in F(R), \mu_Y(x) \in [0,1]\}$ are FP-increasing fuzzy soft rings over ring R, $\Gamma_X \widetilde{\cap} \Gamma_Y$ is not always an FP-increasing fuzzy soft ring over R.

Example 5.4 Let $R = Z_4$, $E = \{x_1, x_2\}$. Let X be a fuzzy set over E defined by
$X = \{0.5/x_1, 0.4/x_2\}$,
$\gamma_X(x_1) = \{\overline{0}/1, \overline{1}/0.6, \overline{2}/0.7, \overline{3}/0.6\}$, $\gamma_X(x_2) = \{\overline{0}/0.7, \overline{1}/0.2, \overline{2}/0.3, \overline{3}/0.2\}$.
Let Y be a fuzzy set over E defined by
$Y = \{0.1/x_1, 0.9/x_2\}$,

$\gamma_Y(x_1) = \{\overline{0}/0.6, \overline{1}/0.4, \overline{2}/0.5, \overline{3}/0.4\}$, $\gamma_Y(x_2) = \{\overline{0}/0.8, \overline{1}/0.5, \overline{2}/0.6, \overline{3}/0.5\}$.
It is clear that Γ_X and Γ_Y are FP-increasing fuzzy soft rings over R. We can see that

$X \cap Y = \{0.1/x_1, 0.4/x_2\}$, but
$\gamma_{(X\cap Y)}(x_1) = \{\overline{0}/0.6, \overline{1}/0.4, \overline{2}/0.5, \overline{3}/0.4\}$,
$\gamma_{(X\cap Y)}(x_2) = \{\overline{0}/0.7, \overline{1}/0.2, \overline{2}/0.3, \overline{3}/0.2\}$.

Then $\Gamma_X \widetilde{\cap} \Gamma_Y$ is not an FP-increasing fuzzy soft ring over R.

Notation 5.5 If $\Gamma_X = \{\mu_X(x)/x, \gamma_X(x)) : x \in E, \gamma_X(x) \in F(R), \mu_X(x) \in [0,1]\}$ and $\Gamma_Y = \{(\mu_Y(x)/x, \gamma_Y(x)) : x \in E, \gamma_Y(x) \in F(R), \mu_Y(x) \in [0,1]\}$ are FP-increasing fuzzy soft rings over ring R with $X \cap Y = \emptyset$, $\Gamma_X \widetilde{\cup} \Gamma_Y$ is not always an FP-increasing fuzzy soft ring over R.

Example 5.6 Let $R = Z_4$, $E = \{x_1, x_2\}$. Let X be a fuzzy set over E defined by

$X = \{0/x_1, 0.5/x_2\}$,
$\gamma_X(x_1) = \emptyset$, $\gamma_X(x_2) = \{\overline{0}/0.6, \overline{1}/0.4, \overline{2}/0.5, \overline{3}/0.4\}$.

Let Y be a fuzzy set over E defined by

$Y = \{0.9/x_1, 0/x_2\}$,
$\gamma_Y(x_1) = \{\overline{0}/0.5, \overline{1}/0.2, \overline{2}/0.3, \overline{3}/0.2\}$, $\gamma_Y(x_2) = \emptyset$.

It is clear that F_X and F_Y are FP-increasing fuzzy soft rings over R and $X \cap Y = \emptyset$. We can see that

$(X \cup Y) = \{0.9/x_1, 0.5/x_2\}$, but
$\gamma_{(X\cup Y)}(x_1) = \{\overline{0}/0.5, \overline{1}/0.2, \overline{2}/0.3, \overline{3}/0.2\}$,
$\gamma_{(X\cup Y)}(x_2) = \{\overline{0}/0.6, \overline{1}/0.4, \overline{2}/0.5, \overline{3}/0.4\}$.

Then $\Gamma_X \widetilde{\cup} \Gamma_Y$ is not an FP-increasing fuzzy soft ring over R.

Theorem 5.7 *Let* $\Gamma_X = \{(\mu_X(x)/x, \gamma_X(x)) : x \in A, \gamma_X(x) \in F(R), \mu_X(x) \in [0,1]\}$ *be an* FP*-increasing fuzzy soft ring over* R *and* $\Gamma_Y = \{(\mu_Y(x)/x, \gamma_Y(x)) : x \in B, \gamma_Y(x) \in F(K), \mu_Y(x) \in [0,1]\}$ *be an* FP*-fuzzy soft set over ring* K*. If* Γ_X *is* FP*-fuzzy soft homomorphic to* Γ_Y*, then* Γ_Y *is an* FP*-increasing fuzzy soft ring over* K*.*

Proof. Let (f, g) be an FP-fuzzy soft homomorphism from Γ_X to Γ_Y. Since Γ_X is an FP-increasing fuzzy soft ring over R, for all $x_1, x_2 \in A, \mu_X(x_1) \leq \mu_X(x_2)$, $\gamma_X(x_1) \subseteq \gamma_X(x_2)$. Now, for all $y_1, y_2 \in B$ and $\mu_Y(y_1) \leq \mu_Y(y_2)$, then there exist $x_1, x_2 \in A$ such that $g(x_1) = y_1$, $g(x_2) = y_2$. Since $\mu_Y(y_1) = \mu_Y(g(x_1)) = \mu_X(x_1)$ and $\mu_Y(y_2) = \mu_Y(g(x_2)) = \mu_X(x_2)$, then $\mu_X(x_1) \leq \mu_X(x_2)$. Hence, $\gamma_Y(y_1) = \gamma_Y(g(x_1)) = f(\gamma_X(x_1)) \subseteq f(\gamma_X(x_2)) = \gamma_Y(g(x_2)) = \gamma_Y(y_2)$ and Γ_Y must be an FP-increasing fuzzy soft ring over K as well. ∎

Corollary 5.8 *If* $\Gamma_X = \{(\mu_X(x)/x, \gamma_X(x)) : x \in E, \gamma_X(x) \in F(R), \mu_X(x) \in [0,1]\}$ *is both a* FP*-equivalent fuzzy soft ring and* FP*-increasing fuzzy soft ring over ring* R*, then* $\Gamma_{\bar{X}} = \{(\mu_{\bar{X}}(\bar{x})/\bar{x}, \gamma_{\bar{X}}(\bar{x})) : \bar{x} \in \bar{E}, \gamma_{\bar{X}}(\bar{x}) \in F(R), \mu_{\bar{X}}(\bar{x}) \in [0,1]\}$ *is an* FP*-increasing fuzzy soft ring over ring* R*.*

6. Aggregate fuzzy subrings

In [3], N. Çağman et al. defined an aggregate fuzzy set of an FP-fuzzy soft set. They also defined $FPFS$-aggregation operator that produced an aggregate fuzzy set from an FP-fuzzy soft set and its fuzzy parameter set.

Definition 6.1 [3] Let $\Gamma_X \in FPFS(U)$. Then $FPFS$-aggregation operator, denoted by $FPFS_{agg}$ is defined by

$$FPFS_{agg} : F(E) \times FPFS(U) \longrightarrow F(U),$$

$$FPFS_{agg}(X, \Gamma_X) = \Gamma_X^*$$

where

$$\Gamma_X^* = \{\mu_{\Gamma_X^*}(u)/u : u \in U\}$$

which is a fuzzy set over U. The value Γ_X^* is called aggregate fuzzy set of the Γ_X. Here the membership degree $\mu_{\Gamma_X^*}(u)$ of u is defined as follows

$$\mu_{\Gamma_X^*}(u) = \frac{1}{\mid E \mid} \sum_{x \in E} \mu_X(x)\mu_{\gamma_X(x)}(u)$$

where $\mid E \mid$ is the cardinality of E.

Theorem 6.2 *Let* $\Gamma_X=\{(\mu_X(x)/x, \gamma_X(x)) : x\in E, \gamma_X(x)\in F(R), \mu_X(x)\in[0,1]\}$ *be an* FP*-fuzzy soft ring over* R*. Then the aggregate fuzzy set* Γ_X^* *of* Γ_X *is a fuzzy subring of* R*.*

Proof. For any $x \in E$, $\gamma_X(x)$ is a fuzzy subring of R. Then for all $r, s \in R$, $\mu_{\gamma_X(x)}(r-s)\geq \min\{\mu_{\gamma_X(x)}(r), \mu_{\gamma_X(x)}(s)\}$ and $\mu_{\gamma_X(x)}(rs)\geq \min\{\mu_{\gamma_X(x)}(r), \mu_{\gamma_X(x)}(s)\}$. Then

$$\begin{aligned} \mu_{\Gamma_X^*}(r-s) &= \frac{1}{\mid E \mid} \sum_{x \in E} \mu_X(x)\mu_{\gamma_X(x)}(r-s) \\ &\geq \min\left\{\frac{1}{\mid E \mid} \sum_{x \in E} \mu_X(x)\mu_{\gamma_X(x)}(r), \frac{1}{\mid E \mid} \sum_{x \in E} \mu_X(x)\mu_{\gamma_X(x)}(s)\right\} \\ &= \min\{\mu_{\Gamma_X^*}(r), \mu_{\Gamma_X^*}(s)\}. \end{aligned}$$

In the same way, we can obtain $\mu_{\Gamma_X^*}(rs) \geq \min\{\mu_{\Gamma_X^*}(r), \mu_{\Gamma_X^*}(s)\}$. Which is to say that Γ_X^* is a fuzzy subring of R. ■

Notation 6.3 Above Γ_X^* is called an aggregate fuzzy subring of FP-fuzzy soft ring Γ_X.

Example 6.4 Let R be a full matrix ring, written by M_n, let A be an upper triangular matrix ring and B a symmetrical matrix ring. And let $E = \{a, b\}$, the parameters a, b stand for "upper triangular" and "symmetrical", respectively. And X be a fuzzy set over E defined by

$$\mu_X(x) = \begin{cases} 1, & x = a, \\ 0.8, & x = b. \end{cases}$$

Let γ_X be defined by

$$\mu_{\gamma_X(a)}(r) = \begin{cases} 0, & r \text{ is not an upper triangular matrix,} \\ 1, & r \text{ is an upper triangular matrix.} \end{cases}$$

$$\mu_{\gamma_X(b)}(r) = \begin{cases} 0, & r \text{ is not symmetrical,} \\ 1, & r \text{ is symmetrical.} \end{cases}$$

It is clear that Γ_X is an FP-fuzzy soft ring over M_n. The aggregate fuzzy set can be found as

$$\Gamma^*_X(m) = \begin{cases} 0.9, & \text{if } m \in A \cap B\ , \\ 0.5, & \text{if } m \in A - B, \\ 0.4, & \text{if } m \in B - A, \\ 0, & \text{otherwise.} \end{cases}$$

We can verify that Γ^*_X is a fuzzy ring of M_n.

Notation 6.5 Let R be a subring of M_n, if Γ_X is defined as in Example 6.4, then R is a diagonal matrix ring if and only if the aggregate fuzzy subring of Γ_X is $\Gamma^*_X = 0.9$.

Notation 6.6 Let R be a subring of M_n, if Γ_X is defined as in Example 6.4, then the Γ^*_X is called a fuzzy diagonal subring of R related to the FP-fuzzy soft ring Γ_X.

Remark 6.7 We can define another fuzzy diagonal subring of R related to another FP-fuzzy soft ring.

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A SECURE AND EFFICIENT AUTHENTICATION WITH KEY AGREEMENT SCHEME BASED ON ELLIPTIC CURVE CRYPTOSYSTEM

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Abstract. Recently, Li et al. [20] proposed an improved authentication with key agreement scheme on elliptic curve cryptosystem for global mobility networks to remedy the weaknesses of Rhee et al.'s scheme. Li et al.'s scheme not only achieves mutual authentication, but also provides the procedure for key agreement and updates of secrets for users and servers. However, we find that Li et al.'s scheme is still insecure and vulnerable to insider attack, impersonation attack and unverifiable password change. In order to eliminate these pitfalls, we propose a new authenticated with key agreement scheme based on elliptic curve cryptosystem. The analysis shows that the proposed scheme is more secure and more suitable for global mobility networks.

Keywords: authentication, elliptic curve cryptosystem, key agreement, impersonation attack

2000 Mathematics Subject Classification: 20C15.

1. Introduction

Mutual authentication between a remote user and a server is the most common approach to ensure that the legal user can access the resources provided by remote systems over unreliable networks. In 1981, Lamport [1] first proposed a password-based authentication scheme to solve the secure communication problem. Since then, some password authentication schemes have been extensively investigated in [2], [3], [4], [5]. However, these schemes have security problems such as password attack, the system overhead of keeping the password tables. To avoid the above problems, smart-card-based password authentication schemes [6], [9], [10], [11], [12], [13], [14], [15] have been proposed. In a smart-card-based password authentication scheme, users insert their smart card into a card reader and input a password for the card. Then, the smart card generates the user's login request, and sends the request to the server. After the user and the server mutual authenticate the identity with each other, they share the common session key for future communication. Although the smart-card-based password authentication scheme improves the system security and solves many security attacks. However, most of user authentication schemes are subject to stolen smart card attack, off-line password guessing attack, impersonation attack and so on. Moreover, the smart-card-based password authentication schemes need the cards and readers which are increasing the cost of deployment.

In order to reduce the deployment cost, the memory device-aided (e.g., USB sticks, mobile phones, PDAs) password authentication protocol has been proposed. In 2009, Rhee et al. [16] first analyzed the security of the existing schemes using smart cards when the tamper-resistant property is eliminated from smart card. Then, Rhee et al. [16] proposed an enhanced scheme based on Khan-Zhang's scheme [17]. In 2012, Chen et al. [18] proposed a password-based remote user authentication and key agreement scheme without using smart cards. They pointed out that their scheme not only could resist off-line dictionary attack, replay, forgery and impersonation attacks but also guaranteed mutual authentication. But, in 2013, Jiang et al. [19] found that Chen et al.'s [18] scheme was insecure against off-line dictionary attacks. To remedy the security flaw, they proposed an improved password authentication protocol without using smart cards. Recently, Li et al. [20] pointed out Rhee et al.'s [16] authentication scheme is not secure against user impersonation attack caused by mathematical homomorphism computed in the finite field based upon the discrete logarithm. And Li et al. [20] proposed a new password-based authentication with key agreement scheme for portable devices on an elliptic curve cryptosystem. However, we find that Li et al.'s scheme is also existing some flaws, such as insider attack, impersonation attack, unverifiable password change. In this paper, to overcome these security flaws, we propose a secure and efficient authentication with key agreement scheme based on elliptic curve cryptosystem.

The rest of this paper is organized as follows. Some preliminaries are given in Section 2. In Section 3, we give a brief review of Li et al.'s scheme. Section 4 describes the cryptanalysis of Li et al.'s scheme. Our scheme is proposed in Section 5, its security is proved in Section 6. Finally, we draw our conclusion in Section 7.

2. Preliminaries

In this section, we will introduce the basic concepts of ECC. In all elliptic curve cryptosystem, the elliptic curve equation is defined as the form of $E_p(a,b)$: $y^2 = x^3 + ax + b(mod\ p)$. Given an integer $s \in F_p^*$ and a point $P \in E_p(a,b)$, the point-multiplication sP over $E_p(a,b)$ can be defined as $s \cdot P = P + P + P + \cdots + P$

(s times). Generally, the security of ECC relies on the difficulties of the following problems.

Definition 1. Given two points P and Q over $E_p(a,b)$, the elliptic curve discrete logarithm problem (ECDLP) is to find an integer $s \in F_p^*$ such that $Q = s \cdot P$.

Definition 2. Given three points P, $s \cdot P$, and $t \cdot P$ over $E_p(a,b)$ for s, $t \in F_p^*$, the computational Diffie-Hellman problem(CDLP) is to find the point $(st)P$ over $E_p(a,b)$.

Definition 3. Given two points P and $Q = s \cdot P + t \cdot P$ over F_p^* for s, $t \in F_p^*$, the elliptic curve factorization problem(ECFP) is to find two points $s \cdot P$ and $t \cdot P$ over $E_p(a,b)$.

3. Review of Li et al.'s scheme

In this section, we briefly review Li et al.'s scheme [20]. The notations used in Li et al.'s scheme are defined in Table 1.

Table 1: Some important notations used in Li et al.'s scheme

p	a large prime number
$E_p(a,b)$	an elliptic curve in the prime finite field F_p
P	the generator of order n
$H(\cdot)$	a key derivation function
ID_i	the identity of the client U_i
pw_i	the password of the client U_i
x_S	server S's secret key
n_i	a large unique number generated by S
m	session identifier
$\|\|$	concatenation operation

3.1. Registration phase

1. A client U_i chooses his/her valid identifier ID_i with password pw_i, then sends ID_i and pw_i to S over a secure channel.

2. Upon receiving the registration request message ID_i and pw_i from U_i, S computes U_i's authentication information $Y_i = (Y_{i,1}, Y_{i,2}) = (ID_i \cdot r_i \cdot n_i \cdot x_S \cdot P + pw_i \cdot P, r_i \cdot P)$ where r_i is a random number only used once in this phase and n_i is a large unique number generated randomly by S for every user.

3. S sends $\{H(\cdot), p, E_p(a,b), P, Y_i\}$ to U_i over a secure(or public) channel and stores the list $ID_i - n_i$ in its database privately.

4. Upon receiving the authentication information, U_i stores it in his/her storage device and remembers his/her ID_i with pw_i.

3.2. Login phase

U_i can perform the following operations to login in to the authentication server:

1. U_i inputs his/her ID_i with pw_i into his/her device.

2. The device chooses temporary secret random numbers $a, b, c, d, k_1 \in F_p^*$. The random numbers mentioned in the scheme are only used once and will not be dropped until the scheme is terminated.

3. Computes $Y'_{i,1} = Y_{i,1} - pw_i \cdot P = ID_i \cdot r_i \cdot n_i \cdot x_S \cdot P$, $C_1 = a \cdot Y'_{i,1} = a \cdot ID_i \cdot r_i \cdot n_i \cdot x_S \cdot P$, $C_2 = a \cdot Y_{i,2} = a \cdot r_i \cdot P$, $C_3 = b \cdot Y_{i,2} = b \cdot r_i \cdot P$, $C_4 = c \cdot Y_{i,2} = c \cdot r_i \cdot P$, $C_5 = c \cdot Y'_{i,1} + k_1 \cdot P = c \cdot ID_i \cdot r_i \cdot n_i \cdot x_S \cdot P + k_1 \cdot P$, $C_6 = d \cdot Y_{i,2} = d \cdot r_i \cdot P$.

4. U_i sends to S the login request message M_1=$\{ID_i, Y_{i,2}, C_1, C_2, C_3, C_4, C_5, C_6\}$.

3.3. Authentication with key agreement phase

1. Upon receiving the login request message, S checks whether the ID_i is valid in the registration table at first and extracts n_i corresponding to ID_i in its database, then verifies if the equation $ID_i \cdot n_i \cdot x_S \cdot C_2 = C_1$ holds. If it holds, S accepts $U_i^{;}$ s login request; otherwise it rejects.

2. S computes $k_1 \cdot P = C_5 - ID_i \cdot n_i \cdot x_S \cdot C_4$. Then S can get the session key $sk = H(K_x)$, where K_x is the x-coordinate of the point $K = k_1 \cdot k_2 \cdot P$ on $E_p(a,b)$, $k_2 \in F_p^*$ is a random number generated by S.

3. S computes $C_7 = ID_i \cdot n_i \cdot x_S \cdot C_3 = ID_i \cdot n_i \cdot x_S \cdot b \cdot r_i \cdot P$, $C_8 = ID_i \cdot n_i \cdot x_S \cdot C_6 + k_2 \cdot P$, $C_9 = E_{sk}(ID_i \parallel m \parallel S)$, where m is a session identifier.

4. Finally, S sends to U_i the message $M_2 = \{C_7, C_8, C_9\}$ for mutual authentication and key confirmation.

3.4. Mutual authentication and key confirmation

Upon receiving the message M_2 from S, U_i performs the following steps:

1. U_i verifies whether the equation $b \cdot Y'_{i,1} = C_7$ holds. If so, U_i believes the response of the message is correct from the responding server; otherwise it rejects.

2. After the mutual authentication process, U_i computes $k_2 \cdot P = C_8 - d \cdot Y'_{i,1}$ and contains the session key $sk = H(K_x)$. Then, U_i can decrypt the message C_9 with sk and confirm the session key if S and ID_i are correct in C_9.

3. U_i computes $C_{10} = E_{sk}(ID_i \parallel m \parallel S)$ and sends $M_3 = \{C_{10}\}$ to S.

4. At the end of the scheme S should execute the final key confirmation by decrypting C_{10} with sk. If the information is correct in C_{10}, the scheme is finished successfully; otherwise it terminates in failure.

3.5. Secret update phase

1. Password update phase: the client U_i could change his/her password offline anytime and anywhere by computing $Y_i^* = (Y_{i,1}^*, Y_{i,2}) = (Y_{i,1} - pw_i \cdot P + pw_i^* \cdot P, Y_{i,2})$ and replacing Y_i by Y_i^* with a new password pw_i^*.

2. Secret number update phase: the server S could change its secret number x_S online by interacting with its client. This phase is executed after the authentication with key agreement procedures and a secure channel based on the session key sk. Thus S and the user U_i can communicate with each other securely using symmetric cryptography algorithm, i.e. all of the following information is encrypted by sk using the symmetric cryptography algorithm. U_i sends the update request. Then S computes the new $Y_{i,1}'^* = ID_i \cdot r_i^* \cdot n_i \cdot x_S^* \cdot P$, $Y_{i,2}'^* = r_i^* \cdot P$ and sends these new values to U_i. Finally, U_i computes $Y_{i,1}^* = Y_{i,1}'^* + pw_i \cdot P$ and replaces the original authentication information $Y_i = (Y_{i,1}, Y_{i,2})$ by $Y_i^* = (Y_{i,1}^*, Y_{i,2}^*)$.

4. Comments on Security Pitfalls of Li et al.'s scheme

In this section, the security of Li et al.'s scheme has been analyzed carefully and we have found some security pitfalls such as insider attack, impersonation attack and unverifiable password change. Now we are going to explore these security flaws.

4.1. Insider attack

The insider attack is defined that any manager of system purposely leaks the secret information, and then leads to serious security flaws of authentication scheme. In the registration phase of Li et al.'s scheme, U_i sends his/her password pw_i to the server S in plain text. Thus, the password of the user U_i will be revealed to the remote system. If the user offers the same password to access the other remote servers for the convenience, it is possible that the privileged insider of the remote server S can successfully impersonate U_i to login to the other remote servers by using pw_i.

4.2. Impersonation attack

In the secure analysis section of Li et al.'s scheme, he said that impersonation attack could not be effective in their scheme. However, we find that a malicious user U_A can be authenticated to remote system even if he or she does not have the valid password pw_i. Assume that the malicious user U_A has intercepted of the legal user U_i's previous login message $\{ID_i, Y_{i,2}, C_1, C_2, C_3, C_4, C_5, C_6\}$ from the public channel. An impersonation attack can be performed as given below:

1. The malicious user U_A computes $C_1' = a' \cdot C_1 = a' \cdot a \cdot ID_i \cdot r_i \cdot n_i \cdot x_S \cdot P$, $C_2' = a' \cdot C_2 = a' \cdot a \cdot r_i \cdot P$.
2. The malicious user sends the fabricated login message $M_1' = \{ID_i, Y_{i,2}, C_1', C_2', C_3, C_4, C_5, C_6\}$ to the S.
3. When the S receives the login request message $M_1' = \{ID_i, Y_{i,2}, C_1', C_2', C_3, C_4, C_5, C_6\}$, S checks whether the ID_i is valid in the registration table at first and extracts n_i corresponding to ID_i in its database, then verifies if the equation $ID_i \cdot n_i \cdot x_S \cdot C_2' = ID_i \cdot n_i \cdot x_S \cdot a \cdot a' \cdot r_i \cdot P$ is equal to C_1'. It is obvious that the equation holds. So, the server S accepts U_i' s login request. From the description above, we know that Li et al.'s scheme suffers from impersonation attack.

4.3. Unverifiable password change

In the secret update phase of Li et al.'s scheme, when U_i wants to change his/her password, he/she chooses a new password pw_i^* by himself/herself, and computes $Y_i^* = (Y_{i,1}^*, Y_{i,2}) = (Y_{i,1} - pw_i \cdot P + pw_i^* \cdot P, Y_{i,2})$, and there is no authentication procedure in password change phase. If the malicious user U_A obtains U_i' s storage device, U_A may arbitrarily key in new and obsolete passwords. Then the storage device will replace Y_i by Y_i^*. Thereupon, even if the original legal user U_i uses his/her own the storage device, he or she cannot access the remote server S anymore.

5. Our proposed scheme

According to our cryptanalysis, some of the cryptanalysis attacks cannot be prevented in Li et al.'s scheme. Therefore, we propose a more secure remote authentication scheme using elliptic curve cryptosystem to remove the security weaknesses existing in Li et al.'s scheme. The proposed scheme has five phases: system initialization phase, the registration phase, the login phase, the authentication with key agreement phase and secret update phase. The details of these phases are as follows.

5.1. System initialization phase

The system initialization phase consists of two steps in our proposed scheme:

1. Let $p > 3$ be a large prime number, and $E_p(a, b)$ be an elliptic curve in the prime finite field F_p. P is a generator of order n and n must be large enough so that the ECDLP is difficult in the cyclic subgroup $< P >$.

2. The server S chooses three one-way secure hash functions $H_1 : \{0,1\}^* \to G_p$, $H_2 : \{0,1\}^* \times G_p \to \{0,1\}^k$, $H_3 : G_p \times G_p \to \{0,1\}^k$, $H_4 : \{0,1\}^* \times \{0,1\}^* \times G_p \times G_p \to \{0,1\}^k$ and the server S selects a random number x_S (which is the master secret of the server S) from $[1, n-1]$.

3. The server S publishes $\{p, E_p(a, b), P,\ H_1(\cdot), H_2(\cdot), H_3(\cdot), H_4(\cdot)\}$ as system parameters and keep the master key x_S secret.

4. All the operation are in F_p, and it omits mod p for the sake of simplicity.

5.2. Registration phase

1. A client U_i chooses his/her ID_i, pw_i and a random number b, then U_i submits ID_i, $H_1(pw_i \parallel b) \cdot P$ to S over a secure channel.

2. Upon receiving the registration request message $\{ID_i, H_1(pw_i \parallel b) \cdot P\}$ from U_i, S computes $X_i = H_2(ID_i \parallel H_1(pw_i \parallel b) \cdot P)$, $Y_i = (Y_{i,1}, Y_{i,2}) = (ID_i \cdot r_i \cdot n_i \cdot x_S \cdot P + H_1(pw_i \parallel b) \cdot P, r_i \cdot P)$, where r_i is a random number only used once in this phase.

3. S sends (X_i, Y_i) to U_i over a secure(public) channel and stores the list $ID_i - n_i$ in its database privately.

4. Upon receiving the authentication information, U_i stores it in his/her storage device and enters b into his/her storage device.

5.3. Login phase

When the client U_i wants to login the authentication server, the user U_i perform the following steps to generate a valid login request message.

1. U_i inputs his/her ID_i, pw_i into his/her device.

2. The device computes $X'_i = H_2(ID_i \parallel H_1(pw_i \parallel b) \cdot P)$ and checks whether $X'_i = X_i$. If it is not equal, the session is terminated. Otherwise, the user's identity ID_i and password pw_i are verified, and the device performs the next steps.

3. The device chooses temporary secret random numbers $a, b, c, d, k_1 \in F_p^*$ and computes $Y'_{i,1} = Y_{i,1} - H_1(pw_i \| b) \cdot P$, $C_1 = H_3(a \cdot Y'_{i,1})$, $C_2 = a \cdot Y_{i,2}$, $C_3 = b \cdot Y_{i,2}$, $C_4 = c \cdot Y_{i,2}$, $C_5 = c \cdot Y'_{i,1} + k_1 \cdot P$, $C_6 = d \cdot Y_{i,2}$.

4. U_i sends to S the login request message $M_1 = \{ID_i, Y_{i,2}, C_1, C_2, C_3, C_4, C_5, C_6\}$.

5.4. Authentication with key agreement phase

1. Upon receiving the login request message, S checks whether the ID_i is valid in the registration table at first and extracts n_i corresponding to ID_i in its database, then verifies if the equation $H_3(ID_i \cdot n_i \cdot x_S \cdot C_2) = C_1$ holds. If it holds, S accepts U_i' s login request; otherwise it rejects.

2. S computes $k_1 \cdot P = C_5 - ID_i \cdot n_i \cdot x_S \cdot C_4$, $C_7 = ID_i \cdot n_i \cdot x_S \cdot C_3 = ID_i \cdot n_i \cdot x_S \cdot b \cdot r_i \cdot P$, $C_8 = ID_i \cdot n_i \cdot x_S \cdot C_6 + k_2 \cdot P$, $C_9 = H_4(ID_i \parallel m \parallel k_1 \cdot P \parallel k_2 \cdot P)$, where m is a session identifier, $k_2 \in F_p^*$ is a random number generated by S.

3. Finally, S sends to U_i the message $M_2 = \{C_7, C_8, C_9\}$ for mutual authentication and key confirmation.

4. Upon receiving the message M_2 from S, U_i performs the following steps: U_i verifies whether the equation $b \cdot Y'_{i,1} = C_7$ holds. If so, U_i computes $k_2 \cdot P = C_8 - d \cdot Y'_{i,1}$, $H_4(ID_i \parallel m \parallel k_1 \cdot P \parallel k_2 \cdot P)$ and verifies whether $H_4(ID_i \parallel m \parallel k_1 \cdot P \parallel k_2 \cdot P) = C_9$. If it is equal, the server S is authenticated by the user U_i. At the end of the scheme, the user U_i and server S can share a session key $sk = k_1 \cdot k_2 \cdot P$ for future confidentiality communication.

5.5. Secret update phase

1. Password update phase: the client U_i inputs his/her ID_i, pw_i into his/her storage device, and request to change his/her password. The device computes $X'_i = H_2(ID_i \parallel H_1(pw_i \parallel b) \cdot P)$ and checks whether $X'_i = X_i$. If it is not equal, the password change request is rejected. Otherwise, the user's identity ID_i and password pw_i are verified, and the user inputs a new password pw_i^*. The device computes $X_i^* = H_2(ID_i \parallel H_1(pw_i^* \parallel b) \cdot P)$, $Y_i^* = (Y_{i,1}^*, Y_{i,2}) = (Y_{i,1} - H_1(pw_i \parallel b) \cdot P + H_1(pw_i^* \parallel b) \cdot P, Y_{i,2})$ and replaces X_i, Y_i by X_i^*, Y_i^* .

2. Secret number update phase: the server S could change its secret number x_S online by interacting with its client. This phase is executed after the authentication with key agreement procedures and a secure channel based on the session key sk. Thus S and the user U_i can communicate with each other securely using symmetric cryptography algorithm, i.e. all of the following information is encrypted by sk using the symmetric cryptography algorithm. U_i sends the update request. Then S computes the new $Y'^*_{i,1} = ID_i \cdot r_i^* \cdot n_i \cdot x_S^* \cdot P$, $Y'^*_{i,2} = r_i^* \cdot P$ and sends these new values to U_i. Finally, U_i computes $Y^*_{i,1} = Y'^*_{i,1} + H_1(pw_i^* \parallel b) \cdot P$ and replaces the original authentication information $Y_i = (Y_{i,1}, Y_{i,2})$ by $Y_i^* = (Y^*_{i,1}, Y^*_{i,2})$.

6. Security analysis and discussion

In this section, we discuss the security properties of our proposed scheme, and make comparisons with some related schemes in functionality and computation cost.

6.1. Insider attack

In the proposed scheme, the server S cannot obtain the user U_i's password pw_i. Since in the registration phase, the user U_i chooses his/her ID_i, pw_i and a random number b, then U_i submits ID_i, $H_1(pw_i \parallel b) \cdot P$ to the server S. It is computationally impossible that to derive the password pw_i from $H_1(pw_i \parallel b) \cdot P$, because of the difficulties of elliptic curve discrete logarithm problem(ECDLP) and the hardness of inverting hash function $H_1(\cdot)$. Therefore, the proposed scheme is secure against insider attack.

6.2. Quickly detect the authorized login

In the login phase of our proposed scheme, when the user inputs identity ID_i and password pw_i, the validity of identity ID_i and password pw_i can be verified by checks whether $X_i' = X_i$. If it is not equal, it means that the user inputs a wrong identity and password, then the storage device terminates the session. On the contrary, if it holds, the device performs the next steps. Thus, our proposed scheme can be quickly detect the wrong password by the device at the beginning of the login phase.

6.3. Impersonation attack

In our proposed scheme, if an adversary U_A wants to impersonation as the legal user U_i to pass the authentication of the server S, he/she must get $Y_{i,1}' = ID_i \cdot r_i \cdot n_i \cdot x_S \cdot P$ to compute the valid authentication message C_1 and C_2. However, an adversary U_A cannot derive $Y_{i,1}'$ without knowing the valid password pw_i of the user U_i. On the other hand, an adversary U_A cannot get $Y_{i,1}'$ from $C_1 = H_3(a \cdot Y_{i,1}')$, since it is protected by ECDLP and hash functions. Therefore, the proposed scheme is secure against impersonation attack.

6.4. Off-line password guessing attack

In the proposed scheme, there is no way for an adversary U_A to guess the user U_i's password based on $X_i = H_2(ID_i \parallel H_1(pw_i \parallel b) \cdot P)$ and $Y_i = (Y_{i,1}, Y_{i,2}) = (ID_i \cdot r_i \cdot n_i \cdot x_s \cdot P + H_1(pw_i \parallel b) \cdot P, r_i \cdot P)$ which are from the storage device. Due to hardness of ECDLP, the adversary U_A cannot obtain U_i's password pw_i from the value X_i . Besides, the adversary U_A cannot launch off-line dictionary attack without the secret random number, the server S's secret key.

6.5. Replay attack

In the proposed scheme, the random numbers a, b, c, d, k_1, k_2 are different in each new session, which make all messages dynamic and valid for that session only. Thus, our proposed scheme is secure against replay attack.

6.6. Server spoofing attack

If an adversary U_A wants to masquerade as the server S to cheat the user U_i. He/She needs to generate the valid response message $M_2 = \{C_7, C_8, C_9\}$. However, he/she cannot correctly compute C_7,C_8,and C_9 without the server's secret key x_S. Therefore, our scheme is secure against server spoofing attack.

6.7. Performance analysis

We analyze the functionary of the proposed scheme and make comparisons with other related schemes. Table 2 shows that our scheme is more secure and robust than other related schemes and achieves more functionality features. Table 3 summarizes the computation cost between our scheme and some related schemes. The following notations are used in Table 3. Besides, Table 3 demonstrates that our scheme does not need symmetric encryption/decryption operations, only needing point multiplication, point addition on ECC and hashing function operations. Hence, our proposed scheme is more secure and efficient than other authentication schemes.

Table 2: Functionality comparisons

	our scheme	Rhee's	Yang's	Li's
Achieves mutual authentication	Yes	No	Yes	Yes
Resist insider attack	Yes	No	Yes	No
Resist replay attack	Yes	No	No	Yes
Resist impersonation attack	Yes	No	No	No
Resist off-line dictionary attack	Yes	No	N/A	Yes
Resist the device stolen attack	Yes	No	N/A	Yes
Resist server spoofing attack	Yes	No	Yes	Yes
Quickly detect the unauthorized login	Yes	No	N/A	No

Table 3: Comparisons of computation cost

	T_{Exp}	T_{ECMul}	T_{ECAdd}	T_h	T_{Sym}	T_{Mul}	Total
our scheme	0	34	9	14	0	0	$31T_{ECMul}$ $+9T_{ECAdd}+11T_h$
Rhee's	7	0	0	6	0	2	$7T_{Exp}+6T_h$ $+2T_{Mul}$
Yang's	0	9	5	9	0	0	$9T_{ECMul}$ $+5T_{ECAdd}+9T_h$
Li's	0	34	9	2	4	0	$34T_{ECMul}+9T_{ECAdd}$ $+2T_h+4T_{Sym}$

T_h	:	the time complexity of hashing operations;
T_{Exp}	:	the time complexity of modular exponentiation in the finite field;
T_{ECAdd}	:	the time complexity of point multiplication on ECC;
T_{ECMul}	:	the time complexity of point addition on ECC;
T_{Sym}	:	the time complexity of symmetric encryption/decryption;
T_{Mul}	:	the time complexity of inverting operation in finite field.

7. Conclusions

We have identified security flaws in the authentication with key agreement scheme on elliptic curve cryptosystem of Li et al.'s scheme. To compensate for these shortcomings, we propose a novel authentication with key agreement scheme. According to our analysis and discussion, the proposed scheme can withstand various attacks and has a lower computation cost.

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DIAMETERS OF SEMI-IDEAL BASED ZERO-DIVISOR GRAPHS FOR FINITE DIRECT PRODUCT OF POSETS

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Abstract. In this paper, we characterize the diameter of zero-divisor graph for direct product $P_1 \times P_2 \times ... \times P_n$ with respect to direct product $I_1 \times I_2 \times ... \times I_n$, where $I_1, I_2, ..., I_n$ are semi-ideals of posets $P_1, P_2, ..., P_n$, respectively.

Keywords: posets, direct product, semi-ideals, prime semi-ideals and diameter.

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1. Preliminaries

Throughout this paper, $(P, \leq)$ denotes a poset with zero element 0 and the graph $G_I(P)$ denotes the semi-ideal based zero-divisor graph of a poset P with respect to a semi-ideal I of P. For $M \subseteq P$, let $(M)^l := \{x \in P : x \leq m \text{ for all } m \in M\}$ denotes the lower cone of M in P. For $A, B \subseteq P$, we write $(A, B)^l$ instead of $(A \cup B)^l$. If $M = \{x_1, ..., x_n\}$ is finite, then we use the notation $(x_1, ..., x_n)^l$ instead of $(\{x_1, ..., x_n\})^l$. By a semi-ideal we mean a non-empty subset I of P such that if $b \in I$ and $a \leq b$, then $a \in I$. A proper semi-ideal I of P is called prime if for any $a, b \in P$, $(a, b)^l \subseteq I$ implies $a \in I$ or $b \in I$.

In [2], I. Beck introduced the concept of a zero-divisor graph of a commutative ring with identity, but this work was mostly concerned with coloring of rings. Later D.F. Anderson and P.S. Livingston in [1] studied the subgraph $\Gamma(R)$ of $G(R)$ whose vertices are the nonzero zero-divisors of R and two distinct vertices x and y are joined by an edge if $xy = 0$. In [11], S.P. Redmond has generalized

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the notion of the zero-divisor graph. For a given ideal I of a commutative ring R, he defined an undirected graph $\Gamma_I(R)$ with vertices $\{x \in R\backslash I : xy \in I$ for some $y \in R\backslash I\}$, where distinct vertices x and y are adjacent if and only if $xy \in I$. The zero-divisor graph of various algebraic structures have been studied by several authors ([4], [5], [6] and [7]).

In [9], Radomr Halas and Marek Jukl have introduced the concept of a graph structure of a posets, let $(P, \leq)$ be a poset with 0. Then the zero-divisor graph of P, denoted by $\Gamma(P)$, is an undirected graph whose vertices are just the elements of P with two distinct vertices x and y are joined by an edge if and only if $L(x, y) = \{0\}$, and proved some interesting results related with clique and chromatic number of this graph structure. In [8], we have studied the semi-ideal-based zero-divisor graph of a poset P. Let P be a poset and I a semi-ideal of P. Then the graph of P with respect to the semi-ideal I, denoted by $G_I(P)$, is the graph whose vertices are the set $\{x \in P\backslash I : (x, y)^l \subseteq I$ for some $y \in P\backslash I\}$ with distinct vertices x and y are adjacent if and only if $(x, y)^l \subseteq I$. If $I = \{0\}$, then $G_I(P) = G(P)$, and I is a prime semi-ideal of P if and only if $G_I(P) = \phi$. And investigated the interplay between the poset properties of P and the graph-theoretic properties of $G_I(P)$.

The direct product of posets P and Q is the poset $P \times Q = \{(x, y) : x \in P, y \in Q\}$ such that $(x, y) \leq (x', y')$ in $P \times Q$ if $x \leq x'$ in P and $y \leq y'$ in Q.

Throughout this paper, let us denote $I_1,\ I_2,\ ....,\ I_n$ be semi-ideals of posets $P_1,\ P_2,\ ...,\ P_n$, respectively and $P = P_1 \times P_2 \times ... \times P_n$, and $I = I_1 \times I_2 \times ... \times I_n$. For $j = 1, 2, ..., m$, if $|P_j| = n_j$, then we can observe that $|V(G_I(P))| \leq n_1 n_2 ... n_m - |I|$, if P_j for each j has a greatest element e_j, then $|V(G_I(P))| < n_1 n_2 ... n_m - |I|$. In this paper, we investigate the relationship between the diameter of $G_I(P)$ and properties of P_i with respect to $V(G_{I_i}(P_i))$ for $i = 1, 2, ..., n$. The notations of graph theory are from [3], the notations of posets from [10].

2. Main results

Lemma 2.1 *Let $I_1, I_2, ..., I_n$ be semi-ideals of posets $P_1, P_2, ..., P_n$, respectively, Then I is a semi-ideal of P.*

Proof. It is trivial. ∎

The following example shows that I is not necessarily to be a prime semi-ideal of P even if $I_1, I_2, ..., I_n$ are prime semi-ideals of posets $P_1, P_2, ..., P_n$, respectively.

Example 2.2 Let $P_1 = \{1, 2, 4\}$ and $P_2 = \{1, 3, 9\}$ be posets with respect to division. Then $I_1 = \{1\}$ and $I_2 = \{1\}$ are prime semi-ideals of P_1 and P_2, respectively. Here $I = I_1 \times I_2$ is a semi-ideal of $P = P_1 \times P_2$, but not prime semi-ideal.

Theorem 2.3 *Let $I_1, I_2, ..., I_n$ be prime semi-ideals of posets $P_1, P_2, ..., P_n$, respectively. Then $V(G_I(P)) \cup I = \bigcup_{j=1}^{n} (P_1 \times P_2 \times ... \times I_j \times ... \times P_n)$ is a prime semi-ideal of P.*

Proof. Let $y = (y_1, ..., y_n)$, $y_i \notin I_i$ for all i. Suppose that $(y, x)^l \subseteq I$ for some $x \in V(G_I(P))$. Then $(y_i, x_i)^l \subseteq I_i$ for all i. Since $I_i's$ are prime semi-ideals of posets $P_i's$, we have $x_i \in I_i$ for all i. So $x \notin V(G_I(P))$, a contradiction. So, $V(G_I(P)) \cup I = \bigcup_{j=1}^{n}(P_1 \times P_2 \times ... \times I_j \times ... \times P_n)$.

Now, we claim that $V(G_I(P)) \cup I$ is a prime semi-ideal of P. Let $a = (a_1, a_2, ..., a_n)$, $b = (b_1, b_2, ..., b_n) \in P$. Suppose that $(a, b)^l \subseteq V(G_I(P)) \cup I$ and $a, b \notin V(G_I(P)) \cup I$. Then $a_i \notin I_i$ and $b_i \notin I_i$ for all i, which implies $(a_i, b_i)^l \nsubseteq I_i$ for all i. So there exists $t_i \in (a_i, b_i)^l$ such that $t_i \notin I_i$. Set $t = (t_1, t_2, ..., t_n)$. Then $t \in (a, b)^l \subseteq V(G_I(P)) \cup I$, a contradiction to $t_i \notin I_i$ for all i. ∎

Theorem 2.4 *Let $I_1, I_2, ..., I_n$ be prime semi-ideals of posets $P_1, P_2, ..., P_n$, respectively. Then $G_I(P)$ is a n-partite graph. Moreover, if $V_1, V_2, ..., V_n$ are partitions of $V(G_I(P))$, then there exists an induced subgraph $K_{|X_1|,|X_2|,...,|X_n|}$, where $\phi \neq X_i \subseteq V_i$. Also there exist $|X_1||X_2|...|X_n|$ number of induced subgraphs $K_n's$ in $K_{|X_1|,|X_2|,...,|X_n|}$.*

Proof. Let

$$\begin{aligned}
V_1 &= \bigcup_{2\leq k\leq n} ((P_1\backslash I_1) \times P_2 \times ... \times I_k ... \times P_n),\\
V_2 &= \bigcup_{3\leq k\leq n} (I_1 \times (P_2\backslash I_2) \times ... \times I_k ... \times P_n),\\
V_3 &= \bigcup_{4\leq k\leq n} (I_1 \times I_2 \times (P_3\backslash I_3) \times ... \times I_k ... \times P_n),\ ...,\\
V_n &= I_1 \times I_2 \times ... \times I_{n-1} \times (P_n\backslash I_n).
\end{aligned}$$

Then $V_1, V_2, ..., V_n$ are nonempty disjoint n-subsets of $V(G_I(P))$.

Let $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n) \in V_j$ for some j. Then we have $x_j, y_j \notin I_j$ which implies $(x_j, y_j) \nsubseteq I_j$, so $(x, y)^l \nsubseteq I$. Thus no two vertices of V_j are adjacent and hence $G_I(P)$ is an n-partite graph.

For moreover case, take

$$\begin{aligned}
X_1 &= (P_1\backslash I_1) \times I_2 \times ... \times I_n,\\
X_2 &= I_1 \times (P_2\backslash I_2) \times ... \times I_n,\\
X_3 &= I_1 \times I_2 \times (P_3\backslash I_3) \times ... \times I_n, ...,\\
X_n &= I_1 \times I_2 \times ... \times (P_n\backslash I_n).
\end{aligned}$$

Then $X_i's$ are subset of $V_i's$ and forms $K_{|X_1|,|X_2|,...,|X_n|}$ and $\{x_1, x_2, ..., x_n\}$ forms K_n for $x_i \in X_i$. ∎

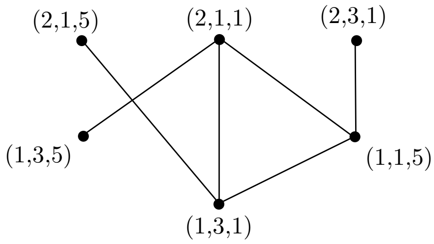
Corollary 2.5 *Let $I_1, I_2, ..., I_n$ be prime semi-ideals of posets $P_1, P_2, ..., P_n$, respectively. Then the clique of $G_I(P)$ is n.*

Theorem 2.6 *Let I_1 and I_2 be prime semi-ideals of posets P_1 and P_2, respectively. Then $G_I(P)$ is complete bipartite graph.*

Proof. By Theorem 2.3 and Theorem 2.4, $G_I(P)$ is bipartite graph with vertex set $V_1 = (P_1 \backslash I_1) \times I_2$ and $V_2 = I_1 \times (P_2 \backslash I_2)$ which forms $K_{|V_1|,|V_2|}$. ∎

The following example shows that $G_I(P)$ need not be a complete n-partite graph if $n > 2$.

Example 2.7 Let $P_1 = \{1, 2\}$, $P_2 = \{1, 3\}$ and $P_3 = \{1, 5\}$ be posets with respect to division and $I_1 = \{1\}$, $I_2 = \{1\}$ and $I_3 = \{1\}$ be prime semi-ideals of posets P_1, P_2 and P_3, respectively. Then $G_I(P)$ with respect to $I = (1, 1, 1)$ is:



Here $G_I(P)$ is 3-partite graph, but not complete 3-partite graph. ∎

Lemma 2.8 *Let I_1 be semi-ideal of poset P_1 with $diam(G_{I_1}(P_1)) = 1$. Then $(x, y)^l \subseteq I_1$ for all x, $y \in V(G_{I_1}(P_1))$, also if $P_1 = V(G_{I_1}(P_1)) \cup I_1$, then $(x, y)^l \subseteq I_1$ for all x, $y \in P_1$.*

Proof. It is trivial. ∎

Theorem 2.9 *Let $I_1, I_2, ..., I_n$ be semi-ideals of posets $P_1, P_2, ..., P_n$, respectively. Then $G_I(P)$ is connected and $diam(G_I(P)) \leq 3$.*

Proof. Let $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n) \in V(G_I(P))$. Then there exist $a = (a_1, a_2, ..., a_n)$, $b = (b_1, b_2, ..., b_n) \in V(G_I(P))$ such that $(x, a)^l \subseteq I$ and $(y, b)^l \subseteq I$. If $(x, y)^l \subseteq I$, then $diam(G_I(P)) = 1$. Suppose that $(x, y)^l \nsubseteq I$. If $(a, b)^l \subseteq I$, then we have a path $x - a - b - y$ of length 3. Suppose that $(a, b)^l \nsubseteq I$. Then $(a_j, b_j)^l \nsubseteq I_j$ for some j, so we can find $t_j \in (a_j, b_j)^l$ with $t_j \notin I_j$ for some $a_j, b_j \in P_j \backslash I_j$. Now for $t = (i_1, i_2 ..., i_{j-1}, t_j, i_{j+1}, ..., i_n) \notin I$, we have $(x, t)^l \subseteq I$ and $(y, t)^l \subseteq I$, which imply $x - t - y$ is a path of length 2. Hence $G_I(P)$ is connected and $diam(G_I(P)) \leq 3$. ∎

Lemma 2.10 *Let $I_1, I_2, ..., I_n$ be semi-ideals of posets $P_1, P_2, ..., P_n$, respectively. If $diam(G_{I_1}(P_1)) = diam(G_{I_2}(P_2)) = ... = diam(G_{I_n}(P_n)) = 1$, then the following hold:*

(i) $diam(G_I(P)) \neq 1$

(ii) $diam(G_I(P)) = 2$ *if and only if* $P_j = V(G_{I_j}(P_j)) \cup I_j$ *for all* $j \in \{1, 2, ..., n\}$.

(iii) $diam(G_I(P)) = 3$ *if and only if* $P_j \neq V(G_{I_j}(P_j)) \cup I_j$ *for some* $j \in \{1, 2, ..., n\}$.

Proof. (i) Let $a = (a_1, a_2, ..., a_n)$ and $b = (a_1, i_2, ..., i_n) \in V(G_I(P))$ with $a \neq b$, where $a_1 \in V(G_{I_1}(P_1))$. Then $(a, b)^l \nsubseteq I$ and hence $diam(G_I(P)) \neq 1$.

(ii) Assume that $P_j = V(G_{I_j}(P_j)) \cup I_j$ for all $j \in \{1, 2, ..., n\}$. If there exist distinct vertices $c = (c_1, c_2, ..., c_n)$, $d = (d_1, d_2, ..., d_n)$ in $V(G_I(P))$ with $a - c - d - b$ is a path of length 3, then $(a_j, d_j)^l \nsubseteq I_j$ for some $a_j, d_j \in P_j \backslash I_j$, a contradiction to $diam(G_{I_j}(P_j)) = 1$. Thus we have a path $a - t - b$ of length 2 for all $t \in V(G_I(P))$. So $diam(G_I(P)) = 2$. Conversely, assume that $diam(G_I(P)) = 2$. Suppose that $P_j \neq V(G_{I_j}(P_j)) \cup I_j$ for some $j \in \{1, 2, ..., n\}$. Then there exists $x_j \in P_j \backslash (V(G_{I_j}(P_j)) \cup I_j)$ for some $j \in \{1, 2, ..., n\}$. Since for each $z_k \in V(G_{I_k}(P_k))$, there exists $z'_k \in V(G_{I_k}(P_k))$ such that $(z_k, z'_k)^l \subseteq I_k$ for all k. So, if $a = (z_1, x_2, ..., x_n)$ and $b = (x_1, z_2, x_3, ..., x_n)$, then $(a, (z'_1, i_2, ..., i_n))^l \subseteq I$ and $(b, (i_1, z'_2, ..., i_n))^l \subseteq I$ which imply a, $b \in V(G_I(P))$. Since $(a, b)^l \nsubseteq I$ and by assumption, there exists $c = (c_1, c_2, ..., c_n) \in V(G_I(P))$ such that $(a, c)^l \subseteq I$ and $(b, c)^l \subseteq I$ which imply $c_j \in I_j$, a contradiction. Thus $P_j = V(G_{I_j}(P_j)) \cup I_j$ for all $j \in \{1, 2, ..., n\}$.

(iii) This follows from (i) and (ii). ∎

Theorem 2.11 *Let $I_1, I_2, ..., I_n$ be semi-ideals of posets $P_1, P_2, ..., P_n$, respectively. If $diam(G_{I_1}(P_1)) = diam(G_{I_2}(P_2)) = ... = diam(G_{I_n}(P_n)) = 2$, then the following hold:*

(i) *$diam(G_I(P)) \neq 1$.*

(ii) *$diam(G_I(P)) = 2$ if and only if $P_j = V(G_{I_j}(P_j)) \cup I_j$, for all $j \in \{1, 2, ..., n\}$.*

(iii) *$diam(G_I(P)) = 3$ if and only if $P_j \neq V(G_{I_j}(P_j)) \cup I_j$, for some $j \in \{1, 2, ..., n\}$.*

Proof. (i) It is clear.

(ii) Let $P_j = V(G_{I_j}(P_j)) \cup I_j$ for all $j \in \{1, 2, ..., n\}$. By (i), there are elements $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n) \in V(G_I(P))$ such that $x \neq y$ and $(x, y)^l \nsubseteq I$ which imply $(x_j, y_j)^l \nsubseteq I_j$ for some j. Since x_j, $y_j \in P_j$ and by assumption, we have $(x_j, z_j)^l \subseteq I_j$ and $(y_j, z_j)^l \subseteq I_j$ for some $z_j \in V(G_{I_j}(P_j))$. Now let $z = (i_1, ...i_{j-1}, z_j, i_{j+1}, ..., i_n)$. Then $z \notin I$ with $(x, z)^l \subseteq I$ and $(y, z)^l \subseteq I$ which imply $x - z - y$ is a path of length 2. So $diam(G_I(P)) = 2$.

Conversely, assume that $diam(G_I(P)) = 2$ and let $P_j \neq V(G_{I_j}(P_j)) \cup I_j$ for some $j \in \{1, 2, ..., n\}$. Then we can find some $m_j \in P_j \backslash (V(G_{I_j}(P_j)) \cup I_j)$. Since for each i, $e_i \in V(G_{I_i}(P_i))$, there is an element e'_i of $V(G_{I_i}(P_i))$ such that $(e_i, e'_i)^l \subseteq I_i$. If $a = (e_1, m_2, ..., m_n)$ and $b = (m_1, e_2, m_3, ..., m_n)$, then $(a, (e'_1, i_2, ..., i_n))^l \subseteq I$ and $(b, (i_1, e'_2, i_3, ..., i_n))^l \subseteq I$. So a, $b \in V(G_I(P))$ and $(a, b)^l \nsubseteq I$. Since $diam(G_I(P)) = 2$, there exists $c = (c_1, ..., c_n) \in V(G_I(P))$ such that $(a, c)^l \subseteq I$ and $(b, c)^l \subseteq I$. Thus $c_j \in I_j$, a contradiction. Thus $P_i = V(G_{I_i}(P_i)) \cup I_i$ for all $i \in \{1, 2..., n\}$.

(iii) It follows from (i) and (ii). ∎

Theorem 2.12 *Let $I_1, I_2, ..., I_n$ be semi-ideals of posets $P_1, P_2, ..., P_n$, respectively. If $diam(G_{I_1}(P_1)) = diam(G_{I_2}(P_2)) = ... = diam(G_{I_n}(P_n)) = 3$, then*

$$diam(G_I(P)) = 3.$$

Proof. Assume that for each $j \in \{1, 2, ..., n\}$, $diam(G_{I_j}(P_j)) = 3$, there exist $x_j,\ y_j \in V(G_{I_j}(P_j))$ with $x_j \neq y_j$, $(x_j, y_j)^l \nsubseteq I_j$ such that there is no $z_j \in V(G_{I_j}(P_j))$ with $(x_j, z_j)^l \subseteq I_j$ and $(y_j, z_j)^l \subseteq I_j$. Consider $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$. For each $j \in \{1, 2, ..., n\}$, there are elements $x'_j, y'_j \in V(G_{I_j}(P_j))$ such that $(x_j, x'_j)^l \subseteq I_j$ and $(y_j, y'_j)^l \subseteq I_j$. So $x,\ y \in V(G_I(P))$ and $(x, y)^l \nsubseteq I$. If $diam(G_I(P)) = 2$, then there exists $a = (a_1, ..., a_n) \in V(G_I(P))$ with $(x, a)^l \subseteq I$ and $(y, a)^l \subseteq I$ which imply $(x_j, a_j)^l \subseteq I_j$ and $(y_j, a_j)^l \subseteq I_j$, a contradiction. So $diam(G_I(P)) = 3$. ∎

Theorem 2.13 *Let $I_1, I_2, ..., I_n$ be semi-ideals of posets $P_1, P_2, ..., P_n$, respectively. If $diam(G_{I_j}(P_j)) = 1$, $diam(G_{I_k}(P_k)) = 2$ for some $j,\ k \in \{1, 2, ..., n\}$ and there is no $m \in \{1, 2, ..., n\}$ with $diam(G_{I_m}(P_m)) = 3$, then the following hold:*

(i) *$diam(G_I(P)) \neq 1$.*

(ii) *$diam(G_I(P)) = 2$ if and only if $P_j = V(G_{I_j}(P_j)) \cup I_j$, for all $j \in \{1, 2, ..., n\}$.*

(iii) *$diam(G_I(P)) = 3$ if and only if $P_j \neq V(G_{I_j}(P_j)) \cup I_j$, for some $j \in \{1, 2, ..., n\}$.*

Proof. (i) It is clear.

(ii) Let $P_j = V(G_{I_j}(P_j)) \cup I_j$ for all $j \in \{1, 2, ..., n\}$. By Lemma 2.8, $(x_j, y_j)^l \subseteq I_j$ for all $x_j,\ y_j \in V(G_{I_j}(P_j)) \cup I_j$. By (i), there are distinct vertices $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$ in $V(G_I(P))$ such that $(x, y)^l \nsubseteq I$.

We divided the proof into two cases.

Case (a): $diam(G_{I_j}(P_j)) = 1$. If $z_j \in V(G_{I_j}(P_j))$, then

$$(x, (i_1, ..., i_{j-1}, z_j, i_{j+1}, ..., i_n))^l \subseteq I \text{ and } (y, (i_1, ...i_{j-1}, z_j, i_{j+1}, ..., i_n))^l \subseteq I.$$

Thus we have a path $x - z - y$ of length 2. So $diam(G_I(P)) = 2$.

Case (b): $diam(G_{I_j}(P_j)) = 2$. Then, for some $x_j,\ y_j \in V(G_{I_j}(P_j))$, there exists $z_j \in V(G_{I_j}(P_j))$ such that $(x_j, z_j)^l \subseteq I_j$ and $(y_j, z_j)^l \subseteq I_j$. Set $z = (i_1, ...i_{j-1}, z_j, i_{j+1}, ..., i_n)$. Then $(x, z)^l \subseteq I$ and $(y, z)^l \subseteq I$. So we must have a path $x - z - y$ of length 2 and hence $diam(G_I(P)) = 2$.

Conversely, assume that $diam(G_I(P)) = 2$. Suppose that $P_j \neq V(G_{I_j}(P_j)) \cup I_j$ for some $j \in \{1, 2, ..., n\}$. Then we can find some elements $m_j \in P_j \backslash (V(G_{I_j}(P_j)) \cup I_j)$. Since for each i, $x_i \in V(G_{I_i}(P_i))$, there is an element x'_i of $V(G_{I_i}(P_i))$ such that $(x_i, x'_i)^l \subseteq I_i$ for all i. If $a = (x_1, m_2, ..., m_n)$ and $b = (m_1, x_2, m_3, ..., m_n)$, then

$(a,(x'_1,i_2,...,i_n))^l \subseteq I$ and $(b,(i_1,x'_2,i_3,...,i_n))^l \subseteq I$. So $a,\ b \in V(G_I(P))$ and $(a,b)^l \nsubseteq I$. Since $diam(G_I(P)) = 2$, there exists $c = (c_1,...,c_n) \in V(G_I(P))$ such that $(a,c)^l \subseteq I$ and $(b,c)^l \subseteq I$ which imply $c_j \in I_j$, a contradiction. Thus $P_j = V(G_{I_j}(P_j)) \cup I_j$ for all $j \in \{1,2...,n\}$.

(iii) It follows from (i) and (ii). ∎

Theorem 2.14 *Let $I_1, I_2, ..., I_n$ be semi-ideals of posets $P_1, P_2, ..., P_n$, respectively. If $diam(G_{I_j}(P_j)) = 1$, $diam(G_{I_k}(P_k)) = 3$ for some $j,\ k \in \{1,2,...,n\}$ and there is no $m \in \{1,2,...,n\}$ with $diam(G_{I_m}(P_m)) = 2$, then the following hold:*

(i) *$diam(G_I(P)) \neq 1$.*

(ii) *$diam(G_I(P)) = 2$ if and only if $diam(G_{I_j}(P_j)) = 1$ and $P_j = V(G_{I_j}(P_j)) \cup I_j$ for all $j \in \{1,2,...,n\}$.*

(iii) *$diam(G_I(P)) = 3$ if and only if there is no $j \in \{1,2,...,n\}$ with $diam(G_{I_j}(P_j)) = 1$ and $P_j = V(G_{I_j}(P_j)) \cup I_j$.*

Proof. (i) It is clear.

(ii) Assume that $diam(G_{I_j}(P_j)) = 1$ and $P_j = V(G_{I_j}(P_j)) \cup I_j$ for all $j \in \{1,2,...,n\}$. Then by Lemma 2.8, $(x_j,y_j)^l \subseteq I_j$ for all $x_j,\ y_j \in V(G_{I_j}(P_j)) \cup I_j$. By (i), there are distinct vertices $x = (x_1,x_2,...,x_n)$ and $y = (y_1,y_2,...,y_n)$ in $V(G_I(P))$ such that $(x,y)^l \nsubseteq I$. Let $a_j \in V(G_{I_j}(P_j))$ and set $a = (i_1,...i_{j-1},a_j,i_{j+1},...,i_n)$. Then $(x,a)^l \subseteq I$ and $(y,a)^l \subseteq I$ which imply $a \in V(G_I(P))$ and $x - a - y$ is a path of length 2. So $diam(G_I(P)) = 2$.

Conversely, assume that $diam(G_I(P))$=2. We now show that $diam(G_{I_i}(P_i))$=1 and $P_i = V(G_{I_i}(P_i)) \cup I_i$ for all $i \in \{1,2,...,n\}$. Suppose not. Let $i_1,i_2,...,i_k$ be such that $diam(G_{I_{i_r}}(P_{i_r})) = 1$ $(1 \leq r \leq k)$, and let $j_1,j_2,...,j_t$ be such that $diam(G_{I_{j_s}}(P_{j_s})) = 3$ $(1 \leq s \leq t)$. Then for each s, there exist distinct vertices $x_{j_s}, y_{j_s} \in V(G_{I_{j_s}}(P_{j_s}))$ such that $(x_{j_s},y_{j_s})^l \nsubseteq I_{j_s}$ and there is no $z_{j_s} \in V(G_{I_j}(P_{j_s}))$ with $(x_{j_s},z_{j_s})^l \subseteq I_{j_s}$ and $(z_{j_s},y_{j_s})^l \subseteq I_{j_s}$. Moreover, for each s $(1 \leq s \leq t)$, there must exist $x'_{j_s}, y'_{j_s} \in V(G_{I_{j_s}}(P_{j_s}))$ with $(x_{j_s},x'_{j_s})^l \subseteq I_{j_s}$ and $(y_{j_s},y'_{j_s})^l \subseteq I_{j_s}$. Now, for each r $(1 \leq r \leq k)$, let $m_{i_r} \in P_{i_r} \backslash (V(G_{I_{i_r}}(P_{i_r})) \cup I_{i_r})$. Set $c = (m_{i_1},...,x_{j_1},...,x_{j_t},..)$ and $d = (m_{i_1},...,y_{j_1},...,y_{j_t},..)$. Then $(c,(i_1,...,x'_{j_1},i_{j+1},...,i_n)) \subseteq I$ and $(d,(i,...,y'_{j_1},i,...,i)) \subseteq I$, which imply $c,d \in V(G_I(P))$ and $(c,d)^l \nsubseteq I$. Since $diam(G_I(P)) = 2$, there exists $e = (e_1,...,e_n) \in V(G_I(P))$ such that $(c,e)^l \subseteq I$ and $(d,e)^l \subseteq I$. Thus $e_i \in I_i$, a contradiction. Thus $diam(G_{I_i}(P_i)) = 1$ and $P_i = V(G_{I_i}(P_i)) \cup I_i$ for all $i \in \{1,2...,n\}$.

(iii) If $diam(G_I(P)) = 2$, then by (ii), we have $diam(G_{I_i}(P_i)) = 1$ and P_i=$V(G_{I_i}(P_i))$∪I_i for all $i \in \{1,2...,n\}$, a contradiction. Thus $diam(G_I(P)) = 3$. ∎

Theorem 2.15 *Let $I_1, I_2, ..., I_n$ be semi-ideals of posets $P_1, P_2, ..., P_n$, respectively. If $diam(G_{I_j}(P_j)) = 2$, $diam(G_{I_k}(P_k)) = 3$ for some $j,\ k \in \{1,2,...,n\}$ and there is no $m \in \{1,2,...,n\}$ with $diam(G_{I_m}(P_m)) = 1$, then the following hold:*

(i) $diam(G_I(P)) \neq 1$.

(ii) $diam(G_I(P)) = 2$ *if and only if* $diam(G_{I_j}(P_j)) = 2$ *and* $P_j = V(G_{I_j}(P_j)) \cup I_j$ *for all* $j \in \{1, 2, ..., n\}$.

(iii) $diam(G_I(P)) = 3$ *if and only if there is no* $j \in \{1, 2, ..., n\}$ *with* $diam(G_{I_j}(P_j)) = 2$ *and* $P_j = V(G_{I_j}(P_j)) \cup I_j$.

Proof. (i) It is clear.

(ii) Assume that $diam(G_{I_j}(P_j)) = 2$ and $P_j = V(G_{I_j}(P_j)) \cup I_j$ for all $j \in \{1, 2, ..., n\}$. By (i), there are elements $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n) \in V(G_I(P))$ such that $(x, y)^l \nsubseteq I$ which implies $(x_j, y_j)^l \nsubseteq I_j$ for some j. Then there exists $a_j \in V(G_{I_j}(P_j))$ such that $(a_j, x_j)^l \subseteq I_j$ and $(a_j, y_j)^l \subseteq I_j$. Set $a = (i_1, ...i_{j-1}, a_j, i_{j+1}, ..., i_n)$. Then $(x, a)^l \subseteq I$ and $(y, a)^l \subseteq I$ which imply $a \in V(G_I(P))$ and $x - a - y$ is a path of length 2, so $diam(G_I(P)) = 2$.

Conversely, assume that $diam(G_I(P))$=2. We now show that $diam(G_{I_i}(P_i))$=2 and $P_i = V(G_{I_i}(P_i)) \cup I_i$ for all $i \in \{1, 2, ..., n\}$. Suppose that for some i $(1 \leq i \leq n)$, if $diam(G_{I_i}(P_i)) = 2$, then $P_i = V(G_{I_i}(P_i)) \cup I_i$. Let $i_1, i_2, ..., i_k$ be such that $diam(G_{I_{i_r}}(P_{i_r})) = 2$ $(1 \leq i \leq k)$, and let $j_1, j_2, ..., j_t$ be such that $diam(G_{I_{j_s}}(P_{j_s})) = 3$ $(1 \leq s \leq t)$. Now for each $r(1 \leq r \leq k)$, $P_{i_r} \neq V(G_{I_{i_r}}(P_{i_r})) \cup I_{i_r}$. For each r $(1 \leq r \leq k)$, let $m_{i_r} \in P_{i_r} \backslash (V(G_{I_{i_r}}(P_{i_r})) \cup I_{i_r})$. Since for each s $(1 \leq s \leq t)$, $diam(G_{I_{j_s}}(P_{j_s})) = 3$, there exist $x_{j_s}, y_{j_s} \in V(G_{I_{j_s}}(P_{j_s}))$ with $x_{j_s} \neq y_{j_s}, (x_{j_s}, y_{j_s})^l \nsubseteq I_{j_s}$ such that there is no $z_{j_s} \in V(G_{I_{j_s}}(P_{j_s}))$ with $(x_{j_s}, z_{j_s})^l \subseteq I_{j_s}$ and $(z_{j_s}, y_{j_s})^l \subseteq I_{j_s}$. Moreover, for each $s(1 \leq s \leq t)$, there must exist $x'_{j_s}, y'_{j_s} \in V(G_{I_{j_s}}(P_{j_s}))$ with $(x_{j_s}, x'_{j_s})^l \subseteq I_{j_s}$ and $(y_{j_s}, y'_{j_s})^l \subseteq I_{j_s}$. Set $c = (m_{i_1}, ..., x_{j_1}, ..., x_{j_t}, ..)$ and d=$(m_{i_1}, ..., y_{j_1}, ..., y_{j_t}, ..)$. Then $(c, (i, ..., x'_{j_1}, i, ..., i)) \subseteq I$ and $(d, (i, ..., y'_{j_1}, i, ..., i)) \subseteq I$, and so $c, d \in V(G_I(P))$. Since $(c, d)^l \nsubseteq I$ and $diam(G_I(P)) = 2$, there must be some $e = (e_1, ..., e_n)$ such that $(c, e)^l \subseteq I$ and $(d, e)^l \subseteq I$. Thus $e_i \in I_i$, a contradiction. Thus $diam(G_{I_i}(P_i)) = 2$ and $P_i = V(G_{I_i}(P_i)) \cup I_i$ for all $i \in \{1, 2..., n\}$.

(iii) It follows from (i) and (ii). ■

Theorem 2.16 *Let* $I_1, I_2, ..., I_n$ *be semi-ideals of posets* $P_1, P_2, ..., P_n$*, respectively. If* $diam(G_{I_j}(P_j)) = 1$, $diam(G_{I_k}(P_k)) = 2$ *and* $diam(G_{I_m}(P_m)) = 3$ *for some* $j, k, m \in \{1, 2, ..., n\}$*, then the following hold:*

(i) $diam(G_I(P)) \neq 1$.

(ii) $diam(G_I(P)) = 2$ *if and only if* $diam(G_{I_j}(P_j)) \leq 2$ *and* $P_j = V(G_{I_j}(P_j)) \cup I_j$ *for some* $j \in \{1, 2, ..., n\}$.

(iii) $diam(G_I(P)) = 3$ *if and only if there is no* $j \in \{1, 2, ..., n\}$ *with* $diam(G_{I_j}(P_j)) \leq 2$ *and* $P_j = V(G_{I_j}(P_j)) \cup I_j$.

Proof. (i) It is clear.

(ii) Let $diam(G_{I_i}(P_i)) \leq 2$ and $P_i = V(G_{I_i}(P_i)) \cup I_i$ for all $i \in \{1, 2, ..., n\}$. We divide the proof into two cases.

Case (a): $diam(G_{I_i}(P_i)) = 1$ and $P_i = V(G_{I_i}(P_i)) \cup I_i$ for all $i \in \{1, 2, ..., n\}$. By a similar argument as in Theorem 2.14 (ii), we get

$$diam(G_I(P)) = 2.$$

Case (b): $diam(G_{I_i}(P_i)) = 2$ and $P_i = V(G_{I_i}(P_i)) \cup I_i$ for all $i \in \{1, 2, ..., n\}$. By a similar argument as in Theorem 2.15 (ii), we get

$$diam(G_I(P)) = 2.$$

Conversely, suppose that $diam(G_I(P)) = 2$. It is easy to see from Theorem 2.15(ii) that

$$diam(G_{I_i}(P_i)) \leq 2 \text{ and } P_i = V(G_{I_i}(P_i)) \cup I_i \text{ for all } i.$$

(iii) It follows from (i) and (ii). ∎

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ON COMMUTING TRACES OF GENERALIZED BIDERIVATIONS OF PRIME RINGS

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Abstract. In this paper, we prove some theorems on symmetric generalized biderivations of a ring, which extend a result of Vukman [9, Theorem 1] and a result of Bresar [3, Theorem 4.1].

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1. Introduction

Throughout the paper all ring will be associative. We shall denote by $Z(R)$ the centre of ring R and by C the extended centroid of R, which is the centre of the two sided Martindale quotients ring Q (we refer the reader [3] for more details). A ring R is said to be prime (resp. semiprime) if $aRb = (0)$ implies that either $a = 0$ or $b = 0$ (resp. $aRa = (0)$ implies that $a = 0$). We shall write for any pair of elements $x, y \in R$ the commutator $xy - yx$ and $x \circ y$ stands for the skew commutator $xy + yx$. We make extensive use of the basic commutator identities (i) $[x, yz] = [x, y]z + y[x, z]$ and (ii) $[xy, z] = [x, z]y + x[y, z]$. An additive mapping $d : R \longrightarrow R$ is called a derivation if $d(xy) = d(x)y + xd(y)$, for all $x, y \in R$. A derivation d is inner if there exists an element $a \in R$ such that $d(x) = [a, x]$ for all $x \in R$. A mapping $D : R \times R \longrightarrow R$ is said to be symmetric if $D(x, y) = D(y, x)$, for all $x, y \in R$. A mapping $f : R \longrightarrow R$ defined by $f(x) = D(x,x)$, where $D : R \times R \longrightarrow R$ is a symmetric mapping, is called the trace of D. It is obvious that in the case $D : R \times R \longrightarrow R$ is a symmetric mapping which is also biadditive (i.e. additive in both arguments). The trace f of D satisfies the relation $f(x+y) = f(x) + f(y) + 2D(x,y)$, for all $x, y \in R$. A biadditive symmetric mapping $D : R \times R \longrightarrow R$ is called a symmetric biderivation if $D(xy,z) = D(x,z)y + xD(y,z)$ for all $x, y, z \in R$. Obviously, in this case the relation $D(x,yz) = D(x,y)z + yD(x,z)$ is also satisfied for all $x, y, z \in R$.

Typical examples are mapping of the form $(x,y) \mapsto \lambda[x,y]$ where $\lambda \in C$. We shall call such maps inner biderivations. In [6] it was shown that every biderivation D of a noncommutative prime ring R is of the form $D(x,y) = \lambda[x,y]$ for some $\lambda \in C$. Further Bresar extended this result for semiprime rings. Some results on biderivations can be found in[2], [6] and [8].

G. Maksa [8] introduced the concept of a symmetric biderivation (see also [9], where an example can be found). It was shown in [8] that symmetric biderivations are related to general solution of some functional equations. Some results on symmetric biderivation in prime and semiprime rings can be found in [5], [11] and [12]. The notion of additive commuting mappings is closely connected with the notion of biderivations. Every commuting additive mapping $f : R \longrightarrow R$ gives rise to a biderivation on R. Namely linearizing $[x, f(x)] = 0$ for all $x, y \in R$ $(x,y) \mapsto [f(x), y]$ is a biderivation (moreover, all derivations appearing are inner).

The notion of generalized symmetric biderivations was introduced by Nurcan [1]. More precisely, a generalized symmetric biderivation is defined as follows: Let R be a ring and $D : R \times R \longrightarrow R$ be a biadditive map. A biadditive mapping $\Delta : R \times R \longrightarrow R$ is said to be generalized biderivation if for every $x \in R$, the map $y \mapsto \Delta(x,y)$ is a generalized derivation of R associated with function $y \mapsto D(x,y)$ as well as if for every $y \in R$, the map $x \mapsto \Delta(x,y)$ is a generalized derivation of R associated with function $x \mapsto D(x,y)$ for all $x, y \in R$. It also satisfies $\Delta(x,yz) = \Delta(x,y)z + yD(x,z)$ and $\Delta(xy,z) = \Delta(x,z)y + xD(y,z)$ for all $x, y, z \in R$. For example consider a biderivation Δ of R and biadditive a function $\alpha : R \times R \longrightarrow R$ such that $\alpha(x,yz) = \alpha(x,y)z$ and $\alpha(xy,z) = \alpha(x,z)y$ for all $x, y, z \in R$. Then $\Delta + \alpha$ is a generalized Δ-biderivation of R.

An additive mapping $h : R \longrightarrow R$ is called left (resp. right) multiplier of R if $h(xy) = h(x)y$ (resp. $h(xy) = xh(y)$) for all $x, y \in R$. A biadditive mapping $D : R \times R \longrightarrow R$ is said to be a left (resp. right) bi-multiplier of R if $D(x,yz) = D(x,y)z$ (resp. $D(xz,y) = xD(z,y)$) for all $x, y, z \in R$.

In this paper, we prove some theorems on symmetric generalized biderivations of a ring which extend a result of Vukman [9, Theorem 1] and a result of Bresar [3, Theorem 4.1].

2. Generalized biderivations on prime rings

The result proved in this section generalizes Theorem 1 in [11]. More precisely, we consider the case when the ring R is prime and replace symmetric biderivations with symmetric generalized biderivations.

In [11], Vukman proved the following result: Let R be a noncommutative prime ring of characteristic different from two and $D : R \times R \longrightarrow R$ be a symmetric biderivation with trace f. If f is commuting on R, then $d = 0$. Vukman [10, Theorem 2] further generalized the result by proving that let R be a noncommutative prime ring of characteristic different from two. Suppose there exists a symmetric biderivation $D : R \times R \longrightarrow R$ with trace f such that the mapping $x \mapsto [f(x), x]$ is centralizing on R. In this case $D = 0$.

Theorem 2.1. *Let R be a prime ring of characteristic different from two and I be a nonzero left ideal of R. If Δ is a symmetric generalized biderivation with associated biderivation D such that $[\Delta(x,x), x] = 0$ for all $x \in I$, then either R is commutative or Δ acts as a left bimultiplier on I.*

Proof. Suppose that

$$[\Delta(x,x), x] = 0, \quad \text{for all } x \in I. \tag{2.1}$$

Linearization of (2.1) yields that

$$\begin{aligned} &[\Delta(x,x), x] + [\Delta(x,x), y] + [\Delta(x,y), x] + [\Delta(x,y), y] + [\Delta(y,x), x] \\ &+[\Delta(y,x), y] + [\Delta(y,y), x] + [\Delta(y,y), y] = 0, \quad \text{for all } x, y \in I. \end{aligned} \tag{2.2}$$

Since Δ is symmetric and using (2.1), we obtain

$$\begin{aligned} &[\Delta(x,x), y] + 2[\Delta(x,y), x] + 2[\Delta(x,y), y] + [\Delta(y,y), x] = 0, \\ &\text{for all } x, y \in I. \end{aligned} \tag{2.3}$$

Substituting $-y$ for y in (2.3), we have

$$\begin{aligned} &-[\Delta(x,x), y] - 2[\Delta(x,y), x] + 2[\Delta(x,y), y] + [\Delta(y,y), x] = 0, \\ &\text{for all } x, y \in I. \end{aligned} \tag{2.4}$$

Adding (2.3) and (2.4) and using char $R \neq 2$, we find

$$2[\Delta(x,y), y] + [\Delta(y,y), x] = 0, \quad \text{for all } x, y \in I. \tag{2.5}$$

Replace x by xz in (2.5) to get

$$\begin{aligned} &2\Delta(x,y)[z,y] + 2[\Delta(x,y), y]z + 2x[D(z,y), y] + 2[x,y]D(z,y) \\ &+[\Delta(y,y), x]z + x[\Delta(y,y), z] = 0, \quad \text{for all } x, y, z \in I. \end{aligned} \tag{2.6}$$

In view of (2.5), (2.6) gives that

$$\begin{aligned} &\Delta(x,y)[z,y] + 2x[D(z,y), y] + 2[x,y]D(z,y) + x[\Delta(y,y), z] = 0, \\ &\text{for all } x, y, z \in I. \end{aligned} \tag{2.7}$$

Substitute y for z to obtain

$$2x[D(y,y), y] + 2[x,y]D(y,y) = 0, \quad \text{for all } x, y \in I. \tag{2.8}$$

Since char R not two, we have

$$x[D(y,y),y]+[x,y]D(y,y)=0, \quad \text{for all } x,y\in I. \tag{2.9}$$

Substitute rx for x in (2.9) and using (2.9), we obtain

$$[r,y]xD(y,y)=0, \quad \text{for all } x,y\in I, \quad \text{for all } r\in R. \tag{2.10}$$

Replace r by rs in (2.10), we find

$$[r,y]RxD(y,y)=0, \quad \text{for all } x,y\in I, \ \text{for all } r\in R. \tag{2.11}$$

Primeness of R yields that either $[y,r]=0$ or $xD(y,y)=0$ for all $x,y\in I$. If $[y,r]=0$ for all $y\in I$ and $r\in R$, then I is contained in $Z(R)$. Since I is a central ideal of R, we have R is commutative by [10]. On the other hand, we have $xD(y,y)=0$ for all $x,y\in I$. Linearization in y yields that $xD(y,z)+xD(z,y)=0$ for all $x,y,z\in I$. Since D is symmetric and using char $R\neq 2$, we get $xD(y,z)=0$ for all $x,y,z\in I$, i.e. Δ acts as a left bimultiplier on I.

Corollary 2.1. *Let R be a prime ring of characteristic different from two and I be a nonzero left ideal of R. If Δ is a symmetric generalized biderivation with associated biderivation D such that $\Delta(x,y)\mp[x,y]\in Z(R)$ for all $x,y\in I$, then either R is commutative or Δ acts as a left bimultiplier on I.*

Corollary 2.2. *Let R be a prime ring of characteristic different from two and I be a nonzero left ideal of R. If Δ is a symmetric generalized biderivation with associated biderivation D such that $\Delta(x,y)\mp x\circ y\in Z(R)$ for all $x,y\in I$, then either R is commutative or Δ acts as a left bimultiplier on I.*

Theorem 2.2. *Let R be a prime ring of characteristic different from two and I be a nonzero left ideal of R. If Δ is a symmetric generalized biderivation with associated biderivation D such that $\Delta(x,x)\circ x=0$ for all $x\in I$, then either R is commutative or Δ acts as a left bimultiplier on I.*

Proof. By assumption, we have

$$\Delta(x,x)\circ x=0 \quad \text{for all } x\in I. \tag{2.12}$$

Linearization of (2.12) yields that

$$\begin{aligned}&\Delta(x,x)x+\Delta(y,y)x+2\Delta(x,y)x+\Delta(x,x)y+\Delta(y,y)y\\ &\quad+2\Delta(x,y)y+x\Delta(x,x)+x\Delta(y,y)+2x\Delta(x,y)+y\Delta(x,x)\\ &\quad+y\Delta(y,y)+2y\Delta(x,y)=0 \quad \text{for all } x,y\in I.\end{aligned} \tag{2.13}$$

In view of (2.12), (2.13), gives that

$$\begin{aligned}&\Delta(y,y)x+2\Delta(x,y)x+\Delta(x,x)y+2\Delta(x,y)y+x\Delta(y,y)\\ &\quad+2x\Delta(x,y)+y\Delta(x,x)+2y\Delta(x,y)=0 \quad \text{for all } x,y\in I.\end{aligned} \tag{2.14}$$

Substituting $-y$ for y in (2.14), we have

$$\begin{aligned}\Delta(y,y)x-2\Delta(x,y)x-\Delta(x,x)y+2\Delta(x,y)y+x\Delta(y,y)\\ -2x\Delta(x,y)-y\Delta(x,x)+2y\Delta(x,y)=0\quad\text{for all } x,y\in I.\end{aligned}\tag{2.15}$$

Adding (2.14) and (2.15) and using the fact that char$R\neq 2$, we get

$$\Delta(y,y)x+2\Delta(x,y)y+x\Delta(y,y)+2y\Delta(x,y)=0,\quad\text{for all } x,y\in I.\tag{2.16}$$

Replacing x by xu in (2.16), we have

$$\begin{aligned}\Delta(y,y)xu+2\Delta(x,y)uy+2xD(u,y)y+xu\Delta(y,y)\\ +2y\Delta(x,y)u+2yxD(u,y)=0\quad\text{for all } x,y\in I.\end{aligned}\tag{2.17}$$

Right multiplying (2.16) by u and then subtracting from (2.17), we obtain

$$\begin{aligned}2\Delta(u,y)[u,y]+2x\Delta(u,y)y+x[u,\Delta(y,y)]+2yxD(u,y)=0,\\ \text{for all } x,y,u\in I.\end{aligned}\tag{2.18}$$

Substituting u by y in (2.18), we get

$$2x\Delta(y,y)y+x[y,\Delta(y,y)]+2yxD(y,y)=0\quad\text{for all } x,y\in I.\tag{2.19}$$

Replacing rx for x in (2.19) and using it, we obtain

$$\begin{aligned}2rx\Delta(y,y)y+rx[y,\Delta(y,y)]+2yrxD(y,y)=0,\\ \text{for all } x,y\in I\quad\text{and}\quad\text{for all } r\in R.\end{aligned}\tag{2.20}$$

Left multiplying (2.19) by r and then subtracting from (2.20), we get

$$2[y,r]xD(y,y)=0,\quad\text{for all } x,y\in I\ \text{ and }\ \text{for all } r\in R.\tag{2.21}$$

This implies that $2[y,r]RxD(y,y)=0$ for all $x,y\in I$ and for all $r\in R$. Since char$R\neq 2$ we have $[y,r]RxD(y,y)=0$ for all $x,y\in I$ and for all $r\in R$. Primeness of R yields that either $[y,r]=0$ or $xD(y,y)=0$ for all $x,y\in I$ and for all $r\in R$. Arguing in the similar manner as in the proof of Theorem 2.1, we get the result.

Theorem 2.3. *Let R be a $2,3$ and 5-torsion free semiprime ring, I an additive subgroup of R such that $x^2\in I$ for all $x\in I$ and $\Delta: R\times R\to R$ be a symmetric generalized biderivation associated with biderivation D with the trace f of Δ. If f is centralizing on I, then f is commuting on I.*

Proof. Let $x\in I$ and take $t=[f(x),x]$, where $f(x)=\Delta(x,x)$. Then $t\in Z(R)$. By our hypothesis, we have

$$[f(x),x]\in Z(R)\quad\text{for all } x\in I.\tag{2.22}$$

Replacing x by $x+y$ in (2.22), we have

$$\begin{aligned}(2.23)\qquad &[f(x),x]+[f(x),y]+[f(y),x]+[f(y),y]+[\Delta(x,y),x]+[\Delta(y,x),x]\\ &+[\Delta(y,x),y]+[\Delta(x,y),y]\in Z(R)\quad\text{for all}\ \ x,y\in I.\end{aligned}$$

Putting $x=-x$ in (2.23) and using (2.22), we get

$$(2.24)\qquad [f(x),y]+2[\Delta(x,y),x]\in Z(R)\quad\text{for all}\ \ x,y\in I.$$

Substituting x^2 for y in (2.24), we have

$$(2.25)\qquad [f(x),x^2]+[\Delta(x,x)x+xD(x,x),x]\in Z(R)\quad\text{for all}\ \ x\in I.$$

We have $[f(x),x^2]=[f(x),x]x+x[f(x),x]=2tx$.
Since $[\Delta(x,x^2),x]=2tx+2x[D(x,x),x]$, the last expression reduces to

$$(2.26)\qquad 2(x[D(x,x),x]+2tx)\in Z(R)\quad\text{for all}\ \ x\in I.$$

Since R is 2-torsion free, we get $x[D(x,x),x]+2tx\in Z(R$.
Let $z=x[D(x,x),x]+2tx\in Z(R)$. This implies that

$$(z-2tx)=x[D(x,x),x].$$

Replacing x by x^2 in our hypothesis, we can write

$$\begin{aligned}[f(x^2),x^2]&=[\Delta(x^2,x^2),x^2]=[\Delta(x^2,x)x+xD(x^2,x),x^2]\\ &=[\Delta(x^2,x),x^2]x+x[D(x^2,x),x^2]x\\ &=[\Delta(x,x)x+xD(x,x),x^2]x+x[D(x,x)x+xD(x,x),x^2]\\ &=[\Delta(x,x),x^2]x^2+x[D(x,x),x^2]x+x[D(x,x),x^2]x+x^2[D(x,x),x^2]\\ &=[\Delta(x,x),x]x^3+x[\Delta(x,x),x]x^2+2x^2[D(x,x),x]x\\ &+2x[D(x,x),x]x^2+x^3[D(x,x),x]+x^2[D(x,x),x]x\\ &=2tx^3+2x(z-2tx)x+2(z-2tx)x^2+x^2(z-2tx)+x(z-2tx)x\\ &=-10tx^3+6zx^2\end{aligned}$$

This implies that $-10tx^3+6zx^2\in Z(R)$. Commuting both sides with $f(x)$, we get $[f(x),-10tx^3+6zx^2]=0$, i.e.,

$$\begin{aligned}&-10t[f(x),x^3]+6z[f(x),x^2]\\ &\quad=-10t[f(x),x]x^2-10tx[f(x),x^2]+6z[f(x),x]x+6zx[f(x),x]\\ &\quad=-10t^2x^2-10tx[f(x),x]x-10tx^2[f(x),x]+12ztx\\ &\quad=-30t^2x^2+12ztx=0.\end{aligned}$$

Again commuting with $f(x)$, we have

$$\begin{aligned}-30t^2[f(x),x^2]+12zt[f(x),x]&=-30t^2[f(x),x]x-30t^2x[f(x),x]+12zt^2\\ &=-60t^3x+12zt^2=0.\end{aligned}$$

Repeating the same argument, we finally arrive at $-60t^4 = 0$. Since R is $2, 3$ and 5 torsion free, we get $t^4 = 0$. Since the center of a semiprime ring contains no nonzero nilpotent elements, we conclude that $t = 0$. This completes the proof.

3. Cocommuting biderivations

In this section, we consider the case in which the mappings $\mu, \phi : R \longrightarrow R$ satisfy $\mu(x)x + x\phi(x) = 0$ for all $x \in R$. Bresar [3, Theorem 4.1] proved that if R is a prime ring, I a nonzero left ideal of R and α and β are nozero derivations of R satisying $\alpha(x)x - x\beta(x) \in Z(R)$ for all $x \in I$, then R is commutative. Argac [1, Theorem 3.5] proved a result for generalized derivation of R.

We extend the aforementioned results by proving the following theorem for a biderivation of R.

Theorem 3.1. *Let R be a prime ring of characteristic not two, I a nonzero left ideal of R and D, G be symmetric biderivations of R with trace f and g respectively. If $D(x,x)x+xG(x,x) = 0$ for all $x \in I$, then either R is commutative or G acts as a left bimultiplier on I. Moreover, in the last case either $D = 0$ or $I[I,I] = 0$.*

Proof. By hypothesis, we have

$$f(x)x + xg(x) = 0 \text{ for all } x \in I, \tag{3.1}$$

where $f(x) = D(x,x)$ and $g(x) = G(x,x)$. Linearization of (3.1) yields that

$$\begin{aligned} f(y)x + f(x)y + 2D(x,y)x + 2D(x,y)y + xg(y) + yg(x) \\ +2xG(x,y) + 2yG(x,y) = 0, \text{ for all } x, y \in I. \end{aligned} \tag{3.2}$$

Substituting $-y$ for y in (3.2), we get

$$\begin{aligned} f(y)x - f(x)y - 2D(x,y)x + 2D(x,y)y + xg(y) - yg(x) \\ -2xG(x,y) + 2yG(x,y) = 0, \text{ for all } x, y \in I. \end{aligned} \tag{3.3}$$

Adding (3.1) and (3.2), we obtain

$$2f(y)x + 4D(x,y)y + 2xg(y) + 4yG(x,y) = 0, \text{ for all } x, y \in I. \tag{3.4}$$

Since char R is not two, we have

$$f(y)x + 2D(x,y)y + xg(y) + 2yG(x,y) = 0, \text{ for all } x, y \in I. \tag{3.5}$$

Replacing x by xz in (3.5), we obtain

$$\begin{aligned} f(y)xz + 2D(x,y)zy + 2xD(z,y)y + xzg(y) + 2yG(x,y)z + 2yxG(z,y) = 0, \\ \text{for all } x, y, z \in I. \end{aligned} \tag{3.6}$$

Comparing (3.5) and (3.6), we obtain

$$\begin{aligned}(3.7)\qquad &-2D(x,y)yz - xg(y)z + 2D(x,y)zy + 2xD(z,y)y + xzg(y)\\ &+2yxG(z,y) = 0, \text{ for all } x,y,z \in I.\end{aligned}$$

This implies that

$$\begin{aligned}(3.8)\qquad &2D(x,y)[z,y] + x[z,g(y)] + 2xD(z,y)y + 2yxG(z,y) = 0,\\ &\text{for all } x,y,z \in I.\end{aligned}$$

Substituting rx for x in (3.8), we get

$$\begin{aligned}(3.9)\qquad &2rD(x,y)[z,y] + 2D(r,y)x[z,y] + rx[z,g(y)]\\ &+2rxD(z,y)y + 2yrxG(z,y) = 0,\\ &\text{for all } x,y,z \in I, \text{ for all } r \in R.\end{aligned}$$

Comparing (3.8) and (3.9), we get

$$\begin{aligned}(3.10)\qquad &2D(r,y)x[z,y] + 2yrxG(z,y) - 2ryxG(z,y) = 0,\\ &\text{for all } x,y,z \in I, \text{ for all } r \in R.\end{aligned}$$

Since R is of characteristic not two, we obtain

$$(3.11)\qquad D(r,y)x[z,y] + [y,r]xG(z,y) = 0, \text{ for all } x,y,z \in I, \text{ for all } r \in R.$$

Replacing y by z in (3.12), we obtain

$$(3.12)\qquad [z,r]xg(z) = 0, \text{ for all } x,z \in I, \text{ for all } r \in R$$

Substituting rx for x in (3.12), we get

$$(3.13)\qquad [z,r]Rxg(z) = 0, \text{ for all } x,z \in I, \text{ for all } r \in R$$

Primeness of R yields that either $[z,r] = 0$ or $xg(z) = 0$. If $[z,r] = 0$ for all $z \in I$ and $r \in R$, then R is commutative by [10]. Suppose $xg(z) = 0$ for all $x,z \in I$. Linearization in z yields that

$$0 = xG(z,y) + xG(y,z) = 2xG(y,z)$$

and using R is not of characteristic two, we get

$$xG(y,z) = 0 \text{ for all } x,y,z \in I.$$

This implies that

$$G(x,yz) = G(x,y)z.$$

Hence G acts as left multiplier. Since $xG(y,z) = 0$ for all $x,y,z \in I$ and using (3.11), we arrive at

$$(3.14)\qquad D(r,y)x[z,y] = 0, \text{ for all } x,y,z \in I, \ r \in R.$$

Replace r by rs in (3.14) to get

$$D(r,y)Rx[z,y]=0, \text{ for all } x,y,z\in I,\ r\in R. \tag{3.15}$$

Primeness of R implies that either $D(r,y)=0$ or $x[z,y]=0$ for all $x,y,z\in I$. Later yields that $I[I,I]=0$ as $D\neq 0$.

Proceeding on the same parallel lines, we can prove the following:

Theorem 3.2. *Let R be a prime ring of characteristic not two, I a nonzero right ideal of R and D, G are symmetric biderivations of R with trace f and g respectively. If $D(x,x)x+xG(x,x)=0$ for all $x\in I$, then then either R is commutative or D acts as a left bimultiplier on I. Moreover in the last case either $G=0$ or $I[I,I]=0$.*

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ON CONNECTIONS BETWEEN VECTOR SPACES AND HYPERCOMPOSITIONAL STRUCTURES

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Abstract. During his sort life, F. Marty, through three articles of his, introduced the notion of hypergroup. W. Prenowitz utilized this structure in the study of Geometry. This paper contributes to the methodology of connecting vector spaces with hypergroups. Convexity is presented in hypercompositional algebra terms and we get to the theorems of Kakutani, Stone, Helly, Randon, Carathéodory and Steinitz, through more general theorems which are valid in hypergroups.

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1. Hypergroups and the theorems of Kakutani and Stone

In 1934, F. Marty, in order to study problems in non-commutative algebra, such as cosets determined by non-invariant subgroups, generalized the notion of the group, thus defining the *hypergroup* [19], [20], [21], an algebraic structure in which the result of the composition of two elements is not an element, but a set of elements. More specifically, an *operation* or *composition* in a non-void set H is a function from $H \times H$ to H, while a *hyperoperation* or *hypercomposition* is a function from $H \times H$ to the powerset $P(H)$ of H. An algebraic structure that satisfies the axioms

(i) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for every $a, b, c \in H$ (associative axiom), and

(ii) $a \cdot H = H \cdot a = H$ for every $a \in H$ (reproductive axiom),

is called *group* if "$\cdot$" is a composition, and *hypergroup* if "$\cdot$" is a hypercomposition [36], [37].

Proposition 1.1. *If a non-void set H is endowed with a composition which satisfies the associative and the reproductive axiom, then H has a bilateral neutral element and any element in H has a bilateral symmetric.*

Proof. Let $x \in H$. Because of reproductive axiom $x \in xH$. Therefore, there exists $e \in H$ such that $xe = x$. Next, let y be an arbitrary element in H. Per reproductive axiom there exists $z \in H$ such that $y = zx$. Consequently, $ye = (zx)e = z(xe) = zx = y$. Hence e is a right neutral element. In an analogous way, there exists a left neutral element e'. Then, the equality $e = e'e = e'$ is valid. Therefore, e is the bilateral neutral element of H. In addition, because of reproductive axiom $e \in xH$. Thus, there exists $x' \in H$, such that $e = xx'$. Hence, any element in H has a right symmetric. Similarly, any element in H has a left symmetric and it is easy to prove that these two symmetric elements coincide. ∎

Remark. An analogous proposition to Proposition 1.1 is not valid when H is endowed with a hypercomposition. In hypergroups there exist different types of neutral elements [34], [53] (e.g., scalar [4], [45], strong [17], [30], [41] etc.). There also exist special types of hypergroups which have a neutral element and each one of their elements has one symmetric element (e.g., canonical hypergroups [45], quasicanonical hypergroups [27], fortified join hypergroups [41], fortified transposition hypergroups [17]) or more symmetric elements (e.g., transposition polysymmetrical hypergroups [30], canonical polysymmetric hypergroups [48], M-polysymmetric hypergroups [33]).

Both equations $a = xb$ and $a = bx$ have a unique solution in groups. On the contrary, in the case of hypergroups, the analogous relations $a \in xb$ and $a \in bx$ do not have unique solutions. Thus, F. Marty in [19] defined the two induced hypercompositions (right and left division) that derive from the hypercomposition of the hypergroup:

$$\frac{a}{|b} = \{x \in H \mid a \in xb\} \qquad \text{and} \qquad \frac{a}{b|} = \{x \in H \mid a \in bx\}.$$

If H is a group, then $\frac{a}{|b} = ab^{-1}$ and $\frac{a}{b|} = b^{-1}a$. It is obvious that if "." is commutative, then the right and the left division coincide. For the sake of notational simplicity, a/b or $a : b$ is used to denote the right division (or right hyperfraction) as well as the division in commutative hypergroups and $b \setminus a$ or $a..b$ is used to denote the left division (or left hyperfraction) [16], [22], [25].

Consequences of axioms (i) and (ii) are [22], [25]:

(i) $ab \neq \emptyset$, for all a, b in H,

(ii) $a/b \neq \emptyset$ and $a \setminus b \neq \emptyset$, for all a, b in H,

(iii) $H = H/a = a/H$ and $H = a \setminus H = H \setminus a$, for all a in H.

Proposition 1.2. [16], [22], [25] *In any hypergroup*

(i) $(a/b)/c = a/(cb)$ *and* $c \setminus (b \setminus a) = (bc) \setminus a$ *(mixed associativity),*

(ii) $(b \setminus a)/c = b \setminus (a/c)$,

(iii) $b \in (a/b) \setminus a$ *and* $b \in a/(b \setminus a)$.

A hypercomposition in a non-void set H is called *closed* if the two participating elements are always included in the result, i.e., if $a, b \in ab$ for all $a, b \in H$. For example, if H is a non-void set and $ab = \{a, b\}$ for all $a, b \in H$ or, if $(H, \cdot)$ is a semigroup and $ab = \{a, b, a \cdot b\}$ for all $a, b \in H$, then these are closed hypercompositions. A hypercomposition is called *right closed* if $a \in ba$ for all $a, b \in H$ and left closed if $a \in ab$ for all $a, b \in H$. A hypercomposition is called *right open* if $a \notin ba$ for all $a, b \in H$ with $b \neq a$. The definition of the *left open* hypercomposition is similar. Obviously, a hypercomposition is *open*, if it is both right and left open.

Proposition 1.3. *The hypercomposition in a hypergroup H is right closed if and only if $a/a = H$ for all $a \in H$, while it is left closed if and only if $a \setminus a = H$ for all $a \in H$.*

Proof. Suppose that the hypercomposition is right closed. Then $a \in xa$ for all $x \in H$. Hence $x \in a/a$ for all $x \in H$. Therefore, $H = a/a$. Conversely now. Let $H = a/a$ for all $a \in H$. Then $a \in ba$ for all $a, b \in H$. Thus the hypercomposition is right closed. ∎

Proposition 1.4. *The hypercomposition in a hypergroup H is right open if and only if $a/a = a$ for all $a \in H$, while it is left open if and only if $a \setminus a = a$ for all $a \in H$.*

Proof. Suppose that the hypercomposition is right open. Let a be an arbitrary element of H. Then $a \notin ba$ for all $b \in H$ with $b \neq a$. Hence $b \notin a/a$ for all $b \in H$ with $b \neq a$. Moreover, because of the reproductive axiom, $a \in Ha$, thus $a \in aa$. Therefore, $a = a/a$. Conversely now. Let $a/a = a$ for all $a \in H$. Then $b \notin a/a$ for all $b \in H$ with $b \neq a$. So $a \notin ba$, for all $b \in H$ with $b \neq a$, i.e., the hypercomposition is right open. ∎

Proposition 1.5. *If the hypercomposition in a hypergroup H is right or left open, then all its elements are idempotent.*

Proof. Suppose that the hypercomposition is right open and that for some $a \in H$ there exists $b \neq a$, such that $b \in aa$. Then, $a/b \subseteq a/aa$. Because of Propositions 1.2(i) and 1.4, $a/(aa) = (a/a)/a = a/a = a$. Thus, $a/b = a$. Therefore, $a \in ab$, which contradicts the assumption. Hence, $aa = a$ for all $a \in H$. ∎

A non-empty subset K of H is called *semi-subhypergroup* when it is stable under the hypercomposition, i.e., it has the property $xy \subseteq K$ for all $x, y \in K$.

Proposition 1.6. *If A, B are semi-subhypergroups of a commutative hypergroup H, then AB is a semi-subhypergroup of H as well.*

K is a subhypergroup of H, if it satisfies the axiom of reproduction, i.e. if the equality $xK = Kx = K$ is valid for all $x \in K$. This means that when K is a subhypergroup, the relations $a \in bx$ and $a \in yb$ can always be solved in K. The non-void intersection of two subhypergroups, although stable under the hypercomposition, usually is not a subhypergroup, since the reproduction is not always valid. In other words the solutions of the relation $a \in yb$ and $a \in bx$ do not lie in the intersection when a and b are elements of the intersection. This led (from the very early steps of hypergroup theory) to the consideration of more special types of subhypergroups. One of them is the *closed subhypergroup*. A subhypergroup K of H is called *left closed* with respect to H, if for any two elements a and b in K all possible solutions of the relation $a \in yb$ lie in K. This means that K is left closed if and only if $a/b \subseteq K$, for all $a, b \in K$. Similarly, K is right closed when all possible solutions of the relation $a \in bx$ lie in K or, equivalently, if $b \setminus a \subseteq K$ for all $a, b \in K$ [24], [25], [37]. Finally, K is *closed* when it is both right and left closed. The non-void intersection of two closed subhypergroups is a closed subhypergroup.

It has been proven ([24], [25]) that the set of the semi-subhypergroups (resp., the set of the closed subhypergroups) which contain a non-void subset E is a complete lattice. Hence, given a non-empty subset E of a hypergroup H, the minimum semi-subhypergroup (in the sense of inclusion) which contains E can be assigned. This semi-subhypergroup is denoted by $[E]$ and it is called the generated by E semi-subhypergroup of H. Similarly, $\langle E \rangle$ is the generated by E closed subhypergroup of H. For notational simplicity, if $E = \{a_1, ..., a_n\}$, $[E] = [a_1, ..., a_n]$ and $\langle E \rangle = \langle a_1, ..., a_n \rangle$ are used instead.

F. Marty's life was short, as he died in a military mission during World War II and [19], [20], [21] are the only works on hypergroups he left behind. However, several papers by other authors began to appear shortly thereafter and until now hundreds of papers have been written on this issue (e.g. see [4], [9]). Moreover since the hypergroup is a very general structure, it was progressively enriched with further axioms, more or less powerful, thus leading to a significant number of special hypergroups – e.g., [4], [9], [11], [16], [17], [18], [28], [29], [30], [33], [41], [45], [47], [52]. Thus, W. Prenowitz enriched hypergroups with an axiom, in order to use them in the study of geometry. More precisely, he introduced into the commutative hypegroup, the *transposition axiom*:

$$a/b \cap c/d \neq \emptyset \text{ implies } ad \cap bc \neq \emptyset \text{ for all } a, b, c, d \in H$$

and named this new hypergroup *join space* [54], [55], [56], [57], [58], [59]. W. Prenowitz utilized this structure in the study of Geometry. Prenowitz was followed by others, such as J. Jantosciak [15], [58],V.W. Bryant, R.J. Webster [2], D. Freni [12], [13] etc. Material from the above mentioned authors, as well as from previous work of the author of this paper, is used in this study in order to make it self-contained.

At this point, it is worth mentioning that a big number of researchers dealt with the further study of the certain hypergroup which W. Prenowitz introduced (see, e.g., [1], [3], [5], [6], [7], [8], [10], [14], [28], [29], [30], [35], [65]).

It is also worth mentioning that the generalization of the vector spaces, which are associated directly with the algebraic study of geometry, attracted the interest of many researchers. So, J. Mittas [46], [50] and M. Scafati-Tallini [60]-[64] presented their approach to the generalization of the vector spaces in the hypercompositional algebra.

Later on, J. Jantosciak generalized the transposition axiom in an arbitrary hypergroup as follows:

$$b \setminus a \cap c/d \neq \emptyset \text{ implies } ad \cap bc \neq \emptyset \text{ for all } a, b, c, d \in H.$$

He named this particular hypergroup *transposition hypergroup* [16]. For the sake of terminology unification, join spaces are also called *join hypergroups.* It has been proven that these hypergroups also comprise a useful tool in the study of languages and automata [31], [38], [40], [43] and a constructive origin for the development of other, new hypercompositional structures [32], [39], [42], [44], [50], [51].

Proposition 1.7. [24], [29] *The following are true in any join hypergroup:*

(i) $a(b/c) \cup b(a/c) \cup a/(c/b) \cup b/(c/a) \subseteq ab/c$,

(ii) $(a/b)(c/d) \cup (a/d)(c/b) \cup (a/b)/(d/c) \cup (a/d)/(b/c) \cup (c/d)/(b/a) \cup (c/b)/(d/a) \subseteq ac/bd$.

Corollary 1.1. *If A, B are semi-subhypergroups of a join hypergroup H, then A/B is a semi-subhypergroup of H.*

Proposition 1.8. *Let V be a vector space over an ordered field F. Then V, when endowed with the hypercomposition*

$$ab = \{\kappa a + \lambda b \mid \kappa, \lambda > 0,\ \kappa + \lambda = 1\},$$

becomes a join hypergroup (join space).

This hypergroup, which was derived from the vector space and is connected with it, was named *attached hypergroup* of V [24], [25]. Observe that the hypercomposition of the attached hypergroup is an open hypercomposition. In [49], one can find some other hypergroups annexed to vector spaces and in [23], [26] more hypecompositional structures connected to vector spaces. A direct consequence of the above proposition is that the convex sets of V are the semi-subhypergroups of the attached hypergroup H_V, while the subspaces of V are the closed subhypergroups of this hypergroup [24], [25].

The following two theorems result in two known propositions of vector spaces, thus showing the importance of the connection of vector spaces with hypergroups,which is achieved through the attached hypergroup.

Theorem 1.1. *Let A, B be two disjoint semi-subhypergroups in a join hypergroup and let x be an idempotent element not in the union $A \cup B$. Then $[A \cup \{x\}] \cap B = \emptyset$ or $[B \cup \{x\}] \cap A = \emptyset$.*

Proof. Suppose that $[A \cup \{x\}] \cap B \neq \emptyset$ and $[B \cup \{x\}] \cap A \neq \emptyset$. Since x is idempotent, the equalities $[A \cup \{x\}] = Ax$ and $[B \cup \{x\}] = Bx$ are valid. Thus, there exists $a \in A$ and $b \in B$, such that $ax \cap B \neq \emptyset$ and $bx \cap A \neq \emptyset$. Hence,

$x \in B/a$ and $x \in A/b$. Thus, $B/a \cap A/b \neq \emptyset$. Next, by application of the transposition axiom, we arrive at $Bb \cap Aa \neq \emptyset$. However, $Bb \subseteq B$ and $Aa \subseteq A$, since A, B are semi-subhypergroups. Therefore, $A \cap B \neq \emptyset$, which contradicts the theorem's assumption. ∎

Corollary 1.2. *Let H be a join hypergroup endowed with an open hypercomposition. If A, B are two disjoint semi-subhypergroups of H and x is an element not in the union $A \cup B$, then $[A \cup \{x\}] \cap B = \emptyset$ or $[B \cup \{x\}] \cap A = \emptyset$.*

Corollary 1.3. (Kakutani's Lemma) *If A, B are disjoint convex sets in a vector space and x is a point not in their union, then either the convex envelope of $A \cup \{x\}$ and B or the convex envelope of $B \cup \{x\}$ and A are disjoint.*

Theorem 1.2. *Let H be a join hypergroup consisting of idempotent elements and suppose that A, B are two disjoint semi-subhypergroups in H. Then, there exist disjoint semi-subhypergroups M, N such that $A \subseteq M$, $B \subseteq N$ and $H = M \cup N$.*

Proof. Suppose that M and N are the maximum disjoint semi-subhypergroups such that $A \subseteq M$, $B \subseteq N$. If we assume that $M \cup N \subset H$, then there exists an element w in H, which does not belong to the union $M \cup N$. Therefore, per Theorem 1.1, either $[M \cup \{w\}] \cap N = \emptyset$ or $[N \cup \{w\}] \cap M = \emptyset$ is valid. This contradicts the hypothesis that M and N are the maximum disjoint semi-subhypergroups with the required property. Hence $H = M \cup N$. ∎

Corollary 1.4. (Stone's Theorem) *If A, B are disjoint convex sets in a vector space V, there exist disjoint convex sets M and N, such that $A \subseteq M$, $B \subseteq N$ and $V = M \cup N$.*

2. Closed subhypergroups and Helly's theorem

As mentioned above, every vector subspace of a vector space V, considered as a subset of the attached hypergroup of V, is a closed subhypergroup of this hypergroup. Therefore, properties of vector subspaces can derive as corollaries of more general properties that are valid in closed subhypergroups. An interesting issue is the construction of closed subhypergroups from a finite set of elements.

Proposition 2.1. *Let H be a commutative hypergroup and $\{a_1, ..., a_n\} \subseteq H$. Then,*

$$[a_1, a_2, ..., a_n] = ([a_1] \cup [a_2] \cup \cdots \cup [a_n]) \cup ([a_1][a_2] \cup \cdots \cup [a_{n-1}][a_n]) \cup \cdots \cup ([a_1] \cdots [a_n]).$$

Proof. It is obvious that the right part of the above equality is a subset of the left part. Inversely, suppose that $x \in [a_{i_1}] \cdots [a_{i_m}]$ and $y \in [a_{j_1}] \cdots [a_{j_n}]$. Then, $xy \subseteq [a_{i_1}] \cdots [a_{i_m}][a_{j_1}] \cdots [a_{j_n}]$ and, through rearrangement of the indices, $xy \subseteq [a_{k_1}] \cdots [a_{k_r}]$. ∎

Proposition 2.2. *Let H be a hypergroup and $a \in H$. Then,* $[a] = a^1 \cup a^2 \cup \cdots \cup a^k \cup \cdots$, *where* $a^1 = \{a\}$, $a^2 = aa$ *and* $a^i = aa^{i-1}$.

Proposition 2.3. *If the hypercomposition in a hypergroup H is right (resp. left) open, then* $a/[a] = a$ (*resp.* $[a] \setminus a = a$).

Proof. Because of mixed associativity and per Proposition 1.4, the equality $a/aa = (a/a)/a = a/a = a$ is valid. The rest follow throw induction. ∎

Proposition 2.4. [25] *In every commutative hypergroup H, the set* $\prod_{i=1}^{n}[a_i]$ *is a semi-subhypergroup of H, which absorbs every element of* $[a_1, ..., a_n]$.

An extensive presentation of properties of semi-subhypergroups of commutative hypergroups can be found in [25].

Definition 2.1. In a hypergroup H the elements $a_1, ..., a_n$ are called *correlated*, if there exist distinct integers $i_1, ..., i_k, j_1, ..., j_n$ that belong to $\{1, ..., n\}$, such that $[a_{i_1}, ..., a_{i_k}] \cap [a_{j_1}, ..., a_{j_m}] \neq \emptyset$. Otherwise, $a_1, a_2, ..., a_n$ are called *non-correlated*.

In a hypergroup endowed with an open hypercomposition, elements $a_1, ..., a_n$ are correlated, if there exist distinct integers $i_1, ..., i_k, j_1, ..., j_m \in \{1, ..., n\}$, such that $a_{i_1} \cdots a_{i_k} \cap a_{j_1} \cdots a_{j_m} \neq \emptyset$. As proven in [24], [25], in the case of the attached hypergroup H_V of a vector space V, a subset of H_V consists of correlated elements if and only if these elements are affinely dependent in V.

Proposition 2.5. *Let A be a semi-subhypergroup of a join hypergroup H. Then, A/A is a closed subhypergroup of H containing A.*

Proof. Let x, y be arbitrary elements in A/A. Then, there exist $a, b, c, d \in A$ such that $x \in a/b$ and $y \in c/d$. Per Proposition 1.7(ii):

$$xy \subseteq (a/b)(c/d) \subseteq ac/bd \subseteq A/A$$

and

$$x/y \subseteq (a/b)/(c/d) \subseteq ad/bc \subseteq A/A.$$

Hence, A/A is stable both under the hypercomposition and the induced hypercomposition. Next, $xA \subseteq A$ is valid for all $x \in A$. Hence $x \in A/A$ for all $x \in A$. Therefore, $A \subseteq A/A$. ∎

Proposition 2.6. *Let H be a join hypergroup and let $\{a_1, ..., a_n\} \subseteq H$. Then,*

$$\langle a_1, a_2, ..., a_n\rangle = [a_1] \cdots [a_n]/[a_1] \cdots [a_n].$$

Proof. Because of Proposition 2.4, $\prod_{i=1}^{n}[a_i] = [a_1]\cdots[a_n]$ is a semi-subhypergroup of H. Therefore, because of Proposition 2.5, $\prod_{i=1}^{n}[a_i]/\prod_{i=1}^{n}[a_i]$ is a closed subhypergroup of H. Since $[a_i]$ is a semi-subhypergroup, the inclusion $a_i[a_i] \subseteq [a_i]$ is valid. Hence, $a_i[a_1]\cdots[a_n] \subseteq [a_1]\cdots[a_n]$. Therefore, $a_i \in [a_1]\cdots[a_n]/[a_1]\cdots[a_n]$, $1 \le i \le n$. ∎

Corollary 2.1. *If H is a join hypergroup endowed with open hypercomposition and $\{a_1, ..., a_n\} \subseteq H$, then $\langle a_1, a_2, ..., a_n\rangle = a_1\cdots a_n/a_1\cdots a_n$.*

Theorem 2.1. *Suppose that elements $a_1, ..., a_n$ of a hypergroup H are correlated. Consider all the semi-subhypergroups of H generated from $n-1$ elements of the above. Then, the intersection of all these semi-subhypergroups is non-void.*

Proof. Since the elements are correlated, there are distinct integers $i_1, ..., i_r, j_1, ..., j_s \in \{1, ..., n\}$ such that $[a_{i_1}, ..., a_{i_r}] \cap [a_{j_1}, ..., a_{j_s}] \neq \emptyset$. But $[a_{i_1}, ..., a_{i_r}]$ or $[a_{j_1}, ..., a_{j_s}]$ is contained in any semi-subhypergroup which is generated by $n-1$ elements from $a_1, ..., a_n$. Thus, the intersection of all these semi-subhypergroups contains the elements of $[a_{i_1}, ..., a_{i_r}] \cap [a_{j_1}, ..., a_{j_s}]$ and, therefore, is non-void. ∎

Theorem 2.2. *Suppose that H is a hypergroup in which every set of cardinality greater than n consists of correlated elements. If $(K_i)_{i\in I}$, $\operatorname{card} I > n$, is a finite family of semi-subhypergroups of H, in which the intersection of every n members is non-void, then all the semi-subhypergroups $(K_i)_{i\in I}$ have a non-void intersection.*

Proof. The theorem will be proven by induction. First, it will be shown that the intersection of every $n+1$ semi-subhypergroups is non-void. Without loss of generality, this will be proven for semi-subhypergroups K_i, $1 \le i \le n+1$. Thus, let $x_i \in \bigcap_{j\neq i} K_j$. Then, $x_1, ..., x_{i-1}, x_{i+1}, ..., x_{n+1} \in K_i$. Therefore, $[x_1, ..., x_{i-1}, x_{i+1}, ..., x_{n+1}] \subseteq K_i$. Since every $n+1$ elements of H are correlated, the elements $x_1, ..., x_{n+1}$ are correlated. Because of Theorem 2.1, the semi-subhypergroups $[x_1, ..., x_{i-1}, x_{i+1}, ..., x_{n+1}]$, $1 \le i \le n+1$, have a non-void intersection. Consequently, the sets K_i, $1 \le i \le n+1$, also have a non-void intersection. Next, suppose that the intersection of the members of each set of $(\operatorname{card} I) - 1$ semi-subhypergroups is non-void. For each $i \in I$, we choose an element x_i of the intersection $\bigcap_{j\neq 1} K_j$. Then, sets $X_i = \{x_j,\ j \in I - \{i\}\} \subseteq K_i$ are constructed. These sets generate the semi-subhypergroups $[X_i]$, $i \in I$. Since every $n+1$ elements of H are correlated, elements $\{x_i,\ i \in I\}$ are correlated and, because of Theorem 2.1, semi-subhypergroups $[X_i]$, $i \in I$, have a non-void intersection. Consequently, semi-subhypergroups K_i, $i \in I$, have a non-void intersection. ∎

In the case of the attached hypergroup of a vector space V, a subset of H_V consists of correlated elements if and only if these elements are affinely dependent [24], [25]. Therefore, we have the corollary:

Corollary 2.2. (Helly's Theorem) *Let us consider a finite family* $(C_i)_{i\in I}$ *of convex sets in* $\mathbb{R}^d$*, with* $d+1 < \operatorname{card} I$*. Then, if any* $d+1$ *of the sets* C_i *have a non-empty intersection, all the sets* C_i *have a non-empty intersection.*

3. Dimension theory in hypergroups and Randon's theorem

During his study on join spaces, Prenowitz introduced a new axiom, which he named "exchange postulate":

$$\text{if } c \in \langle a, b\rangle \text{ and } c \neq a, \text{ then } \langle a, b\rangle = \langle a, c\rangle\,.$$

Consequently, join spaces that satisfy this axiom were named "exchange spaces" [55], [57], [59]. The above axiom enabled Prenowitz to develop a theory of linear independence and dimension of a type familiar to classical geometry. On the other hand, a generalization of this theory has been achieved by Freni, who developed the notions of independence, dimension, etc. in a hypergroup H that satisfies only the axiom:

$$x \in \langle A \cup \{y\}\rangle\,,\ x \notin \langle A\rangle \Longrightarrow y \in \langle A \cup \{x\}\rangle\,, \text{ for every } x, y \in H \text{ and } A \subseteq H.$$

Freni called these hypergroups *cambiste* [4], [12], [13].

A subset B of a hypergroup H is called free or independent if either $B = \emptyset$, or $x \notin \langle B - \{x\}\rangle$ for all $x \in B$, otherwise it is called non-free or dependent. B generates H, if $\langle B\rangle = H$, in which case B is a set of generators of H. If H has a finite set of generators, it is called a finite type hypergroup. A free set of generators is a basis of H. Among the results reached by Freni are:

Proposition 3.1. *Let* B *be a non-empty subset of a cambiste hypergroup* H*.* B *is a basis of* H *if and only if:*

(i) B *is a maximal free set, and*

(ii) B *is a minimal set of generators of* H*.*

Proposition 3.2. *Every cambiste hypergroup has at least one basis.*

Proposition 3.3. *All the bases of a cambiste hypergroup have the same cardinality.*

The *dimension* of a cambiste hypergroup H (denoted by $\dim H$) is the cardinality of any basis of H.

The dimension theory gives very interesting results in convexity hypergroups. A *convexity hypergroup* is a join hypergroup which satisfies the axioms:

(i) the hypercomposition is open,

(ii) $ab \cap ac \neq \emptyset$ implies $b = c$ or $b \in ac$ or $c \in ab$.

Prenowitz, defined this hyperstructure with equivalent axioms to the above, named it *convexity space* and used it, as did Bryant and Webster [2], for generalizing some of the theory of linear spaces. A direct consequence of Propositions 1.4 and 1.5 is the following propositions:

Proposition 3.4. *All the elements of a convexity hypergroup H are idempotent and, moreover, $a/a = a$ for all $a \in H$.*

Proposition 3.5. *The following are true in any convexity hypergroup:*

(i) $ab/ac = (b/c) \cup (ab/c) \cup (b/ac)$,

(ii) $ab/a = \{b\} \cup ab \cup (b/a)$,

(iii) $a/ab = a/b$.

Proof. (i) Let $x \in ab/ac$. Then, $ab \cap a(xc) \neq \emptyset$. Hence, $b \in xc$ or $b \in a(xc) = x(ac)$ or $xc \cap ab \neq \emptyset$. Therefore, $x \in b/c$ or $x \in b/ac$ or $x \in ab/c$. Thus, $ab/ac \subseteq (b/c) \cup (ab/c) \cup (b/ac)$. Next, for the opposite inclusion, suppose that:

(a) $x \in b/c$, then $b \in cx \Longrightarrow ab \in acx \Longrightarrow x \in ab/ac$.

(b) $x \in ab/c$, then $xc \cap ab \neq \emptyset$. Since the hypercomposition is open, $aa = a$ is valid. Therefore, $xac \cap ab \neq \emptyset$. Hence, $x \in ab/ac$.

(c) $x \in b/ac$, then $b \in acx \Longrightarrow ab \cap acx \neq \emptyset$. Hence, $x \in ab/ac$.

From (a), (b) and (c), the desired result follows. Therefore, (i) is valid.

(ii) According to Proposition 1.7(i), we have

$$a(b/a) \cup b(a/a) \cup a/(a/b) \cup b/(a/a) \subseteq ab/a.$$

Since the hypercomposition is open, $a/a = a$ is valid. Also, according to Proposition 1.2(iii), $b \in a/(a/b)$ is valid. From the above inclusion it follows that $ba \cup \{b\} \cup b/a \subseteq ab/a$. The opposite inclusion is easily proven and, therefore, (ii) follows.

(iii) is a direct consequence of mixed associativity.

Corollary 3.1. *If a, b with $a \neq b$ are two elements of a convexity hypergroup H, then $ab/ab = ab \cup a/b \cup b/a \cup \{a, b\}$.*

Remark. The above proposition supplies us with a simplification canon (rule) for hyperfractions in convexity hypergroups.

So, we are naturally led to the following definition:

Definition 3.1. A hyperfraction whose numerator and denominator consist of hyperproducts in which a common factor does not exist, will hereafter be called *irreducible hyperfraction.*

Proposition 3.6. *If K is a closed subhypergroup of a convexity hypergroup H, then* $\langle \{x\} \cup K \rangle = K \cup xK/K \cup K/x$.

Proof. Since x is idempotent, according to Proposition 1.6, xK is a semi-subhypergroup and, therefore, according to Proposition 2.5, xK/xK is a closed subhypergroup. Next, suppose that $t \in K$. Equality $xt = xt$, combined with Proposition 3.4, gives: $xxt = xt \Longrightarrow x \in xt/xt$ and $xtt = xt \Longrightarrow t \in xt/xt$. Hence, $\{x\} \cup K \subseteq xK/xK$. Proposition 3.5 is employed to conclude that $xK/xK = K/K \cup xK/K \cup K/xK$. Since K is closed, $K/K = K$ is valid and mixed associativity gives $K/xK = (K/K)/x = K/x$. Therefore, the proposition is established.

Proposition 3.7. *If $a_1, ..., a_n$ are elements of a convexity hypergroup, then the closed subhypergroup generated by these elements is the union of hyperproducts of the form $a_{i_1}, ..., a_{i_s}$ (where $1 \le i_j \le n$) and irreducible hyperfractions of the form $a_{i_1} \cdots a_{i_k}/a_{i_{k+1}} \cdots a_{i_r}$, $1 \le k < r \le n$.*

Proof. According to Corollary 2.1, $\langle a_1, a_2, ..., a_n \rangle = a_1 a_2 \cdots a_n / a_1 a_2 \cdots a_n$. Next, the previous proposition applies and yields the equality:

$$\begin{aligned} &a_1 a_2 \cdots a_n / a_1 a_2 \cdots a_n \\ &\qquad = a_2 \cdots a_n / a_2 \cdots a_n \cup a_1 a_2 \cdots a_n / a_2 \cdots a_n \cup a_2 \cdots a_n / a_1 a_2 \cdots a_n. \end{aligned}$$

The previous proposition applies again repeatedly to rewrite the sets of the right-hand side of the above equality as union of hyperproducts and irreducible hyperfractions and so the proposition is established. ■

Example. If a, b with $a \ne b$ are two elements in a convexity hypergroup H, then $[a, b] = ab$. Therefore, $\langle a, b \rangle = ab/ab$. Hence, according to Corollary 3.1, $\langle a, b \rangle = ab \cup a/b \cup b/a \cup \{a, b\}$.

Proposition 3.8. *Every convexity hypergroup is a cambiste hypergroup.*

Proof. Suppose that A is a subset of a convexity hypergroup H and that x, y are elements of H such that $x \in \langle A \cup \{y\} \rangle$, $x \notin \langle A \rangle$. Then, the previous proposition applies, yielding $\langle \{y\} \cup A \rangle = A \cup yA/A \cup A/y$. Thus, either $x \in yA/A$ or $x \in A/y$. Hence, $y \in xA/A$ or $y \in A/x$. Therefore, $y \in \langle A \cup \{x\} \rangle$.

Theorem 3.1. *Every $n+1$ elements of an n-dimensional convexity hypergroup H are correlated.*

Proof. Let $A = \{a_1, ..., a_n, a_{n+1}\}$ be a subset of $n+1$ elements of H. Without loss of generality, suppose that $a_{n+1} \in \langle a_1, ..., a_n \rangle$. Then, according to Proposition 3.7, either $a_{n+1} \in a_{i_1} \cdots a_{i_s}$ or $a_{n+1} \in a_{i_1} \cdots a_{i_k}/a_{i_{k+1}} \cdots a_{i_r}$, where $1 \le i_j \le n$, $1 \le k < r \le n$ and $i_j \ne i_\ell$, when $j \ne \ell$. In the first case, $[a_{n+1}] \cap [a_{i_1}, ..., a_{i_s}] \ne \emptyset$ and in the second case, $[a_{n+1}, a_{i_{k+1}}, ..., a_{i_r}] \cap [a_{i_1}, ..., a_{i_k}] \ne \emptyset$. ■

One can easily see that the attached hypergroup of a vector space is a convexity hypergroup and, moreover, if the dimension of the attached hypergroup H_V of a vector space V is n, then the dimension of V is $n-1$.

Corollary 3.2. (Radon's Theorem). *Any set of $d+2$ points in R^d can be partitioned into two disjoint subsets, whose convex hulls intersect.*

4. Carathéodory-type theorems

Proposition 4.1. *Suppose that an element x of a convexity hypergroup belongs both to a hyperproduct $a_1 \cdots a_n$ and to an irreducible hyperfraction $a_{i_1} \cdots a_{i_k}/a_{i_{k+1}} \cdots a_{i_r}$, where $\{a_{i_1}, ..., a_{i_k}\}$ and $\{a_{i_{k+1}}, ..., a_{i_r}\}$ are non-empty subsets of $\{a_1, ..., a_n\}$. Then, x belongs to a hyperproduct with factors from a proper subset of $\{a_1, ..., a_n\}$.*

Proof. The proposition will be proven by induction on the denominator of the hyperfraction. Without loss of generality, suppose that $x \in a_{i_1} \cdots a_{i_k}/a_1$. Then, $a_1 \in a_{i_1} \cdots a_{i_k}/x$. Moreover, $x \in a_1 \cdots a_n$, hence, $a_1 \in x/a_2 \cdots a_n$. The above, in combination with the transposition axiom, lead to $x \in a_2 \cdots a_n$. Thus, the proposition is true, if the denominator of the hyperfraction consists of one element. Next, assume that the proposition holds true if the hyperproduct of the denominator has m factors. Without loss of generality, suppose that $x \in a_{i_1} \cdots a_{i_k}/a_1 \cdots a_m a_{m+1}$. Then, $a_1 \in a_{i_1} \cdots a_{i_k}/x a_2 \cdots a_m a_{m+1}$. Moreover, $a_1 \in x/a_2 \cdots a_n$. Therefore, the transposition axiom implies that $x a_2 \cdots a_m a_{m+1} \cap a_2 \cdots a_n \neq \emptyset$. Thus, $x \in a_2 \cdots a_n/a_2 \cdots a_m a_{m+1}$. Proposition 3.7 applies, yielding either $x \in a_{i_1} \cdots a_{i_s}$, where $\{a_{i_1}, ..., a_{i_s}\} \subset \{a_1, ..., a_n\}$, and so x is written in the desired form, or $x \in a_{i_1} \cdots a_{i_k}/a_{i_{k+1}} \cdots a_{i_r}$, where $\{a_{i_1}, ..., a_{i_k}\}$ and $\{a_{i_{k+1}}, ..., a_{i_r}\}$ are non-empty disjoint subsets of $\{a_1, ..., a_n\}$. In this latter case the number of factors of the denominator is less than $m+1$ and the induction hypothesis implies the result. ∎

Theorem 4.1. *If an element x of an n-dimensional convexity hypergroup H belongs to a hyperproduct of $n+1$ elements, then there exists a proper subset of these elements which contains x in their hyperproduct.*

Proof. Suppose that $a_1, ..., a_n, a_{n+1}$ are $n+1$ elements of H, such that $x \in a_1 \cdots a_n a_{n+1}$. Without loss of generality, suppose that $a_{n+1} \in \langle a_1, ..., a_n \rangle$. Then, according to Proposition 3.7, either $a_{n+1} \in a_{i_1} \cdots a_{i_s}$, or $a_{n+1} \in a_{i_1} \cdots a_{i_k}/a_{i_{k+1}} \cdots a_{i_r}$, where $1 \leq i_j \leq n$, $1 \leq k < r \leq n$ and $i_j \neq i_\ell$, when $j \neq \ell$. In the first case, $x \in a_1 \cdots a_n$. In the second case, Proposition 1.7(i) applies, yielding

$$x \in a_1 \cdots a_n a_{n+1} \Longrightarrow x \in a_1 \cdots a_n (a_{i_1} \cdots a_{i_k}/a_{i_{k+1}} \cdots a_{i_r}) \subseteq a_1 \cdots a_n/a_{i_{k+1}} \cdots a_{i_r}.$$

Hence, according to Proposition 3.7, either $x \in a_{i_1} \cdots i_{i_s}$ or $x \in a_{i_1} \cdots a_{i_k}/a_{i_{k+1}} \cdots a_{i_r}$, where $1 \leq i_j \leq n$, $1 \leq k < r \leq n$ and $i_j \neq i_\ell$, if $j \neq \ell$. In the former case, the theorem is proven. In the latter case, the theorem results by using Proposition 4.1. ∎

Corollary 4.1. (Carathéodory's Theorem) *Any convex combination of points in R^d is a convex combination of at most $d+1$ of them.*

Corollary 4.2. *Let S and T be two finite sets of elements in an n-dimensional convexity hypergroup H. If any semi-subhypergroup generated by $k+1$, $k \leq n$ elements of S is disjoint to any semi-subhypergroup generated by $\ell+1$, $\ell \leq n$ elements of T, then $[S] \cap [T] = \emptyset$.*

Proof. Suppose that $[S] \cap [T] \neq \emptyset$ and let $x \in [S] \cap [T]$. Proposition 2.1 yields $x \in s_1 \cdots s_i \cap t_1 \cdots t_j$, where $\{s_1, ..., s_i\} \subseteq S$ and $\{t_1, ..., t_j\} \subseteq T$. Then, per Theorem 4.1, there exists proper subsets of $\{s_1, ..., s_i\}$ and $\{t_1, ..., t_j\}$ not exceeding n elements, which contains x in their hyperproduct, i.e. $x \in s_1 \cdots s_p \cap t_1 \cdots t_q$, $p, q \leq n$. The contradiction obtained proves the validity of the corollary. ∎

Proposition 4.2. *Suppose that an element x of a convexity hypergroup belongs both to a hyperproduct $a_1 \cdots a_n$ and to an irreducible hyperfraction $ya_{i_1} \cdots a_{i_k}/a_{i_{k+1}} \cdots a_{i_r}$, where $\{a_{i_1}, ..., a_{i_k}\}$ and $\{a_{i_{k+1}}, ..., a_{i_r}\}$ are non-empty subsets of $\{a_1, ..., a_n\}$. Then, there is a hyperproduct containing x with factors from both y and a proper subset of $\{a_1, ..., a_n\}$.*

The proof of the above Proposition is similar to that of Proposition 4.1. Next, using techniques analogous to those used in proving Theorem 4.1, we are led to the following theorem:

Theorem 4.2. *In an n-dimensional convexity hypergroup H, if $A=\{a_1, ..., a_n, a_{n+1}\}$, $x \in a_1 \cdots a_n a_{n+1}$ and $y \in [a_1, ..., a_n, a_{n+1}]$, then there exists a subset B of A containing at most $n-1$ elements of A, such that x belongs to the hyperproduct of y by the elements of B.*

This theorem essentially asserts that one of the n factors of the hyperproduct of Theorem 4.1 may be chosen arbitrarily from the semi-subhypergroup which is generated by the $n+1$ elements, i.e. it can be any element of $[A]$. When this theorem is applied to the attached hypergroup of a vector space, it produces an obvious generalization of Carathéodory's Theorem. Moreover, from the above theorem follows the next theorem which is an extension of Carathéodory's Theorem.

Theorem 4.3. *In an n-dimensional convexity hypergroup H, if A is a subset of H, Y is a subset of $[A]$ and $\operatorname{card} Y \geq 2$, then there exists a subset B of A containing at most $(n-1)\operatorname{card} Y$ elements of A, such that $Y \subseteq [B]$.*

Proof. Let y be an arbitrary element of Y. For each $x \in Y$, let B_x be the subset of A, containing at most $n-1$ elements of A, such that x belongs to the hyperproduct of the fixed element y by the elements of B_x. Note that B_x exists because of Theorem 4.2. Consider the union $C = \bigcup_{x \in Y - \{y\}} B_x$. Then, $\operatorname{card} C \leq (n-1)(\operatorname{card} Y - 1)$ and $x \in [C \cup \{y\}]$ for each $x \neq y$. Next, consider an arbitrary element $b \in C$. According to the above theorem, there exists a subset B_y of A containing at most $n-1$ elements of A, such that y belongs to the hyperproduct of b by the elements of B_y. We define $B = C \cup B_y$. Then, $\operatorname{card} B \leq \operatorname{card} C + \operatorname{card} B_y \leq (n-1)\operatorname{card} Y$. ∎

Definition 4.1. An element a of a semi-subhypergroup S is called *interior element* of S if, for each $x \in S$, $x \neq a$, it exists $y \in S$, $y \neq a$, such that $a \in xy$.

Consequently to the above Definition 4.1, in the case of an n-dimensional cambiste hypergroup H, an element a of a semi-subhypergroup S of H, is interior element of S, if for every closed subhypergroup K, with $\dim K = n-1$ and $a \in K$, the intersections of S with the two disjoint classes K/x and K/y are non-void, i.e. $(K/x) \cap S \neq \emptyset$ and $(K/y) \cap S \neq \emptyset$.

Proposition 4.3. *Let H be a hypergroup endowed with an open hypercomposition and K a subhypergroup of H. Then any element of K is an interior element.*

Proposition 4.4. *Let H be a hypergroup endowed with an open hypercomposition, S a semi-subhypergroup of H and I the subset of the interior elements of S. Then I absorbs S, i.e. $IS \subseteq I$.*

Proof. Suppose that $a \in I$ and $b \in S$. Let r be an element of ab. In order to prove that r is an interior element, we have to show that for any $x \in S$ it exists $y \in S$ such that $r \in xy$. Since a is an interior element, there exists $z \in S$, such that $a \in xz$. Hence, $r \in ab \subseteq (xz)b = x(zb)$. But $zb \subseteq S$. So, there exists $y \in S$ such that $r \in xy$. ∎

Proposition 4.5. *Let H be a hypergroup endowed with an open hypercomposition, S a semi-subhypergroup of H and I the subset of the interior elements of S. Then I is a subhypergroup of H.*

Proof. Suppose that $a \in I$. Because of Proposition 4.4, $aI \subseteq I$. To prove the reverse inclusion, let $b \in I$. Since b is an interior element, there exists $z \in S$, such that $b \in az$. Per Proposition 1.5, $aa = a$, hence $az = (aa)z = a(az)$. Because of Proposition 4.4, $az \subseteq aS \subseteq I$. Thus, there exists $w \in I$, such that $b \in aw$. ∎

An almost direct consequence of Theorem 4.1 and Proposition 4.5 is the following proposition:

Proposition 4.6. *Let a be an interior element of a semi-subhypergroup S of an n-dimensional convexity hypergroup H. Then a is interior element of $[A]$, where A is a subset of S with* $\operatorname{card} A \leq (n+1)^2$.

This proposition states that any interior element of a semi-subhypergroup S of an n-dimensional convexity hypergroup is interior to a finitely generated semi-subhypergroup of S.

A refinement of this proposition is the following theorem:

Theorem 4.4. *Let a be an interior element of a semi-subhypergroup S of an n-dimensional convexity hypergroup H. Then a is interior element of a semi-subhypergroup of S, which is generated by at most $2n$ elements.*

Corollary 4.3. (Steinitz's Theorem) *Any point interior to the convex hull of a set E in R^d is interior to the convex hull of a subset of E, containing $2d$ points at the most.*

In [28], one can see that some of the above landmark theorems are also valid in other types of hypergroups.

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ON THE CONJUGATION INVARIANT PROBLEM IN THE MOD p DUAL STEENROD ALGEBRA

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Abstract. The Leibniz–Hopf algebra $\mathcal{F}$ is the free associative $\mathbf{Z}$-algebra on one generator in each positive degree, with coproduct given by the Cartan formula. Fix an odd prime p, and let $\mathcal{A}$ denote the Bockstein–free part of the mod p Steenrod algebra. We investigate an alternative approach to the conjugation invariant problem in the dual Steenrod algebra $\mathcal{A}^*$ using the conjugation invariants in $\mathcal{F}^* \otimes \mathbf{Z}/p$.

Keywords: antipode, Hopf algebra, Leibniz–Hopf algebra, Steenrod algebra, quasi-symmetric functions.

Mathematics Subject Classification: 55S10, 16T05, 57T05.

1. Introduction

Let p be a fixed odd prime and $F(p) = \mathbb{F}_p\{S^0, S^1, \ldots, S^i, \ldots,\}$ the free associative graded algebra over a field of characteristic p, $\mathbb{F}_p$ on generators S^i of degree i where S^0 is the unit. We may extend $F(p)$ to more rich algebraic structures. Particulary, omitting the above grading and setting $S^i = \mathcal{P}^i$, where $\mathcal{P}^i$, $i \geq 0$, represent the Steenrod reduced powers [19] of degree $2i(p-1)$, we see that the *Bockstein–free part of the* mod p *Steenrod algebra*, which we denote by $\mathcal{A}$ is naturally defined as the quotient of $F(p)$ by the Adem relations [19] and $\mathcal{P}^0 = 1$, the identity element. Topologically, $\mathcal{A}$ is also known as the algebra of stable cohomology operations for ordinary cohomology H^* over $\mathbb{F}_p$. Furthermore, Milnor [17] has showed that $\mathcal{A}$ is a graded connected Hopf algebra.

We now investigate a relationship between $F(p)$ and the Leibniz–Hopf algebra. Precisely, $F(p)$ can be turned into a graded connected Hopf algebra by defining a coproduct to be that given by the *Cartan* formula $\Delta(S^n) = \sum_{i=0}^{n} S^i \otimes S^{n-i}$. This Hopf algebra is cocommutative in the strict (i.e., non-graded) sense. Now let $\mathcal{F}$ denote the *Leibniz–Hopf algebra* [11, Section 1]. In particular, setting $\mathbb{F}_p = \mathbf{Z}/p$, we may see that $F(p)$ is the mod p reduction of the Leibniz–Hopf algebra, $\mathcal{F} \otimes \mathbf{Z}/p$. $\mathcal{F}$ is also known as the *algebra of non-commutative symmetric functions* [10] and has been also studied in [12–16, 18] because of its various connections to other algebraic structures. The graded dual Hopf algebra $\mathcal{F}^*$ is the Hopf algebra of *qua-*

sisymmetric functions and also known as the *overlapping shuffle algebra* [11, Section 1]. $\mathcal{F}^*$ has been of interest to combinatorialists, topologists, algebraists, and studied in [2–4, 13].

We now investigate $\mathcal{F}$ and $\mathcal{F}^*$ in connection with the Steenrod algebra. Recalling the preceding paragraphs, we may see that $\mathcal{A}$ is the quotient algebra of $\mathcal{F} \otimes \mathbf{Z}/p$ by the Adem relations. This quotient structure gives us the surjective Hopf algebra morphism $\pi\colon \mathcal{F} \otimes \mathbf{Z}/p \to \mathcal{A}$, where $\pi(S^i) = \mathcal{P}^i$. Moreover, giving a new grading to $\mathcal{F}$ on generators S^i of degree $2i(p-1)$, $i \geq 0$, π extends to a graded homomorphism (from now on, we use this grading). Following this, we arrive at the graded Hopf algebra inclusion $\pi^*\colon \mathcal{A}^* \to \mathcal{F}^* \otimes \mathbf{Z}/p$ [4, Section 5] (where $\mathcal{A}^*$ denotes the graded dual of $\mathcal{A}$) dual to the homomorphism π. This will be the heart of this paper. Note that this homomorphism is also considered in [22]. The Hopf algebra structure of $\mathcal{A}^*$ admits a unique Hopf algebra conjugation (or "antipode"), $\chi_{\mathcal{A}^*}$. Invariant problem under $\chi_{\mathcal{A}^*}$ has been studied in [8], since it is relevant for the commutativity of ring spectra [1, Lecture 3].

In this paper we reconsider this problem. Let us explain briefly. The conjugation invariants in $\mathcal{A}^*$ form a subvector space, $\mathrm{Ker}(\chi_{\mathcal{A}^*} - 1)$ (where 1 denotes the identity homomorphism). Crossley and Whitehouse [8, Section 1] have given a description of $\mathrm{Ker}(\chi_{\mathcal{A}^*} - 1)$ in some generality. Particularly, it has been showed that how the Poincaré series for $\mathrm{Ker}(\chi_{\mathcal{A}^*} - 1)$ can be determined using Molien's theorem. In Section 3, we give an alternative approach for this (Theorem 3.3). We refer reader to [1, Lecture 3] and [7] for more detailed motivation.

The arguments used to obtain the results in this present paper are similar to those of their mod 2 corresponding parts in [21, Section 5]. However, there are two notable differences that appear in the odd primary case. Let us explain briefly. Firstly, we have to deal with making the homomorphism, π graded, and also modify some of the early results according to this (see Remark 3.2). Secondly, conjugation formula (1) is sign involved. This fact together with mod p binomial coefficients necessitate more careful analysis in Section 3. These difficulties cause the results obtained in this work to be not quite straightforward.

2. Preliminaries

As a vector space, $\mathcal{F} \otimes \mathbf{Z}/p$ has a basis of words $S^{j_1} S^{j_2} \cdots S^{j_r}$ (of finite length) in the letters $S^{j_1}, S^{j_2}, \ldots$, which we denote by $S^{j_1, j_2, \ldots, j_r}$. The *degree* of an element $S^{j_1, j_2, \ldots, j_s}$ is defined to be $2(p-1) \sum_{i=1}^{s} j_i$. We denote the dual basis for $\mathcal{F}^* \otimes \mathbf{Z}/p$ by $\{S_{j_1, j_2, \ldots, j_r}\}$. A conjugation formula for this Hopf algebra is given by the mod p reduction of formula [9, Proposition 3.4] as follows.

$$\chi(S_{j_1, \ldots, j_r}) = (-1)^r \sum S_{b_1, \ldots, b_n} \tag{1}$$

summed over all coarsenings $b_1, \ldots, b_n$ of the *reversed* word $j_r, \ldots, j_1$, i.e., all words $b_1, \ldots, b_n$ that admit $j_r, \ldots, j_1$ as a refinement [6]. For instance,

$$\chi(S_{5,3,2}) = -S_{2,3,5} - S_{5,5} - S_{2,8} - S_{10}.$$

We now recall the *overlapping shuffle product* from [21, Section 2]: the overlapping shuffle product of $S_{a_1,\ldots,a_t}$ and $S_{b_1,\ldots,b_y}$ is defined by

$$\mu(S_{a_1,\ldots,a_t} \otimes S_{b_1,\ldots,b_y}) = \sum_h h(S_{a_1,\ldots,a_t,b_1,\ldots,b_y}),$$

where h first inserts a certain number ℓ of 0s into $a_1, \ldots, a_t$, and inserts a number of ℓ' of 0s into $b_1, \ldots, b_y$, where

$$0 \le \ell \le y, \quad 0 \le \ell' \le t, \quad t + \ell = y + \ell',$$

then it adds the first indices together, then the second, and so on. The sum is over all such h for which the result contains no 0s. For instance,

$$\mu(S_{4,3} \otimes S_2) = S_{4,3,2} + S_{4,2,3} + S_{2,4,3} + S_{6,3} + S_{4,5}.$$

We refer to [11, Section 2] for an alternative definition of this product. We now give some background for $\mathcal{A}^*$. Beside its Hopf algebra structure, Milnor [17] also showed that $\mathcal{A}^*$ is the polynomial part of the mod p dual Steenrod algebra on generators ξ_i $(i \ge 1)$ of degree $2(p^i - 1)$ (see Section 3 of [19, Chapter 6]). Turning to the Hopf algebra homomorphism π^* in Section 1. We now recall the following formulas from [4, Section 5] as follows:

$$\pi^\star(\xi_n) = S_{p^{n-1},p^{n-2},\ldots,p,1}, \tag{2}$$

$$\pi^\star(\xi_n^{p^m}) = S_{p^{m+n-1},p^{m+n-2},\ldots,p^{m+1},p^m}. \tag{3}$$

It is worth pointing out that we obtain calculations in $\mathcal{F}^*$, and π^* is an algebra morphism on the target overlapping shuffle product [11, Section 6].

Now we recall some of the terminology from [6]. A word $S_{j_1,j_2,\ldots,j_n}$ is a *palindrome* if $j_1 = j_n$, $j_2 = j_{n-1}$, and so on. A palindrome is referred to as an *even-length palindrome*, which we denote by ELP, if its length is even. For example, $S_{8,3,3,8}$ is an ELP. A non-palindrome $S_{j_1,\ldots,j_r}$ is referred to as a *higher non-palindrome*, which we denote by HNP if $j_1, \ldots, j_r$ is lexicographically bigger than its reverse $j_r, \ldots, j_1$. For instance, $S_{8,5,4,8}$ is an HNP.

3. A different approach on the conjugation invariant problem in $\mathcal{A}^*$

We now introduce a different approach to determine a basis for $\mathrm{Ker}(\chi_{\mathcal{A}^*} - 1)$.

Theorem 3.1 [5, Theorem 2.5] *In the dual Leibniz–Hopf algebra, $\mathcal{F}^*$, and in the mod p dual $\mathcal{F}^* \otimes Z/p$ for any prime $p > 2$, the submodule* $\mathrm{Ker}(\chi - 1)$ *is equal to* $\mathrm{Im}(\chi + 1)$ *and is free on a basis consisting of: the $(\chi + 1)$-images of all ELPs and HNPs. Thus, in degree n, this module has rank 2^{n-2} if n is even, and $2^{n-2} - 2^{(n-3)/2}$ if n is odd.*

Remark 3.2 In [5, Theorem 2.5], graded algebra structure of $\mathcal{F}$ is obtained by giving S^n, $n \geq 1$, degree n. Thus, when $p > 2$, recalling the modified grading from the Section 1, we give the adapted version of the dimension formula of $\mathrm{Ker}(\chi - 1)$ in Theorem 3.1 as: in $2(p-1)n$, degrees $\mathrm{Ker}(\chi - 1)$ has dimension 2^{n-2} if n is even, and $2^{n-2} - 2^{(n-3)/2}$ if n is odd.

For simplicity, from now on, we denote $\mathcal{F}^* \otimes Z/p$ by $\mathcal{F}^*$. To have a connection between conjugation invariants in $\mathcal{F}^*$ and $\mathcal{A}^*$, we reconsider the graded Hopf algebra inclusion. In particular, π^*, being a Hopf algebra morphism, we see that the following diagram commutes

$$\begin{tikzcd} \mathcal{A}^* \arrow[r, hook, "\pi^*"] \arrow[d, "\chi_{\mathcal{A}^*}"] & \mathcal{F}^* \arrow[d, "\chi"] \\ \mathcal{A}^* \arrow[r, hook, "\pi^*"] & \mathcal{F}^* \end{tikzcd} \tag{4}$$

Moreover, being an injective morphism, in each fixed degree, this gives the following relationship between the conjugation invariants in $\mathcal{A}^*$ and $\mathcal{F}^*$.

Theorem 3.3 $\pi^*(\mathrm{Ker}(\chi_{\mathcal{A}^*} - 1)) = \mathrm{Ker}(\chi - 1) \cap \mathrm{Im}(\pi^*)$. ∎

Proposition 3.4 *Let $S_{p^a,p^b} \in \mathcal{F}^*$. Then*

$$\pi^\star(\xi_1^{p^a}\xi_1^{p^b}) = (\chi + 1)(S_{p^a,p^b}).$$

Proof. Let $S_{p^a,p^b} \in \mathcal{F}^*$. Then, by formula (1), we obtain

$$(\chi + 1)(S_{p^a,p^b}) = S_{p^a,p^b} + S_{p^b,p^a} + S_{p^b+p^a}.$$

On the other hand, as π^* is an algebra morphism, formula (3) gives

$$\pi^\star(\xi_1^{p^a}\xi_1^{p^b}) = (\chi + 1)(S_{p^a,p^b}).$$ ∎

Corollary 3.5 *Let S_{p^a,p^b} be an HNP or an ELP. Then in $p^a + p^b$ degrees*

$$(\chi + 1)(S_{p^a,p^b}) \in \mathrm{Ker}(\chi - 1) \cap \mathrm{Im}(\pi^*).$$

We demonstrate Theorem 3.3 in the following examples at the prime 3.

Example 3.6 In degree 8, $\mathcal{F}^*$ has a basis: $\{S_2, S_{1,1}\}$. By Theorem 3.1, $(\chi + 1)$-images of HNPs and ELPs form a basis for $\mathrm{Ker}(\chi - 1)$, that is $(\chi + 1)(S_{1,1}) = S_2 + 2S_{1,1}$. On the other hand, in the same degree, $\{\xi_1^2\}$ is a basis for $\mathcal{A}^*$. Hence, $\mathrm{Im}(\pi^*)$ has $\pi^*(\xi_1^2)$ as a basis, since π^* is a monomorphism. Following this, by formula (2), we have $\pi^*(\xi_1^2) = S_2 + 2S_{1,1}$ from which we conclude that

$$\mathrm{Ker}(\chi - 1) = \mathrm{Im}(\pi^*).$$

It follows that, by Theorem 3.3, we see that $\{\pi^*(\xi_1^2)\}$ is a basis for $\pi^*(\mathrm{Ker}(\chi_{\mathcal{A}^*} - 1))$, and hence $\{\xi_1^2\}$ is a basis for $\mathrm{Ker}(\chi_{\mathcal{A}^*} - 1)$, since π^* is a monomorphism.

Example 3.7 In degree 12, $\mathcal{F}^*$ has a basis $\{S_3, S_{2,1}, S_{1,2}, S_{1,1,1}\}$. By Theorem 3.1 and Table 2, we see that $\mathrm{Ker}(\chi - 1) \cap \pi^*(\mathcal{A}_p^*) = \emptyset$, and hence $\mathrm{Ker}(\chi_{\mathcal{A}^*} - 1) = \emptyset$.

Example 3.8 We now recall a more efficient method from [21, Section 5]. In degree 16, we first give an order to the monomial basis of $\mathcal{F}^*$ with respect to lexicographic order. We denote this ordered basis by U which is given as follows:

$$U = \{S_4, S_{3,1}, S_{2,2}, S_{2,1,1}, S_{1,3}, S_{1,2,1}, S_{1,1,2}, S_{1,1,1,1}\}.$$

For example, the basis U tells us that $S_{2,2}$ is lexicographically bigger than $S_{1,2,1}$. We now recall linear algebra from [20, pp. 199-200]: if V is the column space of a matrix A, and W is the column space of a matrix B, then $V + W$ is the column space of the matrix $D = [A \;\; B]$ and $\dim(V + W) = \mathrm{rank}\ (D)$ and $\dim(V \cap W) =$ nullity of D which leads the following formula:

$$\dim(V + W) + \dim(V \cap W) = \dim(V) + \dim(W). \tag{5}$$

Table 1: Bases of $\mathrm{Im}(\pi^*)$ in degrees $12, 16, 20$.

Degree 12	$\pi^*(\xi_1^3) = S_3$
Degree 16	$n_1 = \pi^*(\xi_1^4) = S_4 + S_{3,1} + S_{1,3}$ $n_2 = \pi^*(\xi_2) = S_{3,1}$
Degree 20	$n_1' = \pi^*(\xi_1^5) = S_5 + 2S_{4,1} + S_{3,2} + 2S_{3,1,1} + S_{2,3}+$ $2S_{1,4} + 2S_{1,3,1} + 2S_{1,1,3}$ $n_2' = \pi^*(\xi_2\xi_1) = S_{4,1} + S_{3,2} + 2S_{3,1,1} + S_{1,3,1}$

Note that Table 1 is also partially used in another point of view in [22].

Table 2: Bases of $\mathrm{Ker}(\chi - 1)$ in degrees $12, 16, 20$.

Degree 12	$(\chi + 1)(S_{2,1}) = S_3 + S_{2,1} + S_{1,2}$
Degree 16	$t_1 = (\chi + 1)(S_{3,1}) = S_4 + S_{3,1} + S_{1,3}$ $t_2 = (\chi + 1)(S_{2,2}) = S_4 + 2S_{2,2}$ $t_3 = (\chi + 1)(S_{2,1,1}) = -S_4 - S_{2,2} + S_{2,1,1} - S_{1,3} - S_{1,1,2}$ $t_4 = (\chi + 1)(S_{1,1,1,1}) = S_4 + S_{3,1} + S_{2,2} + S_{2,1,1} + S_{1,3}+$ $S_{1,2,1} + S_{1,1,2} + 2S_{1,1,1,1}$
Degree 20	$t_1' = (\chi + 1)(S_{4,1}) = S_5 + S_{4,1} + S_{1,4}$ $t_2' = (\chi + 1)(S_{3,2}) = S_5 + S_{3,2} + S_{2,3}$ $t_3' = (\chi + 1)(S_{3,1,1}) = -S_5 + S_{3,1,1} - S_{2,3} - S_{1,4} - S_{1,1,3}$ $t_4' = (\chi + 1)(S_{2,2,1}) = -S_5 - S_{3,2} + S_{2,2,1} - S_{1,4} - S_{1,2,2}$ $t_5' = (\chi + 1)(S_{2,1,1,1}) = S_5 + S_{3,2} + S_{2,3} + S_{2,1,2} + S_{2,1,1,1}+$ $S_{1,4} + S_{1,2,2} + S_{1,1,3} + S_{1,1,1,2}$ $t_6' = (\chi + 1)(S_{1,2,1,1}) = S_5 + S_{4,1} + S_{2,3} + S_{2,2,1} + S_{1,4}+$ $S_{1,3,1} + S_{1,2,1,1} + S_{1,1,3} + S_{1,1,2,1}$

To use the above argument, using Tables 1 and 2, we first write out the basis matrix of $\mathrm{Im}(\pi^*)$, denoted by $[N]_Y$, and of $\mathrm{Ker}(\chi - 1)$, denoted by $[T]_U$, relative to the basis U in the following:

$$[N]_U = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad [T]_U = \begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 2 & -1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

Let us be precise. The first column of $[N]_U$ represents the coordinate vector of basis element n_1 in Table 1, relative to the basis U. On the other hand, the first column of $[T]_U$ represents the coordinate vector of basis element t_1 in Table 2, relative to the basis U, the second column of $[T]_U$ represents the coordinate vector of basis element t_2 in Table 2, relative to the basis Y, and so on.

It is now clear to see the rank of $D = [[N]_U \ [T]_U]$ is 5. Thus, by formula (5), we obtain: $5 + \dim([N]_U \cap [T]_U) = 6$, from which we can deduce $\dim([M]_Y \cap [N]_Y) = 1$. By Tables 1 and 2, $\mathrm{Im}(\pi^*)$ and $\mathrm{Ker}(\chi - 1)$ have $\pi^*(\xi_1^4)$ as a common basis element. Therefore, by dimension reason, $\{\pi^*(\xi_1^4)\}$ has to be a basis for $\mathrm{Ker}(\chi - 1) \cap \mathrm{Im}(\pi^*)$, and hence $\mathrm{Ker}(\chi_{\mathcal{A}^*} - 1)$ has a basis $\{\xi_1^4\}$ in degree 16.

Example 3.9 In degree 20 we briefly give details of the calculations. We again give lexicographical order to the monomial basis of $\mathcal{F}^*$, which we denote by U' and given in the following:

$$U' = \{S_5, S_{4,1}, S_{3,2}, S_{3,1,1}, S_{2,3}, S_{2,2,1}, S_{2,1,2}, S_{2,1,1,1}, S_{1,4}, S_{1,3,1}, S_{1,2,2}, \\ S_{1,2,1,1}, S_{1,1,3}, S_{1,1,2,1}, S_{1,1,1,2}, S_{1,1,1,1,1}\}.$$

By Tables 1 and 2, writing the basis matrix of $\mathrm{Im}(\pi^*)$, denoted by $[N']_{U'}$, and of $\mathrm{Ker}(\chi - 1)$, denoted by $[T']_{U'}$, relative to the basis U', we see that the rank of $D = \left[[N']_{U'} \ [T']_{U'} \right]$ is 8. Precisely, this is because, both n'_1 and n'_2 in Table 1 have a summand with a coefficient 2 and these do not allow n'_1 and n'_2 to be written as linear combinations of $\{t'_1, \ldots, t'_6\}$. On the other hand, $\mathrm{rank}\left([N']_{U'} \right) = 2$ and $\mathrm{rank}\left([T']_{U'} \right) = 6$. Therefore, $8 + \dim(N' \cap T') = 8$ from which we can deduce

$$\mathrm{Ker}(\chi - 1) \cap \mathrm{Im}(\pi^*) = \emptyset,$$

and hence $\mathrm{Ker}(\chi_{\mathcal{A}^*} - 1) = \emptyset$.

We refer the reader to (http://www.skaji.org/code) for a computer-aided approach to obtain conjugation invariants in the dual Leibniz-Hopf algebra and the dual Steenrod algebra. None of the calculations in this present paper depends on the above computer-aided approach.

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M-FUZZY h-IDEALS IN h-SEMISIMPLE M-Γ-HEMIRINGS

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Abstract. In this paper, the concepts of M-fuzzy h-interior ideals and prime M-fuzzy h- ideals in M-Γ-hemirings are introduced. Some new properties of these kinds of M-fuzzy h-ideals are also given. Finally, some characterizations of the h-semisimple M-Γ-hemirings are investigated by these kinds of M-fuzzy h-ideals.

Keywords: M-Γ-hemiring, M-fuzzy h-interior ideal, prime M-fuzzy h-ideal, h-semisimple M-Γ-hemiring.

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1. Introduction

The concept of Γ-rings was first introduced in 1966 by Barnes [1] which is more a general concept than that of a ring. After the paper of Barnes, many researchers were engaged in studying of some special Γ-rings. Jun and Lee [6] discussed fuzzy Γ-rings, and Jun [5] investigated fuzzy prime ideals of Γ-rings. In particular, Dutta and Chanda [3] studied the structure of fuzzy ideals of a Γ-ring. The concept of Γ-semirings was then introduced by Rao [15], and some properties of such Γ-semirings have been studied by Sardar et al. [16]. Recently, Ma and Zhan [11] investigated fuzzy h-ideals in h-hemiregular and h-semisimple Γ-hemiring, and Zhan and Shum [23] discussed fuzzy h-ideals in Γ-hemirings.

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The notion of semirings was first introduced by H. S. Vandiver in 1934. In the 1980's the theory of semirings contributed to computer science, since the rapid development of computer science need additional theoretical mathematical background. We note that the ideals of semirings also play a crucial role in the structure theory. Although ideals in semirings are useful in their own way, they do not in general coincide with the role of ideals in a ring. For this reason, the usage of ideals in semirings was somewhat limited. By a hemiring, we mean a special semiring with a zero and with a commutative addition. The properties of h-ideals of hemirings were thoroughly investigated by Torre [17]. By using h-ideals in hemirings, Torre established the quotient hemirings which are an exact analog to the ring theory. Recently, Han et al. [4] investigated some characterizations of semiring orders in a semiring, In 2004, Jun [7] defined the fuzzy h-ideals in hemirings. Yin and Li [19] introduced the concepts of fuzzy h-bi-ideals and fuzzy h-quasi-ideals of hemirings. After that, Ma and Zhan [10] introduced the concepts of $(\in, \in_\gamma \vee q_\delta)$-fuzzy h-bi-ideals (resp., h-quasi-ideals) of a hemiring and investigated some of their properties. Recently, Ma et al. [8] introduced the concepts of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-bi-(h-quasi-)ideals of hemirings. In particular, some characterizations of the h-intra-hemiregular and h-quasi-hemiregular hemirings were investigated by these kinds of fuzzy h-ideals. The general properties of fuzzy h-ideals have been considered by Dudek, Kim, Jun, Ma, Zhan, and others. The readers refer to [2], [6], [9], [12], [22] in detail.

In 2007, Zhan and Davvaz [21] gave the fuzzy h-ideals with operators in hemirings and some properties were investigated. Pan [14] gave the concept of M-Γ-hemiring, and established a new fuzzy left h-ideal with operators. The present paper is organized as follows. In Section 2, we recall some basic definitions and properties of M-Γ-hemirings and fuzzy sets. In Sections 3 and 4, we introduce the concepts of M-fuzzy h-interior ideals and prime M-fuzzy h-ideals of M-Γ-hemirings, and we give some related properties. In Section 5, we describe the characterizations of h-semisimple M-Γ-hemirings.

2. Preliminaries

First, we recall some basic notions and results concerning Γ-hemirings, M-Γ-hemirings and fuzzy sets (for more details, see [11, 23]).

2.1. Γ-hemirings

Let S and Γ be two commutative additive semigroups. Then S is said to be a Γ-*semiring* if there exists a mapping $S \times \Gamma \times S \to S$ (images are denoted by $a\alpha b$ for $a, b \in S$ and $\alpha \in \Gamma$) satisfying the following conditions:

(i) $a\alpha(b + c) = a\alpha b + a\alpha c$,

(ii) $(a + b)\alpha c = a\alpha c + b\alpha c$,

(iii) $a(\alpha+\beta)c = a\alpha c + a\beta c$,

(iv) $a\alpha(b\beta c) = (a\alpha b)\beta c$.

By a *zero* of a Γ-semiring S, we mean an element $0 \in S$ such that $0\alpha x = x\alpha 0 = 0$ and $0 + x = x + 0 = x$, for all $x \in S$ and $\alpha \in \Gamma$. A Γ-semiring with a zero is said to be a Γ-*hemiring*.

Throughout this paper, S is a Γ-hemiring and we use the symbol 0_S to denote the zero element of S.

A *left* (resp., *right*) *ideal* of a Γ-hemiring S is a subset A of S which is closed under addition such that $S\Gamma A \subseteq A$ (resp., $A\Gamma S \subseteq A$), where $S\Gamma A = \{x\alpha y \mid x \in S, y \in A, \alpha \in \Gamma\}$. Naturally, a subset A of S is called an *ideal* of S if it is both a left and a right ideal of S. A subset A of S is called an *interior ideal* if A is closed under addition such that $A\Gamma A \subseteq A$ and $S\Gamma A\Gamma S \subseteq A$.

A left ideal (right ideal, ideal) A of S is called a *left h-ideal* (*right h-ideal, h-ideal,*) of S, respectively, if, for any x, $z \in S$ and a, $b \in A$, $x + a + z = b + z$ implies that $x \in A$.

The *h-closure* $\overline{A}$ of A in S is defined by $\overline{A} = \{x \in S \mid x + a_1 + z = a_2 + z$ for some $a_1, a_2 \in A, z \in S\}$.

Clearly, if A is a left ideal of S, then $\overline{A}$ is the smallest left h-ideal of S containing A. We also have $\overline{\overline{A}} = \overline{A}$, for each $A \subseteq S$. Moreover, $A \subseteq B \subseteq S$ implies $\overline{A} \subseteq \overline{B}$.

An interior ideal A of S is called an *h-interior ideal* of S if A is closed under addition such that $\overline{A\Gamma A} \subseteq A, \overline{S\Gamma A\Gamma S} \subseteq A$ and $x + a + z = b + z$ implies that $x \in A$, for all $x, z \in S, a, b \in A$.

Definition 2.1 ([11], [23])

(i) Let μ and ν be fuzzy subsets of S. Then the h-product of μ and ν is defined by

$$(\mu\Gamma_h\nu)(x) = \bigvee_{x+a_1\gamma_1b_1+z=a_2\gamma_2b_2+z} \min\{\mu(a_1), \mu(a_2), \nu(b_1), \nu(b_2)\}$$

$(\mu\Gamma_h\nu)(x) = 0$ if x cannot be expressed as $x + a_1\gamma_1b_1 + z = a_2\gamma_2b_2 + z$.

(ii) Let μ and ν be fuzzy subsets of M-Γ-hemiring S, for any $x \in S$, there exist $a_1, a_2, z \in S, \gamma_1, \gamma_2 \in \Gamma$ and $m_1, m_2, \in M$. Then the M-h-product of μ and ν is defined by

$$(\mu\Gamma_h\nu)(x) = \bigvee_{x+m_1\gamma_1a_1+z=m_2\gamma_2a_2+z} \min\{\mu(m_1), \mu(m_2), \nu(a_1), \nu(a_2)\}$$

$(\mu\Gamma_h\nu)(x) = 0$ if x cannot be expressed as $x + m_1\gamma_1a_1 + z = m_2\gamma_2a_2 + z$.

A fuzzy set is a function μ: $S \to [0, 1]$. For any $A \subseteq S$, we denote the characteristic function of A by χ_A

$$\chi_A = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

Proposition 2.2 ([11], [23]) *Let $A, B \subseteq S$. Then, the following statements hold:*

(1) $A \subseteq B \Leftrightarrow \chi_A \subseteq \chi_B$,

(2) $\chi_A \cap \chi_B = \chi_{A\cap B}$,

(3) $\chi_A \Gamma_h \chi_B = \chi_{\overline{A\Gamma B}}$.

2.2. M-Γ-hemirings

Definition 2.3 ([14]) A Γ-*hemiring S with operators* is an algebraic system consisting of a Γ-hemiring S, a set M and a function defined in the product set $M \times \Gamma \times S$ and having values in S ($M \times \Gamma \times S \to S$) such that, if $m\alpha x$ denotes the elements in S determined by the element m of M , x of S and the elements α, β of Γ, then

$$m\alpha(x + y) = m\alpha x + m\alpha y$$

and

$$m\alpha(x\beta y) = (m\alpha x)\beta(m\alpha y)$$

hold for any $x, y \in S$, $m \in M$ and $\alpha, \beta \in \Gamma$. We usually use the phrase "S is an M-Γ-hemiring" instead of a "Γ-hemiring with operators".

Example 2.4 Let $S = \{0, a, b\}$ be a set with an addition operation $(+)$ and a multiplication operation $(\cdot)$ as follows:

$+$	0	a	b
0	0	a	b
a	a	a	b
b	b	b	b

and

$\cdot$	0	a	b
0	0	0	0
a	0	a	a
b	0	a	a

Then S is an M-Γ-hemiring where $\Gamma = M = S$.

Definition 2.5 ([14]) A left h-ideal I of an M-Γ-hemiring S is called a *left M-h-ideal* of S if $m\alpha x \in I$ for all $m \in M, x \in I$ and $\alpha \in \Gamma$.

Definition 2.6 ([14]) Let S be an M-Γ-hemiring and μ a fuzzy h-ideal of S. If the inequality $\mu(m\alpha x) \geqslant \mu(x)$ holds for any $x \in S$, $m \in M$ and $\alpha \in \Gamma$, then μ is said to be a *fuzzy left h-ideal with operators* of S. We use the phrases "an M-fuzzy left h-ideal of S" instead of "a fuzzy h-ideal with operators of S".

Definition 2.7 ([14]) A fuzzy set μ over M-Γ-hemiring S is called an *M-fuzzy left* (resp., *right*) *h-ideal* over S if it satisfies:

(F1) $\mu(x + y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in S$,

(F2) $\mu(x\alpha y) \geq \mu(y)$ for all $x, y \in S$ and $\alpha \in \Gamma$ (resp., $\mu(x\alpha y) \geq \mu(x)$),

(F3) $x + a + z = b + z$ implies $\mu(x) \geq \min\{\mu(a), \mu(b)\}$ for all $a, b, x, z \in S$,

(F4) $\mu(m\alpha x) \geq \mu(x)$ for all $x \in S$, $m \in M$ and $\alpha \in \Gamma$.

3. M-fuzzy h-interior ideals

It is well known that ideal theory plays a fundamental role in the development of hemirings. In this section, we consider M-fuzzy h-interior ideals of M-Γ-hemirings.

Definition 3.1 An h-interior ideal I of an M-Γ-hemiring S is called a M-h-*interior ideal* of S if $m\alpha x \in I$ for all $m \in M, x \in I$ and $\alpha \in \Gamma$.

Definition 3.2 A fuzzy set μ over M-Γ-hemiring S is called an M-*fuzzy* h-*interior ideal* over S if it satisfies (F1), (F3), (F4) and

(F5) $\mu(x\alpha y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in S, \alpha \in \Gamma$,

(F6) $\mu(x\alpha y\beta z) \geq \mu(y)$ for all $x, y, z \in S$ and $\alpha, \beta \in \Gamma$.

Example 3.3 Assume $S = \mathbb{Z}_3 = \{0, 1, 2\}$, non-negative positive integers module 3. Then S is an M-Γ-hemiring where $\Gamma = M = \{0, 1\}$. Let $\alpha, \beta \in [0, 1)$ be such that $\alpha \geq \beta$. Define a fuzzy set μ over S by $\mu(0) = \alpha, \mu(1) = \mu(2) = \beta$. The one can easily check that μ is an M-h-interior ideal of S.

Note that, if μ is an M-fuzzy h-interior ideal of S, then $\mu(0) \geq \mu(x)$.

For any $t \in [0, 1]$, the set

$$U(\mu; t) = \{x \in S | \mu(x) \geq t\}$$

is called a level subset of μ.

Lemma 3.4 [23] *A fuzzy set μ in a Γ-hemiring S is a fuzzy h-interior ideal of S if and only if the each nonempty level subset $U(\mu; t)$, $t \in (0, 1)$, of μ is an h-interior ideal of S.*

Theorem 3.5 *A fuzzy set μ in an M-Γ-hemiring S is an M-fuzzy h-interior ideal of S if and only if the each nonempty level subset $U(\mu; t)$, $t \in (0, 1)$, of μ is an M-h-interior ideal of S.*

Proof. Let μ be an M-fuzzy h-interior ideal of S. Assume that $U(\mu; t) \neq \emptyset$ for $t \in [0, 1]$. Then by Lemma 3.4, $U(\mu; t)$ is an h-interior ideal of S. For every $x \in U(\mu; t)$, $\alpha \in \Gamma$, $m \in M$, we have

$$\mu(m\alpha x) \geqslant \mu(x) \geqslant t,$$

and hence $m\alpha x \in U(\mu; t)$. Thus $U(\mu; t)$ is an M-fuzzy h-interior ideal of S.

Conversely, suppose that $U(\mu; t) \neq \emptyset$ is an M-h-interior ideal of S. Then μ is a fuzzy h-interior ideal of S by Lemma 3.3. Now assume that there exist $y \in S$, $\gamma \in \Gamma$ and $k \in M$ such that

$$\mu(k\gamma y) < \mu(y).$$

Taking
$$t_0 := \frac{1}{2}(\mu(k\gamma y) + \mu(y)),$$
we obtain $t_0 \in [0,1]$ and
$$\mu(k\gamma y) < t_0 < \mu(y).$$
This implies that $k\gamma y \notin U(\mu; t_0)$ and $y \in U(\mu; t_0)$, which leads a contradiction. Therefore
$$\mu(k\gamma y) \geq \mu(y),$$
for all $y \in S$, $\gamma \in \Gamma$ and $k \in M$. This completes the proof. ∎

Proposition 3.6 *Every M-fuzzy h-ideal of M-Γ-hemiring S is an M-fuzzy h-interior ideal.*

Proof. By the Definitions 2.7 and 3.2, we only prove (F6) holds. Assume μ is an M-fuzzy h-ideal of S. Let $y, z \in S$, $\alpha, \beta \in \Gamma$. Then we have $\mu(x\alpha y\beta z) \geq \mu(y\beta z) \geq \mu(y)$ since μ is an M-fuzzy h-ideal of S. Hence, $\mu(x\alpha y\beta z) \geq \mu(y)$. ∎

4. Prime M-fuzzy h-ideals

In this section, we consider prime M-fuzzy h-ideals of M-Γ-hemirings. A left (right) M-h-ideal P of S is said to be *prime* if $P \neq S$ and for any two left (right) h-ideals A and B of S from $A\Gamma B \subseteq P$ it follows either $A \subseteq P$ or $B \subseteq P$.

Definition 4.1 An M-fuzzy left (resp., right) h-ideal ψ of S is said to be *prime* if ψ is a non-constant function and for any two M-fuzzy left(right) h-ideals μ and ν of S, $\mu\Gamma\nu \subseteq \psi$ implies $\mu \subseteq \psi$ or $\nu \subseteq \psi$.

Example 4.2 Let $(S, +)$ and $(\Gamma, +)$ be two semigroups, where S and Γ are the sets of all non-negative integers and the operations are the usual additive operations. Define a mapping $S \times \Gamma \times S \to S$ by $a\gamma b = a \cdot \gamma \cdot b$, for all $a, b \in S$ and $\gamma \in \Gamma$, where "$\cdot$" is the usual multiplication. Then it can be easily verified that S, under the above multiplication and the structure Γ-mapping, is a Γ-hemiring. If we let $M := \{1\}$, then it is clear that S is an M-Γ-hemiring. Let $r, s \in [0, 1)$ be such that $r \leq s$. Define a fuzzy set μ over S by
$$\mu(x) = \begin{cases} s & \textit{if } x \textit{ is even}, \\ r & \textit{otherwise}. \end{cases}$$
Then μ is a prime M-fuzzy h-ideal over S.

Proposition 4.3 *A fuzzy set χ_P in an M-Γ-hemiring S is a prime M-fuzzy left(right) h-ideal of S if and only if P is a prime left(right) M-h-ideal of S, respectively.*

Proof. Straightforward. ∎

Theorem 4.4 *A fuzzy subset ζ of M-Γ-hemiring S is a prime M-fuzzy left(right) h-ideal of S if and only if*

(1) $\zeta^0 = \{x \in S | \zeta(x) = \zeta(0)\}$ *is a prime left(right) M-h-ideal of S,*

(2) $Im\zeta = \{\zeta(x) | x \in S\}$ *contains exactly two elements,*

(3) $\zeta(0) = 1$.

Proof. We prove only the case of M-fuzzy left h-ideals. The proof for the right h-ideals is similar, and we omit it.

(1) Let ζ be a prime M-fuzzy left h-ideal of S. Then it is easy to check that ζ^0 is a prime left M-h-ideal of S.

(2) Suppose that $Im\zeta$ has more than two values. Then there exist two elements $p, q \in S \setminus \zeta^0$ such that $\zeta(p) \neq \zeta(q)$. Without loss of generality, we can assume that $\zeta(p) < \zeta(q)$. Since ζ is an M-fuzzy left h-ideal and $q \notin \zeta^0$, it follows that $\zeta(p) < \zeta(q) < \zeta(0)$. Hence there exist $r, t \in [0, 1]$ such that

$$\zeta(p) < r < \zeta(q) < t < \zeta(0). \tag{$*$}$$

Let ν and μ be M-fuzzy left h-ideals defined by

$$\nu(x) = \begin{cases} r & \text{if } x \in \langle p \rangle, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{and} \qquad \mu(x) = \begin{cases} t & \text{if } x \in \langle q \rangle, \\ 0 & \text{otherwise} \end{cases}$$

where $\langle p \rangle$ and $\langle q \rangle$ are ideals generated by p and q, respectively.

Then, for any $x \in S$, which can not be expressed in the form

$$x + m_1\gamma_1 b_1 + z = m_2\gamma_2 b_2 + z,$$

where $z \in S, m_1, m_2 \in \langle p \rangle$, $b_1, b_2 \in \langle q \rangle$ and $\gamma_1, \gamma_2 \in \Gamma$, we have

$$(\nu\Gamma\mu)(x) = 0.$$

Otherwise,

$$\begin{aligned} (\nu\Gamma_h\mu)(x) &= \bigvee_{x+m_1\gamma_1 b_1+z=m_2\gamma_2 b_2+z} (\min\{\nu(m_1), \nu(m_2), \mu(b_1), \mu(b_2)\}) \\ &= \min\{r, t\} = r. \end{aligned}$$

Since ζ is an M-fuzzy left h-ideal, from $x + m_1\gamma_1 b_1 + z = m_2\gamma_2 b_2 + z$ it follows that

$$\zeta(x) \geq \zeta(m_1\gamma_1 b_1) \wedge \zeta(m_2\gamma_2 b_2) \geq \zeta(b_1) \wedge \zeta(b_2) \geq r.$$

So, $(\nu\Gamma_h\mu)(x) \leq \zeta(x)$, whence $\nu\Gamma_h\mu \subseteq \zeta$, for ζ is a prime M-fuzzy left h-ideal, we can get $\nu \subseteq \zeta$ or $\mu \subseteq \zeta$. Therefore, $\nu(p) = r \leq \zeta(p)$ or $\mu(q) = t \leq \zeta(q)$ which contradicts to $(*)$. Consequently, $Im\zeta$ contains exactly two elements.

(3) Suppose that $\zeta(0) \neq 1$. Then, according to (2), $Im\zeta = \{a, b\}$, where $0 \leq a < b < 1$. Since $\zeta(0) \geq \zeta(x)$ for all $x \in S$, we have $\zeta(0) = b$. Thus,

$$\zeta(x) = \begin{cases} b & \text{if } x \in \zeta^0, \\ a & \text{otherwise,} \end{cases}$$

Consider, for fixed $p \in \zeta^0$ and $q \in S \setminus \zeta^0$, two fuzzy subsets

$$\mu(x) = \begin{cases} t & \text{if } x \in \langle p \rangle, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{and} \qquad \nu(x) = \begin{cases} r & \text{if } x \in \langle q \rangle, \\ 0 & \text{otherwise} \end{cases}$$

where $0 \leq a < r < b < t \leq 1$.

It is clear that μ and ν are M-fuzzy left h-ideals of S.

Then, for any $x \in S$, if x does not satisfy the equality

$$x + m_1\gamma_1 b_1 + z = m_2\gamma_2 b_2 + z,$$

where $z \in S, m_1, m_2 \in \langle p \rangle$, $b_1, b_2 \in \langle q \rangle$ and $\gamma_1, \gamma_2 \in \Gamma$, we have

$$(\nu\Gamma\mu)(x) = 0.$$

Otherwise,

$$\begin{aligned} (\mu\Gamma_h\nu)(x) &= \bigvee_{x+m_1\gamma_1 b_1+z=m_2\gamma_2 b_2+z} (\min\{\mu(m_1), \mu(m_2), \nu(b_1), \nu(b_2)\}) \\ &= \min\{t, r\} = r. \end{aligned}$$

By (1), ζ^0 is a prime left M-h-ideal. If $a_1, a_2 \in \langle p \rangle$, then $a_1, a_2 \in \zeta^0$, because $p \in \zeta^0$ and $\langle p \rangle \subseteq \zeta^0$. This implies $x \in \zeta^0$. Thus $\zeta(x) = b > r = (\mu\Gamma_h\nu)(x)$. Therefore, $\mu\Gamma_h\nu \subseteq \zeta$. But $\mu(p) = t > b = \zeta(p)$ and $\nu(q) = r > a = \zeta(q)$, which gives $\mu \nsubseteq \zeta$ and $\nu \nsubseteq \zeta$. This contradicts to the assumption that ζ is a prime M-fuzzy left h-ideal of S. Hence $\zeta(0) = 1$. ■

5. h-semisimple M-Γ-hemirings

In this section, we describe the characterizations of h-semisimple M-Γ-hemirings.

Definition 5.1

(1) A subset A of S is said to be Γ-*idempotent* if $A = \overline{A\Gamma A}$.

(2) A fuzzy set μ over S is said to be M-*fuzzy idempotent* if $\mu = \mu\widetilde{\Gamma}_h\mu$.

(3) An M-Γ-hemiring S is said to be h-*semisimple* if every M-h-ideal is Γ-idempotent.

Now, we can give the following lemma.

Lemma 5.2 *Let S be an M-Γ-hemiring. Then the following statements are equivalent:*

(1) *S is h-semisimple,*

(2) *$x \in \overline{M\Gamma x\Gamma S\Gamma x\Gamma S}$, for all $x \in S$,*

(3) *$A \subseteq \overline{M\Gamma A\Gamma S\Gamma A\Gamma S}$, for all $A \in S$.*

Proof. (1) $\Rightarrow$ (2): Let S be an h-semisimple M-Γ-hemiring. Then, for any $x \in S$, we have

$$\overline{M\Gamma S + S\Gamma x + S\Gamma x\Gamma S + \mathbb{N}_0 x},$$

where $\mathbb{N}_0 = \{0, 1, 2, \cdots\}$, is the principle M-h-ideal of S generated by x. Thus,

$$x \in (\overline{M\Gamma S + S\Gamma x + S\Gamma x\Gamma S + \mathbb{N}_0 x})\Gamma(\overline{M\Gamma S + S\Gamma x + S\Gamma x\Gamma S + \mathbb{N}_0 x}) \subseteq \overline{M\Gamma x\Gamma S\Gamma x\Gamma S},$$

which implies $x \in \overline{M\Gamma x\Gamma S\Gamma x\Gamma S}$, for all $x \in \overline{M\Gamma x\Gamma S\Gamma x\Gamma S}$.

(2) $\Rightarrow$ (3) It is obvious.

(3) $\Rightarrow$ (1) Let A be any M-h-ideal of S. Then,

$$A \subseteq \overline{M\Gamma A\Gamma S\Gamma A\Gamma S} \subseteq \overline{A\Gamma S\Gamma SA} \subseteq \overline{A\Gamma A}.$$

Therefore, S is h-semisimple. ∎

Next, we discuss the relationship between M-fuzzy h-ideals and M-fuzzy h-interior ideals in h-semisimple M-Γ-hemirings.

Theorem 5.3 *Let S be an h-semisimple M-Γ-hemiring and let μ be any fuzzy set of S. Then μ is an M-fuzzy h-ideal if and only if it is an M-fuzzy h-interior ideal.*

Proof. If μ is an M-fuzzy h-ideal of S. Then, by Proposition 3.4, we know that μ is an M-fuzzy h-interior ideal.

Conversely, if μ is an M-fuzzy h-interior ideal of S. For any $x, y \in S$ and $\alpha \in \Gamma$. Since S is h-semisimple, by Lemma 5.2, there exist $a_i, a_i', z \in S (i = 1, 2, 3)$, $\beta_i, \beta_i', \in \Gamma(i = 1, 2, 3, 4, 5)$ and $m, m' \in M$ such that

$$x + m\beta_1 x\beta_2 a_1\beta_3 a_2\beta_4 x\beta_5 a_3 + z = m'\beta_1' x\beta_2' a_1'\beta_3' a_2'\beta_4' x\beta_5' a_3' + z,$$

and so

$$x\alpha y + m\beta_1 x\beta_2 a_1\beta_3 a_2\beta_4 x\beta_5 a_3\alpha y + z\alpha y = m'\beta_1' x\beta_2' a_1'\beta_3' a_2'\beta_4' x\beta_5' a_3'\alpha y + z\alpha y.$$

Thus we have

$$\begin{aligned}\mu(x\alpha y) &\geqslant \mu(m\beta_1 x\beta_2 a_1\beta_3 a_2\beta_4 x\beta_5 a_3\alpha y) \wedge \mu(m'\beta_1' x\beta_2' a_1'\beta_3' a_2'\beta_4' x\beta_5' a_3'\alpha y)\\ &\geqslant \mu(x).\end{aligned}$$

This proves that μ is an M-fuzzy right h-ideal of S. Similarly, we can prove that μ is an M-fuzzy left h-ideal of S. Therefore μ is an M-fuzzy h-ideal of S. ■

Finally, we give a characterization of h-semisimple M-Γ-hemirings by M-fuzzy h-interior ideals.

Theorem 5.4 *An M-Γ-hemiring S is h-semisimple if and only if $\mu \cap \nu = \mu\Gamma_h\nu$, for any M-fuzzy h-interior ideals μ and ν.*

Proof. Let S be an h-semisimple M-Γ-hemiring. If μ and ν are M-fuzzy h-interior ideals, then by Proposition 3.4, we know μ and ν are M-fuzzy h-ideals of S. Thus, we have $\mu\Gamma_h\nu \subseteq \mu\Gamma_h\chi_S \subseteq \mu$ and $\mu\Gamma_h\nu \subseteq \chi_S\Gamma_h\nu \subseteq \nu$. So $\mu\Gamma_h\nu \subseteq \mu \cap \nu$.

For any $x \in S$, since S is h-semisimple, by Lemma 5.2, there exist $a_i, a_i', z \in S$ $(i = 1, 2, 3)$, $\beta_i, \beta_i', \in \Gamma$ $(i = 1, 2, 3, 4, 5)$ and $m, m' \in M$ such that

$$x + m\beta_1 x\beta_2 a_1\beta_3 a_2\beta_4 x\beta_5 a_3 + z = m'\beta_1' x\beta_2' a_1'\beta_3' a_2'\beta_4' x\beta_5' a_3' + z,$$

Thus we have

$$\begin{aligned} (\mu\Gamma_h\nu)(x) &= \bigvee_{x+m_1\gamma_1 b_1+z=m_2\gamma_2 b_2+z} (\min\{\mu(m_1), \mu(m_2), \nu(b_1), \nu(b_2)\}) \\ &\geq \min\{\mu(m\beta_1 x\beta_2 a_1), \mu(m'\beta_1' x\beta_2' a_1'), \nu(a_2\beta_4 x\beta_5 a_3), \nu(a_2'\beta_4' x\beta_5' a_3')\} \\ &\geq \min\{\mu(x), \nu(x)\} \\ &= (\mu \cap \nu)(x), \end{aligned}$$

i.e., $\mu \cap \nu \subseteq \mu\Gamma_h\nu$, whence $\mu \cap \nu = \mu\Gamma_h\nu$.

Conversely, let A be any M-h-ideal of S, then it is an M-h-interior ideal. Thus, we have

$$\chi_A = \chi_A \cap \chi_A = \chi_A\Gamma_h\chi_A = \chi_{\overline{A\Gamma A}},$$

which implies, $A = \overline{A\Gamma A}$. Thus S is h-semisimple. ■

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ON ALGEBRAIC AND ANALYTIC CORE II

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Abstract. In this paper, we continue the study of the algebraic core spectrum and the analytic core spectrum of an operator T on the complex Banach space X: $\sigma_{alc}(T) = \{\lambda \in \mathbb{C} : C(\lambda I - T) = \{0\}\}$ and $\sigma_{ac}(T) = \{\lambda \in \mathbb{C} : K(\lambda I - T) = \{0\}\}$ where $C(T)$ and $K(T)$ are respectively the algebraic core and the analytic core for T. We shall be concerned with the relations between $\sigma_{ac}(\cdot)$ $(\sigma_{alc}(\cdot))$ and different classical parts of spectrum: the point spectrum, the approximate point spectrum, the surjectivity spectrum and the Kato spectrum. Moreover, some applications are given.

Keywords: local spectral theory, algebraic core spectrum, analytic core spectrum, Kato resolvent set, quasi-similar operators.

1. Introduction

Throughout, X denotes a complex Banach space and $\mathcal{B}(X)$ denotes the Banach algebra of all bounded linear operators on X, let I be the identity operator, and for $T \in \mathcal{B}(X)$ we denote by T^*, $R(T)$, $R^\infty(T) = \bigcap_{n\geq 0} R(T^n)$, $\rho(T)$, $\sigma(T)$,

$\sigma_p(T)$, $\sigma_{ap}(T)$ and $\sigma_{su}(T)$ respectively the adjoint, the range, the hyper-range, the resolvent set, the spectrum, the point spectrum, the approximate point spectrum and the surjectivity spectrum of T.

Recall that for $T \in \mathcal{B}(X)$ and $x \in X$ the local resolvent of T at x defined as the union of all open subset U of $\mathbb{C}$ for which there is an analytic function $f : U \to X$ such that the equation $(T - \mu I)f(\mu) = x$ holds for all $\mu \in U$. The local spectrum $\sigma_T(x)$ of T at x is defined as $\sigma_T(x) = \mathbb{C} \setminus \rho_T(x)$. Evidently $\rho(T) \subseteq \rho_T(x)$, $\rho_T(x)$ is open and $\sigma_T(x)$ is closed.

Also, an important concept in local spectral theory is the local spectral subspace for an operator $T \in \mathcal{B}(X)$. For subset Ω of $\mathbb{C}$ the local spectral subspace of T associated with Ω is the set $X_T(\Omega) = \{x \in X : \sigma_T(x) \subseteq \Omega\}$, evidently $X_T(\Omega)$ is a hyperinvariant subspace of T not always closed, if $\Omega_1 \subseteq \Omega_2 \subseteq \mathbb{C}$ then $X_T(\Omega_1) \subseteq X_T(\Omega_2)$. We refer the reader to [1], [3], [4], [6] for the properties of the local spectrum and local spectral subspaces .

Next, let $T \in \mathcal{B}(X)$, T is said to have the single valued extension property at $\lambda_0 \in \mathbb{C}$ (SVEP) if for every open neighbourhood $U \subseteq \mathbb{C}$ of λ_0, the only analytic function $f : U \longrightarrow X$ which satisfies the equation $(T - zI)f(z) = 0$ for all $z \in U$ is the function $f \equiv 0$. T is said to have the SVEP if T has the SVEP for every $\lambda \in \mathbb{C}$. Denote by $A(T) = \{\lambda \in \mathbb{C} : T \text{ has the SVEP at } \lambda\}$, by [3, Proposition 1.2.16] $A(T) = \mathbb{C}$ if and only if $X_T(\emptyset) = \{0\}$, if and only if $X_T(\emptyset)$ is closed.

Recall that $T \in \mathcal{B}(X)$ is said to be Kato operator or semi-regular [3], [7] if $R(T)$ is closed and $N(T-\lambda) \subseteq R^\infty(T-\lambda)\}$. Denote by $\rho_K(T)$: $\rho_K(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is Kato }\}$ the Kato resolvent and $\sigma_K(T) = \mathbb{C} \backslash \rho_K(T)$ the Kato spectrum of T. It is well known that $\rho_K(T)$ is on open subset of $\mathbb{C}$ and may be decomposed in connected disjoint open nonempty components [1], $\sigma_K(T)$ play an important role in local spectral theory; in particular, we have

$$\partial\sigma_T(x) \subseteq \sigma_K(T) \subseteq \sigma_{su}(T) \cap \sigma_{ap}(T) \subseteq \sigma(T) \text{ for all } x \in X.$$

According to [1, Definition 1.40], we say that $T \in \mathcal{B}(X)$ admits a generalized Kato decomposition, abbreviated GKD if there exists a pair of T-invariant closed subspaces (M, N) such that $X = M \oplus N$, the restriction $T_{|M}$ is semi-regular, and $T_{|N}$ is quasinilpotent. Obviously, every Kato operator admits a GKD because in this case $M = X$ and $N = \{0\}$, again the quasi-nilpotent operator admits a GKD: Take $M = \{0\}$ and $N = X$. If we suppose that $T_{|N}$ is nilpotent of order $d \in \mathbb{N}$, then T is said to be of Kato type of operator of order d. Finally, T is said essentially semi-regular if it admits a GKD (M, N) such that N is finite-dimensional. Evidently, every essentially semi-regular operator is of Kato type. The Kato type spectrum of T is defined by

$$\sigma_{Kt}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not of Kato type}\},$$

evidently $\sigma_{Kt}(T) \subseteq \sigma_K(T)$. We refer to [1] for more information about the Kato type spectrum.

Let $T \in \mathcal{B}(X)$. The ascent of T is defined by

$$a(T) = \min\{p : N(T^p) = N(T^{p+1})\}.$$

If such p does not exist, we let $a(T) = \infty$. Analogously, the descent of T is $d(T) = \min\{q : R(T^q) = R(T^{q+1})\}$; if such q does not exist, we let $d(T) = \infty$ [4]. It is well known that, if both $a(T)$ and $d(T)$ are finite, then $a(T) = d(T)$ and we have the decomposition $X = R(T^p) \oplus N(T^p)$, where $p = a(T) = d(T)$.

Recall that, for $T \in \mathcal{B}(X)$, the algebraic core $C(T)$ for T is the greatest subspace M of X for which $T(M) = M$. Equivalently,

$$C(T) = \{x \in X : \exists (x_n)_{n\geq 0} \subset X, \text{ such that } x_0 = x, Tx_n = x_{n-1} \text{ for all } n \geq 1\}$$

Moreover, the analytical core for T is a linear subspace of X defined by:

$$K(T) = \{x \in X : \exists (x_n)_{n\geq 0} \subset X \text{ and } \delta > 0 \text{ such that } x_0 = x,\ Tx_n = x_{n-1}, \\ \forall n \geq 1 \text{ and } \|x_n\| \leq \delta^n \|x\|\}$$

There are some relations between the algebraic core and the analytical core, see [1], [3], [5], [9], [11]:

- $T(K(T)) = K(T)$, $T(C(T)) = C(T)$ and $K(T) \subseteq C(T)$.
- If $C(T)$ is closed, then $C(T) = K(T)$.
- $K(T) = X_T(\mathbb{C}\backslash\{0\}) = \{x \in X : 0 \in \rho_T(x)\}$.
- $X_T(\emptyset) \subset K(T) \subseteq C(T) \subset R^\infty(T) \subset R(T)$.
- $N(T - \lambda I) \subseteq K(T - \mu I)$ for all $\lambda \neq \mu$.
- The function : $\lambda \to K(T - \lambda I)$ is constant on component of $\rho_K(T)$.
- If $\lambda \in \rho_K(T)$, then:
$$K(T - \lambda I) = C(T - \lambda I) = X_T(\mathbb{C}\backslash\{\lambda\}) = R^\infty(T - \lambda I).$$

Now, denote

$$\mathcal{R}_{ac}(X) = \{T \in \mathcal{B}(X) : K(T) \neq \{0\}\}$$
$$\mathcal{R}_{alc}(X) = \{T \in \mathcal{B}(X) : C(T) \neq \{0\}\}$$

In [8], we have investigated the study of sets $\mathcal{R}_{ac}(X)$ and $\mathcal{R}_{alc}(X)$, we have showed that these parts of $\mathcal{B}(X)$ are regularities in Kordulla-Müller's sense; consequently

$$\begin{aligned} \sigma_{alc}(T) &= \{\lambda \in \mathbb{C} : \lambda I - T \notin \mathcal{R}_{alc}(X)\} = \{\lambda \in \mathbb{C} : C(\lambda I - T) = \{0\}\} \text{ and} \\ \sigma_{ac}(T) &= \{\lambda \in \mathbb{C} : \lambda I - T \notin \mathcal{R}_{ac}(X)\} = \{\lambda \in \mathbb{C} : K(\lambda I - T) = \{0\}\} \end{aligned}$$

respectively the algebraic core spectrum and the analytic core spectrum satisfie the mapping spectral theorem. We refer the reader to [2], [6], [7] for properties of the regularity theory.

Write $\rho_{alc}(T) = \mathbb{C}\backslash\sigma_{alc}(T)$ and $\rho_{ac}(T) = \mathbb{C}\backslash\sigma_{ac}(T)$ respectively the algebraic core resolvent and the analytic core resolvent of T.

In the following section we continue the study of relations between $\sigma_{alc}(.)$, $\sigma_{ac}(\cdot)$ or $\rho_{alc}(\cdot)$, $\rho_{ac}(\cdot)$ and the classical parts of spectrum: $\sigma_p(\cdot)$, $\sigma_{ap}(\cdot)$, $\sigma_{su}(\cdot)$, $\sigma_K(\cdot)$ respectively the point spectrum, the approximate point spectrum, the surjectivity spectrum and the Kato spectrum. On other hand some results and applications are given.

2. Main results

We begin by the following proposition.

Proposition 2.1 *Let $T \in \mathcal{B}(X)$. Then*

$$\sigma_{alc}(T) \subseteq \sigma_{ac}(T) \subseteq \sigma_{su}(T).$$

Proof. Let $\lambda \in \mathbb{C} \setminus \sigma_{su}(T)$, no loss of generality we can assume that $\lambda = 0$, we have $T(X) = X = K(T)$; hence $K(T) \neq \{0\}$ and consequently $0 \in \mathbb{C} \setminus \sigma_{ca}(T)$. ■

Remarks.

1. We showed already in [11] that $\sigma_{alc}(T) \subseteq \sigma_{ac}(T) \subseteq \sigma_T(x)$ for all $x \in X\setminus\{0\}$; on the other hand, we know that $\sigma_{su}(T) = \bigcup_{x \in X} \sigma_T(x)$. Then, we obtain Proposition 2.1.

2. If, for all $\lambda \in \rho_K(T)$, we have $R^\infty(T - \lambda I) \neq \{0\}$, then

$$\sigma_{alc}(T) \subseteq \sigma_{ac}(T) \subseteq \sigma_K(T) \subseteq \sigma_{su}(T)$$

Proposition 2.2 *Let $T \in \mathcal{B}(X)$. Then $\sigma_{ac}(T) \subseteq A(T)$.*

Proof. Let $\lambda \in \sigma_{ac}(T)$ then $K(T - \lambda I) = \{0\}$. Since $X_{T-\lambda I}(\emptyset) \subseteq K(T - \lambda I)$, then $X_{T-\lambda I}(\emptyset) = \{0\}$; therefore, T satisfies the (SVEP) in λ. ■

Remarks.

1. If $0 \in \sigma_{ac}(T)$, then $\sigma_p(T) \subseteq \{0\}$. Indeed, we have $N(T-\lambda I) \subseteq K(T) = \{0\}$, for all $\lambda \neq 0$, then $\sigma_p(T) \subseteq \{0\}$.

2. Let $T \in \mathcal{B}(X)$, then $\sigma_{alc}(T) \subseteq \sigma_{ac}(T) \subseteq \sigma(T_{|F})$, for all closed subspace $F \neq \{0\}$ of X. Indeed, let $\lambda \in \sigma_{ca}(T)$, then $K(T - \lambda I) = \{0\}$. If $(T - \lambda I)_{|F}$ is invertible, then $(T - \lambda I)(F) = F$, therefore $F \subseteq K(T - \lambda I) = \{0\}$, contradiction.

3. Let $T \in \mathcal{B}(X)$, assume then T is of Kato type, then $K(T) = R^\infty(T)$. If we suppose that, for all $\lambda \in \rho_{Kt}(T)$, we have $R^\infty(T - \lambda I) \neq \{0\}$, then

$$\sigma_{alc}(T) \subseteq \sigma_{ac}(T) \subseteq \sigma_{Kt}(T)$$

Example 1. Let $T \in \mathcal{B}(X)$ a quasi-nilpotent operator, then $\sigma(T) = \{0\}$, by [1, Corollary 2.28] we have $K(T) = \{0\}$, therefore $\sigma_{ac}(T) = \sigma(T) = \{0\}$.

Example 2. Let $T \in \mathcal{B}(X)$ an injective compact operator. It is well known that $\sigma(T)$ contains at most countable set of point, and each nonzero point of $\sigma(T)$ is an isolated eigenvalue, i.e., $\sigma(T) = \{0\} \bigcup \sigma_p(T)$. Moreover, for each $\lambda \in \sigma(T)\backslash\{0\}$, we know that $T - \lambda I$ is Fredholm operator. By [1, Corollary 3.21], we have $K(T - \lambda I) = R^\infty(T - \lambda I) = (T - \lambda I)^p(X)$ where $p = d(T - \lambda I) = a(T - \lambda I)$. Then, we obtain $\sigma_{ac}(T) \subseteq \{0\}$, but if $\sigma_{ac}(T) = \{0\}$ then, by the last remark (1), it follows that $\sigma_p(T) \subseteq \{0\}$, a contradiction. Consequently, $\sigma_{ac}(T) = \emptyset$.

Lemma 2.1 *Let $T \in \mathcal{B}(X)$. Then*

$$\rho_K(T) \cap \sigma_{ap}(T) \subseteq \rho_{ac}(T).$$

Proof. Let $\lambda \in \rho_K(T) \cap \sigma_{ap}(T)$, then $N(T - \lambda I) \neq \{0\}$ and $N(T - \lambda I) \subseteq R^\infty(T - \lambda I)$, hence there exists $x \in N(T - \lambda I) \subseteq R^\infty(T - \lambda I) = K(T - \lambda I)$, this implies that $K(T - \lambda I) \neq \{0\}$ and consequently $\lambda \in \rho_{ac}(T)$. ∎

Lemma 2.2 *Let $T \in \mathcal{B}(X)$. Then*

$$\rho_K(T) \cap \sigma_{su}(T) \subseteq \rho_{ac}(T^*).$$

Proof. We know that $\sigma_{su}(T) = \sigma_{ap}(T^*)$ and $\rho_K(T) = \rho_K(T^*)$. Therefore, $\rho_K(T) \cap \sigma_{su}(T) = \rho_K(T^*) \cap \sigma_{ap}(T^*) \subseteq \rho_{ac}(T^*)$. ∎

Proposition 2.3 *Let $T \in \mathcal{B}(X)$. Then*

$$\rho_K(T) \cap \sigma(T) \subseteq \rho_{ca}(T^*) \cup \rho_{ac}(T).$$

Proof. It is well known that $\sigma(T) = \sigma_{ap}(T) \cup \sigma_{su}(T)$ and $\sigma_p(T) \subseteq \sigma_{ap}(T)$. Apply now Lemmas 2.1 and 2.2. ∎

Proposition 2.4 *Let $T \in \mathcal{B}(X)$. Then*

$$[\sigma_{ap}(T) \cap \sigma_{su}(T)]\backslash[\rho_{ca}(T) \cap \rho_{ca}(T^*)] \subseteq \sigma_K(T).$$

Proof. By Lemmas 2.1 and 2.2, we have

$$\rho_K(T) \cap \sigma_{ap}(T) \cap \sigma_{su}(T) \subseteq \rho_{ac}(T) \cap \rho_{alc}(T^*).$$

Consequently, $[\sigma_{ap}(T) \cap \sigma_{su}(T)]\backslash[\rho_{ac}(T) \cap \rho_{ac}(T^*)] \subseteq \sigma_K(T)$. ∎

Proposition 2.5 *Let $T \in \mathcal{B}(X)$. Then*

1. $\sigma(T)\backslash\sigma_{ap}(T) \subseteq \rho_K(T) \cap \sigma_{su}(T) \subseteq \rho_{ac}(T^*)$
2. $\sigma(T)\backslash\sigma_{su}(T) \subseteq \rho_K(T) \cap \sigma_{ap}(T) \subseteq \rho_{ac}(T)$

Proof. 1. Let $\lambda \in \sigma(T)\backslash\sigma_{ap}(T)$ then $T-\lambda I$ is not invertible, since $N(T-\lambda I) = \{0\}$ and $R(T-\lambda I)$ is closed, then $\lambda \in \sigma_{su}(T)$ and $\lambda \in \rho_K(T)$. It follows that $\sigma(T)\backslash\sigma_{ap}(T) \subseteq \rho_K(T) \cap \sigma_{su}(T)$, by Lemma 2.2 we conclude 1.

2. is immediate by duality $\sigma_{su}(T) = \sigma_{ap}(T^*)$ and $\sigma_{su}(T^*) = \sigma_{ap}(T)$. ∎

Proposition 2.6 *Let $T \in \mathcal{B}(X)$. Then*

1. $\sigma_K(T) \cup \rho_{ca}(T) = \sigma_{ap}(T) \cup \rho_{ac}(T)$;
2. $\sigma_K(T) \cup \rho_{ca}(T^*) = \sigma_{su}(T) \cup \rho_{ac}(T^*)$.

Proof. 1. Since $\sigma_K(T) \subseteq \sigma_{ap}(T)$, then $\sigma_K(T)\cup\rho_{ac}(T) \subseteq \sigma_{ap}(T)\cup\rho_{ac}(T)$. Suppose that $\lambda \notin \sigma_K(T)\cup\rho_{ac}(T)$, then $R(T-\lambda I)$ is closed and $N(T-\lambda I) \subseteq R^{\infty}(T-\lambda I) = K(T-\lambda I) = \{0\}$, therefore $\lambda \notin \sigma_{ap}(T)$ and $\lambda \notin \rho_{ac}(T)$.

2. is clair by duality. ∎

Theorem 2.1 *Let $T \in \mathcal{B}(X)$. Then, for all subsets $\Omega \subseteq$ of $\mathbb{C}$, we have*

$$X_T(\Omega) \neq \{0\} \Rightarrow \sigma_{ac}(T) \subseteq \Omega.$$

Proof. Let $\lambda \notin \Omega$ then

$$\{0\} \neq X_T(\Omega) = X_T(\Omega\backslash\{\lambda\}) \subseteq X_T(\mathbb{C}\backslash\{\lambda\}) = K(T-\lambda I).$$

Therefore, $\lambda \notin \sigma_{ac}(T)$. ∎

Proposition 2.7 *Let $T \in \mathcal{B}(X)$, then*

$$\lambda \in \sigma_p(T) \Longrightarrow \sigma_{ac}(T) \subseteq \{\lambda\}.$$

Proof. Let $\lambda \in \sigma_p(T)$. This implies $\{0\} \neq N(T-\lambda I) \subseteq K(T-\mu)$ for all $\lambda \neq \mu$. Consequently, $K(T-\mu I) \neq \{0\}$ for all $\mu \neq \lambda$, hence $\sigma_{ac}(T) \subseteq \{\lambda\}$. ∎

Theorem 2.2 *Let $T \in \mathcal{B}(X)$, if Ω is a connected component of $\rho_K(T)$ we have*

$$\Omega \subseteq \rho_{ac}(T) \Longleftrightarrow \bigcap_{\lambda\in\Omega} R^{\infty}(T-\lambda I) \neq \{0\}.$$

Proof. Suppose that $\Omega \subseteq \rho_{ac}(T)$. Then we have $K(T-\lambda I) \neq \{0\}$ for all $\lambda \in \Omega$. Since $\Omega \subseteq \rho_{ac}(T)$, then the application $\lambda \to K(T-\lambda I)$ is constant in Ω, and

$$\{0\} \neq K(T-\lambda I) = \bigcap_{\lambda\in\Omega} K(T-\lambda I) = \bigcap_{\lambda\in\Omega} R^{\infty}(T-\lambda I).$$

So, it follows that

$$\bigcap_{\lambda\in\Omega} R^{\infty}(T-\lambda I) \neq \{0\}.$$

Conversely, since $\{0\} \neq \bigcap_{\lambda\in\Omega} R^{\infty}(T-\lambda I) = R^{\infty}(T-\lambda I) = K(T-\lambda I)$, therefore $K(T-\lambda I) \neq \{0\}$ for all $\lambda \in \Omega$. ∎

Corollary 2.1 *Let $T \in \mathcal{B}(X)$, if Ω is a connected component of $\rho_K(T)$ we have*

$$\rho_{ac}(T) \cap \Omega \neq \emptyset \Longrightarrow \Omega \subseteq \rho_{ac}(T)$$

Proof. Let $\lambda_0 \in \Omega \cap \rho_{ac}(T)$, then for all $\lambda \in \Omega$ we obtain

$$R^{\infty}(T - \lambda I) = K(T - \lambda I) = K(T - \lambda_0 I) \neq \{0\}$$

because $\lambda \to K(T - \lambda I)$ is constant, hence $K(T - \lambda I) \neq \{0\}$ for all $\lambda \in \Omega$ and, therefore, $\Omega \subseteq \rho_{ac}(T)$. ■

Remark. Know that $\sigma_{ca}(T)$ is closed; then immediately, by Corollary 2.1,

$$\sigma_{ca}(T) \cap \Omega \neq \emptyset \Longrightarrow \overline{\Omega} \subseteq \sigma_{ca}(T).$$

Theorem 2.3 *Let $T \in \mathcal{B}(X)$ and Ω be connected components of $\rho_K(T)$, such that $G \cap \sigma_{ac}(T) \neq \emptyset$. Then*

1. *$\sigma_p(T)$ is empty;*

2. *$\sigma(T)$ and $\sigma_T(x)$ are connected $\forall x \in X$.*

Proof.

1. Suppose that $\lambda \in \sigma_p(T)$, then $\sigma_{ac}(T) \subseteq \{\lambda\}$, this is a contradiction because $\sigma_{ac}(T) \cap \Omega \neq \emptyset \Longrightarrow \Omega \subseteq \sigma_{ac}(T)$.

2. Suppose that there exists $x_0 \in X$ such that $\sigma_T(x_0)$ is non-connected. Then, there is two non-empty closed subsets σ_1 and σ_2 of $\mathbb{C}$ such that $\sigma_T(x_0) = \sigma_1 \cup \sigma_2$ and $\sigma_1 \bigcap \sigma_2 = \emptyset$. By [1, Theorem 2.17] there exists $x_1, x_2 \in X$ such that $\sigma_T(x_1) \subseteq \sigma_1$ and $\sigma_T(x_2) \subseteq \sigma_2$. Therefore,

$$G \subseteq \sigma_{ac}(T) \subseteq \sigma_T(x_1) \cap \sigma_T(x_2) \subseteq \sigma_1 \cap \sigma_2 = \emptyset,$$

a contradiction. Now, since $\sigma_p(T) = \emptyset$ by 1), then T has the SVEP, hence

$$\sigma(T) = \sigma_{su}(T) = \bigcup_{x \in X} \sigma_T(x).$$

Consequently, $\sigma(T)$ is connected. ■

Example 3. Let $\mathcal{H}$ be a separable Hilbert space with an orthonormal basis $(e_n)_{n \geq 0}$, and let $\omega := (\omega_n)_{n \geq 0}$ be a bounded sequence of strictly positive real numbers. Consider the unilateral weighted right shift operator defined by [3], [10]:

$$Se_n = \omega_n e_{n+1}$$

- The spectrum of S is given by

$$\sigma(S) = \{z \in \mathbb{C} : |z| \leq r(S)\}$$

- The approximate point spectrum of S is the annulus

$$\sigma_{ap}(S) = \{z \in \mathbb{C} : r_1(S) \leq |z| \leq r(S)\}$$

Suppose that $r_1(S) > 0$ and let $\Omega := \{z \in \mathbb{C} : |z| < r_1(S)\}$, then

$$\Omega \subseteq \mathbb{C} \backslash \sigma_{ap}(S) \subseteq \rho_K(S)$$

and G is a connected component of $\rho_K(T)$. We have

$$\bigcap_{n\geq 0} R(S^n) = \{0\},$$

hence $K(T) = \{0\}$ and $0 \in \sigma_{ac}(S)$.

Therefore, $\sigma_{ac}(S) \cap \Omega \neq \emptyset$. By Theorem 2.3, it follows that:

1. $\sigma_p(S)$ is empty;
2. $\sigma_S(x)$ is connected for all $x \in \mathcal{H}$;
3. $\sigma(S)$ is connected.

Let $T, S \in \mathcal{B}(X)$, T and S are said quasi-similarly if there is $R, L \in \mathcal{B}(X)$ injective and has dense range such that $RT = SR$ and $TL = LS$. We said that T and S are similar if there exists $R \in \mathcal{B}(X)$ invertible such that $TR = RS$.

Recall that tow similar operators are some spectral properties (spectrum, approximate point spectrum, essential spectrum...)

In the following result, we show that the algebraic core spectrum and analytic core spectrum are invariant by similarity.

Theorem 2.4 *Let $T, S \in \mathcal{B}(X)$ such that Let T and S are quasi-similar, then*

$$\sigma_{alc}(T) = \sigma_{alc}(S) \quad and \quad \sigma_{ac}(T) = \sigma_{ac}(S).$$

Proof. Since T and S are quasi-similar, then there exists $R, L \in \mathcal{B}(X)$ such that $RT = SR$ and $TL = LS$. Therefore $T - \lambda I$ and $S - \lambda I$ are quasi-similar for all $\lambda \in \mathbb{C}$.

We show that $R(K(T - \lambda I)) \subseteq K(S - \lambda I)$. Indeed, with no loss of the generality we can suppose that $\lambda = 0$. Let $y \in R(K(T))$, then $y = Rx$ such that $x \in K(T)$ or equivalently there exists a sequence $(x_n)_{n\geq 0} \subseteq X$ and $\delta > 0$ satisfying $Tx_n = x_{n-1}$, $x = x_0$ and $||x_n|| < \delta^n ||x||$.

Consider the sequence $(y_n)_{n\geq 0}$, where $y_n = Rx_n$, we have $y_0 = Rx$, $Sy_n = SRx_n = RTx_n = Rx_{n-1} = y_{n-1}$ and $||y_n|| < ||R|| |\delta^n| |x||$, which implies that $R(K(T)) \subseteq K(S)$. And, by similarity, we prove that $L(K(S - \lambda I)) \subseteq K(T - \lambda I)$.

Now, if $K(S - \lambda I) = \{0\}$ then by injectivity of R we have $K(T - \lambda I) = \{0\}$. Let $\lambda \in \sigma_{ac}(S)$, then $K(S - \lambda I) = \{0\}$ and it follows that $K(T - \lambda I) = \{0\}$ and $\lambda \in \sigma_{ac}(T)$, consequently $\sigma_{ac}(S) \subseteq \sigma_{ac}(T)$.

Similarly, we have $\sigma_{ac}(T) \subseteq \sigma_{ac}(S)$. ∎

Theorem 2.5 *For two injective operators $T, S \in \mathcal{B}(X)$, the following statements hold:*

1. $K(ST - \lambda I) \neq \{0\} \Leftrightarrow K(TS - \lambda I) \neq \{0\}$, *for all* $\lambda \in \mathbb{C}$;
2. $C(ST - \lambda I) \neq \{0\} \Leftrightarrow C(TS - \lambda I) \neq \{0\}$, *for all* $\lambda \in \mathbb{C}$;
3. $\sigma_{alc}(TS) = \sigma_{alc}(ST)$ *and* $\sigma_{ac}(TS) = \sigma_{ac}(ST)$.

Proof. 1. We begin by the implication $K(ST-\lambda I) \neq \{0\} \Longrightarrow K(TS-\lambda I) \neq \{0\}$ $\forall \lambda \in \mathbb{C}$. Of course, if $K(ST-\lambda I) \neq \{0\}$, then there exists a sequence $(x_n)_{n\geq 0} \subseteq X$ and $\delta > 0$ such that $x := x_0 \neq 0$, $(ST - \lambda I)x_n = x_{n-1}$ and $||x_n|| < \delta^n ||x||$.

Let $z_n := Tx_n$. We have $(TS - \lambda I)z_n = (TS - \lambda I)Tx_n = T(ST - \lambda)x_n = Tx_{n-1} = z_{n-1}$. Since T is injective, then $z := z_0 = Tx \neq 0$. On the other hand, $||z_n|| < \delta^n ||z||$. Hence $z \in K(TS - \lambda I)$ and consequently $K(TS - \lambda I) \neq \{0\}$.

Conversely, $K(TS - \lambda I) \neq \{0\}$ implies that there is $(x_n)_{n\geq 0} \subseteq X$ and $\delta > 0$ which $x := x_0 \neq 0$, $(TS - \lambda I)x_n = x_{n-1}$ and $||x_n|| < \delta^n ||x||$.

Let $z_n := Sx_n$, then $(ST-\lambda I)z_n = (ST-\lambda I)Sx_n = S(TS-\lambda I)x_n = Sx_{n-1} = z_{n-1}$. But S is injective then $z := z_0 = Sx \neq 0$, and $||z_n|| < \delta^n ||z||$. Therefore, $z \in K(ST - \lambda I)$ and $K(ST - \lambda I) \neq \{0\}$.

2. Similar to 1.

3. Apply 1, 2 and the definition of $\sigma_{alc}(TS)$ and $\sigma_{ac}(TS)$. ∎

Theorem 2.6 *Let $T, S, R \in \mathcal{B}(X)$ such that T is injective and $TST = TRT$. Let $\lambda \in \mathbb{C}$. Then*

1. $K(ST - \lambda I) \neq \{0\} \Longrightarrow K(TR - \lambda I) \neq \{0\}$;
2. $C(ST - \lambda I) \neq \{0\} \Longrightarrow C(TR - \lambda I) \neq \{0\}$.

Either, if $ST^2 = T^2S$, then

$$K(ST - \lambda I) \neq \{0\} \Longleftrightarrow K(TR - \lambda I) \neq \{0\};$$
$$C(ST - \lambda I) \neq \{0\} \Longleftrightarrow C(TR - \lambda I) \neq \{0\}.$$

Proof. 1. Suppose $K(ST - \lambda I) \neq \{0\}$, then there is a sequence $(x_n)_{n\geq 0} \subseteq X$ and $\delta > 0$ such $x := x_0 \neq 0$, $(ST - \lambda I)x_n = x_{n-1}$, $||x_n|| < \delta^n ||x||$.

Let $z_n := Tx_n$, then $(TR-\lambda I)z_n = (TR-\lambda)Tx_n = T(ST-\lambda I)x_n = Tx_{n-1} = z_{n-1}$. Since T is injective, we have $z := z_0 = Tx \neq 0$ and $||z_n|| < \delta^n ||z||$. Hence $z \in K(TR - \lambda I)$ and, therefore, $K(TR - \lambda I) \neq \{0\}$.

If $ST^2 = T^2R$ we shall prove the converse. Indeed, suppose that $K(TR - \lambda I) \neq \{0\}$, then there is $(x_n)_{n\geq 0} \subseteq X$ and $\delta > 0$ which $x := x_0 \neq 0$, $(TR-\lambda I)x_n = x_{n-1}$ and $||x_n|| < \delta^n ||x||$.

Consider $z_n := Tx_n$, then $(ST - \lambda I)z_n = (ST - \lambda I)Tx_n = T(TR - \lambda I)x_n = Tx_{n-1} = z_{n-1}$. But T is injective then $z := z_0 = Tx \neq 0$, we have $||z_n|| < \delta^n ||z||$. Consequently $z \in K(ST - \lambda I)$, this implies $K(ST - \lambda I) \neq \{0\}$.

2. This is a consequence of 1. ∎

Under the conditions of Theorem 2.6, we have the following results.

Corollary 2.2 *Let $T, S, R \in \mathcal{B}(X)$ such that T is injective and $TST = TRT$. Then*

$$\sigma_c(TR) \subseteq \sigma_c(ST) \text{ and } \sigma_{ca}(TR) \subseteq \sigma_{ca}(ST)$$

Either, if $ST^2 = T^2R$:

$$\sigma_{alc}(TR) = \sigma_{alc}(ST) \quad \text{and} \quad \sigma_{ac}(TR) = \sigma_{ca}(ST).$$

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JACOBI FIELDS ON THE MANIFOLD OF FREUND

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Abstract. In this paper, the geometric structures of Freund manifold are considered By defining a Riemannian metric, the curvature tensor and the scalar curvature are given. Then, the Jacobi fields on the Freund manifold have been considered to investigate the instability of the geodesics in view of differential geometry. Moreover, we take submanifold of Freund manifold as an example to illustrate our results.

Keywords: Freund manifold; Riemannian metric; α-connection; α-curvature tensor; Jacobi field.

2000 Mathematics Subject Classification: 53B20, 22E60, 47A64.

1. Introduction

Since people consider all the probability density functions as manifolds and treat the Fisher information matrices as the Riemannian metric, the research concerning the geometric structures of all the statistic manifolds achieves a lot of attention. Based on this fundamental idea, geometry is widely used in several fields says, information theory, computer science and radar imaging. By now the geometric structures of some important statistic manifolds have already been investigated. Further more we use these geometric structures to study other properties of statistic manifolds. Especially the study concerning the stability of Jacobi field on statistic manifold is becoming a hot research field. Cafaro [4], L. Peng [8], [9], C. Li [10] and L. Casetti [5] studied the stability of Jacobi fields on some of statistical manifolds.

In this paper, authors consider the two dimensional Freund distribution as a statistical manifold. First, we define the Riemannian metric on it, also give the corresponding Riemannian connection and curvature tensor then calculate the corresponding geometric variables. Finally, we illustrate the Jacobi field by the submanifold of the Freund manifold and analysis the stability of geodesic.

2. The geometric structure of the Freund manifold

Definition 2.1. We call the set

$$M = \left\{ p \,\middle|\, p(x,y,\zeta) = \left\{ \begin{array}{c} \alpha_1\beta_2 e^{-\beta_2 y-(\alpha_1+\alpha_2-\beta_2)x},\ 0<x<y \\ \alpha_2\beta_1 e^{-\beta_1 x-(\alpha_1+\alpha_2-\beta_1)y},\ 0<y<x \\ \zeta=(\zeta^1,\zeta^2,\zeta^3,\zeta^4)=(\alpha_1,\beta_1,\alpha_2,\beta_2)\in R^4_+ \end{array} \right., \right\}$$

as a Freund manifold, where

$$p(x,y,\zeta) = \left\{ \left\{ \begin{array}{c} \alpha_1\beta_2 e^{-\beta_2 y-(\alpha_1+\alpha_2-\beta_2)x},\ 0<x<y \\ \alpha_2\beta_1 e^{-\beta_1 x-(\alpha_1+\alpha_2-\beta_1)y},\ 0<y<x \\ \zeta=(\zeta^1,\zeta^2,\zeta^3,\zeta^4)=(\alpha_1,\beta_1,\alpha_2,\beta_2)\in R^4_+ \end{array} \right., \right\}$$

is the probability density function of 2-dimensional Freund distribution [6].

Definition 2.2. The fisher information matrix (g_{ij}) is defined as

$$(g_{ij}) = E[\partial_i l \partial_j l], \tag{1}$$

where $l(x,\zeta) = lnp(x,\zeta)$, $\partial_i l = \dfrac{\partial l(x,\zeta)}{\partial \theta^i}$, and E denotes the expectation of $p(x,\zeta)$ [1].

Proposition 2.3. *The Fisher information matrix (g_{ij}) is*

$$(g_{ij}) = \begin{bmatrix} \dfrac{1}{\zeta^1(\zeta^1+\zeta^3)} & 0 & 0 & 0 \\ 0 & \frac{\zeta^3}{(\zeta^2)^2(\zeta^1+\zeta^3)} & 0 & 0 \\ 0 & 0 & \dfrac{1}{\zeta^3(\zeta^1+\zeta^3)} & 0 \\ 0 & 0 & 0 & \dfrac{\zeta^1}{(\zeta^4)^2(\zeta^1+\zeta^3)} \end{bmatrix} \tag{2}$$

Definition 2.4. The Riemannian connection ∇ with respect to Riemannian metric (1) is given by

$$g(\nabla_{\partial_i}\partial_j, \partial_k) = \Gamma_{ijk} = \frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij}), \tag{3}$$

$\alpha-$connection is defined by

$$\Gamma^{(\alpha)}_{ijk} = \Gamma_{ijk} - \frac{\alpha}{2} T_{ijk}, \tag{4}$$

where $T_{ijk} = E[\partial_i l \partial_j l \partial_k l]$.

According to calculations, we get

Proposition 2.5. *The nonzero α-connection coefficients on the Freund manifold are obtained as follows:*

$$\begin{aligned}
\Gamma^{(\alpha)}_{111} &= \frac{2(\alpha-1)\zeta^1 - (1+\alpha)\zeta^3}{2(\zeta^1)^2(\zeta^1+\zeta^3)^2}, & \Gamma^{(\alpha)}_{113} &= \frac{1+\alpha}{2\zeta^1(\zeta^1+\zeta^3)^2},\\
\Gamma^{(\alpha)}_{221} &= \frac{(1+\alpha)\zeta^3}{2(\zeta^2)^2(\zeta^1+\zeta^3)^2}, & \Gamma^{(\alpha)}_{223} &= \frac{(1+\alpha)\zeta^1}{2(\zeta^2)^2(\zeta^1+\zeta^3)^2},\\
\Gamma^{(\alpha)}_{331} &= \frac{1+\alpha}{2\zeta^3(\zeta^1+\zeta^3)^2}, & \Gamma^{(\alpha)}_{333} &= \frac{2(\alpha-1)\zeta^3 - (1+\alpha)\zeta^1}{2(\zeta^3)^2(\zeta^1+\zeta^3)^2},\\
\Gamma^{(\alpha)}_{441} &= \frac{(1+\alpha)\zeta^3}{2(\zeta^4)^2(\zeta^1+\zeta^3)^2}, & \Gamma^{(\alpha)}_{443} &= \frac{(1+\alpha)\zeta^1}{2(\zeta^4)^2(\zeta^1+\zeta^3)^2},\\
\Gamma^{(\alpha)}_{222} &= \frac{(1-\alpha)\zeta^3}{(\zeta^2)^3(\zeta^1+\zeta^3)}, & \Gamma^{(\alpha)}_{444} &= \frac{(1-\alpha)\zeta^1}{(\zeta^4)^3(\zeta^1+\zeta^3)},\\
\Gamma^{(\alpha)}_{131} &= \Gamma^{(\alpha)}_{311} = \frac{1-\alpha}{2\zeta^1(\zeta^1+\zeta^3)^2}, & \Gamma^{(\alpha)}_{133} &= \Gamma^{(\alpha)}_{313} = \frac{1-\alpha}{2\zeta^3(\zeta^1+\zeta^3)^2},\\
\Gamma^{(\alpha)}_{122} &= \Gamma^{(\alpha)}_{212} = \frac{(1-\alpha)\zeta^3}{2(\zeta^2)^2(\zeta^1+\zeta^3)^2}, & \Gamma^{(\alpha)}_{344} &= \Gamma^{(\alpha)}_{434} = \frac{(1-\alpha)\zeta^1}{2(\zeta^4)^2(\zeta^1+\zeta^3)^2},\\
\Gamma^{(\alpha)}_{232} &= \Gamma^{(\alpha)}_{322} = \frac{(1-\alpha)\zeta^1}{2(\zeta^2)^2(\zeta^1+\zeta^3)^2}, & \Gamma^{(\alpha)}_{144} &= \Gamma^{(\alpha)}_{414} = \frac{(1-\alpha)\zeta^3}{2(\zeta^4)^2(\zeta^1+\zeta^3)^2}.
\end{aligned}$$

Definition 2.6. α-connection tensor is defined by

$$R^{(\alpha)}_{ijkm} = (\partial_i \Gamma^{(\alpha)s}_{jk} - \partial_j \Gamma^{(\alpha)s}_{ik}) g_{sm} + \Gamma^{(\alpha)}_{itm}\Gamma^{(\alpha)t}_{jk} - \Gamma^{(\alpha)}_{jtm}\Gamma^{(\alpha)t}_{ik}. \tag{5}$$

where $\Gamma^{(\alpha)k}_{ij} = \Gamma^{(\alpha)}_{ijm} g^{mk}$, α-Ricci curvature $R^{(\alpha)}_{ik}$ and α-scalar curvature $R^{(\alpha)}$ are defined as

$$R^{(\alpha)}_{ik} = R^{(\alpha)}_{ijkl} g^{ji} \tag{6}$$

and

$$R(\alpha) = R^{(\alpha)}_{ik} g^{ik} \tag{7}$$

respectively.

By calculations, we obtain the following proposition

Proposition2.7. *The non-zero α-curvature tensors on the Freund manifold are*

$$\begin{aligned}
R^{(\alpha)}_{1212} &= -\frac{(1-\alpha^2)(\zeta^3)^2}{4\zeta^1(\zeta^2)^2(\zeta^1+\zeta^3)^3},\\
R^{(\alpha)}_{3434} &= -\frac{(1-\alpha^2)(\zeta^1)^2}{4\zeta^3(\zeta^4)^2(\zeta^1+\zeta^3)^3},\\
R^{(\alpha)}_{1414} &= -\frac{(1-\alpha^2)\zeta^3}{4(\zeta^4)^2(\zeta^1+\zeta^3)^3},\\
R^{(\alpha)}_{2323} &= -\frac{(1-\alpha^2)\zeta^1}{4(\zeta^2)^2(\zeta^1+\zeta^3)^3},\\
R^{(\alpha)}_{2424} &= -\frac{(1-\alpha^2)\zeta^1\zeta^3}{4(\zeta^2)^2(\zeta^4)^2(\zeta^1+\zeta^3)^2},\\
R^{(\alpha)}_{2123} &= \frac{(1-\alpha^2)\zeta^3}{4(\zeta^2)^2(\zeta^1+\zeta^3)^3},\\
R^{(\alpha)}_{4341} &= \frac{(1-\alpha^2)\zeta^1}{4(\zeta^4)^2(\zeta^1+\zeta^3)^3}.
\end{aligned}$$

Freund manifold is ± 1 flat which means that all these $\alpha-$curvature vanish when $\alpha = \pm 1$. The non-zero α-Ricci curvature on the Freund manifold are

$$\begin{aligned}
R^{(\alpha)}_{11} &= -\frac{(1-\alpha^2)\zeta^3}{2\zeta^1(\zeta^1+\zeta^3)^2},\\
R^{(\alpha)}_{22} &= -\frac{(1-\alpha^2)\zeta^3}{2(\zeta^2)^2(\zeta^1+\zeta^3)},\\
R^{(\alpha)}_{33} &= -\frac{(1-\alpha^2)\zeta^1}{2\zeta^3(\zeta^1+\zeta^3)^2},\\
R^{(\alpha)}_{44} &= -\frac{(1-\alpha^2)\zeta^1}{2(\zeta^4)^2(\zeta^1+\zeta^3)},\\
R^{(\alpha)}_{13} &= \frac{1-\alpha^2}{2(\zeta^1+\zeta^3)^2}.
\end{aligned}$$

α-scalar curvature satisfies

$$R^{(\alpha)} = -\frac{3}{2}(1-\alpha^2)$$

Definition 2.8. [1] The geodesic on the n-dimensional Riemannian manifold can be denoted by

$$\frac{d^2\zeta^k}{dt^2} + \Gamma^k_{ij}\frac{d\zeta^i}{dt}\frac{d\zeta^j}{dt} = 0,\ k = 1, 2, ..., n. \tag{8}$$

Example 2.9. Consider the 2-dimensional submanifold of the Freund manifold

$$M_1 = \left\{ p \,\middle|\, p(x,y,\zeta) = \left\{ \begin{array}{c} \alpha_1\beta_2 e^{-\beta_2 y-(\alpha_1+\alpha_2-\beta_2)x},\ 0 < x < y, \\ \alpha_2\beta_1 e^{-\beta_1 x-(\alpha_1+\alpha_2-\beta_1)y},\ 0 < y < x, \\ \zeta = (\zeta^2,\zeta^4) = (\beta_1,\beta_2) \in R^2_+. \end{array}\right. , \right\}$$

From (2) we get

$$(g_{ij}) = \begin{bmatrix} \dfrac{\zeta^3}{(\zeta^2)^2(\zeta^1+\zeta^3)} & 0 \\ 0 & \dfrac{\zeta^1}{(\zeta^4)^2(\zeta^1+\zeta^3)} \end{bmatrix}$$

and its inverse matrix is

$$(g^{ij}) = \begin{bmatrix} \dfrac{(\zeta^2)^2(\zeta^1+\zeta^3)}{\zeta^3} & 0 \\ 0 & \dfrac{(\zeta^4)^2(\zeta^1+\zeta^3)}{\zeta^1} \end{bmatrix}$$

the corresponding non-zero α-connection coefficients are

$$\Gamma_{22}^{(\alpha)2} = -\frac{1-\alpha}{\zeta^2}, \ \Gamma_{44}^{(\alpha)4} = -\frac{1-\alpha}{\zeta^4}. \tag{9}$$

From (5) we get $R_{2424}^{(\alpha)} = 0$, so the Gaussian curvature is

$$K = -\frac{R_{2424}^{(\alpha)}}{g_{22}g_{44} - (g_{24})^2} = 0. \tag{10}$$

3. The Jacobi field on the Freund manifold

Now, we consider the Jacobi field on the Freund manifold (M_1, g). Let $\zeta^l : [a,b] \rightarrow M_1$ be the geodesic on M_1, $\zeta^l(t,\beta) : [a,b] \times (-\epsilon, \epsilon) \rightarrow M_1$ is a variation of ζ. For each fixed β, the curvature $\zeta^l(t,\beta)$ is a geodesic, which is called a geodesic variation of ζ. The Jacobi equation along the geodesic satisfies

$$\frac{D^2\mathbf{J}}{Dt^2} + R(\mathbf{J}, v)v = 0, \tag{11}$$

where t is the time, $R(\mathbf{J}, v)$ is the Riemannian curvature tensor. $\dfrac{D}{Dt}$ is the covariant derivative along the geodesic, $v = \dfrac{\partial\theta^k}{\partial t}$ is the velocity of geodesic. $\mathbf{J}$ is called the Jacobi field. The component of Jacobi equation can be denoted by

$$\frac{D^2(\delta\zeta^i)}{Dt^2} + R^i_{kml}\frac{\partial\zeta^k}{\partial t}\frac{\partial\zeta^l}{\partial t}\delta\zeta^m = 0, \tag{12}$$

where $\delta\zeta^k = J^k$ is the component of the Jacobi field. From (12) we get

$$g_{ij}\frac{D^2(\delta\zeta^i)}{Dt^2} + R_{jkml}\frac{\partial\zeta^k}{\partial t}\frac{\partial\zeta^l}{\partial t}\delta\zeta^m = 0, \tag{13}$$

the length of Jacobi field $\mathbf{J}$ is defined by

$$\mathbf{J}^2 = J^i J_j = g_{ij} j^i J^j \tag{14}$$

As an application, we calculate the Jacobi equation in the manifold M_1 given in example and study its stability. From (8) and (9) we get the geodesic equation on manifold M_1 as follows

$$\frac{d^2\beta_k}{dt^2} - \frac{1-\alpha}{\beta_k}\left(\frac{d\beta_k}{dt}\right)^2 = 0,\ k = 1, 2.$$

when $\alpha = 0$, we get the solution

$$\beta_k = \zeta^{2k} = C_{2k-1} e^{c_{2k}t},\ k = 1, 2, \tag{15}$$

where C_i $(i = 1, ..., 4)$ are integration constants. Then we consider the stability of Jacobi field. From (13), we get the Jacobi equation on M_1 as

$$\frac{D^2(\delta\zeta^i)}{Dt^2} = 0.$$

then we get

$$\begin{aligned} &\frac{d^2\delta\zeta^{2k}}{dt^2} + 2\Gamma^k_{kk}\frac{d\delta\zeta^{2k}}{dt}\frac{d\zeta^{2k}}{dt} \\ &\quad + \left[\Gamma^k_{kk}\frac{d^2\zeta^{2k}}{dt^2} + \frac{\partial\Gamma^k_{kk}}{\partial\zeta^{2k}}\left(\frac{d\zeta^{2k}}{dt}\right)^2 + \left(\Gamma^k_{kk}\frac{d\zeta^{2k}}{dt}\right)^2\right]\delta\zeta^{2k} = 0 \end{aligned} \tag{16}$$

where $k = 1, 2$ put (9) and (15) in to (16) we get

$$\frac{d^2\delta\zeta^{2k}}{dt^2} - 2C_{2k}\frac{d\delta\zeta^{2k}}{dt} + (C_{2k})^2\delta\zeta^{2k} = 0. \tag{17}$$

Integrate (17), we obtain

$$\delta\zeta^{2k} = (C_{2k+3}t + C_{2k+4})e^{C_{2k}t},\ k = 1, 2,$$

where C_i $(i = 1, ..., 8)$ are integration constants. Finally, from (14) we get the Jacobi field on M as follows

$$\begin{aligned} \mathbf{J}^2_M &= g_{22}(\delta\zeta^2)^2 + g_{44}(\delta\zeta^4)^2 \\ &= \frac{\alpha_2}{(\alpha_1+\alpha_2)(C_1)^2}(C_5t + C_6)^2 + \frac{\alpha_1}{(\alpha_1+\alpha_2)(C_3)^2}(C_7t + C_8)^2. \end{aligned}$$

then

$$\mathbf{J}^2_M = O(t^2). \tag{18}$$

Equation (18) shows that $\mathbf{J}^2_M$ is divergent when $t \to \infty$ which means Jacobi field is unstable.

Conclusion

We consider the probability density function of the two-dimensional Freund distribution as a statistical manifold, define the Riemannian metric give the α-connection and the $\alpha-$curvature. Moreover, we study the Jacobi field on it and obtain the convergence of the geodesic, which is the foundation of information geometry theory that play crucial role in practical applications.

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LOCAL LARGEST LYAPUNOV EXPONENT IS CRITICAL TO THRESHOLD VOLTAGE AND REFRACTORY PERIODS FOR HODGKIN-HUXLEY MODEL

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Abstract. It is not known whether the spike threshold voltage and refractory periods in neocortical neurons reflects the reliability of spike timing underlying mechanisms. The paper scrutinizes their relationship with local largest Lyapunov exponent ($lLLE$) in an excitatory Hodgkin-Huxley system under either sinusoidal drive or stochastic Poisson drive. The influence of the forcing on the response of the system is examined in the realm of suprathreshold amplitudes. Our results demonstrate that the average $lLLE$ in spike and non-spike regions is different under the physiological threshold voltage and refractory periods. These dynamics contains

(i) The average values of the $lLLE$ in spike region are almost negative, and almost positive in non-spike region under sinusoidal driving.

(ii) The values in spike region are nearly constant under sinusoidal drive with varying frequency; however, the values in non-spike region are different.

(iii) Under low Poisson spike rate and different Poisson inputs strength, the average values of $lLLE$ almost remain the same constant in spike and non-spike region.

Keywords: Hodgkin-Huxley, Lyapunov exponent, refractory periods, threshold voltage.

2010 Mathematics Subject Classification: 34H25, 92B99, 65L20, 65P20.

1. Introduction

Single neuron is the fundamental elements of every nervous system. Understanding the mechanism of these exquisitely structured elements is an important step to explore the mysteries of the brain. Besides, how do single-cell properties contribute to information processing and, ultimately, behavior? [8] is the core of neuroscience. The reliability of spike timing in the cerebral cortex [4] is a classical problem. Meanwhile, a spike produce in the external drive seems that the neurons are unable to adapt their response threshold to change. But it is well known that many sensory neurons do have adaptive capabilities [8], [5]. So, the threshold is very important to the neurons signal processing and system dynamics.

In addition, Berry et al. [3] investigated the relationship between the refractory period of a neuron and its firing precision. The refractoriness of the membrane in the wake of the action potential is another important quantity. The transitory changes make it harder for the axon to produce subsequent action potentials during an interval [15]. Thus, the refractory periods ensure the excitability of neurons more reasonable.

On the other hand, quantifying structures of attractors of dynamical systems has now become an important and common pursuit in scientific fields [20]. The spectrum of Lyapunov exponents (LEs) contains abundant physical information for dynamical system, and it is usually taken as one of the most important and precise dynamical diagnostics to provide characteristics of attractors [12]. Especially, the existence of positive largest Lyapunov exponent (LLE) is an indicator that determines whether the system is chaotic or not. Therefore, the computations of LLE is a very important part for analyze dynamical systems. For smooth dynamical systems [6], [11], the algorithms for computing the LEs have been well established, which often involve the Gram-Schmidt Reorthonormalization (GSR) procedure [2], [19], [13]. In this paper, we will use an accurate and stable numerical algorithm to compute Lyapunov exponents for the Hodgkin-Huxley system.

Scientists and other researchers focus their interests on the dynamics of systems, and hardly ever kick around the relationship between local largest Lyapunov exponent ($lLLE$) and voltage threshold, refractory periods. And we rarely find this point; thus, this is our motive for discuss these problems. Through some numerical results, we find the results of the average of $lLLE$ over samples are very different in spike and non-spike region under different sinusoidal stimuli, and an important observation is that the average values of $lLLE$ in spike region are nearly constant under different sinusoidal drive with varying frequency, even under low rate Poisson drive cases, at an appropriate choice of threshold voltage and refractory periods. Therefore, we can explain the wide application of the threshold voltage and refractory periods are reasonable from another perspective.

The paper is organized as follows. In Section 2, we briefly introduce the Hodgkin-Huxley model and different H-H structure in different stimuli. In Section 3, we discuss the numerical algorithm of calculating LEs for smooth dynamical systems. In Section 4, we apply the method to H-H model and investigate the relationship between the mean $lLLE$ and threshold, refractory periods for this model under different stimuli. Section 5 contains discussion and conclusion.

2. The Hodgkin-Huxley model

Hodgkin-Huxley model [9] is the well-known model of neural excitability. Some researchers show which combination of dynamical variables governs the threshold operation [18] and how adaptation [1] and spike-generation mechanisms [7] influence spike trains. Therefore, we hope to illustrate what $lLLE$ phenomena will emerge under appropriate threshold and refractory periods.

The dynamic equations for the Hodgkin-Huxley model to be considered in this paper are identical to those used by Sun and Zhou [17]:

$$\begin{aligned}
&(2.1) & c\frac{d}{dt}V &= -G_{Na}m^3h(V-V_{Na})-G_Kn^4(V-V_K)-G_L(V-V_L)+I^{ext}\\
&(2.2) & \frac{dm}{dt} &= \alpha_m(V)(1-m)-\beta_m(V)m\\
&(2.3) & \frac{dh}{dt} &= \alpha_h(V)(1-h)-\beta_h(V)h\\
&(2.4) & \frac{dn}{dt} &= \alpha_n(V)(1-n)-\beta_n(V)n
\end{aligned}$$

where C=$1\mu F/cm^2$ is the cell membrane capacitance and V is its membrane potential, m and h are the activation and inactivation variables of the sodium current, respectively, and, n is the activation variable of the potassium current [5], [9]. The parameters G_{Na}=$120mS/cm^2$, G_K=$36mS/cm^2$, and G_L=$0.3mS/cm^2$ are the maximum conductances for the sodium, potassium and leak currents, respectively, $V_{na} = 50mV$, $V_K = -77mV$, and $V_L = -54.387mV$ are the corresponding reversal potentials.

$$\begin{aligned}
&(2.5) & \alpha_m(V) &= 0.1(V+40)/(1-\exp(-V+40)/10))\\
&(2.6) & \beta_m(V) &= 4\exp(-(V+65)/18)\\
&(2.7) & \alpha_h(V) &= 0.07\exp(-(V+65)/20)\\
&(2.8) & \beta_h(V) &= 1/(1+\exp(-(35+V)/10))\\
&(2.9) & \alpha_n(V) &= 0.01(V+55)/(1-\exp(-V+55)/10))\\
&(2.10) & \beta_n(V) &= 0.125\exp(-(V+65)/80)
\end{aligned}$$

There is also a current parameter I^{ext} which stands for an external periodic signal current or poisson current where

$$(2.11) \qquad I^{ext} = I^{sine} = I^{shift} + \sin(2\pi\frac{f}{3}t)$$

or

$$(2.12) \qquad I^{ext} = I^{poisson} = -G(t)(V(t)-V_G)$$

$I^{shift} = 10\mu A/cm^2$, being the amplitude of current shift, and f being the stimulus frequency, $G(t)$ are the conductances, and V_G is the reversal potential ($V_G^E = 0mV$,

$V_G^I = -80mV$). The dynamics of synaptic interactions by using a continuous function. Therefore, the dynamics of $G(t)$ is governed by

$$\frac{d}{dt}G(t) = -\frac{G(t)}{\sigma_r} + \widetilde{G}(t) \tag{2.13}$$

$$\frac{d}{dt}\widetilde{G}(t) = -\frac{\widetilde{G}(t)}{\sigma_d} + \sum_k F\delta\left(t - T_k^F\right) \tag{2.14}$$

Each neuron is either excitatory or inhibitory, as indicated by its type E, I. In the whole work, we consider the coupled H-H systems with only excitatory neurons: a fast rise and a slow decay timescale ,$\sigma_r = 0.5ms$ and $\sigma_d = 3ms$, respectively.

The system is also driven by stochastic inputs: we use a spike train sampled from a Poisson process with rate r as the stimulus. We denote T_k^F as the kth spike from the poisson input to the neuron and its force strength is F.

Additionally, in order to analyze the $lLLE$ relationship with refractory periods, we will discuss the refractory periods of H-H neuronal model at first. Generally, the period from the initiation of the action potential to immediately after the peak is referred to as the absolute refractory period(ARP). During the ARP, a second stimulus (no matter how strong) will not excite the neuron. After the ARP, if strong enough stimuli are given to the neuron, it may respond again by generating action potentials. So, the period during which a stronger normal stimulus is needed in order to elicit an action potentials is referred to as the relative refractory period (RRP) [14].

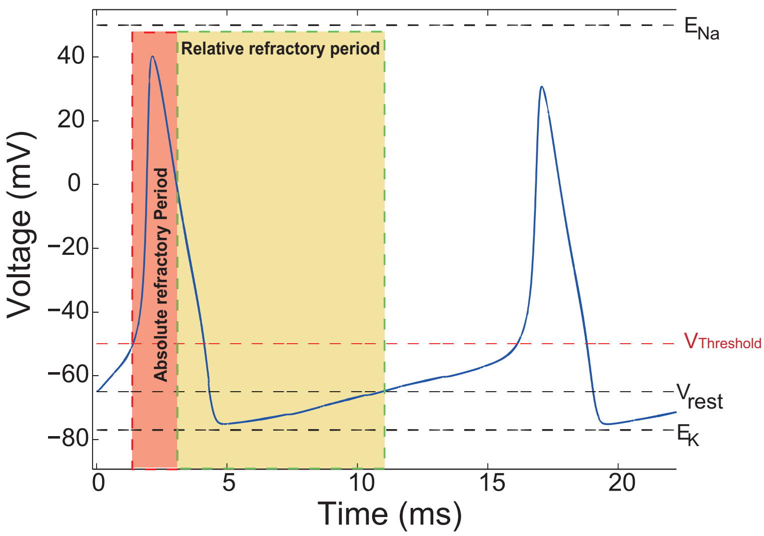

Figure 1: Absolute and relative refractory periods under sine driven at $I^{shift} = 10$, $f = 0$.

The ARP and RRP constitute the refractory periods in this paper. Different neurons have different scales of refractory period. Fig. 1 shows the refractory periods of H-H neuron. The refractory period time is about 10ms.

3. Lyapunov exponents in smooth dynamical systems

Lyapunov exponent is a useful tool for characterizing dynamics in a smooth dynamical system. Especially, the largest Lyapunov exponent (LLE) is very important quantity to measure whether the system is chaotic or not. Generally, the largest Lyapunov exponent λ_1 can be obtained by following two sufficiently close nearby trajectories $\mathbf{X}(t)$ and $\mathbf{X}'(t)$, and set $\mathbf{Z}(t) = \mathbf{X}'(t) - \mathbf{X}(t)$. Sometimes, $\|\mathbf{Z}(t)\|$ may grows unbounded as time interval ΔT is not sufficiently short in the log-ration: $\lambda_1[T_0 + \Delta T] = \frac{1}{\Delta T} \ln(\frac{\|\mathbf{Z}(T_0+\Delta T)\|}{\|\mathbf{Z}(T_0)\|})$. Therefore, a practical approach to avoid numerical overflows is to scale back one of the trajectories, say $\mathbf{X}'(t)$, to the vicinity of the other $\mathbf{X}(t)$ along the direction of separation whenever they become too far apart. We refer to this step as renormalization [13], [16].

To be more specific, the algorithm can be briefly described as Standard algorithm in [20].

In order to obtain that the $lLLE$ is stable, we have compared it with larger running time interval. Therefore, the total iterations of the computation proceeds set as Max_Num in Standard algorithm, the selection of the value of Max_Num is determined by convergence test of the Lyapunov exponent. Here we set the Max_Num is 10^6 when it produces convergent results. Besides, we also compute the LLE by the method described in the paper [10], and find that the result isn't a significant difference between them.

The $lLLE$ obtained by Standard algorithm is very important to analyze the threshold and refractory periods. The detailed analysis will be illustrated as follows.

4. Main results

In the following analysis we focus on the $lLLE$ for the different threshold and refractory periods in spike and non-spike region under either sinusoidal drive or stochastic Poisson drive.

4.1. Periods of external drive

First, we consider a single H-H neuron driven by a sinusoidal external input, which is the frequency f from 0 to 1 in equation (2.11). We perform simulations of this system for different threshold and refractory. A systematic scanning result of the LLE obtained by using the smooth method over a long time interval of $T = 10^5$ ms, as shown in Fig.2, and it demonstrates that there are essentially three dynamical regimes. If LLE is negative, the regime is a stable periodic pattern of spike. At most f, LLE jumps back and forth between zero and positive, it signifies that the dynamics of the system is either quasi-period or chaotic.

We will analyze the average $lLLE$ in these different dynamical regimes under a threshold and refractory which we obtained. Besides, we found that the refractory period time of H-H model is about 10ms (contain absolute refractory periods and relative refractory periods), as appears in Fig. 1.

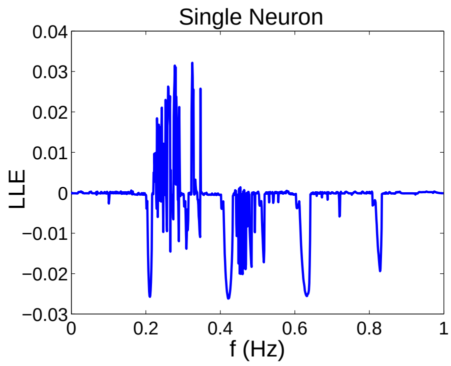

Figure 2: Largest Lyapunov exponent (by standard method) versus parameter f increased with step 0.01/3 in one H-H neuron.

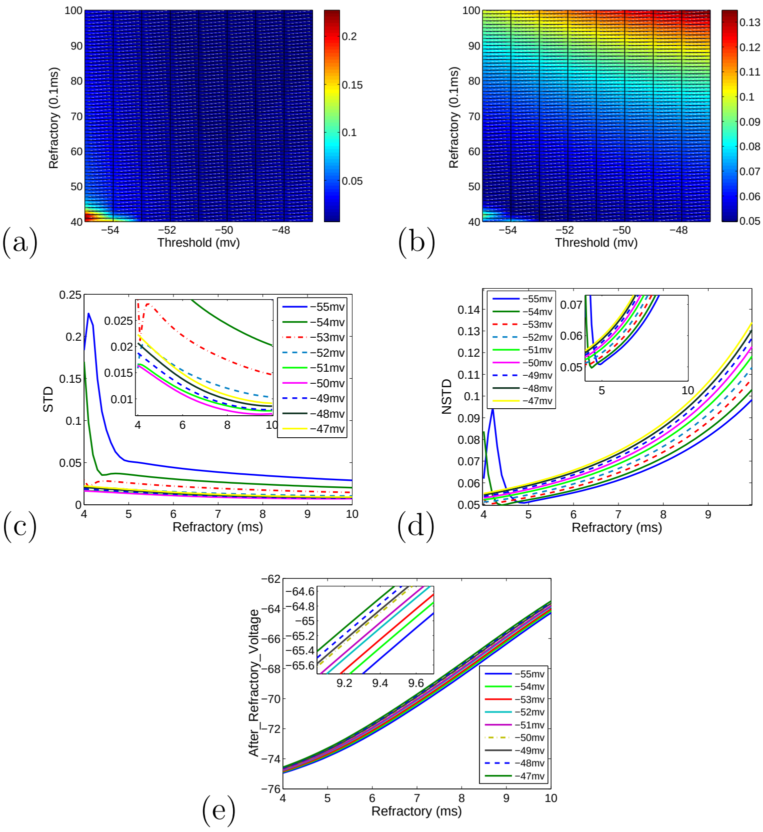

Figure 3: Pcolor plot of the standard deviation of average $lLLE$ in spike region (STD) (a) and non-spike region (NSTD) (b) as functions of refractory periods and spike threshold. The trend of STD and NSTD as functions of the refractory periods with different threshold is (c) and (d) respectively, (e) illustrate the voltage after the refractory periods with different threshold voltage. All the threshold voltage ranges from -55mV to -47mV, and refractory periods from 4ms to 10ms.

We now return to explore the relationship between $lLLE$ and voltage threshold and refractory periods in spike and non-spike region under different sinusoidal stimuli. The frequency f is changed from 0 to 1 regard as the different stimuli. Figs.3(a-d) display the graph of both refractory periods and threshold voltage with the standard deviation of the average $lLLE$ for various values of f (We set STD and $NSTD$ is the standard deviation of average $lLLE$ in spike and non-spike region respectively).

It is seen that the smallest STD in the area of threshold about -50mV, refractory periods from 7 ms to 10 ms, and the STD decreases along the axis from the bottom to the top as Fig.3(a) shows. The deeper the blue color is, the smaller the STD becomes. Fig.3(b) shows that the $NSTD$ is opposite STD except the lower-left corner. For more accurate description, Fig.3(c) shows that the STD decreases with the increase of refractory periods (RPT) under different spike threshold. From the vertical, when the threshold voltage is -50 mV, the smallest STD curve as showed in the enlarged portion of the above screenshot, and the STD reaches a minimum at RPT being 9.5 ms and threshold voltage is -50 mV. Fig.3(d) shows that the $NSTD$ increases with the increase of RPT with different spike threshold, and in most cases, the $NSTD$ is larger than STD, indicating that the average values of $lLLE$ in spike region are more stable than non-spike region. In order to verify whether or not the voltage is at resting state after RPT, we draw a picture as Fig.3(e) shows. In the enlarged portion of the Fig.3(e), the voltage is at resting state. Therefore, the RPT and threshold voltage have been chosen for analysis are reasonable that the RPT is 9.5 ms and threshold voltage is -50 mV respectively. Then, we compute the average values of the $lLLE$ in spike and non-spike region respectively, whose results are shown in Fig.4.

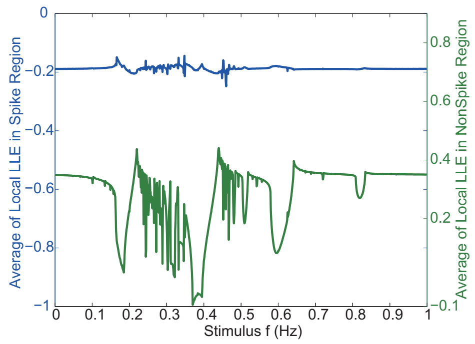

Figure 4: The average $lLLE$ as functions of stimulus f (Hz), the threshold voltage is -50 mV, refractory period is 9.5 ms.

In Fig.4, it reveals that the average values of $lLLE$ are stable about -0.19 in spike region (blue curve in Fig.4), with the STD is about 0.00704, indicating that the attractor of the system is contraction in spike region. However, the average values of $lLLE$ are unstable (dark green curve in Fig.4) and more vibrant in non-spike region, with the $NSTD$ is about 0.10973, and we find the curve is very similar to largest Lyapunov exponent as Fig.2. The average values of $lLLE$ are mostly greater than zero, indicating that attractor of the system is divergent in non-spike region. These phenomena can best demonstrate that the threshold and refractory periods express an especial neuron dynamics that the spike region is miraculous that the local Lyapunov exponent represents.

We will analyze the average local LLE in these three dynamical regimes in spike and non-spike region.

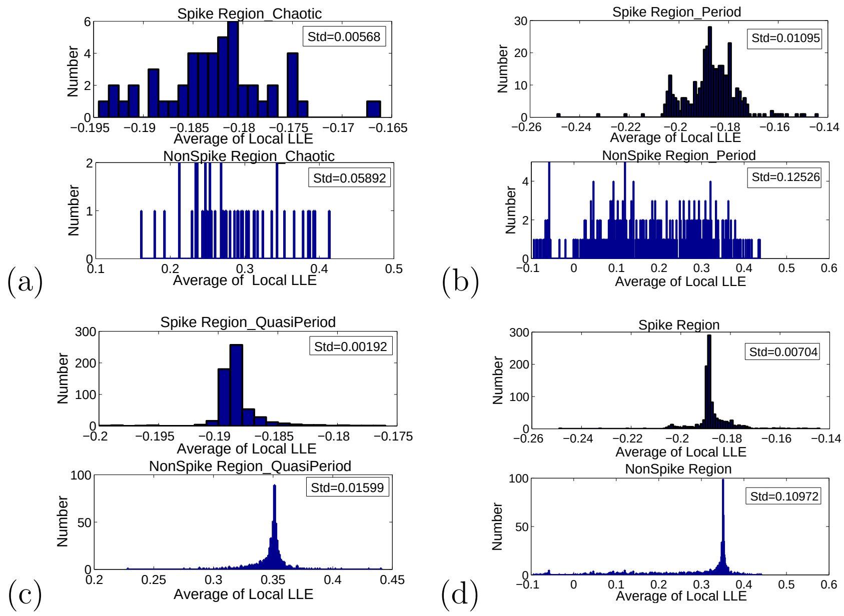

Figure 5: Histogram 1 to 3 graph is the average local LLE in Spike and Non-Spike region in three dynamical regimes (Chaotic, Period, Quasi-Period). The last graph is the average local LLE in Spike and Non-spike region. The standard deviation of spike are smaller than non-spike in chaotic, period and quasi-period. Threshold voltage is -50 mV and refractory period is 9.5 ms.

In Fig.5, we describe the average $lLLE$ in three dynamical regimes(Chaotic, Period, Quasi-Period)with spike and non-spike region. We find the average values of $lLLE$ are more directly concentrated in spike region than non-spike region. Besides, the mean $lLLE$ is smaller than zero in spike region, it is indicating that the attractor is contraction in these three dynamical regimes. Meanwhile, the reason that some value of the mean $lLLE$ in non-spike region is smaller than zero is that the attractor is divergent in period regime, as shown in Fig.5(b). What is the difference between the three dynamical regimes in no spike region? The most significant difference is the average value of $lLLE$ became more concentrated in quasi-period than other regimes.

4.2. Poisson of feedforward input

It has shown that chaos can arise in the dynamics of single H-H neuron, for example, under a periodic external drive as part 4.1. Therefore, there is a natural question: what about a single H-H neuron under a stochastic external Poisson input, and can it be chaotic?

So, the system is also considered as another drive by feedforward inputs. Here, we consider stochastic inputs: we use a spike train sampled from a Poisson process with rate r as the feedforward input. We also hope to find the different dynamical regimes, but only one dynamical regime is period in one neuron model. The results are shown in Fig.6(a). This case is consistent with Sun [17] results.

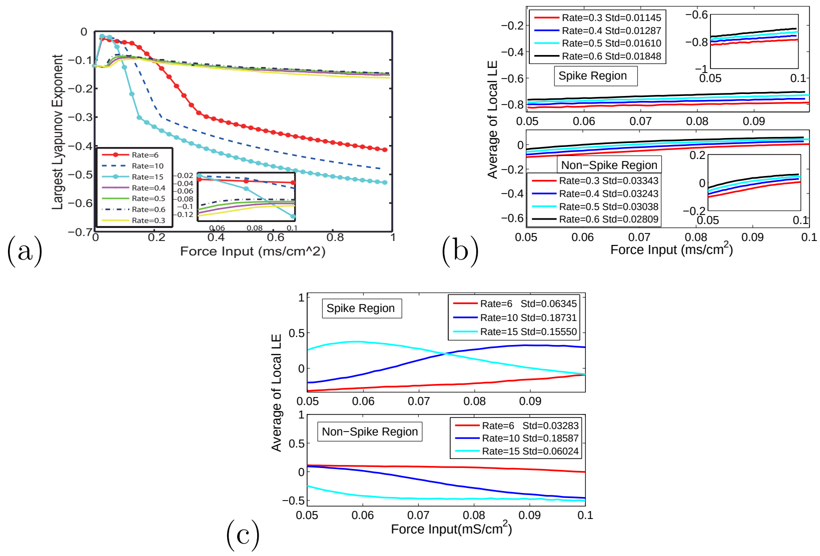

Figure 6: (a) The Largest Lyapunov exponent of the HH neuron system with different rate of poisson input versus the force input strength parameter F is $[0, 1]$, the enlarged portion is the F ranging from 0.05 to 0.1, force strength F. (b) The average local LLE versus the different Poisson input strength in low frequency Poisson spike. (c) The average local LLE versus the different Poisson input strength in high frequency Poisson spike. Threshold is -50 mV and refractory periods is 9.5 ms.

Here, we vary the parameter of the strength of Poisson ranging from 0 to 1 mS/cm^2 to perform seven systematic scanning tests for seven different parameter values of the Poisson rate $r = 0.3, 0.4, 0.5, 0.6, 6, 10, 15$, respectively. We find the largest Lyapunov exponent is negative for any values of Poisson rate r and strength F, which indicates that the dynamics of a single neuron in this system is not chaotic. As shown in Fig.6(a), when the Poisson rate $r = 0.3, 0.4, 0.5, 0.6$, the value increases as the rate increases, but for the other case of large rate $r = 6, 10$ and 15, the value of LLE decreases as the rate increases in the strength F ranging from 0.15 to 1, and we also find the LLE in high frequency is smaller than low frequency when the strength F ranges from 0.25 to 1. In very low force strength of Poisson spike ranging from 0 to 0.05, the LLE increase with increasing Poisson rate in the high Poisson spike frequency region, however, in the low frequency region, the LLE decrease with increasing Poisson rate.

In Fig.6(b), we show that the average $lLLE$ in Spike and Non-Spike region are nearly constant under different stimuli of Poisson force strength input in each small rate of Poisson spike. We also show that the average values of $lLLE$ are negative about -0.8 in spike region; however, the average values of $lLLE$ are monotonically increasing as F increases in non-spike region. The Poisson force strength ranging from 0.05 to 0.1 mS/cm^2 followed as Sun and Zhou [17]. The standard deviation of the average $lLLE$ increases as the rate r increases and expands at large strength in spike region, but the value decreases as the r increases and tightens at large strength in non-spike region in the low frequency Poisson spike. Therefore, these results demonstrated that the average $lLLE$ is stable in different Poisson input strengths in each Poisson spike rate.

In Fig.6(c), we find that the average values of local lyapunov exponent are unstable in spike and non-spike region in high frequency Poisson spike stimuli. And the values sign of the mean $lLLE$ are different with the results of sinusoidal driven; therefore, in high frequency case, the conclusion as other case can not be reached.

5. Discussion and conclusion

We have presented a numerical study of the value of the average l$lLLE$ of HH neuron under Sinusoidal drive and conductance drive with a stochastic nature, such as feedforward Poisson spikes, which are more realistic as an approximation to cortical spike trains. We found three typical dynamical regimes in Sinusoidal drive as the sine frequency varies from weak to strong, and only one dynamical regime in Poisson input as the input strength varies from 0 to 1. We have also discovered the relationship between the $lLLE$ and spike threshold voltage and refractory periods. That is, the average $lLLE$ is stable in spike region and unstable in non-spike region, under different sinusoidal drive, and it has some similar results under Poisson force strength in each fix low Poisson spike rate in spike region. But the value is different with the case at sinusoidal external stimulus in non-spike region. Also, the analysis demonstrates that the reason of choosing the threshold voltage and refractory periods value in physiologically have an intrinsic dynamical structure. This phenomenon estranges us from the unfamiliarity with the threshold voltage and refractory periods from a new perspective, and carves out a specific way to uncover the physical interpretation of the spike encoding and reliability of spike timing.

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AN APPROACH TO THE RELATIVE ORDER BASED GROWTH PROPERTIES OF DIFFERENTIAL MONOMIALS

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Abstract. In this paper an attempt is taken to study the comparative growth properties of composition of entire and meromorphic functions on the basis of relative order and relative lower order of differential monomials generated by transcendental entire and transcendental meromorphic functions.

Keywords and Phrases: transcendental entire function, transcendental meromorphic function, relative order, relative lower order, differential monomial.
AMS Subject Classification (2010): 30D30, 30D35, 30D20.

1. Introduction

Let f be an entire function defined in the open complex plane $\mathbb{C}$. The function $M_f(r)$ on $|z| = r$ known as maximum modulus function corresponding to f is defined as follows:

$$M_f(r) = \max_{|z|=r} |f(z)| \ .$$

When f is meromorphic, $M_f(r)$ cannot be defined as f is not analytic. In this situation, one may define another function $T_f(r)$ known as Nevanlinna's Characteristic function of f, playing the same role as $M_f(r)$ in the following manner:

$$T_f(r) = N_f(r) + m_f(r) \ .$$

Given two meromorphic functions f and g, the ratio $\frac{T_f(r)}{T_g(r)}$ as $r \to \infty$ is called the growth of f with respect to g in terms of their Nevanlinna's Characteristic functions.

When f is entire function, the Nevanlinna's Characteristic function $T_f(r)$ of f is defined as

$$T_f(r) = m_f(r) \ .$$

We call the function $N_f(r,a) \left(\bar{N}_f(r,a) \right)$ as counting function of a-points (distinct a-points) of f. In many occasions $N_f(r,\infty)$ and $\bar{N}_f(r,\infty)$ are denoted by $N_f(r)$ and $\bar{N}_f(r)$ respectively. We put

$$N_f(r,a) = \int_0^r \frac{n_f(t,a) - n_f(0,a)}{t} dt + \bar{n}_f(0,a) \log r \ ,$$

where we denote by $n_f(r,a) \left(\bar{n}_f(r,a) \right)$ the number of a-points (distinct a-points) of f in $|z| \le r$ and an ∞ -point is a pole of f. Also we denote by $n_{f|=1}(r,a)$,the number of simple zeros of $f-a$ in $|z| \le r$.

Accordingly, $N_{f|=1}(r,a)$ is defined in terms of $n_{f|=1}(r,a)$ in the usual way and we set

$$\delta_1(a;f) = 1 - \limsup_{r\to\infty} \frac{N(r,a;f \mid= 1)}{T_f(r)} \quad \text{(cf. [8])}\,,$$

the deficiency of 'a' corresponding to the simple a- points of f i,e. simple zeros of $f-a$. In this connection Yang [7] proved that there exists at most a denumerable number of complex numbers $a \in \mathbb{C} \cup \{\infty\}$ for which

$$\delta_1(a;f) > 0 \text{ and } \sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;f) \le 4.$$

On the other hand, $m\left(r, \frac{1}{f-a}\right)$ is denoted by $m_f(r,a)$ and we mean $m_f(r,\infty)$ by $m_f(r)$, which is called the proximity function of f. We also put

$$m_f(r) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f\left(re^{i\theta}\right) \right| d\theta,$$

where $\log^+ x = \max(\log x, 0)$ for all $x \geqslant 0$.

Further, a meromorphic function $b \equiv b(z)$ is called small with respect to f if $T_b(r) = S_f(r)$ where $S_f(r) = o\{T_f(r)\}$ i.e., $\frac{S_f(r)}{T_f(r)} \to 0$ as $r \to \infty$. Moreover, for any transcendental meromorphic function f , we call $P[f] = bf^{n_0}(f^{(1)})^{n_1}...(f^{(k)})^{n_k}$, to be a differential monomial generated by it where $\sum_{i=0}^{k} n_i \ge 1$ (all $n_i \mid i = 0,1,...,k$ are non-negative integers) and the meromorphic function b is small with respect to f. In this connection the numbers $\gamma_{P[f]} = \sum\limits_{i=0}^{k} n_i$ and $\Gamma_{P[f]} = \sum\limits_{i=0}^{k} (i+1)n_i$ are called the degree and weight of $P[f]$ respectively {cf. [2]}.

The *order* of a meromorphic function f which is generally used in computational purpose is defined in terms of the growth of f with respect to the exponential function as

$$\rho_f = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log T_{\exp z}(r)} = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log\left(\frac{r}{\pi}\right)} = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log r + O(1)} \ .$$

Lahiri and Banerjee [5] introduced the *relative order* (respectively *relative lower order*) of a meromorphic function with respect to an entire function to avoid comparing growth just with $\exp z$. Extending the notion of relative order as cited in the reference, Datta, Biswas and Bhattacharyya [3] gave the definition of *relative order* (respectively *relative lower order*) of *differential monomials* generated by transcendental entire and transcendental meromorphic functions.

For entire and meromorphic functions, the notion of their growth indicators such as *order* and *lower order* are classical in complex analysis and during the past decades, several researchers have already been continuing their studies in the area of comparative growth properties of composite entire and meromorphic functions in different directions using the same. But at that time, the concept of *relative order* and consequently *relative lower order* of entire and meromorphic functions with respect to another entire function was mostly unknown to complex analysis and they are not aware of the technical advantages of using the relative growth indicators of the functions. Therefore the growth of composite entire and meromorphic functions needs to be modified on the basis of their *relative order* and *relative lower type* some of which has been explored in this paper. Actually in this paper we establish some newly developed results based on the growth properties of *relative order* and *relative lower order* of *monomials* generated by transcendental entire and transcendental meromorphic functions.

2. Notation and preliminary remarks

We use the standard notations and definitions of the theory of entire and meromorphic functions which are available in [4] and [6]. Henceforth, we do not explain those in details. Now, we just recall some definitions which will be needed in the sequel.

Definition 1 The *order* ρ_f and *lower order* λ_f of a meromorphic function f are defined as

$$\rho_f = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log r} \text{ and } \lambda_f = \liminf_{r\to\infty} \frac{\log T_f(r)}{\log r} \ .$$

The notion of *order* (*lower order*) to determine the relative growth of two meromorphic functions having same non zero finite order is classical in complex analysis and is given by

Given a non-constant entire function f defined in the open complex plane $\mathbb{C}$, its Nevanlinna's Characteristic function is strictly increasing and continuous. Hence there exists its inverse function $T_g^{-1} : (T_g(0), \infty) \rightarrow (0, \infty)$ with $\lim\limits_{s\rightarrow\infty} T_g^{-1}(s) = \infty$.

Lahiri and Banerjee [5] introduced the definition of *relative order* of a meromorphic function f with respect to an entire function g, denoted by $\rho_g(f)$ as follows:

$$\begin{aligned}\rho_g(f) &= \inf\{\mu > 0 : T_f(r) < T_g(r^\mu) \text{ for all sufficiently large } r\} \\ &= \limsup_{r\rightarrow\infty} \frac{\log T_g^{-1} T_f(r)}{\log r} .\end{aligned}$$

The definition coincides with the classical one [5] if $g(z) = \exp z$. Similarly, one can define the *relative lower order* of a meromorphic function f with respect to an entire g denoted by $\lambda_g(f)$ in the following manner :

$$\lambda_g(f) = \liminf_{r\rightarrow\infty} \frac{\log T_g^{-1} T_f(r)}{\log r}.$$

In this connection, the following two definitions are relevant:

Definition 2 [1] A non-constant entire function f is said to have the property (A) if for any $\delta > 1$ and for all large r, $[M_f(r)]^2 \leq M_f(r^\delta)$ holds. For examples of functions with or without the property (A), one may see [1].

Definition 3 Two entire functions g and h are said to be asymptotically equivalent if there exists l $(0 < l < \infty)$ such that

$$\frac{M_g(r)}{M_h(r)} \rightarrow l \text{ as } r \rightarrow \infty$$

and in this case we write $g \sim h$. Clearly if $g \sim h$ then $h \sim g$.

3. Some examples

In this section, we present some examples in connection with definitions given in the previous section.

Example 1 (Order (lower order)) Given any natural number n, let $f(z) = \exp z^n$. Then $M_f(r) = \exp r^n$. Therefore putting $R = 2$ in the inequality $T_f(r) \leq \log M_f(r) \leq \frac{R+r}{R-r} T_f(R)$ (cf. [4]) we get that $T_f(r) \leq r^n$ and $T_f(r) \geq \frac{1}{3}\left(\frac{r}{2}\right)^n$. Hence

$$\rho_f = \limsup_{r\rightarrow\infty} \frac{\log T_f(r)}{\log r} = n \quad \text{and} \quad \lambda_f = \liminf_{r\rightarrow\infty} \frac{\log T_f(r)}{\log r} = n .$$

Further, if we take $g = \exp^{[2]} z$, then $T_g(r) \sim \frac{\exp r}{(2\pi^3 r)^{\frac{1}{2}}} (r \rightarrow \infty)$. Therefore

$$\rho_f = \lambda_f = \infty .$$

Example 2 (Relative order (relative lower order)) Suppose $f = g = \exp^{[2]} z$ then $T_f(r) = T_g(r) \sim \frac{\exp r}{(2\pi^3 r)^{\frac{1}{2}}}\,(r \to \infty)$. Now we obtain that

$$\begin{aligned} T_g(r) &\leq \log M_g(r) \leq 3T_g(2r) \quad (\text{cf. } [4]) \\ \text{i.e., } T_g(r) &\leq \exp r \leq 3T_g(2r) \ . \end{aligned}$$

Therefore

$$T_g^{-1}T_f(r) \geq \log\left(\frac{\exp r}{(2\pi^3 r)^{\frac{1}{2}}}\right), \quad \text{i.e., } \liminf_{r\to\infty} \frac{\log T_g^{-1}T_f(r)}{\log r} \geq 1$$

and

$$T_g^{-1}T_f(r) \leq 2\log\left(\frac{3\exp r}{(2\pi^3 r)^{\frac{1}{2}}}\right), \quad \text{i.e., } \limsup_{r\to\infty} \frac{\log T_g^{-1}T_f(r)}{\log r} \leq 1 \ .$$

Hence

$$\rho_g(f) = \lambda_g(f) = 1 \ .$$

4. Lemmas

In this section, we present some lemmas which will be needed in the sequel.

Lemma 1 [1] *Let g be an entire function and $\alpha > 1, 0 < \beta < \alpha$. Then*

$$M_g(\alpha r) > \beta M_g(r) \text{ for all sufficiently large } r.$$

Lemma 2 [1] *Let f be an entire function which satisfies Property* (A). *Then for any positive integer n and for all sufficiently large r*

$$[M_f(r)]^n \leq M_f\left(r^\delta\right)$$

holds where $\delta > 1$.

Lemma 3 *Let g be an entire. Then for all sufficiently large values of r,*

$$T_g(r) \leq \log M_g(r) \leq 3T_g(2r) \ .$$

Lemma 3 follows from Theorem 1.6 (cf. [4], p.18), on putting $R = 2r$.

Lemma 4 [4] *Suppose f be a transcendental meromorphic function of finite order or of non-zero lower order and $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;f) = 4$. Also, let g be a transcendental entire function of regular growth having non zero finite order and $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;g) = 4$. Then the relative order and relative lower order of $P[f]$ with respect to $P[g]$ are same as those of f with respect to g.*

Lemma 5 *Let g and h be any two transcendental entire functions of regular growth having non zero finite order with* $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;g)=4$ *and* $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;h)=4$ *respectively. Then for any transcendental meromorphic function f of finite order or of non-zero lower order and* $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;f)=4$,

$$\rho_{P[g]}\left(P[f]\right)=\rho_{P[h]}\left(P[f]\right)$$

and

$$\lambda_{P[g]}\left(P[f]\right)=\lambda_{P[h]}\left(P[f]\right)\ .$$

if g and h have Property (A) *and $g\sim h$.*

Proof. Let $\varepsilon>0$ is arbitrary. Now, we get from Definition 3 and Lemma 1 for all sufficiently large values of r that

$$M_g\left(r\right)<\left(l+\varepsilon\right)M_h\left(r\right)\leq M_h\left(\alpha r\right)\ , \tag{1}$$

where $\alpha>1$ is such that $l+\varepsilon<\alpha$.

Now, from Lemma 3 and in view of definition of relative order, we obtain for all sufficiently large values of r that

$$T_f\left(r\right)\leq T_g\left[\left(r\right)^{\left(\rho_g(f)+\varepsilon\right)}\right],\quad \text{i.e., } T_f\left(r\right)\leq \log M_g\left[\left(r\right)^{\left(\rho_g(f)+\varepsilon\right)}\right]\ .$$

Therefore, in view of (1), Lemma 2 and Lemma 3, it follows from above for any $\delta>1$ that

$$\begin{aligned}
T_f\left(r\right) &\leq \frac{1}{3}\log\left[M_h\left[\left(\alpha r\right)^{\left(\rho_g(f)+\varepsilon\right)}\right]\right]^3\\
\textit{i.e.,}\ T_f\left(r\right) &\leq \frac{1}{3}\log M_h\left[\left(\alpha r\right)^{\delta\left(\rho_g(f)+\varepsilon\right)}\right]\\
\text{i.e., } T_f\left(r\right) &\leq T_h\left[\left(2\alpha r\right)^{\delta\left(\rho_g(f)+\varepsilon\right)}\right]\\
\textit{i.e.,}\ \frac{\log T_h^{-1}T_f\left(r\right)}{\log r} &\leq \delta\left(\rho_g\left(f\right)+\varepsilon\right)\frac{\log\left(2\alpha r\right)}{\log r}\ .
\end{aligned}$$

Letting $\delta\rightarrow 1+$, we get from above that

$$\rho_h\left(f\right)\leq\rho_g\left(f\right)\ . \tag{2}$$

Since $h\sim g$, we also obtain that

$$\rho_g\left(f\right)\leq\rho_h\left(f\right)\ . \tag{3}$$

Now in view of Lemma 4, we obtain from (2) and (3) that

$$\rho_{P[g]}\left(P[f]\right)=\rho_{P[h]}\left(P[f]\right)\ .$$

Similarly, we have

$$\lambda_{P[g]}\left(P[f]\right)=\lambda_{P[h]}\left(P[f]\right)\ .$$

Thus the lemma follows.

5. Theorems

In this section, we present the main results of the paper.

Theorem 1 *Suppose f be a transcendental meromorphic function of finite order or of non-zero lower order and* $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;f)=4$. *Also, let h be a transcendental entire function of regular growth having non zero finite order with* $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;h)=4$ *and g be any entire function such that*

$$0<\lambda_h(f\circ g)\leq\rho_h(f\circ g)<\infty \text{ and } 0<\lambda_h(f)\leq\rho_h(f)<\infty.$$

Then

$$\begin{aligned}\frac{\lambda_h(f\circ g)}{\rho_h(f)} &\leq \liminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_{P[h]}^{-1}T_{P[f]}(r)} \leq \frac{\lambda_h(f\circ g)}{\lambda_h(f)} \\ &\leq \limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_{P[h]}^{-1}T_{P[f]}(r)} \leq \frac{\rho_h(f\circ g)}{\lambda_h(f)}\ .\end{aligned}$$

Proof. From the definition of $\rho_h(f)$ and $\lambda_h(f\circ g)$ and Lemma 4 we have for arbitrary positive ε and for all sufficiently large values of r that

$$\log T_h^{-1}T_{f\circ g}(r)\geqslant(\lambda_h(f\circ g)-\varepsilon)\log r \tag{4}$$

and

$$\begin{aligned}\log T_{P[h]}^{-1}T_{P[f]}(r) &\leq \left(\rho_{P[h]}(P[f])+\varepsilon\right)\log r\\ \text{i.e., } \log T_{P[h]}^{-1}T_{P[f]}(r) &\leq (\rho_h(f)+\varepsilon)\log r\ .\end{aligned} \tag{5}$$

Now, from (4), (5) it follows for all sufficiently large values of r that

$$\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_{P[h]}^{-1}T_{P[f]}(r)}\geqslant\frac{(\lambda_h(f\circ g)-\varepsilon)\log r}{(\rho_h(f)+\varepsilon)\log r}\ .$$

As $\varepsilon\,(>0)$ is arbitrary , we obtain that

$$\liminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_{P[h]}^{-1}T_{P[f]}(r)}\geqslant\frac{\lambda_h(f\circ g)}{\rho_h(f)}\ . \tag{6}$$

Again for a sequence of values of r tending to infinity,

$$\log T_h^{-1}T_{f\circ g}(r)\leq(\lambda_h(f\circ g)+\varepsilon)\log r \tag{7}$$

and for all sufficiently large values of r,

$$\begin{aligned}\log T_{P[h]}^{-1}T_{P[f]}(r) &\geqslant \left(\lambda_{P[h]}(P[f])-\varepsilon\right)\log r\\ \text{i.e., } \log T_{P[h]}^{-1}T_{P[f]}(r) &\geqslant (\lambda_h(f)-\varepsilon)\log r\ .\end{aligned} \tag{8}$$

Combining (7) and (8), we get for a sequence of values of r tending to infinity that

$$\frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{P[h]}^{-1} T_{P[f]}(r)} \leq \frac{(\lambda_h(f \circ g) + \varepsilon) \log r}{(\lambda_h(f) - \varepsilon) \log r} .$$

Since $\varepsilon (> 0)$ is arbitrary, it follows that

$$\liminf_{r \to \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{P[h]}^{-1} T_{P[f]}(r)} \leq \frac{\lambda_h(f \circ g)}{\lambda_h(f)} . \tag{9}$$

Also for a sequence of values of r tending to infinity that

$$\begin{aligned} \log T_{P[h]}^{-1} T_{P[f]}(r) &\leq \left(\lambda_{P[h]}(P[f]) + \varepsilon\right) \log r \\ \text{i.e., } \log T_{P[h]}^{-1} T_{P[f]}(r) &\leq (\lambda_h(f) + \varepsilon) \log r . \end{aligned} \tag{10}$$

Now, from (4) and (10), we obtain for a sequence of values of r tending to infinity that

$$\frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{P[h]}^{-1} T_{P[f]}(r)} \geq \frac{(\lambda_h(f \circ g) - \varepsilon) \log r}{(\lambda_h(f) + \varepsilon) \log r} .$$

As $\varepsilon (> 0)$ is arbitrary, we get from above that

$$\limsup_{r \to \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{P[h]}^{-1} T_{P[f]}(r)} \geq \frac{\lambda_h(f \circ g)}{\lambda_h(f)} . \tag{11}$$

Also for all sufficiently large values of r,

$$\log T_h^{-1} T_{f \circ g}(r) \leq (\rho_h(f \circ g) + \varepsilon) \log r . \tag{12}$$

Now, it follows from (8) and (12) for all sufficiently large values of r that

$$\frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{P[h]}^{-1} T_{P[f]}(r)} \leq \frac{(\rho_h(f \circ g) + \varepsilon) \log r}{(\lambda_h(f) - \varepsilon) \log r} .$$

Since $\varepsilon (> 0)$ is arbitrary, we obtain that

$$\limsup_{r \to \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{P[h]}^{-1} T_{P[f]}(r)} \leq \frac{\rho_h(f \circ g)}{\lambda_h(f)} . \tag{13}$$

Thus the theorem follows from (6), (9), (11) and (13).

The following theorem can be proved in the line of Theorem 1 and so the proof is omitted.

Theorem 2 *Suppose g be a transcendental entire function of finite order or of non-zero lower order and* $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; g) = 4$. *Also let h be a transcendental entire function of regular growth having non zero finite order with* $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; h) = 4$

and f be any meromorphic function such that $0 < \lambda_h(f \circ g) \leq \rho_h(f \circ g) < \infty$ and $0 < \lambda_h(g) \leq \rho_h(g) < \infty$. Then

$$\begin{aligned}\frac{\lambda_h(f \circ g)}{\rho_h(g)} &\leq \liminf_{r \to \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{P[h]}^{-1} T_{P[g]}(r)} \leq \frac{\lambda_h(f \circ g)}{\lambda_h(g)} \\ &\leq \limsup_{r \to \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{P[h]}^{-1} T_{P[g]}(r)} \leq \frac{\rho_h(f \circ g)}{\lambda_h(g)} .\end{aligned}$$

Theorem 3 *Suppose f be a transcendental meromorphic function of finite order or of non-zero lower order and $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; f) = 4$. Also, let h be a transcendental entire function of regular growth having non zero finite order with $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; h) = 4$ and g be any entire function with $0 < \rho_h(f \circ g) < \infty$ and $0 < \rho_h(f) < \infty$. Then*

$$\liminf_{r \to \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{P[h]}^{-1} T_{P[f]}(r)} \leq \frac{\rho_h(f \circ g)}{\rho_h(f)} \leq \limsup_{r \to \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{P[h]}^{-1} T_{P[f]}(r)} .$$

Proof. From the definition of $\rho_{P[h]}(P[f])$ and in view of Lemma 4, we get for a sequence of values of r tending to infinity that

$$\begin{aligned} \log T_{P[h]}^{-1} T_{P[f]}(r) &\geqslant \left(\rho_{P[h]}(P[f]) - \varepsilon\right) \log r \\ \text{i.e., } \log T_{P[h]}^{-1} T_{P[f]}(r) &\geqslant \left(\rho_h(f) - \varepsilon\right) \log r . \end{aligned} \tag{14}$$

Now, from (12) and (14), it follows for a sequence of values of r tending to infinity that

$$\frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{P[h]}^{-1} T_{P[f]}(r)} \leq \frac{(\rho_h(f \circ g) + \varepsilon) \log r}{(\rho_h(f) - \varepsilon) \log r} .$$

As $\varepsilon\,(> 0)$ is arbitrary, we obtain that

$$\liminf_{r \to \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{P[h]}^{-1} T_{P[f]}(r)} \leq \frac{\rho_h(f \circ g)}{\rho_h(f)} . \tag{15}$$

Again for a sequence of values of r tending to infinity,

$$\log T_h^{-1} T_{f \circ g}(r) \geqslant (\rho_h(f \circ g) - \varepsilon) \log r . \tag{16}$$

So combining (5) and 16, we get for a sequence of values of r tending to infinity that

$$\frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{P[h]}^{-1} T_{P[f]}(r)} \geqslant \frac{(\rho_h(f \circ g) - \varepsilon) \log r}{(\rho_h(f) + \varepsilon) \log r} .$$

Since $\varepsilon\,(> 0)$ is arbitrary, it follows that

$$\limsup_{r \to \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{P[h]}^{-1} T_{P[f]}(r)} \geqslant \frac{\rho_h(f \circ g)}{\rho_h(f)} . \tag{17}$$

Thus the theorem follows from (15) and (17).

The following theorem can be carried out in the line of Theorem 3 and therefore we omit its proof.

Theorem 4 *Suppose g be a transcendental entire function of finite order or of non-zero lower order and $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;g)=4$. Also let h be a transcendental entire function of regular growth having non zero finite order with $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;h)=4$ and f be any meromorphic function such that $0<\rho_h(f\circ g)<\infty$ and $0<\rho_h(g)<\infty$. Then*

$$\liminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_{P[h]}^{-1}T_{P[g]}(r)}\leq\frac{\rho_h(f\circ g)}{\rho_h(g)}\leq\limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_{P[h]}^{-1}T_{P[g]}(r)}\ .$$

The following theorem is a natural consequence of Theorem 1 and Theorem 3:

Theorem 5 *Suppose f be a transcendental meromorphic function of finite order or of non-zero lower order and $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;f)=4$. Also let h be a transcendental entire function of regular growth having non zero finite order with $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;h)=4$ and g be any entire function with $0<\lambda_h(f\circ g)\leq\rho_h(f\circ g)<\infty$ and $0<\lambda_h(f)\leq\rho_h(f)<\infty$. Then*

$$\liminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_{P[h]}^{-1}T_{P[f]}(r)}\leq\min\left\{\frac{\lambda_h(f\circ g)}{\lambda_h(f)},\frac{\rho_h(f\circ g)}{\rho_h(f)}\right\}$$

$$\leq\max\left\{\frac{\lambda_h(f\circ g)}{\lambda_h(f)},\frac{\rho_h(f\circ g)}{\rho_h(f)}\right\}\leq\limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_{P[h]}^{-1}T_{P[f]}(r)}\ .$$

The proof is omitted.

Analogously, one may state the following theorem without its proof.

Theorem 6 *Suppose g be a transcendental entire function of finite order or of non-zero lower order and $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;g)=4$. Also let h be a transcendental entire function of regular growth having non zero finite order with $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;h)=4$ and f be any meromorphic function such that $0<\lambda_h(f\circ g)\leq\rho_h(f\circ g)<\infty$ and $0<\lambda_h(g)\leq\rho_h(g)<\infty$ .Then*

$$\liminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_{P[h]}^{-1}T_{P[g]}(r)}\leq\min\left\{\frac{\lambda_h(f\circ g)}{\lambda_h(g)},\frac{\rho_h(f\circ g)}{\rho_h(g)}\right\}$$

$$\leq\max\left\{\frac{\lambda_h(f\circ g)}{\lambda_h(g)},\frac{\rho_h(f\circ g)}{\rho_h(g)}\right\}\leq\limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_{P[h]}^{-1}T_{P[g]}(r)}\ .$$

Theorem 7 *Suppose f be a transcendental meromorphic function of finite order or of non-zero lower order and $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;f)=4$. Also let h be a transcendental entire function of regular growth having non zero finite order with*

$\sum\limits_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;h) = 4$ *and* g *be any entire function with* $0 < \rho_h(f\circ g) < \infty$ *and* $0 < \rho_h(f) < \infty$ *and* $g \sim h$. *Then*

$$\liminf_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log T_{P[h]}^{-1} T_{P[f]}(r)} \leq 1 \leq \limsup_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log T_{P[h]}^{-1} T_{P[f]}(r)} .$$

Proof. From the definition of $\rho_g(f)$, we get for all sufficiently large values of r that

$$\log T_g^{-1} T_f(r) \leq (\rho_g(f) + \varepsilon) \log r \tag{18}$$

and for a sequence of values of r tending to infinity that

$$\log T_g^{-1} T_f(r) \geq (\rho_g(f) - \varepsilon) \log r . \tag{19}$$

Now, from (14) and (18), it follows for a sequence of values of r tending to infinity that

$$\frac{\log T_g^{-1} T_f(r)}{\log T_{P[h]}^{-1} T_{P[f]}(r)} \leq \frac{(\rho_g(f) + \varepsilon) \log r}{(\rho_h(f) - \varepsilon) \log r} .$$

As $\varepsilon\,(> 0)$ is arbitrary, we obtain that

$$\liminf_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log T_{P[h]}^{-1} T_{P[f]}(r)} \leq \frac{\rho_g(f)}{\rho_h(f)} . \tag{20}$$

Now, as $g \sim h$, in view of Lemma 4 and Lemma 5 we obtain from (20) that

$$\liminf_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log T_{P[h]}^{-1} T_{P[f]}(r)} \leq 1 . \tag{21}$$

Again combining (5) and (19), we get for a sequence of values of r tending to infinity that

$$\frac{\log T_g^{-1} T_f(r)}{\log T_{P[h]}^{-1} T_{P[f]}(r)} \geqslant \frac{(\rho_g(f) - \varepsilon) \log r}{(\rho_h(f) + \varepsilon) \log r} .$$

Since $\varepsilon\,(> 0)$ is arbitrary, it follows that

$$\limsup_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log T_{P[h]}^{-1} T_{P[f]}(r)} \geqslant \frac{\rho_g(f)}{\rho_h(f)} . \tag{22}$$

Now as $g \sim h$, in view of Lemma 4 and Lemma 5 we obtain from (22) that

$$\limsup_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log T_{P[h]}^{-1} T_{P[f]}(r)} \geqslant 1 . \tag{23}$$

Thus the theorem follows from (21) and (23).

6. Conclusion

Actually this paper deals with the extension of the works on the growth properties concerning *differential monomials* generated by transcendental entire and transcendental meromorphic functions on the basis of their *relative orders* and *relative lower orders*. These theories can also be modified by the treatment of the notions of *generalized relative orders* (*generalized relative lower orders*) and (p, q)-*th relative orders* ((p, q)-*th relative lower orders*). In addition, some extensions of the same may be done in the light of slowly changing functions. Moreover, the notion of *relative order* and *relative lower order* of *differential monomials* generated by transcendental entire and transcendental meromorphic functions may have a wide range of applications in Complex Dynamics, Factorization Theory of entire functions of single complex variable, the solution of complex differential equations etc. which must be an active area of research.

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UNIFORMLY STABILITY OF IMPULSIVE DELAYED LINEAR SYSTEMS WITH IMPULSE TIME WINDOWS

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Abstract. In this paper, we formulate a new kind of mathematical model of impulsive delayed linear system, which is called *impulsive delayed linear system with impulse time windows*. By constructing a Lyaponov function, we obtain some conditions for the uniformly stability of the system. An example is also given to illustrate the efficiency of the results.

Keywords: uniformly stability, delayed system, impulsive control system, impulse time windows.

1. Introduction

Impulsive control is a control paradigm based on impulsive differential equations. In recent years, many researchers have studied impulsive systems and impulsive control, for example, [1]–[5].

Time delay phenomenon is very common in electric circuit systems. Many researchers have done outstanding works in this area. For instance, Zhang and Sun [3] have studied the stability of impulsive linear differential equations with time delay, Zhou and Wu [4] have given some conditions to ensure the exponential stability of impulsive delayed linear differential equations, Liu et al. [5] have obtained the stability criteria for impulsive systems with time delay, Su et al. [6] have researched the delay-dependent robust H_∞ control for uncertain time-delay systems, Wu et al. [7] have studied the stability and dissipativity analysis of static neural networks with time delay, Shin and Cui in [8] have shown the computing time delay and its effects on real-time control systems, Knospe and Roozbehani [9] have studied the stability of linear systems with interval time delays excluding zero, Zhang et al. [10] have designed a fuzzy controller for nonlinear impulsive fuzzy systems with time delay, Michiels, Van Assche and Niculescu [11] have researched the stabilization of time-delay systems with a controlled time-varying delay and applications.

Impulsive control can provide an efficient method for some cases in which the systems cannot endure continuous disturbance. For the traditional impulsive control system, the impulses are assumed to put at fixed time or the occurrence of the impulses is determined by the state of the system. In the latter situation, the time of the occurrence can also be calculated. How can we input impulses if we don't know or we cannot calculate the the exact occurrence time, but we know that the occurrence time is limited to a small time interval? Can we find some conditions to ensure the system's stability? In this paper, we will answer these questions.

We introduce a delayed impulsive control system with its occurrence time of impulses is limited to a small time interval, which is named by *impulsive delayed linear system with impulse time windows.*

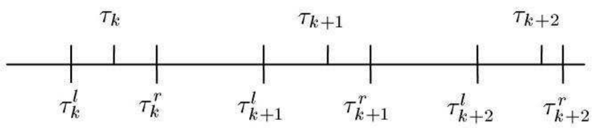

Figure 1: In an impulsive control system with impulse time windows, the occurrence time τ_k $(k = 0, 1, 2, ...)$ of impulses are unknown, but the impulse time windows $[\tau_k^l, \tau_k^r)$ are known, i.e., τ_k^l, τ_k^r $(k = 0, 1, 2, ...)$ are known

From Figure 1, we know that every occurrence time of impulses can be chosen randomly in a small *impulse time window.* So the system is more complicated than the traditional one. To our knowledge, there are seldom papers dealt with delayed impulsive control systems with impulse time windows.

The rest of the paper is organized as follows. In Section 2, we formulate the problem and introduce some notions and definitions. We then obtain, in Section 3, several conditions to ensure that the system is uniformly stable. In Section 4, we give a numerical example. Finally, we conclude our results.

2. Problem statement and preliminaries

Consider the impulsive delayed linear system with impulse time windows

$$\begin{cases} \dot{x}(t) = Ax(t) + Bx(t-\tau), & t \geq t_0, \quad t \neq \tau_k \in [\tau_k^l, \tau_k^r), \\ \Delta x(t) = x(t) - x(t^-) = Cx(t^-), & t = \tau_k, \quad k \in \mathbf{Z}^+, \end{cases} \tag{2.1}$$

where $x \in R^n$, $A, B, C \in R^{n\times n}$, B is nonsingular, $x(t^+) = \lim\limits_{s\to t^+} x(s)$, $x(t^-) = \lim\limits_{s\to t^-} x(s)$, $[\tau_k^l, \tau_k^r)$ $(k = 0, 1, 2, \cdots)$ are impulse time windows and τ_k $(k = 0, 1, 2, \cdots)$ are unknown time points where the impulses occur. We assume that

$$0 = t_0 = \tau_0^l = \tau_0 = \tau_0^r \leq \tau_1^l \leq \tau_1 < \tau_1^r \leq \cdots \leq \tau_k^l \leq \tau_k < \tau_k^r \leq \cdots,$$

and

$$\lim_{k\to\infty} \tau_k = \infty.$$

Obviously, $x(t) = 0$ is a solution of (2.1), which is called *the zero solution*.

Let $PC([-\tau, 0], R^n)$ is a class of piecewise continuous functions $\phi : [-\tau, 0] \to R^n$ and there is at most a finite number of discontinuous points $\hat{t}$, at which both $\phi(\hat{t}^+)$ and $\phi(\hat{t}^-)$ exist and $\phi(\hat{t}^+) = \phi(\hat{t})$.

For $\psi \in PC([-\tau, 0], R^n)$, the norm of ψ is defined by

$$|\psi| = \sup_{-\tau\leq s\leq 0} ||\psi(s)||,$$

where $||\cdot||$ denotes the norm of vector in R^n.

Define

$$PC(\rho) = \{\phi \in PC([-\tau, 0], R^n) : |\phi| < \rho\},$$

for any $\rho > 0$.

For given $\sigma \geq t_0$ and $\varphi \in PC([-\tau, 0], R^n)$, the initial value problem of (2.1) is

$$\begin{cases} \dot{x}(t) = Ax(t) + Bx(t-\tau), & t \geq \sigma, \ t \neq \tau_k \in [\tau_k^l, \tau_k^r), \\ \Delta x(t) = x(t) - x(t^-) = Cx(t^-), & t = \tau_k, \ k \in \mathbf{Z}^+, \\ x(\sigma + t) = \varphi(t) & t \in [-\tau, 0]. \end{cases} \tag{2.2}$$

Definition 1. The zero solution of (2.1) is stable if for any $\sigma \geq t_0$ and $\varepsilon > 0$ there is a $\delta = \delta(\sigma, \varepsilon) > 0$ such that for $t \geq \sigma$ and $\varphi \in PC(\delta)$ we have that

$$||x(t, \sigma, \varphi)|| < \varepsilon.$$

The zero solution of (2.1) is said to be uniformly stable if δ is independent of σ.

Definition 2. [1] For $(t, x) \in (\tau_{i-1}, \tau_i] \times R^n$, we define

$$D^+V(t,x) = \lim_{h \to 0^+} \sup \frac{1}{h}[V(t+h, x+h\dot{x}) - V(t,x)]$$

and

$$D_-V(t,x) = \lim_{h \to 0^-} \inf \frac{1}{h}[V(t+h, x+h\dot{x}) - V(t,x)].$$

Through the rest of the paper, I stands for the identity matrix.

3. Theoretical analysis

Theorem 1. *If there exists a symmetric and positive definite matrix $P \in \mathbf{R}^{n\times n}$, such that, for $k = 0, 1, 2, \cdots$, we have that*

$$\lambda_3(\tau^r_{k+1} - \tau^l_k) < -\ln \lambda_7,$$

where $\lambda_7 = \max\{\lambda_5, \lambda_6\}$, $\lambda_5 \in (0,1)$ is the largest eigenvalue of $P^{-1}(I+C)^T P(I+C)$. $\lambda_6 \in (0,1)$ is the largest eigenvalue of $(B^TB)^{-1}(I+C)^T B^T B(I+C)$ and λ_3 is the largest eigenvalue of $P^{-1}(A^TP + PA + B^TB + P^TP)$, then the zero solution of (1) is uniformly stable.

Proof. Let $\lambda_1 > 0$ and $\lambda_2 > 0$ be the minimum eigenvalue and maximum eigenvalue of P, respectively. Let $\lambda_4 > 0$ is the maximum eigenvalue of B^TB. For any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$, such that $\delta < \sqrt{\frac{\lambda_1\lambda_7}{\lambda_2 + \tau\lambda_4}}\varepsilon$.

Choose the Lyapunov function as

$$V(t, x(t)) = x^T(t)Px(t) + \int_{t-\tau}^{t} x^T(s)B^TBx(s)ds,$$

then $\lambda_1||x(t)||^2 \le V(t, x(t)) \le \lambda_2||x(t)||^2 + \tau\lambda_4 \sup_{-\tau \le s \le 0} ||x(t+s)||^2$.

If $t \neq \tau_k$, $k = 1, 2, \cdots$, we have that

$$\begin{aligned}
D^+V(t, x(t)) &= (x^T(t))'Px(t) + x^T(t)Px'(t) \\
&\quad + x^T(t)B^TBx(t) - x^T(t-\tau)B^TBx(t-\tau) \\
&= x^T(t)(A^TP + PA + B^TB)x(t) + 2x^T(t-\tau)B^TPx(t) \\
&\quad - x^T(t-\tau)B^TBx(t-\tau) \\
&\le x^T(t)(A^TP + PA + B^TB)x(t) + x^T(t-\tau)B^TBx(t-\tau) \\
&\quad + x^T(t)P^TPx(t) - x^T(t-\tau)B^TBx(t-\tau) \\
&= x^T(t)(A^TP + PA + B^TB + P^TP)x(t) \\
&\le \lambda_3 x^T(t)Px(t) \\
&\le \lambda_3(x^T(t)Px(t) + \int_{t-\tau}^{t} x^T(s)B^TBx(s)ds) \\
&= \lambda_3 V(t, x(t)).
\end{aligned}$$

For any $\sigma \geq t_0$ and $\varphi \in PC(\delta)$, set $x(t) = x(t, \sigma, \phi)$ be the solution of (2.1) through (σ, φ).

Suppose that $\sigma \in [\tau^l_{m-1}, \tau^l_m)$ is valid for some $m \in \mathbf{Z}^+$.

Two cases are possible:

Case 1. If $\tau_{m-1} < \sigma < \tau^l_m$, then we have the fact that

$$V(t, x(t)) \leq \frac{\lambda_2 + \tau\lambda_4}{\lambda_7}\delta^2, \ \sigma \leq t < \tau_m. \tag{3.1}$$

Subcase 1 If $t = \sigma$, then

$$\begin{aligned} V(t, x(t)) &= V(\sigma, x(\sigma)) \\ &= V(\sigma, \varphi(0)) \\ &\leq \lambda_2||\varphi(0)||^2 + \tau\lambda_4 \sup_{-\tau \leq s \leq 0} ||x(s)||^2 \\ &\leq (\lambda_2 + \tau\lambda_4)|\varphi|^2 \\ &\leq (\lambda_2 + \tau\lambda_4)\delta^2 \\ &< \frac{\lambda_2 + \tau\lambda_4}{\lambda_7}\delta^2. \end{aligned}$$

Subcase 2 If $\sigma < t < \tau_m$ and suppose that (3.1) is not valid for $t \in (\sigma, \tau_m)$, then there exists $\hat{s} \in (\sigma, \tau_m)$, such that

$$V(\hat{s}, x(\hat{s})) > \frac{\lambda_2 + \tau\lambda_4}{\lambda_7}\delta^2 > (\lambda_2 + \tau\lambda_4)\delta^2 \geq V(\sigma, x(\sigma)).$$

From the continuity of $V(t, x(t))$ in (σ, τ_m), we know that there is a $s_1 \in (\sigma, \hat{s})$ such that

$$\begin{aligned} V(s_1, x(s_1)) &= \frac{\lambda_2 + \tau\lambda_4}{\lambda_7}\delta^2, \\ V(t, x(t)) &\leq \frac{\lambda_2 + \tau\lambda_4}{\lambda_7}\delta^2, \quad \sigma < t \leq s_1, \\ D^+V(s_1, x(s_1)) &\geq 0. \end{aligned}$$

Form the fact that $V(\sigma, x(\sigma)) \leq (\lambda_2 + \tau\lambda_4)\delta^2 < \dfrac{\lambda_2 + \tau\lambda_4}{\lambda_7}\delta^2$, we know that there exists an $s_2 \in [\sigma, s_1)$ such that

$$\begin{aligned} V(s_2, x(s_2)) &= (\lambda_2 + \tau\lambda_4)\delta^2, \\ V(t, x(t)) &\leq (\lambda_2 + \tau\lambda_4)\delta^2, \quad s_2 \leq t \leq s_1, \\ D^+V(s_2, x(s_2)) &\geq 0. \end{aligned}$$

From $D^+V(t, x(t)) \leq \lambda_3 V(t, x(t))$ we know that $\dfrac{D^+V(t, x(t))}{V(t, x(t))} \leq \lambda_3$. Thus

$$\int_{s_2}^{s_1} \frac{D^+V(t, x(t))}{V(t, x(t))}dt \leq \int_{s_2}^{s_1} \lambda_3 dt \leq \int_{\tau^l_{m-1}}^{\tau^r_m} \lambda_3 dt = \lambda_3(\tau^r_m - \tau^l_{m-1}) < -\ln\lambda_7.$$

At the same time,

$$\begin{aligned}\int_{s_2}^{s_1} \frac{D^+V(t,x(t))}{V(t,x(t))}dt &= \int_{V(s_2,x(s_2))}^{V(s_1,x(s_1))} u^{-1}du = \int_{(\lambda_2+\tau\lambda_4)\delta^2}^{\frac{\lambda_2+\tau\lambda_4}{\lambda_7}\delta^2} u^{-1}du \\ &= \ln\left(\frac{\lambda_2+\tau\lambda_4}{\lambda_7}\delta^2\right) - \ln((\lambda_2+\tau\lambda_4)\delta^2) = -\ln\lambda_7.\end{aligned}$$

So, it is a contradiction. Hence, (3.1) is valid for $t \in (\sigma, \tau_m)$.

Next, we will prove that, for any $k = 0, 1, 2, \cdots$, the following is valid

$$V(t,x(t)) \le \frac{\lambda_2+\tau\lambda_4}{\lambda_7}\delta^2, \quad \tau_{m+k}^l \le t < \tau_{m+1}^l. \tag{3.2}$$

Since

$$\begin{aligned}V(\tau_m, x(\tau_m)) &= V(\tau_m, (I+C)x(\tau_m^-)) \\ &= x^T(\tau_m^-)(I+C)^T P(I+C)x(\tau_m^-) \\ &\quad + \int_{-\tau}^0 x^T(s+\tau_m^-)(I+C)^T B^T B(I+C)x(s+\tau_m^-)ds \\ &\le \lambda_5 x^T(\tau_m^-)Px(\tau_m^-) + \lambda_6 \int_{-\tau}^0 x^T(s+\tau_m^-)B^T Bx(s+\tau_m^-)ds \\ &\le \lambda_7 V(\tau_m^-, x(\tau_m^-)) \\ &\le (\lambda_2+\tau\lambda_4)\delta^2,\end{aligned}$$

then similarly to the proof of Case 1, we can easily prove that (3.2) is valid.

Thus we obtain that

$$V(t,x(t)) \le \frac{\lambda_2+\tau\lambda_4}{\lambda_7}\delta^2, \; t \ge \sigma. \tag{3.3}$$

Case 2 If $\tau_{m-1}^l \le \sigma \le \tau_{m-1}$, then similar to the Case 1, we can prove that

$$V(t,x(t)) \le \frac{\lambda_2+\tau\lambda_4}{\lambda_7}\delta^2, \; \sigma \le t < \tau_{m-1}.$$

And finally we can obtain that (3.3) is valid.

Hence, from (3.3) we know that

$$\lambda_1||x(t)||^2 \le V(t,x(t)) \le \frac{\lambda_2+\tau\lambda_4}{\lambda_7}\delta^2, \; t \ge \sigma,$$

which implies that

$$||x(t)|| \le \sqrt{\frac{\lambda_2+\tau\lambda_4}{\lambda_7}}\delta < \varepsilon.$$

Therefore, the zero solution of (2.1) is uniformly stable and we complete the proof. ∎

4. Numerical example

Consider the impulsive delayed linear system with impulse time windows as following.

$$
\begin{cases}
\begin{pmatrix} \dot{x_1}(t) \\ \dot{x_2}(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1(t-\tau) \\ x_2(t-\tau) \end{pmatrix}, \\
\qquad\qquad t \geq t_0,\ t \neq \tau_k \in [\tau_k^l, \tau_k^r), \\
\begin{pmatrix} x_1(\tau_k) \\ x_2(\tau_k) \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} x_1(\tau_k^-) \\ x_2(\tau_k^-) \end{pmatrix}, \qquad k \in \mathbf{Z}^+.
\end{cases}
\tag{4.1}
$$

In the previous system, $A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$ and $I+C = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$.

Set $P = I$, then

$$
\begin{aligned}
&P^{-1}(I+C)^T P(I+C) = \begin{pmatrix} 1/4 & 0 \\ 0 & 1/4 \end{pmatrix}, \\
&(B^T B)^{-1}(I+C)^T B^T B(I+C) = \begin{pmatrix} 1/4 & 0 \\ 0 & 1/4 \end{pmatrix}, \\
&P^{-1}(A^T P + PA + B^T B + P^T P) = \begin{pmatrix} 4 & 2 \\ 2 & 7 \end{pmatrix}.
\end{aligned}
$$

Thus $\lambda_7 = \max\{\lambda_5, \lambda_6\} = \lambda_5 = \lambda_6 = 1/4$, $\lambda_3 = 8$. Apply Theorem 1,

$$(\tau_{k+1}^r - \tau_k^l) < -\frac{\ln \lambda_7}{\lambda_3} = \frac{\ln 2}{4}.$$

So, the zero solution of system (4.1) is uniformly stable, if

$$(\tau_{k+1}^r - \tau_k^l) < \frac{\ln 2}{4}.$$

5. Conclusions

In this paper, we have studied the uniformly stability of impulsive delayed linear systems with impulse time windows. We have obtained some conditions to ensure that the systems are uniformly stable. An example is also given to illustrate the efficiency of the results.

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T-SYSTEMS IN TERNARY SEMIGROUPS

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Abstract. In this paper, we introduce the notions of right T-system transitive, T-homomorphism, semispace in ternary semigroups. We characterize different classes of ternary semigroups by the properties of their right T-system and T-homomorphism.

Keywords: ternary semigroup, right T-system, fixed element, transitive, irreducible, T-homomorphism, semispace.

1. Introduction

The theory of ternary algebraic systems was introduced by Lehmer [4] in 1932, but earlier such structures was studied by Kasner [3] who give the idea of n-ary algebras. Lehmer [4] investigated certain algebraic systems called triplexes which turn out to be commutative ternary groups. Ternary semigroups are universal algebras with one associative ternary operation. Anjaneyulu [1] introduce S-semispace and obtain an isomorphism theorem of semigroup of S-homomorphism on semispaces and deduce the well known Ljapin's [5] theorem on the semigroup of transformations over a set. In this paper we introduce right T-system and T-semispace and study some properties of these T-systems.

To present the main theorems, we first recall the definition of a ternary semigroup which is important here.

Definition. A nonempty set T is called a *ternary semigroup* [3] if there exists a ternary operation $T \times T \times T \to T$ written as $(x_1, x_2, x_3) \to [x_1 x_2 x_3]$, satisfying the following identity, for any $x_1, x_2, x_3, x_4, x_5 \in T$,

$$[(x_1 x_2 x_3)\, x_4 x_5] = [x_1\, (x_2 x_3 x_4)\, x_5] = [x_1 x_2\, (x_3 x_4 x_5)]\,.$$

Example. Let $T = \{-i, 0, i\}$. Then T is a ternary semigroup under the multiplication over complex number while T is not a semigroup under complex number multiplication.

2. T-systems in Ternary semigroups

Definition 2.1. Let T be a ternary semigroup. A non empty set M is called a *right T-system* provided there exists a mapping $(m, n, s) \rightarrow mns$ of $M \times M \times T \rightarrow M$ such that $mn(stu) = m(nst)u = (mns)tu$ for all $m, n \in M$ and $s, t, u \in T$. We denote a right T-system M by M_T.

Let M_T be a right T-system. Then an element $m \in M$ is called a *fixed element* of M_T provided $mmt = m$ for all $t \in T$.

If M_T is a right T-system, then we denote the set

$$F_M = \{m \in M : mnt = m \text{ for all } t \in T\}$$

and F_M is read as the set of fixed (*invariant*) elements of an operand M_T over a ternary semigroup T.

A right T-system M_T is said to be a *transitive* provided for any $m, n, p \in M$, there exists a $t \in T$ such that $mnt = p$.

Let M_T be a right T-system. Then a non empty subset N of M is called a T-subsystem of M_T provided $NNT \subseteq N$, that is, for all $m, n \in N$ and $t \in T$, $mnt \in N$.

A right T-system M_T is said to be *irreducible* provided $MMT \nsubseteq FM$ and the only subsystem of M of cardinality greater than one is M itself.

Theorem 2.2. *Let M_T be a right T-system with $FM = \phi$, that is, M_T has no fixed elements. Then M_T is a transitive T-system if and only if M_T is an irreducible.*

Proof. Let M_T is a transitive T-system. Suppose that if possible M_T is not irreducible. Then $MMT \subseteq FM \Rightarrow$ for all $m \in M, t \in T$, $mmt = m$ and hence M_T is not transitive. We have the contradiction. Therefore $MMT \nsubseteq FM$ implies that M_T is an irreducible.

Conversely, suppose that M_T is an irreducible T-system, i.e., $MMT \subseteq FM \Rightarrow$ for $m \in M$, $mmt \neq m$, for all $t \in T \Rightarrow m, n, p \in M$, there exists a $t \in T$ such that $mnt = p$. Therefore, M_T is a transitive T-system.

Definition 3.3. Let M_T and N_T be two right T-systems. A mapping $f : M \to N$ is called a *T-homomorphism from M_T into N_T* provided $f(mnt) = f(m)\, nt$, for all $m \in M$ and $n, t \in T$.

We denote the set of all T-homomorphism from M_T into N_T by $H_T(M, N)$ and the set of all T-homomorphism from M_T into itself by $H_T(M)$ or simply H.

Definition 3.4. An unital T-system M_T is said to be a *T-semispace* or, simply, a *semispace* provided T is a ternary group such that $mns = mnt$, for some $m, n \in M$, and $s, t \in T$ implies $s = t$. We call T, a *centralizer* of M.

It can be observed that a semispace is a vector set with $FM = \phi$ in the sense of Hoehnke [2].

Let M_T be any semispace. Then the transitive relation on M_T is an equivalence relation and the corresponding equivalence classes as T-equivalence classes. Also, each equivalence class is a transitive T-system and hence an irreducible T-system.

Let $\{C_\alpha\}_{\alpha\in\Delta}$ be the family of T-equivalence classes. By the choice axiom, there exist $\{W_\alpha\}_{\alpha\in\Delta}$ such that $W_\alpha \in C_\alpha$.

In what follows, we fix the family of elements $\{W_\alpha\}_{\alpha\in\Delta}$ and, for simplicity, we write α instead of W_α for each $\alpha \in \Delta$, i.e., we consider Δ as a subset of M.

Let $\alpha \in \Delta$, we define a mapping S_α on M as follows. Let $m \in M$. Then $m = \beta st$, for some $\beta \in \Delta$ and $s, t \in T$. Write for $r \in T$, $ms_\alpha = (\beta st)\, s_\alpha = \alpha st$. Now, clearly, s_α is a T-homoorphism.

Theorem 3.5. *For every $X \in H$, range of X is a union of T-equivalence classes.*

Proof. Let $n \in$ range of X. Then there exist an element $m \in M$ such that $mX = n$. If $n \in C_\alpha$, then $n = \alpha pt$, for some $p \in M$ and $t \in T$.

Let $q \in C_\alpha$. Then $q = \alpha ps$, for some $s \in T$. Consider $(mt^{-1}s)\, X = (mX)\, t^{-1}s = nt^{-1}s = \alpha ptt^{-1}s = \alpha ps = q$. So $q \in$ range of X. Thus range of X is a union of T-equivalence classes.

Definition 3.6. Let $X \in H$. The cardinality of the set of all T-equivalence classes in the range of X is called the *rank* of X.

It is clear that the rank of X is greater than or equal to 1 for all $X \in H$ and, for each $\alpha \in \Delta$, T_α has rank 1. We denote the set of all S-homomorphisms of rank 1 by $\bigcup$. We note that $\bigcup$ does not depend on Δ.

Write
$$V = \left\{ S \in \bigcup : \alpha SS = \alpha \text{ for some } \alpha \in \Delta \right\}.$$

We, now, characterize the idempotents of rank 1 in H.

Theorem 3.7. *V is the set of all idempotents of rank 1 in H.*

Proof. Let $S \in V$. So $\alpha S = \alpha$, for some $\alpha \in \Delta$. Since S has rank 1, the range of S is C_α. Let $m, s \in M$. Then $m = \beta st$ for some $\beta \in \Delta$ and $t \in T$. Assume

$\beta S = \alpha pq$ for some $p, q \in T$. Now, $mS^3 = (\beta st)\, S^3 = (\beta S)\,(stS^2) = (\alpha pq)\, stS^2 = (\alpha S)\, pqstS = \alpha pqstS = (\alpha S)\, pqst = \alpha pqst = (\beta S)\, st = (\beta st)\, S = mS$. Since this is true for all $m \in M$, S is an idempotent.

Conversely, suppose that $S \in \bigcup$ is an idempotent. Suppose range of S is C_α. If $\alpha T = \alpha st$ for some $s, t \in T$, then $\alpha st = \alpha S = \alpha S^3 = (\alpha st)\, SS = (\alpha S)\, stS = (\alpha st)\, stS = (\alpha S)\, stst = \alpha s^3 t^3$. So $s = t = e$ where e is the identity of T. Hence $\alpha S = \alpha$ for some $\alpha \in \Delta$. Therefore, $S \in V$.

In the following theorem, we exhibit a class of primitive idempotents of rank 1 in H.

Theorem 3.8. *For each $\alpha \in \Delta$, the T-homomorphism T_α is a primitive idempotent in H.*

Proof. Let $\alpha \in \Delta$. Clearly S_α is an idempotent in H. Suppose S is an idempotent in H such that $SS_\alpha = S_\alpha S = S$. Let $m, s \in M$. Then $m = \beta st$ for some $\beta \in \Delta$ and $t \in T$. Now, since $mS \in M$ and range of S = rang of $S_\alpha = C_\alpha$, we have $mS = \alpha pq$, for some $p, q \in T$. Now, $(\alpha S)\, st = (\alpha st)\, S = (\beta st)\, S_\alpha S = (\beta st)\, S = (\beta st)\, S^3 = (mS)\, SS = (\alpha pq)\, SS = (\alpha SS)\, pq = (\alpha S)\, pqS = (\alpha pq)\, S = (\alpha S)\, pq$. Since M_T is semispace, it follows that $s = p, t = q$. Therefore,, $mS = \alpha pq = \alpha St = mS_\alpha$. Since this is true for all $m \in M$. We have $S = S_\alpha$. Therefore S_α is a primitive idempotent.

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SOME NEW OPERATIONS ON INTERVAL-VALUED INTUITIONISTIC FUZZY SOFT SETS

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Abstract. Interval-valued intuitionistic fuzzy soft set theory is an intuitionistic fuzzy extension of the interval-valued fuzzy soft set theory or an interval-valued fuzzy extension of the intuitionistic fuzzy soft set theory. In this paper, we further consider interval-valued intuitionistic fuzzy soft sets. Some new operations on interval-valued intuitionistic fuzzy soft sets, i.e., "·", "+" and Cartesian product, are defined, and some related properties are investigated.

Keywords: Soft sets; interval-valued intuitionistic fuzzy soft sets.

1. Introduction

Probability theory, fuzzy sets [35], interval mathematics [11], and other mathematical tools are often useful approaches to dealing with uncertainties [24]. However, all of these theories have their own difficulties, and one of the major reasons is the inadequacy of their parametrization [23]. Soft set theory is a new mathematical tool for modeling uncertainties, which is free from the difficulties existing in those theories. At present, soft set theory has proven useful in many fields, such as prediction [30], [31], rules ming [12], decision making [7], [16], [36], mobile cloud computing [25], data analysis [8], [37]. Recently, many researches focused on theoretical aspect of soft sets. As a continuation of Molodtsov's pioneer work [23], Maji et al. [20] gave a detailed theoretical study on soft sets. Furthermore, Ali et al. [5] proposed some new operations on soft sets, such as restricted intersection, restricted union and restricted difference. Çağman and Enginoğlu [6] defined soft matrices, which are representatives of soft sets. In [10], Gong et al. presented the

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bijective soft sets, which are special soft sets. As an extended concept of bijective soft sets, the exclusive disjunctive soft sets [29] were introduced. Furthermore, Jiang et al. [14] analyzed the existing problems of soft set theory, and presented an extended soft set theory by using the concepts of description logics to act as the parameters of soft sets. Also, the algebraic structures and hyperalgebraic structures of soft sets have been studied increasingly. Aktaş and Çağman [2] introduced soft groups and considered the relationship between fuzzy groups and soft groups. Feng et al. [9] presented soft semirings, Acar et al. [1] defined soft rings, and Jun et al. [15] discussed the soft ordered semigroups. Yamak et al. [33] considered soft hypergroupoids, and we studied soft polygroups [26] and soft hypermodules [28].

Due to the fuzzy characters of parameters, the situation may be more complex in the practical applications of soft sets [34]. By combining fuzzy sets with soft sets, Maji et al. [18] defined fuzzy soft sets. Majumdar and Samanta [22] further generalised the concept of fuzzy soft sets, and introduced generalised fuzzy soft sets, in which a degree is attached with the parametrization of fuzzy sets while defining a fuzzy soft set. Yang et al. [32] presented interval-valued fuzzy soft sets, which is based on a combination of interval-valued fuzzy sets and soft sets. Maji et al. [17], [19], [21] defined intuitionistic fuzzy soft sets by combining intuitionistic fuzzy sets with soft sets. Moreover, Jiang et al. [13] introduced the notion of interval-valued intuitionistic fuzzy soft sets, which is a combination of interval-valued intuitionistic fuzzy sets and soft sets. Also, interval-valued intuitionistic fuzzy soft sets can be considered as intuitionistic fuzzy extension of interval-valued fuzzy soft sets or interval-valued fuzzy extension of intuitionistic fuzzy soft sets. Furthermore, We [27] discussed the necessity and possibility operations on interval-valued intuitionistic fuzzy soft sets, and Zhang et al. [36] developed an adjustable approach to decision making problems based on interval-valued intuitionistic fuzzy soft sets. In this paper, we further consider interval-valued intuitionistic fuzzy soft sets. Some new operations on interval-valued intuitionistic fuzzy soft sets are defined, and some related properties are investigated.

2. Preliminaries

We review some notions about interval-valued intuitionistic fuzzy sets and interval-valued intuitionistic fuzzy soft sets. Let U be an initial universe set, E be the universe set of parameters with respect to U and $A \subseteq E$. $\mathcal{IVIF}(U)$ denotes the set of all interval-valued intuitionistic fuzzy sets of U.

Let $D[0,1]$ denote the set of all closed subintervals of $[0,1]$. An interval-valued fuzzy set [11] A on a universe X is defined by $A = \{\langle x, \mu_A(x)\rangle | x \in X\}$, where $\mu_A : X \to D[0,1]$. For every $x \in X$, $\mu_A(x) = [\underline{\mu}_A(x), \overline{\mu}_A(x)]$ is called the degree of membership of an element x to A. $\underline{\mu}_A(x)$ and $\overline{\mu}_A(x)$ are referred to as the lower and upper degrees of membership of x to A, where $0 \leq \underline{\mu}_A(x) \leq \overline{\mu}_A(x) \leq 1$.

Definition 2.1. ([3]) An interval-valued intuitionistic fuzzy set A over a universe

X is an object having the form $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle | x \in X\}$, where $\mu_A : X \to D[0,1]$ and $\gamma_A : X \to D[0,1]$ satisfying the condition $\overline{\mu}_A(x) + \overline{\gamma}_A(x) \leq 1$ for all $x \in X$. The intervals $\mu_A(x)$ and $\gamma_A(x)$ denote the degree of membership and the degree of nonmembership of an element x to A, respectively.

Definition 2.2. ([4]) Let A and B be two interval-valued intuitionistic fuzzy sets of a universe X, then

(1) $A \subseteq B$ iff $\underline{\mu}_A(x) \leq \underline{\mu}_B(x)$, $\overline{\mu}_A(x) \leq \overline{\mu}_B(x)$, $\underline{\gamma}_A(x) \geq \underline{\gamma}_B(x)$ and $\overline{\gamma}_A(x) \geq \overline{\gamma}_B(x)$, for all $x \in X$;

(2) $A = B$ iff $A \subseteq B$ and $B \subseteq A$;

(3) $A^C = \{\langle x, \gamma_A(x), \mu_A(x)\rangle | x \in X\}$;

(4) $A \cup B = \{\langle x, [\sup(\underline{\mu}_A(x), \underline{\mu}_B(x)), \sup(\overline{\mu}_A(x), \overline{\mu}_B(x))], [\inf(\underline{\gamma}_A(x), \underline{\gamma}_B(x)), \inf(\overline{\gamma}_A(x), \overline{\gamma}_B(x))]\rangle | x \in X\}$;

(5) $A \cap B = \{\langle x, [\inf(\underline{\mu}_A(x), \underline{\mu}_B(x)), \inf(\overline{\mu}_A(x), \overline{\mu}_B(x))], [\sup(\underline{\gamma}_A(x), \underline{\gamma}_B(x)), \sup(\overline{\gamma}_A(x), \overline{\gamma}_B(x))]\rangle | x \in X\}$;

(6) $A + B = \{\langle x, [\underline{\mu}_A(x) + \underline{\mu}_B(x) - \underline{\mu}_A(x) \cdot \underline{\mu}_B(x), \overline{\mu}_A(x) + \overline{\mu}_B(x) - \overline{\mu}_A(x) \cdot \overline{\mu}_B(x)], [\underline{\gamma}_A(x) \cdot \underline{\gamma}_B(x), \overline{\gamma}_A(x) \cdot \overline{\gamma}_B(x)]\rangle | x \in X\}$;

(7) $A \cdot B = \{\langle x, [\underline{\mu}_A(x) \cdot \underline{\mu}_B(x), \overline{\mu}_A(x) \cdot \overline{\mu}_B(x)], [\underline{\gamma}_A(x) + \underline{\gamma}_B(x) - \underline{\gamma}_A(x) \cdot \underline{\gamma}_B(x), \overline{\gamma}_A(x) + \overline{\gamma}_B(x) - \overline{\gamma}_A(x) \cdot \overline{\gamma}_B(x)]\rangle | x \in X\}$;

(8) $\Box A = \{\langle x, \mu_A(x), [\underline{\gamma}_A(x), 1 - \overline{\mu}_A(x)]\rangle | x \in X\}$;

(9) $\Diamond A = \{\langle x, [\underline{\mu}_A(x), 1 - \overline{\gamma}_A(x)], \gamma_A(x)\rangle | x \in X\}$.

If A and B are two interval-valued intuitionistic fuzzy sets over universes X_1 and X_2, respectively, then $A \times B = \{\langle\langle x, y\rangle, [\underline{\mu}_A(x) \cdot \underline{\mu}_B(y), \overline{\mu}_A(x) \cdot \overline{\mu}_B(y)], [\underline{\gamma}_A(x) \cdot \underline{\gamma}_B(y), \overline{\gamma}_A(x) \cdot \overline{\gamma}_B(y)]\rangle | x \in X_1, y \in X_2\}$.

Definition 2.3. ([13]) A pair $\langle F, A\rangle$ is called an interval-valued intuitionistic fuzzy soft set over U, where F is a mapping given by $F : A \to \mathcal{IVIF}(U)$.

An interval-valued intuitionistic fuzzy soft set is a parameterized family of interval-valued intuitionistic fuzzy subsets of U. For any parameter $\varepsilon \in A$, $F(\varepsilon)$ is referred as the interval intuitionistic fuzzy value set of parameter ε. It is actually an interval-valued intuitionistic fuzzy set of U, written as

$$F(\varepsilon) = \{\langle x, \mu_{F(\varepsilon)}(x), \gamma_{F(\varepsilon)}(x)\rangle \mid x \in U\},$$

where $\mu_{F(\varepsilon)}(x)$ is the interval-valued fuzzy membership degree that object x holds on parameter ε, and $\gamma_{F(\varepsilon)}(x)$ is the interval-valued fuzzy membership degree that object x does not hold on parameter ε.

Definition 2.4. ([13]) Let $\langle F,A\rangle$ and $\langle G,B\rangle$ be two interval-valued intuitionistic fuzzy soft sets over U. Then $\langle F,A\rangle$ is called an interval-valued intuitionistic fuzzy soft subset of $\langle G,B\rangle$ denoted by $\langle F,A\rangle \Subset \langle G,B\rangle$, if the following conditions are satisfied: (1) $A \subseteq B$; (2) for all $\varepsilon \in A$, $F(\varepsilon)$ is an interval-valued intuitionistic fuzzy subset of $G(\varepsilon)$, that is, for all $x \in U$ and $\varepsilon \in A$, $\underline{\mu}_{F(\varepsilon)}(x) \leq \underline{\mu}_{G(\varepsilon)}(x)$, $\overline{\mu}_{F(\varepsilon)}(x) \leq \overline{\mu}_{G(\varepsilon)}(x)$, $\underline{\gamma}_{F(\varepsilon)}(x) \geq \underline{\gamma}_{G(\varepsilon)}(x)$ and $\overline{\gamma}_{F(\varepsilon)}(x) \geq \overline{\gamma}_{G(\varepsilon)}(x)$.

Two interval-valued intuitionistic fuzzy soft sets $\langle F,A\rangle$ and $\langle G,B\rangle$ over U are said to be interval-valued intuitionistic fuzzy soft equal, denoted by $\langle F,A\rangle = \langle G,B\rangle$, if $\langle F,A\rangle \Subset \langle G,B\rangle$ and $\langle G,B\rangle \Subset \langle F,A\rangle$.

Let $E = \{e_1,...,e_n\}$ be a set of parameters. The *not* set of E, denoted by $\rceil E$, is defined by $\rceil E = \{\rceil e_1,...,\rceil e_n\}$, where $\rceil e_i = \textit{not } e_i$ for all $i \in \{1,...,n\}$, which holds the opposite meaning of parameter e_i.

Definition 2.5. ([13]) The complement of an interval-valued intuitionistic soft set $\langle F,A\rangle$, denoted by $\langle F,A\rangle^C$, is defined by $\langle F,A\rangle^C = \langle F^C, \rceil A\rangle$, where $F^C : \rceil A \to \mathcal{IVIF}(U)$ is a mapping given by $F^C(\varepsilon) = \{\langle x, \gamma_{F(\rceil\varepsilon)}(x), \mu_{F(\rceil\varepsilon)}(x)\rangle | x \in U\}$ for all $\varepsilon \in \rceil A$.

Definition 2.6. ([13]). The union of two interval-valued intuitionistic fuzzy soft sets $\langle F,A\rangle$ and $\langle G,B\rangle$ over U is the interval-valued intuitionistic fuzzy soft set $\langle H,C\rangle = \langle F,A\rangle \Cup \langle G,B\rangle$, where $C = A \cup B$, and for all $\varepsilon \in C$,

$$H(\varepsilon) = \begin{cases} \{\langle x, \mu_{F(\varepsilon)}(x), \gamma_{F(\varepsilon)}(x)\rangle | x \in U\} & \text{if } \varepsilon \in A - B, \\ \{\langle x, \mu_{G(\varepsilon)}(x), \gamma_{G(\varepsilon)}(x)\rangle | x \in U\} & \text{if } \varepsilon \in B - A, \\ \{\langle x, [\sup(\underline{\mu}_{F(\varepsilon)}(x), \underline{\mu}_{G(\varepsilon)}(x)), \sup(\overline{\mu}_{F(\varepsilon)}(x), \overline{\mu}_{G(\varepsilon)}(x))], & \\ \quad [\inf(\underline{\gamma}_{F(\varepsilon)}(x), \underline{\gamma}_{G(\varepsilon)}(x)), \inf(\overline{\gamma}_{F(\varepsilon)}(x), \overline{\gamma}_{G(\varepsilon)}(x))]\rangle | x \in U\} & \text{if } \varepsilon \in A \cap B. \end{cases}$$

Definition 2.7. ([13]). The intersection of two interval-valued intuitionistic fuzzy soft sets $\langle F,A\rangle$ and $\langle G,B\rangle$ over U is the interval-valued intuitionistic fuzzy soft set $\langle H,C\rangle = \langle F,A\rangle \Cap \langle G,B\rangle$, where $C = A \cup B$, and for all $\varepsilon \in C$,

$$H(\varepsilon) = \begin{cases} \{\langle x, \mu_{F(\varepsilon)}(x), \gamma_{F(\varepsilon)}(x)\rangle | x \in U\} & \text{if } \varepsilon \in A - B, \\ \{\langle x, \mu_{G(\varepsilon)}(x), \gamma_{G(\varepsilon)}(x)\rangle | x \in U\} & \text{if } \varepsilon \in B - A, \\ \{\langle x, [\inf(\underline{\mu}_{F(\varepsilon)}(x), \underline{\mu}_{G(\varepsilon)}(x)), \inf(\overline{\mu}_{F(\varepsilon)}(x), \overline{\mu}_{G(\varepsilon)}(x))], & \\ \quad [\sup(\underline{\gamma}_{F(\varepsilon)}(x), \underline{\gamma}_{G(\varepsilon)}(x)), \sup(\overline{\gamma}_{F(\varepsilon)}(x), \overline{\gamma}_{G(\varepsilon)}(x))]\rangle | x \in U\} & \text{if } \varepsilon \in A \cap B. \end{cases}$$

Definition 2.8. ([13]). Let $\langle F,A\rangle$ and $\langle G,B\rangle$ be two interval-valued intuitionistic fuzzy soft sets over U, then "$\langle F,A\rangle$ or $\langle G,B\rangle$" is an interval-valued intuitionistic fuzzy soft set $\langle H, A\times B\rangle = \langle F,A\rangle \vee \langle G,B\rangle$, where $H(\alpha,\beta) = F(\alpha) \cup F(\beta)$ for all $(\alpha,\beta) \in A\times B$, that is $H(\alpha,\beta) = \{\langle x, [\sup(\underline{\mu}_{F(\alpha)}(x), \underline{\mu}_{G(\beta)}(x)), \sup(\overline{\mu}_{F(\alpha)}(x), \overline{\mu}_{G(\beta)}(x))]$, $[\inf(\underline{\gamma}_{F(\alpha)}(x), \underline{\gamma}_{G(\beta)}(x))$, $\inf(\overline{\gamma}_{F(\alpha)}(x), \overline{\gamma}_{G(\beta)}(x))]\rangle | x \in U\}$ for all $(\alpha,\beta) \in A \times B$.

Definition 2.9. ([13]). Let $\langle F, A\rangle$ and $\langle G, B\rangle$ be two interval-valued intuitionistic fuzzy soft sets over U, then "$\langle F, A\rangle$ and $\langle G, B\rangle$" is an interval-valued intuitionistic fuzzy soft set $\langle H, A \times B\rangle = \langle F, A\rangle \wedge \langle G, B\rangle$, where $H(\alpha, \beta) = F(\alpha) \cap F(\beta)$ for all $(\alpha, \beta) \in A \times B$, that is, $H(\alpha, \beta) = \{\langle x, [\inf(\underline{\mu}_{F(\alpha)}(x), \underline{\mu}_{G(\beta)}(x)), \inf(\overline{\mu}_{F(\alpha)}(x), \overline{\mu}_{G(\beta)}(x))], [\sup(\underline{\gamma}_{F(\alpha)}(x), \underline{\gamma}_{G(\beta)}(x)), \sup(\overline{\gamma}_{F(\alpha)}(x), \overline{\gamma}_{G(\beta)}(x))]\rangle | x \in U\}$ for all $(\alpha, \beta) \in A \times B$.

Definition 2.10. ([27]) The necessity operation on an interval-valued intuitionistic fuzzy soft set $\langle F, A\rangle$ is denoted by $\Box\langle F, A\rangle$ and is defined as $\Box\langle F, A\rangle = \{\langle x, \mu_{\Box F(\varepsilon)}(x), \gamma_{\Box F(\varepsilon)}(x)\rangle | x \in U \text{ and } \varepsilon \in A\}$, where $\mu_{\Box F(\varepsilon)}(x) = [\underline{\mu}_{F(\varepsilon)}(x), \overline{\mu}_{F(\varepsilon)}(x)]$ is the interval-valued fuzzy membership degree that object x holds on parameter ε, and $\gamma_{\Box F(\varepsilon)}(x) = [\underline{\gamma}_{F(\varepsilon)}(x), 1 - \overline{\mu}_{F(\varepsilon)}(x)]$ is the interval-valued fuzzy membership degree that object x does not hold on parameter ε.

Definition 2.11. ([27]) The possibility operation on an interval-valued intuitionistic fuzzy soft set $\langle F, A\rangle$ is denoted by $\Diamond\langle F, A\rangle$ and is defined as $\Diamond\langle F, A\rangle = \{\langle x, \mu_{\Diamond F(\varepsilon)}(x), \gamma_{\Diamond F(\varepsilon)}(x)\rangle | x \in U \text{ and } \varepsilon \in A\}$, where $\gamma_{\Diamond F(\varepsilon)}(x) = [\underline{\gamma}_{F(\varepsilon)}(x), \overline{\gamma}_{F(\varepsilon)}(x)]$ is the interval-valued fuzzy membership degree that object x does not hold on parameter ε, and $\mu_{\Diamond F(\varepsilon)}(x) = [\underline{\mu}_{F(\varepsilon)}(x), 1 - \overline{\gamma}_{F(\varepsilon)}(x)]$ is the interval-valued fuzzy membership degree that object x holds on parameter ε.

3. Some new operations on interval-valued intuitionistic fuzzy soft sets

In this section, we give several new operations on interval-valued intuitionistic fuzzy soft sets, and investigate some related properties.

Definition 3.1. The operation "$\cdot$" of two interval-valued intuitionistic fuzzy soft sets $\langle F, A\rangle$ and $\langle G, B\rangle$ over U is the intuitionistic fuzzy soft set $\langle H, C\rangle = \langle F, A\rangle \cdot \langle G, B\rangle$, where $C = A \cup B$ and for all $\varepsilon \in C$,

$$H(\varepsilon) = \begin{cases} \{\langle x, [\underline{\mu}_{F(\varepsilon)}(x) \cdot \underline{\mu}_{F(\varepsilon)}(x), \overline{\mu}_{F(\varepsilon)}(x) \cdot \overline{\mu}_{F(\varepsilon)}(x)], \\ \quad [\underline{\gamma}_{F(\varepsilon)}(x) + \underline{\gamma}_{F(\varepsilon)}(x) - \underline{\gamma}_{F(\varepsilon)}(x) \cdot \underline{\gamma}_{F(\varepsilon)}(x), \\ \quad \overline{\gamma}_{F(\varepsilon)}(x) + \overline{\gamma}_{F(\varepsilon)}(x) - \overline{\gamma}_{F(\varepsilon)}(x) \cdot \overline{\gamma}_{F(\varepsilon)}(x)]\rangle | x \in U\} & \text{if } \varepsilon \in A - B, \\ \{\langle x, [\underline{\mu}_{G(\varepsilon)}(x) \cdot \underline{\mu}_{G(\varepsilon)}(x), \overline{\mu}_{G(\varepsilon)}(x) \cdot \overline{\mu}_{G(\varepsilon)}(x)], \\ \quad [\underline{\gamma}_{G(\varepsilon)}(x) + \underline{\gamma}_{G(\varepsilon)}(x) - \underline{\gamma}_{G(\varepsilon)}(x) \cdot \underline{\gamma}_{G(\varepsilon)}(x), \\ \quad \overline{\gamma}_{G(\varepsilon)}(x) + \overline{\gamma}_{G(\varepsilon)}(x) - \overline{\gamma}_{G(\varepsilon)}(x) \cdot \overline{\gamma}_{G(\varepsilon)}(x)]\rangle | x \in U\} & \text{if } \varepsilon \in B - A, \\ \{\langle x, [\underline{\mu}_{F(\varepsilon)}(x) \cdot \underline{\mu}_{G(\varepsilon)}(x), \overline{\mu}_{F(\varepsilon)}(x) \cdot \overline{\mu}_{G(\varepsilon)}(x)], \\ \quad [\underline{\gamma}_{F(\varepsilon)}(x) + \underline{\gamma}_{G(\varepsilon)}(x) - \underline{\gamma}_{F(\varepsilon)}(x) \cdot \underline{\gamma}_{G(\varepsilon)}(x), \\ \quad \overline{\gamma}_{F(\varepsilon)}(x) + \overline{\gamma}_{G(\varepsilon)}(x) - \overline{\gamma}_{F(\varepsilon)}(x) \cdot \overline{\gamma}_{G(\varepsilon)}(x)]\rangle | x \in U\} & \text{if } \varepsilon \in A \cap B. \end{cases}$$

We can write $\langle F, A\rangle \cdot \langle F, A\rangle = \langle F, A\rangle^2$. For any positive integer n, $\langle F, A\rangle^n = \{\langle x, \mu_{F(\varepsilon)^n}(x), \gamma_{F(\varepsilon)^n}(x)\rangle | x \in U \text{ and } \varepsilon \in A\}$, where $\mu_{F(\varepsilon)^n}(x) = [(\underline{\mu}_{F(\varepsilon)}(x))^n, (\overline{\mu}_{F(\varepsilon)}(x))^n]$ and $\gamma_{F(\varepsilon)^n}(x) = [1 - (1 - \underline{\gamma}_{F(\varepsilon)}(x))^n, 1 - (1 - \overline{\gamma}_{F(\varepsilon)}(x))^n]$.

Example 3.2. Let $\langle F, A\rangle$ be the interval-valued intuitionistic fuzzy soft set, which describes the "attractiveness of the houses" to Mr. X (say), where U is a set of three houses under the consideration of Mr. X to purchase, denoted by $U = \{h_1, h_2, h_3\}$, $A = \{$convenient traffic, wooden, in good repair$\}$ is a parameter set, and

$$F(\text{convenient traffic}) = \{\langle h_1, [0.5, 0.7], [0.25, 0.3]\rangle, \langle h_2, [0.6, 0.75], [0.15, 0.25]\rangle, \langle h_3, [0.85, 0.9], [0.03, 0.1]\rangle\};$$

$$F(\text{wooden}) = \{\langle h_1, [0.6, 0.75], [0.2, 0.25]\rangle, \langle h_2, [0.73, 0.82], [0.1, 0.15]\rangle, \langle h_3, [0.55, 0.65], [0.26, 0.35]\rangle\};$$

$$F(\text{in good repair}) = \{\langle h_1, [0.76, 0.85], [0.08, 0.15]\rangle, \langle h_2, [0.55, 0.65], [0.2, 0.3]\rangle, \langle h_3, [0.7, 0.8], [0.15, 0.2]\rangle\}.$$

Consider the interval-valued intuitionistic fuzzy soft set $\langle G, B\rangle$ over U, which describes the "attractiveness of the houses" to Mrs. X, where $B = \{$beautiful, wooden, in good repair, in the green surroundings$\}$, and

$$G(\text{beautiful}) = \{\langle h_1, [0.8, 0.9], [0.05, 0.1]\rangle, \langle h_2, [0.65, 0.8], [0.1, 0.2]\rangle, \langle h_3, [0.7, 0.75], [0.2, 0.25]\rangle\};$$

$$G(\text{wooden}) = \{\langle h_1, [0.72, 0.8], [0.15, 0.2]\rangle, \langle h_2, [0.6, 0.7], [0.2, 0.3]\rangle, \langle h_3, [0.65, 0.8], [0.15, 0.2]\rangle\};$$

$$G(\text{in good repair}) = \{\langle h_1, [0.7, 0.8], [0.1, 0.2]\rangle, \langle h_2, [0.6, 0.75], [0.2, 0.25]\rangle, \langle h_3, [0.65, 0.85], [0.1, 0.15]\rangle\};$$

$$G(\text{in the green surroundings}) = \{\langle h_1, [0.75, 0.85], [0.1, 0.15]\rangle, \langle h_2, [0.7, 0.8], [0.05, 0.2]\rangle, \langle h_3, [0.5, 0.6], [0.25, 0.35]\rangle\}.$$

According to Definition 3.1, $\langle H, C\rangle = \langle F, A\rangle \cdot \langle G, B\rangle$, where $C = \{$beautiful, convenient traffic, wooden, in good repair$\}$, and

$$H(\text{convenient traffic}) = \{\langle h_1, [0.25, 0.49], [0.4375, 0.51]\rangle, \langle h_2, [0.36, 0.5625], [0.2775, 0.4375]\rangle, \langle h_3, [0.7225, 0.81], [0.0591, 0.19]\rangle\};$$

$$H(\text{beautiful}) = \{\langle h_1, [0.64, 0.81], [0.0975, 0.19]\rangle, \langle h_2, [0.4225, 0.64], [0.19, 0.36]\rangle, \langle h_3, [0.49, 0.5625], [0.36, 0.4375]\rangle\};$$

$$H(\text{wooden}) = \{\langle h_1, [0.432, 0.6], [0.32, 0.4]\rangle, \langle h_2, [0.438, 0.574], [0.28, 0.405]\rangle, \langle h_3, [0.3575, 0.52], [0.371, 0.48]\rangle\};$$

$$H(\text{in good repair}) = \{\langle h_1, [0.532, 0.68], [0.172, 0.32]\rangle, \langle h_2, [0.33, 0.4875], [0.36, 0.475]\rangle, \langle h_3, [0.455, 0.68], [0.235, 0.32]\rangle\};$$

$$H(\text{in the green surroundings}) = \{\langle h_1, [0.5625, 0.7225], [0.19, 0.2775]\rangle, \langle h_2, [0.49, 0.64], [0.0975, 0.36]\rangle, \langle h_3, [0.25, 0.36], [0.4375, 0.5775]\rangle\}.$$

Theorem 3.3. *Let $\langle F, A\rangle$ and $\langle G, B\rangle$ be two interval-valued intuitionistic fuzzy soft sets over U. For any positive integer m, n, we have the following properties:*

(1) $\langle F, A\rangle \cdot \langle G, B\rangle = \langle G, B\rangle \cdot \langle F, A\rangle$;

(2) $(\langle F, A\rangle \cdot \langle G, B\rangle)^n = \langle F, A\rangle^n \cdot \langle G, B\rangle^n$;

(3) $\langle F, A\rangle^m \cdot \langle F, A\rangle^n = \langle F, A\rangle^{m+n}$.

Proof. (1) It is straightforward.

(2) By Definition 3.1, we have $(\langle F, A\rangle \cdot \langle G, B\rangle)^n = (H, C)^n$, where $C = A \cup B$, and for all $\varepsilon \in C$,

$$H(\varepsilon)^n = \begin{cases} \{\langle x, [(\underline{\mu}_{F(\varepsilon)}(x) \cdot \underline{\mu}_{F(\varepsilon)}(x))^n, (\overline{\mu}_{F(\varepsilon)}(x) \cdot \overline{\mu}_{F(\varepsilon)}(x))^n], \\ \quad [1-(1-(\underline{\gamma}_{F(\varepsilon)}(x)+\underline{\gamma}_{F(\varepsilon)}(x)-\underline{\gamma}_{F(\varepsilon)}(x)\cdot\underline{\gamma}_{F(\varepsilon)}(x)))^n, \\ \quad 1-(1-(\overline{\gamma}_{F(\varepsilon)}(x)+\overline{\gamma}_{F(\varepsilon)}(x)-\overline{\gamma}_{F(\varepsilon)}(x)\cdot\overline{\gamma}_{F(\varepsilon)}(x)))^n]\rangle | x \in U\} \\ \quad \text{if } \varepsilon \in A - B, \\ \{\langle x, [(\underline{\mu}_{G(\varepsilon)}(x) \cdot \underline{\mu}_{G(\varepsilon)}(x))^n, (\overline{\mu}_{G(\varepsilon)}(x) \cdot \overline{\mu}_{G(\varepsilon)}(x))^n], \\ \quad [1-(1-(\underline{\gamma}_{G(\varepsilon)}(x)+\underline{\gamma}_{G(\varepsilon)}(x)-\underline{\gamma}_{G(\varepsilon)}(x)\cdot\underline{\gamma}_{G(\varepsilon)}(x)))^n, \\ \quad 1-(1-(\overline{\gamma}_{G(\varepsilon)}(x)+\overline{\gamma}_{G(\varepsilon)}(x)-\overline{\gamma}_{G(\varepsilon)}(x)\cdot\overline{\gamma}_{G(\varepsilon)}(x)))^n]\rangle | x \in U\} \\ \quad \text{if } \varepsilon \in B - A, \\ \{\langle x, [(\underline{\mu}_{F(\varepsilon)}(x) \cdot \underline{\mu}_{G(\varepsilon)}(x))^n, (\overline{\mu}_{F(\varepsilon)}(x) \cdot \overline{\mu}_{G(\varepsilon)}(x))^n], \\ \quad [1-(1-(\underline{\gamma}_{F(\varepsilon)}(x)+\underline{\gamma}_{G(\varepsilon)}(x)-\underline{\gamma}_{F(\varepsilon)}(x)\cdot\underline{\gamma}_{G(\varepsilon)}(x)))^n, \\ \quad 1-(1-(\overline{\gamma}_{F(\varepsilon)}(x)+\overline{\gamma}_{G(\varepsilon)}(x)-\overline{\gamma}_{F(\varepsilon)}(x)\cdot\overline{\gamma}_{G(\varepsilon)}(x)))^n]\rangle | x \in U\} \\ \quad \text{if } \varepsilon \in A \cap B. \end{cases}$$

Since $\langle F, A\rangle^n = \{\langle x, [(\underline{\mu}_{F(\varepsilon)}(x))^n, (\overline{\mu}_{F(\varepsilon)}(x))^n], [1-(1-\underline{\gamma}_{F(\varepsilon)}(x))^n, 1-(1-\overline{\gamma}_{F(\varepsilon)}(x))^n]\rangle | x \in U \text{ and } \varepsilon \in A\}$ and $\langle G, B\rangle^n = \{\langle x, [(\underline{\mu}_{G(\varepsilon)}(x))^n, (\overline{\mu}_{G(\varepsilon)}(x))^n], [1-(1-\underline{\gamma}_{G(\varepsilon)}(x))^n, 1-(1-\overline{\gamma}_{G(\varepsilon)}(x))^n]\rangle | x \in U \text{ and } \varepsilon \in B\}$, according to Definition 3.1, we can write $\langle F, A\rangle^n \cdot \langle G, B\rangle^n = (O, C)$, where $C = A \cup B$, and for all $\varepsilon \in C$,

$$O(\varepsilon) = \begin{cases} \{\langle x, [\underline{\mu}_{F(\varepsilon)}(x))^n \cdot \underline{\mu}_{F(\varepsilon)}(x))^n, (\overline{\mu}_{F(\varepsilon)}(x))^n \cdot (\overline{\mu}_{F(\varepsilon)}(x))^n], \\ \quad [1-(1-\underline{\gamma}_{F(\varepsilon)}(x))^n + 1-(1-\underline{\gamma}_{F(\varepsilon)}(x))^n \\ \quad -(1-(1-\underline{\gamma}_{F(\varepsilon)}(x))^n)\cdot(1-(1-\underline{\gamma}_{F(\varepsilon)}(x))^n), 1-(1-\overline{\gamma}_{F(\varepsilon)}(x))^n + 1 \\ \quad -(1-\overline{\gamma}_{F(\varepsilon)}(x))^n - (1-(1-\overline{\gamma}_{F(\varepsilon)}(x))^n)\cdot(1-(1-\overline{\gamma}_{F(\varepsilon)}(x))^n)]\rangle | x \in U\} \\ \quad \text{if } \varepsilon \in A - B, \\ \{\langle x, [\underline{\mu}_{G(\varepsilon)}(x))^n\cdot\underline{\mu}_{G(\varepsilon)}(x))^n, (\overline{\mu}_{G(\varepsilon)}(x))^n\cdot(\overline{\mu}_{G(\varepsilon)}(x))^n], [1-(1-\underline{\gamma}_{G(\varepsilon)}(x))^n+1 \\ \quad -(1-\underline{\gamma}_{G(\varepsilon)}(x))^n - (1-(1-\underline{\gamma}_{G(\varepsilon)}(x))^n)\cdot(1-(1-\underline{\gamma}_{G(\varepsilon)}(x))^n), \\ \quad 1-(1-\overline{\gamma}_{G(\varepsilon)}(x))^n + 1-(1-\overline{\gamma}_{G(\varepsilon)}(x))^n \\ \quad -(1-(1-\overline{\gamma}_{G(\varepsilon)}(x))^n)\cdot(1-(1-\overline{\gamma}_{G(\varepsilon)}(x))^n)]\rangle | x \in U\} \\ \quad \text{if } \varepsilon \in B - A, \\ \{\langle x, [\underline{\mu}_{F(\varepsilon)}(x))^n \cdot \underline{\mu}_{G(\varepsilon)}(x))^n, (\overline{\mu}_{F(\varepsilon)}(x))^n \cdot (\overline{\mu}_{G(\varepsilon)}(x))^n], [1-(1-\underline{\gamma}_{F(\varepsilon)}(x))^n \\ \quad +1-(1-\underline{\gamma}_{G(\varepsilon)}(x))^n - (1-(1-\underline{\gamma}_{F(\varepsilon)}(x))^n)\cdot(1-(1-\underline{\gamma}_{G(\varepsilon)}(x))^n), \\ \quad 1-(1-\overline{\gamma}_{F(\varepsilon)}(x))^n + 1-(1-\overline{\gamma}_{G(\varepsilon)}(x))^n \\ \quad -(1-(1-\overline{\gamma}_{F(\varepsilon)}(x))^n)\cdot(1-(1-\overline{\gamma}_{G(\varepsilon)}(x))^n)]\rangle | x \in U\} \\ \quad \text{if } \varepsilon \in A \cap B. \end{cases}$$

We have that

$$\begin{aligned}
&1-(1-\underline{\gamma}_{F(\varepsilon)}(x))^n+1-(1-\underline{\gamma}_{G(\varepsilon)}(x))^n-(1-(1-\underline{\gamma}_{F(\varepsilon)}(x))^n)\cdot(1-(1-\underline{\gamma}_{G(\varepsilon)}(x))^n)\\
&\quad=(1-(1-\underline{\gamma}_{F(\varepsilon)}(x))^n)\cdot(1-(1-(1-\underline{\gamma}_{G(\varepsilon)}(x))^n))+1-(1-\underline{\gamma}_{G(\varepsilon)}(x))^n\\
&\quad=(1-(1-\underline{\gamma}_{F(\varepsilon)}(x))^n)\cdot(1-\underline{\gamma}_{G(\varepsilon)}(x))^n+1-(1-\underline{\gamma}_{G(\varepsilon)}(x))^n\\
&\quad=(1-\underline{\gamma}_{G(\varepsilon)}(x))^n-(1-\underline{\gamma}_{F(\varepsilon)}(x))^n\cdot(1-\underline{\gamma}_{G(\varepsilon)}(x))^n+1-(1-\underline{\gamma}_{G(\varepsilon)}(x))^n\\
&\quad=1-(1-\underline{\gamma}_{F(\varepsilon)}(x))^n\cdot(1-\underline{\gamma}_{G(\varepsilon)}(x))^n\\
&\quad=1-(1-(\underline{\gamma}_{F(\varepsilon)}(x)+\underline{\gamma}_{G(\varepsilon)}(x)-\underline{\gamma}_{F(\varepsilon)}(x)\cdot\underline{\gamma}_{G(\varepsilon)}(x)))^n.
\end{aligned}$$

Consequently, $\langle H, C\rangle^n$ and $\langle O, C\rangle$ are the same interval-valued intuitionistic fuzzy soft set. Therefore, $(\langle F, A\rangle \cdot \langle G, B\rangle)^n = \langle F, A\rangle^n \cdot \langle G, B\rangle^n$.

(3) From Definition 3.1, it follows that

$$\begin{aligned}
\langle F,A\rangle^m\cdot\langle F,A\rangle^n &= \{\langle x,[(\underline{\mu}_{F(\varepsilon)}(x))^m\cdot(\underline{\mu}_{F(\varepsilon)}(x))^n,(\overline{\mu}_{F(\varepsilon)}(x))^m\cdot(\overline{\mu}_{F(\varepsilon)}(x))^n],\\
&\quad[1-(1-\underline{\gamma}_{F(\varepsilon)}(x))^m+1-(1-\underline{\gamma}_{F(\varepsilon)}(x))^n\\
&\quad-(1-(1-\underline{\gamma}_{F(\varepsilon)}(x))^m)\cdot(1-(1-\underline{\gamma}_{F(\varepsilon)}(x))^n),\\
&\quad 1-(1-\overline{\gamma}_{F(\varepsilon)}(x))^m+1-(1-\overline{\gamma}_{F(\varepsilon)}(x))^n\\
&\quad-(1-(1-\overline{\gamma}_{F(\varepsilon)}(x))^m)\cdot(1-(1-\overline{\gamma}_{F(\varepsilon)}(x))^n)]\rangle|x\in U \text{ and } \varepsilon\in A\}\\
&=\{\langle x,[(\underline{\mu}_{F(\varepsilon)}(x))^{m+n},(\overline{\mu}_{F(\varepsilon)}(x))^{m+n}],[1-(1-\underline{\gamma}_{F(\varepsilon)}(x))^m\cdot(1-\underline{\gamma}_{F(\varepsilon)}(x))^n),\\
&\quad 1-(1-\overline{\gamma}_{F(\varepsilon)}(x))^m\cdot(1-\overline{\gamma}_{F(\varepsilon)}(x))^n]\rangle|x\in U \text{ and } \varepsilon\in A\}\\
&=\{\langle x,[(\underline{\mu}_{F(\varepsilon)}(x))^{m+n},(\overline{\mu}_{F(\varepsilon)}(x))^{m+n}],[1-(1-\underline{\gamma}_{F(\varepsilon)}(x))^{m+n},\\
&\quad 1-(1-\overline{\gamma}_{F(\varepsilon)}(x))^{m+n}]\rangle|x\in U \text{ and } \varepsilon\in A\}\\
&=\langle F,A\rangle^{m+n}.\ \blacksquare
\end{aligned}$$

Theorem 3.4. *Let $\langle F, A\rangle$ and $\langle G, B\rangle$ be two interval-valued intuitionistic fuzzy soft sets over U. For any positive integer n, we have the following properties:*

(1) $\Box\langle F, A\rangle^n = (\Box\langle F, A\rangle)^n$;

(2) $\Diamond\langle F, A\rangle^n = (\Diamond\langle F, A\rangle)^n$.

Proof. (1) Since $\langle F, A\rangle^n = \{\langle x, [(\underline{\mu}_{F(\varepsilon)}(x))^n, (\overline{\mu}_{F(\varepsilon)}(x))^n], [1 - (1 - \underline{\gamma}_{F(\varepsilon)}(x))^n, 1-(1-\overline{\gamma}_{F(\varepsilon)}(x))^n]\rangle | x \in U \text{ and } \varepsilon \in A\}$, we have

$$\Box\langle F,A\rangle^n=\{\langle x,[(\underline{\mu}_{F(\varepsilon)}(x))^n,(\overline{\mu}_{F(\varepsilon)}(x))^n],[1-(1-\underline{\gamma}_{F(\varepsilon)}(x))^n,1-(\overline{\mu}_{F(\varepsilon)}(x))^n]\rangle| \\ x\in U \text{ and } \varepsilon\in A\}.$$

Since $\Box\langle F, A\rangle = \{\langle x, [\underline{\mu}_{F(\varepsilon)}(x), \overline{\mu}_{F(\varepsilon)}(x)], [\underline{\gamma}_{F(\varepsilon)}(x), 1-\overline{\mu}_{F(\varepsilon)}(x)]\rangle | x \in U \text{ and } \varepsilon \in A\}$, we have

$$\begin{aligned}(\Box\langle F,A\rangle)^n &= \{\langle x,[(\underline{\mu}_{F(\varepsilon)}(x))^n,(\overline{\mu}_{F(\varepsilon)}(x))^n],[1-(1-\underline{\gamma}_{F(\varepsilon)}(x))^n,\\ &\quad 1-(1-(1-\overline{\mu}_{F(\varepsilon)}(x)))^n]\rangle|x\in U \text{ and } \varepsilon\in A\}\\ &= \{\langle x,[(\underline{\mu}_{F(\varepsilon)}(x))^n,(\overline{\mu}_{F(\varepsilon)}(x))^n],[1-(1-\underline{\gamma}_{F(\varepsilon)}(x))^n,\\ &\quad 1-(\overline{\mu}_{F(\varepsilon)}(x))^n]\rangle|x\in U \text{ and } \varepsilon\in A\}\\ &= \Box\langle F,A\rangle^n.\end{aligned}$$

(2) The proof is similar to that of (1). ∎

Theorem 3.5. *Let $\langle F,A\rangle$ and $\langle G,B\rangle$ be two interval-valued intuitionistic fuzzy soft sets over U. For any positive integer m,n, we have the following properties:*

(1) *if $m\geq n$, then $\langle F,A\rangle^m\subseteq\langle F,A\rangle^n$;*

(2) *if $\langle F,A\rangle\Subset\langle G,B\rangle$, then $\langle F,A\rangle^n\Subset\langle G,B\rangle^n$;*

(3) $(\langle F,A\rangle\Cup\langle G,B\rangle)^n=\langle F,A\rangle^n\Cup\langle G,B\rangle^n$;

(4) $(\langle F,A\rangle\Cap\langle G,B\rangle)^n=\langle F,A\rangle^n\Cap\langle G,B\rangle^n$;

(5) $(\langle F,A\rangle\vee\langle G,B\rangle)^n=\langle F,A\rangle^n\vee\langle G,B\rangle^n$;

(6) $(\langle F,A\rangle\wedge\langle G,B\rangle)^n=\langle F,A\rangle^n\wedge\langle G,B\rangle^n$.

Proof. (1) Since $0\leq\underline{\mu}_{F(\varepsilon)}(x)\leq 1$, $0\leq\overline{\mu}_{F(\varepsilon)}(x)\leq 1$, $0\leq 1-\underline{\gamma}_{F(\varepsilon)}(x)\leq 1$ and $0\leq 1-\overline{\gamma}_{F(\varepsilon)}(x)\leq 1$, we have $(\underline{\mu}_{F(\varepsilon)}(x))^m\leq(\underline{\mu}_{F(\varepsilon)}(x))^n$, $(\overline{\mu}_{F(\varepsilon)}(x))^m\leq(\overline{\mu}_{F(\varepsilon)}(x))^n$, $1-(1-\underline{\gamma}_{F(\varepsilon)}(x))^m\geq 1-(1-\underline{\gamma}_{F(\varepsilon)}(x))^n$ and $1-(1-\overline{\gamma}_{F(\varepsilon)}(x))^m\geq 1-(1-\overline{\gamma}_{F(\varepsilon)}(x))^n$. Thus, we deduce that $\langle F,A\rangle^m\subseteq\langle F,A\rangle^n$.

(2) Since $\langle F,A\rangle\Subset\langle G,B\rangle$, we have $\underline{\mu}_{F(\varepsilon)}(x)\leq\underline{\mu}_{G(\varepsilon)}(x)$, $\overline{\mu}_{F(\varepsilon)}(x)\leq\overline{\mu}_{G(\varepsilon)}(x)$, $\underline{\gamma}_{F(\varepsilon)}(x)\geq\underline{\gamma}_{G(\varepsilon)}(x)$ and $\overline{\gamma}_{F(\varepsilon)}(x)\geq\overline{\mu}_{G(\varepsilon)}(x)$ for all $x\in U$ and $\varepsilon\in A$. It follows that $(\underline{\mu}_{F(\varepsilon)}(x))^n\leq(\underline{\mu}_{G(\varepsilon)}(x))^n$, $(\overline{\mu}_{F(\varepsilon)}(x))^n\leq(\overline{\mu}_{G(\varepsilon)}(x))^n$, $1-(1-\underline{\gamma}_{F(\varepsilon)}(x))^n\geq 1-(1-\underline{\gamma}_{G(\varepsilon)}(x))^n$ and $1-(1-\overline{\gamma}_{F(\varepsilon)}(x))^n\geq 1-(1-\overline{\gamma}_{G(\varepsilon)}(x))^n$ for all $x\in U$ and $\varepsilon\in A$. Therefore, we have $\langle F,A\rangle^n\Subset\langle G,B\rangle^n$.

(3) From Definition 2.6, we can write $\langle F,A\rangle\Cup\langle G,B\rangle=\langle H,C\rangle$, where $C=A\cup B$, and for all $\varepsilon\in C$,

$$H(\varepsilon)=\begin{cases}\{\langle x,\mu_{F(\varepsilon)}(x),\gamma_{F(\varepsilon)}(x)\rangle|x\in U\} & \text{if } \varepsilon\in A-B,\\ \{\langle x,\mu_{G(\varepsilon)}(x),\gamma_{G(\varepsilon)}(x)\rangle|x\in U\} & \text{if } \varepsilon\in B-A,\\ \{\langle x,[\sup(\underline{\mu}_{F(\varepsilon)}(x),\underline{\mu}_{G(\varepsilon)}(x)),\sup(\overline{\mu}_{F(\varepsilon)}(x),\overline{\mu}_{G(\varepsilon)}(x))], & \\ \quad[\inf(\underline{\gamma}_{F(\varepsilon)}(x),\underline{\gamma}_{G(\varepsilon)}(x)),\inf(\overline{\gamma}_{F(\varepsilon)}(x),\overline{\gamma}_{G(\varepsilon)}(x))]\rangle|x\in U\} & \text{if } \varepsilon\in A\cap B.\end{cases}$$

Then $(\langle F,A\rangle \Cup \langle G,B\rangle)^n = \langle H,C\rangle^n$, where $C = A\cup B$, and for all $\varepsilon\in C$,

$$H(\varepsilon)^n = \begin{cases} \{\langle x, [(\underline{\mu}_{F(\varepsilon)}(x))^n, (\overline{\mu}_{F(\varepsilon)}(x))^n], \\ \quad [1-(1-\underline{\gamma}_{F(\varepsilon)}(x))^n, 1-(1-\overline{\gamma}_{F(\varepsilon)}(x))^n]\rangle | x\in U\} & \text{if } \varepsilon\in A-B, \\ \{\langle x, [(\underline{\mu}_{G(\varepsilon)}(x))^n, (\overline{\mu}_{G(\varepsilon)}(x))^n], \\ \quad [1-(1-\underline{\gamma}_{G(\varepsilon)}(x))^n, 1-(1-\overline{\gamma}_{G(\varepsilon)}(x))^n]\rangle | x\in U\} & \text{if } \varepsilon\in B-A, \\ \{\langle x, [(\sup(\underline{\mu}_{F(\varepsilon)}(x), \underline{\mu}_{G(\varepsilon)}(x)))^n, (\sup(\overline{\mu}_{F(\varepsilon)}(x), \overline{\mu}_{G(\varepsilon)}(x)))^n], \\ \quad [1-(1-\inf(\underline{\gamma}_{F(\varepsilon)}(x), \underline{\gamma}_{G(\varepsilon)}(x)))^n, \\ \quad 1-(1-\inf(\overline{\gamma}_{F(\varepsilon)}(x), \overline{\gamma}_{G(\varepsilon)}(x)))^n]\rangle | x\in U\} \\ = \{\langle x, [\sup((\underline{\mu}_{F(\varepsilon)}(x))^n, (\underline{\mu}_{G(\varepsilon)}(x))^n), \sup((\overline{\mu}_{F(\varepsilon)}(x))^n, (\overline{\mu}_{G(\varepsilon)}(x))^n)], \\ \quad [\inf(1-(1-\underline{\gamma}_{F(\varepsilon)}(x))^n, 1-(1-\underline{\gamma}_{G(\varepsilon)}(x))^n), \\ \quad \inf(1-(1-\overline{\gamma}_{F(\varepsilon)}(x))^n, 1-(1-\overline{\gamma}_{G(\varepsilon)}(x))^n)]\rangle | x\in U\} & \text{if } \varepsilon\in A\cap B. \end{cases}$$

Consequently, $(\langle F,A\rangle \Cup \langle G,B\rangle)^n = \langle F,A\rangle^n \Cup \langle G,B\rangle^n$.

(4) The proof is similar to that of (3).

(5) From Definition 2.8, we can write $\langle F,A\rangle \vee \langle G,B\rangle = \langle O, A\times B\rangle$, where

$$\begin{aligned} O(\alpha,\beta) = \{\langle x, &[\sup(\underline{\mu}_{F(\alpha)}(x), \underline{\mu}_{G(\beta)}(x)), \sup(\overline{\mu}_{F(\alpha)}(x), \overline{\mu}_{G(\beta)}(x))], \\ &[\inf(\underline{\gamma}_{F(\alpha)}(x), \underline{\gamma}_{G(\beta)}(x)), \inf(\overline{\gamma}_{F(\alpha)}(x), \overline{\gamma}_{G(\beta)}(x))]\rangle | x\in U\} \end{aligned}$$

for all $(\alpha,\beta)\in A\times B$. Thus, $(\langle F,A\rangle \vee \langle G,B\rangle)^n = \langle O, A\times B\rangle^n$, where for all $(\alpha,\beta)\in A\times B$,

$$\begin{aligned} O(\alpha,\beta)^n &= \{\langle x, [(\sup(\underline{\mu}_{F(\alpha)}(x), \underline{\mu}_{G(\beta)}(x)))^n, (\sup(\overline{\mu}_{F(\alpha)}(x), \overline{\mu}_{G(\beta)}(x)))^n], \\ &[1-(1-\inf(\underline{\gamma}_{F(\alpha)}(x), \underline{\gamma}_{G(\beta)}(x)))^n, 1-(1-\inf(\overline{\gamma}_{F(\alpha)}(x), \overline{\gamma}_{G(\beta)}(x)))^n]\rangle | x\in U\}\cdot \\ &= \{\langle x, [\sup((\underline{\mu}_{F(\alpha)}(x))^n, (\underline{\mu}_{G(\beta)}(x))^n), \sup((\overline{\mu}_{F(\alpha)}(x))^n, (\overline{\mu}_{G(\beta)}(x))^n)], \\ &[\inf(1-(1-\underline{\gamma}_{F(\alpha)}(x))^n, 1-(1-\underline{\gamma}_{G(\beta)}(x))^n), \inf(1-(1-\overline{\gamma}_{F(\alpha)}(x))^n, \\ &1-(1-\overline{\gamma}_{G(\beta)}(x))^n)]\rangle | x\in U\} \end{aligned}$$

Hence, we have $(\langle F,A\rangle \vee \langle G,B\rangle)^n = \langle F,A\rangle^n \vee \langle G,B\rangle^n$.

(6) The proof is similar to that of (5). ∎

Definition 3.6. The operation "+" of two interval-valued intuitionistic fuzzy soft sets $\langle F,A\rangle$ and $\langle G,B\rangle$ over U is the intuitionistic fuzzy soft set $\langle H,C\rangle = \langle F,A\rangle + \langle G,B\rangle$, where $C = A\cup B$ and for all $\varepsilon\in C$,

$$H(\varepsilon) = \begin{cases} \{\langle x, [\underline{\mu}_{F(\varepsilon)}(x) + \underline{\mu}_{F(\varepsilon)}(x) - \underline{\mu}_{F(\varepsilon)}(x) \cdot \underline{\mu}_{F(\varepsilon)}(x), \\ \quad \overline{\mu}_{F(\varepsilon)}(x) + \overline{\mu}_{F(\varepsilon)}(x) - \overline{\mu}_{F(\varepsilon)}(x) \cdot \overline{\mu}_{F(\varepsilon)}(x)], \\ \quad [\underline{\gamma}_{F(\varepsilon)}(x) \cdot \underline{\gamma}_{F(\varepsilon)}(x), \overline{\gamma}_{F(\varepsilon)}(x) \cdot \overline{\gamma}_{F(\varepsilon)}(x)], \rangle | x \in U\} & \text{if } \varepsilon \in A - B, \\ \{\langle x, [\underline{\mu}_{G(\varepsilon)}(x) + \underline{\mu}_{G(\varepsilon)}(x) - \underline{\mu}_{G(\varepsilon)}(x) \cdot \underline{\mu}_{G(\varepsilon)}(x), \\ \quad \overline{\mu}_{G(\varepsilon)}(x) + \overline{\mu}_{G(\varepsilon)}(x) - \overline{\mu}_{G(\varepsilon)}(x) \cdot \overline{\mu}_{G(\varepsilon)}(x)], \\ \quad [\underline{\gamma}_{G(\varepsilon)}(x) \cdot \underline{\gamma}_{G(\varepsilon)}(x), \overline{\gamma}_{G(\varepsilon)}(x) \cdot \overline{\gamma}_{G(\varepsilon)}(x)]\rangle | x \in U\} & \text{if } \varepsilon \in B - A, \\ \{\langle x, [\underline{\mu}_{F(\varepsilon)}(x) + \underline{\mu}_{G(\varepsilon)}(x) - \underline{\mu}_{F(\varepsilon)}(x) \cdot \underline{\mu}_{G(\varepsilon)}(x), \\ \quad \overline{\mu}_{F(\varepsilon)}(x) + \overline{\mu}_{G(\varepsilon)}(x) - \overline{\mu}_{F(\varepsilon)}(x) \cdot \overline{\mu}_{G(\varepsilon)}(x)], \\ \quad [\underline{\gamma}_{F(\varepsilon)}(x) \cdot \underline{\gamma}_{G(\varepsilon)}(x), \overline{\gamma}_{F(\varepsilon)}(x) \cdot \overline{\gamma}_{G(\varepsilon)}(x)]\rangle | x \in U\} & \text{if } \varepsilon \in A \cap B. \end{cases}$$

We can write $\langle F, A\rangle + \langle F, A\rangle = 2\langle F, A\rangle$. For any positive integer n, $n\langle F, A\rangle = \{\langle x, \mu_{nF(\varepsilon)}(x), \gamma_{nF(\varepsilon)}(x)\rangle | x \in U \text{ and } \varepsilon \in A\}$, where $\mu_{nF(\varepsilon)}(x) = [1-(1-\underline{\mu}_{F(\varepsilon)}(x))^n, 1-(1-\overline{\mu}_{F(\varepsilon)}(x))^n]$ and $\gamma_{nF(\varepsilon)}(x) = [(\underline{\gamma}_{F(\varepsilon)}(x))^n, (\overline{\gamma}_{F(\varepsilon)}(x))^n]$.

Example 3.7. Let $\langle F, A\rangle$, $\langle G, B\rangle$ be the interval-valued intuitionistic fuzzy soft sets defined in Example 3.2. According to Definition 3.6, $\langle H, C\rangle = \langle F, A\rangle + \langle G, B\rangle$, where $C = \{$beautiful, convenient traffic, wooden, in good repair$\}$, and

$$H(\text{convenient traffic}) = \{\langle h_1, [0.75, 0.91], [0.0625, 0.09]\rangle, \langle h_2, [0.84, 0.9375], [0.0225, 0.0625]\rangle, \langle h_3, [0.9775, 0.99], [0.0009, 0.01]\rangle\};$$

$$H(\text{beautiful}) = \{\langle h_1, [0.96, 0.99], [0.0025, 0.01]\rangle, \langle h_2, [0.8775, 0.96], [0.01, 0.04]\rangle, \langle h_3, [0.91, 0.9375], [0.04, 0.0625]\rangle\};$$

$$H(\text{wooden}) = \{\langle h_1, [0.888, 0.95], [0.03, 0.05]\rangle, \langle h_2, [0.892, 0.946], [0.02, 0.045]\rangle, \langle h_3, [0.8425, 0.93], [0.039, 0.07]\rangle\};$$

$$H(\text{in good repair}) = \{\langle h_1, [0.928, 0.97], [0.008, 0.03]\rangle, \langle h_2, [0.82, 0.9125], [0.04, 0.075]\rangle, \langle h_3, [0.895, 0.97], [0.015, 0.03]\rangle\};$$

$$H(\text{in the green surroundings}) = \{\langle h_1, [0.9375, 0.9775], [0.01, 0.0225]\rangle, \langle h_2, [0.91, 0.96], [0.0025, 0.04]\rangle, \langle h_3, [0.75, 0.84], [0.0625, 0.1225]\rangle\}.$$

Theorem 3.8. *Let $\langle F, A\rangle$ and $\langle G, B\rangle$ be two interval-valued intuitionistic fuzzy soft sets over U. For any positive integer m, n, we have the following properties:*

(1) $\langle F, A\rangle + \langle G, B\rangle = \langle G, B\rangle + \langle F, A\rangle$;

(2) $n(\langle F, A\rangle + \langle G, B\rangle) = n\langle F, A\rangle + n\langle G, B\rangle$;

(3) $m\langle F, A\rangle + n\langle F, A\rangle = (m + n)\langle F, A\rangle$.

Proof. The proof is similar to that of Theorem 3.3. ■

Theorem 3.9. *Let $\langle F, A\rangle$ and $\langle G, B\rangle$ be two interval-valued intuitionistic fuzzy soft sets over U. For any positive integer n, we have the following properties:*

(1) $\Box n\langle F,A\rangle = n\Box\langle F,A\rangle$;

(2) $\Diamond n\langle F,A\rangle = n\Diamond\langle F,A\rangle$.

Proof. The proof is similar to that of Theorem 3.4. ∎

Theorem 3.10. *Let $\langle F,A\rangle$ and $\langle G,B\rangle$ be two interval-valued intuitionistic fuzzy soft sets over U. For any positive integer m,n, we have the following properties:*

(1) *if $m \le n$, then $m\langle F,A\rangle \subseteq n\langle F,A\rangle$;*

(2) *if $\langle F,A\rangle \Subset \langle G,B\rangle$, then $n\langle F,A\rangle \Subset n\langle G,B\rangle$;*

(3) $n(\langle F,A\rangle \Cup \langle G,B\rangle) = n\langle F,A\rangle \Cup n\langle G,B\rangle$;

(4) $n(\langle F,A\rangle \Cap \langle G,B\rangle) = n\langle F,A\rangle \Cap n\langle G,B\rangle$;

(5) $n(\langle F,A\rangle \vee \langle G,B\rangle) = n\langle F,A\rangle \vee n\langle G,B\rangle$;

(6) $n(\langle F,A\rangle \wedge \langle G,B\rangle) = n\langle F,A\rangle \wedge n\langle G,B\rangle$.

Proof. The proof is similar to that of Theorem 3.5. ∎

Theorem 3.11. *Let $\langle F,A\rangle$ and $\langle G,B\rangle$ be two interval-valued intuitionistic fuzzy soft sets over U, then we have the following properties:*

(1) $\Box(\langle F,A\rangle \cdot \langle G,B\rangle) = \Box\langle F,A\rangle \cdot \Box\langle G,B\rangle$;

(2) $\Box(\langle F,A\rangle + \langle G,B\rangle) = \Box\langle F,A\rangle + \Box\langle G,B\rangle$;

(3) $\Diamond(\langle F,A\rangle \cdot \langle G,B\rangle) = \Diamond\langle F,A\rangle \cdot \Diamond\langle G,B\rangle$;

(4) $\Diamond(\langle F,A\rangle + \langle G,B\rangle) = \Diamond\langle F,A\rangle + \Diamond\langle G,B\rangle$;

(5) $(\langle F,A\rangle \cdot \langle G,B\rangle)^C = \langle F,A\rangle^C + \langle G,B\rangle^C$;

(6) $(\langle F,A\rangle + \langle G,B\rangle)^C = \langle F,A\rangle^C \cdot \langle G,B\rangle^C$.

Proof. (1) From Definition 3.1 and Definition 2.10, we have $\Box(\langle F,A\rangle \cdot \langle G,B\rangle) = \Box\langle H,C\rangle$, where $C = A\cup B$ and for all $\varepsilon \in C$,

$$\Box H(\varepsilon) = \begin{cases} \{\langle x, [\underline{\mu}_{F(\varepsilon)}(x)\cdot\underline{\mu}_{F(\varepsilon)}(x), \overline{\mu}_{F(\varepsilon)}(x)\cdot\overline{\mu}_{F(\varepsilon)}(x)], \\ \quad [\underline{\gamma}_{F(\varepsilon)}(x)+\underline{\gamma}_{F(\varepsilon)}(x)-\underline{\gamma}_{F(\varepsilon)}(x)\cdot\underline{\gamma}_{F(\varepsilon)}(x), \\ \quad 1-\overline{\mu}_{F(\varepsilon)}(x)\cdot\overline{\mu}_{F(\varepsilon)}(x)]\rangle | x\in U\} & \text{if } \varepsilon \in A-B, \\ \{\langle x, [\underline{\mu}_{G(\varepsilon)}(x)\cdot\underline{\mu}_{G(\varepsilon)}(x), \overline{\mu}_{G(\varepsilon)}(x)\cdot\overline{\mu}_{G(\varepsilon)}(x)], \\ \quad [\underline{\gamma}_{G(\varepsilon)}(x)+\underline{\gamma}_{G(\varepsilon)}(x)-\underline{\gamma}_{G(\varepsilon)}(x)\cdot\underline{\gamma}_{G(\varepsilon)}(x), \\ \quad 1-\overline{\mu}_{G(\varepsilon)}(x)\cdot\overline{\mu}_{G(\varepsilon)}(x)]\rangle | x\in U\} & \text{if } \varepsilon \in B-A, \\ \{\langle x, [\underline{\mu}_{F(\varepsilon)}(x)\cdot\underline{\mu}_{G(\varepsilon)}(x), \overline{\mu}_{F(\varepsilon)}(x)\cdot\overline{\mu}_{G(\varepsilon)}(x)], \\ \quad [\underline{\gamma}_{F(\varepsilon)}(x)+\underline{\gamma}_{G(\varepsilon)}(x)-\underline{\gamma}_{F(\varepsilon)}(x)\cdot\underline{\gamma}_{G(\varepsilon)}(x), \\ \quad 1-\overline{\mu}_{F(\varepsilon)}(x)\cdot\overline{\mu}_{G(\varepsilon)}(x)]\rangle | x\in U\} & \text{if } \varepsilon \in A\cap B, \end{cases}$$

Since $\Box\langle F,A\rangle=\{\langle x,[\underline{\mu}_{F(\varepsilon)}(x),\overline{\mu}_{F(\varepsilon)}(x)],[\underline{\gamma}_{F(\varepsilon)}(x),1-\overline{\mu}_{F(\varepsilon)}(x)]\rangle|x\in U \text{ and } \varepsilon\in A\}$ and $\Box\langle G,B\rangle=\{\langle x,[\underline{\mu}_{G(\varepsilon)}(x),\overline{\mu}_{G(\varepsilon)}(x)],[\underline{\gamma}_{G(\varepsilon)}(x),1-\overline{\mu}_{G(\varepsilon)}(x)]\rangle|x\in U \text{ and } \varepsilon\in B\}$, it follows that $\Box\langle F,A\rangle\cdot\Box\langle G,B\rangle=(O,C)$, where $C=A\cup B$, for all $\varepsilon\in C$,

$$O(\varepsilon)=\begin{cases}\{\langle x,[\underline{\mu}_{F(\varepsilon)}(x)\cdot\underline{\mu}_{F(\varepsilon)}(x),\overline{\mu}_{F(\varepsilon)}(x)\cdot\overline{\mu}_{F(\varepsilon)}(x)],\\ \quad[\underline{\gamma}_{F(\varepsilon)}(x)+\underline{\gamma}_{F(\varepsilon)}(x)-\underline{\gamma}_{F(\varepsilon)}(x)\cdot\underline{\gamma}_{F(\varepsilon)}(x),\\ \quad 1-\overline{\mu}_{F(\varepsilon)}(x)+1-\overline{\mu}_{F(\varepsilon)}(x)-(1-\overline{\mu}_{F(\varepsilon)}(x))\cdot(1-\overline{\mu}_{F(\varepsilon)}(x))]\rangle|x\in U\}\\ \quad=\{\langle x,[\underline{\mu}_{F(\varepsilon)}(x)\cdot\underline{\mu}_{F(\varepsilon)}(x),\overline{\mu}_{F(\varepsilon)}(x)\cdot\overline{\mu}_{F(\varepsilon)}(x)],\\ \quad\ [\underline{\gamma}_{F(\varepsilon)}(x)+\underline{\gamma}_{F(\varepsilon)}(x)-\underline{\gamma}_{F(\varepsilon)}(x)\cdot\underline{\gamma}_{F(\varepsilon)}(x),\\ \quad\ 1-\overline{\mu}_{F(\varepsilon)}(x)\cdot\overline{\mu}_{F(\varepsilon)}(x)]\rangle|x\in U\} & \text{if } \varepsilon\in A-B,\\ \{\langle x,[\underline{\mu}_{G(\varepsilon)}(x)\cdot\underline{\mu}_{G(\varepsilon)}(x),\overline{\mu}_{G(\varepsilon)}(x)\cdot\overline{\mu}_{G(\varepsilon)}(x)],\\ \quad[\underline{\gamma}_{G(\varepsilon)}(x)+\underline{\gamma}_{G(\varepsilon)}(x)-\underline{\gamma}_{G(\varepsilon)}(x)\cdot\underline{\gamma}_{G(\varepsilon)}(x),\\ \quad 1-\overline{\mu}_{G(\varepsilon)}(x)+1-\overline{\mu}_{G(\varepsilon)}(x)-(1-\overline{\mu}_{G(\varepsilon)}(x))\cdot(1-\overline{\mu}_{G(\varepsilon)}(x))]\rangle|x\in U\}\\ \quad=\{\langle x,[\underline{\mu}_{G(\varepsilon)}(x)\cdot\underline{\mu}_{G(\varepsilon)}(x),\overline{\mu}_{G(\varepsilon)}(x)\cdot\overline{\mu}_{G(\varepsilon)}(x)],\\ \quad\ [\underline{\gamma}_{G(\varepsilon)}(x)+\underline{\gamma}_{G(\varepsilon)}(x)-\underline{\gamma}_{G(\varepsilon)}(x)\cdot\underline{\gamma}_{G(\varepsilon)}(x),\\ \quad\ 1-\overline{\mu}_{G(\varepsilon)}(x)\cdot\overline{\mu}_{G(\varepsilon)}(x)]\rangle|x\in U\} & \text{if } \varepsilon\in B-A,\\ \{\langle x,[\underline{\mu}_{F(\varepsilon)}(x)\cdot\underline{\mu}_{G(\varepsilon)}(x),\overline{\mu}_{F(\varepsilon)}(x)\cdot\overline{\mu}_{G(\varepsilon)}(x)],\\ \quad[\underline{\gamma}_{F(\varepsilon)}(x)+\underline{\gamma}_{G(\varepsilon)}(x)-\underline{\gamma}_{F(\varepsilon)}(x)\cdot\underline{\gamma}_{G(\varepsilon)}(x),\\ \quad 1-\overline{\mu}_{F(\varepsilon)}(x)+1-\overline{\mu}_{G(\varepsilon)}(x)-(1-\overline{\mu}_{F(\varepsilon)}(x))\cdot(1-\overline{\mu}_{G(\varepsilon)}(x))]\rangle|x\in U\}\\ \quad=\{\langle x,[\underline{\mu}_{G(\varepsilon)}(x)\cdot\underline{\mu}_{G(\varepsilon)}(x),\overline{\mu}_{G(\varepsilon)}(x)\cdot\overline{\mu}_{G(\varepsilon)}(x)],\\ \quad\ [\underline{\gamma}_{G(\varepsilon)}(x)+\underline{\gamma}_{G(\varepsilon)}(x)-\underline{\gamma}_{G(\varepsilon)}(x)\cdot\underline{\gamma}_{G(\varepsilon)}(x),\\ \quad\ 1-\overline{\mu}_{F(\varepsilon)}(x)\cdot\overline{\mu}_{G(\varepsilon)}(x)]\rangle|x\in U\} & \text{if } \varepsilon\in A\cap B.\end{cases}$$

Therefore, we have $\Box(\langle F,A\rangle\cdot\langle G,B\rangle)=\Box\langle F,A\rangle\cdot\Box\langle G,B\rangle$.

The proofs of (2)-(4) are similar to that of (1).

(5) According to Definition 3.1 and Definition 2.5, we have $(\langle F,A\rangle\cdot\langle G,B\rangle)^C=\langle H,A\cup B\rangle^C=(H^C,\rceil(A\cup B))=(H^C,\rceil A\cup\rceil B)$, where for all $\rceil\varepsilon\in\rceil A\cup\rceil B$,

$$H^C(\rceil\varepsilon)=\begin{cases}\{\langle x,[\underline{\gamma}_{F(\varepsilon)}(x)+\underline{\gamma}_{F(\varepsilon)}(x)-\underline{\gamma}_{F(\varepsilon)}(x)\cdot\underline{\gamma}_{F(\varepsilon)}(x),\\ \quad\overline{\gamma}_{F(\varepsilon)}(x)+\overline{\gamma}_{F(\varepsilon)}(x)-\overline{\gamma}_{F(\varepsilon)}(x)\cdot\overline{\gamma}_{F(\varepsilon)}(x)],\\ \quad[\underline{\mu}_{F(\varepsilon)}(x)\cdot\underline{\mu}_{F(\varepsilon)}(x),\overline{\mu}_{F(\varepsilon)}(x)\cdot\overline{\mu}_{F(\varepsilon)}(x)]\rangle|x\in U\} & \text{if } \rceil\varepsilon\in\rceil A-\rceil B,\\ \{\langle x,[\underline{\gamma}_{G(\varepsilon)}(x)+\underline{\gamma}_{G(\varepsilon)}(x)-\underline{\gamma}_{G(\varepsilon)}(x)\cdot\underline{\gamma}_{G(\varepsilon)}(x),\\ \quad\overline{\gamma}_{G(\varepsilon)}(x)+\overline{\gamma}_{G(\varepsilon)}(x)-\overline{\gamma}_{G(\varepsilon)}(x)\cdot\overline{\gamma}_{G(\varepsilon)}(x)],\\ \quad[\underline{\mu}_{G(\varepsilon)}(x)\cdot\underline{\mu}_{G(\varepsilon)}(x),\overline{\mu}_{G(\varepsilon)}(x)\cdot\overline{\mu}_{G(\varepsilon)}(x)]\rangle|x\in U\} & \text{if } \rceil\varepsilon\in\rceil B-\rceil A,\\ \{\langle x,[\underline{\gamma}_{F(\varepsilon)}(x)+\underline{\gamma}_{G(\varepsilon)}(x)-\underline{\gamma}_{F(\varepsilon)}(x)\cdot\underline{\gamma}_{G(\varepsilon)}(x),\\ \quad\overline{\gamma}_{F(\varepsilon)}(x)+\overline{\gamma}_{G(\varepsilon)}(x)-\overline{\gamma}_{F(\varepsilon)}(x)\cdot\overline{\gamma}_{G(\varepsilon)}(x)],\\ \quad[\underline{\mu}_{F(\varepsilon)}(x)\cdot\underline{\mu}_{G(\varepsilon)}(x),\overline{\mu}_{F(\varepsilon)}(x)\cdot\overline{\mu}_{G(\varepsilon)}(x)]\rangle|x\in U\} & \text{if } \rceil\varepsilon\in\rceil A\cap\rceil B.\end{cases}$$

Since $\langle F,A\rangle^C=\langle F^C,\rceil A\rangle$, where $F^C(\rceil\varepsilon)=\{\langle x,[\underline{\gamma}_{F(\varepsilon)}(x),\overline{\gamma}_{F(\varepsilon)}(x)],[\underline{\mu}_{F(\varepsilon)}(x),\overline{\mu}_{F(\varepsilon)}(x)]\rangle|$ $x\in U\}$ for all $\rceil\varepsilon\in\rceil A$, and $\langle G,B\rangle^C=\langle G^C,\rceil B\rangle$, where $G^C(\rceil\varepsilon)=\{\langle x,[\underline{\gamma}_{G(\varepsilon)}(x),\overline{\gamma}_{G(\varepsilon)}(x)],$ $[\underline{\mu}_{G(\varepsilon)}(x),\overline{\mu}_{G(\varepsilon)}(x)]\rangle|x\in U\}$ for all $\rceil\varepsilon\in\rceil B$, we have

$$\langle F,A\rangle^C+\langle G,B\rangle^C=\langle F^C,\rceil A\rangle+\langle G^C,\rceil B\rangle.$$

According to Definition 3.5, we can write $\langle F^C,\rceil A\rangle+\langle G^C,\rceil B\rangle=(O,\rceil A\cup\rceil B)$, where

$$O(\rceil\varepsilon)=\begin{cases}\{\langle x,[\underline{\gamma}_{F(\varepsilon)}(x)+\underline{\gamma}_{F(\varepsilon)}(x)-\underline{\gamma}_{F(\varepsilon)}(x)\cdot\underline{\gamma}_{F(\varepsilon)}(x),\\ \quad\overline{\gamma}_{F(\varepsilon)}(x)+\overline{\gamma}_{F(\varepsilon)}(x)-\overline{\gamma}_{F(\varepsilon)}(x)\cdot\overline{\gamma}_{F(\varepsilon)}(x)],\\ \quad[\underline{\mu}_{F(\varepsilon)}(x)\cdot\underline{\mu}_{F(\varepsilon)}(x),\overline{\mu}_{F(\varepsilon)}(x)\cdot\overline{\mu}_{F(\varepsilon)}(x)]\rangle|x\in U\} & \text{if }\rceil\varepsilon\in\rceil A-\rceil B,\\ \{\langle x,[\underline{\gamma}_{G(\varepsilon)}(x)+\underline{\gamma}_{G(\varepsilon)}(x)-\underline{\gamma}_{G(\varepsilon)}(x)\cdot\underline{\gamma}_{G(\varepsilon)}(x),\\ \quad\overline{\gamma}_{G(\varepsilon)}(x)+\overline{\gamma}_{G(\varepsilon)}(x)-\overline{\gamma}_{G(\varepsilon)}(x)\cdot\overline{\gamma}_{G(\varepsilon)}(x)],\\ \quad[\underline{\mu}_{G(\varepsilon)}(x)\cdot\underline{\mu}_{G(\varepsilon)}(x),\overline{\mu}_{G(\varepsilon)}(x)\cdot\overline{\mu}_{G(\varepsilon)}(x)]\rangle|x\in U\} & \text{if }\rceil\varepsilon\in\rceil B-\rceil A,\\ \{\langle x,[\underline{\gamma}_{F(\varepsilon)}(x)+\underline{\gamma}_{G(\varepsilon)}(x)-\underline{\gamma}_{F(\varepsilon)}(x)\cdot\underline{\gamma}_{G(\varepsilon)}(x),\\ \quad\overline{\gamma}_{F(\varepsilon)}(x)+\overline{\gamma}_{G(\varepsilon)}(x)-\overline{\gamma}_{F(\varepsilon)}(x)\cdot\overline{\gamma}_{G(\varepsilon)}(x)],\\ \quad[\underline{\mu}_{F(\varepsilon)}(x)\cdot\underline{\mu}_{G(\varepsilon)}(x),\overline{\mu}_{F(\varepsilon)}(x)\cdot\overline{\mu}_{G(\varepsilon)}(x)]\rangle|x\in U\} & \text{if }\rceil\varepsilon\in\rceil A\cap\rceil B.\end{cases}$$

Hence, we have $(\langle F,A\rangle\cdot\langle G,B\rangle)^C=\langle F,A\rangle^C+\langle G,B\rangle^C$.

(6) The proof is similar to that of (5). ∎

Definition 3.12. Let $\langle F,A\rangle$ and $\langle G,B\rangle$ be two interval-valued intuitionistic fuzzy soft sets over U_1 and U_2, respectively. The Cartesian product of $\langle F,A\rangle$ and $\langle G,B\rangle$ is the intuitionistic fuzzy soft set $\langle H,A\times B\rangle=\langle F,A\rangle\times\langle G,B\rangle$, where $H(\alpha,\beta)=\{\langle\langle x,y\rangle,[\underline{\mu}_{F(\alpha)}(x)\cdot\underline{\mu}_{G(\beta)}(y),\overline{\mu}_{F(\alpha)}(x)\cdot\overline{\mu}_{G(\beta)}(y)],[\underline{\gamma}_{F(\alpha)}(x)\cdot\underline{\gamma}_{G(\beta)}(y),\overline{\gamma}_{F(\alpha)}(x)\cdot\overline{\gamma}_{G(\beta)}(y)]\rangle|\ x\in U_1,y\in U_2\}$, for all $\alpha\in A$ and $\beta\in B$.

Example 3.13. Consider the interval-valued intuitionistic fuzzy soft set $\langle F,A\rangle$ over U, defined in Example 3.2, which describes the "attractiveness of the houses", and the interval-valued intuitionistic fuzzy soft set $\langle G,B\rangle$ over V, which describes the "capacity of the building companies", where V is a set of three building companies, denoted by $V=\{c_1,c_2,c_3\}$, $B=\{\text{high quality, good service}\}$ is a parameter set, and

$$\begin{aligned}G(\text{high quality})&=\{\langle c_1,[0.73,0.8],[0.1,0.18]\rangle,\langle c_2,[0.55,0.6],[0.2,0.35]\rangle,\\&\quad\langle c_3,[0.65,0.75],[0.16,0.23]\rangle\};\\ G(\text{good service})&=\{\langle c_1,[0.5,0.6],[0.28,0.35]\rangle,\langle c_2,[0.75,0.85],[0.1,0.15]\rangle,\\&\quad\langle c_3,[0.63,0.78],[0.1,0.2]\rangle\}.\end{aligned}$$

According to Definition 3.12, $\langle H,A\times B\rangle=\langle F,A\rangle\times\langle G,B\rangle$, where

$$\begin{aligned}&H(\text{convenient traffic, high quality})=\{\langle\langle h_1,c_1\rangle,[0.365,0.56],[0.025,0.054]\rangle,\\&\quad\langle\langle h_1,c_2\rangle,[0.275,0.42],[0.05,0.105]\rangle,\langle\langle h_1,c_3\rangle,[0.325,0.525],[0.04,0.069]\rangle,\\&\quad\langle\langle h_2,c_1\rangle,[0.438,0.6],[0.015,0.045]\rangle,\langle\langle h_2,c_2\rangle,[0.33,0.45],[0.03,0.0875]\rangle,\\&\quad\langle\langle h_2,c_3\rangle,[0.39,0.5625],[0.024,0.0575]\rangle,\\&\quad\langle\langle h_3,c_1\rangle,[0.6205,0.72],[0.003,0.018]\rangle,\\&\quad\langle\langle h_3,c_2\rangle,[0.4675,0.54],[0.006,0.035]\rangle,\\&\quad\langle\langle h_3,c_3\rangle,[0.5525,0.675],[0.0048,0.023]\rangle\};\end{aligned}$$

$H(\text{convenient traffic, good service}) = \{\langle\langle h_1, c_1\rangle, [0.25, 0.42], [0.07, 0.105]\rangle,$
$\langle\langle h_1, c_2\rangle, [0.375, 0.595], [0.025, 0.045]\rangle, \langle\langle h_1, c_3\rangle, [0.315, 0.546], [0.025, 0.06]\rangle,$
$\langle\langle h_2, c_1\rangle, [0.3, 0.45], [0.042, 0.0875]\rangle, \langle\langle h_2, c_2\rangle, [0.45, 0.6375], [0.015, 0.0375]\rangle,$
$\langle\langle h_2, c_3\rangle, [0.378, 0.585], [0.015, 0.05]\rangle,$
$\langle\langle h_3, c_1\rangle, [0.425, 0.54], [0.0084, 0.035]\rangle,$
$\langle\langle h_3, c_2\rangle, [0.6375, 0.765], [0.003, 0.015]\rangle,$
$\langle\langle h_3, c_3\rangle, [0.5355, 0.702], [0.003, 0.02]\rangle\};$

$H(\text{wooden, high quality}) = \{\langle\langle h_1, c_1\rangle, [0.438, 0.6], [0.02, 0.045]\rangle,$
$\langle\langle h_1, c_2\rangle, [0.33, 0.45], [0.04, 0.0875]\rangle,$
$\langle\langle h_1, c_3\rangle, [0.39, 0.5625], [0.032, 0.0575]\rangle,$
$\langle\langle h_2, c_1\rangle, [0.5329, 0.656], [0.01, 0.027]\rangle,$
$\langle\langle h_2, c_2\rangle, [0.4015, 0.492], [0.02, 0.0525]\rangle,$
$\langle\langle h_2, c_3\rangle, [0.4745, 0.615], [0.016, 0.0345]\rangle,$
$\langle\langle h_3, c_1\rangle, [0.4015, 0.52], [0.026, 0.063]\rangle,$
$\langle\langle h_3, c_2\rangle, [0.3025, 0.39], [0.052, 0.1225]\rangle,$
$\langle\langle h_3, c_3\rangle, [0.3575, 0.4875], [0.0416, 0.0805]\rangle\};$
$H(\text{wooden, good service}) = \{\langle\langle h_1, c_1\rangle, [0.3, 0.45], [0.056, 0.0875]\rangle,$
$\langle\langle h_1, c_2\rangle, [0.45, 0.6375], [0.02, 0.0375]\rangle,$
$\langle\langle h_1, c_3\rangle, [0.378, 0.585], [0.02, 0.05]\rangle,$
$\langle\langle h_2, c_1\rangle, [0.365, 0.492], [0.028, 0.0525]\rangle,$
$\langle\langle h_2, c_2\rangle, [0.5475, 0.697], [0.01, 0.0225]\rangle,$
$\langle\langle h_2, c_3\rangle, [0.4599, 0.6396], [0.01, 0.03]\rangle,$
$\langle\langle h_3, c_1\rangle, [0.275, 0.39], [0.0728, 0.1225]\rangle,$
$\langle\langle h_3, c_2\rangle, [0.4125, 0.5525], [0.026, 0.0525]\rangle,$
$\langle\langle h_3, c_3\rangle, [0.3465, 0.507], [0.026, 0.07]\rangle\};$
$H(\text{in good repair, high quality}) = \{\langle\langle h_1, c_1\rangle, [0.5548, 0.68], [0.008, 0.027]\rangle,$
$\langle\langle h_1, c_2\rangle, [0.418, 0.51], [0.016, 0.0525]\rangle,$
$\langle\langle h_1, c_3\rangle, [0.494, 0.6375], [0.0128, 0.0345]\rangle,$
$\langle\langle h_2, c_1\rangle, [0.4015, 0.52], [0.02, 0.054]\rangle,$
$\langle\langle h_2, c_2\rangle, [0.3025, 0.39], [0.04, 0.105]\rangle,$
$\langle\langle h_2, c_3\rangle, [0.3575, 0.4875], [0.032, 0.069]\rangle,$
$\langle\langle h_3, c_1\rangle, [0.511, 0.64], [0.015, 0.036]\rangle,$
$\langle\langle h_3, c_2\rangle, [0.385, 0.48], [0.03, 0.07]\rangle,$
$\langle\langle h_3, c_3\rangle, [0.455, 0.6], [0.024, 0.046]\rangle\};$
$H(\text{in good repair, good service}) = \{\langle\langle h_1, c_1\rangle, [0.38, 0.51], [0.0224, 0.0525]\rangle,$
$\langle\langle h_1, c_2\rangle, [0.57, 0.7225], [0.008, 0.0225]\rangle,$
$\langle\langle h_1, c_3\rangle, [0.4788, 0.663], [0.008, 0.03]\rangle,$
$\langle\langle h_2, c_1\rangle, [0.275, 0.39], [0.056, 0.105]\rangle,$
$\langle\langle h_2, c_2\rangle, [0.4125, 0.5525], [0.02, 0.045]\rangle,$
$\langle\langle h_2, c_3\rangle, [0.3465, 0.507], [0.02, 0.06]\rangle,$
$\langle\langle h_3, c_1\rangle, [0.35, 0.48], [0.042, 0.07]\rangle,$
$\langle\langle h_3, c_2\rangle, [0.525, 0.68], [0.015, 0.03]\rangle,$
$\langle\langle h_3, c_3\rangle, [0.441, 0.624], [0.015, 0.04]\rangle\}.$

4. Conclusions

In this paper, some new operations on interval-valued intuitionistic fuzzy soft sets, i.e., "$\cdot$", "+"and Cartesian product, are introduced, and some basic properties are investigated. In the following work, we will consider the entropy measure, similarity measure and inclusion measure of interval-valued intuitionistic fuzzy soft sets and their relations.

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FUZZY HYPER KS-SEMIGROUPS

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Abstract. Hyperstructure theory is applied to KS-semigroups, an algebra related to BCK-algebra and semigroup and thus, the notion of hyper KS-semigroups is introduced.

In this paper, the concept of fuzzy sets is applied to hyper KS-semigroups. In this fuzzification, the notions of fuzzy hyper subKS-semigroups and fuzzy hyper KS-ideals are introduced and relationships among them are investigated. Using the concept of upper level subsets, relationships between hyper subKS-semigroups (resp. hyper KS-ideals) and fuzzy hyper subKS-semigroups (resp. fuzzy hyper KS-ideals) are established. Finally, under a homomorphism $f : G \to H$ of hyper KS-semigroups, it is shown that the pre-image of a fuzzy hyper KS-ideal of H is a fuzzy hyper KS-ideal of G.

1. Introduction

KS-semigroups was introduced by K. H. Kim [6] which is a combination of BCK-algebra and semigroup.

Hyperstructure theory (also called multivalued algebras) was introduced by F. Marty at the 8th congress of Scandinavian Mathematicians in 1934. Recall that in a classical algebraic structure, the image of two elements of a set is an element of the set, while in an algebraic hyperstructure, the image of two elements is a set. Thus, it is considered as a generalization of classical algebraic structures.

In [8], hyperstructure theory was applied to KS-semigroups and so a new class of algebra, called hyper KS-semigroups, was introduced. In this paper, we apply the concept fuzzy sets to hyper KS-semigroups.

2. Preliminaries

Let H be a non-empty set endowed with a *hyperoperation* " $*$ " that is, " $*$ " is a function from $H \times H$ to $P^*(H) = P(H) \setminus \{\varnothing\}$. For two subsets A and B of H, $A * B = \bigcup\limits_{a\in A, b\in B} a * b$. We shall use $x * y$ instead of $x * \{y\}, \{x\} * y$, or $\{x\} * \{y\}$. When $A \subseteq H$ and $x \in H$, we agree to write $A * x$ instead of $A * \{x\}$. Similarly, we write $x * A$ for $\{x\} * A$. In effect, $A * x = \bigcup\limits_{a\in A} a * x$ and $x * A = \bigcup\limits_{a\in A} x * a$. The structure $(H, \cdot)$ is called a *semihypergroup* if $(x \cdot y) \cdot z = x \cdot (y \cdot z)$, for all $x, y, z \in H$.

Definition 2.1 [5] A *hyper BCK-algebra* is a nonempty set H endowed with a hyperoperation " $*$ " and a constant $\mathbf{0}$ satisfying the following axioms: for all $x, y, z \in H$,

(H1) $(x * z) * (y * z) < x * y$,

(H2) $(x * y) * z = (x * z) * y$,

(H3) $x * H \ll x$,

(H4) $x \ll y$ and $y \ll x$ imply $x = y$,

where (a) $x \ll y$ is defined by $\mathbf{0} \in x * y$, and (b) for every $A, B \subseteq H$, $A \ll B$ is defined as follows: for all $a \in A$, there exists $b \in B$ such that $a \ll b$. In such case, we call " $\ll$ " the *hyper order* in H.

A nonempty subset I of a hyper BCK-algebra $(H, *, \mathbf{0})$ is called a hyper BCK-ideal if $\mathbf{0} \in I$ and for any $x, y \in H$, $x * y \ll I$ and $y \in I$ imply that $x \in I$.

In any hyper BCK-algebra $(H, *, \mathbf{0})$, the following hold (see [5]) for all $x \in H$.

(A1) $x * H \ll x$ if and only if $x * y \ll \{x\}$ for all $y \in H$,

(A2) $x \ll x$ and

(A3) $x * \mathbf{0} = \{x\}$.

Definition 2.2 [8] A *hyper KS-semigroup* is a nonempty set H together with two hyperoperations "$*$" and "$\cdot$" and a constant $\mathbf{0}$ satisfying the following conditions:

(i) $(H, *, \mathbf{0})$ is a hyper BCK-algebra.

(ii) $(H, \cdot)$ is a semihypergroup having zero as a bilaterally absorbing element, that is, $x \cdot \mathbf{0} = \mathbf{0} \cdot x = \{\mathbf{0}\}$ for all $x \in H$; and

(iii) " $\cdot$ " is left and right distributive over " $*$ ", that is, for any $x, y, z \in H$,

$x \cdot (y * z) = (x \cdot y) * (x \cdot z)$ and $(x * y) \cdot z = (x \cdot z) * (y \cdot z)$.

From now on, a hyper KS-semigroup $(H, *, \cdot, \mathbf{0})$ shall be denoted by H and for all $x, y \in H$, we agree to write $x \cdot y$ as xy.

Example 2.3 [8] Let $H = \{0, 1, 2\}$. Then $(H, *, \cdot, 0)$ is a hyper KS-semigroup with hyperoperations " $*$ " and " $\cdot$ " defined as follows.

$*$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{0,1\}$
2	$\{2\}$	$\{1,2\}$	$\{0,1,2\}$

$\cdot$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{0\}$	$\{1\}$	$\{0,1\}$
2	$\{0\}$	$\{0,1\}$	$\{0,1,2\}$

Let I be a nonempty subset of a hyper KS-semigroup $(H, *, \cdot, \mathbf{0})$. Then I is said to be a *hyper subKS-semigroup* of H if for all $x, y \in I$, $x * y \subseteq I$ and $xy \subseteq I$. I is said to be a *hyper left* (resp. *hyper right*) *stable* if $xa \subseteq I$ (resp. $ax \subseteq I$) for all $x \in H$ and for all $a \in I$. I is said to be a *hyper stable* if I is both hyper left and right stable. I is said to be a *hyper left* (resp. *hyper right*) *KS-ideal* if (i) I is a hyper left (resp. hyper right) stable and (ii) for any $x, y \in H$, $x * y \ll I$ and $y \in I$ imply that $x \in I$. I is a *hyper KS-ideal* if I is both a hyper left and a hyper right KS-ideal.

Remark 2.4 A hyper KS-ideal contains $\mathbf{0}$ and hence, it is a hyper BCK-ideal.

3. Fuzzy sets in hyper KS-semigroups

In this section, we introduce the notions of fuzzy hyper subKS-semigroups and fuzzy hyper KS-ideals and provide some characterizations with respect to their upper level subsets.

Throughout this paper, a hyper KS-semigroup $(H, *, \cdot, \mathbf{0})$ shall be denoted by H with hyper order denoted by $\ll$. Note that the symbols $\leq, \geq, <$ and $>$ refer to the usual inequalities in real numbers.

Definition 3.1 A fuzzy set μ in a hyper KS-semigroup H is called a *fuzzy hyper subKS-semigroup* of H if it satisfies the following conditions: for all $x, y \in H$,

(i) $\inf\limits_{a \in x*y} \mu(a) \geq \min\{\mu(x), \mu(y)\}$ and

(ii) $\inf\limits_{a \in xy} \mu(a) \geq \min\{\mu(x), \mu(y)\}$.

Lemma 3.2 *Let μ be a fuzzy hyper subKS-semigroup of a hyper KS-semigroup H. Then for all $x \in H$, $\mu(\boldsymbol{0}) \geq \mu(x)$. Moreover, if μ is onto, $\mu(\boldsymbol{0}) = 1$.*

Proof. Let μ be a fuzzy hyper subKS-semigroup of H and let $x \in H$. By (A2), $x \ll x$ and so $\mathbf{0} \in x * x$. Thus,

$$\mu(\mathbf{0}) \geq \inf_{t \in x*x} \mu(t) \geq \min\{\mu(x), \mu(x)\} = \mu(x).$$

If μ is onto, then $\mu(x) = 1$ for some $x \in H$. Hence, $1 = \mu(x) \leq \mu(\mathbf{0}) \leq 1$. Therefore, $\mu(\mathbf{0}) = 1$. ∎

Definition 3.3 A fuzzy set μ in a hyper KS-semigroup H is called a *left* (resp. *right*) *fuzzy hyper KS-ideal* of H if it satisfies the following conditions: for all $x, y \in H$,

(F1) $x \ll y$ implies $\mu(x) \geq \mu(y)$,

(F2) $\mu(x) \geq \min\{\inf_{a \in x*y} \mu(a), \mu(y)\}$, and

(F3) $\inf_{a \in xy} \mu(a) \geq \mu(y)$ (resp. $\inf_{a \in xy} \mu(a) \geq \mu(x)$).

A fuzzy set μ is a *fuzzy hyper KS-ideal* if it is both a left and a right fuzzy hyper KS-ideal of H.

In a hyper BCK-algebra, a fuzzy hyper BCK-ideal satisfies (F1) and (F2).

Example 3.4 Let H be the hyper KS-semigroup in Example 2.3. Define a fuzzy set μ in H by $\mu(0) = t_1$, $\mu(1) = t_2$ and $\mu(2) = t_3$ where $t_1, t_2, t_3 \in [0, 1]$ and $t_1 > t_2 > t_3$. Then it can be shown that μ is a fuzzy hyper KS-ideal of H.

Theorem 3.5 *A fuzzy hyper KS-ideal of a hyper KS-semigroup is a fuzzy hyper subKS-semigroup.*

Proof. Let μ be a fuzzy hyper KS-ideal in H and let $x, y \in H$. By (A1), $x * y \ll \{x\}$. Thus, for all $a \in x * y$, $a \ll x$. By Definition 3.3(F1), $\mu(x) \leq \mu(a)$ for all $a \in x * y$ and so $\mu(x) \leq \inf_{a \in x*y} \mu(a)$. This implies that

$$\min\{\inf_{a \in x*y} \mu(a), \mu(y)\} \geq \min\{\mu(x), \mu(y)\}$$

and so it follows that $\inf_{a \in x*y} \mu(a) \geq \min\{\inf_{a \in x*y} \mu(a), \mu(y)\} \geq \min\{\mu(x), \mu(y)\}$. Also, $\inf_{a \in xy} \mu(a) \geq \mu(y)$ and $\inf_{a \in xy} \mu(a) \geq \mu(x)$. Thus, $\inf_{a \in xy} \mu(a) \geq \min\{\mu(x), \mu(y)\}$. Hence, μ is a fuzzy hyper subKS-semigroup in H. ∎

The converse of the preceding theorem may not be true in general. Consider the following example.

Example 3.6 Consider the hyper KS-semigroup $(H, *, \cdot, 0)$ with hyperoperations "$*$" and "$\cdot$" defined as follows.

$*$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{0,1\}$
2	$\{2\}$	$\{1,2\}$	$\{0,1,2\}$

$\cdot$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{0\}$	$\{1\}$	$\{0,1,2\}$
2	$\{0\}$	$\{0,1\}$	$\{0,2\}$

Define a fuzzy set μ in H by $\mu(0) = t_1$, $\mu(1) = t_2$ and $\mu(2) = t_3$, where $t_1, t_2, t_3 \in [0,1]$ and $t_1 > t_2 > t_3$. Then by routine calculations, we can show that μ is a fuzzy hyper subKS-semigroup but not a fuzzy hyper KS-ideal since $t_3 = \mu(2) = \inf\limits_{a\in 1\cdot 2=\{0,1,2\}} \mu(a) < \mu(1) = t_2$.

For two fuzzy sets μ and ν in H, $(\mu \cap \nu)(x) = \min\{\mu(x), \nu(x)\}$ for all $x \in H$.

Theorem 3.7 *If μ and ν are fuzzy hyper subKS-semigroups of a hyper KS-semigroup H, then $\mu \cap \nu$ is also a fuzzy hyper subKS-semigroup of H.*

Proof. Since μ and ν are fuzzy hyper subKS-semigroups of H, it follows that for all $x, y \in H$ and $a \in x * y$, $\mu(a) \geq \inf\limits_{a\in x*y} \mu(a) \geq \min\{\mu(x), \mu(y)\}$ and $\nu(a) \geq \inf\limits_{a\in x*y} \nu(a) \geq \min\{\nu(x), \nu(y)\}$ Thus, for all $a \in x * y$, we have

$$\begin{aligned}(\mu\cap\nu)(a) &= \min\{\mu(a), \nu(a)\}\\ &\geq \min\{\min\{\mu(x), \mu(y)\}, \min\{\nu(x), \nu(y)\}\}\\ &= \min\{\min\{\mu(x), \nu(x)\}, \min\{\mu(y), \nu(y)\}\}\\ &= \min\{(\mu\cap\nu)(x), (\mu\cap\nu)(y)\}\end{aligned}$$

and so $\inf\limits_{a\in x*y} (\mu\cap\nu)(a) \geq \min\{(\mu\cap\nu)(x), (\mu\cap\nu)(y)\}$. Also, for all $b \in xy$,

$$\begin{aligned}(\mu\cap\nu)(b) &= \min\{\mu(b), \nu(b)\}\\ &\geq \min\{\min\{\mu(x), \mu(y)\}, \min\{\nu(x), \nu(y)\}\}\\ &= \min\{\min\{\mu(x), \nu(x)\}, \min\{\mu(y), \nu(y)\}\}\\ &= \min\{(\mu\cap\nu)(x), (\mu\cap\nu)(y)\}\end{aligned}$$

and so $\inf\limits_{b\in xy} (\mu\cap\nu)(b) \geq \min\{(\mu\cap\nu)(x), (\mu\cap\nu)(y)\}$. Therefore, $\mu\cap\nu$ is a fuzzy hyper subKS-semigroup of H.

For a fuzzy set μ in H, $H_\mu = \{x \in H | \mu(x) = \mu(\mathbf{0})\}$.

Theorem 3.8 *Let μ be a fuzzy set in a hyper KS-semigroup H.*

(i) *If μ is a fuzzy hyper subKS-semigroup of H, then H_μ is a hyper subKS-semigroup of H.*

(ii) *If μ is a fuzzy hyper KS-ideal of H, then H_μ is a hyper KS-ideal of H.*

Proof. Let $H_\mu = \{x \in H | \mu(x) = \mu(\mathbf{0})\}$.

(i) Since $\mathbf{0} \in H_\mu$, $H_\mu \neq \varnothing$. Clearly $H_\mu \subseteq H$. Suppose that μ is a fuzzy hyper subKS-semigroup of H. Let $x, y \in H_\mu$. Then $\mu(x) = \mu(\mathbf{0}) = \mu(y)$ and so for all $a \in x * y, b \in xy$,

$$\mu(a) \geq \inf_{a \in x*y} \mu(a) \geq \min\{\mu(x), \mu(y)\} = \min\{\mu(\mathbf{0}), \mu(\mathbf{0})\} = \mu(\mathbf{0}) \text{ and}$$

$$\mu(b) \geq \inf_{b \in xy} \mu(b) \geq \min\{\mu(x), \mu(y)\} = \min\{\mu(\mathbf{0}), \mu(\mathbf{0})\} = \mu(\mathbf{0}).$$

By Lemma 3.2, $\mu(a) \leq \mu(\mathbf{0})$ and $\mu(b) \leq \mu(\mathbf{0})$. Hence, $\mu(a) = \mu(\mathbf{0})$ and $\mu(b) = \mu(\mathbf{0})$. Thus, $a, b \in H_\mu$. Therefore, $x * y, xy \subseteq H_\mu$ and so H_μ is a hyper subKS-semigroup of H.

(ii) Suppose that μ is a fuzzy hyper KS-ideal of H. From [4], H_μ satisfies condition (ii) of a hyper KS-ideal. We only need to show that H_μ is hyper stable. Let $x \in H$ and $a \in H_\mu$. Then $\mu(a) = \mu(\mathbf{0})$. Let $b \in ax$ and $c \in xa$. Then

$$\mu(b) \geq \inf_{b \in ax} \mu(b) \geq \mu(a) = \mu(\mathbf{0}) \text{ and } \mu(c) \geq \inf_{c \in xa} \mu(c) \geq \mu(a) = \mu(\mathbf{0}).$$

By Lemma 3.2, $\mu(\mathbf{0}) \geq \mu(b)$ and $\mu(\mathbf{0}) \geq \mu(c)$. Thus, $\mu(\mathbf{0}) = \mu(b)$ and $\mu(\mathbf{0}) = \mu(c)$ and so, $b, c \in H_\mu$. Hence, $ax, xa \subseteq H_\mu$ so that H_μ is hyper stable. Therefore, H_μ is a hyper KS-ideal of H. ∎

For a fuzzy set μ in H and $t \in [0, 1]$, the *upper level subset* of μ is given by $\mu_t = \{x \in H | \mu(x) \geq t\}$.

Remark 3.9 $\mu_t = H$ if $t = 0$.

Theorem 3.10 *Let μ be a fuzzy set in a hyper KS-semigroup H. Then μ is a fuzzy hyper subKS-semigroup of H if and only if the upper level subset μ_t is a hyper subKS-semigroup of H whenever $\mu_t \neq \varnothing$ for $t \in [0, 1]$.*

Proof. Suppose that μ is a fuzzy hyper subKS-semigroup of H and assume that $\mu_t \neq \varnothing$ where $t \in [0, 1]$. Let $x, y \in \mu_t$. Then $\mu(x) \geq t$ and $\mu(y) \geq t$. Thus, for all $a \in x * y$ and for all $b \in xy$,

$$\mu(a) \geq \inf_{a \in x*y} \mu(a) \geq \min\{\mu(x), \mu(y)\} \geq \min\{t, t\} = t \text{ and}$$

$$\mu(b) \geq \inf_{b \in xy} \mu(b) \geq \min\{\mu(x), \mu(y)\} \geq \min\{t, t\} = t.$$

Hence, μ_t is a hyper subKS-semigroup of H.

Conversely, assume that for each $t \in [0, 1]$, $\mu_t \neq \varnothing$ is a hyper subKS-semigroup of H. Let $x, y \in H$ and $t = \min\{\mu(x), \mu(y)\}$. Then $\mu(x) \geq t$ and $\mu(y) \geq t$. Thus, $x, y \in \mu_t$. Since μ_t is a hyper subKS-semigroup of H, $x*y, xy \subseteq \mu_t$ and so for all $a \in x * y$ and for all $b \in xy$, $\mu(a) \geq t$ and $\mu(b) \geq t$. Hence, $\inf_{a \in x*y} \mu(a) \geq t = \min\{\mu(x), \mu(y)\}$ and $\inf_{b \in xy} \mu(b) \geq t = \min\{\mu(x), \mu(y)\}$. Therefore, μ is a fuzzy hyper subKS-semigroup of H. ∎

Theorem 3.11 *Let μ be a fuzzy set in a hyper KS-semigroup H. Then μ is a fuzzy hyper KS-ideal of H if and only if the upper level subset μ_t is a hyper KS-ideal of H whenever $\mu_t \neq \varnothing$ for $t \in [0,1]$.*

Proof. Suppose that μ is a fuzzy hyper KS-ideal of H and assume that $\mu_t \neq \varnothing$ where $t \in [0,1]$. From [3], μ_t satisfies condition (ii) of a hyper KS-ideal. Thus, we only need to show that $ax, xa \subseteq \mu_t$ for all $x \in H$ and for all $a \in \mu_t$. Let $x \in H$ and $a \in \mu_t$. Then $\mu(a) \geq t$. Now, by Definition 3.3(F3), we have $\mu(z) \geq \inf_{z \in ax} \mu(z) \geq \mu(a) \geq t$ and $\mu(w) \geq \inf_{w \in xa} \mu(w) \geq \mu(a) \geq t$. Hence, $z, w \in \mu_t$ and so $ax, xa \subseteq \mu_t$. Thus, μ_t is hyper stable. Therefore, μ_t is a hyper KS-ideal of H.

Conversely, assume that for each $t \in [0,1]$, $\mu_t \neq \varnothing$ is a hyper KS-ideal of H. From Remark [**?**] and [3], μ satisfies (F1) and (F2). We only need to show Definition 3.3(F3). Let $x, y \in H$ and take $t = \mu(y)$. Then $y \in \mu_t$ and since μ_t is hyper stable, it follows that $xy \in \mu_t$. Thus, for all $a \in xy$, $a \in \mu_t$. That is, $\mu(a) \geq t = \mu(y)$. Hence, $\inf_{a \in xy} \mu(a) \geq t = \mu(y)$. Similarly, take $t = \mu(x)$ so that $xy \subseteq \mu_t$ and thus, for all $a \in xy$, $a \in \mu_t$, that is, $\mu(a) \geq t = \mu(x)$. Hence, $\inf_{a \in xy} \mu(a) \geq t = \mu(x)$. Therefore, μ is a fuzzy hyper KS-ideal of H. ∎

For any nonempty subset I of a hyper KS-semigroup H, we define a fuzzy set μ_I in H by

$$\mu_I(x) = \begin{cases} 1, & \text{if } x \in I, \\ 0, & \text{otherwise} \end{cases}$$

that is, μ_I is the *characteristic function* of I.

Corollary 3.12 *Let I be a nonempty subset I of a hyper KS-semigroup H and μ_I be the characteristic function of I.*

(i) *I is a hyper subKS-semigroup of H if and only if μ_I is a fuzzy hyper subKS-semigroup of H.*

(ii) *I is a hyper KS-ideal of H if and only if μ_I is a fuzzy hyper KS-ideal of H.*

Proof. Observe the level subset of μ_I

$$(\mu_I)_t = \begin{cases} I, & \text{if } t \in (0,1], \\ H, & \text{if } t = 0. \end{cases}$$

The results follow directly from Theorems 3.10 and 3.11. ∎

Let $(H_1, *_1, \cdot_1, 0_1)$ and $(H_2, *_2, \cdot_2, 0_2)$ be hyper KS-semigroups and $f : H_1 \to H_2$ be a map. Then f is called a *hyper KS-semigroup homomorphism* if $f(x *_1 y) = f(x) *_2 f(y)$ and $f(x \cdot_1 y) = f(x) \cdot_2 f(y)$ for all $x, y \in H_1$.

Theorem 3.13 *Let $f : G \to H$ be an epimorphism of hyper KS-semigroups. If ν is a fuzzy hyper KS-ideal of H, then the homomorphic pre-image μ of ν under f is a fuzzy hyper KS-ideal of G.*

Proof. Let $f : G \to H$ be an epimorphism of hyper KS-semigroups. From Remark 2.4 and [4], μ satisfies (F1) and (F2). So, we only need to show (F3). Let $x, y \in G$. Then

$$\inf_{a \in xy} \mu(a) = \inf_{f(a) \in f(x)f(y)} \nu(f(a)) \geq \nu(f(x)) = \mu(x) \text{ and}$$

$$\inf_{a \in xy} \mu(a) = \inf_{f(a) \in f(x)f(y)} \nu(f(a)) \geq \nu(f(y)) = \mu(y).$$

Thus, μ is a fuzzy hyper KS-ideal of G. ■

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ENGEL FUZZY SUBGROUPS

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Abstract. In this paper we introduce and study Engel fuzzy subgroups. We will proceed by introduce and study soluble and nilpotent fuzzy subgroups. In particular, we show that if $x \in L_3(\mu)$ and $\mu(x^{p^n}) = \mu(e)$ for some integer $n \geq 2$, then μ is fuzzy soluble.

Keywords: fuzzy subgroups, nilpotent fuzzy subgroup, Engel fuzzy subgroup, soluble fuzzy subgroup.

1. Introduction

Let G be an arbitrary group and $x, y \in G$. Define inductively the n-commutator

$$[x,_0 y] = x, [x,_1 y] = x^{-1}y^{-1}xy$$

and, for all $n > 0$,

$$[x,_n y] = [[x,_{n-1} y], y].$$

Definition 1.1. A group G is called an *Engel group* if for each ordered pair (x, y) of elements in G there exists positive integer $n(x, y)$ such that $[x,_n, y] = e$; $[x, y] = x^{-1}y^{-1}xy$.

Suppose $n = n(x, y)$ can be chosen independently of x, y then we say that G is an *n-Engel group*. In this definition we have used bracket from the left. But since

$$[y,_n x] = [_n x^{-1}, y]^{x^n},$$

it does not matter whether we use bracketing from the right or from the left. If $n = 1$ then the 1-Engel group is abelian. Levi [3] proved that a group G is a 2-Engel group if and only if the normal closure x^G of arbitrary element x is abelian. Furthermore, we have 2-Engel groups are nilpotent of class at most 3. Also, he has shown that every group of exponent 3 is a 2-Engel group. Heinken [6] shown that every 3-Engel group G is nilpotent of class at most 4 if G has no element of orders 2, or 5. L. Kappe and W. Kappe [7] gave a characterization for 3−Engel groups which is analogous to Levi's theorem on 2−Engel groups. They shown that the following are equivalent:

1) G is a 3-Engel group.

2) x^G is a 2−Engel group for all $x \in G$.

3) for all $x \in G$, x^G is nilpotent of class at most 2.

We do not have a corresponding characterization for 4- Engel groups. Traustason [8] studied 4-Engel groups. The origin of Engel groups lies in the theory of lie algebras. In fact, they are a group theoretic analog of Engel lie algebras. According to Engel's theorem every finite dimensional Engel lie algebra over a field is nilpotent. In 1936 Zorn [5] proved a corresponding theorem for Engel groups.

Zorn's Theorem. *A finite Engel group is nilpotent.*

Definition 1.2. [9] Let μ be a fuzzy subset of a semigroup G. Let

$$Z(\mu) = \{x \in G; \mu(xy) = \mu(yx) \quad \text{and } \mu(xyz) = \mu(yxz) \ \text{ for all } \quad y, z \in G\}.$$

Then μ is called *commutative* in G if $Z(\mu) = G$.

We recall the notion of the ascending central series of a fuzzy subgroup and a nilpotent fuzzy subgroup of a group [9]. Let μ be a fuzzy subgroup of a group G. Let $Z^0(\mu) = \{e\}$ and π_0 be the natural homomorphism of G onto $G/Z^0(\mu)$. Suppose that $Z^i(\mu)$ has been defined and that $Z^i(\mu)$ is a normal subgroup of G for $i \in N \cup \{0\}$. Let π_i be the natural homomorphism of G onto $G/Z^i(\mu)$. Define $Z^{i+1}(\mu) = \pi_i^{-1}(Z(\pi_i(\mu)))$. Then $Z^{i+1}(\mu) \supseteq Ker(\pi_i) = Z^i(\mu)$ for $i = 0, 1, ...$ The normality of $Z^{i+1}(\mu)$ in G is proved.

Definition 1.3. Let μ be a fuzzy subgroup of a group G. The *ascending central series* of μ is defined to be the ascending chain of normal subgroups of G,

$$Z^0(\mu) \subseteq Z^1(\mu) \subseteq ....$$

Definition 1.4. A fuzzy subgroup μ of a group G is called *nilpotent* if there exist a nonnegative integer such that $Z^m(\mu) = G$. The smallest such integer is called the class of μ.

In [9] we have the following main results, that will be used throughout this paper.

Theorem 1.5. *Let μ be a fuzzy subgroup of a group G. If G is nilpotent of class m, then μ is nilpotent of class n for some nonnegative integer $n \leq m$.*

Theorem 1.6. *Let λ be a nilpotent fuzzy subgroup of G. If μ is a fuzzy subgroup of G such that $\mu \subseteq \lambda$, then μ is nilpotent.*

Theorem 1.7. *Let μ be a fuzzy subgroup of a group G. Then for all $x, y \in G$, $\mu(x) \neq \mu(y)$ implies $\mu(xy) = \mu(x) \wedge \mu(y)$.*

Theorem 1.8. *Let μ be a fuzzy subgroup of a group G. Let $i \in N$. If $xyx^{-1}y^{-1} \in Z^{i-1}(\mu)$ for all $y \in G$, then $x \in Z^i(\mu)$.*

Theorem 1.9. *Let μ be a fuzzy subgroup of a group G. Let $T = \{x \in G;$ $\mu(xyx^{-1}y^{-1}) = \mu(e)$ for all $y \in G\}$. Then $T = Z(\mu)$.*

2. Engel fuzzy subgroups

In this section we introduce the concept of Engel fuzzy subgroups and investigate some basic properties of Engel fuzzy subgroups.

Definition 2.1. Let G be a group and $\mu : G \longrightarrow [0,1]$ be a fuzzy subgroup. Then $\mu : G \longrightarrow [0,1]$ is called an *n-Engel fuzzy subgroup* if for all $x, y \in G$, $\mu[x,_n y] = \mu(e)$, where e is identity element of G.

Example 2.2. Let $D_3 = \langle a, b; a^3 = b^2 = e, ba = a^2b\rangle$ be the dihedral group with six element. Define a fuzzy subgroup μ of D_3 by the following:

$$\mu(x) = \begin{cases} t_0 & \text{if } x \in \langle a\rangle \\ t_1 & \text{if } x \notin \langle a\rangle, \end{cases}$$

where $t_0 > t_1$. It is easy to see that μ is an 1-Engel fuzzy subgroup while D_3 is not an Engel group, since $[a,_n b] = a \neq e$.

Theorem 2.3. *Let μ be a fuzzy subgroup of a group G. If the non-empty α-level cut μ_t is Engel group, for all $t \in [0,1]$, then μ is an Engel fuzzy subgroup of G. If $\mu(x) = \mu(e)$, implies that $x = e$, then the converse of the theorem is true.*

Proof. Let $x, y \in G$, and $t = \min\{\mu(x), \mu(y)\}$. Then $\mu(x) \geq t$ and $\mu(y) \geq t$, so $x, y \in \mu_t$. Now, by hypotheses, $[x,_n y] = e$. Therefore, $\mu[x,_n y] = \mu(e)$. The converse is clear. ∎

Theorem 2.4. [9] *Let μ be a fuzzy subgroup of a group G. Then $\mu(xyx^{-1}y^{-1}) = \mu(e)$, for all $x, y \in G$, if and only if μ is commutative in G.*

Now, as a consequence of this theorem, we have that:

Theorem 2.5. *Let $\mu : G \longrightarrow [0,1]$ be a fuzzy subgroup. Then μ is commutative if and only if μ is 1-Engel fuzzy subgroup. Moreover every 1-Engel fuzzy subgroup is nilpotent of class at most 1.*

Theorem 2.6. *Let μ be a fuzzy subgroup of a group G. Then μ is nilpotent of class at most 3 if G is a 2-Engel group.*

Proof. It is the immediate result of Theorem 1.5 and Levi's theorem. ∎

Theorem 2.7. *Let G and H be two groups and $f : G \longrightarrow H$ be a group homomorphism. If μ is an n-Engel fuzzy subgroup of H, then $f^{-1}(\mu)$ is an n-Engel fuzzy subgroup of G.*

Proof. Clearly, $f^{-1}(\mu)$ is a fuzzy subgroup of G. Let $x, y \in G$ and e, e' be the identity elements of G, H, respectively. Then

$$f^{-1}\mu[x,_n y] = \mu(f[x,_n y]) = \mu[f(x),_n f(y)] = \mu(e^{'}) = f^{-1}\mu(e).$$ ∎

Theorem 2.8. *Let μ be a n-Engel fuzzy subgroup of G and H be a group. Suppose that $f : G \longrightarrow H$ is an onto homomorphism. Then $f(\mu)$ is an n-Engel fuzzy subgroup of H.*

Proof. Clearly, $f(\mu)$ is a fuzzy subgroup of H. Let $u, v \in H$ and $e, e^{'}$ be the identity elements of G, H, respectively. Then $u = f(x)$, $v = f(y)$ for some $x, y \in G$. Then

$$\begin{aligned} f(\mu)[u,_n v] &= \sup\{\mu(z), z \in f^{-1}[u,_n v]\} \\ &\geq \sup\{\mu[x,_n y], u = f(x), v = f(y)\} \\ &= \mu(e) \\ &= (f\mu)(e^{'}) \end{aligned}$$

This completes the proof. ∎

Theorem 2.9. *Let μ, η be two Engel fuzzy subgroups then $\mu \cap \eta$ and $\mu \times \eta$ are Engel fuzzy subgroups too.*

Proof.

$$\begin{aligned} \mu \times \eta[(x_1, y_1),_n (x_2, y_2)] &= \mu \times \eta([x_1,_n x_2], [y_1,_n y_2]) \\ &= \min\{\mu([x_1,_n x_2]), \eta([y_1,_n y_2])\} \\ &= \min\{\mu(e_1), \eta(e_2)\} \\ &= \mu \times \eta(e_1, e_2). \end{aligned}$$

Also

$$(\mu \cap \eta)[x,_n y] = \min\{\mu[x,_n y], \eta[x,_n y]\} = \min\{\mu(e), \eta(e)\} = (\mu \cap \eta)(e)$$

This completes the proof. ∎

Theorem 2.10. *Let μ be a normal $(n-1)$-Engel fuzzy subgroup of G, then μ is an n-Engel fuzzy subgroup.*

Proof. Let $x, y \in G$. Then $\mu[x,_n y] = \mu[[x,_{n-1} y], y] = \mu([x,_{n-1} y]^{-1}[x,_{n-1} y]^y) \geq \min\{\mu([x,_{n-1} y]^{-1}), \mu([x,_{n-1} y]^y)\}$. Since μ is a normal $(n-1)$-Engel fuzzy subgroup so $\mu[x,_{n-1} y] = \mu(e)$. Therefore μ is an n-Engel fuzzy subgroup. ∎

Theorem 2.11. *If G is an n-Engel group then G/μ is n-Engel, where $G/\mu = \{x\mu, x \in G\}$, $(x\mu)o(y\mu) = (xy)\mu$, $(x\mu)^{-1} = x^{-1}\mu$.*

Proof. First let $n = 1$ so

$$[x\mu, y\mu] = (x^{-1}\mu)o(y^{-1}\mu)o(x\mu)o(y\mu) = [x, y]\mu.$$

And, by hypotheses, $[x\mu, y\mu] = e\mu$. Therefore, G/μ is an 1-Engel group.

Now, by induction on n, we have

$$\begin{aligned}
[x\mu,_n y\mu] &= [[x\mu,_{n-1} y\mu], y\mu] \\
&= ([x\mu,_{n-1} y\mu]^{-1})o(y^{-1}\mu)o([x\mu,_{n-1} y\mu])o(y\mu) \\
&= ([x,_{n-1} y]^{-1}\mu)o(y^{-1}\mu)o[(x,_{n-1} y]\mu)o(y\mu) \\
&= [[x,_{n-1} y], y]\mu \\
&= [x,_n y]\mu \\
&\Longrightarrow [x\mu,_n y\mu] = e\mu
\end{aligned}$$

This completes the proof. ∎

Theorem 2.12. *Let μ be a normal fuzzy subgroup. Then μ is an n-Engel fuzzy subgroup if and only if G/μ is an n-Engel group.*

Proof. By (1.3.11) [9], $\mu(x) = \mu(e)$ if $x\mu = e\mu$ for a normal fuzzy subgroup μ. Also, if $\mu(x) = \mu(e)$, then, for all $z \in G$, we have

$$(x\mu)z = \mu(x^{-1}z) \geq \min\{\mu(x), \mu(z)\} = \mu(z) = (e\mu)z.$$

Therefore, $x\mu = e\mu$. If G/μ is n-Engel, then using the method of the last theorem

$$[x\mu,_n y\mu] = [x,_n y]\mu = e\mu$$

if and only if $\mu([x,_n y]) = \mu(e)$. ∎

Theorem 2.13. *Let $\dfrac{G}{\mu}$ be a nilpotent group of class n. Then μ is nilpotent of class n.*

Proof. Since $\dfrac{G}{\mu}$ is nilpotent of class n, then

$$\frac{G}{\mu} = \{x\mu; [x\mu, y_1\mu, ..., y_n\mu] = e\mu \text{ for all } y_1, ..., y_n \in G\}.$$

Let x be an arbitrary element of G. Then $x\mu \in \dfrac{G}{\mu}$ implies that, for all $y_i\mu$, $[x\mu, y_1\mu, ..., y_n\mu] = e\mu$. Consequently, $[x, y_1, ..., y_n]\mu = e\mu$. Thus $[x, y_1, ..., y_{n-1}] \in Z(\mu)$. By Theorem 1.8 we have , $[x, y_1, ..., y_{n-2}] \in Z^2(\mu)$. By a similar method, $[x, y_1] \in Z^{n-1}(\mu)$. Thus $x \in Z^n(\mu)$. Consequently, $Z^n(\mu) = G$. This completes the result. ∎

Theorem 2.14. *Let μ be a normal finite Engel fuzzy subgroup. Then μ is nilpotent.*

Proof. By 2.12, $\dfrac{G}{\mu}$ is an Engel group. Now, Zorn's theorem implies that $\dfrac{G}{\mu}$ is nilpotent of class, say n. The result follows by the previous theorem. ∎

Theorem 2.15. *Let μ be a normal fuzzy subgroup. Then $\eta = \mu\mid_{y^G}$ is commutative for all y if and only if μ is a 2-Engel fuzzy subgroup.*

Proof. By hypotheses, $Z(\mu\mid_{y^G}) = y^G$. So, by 1.9, $\mu\mid_{y^G}[t, s] = \mu\mid_{y^G}(e) = \mu(e)$ for all $s, t \in y^G$. Therefore,

$$\mu[[x, y], y] = \mu[y^{-x}y, y] = \mu\mid_{y^G}([y^{-x}y, y]) = \mu\mid_{y^G}(e) = \mu(e).$$

Conversely, let μ be a 2-Engel fuzzy subgroup so

$$\begin{aligned}\mu\mid_{x^G}([x, x^y]) &= \mu([x, x^y]) = \mu([x, x[x, y]]) \\ &= \mu([x, [x, y]) \\ &= \mu(e).\end{aligned}$$

Also

$$\begin{aligned}\mu\mid_{x^G}([x^y x^z, x^s]) &= \mu\mid_{x^G}([x^y, x^s]^{x^z}[x^z, x^s]) \\ &\geq \mu\mid_{x^G}([x, x^{sy^{-1}}]^{yx^z}) \wedge \mu\mid_{x^G}([x, x^{sz^{-1}}]^z) \\ &= \mu(e).\end{aligned}$$

This completes the proof. ∎

Theorem 2.16. *Let μ be a normal 2-Engel fuzzy subgroup and x, y, z, t, be elements of G. Then the followings are equivalent:*

(1) *μ is 2-Engel,*

(2) *$\mu \mid x^G$ is commutative,*

(3) *$\mu[x, y, z] = \mu[z, x, y]$.*

Proof. By the last theorem, it is enough to show that parts (2) and (3) are equivalent. Let $A = x^G$. If part (2) holds, then $\mu[a_1, a_2] = \mu(e)$ for all $a_1, a_2 \in A$. Now, since $\mu[x, y] \geq \mu(x)$, then

$$\mu[a, y, y^{-1}] = \mu[[a, y], y]^{-y^{-1}} \geq \mu[[a, y], y] \wedge \mu[[a, y], y] = \mu(e),$$

which implies that

$$\begin{aligned}\mu(e) = \mu([a, yz, z^{-1}y^{-1}])\\ &= \mu([a, z, y^{-1}][a, z, y^{-1}, [a, y]^z][[a, y][a, y, z], y^{-1}]\\ &\quad [a, z, z^{-1}][[a, y]^z, z^{-1}][[a, z, z^{-1}][[a, y]^z, z^{-1}], y^{-1}])\\ &= \mu([a, z, y^{-1}][a, z, y^{-1}, [a, y]^z][a, y, y^{-1}]^{[a,y,z]}[a, y, z, y^{-1}]\\ &\quad [a, z, z^{-1}][a, y, z^{-1}]^{[a,y,z]}[a, y, z, z^{-1}]).\end{aligned}$$

By part (2), we have $\mu[a, z, y^{-1}, [a, y]^z]$=$\mu(e)$=$\mu[a, y, z, y^{-1}]$. Also $e\mu$=$[a, y, y^{-1}]\mu$, $e\mu = [a, y, y]\mu$. Consequently, $\mu[a, y, z] = \mu[a, z, y_{-1}]$, which implies that $\mu[a, y, z] = \mu[a, z, y^{-1}]$. Thus $\mu[x, y, z] = \mu[x, z, y^{-1}] = \mu[x, z, y]^{-y^{-1}} = \mu[x, z, y][x, z, y, y^{-1}]$. Now, if $\mu[x, z, y] = \mu[x, z, y, y^{-1}]$ then $\mu[x, y, z] \geq \mu(e)$ which implies that $\mu[x, y, z] = \mu[x, z, y] = \mu(e)$. Also, if $\mu[x, z, y] \neq \mu[x, z, y, y^{-1}]$, then, by Theorem 1.7 and $\mu[a, y, y^{-1}] = \mu(e)$, we have $\mu[x, y, z] = \mu[x, z, y]$.

On the other hand, $\mu[[x, z]^{-1}, y] = \mu[[x, z, y]^{-[x,z]^{-1}}$. Therefore, since μ is normal, we have $\mu[[x, z]^{-1}, y] = \mu[x, z, y]$. Hence $\mu[x, y, z] = \mu[z, x, y]$.

If (3) is satisfied, then $\mu[x^b, x] = \mu([x, x]^{[x,b]}[x, b, x]) = \mu[x, b, x] = \mu[x, x, b] = \mu(e)$. Thus (2) holds. ∎

Theorem 2.17. *Let μ be a fuzzy subgroup of G. Then the following are equivalent:*

(1) *μ is a fuzzy 3-Engel subgroup,*

(2) *$\mu \mid_{x^G}$ is a 2-Engel fuzzy subgroup for all $x \in G$,*

(3) *for all $s, t \in x^G$, $[t, s] \in Z(\mu \mid_{x^G})$.*

Proof. (2) $\to$ (1) $\mu[x, y, y, y]$=$\mu[y^{-x}y, y, y]$=$\mu \mid_{y^G} ([y^{-x}y, y, y])$=$\mu \mid_{y^G} (e) = \mu(e)$.
(3) $\to$ (2) Since for all $t, s \in x^G$, $[t, s] \in Z(\mu \mid_{x^G})$, so $\mu \mid_{x^G} [[t, s], r] = \mu(e)$ for all $r \in x^G$. Hence, the result follows.
(1) $\to$ (3) Since μ is 3-Engel then $\frac{G}{\mu}$ is 3-Engel. Now, by Kappe's theorem, $(x\mu)^{\frac{G}{\mu}}$ is nilpotent of class 2. Thus, for all g_1, g_2, g_3 in G, we have:

$$\begin{aligned}[(x\mu)^{g_1\mu}, (x\mu)^{g_2\mu}, (x\mu)^{g_3\mu}] = e\mu &\implies [x^{g_1}, x^{g_2}, x^{g_3}]\mu = e\mu\\ &\implies \mu[x^{g_1}, x^{g_2}, x^{g_3}] = \mu(e)\\ &\implies [x^{g_1}, x^{g_2}] \in Z(\mu \mid_{x^G}).\end{aligned}$$ ∎

3. Right and left Engel fuzzy subgroups

In this section, we will define right and left fuzzy Engel elements. Also we get some results which are similar to theorems of right and left Engel elements.

Definition 3.1. Let $\mu : G \longrightarrow [0, 1]$ be a fuzzy subgroup. Then we call $x \in G$ a *right fuzzy n-Engel element* if $\mu[x,_n y] = \mu(e)$ for all $y \in G$. The set of all right(left) fuzzy n-Engel elements is called a *right(left) fuzzy n-Engel set*. We denote the set of all right(left) fuzzy n-Engel elements by $R_n(\mu)$, $(L_n(\mu))$.

Theorem 3.2. *Let μ be a normal fuzzy subgroup. Then*

$$L_2(\mu) = \{x \in G, \mu\,|_{x^G} \ \textit{ is commutative}\}.$$

Proof. Let $a, b \in G$. Then

$$\begin{aligned}
x \in L_2(\mu) &\iff \mu[ab^{-1}, x, x] = \mu(e) \\
&\iff \mu[x^{-ab^{-1}}x, x] = \mu(e) \\
&\iff \mu([x^{-ab^{-1}}, x]^x [x, x]) = \mu(e) \\
&\iff \mu([x^{-ab^{-1}}, x]^x) = \mu(e) \\
&\iff \mu([x^{ab^{-1}}, x]^{-ab^{-1}}) = \mu(e) \\
&\iff \mu([x^{ab^{-1}}, x]) = \mu(e) \\
&\iff \mu([x^a, x^b]^{b^{-1}}) = \mu(e) \\
&\iff \mu([x^a, x^b]) = \mu(e).
\end{aligned}$$

By Theorem 2.4 $\mu\,|_{x^G}$ is commutative. This completes the proof. ∎

Corollary 3.3. *Let μ be a normal fuzzy subgroup. Then*

$$L_2(\mu) = \{x \in G, \mu\,|_{x^G} \ \textit{ is nilpotent}\}.$$

Proof. It is clear by Theorem 3.2 and the definition of commutative and nilpotent fuzzy subgroups.

Theorem 3.4. *Let μ be a normal fuzzy subgroup of G, $a, g \in G$ and $a \in R_n(\mu)$. Then $a^g \in R_n(\mu)$.*

Proof. $\mu[a,_n x] = \mu(e)$ for all $x \in G$. Thus

$$\begin{aligned}
\mu[a^g,_n x] &= \mu([[a^g, x],_{n-1} x]) \\
&= \mu([[a, x^{g^{-1}}]^g,_{n-1} x]) \\
&= \mu([a, x^{g^{-1}},_{n-1} x^{g^{-1}}]^g) \\
&= \mu([a,_n x^{g^{-1}}]) = \mu(e).
\end{aligned}$$

This completes the proof. ∎

Theorem 3.5. *Let μ be a normal fuzzy subgroup. Then $(R_n(\mu))^{-1} \subseteq L_{n+1}(\mu)$.*

Proof.

$$\begin{aligned}
\mu[x,_{n+1} g] &= \mu([[x, g],_n g]) \\
&= \mu([x^{-1}g^{-1}xg,_n g]) \\
&= \mu([(g^{-1})^x g,_n g]) \\
&= \mu([[(g^{-1})^x g, g],_{n-1} g]) \\
&= \mu([[(g^{-1})^x, g]^g,_{n-1} g]) \\
&= \mu([(g^{-1})^x,_n g]^g) \\
&= \mu([(g^{-1})^x,_n g]).
\end{aligned}$$

If $g \in (R_n(\mu))^{-1}$, then $g^{-1} \in (R_n(\mu))$. Therefore, by Theorem 3.4, $(g^{-1})^x \in (R_n(\mu))$. Hence $\mu[x,_{n+1} g] = \mu[(g^{-1})^x,_n g] = \mu(e)$. This completes the proof. ∎

Theorem 3.6. *Let μ be a fuzzy subgroup of G. Then $G = R(\mu)$ if and only if $G = L(\mu)$, where $R(\mu)$ $(L(\mu))$ is the set of all right (left) Engel fuzzy elements.*

Proof. Let $g \in G = R(\mu)$. Then $g^{-1} \in G = R(\mu)$. By Theorem 3.6 $g \in L(\mu)$. Conversely, let $G = L(\mu)$. Then, for all $x \in G = L(\mu)$ is a left Engel fuzzy subgroup. Thus, $\forall x$, $\forall g$, $\mu[g,_n x] = \mu(e)$. Thus for all g, g is a right Engel fuzzy element. Now by Theorem 3.5 for all g, g^{-1} is a left Engel fuzzy element. Thus, for all $g \in G$, we have $\mu[x,_{n+1} g^{-1}] = \mu(e)$, $\forall x$, which implies that x is a right Engel fuzzy element. Therefore, $G = L(\mu) \subseteq R(\mu)$. Consequently, $G = R(\mu)$. ∎

Remark 3.7. If μ is an n-Engel fuzzy subgroup of G, then every element of G is both left and right n-Engel fuzzy element.

Theorem 3.8. *Suppose μ be a normal fuzzy subgroup. Then $x \in L_3(\mu)$ if and only if $\mu|_{<x,x^y>}$ is nilpotent of class at most 2, $x, y \in G$.*

Proof. Since $[y^\epsilon,_3 x] = [x^{-1}, [x^{-1}, [x^{-1}, y^\epsilon]]]^{x^3}$, where $\epsilon \in \{-1, 1\}$, we have

$$\begin{aligned} x \in L_3(\mu) &\iff \mu([y,_3 x]) = \mu([y^{-1},_3 x]) = \mu(e) \\ &\iff \mu([x^{-1}, [x^{-1}, [x^{-1}, y]]]) = \mu([x^{-1}, [x^{-1}, [x^{-1}, y^{-1}]]]) = \mu(e). \end{aligned}$$

Therefore,

$$\begin{aligned} \mu([x^{-x^{-y}}, x^{-1}]) &= \mu([(x^{-1})^{x^{-y}}, x^{-1}]) \\ &= \mu([x^{-1}[x^{-1}, x^{-y}], x^{-1}]) \\ &= \mu([[x^{-1}, x^{-1}]^{[x^{-1}, x^{-y}]}[[x^{-1}, x^{-y}], x^{-1}]]) \\ &= \mu([[x^{-1}, x^{-1}[x^{-1}, y]], x^{-1}]) \\ &= \mu([[[x^{-1}, [x^{-1}, y]][x^{-1}, x^{-1}]^{[x^{-1}, y]}, x^{-1}]) \\ &= \mu([x^{-1}, [x^{-1}, x^{-1}, y]]^{-1}) \\ &= \mu([x^{-1}, x^{-1}, x^{-1}, y]) \\ &= \mu(e). \end{aligned}$$

Thus

$$\mu(e) = \mu([x^{-x^{-y}}, x^{-1}]) = \mu([(x^{x^{-y}})^{-1}, x^{-1}]) = \mu([(x^{x^{-y}}), x^{-1}]^{-x^{-x^{-y}}}) = \mu([(x^{x^{-y}}), x^{-1}]).$$

On the other hand

$$\begin{aligned} \mu([[x^{-y}, x^{-1}], x^{-1}]) &= \mu([x^y x x^{-y} x^{-1}, x^{-1}]) \\ &= \mu([(x^{x^{-y}})x^{-1}, x^{-1}]) \\ &= \mu([(x^{x^{-y}}), x^{-1}]^{x^{-1}}[x^{-1}, x^{-1}]). \end{aligned}$$

Since μ is normal and $\mu([(x^{x^{-y}}), x^{-1}]) = \mu(e)$, therefore,

$$\mu([[x^{-y}, x^{-1}], x^{-1}]) = \mu(e). \tag{I}$$

Also

$$\begin{aligned}
\mu([x^{-1},[x^{-1},[x^{-1},y^{-1}]]])=\mu(e) &\Longrightarrow \mu([[x^{-1},[x^{-1},y^{-1}]],x^{-1}]^{-1})=\mu(e)\\
&\Longrightarrow \mu([[x^{-1},[x^{-1},y^{-1}]],x^{-1}])=\mu(e)\\
&\Longrightarrow \mu([x^{-x^{-y^{-1}}},x^{-1}])=\mu(e)\\
&\Longrightarrow \mu([x^{-1}[x^{-1},x^{-y^{-1}}],x^{-1}])=\mu(e)\\
&\Longrightarrow \mu([x^{-1},x^{-y^{-1}},x^{-1}])=\mu(e)\\
&\Longrightarrow \mu([[x^{-y^{-1}},x^{-1}]^{-1},x^{-1}])=\mu(e)\\
&\Longrightarrow \mu([[x^{-y^{-1}},x^{-1}],x^{-1}]^{-[x^{-y^{-1}},x^{-1}]^{-1}})=\mu(e)\\
&\Longrightarrow \mu([[x^{-y^{-1}},x^{-1}],x^{-1}])=\mu(e). \qquad \text{(II)}\\
&\Longrightarrow \mu([x^{-1},x^{-y},x^{-y}])=\mu(e). \qquad \text{(III)}
\end{aligned}$$

Now, we can show that $\mu|_{<x,x^y>}$ is nilpotent of class at most 2, since, by Theorem 1.8, if for all $z\in <x,x^y>$, $[x^{-1},z]\in Z(\mu|_{<x,x^y>})$, then $x^{-1}\in Z^2(\mu|_{<x,x^y>})$. Similarly, $x^y\in Z^2(\mu|_{<x,x^y>})$. Therefore, $Z^2(\mu|_{<x,x^y>})=<x,x^y>$. Hence $\mu|_{<x,x^y>}$ is nilpotent of class at most 2. ∎

Corollary 3.9. *Let μ be a normal fuzzy subgroup of G. Then $L_3(\mu)=\{x\in G,$ $\mu|_{<x,x^y>}$ is nilpotent of class at most 2 for all $y\in G\}$.*

Theorem 3.10. *Let μ be a normal fuzzy subgroup. Then $R_2(\mu)\subseteq L_2(\mu)$.*

Proof. Let $a\in R_2(\mu)$. Then, for all $x\in G$, we have

$$\begin{aligned}
\mu[a,ax,ax]=\mu(e) &\Longrightarrow \mu[[a,a]^x[a,x],ax]=\mu(e)\\
&\Longrightarrow \mu[[a,x],ax]=\mu(e)\\
&\Longrightarrow \mu([a,x,a]^x[a,x,x])=\mu(e).
\end{aligned}$$

i) If $\mu([a,x,a]^x)$=$\mu([a,x,x])$, then $\mu([a,x,x])$=$\mu(e)$ implies that $\mu([a,x,a]^x)$=$\mu(e)$. Then by hypotheses $\mu([a,x,a])$=$\mu(e)$. But $\mu(e)=\mu([a,x,a])=\mu([[x,a]^{-1},a])$ $=\mu([x,a,a]^{-[x,a]^{-1}})$. Since μ is normal, then $\mu(e)$=$\mu([x,a,a]^{-[x,a]^{-1}})$=$\mu([x,a,a]^{-1})$ $=\mu([x,a,a])$. Therefore, $a\in L_2(\mu)$.

ii) if $\mu([a,x,a]^x)\neq\mu([a,x,x])$, then by Theorem 1.7 $\mu([a,x,a]^x)$=$\mu([a,x,x])$=$\mu(e)$. Now, by the similar method of part (i), the result follows. ∎

Theorem 3.11. *Let μ be a normal fuzzy subgroup, $x\in L_3(\mu)$ and p be a prime number. If $\mu(x^{p^n})=\mu(e)$ for some integer $n\geq 2$, then $x^{p^{n-1}}\in L_2(\mu)$.*

Proof. Let y be an arbitrary element of G. By the proof of Theorem 3.8, $\mu([x_1,x_2,x_3])=\mu(e)$ for all $x_i\in\langle x,x^y\rangle$. Thus

$$\mu([(x^{-y})^{p^{n-1}},x^{p^{n-1}}])=\mu([(x^{-y})^{p^{n-2}}p,x^{p^{n-1}}])=\mu([(x^{-y})^{p^{n-2}},x^{p^{n-1}}]^pa), \qquad \text{(I)}$$

in which $\mu(a)=\mu(e)$.

i) If $\mu([(x^{-y})^{p^{n-2}}, x^{p^{n-1}}]^p) = \mu(a) = \mu(e)$, then

$$\mu([(x^{-y})^{p^{n-1}}, x^{p^{n-1}}]) \geq \mu([(x^{-y})^{p^{n-2}}, x^{p^{n-1}}]^p) \wedge \mu(a) = \mu(e).$$

Therefore, since $\mu([y, x^{p^{n-1}}, x^{p^{n-1}}]) = \mu([(x^{-y})^{p^{n-1}}, x^{p^{n-1}}])$, then $x^{p^{n-1}} \in L_2(\mu)$.
ii) If $\mu([(x^{-y})^{p^{n-2}}, x^{p^{n-1}}]^p) \neq \mu(a)$, then, by Theorem 1.7, we have

$$\mu([(x^{-y})^{p^{n-1}}, x^{p^{n-1}}]) = \mu([(x^{-y})^{p^{n-2}}, x^{p^{n-1}}]^p).$$

Similarly, $\mu([(x^{-y})^{p^{n-2}}, (x^{p^{n-1}})^p]) = \mu([(x^{-y})^{p^{n-2}}, x^{p^{n-1}}]^p b)$, in which $\mu(b) = \mu(e)$. If $\mu([(x^{-y})^{p^{n-2}}, x^{p^{n-1}}]^p) = \mu(b) = \mu(e)$ then $\mu([(x^{-y})^{p^{n-1}}, x^{p^{n-1}}]) = \mu(e)$. Also, if $\mu([(x^{-y})^{p^{n-2}}, x^{p^{n-1}}]^p) \neq \mu(b)$, then, by Theorem 1.7, we have

$$\mu([(x^{-y})^{p^{n-2}}, (x^{p^{n-1}})^p]) = \mu([(x^{-y})^{p^{n-2}}, (x^{p^{n-1}})]^p).$$

Thus $\mu([(x^{-y})^{p^{n-1}}, x^{p^{n-1}}]) = \mu([(x^{-y})^{p^{n-2}}, (x^{p^{n-1}})^p]) = \mu([(x^{-y})^{p^{n-2}}, x^{p^n}]) \geq \mu(x^{p^n}) = \mu(e)$. Hence $x^{p^{n-1}} \in L_2(\mu)$. ■

Definition 3.12. Let μ be a fuzzy subgroup of G.

$$1 \trianglelefteq H_0 ... \trianglelefteq H_n = H$$

has a fuzzy commutative factor, if $\mu| \frac{H_{i+1}}{H_i}$ is commutative, it means that $\mu[xH_i, yH_i] = \mu(H_i)$ for all $x, y \in H_{i+1}$, $0 \leq i \leq n$.

Definition 3.13. Let μ be a fuzzy subgroup of G. μ is called fuzzy soluble if there exist $H \subseteq G$ such that its normal series have fuzzy commutative factors.

Example 3.14. Each fuzzy subgroup of a soluble group is a fuzzy soluble.

Theorem 3.15. *Let μ, ν be two normal fuzzy subgroups of G, $\mu \subseteq \nu$ and $\mu(e) = \nu(e)$. If μ is a fuzzy soluble, then ν is.*

Proof. By hypotheses there is $H \subseteq G$ such that its normal series have fuzzy commutative factors with respect to μ. Now it is easy to see that this normal series have fuzzy commutative factors with respect to ν. ■

Theorem 3.16. *Let μ be a normal fuzzy subgroup and p be a prime number. If $x \in L_3(\mu)$ and $\mu(x^{p^n}) = \mu(e)$ for some integer $n \geq 2$, then μ is fuzzy soluble.*

Proof. By Theorem 3.11, $x^{p^{n-1}} \in L_2(\mu)$. Therefore, using Theorem 3.3, $\mu|_{(\langle x^{p^{n-1}} \rangle)^G}$ is commutative. Also $\mu \mid_{\frac{<x^{p^1}>^G}{<x^{p^2}>^G}}$ is commutative, since by the same manipulation of Theorem 3.15, for $a, b \in G$, we have $\mu([(x^a)^p < x^{p^2} >^G, (x^b)^p < x^{p^2} >^G]) = \mu([(x^a) < x^{p^2} >^G, (x^b)^p < x^{p^2} >^G]^p (a < x^{p^2} >^G))$, in which

$$\mu(a < x^{p^2} >^G) = \mu(e < x^{p^2} >^G). \qquad (*)$$

Now
1) if $\mu(a < x^{p^2} >^G) = \mu(e < x^{p^2} >^G) = [(x^a) < x^{p^2} >^G, (x^b)^p < x^{p^2} >^G]^p$, then

$\mu[(x^a)^p < x^{p^2} >^G, (x^b)^p < x^{p^2} >^G] = \mu(e < x^{p^2} >^G)$. Therefore, we get the result.
2) If $\mu([(x^a) < x^{p^2} >^G, (x^b)^p < x^{p^2} >^G]^p) \neq (a < x^{p^2} >^G)$, then we have

$$\mu([(x^a)^p < x^{p^2} >^G, (x^b)^p < x^{p^2} >^G]) = \mu([(x^a) < x^{p^2} >^G, (x^b)^p < x^{p^2} >^G]^p). \qquad (**)$$

Now, we have two cases:

(i) If $\mu([(x^a) < x^{p^2} >^G, (x^b)^p < x^{p^2} >^G]^p) = \mu(e < x^{p^2} >^G)$, then by $(**)$ we have $\mu([(x^a)^p < x^{p^2} >^G, (x^b)^p < x^{p^2} >^G]) = \mu(e < x^{p^2} >^G)$.

(ii) If $\mu[(x^a) < x^{p^2} >^G, (x^b)^p < x^{p^2} >^G]^p \neq \mu(e < x^{p^2} >^G)$, then by the same manipulation of Theorem 3.11, and (I) in the proof of Theorem 3.11, we have $\mu[(x^a) < x^{p^2} >^G, (x^b)^p < x^{p^2} >^G]^p = \mu[(x^a) < x^{p^2} >^G, (x^b)^p < x^{p^2} >^G]^p(b < x^{p^2} >^G) = \mu[(x^a) < x^{p^2} >^G, (x^b)^p < x^{p^2} >^G]^p$ in which $\mu(b < x^{p^2} >^G) = \mu(e < x^{p^2} >^G)$. Therefore, by $(**)$, we have $\mu[(x^a)^p < x^{p^2} >^G, (x^b)^p < x^{p^2} >^G] = \mu[(x^a) < x^{p^2} >^G, (x^b)^p < x^{p^2} >^G] = \mu[(x^a) < x^{p^2} >^G, e < x^{p^2} >^G] = e < x^{p^2} >^G$.

Similarly, $\mu \mid_{\frac{<x^{p^{n-i}}>^G}{<x^{p^{n-(i-1)}}>^G}}$ is commutative, which implies that $1 \trianglelefteq (\langle x^{p^{n-1}} \rangle)^G \trianglelefteq ... \trianglelefteq \langle x^p \rangle^G$ is a series of normal subgroups of G with fuzzy commutative factors. Thus μ is soluble. ■

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ON 2-ABSORBING PRIMARY AND WEAKLY 2-ABSORBING ELEMENTS IN MULTIPLICATIVE LATTICES

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Abstract. In this paper, we introduce the concept of 2-absorbing primary and weakly 2-absorbing primary elements which are generalizations of primary and weakly primary elements in multiplicative lattices. Let L be a multiplicative lattice. A proper element q of L is said to be a (weakly) 2-*absorbing primary element* of L if whenever $a, b, c \in L$ with $(0 \neq abc \leq q)$ $abc \leq q$ implies either $ab \leq q$ or $ac \leq \sqrt{q}$ or $bc \leq \sqrt{q}$. Some properties of 2-absorbing primary and weakly 2-absorbing primary elements are presented and relations among prime, primary, 2-absorbing, weakly 2-absorbing, 2-absorbing primary and weakly 2-absorbing primary elements are investigated. Furthermore, we determine 2-absorbing primary elements in some special lattices and give a new characterization for principal element domains in terms of 2-absorbing primary elements.

Keywords: prime element, primary element, 2-absorbing element, 2-absorbing primary element, multiplicative lattice.

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1. Introduction

The concept of 2-absorbing ideal in a commutative ring with identity, which is a generalization of prime ideal, was introduced by Badawi in [7] and studied in [8], [12], and [1]. Various generalizations of prime ideals are also studied in

[5], [11], [14] and [6]. As a generalization of primary ideals the concept of 2-absorbing primary ideals and weakly 2-absorbing primary ideals are introduced in [9] and [10]. Our aim is to extend the concept of 2-absorbing primary ideals of commutative rings to 2-absorbing primary elements of non modular multiplicative lattices and give a characterization for principal element domains in terms of 2-absorbing primary elements.

A *multiplicative lattice* is a complete lattice L with the least element 0 and compact greatest element 1, on which there is defined a commutative, associative, completely join distributive product for which 1 is a multiplicative identity. An element a of L is said to be compact if whenever $a \leq \bigvee_{\alpha \in I} a_\alpha$ implies $a \leq \bigvee_{\alpha \in I_0} a_\alpha$ for some finite subset I_0 of I. By a *C-lattice* we mean a (not necessarily modular) multiplicative lattice which is generated under joins by a multiplicatively closed subset C of compact elements. C-lattices can be localized. For any prime element p of L, L_p denotes the localization at $F = \{x \in C \mid x \nleq p\}$. For details on C-lattices and their localization theory, the reader is referred to [15] and [19]. We note that in a C-lattice, a finite product of compact elements is again compact. Throughout this paper, L denotes a C-lattice and the set of all compact elements of L is shown by L_*. An element $e \in L$ is said to be *principal* [13], if it satisfies the meet principal property (i) $a \wedge be = ((a : e) \wedge b)e$ and join principal property (ii) $(ae \vee b) : e = (b : e) \vee a$. A finite product of meet (join) principal elements of L is again meet (join) principal from [13, Lemma 3.3 and Lemma 3.4].

If every element of L is principal, then L is called a *principal element lattice*. For more information about principal element lattices, the reader is referred to [3], [16] and [17]. L is called a *totally ordered lattice*, if any two elements of L are comparable. L is said to be a Prüfer lattice if every compact element is principle.

An element $a \in L$ is said to be *proper* if $a < 1$. A proper element p of L (*weakly, [4]) prime* if $(0 \neq ab \leq p)$ $ab \leq p$ implies either $a \leq p$ or $b \leq p$. If 0 is prime, then L is said to be a *domain*. An element $m < 1$ in L is said to be *maximal* if $m < x \leq 1$ implies $x = 1$. It can be easily shown that maximal elements are prime. A maximal element m of L is said to be *simple*, if there is no element $a \in L$ such that $m^2 < a < m$. L is said to be *quasi-local* if it contains a unique maximal element. If $L = \{0, 1\}$, then L is called a *field*. An element $a \in L$ is said to be a *strong compact* element if both a and $a^\omega = \bigwedge_{n=1}^{\infty} a^n$ are compact elements of L. Strong compact elements have been studied in [16]. For $a \in L$, we define radical of a as $\sqrt{a} = \wedge\{p \in L : p \text{ is prime and } a \leq p\}$. Note that in a C-lattice L, $\sqrt{a} = \wedge\{p \in L : p \text{ is prime and } a \leq p\} = \vee\{x \in L_* \mid x^n \leq a \text{ for some } n \in \mathbb{Z}^+\}$. (See also Theorem 3.6 of [21]). A proper element q is said to be (weakly) *primary* if for every $a, b \in L$, $(0 \neq ab \leq q)$ $ab \leq q$ implies either $a \leq q$ or $b^n \leq q$ for some $n \in \mathbb{Z}^+$, [6]. If q is primary and if $\sqrt{q} = p$ is a prime element, then q is called a p-primary element. A principally generated C-lattice domain L is said to be a *Dedekind domain*, if every element of L is a finite product of prime elements of L.

Recall from [18] that a proper element q of L is called a (weakly) 2-*absorbing* element of L if whenever $a, b, c \in L$ with $(0 \neq abc \leq q)$ $abc \leq q$, then $ab \leq q$ or

$ac \leq q$ or $bc \leq q$. In this paper, we introduce the concepts of 2-absorbing primary and weakly 2-absorbing primary element which are generalizations of primary and weakly primary elements. A proper element q of L is said to be a (weakly) 2-*absorbing primary* element of L if whenever $a, b, c \in L$ with $(0 \neq abc \leq q)$ $abc \leq q$, then $ab \leq q$ or $ac \leq \sqrt{q}$ or $bc \leq \sqrt{q}$.

Among many results in this paper, it is shown (Theorem 2.4) that the radical of a 2-absorbing primary element of L is a 2-absorbing element of L. It is shown (Theorem 2.6) that if q_1 is a p_1-primary element of L for some prime element p_1 of L and q_2 is a p_2-primary element of L for some prime element p_2 of L, then q_1q_2 and $q_1 \wedge q_2$ are 2-absorbing primary elements of L. It is shown (Theorem 2.7) that if radical of q is primary, then q is a 2-absorbing primary element. 2-absorbing primary and weakly 2-absorbing primary elements of cartesian product of multiplicative lattices are presented (Theorem 2.20-2.24). A new characterization for principal element domains in terms of 2-absorbing primary elements is established (Theorem 3.30).

2. 2-absorbing primary and Weakly 2-absorbing primary elements

Definition 2.1

(1) A proper element q of L is called a 2-absorbing primary element of L if whenever $a, b, c \in L$ and $abc \leq q$, then $ab \leq q$ or $bc \leq \sqrt{q}$ or $ac \leq \sqrt{q}$.

(2) A proper element q of L is called a *weakly 2-absorbing primary* element of L if whenever $a, b, c \in L$ and $0 \neq abc \leq q$, then $ab \leq q$ or $ac \leq \sqrt{q}$ or $bc \leq \sqrt{q}$.

The following theorem is obvious from the definitions, so the proof is omitted.

Theorem 2.2 *Let q be a proper element of L. Then*

(1) *If q is a (weakly) prime element, then q is a (weakly) 2-absorbing primary element.*

(2) *If q is a (weakly) primary element, then q is a (weakly) 2-absorbing primary element.*

(3) *If q is a (weakly) 2-absorbing element, then q is a (weakly) 2-absorbing primary element.*

(4) *If q is a 2-absorbing primary element, then q is a weakly 2-absorbing primary element.*

It is known from [Theorem 1, [15]] that if L is a Prüfer lattice and p is a prime element of L, then p^n is p-primary element. Thus p^n is a 2-absorbing primary element of L for all $n > 0$.

Theorem 2.3

(1) *An element $q \in L$ is a 2-absorbing primary element if and only if for any $a, b, c \in L_*$, $abc \leq q$ implies either $ab \leq q$ or $bc \leq \sqrt{q}$ or $ac \leq \sqrt{q}$.*

(2) *An element $q \in L$ is a weakly 2-absorbing primary element if and only if for any $a, b, c \in L_*$, $0 \neq abc \leq q$ implies either $ab \leq q$ or $bc \leq \sqrt{q}$ or $ac \leq \sqrt{q}$.*

Proof. (1) Assume that for any $a, b, c \in L_*$, $abc \leq q$ implies either $ab \leq q$ or $bc \leq \sqrt{q}$ or $ac \leq \sqrt{q}$. Let $a, b, c \in L$, $abc \leq q$, $bc \nleq \sqrt{q}$ and $ac \nleq \sqrt{q}$. Then there exist compact elements $a' \leq a$, $b' \leq b$ and $c' \leq c$ such that $a'b'c' \leq q$. Since $ac \nleq \sqrt{q}$ and $bc \nleq \sqrt{q}$, there exist compact elements $a_1 \leq a$, $c_1 \leq c$, $c_2 \leq c$ and $b_1 \leq b$ such that $a_1c_1 \nleq \sqrt{q}$ and $b_1c_2 \nleq \sqrt{q}$. Put $c_3 = c_1 \vee c_2 \vee c'$, $a_2 = a_1 \vee a'$, $b_2 = b_1 \vee b'$. We show that $ab \leq q$. Choose compact elements $a_\alpha \leq a$ and $b_\alpha \leq b$. Then $(a_2 \vee a_\alpha)c_3(b_2 \vee b_\alpha) \leq q$, $(a_2 \vee a_\alpha)c_3 \nleq \sqrt{q}$, $c_3(b_2 \vee b_\alpha) \nleq \sqrt{q}$ and hence by the hypothesis, $(a_2 \vee a_\alpha)(b_2 \vee b_\alpha) \leq q$. So $a_\alpha b_\alpha \leq q$. Consequently, $ab \leq q$. Therefore q is a 2-absorbing element of L. The converse part is obvious.

(2) It can be easily shown similar to (1). ∎

Theorem 2.4 *If q is a 2-absorbing primary element of L, then $\sqrt{q}$ is a 2-absorbing element of L.*

Proof. Let $a, b, c \in L$ such that $abc \leq \sqrt{q}$, $ac \nleq \sqrt{q}$ and $bc \nleq \sqrt{q}$. Since $abc \leq \sqrt{q}$, there exists a positive integer n such that $(abc)^n = a^n b^n c^n \leq q$. We obtain $a^n c^n \nleq \sqrt{q}$ and $b^n c^n \nleq \sqrt{q}$. Since q is 2-absorbing primary, we conclude that $a^n b^n = (ab)^n \leq q$, and hence $ab \leq \sqrt{q}$. Thus $\sqrt{q}$ is a 2-absorbing element of L. ∎

Theorem 2.5 *Let q be a proper element of L. Then $\sqrt{q}$ is a (weakly) 2-absorbing element of L if and only if $\sqrt{q}$ is a (weakly) 2-absorbing primary element of L.*

Proof. Since $\sqrt{\sqrt{q}} = \sqrt{q}$, the proof is clear. ∎

Theorem 2.6 *If q is a 2-absorbing primary element of L, then one of the following statements must hold.*

(1) *$\sqrt{q} = p$ is a prime element,*

(2) *$\sqrt{q} = p_1 \wedge p_2$, where p_1 and p_2 are the only distinct prime elements of L that are minimal over q.*

Proof. Suppose that q is a 2-absorbing primary element of L. Then $\sqrt{q}$ is a 2-absorbing element by Theorem 2.4. Since $\sqrt{\sqrt{q}} = \sqrt{q}$, the claim follows from Theorem 3 in [18]. ∎

Let q be a proper element of L. It is known that if $\sqrt{q}$ is a maximal element of L, then q is a primary element of L. The following theorem states that it is sufficient that if $\sqrt{q}$ is a primary element of L, then q is a 2-absorbing primary element of L. Note that $\sqrt{q}$ is a (weakly) prime element of L if and only if $\sqrt{q}$ is a (weakly) primary element of L as $\sqrt{q} = \sqrt{\sqrt{q}}$.

Theorem 2.7 *Let q be a proper element of L.*

(1) *If $\sqrt{q}$ is a primary element of L, then q is a 2-absorbing primary element of L.*

(2) *If $\sqrt{q}$ is a weakly primary element of L, then q is a weakly 2-absorbing primary element of L.*

Proof. (1) Suppose that $abc \leq q$ for some $a, b, c \in L$ and $ab \nleq q$. Since $(ac)(bc) = abc^2 \leq q \leq \sqrt{q}$ and $\sqrt{q}$ is a primary element of L, we have $bc \leq \sqrt{q}$ or $ac \leq \sqrt{q}$. Hence q is a 2-absorbing primary element of L.

(2) Suppose that $0 \neq abc \leq q$ for some $a, b, c \in L$ and $ab \nleq q$. Suppose that $ab \nleq \sqrt{q}$. Since $\sqrt{q}$ is a weakly primary element of L, we have $c \leq \sqrt{q}$, and thus $ac \leq \sqrt{q}$. Suppose that $ab \leq \sqrt{q}$. Since $0 \neq abc \leq q$ and $ab \leq \sqrt{q}$, we have $0 \neq ab \in \sqrt{q}$. Since $\sqrt{q}$ is a weakly primary element of L and $0 \neq ab \leq \sqrt{q}$, we have $a \leq \sqrt{q}$ or $b \leq \sqrt{q}$. Thus $ac \leq \sqrt{q}$ or $bc \leq \sqrt{q}$. Thus q is a weakly 2-absorbing primary element of L. ∎

Definition 2.8 Let q be a 2-absorbing primary element of L. Then $p = \sqrt{q}$ is a 2-absorbing element by Theorem 2.2. We say that q is a p-2-absorbing primary element of L.

Theorem 2.9 *Let q_1 is a p_1-primary element of L and q_2 is a p_2-primary element of L for some prime elements p_1 and p_2 of L. Then the following statements hold.*

(1) *q_1q_2 is a 2-absorbing primary element of L.*

(2) *$q_1 \wedge q_2$ is a 2-absorbing primary element of L.*

Proof. (1). Suppose that $abc \leq q_1q_2$ for some $a, b, c \in L$, $ac \nleq \sqrt{q_1q_2}$, and $bc \nleq \sqrt{q_1q_2}$. Then $a, b, c \nleq \sqrt{q_1q_2}$. As $\sqrt{q_1q_2} = p_1 \wedge p_2$, $\sqrt{q_1q_2}$ is a 2-absorbing element of L by [18]. Since $ac, bc \nleq \sqrt{q_1q_2}$, we have $ab \leq \sqrt{q_1q_2}$. We show that $ab \leq q_1q_2$. Since $ab \leq \sqrt{q_1q_2} \leq p_1$, we may assume that $a \leq p_1$. Since $a \nleq \sqrt{q_1q_2} = p_1 \wedge p_2$ and $ab \leq \sqrt{q_1q_2} \leq p_2$, we conclude that $a \nleq p_2$ and $b \leq p_2$. Since $b \leq p_2$ and $b \nleq \sqrt{q_1q_2}$, we have $b \nleq p_1$. If $a \leq q_1$ and $b \leq q_2$, then $ab \leq q_1q_2$, so we are done. Thus assume that $a \nleq q_1$. Since q_1 is a p_1-primary element of L and $a \nleq q_1$, we have $bc \leq p_1$. Since $b \leq p_2$ and $bc \leq p_1$, we have $bc \leq \sqrt{q_1q_2}$, a contradiction. Thus $a \leq q_1$. Similarly, if $b \nleq q_2$, we conclude $ac \leq \sqrt{q_1q_2}$, which is again a contradiction. So $a \leq q_1$ and $b \leq q_2$ and thus $ab \leq q_1q_2$.

(2). Let $q = q_1 \wedge q_2$. Then $\sqrt{q} = p_1 \wedge p_2$ is a 2-absorbing element of L. Suppose that $abc \leq q$ for some $a, b, c \in L$, $ac \nleq \sqrt{q}$, and $bc \nleq \sqrt{q}$. Then $a, b, c \nleq \sqrt{q} = p_1 \wedge p_2$ and $ab \leq \sqrt{q} \leq p_1$. We show that $ab \leq q$. Since $ab \leq \sqrt{q} \leq p_1$, we may assume that $a \leq p_1$. Since $a \nleq \sqrt{q}$ and $ab \leq \sqrt{q} \leq p_2$, we conclude that $a \nleq p_2$ and $b \leq p_2$. Since $b \leq p_2$ and $b \nleq \sqrt{q}$, we get $b \nleq p_1$. If $a \leq q_1$ and $b \leq q_2$, then $ab \leq q$ and we are done. So suppose that $a \nleq q_1$. Since q_1 is a p_1-primary element of L and $a \nleq q_1$, we have $bc \leq p_1$. Since $b \leq p_2$ and $bc \leq p_1$, we have $bc \leq \sqrt{q}$, a contradiction. Hence we have $a \leq q_1$. By the similar argument, we conclude $a \leq q_1$ and $b \leq q_2$. Thus $ab \leq q$. ■

As a consequence of Theorem 2.9, we have the following corollary.

Corollary 2.10 *Let p_1, p_2 be prime elements of L. If p_1^n is a p_1-primary element of L and p_2^m is a p_2-primary element of L for some positive integers n, m, then $p_1^n p_2^m$ and $p_1^n \wedge p_2^m$ are 2-absorbing primary elements of L.*

Theorem 2.11 *Let $q_1, q_2, ..., q_n$ be p-2-absorbing primary elements of L for some 2-absorbing element p of L. Then $q = \bigwedge_{i=1}^{n} q_i$ is a p-2-absorbing primary element of L.*

Proof. Let $a, b, c \in L$ with $abc \leq q$. Suppose that $ab \nleq q$. Then $ab \nleq q_i$ for some $i \in \{1, 2, ..., n\}$. It implies either $bc \leq \sqrt{q_i} = p$ or $ac \leq \sqrt{q_i} = p$. Since $\sqrt{q} = \bigwedge_{i=1}^{n} \sqrt{q_i} = p$, we are done. ■

Definition 2.12 Let q be a weakly 2-absorbing primary element of L. We say (a, b, c) is a *triple-zero* of q if $abc = 0$, $ab \nleq q$, $bc \nleq \sqrt{q}$, and $ac \nleq \sqrt{q}$.

Note that if q is a weakly 2-absorbing primary element of L that is not 2-absorbing primary element, then there exists a triple-zero (a, b, c) of q for some $a, b, c \in L$.

Theorem 2.13 *Let q be a weakly 2-absorbing primary element of L and suppose that (a, b, c) is a triple-zero of q for some $a, b, c \in L$. Then*

(1) $abq = bcq = acq = 0$,

(2) $aq^2 = bq^2 = cq^2 = 0$.

Proof. (1) Suppose that $abq \neq 0$. Then there exists a compact element $x \leq q$ such that $abx \neq 0$. Hence $0 \neq ab(c \vee x) \leq q$. Since $ab \nleq q$ and q is weakly 2-absorbing primary, we have $a(c \vee x) \leq \sqrt{q}$ or $b(c \vee x) \leq \sqrt{q}$. So $ac \leq \sqrt{q}$ or $bc \leq \sqrt{q}$, a contradiction. Thus $abx = 0$, and so $abq = 0$. Similarly, it can be easily verified that $bcq = acq = 0$.

(2) Suppose that $aq_1q_2 \neq 0$ for some compact elements $q_1, q_2 \leq q$. Hence from (1) we have $0 \neq a(b \vee q_1)(c \vee q_2) = aq_1q_2 \leq q$. It implies either $a(b \vee q_1) \leq q$ or $a(c \vee q_2) \leq \sqrt{q}$ or $(b \vee q_1)(c \vee q_2) \leq \sqrt{q}$. Thus $ab \leq q$ or $ac \leq \sqrt{q}$ or $bc \leq \sqrt{q}$, a contradiction. Therefore $aq^2 = 0$. Similarly, one can easily show that $bq^2 = cq^2 = 0$. ■

Theorem 2.14 *If q is a weakly 2-absorbing primary element of L that is not 2-absorbing primary, then $q^3 = 0$.*

Proof. Suppose that q is a weakly 2-absorbing primary element that is not a 2-absorbing primary element of L. Then there exists (a, b, c) a triple-zero of q for some $a, b, c \in L$. Assume that $q^3 \neq 0$. Hence $q_1q_2q_3 \neq 0$, for some compact elements $q_1, q_2, q_3 \leq q$. By Theorem 2.13, we obtain $(a \vee q_1)(b \vee q_2)(c \vee q_3) = q_1q_2q_3 \neq 0$. This implies that $(a \vee q_1)(b \vee q_2) \leq q$ or $(a \vee q_1)(c \vee q_3) \leq \sqrt{q}$ or $(b \vee q_2)(c \vee q_3) \leq \sqrt{q}$. Thus we have $ab \leq q$ or $ac \leq \sqrt{q}$ or $bc \leq \sqrt{q}$, a contradiction. Thus $q^3 = 0$. ∎

Corollary 2.15 *If q is a weakly 2-absorbing primary element of L that is not 2-absorbing primary, then $\sqrt{q} = \sqrt{0}$.*

Theorem 2.16 *Let $q_1, q_2, ..., q_n$ be weakly 2-absorbing primary elements of L that are not 2-absorbing primary. Then $q = \bigwedge_{i=1}^{n} q_i$ is a weakly 2-absorbing primary element of L.*

Proof. Since q_i's are weakly 2-absorbing primary that are not 2-absorbing primary, we get $\sqrt{q_i} = \sqrt{0}$ for each $1 \leq i \leq n$ by Corollary 2.15. So the result is obtained easily similar to the argument in the proof of Theorem 2.11. ∎

Theorem 2.17 *Suppose that 0 has a triple-zero (a, b, c) for some $a, b, c \in L$ such that $ab \nleq \sqrt{0}$. Let q be a weakly 2-absorbing primary element of L. Then q is not a 2-absorbing primary element of L if and only if $q \leq \sqrt{0}$.*

Proof. Suppose that q is not a 2-absorbing primary element of L. Then $q \leq \sqrt{0}$ by Corollary 2.15. Conversely, suppose that $q \leq \sqrt{0}$. By hypothesis, we conclude that $ab \nleq q$, $ac \nleq \sqrt{0}$, and $bc \nleq \sqrt{0}$. Thus (a, b, c) is a triple-zero of q. Hence q is not a 2-absorbing primary element of L. ∎

Recall that L is said to be *reduced* if $\sqrt{0} = 0$.

Corollary 2.18 *Let L be a reduced lattice and $q \neq 0$ be a proper element of L. Then q is a weakly 2-absorbing primary element if and only if q is a 2-absorbing primary element of L.*

Theorem 2.19 *Let m be a maximal element of L and q be a proper element of L. If q is a 2-absorbing primary element of L, then q_m is a 2-absorbing primary element of L_m.*

Proof. Let $a, b, c \in L_*$ such that $a_mb_mc_m \leq q_m$. Then $abc \leq q_m$, so $uabc \leq q$ for some $u \nleq m$. Hence we get either $uab \leq q$ or $bc \leq \sqrt{q}$ or $uac \leq \sqrt{q}$. Since $(\sqrt{q})_m = \sqrt{q_m}$ by [15], and $u_m = 1_m$, it follows either $a_mb_m \leq q_m$ or $b_mc_m \leq \sqrt{q_m}$ or $a_mc_m \leq \sqrt{q_m}$.It completes the proof. ∎

Recall that for any $a \in L$, $L/a = \{b \in L : a \leq b\}$ is a multiplicative lattice with multiplication $c \circ d = cd \vee a$. For more details, the reader is referred to [2].

Lemma 1 *Let a and q be proper elements of L with $a \leq q$. If q is a 2-absorbing primary element of L, then $\overline{q}$ is a weakly 2-absorbing primary element of L/a.*

Proof. The proof is clear. ∎

Theorem 2.20 *Let $L = L_1 \times L_2$, where L_1 and L_2 are C- lattices. Then a proper element q is a 2-absorbing primary element of L if and only if it has one of the following three forms.*

(1) *$q = (q_1, 1_{L_2})$ for some 2-absorbing primary element q_1 of L_1,*

(2) *$q = (1_{L_1}, q_2)$ for some 2-absorbing primary element q_2 of L_2,*

(3) *$q = (q_1, q_2)$ for some primary element q_1 of L_1 and some primary element q_2 of L_2.*

Proof. If $q = (q_1, 1_{L_2})$ for some 2-absorbing primary element q_1 of L_1 or $q = (1_{L_1}, q_2)$ for some 2-absorbing primary element q_2 of L_2, then it is clear that q is a 2-absorbing primary element of L. Hence assume that $q = (q_1, q_2)$ for some primary element q_1 of L_1 and some primary element q_2 of L_2. Then $q_1' = (q_1, 1_{L_2})$ and $q_2' = (1_{L_1}, q_2)$ are primary elements of L. Hence $q_1' \wedge q_2' = (q_1, q_2) = q$ is a 2-absorbing primary element of L by Theorem 2.9.

Conversely, suppose that q is a 2-absorbing primary element of L. Then $q = (q_1, q_2)$ for some element q_1 of L_1 and some element q_2 of L_2. Suppose that $q_2 = 1_{L_2}$. Since q is a proper element of L, $q_1 \neq 1_{L_1}$. Let $L' = L/\{0\} \times L_2$. Then $\overline{q} = (q_1, 1_{L_2})$ is a 2-absorbing primary element of L' by Lemma 1. Now, we show that q_1 is a 2-absorbing primary element of L_1. Let $abc \leq q_1$ for some $a, b, c \in L_1$. Hence $(a, 1_{L_2})(b, 1_{L_2})(c, 1_{L_2}) = (abc, 1_{L_2}) \leq \overline{q}$, which implies that $(a, 1_{L_2})(b, 1_{L_2}) \leq \overline{q}$ or $(b, 1_{L_2})(c, 1_{L_2}) \leq \sqrt{\overline{q}}$ or $(a, 1_{L_2})(c, 1_{L_2}) \leq \sqrt{\overline{q}}$. It means that either $ab \leq q_1$ or $bc \leq \sqrt{q_1}$ or $ac \leq \sqrt{q_1}$. Thus q_1 is a 2-absorbing primary element of L_1.

If $q_1 = 1_{L_1}$, then q_2 can be obtained as a 2-absorbing primary element of L_2 by the similar way. Hence assume that $q_1 \neq 1_{L_1}$ and $q_2 \neq 1_{L_2}$. Then $\sqrt{q} = (\sqrt{q_1}, \sqrt{q_2})$. On the contrary, suppose that q_1 is not a primary element of L_1. Then there are a, $b \in L_1$ such that $ab \leq q_1$ but neither $a \leq q_1$ nor $b \leq \sqrt{q_1}$. Let $x = (a, 1)$, $y = (1, 0)$, and $z = (b, 1)$. Then $xyz = (ab, 0) \leq q$ implies that either $xy = (a, 0) \leq q$ and $xz = (ab, 1) \leq \sqrt{q}$ and $yz = (b, 0) \leq \sqrt{q}$, a contradiction. Therefore q_1 is a primary element of L_1. Similarly it can be easily seen that q_2 is a primary element of L_2, as needed. ∎

Theorem 2.21 *Let L_1 and L_2 be C-lattices, q be a proper element of L_1, and $L = L_1 \times L_2$. Then the following statements are equivalent.*

(1) *$(q, 1_{L_2})$ is a weakly 2-absorbing primary element of L.*

(2) *$(q, 1_{L_2})$ is a 2-absorbing primary element of L.*

(3) *q is a 2-absorbing primary element of L_1.*

Proof. (1)⇒(2) Since $(q, 1_{L_2}) \not\leq \sqrt{0}$, we conclude that $(q, 1_{L_2})$ is a 2-absorbing primary element of L by Corollary 2.15.

(2)⇒(3) Suppose that q is not a 2-absorbing primary element of L_1. Then there exist $a, b, c \in L_1$ such that $abc \leq q$, but $ab \not\leq q$, $bc \not\leq \sqrt{q}$, and $ac \not\leq \sqrt{q}$. Since $(a, 1_{L_2})(b, 1_{L_2})(c, 1_{L_2}) \leq (q, 1_{L_2})$, we have $(a, 1_{L_2})(b, 1_{L_2}) = (ab, 1_{L_2}) \leq (q, 1_{L_2})$ or $(a, 1_{L_2})(c, 1_{L_2}) = (ac, 1_{L_2}) \leq \sqrt{(q, 1_{L_2})} = (\sqrt{q}, 1_{L_2})$ or $(b, 1_{L_2})(c, 1_{L_2}) = (bc, 1_{L_2}) \leq \sqrt{(q, 1_{L_2})} = (\sqrt{q}, 1_{L_2})$. It follows that $ab \leq q$ or $bc \leq \sqrt{q}$ or $ac \leq \sqrt{q}$, a contradiction. Thus q is a 2-absorbing primary element of L_1.

(3)⇒(1) Let q be a 2-absorbing primary element of L_1. Then it can be easily shown that $(q, 1_{L_2})$ is a 2-absorbing primary element of L, therefore (1) holds. ■

Theorem 2.22 *Let L_1 and L_2 be C-lattices, q_1, q_2 be nonzero elements of L_1 and L_2, respectively, and let $L = L_1 \times L_2$. If (q_1, q_2) is a proper element of L, then the following statements are equivalent.*

(1) *(q_1, q_2) is a weakly 2-absorbing primary element of L.*

(2) *$q_1 = 1_{L_1}$ and q_2 is a 2-absorbing primary element of L_1 or $q_2 = 1_{L_2}$ and q_1 is a 2-absorbing primary element of L_1 or q_1, q_2 are primary elements of L_1 and L_2, respectively.*

(3) *(q_1, q_2) is a 2-absorbing primary element of L.*

Proof. (1)⇒(2) Assume that (q_1, q_2) is a weakly 2-absorbing primary element of L. If $q_1 = 1_{L_1}$ $(q_2 = 1_{L_2})$, then q_2 is a 2-absorbing primary element of L_2 (q_1 is a 2-absorbing primary element of L_1) by Theorem 2.21. So we may assume that $q_1 \neq 1_{L_1}$ and $q_2 \neq 1_{L_2}$. Let $a, b \in L_2$ such that $ab \leq q_2$ and let $x \in L_*$ with $0 \neq x \leq q_1$. Then $0 \neq (x, 1)(1, a)(1, b) = (x, ab) \leq (q_1, q_2)$. Since q_1 is proper, $(1, a)(1, b) = (1, ab) \not\leq \sqrt{(q_1, q_2)}$. Hence we have $(x, 1)(1, a) = (x, a) \leq (q_1, q_2)$ or $(x, 1)(1, b) = (x, b) \leq \sqrt{(q_1, q_2)}$, and so $a \leq q_2$ or $b \leq \sqrt{q_2}$. Thus q_2 is a primary element of L_2. Similarly, it can be easily shown that q_1 is a primary element of L_1.

(2)⇒(3) The proof is clear by Theorem 2.20.

(3)⇒(1) It is clear. ■

Theorem 2.23 *Let L_1 and L_2 be C-lattices and $L = L_1 \times L_2$. Then a nonzero proper element q of L is a weakly 2-absorbing primary element of L that is not 2-absorbing primary if and only if one of the following conditions holds.*

(1) *$q = (q_1, q_2)$, where q_1 is a nonzero weakly primary element of L_1 that is not primary and $q_2 = 0$ is a primary element of L_2.*

(2) *$q = (q_1, q_2)$, where q_2 is a nonzero weakly primary element of L_2 that is not primary and $q_1 = 0$ is a primary element of L_1.*

Proof. Suppose that q is a nonzero weakly 2-absorbing primary element of L that is not 2-absorbing primary element. Then $q = (q_1, q_2)$ for some elements q_1, q_2 of L_1 and L_2 respectively. Assume that $q_1 \neq 0$ and $q_2 \neq 0$. Then q is a 2-absorbing primary element of L by Theorem 2.22, a contradiction. Therefore $q_1 = 0$ or $q_2 = 0$. Without loss of generality we may assume that $q_2 = 0$. We show that $q_2 = 0$ is a primary element of L_2. Let $a, b \in L_2$ such that $ab \leq q_2$, and let $x \in L_*$ such that $0 \neq x \leq q_1$. Since $0 \neq (x, 1)(1, a)(1, b) = (x, ab) \leq q$ and $(1, a)(1, b) = (1, ab) \nleq \sqrt{q}$, we obtain $(x, a) = (x, 1)(1, a) \leq q$ or $(x, b) = (x, 1)(1, b) \leq \sqrt{q}$, and so $a \leq q_2$ or $b \leq \sqrt{q_2}$. Thus $q_2 = 0$ is a primary element of L_2. Next, we show that q_1 is a weakly primary element of L_1. Let $0 \neq ab \leq q_1$, for some $a, b \in L_1$. Since $0 \neq (a, 1)(b, 1)(1, 0) \leq (q_1, 0)$ and $(ab, 1) \nleq (q_1, 0)$, we conclude $(a, 0) = (a, 1)(1, 0) \leq \sqrt{(q_1, 0)} = \sqrt{q}$ or $(b, 0) = (b, 1)(1, 0) \leq \sqrt{(q_1, 0)} = \sqrt{q}$. Thus $a \leq q_1$ or $b \leq \sqrt{q_1}$, and therefore q_1 is a weakly primary element of L_1. Now, we show that q_1 is not primary. Suppose that q_1 is a primary element of L_1. Since $q_2 = 0$ is a primary element of L_2, we conclude that $q = (q_1, q_2)$ is a 2-absorbing primary element of L by Theorem 2.20, a contradiction. Thus q_1 is a weakly primary element of L_1 that is not primary.

Conversely, suppose that (1) holds. Assume that $(0, 0) \neq (a, a')(b, b')(c, c') \leq q = (q_1, 0)$. Since $a'b'c' = 0$ and $(0, 0) \neq (a, a')(b, b')(c, c') \leq (q_1, 0)$, we conclude that $abc \neq 0$. Assume $(a, a')(b, b') \nleq q$. We consider three cases.

Case one: Suppose that $ab \nleq q_1$, but $a'b' = 0$. Since q_1 is a weakly primary element of L_1, we have $c \leq \sqrt{q_1}$. Since $q_2 = 0$ is a primary element of L_2, we have $a' = 0$ or $b' \leq \sqrt{q_2}$. Thus $(a, a')(c, c') \leq \sqrt{q}$ or $(b, b')(c, c') \leq \sqrt{q}$.

Case two: Suppose that $ab \nleq q_1$ and $a'b' \neq 0$. Then $(c, c') \leq (\sqrt{q_1}, \sqrt{0}) = \sqrt{q}$. Thus $(a, a')(c, c') \leq \sqrt{q}$ or $(b, b')(c, c') \leq \sqrt{q}$.

Case three: Suppose that $ab \leq q_1$, but $a'b' \neq 0$. Since $0 \neq ab \leq q_1$ and q_1 is a weakly primary element of L_1, we have $a \leq q_1$ or $b \leq \sqrt{q_1}$. Since $a'b' \neq 0$ and $q_2 = 0$ is a primary element of L_2, we have $c' \leq \sqrt{q_2}$. Thus $(a, a')(c, c') \leq \sqrt{q}$ or $(b, b')(c, c') \leq \sqrt{q}$. Hence q is a weakly 2-absorbing primary element of L. Since q_1 is not a primary element of L_1, q is not a 2-absorbing primary element of L by Theorem 2.22. ∎

Theorem 2.24 *Let $L = L_1 \times L_2 \times ... \times L_n$, where $2 < n < \infty$, and $L_1, L_2, ..., L_n$ are C-lattices and let q be a nonzero proper element of L. Then the following statements are equivalent.*

(1) *q is a weakly 2-absorbing primary element of L.*

(2) *q is a 2-absorbing primary element of L.*

(3) *Either $q = (q_t)_{t=1}^{n}$ such that for some $k \in \{1, 2, ..., n\}$, q_k is a 2-absorbing primary element of L_k, and $q_t = 1_{L_t}$ for every $t \in \{1, 2, ..., n\} \setminus \{k\}$ or $q = (q_t)_{t=1}^{n}$ such that for some $k, m \in \{1, 2, ..., n\}$, q_k is a primary element of L_k, q_m is a primary element of L_m, and $q_t = 1_{L_t}$ for every $t \in \{1, 2, ..., n\} \setminus \{k, m\}$.*

Proof. (1)$\Leftrightarrow$ (2) Since q is a proper element of L, we have $q = (q_1, \cdots, q_n)$, where every q_i's are element of L_i, and $q_j \neq 1_{L_j}$ for some $j \in \{1, ..., n\}$. Suppose that $q = (q_1, q_2, ..., q_n) \neq 0$ is a weakly 2-absorbing primary element of L. Then there is a compact element $0 \neq (a_1, a_2, ..., a_n) \leq q$. Hence $0 \neq (a_1, a_2, ..., a_n) = (a_1, 1, 1, ..., 1)(1, a_2, 1, ..., 1)...(1, 1, ..., a_n) \leq q$ implies there is a $j \in \{1, ..., n\}$ such that $b_j = 1_{L_j}$ and $(b_1, ..., b_n) \leq \sqrt{q} = (\sqrt{q_1}, ..., \sqrt{q_n})$, where $b_1, ..., b_n \in \{a_1, ..., a_n\}$. Hence $\sqrt{q_j} = 1_{L_j}$, and so $q_j = 1_{L_j}$. Thus $\sqrt{q} \neq \sqrt{0}$, and hence by Corollary 2.15, q is a 2-absorbing primary element. The converse is obvious.

(2)$\Leftrightarrow$ (3) We use induction on n. If $n = 2$, then we are done by Theorem 2.22. Hence let $3 \leq n < \infty$ and assume that the result is satisfied when $S = L_1 \times \cdots \times L_{n-1}$. Thus $L = S \times L_n$. Theorem 2.22 implies that q is a 2-absorbing primary element of L if and only if either $q = (s, 1_{L_n})$ for some 2-absorbing primary element s of S or $q = (1_s, t)$ for some 2-absorbing primary element t of L_n or $q = (s, t)$ for some primary element s of S and some primary element t of L_n. Since a proper element s of S is a primary element of S if and only if $s = (q_k)_{k=1}^{n-1}$ such that for some $k \in \{1, 2, ..., n-1\}$, we conclude that q_k is a primary element of L_k, and $q_t = 1_{L_t}$ for every $t \in \{1, 2, ..., n-1\} \setminus \{k\}$. So this completes the proof of the theorem. ∎

3. 2-absorbing primary elements in some special lattices

Theorem 3.25 *Suppose that $\sqrt{0}$ is a prime (primary) element of L. Let q be a proper element of L. Then q is a weakly 2-absorbing primary element of L if and only if q is a 2-absorbing primary element of L.*

Proof. Suppose that q is a weakly 2-absorbing primary element of L. Assume that $abc \leq q$ for some $a, b, c \in L$. If $0 \neq abc \leq q$, then $ab \leq q$ or $ac \leq \sqrt{q}$ or $bc \leq \sqrt{q}$. Hence assume that $abc = 0$ and $ab \nleq q$. Since $abc = 0 \leq \sqrt{0}$ and $\sqrt{0}$ is a prime element of L, we conclude that $a \leq \sqrt{0}$ or $b \leq \sqrt{0}$ or $c \leq \sqrt{0}$. Since $\sqrt{0} \leq \sqrt{q}$, we conclude that $ac \leq \sqrt{0} \leq \sqrt{q}$ or $bc \in \sqrt{0} \leq \sqrt{q}$. Thus q is a 2-absorbing primary element of L. The converse is clear. ∎

Recall that L is called *quasilocal* if it has exactly one maximal element.

Theorem 3.26 *Let L be a quasilocal lattice with maximal element $\sqrt{0}$. The following statements hold.*

(1) *Every element of L is a weakly 2-absorbing primary element of L.*

(2) *A proper element q of L is a weakly 2-absorbing primary element if and only if q is a 2-absorbing primary element.*

Proof. It is obvious by Theorem 3.25. ∎

Theorem 3.27 *Let L_1, L_2 and L_3 be C-lattices and let $L = L_1 \times L_2 \times L_3$. Then every proper element of L is a weakly 2-absorbing primary element of L if and only if L_1, L_2 and L_3 are fields.*

Proof. Suppose that every proper element of L is a weakly 2-absorbing primary element of L. Without loss of generality, we may assume that L_1 is not a field. Then there exists a nonzero proper element q of L_1. Thus $a = (q, 0, 0)$ is a weakly 2-absorbing primary element of L, which contradicts with Theorem 2.24.

Conversely, suppose that L_1, L_2, L_3 are fields. Then every nonzero proper element of L is a 2-absorbing element by Theorem 2.24. Since 0 is always weakly 2-absorbing primary, the proof is completed. ∎

Theorem 3.28 *Suppose that every proper element of L is a weakly 2-absorbing primary element. Then L has at most three incomparable prime elements.*

Proof. Assume that there are p_1, p_2, p_3 and p_4 incomparable prime elements of L. Let $q = p_1 \wedge p_2 \wedge p_3$. Hence $\sqrt{q} = \sqrt{p_1} \wedge \sqrt{p_2} \wedge \sqrt{p_3}$. Thus $\sqrt{q}$ is not a 2-absorbing element of L by Theorem 2.6. So q is not a 2-absorbing primary element of L by Theorem 2.2. Hence $q^3 = 0$ by Theorem 2.14. Thus $q^3 = p_1^3 p_2^3 p_3^3 = 0 < p_4$ implies that $p_1 < p_4$ or $p_2 < p_4$ or $p_3 < p_4$, a contradiction. Thus L has at most three incomparable prime elements. ∎

In view of Theorem 3.28, we have the following result.

Corollary 3.29 *Suppose that every proper element of L is a weakly 2-absorbing primary element. Then L has at most three maximal elements.*

Theorem 3.30 *Let L is a principally generated domain that is not a field. Then the following statements are equivalent.*

(1) *L is a principal element domain.*

(2) *Every maximal element is strong compact and a nonzero proper element q of L is a 2-absorbing primary element of L if and only if either $q = m^n$ for some maximal element m of L and some positive integer n or $q = m_1^n m_2^k$ for some maximal elements m_1, m_2 of L and some positive integers n, k.*

(3) *Every maximal element is strong compact and a nonzero proper element q of L is a 2-absorbing primary element of L if and only if either $q = p^n$ for some prime element p of L and some positive integer n or $q = p_1^n p_2^k$ for some prime elements p_1, p_2 of L and some positive integers n, k.*

Proof. (1) $\Rightarrow$ (2). Let L be a principal element domain. Then every maximal element is strong compact by [16, Theorem 2]. Suppose q is a nonzero 2-absorbing primary element of L that is not maximal. Then $q = m_1^{n_1} m_2^{n_2} . \cdots . m_k^{n_k}$ for some distinct maximal elements $m_1, ..., m_k$ of L and some integers $n_1, ..., n_k \geq 1$. Since every nonzero prime element of L is maximal and $\sqrt{q}$ is either a maximal element of L or $q_1 \wedge q_2$ for some maximal elements q_1, q_2 of L by Theorem 2.6, we conclude that either $q = m^n$ for some maximal element m of L and some $n \geq 1$ or $q = m_1^n m_2^k$ for some maximal elements m_1, m_2 of L and some $n, m \geq 1$. Conversely, suppose that $q = m^n$ for some maximal element m of L and some positive integer $n \geq 1$

or $q = m_1^n m_2^k$ for some maximal elements m_1, m_2 of L and some integers $n, k \geq 1$. Then q is a 2-absorbing primary element of L by Theorem 2.9 and Corollary 2.10.

(2)$\Rightarrow$(3) It is clear.

(3)$\Rightarrow$(1) Suppose that m is a maximal element of L and $q \in L$ with $m^2 \leq q \leq m$. Then q is an m-primary element. Hence q is a 2-absorbing primary element. From the hypothesis (3), either $q = m$ or $q = m^2$, so there is no element $a \in L$ such that $m^2 < a < m$ which shows that m is simple. Therefore, by [16, Theorem 2], L is a principal element domain. ∎

Suppose that L is principally generated. Then L is a Dedekind domain if and only if L is a principal element lattice by Theorem 2.7 in [3]. So we have the following result as a consequence of Theorem 3.30.

Corollary 3.31 *Let L be a principally generated domain. If L is a Dedekind domain, then $1_L \neq q \in L$ is 2-absorbing primary if and only if $q = p^n$ for some prime element p of L, a positive integer n or $q = p_1^n p_2^m$ for some prime elements p_1, p_2 of L, some positive integers n, m.*

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FUZZY IDEALS OF IMPLICATION GROUPOIDS

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Abstract. In this paper, we introduce the concept of fuzzy ideals in implication groupoids and investigate its properties.

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1. Introduction

In 50-ties, L. Henkin and T. Skolem introduced the notion of Hilbert algebra as an algebraic counterpart of intuitionistic logic. The structure of Hilbert algebras has been later studied by by D. Busneag [2] and Y.B. Jun [13]. It is well known that the filters of a Hilbert algebra forms a deductive system. Since there exist various modifications of the Hilbert algebra, we now cite the one given in [2]. Recall that a Hilbert algebra is an algebra $\mathcal{H} = (H, *, 1)$ of type $(2, 0)$ satisfying the following axioms.

$(H1)$ $x * (y * x) = 1$.
$(H2)$ $(x * (y * z)) * ((x * y) * (x * z)) = 1$.
$(H3)$ $x * y = 1$ and $y * x = 1$ imply $x = y$.

In [6], I. Chajda and R. Halas further studied the properties of ideals and congruences of Hilbert algebras. Later, I. Chajda and R. Halas [7] introduced the concept of implication groupoid as a generalization of the implication reduct of intuitionistic logic, i.e. a Hilbert algebra and studied some connections among ideals, deductive systems and congruence kernels whenever the implication groupoid is distributive. In [10], [11], [13], [12], W.A. Dudek, Y.B. Jun et al studied the concept of fuzzy ideal, fuzzy deductive systems in Hilbert algebras and discuss the relation between the fuzzy ideals and fuzzy deductive systems.

In this paper, we give a characterization theorem of fuzzy ideals of a distributive implication groupoid. We also consider to characterize the fuzzy ideals of a distributive implication groupoid in terms of their level ideals. Our results strengthen and enrich many known results in the literature concerning fuzzy ideals and fuzzy filters of implicative semigroups, for example, see[14], [9], [16], [15]. It is noted that some results given in this paper are extended results of implicative fuzzy ideals of a distributive implication groupoid recently given by Bandaru and Shum in [3].

2. Preliminaries

We first recall some definitions and basic results which were discussed in [9], [7], [4] for the development of the paper.

Definition 2.1. An algebra $(A, *, 1)$ of type (2,0) is called an Implication groupoid if it satisfies the following identities:
(1) $x * x = 1$
(2) $1 * x = x$ for all $x, y \in A$.

Example 2.2. Let $A = \{1, a, b\}$ in which $*$ is defined by

$*$	1	a	b
1	1	a	b
a	a	1	b
b	a	b	1

Then $(A, *, 1)$ is an implication groupoid.

Example 2.3. Let $A = \{1, a, b, c\}$ in which $*$ is defined by

$*$	1	a	b	c
1	1	a	b	c
a	1	1	b	b
b	1	a	1	a
c	1	a	b	1

Then $(A, *, 1)$ is an implication groupoid.

Definition 2.4. An Implication groupoid $(A, *, 1)$ of type (2,0) is called a distributive implication groupoid if it satisfies the following identity:

(LD) $$x * (y * z) = (x * y) * (x * z) \quad \text{(left distributivity)}$$

for all $x, y, z \in A$.

Example 2.5. Let $A = \{1, a, b, c, d\}$ in which $*$ is defined by

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	b	1
b	1	a	1	1	d
c	1	a	1	1	d
d	1	1	c	c	1

Then $(A, *, 1)$ is a distributive implication groupoid.

In every implication groupoid, one can introduce the so called induced relation $\leq$ by the setting

$$x \leq y \text{ if and only if } x * y = 1.$$

Lemma 2.6. *Let $(A, *, 1)$ be a distributive implication groupoid. Then A satisfies the identities*

$$x * 1 = 1 \textit{ and } x * (y * x) = 1$$

Moreover, the induced relation $\leq$ is a quasiorder on A and the following relationships are satisfied:

(i) $x \leq 1$
(ii) $x \leq y * x$
(iii) $x * ((x * y) * y) = 1$
(iv) $1 \leq x$ *implies* $x = 1$
(v) $y * z \leq (x * y) * (x * z)$
(vi) $x \leq y$ *implies* $y * z \leq x * z$
(vii) $x * (y * z) \leq y * (x * z)$
(viii) $x * y \leq (y * z) * (x * z)$

Definition 2.7. Let $\mathcal{A} = (A, *, 1)$ be an implication groupoid. A subset $I \subseteq A$ is called an ideal of $\mathcal{A}$ if

(I1) $1 \in I$
(I2) $x \in A, y \in I$ imply $x * y \in I$.
(I3) $x \in A, y_1, y_2 \in I$ imply $(y_2 * (y_1 * x)) * x \in I$

Remark 2.8. If I is an ideal of an implication groupoid $\mathcal{A} = (A, *, 1)$ and $a \in I$, $x \in A$, then $(a * x) * x \in I$.

Definition 2.9. Let $\mathcal{A} = (A, *, 1)$ be an implication groupoid. A subset $D \subseteq A$ is called a deductive system of $\mathcal{A}$ if

(D1) $1 \in D$
(D2) $x \in D$ and $x * y \in D$ imply $y \in D$.

Lemma 2.10. *Let $\mathcal{A}$ be an implication groupoid. Then every ideal of $\mathcal{A}$ is a deductive system of $\mathcal{A}$.*

It is noted that the converse of the above lemma does not hold in general.

Example 2.11. From Example 2.2, we can easily see that $\{1, a\}$ is its deductive system which is not an ideal since $b * a = b \notin \{1, a\}$.

Theorem 2.12. *A nonempty subset I of a distributive implication groupoid $\mathcal{A}$ is an ideal if and only if it is a deductive system of $\mathcal{A}$.*

Definition 2.13. Let X be a set. A fuzzy set in X is a function $\mu : X \longrightarrow [0, 1]$.

Definition 2.14. Let μ be a fuzzy set in a set X. For $\alpha \in [0, 1]$, the set $\mu_\alpha = \{x \in X \mid \mu(x) \geq \alpha\}$ is called a level subset of μ.

Definition 2.15. If μ is a fuzzy relation on a set X and ν is a fuzzy set in X, then μ is called a fuzzy relation on ν if

$$\mu(x, y) \leq \min\{\nu(x), \nu(y)\} \text{ for all } x, y \in X.$$

Definition 2.16. The Cartesian product of two fuzzy sets μ and ν in X is defined by

$$(\mu \times \nu)(x, y) = \min\{\nu(x), \nu(y)\} \text{ for all } x, y \in X.$$

Lemma 2.17. *Let μ and ν be fuzzy sets in a set X. Then*
(i) $\mu \times \nu$ *is a fuzzy relation on* X.
(ii) $(\mu \times \nu)_\alpha = \mu_\alpha \times \nu_\alpha$ *for all* $\alpha \in [0, 1]$.

Definition 2.18. Let ν be a fuzzy set in a set X. The strongest fuzzy relation on X is a fuzzy relation μ_ν defined by μ_ν defined by

$$\mu_\nu(x, y) = \min\{\nu(x), \nu(y)\} \text{ for all } x, y \in X.$$

Lemma 2.19. *For a given fuzzy set ν in a set X, let μ_ν be the strongest fuzzy relation on X. Then for $\alpha \in [0, 1]$, we have $(\mu_\nu)_\alpha = \nu_\alpha \times \nu_\alpha$.*

3. Fuzzy ideals

In this section we introduce the concept of fuzzy ideal in a distributive implication groupoid and study their properties.

In what follows, X is a distributive implication groupoid unless otherwise specified.

Definition 3.1. A fuzzy set μ in X is called a fuzzy ideal of X if it satisfies the following conditions:
(i) $\mu(1) \geq \mu(x)$
(ii) $\mu(y) \geq \min\{\mu(x), \mu(x * y)\}$, for all $x, y \in X$.

Example 3.2. Let $A = \{1, a, b, c, d\}$ in which $*$ is defined by

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	1	d
b	1	1	1	1	d
c	1	1	1	1	d
d	1	a	b	c	1

Then $(A, *, 1)$ is a distributive implication groupoid. Let $t_1, t_2 \in [0, 1]$ be such that $t_1 > t_2$. Define a mapping $\mu : X \longrightarrow [0, 1]$ by $\mu(1) = \mu(d) = t_1$ and $\mu(a) = \mu(b) = \mu(c) = t_2$. Then μ is a fuzzy ideal of X.

We now give a characterization theorem of fuzzy ideals of a distributive implication groupoid.

Theorem 3.3. *Let μ be a fuzzy set in a distributive implication groupoid X. Then μ is a fuzzy ideal of X if and only if for every $\alpha \in [0, 1]$, the level subset μ_α is an ideal of X, when $\mu_\alpha \neq \emptyset$.*

Proof. Let μ be a fuzzy ideal of X. Then $\mu(1) \geq \mu(x)$ for all $x \in X$.

In particular, $\mu(1) \geq \mu(x) \geq \alpha$ for every $x \in \mu_\alpha$. Hence $1 \in \mu_\alpha$.

Let $x, x * y \in \mu_\alpha$. Then $\mu(x) \geq \alpha$ and $\mu(x * y) \geq \alpha$ and hence $\mu(y) \geq \min\{\mu(x), \mu(x * y)\} \geq \alpha$. Therefore $y \in \mu_\alpha$. Hence μ_α is an ideal of X.

Conversely, assume that μ_α is an ideal of X for every $\alpha \in [0, 1]$ with $\mu_\alpha \neq \emptyset$. Let $x, y \in X$ and $\mu(x * y) = \alpha_1$ and $\mu(x) = \alpha_2$. Then $x * y \in \mu_{\alpha_1}$ and $x \in \mu_{\alpha_2}$. Without loss of generality, we may assume that $\alpha_1 \leq \alpha_2$. Then $\mu_{\alpha_2} \subseteq \mu_{\alpha_1}$ and so $x \in \mu_{\alpha_1}$. Since μ_{α_1} is a ideal of X, we have $y \in \mu_{\alpha_1}$. Hence, $\mu(y) \geq \alpha_1 = \min\{\mu(x * y), \mu(x)\}$.

Suppose $\mu(1) < \mu(x_0)$ for some $x_0 \in X$. Let $\alpha_0 = \frac{1}{2}(\mu(1) + \mu(x_0))$. Then $\mu(1) < \alpha_0$ and $0 \leq \alpha_0 < \mu(x_0) \leq 1$. Hence $x_0 \in \mu_{\alpha_0}$ and $\mu_{\alpha_0} \neq \emptyset$. Since μ_{α_0} is a ideal of X, we have $1 \in \mu_{\alpha_0}$ and so $\mu(1) \geq \alpha_0$. This is a contradiction and hence $\mu(1) \geq \mu(x)$ for all $x \in X$. Therefore, μ is a fuzzy ideal of X. ∎

Definition 3.4. Let μ be a fuzzy ideal of X. Then for each $\alpha \in [0, 1]$,the ideal μ_α of X, $\alpha \in [0, 1]$, is called a level ideals of μ, when $\mu_\alpha \neq \emptyset$.

Now, we give a crucial lemma concerning the level ideals of a distributive implication groupoid.

Lemma 3.5. *Any ideal of a distributive implication groupoid X can be realized as a level ideal of some fuzzy ideal of X.*

Proof. Let A be an ideal of X and $\mu : X \longrightarrow [0, 1]$ be a fuzzy set defined by

$$\mu(x) = \begin{cases} \alpha & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

where α is a fixed number in $(0, 1)$. Note that $1 \in A$, so that $\mu(1) = \alpha \geq \mu(x)$ for all $x \in A$. Let $x, y \in X$. Now, we verify condition (ii) of Definition 3.1.

If $x \in A$ and $x * y \in A$ then $y \in A$ and whence $\mu(y) = \mu(x) = \mu(x * y) = \alpha$. Hence, we have

$$\mu(y) \geq \min\{\mu(x), \mu(x * y)\}.$$

If $x \notin A$ and $x * y \notin A$ then $\mu(x) = \mu(x * y) = 0$. This shows that

$$\mu(y) \geq \min\{\mu(x), \mu(x * y)\}.$$

If exactly one of x and $x * y \in A$ then exactly one of $\mu(x)$ and $\mu(x * y)$ is equal to 0. Hence,

$$\mu(y) \geq \min\{\mu(x), \mu(x * y)\}$$

Therefore,

$$\mu(y) \geq \min\{\mu(x), \mu(x * y)\} \text{ for all } x, y \in X.$$

This proves that μ is fuzzy ideal of X and $\mu_\alpha = A$. ∎

In the following theorems, we consider the level ideals of a distributive implication groupoid X.

Theorem 3.6. *Let μ be a fuzzy ideal of a distributive implication groupoid X. Then two level ideals $\mu_{\alpha_1}, \mu_{\alpha_2}$ (with $\alpha_1 < \alpha_2$) of μ are equal if and only if there is no $x \in X$ such that $\alpha_1 \leq \mu(x) < \alpha_2$.*

Proof. Assume that $\mu_{\alpha_1} = \mu_{\alpha_2}$ for $\alpha_1 < \alpha_2$. If there exists $x \in X$ such that $\alpha_1 \leq \mu(x) < \alpha_2$ then μ_{α_2} is a proper subset of μ_{α_1}. This is impossible.

Conversely, suppose that there is no $x \in X$ such that $\alpha_1 \leq \mu(x) < \alpha_2$. Note that $\alpha_1 < \alpha_2$ implies $\mu_{\alpha_2} \subseteq \mu_{\alpha_1}$. If $x \in \mu_{\alpha_1}$, then $\mu(x) \geq \alpha_1$ and so $\mu(x) \geq \alpha_2$ because $\mu(x) \not< \alpha_2$. Hence $x \in \mu_{\alpha_2}$ which says that $\mu_{\alpha_1} \subseteq \mu_{\alpha_2}$. Thus $\mu_{\alpha_1} = \mu_{\alpha_2}$. This completes the proof. ∎

Let μ be a fuzzy set in X and denote the image of μ by $Im(\mu)$.

Theorem 3.7. *Let μ be a fuzzy ideal of a distributive implication groupoid X. If $Im(\mu) = \{\alpha_1, \alpha_2, ..., \alpha_n\}$, where $\alpha_1 < \alpha_2 < \cdots < \alpha_n$, then the family of ideals $\mu_{\alpha_i}(i = 1, 2, ..., n)$ constitutes all the level ideals of μ.*

Proof. Let $\alpha \in [0, 1]$ and $\alpha \notin Im(\mu)$. If $\alpha < \alpha_1$, then $\mu_{\alpha_1} \subseteq \mu_\alpha$. Since $\mu_{\alpha_1} = X$, we have $\mu_\alpha = X$ and $\mu_\alpha = \mu_{\alpha_1}$. If $\alpha_i < \alpha < \alpha_{i+1} (1 \leq i \leq n-1)$, then there is no $x \in X$ such that $\alpha \leq \mu(x) < \alpha_{i+1}$. Using Theorem 3.7, we obtain $\mu_\alpha = \mu_{\alpha_i+1}$. This shows that for any $\alpha \in [0, 1]$ with $\alpha \leq \mu(1)$, the level ideals μ_α is in $\{\mu_{\alpha_i} \mid 1 \leq i \leq n\}$. ∎

The following lemma is obvious and we omit the proof.

Lemma 3.8. *Let X be a distributive implication groupoid and μ a fuzzy ideal of X. If α and β belong to $Im(\mu)$ such that $\mu_\alpha = \mu_\beta$ then $\alpha = \beta$.*

Theorem 3.9. *Let μ and ν be two fuzzy ideals of a distributive implication groupoid X such that μ and ν have the finite images and have the identical family of level ideals. If $Im(\mu) = \{\alpha_1, \alpha_2, ..., \alpha_m\}$ and $Im(\nu) = \{\beta_1, \beta_2, ..., \beta_n\}$ where $\alpha_1 > \alpha_2 > \cdots > \alpha_m$ and $\beta_1 > \beta_2 > \cdots > \beta_n$ then*

(i) $m = n$,

(ii) $\mu_{\alpha_i} = \nu_{\beta_i}$ *for* $i = 1, 2, ..., m$,

(iii) *if $x \in X$ such that $\mu(x) = \alpha_i$ then $\nu(x) = \beta_i$ for $i = 1, 2, ..., m$.*

Proof. (i) By Theorem 3.7, we can say that the only level ideals of μ and ν are μ_{α_i} and ν_{β_i} respectively. Since μ and ν have the identical family of level ideals, it follows that $m = n$ and so (i) holds.

(ii) Again, by Theorem 3.7, we get that

$$\{\mu_{\alpha_1}, \mu_{\alpha_2}, ..., \mu_{\alpha_m}\} = \{\nu_{\beta_1}, \nu_{\beta_2}, ..., \nu_{beta_m}\},$$

and, by Theorem 3.6, we have

$$\mu_{\alpha_1} \subset \mu_{\alpha_2} \subset \cdots \subset \mu_{\alpha_m} = A \text{ and } \nu_{\beta_1} \subset \nu_{\beta_2} \subset \cdots \subset \nu_{\beta_m} = A.$$

Hence $\mu_{\alpha_i} = \beta_i$ for $i = 1, 2, ..., m$ and (ii) holds.

(iii) Let $x \in A$ be such that $\mu(x) = \alpha_i$ and let $\nu(x) = \beta_j$. Then $x \in \mu_{\alpha_i} = \nu_{\beta_i}$ and so $\nu(x) \geq \beta_i$. Hence $\beta_j \geq \beta_i$ which implies $\nu_{\beta_j} \subseteq \nu_{\beta_i}$. Since $x \in \nu_{\beta_j} = \mu_{\alpha_j}$, therefore $\alpha_i = \mu(x) \geq \alpha_j$. It follows that $\mu_{\alpha_i} \subseteq \mu_{\alpha_j}$. By (ii), $\nu_{\beta_i} = \mu_{\alpha_i}, \mu_{\alpha_j} = \nu_{\beta_j}$. Consequently $\nu_{\beta_i} = \nu_{\beta_j}$ and by Lemma 3.8 we have $\beta_i = \beta_j$. Thus $\nu(x) = \beta_i$. ■

The following theorem can be proved easily.

Theorem 3.10. *Let μ and ν be as in Theorem* 3.9. *Then $\mu = \nu$ if and only if $Im(\mu) = Im(\nu)$.*

Theorem 3.11. *Let X be a distributive implication groupoid and let μ be a fuzzy set in X with $Im(\mu) = \{\alpha_0, \alpha_1, ..., \alpha_k\}$ where $\alpha_0 > \alpha_1 > \cdots > \alpha_k$. Suppose that there exists a chain of ideals of $X : A_0 \subset A_1 \subset \cdots \subset A_k = A$ such that $\mu(\overline{A_n}) = \alpha_n$ where $\overline{A_n} = A_n - A_{n-1}, A_{-1} = \emptyset$ for $n = 0, 1, ..., k$. Then μ is a fuzzy ideal of X.*

Proof. Since $1 \in A_0$, we have $\mu(1) = \alpha_0 \geq \mu(x)$ for all $x \in A$. In order to prove that μ satisfies the condition (ii) of Definition 3.1, we divide into the following cases:

If x and y belong to the same $\overline{A_n}$, then $\mu(x) = \mu(y) = \alpha_n$ and so

$$\mu(y) \geq \min\{\mu(x), \mu(x * y)\}$$

Assume that $x \in \overline{A_i}$ and $y \in \overline{A_j}$ for every $i \neq j$. Without loss of generality, we may assume that $i < j$. Then $\mu(x) = \alpha_i > \alpha_j = \mu(y)$ and so

$$\min\{\mu(y), \mu(y * x)\} \leq \mu(y) < \mu(x).$$

Since $x \in \overline{A_i}$, we have $x \in A_i$. It follows that $x \in A_{j-1}$ as $i \leq j-1$. Now, we assert that $x * y \notin D_{j-1}$. In fact, if not, then $x * y \in A_{j-1}$ and $x \in A_{j-1}$ imply $y \in A_{j-1}$, which contradicts to $y \in \overline{A_j} = A_j - A_{j-1}$. Hence $\mu(x * y) \leq \alpha_j$ and so

$$\mu(y) \geq \min\{\mu(x), \mu(x * y)\}.$$

Summarizing the above results, we obtain that $\mu(y) \geq \min\{\mu(x), \mu(x * y)\}$ for all $x, y \in X$. Therefore, μ is a fuzzy ideal of X. ∎

Theorem 3.12. *Let μ be a fuzzy ideal of a distributive implication groupoid X. If $Im(\mu) = \{\alpha_0, \alpha_1, ..., \alpha_k\}$ with $\alpha_0 > \alpha_1 > \cdots > \alpha_k$, then $A_n = \mu_{\alpha_n}, n = 0, 1, ..., k$ are ideals of X and $\mu(\overline{A_n}) = \alpha_n, n = 0, 1, 2, ..., k$ where $\overline{A_n} = A_n - A_{n-1}$ and $A_{-1} = \emptyset$.*

Proof. By Theorem 3.7, $A_n = \mu_{\alpha_n} (n = 0, 1, ..., k)$ is an ideal of X. Clearly, $\mu(A_0) = \alpha_0$. Since $\mu(A_1) = \{\alpha_0, \alpha_1\}$ for $x \in \overline{A_1}$ we have $\mu(x) = \alpha_1$, namely $\mu(\overline{A_1}) = \alpha_1$. Repeating the above argument, we have $\mu(\overline{A_n}) = \alpha_n (0 \leq n \leq k)$. ∎

Theorem 3.13. *If μ is a fuzzy ideal of a distributive implication groupoid X, then the set $X_\mu = \{x \in X \mid \mu(x) = \mu(1)\}$ is an ideal of X.*

Proof. Clearly, $1 \in X_\mu$. Assume that $x \in X_\mu$ and $x * y \in X_\mu$. Then

$$\mu(x) = \mu(1) = \mu(x * y).$$

Since μ is a fuzzy ideal of X, we have

$$\mu(y) \geq \min\{\mu(x), \mu(x * y)\} = \mu(1).$$

Therefore, $\mu(y) = \mu(1)$. Hence $y \in X_\mu$. ∎

Using a given fuzzy ideal, we construct a new fuzzy ideal.

Let $\alpha \geq 0$ be a real number. If $m \in [0, 1], m^\alpha$ shall mean the positive root in case $\alpha < 1$. We define $\mu^\alpha : X \to [0, 1]$ by $\mu^\alpha(x) = (\mu(x))^\alpha$.

Finally, we conclude this section with the following theorem.

Theorem 3.14. *If μ is a fuzzy ideal of a distributive implication groupoid X, then μ^α is also a fuzzy ideal of X and $X_{\mu^\alpha} = X_\mu$.*

Proof. We have that $\mu^\alpha(1) = (\mu(1))^\alpha \geq (\mu(x))^\alpha = \mu^\alpha(x)$ for all $x \in X$. Let $x, y \in X$. We assert that $\mu^\alpha(y) \geq \min\{\mu^\alpha(x), \mu^\alpha(x * y)\}$. In fact, suppose that $\mu(x) \leq \mu(x * y)$. It follows from Definition 3.1(ii) that

$$\mu(y) \geq \mu(x).$$

Hence $\mu^\alpha(x) \leq \mu^\alpha(x * y)$ and $\mu^\alpha \leq \mu^\alpha(y)$ which imply that

$$\mu^\alpha(y) \geq \min\{\mu^\alpha(x), \mu^\alpha(x * y)\}.$$

The argument is similar if $\mu(x) \geq \mu(x * y)$. Finally,

$$\begin{aligned} X_{\mu^\alpha} &= \{x \in X \mid \mu^\alpha(x) = \mu^\alpha(1)\} \\ &= \{x \in X \mid (\mu(x))^\alpha = (\mu(1))^\alpha\} \\ &= \{x \in X \mid \mu(x) = \mu(1)\} \\ &= X_\mu \end{aligned}$$

■

4. Cartesian product of fuzzy ideals

Let $(X, *, 1)$ and $(Y, *, 1)$ be distributive implication groupoids. Define an operation $\rightarrow$ on $X \times Y$ by

$$(x, y) \rightarrow (s, t) = (x * s, y * t) \text{ for all } (x, y), (s, t) \in X \times Y.$$

Then we can easily verify that $(X \times Y, \rightarrow, (1, 1))$ is a distributive implication groupoid.

The following proposition can be proved easily.

Proposition 4.1. *Let A_1 and A_2 be ideals of distributive implication groupoids X and Y respectively. Then $A_1 \times A_2$ is a ideal of $X \times Y$.*

Proposition 4.2. *For a given fuzzy set ν in a distributive implication groupoid X, let μ_ν be the strongest fuzzy relation on X. If μ_ν is a fuzzy ideal of $X \times X$ then $\nu(x) \leq \nu(1)$ for all $x \in X$.*

Proof. Since μ_ν is a fuzzy ideal of $X \times X$, we have

$$\mu_\nu(x, y) \leq \mu_\nu(1, 1) \text{ for all } (x, y) \in X \times X.$$

Hence $\min\{\nu(x), \nu(y)\} \leq \min\{\nu(1), \nu(1)\}$ which implies that $\nu(x) \leq \nu(1)$ for all $x \in X$. ■

The following proposition follows from Lemma 2.19 and we omit the proof.

Proposition 4.3. *If ν is a fuzzy ideal of a distributive implication groupoid X then the level ideals of μ_ν are given by $(\mu_\nu)_\alpha = \nu_\alpha \times \nu_\alpha$ for all $\alpha \in [0, 1]$.*

Theorem 4.4. *Let μ and ν be fuzzy ideals of a distributive implication groupoid X. Then $\mu \times \nu$ is a fuzzy ideal of $X \times X$.*

Proof. For any $(x, y) \in X \times X$, we have

$$(\mu \times \nu)(1, 1) = \min\{\mu(1), \nu(1)\} \geq min\{\mu(x), \nu(y)\} = (\mu \times \nu)(x, y).$$

Now, let $(x, y), (r, s) \in X \times X$. Then

$$\begin{aligned}
&\min\{(\mu\times\nu)(x,y),(\mu,\nu)((x,y)\to(r,s))\}\\
&\qquad=\min\{(\mu\times\nu)(x,y),(\mu,\nu)((x*r,y*s))\}\\
&\qquad=\min\{\min\{\mu(x),\nu(y)\},min\{\mu(x*r),\nu(y*s)\}\}\\
&\qquad=\min\{\min\{\mu(x),\mu(x*r)\},min\{\nu(y),\nu(y*s)\}\}\\
&\qquad\le\min\{\mu(r),\nu(s)\}\\
&\qquad=(\mu\times\nu)(r,s).
\end{aligned}$$

■

Theorem 4.5. *Let μ and ν be fuzzy sets in a distributive implication groupoid X such that $\mu\times\nu$ is a fuzzy ideal of $X\times X$. Then*

(i) *either $\mu(1)\ge\mu(x)$ or $\nu(1)\ge\nu(x)$ for all $x\in X$.*
(ii) *if $\mu(1)\ge\mu(x)$ for all $x\in X$ then $\nu(1)\ge\mu(x)$ or $\nu(1)\ge\nu(x)$ for all $x\in X$.*
(iii) *if $\nu(1)\ge\nu(x)$ for all $x\in X$ then $\mu(1)\ge\nu(x)$ or $\mu(1)\ge\mu(x)$ for all $x\in X$.*
(iv) *either μ or ν is a fuzzy ideal of X.*

Proof. (i) If both μ and ν do not satisfy $\mu(1)\ge\mu(x)$ and $\nu(1)\ge\nu(x)$ for all $x\in X$ then there exist $x,y\in X$ such that $\mu(x)>\mu(1)$ and $\nu(y)>\nu(1)$. Then

$$(\mu\times\nu)(x,y)=\min\{\mu(x),\nu(y)\}>\min\{\mu(1),\nu(1)\}=(\mu\times\nu)(1,1)$$

which is contradiction. Hence (i) proved.

(ii) Again, we use reduction to absurdity. Let $x,y\in X$ be such that $\mu(x)>\nu(1)$ and $\nu(y)>\nu(1)$. Then

$$(\mu\times\nu)(1,1)=\min\{\mu(1),\nu(1)\}=\nu(1)$$

and

$$(\mu\times\nu)(x,y)=\min\{\mu(x),\nu(y)\}>\nu(1)=(\mu\times\nu)(1,1)$$

which is a contradiction. Hence (ii) is proved.

(iii) The proof is similar to (ii).

(iv) Since, by (i), either μ or ν satisfies Definition 3.1(i), without loss of generality we may assume that μ satisfies Definition 3.1(i). Using (ii), we have that either $\mu(x)\le\nu(1)$ or $\nu(x)\le\nu(1)$ for all $x\in X$.

If $\mu(x)\le\nu(1)$ for all $x\in X$ then

$$(\mu\times\nu)(x,1)=\min\{\mu(x),\nu(1)\}=\mu(x)\text{ for all }x\in X.$$

Let $(x,y),(r,s)\in X\times X$. Since $\mu\times\nu$ is a fuzzy ideal of $X\times X$ by Definition 3.1(ii) we have

$$\begin{aligned}
(\mu\times\nu)(r,s)&\ge\min\{(\mu\times\nu)(x,y),(\mu\times\nu)((x,y)\to(r,s))\}\\
&=\min\{(\mu\times\nu)(x,y),(\mu\times\nu)(x*r,y*s)\}.
\end{aligned}\tag{I}$$

If we take $y = s = 1$, then

$$\begin{aligned}\mu(r) &= (\mu \times \nu)(r, 1) \\ &\geq \min\{(\mu \times \nu)(x, 1), (\mu \times \nu)(x * r, 1 * 1)\} \\ &= \min\{(\mu \times \nu)(x, 1), (\mu \times \nu)(x * r, 1)\} \\ &= \min\{\min\{\mu(x), \nu(1)\}, \min\{\mu(x * r), \nu(1)\}\} \\ &= \min\{\mu(x), \mu(x * r)\}\end{aligned}$$

showing that μ satisfies Definition 3.1(ii). Hence μ is a fuzzy ideal of X.

Now, we consider the case $\nu(x) \leq \nu(1)$ for all $x \in X$. Suppose that $\mu(y) > \nu(1)$ for some $y \in X$. Then $\mu(1) \geq \mu(y) > \nu(1)$. Since $\nu(x) \leq \nu(1)$ for all $x \in X$, it follows that $\mu(1) > \nu(x)$ for all $x \in X$. Hence $(\mu \times \nu)(1, x) = \min\{\mu(1), \nu(x)\} = \nu(x)$ for all $x \in X$.

Taking $x = r = 1$ in (I), then

$$\begin{aligned}\nu(s) &= (\mu \times \nu)(1, s) \\ &\geq min\{(\mu \times \nu)(1, y), (\mu \times \nu)(1 * 1, y * s)\} \\ &= \min\{(\mu \times \nu)(1, y), (\mu \times \nu)(1, y * s)\} \\ &= \min\{\min\{\mu(1), \nu(y)\}, \min\{\mu(1), \nu(y * s)\}\} \\ &= \min\{\nu(y), \nu(y * s)\},\end{aligned}$$

which proves that ν satisfies Definition 3.1(ii). Hence ν is a fuzzy ideal of X. ∎

Now, we give an example to show that if $\mu \times \nu$ is a fuzzy ideal of $X \times X$ then μ and ν both need not be fuzzy ideals of X.

Example 4.6. Let X be a distributive implication groupoid with $|A| \geq 2$ and let $\alpha, \beta \in [0, 1]$ be such that $0 \leq \alpha \leq \beta < 1$. Define the fuzzy sets μ and $\nu : X \to [0, 1]$ by $\mu(x) = \alpha$ and

$$\nu(x) = \begin{cases} \beta, & \text{if } x = 1; \\ 1, & \text{if } x \neq 1. \end{cases}$$

for all $x \in X$, respectively. Then $(\mu \times \nu)(x, y) = min\{\mu(x), \nu(y)\} = \alpha$ for all $(x, y) \in X \times X$ that is $\mu \times \nu : X \times X \to [0, 1]$ is a constant function. Hence $\mu \times \nu$ is a fuzzy ideal of $X \times X$. Now μ is a fuzzy ideal of X but ν is not a fuzzy ideal of X because ν does not satisfy Definition 3.1(i).

In the following theorem, we characterize the fuzzy ideal of a distributive implication groupoid X.

Theorem 4.7. *Let ν be a fuzzy set in a distributive implication groupoid X and let μ_ν be the strongest fuzzy relation on X. Then ν is a fuzzy ideal of X if and only if μ_ν is a fuzzy ideal of $X \times X$.*

Proof. Assume that ν is a fuzzy ideal of X. We note from Definition 3.1(i) that for all $(x,y)\in X\times X$,

$$\mu_\nu(x,y)=\min\{\nu(x),\nu(y)\}\leq\min\{\nu(1),\nu(1)\}=\mu_\nu(1,1)$$

showing that μ_ν satisfies Definition 3.1(i). Let $(x,y),(r,s)\in X\times X$. Then

$$\begin{aligned}\min\{\mu_\nu(x,y),&\mu_\nu((x,y)\to(r,s))\}\\&=\min\{\mu_\nu(x,y),\mu_\nu((x*r,y*s))\}\\&=\min\{\min\{\nu(x),\nu(y)\},\min\{\nu(x*r),\nu(y*s)\}\}\\&=\min\{\min\{\nu(x),\nu(x*r)\},\min\{\nu(y),\nu(y*s)\}\}\\&\leq\min\{\nu(r),\nu(s)\}\\&=\mu_\nu(r,s)\end{aligned}$$

This proves that μ_ν satisfies Definition 3.1(ii). Hence μ_ν is a fuzzy ideal of $X\times X$.

Conversely, suppose that μ_ν is a fuzzy ideal of $X\times X$. Then

$$min\{\nu(x),\nu(y)\}=\mu_\nu(x,y)\leq\mu_\nu(1,1)=min\{\nu(1),\nu(1)\}=\nu(1),$$

for all $x,y\in X$. It follows that $\nu(x)\leq\nu(1)$ for all $x\in X$.

For any $(x,y),(r,s)\in X\times X$, we have

$$\begin{aligned}\min\{\nu(r),\nu(s)\}&=\mu_\nu(r,s)\\&\geq\min\{\mu_\nu(x,y),\mu_\nu((x,y)\to(r,s))\}\\&=\min\{\mu_\nu(x,y),\mu_\nu((x*r,y*s))\}\\&=\min\{\min\{\nu(x),\nu(y)\},\min\{\nu(x*r),\nu(y*s)\}\}\\&=\min\{\min\{\nu(x),\nu(x*r)\},\min\{\nu(y),\nu(y*s)\}\}.\end{aligned}$$

In particular, if we take $y=s=1$ (resp. $x=r=1$), then

$$\nu(r)\geq\min\{\nu(x),\nu(x*r)\}(\text{ resp. }\nu(s)\geq\min\{\nu(y),\nu(y*s)\}).$$ ■

Definition 4.8. Let $(X,*,1)$ and $(Y,\Delta,1')$ be two distributive implication groupoids. Then a mapping $f:X\to Y$ is called a homomorphism if $f(x*y)=f(x)\Delta f(y)$ for all $x,y\in X$.

Note that if $f:X\to Y$ is homomorphism of distributive implication groupoids, then $f(1)=1'$.

Definition 4.9. Let $f:X\to Y$ be a mapping of distributive implication groupoids and μ be a fuzzy set of Y. The map μ^f is the pre-image of μ under f, if $\mu^f(x)=\mu(f(x))$ for all $x\in X$.

Theorem 4.10. *Let $f:X\to Y$ be a homomorphism of distributive implication groupoids. If μ is a fuzzy ideal of Y then μ^f is a fuzzy ideal of X.*

Proof. For any $x \in X$, we have

$$\mu^f(x) = \mu(f(x)) \leq \mu(1') = \mu(f(1)) = \mu^f(1).$$

Let $x, y \in X$. Then

$$\begin{aligned}\min\{\mu^f(x*y), \mu^f(x)\} &= \min\{\mu(f(x*y)), \mu(f(x))\}\\ &= \min\{\mu(f(x)*f(y)), \mu(f(x))\}\\ &\leq \mu(f(y)) = \mu^f(y).\end{aligned}$$

Hence μ^f is a fuzzy ideal of X. ∎

We conclude this paper with the following theorem.

Theorem 4.11. *Let $f : X \to Y$ be an onto homomorphism of distributive implication groupoids. If μ^f is fuzzy ideal of X, then μ is a fuzzy ideal of Y.*

Proof. Let $y \in Y$. Then there exists $x \in X$ such that $f(x) = y$. Then

$$\mu(y) = \mu(f(x)) = \mu^f(x) \leq \mu^f(1) = \mu(f(1)) = \mu(1').$$

Let $x, y \in Y$. Then there exist $a, b \in X$ such that $f(a) = x$ and $f(b) = y$. It follows that

$$\begin{aligned}\mu(y) &= \mu(f(b))\\ &= \mu^f(b)\\ &\geq \min\{\mu^f(a*b), \mu^f(a)\}\\ &= \min\{\mu(f(a*b)), \mu(f(a))\}\\ &= \min\{\mu(f(a)*f(b)), \mu(f(a))\}\\ &= \min\{\mu(x*y), \mu(x)\}.\end{aligned}$$

Hence μ is a fuzzy ideal of Y. ∎

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SOME REFINEMENTS OF THE HEINZ INEQUALITIES

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Abstract. This paper aims to discuss Heinz inequalities for unitarily invariant norms. We present some refinements of the Heinz inequalities for matrices due to Kittaneh [Integr. Equ. Oper. Theory, 68:519-527, 2010]. Our results generalize the results shown by Feng [J. Inequal. Appl., 2012:18, 2012], Wang [J. Inequal. Appl., 2013:424, 2013] and Yan et al. [J. Inequal. Appl. 2014:50, 2014].

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1. Introduction

Let $M_{m,n}$ be the space of $m \times n$ complex matrices and $M_n = M_{n,n}$. Let $|||\cdot|||$ denote any unitarily invariant norm on M_n. So, $|||UAV||| = |||A|||$ for all $A \in M_n$ and for all unitary matrices $U, V \in M_n$. The *Ky Fan k-norm* $|||\cdot|||_{(k)}$ is defined as

$$|||A|||_{(k)} = \sum_{j=1}^{k} s_j(A), \quad k = 1, \cdots, n,$$

where $s_1(A) \geq s_2(A) \geq \cdots \geq s_{n-1}(A) \geq s_n(A)$ are the singular values of A, that is, the eigenvalues of the positive semidefinite matrix $|A| = (AA^*)^{\frac{1}{2}}$, arranged in decreasing order and repeated according to multiplicity. The *Schattenp-norm* $|||\cdot|||_p$ is defined as

$$|||A|||_p = \left(\sum_{j=1}^{n} s_j^p(A) \right)^{1/p} = (\operatorname{tr}|A|^p)^{1/p}, \quad 1 \leq p < \infty.$$

It is known that these norms are unitarily invariant [1].

Let $A, B, X \in M_n$ such that A and B are positive semidefinite. Then, for every unitarily invariant norm, the function

$$\varphi(v) = |||A^v X B^{1-v} + A^{1-v} X B^v|||$$

is convex on $[0, 1]$, attains its minimum at $v = \frac{1}{2}$ and attains its maximum at $v = 0$ and $v = 1$. Moreover, $\varphi(v) = \varphi(1 - v)$ for $0 \le v \le 1$.

Bhatia and Davis proved Heinz inequalities in [2] that if $A, B, X \in M_n$ such that A and B are positive semidefinite, for $0 \le v \le 1$ and for every unitarily invariant norm, then

$$\begin{aligned} 2|||A^{\frac{1}{2}} X B^{\frac{1}{2}}||| &\le |||A^v X B^{1-v} + A^{1-v} X B^v||| \\ &\le |||AX + AB|||. \end{aligned} \tag{1}$$

For more information on Heinz inequality for matrices, the reader is referred to [2]-[7].

By the convexity of function $\varphi(v) = |||A^v X B^{1-v} + A^{1-v} X B^v|||$, Kittaneh [3], Feng [4], Wang [5] and Yan et al [6] got some refinements of (1). In this paper, we also present several refinements of (1). Our results are generalization of results shown in [3]-[6].

2. Main results

In this section, we present several improvement refinements of the Heinz inequalities, to do this, we need the following lemmas.

Lemma 1. (Hermite-Hadamard Integral Inequality) [3] *Let f be a real valued convex function on the interval $[a, b]$. Then*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a} \int_a^b f(t)\, dt \le \frac{f(a) + f(b)}{2}.$$

Lemma 2. *Let f be a real valued convex function on the interval $[a, b]$. Then*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\le \frac{1}{b-a} \int_a^b f(t)\, dt \\ &\le \frac{1}{2n} \left[(n-1) f(a) + 2f\left(\frac{a+b}{2}\right) + (n-1) f(b)\right] \\ &\le \frac{f(a) + f(b)}{2}, \end{aligned}$$

where $n \ge 2$ is an integer.

Proof. By Lemma 1, we can easily verify the inequality

$$\frac{1}{2n}\left[(n-1)\,f\,(a)+2f\left(\frac{a+b}{2}\right)+(n-1)\,f(b)\right]\leq\frac{f\,(a)+f\,(b)}{2}.$$

Then, we will prove the following inequality:

$$\frac{1}{b-a}\int_a^b f\,(t)\,dt\leq\frac{1}{2n}\left[(n-1)\,f\,(a)+2f\left(\frac{a+b}{2}\right)+(n-1)\,f(b)\right].$$

Using Lemma 1, we have

$$\begin{aligned}\frac{1}{b-a}\int_a^b f\,(t)dt &= \frac{1}{b-a}\int_a^{\frac{a+b}{2}} f\,(t)dt+\frac{1}{b-a}\int_{\frac{a+b}{2}}^b f\,(t)dt\\
&\leq\frac{1}{b-a}\left[\frac{f\,(a)+f\left(\frac{a+b}{2}\right)}{2}\cdot\frac{b-a}{2}+\frac{f\left(\frac{a+b}{2}\right)+f\,(b)}{2}\cdot\frac{b-a}{2}\right]\\
&=\frac{1}{4}\left[f\,(a)+2f\left(\frac{a+b}{2}\right)+f\,(b)\right]\\
&=\frac{1}{2n}\left[\frac{n}{2}f\,(a)+nf\left(\frac{a+b}{2}\right)+\frac{n}{2}f\,(b)\right]\\
&\leq\frac{1}{2n}\left[\frac{n}{2}f\,(a)+2f\left(\frac{a+b}{2}\right)+\frac{n-2}{2}\left(f\,(a)+f\,(b)\right)+\frac{n}{2}f\,(b)\right]\\
&=\frac{1}{2n}\left[(n-1)\,f\,(a)+2f\left(\frac{a+b}{2}\right)+(n-1)\,f(b)\right].\end{aligned}$$

This completes the proof. ∎

Applying Lemma 2 to the function $\varphi\,(v)=|||A^vXB^{1-v}+A^{1-v}XB^v|||$ on the interval $[u,1-u]$ when $0\leq u<\frac{1}{2}$, and on the interval $[1-u,u]$ when $\frac{1}{2}<u\leq 1$, we achieve a refinement of the first inequality in (1).

Theorem 1. *Let $A,B,X\in M_n$ such that A and B are positive definite, for $0\leq u\leq 1$ and for every unitarily invariant norm. Then*

$$\begin{aligned}2|||A^{\frac{1}{2}}XB^{\frac{1}{2}}||| &\leq\frac{1}{|1-2u|}\left|\int_u^{1-u}|||A^vXB^{1-v}+A^{1-v}XB^v|||dv\right|\\
&\leq\frac{1}{n}\left[(n-1)|||A^uXB^{1-u}+A^{1-u}XB^u|||+2|||A^{\frac{1}{2}}XB^{\frac{1}{2}}|||\right]\\
&\leq|||A^uXB^{1-u}+A^{1-u}XB^u|||,\end{aligned}\tag{2}$$

where $n\geq 2$ is an integer.

Proof. If $0 \leq u < \dfrac{1}{2}$, then, by Lemma 2, we have

$$\begin{aligned}
\varphi\left(\frac{1-u+u}{2}\right) &\leq \frac{1}{1-2u}\int_u^{1-u}\varphi(v)\,dv \\
&\leq \frac{1}{2n}\left[(n-1)\,\varphi(u) + 2\varphi\left(\frac{1-u+u}{2}\right) + (n-1)\,\varphi(1-u)\right] \\
&\leq \frac{\varphi(u)+\varphi(1-u)}{2}.
\end{aligned}$$

That is,

$$\begin{aligned}
\varphi\left(\frac{1}{2}\right) &\leq \frac{1}{1-2u}\int_u^{1-u}\varphi(v)\,dv \\
&\leq \frac{1}{n}\left[(n-1)\varphi(u) + \varphi\left(\frac{1}{2}\right)\right] \\
&\leq \varphi(u),
\end{aligned}$$

where $\varphi(v) = |||A^v XB^{1-v} + A^{1-v}XB^v|||$. Thus

$$\begin{aligned}
2|||A^{\frac{1}{2}}XB^{\frac{1}{2}}||| &\leq \frac{1}{1-2u}\int_u^{1-u}|||A^vXB^{1-v}+A^{1-v}XB^v|||dv \\
&\leq \frac{1}{n}\left[(n-1)|||A^uXB^{1-u}+A^{1-u}XB^u||| + 2|||A^{\frac{1}{2}}XB^{\frac{1}{2}}|||\right] \\
&\leq |||A^uXB^{1-u}+A^{1-u}XB^u|||.
\end{aligned} \tag{3}$$

If $\dfrac{1}{2} < u \leq 1$, then the proof is similar to the case $0 \leq u < \dfrac{1}{2}$, so we obtain

$$\begin{aligned}
2|||A^{\frac{1}{2}}XB^{\frac{1}{2}}||| &\leq \tfrac{1}{2u-1}\textstyle\int_{1-u}^{u}|||A^vXB^{1-v}+A^{1-v}XB^v|||dv \\
&\leq \frac{1}{n}\left[(n-1)|||A^uXB^{1-u}+A^{1-u}XB^u||| + 2|||A^{\frac{1}{2}}XB^{\frac{1}{2}}|||\right] \\
&\leq |||A^uXB^{1-u}+A^{1-u}XB^u|||.
\end{aligned} \tag{4}$$

Hence,

$$\begin{aligned}
&\lim_{u\to\frac{1}{2}}\frac{1}{|1-2u|}\left|\int_u^{1-u}|||A^vXB^{1-v}+A^{1-v}XB^v|||dv\right| \\
&= \lim_{u\to\frac{1}{2}}\frac{1}{n}\left[(n-1)|||A^uXB^{1-u}+A^{1-u}XB^u||| + 2|||A^{\frac{1}{2}}XB^{\frac{1}{2}}|||\right] \\
&= 2|||A^{\frac{1}{2}}XB^{\frac{1}{2}}|||.
\end{aligned}$$

The inequalities in (2) follow by combining the inequalities (3) and (4). This completes the proof. ∎

Applying Lemma 2 to the function $\varphi(v) = |||A^v XB^{1-v} + A^{1-v}XB^v|||$ on the interval $\left[u, \frac{1}{2}\right]$ when $0 \le u < \frac{1}{2}$, and on the interval $\left[\frac{1}{2}, u\right]$ when $\frac{1}{2} < u \le 1$, we obtain the following result.

Theorem 2. *Let $A, B, X \in M_n$ such that A and B are positive definite. For $0 \le u \le 1$ and for every unitarily invariant norm. Then*

$$\begin{aligned}
&|||A^{\frac{1+2u}{4}}XB^{\frac{3-2u}{4}} + A^{\frac{3-2u}{4}}XB^{\frac{1+2u}{4}}||| \\
&\le \frac{2}{|1-2u|}\left|\int_u^{\frac{1}{2}} |||A^v XB^{1-v} + A^{1-v}XB^v|||dv\right| \\
&\le \frac{1}{2n}\big[(n-1)|||A^u XB^{1-u} + A^{1-u}XB^u||| \\
&+2|||A^{\frac{1+2u}{4}}XB^{\frac{3-2u}{4}} + A^{\frac{3-2u}{4}}XB^{\frac{1+2u}{4}}||| + 2(n-1)|||A^{\frac{1}{2}}XB^{\frac{1}{2}}|||\big] \\
&\le \frac{1}{2}\left(|||A^u XB^{1-u} + A^{1-u}XB^u||| + 2|||A^{\frac{1}{2}}XB^{\frac{1}{2}}|||\right),
\end{aligned} \tag{5}$$

where $n \ge 2$ is an integer.

Proof. The proof is similar to Theorem 1, so we omit it. ∎

Inequalities (5) and the first inequality in (1) yield the following refinement of the first inequality in (1).

Corollary 1. *Let $A, B, X \in M_n$ such that A and B are positive definite. For $0 \le u \le 1$ and for every unitarily invariant norm. Then*

$$\begin{aligned}
2|||A^{\frac{1}{2}}XB^{\frac{1}{2}}||| &\le |||A^{\frac{1+2u}{4}}XB^{\frac{3-2u}{4}} + A^{\frac{3-2u}{4}}XB^{\frac{1+2u}{4}}||| \\
&\le \frac{2}{|1-2u|}\left|\int_u^{\frac{1}{2}} |||A^v XB^{1-v} + A^{1-v}XB^v|||dv\right| \\
&\le \frac{1}{2n}[(n-1)|||A^u XB^{1-u} + A^{1-u}XB^u||| \\
&+2|||A^{\frac{1+2u}{4}}XB^{\frac{3-2u}{4}} + A^{\frac{3-2u}{4}}XB^{\frac{1+2u}{4}}||| + 2(n-1)|||A^{\frac{1}{2}}XB^{\frac{1}{2}}|||] \\
&\le \frac{1}{2}\left(|||A^u XB^{1-u} + A^{1-u}XB^u||| + 2|||A^{\frac{1}{2}}XB^{\frac{1}{2}}|||\right) \\
&\le |||A^u XB^{1-u} + A^{1-u}XB^u|||,
\end{aligned} \tag{6}$$

where $n \ge 2$ is an integer.

It should be noticed here that in the inequalities (5) and (6)

$$\lim_{u\to\frac{1}{2}} \frac{1}{|1-2u|}\left|\int_u^{\frac{1}{2}} |||A^v XB^{1-v} + A^{1-v}XB^v|||dv\right| = |||A^{\frac{1}{2}}XB^{\frac{1}{2}}|||.$$

In the sequel, we get another refinement of the second inequality in (1).

Applying Lemma 2 to the function $\varphi(v) = |||A^v X B^{1-v} + A^{1-v} X B^v|||$ on the interval $[0, u]$ when $0 < u \le \frac{1}{2}$, and on the interval $[u, 1]$ when $\frac{1}{2} \le u < 1$, we obtain the following theorem.

Theorem 3. *Let $A, B, X \in M_n$ such that A and B are positive definite. Then*

1. *for $0 \le u \le \frac{1}{2}$ and for every unitarily invariant norm,*

$$
\begin{aligned}
& |||A^{\frac{u}{2}} X B^{1-\frac{u}{2}} + A^{1-\frac{u}{2}} X B^{\frac{u}{2}}||| \\
& \le \frac{1}{u} \int_0^u |||A^v X B^{1-v} + A^{1-v} X B^v||| dv \\
& \le \frac{1}{2n} [(n-1)|||AX + XB||| + 2|||A^{\frac{u}{2}} X B^{1-\frac{u}{2}} + A^{1-\frac{u}{2}} X B^{\frac{u}{2}}||| \\
& +(n-1)|||A^u X B^{1-u} + A^{1-u} X B^u|||] \\
& \le \frac{1}{2} \left(|||AX + XB||| + |||A^u X B^{1-u} + A^{1-u} X B^u||| \right),
\end{aligned}
\tag{7}
$$

where $n \ge 2$ is an integer,

2. *for $\frac{1}{2} \le u \le 1$ and for every unitarily invariant norm,*

$$
\begin{aligned}
& |||A^{\frac{1+u}{2}} X B^{\frac{1-u}{2}} + A^{\frac{1-u}{2}} X B^{\frac{1+u}{2}}||| \\
& \le \frac{1}{1-u} \int_u^1 |||A^v X B^{1-v} + A^{1-v} X B^v||| dv \\
& \le \frac{1}{2n} [(n-1)|||AX + XB||| + 2|||A^{\frac{1+u}{2}} X B^{\frac{1-u}{2}} + A^{\frac{1-u}{2}} X B^{\frac{1+u}{2}}||| \\
& +(n-1)|||A^u X B^{1-u} + A^{1-u} X B^u|||] \\
& \le \frac{1}{2} \left(|||AX + XB||| + |||A^u X B^{1-u} + A^{1-u} X B^u||| \right),
\end{aligned}
\tag{8}
$$

where $n \ge 2$ is an integer.

Proof. The proof is similar to Theorem 1, so we omit it. ∎

In view of the fact that the function $\varphi(v) = |||A^v X B^{1-v} + A^{1-v} X B^v|||$ is decreasing on the interval $\left[0, \frac{1}{2}\right]$ and increasing on the interval $\left[\frac{1}{2}, 1\right]$, by Theorem 3, we have the following result, which is a refinement of the second inequality in (1).

Corollary 2. *Let $A, B, X \in M_n$ such that A and B are positive definite. Then*

1. *for* $0 \leq u \leq \frac{1}{2}$ *and for every unitarily invariant norm*

$$\begin{aligned} & |||A^u X B^{1-u} + A^{1-u} X B^u||| \\ & \quad \leq |||A^{\frac{u}{2}} X B^{1-\frac{u}{2}} + A^{1-\frac{u}{2}} X B^{\frac{u}{2}}||| \\ & \quad \leq \frac{1}{u} \int_0^u |||A^v X B^{1-v} + A^{1-v} X B^v||| dv \\ & \quad \leq \frac{1}{2n} [(n-1)|||AX + XB||| + 2|||A^{\frac{u}{2}} X B^{1-\frac{u}{2}} + A^{1-\frac{u}{2}} X B^{\frac{u}{2}}||| \\ & \quad +(n-1)|||A^u X B^{1-u} + A^{1-u} X B^u|||] \\ & \quad \leq \frac{1}{2} \left(|||AX + XB||| + |||A^u X B^{1-u} + A^{1-u} X B^u||| \right) \\ & \quad \leq |||AX + XB|||, \end{aligned} \tag{9}$$

where $n \geq 2$ is an integer.

2. *for* $\frac{1}{2} \leq u \leq 1$ *and for every unitarily invariant norm*

$$\begin{aligned} & |||A^u X B^{1-u} + A^{1-u} X B^u||| \\ & \quad \leq |||A^{\frac{1+u}{2}} X B^{\frac{1-u}{2}} + A^{\frac{1-u}{2}} X B^{\frac{1+u}{2}}||| \\ & \quad \leq \frac{1}{1-u} \int_u^1 |||A^v X B^{1-v} + A^{1-v} X B^v||| dv \\ & \quad \leq \frac{1}{2n} [(n-1)|||AX + XB||| + 2|||A^{\frac{1+u}{2}} X B^{\frac{1-u}{2}} + A^{\frac{1-u}{2}} X B^{\frac{1+u}{2}}||| \\ & \quad +(n-1)|||A^u X B^{1-u} + A^{1-u} X B^u|||] \\ & \quad \leq \frac{1}{2} \left(|||AX + XB||| + |||A^u X B^{1-u} + A^{1-u} X B^u||| \right) \\ & \quad \leq |||AX + XB|||, \end{aligned} \tag{10}$$

where $n \geq 2$ is an integer.

It should be noticed that in the inequalities (7) to (10), we have

$$\begin{aligned} & \lim_{u \to 0} \frac{1}{u} \int_0^u |||A^v X B^{1-v} + A^{1-v} X B^v||| dv \\ & \quad = \lim_{u \to 1} \frac{1}{1-u} \int_u^1 |||A^v X B^{1-v} + A^{1-v} X B^v||| dv \\ & \quad = |||AX + XB|||. \end{aligned}$$

Remark 1. The three special values $n = 2$, $n = 16$ and $n = 4$ give the refinements of Heinz inequalities obtained in [4], [5] and [6], respectively.

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COMPUTATION OF TOPOLOGICAL INDICES OF NON-COMMUTING GRAPHS

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Abstract. Let G be a non-abelian finite group. The non-commuting graph Γ_G of G is defined as a graph with vertex set $G - Z(G)$ in which two distinct vertices x and y are joined if and only if $xy \neq yx$. Various topological indices have been defined for simple and connected graphs. Since non-commuting graph is a simple and connected graph, topological indices could be defined for it. The main object of this article is to calculate various indices like Wiener index, Hyper-Wiener index, Schultz index and Gutman index for the non-commuting graph of the group G.

Keywords: non-commuting graph, Wiener index, hyper-Wiener index, Schultz index, Gutman index.

Mathematics Subject Classification: 05C12.

1. Introduction

Let G be a non-abelian finite group. Various graphs could be attributed to G, one of which is the non-commuting graph denoted by Γ_G. The set of vertices and edges of Γ_G are $V(\Gamma_G)$ and $E(\Gamma_G)$ respectively so that $V(\Gamma_G) = G - Z(G)$ in which $Z(G)$ is the center of G and for every $x, y \in V(\Gamma_G)$ we have:

$$\{x, y\} \in E(\Gamma_G) \Longleftrightarrow xy \neq yx$$

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It is apparent that if G is an abelian group, Γ_G would turn to a null graph. For this, G is assumed to be a non-abelian group. The centralizer of x within G which is denoted by $C_G(x)$ is a subset of G which is defined as $\{g \in G|\ gx = xg\}$.

According to [8] the non-commuting graph of a finite group G was first introduced by Paul Erdos in connection with the following problem: let G be a group whose non-commuting graph Γ_G has no infinite complete subgraphs. Is it true that there is a finite bound on the cardinalities of complete subgraphs of Γ_G? By [8], the answer to this question is positive and this was the origin of many similar questions and research.

Assume that $\Gamma = (V, E)$ is a graph in which V is the set of vertices and E is the set of edges. This graph is assumed to be a finite graph whenever $|V|, |E|$ are finite. We assume this graph is connected, the distance between two x and y the vertex is shown as $d(x, y)$. It is the shortest path between the two vertices x and y. The degree of the vertex x which is shown by $\rho(x)$ equals to the number of edges through x. The largest distance between all pairs of the vertices of G is called the diameter of G.

The Wiener index of the graph G which is shown as $W(G)$ is defined as follows:

$$W(G) = \frac{1}{2} \sum_{x,y \in V} d(x, y) = \frac{1}{2} \sum_{x \in V} d(x).$$

Where $d(x) = \sum_{v \in V} d(x, v)$.

The Wiener index is one of the oldest descriptors concerned with the molecular graph. This index first was proposed by H. Wiener [9] and it is concerned with the determination of the boiling points of Paraffins. In mathematical research, the Wiener index has been first studied in [4]. It is an invariant of the graph, it is invariant under the automorphism group of the graph.

The Hyper-Wiener index of the graph G which is shown as $WW(G)$ is defined as follows:

$$WW(G) = \frac{1}{2} \sum_{\{x,y\} \subset V} (d(x, y) + d^2(x, y)).$$

The Hyper-Wiener index is one of the recently introduced distance-based molecular structure-descriptors. It was put forward in 1993 and since then it has attracted much attention of theoretical chemists. In parallel with the symbol W for the Wiener index the hyper-Wiener index is traditionally denoted by WW [3], [6].

Schultz in 1989 introduced a graph-theoretical descriptor for characterizing alkanes by an integer, namely the Schultz index, defined as

$$S(G) = \sum_{\{x,y\} \subset V} (\rho(x) + \rho(y)) d(x, y).$$

The Gutman index (also known as Schultz index of the second kind [2], [3], [5], [7]) of a graph G is defined as

$$Gut(G) = \sum_{\{x,y\} \subset V} \rho(x)\rho(y) d(x, y).$$

Our main goal is to calculate the above mentioned indices for the non-commuting graph of G in terms of the order of $G, Z(G)$ and the number of conjugacy classes of G. The following lemmas will be used repeatedly in calculating process:

Lemma 1.1. *Let G be a non-abelian finite group, then* $\operatorname{diam}(\Gamma_G) = 2$.

Proof. This is Proposition 2.1 in [1]. ∎

Lemma 1.2. *Let G be a non-abelian finite group and k be the number of conjugacy classes of G, then*

$$|E(\Gamma_G)| = \frac{1}{2}|G|(|G| - k(G)).$$

Proof. See [1] Lemma 3.27. ∎

Lemma 1.3. *Let G be a non-abelian finite group. If x is one of the vertices of Γ_G, then*

$$\rho(x) = |G| - |C_G(x)|.$$

Proof. See Lemma 3.1 in [1]. ∎

2. The Wiener index of the non-commuting graph of a group

Before we calculate the Wiener index, we prove the following lemma.

Lemma 2.1. *Let G be a finite group and k be the number of conjugacy classes of G, then*

$$\sum_{x \notin Z(G)} |C_G(x)| = |G|(k - |Z(G)|).$$

Proof. We know that G is the union of its conjugacy classes and assume that $\{x_i\}_{i=1}^k$ is the set of the representatives of the conjugacy classes of G. Then we have:

$$G = \bigcup_{i=1}^{k} \operatorname{class}(x_i)$$

Now, let $\{x_i\}_{i=1}^t$ be the set of non-central of G class representatives and then we have $k = t + |Z(G)|$. Every x which is not placed within would be placed within one of $Z(G)$ in which class$(x_i)s$ in which $1 \le i \le t$. Therefore we have:

$$\sum_{x \notin Z(G)} |C_G(x)| = \sum_{i=1}^{t} |\operatorname{class}(x_i)|\ |C_G(x_i)| = |G|t = |G|(k - |Z(G)|).$$ ∎

Now, we calculate the Wiener index of the non-commuting graph of a group G.

Assume that $x, y \in G - Z(G)$ are two arbitrary distinct vertices of the graph Γ_G. According to the Lemma 1.1 we have $d(x, y) = 1$ or 2.

If $d(x, y) = 1$, then $xy \neq yx$ and $y \in G - C_G(x)$.

If $d(x, y) = 2$, then $xy = yx$ and $x \neq y \in C_G(x) - Z(G)$.

So, we have

$$W(\Gamma_G) = \frac{1}{2} \sum_{x \in G - Z(G)} d(x)$$

for all $x \in G - Z(G)$. Therefore:

$$\begin{aligned} d(x) = 2 & \text{ (The number of vertices whose distance from } x \text{ is 2)} \\ & + 1 \text{(The number of vertices whose distance from } x \text{ is 1).} \end{aligned}$$

Then

$$d(x) = 2(|C_G(x)| - |Z(G)| - 1) + (|G| - |C_G(x)|) = |G| + |C_G(x)| - 2|Z(G)| - 2$$

Now, we can calculate the Wiener index:

$$\begin{aligned} W(\Gamma_G) &= \frac{1}{2} \sum_{x \in G - Z(G)} d(x) \\ &= \frac{1}{2} \left(\sum_{x \in G - Z(G)} (|G| - 2|Z(G)| - 2) + |C_G(x)| \right) \\ &= \frac{1}{2} \left[(|G| - 2|Z(G)| - 2)(|G| - |Z(G)|) + \sum_{x \in G - Z(G)} |C_G(x)| \right] \end{aligned}$$

By Lemma 2.1, we have

$$W(\Gamma_G) = \frac{1}{2}[(|G| - 2|Z(G)| - 2)(|G| - |Z(G)|) + |G|(k - |Z(G)|)]$$

So

$$W(\Gamma_G) = \frac{1}{2}[(|G| - 2|Z(G)|)^2 - |G|(|G| - k) + |G|(|G| - 2) - 2|Z(G)|(|Z(G)| - 1)]$$

or

$$W(\Gamma_G) = \frac{1}{2}[(|G| - 2|Z(G)|)^2 - 2|E(\Gamma_G)| + |G|(|G| - 2) - 2|Z(G)|(|Z(G)| - 1)].$$

Therefore, we have proved the following:

Theorem 2.2. *Let G be a non-abelian finite group and Γ_G be its non-commuting graph. Then*

$$W(\Gamma_G) = \frac{1}{2}[(|G| - 2|Z(G)|)^2 - 2|E(\Gamma_G)| + |G|(|G| - 2) - 2|Z(G)|(|Z(G)| - 1)].$$

3. The Hyper-Wiener index of non-commuting graph of a group

Assume that x, y are two arbitrary vertices of the non-commuting graph of the group G. The Hyper-Wiener index of this graph is defined as follows:

$$WW(G) = \frac{1}{4} \sum_{x,y \in G-Z(G)} (d(x,y) + d^2(x,y)).$$

In order to calculate the Hyper-Wiener index; first we calculate $d^2(x,y)$ for every $x, y \in G - Z(G)$.

Let us set $G - Z(G) = \{x_i\}_{i=1}^m$ where $m = |G| - |Z(G)|$. So we have:

$$\sum_{x,y \in G-Z(G)} d^2(x,y) = \sum_{x_i \in G-Z(G)} d^2(x_1, x_i) + \ldots + \sum_{x_i \in G-z(G)} d^2(x_m, x_i)$$

Without loss of generality, we calculate $\sum\limits_{x_i \in G-Z(G)} d^2(x_j, x_i)$ for a constant x_j.

1) If $d^2(x_j, x_i) = 1$, then $x_j x_i \neq x_i x_j$ and $x_i \in G - C_G(x_j)$.

2) If $d^2(x_j, x_i) = 4$, then $x_j x_i = x_i x_j$ and $x_j \neq x_i \in C_G(x_j) - Z(G)$.

Therefore:

$$\begin{aligned}
\sum_{x_i \in G-Z(G)} d^2(x_j, x_i) &= 4 \text{ (The number of vertices whose distances from } x \text{ is 2)} \\
&\quad + \text{(The number of vertices whose distances from } x \text{ is 1)} \\
&= 4(|C_G(x_1)| - |Z(G)| - 1) + (|G| - |C_G(x_j)|) \\
&= |G| + 3|C_G(x_j)| - 4|Z(G)| - 4
\end{aligned}$$

But, x_j is an arbitrary vertex. So we can write this formula for all x_i in which $1 \leq i \leq m$. Now we calculate $\sum\limits_{x,y \in G-Z(G)} d^2(x,y)$.

$$\begin{aligned}
\sum_{x,y \in G-Z(G)} d^2(x,y) &= \sum_{i=1}^{m} (|G| + 3|C_G(x_i)| - 4|Z(G)| - 4) \\
&= (|G| - 4|Z(G)| - 4)(|G| - |Z(G)|) + 3\sum_{i=1}^{m} |C_G(x_i)| \\
&= (|G| - 4|Z(G)| - 4)(|G| - |Z(G)|) + 3|G|(k - |Z(G)|) \\
&= (|G| - 4|Z(G)|)^2 + |G|(3|G| - 4) \\
&\quad - 6|E(\Gamma_G)| - 4|Z(G)|(3|Z(G)| - 1).
\end{aligned}$$

So, the Hyper-Wiener index is as follows:

$$\begin{aligned}
WW(\Gamma_G) = \frac{1}{2} W(\Gamma_G) + \frac{1}{4}[&(|G| - 4|Z(G)|)^2 + |G|(3|G| - 4) \\
&-6|E(\Gamma_G)| - 4|Z(G)|(3|Z(G)| - 1)].
\end{aligned}$$

By Lemma 2.1, we have:

$$WW(\Gamma_G) = \frac{1}{4}[(|G| - 4|Z(G)|)^2 + (|G| - 2|Z(G)|)^2 - 8|E(\Gamma_G)| \\ + 2|G|(2|G| - 3) - 2|Z(G)|(7|Z(G)| - 3)].$$

Therefore, we have proved the following:

Theorem 3.1. *Let G be a non-abelian finite group and Γ_G be its non-commuting graph. Then*

$$WW(\Gamma_G) = \frac{1}{4}[(|G| - 4|Z(G)|)^2 + (|G| - 2|Z(G)|)^2 - 8|E(\Gamma_G)| \\ + 2|G|(2|G| - 3) - 2|Z(G)|(7|Z(G)| - 3)].$$

4. The Schultz index of the non-commuting graph of a group

The Schultz index of a general graph $\Gamma = (V, E)$ is as follows:

$$S(\Gamma_G) = \sum_{\{x,y\}\subset V} (\rho(x) + \rho(y))d(x, y).$$

Without loss of generality we calculate $\sum_{x_i\in G-Z(G)} (\rho(x_j) + \rho(x_i))d(x_j, x_i)$ for a fixed x_j.

$$\begin{aligned}\sum_{x_i\in G-Z(G)} (\rho(x_j) + \rho(x_i))d(x_j, x_i) &= \sum_{x_i\in G-C_G(x_j)} \rho(x_j) + \rho(x_i) + 2 \\ &= \sum_{x_j\neq x_i\in C_G(x_j)-Z(G)} \rho(x_j) + \rho(x_i)\end{aligned}$$

Therefore, we have:

$$\sum_{x_i\in G-Z(G)} (\rho(x_j) + \rho(x_i))d(x_j, x_i) = \sum_{x_i\in G-C_G(x_j)} (2|G| - |C_G(x_j)| - |C_G(x_i)|)$$

$$+ 2 \sum_{x_j\neq x_i\in C_G(x_j)-Z(G)} (2|G| - |C_G(x_j)| - |C_G(x_i)|)$$

$$= (2|G| - |C_G(x_j)|)(|G| + |C_G(x_j)| - 2|Z(G)| - 2)$$

$$- \left(\sum_{x_i\in G-C_G(x_j)} |C_G(x_j)| + \sum_{x_j\neq x_i\in C_G(x_j)-Z(G)} |C_G(x_i)|\right) - \sum_{x_j\neq x_i\in C_G(x_j)-Z(G)} |C_G(x_i)|$$

Such that $G - Z(G) = \{x_i\}_{i=1}^{m}$ by Lemma 2.1, we have:

$$\begin{aligned}\sum_{x_i\in G-Z(G)}(\rho(x_j)+\rho(x_i))d(x_j,x_i) &= (|G|+\rho(x_j))(2|G|-2|Z(G)|-2)\\ &+|G|(|Z(G)|+1-|G|)-\rho(x_j)(|G|+1+\rho(x_j))\\ &+2|E(\Gamma_G)|-\sum_{x_j\neq x_i\in C_G(x_j)-Z(G)}|C_G(x_i)|\end{aligned}$$

x_j is arbitrary. So we can write this formula for all x_i in which $1\leq i\leq m$.

Now, we continue to calculate the Schultz index of the non-commuting graph:

$$\begin{aligned}S(\Gamma_G) &= \frac{1}{2}\left[\sum_{x_i\in G-Z(G)}(\rho(x_1)+\rho(x_i))d(x_1,x_i)+\dots\right.\\ &\left.+\sum_{x_i\in G-Z(G)}(\rho(x_m)+\rho(x_i))d(x_m,x_i)\right]\\ &=\frac{1}{2}[(2|G|-2|Z(G)|-2)(|G|(|G|-|Z(G)|)+2|E(\Gamma_G)|)\\ &+|G|(|G|-|Z(G)|)(|Z(G)|+1-|G|)+2|E(\Gamma_G)|(|G|-|Z(G)|)\\ &-(|G|+1)(2|E(\Gamma_G)|)-\sum_{i=1}^{m}(\rho(x_i))^2\\ &-\left(\sum_{x_1\neq x_i\in C_G(x_1)-Z(G)}|C_G(x_i)|+\dots+\sum_{x_m\neq x_i\in C_G(x_m)-Z(G)}|C_G(x_i)|\right)]\end{aligned}$$

We can calculate $\sum_{i=1}^{m}(\rho(x_i))^2$ as follows:

$$\begin{aligned}\sum_{i=1}^{m}(\rho(x_i))^2 &= \sum_{i=1}^{m}(|G|-|C_G(x_i)|)^2\\ &=4|G|\;|E(\Gamma_G)|-|G|^2(|G|-|Z(G)|)+\sum_{i=1}^{m}|C_G(x_i)|^2\end{aligned}$$

So, we have:

$$\begin{aligned}S(\Gamma_G) &= \frac{1}{2}[(2|G|-2|Z(G)|-2)(|G|(|G|-|Z(G)|)+2|E(\Gamma_G)|)\\ &+|G|(|G|-|Z(G)|)(|Z(G)|+1)-2|E(\Gamma_G)|(2|G|\\ &+|Z(G)|+1)-\sum_{i=1}^{m}|C_G(x_i)|^2\\ &-\left(\sum_{x_1\neq x_i\in C_G(x_1)-Z(G)}|C_G(x_i)|+\dots+\sum_{x_m\neq x_i\in C_G(x_m)-Z(G)}|C_G(x_i)|\right)].\end{aligned}$$

Therefore, we have the following theorem:

Theorem 4.1. *Let G be a non-abelian finite group and Γ_G be its non-commuting graph. Then*

$$\begin{aligned} S(\Gamma_G) = \frac{1}{2}[&(2|G| - 2|Z(G)| - 2)(|G|(|G| - |Z(G)|) + 2|E(\Gamma_G)|) \\ &+ |G|(|G| - |Z(G)|)(|Z(G)| + 1) - 2|E(\Gamma_G)|(2|G| \\ &+ |Z(G)| + 1) - \sum_{i=1}^{m} |C_G(x_i)|^2 \\ &- \left(\sum_{i=1}^{m} \sum_{x_i \neq x_j \in C_G(x_i) - Z(G)} |C_G(x_j)|\right)]. \end{aligned}$$

Definition 4.2. Let G be a non-abelian group. G is called an AC-group if $C_G(x)$ is abelian for all $x \in G - Z(G)$.

The following characterization of AC-group may be useful in some points.

Theorem 4.3. *Let G be an AC-group, then*

$$\begin{aligned} S(\Gamma_G) = \frac{1}{2}\Bigg[&(2|G| - 2|Z(G)| - 2)|G|(|G| - |Z(G)|) \\ &+ 2|G|(|G| - |Z(G)|)(|Z(G)| + 1) \\ &- 4|E(\Gamma_G)|(|G| + |Z(G)| + 1) - 2\sum_{x \in G - Z(G)} (|G| - \rho(x))^2\Bigg]. \end{aligned}$$

Proof. We have:

$$\begin{aligned} S(\Gamma_G) = \frac{1}{2}\Bigg[&(2|G| - 2|Z(G)| - 2)(|G|(|G| - |Z(G)|) + 2|E(\Gamma_G)|) \\ &+ |G|(|G| - |Z(G)|)(|Z(G)| + 1) - 2|E(\Gamma_G)|(2|G| \\ &+ |Z(G)| + 1) - \sum_{i=1}^{m} |C_G(x_i)|^2 \\ &- \Big(\sum_{i=1}^{m} \sum_{x_i \neq x_j \in C_G(x_i) - Z(G)} |C_G(x_j)|\Big)\Bigg]. \end{aligned}$$

G is an AC-group. So we can calculate $\sum_{x_j \neq x_i \in C_G(x_j) - Z(G)} |C_G(x_i)|$ for all x_i. It is easy to prove that for all $x_j \neq x_i \in C_G(x_j) - Z(G)$ that $|C_G(x_i)| = |C_G(x_j)|$.

So, we have:

$$\begin{aligned}
&\sum_{x_1\neq x_i\in C_G(x_1)-Z(G)} |C_G(x_i)| + \ldots + \sum_{x_m\neq x_i\in C_G(x_m)-Z(G)} |C_G(x_i)| \\
&= |C_G(x_1)|(|C_G(x_1)| - |Z(G)| - 1) + \ldots \\
&+ |C_G(x_1)|(|C_G(x_1)| - |Z(G)| - 1) \\
&= \sum_{i=1}^{m} |C_G(x_i)|^2 - (|Z(G)| + 1)\sum_{i=1}^{m} |C_G(x_i)| \\
&= \sum_{i=1}^{m} |C_G(x_i)|^2 - (|Z(G)| + 1)|G|(k - |Z(G)|) \\
&= \sum_{i=1}^{m} |C_G(x_i)|^2 - (|Z(G)| + 1) - 2|E(\Gamma_G)| + |G|(|G| - |Z(G)|).
\end{aligned}$$

By using this formulation, we calculate the Schultz index of the non-commuting graph of G.

$$\begin{aligned}
S(\Gamma_G) = \frac{1}{2}\Big[&(2|G| - 2|Z(G)| - 2)(|G|(|G| - |Z(G)|) + 2|E(\Gamma_G)|) \\
&+ |G|(|G| - |Z(G)|)(|Z(G)| + 1) - 2|E(\Gamma_G)|(2|G| \\
&+ |Z(G)| + 1) - 2\sum_{i=1}^{m} |C_G(x_i)|^2 \\
&+ (|Z(G)| + 1) - 2|E(\Gamma_G)| + |G|(|G| - |Z(G)|)\Big].
\end{aligned}$$

By easy calculation, we have:

$$\begin{aligned}
S(\Gamma_G) \;=\; \frac{1}{2}\Big[&(2|G| - 2|Z(G)| - 2)|G|(|G| - |Z(G)|) \\
&+2|G|(|G| - |Z(G)|)(|Z(G)| + 1) \\
&-4|E(\Gamma_G)|(|G| + |Z(G)| + 1) - 2\sum_{x\in G-Z(G)} (|G| - \rho(x))^2\Big].
\end{aligned}$$

■

5. The Gutman index of non-commuting graph of a group

The Gutman index of non-commuting graph is as follows:

$$\mathrm{Gut}(\Gamma_G) = \sum_{\{x,y\}\subset V(\Gamma_G)} \rho(x)\rho(y)d(x,y).$$

We have

$$G - Z(G) = \{x_i\}_{i=1}^{m}.$$

We can write:

$$\mathrm{Gut}(\Gamma_G)=\frac{1}{2}\left[\sum_{i=1}^{m}\rho(x_1)\rho(x_i)d(x_1,x_i)+\ldots+\sum_{i=1}^{m}\rho(x_m)\rho(x_i)d(x_m,x_i)\right]$$

Without loss of generality, we calculate $\sum\limits_{i=1}^{m}\rho(x_j)\rho(x_i)d(x_j,x_i)$ for a fixed x_j.

$$\begin{aligned}\sum_{i=1}^{m}\rho(x_j)\rho(x_i)d(x_j,x_i)&=\sum_{x_i\in G-C_G(x_j)}\rho(x_j)\rho(x_i)+2\sum_{x_j\neq x_i\in C_G(x_j)-Z(G)}\rho(x_j)\rho(x_i)\\
&=\rho(x_j)\left[\sum_{x_i\in G-C_G(x_j)}\rho(x_i)+2\sum_{x_j\neq x_i\in C_G(x_j)-Z(G)}\rho(x_i)\right]\\
&=\rho(x_j)\left[\sum_{x_i\in G-C_G(x_j)}(|G|-|C_G(x_i)|)\right.\\
&\left.+2\sum_{x_j\neq x_i\in C_G(x_j)-Z(G)}(|G|-|C_G(x_i)|)\right]\\
&=\rho(x_j)[|G|(|G|-|Z(G)|-1)-\rho(x_j)(|G|+1)+2|E(\Gamma_G)|\\
&-\sum_{x_j\neq x_i\in C_G(x_j)-Z(G)}|C_G(x_i)|]\end{aligned}$$

We can write this formula for all x_i in which $1\leq i\leq m$. Now, we calculate the Gutman index:

$$\begin{aligned}&\mathrm{Gut}(\Gamma_G)=\frac{1}{2}\left[\sum_{i=1}^{m}\rho(x_1)\rho(x_i)d(x_1,x_i)+\ldots+\sum_{i=1}^{m}\rho(x_m)\rho(x_i)d(x_m,x_i)\right]\\
&=\frac{1}{2}\left[2|E(\Gamma_G)|\ |G|(|G|-|Z(G)|-1)+(2|E(\Gamma_G)|)^2-(|G|+1)\sum_{i=1}^{m}(\rho(x_i))^2\right.\\
&\left.-\left(\rho(x_1)\sum_{x_1\neq x_i\in C_G(x_1)-Z(G)}|C_G(x_i)|+\ldots+\rho(x_m)\sum_{x_m\neq x_i\in C_G(x_m)-Z(G)}|C_G(x_i)|\right)\right]\end{aligned}$$

By using the quality of $\sum\limits_{i=1}^{m}(\rho(x_i))^2$ and $\rho(x_j)$, we have:

$$\begin{aligned}\mathrm{Gut}(\Gamma_G) = \frac{1}{2}[&(2|E(\Gamma_G)|)^2 + |G|^2(|G|+1)(|G|-|Z(G)|)\\
&- 2|E(\Gamma_G)|\ |G|(|G|+|Z(G)|+3) - (|G|+1)\sum_{i=1}^{m}|C_G(x_i)|^2\\
&- |G|\left(\sum_{x_1\neq x_i\in C_G(x_1)-Z(G)}|C_G(x_i)| + \ldots + \sum_{x_m\neq x_i\in C_G(x_m)-Z(G)}|C_G(x_i)|\right)\\
&+ |C_G(x_1)|\sum_{x_1\neq x_i\in C_G(x_1)-Z(G)}|C_G(x_i)| + \ldots\\
&+ |C_G(x_m)|\sum_{x_m\neq x_i\in C_G(x_m)-Z(G)}|C_G(x_i)|].\end{aligned}$$

Theorem 5.1. *Let G ba a non-abelian finite group and Γ_G be its non-commuting graph. Then*

$$\begin{aligned}\mathrm{Gut}(\Gamma_G) = \frac{1}{2}\Bigg[&(2|E(\Gamma_G)|)^2 + |G|^2(|G|+1)(|G|-|Z(G)|)\\
&- 2|E(\Gamma_G)|\ |G|(|G|+|Z(G)|+3) - (|G|+1)\sum_{i=1}^{m}|C_G(x_i)|^2\\
&- |G|\Big(\sum_{x_1\neq x_i\in C_G(x_1)-Z(G)}|C_G(x_i)| + \ldots + \sum_{x_m\neq x_i\in C_G(x_m)-Z(G)}|C_G(x_i)|\Big)\\
&+ \sum_{i=1}^{m}|C_G(x_i)|\sum_{x_i\neq x_j\in C_G(x_i)-Z(G)}|C_G(x_j)|\Bigg].\end{aligned}$$

Theorem 5.2. *Let G be an AC-group, then*

$$\begin{aligned}\mathrm{Gut}(\Gamma_G) = \frac{1}{2}\Big[&(2|E(\Gamma_G)|)^2 + |G|^2(|G|+1)(|G|-|Z(G)|)\\
&- 2|E(\Gamma_G)|\ |G|(|G|+|Z(G)|+3)\\
&+ |G|(|Z(G)|+1)(|G|(|G|-|Z(G)|) - 2|E(\Gamma_G)|) + \sum_{i=1}^{m}|C_G(x_i)|^3\\
&- (2|G|+|Z(G)|+2)\sum_{i=1}^{m}|C_G(x_i)|^2\Big].\end{aligned}$$

Proof. Using Theorem 5.1 and Definition 4.2 the result follows easily. ∎

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CHARACTERIZATION OF BI Γ-TERNARY SEMIGROUPS BY THEIR IDEALS

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Abstract. In this paper, the concept of biΓ-ternary semigroup has been introduced. The notion of biΓ-ternary subsemigroup, biΓ left (right, lateral) ideals, biΓ-quasi and biΓ-bi-ideals of this newly defined structure has been introduced. Also the regular biΓ-ternary semigroups have been studied in terms of biΓ-ideals.

Keywords: ternary semigroup, Γ-semigroup, biΓ-ternary smigroup, biΓ-ideal, regular biΓ-ternary smigroup.

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1. Introduction

The concept of a semigroup is very simple but it plays a key role in the development of Mathematics. The formal study of semigroups began in the early 20^{th} century. The semigroups are significantly important in many areas of mathematics because they are the abstract algebraic underpinning of "memoryless" systems: time-dependent systems that start from scratch at each iteration. In applied mathematics, semigroups are fundamental models for linear time-invariant systems. In partial differential equations, a semigroup is associated to any equation whose spatial evolution is independent of time. The theory of finite semigroups has been of particular importance in theoretical computer science since 1950s because of the natural link between finite semigroups and finite automata. In probability theory, semigroups are associated with Markov process.

The algebraic theory of semigroups was widely studied by Clifford and Preston [1], [2], Petrich [15], [16], [17] and Ljapin [14]. They all discussed the notion of an ideal in semigroups. Good and Hughes [6] and Lajos [11] presented the idea of bi-ideals in the semigroup. Lajos [12] and Szasz [26], [27] gave the notion of interior ideals in the semigroup. Steinfeld [25] introduced the notion of quasi-ideals in the semigroups.

Lehmer [13], gave the formal definition of a ternary semigroup in 1932 but Kasner and Prüfer [10], [18] studied such structures earlier. Sioson [24] developed the ideal theory of ternary semigroups. Dixit and Dewan [5] enhanced the theory of quasi-ideal and bi-ideal of the ternary semigroups. Santiago [21], worked on the theory of ternary semigroups and semiheaps. Dutta et al. [4] studied regular ternary semigroups.

As a generalization of semigroup and ternary semigroup, Sen [22] introduced the notion of Γ-semigroup in 1981 and developed a theory on Γ-semigroups [23]. Many classical notions of semigroups have been extended to Γ-semigroups by Saha and Sen in [19, 20, 23]. The notion of bi-ideal in Γ-semigroup was introduced by Chinram and Jirojkul [3, 9]. Iampan [7] and Islam [8] extended the work on bi-ideals in Γ-semigroups.

In this paper we inspired from the concept of ternary semigroup and Γ-semigroup and obtain a new algebraic structure called biΓ-ternary semigroup. The word biΓ is used due to the double appearance of the nonempty set Γ in the structure. Here the notions of biΓ-ternary subsemigroup, biΓ-left (right, lateral) ideal, biΓ-quasi ideal and biΓ-bi-ideal have been presented with the characterization of regular biΓ-ternary semigroup by these ideals.

2. Preliminaries

2.1. Semigroup

A semigroup is a set S along with a binary operation "$*$" (that is, a function $* : S \times S \to S$) that satisfies the associative property. For all $a, b, c \in S$, the equation $(a * b) * c = a * (b * c)$ holds. Generally, we write this as $(ab)c = a(bc)$. The semigroup operation induces an operation on the collection of its subsets: given subsets A and B of a semigroup S, their product $A * B$, written commonly as AB, is the set $\{ab \mid a \in A \text{ and } b \in B\}$. In terms of this operations, a subset A of S is called a subsemigroup of S if $AA \subseteq A$, a right ideal if $AS \subseteq A$, and a left ideal if $SA \subseteq A$. If A is both a left ideal and a right ideal then it is called an ideal (or a two-sided ideal). A subsemigroup A of S is called a bi-ideal of S if $ASA \subseteq A$. A nonempty subset A of S is called an interior ideal of S if $SAS \subseteq A$.

2.2. Ternary semigroup

A ternary semigroup T is a nonempty set whose elements are closed under the ternary operation of multiplication and satisfies the associative law defined as

$$[[abc]\,de] = [a\,[bcd]\,e] = [ab\,[cde]], \quad \text{for all } a, b, c, d, e \in T.$$

For simplicity we shall write $[abc]$ as abc. A nonempty subset A of a ternary semigroup T is called a ternary subsemigroup of T if $AAA \subseteq A$ and is called an idempotent if $AAA = A^3 = A$. A left (right, lateral) ideal of a ternary semigroup T is a nonempty subset A of T such that $TTA \subseteq A$ ($ATT \subseteq A$, $TAT \subseteq A$). A nonempty subset of T is called an ideal if it is a left, a right and a lateral ideal of T. A subsemigroup B of a ternary semigroup T is called a bi-ideal of T if $BTBTB \subseteq B$. A nonempty subset A of T is called an interior ideal of T if $TTATT \subseteq A$.

2.3. Γ-Semigroup

Let $S = \{x, y, z, ...\}$ and $\Gamma = \{\alpha, \beta, \gamma, ...\}$ be two nonempty sets. Then S is called a Γ-semigroup if it satisfies,

(i) $x\gamma y \in S$

(ii) $(x\beta y)\gamma z = x\beta(y\gamma z)$, for all $x, y, z \in S$ and $\beta, \gamma \in \Gamma$.

A nonempty subset 'A' of a Γ-semigroup S is called Γ-subsemigroup of S if $A\Gamma A \subseteq A$. By a left (right) Γ-ideal of a Γ-semigroup S we mean a nonempty subset A of S such that $S\Gamma A \subseteq A$ ($A\Gamma S \subseteq A$) and a two sided Γ-ideal or simply a Γ-ideal is that which is both a left and a right Γ-ideal of S. A Γ-subsemigroup B of a Γ-semigroup S is called a Γ-bi-ideal of S if $B\Gamma S\Gamma B \subseteq B$. A nonempty subset A of T is called an interior ideal of T if $TTATT \subseteq A$.

3. BiΓ-ternary semigroup

3.1. Basic concepts

Here, we define the basic concepts of BiΓ-ternary semigroup.

Definition 3.1.1 Let $T = \{x, y, z, ...\}$ and $\Gamma = \{\alpha, \beta, \gamma, ...\}$ be two nonempty sets. Then we call T as a BiΓ-ternary semigroup if it satisfies,

(i) $(x\alpha y)\beta z \in T$

(ii) $((v\alpha w\beta x)\gamma y)\delta z = (v\alpha(w\beta x\gamma y))\delta z = v\alpha(w\beta(x\gamma y\delta z),$

for all $x, y, z, v, w \in S$ and $\alpha, \beta, \gamma, \delta \in \Gamma$.

Example 3.1.2 Let $T = \{4n + 3, n \in N\}$ and $\Gamma = \{4n + 1, n \in N\}$. Define the mapping $T \times \Gamma \times T \times \Gamma \times T \to T$ as $(x\gamma y)\delta z = x + \gamma + y + \delta + z$. Let $x, y, z \in T$ and $\gamma, \delta \in \Gamma$, then

$$\begin{aligned}
(x\alpha y)\beta z &= x + \alpha + y + \beta + z \\
&= 4n_1 + 3 + 4n' + 1 + 4n_2 + 3 + 4n'' + 1 + 4n_3 + 3 \\
&= 4(n_1 + n' + n_2 + n'' + n_3 + 2) + 3 \\
&= 4n + 3, \\
(\text{where, } n &= n_1 + n' + n_2 + n'' + n_3 + 2 \in N, \text{ for } n_1, n', n_2, n'', n_3 \in N\ )
\end{aligned}$$

Also it is clear that $((v\alpha w\beta x)\gamma y)\delta z = (v\alpha(w\beta x\gamma y))\delta z = v\alpha(w\beta(x\gamma y\delta z)$, for all $x, y, z, v, w \in S$ and $\alpha, \beta, \gamma, \delta \in \Gamma$. Hence T is a biΓ-ternary semigroup.

Example 3.1.3 Let $T = \{2n, n \in N\}, \Gamma = \{\alpha, \beta, \gamma ...\}$. Define $(x\gamma y)\delta z = x + y + z$, for all, $x, y, z \in T$ and $\gamma, \delta \in \Gamma$. Then T is a biΓ-ternary semigroup.

Example 3.1.4 Let $S = \{0, a, b, c\}$ and $\Gamma = \{\alpha, \beta\}$, consider the operation defined bellow

α	0	a	b	c
0	0	0	0	0
a	0	b	0	a
b	0	b	0	c
c	0	0	0	b

and

β	0	a	b	c
0	0	0	0	0
a	a	a	a	a
b	0	0	0	0
c	a	a	a	c

Then S is a neither a Γ-semigroup nor a biΓ-ternary semigroup, as we can see,

$$\begin{aligned}
(a\alpha c)\alpha c &= a \neq 0 = a\alpha(c\alpha c) \text{ and} \\
((a\alpha c)\beta b)\alpha a &= (a\beta b)\alpha a = a\alpha a = b \\
(a\alpha(c\beta b))\alpha a &= b \\
a\alpha((c\beta b)\alpha a) &= 0 \neq b \\
a\alpha(c\beta(b\alpha a) &= b \\
0 &\neq b.
\end{aligned}$$

which implies that S is not a biΓ-ternary semigroup.

Remark 3.1.5 Every Γ-semigroup is a biΓ-ternary semigroup but the converse is not true.

Example 3.1.6 Let $T = Z^-$ and $\Gamma \subseteq Z^+$. Define $(x\gamma y)\delta z$, for $x, y, z \in T$ and $\gamma, \delta \in \Gamma$ as the usual multiplication of integers. Then for $x, y, z \in T$ and $\gamma, \delta \in \Gamma, (x\gamma y)\delta z \in T$ and $((v\alpha w\beta x)\gamma y)\delta z = (v\alpha(w\beta x\gamma y))\delta z = v\alpha(w\beta(x\gamma y\delta z)$, for all $x, y, z, v, w \in S$ and $\alpha, \beta, \gamma, \delta \in \Gamma$. Hence T is a biΓ-ternary semigroup.

Now for $x, y \in T = Z^-$ and $\alpha \in \Gamma = Z^+$, $x\alpha y \notin T = Z^-$. Which shows that $T = Z^-$ is not a Γ-semigroup.

Example 3.1.7 Let $T = iR$,where, $i = \sqrt{-1}$ and R is the set of real numbers. If $\Gamma \subseteq R$ and $(x\alpha y)\beta z$ is defined as the usual multiplication of complex numbers. Then, for $x, y, z \in T$ there exist $a, b, c \in R$ so that $x = ai$, $y = bi$ and $z = ci$. For, $\alpha, \beta \in \Gamma$,

$$(x\alpha y)\beta z = (ai\alpha bi)\beta ci = abc\alpha\beta i^3 = -abc\alpha\beta i = ri, \text{ where } r = -abc\alpha\beta \in R.$$

Also,

$$((v\alpha w\beta x)\gamma y)\delta z = (v\alpha(w\beta x\gamma y))\delta z = v\alpha(w\beta(x\gamma y\delta z),$$

for all $x, y, z, v, w \in S$ and $\alpha, \beta, \gamma, \delta \in \Gamma$. Hence T is a biΓ-ternary semigroup. But, for $x = ai, y = bi \in T = iR$ and $\alpha \in \Gamma$,

$$x\alpha y = ai\alpha bi = ab\alpha i^2 = -ab\alpha \notin T = iR,$$

which shows that T is not a Γ-semigroup.

Definition 3.1.8 Let T be a biΓ-ternary semigroup and A be a nonempty subset of T. Then A is called a biΓ-ternary subsemigroup of T if,

$$A\Gamma A\Gamma A \subseteq A.$$

Example 3.1.9 let $T = N = \{1, 2, 3, ...\}$ and $\Gamma = \{4n + 2, n \in N\}$. Define $(x\alpha y)\beta z = x+\alpha+y+\beta+z$. Under this operation T is a biΓ-ternary semigroup.

Let $A = \{4n, n \in N\}$ be a nonempty subset of T. For $x, y, z \in A$ and $\alpha, \beta \in \Gamma$,

$$\begin{aligned}
(x\alpha y)\beta z &= (x+\alpha+y)+\beta+z \\
&= (4n_1+4n'+2+4n_2)+4n''+2+4n_3 \\
&= 4(n_1+n'+n_2+n''+n_3+1) \\
&= 4n \in A \\
\text{Where, } n &= n_1+n'+n_2+n''+n_3+1 \in N, \text{ for } n_1, n', n_2, n'', n_3 \in N.
\end{aligned}$$

which implies that $A\Gamma A\Gamma A \subseteq A$. Hence A is a biΓ-ternary subsemigroup.

Definition 3.1.10 Let T be a biΓ-ternary semigroup and A a nonempty subset of T. Then A is called a biΓ-left (right, lateral) ideal of T if

$$T\Gamma T\Gamma A \subseteq A \ (A\Gamma T\Gamma T \subseteq A, T\Gamma A\Gamma T \subseteq A)$$

A is called a biΓ-ideal of T if it is a biΓ-left, a biΓ-right and a biΓ-lateral ideal of T.

Example 3.1.11 Let $T = \{2n, n \in N\}, \Gamma = \{\alpha, \beta, \gamma, ...\}$ and $A = \{4n, n \in N\}$. Define, $(x\gamma y)\delta z = (2x + 2y) + z$, for $x, y, z \in T$ and $\gamma, \delta \in \Gamma$. Then T is a biΓ-ternary semigroup. For $x, y \in T$, $a \in A$ and $\alpha, \beta \in \Gamma$, we have

$$\begin{aligned}
(x\gamma y)\delta a &= (2x+2y)+a \\
&= 2(2n_1+2n_2)+4n', x=2n_1, y=2n_2 \text{ and } a=4n' \\
&= 4(n_1+n_2+n') \\
&= 4n \in A \text{ (where, } n=n_1+n_2+n' \in N, \text{ for } n_1, n_2, n' \in N.
\end{aligned}$$

which implies that $T\Gamma T\Gamma A \subseteq A$. Hence A is a biΓ-left ideal of T.
Now, consider

$$\begin{aligned}
(a\gamma x)\delta y &= (2a+2x)+y \\
&= (8n'+4n_1)+2n_2, x=2n_1, y=2n_2 \text{ and } a=4n' \\
&= 4(2n'+n_1)+2n_2. \\
\text{Taking } n' &= n_1=n_2=1, \Rightarrow (a\gamma x)\delta y = 4(2.1+1)+2.1 = 14 \notin A.
\end{aligned}$$

which implies that $A\Gamma T\Gamma T \notin A$. Similarly we can show that $T\Gamma A\Gamma T \notin A$. Hence A is neither a biΓ-right nor a biΓ-lateral ideal of T.

Remark 3.1.12 If we define, $(x\gamma y)\delta z = (x+2y)+2z$ and $(x\gamma y)\delta z = (2x+y)+2z$ respectively, then A is a biΓ-right and a biΓ-lateral ideal of T.

Example 3.1.13 In the above example if we define, $(x\gamma y)\delta z = (2x + 2y) + 2z$, then A is a biΓ-left, a biΓ-right and a biΓ-lateral ideal of T. Hence A is a bi Γ-ideal of T.

3.2. Main results

In what follows, let T denotes a biΓ-ternary semigroup, unless otherwise it is stated. In short, we shall use $B\Gamma TS(s)$ for biΓ-ternary semigroup(s), $B\Gamma TSS(s)$ for biΓ-ternary subsemigroup(s), $B\Gamma LI(s), B\Gamma RI(s), B\Gamma MI(s)$ and $B\Gamma I(s)$ for biΓ-left ideal(s), biΓ-right ideal(s), biΓ-lateral ideal(s) and biΓ-ideal(s) of a biΓ-ternary semigroup.

Proposition 3.2.1 *Let T be a $B\Gamma TS$ and $\phi \neq X \subseteq T$, then*

(i) $T\Gamma T\Gamma X$ *be a* $B\Gamma LI$ *of* T.

(ii) $X\Gamma T\Gamma T$ *be a* $B\Gamma RI$ *of* T.

(iii) $T\Gamma X\Gamma T \cup T\Gamma T\Gamma X\Gamma T\ \Gamma T$ *be a* $B\Gamma MI$ *of* T.

Proof. It follows directly from the definitions of $B\Gamma LI, B\Gamma RI$ and $B\Gamma MI$. ∎

Lemma 3.2.2 *Let T be a $B\Gamma TS$, for any $t \in T$, define,*

(i) $(t)_l = t \cup T\Gamma T\Gamma t$

(ii) $(t)_r = t \cup t\Gamma T\Gamma T$

(iii) $(t)_m = t \cup T\Gamma t\Gamma T \cup T\Gamma T\Gamma t\Gamma T\Gamma T$

(iv) $(t) = t \cup T\Gamma T\Gamma t \cup t\Gamma T\Gamma T \cup T\Gamma t\Gamma T \cup T\Gamma T\Gamma t\Gamma T\Gamma T$.

Then $(t)_l, (t)_r, (t)_m$ and (t) are $B\Gamma LI$, $B\Gamma RI$, $B\Gamma MI$ and $B\Gamma I$ of T respectively.

Proof. (i) Since $(t)_l = t \cup T\Gamma T\Gamma t$, then

$$\begin{aligned}
T\Gamma T\Gamma(t)_l &= T\Gamma T\Gamma(t \cup T\Gamma T\Gamma t)\\
&= T\Gamma T\Gamma t \cup T\Gamma T\Gamma T\Gamma T\Gamma t\\
&\subseteq T\Gamma T\Gamma t \cup T\Gamma T\Gamma t, \quad \text{since } T\Gamma T\Gamma T \subseteq T.\\
&= T\Gamma T\Gamma t \subseteq t \cup T\Gamma T\Gamma t = (t)_l\\
T\Gamma T\Gamma(t)_l &\subseteq (t)_l, \text{ implies that } (t)_l \text{ is bi}\Gamma\text{-left ideal of } T.
\end{aligned}$$

(ii) and (iii). Proof is similar as (i).

(iv) $(t) = t \cup T\Gamma T\Gamma t \cup t\Gamma T\Gamma T \cup T\Gamma t\Gamma T \cup T\Gamma T\Gamma t\Gamma T\Gamma T$. As,

$$\begin{aligned}
T\Gamma T\Gamma(t) &= T\Gamma T\Gamma(t \cup T\Gamma T\Gamma t \cup t\Gamma T\Gamma T \cup T\Gamma t\Gamma T \cup T\Gamma T\Gamma t\Gamma T\Gamma T) \\
&= T\Gamma T\Gamma t \cup T\Gamma T\Gamma T\Gamma T\Gamma t \cup T\Gamma T\Gamma t\Gamma T\Gamma T \cup T\Gamma T\Gamma T\Gamma t\Gamma T \\
&\quad \cup T\Gamma T\Gamma T\Gamma T\Gamma t\Gamma T\Gamma T \\
&\subseteq T\Gamma T\Gamma t \cup T\Gamma T\Gamma t \cup T\Gamma T\Gamma t\Gamma T\Gamma T \cup T\Gamma t\Gamma T \cup T\Gamma T\Gamma t\Gamma T\Gamma T \\
&= T\Gamma T\Gamma t \cup T\Gamma t\Gamma T \cup T\Gamma T\Gamma t\Gamma T\Gamma T \\
&\subseteq t \cup T\Gamma T\Gamma t \cup t\Gamma T\Gamma T \cup T\Gamma t\Gamma T \cup T\Gamma T\Gamma t\Gamma T\Gamma T = (t) \\
T\Gamma T\Gamma(t) &\subseteq (t),
\end{aligned}$$

implies that (t) is biΓ-left ideal. Similarly, we can show that it is biΓ-right ideal. Now consider,

$$\begin{aligned}
T\Gamma(t)\Gamma T &= T\Gamma(t \cup T\Gamma T\Gamma t \cup t\Gamma T\Gamma T \cup T\Gamma t\Gamma T \cup T\Gamma T\Gamma t\Gamma T\Gamma T)\Gamma T \\
&= T\Gamma t\Gamma T \cup T\Gamma T\Gamma T\Gamma t\Gamma T \cup T\Gamma t\Gamma T\Gamma T\Gamma T \cup T\Gamma T\Gamma t\Gamma T\Gamma T \\
&\quad \cup T\Gamma T\Gamma T\Gamma t\Gamma T\Gamma T\Gamma T \\
&\subseteq T\Gamma t\Gamma T \cup T\Gamma t\Gamma T \cup T\Gamma t\Gamma T \cup T\Gamma T\Gamma t\Gamma T\Gamma T \cup T\Gamma t\Gamma T \\
&= T\Gamma t\Gamma T \cup T\Gamma T\Gamma t\Gamma T\Gamma T \\
&\subseteq t \cup T\Gamma T\Gamma t \cup t\Gamma T\Gamma T \cup T\Gamma t\Gamma T \cup T\Gamma T\Gamma t\Gamma T\Gamma T = (t) \\
T\Gamma(t)\Gamma T &\subseteq (t), \text{ implies that } (t) \text{ is bi}\Gamma\text{-lateral ideal.}
\end{aligned}$$

Hence (t) is biΓ-ideal of T. ∎

Remark 3.2.3 The ideals $(t)_l, (t)_m, (t)_r, (t)$ are called principal biΓ-left, biΓ-right, biΓ-lateral and biΓ-ideal of T generated by t. Note that for any $a \in A \subseteq T$, $\underset{a\in A}{\cup}(a)_l = (A)_l$, $\underset{a\in A}{\cup}(a)_m = (A)_m$, $\underset{a\in A}{\cup}(a)_r = (A)_r$ and $\underset{a\in A}{\cup}(a) = (A)$ are biΓ-left ideal, biΓ-right ideal, biΓ-lateral ideal and biΓ-ideal of T generated by A.

Lemma 3.2.4 *Let T be a $B\Gamma TS$. Then*

(i) *The orbitrary intersection of $B\Gamma TSS(s)$ of T is again a $B\Gamma TSS$ of T.*

(ii) *The orbitrary intersection of $B\Gamma LI(s)$ ($B\Gamma RI(s)$, $B\Gamma MI(s)$, $B\Gamma I(s)$) of T is a $B\Gamma LI$ ($B\Gamma RI$, $B\Gamma MI$, $B\Gamma I$) of T.*

Proof. (i) Let $\{A_i, i \in I\}$ be a collection of biΓ-ternary subsemigroups of T, then $A_i\Gamma A_i\Gamma A_i \subseteq A_i$, for all $i \in I$. Also $\underset{i\in I}{\cap} A_i \subseteq A_i$ for all $i \in I$ then,

$$(\underset{i\in I}{\cap} A_i)\Gamma(\underset{i\in I}{\cap} A_i)\Gamma(\underset{i\in I}{\cap} A_i) \subseteq A_i\Gamma A_i\Gamma A_i \subseteq A_i, \text{ for all } i \in I.$$

implies that

$$(\underset{i\in I}{\cap} A_i)\Gamma(\underset{i\in I}{\cap} A_i)\Gamma(\underset{i\in I}{\cap} A_i) \subseteq \underset{i\in I}{\cap} A_{i.}$$

Hence $\bigcap_{i\in I} A_i$ is a biΓ-ternary subsemigroup of T.

(ii) Let $\{L_i, i \in I\}$ be a collection of biΓ-left ideals of Tthen $T\Gamma T\Gamma L_i \subseteq L_i$, for all $i \in I$. Also $\bigcap_{i\in I} L_i \subseteq L_i$ for all $i \in I$ then,

$$\begin{aligned} T\Gamma T\Gamma(\bigcap_{i\in I} L_i) &\subseteq T\Gamma T\Gamma L_i \subseteq L_i, \text{ for all } i \in I. \\ T\Gamma T\Gamma(\bigcap_{i\in I} L_i) &\subseteq L_i, \text{ for all } i \in I. \text{ Implies that} \\ T\Gamma T\Gamma(\bigcap_{i\in I} L_i) &\subseteq \bigcap_{i\in I} L_i. \end{aligned}$$

Hence $\bigcap_{i\in I} L_i$ is a biΓ-left ideal of T. Similarly, we can prove for biΓ-right and biΓ-lateral ideal of T. ∎

Definition 3.2.5 A nonempty subset Q of a biΓ-ternary semigroup T is called a biΓ-quasi-ideal of T if

$$Q\Gamma T\Gamma T \cap T\Gamma Q\Gamma T \cap T\Gamma T\Gamma Q \subseteq Q$$

and

$$Q\Gamma T\Gamma T \cap T\Gamma T\Gamma Q\Gamma T\Gamma T \cap T\Gamma T\Gamma Q \subseteq Q.$$

Definition 3.2.6 A biΓ-bi-ideal B of a biΓ-ternary semigroup T is a biΓ-ternary subsemigroup of T satisfying,

$$B\Gamma T\Gamma B\Gamma T\Gamma B \subseteq B.$$

We will write $B\Gamma QI(s)$ and $B\Gamma BI(s)$ for biΓ-quasi-ideal(s) and biΓ-bi-ideal(s), respectively.

Proposition 3.2.7 *Let T be a $B\Gamma TS$. Then every $B\Gamma QI$ of T is a $B\Gamma TSS$ of T.*

Proof. We suppose that Q is a biΓ-quasi-ideal of T. Since

$$\begin{aligned} Q\Gamma Q\Gamma Q &\subseteq Q\Gamma T\Gamma T, \ Q\Gamma Q\Gamma Q \subseteq T\Gamma Q\Gamma T \text{ and } Q\Gamma Q\Gamma Q \subseteq T\Gamma T\Gamma Q. \\ &\Rightarrow Q\Gamma Q\Gamma Q \subseteq Q\Gamma T\Gamma T \cap T\Gamma Q\Gamma T \cap T\Gamma T\Gamma Q \subseteq Q. \\ &\Rightarrow Q\Gamma Q\Gamma Q \subseteq Q, \text{ since } Q \text{ is bi}\Gamma\text{-quasi-ideal}. \end{aligned}$$

Implies that Q is biΓ-ternary subsemigroup of T. ∎

Proposition 3.2.8 *The arbitrary intersection of $B\Gamma QI(s)$ of T is a $B\Gamma QI$ of T.*

Proof. Straightforward. ∎

Remark 3.2.9 Note that a $B\Gamma LI$ ($B\Gamma RI$, $B\Gamma MI$)of T is also $B\Gamma QI$ of T but any $B\Gamma QI$ of T may not be a $B\Gamma LI$ ($B\Gamma RI$, $B\Gamma MI$)of T, so we have following lemma.

Lemma 3.2.10 *Let T be a $B\Gamma TS$. Then every $B\Gamma LI$ ($B\Gamma RI$, $B\Gamma MI$) of T is a $B\Gamma QI$ of T.*

Proof. Let L be a biΓ-left ideal of T, then

$$T\Gamma T\Gamma L \subseteq L,$$

which implies that

$$L\Gamma T\Gamma T \cap T\Gamma L\Gamma T \cap T\Gamma T\Gamma L \subseteq L,$$

also

$$L\Gamma T\Gamma T \cap T\Gamma T\Gamma L\Gamma T\Gamma T \cap T\Gamma T\Gamma L \subseteq L.$$

Hence L is biΓ-quasi-ideal of T. Other cases are similar. ∎

Lemma 3.2.11 *A nonempty subset Q of T is a $B\Gamma QI$ of T if and only if it is an intersection of a $B\Gamma LI$, a $B\Gamma MI$ and a $B\Gamma RI$ of T.*

Proof. Let L, M and R be the biΓ-left, biΓ-lateral and biΓ-right ideals of T. Let $Q = R \cap M \cap L$, then

$$\begin{aligned}
&Q\Gamma T\Gamma T \cap T\Gamma Q\Gamma T \cap T\Gamma T\Gamma Q\\
&= (R \cap M \cap L)\Gamma T\Gamma T \cap T\Gamma(R \cap M \cap L)\Gamma T \cap T\Gamma T\Gamma(R \cap M \cap L)\\
&\subseteq R\Gamma T\Gamma T \cap T\Gamma M\Gamma T \cap T\Gamma T\Gamma L\\
&\subseteq R \cap M \cap L,\ \text{since, } L, M\ \text{ and } R,\ \text{are bi}\Gamma\text{-left, bi}\Gamma\text{-lateral, bi}\Gamma - right\ \text{ideals.}\\
&= Q.
\end{aligned}$$

Similarly, $Q\Gamma T\Gamma T \cap T\Gamma T\Gamma Q\Gamma T\Gamma T \cap T\Gamma T\Gamma Q \subseteq Q$. Hence Q is a biΓ-quasi-ideal of T.

Conversely, let Q be a biΓ-quasi-ideal of T. For any $q \in Q$, $(q)_l, (q)_m, (q)_r$, be the biΓ-left, biΓ-lateral and biΓ-right ideals of T generated by q, then

$$\begin{aligned}
q &\in (q)_r \cap (q)_m \cap (q)_l\\
\bigcup_{q\in Q}\{q\} &\subseteq \bigcup_{a\in Q}(q)_r \cap \bigcup_{q\in Q}(q)_m \cap \bigcup_{q\in Q}(q)_l\\
Q &\subseteq (Q)_r \cap (Q)_m \cap (Q)_l.
\end{aligned}$$

Since, $(Q)_l = Q \cup Q\Gamma T\Gamma T, (Q)_m = Q \cup T\Gamma Q\Gamma T \cup T\Gamma T\Gamma Q\Gamma T\Gamma T$ and $(Q)_r = Q \cup T\Gamma T\ \Gamma Q$, then

$$\begin{aligned}
&(Q)_r \cap (Q)_m \cap (Q)_l\\
&= (Q \cup Q\Gamma T\Gamma T) \cap (Q \cup T\Gamma Q\Gamma T \cup T\Gamma T\Gamma Q\Gamma T\Gamma T) \cap Q \cup T\Gamma T\Gamma Q\\
&= Q \cup (Q\Gamma T\Gamma T \cap T\Gamma Q\Gamma T \cap T\Gamma T\Gamma Q) \cup (Q\Gamma T\Gamma T \cap T\Gamma T\Gamma Q\Gamma T\Gamma T \cap T\Gamma T\Gamma Q)\\
&\subseteq Q,\ \text{Since } Q \text{ is bi}\Gamma\text{-quasi-ideal of } T,
\end{aligned}$$

which implies that $Q = (Q)_r \cap (Q)_m \cap (Q)_l$. Where, $(Q)_r$, $(Q)_m$, and $(Q)_l$ are biΓ-left ideal, biΓ-lateral ideal and a biΓ-right ideal of T. Hence the proof. ∎

Lemma 3.2.12 *Let T be a $B\Gamma TS$ and L_s, M_s, R_s be the smallest $B\Gamma LI$, $B\Gamma MI$, $B\Gamma RI$ of T. The $R_s \cap M_s \cap L_s$ is the smallest $B\Gamma QI$ of T.*

Proof. Straightforward. ∎

Lemma 3.2.13 *Let T be a $B\Gamma TS$. If Q be a $B\Gamma QI$ of T and S be a $B\Gamma TSS$ of T, then $Q \cap S$ is a $B\Gamma QI$ of S.*

Proof. Let Q be the biΓ-quasi-ideal and S be a biΓ-ternary subsemigroup of T. If $Q \cap S \neq \phi$, then as

$$\begin{aligned} & (Q \cap S)\Gamma S\Gamma S \cap S\Gamma(Q \cap S)\Gamma S \cap S\Gamma S\Gamma(Q \cap S) \\ \subseteq\ & S\Gamma S\Gamma S \cap S\Gamma S\Gamma S \cap S\Gamma S\Gamma S, \text{ since } Q \cap S \subseteq S. \\ =\ & S\Gamma S\Gamma S \subseteq S. \end{aligned}$$

Also,

$$\begin{aligned} & (Q \cap S)\Gamma S\Gamma S \cap S\Gamma(Q \cap S)\Gamma S \cap S\Gamma S\Gamma(Q \cap S) \\ \subseteq\ & Q\Gamma S\Gamma S \cap S\Gamma Q\Gamma S \cap S\Gamma S\Gamma Q, \text{ since } Q \cap S \subseteq Q. \\ \subseteq\ & Q\Gamma T\Gamma T \cap T\Gamma Q\Gamma T \cap T\Gamma T\Gamma Q, \text{ since } S \subseteq T. \\ \subseteq\ & Q, \text{ since } Q \text{ is bi}\Gamma\text{-quasi-ideal of } T, \end{aligned}$$

which implies that

$$(Q \cap S)\Gamma S\Gamma S \cap S\Gamma(Q \cap S)\Gamma S \cap S\Gamma S\Gamma(Q \cap S) \subseteq Q \cap S.$$

Similarly,

$$(Q \cap S)\Gamma S\Gamma S \cap S\Gamma S\Gamma(Q \cap S)\Gamma S\Gamma S \cap S\Gamma S\Gamma(Q \cap S) \subseteq Q \cap S.$$

Hence, $Q \cap S$ is a biΓ-quasi-ideal of S. ∎

Proposition 3.2.14 *Let T be a $B\Gamma TS$ and $X, Y(\neq \phi) \subseteq T$, then $X\Gamma T\Gamma Y$ is a $B\Gamma BI$ of T.*

Proof. Let $B = X\Gamma T\Gamma Y$, as

$$\begin{aligned} B\Gamma B\Gamma B &= (X\Gamma T\Gamma Y)\Gamma(X\Gamma T\Gamma Y)\Gamma(X\Gamma T\Gamma Y) \\ &= X\Gamma T\Gamma Y\Gamma X\Gamma T\Gamma Y\Gamma X\Gamma T\Gamma Y \\ &\subseteq X\Gamma T\Gamma T\Gamma T\Gamma T\Gamma T\Gamma T\Gamma T\Gamma Y \\ &\subseteq X\Gamma T\Gamma Y = B \end{aligned}$$

which implies that $B = X\Gamma T\Gamma Y$ is biΓ-ternary subsemigroup of T. Also

$$\begin{aligned} B\Gamma T\Gamma B\Gamma T\Gamma B &= (X\Gamma T\Gamma Y)\Gamma T\Gamma(X\Gamma T\Gamma Y)\Gamma T\Gamma(X\Gamma T\Gamma Y) \\ &= X\Gamma T\Gamma Y\Gamma T\Gamma X\Gamma T\Gamma Y\Gamma T\Gamma X\Gamma T\Gamma Y \\ &\subseteq X\Gamma T\Gamma T\Gamma T\Gamma T\Gamma T\Gamma T\Gamma T\Gamma T\Gamma T\Gamma Y \\ &\subseteq X\Gamma T\Gamma Y = B \end{aligned}$$

Hence B is a biΓ-bi-ideal of T. ∎

Theorem 3.2.15 *Let* $X, Y, Z(\neq \phi) \subseteq T$, *then* $X\Gamma Y\Gamma Z$ *is a bi*Γ*-bi-ideal of* T *if any one of* X, Y *or* Z *is either a bi*Γ*-left ideal or a bi*Γ*-right ideal or a bi*Γ*-lateral ideal of* T.

Proof. We suppose that Z is biΓ-left ideal of T then $T\Gamma T\Gamma Z \subseteq Z$. Let $B = X\Gamma Y\Gamma Z$ then as,

$$\begin{aligned} B\Gamma B\Gamma B &= (X\Gamma Y\Gamma Z)\Gamma(X\Gamma Y\Gamma Z)\Gamma(X\Gamma Y\Gamma Z) \\ &\subseteq X\Gamma Y\Gamma T\Gamma T\Gamma T\Gamma T\Gamma T\Gamma T\Gamma Z \\ &\subseteq X\Gamma Y\Gamma T\Gamma T\Gamma Z, \\ &\subseteq X\Gamma Y\Gamma Z = B, \text{ since } T\Gamma T\Gamma Z \subseteq Z. \end{aligned}$$

Implies that, $B = X\Gamma Y\Gamma Z$ is a biΓ-ternary subsemigroup of T.

Also

$$\begin{aligned} B\Gamma T\Gamma B\Gamma T\Gamma B &= (X\Gamma Y\Gamma Z)\Gamma T\Gamma(X\Gamma Y\Gamma Z)\Gamma T\Gamma(X\Gamma Y\Gamma Z) \\ &\subseteq X\Gamma Y\Gamma T\Gamma T\Gamma T\Gamma T\Gamma T\Gamma T\Gamma T\Gamma T\Gamma Z \\ &\subseteq X\Gamma Y\Gamma T\Gamma T\Gamma Z \\ &\subseteq X\Gamma Y\Gamma Z = B, \text{ since } T\Gamma T\Gamma Z \subseteq Z. \end{aligned}$$

Hence $B = X\Gamma Y\Gamma Z$ is a biΓ-bi-ideal of T. ∎

Lemma 3.2.16 *Let* T *be a* $B\Gamma TS$ *then every* $B\Gamma QI$ *of* T *is a* $B\Gamma BI$ *of* T.

Proof. Let Q be a biΓ-quasi-ideal of a biΓ-ternary semigroup T then

$$\begin{aligned} Q\Gamma T\Gamma T \cap T\Gamma Q\Gamma T \cap T\Gamma T\Gamma Q &\subseteq Q \text{ and} \\ Q\Gamma T\Gamma T \cap T\Gamma T\Gamma Q\Gamma T\Gamma T \cap T\Gamma T\Gamma Q &\subseteq Q. \end{aligned}$$

Now, as

$$\begin{aligned} Q\Gamma T\Gamma Q\Gamma T\Gamma Q &\subseteq Q\Gamma T\Gamma T\Gamma T\Gamma T \subseteq Q\Gamma T\Gamma T, \text{ and} \\ Q\Gamma T\Gamma Q\Gamma T\Gamma Q &\subseteq T\Gamma T\Gamma T\Gamma T\Gamma Q \subseteq T\Gamma T\Gamma Q, \text{ also} \\ Q\Gamma T\Gamma Q\Gamma T\Gamma Q &\subseteq T\Gamma T\Gamma Q\Gamma T\Gamma T, \end{aligned}$$

which implies that

$$Q\Gamma T\Gamma Q\Gamma T\Gamma Q \subseteq Q\Gamma T\Gamma T \cap T\Gamma T\Gamma Q\Gamma T\Gamma T \cap T\Gamma T\Gamma Q,$$

$$\Rightarrow Q\Gamma T\Gamma Q\Gamma T\Gamma Q \subseteq Q.$$

Hence Q is a biΓ-bi-ideal of T. ∎

Note that the converse of above lemma is not true (see Example 3.3.10).

Corollary 3.2.17 *Let* T *be a* $B\Gamma TS$ *then every* $B\Gamma LI$ $(B\Gamma RI$, $B\Gamma MI)$*of a* T *is a* $B\Gamma BI$ *of* T.

Proof. Follows from *Lemma* 3.2.10 and *Lemma* 3.2.16. ∎

Theorem 3.2.18 *Let T be a $B\Gamma TS$ and A be a $B\Gamma I$ and Q be a $B\Gamma QI$ of T then $A\cap Q$ is a $B\Gamma BI$ of T.*

Proof. Since $A\cap Q\subseteq A$ and $A\cap Q\subseteq Q$, where A is a biΓ-ternary subsemigroup of T and Q is a biΓ-quasi-ideal of T then as,

$$\begin{aligned}(A\cap Q)\Gamma(A\cap Q)\Gamma(A\cap Q) &\subseteq A\Gamma A\Gamma A\subseteq A, \text{ and}\\ (A\cap Q)\Gamma(A\cap Q)\Gamma(A\cap Q) &\subseteq Q\Gamma Q\Gamma Q\subseteq Q,\end{aligned}$$

$$\Rightarrow (A\cap Q)\Gamma(A\cap Q)\Gamma(A\cap Q)\subseteq A\cap Q,$$

implies that $A\cap Q$ is a biΓ-ternary subsemigroup of T.

Since Q is biΓ-quasi-ideal and hence a biΓ-bi-ideal then,

$$(A\cap Q)\Gamma T\Gamma(A\cap Q)\Gamma T\Gamma(A\cap Q)\subseteq Q\Gamma T\Gamma Q\Gamma T\Gamma Q\subseteq Q.$$

Also, since A is a biΓ-ideal and hence a biΓ-lateral ideal of T then

$$(A\cap Q)\Gamma T\Gamma(A\cap Q)\Gamma T\Gamma(A\cap Q)\subseteq A\Gamma(T\Gamma A\Gamma T)\Gamma A\subseteq A\Gamma A\Gamma A\subseteq A.$$

This implies that $(A\cap Q)\Gamma T\Gamma(A\cap Q)\Gamma T\Gamma(A\cap Q)\subseteq A\cap Q$. Hence $A\cap Q$ is biΓ-bi-ideal of T. ∎

Lemma 3.2.19 *Let T be a $B\Gamma TS$, then the arbitrary intersection of $B\Gamma BI(s)$ of T is a $B\Gamma BI$ of T.*

Proof. Straightforward. ∎

3.3. Regular biΓ-ternary semigroup

Definition 3.3.1 Let T be a $B\Gamma TS$. An element $a\in T$ is called a biΓ-regular element of T if $a\in a\Gamma T\Gamma a\Gamma T\Gamma a$, i.e. there exists $x,y\in T$ and $\alpha,\beta,\gamma,\delta\in\Gamma$ such that $a=a\alpha x\beta a\gamma y\delta a$. A $B\Gamma TS$, T is called a regular biΓ-ternary semigroup if its every element is a biΓ-regular element.

Lemma 3.3.2 *Every $B\Gamma MI$ ideal of a regular $B\Gamma TS$ is a regular $B\Gamma TS$.*

Proof. Let T be a regular $B\Gamma TS$ and M be a $B\Gamma MI$ of T then $T\Gamma M\Gamma T\subseteq M$. Let $a\in M$ then $a\in T$ and T is regular, so there exist $x,y\in T$, $\alpha,\beta,\gamma,\delta\in T$, such that

$$\begin{aligned} a &= a\alpha x\beta a\gamma y\delta a\\ &= a\alpha x\beta a\gamma y\delta a\alpha x\beta a\gamma y\delta a = a\alpha(x\beta a\gamma y)\delta a\alpha(x\beta a\gamma y)\delta a\\ &= a\alpha m\delta a\alpha m\delta a, \text{ where } m=x\beta a\gamma y\in T\Gamma M\Gamma T\subseteq M.\\ &\in a\Gamma M\Gamma a\Gamma M\Gamma a,\end{aligned}$$

which implies that a is regular in M. Hence M is regular biΓ-ternary semigroup. ∎

Note that a $B\Gamma LI$ and a $B\Gamma RI$ of a regular $B\Gamma TS$ may not be a regular $B\Gamma TS$.

Corollary 3.3.3 *Every $B\Gamma I$ of a regular $B\Gamma TS$ is a regular $B\Gamma TS$.*

Proof. Straightforward. ■

Definition 3.3.4 Let T be a $B\Gamma TS$ and I be a $B\Gamma I$ of T. Then I is called an idempotent $B\Gamma I$ of T if $I\Gamma I\Gamma I = I$.

Lemma 3.3.5 *Let T be a regular $B\Gamma TS$. Then every $B\Gamma MI$ of T is an idempotent $B\Gamma I$ of T.*

Proof. let M be a biΓ-lateral ideal of a regular biΓ-ternary semigroup T then $M\Gamma M\Gamma M \subseteq T\Gamma M\Gamma T \subseteq M$. For any $m \in M$, $m \in T$, (Since $M \subseteq T$) and T is regular, $m \in m\Gamma T\Gamma m\Gamma T\Gamma m$ implies that

$$\begin{aligned}
m &= m\alpha x\beta m\gamma y\delta m, \text{ for, } x, y \in T \text{ and } \alpha, \beta, \gamma, \delta \in \Gamma. \\
&= m\alpha(x\beta m\gamma y)\delta m \\
&\in M\Gamma M\Gamma M, \text{ implies that} \\
M &\subseteq M\Gamma M\Gamma M.
\end{aligned}$$

Hence $M\Gamma M\Gamma M = M$, implies that M is idempotent. ■

Theorem 3.3.6 *Let T be a $B\Gamma TS$, then the following statements are equivalent,*

(i) *T is regular.*

(ii) *$R\Gamma M\Gamma L = R \cap M \cap L$, where, L, R and M are $B\Gamma LI$, $B\Gamma RI$ and $B\Gamma MI$ of T.*

(iii) *$(a)_r\Gamma(b)_m\Gamma(c)_l = (a)_r \cap (b)_m \cap (c)_l$, for every $a, b, c \in T$.*

(iv) *$(t)_r\Gamma(t)_m\Gamma(t)_l = (t)_r \cap (t)_m \cap (t)_l$, for each $t \in T$.*

Proof. (i) ⇒ (ii) Let T be a regular $B\Gamma TS$ and R, M, L be the biΓ-right, biΓ-lateral and biΓ-left ideals of T then as

$$\begin{aligned}
R\Gamma M\Gamma L &\subseteq R\Gamma T\Gamma T \subseteq R \\
R\Gamma M\Gamma L &\subseteq T\Gamma M\Gamma T \subseteq M \text{ and} \\
R\Gamma M\Gamma L &\subseteq T\Gamma T\Gamma L \subseteq L, \text{ implies that} \\
R\Gamma M\Gamma L &\subseteq R \cap M \cap L.
\end{aligned}$$

Now let $a \in R \cap M \cap L \subseteq T$ and T is regular then there exist $x, y \in T, \alpha, \beta, \gamma, \delta \in \Gamma$ such that $a = a\alpha x\beta a\gamma y\delta a$. Also

$$\begin{aligned}
a &= a\alpha x\beta a\gamma y\delta a = a\alpha(x\beta a\gamma y)\delta a \in R\Gamma M\Gamma L \\
&\Rightarrow R \cap M \cap L \subseteq R\Gamma M\Gamma L.
\end{aligned}$$

Hence $R \cap M \cap L = R\Gamma M\Gamma L$.

(ii) $\Rightarrow$ (iii) Let $R \cap M \cap L = R\Gamma M\Gamma L$, for every bi$\Gamma$-right R, biΓ-lateral M and biΓ-left ideal L of T. For $a, b, c \in T$, taking $R = (a)_r$, $M = (b)_m$ and $L = (c)_l$, by (ii), we have $(a)_r\Gamma(b)_m\Gamma(c)_l = R\Gamma M\Gamma L = R \cap M \cap L = (a)_r \cap (b)_m \cap (c)_l$.

(iii) $\Rightarrow$ (iv) Taking $a = b = c = t$, then (iii) becomes $(t)_r\Gamma(t)_m\Gamma(t)_l = (t)_r \cap (t)_m \cap (t)_l$.

(iv) $\Rightarrow$ (i) For any $t \in T$, the biΓ-right ideal, biΓ-lateral ideal and biΓ-left ideal of T generated by t are given as,

$$\begin{aligned} (t)_r &= t \cup t\Gamma T\Gamma T, \\ (t)_m &= t \cup T\Gamma t\Gamma T \cup T\Gamma T\Gamma t\Gamma T\Gamma T \\ (t)_l &= t \cup T\Gamma T\Gamma t. \end{aligned}$$

By given condition

$$\begin{aligned} (t)_r \cap (t)_m \cap (t)_l &= (t)_r\Gamma(t)_m\Gamma(t)_l \\ &= (t \cup t\Gamma T\Gamma T)\Gamma(t \cup T\Gamma t\Gamma T \cup T\Gamma T\Gamma t\Gamma T\Gamma T)\Gamma t \cup T\Gamma T\Gamma t \\ &= (t\Gamma t\Gamma t) \cup (t\Gamma T\Gamma t\Gamma T\Gamma t) \cup (t\Gamma T\Gamma T\Gamma t\Gamma t) \cup (t\Gamma t\Gamma T\Gamma T\Gamma t) \\ &\quad \cup (t\Gamma T\Gamma T\Gamma t\Gamma T\Gamma T\Gamma t). \end{aligned}$$

Since, $t \in (t)_r \cap (t)_m \cap (t)_l$.

If $t \in t\Gamma t\Gamma t$, then

$$\begin{aligned} t &= t\alpha t\beta t, \text{ for } \alpha, \beta \in \Gamma. \\ &= t\alpha t\beta t\alpha t\beta t \in t\Gamma T\Gamma t\Gamma T\Gamma t, \ t \text{ is regular.} \end{aligned}$$

If $t \in t\Gamma T\Gamma t\Gamma T\Gamma t$, then t is regular.

If $t \in t\Gamma T\Gamma T\Gamma t\Gamma t$, then

$$\begin{aligned} t &= t\alpha x\beta y\gamma t\delta t, \text{ for } x, y \in T, \ \alpha, \beta, \gamma, \delta \in \Gamma. \\ &= t\alpha(x\beta y\gamma t)\delta t\alpha(x\beta y\gamma t)\delta t \\ &\in t\Gamma T\Gamma t\Gamma T\Gamma t, \text{ since, } x\beta y\gamma t \in T, \ \Rightarrow t \text{ is regular.} \end{aligned}$$

If $t \in t\Gamma t\Gamma T\Gamma T\Gamma t$, then

$$\begin{aligned} t &= t\alpha t\beta x\gamma y\delta t, \text{ for } x, y \in T, \ \alpha, \beta, \gamma, \delta \in \Gamma. \\ &= t\alpha(t\beta x\gamma y)\delta t\alpha(t\beta x\gamma y)\delta t \\ &\in t\Gamma T\Gamma t\Gamma T\Gamma t, \text{ since, } t\beta x\gamma y \in T, \ \Rightarrow t \text{ is regular.} \end{aligned}$$

If $t \in t\Gamma T\Gamma T\Gamma t\Gamma T\Gamma T\Gamma t$, then as

$$\begin{aligned} t\Gamma T\Gamma T\Gamma t\Gamma T\Gamma T\Gamma t &\subseteq t\Gamma T\Gamma T\Gamma T\Gamma T\Gamma T\Gamma t, \text{ since, } t \in T. \\ &\subseteq t\Gamma T\Gamma t, \text{ since } T\Gamma T\Gamma T \subseteq T. \\ t &\in t\Gamma T\Gamma t, \text{then} \\ t &= t\alpha x\beta t, \ x \in T, \ \alpha, \beta \in \Gamma. \\ &= t\alpha x\beta t\alpha x\beta t \in t\Gamma T\Gamma t\Gamma T\Gamma t, \ \Rightarrow t \text{ is regular.} \end{aligned}$$

Since $t \in T$ is arbitrary. Hence T is regular biΓ-ternary semigroup. ∎

Theorem 3.3.7 *Let T be a $B\Gamma TS$, then the following statements are equivalent,*

(i) *T is regular*

(ii) *$R\Gamma T\Gamma L = R \cap L$, for every $B\Gamma RI$, R and $B\Gamma LI$, L of T.*

(iii) *$(s)_r \Gamma T \Gamma (t)_l = (s)_r \cap (t)_l$, for every $s, t \in T$.*

(iv) *$(t)_r \Gamma T \Gamma (t)_l = (t)_r \cap (t)_l$, for each $t \in T$.*

Proof. Straightforward. ■

Theorem 3.3.8 *Let T be a $B\Gamma TS$ then the following statements are equivalent,*

(i) *T is regular.*

(ii) *$B\Gamma T\Gamma B\Gamma T\Gamma B = B$, for every $B\Gamma BI$, B of T.*

(iii) *$Q\Gamma T\Gamma Q\Gamma T\Gamma Q = Q$, for every $B\Gamma QI$, Q of T.*

Proof. (i) $\Rightarrow$ (ii) Let T be a $B\Gamma TS$ and B be a $B\Gamma BI$ of T then $B\Gamma T\Gamma B\Gamma T\Gamma B \subseteq B$. Now, for $b \in B \subseteq T$, where T is regular, $b \in b\Gamma T\Gamma b\Gamma T\Gamma b \subseteq B\Gamma T\Gamma B\Gamma T\Gamma B$, implies that, $B \subseteq B\Gamma T\Gamma B\Gamma T\Gamma B$. Hence $B\Gamma T\Gamma B\Gamma T\Gamma B = B$.

(ii) $\Rightarrow$ (iii) We suppose that *(ii)* holds and Q be a *bi*Γ-*quasi-ideal* of T then by *Lemma* 3.2.16, Q is a *bi*Γ-*bi-ideal* of T and by *(ii)*, $Q\Gamma T\Gamma Q\Gamma T\Gamma Q = Q$, holds.

(iii) $\Rightarrow$ (i) We suppose that for any *bi*Γ-*quasi-ideal* Q of T, $Q\Gamma T\Gamma Q\Gamma T\Gamma Q = Q$ holds. Let R, M and L be the the biΓ-right, biΓ-lateral and biΓ-left ideals of T respectively. Then $R \cap M \cap L = Q_1$ be a *bi*Γ-*quasi-ideal* of T and by the supposition

$$\begin{aligned} Q_1\Gamma T\Gamma Q_1\Gamma T\Gamma Q_1 &= Q_1 = R \cap M \cap L, \text{ and} \\ Q_1\Gamma T\Gamma Q_1\Gamma T\Gamma Q_1 &\subseteq R\Gamma T\Gamma M\Gamma T\Gamma L \subseteq R\Gamma M\Gamma L, \text{ since } M \text{ is lateral ideal.} \\ \text{This implies that, } R \cap M \cap L &\subseteq R\Gamma M\Gamma L. \end{aligned}$$

Also,

$$\begin{aligned} R\Gamma M\Gamma L &\subseteq R\Gamma T\Gamma T \subseteq R,\ R\Gamma M\Gamma L \subseteq M \text{ and } R\Gamma M\Gamma L \subseteq L, \\ \text{implies that, } R\Gamma M\Gamma L &\subseteq R \cap M \cap L. \end{aligned}$$

Hence $R\Gamma M\Gamma L = R \cap M \cap L$, which implies that by *Theorem* 3.3.6, T is regular. ■

Lemma 3.3.9 *Let T be a $B\Gamma TS$. Then T is regular if and only if every $B\Gamma I$ of T is an idempotent $B\Gamma I$.*

Proof. Let T be a regular $B\Gamma TS$ and A be a biΓ-ideal of T. Then $A\Gamma A\Gamma A \subseteq A$. Now, let $a \in A \subseteq T$ and T is regular then there exist $x, y \in T$ and $\alpha, \beta, \gamma, \delta \in \Gamma$ such that

$$\begin{aligned} a &= a\alpha(x\beta a\gamma y)\delta a, \\ &\in A\Gamma A\Gamma A, \text{ since, } x\beta a\gamma y \in T\Gamma A\Gamma T \subseteq A. \\ A &\subseteq A\Gamma A\Gamma A. \end{aligned}$$

Hence $A\Gamma A\Gamma A = A$ i.e. A is idempotent.

Conversely, we suppose that every biΓ-ideal of T is idempotent. Let A, B, C be three biΓ-ideals of T then $A \cap B \cap C$ is also a biΓ-ideal of T and hence by supposition

$$(A \cap B \cap C)\Gamma(A \cap B \cap C)\Gamma(A \cap B \cap C) = (A \cap B \cap C).$$

Since, A, B, C are biΓ-ideals of T then $A\Gamma B\Gamma C \subseteq A\Gamma T\Gamma T \subseteq A$, $A\Gamma B\Gamma C \subseteq T\Gamma B\Gamma T \subseteq B$ and $A\Gamma B\Gamma C \subseteq T\Gamma T\Gamma C \subseteq C$, implies that $A\Gamma B\Gamma C \subseteq A \cap B \cap C$.

Also, $A \cap B \cap C \subseteq A, A \cap B \cap C \subseteq B$ and $A \cap B \cap C \subseteq C$, implies that

$$\begin{aligned} (A \cap B \cap C)\Gamma(A \cap B \cap C)\Gamma(A \cap B \cap C) &\subseteq A\Gamma B\Gamma C, \\ \text{implies that, } A \cap B \cap C &\subseteq A\Gamma B\Gamma C. \end{aligned}$$

Hence $A\Gamma B\Gamma C = A \cap B \cap C$ and by *Theorem* 3.3.6, T is regular. ∎

Example 3.3.10 Let T be a $B\Gamma TS$. Then a $B\Gamma BI$ of T may not be a $B\Gamma QI$ of T.

Proof. Let T be a $B\Gamma TS$, which is not regular. Let L_s, M_s and R_s be the smallest *biΓ-left ideal, biΓ-lateral ideal and biΓ-right ideal of T* then by Lemma 3.2.12 and 3.2.16, $R_s\Gamma M_s\Gamma L_s$ is a *biΓ-bi-ideal of T*. We claim that $R_s\Gamma M_s\Gamma L_s$ is not a *biΓ-quasi ideal of T*, otherwise, consider as

$$\begin{aligned} R_s\Gamma M_s\Gamma L_s &\subseteq R_s\Gamma T\Gamma T \subseteq R_s, \text{ since } R_s \text{ is bi}\Gamma\text{-right ideal.} \\ R_s\Gamma M_s\Gamma L_s &\subseteq T\Gamma M_s\Gamma T \subseteq M_s, \text{ since } M_s \text{ is bi}\Gamma\text{-lateral ideal.} \\ R_s\Gamma M_s\Gamma L_s &\subseteq T\Gamma T\Gamma L_s \subseteq L_s, \text{ since } L_s \text{ is bi}\Gamma\text{-left ideal,} \end{aligned}$$

implies that, $R_s\Gamma M_s\Gamma L_s \subseteq R_s \cap M_s \cap L_s$.

Now, if $R_s\Gamma M_s\Gamma L_s$ is a *biΓ-quasi ideal of T* then by Lemma 3.2.12, $R_s \cap M_s \cap L_s$ is the smallest *biΓ-quasi ideal of T*. Which implies that $R_s \cap M_s \cap L_s \subseteq R_s\Gamma M_s\Gamma L_s$. Hence $R_s \cap M_s \cap L_s = R_s\Gamma M_s\Gamma L_s$, where L_s, M_s and R_s be the biΓ-left ideal , biΓ-lateral ideal and biΓ-right ideal of T. But this hold only if T is a regular biΓ-ternary semigroup, which is a contradiction. Hence $R_s\Gamma M_s\Gamma L_s$ is a *biΓ-bi-ideal of T* but not a *biΓ-quasi ideal of T*. ∎

From the above example, we can write the following lemma.

Lemma 3.3.11 *Let T be a regular $B\Gamma TS$. Then every $B\Gamma BI$ of T is a $B\Gamma QI$ of T.*

Proof. Straightforward. ∎

By combining Lemmas 3.2.16 and 3.3.11, we can write the following theorem.

Theorem 3.3.12 *Let T be a regular $B\Gamma TS$. Then a nonempty subset A of T is a $B\Gamma BI$ of T if and only if it is a $B\Gamma QI$ of T.*

Also, in view of Lemmas 3.2.11 and 3.3.11, we can write the following theorem.

Theorem 3.3.13 *Let T be a regular $B\Gamma TS$. Then a $B\Gamma TSS$ of T is a $B\Gamma BI$ of T if and only if it is an intersection of a $B\Gamma LI$, a $B\Gamma MI$ and a $B\Gamma RI$ of T.*

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(i, j)-ω-b-OPEN SETS AND (i, j)-ω-b-CONTINUITY IN BITOPOLOGICAL SPACES

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Abstract. As a generalization of (i, j)-b-open sets in bitopological spaces, we introduce and explore the notions of (i, j)-ω-b-open sets. We also develop its relationship with already defined generalizations of b-open sets. Moreover we define and discuss the properties of (i, j)-ω-b-continuous functions.

Keywords: bitopological spaces, (i, j)-ω-semiopen sets, (i, j)-ω-semiclosed sets.

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1. Introduction

In [5], Kelly initiated the study of bitopological spaces. Thereafter a lot of work have been done to generalize the topological concepts to bitopological setting.

Andrejevic [2] introduced the concept of b- open sets and extended this notions to a bitopological spaces. Recently in [4], Hdeib introduced the notions ω-closed set as generalization of closed sets. A point $x \in X$ is called a condensation point of A, if for each $U \in \tau$ with $x \in U$, the set $U \cap A$ is uncountable. A is said to be ω-closed [4], if it contains all its condensation points. The complement of a ω-closed set is said to be ω-open. It is well known that a subset W of a space (X, τ) is ω-open if and only if for each $x \in W$, there exists $U \in \tau$ such that $x \in U$ and $U - W$ is countable. The set of all ω-open sets in (X, τ) is denoted by τ_ω, τ_ω is a topology on X finer than τ. In this paper, as a generalization of (i, j)-b-open sets in bitopological spaces, we introduce and explore the notions of (i, j)-ω-b open sets. We also develop its relationship with already defined generalizations of b-open sets. Moreover we define and discuss the properties of (i, j)-ω-b-continuous functions. For a subset A of X, the closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A of a bitopological space (X, τ_1, τ_2) is said to be (i, j)-b- open, if $A \subseteq \tau_i\text{-}cl(\tau_j\text{-}Int(A)) \cup \tau_i\text{-}Int(\tau_j\text{-}Cl(A))$, where $i \neq j$, $i, j = 1, 2$. The complement of a (i, j)-b-open set is said to be a (i, j)-b-closed. The (i, j)-b-closure of A, denoted by (i, j)-b-$cl(A)$ is defined to be the intersection of all (i, j)-b-closed sets containing A. The (i, j)-b-interior of A, denoted by (i, j)-b-$Int(A)$ is defined to be the union of all (i, j)-b-open sets contained in A. The family of all (i, j)-b-open (respectively (i, j)-b-closed) subsets of a space (X, τ_1, τ_2) is denoted by (i, j)-$BO(X)$, (respectively (i, j)-$BC(X)$). A function $f : (X, \tau_1, \tau_2) \mapsto (Y, \sigma_1, \sigma_2)$ is said to be (i, j)-b- continuous, if the inverse image of every σ_i-open set in (Y, σ_1, σ_2) is (i, j)-b-open in (X, τ_1, τ_2), where $i \neq j$, $i, j = 1, 2$. Observe that a σ_i-open set U in (Y, σ_1, σ_2) means $U \in \sigma_i$.

2. (i, j)-ω-b-open sets

A set X equipped with two topologies is called a bitopological space. Throughout this paper, spaces (X, τ_1, τ_2) (or simply X) always means a bitopological spaces on which no separation axioms are assumed unless explicitly stated.

Definition 1 A subset A of a bitopological space X is (i, j)-ω-b-open, if for each $x \in A$ there exists a (i, j)-b- open subset U_x containing x such that $U_x - A$ is a countable set. The complement of a (i, j)-ω-b-open is said to be (i, j)-ω-b-closed set.

The family of all (i, j)-ω-b-open (respectively (i, j)-ω-b-closed) subsets of a space (X, τ_1, τ_2) is denoted by (i, j)-ω-$BO(X)$, (respectively (i, j)-ω-$BC(X)$). Also the family of all (i, j)-ω-b-open sets of (X, τ_1, τ_2) containing x is denoted by (i, j)-ω-$BO(X, x)$.

Example 2.1 Let $X = \{a, b, c\}$, $\tau_1 = \{\emptyset, \{a, b\}, X\}$, $\tau_2 = \{\emptyset, \{b, c\}, X\}$. Then $\{a\}$ is a (i, j)-ω-b-open but not (i, j)-b-open.

Example 2.2 Let X be the real line, $\tau_1 = \tau_2 =$ the usual topology. Q is (1,2)-ω-b-open set but not either 12-bω-open neither 12-ωb-open, see [6], for the definition of ij-bω-open and ij-ωb-open.

Example 2.3 Let $X = A \cup B \cup C \cup D$, where A, B, C, D are disjoint uncountable sets, $\tau_1 = \tau_2 = \{\emptyset, A, B, A \cup B, A \cup B \cup C, X\}$. Then $A \cup C$ is a (i,j)-ω-b-open but not (i,j)-ωb-open set.

It is well known that every semiopen (respectively preopen) set is a b-open set, in consequence, every (i,j)-ω-semiopen (respectively (i,j)-ω-propen) set is an (i,j)-ω-b-open set and therefore the results obtained in this article generalize the results obtained in [3] (respectively [8]).

Remark 2.4 It is easy to see in Example 2.3, the set $A \cup C$ is (i,j)-ω-b-open but is not (i,j)-ω-preopen set.

Theorem 2.5 *Let A be a subset of a bitopological space X. A is an (i,j)-ω-b-open if and only if for every $x \in A$, there exists a (i,j)-b-open set U_x containing x and a countable subset C such that $U_x - C \subseteq A$.*

Proof. Let A be an (i,j)-ω-b-open set and $x \in A$, then by Definition 1, there exists an (i,j)-b-open subset U_x containing x such that $U_x - A$ is countable. Let $C = U_x - A = U_x \cap (X - A)$. Then $U_x - C \subseteq A$. Conversely, let $x \in A$. Then by hypothesis, there exists a (i,j)-b-open subset U_x containing x and a countable subset C such that $U_x - C \subseteq A$. Thus $U_x - A \subseteq C$ and $U_x - A$ is countable and the result follows. ■

Theorem 2.6 *Let C be a subset of a bitopological space X. If C is an (i,j)-ω-b-closed set, then $C \subseteq K \cup B$, for some (i,j)-b-closed subset K and a countable subset B.*

Proof. If C is a (i,j)-ω-b-closed set, then its complement $X - C$ is a (i,j)-ω-b-open set and therefore by Theorem 2.5, for every $x \in X - C$, there exists a (i,j)-b-open set U containing x and a countable set B such that $U - B \subseteq X - C$. Thus $C \subseteq X - (U - B) = X - (U \cap (X - B)) = (X - U) \cup B$, let $K = X - U$. Follows that $C \subseteq K \cup B$ and K is an (i,j)-b-closed set. ■

Theorem 2.7 *The union of any family of (i,j)-ω-b-open sets is an (i,j)-ω-b-open set.*

Proof. Let $\{A_\alpha : \alpha \in I\}$ be a collection of (i,j)-ω-b-open subsets of X, then for every $x \in \bigcup_{\alpha \in I} A_\alpha$, $x \in A_\alpha$, for some $\alpha \in I$. Hence, using Definition 1, there exists a (i,j)-b-open subset U containing x, such that $U - A_\alpha$ is countable. Now as $U - (\bigcup_{\alpha \in I} A_\alpha) \subseteq U - A_\alpha$, it follows that $U - (\bigcup_{\alpha \in I} A_\alpha)$ is countable. In consequence, $\bigcup_{\alpha \in I} A_\alpha$ is an (i,j)-ω-b-open set. ■

Definition 2 Let A be a subset of a bitopological space X, the union of all (i,j)-ω-b-open sets contained in A is called the (i,j)-ω-b-interior of A and is denoted by $(i,j) - \omega$-b-Int(A). The intersection of all (i,j)-ω-b-closed sets of X containing A is called the (i,j)-ω-b-closure of A and is denoted by (i,j)-ω−b−$Cl(A)$.

Remark 2.8 By Theorem 2.7, The (i,j)-ω-b-$Cl(A)$ is a (i,j)-ω-b-closed set and the (i,j)-ω-$b - Int(A)$ is a (i,j)-ω-b-open set.

Theorem 2.9 *Let X be a bitopological space and $A, B \subseteq X$. Then the following properties hold:*

(1) (i,j)-ω-b-$Int((i,j)$-ω-b-$Int(A)) = (i,j)$-ω-b-$Int(A)$.

(2) *If* $A \subseteq B$, *then* (i,j)-ω-b-$Int(A) \subseteq (i,j)$-ω-b-$Int(B)$.

(3) (i,j)-ω-b-$Int(A \cap B) \subseteq (i,j)$-$\omega$-$b$-$Int(A) \cap (i,j)$-$\omega$-$b$-$Int(B)$.

(4) (i,j)-ω-b-$Int(A) \cup (i,j)$-ω-b-$Int(B) \subseteq (i,j)$-ω-b-$Int(A \cup B)$.

(5) (i,j)-ω-b-$Int(A)$ *is the largest* (i,j)-ω-b-*open subset of* X *contained in* A.

(6) A *is* (i,j)-ω-b-*open if and only if* $A = (i,j)$-ω-b-$Int(A)$.

(7) (i,j)-ω-b-$Cl((i,j)$-ω-b-$Cl(A)) = (i,j)$-ω-b-$Cl(A)$.

(8) *If* $A \subseteq B$, *then* (i,j)-ω-b-$Cl(A) \subseteq (i,j)$-ω-b-$Cl(B)$.

(9) (i,j)-ω-b-$Cl(A) \cup (i,j)$-ω-b-$Cl(B) \subseteq (i,j)$-ω-b-$Cl(A \cup B)$.

(10) (i,j)-ω-b-$Cl(A \cap B) \subseteq (i,j)$-$\omega$-$b$-$Cl(A) \cap (i,j)$-$\omega$-$b$-$Cl(B)$.

Proof. (1), (2), (6), (7) and (8) follow directly from the definition 1 of (i,j)-ω-b-open and (i,j)-ω-b-closed sets. (3), (4) and (5) follow from (2). (9) and (10) follow by applying (8). ∎

Example 2.10 Let X be the real line, $\tau_1 = \{\emptyset, \Re, Q^c\}$ and $\tau_2 = \{\emptyset, \Re, Q, Q^c\}$. Take $A = (0,1)$, $B = (1,2)$, then (i,j)-ω-b-$Cl(A \cap B) \subset (i,j)$-$\omega$-$b$-$Cl(A) \cap (i,j)$-$\omega$-$b$-$Cl(B)$.

Example 2.11 Let X be the real line, $\tau_1 = \{\emptyset, \Re, Q\}$ and $\tau_2 = \{\emptyset, \Re, Q\}$. The collection of (i,j)-$BO(X)$ is $\{\emptyset, \Re, Q\}$. take $A = Q$, $B = \{\pi\}$. Then (i,j)-ω-b-$Cl(A) = Q$, (i,j)-ω-b-$Cl(B) = \{\pi\}$ and (i,j)-ω-b-$Cl(A) \cup (i,j)$-ω-b-$Cl(B) \subset (i,j)$-ω-b-$Cl(A \cup B)$.

Remark 2.12 Observe that the collection (i,j)-ω-$BO(X)$ forms a minimal structure.

The following theorem give a characterization of the (i,j)-ω-b-closure of a set.

Theorem 2.13 *Let A be a subset of a bitopological space X and $x \in X$. Then* $x \in (i,j)$-ω-b-$Cl(A)$ *if and only if* $U \cap A \neq \emptyset$ *for every* $U \in (i,j)$-ω-$BO(X,x)$.

Proof. Suppose that $x \in (i,j)$-ω-b-$Cl(A)$ and we show that $U \cap A \neq \emptyset$, for all $U \in (i,j)$-ω-$BO(X,x)$. Suppose on the contrary that there exists $U \in (i,j)$-ω-$BO(X,x)$ such that $U \cap A = \emptyset$, then $A \subseteq X - U$ and $X - U$ is a (i,j)-ω-b-closed set. This follows that (i,j)-ω-b-$Cl(A) \subseteq (i,j)$-ω-b-$Cl(X - U) = X - U$. Since $x \in (i,j)$-ω-b-$Cl(A)$, we have $x \in X - U$ and hence $x \notin U$. Which contradicts the fact that $x \in U$. Therefore, $U \cap A \neq \emptyset$.

Conversely, suppose on the contrary that $U \cap A \neq \emptyset$ for every $U \in (i, j)$-ω-$BO(X, x)$. We shall prove that $x \in (i, j)$-ω-b-$Cl(A)$. Suppose that $x \notin (i, j)$-ω-b-$Cl(A)$, let $U = X - (i, j)$-ω-b-$Cl(A)$, then $U \in (i, j)$-ω-$BO(X, x)$ and $U \cap A = (X - ((i, j)$-ω-b-$Cl(A))) \cap A \subseteq (X - A) \cap A = \emptyset$. This is a contradiction to the fact that $U \cap A \neq \emptyset$. Hence $x \in (i, j)$-ω-b-$Cl(A)$. ∎

The following theorem give the duality between the (i, j)-ω-b-closure and the (i, j)-ω-b-interior of a set.

Theorem 2.14 *Let A be a subset of a bitopological space X. The following property holds:*

(1) (i, j)-ω-b-$Cl(X \backslash A) = X \backslash (i, j)$-$\omega$-$b$-$Int(A)$.

Proof. (1). Let $x \in X \backslash (i, j)$-$\omega$-$b$-$Cl(A)$. Then by Teorem 2.13, there exists $V \in (i, j)$-ω-$BO(X, x)$ such that $V \cap A = \emptyset$ and hence we obtain $x \in (i, j)$-ω-b-$Int(X \backslash A)$. This shows that $X \backslash (i, j)$-ω-b-$Cl(A) \subset (i, j)$-ω-b-$Int(X \backslash A)$. Now consider $x \in (i, j)$-ω-b-$Int(X \backslash A)$. Since (i, j)-ω-b-$Int(X \backslash A) \cap A = \emptyset$, we obtain $x \notin (i, j)$-ω-b-$Cl(A)$. Therefore, we have, (i, j)-ω-b-$Cl(X \backslash A) = X \backslash (i, j)$-$\omega$-$b$-$Int(A)$. ∎

Definition 3 Let A be a subset of a bitopological space X. A is said an (i, j)-ω-b-neighborhood of a point $x \in X$ if there exists an (i, j)-ω-b-open set W such that $x \in W \subset A$.

Theorem 2.15 *Let A be a subset of a bitopological space X. A is an (i, j)-ω-b-open set if and only if it is a (i, j)-ω-b-neighborhood of each of its points.*

Proof. Let A be an (i, j)-ω-b-open set of X. Then by definition 3, A is an (i, j)-ω-b-neighborhood of each of its points. Conversely, suppose that A is an (i, j)-ω-b-neighborhood of each of its points. Then for each $x \in A$, there exists $S_x \in (i, j)$-ω-$BO(X, x)$ such that $S_x \subset A$. Then $A = \bigcup\{S_x : x \in A\}$. Since each S_x is an (i, j)-ω-b-open, using Theorem 2.7, A is an (i, j)-ω-b-open in X. ∎

Theorem 2.16 *Let X be a bitopological space. If each nonempty (i, j)-ω-b-open set of X is uncountable, then (i, j)-b-$Cl(A) = (i, j)$-ω-b-$Cl(A)$, for each subset $A \in \tau_1 \cap \tau_2$.*

Proof. Always, (i, j)-ω-$b-Cl(A) \subseteq (i, j)$-$b-Cl(A)$. Conversely, let $x \in (i, j)$-$b-Cl(A)$ and B an (i, j)-ω-b-open set containing x. Using Theorem 2.5, there exists an (i, j)-b-open set V containing x and a countable set C such that $V - C \subseteq B$. Follows that $(V - C) \cap A \subseteq B \cap A$ and so $(V \cap A) - C \subseteq B \cap A$. Now $x \in V$, $x \in (i, j)$-$b-Cl(A)$ such that $V \cap A \neq \emptyset$ where $V \cap A$ is a (i, j)-ω-b-open set, since V is a (i, j)-b-open set and $A \in \tau_1 \cap \tau_2$. Using the hypothesis, each nonempty (i, j)-ω-b-open set of X is uncountable and so is $(V \cap A) \backslash C$. Thus $B \cap A$ is uncountable. Therefore, $B \cap A \neq \emptyset$ implies that $x \in (i, j)$-ω-$b - Cl(A)$. ∎

The following theorem give under some conditions, the collection (i, j)-ω-$BO(X))$ is a topology.

Theorem 2.17 *Let X be a bitopological space. If every (i,j)-b-open subset of X is τ_i-open in X. Then $(X,(i,j)$-ω-$BO(X))$ is a topological space.*

Proof. 1. $\emptyset, X$ belong to (i,j)-ω-$BO(X)$.

2. Let U, V be element of (i,j)-ω-$BO(X)$ and suppose that $x \in U \cap V$. Then by Definition 1, there exist (i,j)-b-open sets G, H in X containing x such that $G \backslash U$ and $H \backslash V$ are countable. Since $(G \cap H) \backslash (U \cap V) = (G \cap H) \cap ((X \backslash U) \cup (X \backslash V)) \subseteq (G \cap (X \backslash U)) \cup (H \cap (X \backslash V))$ implies that $(G \cap H) \backslash (U \cap V)$ is a countable set and by hypothesis, the intersection of two (i,j)-b-open set is (i,j)-b-open. Hence $U \cap V \in (i,j)$-ω-$BO(X))$.

3. The union follows directly from Theorem 2.7. ∎

The following example shows that the converse of the Theorem 2.17 not necessarily is true.

Example 2.18 In the Example 2.1, the collection of $(1,2)$-ω-$BO(X) = P(X)$, in consequence, is a topology on X, but the set $\{a\}$ is $(1,2)$-b-open and $\{a\} \notin \tau_1$

3. (i,j)-ω-b-continuous functions

Definition 4 Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be bitopological spaces. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is:

(1) (i,j)-ω-b-continuous, if the inverse image of every σ_i-open set of Y is (i,j)-ω-b-open in (X, τ_1, τ_2), where $i \neq j$, i, j=1, 2.

(2) (i,j)-b-continuous, if the inverse image of every σ_i-open set of Y is (i,j)-b-open in (X, τ_1, τ_2), where $i \neq j$, i, j=1, 2.

Theorem 3.1 *Every (i,j)-b-continuous function is (i,j)-ω-b-continuous.*

Proof. The proof follows from the fact that every (i,j)-b-open set is (i,j)-ω-b-open. ∎

However, the converse may be not true.

Example 3.2 Let $X = \{a, b, c\}$, $\tau_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$, $\tau_2 = \{\emptyset, \{a\}, X\}$, $\sigma_1 = \{\emptyset, \{a, b\}, X\}$, $\sigma_2 = \{\emptyset, \{a, c\}, X\}$. Then the identity function $f : (X, \tau_1, \tau_2) \to (X, \sigma_1, \sigma_2)$ is (i,j)-ω-b-continuous but not (i,j)-b-continuous.

Remark 3.3 Since every (i,j)-ω-preopen set is (i,j)-ω-b-open, then every (i,j)-ω-precontinuous function [8] is (i,j)-ω-b-continuous but not conversely.

Theorem 3.4 *Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be a bitopological spaces and $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ a function, the following statements are equivalent:*

(1) *f is (i,j)-ω-b-continuous;*

(2) *For each point $x \in X$ and each σ_i-open set F in Y such that $f(x) \in F$, there is a (i,j)-ω-b-open set A in X such that $x \in A$, and $f(A) \subset F$;*

(3) *The inverse image of each σ_i-closed set in Y is a (i,j)-ω-b-closed in X;*

(4) $f((i,j)\text{-}\omega\text{-}b - Cl(A)) \subseteq \sigma_i\text{-}cl(f(A))$*, for every* $A \subseteq X$*;*

(5) $(i,j)\text{-}\omega\text{-}b\text{-}Cl(f^{-1}(B)) \subseteq f^{-1}(\sigma_i\text{-}cl(B))$*, for every* $B \subseteq Y$*;*

(6) $f^{-1}(\sigma_i\text{-}Int(C)) \subseteq (i,j)\text{-}\omega\text{-}b\text{-}Int(f^{-1}(C))$*, for every* $C \subseteq Y$.

Proof. (1)⇒(2): Let $x \in X$ and F be a σ_i-open set of Y containing $f(x)$. By (1), $f^{-1}(F)$ is (i,j)-ω-b-open in X. Let $A = f^{-1}(F)$. Then $x \in A$ and $f(A) \subset F$.

(2)⇒(1): Let F be σ_i-open in Y and let $x \in f^{-1}(F)$. Then $f(x) \in F$. By (2), there is a (i,j)-ω-b-open set U_x in X such that $x \in U_x$ and $f(U_x) \subseteq F$ implies $x \in U_x \subseteq f^{-1}(F)$. Hence $f^{-1}(F)$ is a (i,j)-ω-b-open in X.

(1)⇔(3): This follows from the fact that for any subset B of Y, $f^{-1}(Y\backslash B) = X\backslash f^{-1}(B)$.

(3)⇒(4): Let A be a subset of X. Since $A \subseteq f^{-1}(f(A))$, we have $A \subseteq f^{-1}(\sigma_i\text{-Cl}(f(A)))$. By hypothesis $f^{-1}(\sigma_i\text{-Cl}(f(A)))$ is a (i,j)-ω-b-closed set in X and hence $(i,j)\text{-}\omega\text{-}b\text{-}Cl(A)) \subseteq f^{-1}(\sigma_i\text{-Cl}(f(A)))$. Follows $f((i,j)\text{-}\omega\text{-}b\text{-}Cl(A))) \subseteq f(f^{-1}(\sigma_i\text{-Cl}(f(A))) \subseteq \sigma_i\text{-Cl}(f(A))$.

(4)⇒(3): Let F be any σ_i-closed subset of Y. Then $f((i,j)\text{-}\omega\text{-}b\text{-}Cl(f^{-1}(F)) \subseteq \sigma_i\text{-}cl(f(f^{-1}(F))) \subseteq \sigma_i\text{-}cl(F) = F$. Therefore, the $(i,j)\text{-}\omega\text{-}b\text{-}Cl(f^{-1}(F)) \subseteq f^{-1}(F)$. Consequently, $f^{-1}(F)$ is a (i,j)-ω-b-closed set in X.

(4)⇒(5): Let $B \subseteq Y$. Now, $f((i,j)\text{-}\omega\text{-}b\text{-}Cl(f^{-1}(B))) \subseteq \sigma_i\text{-Cl}(f(f^{-1}(B))) \subseteq \sigma_i\text{-Cl}(B)$. Consequently, $(i,j)\text{-}\omega\text{-}b - Cl(f^{-1}(B)) \subseteq f^{-1}(\sigma_i\text{-Cl}(B))$.

(5)⇒(4): Let $B = f(A)$ where $A \subseteq X$. Then, $(i,j)\text{-}\omega\text{-}b\text{-}Cl(A) \subseteq (i,j)\text{-}\omega\text{-}b - Cl(f^{-1}(B)) \subseteq f^{-1}(\sigma_i\text{-Cl}(B)) = f^{-1}(\sigma_i\text{-Cl}(f(A)))$, and hence $f((i,j)\text{-}\omega\text{-}b\text{-}Cl(A)) \subseteq \sigma_i\text{-Cl}(f(A))$.

(1)⇒(6): Let $B \subseteq Y$. Clearly, $f^{-1}(\sigma_i\text{-Int}(B))$ is a (i,j)-ω-b-open and we have $f^{-1}(\sigma_i\text{-Int}(B)) \subseteq (i,j)\text{-}\omega\text{-}b\text{-}Int(f^{-1}\sigma_i\text{-Int}(B)) \subseteq (i,j)\text{-}\omega\text{-}b\text{-}Int(f^{-1}B)$.

(6)⇒(1): Let B be a σ_i-open set in Y. Then $\sigma_i\text{-Int}(B) = B$ and $f^{-1}(B) \subseteq f^{-1}(\sigma_i\text{-Int}(B)) \subseteq (i,j)\text{-}\omega\text{-}b\text{-}Int(f^{-1}(B))$. Hence, we have $f^{-1}(B) = (i,j)\text{-}\omega\text{-}b\text{-}Int(f^{-1}(B))$. This implies that $f^{-1}(B)$ is a (i,j)-ω-b-open in X. ∎

Definition 5 Let (X,τ_1,τ_2) and (Y,σ_1,σ_2) be bitopological spaces and $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ a function. The graph $G(f)$ of $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ is said to be (i,j)-ω-b-closed in $X \times Y$, if for each $(x,y) \in (X \times Y) \setminus G(f)$, there exists $U \in (i,j)\text{-}\omega\text{-}BO(X,x)$, $i,j = 1,2$ with $i \neq j$ and a σ_i-open set V of Y containing y such that $(U \times V) \cap G(f) = \emptyset$.

Lemma 3.5 *The graph $G(f)$ of $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ is (i,j)-ω-b-closed in $X \times Y$ if and only if for each $(x,y) \in (X \times Y) \setminus G(f)$, there exists $U \in (i,j)\text{-}\omega\text{-}BO(X,x)$, $i,j = 1,2$ and $i \neq j$ and a σ_i-open set V of Y containing y such that $f(U) \cap V = \emptyset$.*

Proof. The proof is an immediate consequence of Definition 5. ∎

Theorem 3.6 *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is an (i,j)-ω-b-continuous function and (Y, σ_i) is a T_2-space, $i = 1, 2$, then $G(f)$ is (i,j)-ω-b-closed.*

Proof. Let $(x, y) \in (X \times Y) \setminus G(f)$. Then $y \neq f(x)$. Since (Y, σ_i) is T_1, there exist a σ_i-open set V and W of Y such that $f(x) \in V$ and $y \notin W$ and $V \cap W = \emptyset$. Since f is (i,j)-ω-b-continuous, there exists $U \in (i,j)$-ω-$BO(X, x)$ such that $f(U) \subset V$. Therefore, $f(U) \cap W = \emptyset$. Therefore, by Lemma 3.5, $G(f)$ is (i,j)-ω-b-closed. ∎

Definition 6 A bitopological space X is said to be a (i,j)-ω-b-T_2 space, if for each pair of distinct points $x, y \in X$, there exist $U, V \in (i,j)$-ω-$BO(X)$ containing x and y, respectively, such that $U \cap V = \emptyset$.

Theorem 3.7 *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a (i,j)-ω-b-continuous injective function and (Y, σ_i) is a T_2 space, then (X, τ_1, τ_2) is a ω-b-T_2 space.*

Proof. The proof follows from the Definition 4 and 6. ∎

Theorem 3.8 *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is an injective (i,j)-ω-b-continuous function with (i,j)-ω-b-closed graph, then X is an (i,j)-ω-b-T_2 space.*

Proof. Let x_1 and x_2 be any pair of distinct points of X. Then $f(x_1) \neq f(x_2)$, so $(x_1, f(x_2)) \in (X \times Y) \backslash G(f)$. Since the graph $G(f)$ is (i,j)-ω-b-closed, then by Lemma 3.5, there exist an (i,j)-ω-b-open set U containing x_1 and $V \in \sigma_i$ containing $f(x_2)$ such that $f(U) \cap V = \emptyset$. Since f is (i,j)-ω-b-continuous, $f^{-1}(V)$ is an (i,j)-ω-b-open set containing x_2 such that $U \cap f^{-1}(V) = \emptyset$. Hence X is (i,j)-ω-b-T_2. ∎

Definition 7 Let A be a subset of a bitopological space X. A collection $\{U_\alpha : \alpha \in I\}$ of (i,j)-b-open subsets of X is called an (i,j)-b-open cover of A, if $A \subseteq \bigcup_{\alpha \in I} U_\alpha$.

Definition 8 A bitopological space X is said to be (i,j)-b-Lindeloff, if every (i,j)-b-open cover of X has a countable subcover. A subset A of bitopological space X is said to be (i,j)-b-Lindeloff relative to X, if every cover of A by (i,j)-b-open sets of X has a countable subcover.

Theorem 3.9 *Let X be a bitopological space. If every (i,j)-b-open subset is (i,j)-b-Lindeloff relative to X. Then every subset is (i,j)-b-Lindeloff relative to X*

Theorem 3.10 *For a bitopological space X. The following properties are equivalent:*

(1) *X is (i,j)-b-Lindeloff.*

(2) *Every countable cover of X by (i,j)-b-open sets has a countable subcover.*

Proof. (2)⇒(1): Since every (i,j)-b-open set is (i,j)-ω-b-open, the proof follows. (1)⇒(2): Let $\{U_\alpha : \alpha \in I\}$ be any cover of X by (i,j)-ω-b-open sets of X. For each $x \in X$, there exists an $\alpha_x \in I$ such that $x \in U_{\alpha_x}$. Since U_{α_x} is an (i,j)-ω-b-open, then using Definition 1, there exists a (i,j)-b-open set V_{α_x} such that $x \in V_{\alpha_x}$ and $V_{\alpha_x} - U_{\alpha_x}$ is countable. The family $\{V_\alpha : \alpha \in I\}$ is a (i,j)-b-open cover of X and X is (i,j)-b-Lindeloff. By Definition 8, the collection $\{V_\alpha : \alpha \in I\}$ has a countable subcover $\{U_{\alpha_{x_i}}\}_{i\in N}$ such that $X = \bigcup_{i\in N} V_{\alpha_{x_i}}$. Since $X = \bigcup_{i\in N}[(V_{\alpha_{x_i}} - U_{\alpha_{x_i}}) \cup U_{\alpha_{x_i}}] = \bigcup_{i\in N}[(V_{\alpha_{x_i}} - U_{\alpha_{x_i}}) \bigcup_{i\in N} U_{\alpha_{x_i}}]$ and $V_{\alpha_{x_i}} - U_{\alpha_{x_i}}$ is a countable set, for each α_{x_i}, there exists a countable subset $I_{\alpha(x_i)}$ of I such that $V_{\alpha_{x_i}} - U_{\alpha_{x_i}} \subseteq \bigcup_{I_{\alpha(x_i)}} U_\alpha$ and therefore $X = \bigcup_{i\in N}(\bigcup_{\alpha\in I_{\alpha(x_i)}} U_\alpha) \cup (\bigcup_{i\in N} U_{\alpha(x_i)})$. ∎

Definition 9 A bitopological space X is called pairwise Lindeloff if each pairwise open cover of X has a countable subcover.

Theorem 3.11 *Let $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ be a surjective an (i,j)-ω-b-continuous function. If X is (i,j)-b-Lindeloff, then Y is pairwise Lindeloff.*

Proof. Let $\{U_\alpha : \alpha \in I\}$ be any pairwise open cover of Y by σ_i-open sets. Then $\{f^{-1}(U_\alpha) : \alpha \in I\}$ is a (i,j)-ω-b-open cover of X. Since X is (i,j)-b-Lindeloff, there exists a countable subset I_0 of I such that $X = \bigcup_{\alpha\in I_0} U_\alpha$. Therefore, Y is a pairwise Lindeloff. ∎

Definition 10 Let (X,τ_1,τ_2) and (Y,σ_1,σ_2) be a bitopological spaces. A function $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ is called:

1 (i,j)-ω-b-open if $f(U)$ is an (i,j)-ω-b-open set in Y for every τ_i-open set U of X.

2 (i,j)-ω-b-closed if $f(U)$ is an (i,j)-ω-b-closed set in Y for every τ_i-closed set U of X.

The following theorem give a characterization of (i,j)-ω-b-open functions.

Theorem 3.12 *Let (X,τ_1,τ_2) and (Y,σ_1,σ_2) be a bitopological spaces and $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ a function, the following properties are equivalent:*

(1) *f is an (i,j)-ω-b-open.*

(2) *$f(\tau_i - Int(U)) \subseteq (i,j)$-$\omega$-$b - Int(f(U))$, for each subset U of X.*

(3) *$\tau_i - Int(f^{-1}(V)) \subseteq f^{-1}((i,j)$-$\omega$-$b - Int(V))$, for each subset V of Y.*

Proof. (1)⇒(2): Let U be any subset of X. Then $\tau_i - Int(U)$ is a τ_i-open set of X. Then $f(\tau_i - Int(U))$ is a (i,j)-ω-b-open set of Y. Since $f(\tau_i - Int(U)) \subseteq f(U)$, $f(\tau_i - Int(U)) = (i,j)$-$\omega$-$b - Int(f(\tau_i - Int(U))) \subseteq (i,j)$-$\omega$-$b - Int(f(U))$.

(2)⇒(3): Let V be any subset of Y. Then $f(\tau_i - Int(f^{-1}(V))) \subseteq (i,j)$-$\omega$-$b$-$Int(f(f^{-1}(V)))$. Hence τ_i-$Int(f^{-1}(V)) \subseteq f^{-1}((i,j)$-$\omega$-$b - Int(V))$.

(3)⇒(1): Let U be any τ_i-open set of X. Then $\tau_i - Int(U) = U$. Now, $V = \tau_i - Int(V) \subseteq \tau_i - Int(f^{-1}(f(V)) \subseteq f^{-1}((i,j)$-$\omega$-$b - Int(f(V)))$. Which implies that $f(V) \subseteq f(f^{-1}((i,j)$-$\omega$-$b - Int(f(V)))) \subseteq (i,j)$-$\omega$-$b - Int(f(V))$. Hence $f(V)$ is a (i,j)-ω-b-open set of Y. Thus f is (i,j)-ω-b-open. ∎

Theorem 3.13 *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a function, then f is a (i,j)-ω-b-closed function if and only if the (i,j)-ω-$b - Cl(f(V)) \subseteq f(\tau_i - Cl(V)))$ for each subset V of X.*

Proof. Let f be an (i,j)-ω-b-closed function and V be any subset of X. Then $f(V) \subseteq f(\tau_i{-}Cl(V))$ and $f(\tau_i{-}Cl(V))$ is an (i,j)-ω-b-closed set of Y. Hence (i,j)-ω-b-$Cl(f(V)) \subseteq (i,j)$-ω-b-$Cl(f(\tau_i - Cl(V))) = f(\tau_i - Cl(V)))$.

Conversely, let V be a τ_i-closed set of X. Then $f(V) \subseteq (i,j)$-ω-$b{-}Cl(f(V)) \subseteq f(\tau_i - Cl(V))) = f(V)$. Hence $f(V)$ is an (i,j)-ω-b-closed set of Y. Therefore, f is an (i,j)-ω-b-closed function. ∎

Definition 11 A bitopological space X is said to be (i,j)-ω-b-connected, if X cannot be expressed as the union of two nonempty disjoint (i,j)-ω-b-open sets.

Example 3.14 The bitopological space defined in Example 2.2 is not (i,j)-ω-b-connected but the bitopological space defined in Example 2.3 is (i,j)-ω-b-connected

Definition 12 A bitopological space X is said to be pairwise connected [7], if it cannot be expressed as the union of two nonempty disjoint sets U and V such that U is τ_i-open and V is τ_j-open, where $i, j = 1, 2$ and $i \neq j$.

Example 3.15 The bitopological space defined in Example 2.3 is pairwise connected

Theorem 3.16 *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a (i,j)-ω-b-continuous function. If X is an (i,j)-ω-b-connected space then $f(X)$ is pairwise connected.*

Proof. The proof is clear. ∎

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INTUITIONISTIC FUZZY SETS IN UP-ALGEBRAS[1]

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Abstract. The concept of intuitionistic fuzzy sets was first introduced by Atanassov, which is a generalization of the concept of fuzzy sets. In this paper, we apply the concept of intuitionistic fuzzy sets to UP-algebras. The notions of intuitionistic fuzzy UP-ideals and intuitionistic fuzzy UP-subalgebras of UP-algebras are introduced and their basic properties are investigated. Upper t-(strong) level subsets and lower t-(strong) level subsets are derived from some intuitionistic fuzzy sets.

Keywords: UP-algebra, intuitionistic fuzzy set, intuitionistic fuzzy UP-ideal, intuitionistic fuzzy UP-subalgebra, upper t-(strong) level subset, lower t-(strong) level subset.

Mathematics Subject Classification: 03G25.

1. Introduction and preliminaries

Among many algebraic structures, algebras of logic form important class of algebras. Examples of these are BCK-algebras [6], BCI-algebras [7], BCH-algebras [4], KU-algebras [18], SU-algebras [9] and others. They are strongly connected with logic. For example, BCI-algebras introduced by Iséki [7] in 1966 have connections with BCI-logic being the BCI-system in combinatory logic which has application in the language of functional programming. BCK and BCI-algebras are two classes of logical algebras. They were introduced by Imai and Iséki [6], [7] in 1966 and have been extensively investigated by many researchers. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras.

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The fundamental concept of fuzzy sets in a set was first introduced by Zadeh [27] in 1965. The fuzzy set theories developed by Zadeh and others have found many applications in the domain of mathematics and elsewhere. The concept of intuitionistic fuzzy sets was first published by Atanassov in his pioneer papers [2], [3], as generalization of the notion of fuzzy sets. Several researches were conducted on the generalizations of the notion of intuitionistic fuzzy sets and application to many logical algebras such as: In 2000, Jun and Kim [8] introduced the notion of equivalence relations on the family of all intuitionistic fuzzy ideals of BCK-algebras. In 2004, Zhan and Z. Tan [30] introduced the notion of intuitionistic fuzzy α-ideals of BCI-algebras. In 2005, Kim and Jeong [12] introduced the notion of intuitionistic fuzzy o-subalgebra of BCK-algebras with condition (S). Xueling and Jianming [26] introduced the notion of intuitionistic Ω-fuzzy ideals of BCK-algebras. Zahedi and Torkzadeh [28] introduced the notions of intuitionistic fuzzy dual positive implicative hyper K-ideals of types 1,2,3,4 and intuitionistic fuzzy dual hyper K-ideals. In 2006, Kim and Jeong [10] introduced the notion of intuitionistic fuzzy subalgebras of B-algebras which is related to several classes of algebras such as BCI/BCK-algebras. In 2007, Kim [11] introduced the notion of intuitionistic (T, S)-normed fuzzy subalgebras in BCK/BCI-algebras. Zarandi and A. B. Saeid [29] studied the intuitionistic fuzzification of the concept of subalgebras and ideals of BG-algebras. In 2008, Akram, Dar, Meng and Shum [1] introduced the notion of interval-valued intuitionistic fuzzy ideals of K-algebras. In 2011, Mostafa, Naby and Elgendy [14] introduced the intuitionistic fuzzification of the concept of KU-ideals and the image (preimage) of KU-ideals in KU-algebras. Satyanarayana and Prasad [21] studied the intuitionistic fuzzy implicative ideals, intuitionistic fuzzy positive implicative ideals and intuitionistic fuzzy commutative ideals in BCK-algebras. In 2012, Malik and Touqeer [13] introduced the intuitionistic fuzzification of the concept of BCI-commutative ideals of BCI-algebras. Palaniappan, Veerappan and Devi [17] introduced the notion of interval valued intuitionistic fuzzy H-ideals of BCI-algebras. Senapati, Bhowmik and Pal [22] introduced the notion of interval-valued intuitionistic fuzzy closed ideals of BG-algebras. In 2013, Nezhad, Rayeni and Rezaei [15] introduced the notion of intuitionistic fuzzy soft subalgebras (filters) of BE-algebras. Palaniappan, Devi and Veerappan [16] introduced the notion of intuitionistic fuzzy n-fold positive implicative ideals of BCI-algebras. In 2014, Ragavan, Solairaju and Balamurugan [19] introduced the notion of interval valued Intuitionistic Fuzzy R-ideals of BCI-algebras. Satyanarayana, Krishna and Prasad [20] introduced the notions of intuitionistic fuzzy (weak) implicative hyper BCK-ideals of hyper BCK-algebras. Senapati, Bhowmik and Pal [23] introduced the notions of fuzzy dot subalgebras, fuzzy normal dot subalgebras and fuzzy dot ideals of B-algebras. Sun and Li [25] introduced the notions of intuitionistic fuzzy subalgebras with thresholds (λ, μ) and intuitionistic fuzzy ideals with thresholds (λ, μ) of BCI-algebras.

Iampan [5] now introduced a new algebraic structure, called a UP-algebra and a concept of UP-ideals and UP-subalgebras of UP-algebras. The notions of intuitionistic fuzzy UP-ideals and intuitionistic fuzzy UP-subalgebras play an important role in studying the many logical algebras. In this paper, we introduce the notions of intuitionistic fuzzy UP-ideals and intuitionistic fuzzy UP-subalgebras of UP-algebras, and their properties are investigated.

Before we begin our study, we will introduce to the definition of a UP-algebra.

Definition 1.1. [5] An algebra $A = (A; \cdot, 0)$ of type $(2, 0)$ is called a *UP-algebra* if it satisfies the following axioms: for any $x, y, z \in A$,

(UP-1) $(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0$,

(UP-2) $0 \cdot x = x$,

(UP-3) $x \cdot 0 = 0$, and

(UP-4) $x \cdot y = y \cdot x = 0$ implies $x = y$.

Example 1.2. [5] Let X be a set. Define a binary operation $\cdot$ on the power set of X by putting $A \cdot B = B \cap A'$ for all $A, B \in \mathcal{P}(X)$. Then $(\mathcal{P}(X); \cdot, \emptyset)$ is a UP-algebra.

We can easily show the following example.

Example 1.3. [5] Let $A = \{0, a, b, c\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

(1.1)

$\cdot$	0	a	b	c
0	0	a	b	c
a	0	0	0	0
b	0	a	0	c
c	0	a	b	0

Then $(A; \cdot, 0)$ is a UP-algebra.

In what follows, let A denote a UP-algebra unless otherwise specified. The following proposition is very important for the study of UP-algebras.

Proposition 1.4. [5] *In a UP-algebra A, the following properties hold: for any $x, y \in A$,*

(1) $x \cdot x = 0$,

(2) $x \cdot y = 0$ *and* $y \cdot z = 0$ *imply* $x \cdot z = 0$,

(3) $x \cdot y = 0$ *implies* $(z \cdot x) \cdot (z \cdot y) = 0$,

(4) $x \cdot y = 0$ *implies* $(y \cdot z) \cdot (x \cdot z) = 0$,

(5) $x \cdot (y \cdot x) = 0$,

(6) $(y \cdot x) \cdot x = 0$ *if and only if* $x = y \cdot x$, *and*

(7) $x \cdot (y \cdot y) = 0$.

On a UP-algebra $A = (A; \cdot, 0)$, we define a binary relation $\leq$ on A as follows: for all $x, y \in A$,

$$x \leq y \text{ if and only if } x \cdot y = 0. \tag{1.2}$$

Proposition 1.5 obviously follows from Proposition 1.4.

Proposition 1.5. *[5] In a UP-algebra A, the following properties hold: for any $x, y \in A$,*

(1) $x \leq x$,

(2) $x \leq y$ *and* $y \leq x$ *imply* $x = y$,

(3) $x \leq y$ *and* $y \leq z$ *imply* $x \leq z$,

(4) $x \leq y$ *implies* $z \cdot x \leq z \cdot y$,

(5) $x \leq y$ *implies* $y \cdot z \leq x \cdot z$,

(6) $x \leq y \cdot x$, *and*

(7) $x \leq y \cdot y$.

From Proposition 1.5 and UP-3, we have Proposition 1.6.

Proposition 1.6. [5] *Let A be a UP-algebra with a binary relation $\leq$ defined by* (1.2)*. Then $(A, \leq)$ is a partially ordered set with* 0 *as the greatest element.*

We often call the partial ordering $\leq$ defined by (1.2) the *UP-ordering* on A. From now on, the symbol $\leq$ will be used to denote the UP-ordering, unless specified otherwise.

Definition 1.7. [5] A nonempty subset B of A is called a *UP-ideal* of A if it satisfies the following properties:

(1) the constant 0 of A is in B, and

(2) for any $x, y, z \in A, x \cdot (y \cdot z) \in B$ and $y \in B$ imply $x \cdot z \in B$.

Clearly, A and $\{0\}$ are UP-ideals of A.

Theorem 1.8. [5] *Let A be a UP-algebra and $\{B_i\}_{i \in I}$ a family of UP-ideals of A. Then $\bigcap_{i \in I} B_i$ is a UP-ideal of A.*

Definition 1.9. [5] A subset S of A is called a *UP-subalgebra* of A if it constant 0 of A is in S, and $(S; \cdot, 0)$ itself forms a UP-algebra. Clearly, A and $\{0\}$ are UP-subalgebras of A.

Applying Proposition 1.4 1.4, we can then easily prove the following proposition.

Proposition 1.10. [5] *A nonempty subset S of a UP-algebra $A = (A; \cdot, 0)$ is a UP-subalgebra of A if and only if S is closed under the $\cdot$ multiplication on A.*

Theorem 1.11. [5] *Let A be a UP-algebra and $\{B_i\}_{i\in I}$ a family of UP-subalgebras of A. Then $\bigcap_{i\in I} B_i$ is a UP-subalgebra of A.*

Theorem 1.12. [5] *Let A be a UP-algebra and B a UP-ideal of A. Then $A\cdot B \subseteq B$. In particular, B is a UP-subalgebra of A.*

We can easily show the following example.

Example 1.13. [5] Let $A = \{0, a, b, c, d\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

(1.3)

$\cdot$	0	a	b	c	d
0	0	a	b	c	d
a	0	0	b	c	d
b	0	0	0	c	d
c	0	0	b	0	d
d	0	0	0	0	0

Using the following program in the software "MATLAB", we know that $(A; \cdot, 0)$ is a UP-algebra, where we use numbers $1, 2, 3, 4$ and 5 instead of $0, a, b, c$ and d, respectively.

Program for test UP-1

```
display(['Input n = 4 or n = 5']);
n = input('n = ');
b = zeros(n,n);
if n == 4
    b = [   1   2   3   4;
            1   1   1   1;
            1   2   1   4;
            1   2   3   1   ];
else
     b = [  1   2   3   4   5;
            1   1   3   4   5;
            1   1   1   4   5;
            1   1   3   1   5;
            1   1   1   1   1   ];
end
tc = 0;
cp = 0;
np = 0;
for i = 1:n
    for j = 1:n
```

```
        for k = 1:n
            tc = tc + 1;
            rc = b(b(j,k),b(b(i,j),b(i,k)));
            if rc == 1
                cp = cp + 1;
            else
                np = np + 1;
             end
        end
    end
end
```

We can check condition 1.7 in Definition 1.7 that the set $\{0, a, c\}$ is a UP-ideal of A by using the following program.

Program for test Definition 1.7 1.7

```
clc,clear
display(['Input n = 4 or n = 5']);
n = input('n = ');
b = zeros(n,n);
if n == 4
    b = [   1   2   3   4;
            1   1   1   1;
            1   2   1   4;
            1   2   3   1   ];
else
     b = [  1   2   3   4   5;
            1   1   3   4   5;
            1   1   1   4   5;
            1   1   3   1   5;
            1   1   1   1   1   ];
end
tc  = 0;
cp  = 0;
scp = 0;
ncp = 0;
np  = 0;
for i = 1:n
    for j = 1:4
        for k = 1:n
            rc = b(i,b(j,k));
            if (rc <= 2) | (rc == 4)
                tc = tc + 1;
                   if j ~= 3
                    cp = cp + 1;
```

```
                    src = b(i,k);
                    if (src <= 2) | (src == 4)
                        scp = scp + 1;
                    else
                        ncp = ncp + 1;
                    end
                end
            end
            if ((rc == 3) | (rc ==5)) & (j == 3)
                np = np + 1;
            end
        end
    end
end
```

We can check that the set $\{0, a, b\}$ is a UP-ideal of A.

By Proposition 1.10, we can check that the set $\{0, a, b, c\}$ is a UP-subalgebra of A.

2. Main results

In this section, firstly, we recall the definition of a fuzzy set in a nonempty set and the definitions of a fuzzy UP-ideal and a fuzzy UP-subalgebra of a UP-algebra. Secondly, we introduce the notions of a intuitionistic fuzzy UP-ideal and a intuitionistic fuzzy UP-subalgebra of a UP-algebra and study some of their basic properties.

Definition 2.1. [27] A *fuzzy set* in a nonempty set X (or a fuzzy subset of X) is an arbitrary function $f\colon X \to [0, 1]$ where $[0, 1]$ is the unit segment of the real line. If $A \subseteq X$, the *characteristic function* f_A of X is a function of X into $\{0, 1\}$ defined as follows:

$$f_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

By the definition of the characteristic function, f_A is a function of X into $\{0, 1\} \subset [0, 1]$. Hence, f_A is a fuzzy set in X.

Definition 2.2. Let f be a fuzzy set in A. The fuzzy set $\overline{f}$ defined by $\overline{f}(x) = 1 - f(x)$ for all $x \in A$ is called the *complement* of f in A.

Definition 2.3. [24] A fuzzy set f in A is called a *fuzzy UP-ideal* of A if it satisfies the following properties: for any $x, y, z \in A$,

(1) $f(0) \geq f(x)$, and

(2) $f(x \cdot z) \geq \min\{f(x \cdot (y \cdot z)), f(y)\}$.

Example 2.4. By Example 1.13, we get $\{0, a, b\}$ is a UP-ideal of A. Then

$$f(x) = \begin{cases} 1 & \text{if } x \in \{0, a, b\}, \\ 0 & \text{if } x \in \{c, d\} \end{cases}$$

is a fuzzy UP-ideal of A by using the following program.

```
clc,clear
display(['Input n = 4 or n = 5']);
n = input('n = ');
b = zeros(n,n);
f = zeros(n,n);
if n == 4
      b = [ 1   2   3   4;
            1   1   1   1;
            1   2   1   4;
            1   2   3   1   ];
     f = [  1   1   0.3 0.4;
            1   1   1   1;
            1   1   1   0.4;
            1   1   0.3 1  ];
else
    b = [   1   2   3   4   5;
            1   1   3   4   5;
            1   1   1   4   5;
            1   1   3   1   5;
            1   1   1   1   1   ];
    f = [   1   1   1   0   0;
            1   1   1   0   0;
            1   1   1   0   0;
            1   1   1   1   0;
            1   1   1   1   1   ];
end
tc  = 0;
cp  = 0;
ncp = 0;
az  = 1;
bz  = 1;
cz  = 1;
dz  = 0;
ez  = 0;
for i = 1:n
    for j = 1:n
        for k = 1:n
            re = b(j,k);
            rc = f(i,re);
```

```
            rm = b(i,k);
            rd = f(i,k);
            if(j==1)
                tc = tc + 1;
              if(rd >= min(rc,az))
                cp=cp+1;
              else
                 ncp=ncp+1;
              end
            end
            if(j==2)
                tc = tc + 1;
              if(rd >= min(rc,bz))
                  cp=cp+1;
              else
                  ncp=ncp+1;
               end
            end
            if(j==3)
                tc = tc + 1;
              if(rd >= min(rc,cz))
                  cp=cp+1;
              else
                  ncp=ncp+1;
              end
            end
            if(j==4)
                tc = tc + 1;
              if(rd >= min(rc,dz))
                  cp=cp+1;
              else
                  ncp=ncp+1;
              end
            end
            if(j==5)
                tc = tc + 1;
              if(rd >= min(rc,ez))
                  cp=cp+1;
              else
                  ncp=ncp+1;
              end
            end
        end
    end
end
```

Definition 2.5. [24] A fuzzy set f in A is called a *fuzzy UP-subalgebra* in A if for any $x, y \in A$,

$$f(x \cdot y) \geq \min\{f(x), f(y)\}. \tag{2.1}$$

Example 2.6. By Example 1.13, we get $\{0, a, b, c\}$ is a UP-subalgebra of A. Then

$$f(x) = \begin{cases} 1 & \text{if } x \in \{0, a, b, c\}, \\ 0 & \text{if } x \in \{d\} \end{cases}$$

is a fuzzy UP-subalgebra of A by using the following program.

```
clc,clear
display(['Input n = 4 or n = 5']);
n = input('n = ');
g = zeros(n,n);
b = zeros(n,n);
f = zeros(n,n);
if n == 4
    b = [   0.7 0.7 0.7 0.3;
            0.7 0.7 0.7 0.7;
            0.7 0.7 0.7 0.3;
            0.7 0.7 0.7 0.7 ];
    f = [   0.7 0.7 0.7 0.3;
            0.7 0.7 0.7 0.3;
            0.7 0.7 0.7 0.3;
            0.7 0.7 0.7 0.3 ];
else
    g = [   1   2   3   4   5;
            1   1   3   4   5;
            1   1   1   4   5;
            1   1   3   1   5;
            1   1   1   1   1   ];
    b = [   1   1   1   1   0;
            1   1   1   1   0;
            1   1   1   1   0;
            1   1   1   1   0;
            1   1   1   1   1   ];
    f = [   1   1   1   1   0;
            1   1   1   1   0;
            1   1   1   1   0;
            1   1   1   1   0;
            1   1   1   1   0   ];
end
tc  = 0;
cp  = 0;
```

```
ncp = 0;
az  = 0.7;
bz  = 0.7;
cz  = 0.7;
dz  = 0.3;
ez  = 0.2;
for i = 1:n
    for j = 1:n
             rc = b(i,j);
             rd = f(i,j);
             if(i==1)
                 tc = tc + 1;
              if(rc >= min(az,rd))
                  cp = cp + 1;
              else
                  ncp = ncp + 1;
              end
             end
             if(i==2)
                 tc = tc + 1;
              if(rc >= min(bz,rd))
                  cp = cp + 1;
              else
                  ncp = ncp + 1;
              end
             end
             if(i==3)
                 tc = tc + 1;
              if(rc >= min(cz,rd))
                  cp = cp + 1;
              else
                  ncp = ncp + 1;
              end
             end
             if(i==4)
                 tc = tc + 1;
              if(rc >= min(dz,rd))
                  cp = cp + 1;
              else
                  ncp = ncp + 1;
              end
             end
             if(i==5)
                 tc = tc + 1;
              if(rc >= min(ez,rd))
```

```
                cp = cp + 1;
            else
                ncp = ncp + 1;
            end
      end
   end
end
```

Definition 2.7. [2], [3] An *intuitionistic fuzzy set* (briefly, IFS) in a nonempty set X is an object F having the form

$$F = \{(x, \mu_F(x), \gamma_F(x)) \mid x \in X\} \tag{2.2}$$

where the fuzzy sets $\mu_F\colon X \to [0,1]$ and $\gamma_F\colon X \to [0,1]$ denote the degree of membership and the degree of nonmembership, respectively, and for all $x \in X$,

$$0 \leq \mu_F(x) + \gamma_F(x) \leq 1. \tag{2.3}$$

An intuitionistic fuzzy set $F = \{(x, \mu_F(x), \gamma_F(x)) \mid x \in X\}$ in X can be identified to an ordered pair (μ_F, γ_F) in $[0,1]^X \times [0,1]^X$. For the sake of simplicity, we shall use the symbol $F = (\mu_F, \gamma_F)$ for the IFS $F = \{(x, \mu_F(x), \gamma_F(x)) \mid x \in X\}$.

Definition 2.8. An IFS $F = (\mu_F, \gamma_F)$ in A is called an *intuitionistic fuzzy UP-ideal* of A if it satisfies the following properties: for any $x, y, z \in A$,

(1) $\mu_F(0) \geq \mu_F(x)$,

(2) $\gamma_F(0) \leq \gamma_F(x)$,

(3) $\mu_F(x \cdot z) \geq \min\{\mu_F(x \cdot (y \cdot z)), \mu_F(y)\}$, and

(4) $\gamma_F(x \cdot z) \leq \max\{\gamma_F(x \cdot (y \cdot z)), \gamma_F(y)\}$.

Definition 2.9. An IFS $F = (\mu_F, \gamma_F)$ in A is called an *intuitionistic fuzzy UP-subalgebra* of A if it satisfies the following properties: for any $x, y \in A$,

(1) $\mu_F(x \cdot y) \geq \min\{\mu_F(x), \mu_F(y)\}$, and

(2) $\gamma_F(x \cdot y) \leq \max\{\gamma_F(x), \gamma_F(y)\}$.

Example 2.10. Consider a UP-algebra $A = \{0, a, b, c\}$ with the following Cayley table:

$\cdot$	0	a	b	c
0	0	a	b	c
a	0	0	0	0
b	0	a	0	c
c	0	a	b	0

Let $F = (\mu_F, \gamma_F)$ be an IFS in A defined by

$$\mu_F(x) = \begin{cases} 0.3 & \text{if } x = c, \\ 0.7 & \text{if } x \neq c \end{cases}$$

and

$$\gamma_F(x) = \begin{cases} 0.5 & \text{if } x = c, \\ 0.2 & \text{if } x \neq c. \end{cases}$$

Then $F = (\mu_F, \gamma_F)$ is an intuitionistic fuzzy UP-subalgebra of A by using the following programs.

Program for test μ_F

```
clc,clear
display(['Input n = 4 or n = 5']);
n = input('n = ');
b = zeros(n,n);
f = zeros(n,n);
if n == 4
    b = [   0.7 0.7 0.7 0.3;
            0.7 0.7 0.7 0.7;
            0.7 0.7 0.7 0.3;
            0.7 0.7 0.7 0.7 ];
    f = [   0.7 0.7 0.7 0.3;
            0.7 0.7 0.7 0.3;
            0.7 0.7 0.7 0.3;
            0.7 0.7 0.7 0.3 ];
else
    b = [   1   2   3   4   5;
            1   1   3   4   5;
            1   1   1   4   5;
            1   1   3   1   5;
            1   1   1   1   1   ];
end
tc  = 0;
cp  = 0;
ncp = 0;
az  = 0.7;
bz  = 0.7;
cz  = 0.7;
dz  = 0.3;
for i = 1:n
    for j = 1:n
             rc = b(i,j);
             rd = f(i,j);
             if(i==1)
                 tc = tc + 1;
              if(rc >= min(az,rd))
                  cp = cp + 1;
              else
```

```
                  ncp = ncp + 1;
              end
             end
             if(i==2)
                 tc = tc + 1;
              if(rc >= min(bz,rd))
                  cp = cp + 1;
              else
                  ncp = ncp + 1;
              end
             end
             if(i==3)
                 tc = tc + 1;
              if(rc >= min(cz,rd))
                  cp = cp + 1;
              else
                  ncp = ncp + 1;
              end
             end
             if(i==4)
                 tc = tc + 1;
              if(rc >= min(dz,rd))
                  cp = cp + 1;
              else
                  ncp = ncp + 1;
              end
        end
     end
end
```

Program for test γ_F

```
clc,clear
display(['Input n = 4 or n = 5']);
n = input('n = ');
b = zeros(n,n);
f = zeros(n,n);
if n == 4
    b = [   0.2 0.2 0.2 0.5;
            0.2 0.2 0.2 0.2;
            0.2 0.2 0.2 0.5;
            0.2 0.2 0.2 0.2 ];
    f = [   0.2 0.2 0.2 0.5;
            0.2 0.2 0.2 0.5;
            0.2 0.2 0.2 0.5;
            0.2 0.2 0.2 0.5 ];
```

```
else
    b = [   1   2   3   4   5;
            1   1   3   4   5;
            1   1   1   4   5;
            1   1   3   1   5;
            1   1   1   1   1   ];
end
tc  = 0;
cp  = 0;
ncp = 0;
az  = 0.2;
bz  = 0.2;
cz  = 0.2;
dz  = 0.5;
for i = 1:n
    for j = 1:n
             rc = b(i,j);
             rd = f(i,j);
             if(i==1)
                 tc = tc + 1;
              if(rc <= max(az,rd))
                  cp = cp + 1;
              else
                  ncp = ncp + 1;
              end
             end
             if(i==2)
                 tc = tc + 1;
              if(rc <= max(bz,rd))
                  cp = cp + 1;
              else
                  ncp = ncp + 1;
              end
             end
             if(i==3)
                 tc = tc + 1;
              if(rc <= max(cz,rd))
                  cp = cp + 1;
              else
                  ncp = ncp + 1;
              end
             end
             if(i==4)
                 tc = tc + 1;
              if(rc <= max(dz,rd))
```

```
                cp = cp + 1;
            else
                ncp = ncp + 1;
            end
        end
    end
end
```

Lemma 2.11. *Every intuitionistic fuzzy UP-subalgebra $F = (\mu_F, \gamma_F)$ of A satisfies the inequalities: for all $x \in A$,*

(1) $\mu_F(0) \geq \mu_F(x)$, *and*

(2) $\gamma_F(0) \leq \gamma_F(x)$.

Proof. Let $x \in A$. Then

(By Proposition 1.4 1.4)
$$\begin{aligned} \mu_F(0) &= \mu_F(x \cdot x) \\ &\geq \min\{\mu_F(x), \mu_F(x)\} \\ &= \min\{\mu_F(x)\} \\ &= \mu_F(x) \end{aligned}$$

and

(By Proposition 1.4 1.4)
$$\begin{aligned} \gamma_F(0) &= \gamma_F(x \cdot x) \\ &\leq \max\{\gamma_F(x), \gamma_F(x)\} \\ &= \max\{\gamma_F(x)\} \\ &= \gamma_F(x). \blacksquare \end{aligned}$$

Lemma 2.12. *Let an IFS $F = (\mu_F, \gamma_F)$ in A be an intuitionistic fuzzy UP-ideal of A. If $x, y \in A$ is such that $y \leq x$ in A, then*

(1) $\mu_F(y) \leq \mu_F(x)$, *and*

(2) $\gamma_F(y) \geq \gamma_F(x)$.

Proof. Let $x, y \in A$ be such that $y \leq x$ in A. Then $y \cdot x = 0$. Thus

$$\begin{aligned} &\text{(By UP-2)} & \mu_F(x) &= \mu_F(0 \cdot x) \\ && &\geq \min\{\mu_F(0 \cdot (y \cdot x)), \mu_F(y)\} \\ &\text{(By UP-2)} && = \min\{\mu_F(y \cdot x), \mu_F(y)\} \\ && &= \min\{\mu_F(0), \mu_F(y)\} \\ && &= \mu_F(y) \end{aligned}$$

and

$$
\begin{aligned}
\text{(By UP-2)} \qquad \gamma_F(x) &= \gamma_F(0 \cdot x) \\
&\leq \max\{\gamma_F(0 \cdot (y \cdot x)), \gamma_F(y)\} \\
\text{(By UP-2)} \qquad &= \max\{\gamma_F(y \cdot x), \gamma_F(y)\} \\
&= \max\{\gamma_F(0), \gamma_F(y)\} \\
&= \gamma_F(y).
\end{aligned}
$$

Hence, μ_F is an order preserving fuzzy set and γ_F is an anti order preserving fuzzy set in A. ∎

Lemma 2.13. *Let an IFS $F = (\mu_F, \gamma_F)$ in A be an intuitionistic fuzzy UP-ideal of A. If $w, x, y, z \in A$ is such that $x \leq w \cdot (y \cdot z)$ in A, then*

(1) $\mu_F(x \cdot z) \geq \min\{\mu_F(w), \mu_F(y)\}$, *and*

(2) $\gamma_F(x \cdot z) \leq \max\{\gamma_F(w), \gamma_F(y)\}$.

Proof. Let $w, x, y, z \in A$ be such that $x \leq w \cdot (y \cdot z)$ in A. Then $x \cdot (w \cdot (y \cdot z)) = 0$. Hence,

(By Definition 2.8 2.8)

$$
\begin{aligned}
\mu_F(x \cdot z) &\geq \min\{\mu_F(x \cdot (y \cdot z)), \mu_F(y)\} \\
\text{(By Definition 2.8 2.8)} \qquad &\geq \min\{\min\{\mu_F(x \cdot (w \cdot (y \cdot z))), \mu_F(w)\}, \mu_F(y)\} \\
&= \min\{\min\{\mu_F(0), \mu_F(w)\}, \mu_F(y)\} \\
\text{(By Definition 2.8 2.8)} \qquad &= \min\{\mu_F(w), \mu_F(y)\}
\end{aligned}
$$

and

(By Definition 2.8 2.8)

$$
\begin{aligned}
\gamma_F(x \cdot z) &\leq \max\{\gamma_F(x \cdot (y \cdot z)), \gamma_F(y)\} \\
\text{(By Definition 2.8 2.8)} \qquad &\leq \max\{\max\{\gamma_F(x \cdot (w \cdot (y \cdot z))), \gamma_F(w)\}, \gamma_F(y)\} \\
&= \max\{\max\{\gamma_F(0), \gamma_F(w)\}, \gamma_F(y)\} \\
\text{(By Definition 2.8 2.8)} \qquad &= \max\{\gamma_F(w), \gamma_F(y)\}..
\end{aligned}
$$

∎

Corollary 2.14. *Let an IFS $F = (\mu_F, \gamma_F)$ in A be an intuitionistic fuzzy UP-ideal of A. If $x, y, z \in A$ is such that $x \leq y \cdot z$ in A, then*

(1) $\mu_F(x \cdot z) \geq \mu_F(y)$, *and*

(2) $\gamma_F(x \cdot z) \leq \gamma_F(y)$.

Proof. Let $x, y, z \in A$ be such that $x \leq y \cdot z$ in A. By Lemma 2.13, put $w = 0$. By UP-2, we have that $x \leq 0 \cdot (y \cdot z)$. Hence,

$$\mu_F(x \cdot z) \geq \min\{\mu_F(0), \mu_F(y)\} = \mu_F(y)$$

and

$$\gamma_F(x \cdot z) \leq \max\{\gamma_F(0), \gamma_F(y)\} = \gamma_F(y).$$

∎

Theorem 2.15. *Every intuitionistic fuzzy UP-ideal of A is an intuitionistic fuzzy UP-subalgebra of A.*

Proof. Let $F = (\mu_F, \gamma_F)$ be an intuitionistic fuzzy UP-ideal of A and let $x, y \in A$. By Proposition 1.5 1.5, we have $x \leq y \cdot x$. It follows from Lemma 2.12 that

$$\mu_F(y \cdot x) \geq \mu_F(x) \geq \min\{\mu_F(y), \mu_F(x)\}$$

and

$$\gamma_F(y \cdot x) \leq \gamma_F(x) \leq \max\{\gamma_F(y), \gamma_F(x)\}.$$

Hence, $F = (\mu_F, \gamma_F)$ is an intuitionistic fuzzy UP-subalgebra of A. ∎

The converse of Theorem 2.15 may not be true. For example, the intuitionistic fuzzy UP-subalgebra $F = (\mu_F, \gamma_F)$ in Example 2.10 is not an intuitionistic fuzzy UP-ideal of A since

$$\gamma_F(b \cdot c) = 0.5 > 0.2 = \max\{\gamma_F(b \cdot (a \cdot c)), \gamma_F(a)\}. \tag{2.4}$$

Lemma 2.16. *Let f be a fuzzy set in A. Then the following statements hold: for any $x, y \in A$,*

(1) $1 - \max\{f(x), f(y)\} = \min\{1 - f(x), 1 - f(y)\}$, *and*

(2) $1 - \min\{f(x), f(y)\} = \max\{1 - f(x), 1 - f(y)\}$.

Proof. 2.16 If $\max\{f(x), f(y)\} = f(x)$, then $f(y) \leq f(x)$. Thus $1 - f(y) \geq 1 - f(x)$, so $\min\{1 - f(x), 1 - f(y)\} = 1 - f(x) = 1 - \max\{f(x), f(y)\}$. Similarly, if $\max\{f(x), f(y)\} = f(y)$, then

$$\min\{1 - f(x), 1 - f(y)\} = 1 - f(y) = 1 - \max\{f(x), f(y)\}.$$

2.16 If $\min\{f(x), f(y)\} = f(x)$, then $f(x) \leq f(y)$. Thus $1 - f(x) \geq 1 - f(y)$, so $\max\{1 - f(x), 1 - f(y)\} = 1 - f(x) = 1 - \min\{f(x), f(y)\}$. Similarly, if $\min\{f(x), f(y)\} = f(y)$, then

$$\max\{1 - f(x), 1 - f(y)\} = 1 - f(y) = 1 - \min\{f(x), f(y)\}.$$ ∎

Theorem 2.17. *An IFS $F = (\mu_F, \gamma_F)$ is an intuitionistic fuzzy UP-ideal of A if and only if the fuzzy sets μ_F and $\overline{\gamma}_F$ are fuzzy UP-ideals of A.*

Proof. Assume that an IFS $F = (\mu_F, \gamma_F)$ is an intuitionistic fuzzy UP-ideal of A. Then for any $x, y, z \in A$, we have

$$\mu_F(0) \geq \mu_F(x) \text{ and } \mu_F(x \cdot z) \geq \min\{\mu_F(x \cdot (y \cdot z)), \mu_F(y)\}.$$

Hence, μ_F is a fuzzy UP-ideal of A. Now, for any $x, y, z \in A$, we have

$$\gamma_F(0) \leq \gamma_F(x) \text{ and } \gamma_F(x \cdot z) \leq \max\{\gamma_F(x \cdot (y \cdot z)), \gamma_F(y)\}.$$

Thus $\overline{\gamma}_F(0) = 1 - \gamma_F(0) \geq 1 - \gamma_F(x) = \overline{\gamma}_F(x)$ and

$$\begin{aligned} \overline{\gamma}_F(x \cdot z) &= 1 - \gamma_F(x \cdot z) \\ &\geq 1 - \max\{\gamma_F(x \cdot (y \cdot z)), \gamma_F(y)\} \\ \text{(By Lemma 2.16 2.16)} \qquad &= \min\{1 - \gamma_F(x \cdot (y \cdot z)), 1 - \gamma_F(y)\} \\ &= \min\{\overline{\gamma}_F(x \cdot (y \cdot z)), \overline{\gamma}_F(y)\}. \end{aligned}$$

Hence, $\overline{\gamma}_F$ is a fuzzy UP-ideal of A.

Conversely, assume that μ_F and $\overline{\gamma}_F$ are fuzzy UP-ideals of A. Then for any $x, y, z \in A$, we have

$$\mu_F(0) \geq \mu_F(x) \text{ and } \mu_F(x \cdot z) \geq \min\{\mu_F(x \cdot (y \cdot z)), \mu_F(y)\}.$$

Now, for any $x, y, z \in A$, we have

$$\overline{\gamma}_F(0) \geq \overline{\gamma}_F(x) \text{ and } \overline{\gamma}_F(x \cdot z) \geq \min\{\overline{\gamma}_F(x \cdot (y \cdot z)), \overline{\gamma}_F(y)\}.$$

Thus $1 - \gamma_F(0) \geq 1 - \gamma_F(x)$, so $\gamma_F(0) \leq \gamma_F(x)$. Now,

$$\begin{aligned} 1 - \gamma_F(x \cdot z) &\geq \min\{1 - \gamma_F(x \cdot (y \cdot z)), 1 - \gamma_F(y)\} \\ \text{(By Lemma 2.16 2.16)} \qquad &= 1 - \max\{\gamma_F(x \cdot (y \cdot z)), \gamma_F(y)\}, \end{aligned}$$

so $\gamma_F(x \cdot z) \leq \max\{\gamma_F(x \cdot (y \cdot z)), \gamma_F(y)\}$. Hence, $F = (\mu_F, \gamma_F)$ is an intuitionistic fuzzy UP-ideal of A. ∎

Theorem 2.18. *An IFS $F = (\mu_F, \gamma_F)$ is an intuitionistic fuzzy UP-subalgebra of A if and only if the fuzzy sets μ_F and $\overline{\gamma}_F$ are fuzzy UP-subalgebras of A.*

Proof. Assume that an IFS $F = (\mu_F, \gamma_F)$ is an intuitionistic fuzzy UP-subalgebra of A. Then for any $x, y \in A$, we have

$$\mu_F(x \cdot y) \geq \min\{\mu_F(x), \mu_F(y)\}.$$

Hence, μ_F is a fuzzy UP-subalgebra of A. Now, for any $x, y \in A$, we have

$$\gamma_F(x \cdot y) \leq \max\{\gamma_F(x), \gamma_F(y)\}.$$

Thus

$$\begin{aligned} \overline{\gamma}_F(x \cdot y) &= 1 - \gamma_F(x \cdot y) \\ &\geq 1 - \max\{\gamma_F(x), \gamma_F(y)\} \\ \text{(By Lemma 2.16 2.16)} \qquad &= \min\{1 - \gamma_F(x), 1 - \gamma_F(y)\} \\ &= \min\{\overline{\gamma}_F(x), \overline{\gamma}_F(y)\}. \end{aligned}$$

Hence, $\overline{\gamma}_F$ is a fuzzy UP-subalgebra of A.

Conversely, assume that μ_F and $\overline{\gamma}_F$ are fuzzy UP-subalgebras of A. Then for any $x, y \in A$, we have

$$\mu_F(x \cdot y) \geq \min\{\mu_F(x), \mu_F(y)\}.$$

Now, for any $x, y \in A$, we have

$$\overline{\gamma}_F(x \cdot y) \geq \min\{\overline{\gamma}_F(x), \overline{\gamma}_F(y)\}.$$

Thus

$$\begin{aligned} 1 - \gamma_F(x \cdot y) &\geq \min\{1 - \gamma_F(x), 1 - \gamma_F(y)\} \\ \text{(By Lemma 2.16 2.16)} \qquad &= 1 - \max\{\gamma_F(x), \gamma_F(y)\}, \end{aligned}$$

so $\gamma_F(x \cdot y) \leq \max\{\gamma_F(x), \gamma_F(y)\}$. Hence, $F = (\mu_F, \gamma_F)$ is an intuitionistic fuzzy UP-subalgebra of A. ∎

Theorem 2.19. *An IFS $F = (\mu_F, \gamma_F)$ is an intuitionistic fuzzy UP-ideal of A if and only if the IFSs $\Box F = (\mu_F, \overline{\mu}_F)$ and $\Diamond F = (\overline{\gamma}_F, \gamma_F)$ are intuitionistic fuzzy UP-ideals of A.*

Proof. Assume that $F = (\mu_F, \gamma_F)$ is an intuitionistic fuzzy UP-ideal of A. Then for any $x, y, z \in A$, we have

$$\mu_F(0) \geq \mu_F(x) \text{ and } \mu_F(x \cdot z) \geq \min\{\mu_F(x \cdot (y \cdot z)), \mu_F(y)\}.$$

Thus for any $x, y, z \in A$, we have $\overline{\mu}_F(0) = 1 - \mu_F(0) \leq 1 - \mu_F(x) = \overline{\mu}_F(x)$ and

$$\begin{aligned} \overline{\mu}_F(x \cdot z) &= 1 - \mu_F(x \cdot z) \\ &\leq 1 - \min\{\mu_F(x \cdot (y \cdot z)), \mu_F(y)\} \\ \text{(By Lemma 2.16 2.16)} \qquad &= \max\{1 - \mu_F(x \cdot (y \cdot z)), 1 - \mu_F(y)\} \\ &= \max\{\overline{\mu}_F(x \cdot (y \cdot z)), \overline{\mu}_F(y)\}. \end{aligned}$$

Hence, $\Box F = (\mu_F, \overline{\mu}_F)$ is an intuitionistic fuzzy UP-ideal of A. Now, for any $x, y, z \in A$, we have

$$\gamma_F(0) \leq \gamma_F(x) \text{ and } \gamma_F(x \cdot z) \leq \max\{\gamma_F(x \cdot (y \cdot z)), \gamma_F(y)\}.$$

Thus for any $x, y, z \in A$, we have $\overline{\gamma}_F(0) = 1 - \gamma_F(0) \geq 1 - \gamma_F(x) = \overline{\gamma}_F(x)$ and

$$\begin{aligned} \overline{\gamma}_F(x \cdot z) &= 1 - \gamma_F(x \cdot z) \\ &\geq 1 - \max\{\gamma_F(x \cdot (y \cdot z)), \gamma_F(y)\} \\ \text{(By Lemma 2.16 2.16)} \qquad &= \min\{1 - \gamma_F(x \cdot (y \cdot z)), 1 - \gamma_F(y)\} \\ &= \min\{\overline{\gamma}_F(x \cdot (y \cdot z)), \overline{\gamma}_F(y)\}. \end{aligned}$$

Hence, $\Diamond F = (\overline{\gamma}_F, \gamma_F)$ is an intuitionistic fuzzy UP-ideal of A.

Conversely, assume that $\Box F = (\mu_F, \overline{\mu}_F)$ and $\Diamond F = (\overline{\gamma}_F, \gamma_F)$ are intuitionistic fuzzy UP-ideals of A. Then for any $x, y, z \in A$, we have

$$\mu_F(0) \geq \mu_F(x) \text{ and } \mu_F(x \cdot z) \geq \min\{\mu_F(x \cdot (y \cdot z)), \mu_F(y)\},$$

and

$$\gamma_F(0) \leq \gamma_F(x) \text{ and } \gamma_F(x \cdot z) \leq \max\{\gamma_F(x \cdot (y \cdot z)), \gamma_F(y)\}.$$

Hence, $F = (\mu_F, \gamma_F)$ is an intuitionistic fuzzy UP-ideal of A. ∎

Theorem 2.20. *An IFS $F = (\mu_F, \gamma_F)$ is an intuitionistic fuzzy UP-subalgebra of A if and only if the IFSs $\Box F = (\mu_F, \overline{\mu}_F)$ and $\Diamond F = (\overline{\gamma}_F, \gamma_F)$ are intuitionistic fuzzy UP-subalgebras of A.*

Proof. Assume that $F = (\mu_F, \gamma_F)$ is an intuitionistic fuzzy UP-subalgebra of A. Then for any $x, y \in A$, we have

$$\mu_F(x \cdot y) \geq \min\{\mu_F(x), \mu_F(y)\}.$$

Thus for any $x, y \in A$, we have

$$\begin{aligned} \overline{\mu}_F(x \cdot y) &= 1 - \mu_F(x \cdot y) \\ &\leq 1 - \min\{\mu_F(x), \mu_F(y)\} \\ \text{(By Lemma 2.16 2.16)} \qquad &= \max\{1 - \mu_F(x), 1 - \mu_F(y)\} \\ &= \max\{\overline{\mu}_F(x), \overline{\mu}_F(y)\}. \end{aligned}$$

Hence, $\Box F = (\mu_F, \overline{\mu}_F)$ is an intuitionistic fuzzy UP-subalgebra of A. Now, for any $x, y \in A$, we have

$$\gamma_F(x \cdot y) \leq \max\{\gamma_F(x), \gamma_F(y)\}.$$

Thus for any $x, y \in A$, we have

$$\begin{aligned} \overline{\gamma}_F(x \cdot y) &= 1 - \gamma_F(x \cdot y) \\ &\geq 1 - \max\{\gamma_F(x), \gamma_F(y)\} \\ \text{(By Lemma 2.16 2.16)} \qquad &= \min\{1 - \gamma_F(x), 1 - \gamma_F(y)\} \\ &= \min\{\overline{\gamma}_F(x), \overline{\gamma}_F(y)\}. \end{aligned}$$

Hence, $\Diamond F = (\overline{\gamma}_F, \gamma_F)$ in an intuitionistic fuzzy UP-subalgebra of A.

Conversely, assume that $\Box F = (\mu_F, \overline{\mu}_F)$ and $\Diamond F = (\overline{\gamma}_F, \gamma_F)$ are intuitionistic fuzzy UP-subalgebra of A. Then for any $x, y \in A$, we have

$$\mu_F(x \cdot y) \geq \min\{\mu_F(x), \mu_F(y)\} \text{ and } \gamma_F(x \cdot y) \leq \max\{\gamma_F(x), \gamma_F(y)\}.$$

Hence, $F = (\mu_F, \gamma_F)$ is an intuitionistic fuzzy UP-subalgebra of A. ■

Definition 2.21. Let f be a fuzzy set in A. For any $t \in [0, 1]$, the set

$$U(f; t) = \{x \in A \mid f(x) \geq t\} \text{ and } U^+(f; t) = \{x \in A \mid f(x) > t\}$$

are called an *upper t-level subset* and an *upper t-strong level subset* of f, respectively. The set

$$L(f; t) = \{x \in A \mid f(x) \leq t\} \text{ and } L^-(f; t) = \{x \in A \mid f(x) < t\}$$

are called a *lower t-level subset* and a *lower t-strong level subset* of f, respectively.

Theorem 2.22. *An IFS $F = (\mu_F, \gamma_F)$ is an intuitionistic fuzzy UP-ideal of A if and only if for all $s, t \in [0, 1]$, the sets $U(\mu_F; t)$ and $L(\gamma_F; s)$ are either empty or UP-ideals of A.*

Proof. Assume that $F = (\mu_F, \gamma_F)$ is an intuitionistic fuzzy UP-ideal of A. Let $U(\mu_F; t)$ and $L(\gamma_F; s)$ be nonempty subsets of A for all $s, t \in [0, 1]$. Then there exist $a \in U(\mu_F; t)$ and $b \in L(\gamma_F; s)$, that is, $\mu_F(a) \geq t$ and $\gamma_F(b) \leq s$. Since $F = (\mu_F, \gamma_F)$ is an intuitionistic fuzzy UP-ideal of A, we have $\mu_F(0) \geq \mu_F(x)$ and $\gamma_F(0) \leq \gamma_F(x)$ for all $x \in A$. Thus $\mu_F(0) \geq \mu_F(a) \geq t$ and $\gamma_F(0) \leq \gamma_F(b) \leq s$, so $0 \in U(\mu_F; t)$ and $0 \in L(\gamma_F; s)$. Let $x, y, z \in A$ be such that $x \cdot (y \cdot z) \in U(\mu_F; t)$ and $y \in U(\mu_F; t)$. Then $\mu_F(x \cdot (y \cdot z)) \geq t$ and $\mu_F(y) \geq t$. Thus

(By Definition 2.8 2.8)
$$\begin{aligned} \mu_F(x \cdot z) &\geq \min\{\mu_F(x \cdot (y \cdot z)), \mu_F(y)\} \\ &\geq \min\{t, t\} \\ &= t, \end{aligned}$$

so $x \cdot z \in U(\mu_F; t)$. Hence, $U(\mu_F; t)$ is a UP-ideal of A. Finally, let $x, y, z \in A$ be such that $x \cdot (y \cdot z) \in L(\gamma_F; s)$ and $y \in L(\gamma_F; s)$. Then $\gamma_F(x \cdot (y \cdot z)) \leq s$ and $\gamma_F(y) \leq s$. Thus

(By Definition 2.8 2.8)
$$\begin{aligned} \gamma_F(x \cdot z) &\leq \max\{\gamma_F(x \cdot (y \cdot z)), \gamma_F(y)\} \\ &\leq \max\{s, s\} \\ &= s, \end{aligned}$$

so $x \cdot z \in L(\gamma_F; s)$. Hence, $L(\gamma_F; s)$ is a UP-ideal of A.

Conversely, assume that for any $s, t \in [0, 1]$, the sets $U(\mu_F; t)$ and $L(\gamma_F; s)$ are either empty or UP-ideals of A. For any $x \in A$, let $\mu_F(x) = t$ and $\gamma_F(x) = s$. Then $x \in U(\mu_F; t) \neq \emptyset$ and $x \in L(\gamma_F; s) \neq \emptyset$. By assumption, we have $U(\mu_F; t)$ and $L(\gamma_F; s)$ are UP-ideals of A. Thus $0 \in U(\mu_F; t)$ and $0 \in L(\gamma_F; s)$, so $\mu_F(0) \geq t = \mu_F(x)$ and $\gamma_F(0) \leq s = \gamma_F(x)$ for all $x \in A$. Suppose that there exist $x, y, z \in A$ such that $\mu_F(x \cdot z) < \min\{\mu_F(x \cdot (y \cdot z)), \mu_F(y)\}$. Put

$$t_0 = \frac{1}{2}[\mu_F(x \cdot z) + \min\{\mu_F(x \cdot (y \cdot z)), \mu_F(y)\}].$$

Thus $t_0 \in [0, 1]$ and $\mu_F(x \cdot z) < t_0 < \min\{\mu_F(x \cdot (y \cdot z)), \mu_F(y)\}$. This implies that $x \cdot z \notin U(\mu_F; t_0), x \cdot (y \cdot z) \in U(\mu_F; t_0)$ and $y \in U(\mu_F; t_0)$. Thus $U(\mu_F; t_0)$ is not a UP-ideal of A. Now, suppose that there exist $a, b, c \in A$ such that $\gamma_F(a \cdot c) > \max\{\gamma_F(a \cdot (b \cdot c)), \gamma_F(b)\}$. Put

$$s_0 = \frac{1}{2}[\gamma_F(a \cdot c) + \max\{\gamma_F(a \cdot (b \cdot c)), \gamma_F(b)\}].$$

Thus $s_0 \in [0, 1]$ and $\max\{\gamma_F(a \cdot (b \cdot c)), \gamma_F(b)\} < s_0 < \gamma_F(a \cdot c)$. This implies that $a \cdot c \notin L(\gamma_F; s_0), a \cdot (b \cdot c) \in L(\gamma_F; s_0)$ and $b \in L(\gamma_F; s_0)$. Thus $L(\gamma_F; s_0)$ is not a UP-ideal of A. By assumption, we have $U(\mu_F; t_0)$ and $L(\gamma_F; s_0)$ are empty. This is a contradiction to the fact that $y \in U(\mu_F; t_0) \neq \emptyset$ and $b \in L(\gamma_F; s_0) \neq \emptyset$. Hence, $\mu_F(x \cdot z) \geq \min\{\mu_F(x \cdot (y \cdot z)), \mu_F(y)\}$ and $\gamma_F(x \cdot z) \leq \max\{\gamma_F(x \cdot (y \cdot z)), \gamma_F(b)\}$ for all $x, y, z \in A$. Therefore, $F = (\mu_F, \gamma_F)$ is an intuitionistic fuzzy UP-ideal of A. ∎

Theorem 2.23. *An IFS $F = (\mu_F, \gamma_F)$ is an intuitionistic fuzzy UP-subalgebra of A if and only if for all $s, t \in [0, 1]$, the sets $U(\mu_F; t)$ and $L(\gamma_F; s)$ are either empty or UP-subalgebras of A.*

Proof. Assume that $F = (\mu_F, \gamma_F)$ is an intuitionistic fuzzy UP-subalgebra of A. Let $U(\mu_F; t)$ and $L(\gamma_F; s)$ be nonempty subsets of A for all $s, t \in [0, 1]$. Let $x, y \in U(\mu_F; t)$. Then $\mu_F(x) \geq t$. Thus

(By Definition 2.9 2.9)
$$\begin{aligned} \mu_F(x \cdot y) &\geq \min\{\mu_F(x), \mu_F(y)\} \\ &\geq \min\{t, t\} \\ &= t, \end{aligned}$$

so $x \cdot y \in U(\mu_F; t)$. It follows from Proposition 1.10 that $U(\mu_F; t)$ is a UP-subalgebra of A. Finally, let $x, y \in L(\mu_F; t)$. Then $\mu_F(y) \geq t$ and

(By Definition 2.9 2.9)
$$\begin{aligned} \gamma_F(x \cdot y) &\leq \max\{\gamma_F(x), \gamma_F(y)\} \\ &\leq \max\{s, s\} \\ &= s, \end{aligned}$$

so $x \cdot y \in L(\gamma_F; s)$. It follows from Proposition 1.10 that $L(\gamma_F; s)$ is a UP-subalgebra of A. Conversely, assume that for any $s, t \in [0, 1]$, the set $U(\mu_F; t)$ and $L(\gamma_F; s)$ are either empty or UP-subalgebras of A. For any $x, y \in A$, let $\min\{\mu_F(x), \mu_F(y)\} = t$ and $\max\{\gamma_F(x), \gamma_F(y)\} = s$. Then $x, y \in U(\mu_F; t) \neq \emptyset$ and $x, y \in L(\gamma_F; s) \neq \emptyset$. By assumption, we have $U(\mu_F; t)$ and $L(\gamma_F; s)$ are UP-subalgebras of A and so $x \cdot y \in U(\mu_F; t)$ and $x \cdot y \in L(\gamma_F; s)$. It follows that $\mu_F(x \cdot y) \geq t = \min\{\mu_F(x), \mu_F(y)\}$ and $\gamma_F(x \cdot y) \leq s = \max\{\gamma_F(x), \gamma_F(y)\}$. Hence, $F = (\mu_F, \gamma_F)$ is an intuitionistic fuzzy UP-subalgebra of A. ∎

Theorem 2.24. *If an IFS $F = (\mu_F, \gamma_F)$ is an intuitionistic fuzzy UP-ideal of A, then for all $s, t \in [0, 1]$, the sets $U^+(\mu_F; t)$ and $L^-(\gamma_F; s)$ are either empty or UP-ideals of A.*

Proof. Assume that $F = (\mu_F, \gamma_F)$ is an intuitionistic fuzzy UP-ideal of A. Let $s, t \in [0, 1]$ be such that $U^+(\mu_F; t)$ and $L^-(\gamma_F; s)$ are nonempty subsets of A. Then there exist $a \in U^+(\mu_F; t)$ and $b \in L^-(\gamma_F; s)$, that is, $\mu_F(a) > t$ and $\gamma_F(b) < s$. Since $F = (\mu_F, \gamma_F)$ is an intuitionistic fuzzy UP-ideal of A, we have $\mu_F(0) \geq \mu_F(x)$ and $\gamma_F(0) \leq \gamma_F(x)$ for all $x \in A$. Thus $\mu_F(0) \geq \mu_F(a) > t$ and $\gamma_F(0) \leq \gamma_F(b) < s$, so $0 \in U^+(\mu_F; t)$ and $0 \in L^-(\gamma_F; s)$. Let $x, y, z \in A$ be such that $x \cdot (y \cdot z) \in U^+(\mu_F; t)$ and $y \in U^+(\mu_F; t)$. Then $\mu_F(x \cdot (y \cdot z)) > t$ and $\mu_F(y) > t$. Thus

(By Definition 2.8 2.8)
$$\begin{aligned} \mu_F(x \cdot z) &\geq \min\{\mu_F(x \cdot (y \cdot z)), \mu_F(y)\} \\ &> \min\{t, t\} \\ &= t, \end{aligned}$$

so $x \cdot z \in U^+(\mu_F; t)$. Hence, $U^+(\mu_F; t)$ is a UP-ideal of A. Finally, let $x, y, z \in A$ be such that $x \cdot (y \cdot z) \in L^-(\gamma_F; s)$ and $y \in L^-(\gamma_F; s)$. Then $\gamma_F(x \cdot (y \cdot z)) < s$ and $\gamma_F(y) < s$. Thus

(By Definition 2.8 2.8)
$$\begin{aligned} \gamma_F(x \cdot z) &\leq \max\{\gamma_F(x \cdot (y \cdot z)), \gamma_F(y)\} \\ &< \max\{s, s\} \\ &= s, \end{aligned}$$

so $x \cdot z \in L^-(\gamma_F; s)$. Hence, $L^-(\gamma_F; s)$ is a UP-ideal of A. ∎

Theorem 2.25. *If for all $s, t \in [0,1]$, the sets $U^+(\mu_F; t)$ and $L^-(\gamma_F; s)$ are UP-ideals of A, then an IFS $F = (\mu_F, \gamma_F)$ is an intuitionistic fuzzy UP-ideal of A.*

Proof. Assume that for all $s, t \in [0,1]$, the sets $U^+(\mu_F; t)$ and $L^-(\gamma_F; s)$ are UP-ideals of A. For any $x \in A$, we have $\mu_F(x) \in [0,1]$ and $\gamma_F(x) \in [0,1]$. By assumption, we have $U^+(\mu_F; \mu_F(x))$ and $L^-(\gamma_F; \gamma_F(x))$ are UP-ideals of A. Thus $0 \in U^+(\mu_F; \mu_F(x))$ and $0 \in L^-(\gamma_F; \gamma_F(x))$, that is, $\mu_F(0) > \mu_F(x)$ and $\gamma_F(0) < \gamma_F(x)$. Suppose that there exist $x, y, z \in A$ such that $\mu_F(x \cdot z) < \min\{\mu_F(x \cdot (y \cdot z)), \mu_F(y)\}$. Put $t_0 = \frac{1}{2}[\mu_F(x \cdot z) + \min\{\mu_F(x \cdot (y \cdot z)), \mu_F(y)\}]$. Thus $t_0 \in [0,1]$ and $\mu_F(x \cdot z) < t_0 < \min\{\mu_F(x \cdot (y \cdot z)), \mu_F(y)\}$. This implies that $x \cdot z \notin U^+(\mu_F; t_0), x \cdot (y \cdot z) \in U^+(\mu_F; t_0)$ and $y \in U^+(\mu_F; t_0)$. Thus $U^+(\mu_F; t_0)$ is not a UP-ideal of A. Now, suppose that there exist $a, b, c \in A$ such that $\gamma_F(a \cdot c) > \max\{\gamma_F(a \cdot (b \cdot c)), \gamma_F(b)\}$. Put $s_0 = \frac{1}{2}[\gamma_F(a \cdot c) + \max\{\gamma_F(a \cdot (b \cdot c)), \gamma_F(b)\}]$. Thus $s_0 \in [0,1]$ and $\max\{\gamma_F(a \cdot (b \cdot c)), \gamma_F(b)\} < s_0 < \gamma_F(a \cdot c)$. This implies that $a \cdot c \notin L^-(\gamma_F; s_0), a \cdot (b \cdot c) \in L^-(\gamma_F; s_0)$ and $b \in L^-(\gamma_F; s_0)$. Thus $L^-(\gamma_F; s_0)$ is not a UP-ideal of A. This is a contradiction to the fact that for all $s, t \in [0,1]$, the sets $U^+(\mu_F; t)$ and $L^-(\gamma_F; s)$ are UP-ideals of A. Hence, $\mu_F(x \cdot z) \geq \min\{\mu_F(x \cdot (y \cdot z)), \mu_F(y)\}$ and $\gamma_F(x \cdot z) \leq \max\{\gamma_F(x \cdot (y \cdot z)), \gamma_F(b)\}$ for all $x, y, z \in A$. Therefore, $F = (\mu_F, \gamma_F)$ is an intuitionistic fuzzy UP-ideal of A. ■

Theorem 2.26. *If an IFS $F = (\mu_F, \gamma_F)$ is an intuitionistic fuzzy UP-subalgebra of A, then for all $s, t \in [0,1]$, the sets $U^+(\mu_F; t)$ and $L^-(\gamma_F; s)$ are either empty or UP-subalgebras of A.*

Proof. Assume that $F = (\mu_F, \gamma_F)$ is an intuitionistic fuzzy UP-subalgebra of A. Let $s, t \in [0,1]$ be such that $U^+(\mu_F; t)$ and $L^-(\gamma_F; s)$ are nonempty subsets of A. Let $x, y \in U^+(\mu_F; t)$. Then $\mu_F(x) > t$ and $\mu_F(y) > t$. Thus

(By Definition 2.9 2.9)
$$\begin{aligned} \mu_F(x \cdot y) &\geq \min\{\mu_F(x), \mu_F(y)\} \\ &> \min\{t, t\} \\ &= t, \end{aligned}$$

so $x \cdot y \in U^+(\mu_F; t)$. It follows from Proposition 1.10 that $U^+(\mu_F; t)$ is a UP-subalgebra of A. Finally, let $x, y \in L^-(\gamma_F; s)$. Then $\gamma_F(x) < s$ and $\gamma_F(y) < s$. Thus

(By Definition 2.9 2.9)
$$\begin{aligned} \gamma_F(x \cdot y) &\leq \max\{\gamma_F(x), \gamma_F(y)\} \\ &< \max\{s, s\} \\ &= s, \end{aligned}$$

so $x \cdot y \in L^-(\gamma_F; s)$. It follows from Proposition 1.10 that $L^-(\gamma_F; s)$ is a UP-subalgebra of A. ■

Theorem 2.27. *If for all $s, t \in [0,1]$, the sets $U^+(\mu_F; t)$ and $L^-(\gamma_F; s)$ are UP-subalgebras of A, then an IFS $F = (\mu_F, \gamma_F)$ is an intuitionistic fuzzy UP-subalgebra of A.*

Proof. Assume that for all $s, t \in [0,1]$, the sets $U^+(\mu_F; t)$ and $L^-(\gamma_F; s)$ are UP-subalgebras of A. Suppose that there exist $x, y \in A$ such that $\mu_F(x \cdot y) < \min\{\mu_F(x),\ \mu_F(y)\}$. Put $t_0 = \frac{1}{2}[\mu_F(x \cdot y) + \min\{\mu_F(x), \mu_F(y)\}]$. Thus $t_0 \in [0,1]$ and $\mu_F(x \cdot y) < t_0 < \min\{\mu_F(x), \mu_F(y)\}$. This implies that $x \cdot y \notin U^+(\mu_F; t_0)$, $x \in U^+(\mu_F; t_0)$ and $y \in U^+(\mu_F; t_0)$. Thus $U^+(\mu_F; t_0)$ is not a UP-subalgebra of A. Now, suppose that there exist $a, b \in A$ such that $\gamma_F(a \cdot b) > \max\{\gamma_F(a), \gamma_F(b)\}$. Put $s_0 = \frac{1}{2}[\gamma_F(a \cdot b) + \max\{\gamma_F(a), \gamma_F(b)\}]$. Thus $s_0 \in [0,1]$ and $\max\{\gamma_F(a), \gamma_F(b)\} < s_0 < \gamma_F(a \cdot b)$. This implies that $a \cdot b \notin L^-(\gamma_F; s_0), a \in L^-(\gamma_F; s_0)$ and $b \in L^-(\gamma_F; s_0)$. Thus $L^-(\gamma_F; s_0)$ is not a UP-subalgebra of A. This is a contradiction to the fact that for all $s, t \in [0,1]$, the sets $U^+(\mu_F; t)$ and $L^-(\gamma_F; s)$ are UP-subalgebras of A. Hence, $\mu_F(x \cdot y) \geq \min\{\mu_F(x), \mu_F(y)\}$ and $\gamma_F(x \cdot y) \leq \max\{\gamma_F(x), \gamma_F(y)\}$ for all $x, y \in A$. Therefore, $F = (\mu_F, \gamma_F)$ is an intuitionistic fuzzy UP-subalgebra of A. ■

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EQUITABLE COLORINGS OF CARTESIAN PRODUCTS OF FANS WITH BIPARTITE GRAPHS[1]

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Abstract. In this paper, by the sorting method of vertices, it is obtained that the equitable chromatic number and the equitable chromatic threshold of the Cartesian products of fans with bipartite graphs.

Keywords: Cartesian product, equitable coloring, equitable chromatic number, equitable chromatic threshold.

MR (2000) Subject Classification: 05C70, 05C15.

1 Introduction

All graphs considered in this paper are finite, undirected, loopless and without multiple edges. For a positive integer k and a real number x, let $[k] = \{1, 2, ..., k\}$, $\lceil x \rceil$ and $\lfloor x \rfloor$ denote the smallest integer not less than x and the largest integer not greater than x, respectively.

A graph G is said to be k-colorable if there is a map $c : V(G) \to [k]$ such that adjacent vertices are mapped to distinct numbers. The map c is called a proper k-coloring of G, and all pre-images of a fixed number form a so-called color class. No two vertices are adjacent in each color class. The chromatic number of G, denoted by $\chi(G)$, is the smallest number k such that G is k-colorable.

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A graph $G = (V, E)$ is said to be equitably k-colorable if $V(G)$ can be divided into k independent sets $V_1, V_2, ..., V_k$ such that $||V_i| - |V_j|| \leq 1$ for all $i, j \in [k]$. The smallest integer k for which G is equitably k-colorable is called equitable chromatic number of a graph G, and denoted by $\chi_=(G)$. The equitable chromatic threshold of a graph G, denoted by $\chi^*_=(G)$, is the minimum t such that G is equitably k-colorable for all $k \geq t$. It is evident from the definition that

$$\chi(G) \leq \chi_=(G) \leq \chi^*_=(G)$$

for any graph G.

Equitable coloring as a special vertex coloring on graphs was first introduced by Meyer[1]. His motivation came from the problem of assigning one of the six days of the work week to each garbage collection route. Here, the vertices represent garbage collection routes and two such vertices are joined by an edge when the corresponding routes should not be run on the same day. The problem of assigning one of the six days of the work week to each route becomes the problem of 6-coloring of G. On practical grounds it might also be desirable to have an approximately equal number of routes run on each of the six days, so we have to color the graph in the equitable way.

Another application of equitable coloring is in scheduling and timetabling. Consider, for example, a problem of constructing university timetables. It is known that this problem can be modeled as coloring the vertices of a graph G whose nodes correspond to classes, edges correspond to time conflicts between classes, and colors to hours. If the set of available rooms is restricted, then we may be forced to partition the vertex set into independent subsets of as near equal size as possible, since then the room usage is the highest. For applications of equitable coloring such as scheduling and constructing timetables, please see [1], [5], [11], [12], [13].

In [3], by Lin and Chang, it is obtained that the exact values or upper bounds of the equitable chromatic number on Kronecker products of G and H, when G and H are complete graphs, bipartite graphs, paths or cycles, and in [4], it is studied that the equitable colorings of Cartesian product of paths and cycles, respectively, with bipartite graphs. In [16], Lih and Wu studied the equitable colorings of bipartite graphs, and in [17], Lih gave a good survey for this coloring. In [23], Zhu gave a survey for Hedetniemi's conjecture about the tensor product of graphs. The general problem of deciding if $\chi_=(G) \leq 3$ is NP-complete [10]. If, however, G has a regular or simplified structure we are sometimes able to provide a polynomial algorithm coloring it in the equitable way. For more details about this coloring, please see [1], [2], [6], [7], [8], [14], [20], [21], [22].

The Cartesian product of graphs $G = (V_1, E_1)$ and $H = (V_2, E_2)$ is the graph $G \square H$ with vertex set $\{(u, x) \mid u \in V_1, x \in V_2\}$ and edge set

$$\{(u,x)\ (v,y)\ | u = v \ \text{ with } \ xy \in E_2 \ \text{ or } \ x = y \ \text{ with } \ uv \in E_1\}.$$

Graph products are interesting and useful in many situations. For example, Sabidussi [19] showed that any graph has the unique decomposition into prime factors under the Cartesian product. The complexity of many problems, also equitable coloring, that deal with very large and complicated graphs is reduced greatly if one is able to fully characterize the properties of less complicated prime factors.

In the present paper, we study the equitable colorings of Cartesian products of fans with complete bipartite graphs.

2. Main results

In the following, let s, l, m, n, n' be all nonnegative integers, $F_{n'+1}$ represent the Fan with vertex set $V(F_{n'+1}) = \{x, x_1, x_2, ..., x_{n'}\}$, and H represent a complete bipartite graph with two parts $Y = \{y_1, y_2, ..., y_m\}$ and $Z = \{z_1, z_2, ..., z_n\}$ where $m \geq n$. We will study the equitable chromatic number and the equitable chromatic threshold of the Cartesian product $F_{n'+1} \Box H$ according to the parity of n' and m.

On the other hand, if $n = 1$, then H is a star and denoted by $K_{m,1}$. If $n' + 1 = 2$ or $n' + 1 = 3$, then $F_{n'+1}$ is a path or a cycle. In this paper, we always suppose that $n' + 1 > 3$. Clearly, $\chi_=(F_{n'+1} \Box H) \geq \chi(F_{n'+1}) = 3$.

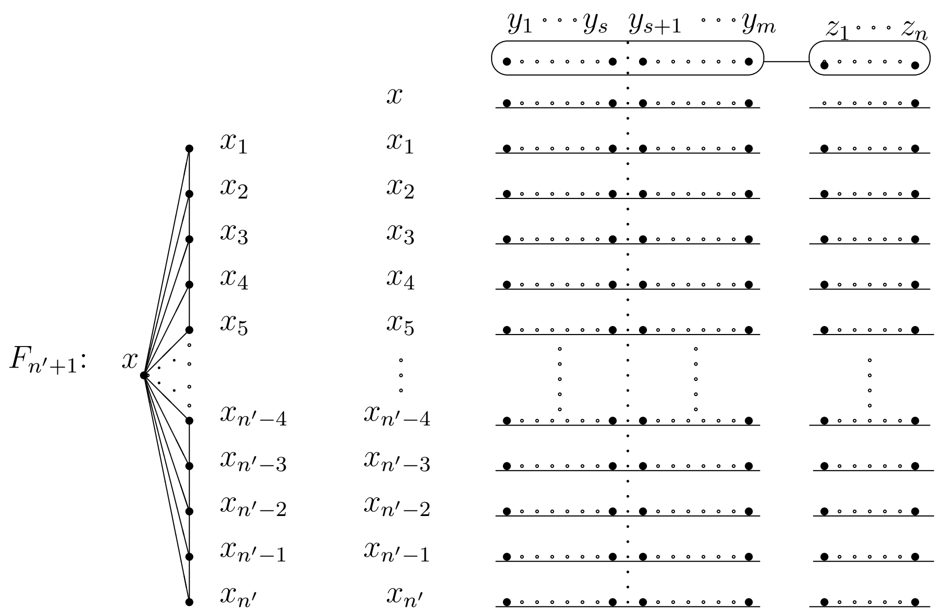

Figure 1. The Cartesian product $F_{n'+1} \Box H$

Theorem 2.1. *Suppose that $m \geq n \geq 1$, $l \geq 2$ and $k \geq 4$. If $n' = 2l$, then $F_{n'+1} \Box H$ is equitably k-colorable.*

Proof. The structure of the Cartesian product graph $F_{n'+1} \Box H$ is represented in Figure 1.

For $m = 2p+1$, $p \geq 0$ and $s = p+1$ (or for $m = 2p$, $p \geq 1$ and $s = p$), we will sort the vertices of $F_{n'+1} \square H$ as following.

$$\begin{aligned}
&(x, y_1), (x, y_2), \ldots, (x, y_s), (x_1, z_1), (x_3, z_1), \ldots, (x_{2h-1}, z_1), \ldots, (x_{n'-1}, z_1),\\
&(x_1, z_2), (x_3, z_2), \ldots, (x_{2h-1}, z_2), \ldots, (x_{n'-1}, z_2), \ldots, (x_1, z_n), (x_3, z_n), \ldots,\\
&(x_{2h-1}, z_n), \ldots, (x_{n'-1}, z_n), (x_2, y_{s+1}), (x_4, y_{s+1}), \ldots, (x_{2h}, y_{s+1}), \ldots, (x_{n'}, y_{s+1}),\\
&(x_2, y_{s+2}), (x_4, y_{s+2}), \ldots, (x_{2h}, y_{s+2}), \ldots, (x'_n, y_{s+2}), \ldots, (x_2, y_m), (x_4, y_m), \ldots,\\
&(x_{2h}, y_m), \ldots, (x_{n'}, y_m), (x_2, y_1), (x_2, y_2) \ldots, (x_2, y_s), (x_4, y_1), (x_4, y_2) \ldots, (x_4, y_s),\\
&\ldots, (x_{2h}, y_1), (x_{2h}, y_2) \ldots, (x_{2h}, y_s), \ldots, (x_{n'}, y_1), (x_{n'}, y_2), \ldots, (x_{n'}, y_s),\\
&(x, z_1), (x, z_2), \ldots, (x, z_n), (x_1, y_{s+1}), (x_3, y_{s+1}), \ldots,\\
&(x_{2h-1}, y_{s+1}), \ldots, (x_{n'-1}, y_{s+1}), (x_1, y_{s+2}), (x_3, y_{s+2}), \ldots,\\
&(x_{2h-1}, y_{s+2}), \ldots, (x_{n'-1}, y_{s+2}), \ldots, (x_1, y_m), (x_3, y_m), \ldots,\\
&(x_{2h-1}, y_m), \ldots, (x_{n'-1}, y_m), (x_1, y_1), (x_1, y_2) \ldots, (x_1, y_s),\\
&(x_3, y_1), (x_3, y_2) \ldots, (x_3, y_s), \ldots, (x_{2h-1}, y_1), (x_{2h-1}, y_2) \ldots, (x_{2h-1}, y_s),\\
&\ldots, (x_{n'-1}, y_1), (x_{n'-1}, y_2), \ldots, (x_{n'-1}, y_s), (x_2, z_1), (x_2, z_2), \ldots, (x_2, z_n),\\
&(x_4, z_1), (x_4, z_2), \ldots, (x_4, z_n), \ldots, (x_{2h}, z_1), (x_{2h}, z_2), \ldots, (x_{2h}, z_n), \ldots,\\
&(x_{n'}, z_1), (x_{n'}, z_2), \ldots, (x_{n'}, z_n), (x, y_{s+1}), (x, y_{s+2}), \ldots, (x, y_m),
\end{aligned}$$

where h is a positive integer and $1 \leq h \leq l$. It is not difficult to verify that the smallest cardinality of independent set consisting of consecutive vertices is at least $\min\{lp + nl + p + 1, 2pl + l + n\}$ for $m = 2p + 1$ and $\min\{2lp + n, nl + pl + p\}$ for $m = 2p$.

Let

$$\sigma_t = \left\lfloor \frac{(2l+1)(m+n) + t - 1}{k} \right\rfloor,$$

where $t \in [k]$. By $l \geq 2$ and $k \geq 4$, we have

$$\sigma_1 = \left\lfloor \frac{(2l+1)(m+n)}{k} \right\rfloor$$

$$\leq \sigma_k = \left\lfloor \frac{(2l+1)(m+n) + k - 1}{k} \right\rfloor = \left\lceil \frac{(2l+1)(m+n)}{k} \right\rceil \leq \left\lceil \frac{(2l+1)(m+n)}{4} \right\rceil.$$

If $m = 2p + 1$, then

$$\sigma_t \leq \left\lceil \frac{(2l+1)(m+n)}{4} \right\rceil \leq \min\{lp + nl + p + 1, 2pl + l + n\}.$$

If $m = 2p$, then

$$\sigma_t \leq \left\lceil \frac{(2l+1)(m+n)}{4} \right\rceil < \min\{2lp + n, nl + pl + p\}.$$

Hence, according to the vertex sorting above, the vertex set of $F_{n'+1} \square H$ can be partitioned into k independent sets with cardinality $\sigma_1, \sigma_2, \ldots, \sigma_k$, respectively. Therefore $F_{n'+1} \square H$ is equitably k-colorable. ∎

Theorem 2.2. *Suppose that $m \geq n \geq 1$.*

(1) *For $n' = 4$ and $2n - m \in \{0, \pm 1, \pm 2\}$, or*

(2) *For $n' = 2l$, $l \geq 3$ and $m = 2n$,*

we have that $F_{n'+1} \square H$ is equitably 3-colorable, and then $\chi^*_{=}(F_{n'+1} \square H) = 3$.

Proof. (1) Assume that s is a nonnegative integer, then we can partition the vertex set $V(F_{n'+1} \square H)$ into the following three parts:

$$V_1 = \left\{ \begin{array}{l} (x, y_1), (x, y_2), ..., (x, y_s), (x_1, z_1), (x_3, z_1), (x_1, z_2), (x_3, z_2), ..., (x_1, z_n), \\ (x_3, z_n), (x_2, y_{s+1}), (x_4, y_{s+1}), (x_2, y_{s+2}), (x_4, y_{s+2}), ..., (x_2, y_m), (x_4, y_m) \end{array} \right\},$$

$$V_2 = \left\{ \begin{array}{l} (x_1, y_1), (x_1, y_2), ..., (x_1, y_s), (x_3, y_1), (x_3, y_2), ..., (x_3, y_s), (x_2, z_1), (x_2, z_2), \\ ..., (x_2, z_n), (x_4, z_1), (x_4, z_2), ..., (x_4, z_n), (x, y_{s+1}), (x, y_{s+2}), ..., (x, y_m) \end{array} \right\},$$

and

$$V_3 = \left\{ \begin{array}{l} (x_2, y_1), (x_2, y_2), ..., (x_2, y_s), (x_4, y_1), (x_4, y_2), ..., (x_4, y_s), (x, z_1), (x, z_2), ..., \\ (x, z_n), (x_1, y_{s+1}), (x_3, y_{s+1}), (x_1, y_{s+2}), (x_3, y_{s+2}), ..., (x_1, y_m), (x_3, y_m) \end{array} \right\},$$

where $|V_1| = 2n + 2m - s$, $|V_2| = 2n + m + s$, and $|V_3| = n + 2m$.

If $m = 2n + 1$ and $s = n + 1$, then $|V_1| = 2n + 2m - s = 5n + 1$, $|V_2| = 2n + m + s = 5n + 2$, and $|V_3| = n + 2m = 5n + 2$.If $m = 2n - 1$ and $s = n$, then $|V_1| = 5n - 2$, $|V_2| = 5n - 1$, and $|V_3| = 5n - 2$.

If $m = 2n$ and $s = n$, then $|V_1| = 2n + 2m - s = 5n$, $|V_2| = 2n + m + s = 5n$, and $|V_3| = n + 2m = 5n$.

If $m = 2n + 2$ and $s = n + 1$, then $|V_1| = 2n + 2m - s = 5n + 3$, $|V_2| = 2n + m + s = 5n + 3$, and $|V_3| = n + 2m = 5n + 4$.

If $m = 2n - 2$ and $s = n - 1$, then $|V_1| = 2n + 2m - s = 5n - 3$, $|V_2| = 2n + m + s = 5n - 3$, and $|V_3| = n + 2m = 5n - 4$.

Therefore, $F_{n'+1} \square H$ is equitably 3-colorable. By Theorem 2.1, we have $\chi^*_{=}(F_{n'+1} \square H) = 3$.

(2) Assume that $s = n$ and $1 \leq h \leq l$, then we will partition the vertex set $V(F_{n'+1} \square H)$ into the following three parts:

$$V_1 = \left\{ \begin{array}{l} (x, y_1), (x, y_2), \ldots, (x, y_s), (x_1, z_1), (x_3, z_1), \ldots, (x_{2h-1}, z_1), \ldots, \\ (x_{n'-1}, z_1), (x_1, z_2), (x_3, z_2), \ldots, (x_{2h-1}, z_2), \ldots, (x_{n'-1}, z_2), \ldots, \\ (x_1, z_n), (x_3, z_n), \ldots, (x_{2h-1}, z_n), \ldots, (x_{n'-1}, z_n), (x_2, y_{s+1}), (x_4, y_{s+1}), \\ \ldots, (x_{2h}, y_{s+1}), \ldots, (x_{n'}, y_{s+1}), (x_2, y_{s+2}), (x_4, y_{s+2}), \ldots, (x_{2h}, y_{s+2}), \\ \ldots, (x_{n'}, y_{s+2}), \ldots, (x_2, y_m), (x_4, y_m), \ldots, (x_{2h}, y_m), \ldots, (x_{n'}, y_m) \end{array} \right\},$$

$$V_2 = \left\{ \begin{array}{l} (x_1, y_1), (x_1, y_2) \ldots, (x_1, y_s), (x_3, y_1), (x_3, y_2) \ldots, (x_3, y_s), \\ \ldots, (x_{2h-1}, y_1), (x_{2h-1}, y_2), \ldots, (x_{2h-1}, y_s), \ldots, (x_{n'-1}, y_1), \\ (x_{n'-1}, y_2) \ldots, (x_{n'-1}, y_s), (x_2, z_1), (x_2, z_2), \ldots, (x_2, z_n), \\ (x_4, z_1), (x_4, z_2), \ldots, (x_4, z_n), \ldots, (x_{2h}, z_1), (x_{2h}, z_2), \ldots, (x_{2h}, z_n), \\ \ldots, (x_{n'}, z_1), (x_{n'}, z_2), \ldots, (x_{n'}, z_n), (x, y_{s+1}), (x, y_{s+2}), \ldots, (x, y_m) \end{array} \right\},$$

and

$$V_3=\left\{\begin{array}{l}(x_2,y_1),(x_2,y_2)\ldots,(x_2,y_s),(x_4,y_1),(x_4,y_2)\ldots,(x_4,y_s),\ldots,(x_{2h},y_1),\\ (x_{2h},y_2),\ldots,(x_{2h},y_s),\ldots,(x_{n'},y_1),(x_{n'},y_2),\ldots,(x_{n'},y_s),(x,z_1),\\ (x,z_2),\ldots,(x,z_n),(x_1,y_{s+1}),(x_3,y_{s+1}),\ldots,(x_{2h-1},y_{s+1}),\\ \ldots,(x_{n'-1},y_{s+1}),(x_1,y_{s+2}),(x_3,y_{s+2}),\ldots,(x_{2h-1},y_{s+2}),\ldots,\\ (x_{n'-1},y_{s+2}),\ldots,(x_1,y_m),(x_3,y_m),\ldots,(x_{2h-1},y_m),\ldots,(x_{n'-1},y_m)\end{array}\right\}.$$

It is easy to see that $|V_1|=(2l+1)n$, $|V_2|=(2l+1)n$, and $|V_3|=(2l+1)n$. Hence $F_{n'+1}\square H$ is equitably 3-colorable. By Theorem 2.1, we have $\chi^*_=(F_{n'+1}\square H)=3$. ∎

Theorem 2.3. *Suppose that $m\geq n\geq 1$, $l\geq 1$ and $k\geq 4$. If $n'=2l+1$, then $F_{n'+1}\square H$ is equitably k-colorable.*

Proof. For $m=2p+1$, $p\geq 0$ and $s=p+1$ (or $m=2p$, $p\geq 1$ and $s=p$), we sort the vertex set of $F_{n'+1}\square H$ as following:

$$\begin{array}{l}(x,y_1),(x,y_2),\ldots,(x,y_s),(x_2,z_1),(x_4,z_1),\ldots,(x_{2h+2},z_1),\ldots,(x_{n'-1},z_1),\\ (x_2,z_2),(x_4,z_2),\ldots,(x_{2h+2},z_2),\ldots,(x_{n'-1},z_2),\ldots,(x_2,z_n),(x_4,z_n),\\ \ldots,(x_{2h+2},z_n),\ldots,(x_{n'-1},z_n),(x_1,y_{s+1}),(x_3,y_{s+1}),\ldots,(x_{2h'+1},y_{s+1}),\\ \ldots,(x_{n'},y_{s+1}),(x_1,y_{s+2}),(x_3,y_{s+2}),\ldots,(x_{2h'+1},y_{s+2}),\ldots,\\ (x_{n'},y_{s+2}),\ldots,(x_1,y_m),(x_3,y_m),\ldots,(x_{2h'+1},y_m),\ldots,(x_{n'},y_m),\\ (x_1,y_1),(x_1,y_2)\ldots,(x_1,y_s),(x_3,y_1),(x_3,y_2)\ldots,(x_3,y_s),\ldots,\\ (x_{2h'+1},y_1),(x_{2h'+1},y_2)\ldots,(x_{2h'+1},y_s),\ldots,(x_{n'},y_1),(x_{n'},y_2),\\ \ldots,(x_{n'},y_s),(x,z_1),(x,z_2),\ldots,(x,z_n),(x_2,y_{s+1}),(x_4,y_{s+1}),\ldots,\\ (x_{2h+2},y_{s+1}),\ldots,(x_{n'-1},y_{s+1}),(x_2,y_{s+2}),(x_4,y_{s+2}),\ldots,(x_{2h+2},y_{s+2}),\\ \ldots,(x_{n'-1},y_{s+2}),\ldots,(x_2,y_m),(x_4,y_m),\ldots,(x_{2h+2},y_m),\ldots,(x_{n'-1},y_m),\\ (x_2,y_1),(x_2,y_2)\ldots,(x_2,y_s),(x_4,y_1),(x_4,y_2)\ldots,(x_4,y_s),\ldots,(x_{2h+2},y_1),\\ (x_{2h+2},y_2)\ldots,(x_{2h+2},y_s),\ldots,(x_{n'-1},y_1),(x_{n'-1},y_2)\ldots,(x_{n'-1},y_s),\\ (x_1,z_1),(x_1,z_2),\ldots,(x_1,z_n),(x_3,z_1),(x_3,z_2),\ldots,(x_3,z_n),\ldots,\\ (x_{2h'+1},z_1),(x_{2h'+1},z_2),\ldots,(x_{2h'+1},z_n),\ldots,(x_{n'},z_1),(x_{n'},z_2),\ldots,\\ (x_{n'},z_n),(x,y_{s+1}),(x,y_{s+2}),\ldots,(x,y_m),\end{array}$$

where h,h' are all nonnegative integers, $0\leq h\leq l-1$, and $0\leq h'\leq l$. It is obvious that the smallest cardinality of independent set consisting of consecutive vertices in the order above is at least

$$\min\{lp+nl+2p+1,2lp+l+n,pl+nl+p+l+n\}$$

when $m=2p+1$ and

$$\min\{2lp+n,nl+p(l+2),(l+1)p+nl+n\}$$

when $m=2p$.

Let

$$\sigma_t=\left\lfloor\frac{(2l+2)(m+n)+t-1}{k}\right\rfloor,$$

where $t \in [k]$. By $l \geq 1$ and $k \geq 4$, we can obtain that

$$\sigma_1 = \left\lfloor \frac{(2l+2)(m+n)}{k} \right\rfloor \leq \sigma_k = \left\lfloor \frac{(2l+2)(m+n)+k-1}{k} \right\rfloor = \left\lceil \frac{(2l+2)(m+n)}{k} \right\rceil \leq \left\lceil \frac{(2l+2)(m+n)}{4} \right\rceil .$$

If $m = 2p+1$, then

$$\sigma_t \leq \left\lceil \frac{(2l+2)(m+n)}{4} \right\rceil \leq \min\{lp+nl+2p+1, 2lp+l+n, pl+nl+p+l+n\}.$$

If $m = 2p$, then

$$\sigma_t \leq \left\lceil \frac{(2l+2)(m+n)}{4} \right\rceil \leq \min\{2lp+n, nl+p(l+2), (l+1)p+nl+n\}.$$

Therefore, according to the vertex ordering above, the vertex set of $F_{n'+1}\square H$ can be partitioned into k independent sets with cardinality $\sigma_1, \sigma_2, \ldots, \sigma_k$, respectively. Hence $F_{n'+1}\square H$ is equitably k-colorable for $k \geq 4$. ∎

Theorem 2.4. *Suppose that $m \geq n \geq 1$.*
(1) *If $n' = 3$, and $2n - m \in \{0, \pm 1, \pm 2\}$, or*
(2) *if $l \geq 2$, $n' = 2l+1$ and $m = 2n$,*
then $F_{n'+1}\square H$ is equitably 3-colorable, and $\chi^*_=(F_{n'+1}\square H) = 3$.

Proof. (1) Assume that s is a nonnegative integer, then we can partition the vertex set $V(F_{n'+1}\square H)$ into the following three parts:

$$V_1 = \left\{ \begin{array}{l} (x,y_1),(x,y_2),\ldots,(x,y_s),(x_2,z_1),(x_2,z_2),\ldots,(x_2,z_n), \\ (x_1,y_{s+1}),(x_3,y_{s+1}),(x_1,y_{s+2}),(x_3,y_{s+2}),\ldots,(x_1,y_m),(x_3,y_m) \end{array} \right\},$$

$$V_2 = \left\{ \begin{array}{l} (x_2,y_1),(x_2,y_2)\ldots,(x_2,y_s),(x_1,z_1),(x_1,z_2),\ldots,(x_1,z_n), \\ (x_3,z_1),(x_3,z_2),\ldots,(x_3,z_n),(x,y_{s+1}),(x,y_{s+2}),\ldots,(x,y_m) \end{array} \right\},$$

and

$$V_3 = \left\{ \begin{array}{l} (x_1,y_1),(x_1,y_2)\ldots,(x_1,y_s),(x_3,y_1),(x_3,y_2)\ldots,(x_3,y_s), \\ (x,z_1),(x,z_2),\ldots,(x,z_n),(x_2,y_{s+1}),(x_2,y_{s+2}),\ldots,(x_2,y_m) \end{array} \right\},$$

where $|V_1| = n+2m-s$, $|V_2| = 2n+m$, and $|V_3| = n+m+s$.

If $m = 2n+1$ and $s = n+1$, then $|V_1| = n+2m-s = 4n+1$, $|V_2| = 2n+m = 4n+1$, and $|V_3| = n+m+s = 4n+2$. If $m = 2n-1$ and $s = n$, then $|V_1| = 4n-2$, $|V_2| = 4n-1$, and $|V_3| = 4n-1$.

If $m = 2n$ and $s = n$, then $|V_1| = n+2m-s = 4n$, $|V_2| = 2n+m = 4n$, and $|V_3| = n+m+s = 4n$. If $m = 2n-2$ and $s = n-1$, then $|V_1| = 4n-3$, $|V_2| = 4n-2$, and $|V_3| = 4n-3$. If $m = 2n+2$ and $s = n+1$, then $|V_1| = 4n+3$, $|V_2| = 4n+2$, and $|V_3| = 4n+3$.

Hence $F_{n'+1}\Box H$ is equitably 3-colorable. Applying Theorem 2.3, we have $\chi^*_=(F_{n'+1}\Box H) = 3$.

(2) Assume that $s = n$, $0 \le h \le l-1$, and $0 \le h' \le l$. Then we can partition the vertex set $V(F_{n'+1}\Box H)$ into the following three parts:

$$V_1 = \left\{ \begin{array}{l} (x,y_1),(x,y_2),\ldots,(x,y_s),(x_2,z_1),(x_4,z_1),\ldots,(x_{2h+2},z_1),\ldots, \\ (x_{n'-1},z_1),(x_2,z_2),(x_4,z_2),\ldots,(x_{2h+2},z_2),\ldots,(x_{n'-1},z_2),\ldots, \\ (x_2,z_n),(x_4,z_n),\ldots,(x_{2h+2},z_n),\ldots,(x_{n'-1},z_n),(x_1,y_{s+1}), \\ (x_3,y_{s+1}),\ldots,(x_{2h'+1},y_{s+1}),\ldots,(x_{n'},y_{s+1}),(x_1,y_{s+2}),(x_3,y_{s+2}), \\ \ldots,(x_{2h'+1},y_{s+2}),\ldots,(x_{n'},y_{s+2}),\ldots,(x_1,y_m),(x_3,y_m), \\ \ldots,(x_{2h'+1},y_m),\ldots,(x_{n'},y_m) \end{array} \right\},$$

$$V_2 = \left\{ \begin{array}{l} (x_2,y_1),(x_2,y_2)\ldots,(x_2,y_s),(x_4,y_1),(x_4,y_2)\ldots,(x_4,y_s),\ldots, \\ (x_{2h+2},y_1),(x_{2h+2},y_2)\ldots,(x_{2h+2},y_s),\ldots,(x_{n'-1},y_1),(x_{n'-1},y_2), \\ \ldots,(x_{n'-1},y_s),(x_1,z_1),(x_1,z_2),\ldots,(x_1,z_n),(x_3,z_1),(x_3,z_2), \\ \ldots,(x_3,z_n),\ldots,(x_{2h'+1},z_1),(x_{2h'+1},z_2),\ldots,(x_{2h'+1},z_n),\ldots, \\ (x_{n'},z_1),(x_{n'},z_2),\ldots,(x_{n'},z_n),(x,y_{s+1}),(x,y_{s+2}),\ldots,(x,y_m) \end{array} \right\},$$

and

$$V_3 = \left\{ \begin{array}{l} (x_1,y_1),(x_1,y_2),\ldots,(x_1,y_s),(x_3,y_1),(x_3,y_2)\ldots,(x_3,y_s), \\ \ldots,(x_{2h'+1},y_1),(x_{2h'+1},y_2),\ldots,(x_{2h'+1},y_s),\ldots, \\ (x_{n'},y_1),(x_{n'},y_2),\ldots,(x_{n'},y_s),(x,z_1),(x,z_2),\ldots,(x,z_n), \\ (x_2,y_{s+1}),(x_4,y_{s+1}),\ldots,(x_{2h+2},y_{s+1}),\ldots,(x_{n'-1},y_{s+1}), \\ (x_2,y_{s+2}),(x_4,y_{s+2}),\ldots,(x_{2h+2},y_{s+2}),\ldots,(x_{n'-1},y_{s+2}), \\ \ldots,(x_2,y_m),(x_4,y_m),\ldots,(x_{2h+2},y_m),\ldots,(x_{n'-1},y_m) \end{array} \right\}.$$

It is obvious that
$$|V_1| = |V_2| = |V_3| = (2l+2)n.$$
Hence $F_{n'+1}\Box H$ is equitably 3-colorable.

By Theorem 2.3, we have $\chi^*_=(F_{n'+1}\Box H) = 3$. ∎

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SUBORDINATION RESULTS FOR A CERTAIN SUBCLASS OF NON-BAZILEVIC ANALYTIC FUNCTIONS DEFINED BY LINEAR OPERATOR

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Abstract. In this work, by making use of the principle of subordination, we introduce a certain subclass of non-Bazilevic analytic functions defined by linear operator. Such results as subordination and superordination, sandwich theorem and inequality properties are given.

1. Introduction

Let A_s denote the class of the functions f of the form

$$f(z) = z + \sum_{n=s+1}^{\infty} a_n z^n, \quad (s \in \mathbb{N} = \{1, 2, 3, ...\}), \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$.

If $f(z)$ and $F(z)$ are analytic in $\mathbb{U}$. Then we say that the function $f(z)$ is subordinate to $F(z)$ in $\mathbb{U}$ if there exists an analytic function $w(z)$ in $\mathbb{U}$ such that $|w(z)| \leq 1$ and $f(z) = F(w(z))$, denoted by $f \prec F$ or $f(z) \prec F(z)$. Furthermore, if the function $F(z)$ is univalent in $\mathbb{U}$, then we have the following equivalence (see [10]):

$$f(z) \prec F(z) \Leftrightarrow f(0) = F(0) \ \text{ and } \ f(\mathbb{U}) \subset F(\mathbb{U}).$$

Let $\psi : \mathbb{C}^2 \times \mathbb{U} \to \mathbb{C}$ and $h(z)$ be univalent in $\mathbb{U}$. If $p(z)$ is analytic in $\mathbb{U}$ and satisfies the first order differential subordination:

$$\varphi\left(p(z), zp'(z); z\right) \prec h(z), \tag{2}$$

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then $p(z)$ is a solution of the differential subordination (2). The univalent function $q(z)$ is called a dominant of the solutions of the differential subordination (2) if $p(z) \prec q(z)$ for all $p(z)$ satisfying (2). A univalent dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants of (2) is called the best dominant. If $p(z)$ and $\varphi(p(z), zp'(z))$ are univalent in $\mathbb{U}$ and if $p(z)$ satisfies first order differential superordination:

$$h(z) \prec \varphi\left(p(z), zp'(z); z\right), \tag{3}$$

then $p(z)$ is a solution of the differential superordination (3). An analytic function $q(z)$ is called a subordinant of the solutions of the differential superordination (3) if $q(z) \prec p(z)$ for all $p(z)$ satisfying (3). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants of (3) is called the best subordinant. For further properties of subordination and superordination, see [10] and [11].

For functions $f, g \in A_s$, where f is given by (1) and g is defined by $g(z) = z + \sum\limits_{n=s+1}^{\infty} b_n z^n$, then the Hadamard product (or convolution) $f * g$ of the functions f and g is defined by

$$(f * g)(z) = f(z) * g(z) = z + \sum_{n=s+1}^{\infty} a_n b_n z^n.$$

For the functions $f, g \in A_s$, we define the linear operator $D^k_{\alpha,\beta,\lambda} : A_k \to A_k$ (for $k = 0, 1, 2, ...$), $0 < \alpha \leq 1$, $0 < \beta \leq 1, \lambda \geq 0$, and $z \in \mathbb{U}$ by:

$$\begin{aligned}
D^0_{\alpha,\beta,\lambda}(f * g)(z) &= (f * g)(z), \\
D^1_{\alpha,\beta,\lambda}(f * g)(z) &= D_{\alpha,\beta,\lambda}(f * g)(z) \\
&= [1 - \lambda(\alpha + \beta - 1)](f * g)(z) + z\lambda(\alpha + \beta - 1)[(f * g)(z)]' \\
&= z + \sum_{n=s+1}^{\infty} [\lambda(\alpha + \beta - 1)(n - 1) + 1]\, a_n b_n z^n,
\end{aligned}$$

and (in general)

$$\begin{aligned}
D^k_{\alpha,\beta,\lambda}(f * g)(z) &= D_{\alpha,\beta,\lambda}\left(D^{k-1}_{\alpha,\beta,\lambda}(f * g)(z)\right) \\
&= z + \sum_{n=s+1}^{\infty} [\lambda(\alpha + \beta - 1)(n - 1) + 1]^k\, a_n b_n z^n, \quad (\lambda \geq 0).
\end{aligned} \tag{4}$$

Using (4), it is easy to verify that

$$\begin{aligned}
&\lambda(\alpha + \beta - 1)z[D^k_{\alpha,\beta,\lambda}(f * g)(z)]' \\
&\quad = D^{k+1}_{\alpha,\beta,\lambda}(f * g)(z) + [1 - \lambda(\alpha + \beta - 1)]D^k_{\alpha,\beta,\lambda}(f * g)(z).
\end{aligned} \tag{5}$$

Remark 1. For $b_n = C(\delta, n)$, the operator $D^k_{\alpha,\beta,\lambda}(f * g)(z)$ extends to $D^k_{\alpha,\beta,\delta,\lambda} f(z)$, where the operator $D^k_{\alpha,\beta,\delta,\lambda} f(z)$ was introduced and studied by Alamoush and Darus, which generalizes many other operators (see [1]), where

$$C(\delta, n) = \begin{pmatrix} n + \delta - 1 \\ \delta \end{pmatrix}.$$

Definition 1. A function $f \in A_s$ is said to be in the class $N^k_{\alpha,\beta,\lambda}(g,\rho,\mu;A,B)$ if it satisfies the following subordination condition:

$$(1+\rho)\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu} - \rho\frac{D^{k+1}_{\alpha,\beta,\lambda}(f*g)(z)}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu} \prec \frac{1+Az}{1+Bz}, \tag{6}$$

where ($g \in A_s$, $\rho \in \mathbb{C}$, $0 < \mu < 1$, $-1 \leq B < A \leq 1$, $A \neq B$, $A \in \mathbb{R}$, and $D^k_{\alpha,\beta,\lambda}f(z)$ as defined on (4)). Here all the powers are the principal values.

Furthermore, the function $f \in N^k_{\alpha,\beta,\lambda}(g,\rho,\mu;\varpi)$ if and only if $f, g \in A_s$ and

$$Re\left\{(1+\rho)\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu} - \rho\frac{D^{k+1}_{\alpha,\beta,\lambda}(f*g)(z)}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu}\right\} > \varpi,$$

where ($0 \leq \varpi < 1$; $z \in \mathbb{U}$).

We note that:

If $k = 0$, and $b_n = 1$, then the class $N^k_{\alpha,\beta,\lambda}(g,\rho,\mu;A,B)$ reduces to the class $N(\rho,\mu;A,B)$ which is defined by Wang el. at [5]. If $k = 0$, $\rho = -1$, $n = 1$, $A = 1$, $B = -1$ and $b_n = 1$, then the class $N^k_{\alpha,\beta,\lambda}(g,\rho,\mu;A,B)$ reduces to the class of non-Bazilevic functions which introduced by Obradovic [13]. If $k = 0$, $\rho = -1$, $n = 1$, $A = 1-2\varpi$, $B = -1$ and $b_n = 1$, then the class $N^k_{\alpha,\beta,\lambda}(g,\rho,\mu;A,B)$ reduces to the class of non-Bazilevic functions of order $\varpi (0 \leq \varpi < 1)$ which was given by Tuneski and Darus [12]. Other works related to non-Bazilevic can be found in ([2]-[7]).

In the present paper, we discuss and prove the subordination and superordination properties, sandwich theorem and inequality properties for the class $N^k_{\alpha,\beta,\lambda}(g,\rho,\mu;A,B)$.

2. Preliminary results

In order to establish our main results, we need the following definition and lemmas.

Definition 2. [9]. Denote by Q the set of all functions f that are analytic and injective on $\overline{\mathbb{U}} \setminus E(f)$, where

$$E(f) = \left\{\zeta \in \partial\mathbb{U} : \lim_{z\to\zeta} f(z) = \infty\right\},$$

and such that $f'(\zeta) \neq 0$ for $\zeta \in \overline{\mathbb{U}} \setminus E(f)$

Lemma 1. [10] *Let the function $h(z)$ be analytic and convex (univalent) in $\mathbb{U}$ with $h(0) = 1$. Suppose also that the function $g(z)$ given by*

$$g(z) = 1 + c_k z^k + c_{k+1}z^{k+1} + ... \tag{7}$$

is analytic in $\mathbb{U}$. *If*

$$g(z)+\frac{zg'(z)}{\gamma}\prec h(z), \quad (Re(\gamma)>0;\gamma\neq 0; z\in\mathbb{U}), \tag{8}$$

then

$$g(z)\prec q(z)=\frac{\gamma}{k}z^{-\frac{\gamma}{k}}\int h(t)t^{\frac{\gamma}{k}-1}dt\prec h(z),$$

and $q(z)$ *is the best dominant of* (8).

Lemma 2. [8] *Let* $q(z)$ *be a convex univalent function in* $\mathbb{U}$ *and let* $\sigma\in\mathbb{C}$, $\eta\in\mathbb{C}^*=\mathbb{C}\setminus\{0\}$ *with*

$$Re\left(1+\frac{zq''(z)}{q(z)}\right)>\max\left\{0,-Re\left(\frac{\sigma}{\eta}\right)\right\}.$$

If the function $g(z)$ *is analytic in* $\mathbb{U}$ *and*

$$\sigma g(z)+\eta zg'(z)\prec\sigma q(z)+\eta zq'(z),$$

then $g(z)\prec q(z)$ *and* $q(z)$ *is the best dominant.*

Lemma 3. [9] *Let* $q(z)$ *be a convex univalent function in* $\mathbb{U}$ *and let* $k\in\mathbb{C}$. *Further assume that* $Re(k)>0$. *If*

$$g(z)\in H[q(0),1]\cap Q,$$

and

$$g(z)+kzg'(z)$$

is univalent in $\mathbb{U}$, *then*

$$q(z)+kzq'(z)\prec g(z)+kzg'(z),$$

implies $q(z)\prec g(z)$ *and* $q(z)$ *and* q *is the best subordinant.*

Lemma 4. [14] *Let* F *be analytic and convex in* $\mathbb{U}$. *If*

$$f,g\in A \quad and \quad f,g\prec F$$

then

$$\lambda f+(1-\lambda)g\prec F,\ (0\leq\lambda\leq 1).$$

Lemma 5. [15] *Let*

$$f(z)=1+\sum_{n=1}^{\infty}a_nz^n$$

be analytic in U *and*

$$g(z)=1+\sum_{n=1}^{\infty}b_nz^n$$

be analytic and convex in U. *If* $f(z)\prec g(z)$, *then*

$$|a_n|<|b_1| \quad (n\in\mathbb{N}).$$

3. Main results

We begin by presenting our first subordination property given by Theorem 1.

Theorem 1. *For $g \in A_s$, $\rho \in \mathbb{C}$, $0 < \mu < 1$, $-1 \le B < A \le 1$, $A \ne B$, $A \in \mathbb{R}$, and $D^k_{\alpha,\beta,\lambda}(f * g)$ as defined by (4). Let $f(z) \in N^k_{\alpha,\beta,\lambda}(g, \rho, \mu; A, B)$ with $Re(\rho) > 0$. Then*

$$\begin{aligned}(9)\qquad &\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f * g)(z)}\right)^{\mu} \prec q(z) \\ &= \frac{\mu}{\lambda(\alpha+\beta-1)s\rho}\int_0^1 u^{\frac{\mu}{\lambda(\alpha+\beta-1)s\rho}-1}\frac{1+Azu}{1+Bzu}du \prec \frac{1+Az}{1+Bz}\end{aligned}$$

and $q(z)$ is the best dominant.

Proof. Define the function $g(z)$ by

$$g(z) = \left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f * g)(z)}\right)^{\mu} \quad (z \in \mathbb{U}). \tag{10}$$

Then $g(z)$ is of the form (7) and analytic in $\mathbb{U}$ with $g(0) = 1$. Taking logarithmic differentiation of (10) with respect to z and using (5), we deduce that

$$\begin{aligned}(11)\qquad &(1+\rho)\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f * g)(z)}\right)^{\mu} - \rho\frac{D^{k+1}_{\alpha,\beta,\lambda}(f * g)(z)}{D^k_{\alpha,\beta,\lambda}(f * g)(z)}\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f * g)(z)}\right)^{\mu} \\ &= g(z) + \frac{\lambda(\alpha+\beta-1)\rho}{\mu}zg'(z).\end{aligned}$$

Since $f(z) \in N^k_{\alpha,\beta,\lambda}(g, \rho, \mu; A, B)$, we have

$$g(z) + \frac{\lambda(\alpha+\beta-1)\rho}{\mu}zg'(z) \prec \frac{1+Az}{1+Bz}.$$

Applying Lemma 1 to (11) with $\gamma = \dfrac{\mu}{\lambda(\alpha+\beta-1)\rho}$, we get

$$\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f * g)(z)}\right)^{\mu} \prec q(z) = \frac{\mu}{\lambda(\alpha+\beta-1)s\rho}z^{-\frac{\mu}{\lambda(\alpha+\beta-1)s\rho}}\int_z^1 t^{\frac{\mu}{\lambda(\alpha+\beta-1)s\rho}-1}\frac{1+At}{1+Bt}dt$$

$$= \frac{\mu}{\lambda(\alpha+\beta-1)s\rho}\int_0^1 u^{\frac{\mu}{\lambda(\alpha+\beta-1)s\rho}-1}\frac{1+Azu}{1+Bzu}du \prec \frac{1+Az}{1+Bz}, \tag{12}$$

and $q(z)$ is the best dominant. The proof of Theorem 1 is thus complete. ∎

Theorem 2. *Let $q(z)$ be univalent in $\mathbb{U}$, $\rho \in \mathbb{C}^*$. Suppose also that $q(z)$ satisfies the following inequality:*

$$Re\left(1+\frac{zq''(z)}{q(z)}\right) > \max\left\{0, -Re\left(\frac{\mu}{\lambda(\alpha+\beta-1)\rho}\right)\right\}. \tag{13}$$

If $f \in A_s$ satisfies the following subordination condition:

$$\begin{aligned}(1+\rho)\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu} &-\rho\frac{D^{k+1}_{\alpha,\beta,\lambda}(f*g)(z)}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu}\\ &\prec q(z)+\frac{\lambda(\alpha+\beta-1)\rho}{\mu}zq'(z),\end{aligned} \tag{14}$$

then

$$\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu} \prec q(z)$$

and $q(z)$ is the best dominant.

Proof. Let the function $g(z)$ be defined by (10). We know that (11) holds true. Combining (11) and (14), we find that

$$g(z)+\frac{\lambda(\alpha+\beta-1)\rho}{\mu}zg'(z) \prec q(z)+\frac{\lambda(\alpha+\beta-1)\rho}{\mu}zq'(z). \tag{15}$$

By using Lemma 2 and (15), we easily get the assertion of Theorem 2. ∎

Taking $q(z)=\dfrac{1+Az}{1+Bz}$ in Theorem 2, we get the following result.

Corollary 1. *Let $\rho \in \mathbb{C}$ and $-1 \le B < A \le 1$. Suppose also that*

$$Re\left(\frac{1-Bz}{1+Bz}\right) > \max\left\{0, -Re\left(\frac{\mu}{\lambda(\alpha+\beta-1)\rho}\right)\right\}.$$

If $f \in A_s$ satisfies the following subordination:

$$(1+\rho)\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu} -\rho\frac{D^{k+1}_{\alpha,\beta,\lambda}(f*g)(z)}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu}$$

$$\prec \frac{1+Az}{1+Bz}+\frac{\lambda(\alpha+\beta-1)\rho}{\mu}\frac{(A-B)z}{(1+Bz)^2},$$

then

$$\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu} \prec \frac{1+Az}{1+Bz},$$

and the function $\dfrac{1+Az}{1+Bz}$ is the best dominant.

Now, by making use of Lemma 3, we now derive the following superordination result.

Theorem 3. *Let $q(z)$ be convex univalent in $\mathbb{U}$, $\rho \in \mathbb{C}$ with $Re(\rho) > 0$. Also let*

$$\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^\mu \in H[q(0),1] \cap Q$$

and

$$(1+\rho)\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^\mu - \rho\frac{D^{k+1}_{\alpha,\beta,\lambda}(f*g)(z)}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^\mu$$

be univalent in $\mathbb{U}$. If $f \in A_s$ satisfies the following superordination:

$$q(z) + \frac{\lambda(\alpha+\beta-1)\rho}{\mu} zq'(z)$$

$$\prec (1+\rho)\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^\mu - \rho\frac{D^{k+1}_{\alpha,\beta,\lambda}(f*g)(z)}{D^k_{\alpha,\beta,\delta,\lambda}(f*g)(z)}\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^\mu,$$

then

$$q(z) \prec \left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^\mu$$

and the function $q(z)$ is the best subordinant.

Proof. Let the function $g(z)$ be defined by (10). Then

$$q(z) + \frac{\lambda(\alpha+\beta-1)\rho}{\mu} zq'(z) \prec (1+\rho)\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^\mu$$

$$-\rho\frac{D^{k+1}_{\alpha,\beta,\lambda}(f*g)(z)}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^\mu = g(z) + \frac{\lambda(\alpha+\beta-1)\rho}{\mu} zg'(z).$$

An application of Lemma 3 yields the assertion of Theorem 3.

Taking $q(z) = \dfrac{1+Az}{1+Bz}$ in Theorem 3, we get the following result.

Corollary 2. *Let $\rho \in \mathbb{C}$ and $-1 \le B < A \le 1$ with $Re(\rho) > 0$. Suppose also that*

$$\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^\mu \in H[q(0),1] \cap Q,$$

and

$$(1+\rho)\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^\mu - \rho\frac{D^{k+1}_{\alpha,\beta,\lambda}(f*g)(z)}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^\mu$$

be univalent in $\mathbb{U}$. *If* $f \in A_s$ *satisfies the following superordination:*

$$\frac{1+Az}{1+Bz}+\frac{\lambda(\alpha+\beta-1)\rho}{\mu}\frac{(A-B)z}{(1+Bz)^2}$$

$$\prec (1+\rho)\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu}-\rho\frac{D^{k+1}_{\alpha,\beta,\lambda}(f*g)(z)}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu},$$

then

$$\frac{1+Az}{1+Bz}\prec\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu}$$

and the function $\dfrac{1+Az}{1+Bz}$ *is the best subordinant.*

Combining Theorems 2 and 3, we easily get the following "Sandwich-type result".

Theorem 4. *Let* q_1 *be convex univalent and let* q_2 *be univalent in* $\mathbb{U}$, $\rho \in \mathbb{C}$ *with* $Re(\rho) > 0$. *Let* q_2 *satisfies* (13). *If*

$$\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu}\in H[q_1(0),1]\cap Q$$

and

$$(1+\rho)\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu}-\rho\frac{D^{k+1}_{\alpha,\beta,\lambda}(f*g)(z)}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu}$$

be univalent in $\mathbb{U}$, *also*

$$q_1(z)+\frac{\lambda(\alpha+\beta-1)\rho}{\mu}zq_1'(z)\prec(1+\rho)\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu}$$

$$-\rho\frac{D^{k+1}_{\alpha,\beta,\lambda}(f*g)(z)}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu}=q_2(z)+\frac{\lambda(\alpha+\beta-1)\rho}{\mu}zq_2'(z),$$

then

$$q_1(z)\prec\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu}\prec q_2(z).$$

and $q_1(z)$ *and* $q_2(z)$ *are, respectively, the best subordinant and dominant.*

Next, we consider the following:

Theorem 5. *If* $\rho > 0$ *and* $f \in N^k_{\alpha,\beta,\lambda}(g,\mu,\varpi)$ $(0 \le \varpi < 1)$, *then* $f \in N^k_{\alpha,\beta,\lambda}(g,\rho,\mu;\varpi)$ *for* $|z| < R$, *where*

$$R=\left(\sqrt{\left(\frac{\lambda(\alpha+\beta-1)s\rho}{\mu}\right)^2+1}-\frac{\lambda(\alpha+\beta-1)s\rho}{\mu}\right)^{\frac{1}{s}}. \tag{16}$$

The bound R *is the best possible.*

Proof. We begin by writing

$$\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu} = \varpi + (1-\varpi)g(z) \quad (z \in \mathbb{U},\ 0 \le \varpi < 1). \tag{17}$$

Then, clearly, the function $g(z)$ is of the form (7), is analytic and has a positive real part in $\mathbb{U}$. By taking the derivatives of both sides of (17), we get

$$\frac{1}{1-\varpi}\left\{(1+\rho)\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu} - \rho\frac{D^{k+1}_{\alpha,\beta,\lambda}(f*g)(z)}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu} - \varpi\right\}$$

$$= g(z) + \frac{\lambda(\alpha+\beta-1)\rho}{\mu}zg'(z). \tag{18}$$

By making use of the following well-known estimate (see [16], Theorem 1):

$$\frac{|zg'(z)|}{Re\,\{g(z)\}} \le \frac{2sr^s}{1-2r^{2s}} \quad (|z| = r < 1)$$

in (18), we obtain

$$Re\left(\frac{1}{1-\varpi}\left\{(1+\rho)\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu} - \rho\frac{D^{k+1}_{\alpha,\beta,\lambda}(f*g)(z)}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu} - \varpi\right\}\right)$$

$$\ge Re\,\{g(z)\}\left(1 - \frac{2\lambda(\alpha+\beta-1)\rho sr^s}{\mu(1-r^{2s})}\right). \tag{19}$$

It is seen that the right-hand side of (19) is positive, provided that $r < R$, where R is given by (16). In order to show that the bound R is the best possible, we consider the function $f(z) \in A_s$ defined by

$$\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu} = \varpi + (1-\varpi)\frac{1+z^s}{1-z^s} \ (z \in U,\ 0 \le \varpi < 1).$$

Noting that

$$\frac{1}{1-\varpi}\left\{(1+\rho)\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu} - \rho\frac{D^{k+1}_{\alpha,\beta,\lambda}(f*g)(z)}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu} - \varpi\right\}$$

$$= \frac{1+z^s}{1-z^s} + \frac{2\lambda(\alpha+\beta-1)\rho sz^s}{\mu(1-z^s)^2} = 0$$

for $|z| = R$, we conclude that the bound is the best possible. Theorem 5 is thus proved.

Now, we give the inclusion properties:

Theorem 6. *Let $\rho_2 \geq \rho_1 \geq 0$ and $-1 \leq B_1 \leq B_2 < A_2 \leq A_1 \leq 1$. Then*

$$N^k_{\alpha,\beta,\lambda}(g, \rho_2, \mu; A_2, B_2) \subset N^k_{\alpha,\beta,\lambda}(g, \rho_1, \mu; A_1, B_1). \tag{20}$$

Proof. Let $f \in N^k_{\alpha,\beta,\lambda}(g, \rho_2, \mu; A_2, B_2)$. Then we have

$$(1+\rho_2)\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^\mu - \rho_2 \frac{D^{k+1}_{\alpha,\beta,\lambda}(f*g)(z)}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^\mu \prec \frac{1+A_2 z}{1+B_2 z}.$$

Since $-1 \leq B_1 \leq B_2 < A_2 \leq A_1 \leq 1$, we easily find that

$$\begin{aligned}(1+\rho_2)\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^\mu - \rho_2 \frac{D^{k+1}_{\alpha,\beta,\lambda}(f*g)(z)}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^\mu \\ \prec \frac{1+A_2 z}{1+B_2 z} \prec \frac{1+A_1 z}{1+B_1 z},\end{aligned} \tag{21}$$

that is $f \in N^k_{\alpha,\beta,\lambda}(g, \rho_2, \mu; A_1, B_1)$. Thus the assertion of Theorem 6 holds for $\rho_2 = \rho_1 \geq 0$. If $\rho_2 > \rho_1 \geq 0$, by Theorem 1 and (21), we know that $f \in N^k_{\alpha,\beta,\lambda}(g, 0, \mu; A_1, B_1)$, that is,

$$\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^\mu \prec \frac{1+A_1 z}{1+B_1 z}. \tag{22}$$

At the same time, we have

$$\begin{aligned}&(1+\rho_1)\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^\mu - \rho_1 \frac{D^{k+1}_{\alpha,\beta,\lambda}(f*g)(z)}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^\mu \\ &= \left(1-\frac{\rho_1}{\rho_2}\right)\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^\mu \\ &+\frac{\rho_1}{\rho_2}\left[(1+\rho_2)\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^\mu - \rho_2 \frac{D^{k+1}_{\alpha,\beta,\lambda}(f*g)(z)}{D^k_{\alpha,\beta,\delta,\lambda}(f*g)(z)}\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^\mu\right].\end{aligned} \tag{23}$$

Moreover, since $0 \leq \frac{\rho_1}{\rho_2} < 1$ and the function $\frac{1+A_1 z}{1+B_1 z}$ $(-1 \leq B_1 < A_1 \leq 1)$ is analytic and convex in $\mathbb{U}$. Combining (21)-(23) and Lemma 4, we find that

$$(1+\rho_1)\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^\mu - \rho_1 \frac{D^{k+1}_{\alpha,\beta,\lambda}(f*g)(z)}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^\mu \prec \frac{1+A_1 z}{1+B_1 z},$$

that is $f \in N^k_{\alpha,\beta,\lambda}(g, \rho_1, \mu; A_1, B_1)$, which implies that the assertion (20) of Theorem 6 holds.

Theorem 7. *Let* $f \in N^k_{\alpha,\beta,\lambda}(g,\rho,\mu;A,B)$ *with* $\rho > 0$ *and* $-1 \leq B < A \leq 1$. *Then*

$$\begin{aligned} &\frac{\mu}{\lambda(\alpha+\beta-1)s\rho}\int_0^1 u^{\frac{\mu}{\lambda(\alpha+\beta-1)s\rho}-1}\frac{1-Au}{1-Bu}du \\ &< \Re\left\{\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu}\right\} < \frac{\mu}{\lambda(\alpha+\beta-1)s\rho}\int_0^1 u^{\frac{\mu}{\lambda(\alpha+\beta-1)s\rho}-1}\frac{1+Au}{1+Bu}du. \end{aligned} \tag{24}$$

The extremal function of (24) *is defined by*

$$\begin{aligned} F(z) &= D^k_{\alpha,\beta,\lambda}(f*g)(z) \\ &= z\left(\frac{\mu}{\lambda(\alpha+\beta-1)s\rho}\int_0^1 u^{\frac{\mu}{\lambda(\alpha+\beta-1)s\rho}-1}\frac{1+Auz^s}{1+Buz^s}du\right)^{\frac{-1}{\mu}}. \end{aligned} \tag{25}$$

Proof. Let $f \in N^k_{\alpha,\beta,\lambda}(g,\rho,\mu;A,B)$ with $\rho > 0$. From Theorem 1, we know that (9) holds, which implies that

$$\begin{aligned} &\Re\left\{\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu}\right\} \\ &< \sup_{z\in U}\Re\left\{\frac{\mu}{\lambda(\alpha+\beta-1)s\rho}\int_0^1 u^{\frac{\mu}{\lambda(\alpha+\beta-1)s\rho}-1}\frac{1+Auz}{1+Buz}du\right\} \\ &\leq \frac{\mu}{\lambda(\alpha+\beta-1)s\rho}\int_0^1 u^{\frac{\mu}{\lambda(\alpha+\beta-1)s\rho}-1}\sup_{z\in U}\left(\frac{1+Auz}{1+Buz}\right)du \\ &< \frac{\mu}{\lambda(\alpha+\beta-1)s\rho}\int_0^1 u^{\frac{\mu}{\lambda(\alpha+\beta-1)s\rho}-1}\frac{1+Au}{1+Bu}du \end{aligned} \tag{26}$$

and

$$\begin{aligned} &\Re\left\{\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu}\right\} \\ &> \inf_{z\in U}\Re\left\{\frac{\mu}{\lambda(\alpha+\beta-1)s\rho}\int_0^1 u^{\frac{\mu}{\lambda(\alpha+\beta-1)s\rho}-1}\frac{1+Auz}{1+Buz}du\right\} \\ &\geq \frac{\mu}{\lambda(\alpha+\beta-1)s\rho}\int_0^1 u^{\frac{\mu}{\lambda(\alpha+\beta-1)s\rho}-1}\inf_{z\in U}\left(\frac{1+Auz}{1+Buz}\right)du \\ &> \frac{\mu}{\lambda(\alpha+\beta-1)s\rho}\int_0^1 u^{\frac{\mu}{\lambda(\alpha+\beta-1)s\rho}-1}\frac{1-Au}{1-Bu}du. \end{aligned} \tag{27}$$

Combining (26) and (27), we get (24). By noting that the function $F(z)$ defined by (25) belongs to the class $N^k_{\alpha,\beta,\lambda}(g,\rho,\mu;A,B)$, we obtain that equality (24) is sharp. The proof of Theorem 7 is evidently complete. ∎

Similarly, by applying the method of proof of Theorem 7, we easily get the following result.

Corollary 3. *Let* $f \in N^k_{\alpha,\beta,\lambda}(g,\rho,\mu;A,B)$ *with* $\rho > 0$ *and* $-1 \leq B < A \leq 1$. *Then*

$$\frac{\mu}{\lambda(\alpha+\beta-1)s\rho}\int_0^1 u^{\frac{\mu}{\lambda(\alpha+\beta-1)s\rho}-1}\frac{1+Au}{1+Bu}du$$

$$< \Re\left\{\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu}\right\} \tag{28}$$

$$< \frac{\mu}{\lambda(\alpha+\beta-1)s\rho}\int_0^1 u^{\frac{\mu}{\lambda(\alpha+\beta-1)s\rho}-1}\frac{1-Au}{1-Bu}du.$$

The extremal function of (28) *is defined by* (25).

Theorem 8. *Let*

$$f(z) = z + \sum_{n=s+1}^{\infty} a_n z^n \in N^k_{\alpha,\beta,\lambda}(g,\rho,\mu;A,B), \quad (s \in \mathbb{N} = \{1,2,3,...\}). \tag{29}$$

Then

$$|a_{s+1}| \leq [\lambda(\alpha+\beta-1)+1]^{-k}\,\frac{(A-B)}{|\mu+\lambda(\alpha+\beta-1)\rho||b_{s+1}|}. \tag{30}$$

The inequality (30) *is sharp, with the extremal function defined by* (25).

Proof. Combining (6) and (29), we obtain

$$\begin{aligned}(1+\rho)&\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu} - \rho\frac{D^{k+1}_{\alpha,\beta,\lambda}(f*g)(z)}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu}\\ &= 1 + (\mu+\lambda(\alpha+\beta-1)\rho)\,[\lambda(\alpha+\beta-1)+1]^k\,a_{s+1}b_{s+1}z + ... \prec \frac{1+Az}{1+Bz}.\end{aligned} \tag{31}$$

An application of Lemma 5 to (31) yields

$$\left|(\mu+\lambda(\alpha+\beta-1)\rho)\,[\lambda(\alpha+\beta-1)+1]^k\,a_{s+1}b_{s+1}\right| < A-B. \tag{32}$$

Thus, from (32), we easily arrive at (30) asserted by Theorem 8. ∎

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ON FULLY STABLE ACTS

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Abstract. The purpose of this paper is to introduce and investigate the fully stable acts as a concept generalizing fully stable modules but is stronger than that of duo acts. In this study, we consider some properties and characterizations of the class of fully stable acts, and the relations between this class and other well studied classes of acts, like quasi-injective acts and acts satisfying Baer's criterion.

Keywords: fully stable, Baer's criterion, quasi-injective, right S-act.

Mathematics Subject Classification: 20M30.

1. Preliminaries

Let S be a monoid. A right S-act M_S is a nonempty set M together with a map (written multiplicatively) from $M \times S$ into M satisfying $m1 = m$ and $m(st) = (ms)t$, for all $m \in M$ and $s, t \in S$.

A nonempty subset N of an S-act M_S is S-subact if $ns \in N$ for all $s \in S$ and $n \in N$. We say that M_S is a cyclic S-act if $M_S = uS$ for some $u \in M_S$.

An element $z \in M_S$ is called a *fixed element* of M_S if $zs = z$ for all $s \in S$. The set of all fixed elements of M_S will be denoted by $\mathcal{F}(M)$.

If M_S has a unique fixed element z, then z is called *zero element* of M_S. We will denote the zero element of M_S by $\mathcal{O}$. Every S-act M_S can be extended to an S-act with fixed element z by taking the disjoint union: $M_S \dot\cup \{z\}_S$.

A nonempty subset $K \subseteq S$ is called *left ideal* of a monoid S if $SK \subseteq K$; a *right ideal* of S if $KS \subseteq K$; an ideal of S if $KS \subseteq K$ and $SK \subseteq K$.

Recall that, for two S-acts A_S, B_S a mapping $\theta : A_S \to B_S$ is called a *homomorphism of S-acts* or just an *S-homomorphism* if $\theta(as) = \theta(a)s$ for all $a \in A_S$, $s \in S$. The set of all S-homomorphisms from A_S into B_S will be denoted by $\mathrm{Hom}(A_S, B_S)$ or sometimes by $\mathrm{Hom}_S(A, B)$.

Note that if $\theta : A_S \to B_S$ is an S-homomorphism then $\operatorname{Im}\theta = \theta(A_S)$ is a subact of B_S, and the S-homomorphism $f : M_S \to M_S$ is called an endomorphism of M_S.

The set $\mathrm{Hom}_S(M, M)$ which forms a monoid under composition of mappings is denoted by $\mathrm{End}_S(M)$ and is called the endomorphism monoid of M_S.

An equivalence relation ρ on an S-act M_S is called an *S-act congruence* or a *congruence* on M_S, if $(m, n) \in \rho$ implies $(ms, ns) \in \rho$ for $m, n \in M_S$, $s \in S$. If S is a monoid then any right (semigroup) congruence ρ on S is an act congruence on S_S. Also, for an S-act M_S, $H \subset S$, $K \subset M \times M$, $T \subset M$, $J \subset S \times S$.

$$\begin{aligned}
\mathcal{L}_M(H) &= \{(m, n) \in M \times M \mid mx = nx \text{ for all } x \in H\};\\
\mathcal{R}_S(K) &= \{s \in S \mid as = bs \text{ for all } (a, b) \in K\};\\
\mathcal{R}_S(T) &= \{(a, b) \in S \times S \mid ma = mb \text{ for all } m \in T\};\\
\mathcal{L}_M(J) &= \{m \in M \mid ma = mb \text{ for all } (a, b) \in J\}.
\end{aligned}$$

The above is a kind of *annihilator* in S-act. Where $\mathcal{L}_M(H)$ (resp. $\mathcal{L}_M(J)$) are called the left annihilator of H (resp. J) and $\mathcal{R}_S(K)$ (resp. $\mathcal{R}_S(T)$) are called the right annihilator of K (resp. T).

Clearly,

$$\mathcal{R}_S(M) = \{(a, b) \in S \times S \mid ma = mb, \text{ for all } m \in M\}$$

is a right semigroup congruence on S and $\mathcal{R}_S(K)$ is a right ideal of S. If S is commutative then the set

$$\mathcal{L}_M(S) = \{(m, n) \in M \times M \mid mx = nx \text{ for all } x \in S\}$$

is a congruence on M_S and, if $\mathcal{L}_M(J) \neq \emptyset$, then it is a subact of M_S.

Recall that for a family $\{A_i\}$, $i \in I$, of right S-acts their Cartesian product $\prod_{i \in I} A_i$ with the S-action (multiplication) defined by $(a_i)s = (a_i s)$ is the *product* of a family of $\{A_i\}$, $i \in I$ of a right S-act.

The *coproduct* of a family of $\{A_i\}$, $i \in I$ of a right S-act is their disjoint union

$$\coprod_{i \in I} A_i = \left(\bigcup_{i \in I} A_i \times \{i\}\right)$$

with the action of S defined by $(a, i)s = (as, i)$ for $a \in A_i$ and $s \in S$.

For the family $\{A_i : i \in I\}$ of S-acts with a unique fixed element (zero element $\mathcal{O}$), the *direct sum* $\bigoplus_{i \in I} A_i$ is defined as the subact of the product $\prod_{i \in I} A_i$ consisting of all $(a_i)_{i \in I}$ such that $a_i = \mathcal{O}$ for all $i \in I$ except a finite number. We use $\bigoplus_{i \in I} A_i$ only when the S-acts A_i have unique fixed elements.

An S-act M_S is called *injective* if for each S-monomorphism $g : A_S \to B_S$ (where A_S, B_S are any two S-acts) and each S-homomorphism $f : A_S \to M_S$, there exists an S-homomorphism $h : B_S \to M_S$ such that $hg = f$.

A subact B_S is *essential* in an S-act M_S if for any S-act A_S and any S-homomorphism $f : M_S \to A_S$ whose restriction to B is one-to-one, the map f is itself one-to-one. In such a case, we say that M_S is an essential extension of B_S. The minimal injective extension of M_S is called the injective hull of M_S and is denoted by $E(M)$. Note that $E(M)$ is the injective hull of M_S if and only if M_S is essential in $E(M)$ and $E(M)$ is injective [3].

The *Jacobson radical* of an S-act M_S (denoted $\mathcal{J}(M_S)$) is defined by:

$$\mathcal{J}(M_S) = \{m \in M_S \mid \lambda_m \text{ is one-to-one only on one element right ideals of } S\},$$

where the mappings $\lambda_m : S_S \to M_S$ are given by $s \mapsto ms$ for each $s \in S$.

For an S-act M_S with zero element $\mathcal{O}$, the Jacobson radical $\mathcal{J}(M_S)$ is a subact of M_S [4].

2. Fully stable acts

In 1990, M.S. Abbas introduced a class of modules is called a fully stable as follows, a submodule N of an R-module M_R is called stable if $f(N) \subseteq N$ for each R-homomorphism $f : N \to M$, M is called fully stable module in case each submodule of M is stable [1].

In this section, we introduced the fully stable concept as a class of acts, and give several characterizations of fully stable acts. Also we consider the relations between this class and acts satisfying Baer's criterion.

Definition 2.1. Let M_S be an S-act. A subact N_S of M_S is called stable, if $f(N) \subseteq N$ for each S-homomorphism $f : N \to M$. The act M is called fully stable in case each subact of M is stable. A monoid S is fully stable if it is a fully stable S-act.

We have directly from the definition that every fully stable act is duo act, where by a duo S-act M_S we mean an S-act in which every subact N_S is fully invariant (i.e. $f(N) \subseteq N$ for any S-homomorphism $f : M \to M$ [2]).

However, the converse need not to be true in general; for example, it is easy to see that the act $\mathbb{Z}_{(\mathbb{Z},.)}$ of all integers is duo but not fully stable. For, if we define $\alpha : 2\mathbb{Z} \to \mathbb{Z}$ by $2n \mapsto 3n$, then, clearly, α is a $\mathbb{Z}$-homomorphism for which $\alpha(2\mathbb{Z}) \nsubseteq 2\mathbb{Z}$ since $\alpha(2.1) = 3.1 = 3 \notin 2\mathbb{Z}$.

Remarks 2.2.

1. Every subact of a fully stable act is fully stable.

2. The direct sum (hence, product) of fully stable acts need not be fully stable. For instance, let M_S be a fully stable S-act with a unique fixed element (zero element $\mathcal{O}$). The map $f : M \oplus \{\mathcal{O}\} \to M \oplus M$ defined by $f((m, \mathcal{O})) = (\mathcal{O}, m)$ is an S-homomorphism. Hence from the definition of direct sum, there is an element $\mathcal{O} \neq a \in M$ with $f((a, \mathcal{O})) = (\mathcal{O}, a) \notin M \oplus \{\mathcal{O}\}$. Thus $f(M \oplus \{\mathcal{O}\}) \nsubseteq M \oplus \{\mathcal{O}\}$.

3. The coproduct of any family of fully stable acts need not be fully stable. For example, let M_S be a fully stable S-act, $N \times \{1\}$ be a subact of $M \amalg M = M \times \{1\} \dot{\cup} M \times \{2\}$. Define $\theta : N \times \{1\} \to M \amalg M$ by $\theta((n,1)) = (n,2)$. Clearly, θ is an S-homomorphism but $\theta(N \times \{1\}) \nsubseteq N \times \{1\}$, since for any $n \in N$ $(n,2) \notin N \times \{1\}$.

In the following corollary, it is seen that to determine whether an act is fully stable it suffices to consider stability of a very restricted class of subacts.

Corollary 2.3. *An S-act M_S is fully stable if and only if every cyclic subact is stable.*

In the following proposition we give another characterization of fully stable acts which will be used later, when a monoid S is commutative.

Proposition 2.4. *An S-act M_S is fully stable if and only if for each x, y in M, $y \notin xS$ implies $\mathcal{R}_S(x) \nsubseteq \mathcal{R}_S(y)$.*

Proof. Suppose that M is fully stable and that there exist two elements x,$y \in M$ with $y \notin xS$ and $\mathcal{R}_S(x) \subseteq \mathcal{R}_S(y)$, define $f : xS \to M$ by $f(xr) = yr$ for $r \in S$. If $xr_1 = xr_2$ where $r_1, r_2 \in S$, then $(r_1, r_2) \in \mathcal{R}_S(x) \subseteq \mathcal{R}_S(y)$. This implies that $yr_1 = yr_2$, hence $f(xr_1) = f(xr_2)$, and f is well-defined. Clearly, f is an S-homomorphism. Since M is fully stable, we have $f(xS) \subseteq xS$ and $y = f(x) \in xS$ which is a contradiction.

Conversely, assume that there exists a cyclic subact xS of M and an S-homomorphism $\theta : xS \to M$ such that $\theta(xS) \nsubseteq xS$. Then, there exists an element $y \in xS$ such that $\theta(y) \notin xS$. Let $(s,t) \in \mathcal{R}_S(x)$, hence $xs = xt$. So

$$\theta(y)s = \theta(ys) = \theta(xrs) = \theta(xsr) = \theta(xtr) = \theta(xrt) = \theta(yt) = \theta(y)t.$$

Therefore, $(s,t) \in \mathcal{R}_S(y)$ and $\mathcal{R}_S(x) \subseteq \mathcal{R}_S(y)$, which is a contradiction. ∎

It is well-known that the Jacobson radical $\mathcal{J}(M)$ of an S-act is a fully invariant subact [3].

The following proposition gives a kind of subact which is always stable in any act.

Proposition 2.5. *The Jacobson radical of any act is a stable subact.*

Proof. Let M_S be an S-act and $f : \mathcal{J}(M) \to M$ an S-homomorphism. If A is a right ideal of S with more than one element i.e. $|A| \geq 2$, then there exist $a_1 \neq a_2 \in A$ such that $ma_1 = ma_2$. Hence

$$f(m)a_1 = f(ma_1) = f(ma_2) = f(m)a_2, \text{ for } m \in \mathcal{J}(M).$$

So $\lambda_{f(m)}$ is not one-to-one on A. Thus $f(m) \in \mathcal{J}(M)$. ∎

Definition 2.6. Let N_S be a subact of some act M_S. We say that N_S satisfies Baer criterion, if for every S-homomorphism $f : N_S \to M_S$, there exists an element $s \in S$ such that $f(n) = ns$ for each $n \in N_S$. An S-act M_S is said to satisfy Baer criterion if every subact of M_S satisfies Baer criterion.

Proposition 2.7. *If M_S is a fully stable S-act, then M_S satisfies Baer criterion for cyclic subacts* (*where S is a commutative monoid*).

Proof. Let xS be a cyclic subact of M_S and $f : xS \to M$ an S-homomorphism. Since xS is stable, we have $f(xS) \subseteq xS$ and hence $f(x) \in xS$ i.e. there is $t \in S$ such that $f(x) = xt$. Let $w \in xS$, hence $w = xr$ for some $r \in S$ and hence $f(w) \in xS$. So

$$f(w) = f(xr) = f(x)r = (xt)r = x(tr) = x(rt) = (xr)t = wt.$$

Hence there is $t \in S$ such that $f(w) = wt$ for every $w \in xS$. Thus Baer criterion holds for cyclic subacts. ∎

In the following proposition and its corollary, we obtain another characterization of fully stable acts. We assume the monoid S is commutative.

Proposition 2.8. *For an S-act M_S, Baer criterion holds for cyclic subacts if and only if $\mathcal{L}_M(\mathcal{R}_S(x)) = xS$ for all $x \in M$.*

Proof. Assume that the Baer criterion holds for cyclic subacts of M_S. Let $y \in \mathcal{L}_M(\mathcal{R}_S(x))$ and define $\theta : xS \to M$ by $\theta(xr) = yr$ for each $r \in S$. If $xr_1 = xr_2$ where $r_1, r_2 \in S$, then $(r_1, r_2) \in \mathcal{R}_S(x)$, hence $yr_1 = yr_2$ (since $y \in \mathcal{L}_M\mathcal{R}_S(x)$). Thus θ is well-defined. It is clear that θ is an S-homomorphism. By the assumption, there exists an element $t \in S$ such that $\theta(w) = wt$ for each $w \in xS$.

In particular,

$$y = \theta(x) = xt \in xS.$$

This implies that $\mathcal{L}_M(\mathcal{R}_S(x)) \subseteq xS$; since the inclusion $xS \subseteq \mathcal{L}_M(\mathcal{R}_S(x))$ is always true. Hence

$$\mathcal{L}_M(\mathcal{R}_S(x)) = xS.$$

Conversely, assume that $\mathcal{L}_M(\mathcal{R}_S(x)) = xS$ for each $x \in M$. Then, for each S-homomorphism $f : xS \to M$ and $(s, t) \in \mathcal{R}_S(x)$, we have

$$xs = xt \text{ and } f(x)s = f(xs) = f(xt) = f(x)t.$$

Thus $f(x) \in \mathcal{L}_M(\mathcal{R}_S(x)) = xS$. Therefore, $f(x) = xt$ for some $t \in S$. Now, for each $w \in xS$ there exists $r \in S$ such that $w = xr$, hence

$$f(w) = f(xr) = f(x)r = (xt)r = x(tr) = x(rt) = (xr)t = wt.$$

So there exists $t \in S$ such that $f(w) = wt$ for each $w \in xS$. ∎

As we have mentioned earlier, any fully stable S-act satisfies Baer criterion for cyclic subacts, thus we have the following corollary.

Corollary 2.9. *An S-act M_S is fully stable if and only if $\mathcal{L}_M(\mathcal{R}_S(x)) = xS$ for each $x \in M$.*

The results of this section can be summarized together with those of section one, in the following theorem.

Theorem 2.10. *The following statements are equivalent for an S-act M_S.*

1. *M_S is a fully stable act.*
2. *Every cyclic subact of M_S is stable.*
3. *For each x, y in M_S, $y \notin xS$ implies $\mathcal{R}_S(x) \nsubseteq \mathcal{R}_S(y)$.*
4. *M_S satisfies Baer criterion for cyclic subacts.*
5. *For each x in M_S, $\mathcal{L}_M(\mathcal{R}_S(x)) = xS$.*

Another characterization of fully stable acts is given here.

Remark 2.11. An S-act M_S is fully stable if and only if for each S-act A_S and for any two homomorphisms $f, g : A \to M$, with g injective (one-to-one mapping), we have $\operatorname{Im} f \subseteq \operatorname{Im} g$.

Proof. ($\Rightarrow$) Let A_S be an S-act and $f, g : A \to M$ S-homomorphisms. By injectivity of g, there exists an S-homomorphism $h : g(A) \to A$ such that $h \circ g = id_A$. Since $g(A)$ is a subact of M, we have $g(A)$ is stable. Hence $f \circ h(g(A)) \subseteq g(A)$. So $f(h \circ g(A)) \subseteq g(A)$ and $f(A) \subseteq g(A)$. ($\Leftarrow$) Let N_S be a subact of M_S and $f : N \to M$ an S-homomorphism. Since the inclusion $i : N \to M$ is an injective homomorphism, we get $f(N) \subseteq i(N) = N$. Thus, M_S is fully stable. ∎

3. Fully stable and quasi-injective acts

Recall that an S-act A_S is called quasi-injective [3] if for each subact B_S of A_S and any S-homomorphism $f : B_S \to A_S$ there exists an S-homomorphism $g : A_S \to A_S$ extending f. We will discuss the relation between quasi-injective and fully stable acts under the assumption that the monoid S is commutative. First, we recall some concepts needed. Given some concrete category C, an object $K \in C$ is called a *cofree* object in C if there exists $I \in Set$ and a mapping $\psi : \lfloor K \rfloor \to I$ such that the following universal property is valid: for every $X \in C$ and every mapping $\xi : \lfloor X \rfloor \to I$ there exists exactly one $\xi^* \in \operatorname{Mor}_C(X, K)$ such that the following diagram in Set is commutative:

$$\begin{tikzcd}
I & \lfloor X \rfloor \arrow[l, "\xi"'] \arrow[dl, "\lfloor \xi^* \rfloor"] \\
\lfloor K \rfloor \arrow[u, "\psi"] &
\end{tikzcd}$$

We write $Cof(I)$ for K and say that K is *I-cofree.* The set I is called a *cobasis* for K.

For the cofree concept in S-Act, we have the following proposition. But, first, recall that $I^S = \mathrm{Hom}({}_SS_{\{1\}}, {}_{\{1\}}I_{\{1\}})$ is a right S-act and fs for $f \in I^S$, $s \in S$ is defined by $(fs)(t) = f(st)$ for every $t \in S$, $I \neq \emptyset$, see [3, Remark 1.7.20].

Proposition 3.1. [2, p.151] *Let $I \neq \emptyset$. The S-act I^S with $\psi(f) = f(1)$ for all $f \in I^S$ is an $|I|$-cofree object in Act-S.*

The next proposition shows that cofree of a fully stable act is itself a fully stable act.

Proposition 3.2. *If the S-act M_S is a fully stable act, then $(M^S)_S$ is fully stable (where S is a commutative monoid, i.e., the left S-act S is right).*

Proof. Let $f, g \in M^S$ such that $\mathcal{R}_S(g) \subseteq \mathcal{R}_S(f)$, where

$$\mathcal{R}_S(g) = \{(s,t) \in S \times S \mid gs = gt\} \text{ and } \mathcal{R}_S(f) = \{(s,t) \in S \times S \mid fs = ft\}.$$

Since M is a cobasis of M^S, there exists an S-homomorphism $\psi : M^S \to M$ such that $\psi(f) = f(1)$, for each $f \in M^S$. Hence $f(1), g(1) \in M$ and

$$\mathcal{R}_S(g(1)) \subseteq \mathcal{R}_S(f(1)).$$

Since, if $(s_1, s_2) \in \mathcal{R}_S(g(1))$, then $g(1)s_1 = g(1)s_2$, hence $g(s_1) = g(s_2)$ and hence $g(s_1)(1) = g(s_2)(1)$. Now, for each $t \in S$, we have that

$$g(s_1)(t) = g(s_1)(1)t = g(s_2)(1)t = g(s_2)(t)$$

by commutativity of S, hence $(s_1, s_2) \in \mathcal{R}_S(g) \subseteq \mathcal{R}_S(f)$, so that $(s_1, s_2) \in \mathcal{R}_S(f(1))$. Thus,

$$\mathcal{R}_S(g(1)) \subseteq \mathcal{R}_S(f(1)).$$

By full stability of M, we have

$$f(1)S \subseteq g(1)S.$$

Therefore, $f \in gS$ and, by Proposition 2.4, we have that M^S is fully stable S-act. ∎

Now, we ask the following question. Is there a relation between fully stable acts and quasi-injective acts? The following theorem answers this question.

Theorem 3.3. *Every fully stable act is quasi-injective.*

Proof. Let M_S be a fully stable act. Hence, for any subact N_S of M_S and S-homomorphism $\alpha : N \to M$, we have that $\alpha(N) \subseteq N$, that is $\alpha : N \to N$.

By the injectivity of $E(M)$, the map α extends to an S-homomorphism $\beta : E(M) \to E(M)$. But $(M^S)_S$ is a cofree fully stable S-act, hence $(M^S)_S$ is injective fully stable act see Theorem 3.1.5 in [3], but $E(M)$ is a minimal injective extension of M_S, hence $E(M)$ is a subact of $(M^S)_S$ and since every subact of fully stable is fully stable [Remark 2.2.1], hence $E(M)$ is a fully stable act and then $\hat{\beta} : M \to M$ is an extension of α where $\hat{\beta} = \beta|_M$. Therefore, M a quasi-injective. ∎

Corollary 3.4. *The injective hull of fully stable act is fully stable.*

The converse of Theorem 3.3 is not true in general as in the following example.

Example 3.5. Let $S = \{0, 1\}$. Consider the S-act $A = \{\mathcal{O}, a, b, c\}$ with multiplication $0 = b0 = c0 = \mathcal{O}$. The act A_S is injective, so it is must be quasi-injective. But it is not fully stable, because $aS = \{\mathcal{O}, a\} \neq \mathcal{L}_A(\mathcal{R}_S(a)) = A$.

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AN EFFECTIVE BOUNDARY INTEGRAL APPROACH FOR THE SOLUTION OF NONLINEAR TRANSIENT THERMAL DIFFUSION PROBLEMS

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Abstract. Numerical calculations of nonlinear transient thermal diffusion problems have been carried out with a modified 'simple' boundary integral formulation known as the Green element method (GEM). The theory of the formulation is based on the singular integral equation of the boundary element method (BEM) but its implementation is element-by-element like the finite element method (FEM). Domain integrals resulting from nonlinearity of the problems as well as those arising from the approximation of the time derivative are encountered but unlike the classical approach, they are resolved within the element domain. Comparisons of GEM results with those obtained analytically or from the finite difference Newton-Richtmeyer's and the finite element method (FEM) serve to confirm the usefulness of the proposed formulation in handling nonlinearity in an unambiguous, straightforward and elegant manner without transforming or complicating the governing equations.
Keywords: nonlinearity, boundary element method, finite element method, finite difference method, Green element method, Newton-Richtmeyer, transient, thermal, diffusion.

1. Introduction

The overall conception that the boundary element method (BEM) is capable of solving many complex numerical problems in engineering and science is founded on the volumes of published work in this field that have found their way into scientific literature for the past few decades. In addition the ease with which BEM handles the aspect ratio degradation, its pointwise application of the discretized governing equation which not only facilitates its handling of high gradient scalar fields but also enhances the use of coarser grids around the vicinity of point loads and singularities, its ability to compute both the dependent variable and its flux simultaneously with the same level of accuracy, its relative ease of formulation and its boundary-only discretization which leads to a reduction in problem dimensionality are among one of those attractive features which lend the method its unique qualities. BEM superiority over other traditional numerical methods has been demonstrated in the way it handles the Laplace equation and in the solution of those nonlinear problems that are amenable to transformations of the type that enhance domain-avoidance. In all these demonstrations the full BEM coefficient matrix equation has always been put to task[1]-[9]. In as much as the relative advantages of BEM attract more and more users there are still some issues concerning its application that have not been fully addressed. For example some of the relatively simple however extremely challenging problems that are yet to be resolved include time-dependent heat diffusion problems, problems involving nonlinearity, heterogeneity, non-smooth problems. Although some of these problems have been used to validate BEM codes, the ponderous mathematical rigor involved in inventing techniques and artifices designed to contain body-force terms and deal with the problem domain have led to various types of BEM techniques [10]-[19]. Extensions to some nonlinear problems like the Navier-Stokes equations are not straightforward and are still in their elementary stages of implementation. It still remains a concern how singularity in heat flux as well as issues related to nonlinearity and heterogeneity have not been satisfactorily and straightforwardly dealt with by BEM approach. Neither is it clear why there is a noticeable scarcity of 1-D BEM codes specifically written to address 1-D type problems like in finite difference and finite element methods. As a consequence, optimism about the accuracy and advantages of BEM remain tenuous as we deal with a variety of the problems mentioned above. This is indicative BEM's restricted ability to only handle steady state problems or any problem for that matter that does not involve any numerical calculations involving the problem domain. It is worthy of note that the reason for this dramatic loss of accuracy remains unclear and has not been fully addressed in boundary element literature. Our primary aim in this paper is to further explore numerically the adaptation of the boundary integral formulation to handle nonlinearity and domain discretization [20]-[29]. The singular integral equation which results from applying the Green's second identity to the stationary part of the Laplace operator (the linear diffusion operator) is applied to the problem domain in a way that is akin to the finite element implementation. This approach though boundary integral based adopts domain discretization unreservedly and gains immensely from the finite element handling of the problem domain especially for those problems whose physics dictate an encounter with domain discretization. Past experience [1]-[5] clearly indicates that avoiding the problem domain at all costs or devising rigorous techniques to transfer all domain integrals to the boundary has not only met with mixed fortunes but in actual fact considerably slowed down the development of BEM into a highly efficient and competitive numerical tool.

2. Theoretical background

Let us consider a heat conductor with a nonlinear constitutive equation for the heat flux. The heat conduction equation to be satisfied by the temperature field for this specification is give by:

$$\frac{\partial}{\partial x}\left(D(\theta)\tfrac{\partial \theta}{\partial x}\right) = \rho c \frac{\partial \theta}{\partial t} + F(x,t,\theta) \tag{1}$$

where θ is the temperature, D, ρ and c are are the temperature dependent thermal conductivity, density and specific heat respectively and F is the heat source function. Nonlinearity of the transient-state heat conduction is contributed generally by: nonlinear boundary conditions as well as the thermal conductivity dependence on the scalar variable. Such problems have found great relevance in various areas of engineering, mathematical physics, and applied science, especially in areas such as thermo-structural design of nuclear reactors and microwave heating. The difficulties arising from the nonlinearities associated with equation (1) informed the choice of numerical technique for the solution. In boundary element applications most of the approaches adopted relies heavily on the transformation of the governing equations into corresponding analogs that help to eliminate or obfuscate all impacts of domain integrations on the solution profile. Though some of the earlier attempts in this field were beset by errors encountered in transforming from one plane of computation to the other recent improvements have resulted in encouraging results. One of the earliest attempts to deal with nonlinearity by adopting a boundary integral procedure can be found in [30-34]. Attempts to improve on this body of work can only add to the competitiveness of BEM by investigating and clarifying in a realistic way specific problems that are peculiar to BEM formulation.

2.1. The Green element formulation by linear interpolation basis functions

A unique solution to the mathematical statement expressed by equation (1) can be obtained when appropriate conditions for the dependent variable θ as well as its flux $q = -D\nabla\theta.n$ are specified on the boundary of the problem domain. There can be three of this namely: the Dirichlet-type condition specifies the temperature on the boundary:

$$\theta(x,t) = \theta_1(t) \tag{2a}$$

The heat flux can be specified across another part of the problem domain to give the Neumann-type boundary condition.

$$-D\nabla\theta.n = q_2(t) \tag{2b}$$

The Robbin or Cauchy-type boundary condition can be specified on a boundary to give:

$$\vartheta_1\theta + \vartheta_2 D\nabla\theta.n = \Psi_3(t) \tag{2c}$$

GEM formulation starts by putting equation (1) into its Poisson form:

$$\frac{\partial^2\theta}{\partial x^2} = \frac{1}{D(\theta)}\left[-\frac{\partial D}{\partial\theta}\frac{\partial\theta}{\partial x} + \rho c\frac{\partial\theta}{\partial x} + f(x,t,\theta)\right] \tag{3a}$$

or

$$\frac{\partial^2\theta}{\partial x^2} = -\frac{\partial LnD}{\partial x}\vartheta + \frac{1}{D(\theta)}\left[\frac{\partial((\beta)\theta)}{\partial t} + f\right] \tag{3b}$$

where $\beta(\theta) = \rho(\theta)c(\theta)$ is the heat capacity of the medium and $\vartheta = \partial\theta/\partial x$.

The auxiliary differential equation $\frac{d^2G}{dx^2} = \delta(x - x_1)$ as well as the Green's second identity are adopted to convert equation(1) into its integral analog:

$$\begin{aligned}\lambda\theta(x_i, t) &+ G^*(x_2, t)\theta x_2, t - G^*(x_1, t)\theta(x_1, t) \\ &- G(x_2, t)\vartheta(x_2, t) + G(x_1, t)\vartheta(x_1, t) \\ &+ \int_{x_1}^{x_2} G(x, x_1)\left[-\frac{\partial LnD}{\partial x}\vartheta(x, t) + \frac{1}{D(\theta)}\left(\beta(\theta)\frac{\partial\theta}{\partial t} + f\right)\right] dx = 0\end{aligned} \tag{4}$$

where the subscript I denotes the source point, λ is the Cauchy integration of the Dirac delta function and is given a value of 0.5 when situated at the boundary of the problem domain otherwise it is 0.5, $G(x, x_1) = \frac{(|x - x_1| + p)}{2}$ is the Green's function and $G^*(x, x_1) = \frac{dG(x, x_i)}{dx}$ is the derivative of the Green's function. It is worthwhile to comment that equation (4) is a boundary integral formulation and applies to both the problem domain as well as its boundary. The finite element implementation of equation(4) is the core of GEM and in line with this Lagrange type interpolation function are prescribed for the dependent variable θ and its functions: $LnD(\theta)$, $\frac{1}{D(\theta)}$, $\vartheta(\theta)$. This is put in the general form:

$$\xi \approx \omega_j\xi_j \tag{5}$$

where ω_j is the interpolating function with respect to node j and the Einstein summation for the repeating index indicates summation for all the nodes in a particular element of the problem domain. Substituting the interpolation function as well as the expressions for $G(v, x_i)$ and $G^*(x, x_i)$ yields a system of discrete element equations:

$$R_{ij}\theta_j + (L_{ij} - U_{inj}\Theta_n)\vartheta_j + T_{inj}\left(\psi_n\frac{d\theta_j}{dt}\right) = 0,\ i, j, n = 1, 2 \tag{6}$$

where the discrete element matrices R_{ij},L_{ij},U_{ij}, T_{inj} have all been defined [27]. In GEM computation we propose that the problem domain be divided into a finite number of arbitrary sub-regions or elements where a continuous function is approximated by a piecewise function in such a way that at the nodes, the values of the approximating function coincide with those of the approximated function. Furthermore, the approximating functions are chosen in such a way as to satisfy the continuity requirement along the surfaces separating the adjacent elements. Given this procedure, we may look at the approximations of the scalar field within a particular element independent of what happens in adjacent or neighboring elements. It is this unique aspect of the finite element technique that enhances GEM's handling of nonlinearity and heterogeneity.

Equation (6) represents a system of nonlinear equations that describes heat conduction in an element in terms of the values of the temperature at the nodes of that element. On assembling the whole system of element equations for the entire problem domain, the node points of each element, become the interior nodes except those on the boundaries which are referred to as the boundary nodes. A finite difference approximation of the temporal derivative yields:

$$\left.\frac{d\theta_i}{dt}\right|_{t=t_{m+\alpha\Delta t}} = \frac{\theta_i^m - \theta_i^{m+1}}{\Delta t}$$

where $\Delta t = t^{(m+1)-t^m}$ and α is a time weighting factor over a range $0 \leq \alpha \leq 1$. As a result, equation (6) becomes:

$$\begin{aligned}
&\left[\alpha R_{ij} + \frac{T_{inj}(\alpha\psi_n^{(m+1)} + (1-\alpha)\psi_n^m)}{\Delta t}\right]\theta_j^{m+1} + \alpha\left[L_{ij} - U_{inj}\Theta_n\right]\vartheta^{(m+1)_j} \\
&+\left[(1-\alpha)R_{ij} + \frac{T_{inj}(\alpha\psi_n^{(m+1)}+(1-\alpha)\psi_n^m)}{\Delta t}\right]\theta_j^{m+1} + (1-\alpha)[L_{ij}-U_{inj}\Theta_n]\vartheta^{(m)_j} \\
&+ T_{inj}[\alpha k_n^{m+1} + (1-\alpha)k_n^m][\alpha f_n^{(m+1)} + (1-\alpha)f_n^m] \equiv g_i = 0,\ i,j,n = 1,2.
\end{aligned} \tag{7}$$

Equation (7) is a system of nonlinear element discrete equations. The Newton-Raphson technique is adopted for the linearization process to yield:

$$J_{ij}^{m+1} = \begin{cases} \left.\dfrac{\partial g_i}{\partial\theta_j}\right|_{\theta_j = \theta_j^{m+1}} & = \alpha R_{ij} + \dfrac{T_{inj}[\alpha\psi_n^{m+1} + (1-\alpha)\psi_n^m]}{\Delta t} \\ & \quad + \dfrac{\alpha T_{inj}[\alpha\theta_n^{m+1} + (1-\alpha)\theta_n^{(m)}]}{\Delta t}\dfrac{d\psi}{d\theta_j} \\ & \quad -\alpha U_{inj}\varphi_n^{m+1,k}\dfrac{\Theta_l}{d\theta_j} \\ \left.\dfrac{\partial g_i}{\partial\varphi_j}\right|_{\varphi_j=\varphi_j^{m+1}} & = \alpha[L_{ij} - U_{inj}\Theta_n^{m+1,k}] \end{cases} \tag{8}$$

The computation is initiated by an estimate of the unknown dependent variables $\{\theta_j^{m+1,k}, \varphi_j^{m+1,k}\}^T$ and is updated according to $\{\theta_j^{m+1,k}+\Delta\theta_j^{m+1,k}+\varphi_j^{m+1,k}+\Delta\varphi_j^{m+1,k}\}$, where the incremental values $\{\Delta\theta_j^{m+1}, \Delta_j^{m+1}\}$ are obtained by solving the matrix equation

$$[J_{ij}^m]\left\{\begin{array}{c} \Delta\theta_j^{(m+1)} \\ \Delta\varphi_j^{(m+1)} \end{array}\right\} = -g_i^{(m+1)} \tag{9}$$

where the superscript k represents the iteration counter. Equation (8) is solved iteratively until the difference between subsequent values falls within a predetermined value of error tolerance. We refer to this model as mod-1.

2.2. Green element formulation by cubic Hermitian basis function

In mod-1, line segments (elements) have been used to discretize the problem domain and linear interpolation polynomials are applied to approximate the dependent variables and their functions within those line segments. These interpolating procedure guarantees what is known FEM speak as zero-order continuity in the sense that only the dependent variables and their functions are inter-element continuous but not their first derivatives. For the second model we employ first-order cubic Hermitian polynomials which ensures that both the dependent variables as well as their first derivatives are continuous across an element. We expect an improvement in accuracy but at a price of more tedious computation. The line integral in equation (4) is evaluated by applying the cubic Hermitian interpolation function to approximate the dependent variable and its spatial derivative

$$\theta(x,t) \approx \Omega_j(\xi)\theta(t) + \hat{\Omega}_j(\xi)\,\frac{\partial\theta_j(t)}{\partial x} = \Omega(\xi)\theta(t) + \hat{\Omega}_j(\xi)\varphi_j(t) \tag{10}$$

where Ω_j and $\hat{\Omega}_j$ are defined in terms of a local coordinate $\xi{=}\frac{(x-x_1)}{(x_2-x_1)} = \frac{(x-x_1)}{l}$ as

$$\Omega_1(\xi) = 1.0 - 3\xi^2 + 2\xi^3 \tag{11a}$$

$$\Omega_2(\xi) = 3\xi^2 - 2\xi^3 \tag{11b}$$

$$\Omega_1(\xi) = l\xi(\xi - 1) \tag{11c}$$

$$\Omega_2(\xi) = l\xi(\xi - 1) \tag{11d}$$

Applying the cubic Hermitian interpolation, equation (4) becomes

$$\begin{aligned} &R_{ij}\theta_j - V_{inj}\Theta_n\theta_j - \hat{V}_{inj}\left(\frac{d\Theta}{d\theta}\varphi\right)_n\theta_j + S_{inj}\psi_n\left(\frac{d\theta}{dt}\right)_j + \hat{S}_{inj}\psi_n\left(\frac{d\psi}{d\theta}\right)_j\left(\frac{d\theta}{dt}\right)_n \\ &+ L_{ij}\theta_j - \hat{V}_{inj}\Theta_n\theta_j - \check{V}_{inj}\left(\frac{d\Theta}{d\theta}\varphi\right)_n\theta_j + \hat{S}_{inj}\psi_n\left(\frac{d\theta}{dt}\right)_j + \check{S}_{inj}\psi_n\left(\frac{d\psi}{d\theta}\right)_j\left(\frac{d\varphi}{dt}\right)_n \\ &+ S_{inj}\psi_n f_j + \hat{S}_{inj}\psi_n\left(\frac{dF}{d\theta}\varphi\right)_j + \hat{S}_{inj}\left(\frac{d\psi}{d\theta}\right)_j f_n + \check{S}_{inj}\left(\frac{d\psi}{d\theta}\varphi\right)_n\left(\frac{dF}{d\theta}\varphi\right)_j = 0 \end{aligned} \tag{12}$$

Equation(12) is a nonlinear system of discrete first-order differential equations in which the element matrices have the following expressions.

$$R_{ij} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} l_{max} & -(l+l_{max}) \\ (l+l_{max}) & -l_{max} \end{bmatrix} \tag{13a}$$

$$V_{inj} = \tfrac{1}{l}\int_0^1 G(\xi,\xi_i)\tfrac{d\Omega_n}{d\xi}\tfrac{d\Omega_j}{d\xi}d\xi \tag{13b}$$

$$\hat{V}_{inj} = \tfrac{1}{l}\int_0^1 G(\xi,\xi_i)\tfrac{d\Omega_n}{d\xi}\tfrac{d\hat{\Omega}_j}{d\xi}d\xi \tag{13c}$$

$$\check{V}_{inj} = \tfrac{1}{l}\int_0^1 G(\xi,\xi_i)\tfrac{d\hat{\Omega}_n}{d\xi}\tfrac{d\hat{\Omega}_j}{d\xi}d\xi \tag{13d}$$

$$U_{inj} = \tfrac{1}{l}\int_0^1 G(\xi,\xi_i)\Omega_n\tfrac{d\Omega_j}{d\xi}d\xi \tag{13e}$$

$$\hat{U}_{inj} = \tfrac{1}{l}\int_0^1 G(\xi,\xi_i)\Omega_n\tfrac{d\hat{\Omega}_j}{d\xi}d\xi \tag{13f}$$

$$W_{inj} = \tfrac{1}{l}\int_0^1 G(\xi,\xi_i)\hat{\Omega}_n\tfrac{d\Omega_j}{d\xi}d\xi \tag{13g}$$

$$\hat{W}_{inj} = \tfrac{1}{l}\int_0^1 G(\xi,\xi_i)\hat{\Omega}_n\tfrac{d\hat{\Omega}_j}{d\xi}d\xi \tag{13h}$$

$$S_{inj} = \tfrac{1}{l}\int_0^1 G(\xi,\xi_i)\Omega_n\Omega_j d\xi \tag{13i}$$

$$\text{(13j)} \qquad \hat{S}_{inj} = \tfrac{1}{l}\int_0^1 G(\xi,\xi_i)\Omega_n\hat{\Omega}_j d\xi$$

$$\text{(13k)} \qquad \check{S}_{inj} = \tfrac{1}{l}\int_0^1 G(\xi,\xi_i)\hat{\Omega}_n\hat{\Omega}_j d\xi$$

A two-level discretization is applied to approximate the temporal derivatives and the Picard scheme is applied to linearize the nonlinear terms to yield

$$\left\{\begin{array}{l} \alpha\left[L_{ij}\varphi_j - \hat{V}_{inj}\{\alpha\Theta_n^{(m+i,k)} + \omega\Theta_n^{(m)}\} - \overline{V}_{inj}\left\{\alpha\left(\dfrac{d\Theta}{d\theta}\,\varphi\right)_n^{(m+1,k)} + \omega\left(\dfrac{d\Theta}{d\theta}\,\varphi\right)_n^{(m)}\right\}\right] \\ +\dfrac{\hat{S}_{inj}}{\Delta t}\,(\alpha\psi_n^{(m+1,k)} + \omega\psi_n^{(m)}) + \dfrac{\dot{S}_{inj}}{\Delta t}\left(\alpha\left(\dfrac{d\psi}{d\theta}\right)_n^{(m+1,k)} + \omega\left(\dfrac{d\psi}{d\theta}\right)_n^{(m)}\right) \end{array}\right\}\varphi_j^{(m+1,k+1)}$$

(14)

$$= -\left[\begin{array}{l} \left\{\begin{array}{l} \omega\left[R_{ij}\theta_j - V_{inj}\{\alpha\Theta_n^{(m+1,k)} + \omega\Theta_n^{(m)}\} - \hat{V}_{inj}\left\{\alpha\left(\frac{d\Theta}{d\theta}\varphi\right)_n^{(m+1,k)} + \omega\left(\frac{d\Theta}{d\theta}\varphi\right)_n^{(m)}\right\}\right] \\ -\frac{S_{inj}}{\Delta t}(\alpha\Psi_n^{(m+1,k)} + \omega\psi_n^{(m)}) - \frac{\hat{S}_{inj}}{\Delta t}\left(\alpha\left(\frac{d\Psi}{d\theta}\right)_n^{(m+1,k)} + \omega\left(\frac{d\Psi}{d\theta}\right)_n^{(m)}\right) \end{array}\right\}\theta_j^{(m)} \\ + \left\{\begin{array}{l} \omega\left[L_{ij}\varphi_j - \hat{V}_{inj}\{\alpha\Theta_n^{(m+1,k)} + \omega\Theta_n^{(m)}\} - \bar{V}_{inj}\left\{\alpha\left(\frac{d\Theta}{d\theta}\varphi\right)_n^{(m+1,k)} + \omega\left(\frac{d\Theta}{d\theta}\varphi\right)_n^{(m)}\right\}\right] \\ -\frac{\hat{S}_{inj}}{\Delta t}(\alpha\Psi_n^{(m+1,k)} + \omega\psi_n^{(m)}) - \frac{\ddot{S}_{inj}}{\Delta t}\left(\alpha\left(\frac{d\Psi}{d\theta}\right)_n^{(m+1,k)} + \omega\left(\frac{d\Psi}{d\theta}\right)_n^{(m)}\right) \end{array}\right\}\varphi_j^{(m)} \\ + S_{inj}[\alpha\Psi_n^{(m+1,k)} + \omega\Psi_n^{(m)}][\alpha F_n^{(m+1,k)} + \omega F_n^{(m)}] \\ + \hat{S}_{inj}[\alpha\Psi_n^{(m+1,k)} + \omega\Psi^{(m)_n}]\left[\alpha\left(\frac{dF}{d\theta}\varphi\right)_n^{(m+1,k)} + \omega\left(\frac{dF}{d\theta}\varphi\right)_n^{(m)}\right] \\ + \hat{S}_{inj}\left[\alpha\left(\frac{d\Psi}{d\theta}\varphi\right)_n^{(m+1,k)} + \omega\left(\frac{d\Psi}{d\theta}\varphi\right)_n^{(m)}\right][\alpha F_n^{(m+1,k)} + \omega F_n^{(m)}] \\ + \ddot{S}_{inj}\left[\alpha\left(\frac{d\Psi}{d\theta}\varphi\right)_n^{(m+1,k)} + \omega\left(\frac{d\Psi}{d\theta}\varphi\right)_n^{(m)}\right]\left[\alpha\left(\frac{dF}{d\theta}\right)_n^{(m+1,k)} + \omega\left(\frac{dF}{d\theta}\right)_n^{(m)}\right] \end{array}\right]$$

where α is a temporal derivative approximation parameter which determines the level of the difference scheme. It takes a value between zero and unity. $\omega = 1-\alpha$. The superscript k shows the iteration number while $m+1$ and m denote the current(t_{m+1}) and the previous (t_m) levels. Equation (13) is solved iteratively to yield the current values of the dependent variables: $\theta_j^{m+1,k}$ and $\varphi_j^{m+1,k}$. We refer to this formulation as Mod-2.

2.3. The finite difference Newton-Richtmeyer Scheme

We adopt a model based on the finite difference Newton-Richtmeyer linearization scheme in order to compare the relative performance of the Green element formulation against one of the traditional methods. A key issue of this method which is worth mentioning is the ease with which it deals with nonlinearity and its eventual resolution of the coefficients into a tridiagonal form which enhances numerical implementation. The solution algorithm can be formally stated as:

$$\text{(15)} \qquad \begin{aligned} \tfrac{\Delta\theta}{\Delta t} &= \frac{[D(\theta)]_{i+1/2}^{j}(\theta_{i+1}^{j} - \theta_i^{j}) - [D(\theta)]_{i-1/2}^{j}(\theta_i^{j} - \theta_{i-1}^{j})}{\Delta x^2} \\ &+ \tfrac{[D(\theta)]_{i+1/2}^{j}(\Delta\theta_{i+1}^{j} - \Delta\theta_i^{j}) - [D(\theta)]_{i-1/2}^{j}(\Delta\theta_i^{j} - \Delta\theta_{i-1}^{j})}{\Delta x^2} \\ &+ \tfrac{\left[\frac{\partial D(\theta)}{\partial\theta}\right]_{i+1/2}^{j}(\Delta\theta_{i+1}^{j+1} - \Delta\theta_i^{j+1})(\theta_{i+1}^{j} - \theta_i^{j}) - \left[\frac{\partial D(\theta)}{\partial\theta}\right]_{1+1/2}^{j}(\Delta\theta_i^{j+1} - \Delta\theta_{i-1}^{j+1})(\theta_i^{j} - \theta_{i-1}^{j})}{2\Delta x^2} \end{aligned}$$

And any nonlinearity resulting from source, sink or reaction terms is linearized according to:

$$[F]_i^{j+1} = [F]_i^j + \frac{\partial}{\partial t}[F]\Delta t \tag{16}$$

where i and j refer to the space and time coordinates and is the immediate dependent variable . The process is executed iteratively until the difference between the old and new values falls within a prescribed error tolerance. This is third model mod-3 .

3. Numerical calculations

Examples of transient heat conduction in nonlinear materials are used to validate and compare the current models. The first example is a nonlinear heat conduction problem with a constant heat capacity but without any exact solution. The second example involves temperature dependence on both the conductivity and heat capacity in addition, it possesses a closed form solution. In the third set of calculations we shall verify the ability of the models to handle cases involving nonlinear conductivity and reaction terms.

Example 3.1 [35] involves a semi-infinite bar 20.0cm long at an initial temperature of and subjected to a temperature jump expressed by the following boundary conditions at $x = 0$

$$\begin{array}{ll} \theta = 200.0 & 0 < t \leq 10.0 \\ \theta = 100.0 & t > 100 \end{array}$$

and at $x = L : \theta = 100^0C\ t > 0$.

The thermophysical properties are for both the heat capacity and the temperature dependent thermal diffusion are given as: $D(\theta) = 2.0 + 0.01\theta$, $\rho c = 8.0$. The numerical calculations for the three models are executed using 21 grid points in a 1-D domain and a uniform time step of $\Delta t = 0.1$ in line with the parameters used in [35] in their FEM calculations. In order to enhance comparison and validation, mod-1, mod-2 and mod-3 calculations are plotted alongside the FEM solutions at time 10 and 13. Figure 1 illustrates the temperature histories predicted by the three models in comparison with that of FEM. While those of mod-1 and mod-2 are in close agreement with FEM solutions, mod-3 shows a minor deviation with the rest especially in those areas which exhibit maximum slope in the solution profile. We carried the numerical experimentation a step further by decreasing the time step ($\Delta t = 0.01$) for the three models and comparing the numerical results with those of FEM at $\Delta t = 0.1$.

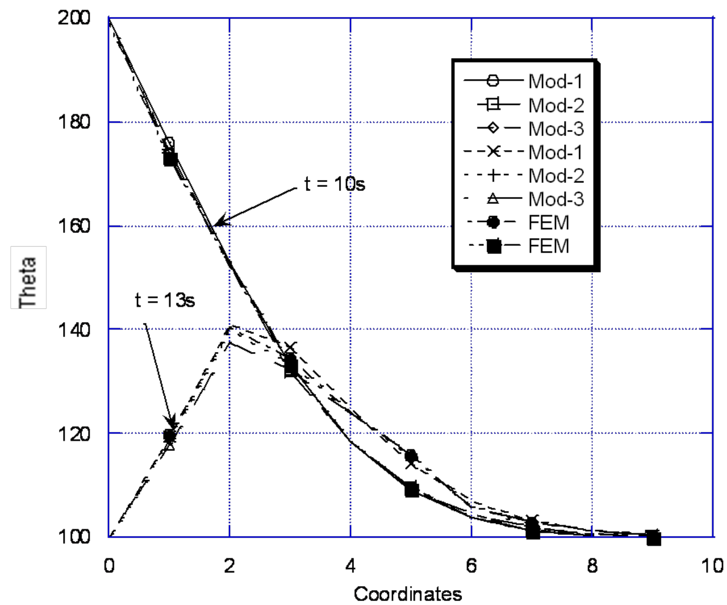

Figure 1: Temperature history predicted by the 3 models($\Delta t = 0.1$)

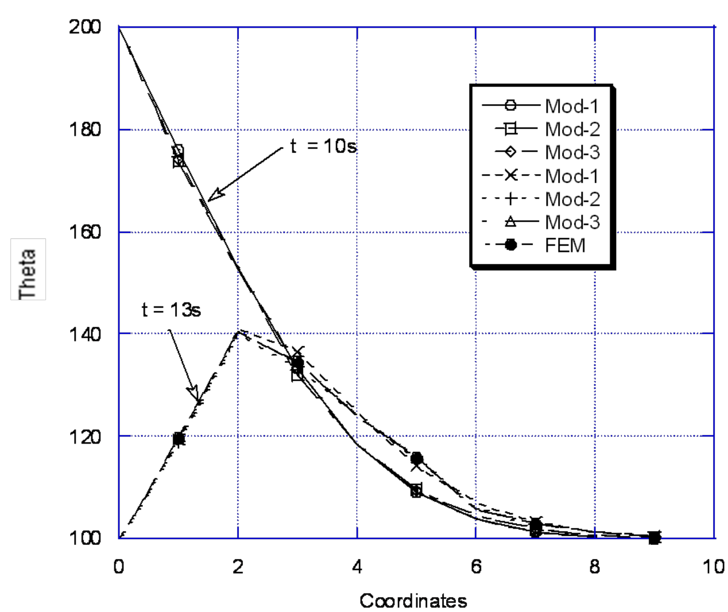

Figure 2: Temperature history predicted by the 3 models($\Delta t = 0.01$)

Figure 2 shows that while there is hardly any change in the profiles for mod-1 and mod-2, results obtained from mod-3 appear to be in closer agreement with the other profiles. These results not only validate the formulations and accuracy of the models but also confirms the higher convergence rate of the GEM models.

Example 3.2 In this example, we consider a nonlinear transient heat conduction for a 2.0 length semi-infinite bar with a constant heat input. The bar initially at a temperature of 0^0C throughout its entire length is subjected to a constant heat input $-D(\partial\theta\partial n) = 1.0$ at its left end boundary and a zero flux at the right end$(x = L)$. Both the heat capacity and thermal conductivity are expressed as $D(\theta) = \rho(\theta) = 1.0 + 0.5\theta$. The analytical solution of this problem can be found in [35] and is given as:

$$\theta(x,t) = 2\left\{\sqrt{\left[1.0 + 2.0\frac{\sqrt{t}}{\sqrt{\pi}}exp\left(\frac{-x^2}{4t}\right) - xerfc\left(\frac{x}{2\sqrt{t}}\right)\right] - 1.0}\right\}$$

Using a 0.1 length spatial elements for the problem domain and a time step of 0.5 the scalar history for mod-1, mod2 and mod-3 together with the analytical solutions are presented in Figure 3.

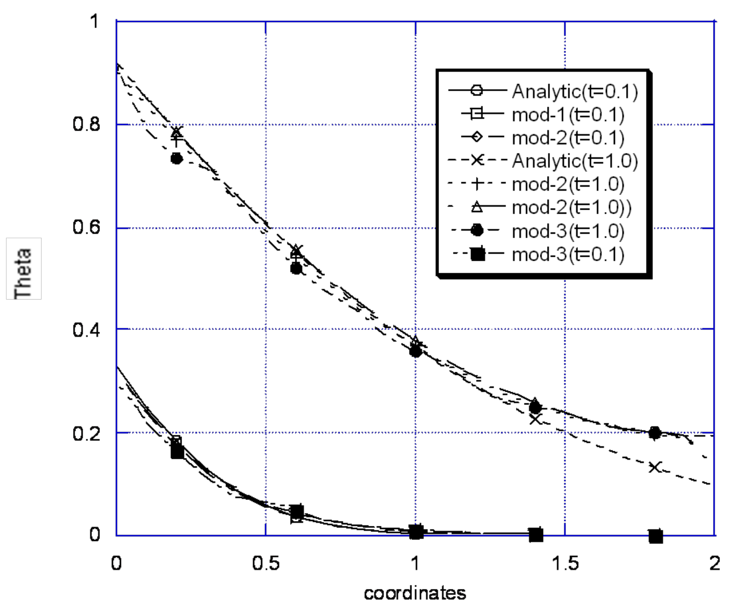

Figure 3: Comparison of analytical and numerical results at $t = 0.1$ and $t = 1.0$

The results appear to be close. However, a more detailed analysis of the error norms generated by the solutions as presented in Table 1 shows the overall superiority of mod-2 over the other two models.

Error calculation For Example 2 ($t = 0.1$)			
Models	RMs	$L_2 norm$	$L_{\inf}$ norm
Mod-1	0.566361e-02	0.2595393e-03	0.145253e-01
Mod-2	0.54185e-02	0.248309e-01	0.12457e-01
Mod-3	0.50504e-02	0.229876e-01	0.175463e-01
Error calculation For Example 2 ($t = 0.5$)			
Models	RMs	$L_2 norm$	$L_{\inf}$ norm
Mod-1	0.964653e-02	0.513488e-01	0.233718e-01
Mod-2	0.696495e-02	0.319174e-01	0.180562e-01
Mod-3	0.816381e-02	0.665743e-01	0.300975e-01
Error calculation For Example 2 ($t = 1$)			
Models	RMs	$L_2 norm$	$L_{\inf}$ norm
Mod-1	0.370983e-01	0.170000e+00	0.934392e-01
Mod-2	0.358708e-01	0.164381e+00	0.911992e-01
Mod-3	0.401325e-01	0.199785e+00	1.00875e-01

Table 1: Comparison of errors for the three models

Table 2 shows the temperature as a function of time at the left end boundary brought about by a constant heat supply. Mod-2 displays a marginal superiority over the other models followed by mod-1. This is an indication of the ability of the GEM technique to handle nonlinear boundary conditions in a straightforward manner.

Example 3.3 Our aim here is to check how the models can correctly represent the effects of nonlinear source terms on the scalar profile of a heat conduction process involving a thermally dependent conductivity term or variable properties [36]. Let us consider the following problem for the governing differential equation as represented by equation (1):

$$\begin{array}{rclrcl} D(\theta) & = & \theta & F(x,t,\theta) & = & \theta^2 \\ \theta(0,t) & = & 1 & \theta(1,t) & = & 0.0 \\ \theta(x,0) & = & 0.0 & & & \end{array}$$

The steady state solution of the above problem is given as:

$$\theta(x,t) = \left\{cosh[x\sqrt{2}] - coth[\sqrt{2}]sinh[x\sqrt{2}]\right\}^{1/2}$$

Table 3 shows that the numerical results for all the models are almost the same for this problem at steady state. Whatever gains made by Mod-2 because of its Hermite interpolation procedure are hardly noticeable at steady state.

Temperature at the left end boundary ($x = 0$)				
Time	Analytical	mod-1	mod-2	mod-3
0.05	0.2381e+00	0.2523e+00	0.2275e+00	0.2514e+00
0.10	0.3297e+00	0.3145e+00	0.3217e+00	0.3154e+00
0.15	0.3975e+00	0.3902e+00	0.3920e+00	0.3892e+00
0.20	0.4533e+00	0.4458e+00	0.4449e+00	0.4417e+00
0.25	0.5014e+00	0.4934e+00	0.4946e+00	0.4891e+00
0.30	0.5440e+00	0.5355e+00	0.53272e+00	0.5318e+00
0.35	0.5827e+00	0.5734e+00	0.5763e+00	0.5694e+00
0.40	0.6181e+00	0.6181e+00	0.6181e+00	0.6111e+00
0.45	0.6510e+00	0.6402e+00	0.6452e+00	0.6382e+00
0.50	0.6817e+00	0.6701e+00	0.6761e+00	0.6681e+00
0.55	0.7106e+00	0.6983e+00	0.7052e+00	0.6914e+00
0.60	0.7379e+00	0.7249e+00	0.7328e+00	0.7209e+00
0.65	0.7369e+00	0.7504e+00	0.7590e+00	0.7484e+00
0.70	0.7886e+00	0.7746e+00	0.7840e+00	0.7703e+00
0.75	0.8123e+00	0.7979e+00	0.8080e+00	0.7935e+00
0.80	0.8350e+00	0.8202e+00	0.8322e+00	0.7982e+00
0.85	0.8568e+00	0.8418e+00	0.8551e+00	0.8412e+00
0.90	0.8778e+00	0.8627e+00	0.8771e+00	0.8673e+00
0.95	0.8981e+00	0.8830e+00	0.8983e+00	0.8801e+00
1.00	0.9178e+00	0.9027e+00	0.8983e+00	0.8927e+00

Table 2: Temperature as a function of time at the left boundary

Numerical and analytic results at steady state				
Coordinate	Analytical	mod-1	mod-2	mod-3
0.1	0.9219e+00	0.9220e+00	0.9219e+00	0.9221e+00
0.2	0.8467e+00	0.8465e+00	0.8465e+00	0.8471e+00
0.3	0.7735e+00	0.7735e+00	0.7735e+00	0.7742e+00
0.4	0.7013e+00	0.6994e+00	0.7016e+00	0.7022e+00
0.5	0.6287e+00	0.6282e+00	0.6285e+00	0.6298e+00
0.6	0.5539e+00	0.5528e+00	0.5538e+00	0.5551e+00
0.7	0.4739e+00	0.4732e+00	0.4736e+00	0.4735e+00
0.8	0.3835e+00	0.3832e+00	0.3835e+00	0.3849e+00
0.9	0.2697e+00	0.2695e+00	0.2695e+00	0.2708e+00

Table 3: Comparison of numerical and analytic solutions at steady state

Since all the models converge to the right results at steady state, we took a closer look at the scalar evolution of all the models at different times figure 4 by allowing the source term to vary from linear, quadratic and cubic.
It is noticeable here that despite the power law variation of the source terms, the overall temperature approaches steady state at a relatively fast rate. This is

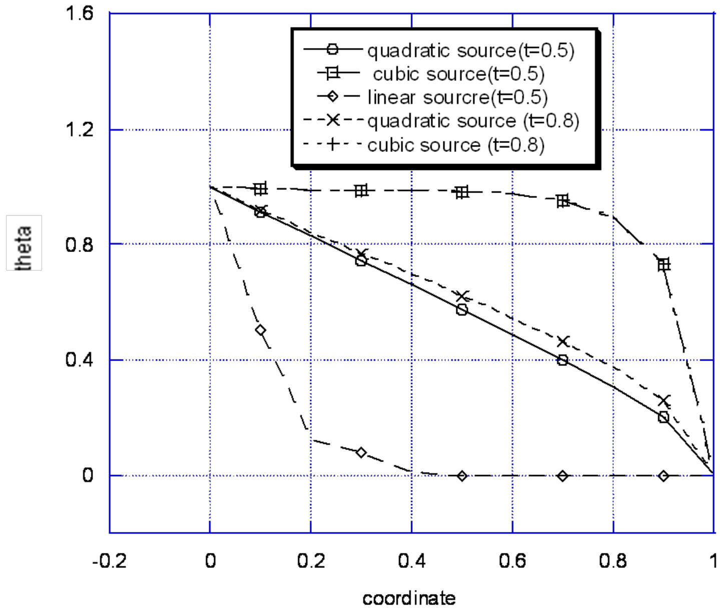

Figure 4: Solution Profiles at different times

attributed to the nonlinearity of this problem caused by its temperature-dependent thermophysical properties. As a result, heat is prevented from being conducted as rapidly as would have been the case if it were a for a constant property problem.

4. Conclusion

In this paper, the usefulness of using a one-dimensional hybrid boundary element method has been presented. Three models have been created and one of them uses a cubic hermitian formulation to interpolate the dependent variables. The relative gains made by this more complicated formulation have been shown to be marginal. This may have a bearing to the class of problems addressed in this work. It may prove to be the case, that when the steepness of the scalar variable is not so profound, that BEM hermitian interpolation may not be the direction to go. At the same time this work raises the issues involved in formulating robust one-dimensional boundary element algorithms that can straightforwardly handle problems that have continuously raised concerns to the BEM community especially those that are transient, heterogeneous, and nonlinear. The usual practice of using two-dimensional formulation to solve problems that are typically one-dimensional by imposing no-flux conditions at the boundaries are issues that need be addressed as relevant details concerning both the physics of the problem and its application stand the risk of being obfuscated. My current work points in this direction.

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ON IMPROVED YOUNG TYPE INEQUALITIES FOR MATRICES

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Abstract. This paper aims to give improved Young type inequalities which are due to Hu [2]. Then we use these inequalities to establish corresponding Young type inequalities for matrices.

Keywords: unitarily invariant norms; Young type inequality; positive semidefinite matrices; singular values.

2010 Mathematical Subject Classification: 47A30; 15A42; 15A60.

1. Introduction

Let M_n be the space of $n \times n$ complex matrices. Let $\|\cdot\|$ denote any unitarily invariant norm on M_n. So, $\|UAV\| = \|A\|$ for all $A \in M_n$ and for all unitary matrices $U, V \in M_n$. For $A = (a_{ij}) \in M_n$, the Hilbert-Schmidt norm and the trace norm of A are defined by

$$\|A\|_2 = \sqrt{\sum_{j=1}^{n} s_j^2(A)}, \quad \|A\|_1 = \sum_{j=1}^{n} s_j(A), \text{ respectively,}$$

where $s_i(A)\,(i = 1, ..., n)$ are the singular values of A with $s_1(A) \geq \cdots \geq s_n(A)$, which are the eigenvalues of the positive semidefinite matrix $|A| = (AA^*)^{\frac{1}{2}}$, arranged in decreasing order and repeated according to multiplicity.

The classical Young inequality says that if $a, b \geq 0$ and $0 \leq v \leq 1$, then

$$a^v b^{1-v} \leq va + (1-v)\,b \tag{1.1}$$

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with equality if and only if $a = b$.

The Kontorovich constant is defined as

$$K\left(t,2\right) = \frac{\left(t+1\right)^2}{4t}, \quad t > 0.$$

Zuo, Shi and Fujii [1] obtained an improvement of inequality (1.1) which can be stated as follows:

$$K\left(h,2\right)^r a^v b^{1-v} \le va + \left(1-v\right)b, \tag{1.2}$$

where $h = \dfrac{a}{b}$ and $r = \min\left\{v, 1-v\right\}$.

In a recent work, Hu [2] gave the following Young type inequalities:

$$\left[\left(va\right)^v b^{1-v}\right]^2 + v^2\left(a-b\right)^2 \le v^2a^2 + \left(1-v\right)^2 b^2, \quad 0 \le v \le \frac{1}{2}, \tag{1.3}$$

and

$$\{a^v[(1-v)b]^{1-v}\}^2 + (1-v)^2(a-b)^2 \le v^2a^2 + (1-v)^2b^2, \ \frac{1}{2} \le v \le 1. \tag{1.4}$$

Based on the scalar Young type inequalities (1.3) and (1.4), Hu proved in [2] that if $A, B, X \in M_n$ such that A and B are positive semidefinite, then

$$\begin{aligned} \left\|vAX + \left(1-v\right)XB\right\|_2^2 \ge{}& v^2\left\|AX - XB\right\|_2^2 + v^{2v}\left\|A^vXB^{1-v}\right\|_2^2 \\ &+2v\left(1-v\right)\left\|A^{1/2}XB^{1/2}\right\|_2^2, \quad 0 \le v \le \frac{1}{2} \end{aligned} \tag{1.5}$$

and

$$\begin{aligned} \left\|vAX + \left(1-v\right)XB\right\|_2^2 \ge{}& \left(1-v\right)^2\left\|AX - XB\right\|_2^2 \\ &+\left(1-v\right)^{2-2v}\left\|A^vXB^{1-v}\right\|_2^2 \\ &+2v\left(1-v\right)\left\|A^{1/2}XB^{1/2}\right\|_2^2, \quad \frac{1}{2} \le v \le 1. \end{aligned} \tag{1.6}$$

These are the Hilbert-Schmidt norm versions of Young type inequalities.

At the same time, Hu [2] obtained that if $A, B \in M_n$ are positive definite, then

$$\begin{aligned} \det\left(vA + \left(1-v\right)B\right)^2 \ge{}& v^{2nv}\det\left(A^vB^{1-v}\right)^2 + v^{2n}\det\left(A-B\right)^2 \\ &+\left(2v\left(1-v\right)\right)^n \det B^{1/2}AB^{1/2}, \quad 0 \le v \le \tfrac{1}{2} \end{aligned} \tag{1.7}$$

and

$$\begin{aligned} \det\left(vA + \left(1-v\right)B\right)^2 \ge{}& \left(1-v\right)^{2n(1-v)}\det\left(A^vB^{1-v}\right)^2 \\ &+\left(1-v\right)^{2n}\det\left(A-B\right)^2 \\ &+\left(2v\left(1-v\right)\right)^n \det B^{1/2}AB^{1/2}, \quad \frac{1}{2} \le v \le 1. \end{aligned} \tag{1.8}$$

These are the determinant versions of Young type inequalities.

For more information on matrix versions of the Young inequality the reader is referred to [3]-[7]. In this paper, we present improvements of inequalities (1.5), (1.6), (1.7) and (1.8).

2. Young type inequalities for scalars

We begin this section with the Young type inequalities for scalars.

Theorem 1. *Let $a, b \geq 0$. If $0 \leq v \leq \frac{1}{2}$, then*

$$K(h,2)^r \left[(va)^v b^{1-v}\right]^2 + v^2 (a-b)^2 \leq v^2 a^2 + (1-v)^2 b^2, \tag{2.1}$$

where $h = \dfrac{va}{b}$, $r = \min\{2v, 1-2v\}$.

If $\dfrac{1}{2} \leq v \leq 1$, then

$$K(h,2)^r \left\{a^v [(1-v)b]^{1-v}\right\}^2 + (1-v)^2 (a-b)^2 \leq v^2 a^2 + (1-v)^2 b^2, \tag{2.2}$$

where $h = \dfrac{a}{(1-v)b}$, $r = \min\{2v-1, 2-2v\}$.

Proof. If $0 \leq v \leq \dfrac{1}{2}$. Then, by inequality (1.2), we have

$$\begin{aligned} v^2a^2 + (1-v)^2 b^2 - v^2 (a-b)^2 &= b\left[2v(va) + (1-2v)b\right] \\ &\geq bK(h,2)^r (va)^{2v} b^{1-2v} \\ &= K(h,2)^r \left[(va)^v b^{1-v}\right]^2, \end{aligned}$$

and so

$$v^2a^2 + (1-v)^2 b^2 \geq K(h,2)^r \left[(va)^v b^{1-v}\right]^2 + v^2 (a-b)^2.$$

If $\dfrac{1}{2} \leq v \leq 1$, then

$$\begin{aligned} v^2a^2 + (1-v)^2 b^2 - (1-v)^2 (a-b)^2 &= a\left[(2v-1)a + 2(1-v)^2 b\right] \\ &\geq aK(h,2)^r a^{2v-1} [(1-v)b]^{2-2v} \\ &= K(h,2)^r \left\{a^v [(1-v)b]^{1-v}\right\}^2, \end{aligned}$$

and so

$$v^2a^2 + (1-v)^2 b^2 \geq K(h,2)^r \left\{a^v [(1-v)b]^{1-v}\right\}^2 + (1-v)^2 (a-b)^2.$$

This completes the proof. ∎

Remark 1. Obviously, (2.1) and (2.2) are improvement of the scalar Young type inequalities (1.3) and (1.4).

3. Young type inequalities for matrices

Based on the scalar Young type inequalities (2.1) and (2.2), we obtain the Hilbert-Schmidt norm, the trace norm and the determinant versions of Young type inequalities.

Theorem 2. *Let $A, B, X \in M_n$ such that A and B are positive semidefinite. If $0 \le v \le \frac{1}{2}$, then*

$$\begin{aligned} \|vAX + (1-v)XB\|_2^2 \ge v^2 \|AX - XB\|_2^2 + K^r v^{2v} \|A^v XB^{1-v}\|_2^2 \\ +2v(1-v)\left\|A^{1/2}XB^{1/2}\right\|_2^2, \end{aligned} \tag{3.1}$$

where $K = \min\left\{K\left(\frac{v\lambda_i}{\mu_j}, 2\right), i, j = 1, ..., n\right\}$, $r = \min\{2v, 1-2v\}$.

If $\frac{1}{2} \le v \le 1$, *then*

$$\begin{aligned} &\|vAX + (1-v)XB\|_2^2 \ge (1-v)^2 \|AX - XB\|_2^2 \\ &+K^r (1-v)^{2-2v} \|A^v XB^{1-v}\|_2^2 + 2v(1-v)\left\|A^{1/2}XB^{1/2}\right\|_2^2, \end{aligned} \tag{3.2}$$

where $K = \min\left\{K\left(\frac{\lambda_i}{(1-v)\mu_j}, 2\right), i, j = 1, ..., n\right\}$, $r = \min\{2v-1, 2-2v\}$.

Proof. Since every positive semidefinite matrix is unitarily diagonalizable, it follows that there are unitary matrices $U, V \in M_n$ such that $A = UDU^*$ and $B = VEV^*$, where

$$D = diag(\lambda_1, ..., \lambda_n), \quad E = diag(\mu_1, ..., \mu_n), \text{ and } \lambda_i, \mu_i \ge 0, \ i = 1, ..., n.$$

Let $Y = U^*XV = (y_{ij})$. Then

$$vAX + (1-v)XB = U(vDY + (1-v)YE)V^* = U((v\lambda_i + (1-v)\mu_j)y_{ij})V^*,$$

$$AX - XB = U((\lambda_i - \mu_j)y_{ij})V^*, \ A^{1/2}XB^{1/2} = U(\lambda_i^{1/2}\mu_j^{1/2}y_{ij})V^*$$

and

$$A^v XB^{1-v} = U\left(\lambda_i^v \mu_j^{1-v} y_{ij}\right)V^*.$$

If $0 \le v \le \frac{1}{2}$, by inequality (2.1), we have

$$\begin{aligned} &\|vAX + (1-v)XB\|_2^2 = \sum_{i,j=1}^n (v\lambda_i + (1-v)\mu_j)^2 |y_{ij}|^2 \\ &= \sum_{i,j=1}^n \left(v^2\lambda_i^2 + (1-v)^2\mu_j^2 + 2v(1-v)\lambda_i\mu_j\right)|y_{ij}|^2 \\ &\ge v^2 \sum_{i,j=1}^n (\lambda_i - \mu_j)^2 |y_{ij}|^2 + K^r v^{2v} \sum_{i,j=1}^n \left(\lambda_i^v \mu_j^{1-v}\right)^2 |y_{ij}|^2 + \sum_{i,j=1}^n 2v(1-v)\lambda_i\mu_j |y_{ij}|^2 \\ &\ge v^2 \|AX - XB\|_2^2 + K^r v^{2v} \|A^v XB^{1-v}\|_2^2 + 2v(1-v)\left\|A^{1/2}XB^{1/2}\right\|_2^2. \end{aligned}$$

If $\frac{1}{2} \le v \le 1$, then by the inequality (2.2) and the same method above, we have the inequality (3.2). This completes the proof. ■

Remark 2. Obviously, (3.1) and (3.2) are improvement matrix Young type inequalities (1.5) and (1.6).

To obtain refinements of the trace norm versions of Young type inequalities, we need the following lemmas.

Lemma 1. (Cauchy-Schwarz Inequality) [8] *Let $a_i \geq 0, b_i \geq 0$ for $i = 1, 2, ..., n$, then*

$$\sum_{i=1}^{n} a_i b_i \leq \left(\sum_{i=1}^{n} a_i^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} b_i^2\right)^{\frac{1}{2}}.$$

Lemma 2. [8] *Let $A, B \in M_n$, then*

$$\sum_{j=1}^{n} s_j\left(AB\right) \leq \sum_{j=1}^{n} s_j\left(A\right) s_j\left(B\right).$$

Theorem 3. *Let $A, B \in M_n$ be positive semidefinite. If $0 \leq v \leq \frac{1}{2}$, then*

$$K^r v^{2v} \|A^v B^{1-v}\|_1 \leq v^2 \|A\|_1 + (1-v)^2 \|B\|_1 - v^2 \Big(\sqrt{\|A\|_1} - \sqrt{\|B\|_1}\Big)^2, \tag{3.3}$$

where $K = \min\left\{K\Big(\frac{v\sqrt{s_j(A)}}{\sqrt{s_j(B)}}, 2\Big),\ j = 1, ..., n\right\}$, $r = \min\{2v, 1-2v\}$.

If $\frac{1}{2} \leq v \leq 1$, then

$$\begin{aligned} K^r (1-v)^{2(1-v)} \|A^v B^{1-v}\|_1 &\leq v^2 \|A\|_1 + (1-v)^2 \|B\|_1 \\ &\quad -(1-v)^2 (\sqrt{\|A\|_1} - \sqrt{\|B\|_1})^2, \end{aligned} \tag{3.4}$$

where $K = \min\left\{K\Big(\frac{\sqrt{s_j(A)}}{(1-v)\sqrt{s_j(B)}}, 2\Big), j = 1, ..., n\right\}$, $r = \min\{2v-1, 2-2v\}$.

Proof. If $0 \leq v \leq \frac{1}{2}$, then using Lemma 1, Lemma 2 and the inequality (2.1), we have

$$\begin{aligned} \operatorname{tr}\left(v^2 A + (1-v)^2 B\right) &= v^2 \operatorname{tr} A + (1-v)^2 \operatorname{tr} B \\ &= \sum_{j=1}^{n} \left(v^2 s_j\left(A\right) + (1-v)^2 s_j\left(B\right)\right) \\ &\geq \sum_{j=1}^{n} K\left(\frac{v\sqrt{s_j(A)}}{\sqrt{s_j(B)}}, 2\right)^r \left[\left(v\sqrt{s_j\left(A\right)}\right)^v \sqrt{s_j\left(B\right)}^{1-v}\right]^2 \\ &\quad + v^2 \left(\sum_{j=1}^{n} s_j\left(A\right) + \sum_{j=1}^{n} s_j\left(B\right) - 2\sum_{j=1}^{n} \sqrt{s_j\left(A\right) s_j\left(B\right)}\right) \end{aligned}$$

$$\geq K^{r}\sum_{j=1}^{n}\left[v^{2v}s_{j}\left(A^{v}\right)s_{j}\left(B^{1-v}\right)\right]$$
$$+v^{2}\left(\left\|A\right\|_{1}+\left\|B\right\|_{1}-2\left(\sum_{j=1}^{n}s_{j}\left(A\right)\right)^{\frac{1}{2}}\left(\sum_{j=1}^{n}s_{j}\left(B\right)\right)^{\frac{1}{2}}\right)$$
$$=K^{r}v^{2v}\sum_{j=1}^{n}\left[s_{j}\left(A^{v}\right)s_{j}\left(B^{1-v}\right)\right]+v^{2}\left(\sqrt{\left\|A\right\|_{1}}-\sqrt{\left\|B\right\|_{1}}\right)^{2}.$$

Thus

$$v^{2}\left\|A\right\|_{1}+(1-v)^{2}\left\|B\right\|_{1}-v^{2}\left(\sqrt{\left\|A\right\|_{1}}-\sqrt{\left\|B\right\|_{1}}\right)^{2}\geq K^{r}v^{2v}\sum_{j=1}^{n}\left[s_{j}\left(A^{v}\right)s_{j}\left(B^{1-v}\right)\right].$$

Then

$$K^{r}v^{2v}\left\|A^{v}B^{1-v}\right\|_{1}\leq v^{2}\left\|A\right\|_{1}+(1-v)^{2}\left\|B\right\|_{1}-v^{2}\left(\sqrt{\left\|A\right\|_{1}}-\sqrt{\left\|B\right\|_{1}}\right)^{2}.$$

If $\frac{1}{2}\leq v\leq 1$, then by the inequality (2.2) and the same method above, we have the inequality (3.4). This completes the proof. ∎

Theorem 4. *Let $A, B\in M_n$ be positive definite. If $0\leq v\leq \frac{1}{2}$, then*

$$\begin{aligned}\det\left(vA+(1-v)B\right)^{2}\geq{}& K^{nr}v^{2nv}\det\left(A^{v}B^{1-v}\right)^{2}+v^{2n}\det\left(A-B\right)^{2}\\&+\left(2v\left(1-v\right)\right)^{n}\det B^{1/2}AB^{1/2},\end{aligned}\tag{3.5}$$

where $K=\min\left\{K\left(vs_{j}\left(B^{-1/2}AB^{-1/2}\right),2\right), j=1,...,n\right\}$, $r=\min\left\{2v,1-2v\right\}$.

If $\frac{1}{2}\leq v\leq 1$, then

$$\begin{aligned}\det\left(vA+(1-v)B\right)^{2}\geq{}& K^{nr}\left(1-v\right)^{2n(1-v)}\det\left(A^{v}B^{1-v}\right)^{2}\\&+\left(1-v\right)^{2n}\det\left(A-B\right)^{2}\\&+\left(2v\left(1-v\right)\right)^{n}\det B^{1/2}AB^{1/2},\end{aligned}\tag{3.6}$$

where

$$K=\min\left\{K\left(\frac{s_{j}\left(B^{-1/2}AB^{-1/2}\right)}{1-v},2\right), j=1,...,n\right\},\ r=\min\left\{2v-1,2-2v\right\}.$$

Proof. If $0 \leq v \leq \frac{1}{2}$, then

$$\det\left(vB^{-1/2}AB^{-1/2} + (1-v)I\right)^2 = \prod_{j=1}^{n}\left(vs_j\left(B^{-1/2}AB^{-1/2}\right) + 1 - v\right)^2$$

$$= \prod_{j=1}^{n}\left(v^2 s_j^2\left(B^{-1/2}AB^{-1/2}\right) + (1-v)^2 + 2v(1-v)s_j\left(B^{-1/2}AB^{-1/2}\right)\right)$$

$$\geq \prod_{j=1}^{n}(K^r v^{2v} s_j^{2v}\left(B^{-1/2}AB^{-1/2}\right) + v^2\left(s_j\left(B^{-1/2}AB^{-1/2}\right) - 1\right)^2$$
$$+2v(1-v)s_j\left(B^{-1/2}AB^{-1/2}\right))$$

$$\geq K^{nr}v^{2nv}\prod_{j=1}^{n}s_j^{2v}\left(B^{-1/2}AB^{-1/2}\right) + v^{2n}\prod_{j=1}^{n}\left(s_j\left(B^{-1/2}AB^{-1/2}\right) - 1\right)^2$$
$$+ (2v(1-v))^n\prod_{j=1}^{n}s_j\left(B^{-1/2}AB^{-1/2}\right)$$

$$= K^{nr}v^{2nv}\det\left(B^{-1/2}AB^{-1/2}\right)^{2v} + v^{2n}\det\left(B^{-1/2}AB^{-1/2} - I\right)^2$$
$$+ (2v(1-v))^n \det B^{-1/2}AB^{-1/2}.$$

Thus, we have

$$\det(vA + (1-v)B)^2 \geq K^{nr}v^{2nv}\det\left(A^v B^{1-v}\right)^2 + v^{2n}\det(A-B)^2$$
$$+ (2v(1-v))^n \det B^{1/2}AB^{1/2}.$$

If $\frac{1}{2} \leq v \leq 1$, then by the inequality (2.2) and the same method above, we have the inequality(3.6). This completes the proof. ∎

Remark 3. Obviously, (3.5) and (3.6) are improvement determinant versions of Young type inequalities (1.7) and (1.8).

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SOFT ISOMORPHISM THEOREMS FOR SOFT HEMIRINGS

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Abstract. In this paper, the concepts of soft strong h-ideals and strong h-idealistic soft hemirings are introduced. Some properties of soft hemirings and strong h-idealistic soft hemirings are given. In particular, we construct a novel soft quotient structure of an idempotent hemiring. By means of a kind of new way, soft isomorphism theorems of soft hemirings are established, which are different from soft isomorphism theorems of soft rings.

Keyword: soft strong h-ideal; strong h-idealistic soft hemiring; soft isomorphism theorems.

2000 Mathematics Subject Classification: 16Y60; 03G25.

1. Introduction

In 1999, Molodtsov [15] put forward the concept of soft sets as a new mathematical tool for dealing with uncertainties. And then, the research on the soft set theory has been extensively studied by many authors. Recently, some basic operations on soft sets were defined by Maji [13] and Ali [2]. What's more, Çağman [3], [4], [14] applied soft set theory to decision making. We also know that soft sets can also be applied in computer science and information science, which refereed to [13].

It is noted that some soft algebras were also discussed, such as [1], [8], [9]. In 2005, a new definition of soft sets called the parametrization reduction was introduced by Chen[5]. By comparing their definition with the related concept of attributed reduction on rough set theory, the theory of soft sets has been developed.

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The applications of soft set in the ideal theory of BCK/BCI-algebras was investigated by Jun and Park [9], and then Feng [6] started to investigate the structure of soft semirings. It is pointed out that some characterizations of hemirings by soft set theory were investigated by Ma, Zhan and others, which refereed to [10], [11], [12], [17], [18].

In this paper, we construct a novel soft quotient structure of an idempotent hemiring and then soft isomorphism theorems of soft hemirings are established. The remaining part of this paper is organized as follows. In section 2, we first recall some concepts and results on soft sets. In section 3, some properties of soft hemirings will be given. Further, soft isomorphism theorems of soft hemirings are established. In the final section, we give a brief conclusion.

2. Preliminaries

By a hemiring, we mean an additively commutative semiring with zero. By zero of a semiring $(S, +, \cdot)$ we mean an element $0 \in S$ such that $0 \cdot x = x \cdot 0 = 0$ and $0 + x = x + 0 = x$ for all $x \in S$. Throughout this paper, S is a hemiring.

A non-empty subset A in S is called a subhemiring of S if A is closed under addition and multiplication. A non-empty subset A in S is called a left (resp. right) ideal of S if A is closed under addition and $SA \subseteq A$ (resp. $AS \subseteq A$). Further, A is called an ideal of S if it is both a left ideal and a right ideal of S.

An ideal I of S is called an h-ideal if $x, z \in S$, $a, b \in I$ and $x + a + z = b + z$ implies $x \in I$. An ideal I of S is called a strong h-ideal if $x, y, z \in S$, $a, b \in I$ and $x + a + z = y + b + z$ implies $x \in y + I$ [9], [16].

The strong h-closure $\widetilde{A}$ in S is defined by

$$y + \widetilde{A} = \{x \in S\}$$

satisfying $x + a_1 + z = y + a_2 + z$ for some $a_1, a_2 \in A, y, z \in S$.

Let ρ be a congruence relation on S, that is, ρ is an equivalence relation on S such that $(a, b) \in \rho$ and $(c, d) \in \rho$ in R implies $(a+c, b+d) \in \rho$ and $(ac, bd) \in \rho$ for all $a, b, c, d \in S$.

Let I be a strong h-ideal of S, $x, y \in S$. We call x is congruent to y mod I, if and only if there exist $a, b \in I$ and $z \in S$ be such that $x + a + z = y + b + z$. It is checked that the relation $x \equiv y(\text{mod } I)$ is a congruence relation on S.

Definition 2.1 [15] A pair $\mathfrak{S} = (F, A)$ is called a soft set over U, where $A \subseteq E$ and $F : A \to \mathscr{P}(U)$ is a set-valued mapping.

For a soft set (F, A), the set $\text{Supp}(F, A) = \{x \in A | F(x) \neq \emptyset\}$ is called a soft support of the soft set (F, A). Thus a null soft set is indeed a soft set with an empty support, and we say that a soft set (F, A) is non-null if $\text{Supp}(F, A) \neq \emptyset$.

Definition 2.2 [6] Let (F, A) be a non-null soft set over S. Then

(1) (F, A) is called a soft hemiring over S if $F(x)$ is a subhemiring of S for all $x \in \mathrm{Supp}(F, A)$;

(2) (F, A) is called an idealistic soft hemiring over S if $F(x)$ is an ideal of S for all $x \in \mathrm{Supp}(F, A)$. The *bi*-idealistic ($k$-idealistic, h-idealistic) soft hemiring are defined similarly.

Definition 2.3 [7] Let (F, A) and (G, B) be two soft sets over a common universe U. The inclusion symbol "$\subseteq$" of (F, A) and (G, B), denoted by $(F, A) \subseteq (G, B)$, is defined as

(1) $A \subseteq B$;

(2) $F(x) \subseteq G(x)$ for all $x \in A$.

Definition 2.4 [2] Let (F, A) and (G, B) be two soft sets over a common universe U.

(1) The *bi*-intersection of (F, A) and (G, B), is defined to the soft set (H, C), where $C = A \cap B$, and $H : C \to \mathscr{P}(U)$ is a mapping given by $H(c) = F(c) \cap G(c)$ for all $c \in C$. This is denoted by $(F, A)\widetilde{\sqcap}(G, B) = (H, C)$.

(2) "(F, A) AND (G, B)", denoted by $(F, A)\widetilde{\wedge}(G, B)$, is defined by $(F, A)\widetilde{\wedge}(G, B)$ $= (H, A \times B)$, where $H(x, y) = F(x) \cap G(y)$ for all $(x, y) \in A \times B$.

(3) The union of (F, A) and (G, B), denoted by $(F, A)\widetilde{\cup}(G, B)$, is defined as the soft set (H, C), where $C = A \cup B$, and $\forall e \in C$,

$$H(e) = \begin{cases} F(e), & \text{if } \ e \in A - B, \\ G(e), & \text{if } \ e \in B - A, \\ F(e) \cup G(e), & \text{if } \ e \in A \cap B. \end{cases}$$

Definition 2.5 [6] Let (η, A) be a soft hemiring over S. A soft set (γ, I) over S is called a soft ideal of (η, A), denote by $(\gamma, I)\widetilde{\triangleleft}(\eta, A)$, if it satisfies:

(1) $I \subseteq A$;

(2) $\gamma(x)$ is an ideal of $\eta(x)$ for all $x \in \mathrm{Supp}(\gamma, I)$.

Definition 2.6 Let (F, A) be a soft hemiring over S. A soft set (G, B) over S is called a soft strong h-ideal of (F, A), denote by $(G, B)\widetilde{\triangleleft}(F, A)$, if it satisfies:

(1) $B \subseteq A$;

(2) $G(x)$ is a strong h-ideal of $F(x)$ for all $x \in B$.

Theorem 2.7 *Let (F, A) and (G, B) be strong h-ideals of a soft hemiring (H, C) over S. Then the soft set $(F, A)\widetilde{\sqcap}(G, B)$ is a soft strong h-ideal of (H, C).*

Proof. Assume that $(F, A)\widetilde{\triangleleft}(H, C)$ and $(G, B)\widetilde{\triangleleft}(H, C)$. By Definition 2.4 (1), we can write $(F, A)\widetilde{\sqcap}(G, B) = (\gamma, I)$, where $I = A \cap B$ and $\gamma(x) = F(x) \cap G(x)$ for all $x \in I$. Obviously, we have $I \subseteq C$. Suppose that the soft set (γ, I) is non-null. If $x \in I$, then $\gamma(x) = F(x) \cap G(x) \neq \emptyset$. Since $(F, A)\widetilde{\triangleleft}(H, C)$ and $(G, B)\widetilde{\triangleleft}(H, C)$, we deduce that the nonempty sets $F(x)$ and $G(x)$ are both strong h-ideals of $H(x)$. It follows that $\gamma(x)$ is a strong h-ideal of $H(x)$ for all $x \in I$. Therefore $(F, A)\widetilde{\sqcap}(G, B) = (\gamma, I)$ is a soft strong h-ideal of (H, C) as required. ∎

Theorem 2.8 *Let (F, A) and (G, B) be strong h-ideals of a soft hemiring (H, C) over S. If A and B are disjoint, then the soft set $(F, A)\widetilde{\cup}(G, B)$ is a soft strong h-ideal of (H, C).*

Proof. Assume that $(F, A)\widetilde{\triangleleft}(H, C)$ and $(G, B)\widetilde{\triangleleft}(H, C)$. According 2.4 (3), we can write $(F, A)\widetilde{\cup}(G, B) = (\gamma, I)$, where $I = A \cup B$ and for every $x \in I$,

$$\gamma(x) = \begin{cases} F(x), & \text{if } \ e \in A - B, \\ G(x), & \text{if } \ e \in B - A, \\ F(x) \cup G(x), & \text{if } \ e \in A \cap B. \end{cases}$$

Clearly, we have $I \subseteq C$. Suppose that A and B are disjoint, i.e., $A \cap B = \emptyset$. Then, for every $x \in I$, we know that either $x \in A - B$ or $x \in B - A$. If $x \in A - B$, then $\gamma(x) = F(x) \neq \emptyset$ is a strong h-ideal of $H(x)$ since $(F, A)\widetilde{\triangleleft}(H, C)$. Similarly, if $x \in B - A$, then $\gamma(x) = G(x) \neq \emptyset$ is a strong h-ideal of $H(x)$ since $(G, B)\widetilde{\triangleleft}(H, C)$. Thus we conclude $\gamma(x)$ is a strong h-ideal of $H(x)$ for all $x \in I$, and so $(F, A)\widetilde{\cup}(G, B)$ is a soft strong h-ideal of (H, C). ∎

3. Strong h-idealistic soft hemirings and soft isomorphism theorems

In this section, we define the notion of strong h-idealistic soft hemirings, and then construct a soft quotient structure of an idempotent hemiring. Further, soft isomorphism theorems of soft hemirings are established.

Definition 3.1 Let (F, A) be a soft set over S. Then (F, A) is said to be a strong left(right) h-idealistic soft hemiring over S if and only if $F(x)$ is a strong left(right) h-ideal of S for all $x \in A$. (F, A) is said to be a strong h-idealistic soft hemiring over S if and only if (F, A) is both a strong right h-idealistic soft hemiring over S and a strong left h-idealistic soft hemiring over S.

Example 3.2 Let S and A be the hemirings of all non-negative integers with respect to the usual addition and multiplication of integers. $\forall x \in A$, let $F(x) = \{y \mid y\rho x \iff y = 2xa, a \in A\}$. If $y_1, y_2 \in F(x)$, then there exist $a_1, a_2 \in A$ such that $y_1 = 2xa_1, y_2 = 2xa_2, y_1+y_2 = 2xa_a+2xa_2 = 2x(a_1+a_2)$, then $y_1+y_2 \in F(x)$.

Let $z_1 \in S, y_1z_1 = 2xa_1z_1 = 2x(a_1z_1) \in F(x)$. Similarly, $z_1y_1 \in F(x)$. So $F(x)$ is an ideal of S. It is easy to check that $x + a + z = y + b + z$ implies $x \in y + F(x)$ for any $x, y, z \in S$ and $a, b \in F(x)$, then $F(x)$ is a strong h-ideal of S and (F, A) is a strong h-idealistic soft hemiring.

Proposition 3.3 *Let (F, A) be a soft set over S and let $B \subseteq A$. If (F, A) is a strong h-idealistic soft hemiring over S, then so is (F, B) whenever it is non-null.*

Proof. Straightforward. ∎

Theorem 3.4 *Let (F, A) and (G, B) be two strong h-idealistic soft hemirings over S. Then $(F, A)\widetilde{\sqcap}(G, B)$ is a strong h-idealistic soft hemiring over S if it is non-null.*

Proof. By Definition 2.4 (1), we can write $(F, A)\widetilde{\sqcap}(G, B) = (\gamma, I)$, where $I = A \cap B$ and $\gamma(x) = F(x) \cap G(x)$ for all $x \in I$. Suppose that (γ, I) is a non-null soft set over S. If $x \in I$, then $\gamma(x) = F(x) \cap G(x) \neq \emptyset$. Thus the nonempty sets $F(x)$ and $G(x)$ are strong h-ideals of S. It follows that $\gamma(x)$ is a strong h-ideal of S for all $x \in I$. Hence, $(\gamma, I) = (F, A)\widetilde{\sqcap}(G, B)$ is a strong h-idealistic soft hemiring over S. ∎

Theorem 3.5 *Let (F, A) and (G, B) be two strong h-idealistic soft hemirings over S. If A and B are disjoint, then the union $(F, A)\widetilde{\cup}(G, B)$ is a strong h-idealistic soft hemiring over S.*

Proof. According 2.4 (3), we can write $(F, A)\widetilde{\cup}(G, B) = (\gamma, I)$, where $I = A \cup B$ and for every $x \in I$,

$$\gamma(x) = \begin{cases} F(x), & \text{if } \ e \in A - B, \\ G(x), & \text{if } \ e \in B - A, \\ F(x) \cup G(x), & \text{if } \ e \in A \cap B. \end{cases}$$

Suppose that $A \cap B = \emptyset$. Then, for every $x \in I$, we know that either $x \in A - B$ or $x \in B - A$. If $x \in A - B$, then $\gamma(x) = F(x)$ is a strong h-ideal of S since (F, A) is a strong h-idealistic soft hemirings over S. Similarly, if $x \in B - A$, then $\gamma(x) = G(x)$ is a strong h-ideal of S since (G, B) is a strong h-idealistic soft hemirings over S. Thus we conclude that $\gamma(x)$ is a strong h-ideal of S for all $x \in I$, and so $(\gamma, I) = (F, A)\widetilde{\cup}(G, B)$ is a strong h-idealistic soft hemirings over S. ∎

Theorem 3.6 *Let (F, A) and (G, B) be two strong h-idealistic soft hemirings over S. Then $(F, A)\widetilde{\wedge}(G, B)$ is a strong h-idealistic soft hemiring over S if it is non-null.*

Proof. According 2.4 (2), we can write $(F, A)\widetilde{\wedge}(G, B) = (\gamma, C)$, where $C = A \times B$ and $\gamma(x, y) = F(x) \cap G(y)$ for all $(x, y) \in C$. Suppose that (γ, C) is a non-null soft set over S. If $(x, y) \in C$, then $\gamma(x, y) = F(x) \cap G(y) \neq \emptyset$. Since (F, A) and (G, B) are strong h-idealistic soft hemirings over S, we deduce that the nonempty sets $F(x)$ and $G(y)$ are both strong h-ideals of S. Hence, $\gamma(x, y)$ is a strong h-ideal of S for all $(x, y) \in C$, and so we conclude that $(\gamma, C) = (F, A)\widetilde{\wedge}(G, B)$ is a strong h-idealistic soft hemirings over S. ∎

Definition 3.7 Let (F, A) and (G, B) be soft hemirings over two hemirings R and S, respectively. Let $f : R \to S$ and $g : A \to B$ be two mappings. Then the pair (f, g) is called a soft hemiring homomorphism if it satisfies the following conditions:

(1) f is an epimorphism of hemirings.

(2) g is a surjective mapping.

(3) $f(F(x)) = G(g(x))$ for all $x \in A$.

If there exists a soft hemiring homomorphism between (F, A) and (G, B), we say that (F, A) is soft homomorphic to (G, B), which is denoted by $(F, A) \sim (G, B)$. Moreover, if f is an isomorphism of hemirings and g is a bijective mapping, then (f, g) is called a soft hemiring isomorphism. In this case, we say that (F, A) is soft isomorphic to (G, B), which is denoted by $(F, A) \simeq (G, B)$.

Example 3.8 Denote by $\mathbb{Z}$ and Z_n the hemiring of integers and the hemiring of integers module(a positive integer) n, respectively. Let $f : \mathbb{Z} \to Z_n$ be the natural mapping defined by $f(x) = [x]$ for all $x \in \mathbb{Z}$. Evidently, f is an epimorphism of hemirings. Let $\mathbb{Z}^+$ be the set of positive integers and define a mapping $g : \mathbb{Z}^+ \to Z_n$ by $g(x) = [x]$ for all $x \in \mathbb{Z}^+$, then it is easy to see that the mapping g is surjective. Let $(\alpha, \mathbb{Z}^+)$ be a soft set over $\mathbb{Z}$, where $\alpha : \mathbb{Z} \to \mathscr{P}(\mathbb{Z})$ is a set-valued function defined by $\alpha(x) = \{3xk|k \in \mathbb{Z}\}$ for all $x \in \mathbb{Z}^+$. One easily verifies that $\alpha(x) = 3x\mathbb{Z}$ is a subhemiring of $\mathbb{Z}$ for all $x \in \mathbb{Z}^+$. Thus (α, Z^+) is a soft hemiring over $\mathbb{Z}$. Let (β, Z_n) be a soft set over Z_n, where $\beta : Z_n \to \mathscr{P}(Z_n)$ is a set-valued function given by $\beta([x]) = \{[3xk]|k \in \mathbb{Z}\}$ for all $[x] \in Z_n$. Then one can also prove that (β, Z_n) is a soft hemiring over Z_n. Moreover, since $f(\alpha(x)) = f(3x\mathbb{Z}) = \{[3xk]|k \in \mathbb{Z}\}$ and $\beta(g(x)) = \beta([x]) = \{[3xk]|k \in \mathbb{Z}\}$ for all $x \in \mathbb{Z}^+$, we deduce that $f(\alpha(x)) = \beta(g(x))$ for all $x \in \mathbb{Z}^+$. Hence (f, g) is a soft hemiring homomorphism and $(\alpha, \mathbb{Z}^+) \sim (\beta, Z_n)$.

Lemma 3.9 [16] *Let I be a strong h-ideal of S. If $x, y \in S$, then*

[(1) $x \in [y]_I$ *if and only if* $x \in y + I$,

(2) $[x]_I + [y]_I = [x + y]_I$,

(3) $\{ab|a \in [x]_I, b \in [y]_I\} \subseteq [xy]_I$.

Next, S is always an idempotent hemiring, we introduce the concepts of soft quotient structure over an idempotent hemiring.

Lemma 3.10 *Let (F, A) be a strong h-idealistic soft hemiring over S, and $S/F(\alpha) = \{[x]_{F(\alpha)} : x \in S\}$, where $\alpha \in A$. Then, for any $\alpha \in A$, $S/F(\alpha)$ is a hemiring under the binary operation induced by S, which is given by*

$$\begin{aligned} [x]_{F(\alpha)} + [y]_{F(\alpha)} &= [x + y]_{F(\alpha)}, \\ [x]_{F(\alpha)}[y]_{F(\alpha)} &= [xy]_{F(\alpha)} \end{aligned}$$

for all $x, y \in S$.

Proof. Firstly, we show that the above binary operations are well defined. In fact, if $[a]_{F(\alpha)} = [a']_{F(\alpha)}$ and $[b]_{F(\alpha)} = [b']_{F(\alpha)}$ for all $a, a', b, b' \in S, \alpha \in A$. Since (F, A) is a strong h-idealistic soft hemiring over S, then, by Definition 3.1, we know that $F(\alpha)$ is a strong h-ideal of S for all $\alpha \in A$.

By Lemma 3.8,

$$[a]_{F(\alpha)} = a + F(\alpha), \quad [a']_{F(\alpha)} = a' + F(\alpha),$$
$$[b]_{F(\alpha)} = b + F(\alpha), \quad [b']_{F(\alpha)} = b' + F(\alpha),$$

then we have

$$\begin{aligned}
[a+b]_{F(\alpha)} &= a + b + F(\alpha) = a + F(\alpha) + b + F(\alpha) \\
&= a' + F(\alpha) + b' + F(\alpha) = a' + b' + F(\alpha) \\
&= [a' + b']_{F(\alpha)}, \\
[ab]_{F(\alpha)} &= ab + F(\alpha) = ab + aF(\alpha) + F(\alpha)b + F(\alpha)^2 \\
&= (a' + F(\alpha))(b' + F(\alpha)) = [a'b']_{F(\alpha)}.
\end{aligned}$$

Now, it is easy to check that $S/F(\alpha)$ is a hemiring. ∎

Lemma 3.11 *If A is an ideal of S, then $\widetilde{A}$ is a strong h-ideal of S containing A.*

Proof. Let $a, b \in \widetilde{A}$, then there exist $y_1, y_2 \in S$ such that

$$y_1 + a \in y_1 + \widetilde{A}$$

and

$$y_2 + b \in y_2 + \widetilde{A}$$

satisfying

$$y_1 + a + a_1 + z_1 = y_1 + a_2 + z_1$$

and

$$y_2 + b + b_1 + z_2 = y_2 + b_2 + z_2$$

for some $a_1, a_2, b_1, b_2 \in A, z \in S$. Then we have

$$y_1 + y_2 + a + b + a_1 + b_1 + z_1 + z_2 = y_1 + y_2 + a_2 + b_2 + z_1 + z_2,$$

that is

$$a + b + a_1 + b_1 + z' = a_2 + b_2 + z',$$

where $z' = y_1 + y_2 + z_1 + z_2$.

Since $a_1, a_2, b_1, b_2 \in A$ and A is an ideal, then we have $a_1 + b_1 \in A, a_2 + b_2 \in A$, so $a + b \in 0 + \widetilde{A}$. In a similar way, we have $ra, ar \in \widetilde{A}$ for $r \in S$. Thus $\widetilde{A}$ is a ideal. By the definition of strong h-closure, we know that $\widetilde{A}$ has the strong h-property, so $\widetilde{A}$ is a strong h-ideal. ∎

Theorem 3.12 *Let (F,A) be a strong h-idealistic soft hemiring over S. If (H,B) and (I,C) are soft strong h-ideals of (F,A), then $(P,B) \simeq (Q,B)$ and $(S,C) \simeq (T,C)$, where $P(x) = H(x)/(M \cap N)$, $Q(x) = (H(x)+N)/N$, $S(x) = I(x)/(M \cap N)$, $T(x) = (I(x)+M)/M$, $M = \bigcap_{x \in B} H(x)$ and $N = \bigcap_{x \in C} I(x)$.*

Proof. We first write

$$K = <\widetilde{\bigcup_{x \in B} H(x)}> \text{ and } L = <\widetilde{\bigcup_{x \in C} I(x)}>.$$

By Lemma 3.11, we know that K and L are strong h-ideals of S. Then $M = \bigcap_{x \in B} H(x)$ is a strong h-ideal of S. It is clear that M is also a strong h-ideal of K so that $M \cap N$ is a strong h-ideal of K, and hence, (P,B) is a soft hemiring over $K/(M \cap N)$. Similarly, (Q,B) is a soft hemiring over $(K+N)/N$.

Now, we define $f : K/(M \cap N) \to (K+N)/N$ by $f([k]_{M \cap N}) = [k]_N$ for $k \in K$ and define $g : B \to B$ by $g(x) = x$. By Lemma 3.9, we can check that f is an isomorphism from $K/(M \cap N)$ to $(K+N)/N$. Obviously, g is a bijective mapping and $f(P(x)) = f(H(x)/(M \cap N)) = (H(x)+N)/N = Q(x) = Q(g(x))$. This shows that $(P,B) \simeq (Q,B)$. Similarly, we can prove that $(S,C) \simeq (T,C)$. ∎

Theorem 3.13 *Let (F,A) be a strong h-idealistic soft hemiring over S. If (H,B) and (I,C) are soft strong h-ideals of (F,A) with $B \cap C \neq \emptyset$ and $I(x) \subset H(x)$ for all $x \in B \cap C$, then $(P, B \cap C) \simeq (Q, B \cap C)$, where $P(x) = (F(x)/N)/(M/N)$ and $Q(x) = F(x)/M$ with $M = \bigcap_{x \in B \cap C} H(x)$ and $N = \bigcap_{x \in B \cap C} I(x)$.*

Proof. It can be easily verified that $M = \bigcap_{x \in B \cap C} H(x)$ and $N = \bigcap_{x \in B \cap C} I(x)$ are strong h-ideals of S, and N is a strong h-ideal of M. Now it is easy to see that $(P, B \cap C)$ is a soft hemiring over the hemiring $(S/N)/(M/N)$ and so $(Q, B \cap C)$ is a soft hemiring over S/M.

Define the mapping $f : (S/N)/(M/N) \to S/M$ by $f(\{[r]_N\}_{M/N}) = [r]_M$ for $r \in S$ and define $g : B \cap C \to B \cap C$ by $g(x) = x$. Then by Lemma 3.9, we can check that f is an isomorphism from $(S/N)/(M/N)$ to S/M. Obviously, g is a bijective mapping and $f(P(x)) = f((F(x)/N)/(M/N)) = F(x)/M = Q(g(x))$. Hence, $(P, B \cap C) \simeq (Q, B \cap C)$. ∎

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FINITE p-GROUPS IN WHICH NORMAL CLOSURES FOR EVERY NONNORMAL SUBGROUPS ARE MINIMAL NONABELIAN

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Abstract. The authors study finite p-groups G such that A^G is minimal non-abelian for all non-normal subgroup $A < G$. This topic is Problem 805 posed by Berkovich and Janko in [4]. The authors give the complete classification of such kind of p-groups.

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1. Introduction

Let G be a finite group, H a subgroup of G. It is well-known fact that the normal closure H^G of H in G plays a very important role in determining the structure of the group G, especially for a p-group. For example, let G be a p-group, for every $a \in G$ if $\langle a\rangle^G$ is abelian, then the nilpotence class of G is less or equal to 3. Moreover the nilpotence class of $\langle a\rangle^G$ for every $a \in G$ is equal or less than 2 if and only if G is 3-Engel group. In [1] and [2], the authors studied p-groups G such that $\langle a\rangle^G$ having a cyclic subgroup maximal subgroup for any $a \in G$.

Berkovich and Janko in [4] posed an open Problem 805: Study the p-groups G such that A^G is minimal nonabelian for all nonnormalabelian $A < G$.

In this paper, we study the above problem, i.e., finite p-groups G such that A^G is a minimal non-abelian group for all non-normal subgroup $A < G$ and come to the classification of such kinds of p-groups. For convenience, we say such a p-group satisfies A^G-MNA-property.

All notations are the same as in [3] and [6].

2. Preliminaries

Lemma 2.1 *Let G be a p-group satisfying A^G-MNA-property. Then the following holds:*

(1) *If an abelian subgroup $A \lhd G$, then the subgroup of A is normal in G;*

(2) *If G is not a Dedekind group, then $Cl(G) \geq 3$;*

(3) *For every $a \in G$, $\langle a^p\rangle$ is normal in G;*

(4) *If $p \geq 3$, then every element of order p is contained $Z(G)$, i.e., $\Omega_1(G) \leq Z(G)$.*

Proof. (1) If there exists a subgroup $B < A$ such that B is non-normal G, then $B^G \leq A$ is a minimal non-abelian subgroup, a contradiction.

(2) If $Cl(G) \leq 2$, then $G' \leq Z(G)$. Since G is not a Dedekind group, then there exists a cyclic subgroup A of G such that $A \ntrianglelefteq G$. It is easy to see that $A^G \leq AG'$, hence A^G is abelian, a contradiction.

(3) Suppose that $\langle a^p\rangle$ is not a normal subgroup in G. Then $\langle a\rangle \ntrianglelefteq G$. Thus $\langle a\rangle^G$ is a minimal non-abelian subgroup. We have that $\langle a^p\rangle \leq \Phi(\langle a\rangle^G) \trianglelefteq G$, where $\Phi(\langle a\rangle^G)$ is the Frattini subgroup of $\langle a\rangle^G$. Clearly $\Phi(\langle a\rangle^G)$ is abelian. Hence $\langle a^p\rangle \trianglelefteq G$, a contradiction.

(4) It is enough to show $< a > \lhd G$ for every element a of order p. Otherwise, there exists $\langle a\rangle$ not normal in G. Hence $\langle a\rangle^G$ is a minimal non-abelian subgroup and $|\langle a\rangle^G| = p^3$. Since $p \geq 3$, it follows that $\langle a\rangle^G$ has an abelian G-invariant subgroup N of type (p, p) by Lemma 1.4 in [3]. Now, by (1), we get $N \leq Z(G)$ and $\langle a\rangle^G$ is abelian, a contradiction. ∎

Lemma 2.2 *Suppose that G is not a Dedekind p-group and satisfies A^G-MNA-property. Then $Cl(G) \leq 3$ if $p \neq 3$, but $Cl(G) \leq 4$ if $p = 3$.*

Proof. Consider $\overline{G} = G/Z(G)$. Let $\langle \overline{x} \rangle \ntrianglelefteq \overline{G}$, where $\overline{x} \in \overline{G}$. Obviously, $\langle x \rangle$ is not normal in G. Then $\langle x \rangle^G$ is a minimal non-abelian group. If $[x, x^g] \neq 1$ for $g \in G$, then $\langle x \rangle^G = \langle x, x^g \rangle = H$. Since $H' = [x, x^g]$ is of order p, it follows that $[x, x^g] \in Z(G)$. Hence $\langle \overline{x} \rangle^{\overline{G}}$ is abelian, thus $\overline{G}$ is a 2-Engel group. Therefore, $Cl(\overline{G}) \leq 2$ if $p \neq 3$, but $Cl(\overline{G}) \leq 3$ if $p = 3$, which concludes the lemma. ∎

Proposition 2.3 *Suppose that S is not a Dedekind p-group G and satisfies A^G-MNA-property. Then $p = 2$ and $Cl(G) = 3$.*

Proof. At first we show that $p = 2$. Otherwise, let $\langle y \rangle \ntrianglelefteq G$. Then $\langle y \rangle^G$ is a minimal non-abelian group. By Lemma 2.1(4), we have $|y| \geq p^2$. But by Lemma 2.1(3) it follows that $\langle y^p \rangle \trianglelefteq G$, hence $\langle y^p \rangle^g = \langle y^p \rangle = \langle (y^g)^p \rangle$. Therefore $(y^g)^p = y^{dp}$, where $(d, p) = 1$. By regularity of $\langle y \rangle^G$, we have $(y^{-d} y^g)^p = 1$. Let $y_1 = y^{-d} y^g$, then there exists an element y_1 of order p such that $\langle y \rangle^G = \langle y, y^g \rangle = \langle y, y_1 \rangle$. By Lemma 2.1(4), we have $y_1 \in Z(G)$, consequently $\langle y \rangle^G = \langle y, y_1 \rangle$ is abelian, a contradiction.

Now, by Lemma 2.1(2) and Lemma 2.2, we have $Cl(G) = 3$ and $p = 2$. ∎

3. Classification of p-groups satisfying A^G-MNA-property

Theorem 3.1 *Assume that a p-group G is not a Dedekind group and satisfies A^G-MNA-property. Then one of the following holds:*

(1) $G \cong D_{2^4}$;

(2) $G \cong Q_{2^4}$;

(3) $G \cong SD_{2^4}$;

(4) $G \cong \langle x, y \rangle$ *and* $|G| = 2^5$, *where* $|x| = 8$ *and* $|y| = 4$, $\langle x \rangle \cap \langle y \rangle = 1$, $Cl(G) = 3$.

Proof. (a) We assume $\Omega_1(G) \nleq Z(G)$ and prove (1) or (3) holds.

At first we have $p = 2$ by Lemma 2.1 (4). Since G is not a Dedekind group, there exists $\langle x \rangle \ntrianglelefteq G$. Hence $H = \langle x \rangle^G$ is a minimal non-abelian subgroup by G satisfying A^G-MNA-property, which implies $H \cong D_8$. Because $Aut(D_8) \cong D_8$ and $H/Z(H) \cong Inn(H) \cong C_2 \times C_2$, we have $|G/HC_G(H)| \leq 2$.

If $|G/HC_G(H)| = 2$. We assert that $C_G(H) \leq H$. Otherwise, there exists $y \in C_G(H) - H$. Let $A = \langle x, y \rangle$, then it must hold that $A \ntrianglelefteq G$. In fact, if $A \trianglelefteq G$ then $\langle x \rangle \trianglelefteq G$ for A is abelian and by Lemma 2.1(1), a contradiction. Hence $A^G \leq HC_G(H)$ is a minimal non-abelian subgroup. Since $H \leq A^G$, we have $H = A^G$, and then $y \in H$, which contradicts the fact $y \in C_G(H) \setminus H$. Therefore $G = HC_G(H)$, consequently $G \cong D_{2^4}$ or $G \cong SD_{2^4}$, i.e., (1) or (3) holds.

(b) Now, we assume $\Omega_1(G) \leq Z(G)$ and shall prove (2) or (4) holds.

(I) At first, we assume $|\Omega_1(G)| = 2$. Since G has no abelian G-invariant subgroups of type (p, p), G is a 2-group maximal class by Lemma 1.4 in [3]. Then, $G \cong Q_{2^4}$, that is, G is as in (2).

(II) Now assume $|\Omega_1(G)| \geq 4$. We have divided the proof into two subcases.

(i) If $exp(G) \leq 4$. By $\Omega_1(G) \leq Z(G)$ and G is not a Dedekind group, there exists some $\langle a\rangle \ntrianglelefteq G$ and $|a| = 4$. Let $\langle a\rangle^G = \langle a, a^b\rangle$. Since $\langle a^2\rangle \trianglelefteq G$ by Lemma 2.1 (3), we get that $(a^2)^b = a^2 = (a^b)^2$. Hence $|\langle a\rangle^G| = 8$. But $\langle a\rangle^G$ is a minimal non-abelian subgroup by hypothesis, it follows that $\langle a\rangle^G \cong D_8$. Also $\langle a\rangle^G$ can be generated by two elements of order 2 and $\Omega_1(G) \leq Z(G)$, so that $\langle a\rangle^G$ is abelian, a contradiction. Therefore G is a Dedekind group, a last contradiction.

(ii) Suppose $exp(G) \geq 8$. At first we claim that if $x \in G$ of order ≥ 8 then $\langle x\rangle \trianglelefteq G$.

If there exists some $x \in G$ such that $|x| \geq 8$ and $\langle x\rangle \ntrianglelefteq G$, then we may set $\langle x\rangle^G = \langle x, x_g\rangle = H$ for some $g \in G$. It follows that $\langle x^2\rangle \trianglelefteq G$ by Lemma 2.1 (3), hence $(\langle x^2\rangle)^g = \langle x^2\rangle = \langle {x_1}^2\rangle$. Therefore there exists k such that $x^2 = {x_1}^{2k}$ where $(k, 2) = 1$. By Hall-Petrescu formula and H is a minimal non-abelian subgroup, we have $(x{x_1}^{-k})^4 = x^4({x_1}^{-k})^4[x, {x_1}^{-k}]^6 = 1$. If $|x{x_1}^{-k}| = 2$. Then it follows by $H = \langle x{x_1}^{-k}, x_1\rangle$ and $\Omega_1(G) \leq Z(G)$ that H is abelian, a contradiction. Thus $|x{x_1}^{-k}| = 4$. Let $x_2 = xx_1^{-k}$, then $H = \langle x, x_2\rangle$. Since $x_2^2 = (xx_1^{-k})^2 \in \langle x\rangle$ and $|x_2| = 4$, we come to $x_2^2 = [x, x_1]$ and $\langle x_2\rangle \trianglelefteq H$. Because $Cl(H) = 2$ and $exp(H) = |x|$, one has that $x_2^2 \in \langle x\rangle \cap \langle x_2\rangle \neq 1$. Hence $\langle x\rangle$ is a cyclic subgroup of H having index 2 in G, so there exists x_3 of order 2 such that $H = \langle x, x_3\rangle$. But $x_3 \in Z(G)$, we get that H is abelian, a contradiction. The claim follows.

Let x be an element of order 8, and $y \in G$ an element of order 4 such that $\langle y\rangle$ is not normal in G. Then $\langle x\rangle \trianglelefteq G$ by above argument, thus $K = \langle x, y\rangle = \langle x\rangle\langle y\rangle$ is a subgroup of order at most 32. In the following, we discuss the structure of K case by case.

Case 1. Assume that $\langle x\rangle \cap \langle y\rangle \neq 1$. By $\Omega_1(G) \leq Z(G)$, we have $\langle y^2\rangle \trianglelefteq G$. If $Cl(K) \leq 2$, since $[x, y]^2 = [x, y^2] = [x^2, y] = 1$, then K is minimal non-abelian. By $\langle x\rangle \cap \langle y\rangle \neq 1$, then there exists l such that $x^{4l} = y^2$ where $(l, 2) = 1$. It follows that $(x^{-2l}y)^2 = x^{-4l}y^2 = 1$. Hence $|x^{-2l}y| = 2$. Obviously, since $K = \langle x, x^{-2l}y\rangle$ is minimal non-abelian, we get $\langle x^{-2l}y\rangle \ntrianglelefteq G$, which contradicts $\Omega_1(G) \leq Z(G)$. Therefore, $Cl(K) = 3$. Notice that K has a maximal and cyclic subgroup $\langle x\rangle$ and $\Omega_1(G) \leq Z(G)$, we get that $K \cong Q_{2^4}$. It follows that $\langle y\rangle^G = \langle y\rangle^K \cong Q_8 \trianglelefteq G$. Let $T = \langle y, \Omega_1(G)\rangle$, since $\langle y\rangle^G \leq \langle T\rangle^G$, we have that $\langle T\rangle^G$ is a non-abelian subgroup and $T \ntrianglelefteq G$. Now because T is abelian, it follows that $\langle T\rangle^G$ is a minimal non-abelian subgroup. Hence $\langle y\rangle^G = \langle T\rangle^G$. Otherwise, $\langle y\rangle^G$ is abelian, a contradiction. By $\Omega_1(G) \leq \langle T\rangle^G$ and $|\Omega_1(\langle T\rangle^G)| = |\Omega_1(\langle y\rangle^G)| = |\Omega_1(Q_8)|=2$, it follows that $|\Omega_1(G)| = 2$, which contradicts $|\Omega_1(G)| \geq 4$.

Case 2. Now, assume that $\langle x\rangle \cap \langle y\rangle = 1$. If $Cl(K) = 3$. Take $a \in G \setminus K$, set $T = \langle y, a\rangle$. Because $|K| = 2^5$ and $K' \leq \langle x\rangle$, if $|K : K'| = 4$, then K is of maximal nilpotent class, which contradicts $Cl(K) = 3$. Thus $K' = \langle x^2 \rangle$. If $|\langle y\rangle^K| = 2^3$, then $\langle y\rangle^K$ is abelian by $\langle y\rangle^K \leq \langle y\rangle K'$, so K is a 2-Engle group. Thus $Cl(K) \leq 2$, a contradiction. It follows that $\langle y\rangle^K = \langle x^2, y\rangle$ is a minimal non-abelian subgroup. If $\langle y\rangle^K < \langle y\rangle^G$, since $\langle y\rangle^G$ is a minimal non-abelian subgroup, then $\langle y\rangle^K$ is abelian, a contradiction too. Hence $\langle y\rangle^K = \langle y\rangle^G = \langle x^2, y\rangle$. In order to prove that $G = K$, we divide the following proof into two subcases.

Subcase 1. While $|a| = 2$. By $\Omega_1(G) \leq Z(G)$, it follows that T is abelian. If $T \trianglelefteq G$, then $\langle y\rangle$ is normal in G by Lemma 2.1(1), a contradiction. Hence $T \ntrianglelefteq G$. Consequently T^G is a minimal non-abelian subgroup, it follows that $\langle y\rangle^G = T^G = \langle a\rangle\langle y\rangle^G = \langle y\rangle^K$, so that $a \in K$, a contradiction.

Subcase 2. While $|a| = 4$. Since $T^G = \langle a\rangle^G\langle y\rangle^G = \langle a\rangle^T\langle y\rangle^T = T$, then $T \trianglelefteq G$. Thus $T = \langle a\rangle\langle y\rangle^G$. By $a^2 \in \Omega_1(\langle y\rangle^G)$, it follows $|T| = 2^5$. If $exp(G) = 4$, then $Cl(G) \leq 2$ by the same argument in the proof of (i). Since $T' = \langle [a, y]\rangle$ and $[a, y]^2 = [a^2, y] = 1$, we have $|T'| = 2$. It follows that T is a minimal non-abelian subgroup. It follows by $\langle y\rangle^G < T$ that $\langle y\rangle^G$ is abelian, a contradiction. Hence there exists $a_1 \in T \setminus \langle y\rangle^G$ with $|a_1| = 8$. If $a_1 \in K$, then $|\langle a_1, \langle y\rangle^G\rangle| = 2^5$. It follows that $\langle a_1, \langle y\rangle^G\rangle = T \leq K$, which contradicts $a \in G \setminus K$. Thus $a_1 \notin K$. Let $H = \langle a_1, x\rangle$. Because of $\langle a_1\rangle \trianglelefteq G$ and $\langle x\rangle \trianglelefteq G$, we get $Cl(H) \leq 2$ by Theorem 21 in [3]. By $|T : \langle y\rangle^G| = 2$, it follows that ${a_1}^2 \in \langle y\rangle^G$. Hence $|\langle a_1, x, y\rangle| = 2^6$.

If $\langle a_1\rangle \cap \langle x\rangle = 1$, then $|\langle a_1, x\rangle| = 2^6$. We get $\langle a_1, x\rangle = \langle a_1, x, y\rangle$, which contradicts $Cl(\langle x, y\rangle) = 3$. Therefore, $\langle a_1\rangle \cap \langle x\rangle \neq 1$.

Now, assume that $\langle a_1\rangle \cap \langle x\rangle = \langle {a_1}^2\rangle$. Since $Cl(\langle a_1, x\rangle) \leq 2$, one has $\langle a_1, x\rangle = \langle a_2, x\rangle$, where $|a_2| = 2$. By $a_2 \in \Omega_1(\langle a_1, x, y\rangle) = \Omega_1(\langle x, y\rangle)$, it follows that $a_1 \in \langle a_1, x\rangle \leq \langle x, y\rangle = K$, a contradiction. Hence $\langle a_1\rangle \cap \langle x\rangle = \langle {a_1}^4\rangle = \langle x^4\rangle$. By $Cl(\langle a_1, x\rangle) \leq 2$, there exists a_3 of order 4 such that $\langle a_1, x\rangle = \langle a_3, x\rangle = \langle x\rangle \rtimes \langle a_3\rangle$. Noticing $\langle a_3\rangle^G \leq \Omega_2(\langle x, a_1\rangle)$ and $\Omega_2(\langle x, a_1\rangle) = \langle x^2\rangle \times \langle a_3\rangle$ is abelian, we assert $\langle a_3\rangle \trianglelefteq G$. Otherwise, $\langle a_3\rangle^G$ is abelian, a contradiction. Hence $\langle a_1, x\rangle$ is abelian. If $a_3 \in K$, then $a_1 \in \langle a_1, x\rangle = \langle a_3, x\rangle \leq K$, a contradiction. Thus $a_3 \notin K$. Let $T_2 = \langle y, a_3\rangle$. Since ${T_2}^G = \langle a_3\rangle^G\langle y\rangle^G = \langle a_3\rangle\langle y\rangle^{T_2} = T_2$, $T_2 \trianglelefteq G$. Let $t \in T_2$, then $t = m_1m_2$, where $m_1 \in \langle a_3\rangle$ and $m_2 \in \langle y\rangle^G$. By Hall-Petrescu formula, we get $t^4 = (m_1m_2)^4 = {m_1}^4{m_2}^4{c_2}^6{c_3}^4 = 1$, where $c_i \in K_i(\langle m_1, m_2\rangle)$, $i = 2, 3$. Hence $exp(T_2) = 4$, it is impossible by the same argument in the proof of (i). Therefore $|a| \neq 4$.

Subcase 3. Assume that $|a| \geq 8$. By $T = \langle y, a\rangle$, where $a \in G \setminus K$ and $|y| = 4$. Similarly, we have $T^G = \langle a\rangle\langle y\rangle^G = T \trianglelefteq G$. By the same reasoning as above, we have $\langle a\rangle\cap\langle y\rangle = 1$ and $Cl(\langle y, a\rangle) = 3$. Since $[a, y^2] = 1$, y induces an automorphism of order 2 of $\langle a\rangle$. We have $a^y = a^{1+2^{n-1}}$ or $a^y = a^{-1+k2^{n-1}}$. Since $Cl(\langle y, a\rangle) = 3$, then $a^y = a^{-1+k2^{n-1}}$ and $|a| = 8$. This case is subcase 2. Hence it is impossible if $|a| \geq 8$. Therefore, $G = K$.

If $Cl(K) = 2$, it follows that $[x, y^2] = [x^2, y] = [x, y]^2 = 1$ by $y^2 \in Z(G)$. Since $K' = \langle [x, y], \gamma_3(K)\rangle = \langle [x, y]\rangle$, we have $|K'| = 2$, hence K is a minimal non-abelian subgroup. Therefore $x \in K = \langle y\rangle^G$. But $\langle y\rangle^G$ is a minimal non-abelian subgroup, we get $exp(\langle y\rangle^G) = 4$, which contradicts $|x| = 8$. ∎

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SCOTT CLOSED INJECTIVITY AND RETRACTNESS OF DIRECTED COMPLETE POSET ACTS

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Abstract. Domain theory, which studies directed complete partially ordered sets, was introduced by Scott in the 1970s as a foundation for programming semantics and provides an abstract model of computation, and has grown into a respected field on the borderline between mathematics and computer science.

In this paper, we consider actions of a semigroup (monoid or group) on directed complete posets and study the algebraic notions of injectivity and retractness with respect to Scott closed embeddings in the categories so obtained.

Keywords: Dcpo, S-Dcpo, Scott closed embedding, injective object, retract.

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1. Introduction and preliminaries

Injectivity and retractness are crucial notions in many branches of mathematics. Many mathematicians studied these notions in different categories with respect to different classes of monomorphisms and investigated their relations, see for example [5], [8], [6], [11], [9]. In this paper we study these notions with respect to the class of Scott closed embeddings in the category of actions of a monoid on directed complete posets.

First we recall some preliminaries needed in the sequel. The reader can find more details in [2], [4], [10], [13]. Let **Pos** denote the category of all partially ordered sets (posets) with order-preserving (monotone) maps between them. A non-empty subset D of a partially ordered set is called *directed*, denoted by $D \subseteq^d P$, if for every $a, b \in D$ there exists $c \in D$ such that $a, b \leq c$; and P is called *directed complete*, or briefly a *dcpo*, if for every $D \subseteq^d P$, the directed join $\bigvee^d D$ exists in P.

A *dcpo map* or a *continuous map* $f : P \to Q$ between dcpo's is a map with the property that for every $D \subseteq^d P$, $f(D)$ is a directed subset of Q and $f(\bigvee^d D) = \bigvee^d f(D)$. Thus we have the category **Dcpo** of all dcpo's with continuous maps between them.

A *po-monoid* (*po-semigroup, po-group*) S is a monoid (semigroup, group) with a partial order $\leq$ which is compatible with its binary operation (that is, for $s, t, s', t' \in S$, $s \leq t$ and $s' \leq t'$ imply $ss' \leq tt'$). Similarly, a *dcpo-monoid* (*group*) is a monoid (group) which is also a dcpo whose binary operation is a continuous map.

Recall that a (*right*) *S-act* or *S-set* for a monoid S is a set A equipped with an *action* $A \times S \to A$, $(a, s) \rightsquigarrow as$, such that $ae = a$ (e is the identity element of S) and $a(st) = (as)t$, for all $a \in A$ and $s, t \in S$. Let **Act**-S denote the category of all S-acts with action preserving maps ($f : A \to B$ with $f(as) = f(a)s$, for all $a \in A, s \in S$). Let A be an S-act. An element $a \in A$ is called a *zero, fixed,* or *a trap element* if $as = a$, for all $s \in S$.

For a po-monoid S, a (*right*) *S-poset* is a poset A which is also an S-act whose action $\lambda : A \times S \to A$ is order-preserving, where $A \times S$ is considered as a poset with componentwise order. The category of all S-posets with action preserving monotone maps between them is denoted by **Pos**-S.

Also, for a dcpo-monoid S, a (*right*) *S-dcpo* is a dcpo A which is also an S-act whose action $\lambda : A \times S \to A$ is a continuous map.

A non-empty subset B of an S-dcpo A is called a *sub S-dcpo* of A if B is a sub dcpo (a subset which is closed under directed joins) and subact of A. In this case, A is said to be an *extension* of B.

By an *S-dcpo map* between S-dcpo's, we mean a map $f : A \to B$ which is both continuous and action preserving. We denote the category of all S-dcpo's and S-dcpo maps between them by **Dcpo**-S.

A morphism $f : A \to B$ in the category of all S-dcpo's is called *order-embedding*, or briefly *embedding*, if for all $x, y \in A$, $f(x) \leq f(y)$ if and only if $x \leq y$.

Because of the fact that sub S-dcpo's are exactly subsets for which the inclusion map is order-embedding, we consider an arbitrary order-embedding as an inclusion from a sub S-dcpo.

Notice that order-embeddings are monomorphisms in the category of all S-dcpo's. But the converse is not necessarily true. For example, take S to be the one element dcpo-monoid, $A = \{\perp, a, a'\}$ with the order $\perp \leq a, a'$, $a \parallel a'$, and B the three element chain $\mathbf{3} = \{0, 1, 2\}$. Let $g : A \to B$ be defined as $g(\perp) = 0,\ g(a) = 1,\ g(a') = 2$. Then, g is one-one and hence a monomorphism in **Dcpo**-S, but it is not an embedding.

Finally, recall that for a class $\mathcal{M}$ of monomorphisms in a category $\mathcal{C}$, an object $A \in \mathcal{C}$ is called $\mathcal{M}$-*injective* if for each $\mathcal{M}$-*morphism* $f : B \to C$ and morphism $g : B \to A$ there exists a morphism $h : C \to A$ such that $hf = g$.

Also, an object A of a category $\mathcal{C}$ is called $\mathcal{M}$-*absolute retract* if it is a retract of each of its $\mathcal{M}$-extensions; that is, for each $\mathcal{M}$-morphism $f : A \to C$ there exists a morphism $h : C \to A$ such that $hf = id_A$, in which case h is said to be a *retraction*.

We say that $\mathcal{C}$ has enough $\mathcal{M}$-injective objects if for each $A \in \mathcal{C}$ there exists an $\mathcal{M}$-injective $\mathcal{M}$-extension of A.

In this paper, we consider $\mathcal{M}$ to be the class *sc* of all Scott closed embeddings (will be introduced in the following) in the above mentioned category of dcpo-monoid actions. Then, we investigate injectivity and retractness with respect to Scott closed embeddings in the category **Dcpo**-S and some of its subcategories.

2. Scott closed injectivity in Dcpo-S

In this section, we investigate injectivity with respect to Scott closed embeddings in the category **Dcpo**-S and give a non-trivial *sc*-injective object. Then, we consider the behaviour of *sc*-injective objects with products and coproducts.

Definition 2.1. A possibly empty subset B of a dcpo C is said to be *Scott closed* in C if B is a downward closed subset of C and for every $D \subseteq^d B$, $\bigvee^d D \in B$. An embedding $f : B \to C$ is said to be *Scott closed* or *sc-embedding* if B is Scott closed in C.

Definition 2.2. An S-dcpo A is said to be *Scott closed injective* or *sc-injective* if it is injective with respect to *sc*-embeddings.

Lemma 2.3. *If A is an sc-injective object in* **Dcpo**-S*, then it has a zero top element.*

Proof. Let A be an *sc*-injective in **Dcpo**-S. Consider the S-dcpo $C = A \oplus \{\theta\}$ where θ is taken to be a zero element. Since A is *sc*-injective, there exists a retraction $h : C \to A$. Then, the zero element $h(\theta)$ is the top element of A. ∎

Recall from [13] that, for a dcpo A and dcpo-monoid S, the cofree S-dcpo on A is the set $A^{(S)}$, of all dcpo maps from S to A, with pointwise order and the action given by $(fs)(t) = f(st)$, for $s, t \in S$ and $f \in A^{(S)}$.

Theorem 2.4. *Let A be a dcpo which has a top element. Then $A^{(S)}$ is an sc-injective S-dcpo.*

Proof. Let A be a dcpo with the top element T_A, $i : B \hookrightarrow C$ be an *sc*-embedding in **Dcpo**-S and $g : B \to A^{(S)}$ be an S-dcpo map. We must find an S-dcpo map $h : C \to A^{(S)}$ which extends g. Define $h : C \to A^{(S)}$ where

$$h(x)(s) = \begin{cases} g(xs)(e) & \text{if } \ xs \in B, \\ \top_A & \text{otherwise.} \end{cases}$$

Now, we show that h is an S-dcpo map. First, we prove that h is well-defined and $h(x)$ is a dcpo-map. We know $h(x)$ preserves the order. To see this, let $s, s' \in S$

where $s \leq s'$. So $xs \leq xs'$. If $xs' \in B$ then, since B is Scott closed in C, we get $xs \in B$. Therefore, since g preserves the order, we have $g(xs) \leq g(xs')$ and so

$$h(x)(s) = g(xs)(e) \leq g(xs')(e) = h(x)(s').$$

Otherwise, if $xs' \notin B$ then, $h(x)(s') = \top_A$ and it is clear that $h(x)(s) \leq h(x)(s') = \top_A$. Now, we show that $h(x)$ is continuous. To see this, let $T \subseteq^d S$. Since $h(x)$ preserves the order, we get $h(x)(T) \subseteq^d A$. Now, if $x \bigvee^d T \in B$ then, since B is Scott closed in C and $xt \leq x \bigvee^d T \in B$ for every $t \in T$, we get $xt \in B$, for every $t \in T$. Since g is continuous and B is an S-dcpo, we get

$$h(x)\left(\bigvee^d T\right) = g\left(x \bigvee^d T\right)(e) = g\left(\bigvee_{t\in T}^d xt\right)(e) = \bigvee_{t\in T}^d g(xt)(e) = \bigvee_{t\in T}^d h(x)(t).$$

Now, if $x \bigvee^d T \notin B$, then there exists $t_0 \in T$ such that $xt_0 \notin B$. The latter is because if, on the contrary, for every $t \in T$, $xt \in B$ then, since B is Scott closed in C, we get $x \bigvee^d T \in B$, which is a contradiction. So

$$\top_A = h(x)(t_0) \leq \bigvee_{t\in T}^d h(x)(t) \quad \text{and} \quad h(x)\left(\bigvee^d T\right) = \top_A = \bigvee_{t\in T}^d h(x)(t).$$

Hence $h(x) \in A^{(S)}$ and h is well-defined. Now, we prove that h is an S-dcpo map. First, we show that h preserves the order. To see this, let $x, x' \in C$ where $x \leq x'$. So for every $s \in S$, we get $xs \leq x's$. If $x's \in B$, then, $xs \in B$. Since g preserves the order, $g(xs) \leq g(x's)$ and so $h(x)(s) = g(xs)(e) \leq g(x's)(e) = h(x')(s)$. Otherwise, if $x's \notin B$ then, two cases may occur.

Case (i): If $xs \in B$, then $h(x)(s) = g(xs)(e) \leq \top_A = h(x')(s)$.

Case (ii): If $xs \notin B$, then $h(x)(s) = \top_A = h(x')(s)$.

Therefore, $h(x) \leq h(x')$ and h preserves the order. Now, to show that h is continuous, let $D \subseteq^d C$. Then, since h preserves the order, we have $h(D) \subseteq^d A^{(S)}$. Now, two cases may occur.

Case (i): $D \subseteq B$. In this case, $\bigvee^d D \in B$. This is because B is Scott closed in C. So, for every $s \in S$, $(\bigvee^d D)s \in B$. On the other hand,

$$ds \leq \bigvee_{d\in D}^d ds = \left(\bigvee^d D\right) s \in B,$$

for every $d \in D$. Since B is Scott closed in C, we get $ds \in B$, for every $d \in D$. Thus, for every $s \in S$, we have:

$$h\left(\bigvee^d D\right)(s) = g\left(\left(\bigvee^d D\right) s\right)(e) = g\left(\bigvee_{d\in D}^d ds\right)(e) = \bigvee_{d\in D}^d g(ds)(e) = \bigvee_{d\in D}^d h(d)(s).$$

So

$$h\left(\bigvee^d D\right) = \bigvee^d_{d\in D} h(d).$$

Case (ii): $D \not\subseteq B$. In this case, $\bigvee^d D \notin B$. This is because, on the contrary, if $\bigvee^d D \in B$, then, since B is Scott closed in C and for every $d \in D$, $d \leq \bigvee^d D \in B$ we get $d \in B$, for every $d \in D$. It contradicts $D \not\subseteq B$. Now, two cases may occur. If $(\bigvee^d D)s \in B$ then, for every $d \in D$ we get $ds \in B$. This is because, for every $d \in D$, $ds \leq (\bigvee^d D)s \in B$ and B is Scott closed in C. So

$$h\left(\bigvee^d D\right)(s) = g\left(\left(\bigvee^d D\right) s\right)(e) = g\left(\bigvee^d_{d\in D} ds\right)(e) = \bigvee^d_{d\in D} g(ds)(e) = \bigvee^d_{d\in D} h(d)(s).$$

Otherwise, if $(\bigvee^d D)s \notin B$ then, there exists $d_0 \in D$ such that $d_0 s \notin B$. The latter is because if, on the contrary, for every $d \in D$, $ds \in B$ then, since B is Scott closed in C, we have

$$\bigvee^d_{d\in D} ds = \left(\bigvee^d D\right) s \in B,$$

which is a contradiction. So

$$\top_A = h(d_0)(s) \leq \bigvee^d_{d\in D} h(d)(s) \quad \text{and} \quad h\left(\bigvee^d D\right)(s) = \top_A = \bigvee^d_{d\in D} h(d)(s).$$

Now, we show that h preserves the action. To see this, let $x \in C$ and $t \in S$. We prove that for every $s \in S$, $h(xt)(s) = (h(x)t)(s)$. Let $s \in S$. Then, two cases may occur. If $(xt)s = x(ts) \in B$ then, since g preserves the action, we get $h(xt)(s) = g((xt)s)(e) = g(x(ts))(e) = h(x)(ts) = (h(x)t)(s)$. Otherwise, if $(xt)s = x(ts) \notin B$, then $h(xt)(s) = \top_A$. On the other hand, $(h(x)t)(s) = h(x)(ts) = \top_A$. Hence $h(xt)(s) = (h(x)t)(s)$, for every $s \in S$ and h preserves the action.

Finally, $h|_B = g$. To prove this, let $x \in B$. Then, for every $s \in S$ we have $h(x)(s) = g(xs)(e) = (g(x)s)(e) = g(x)(s)$, as required. ■

In the following, we consider the behaviour of *sc*-injective S-dcpo's with products and coproducts. First, we recall the following remark from [12].

Remark 2.5. Let $\{A_i\}_{i\in I}$ be a family of S-dcpo's. Then:

(i) The product of A_i's in the category **Dcpo**-S, is their cartesian product $\prod_{i\in I} A_i$ with componentwise action and order.

(ii) The coproduct of A_i's in the category **Dcpo**-S, is their disjoint union $\dot{\bigcup}_{i\in I} A_i$ with the order given by $x \leq y$ in $\dot{\bigcup}_{i\in I} A_i$ if and only if $x, y \in A_i$ and $x \leq y$ in A_i, for some $i \in I$; and with the action as A_i, for $a \in A_i$ and $s \in S$.

Theorem 2.6. *Let $\{A_i : i \in I\}$ be a family of S-dcpo's. Then, the product $\prod_{i\in I} A_i$ is sc-injective if and only if each A_i is sc-injective.*

Proof. If each A_i is sc-injective then, by the universal property of products, it is clear that $\prod_{i\in I} A_i$ is sc-injective. For the converse, let $\prod_{i\in I} A_i$ be sc-injective, and $j \in I$. Then A_j is sc-injective. To see this, consider the diagram where f is an sc-embedding and g is an S-dcpo map. Define $\bar{g} : B \to \prod_{i\in I} A_i$ where

$$\bar{g}(b)(i) = \begin{cases} g(b) & \text{if} \quad i = j, \\ \theta_i & \text{if} \quad i \neq j, \end{cases}$$

where for $i \in I$, θ_i is a zero element of A_i which exists since $\prod_{i\in I} A_i$ has a zero top element by Remark 2.3, and the i-th component of that zero element is a zero element of A_i, $i \in I$. We show that $\bar{g}$ is an S-dcpo map. Since g preserves the order, it is clear that $\bar{g}$ preserves the order. Now, we prove that $\bar{g}$ is continuous. To see this, let $D \subseteq^d B$. Then we have $\bar{g}(D) \subseteq^d \prod_{i\in I} A_i$, since $\bar{g}$ preserves the order. Thus we get:

$$\bar{g}\left(\bigvee^d D\right)(i) = \begin{cases} g\left(\bigvee^d D\right) = \bigvee^d_{d\in D} g(d) & \text{if} \quad i = j, \\ \theta_i & \text{if} \quad i \neq j, \end{cases}$$

and

$$\bar{g}(d)(i) = \begin{cases} g(d) & \text{if} \quad i = j, \\ \theta_i & \text{if} \quad i \neq j, \end{cases}$$

for every $d \in D$. So, for all $i \in I$,

$$\bar{g}\left(\bigvee^d D\right)(i) = \bigvee^d_{d\in D} (\bar{g}(d)(i)) = \left(\bigvee^d_{d\in D} \bar{g}(d)\right)(i).$$

Therefore,

$$\bar{g}\left(\bigvee^d D\right) = \bigvee^d_{d\in D} \bar{g}(d).$$

Finally, $\bar{g}$ preserves the action. This is because, for every $s \in S$ we have:

$$\bar{g}(bs)(i) = \begin{cases} g(bs) = g(b)s & \text{if} \quad i = j, \\ \theta_i & \text{if} \quad i \neq j. \end{cases}$$

Also

$$(\bar{g}(b)(i))s = \begin{cases} g(b)s & \text{if} \quad i = j, \\ \theta_i s = \theta_i & \text{if} \quad i \neq j, \end{cases}$$

and hence, for all $i \in I$,

$$\bar{g}(bs)(i) = (\bar{g}(b)(i))s = (\bar{g}(b)s)(i) \quad \text{and} \quad \bar{g}(bs) = \bar{g}(b)s.$$

Now, since $\prod_{i\in I} A_i$ is sc-injective, there exists an S-dcpo map $\bar{h} : C \to \prod_{i\in I} A_i$, where $\bar{h}f = \bar{g}$. It is clear that $\pi_j\bar{h} : C \to A_j$ extends g, where $\pi_j : \prod_{i\in I} A_i \to A_j$ is the j-th projection map. So A_j is sc-injective.

■

Theorem 2.7. *Let $\{A_i : i \in I, |I| > 1\}$ be an arbitrary family of S-dcpo's. Then, $\coprod_{i\in I} A_i$ is not sc-injective.*

Proof. By Remark 2.5, $\coprod_{i\in I} A_i$ is not bounded and so by Lemma 2.3, it is not sc-injective. ■

3. Scott closed injectivity versus absolute retract in Dcpo-S

In this section, we investigate the relation between sc-injectivity and sc-absolute retractness in the category **Dcpo**-S and its full subcategories **R**-**Dcpo**-S and **SR**-**Dcpo**-S of reversible and strongly reversible S-dcpo's, respectively.

First, we mention that, similar to Lemma 2.3, we have:

Lemma 3.1. *If A is an sc-absolute retract in* **Dcpo**-*S, then it has a zero top element.*

Theorem 3.2. *Let S be a dcpo-monoid with any one of the following properties:*

(1) $\forall s \in S$, $\exists t \in S$, $e \leq st$.

(2) $\forall s \in S$, $e \leq s^2$.

(3) *S is a dcpo-group.*

(4) $\perp_S = e$.

(5) *S is a right zero semigroup with an adjoined identity and has a top element.*

Then, for object A in **Dcpo**-*S, the following statements are equivalent:*

(i) *A has a zero top element.*

(ii) *A is sc-injective.*

(iii) *A is sc-absolute retract.*

Proof. (i)$\Rightarrow$(ii) (1) Let S be a dcpo-monoid with the property that for every $s \in S$, there exists $t \in S$ such that $e \leq st$ and A be an S-dcpo with zero top element $\top$. Then, we prove that A is an sc-injective S-dcpo. To see this, let $f : B \to C$ be an sc-embedding in **Dcpo**-S and $g : B \to A$ be an S-dcpo map. Then, define $h : C \to A$ as:

$$h(x) = \begin{cases} g(x) & \text{if } x \in B, \\ \top & \text{otherwise.} \end{cases}$$

We prove that h is an S-dcpo map. First, we show that h preserves the order. To see this, take $x, x' \in C$ where $x \leq x'$. Two cases may occur. If $x' \in B$ then, since B is Scott closed in C, we get $x \in B$ and so $h(x) = g(x) \leq g(x') = h(x')$. Otherwise, if $x' \notin B$, then we have $h(x) \leq h(x') = \top$. Now, we show that h is continuous. Let $D \subseteq^d B$. Then, since h preserves the order, we get $h(D) \subseteq^d A$. Two cases may occur:

Case (i): $D \subseteq B$.
In this case, since B is Scott closed in C, we have $\bigvee^d D \in B$. So,

$$h\left(\bigvee^d D\right) = g\left(\bigvee^d D\right) = \bigvee^d_{d\in D} g(d) = \bigvee^d_{d\in D} h(d).$$

Case (ii): $D \nsubseteq B$.
In this case $\bigvee^d D \notin B$. This is because on the contrary, if $\bigvee^d D \in B$ then, since B is Scott closed in C we get $d \in B$, for all $d \in D$. It contradicts $D \nsubseteq B$. By the assumption, there exists $d_0 \in D$ where $d_0 \notin B$. So,

$$\top = h(d_0) \leq \bigvee^d_{d\in D} h(d) \quad \text{and} \quad h\left(\bigvee^d D\right) = \top = \bigvee^d_{d\in D} h(d).$$

Finally, h preserves the action. To prove this, let $x \in C$ and $s \in S$. Then, two cases may occur. If $xs \in B$ then, since for $s \in S$, there exists $t \in S$ such that $x \leq xst \in B$ and B is Scott closed in C, we get $x \in B$. So, $h(xs) = g(xs) = g(x)s = h(x)s$. Otherwise, if $xs \notin B$ then, $x \notin B$. The latter is because, on the contrary, if $x \in B$ then, since B is an S-dcpo, we have $xs \in B$ which is a contradiction. Therefore, $h(xs) = \top = \top s = h(x)s$.

Notice that if S is a dcpo-monoid with one of the properties (2)-(3) or (4), then, S is a dcpo-monoid with the property (1). So we get the result.

(5) Let S be a right zero semigroup with an adjoined identity and has a top element. Also, let A be an S-dcpo with zero top element $\top$, $f : B \to C$ be an sc-embedding in **Dcpo**-S and $g : B \to A$ be an S-dcpo map. Then, h is defined similar to the proof of case (1). The only part which needs to be changed, is showing that h preserves the action. Let $x \in C$ and $s \in S$. Then, if $xs \in B$ we get $x\top_S = (xs)\top_S \in B$. On the other hand, $x = xe \leq x\top_S$. Therefore,

$x \in B$, since B is Scott closed in C. Hence, $h(xs) = g(xs) = g(x)s = h(x)s$. Otherwise, if $xs \notin B$ then, $x \notin B$. The latter is because, on the contrary, if $x \in B$ then, since B is an S-dcpo, we have $xs \in B$, which is a contradiction. Therefore, $h(xs) = \top = \top s = h(x)s$.

(ii)⇒(iii) It is clear.

(iii)⇒(i) By Lemma 3.1, we get the result. ∎

Corollary 3.3. *If S is a dcpo-monoid with any one of the properties mentioned in Theorem 3.2, then* **Dcpo**-S *has enough injective objects with respect to sc-embeddings.*

Proof. Let S be a dcpo-monoid with any one of the properties mentioned in Theorem 3.2 and A be an S-dcpo. Then, consider the sc-embedding $i : A \hookrightarrow A \oplus \top$ in **Dcpo**-S, where $\top$ is a zero element. Now by Theorem 3.2, we get the result. ∎

Definition 3.4. An S-dcpo A is called *reversible* if for every $a \in A$ and $s \in S$, there exists $t \in S$ such that $ast = a$.

So, we have the category **R-Dcpo**-S of all reversible S-dcpo's and S-dcpo maps between them.

Theorem 3.5. *For object A in* **R-Dcpo**-S*, the following conditions are equivalent:*

(i) *A has a zero top element.*

(ii) *A is sc-injective.*

(iii) *A is sc-absolute retract.*

Proof. (i)⇒ (ii) Suppose that B and C are reversible S-dcpo's, A is a reversible S-dcpo with zero top element $\top$, $f : B \to C$ is an sc-embedding in **R-Dcpo**-S and $g : B \to A$ is an S-dcpo map. Define $h : C \to A$ where:

$$h(x) = \begin{cases} g(x) & \text{if } x \in B, \\ \top & \text{otherwise.} \end{cases}$$

We show that h is an S-dcpo map. The proof of the fact that h is continuous, is similar to the proof of Theorem 3.2. The only part which needs to be changed, is that h preserves the action. To show this, let $x \in C$ and $s \in S$. Then, two cases may occur. If $xs \in B$ then, since B is a reversible S-dcpo, there exists $t \in S$ such that $x = xst \in B$. Therefore, $h(xs) = g(xs) = g(x)s = h(x)s$. Otherwise, if $xs \notin B$ then, $x \notin B$. The latter is because, on the contrary, if $x \in B$ then, since B is an S-dcpo, we have $xs \in B$ which is a contradiction. Therefore, $h(xs) = \top = \top s = h(x)s$.

(ii)⇒(iii) is clear, and the proof of (iii)⇒(i) is similar to the proof of Lemma 3.1. ∎

Now, similar to Corollary 3.3, we have:

Corollary 3.6. **R-Dcpo**-S *has enough injective objects with respect to sc-embeddings.*

Definition 3.7. An S-dcpo A is called *strongly reversible* if for every $a \in A$ and $s \in S$, we have $as^2 = a$.

So, we have the category **SR-Dcpo**-S of all strongly reversible S-dcpo's and S-dcpo maps between them.

Theorem 3.8. *For object A in* **SR-Dcpo**-S*, the following conditions are equivalent:*

(i) *A has a zero top element.*

(ii) *A is sc-injective.*

(iii) *A is sc-absolute retract.*

Proof. (i)⇒(ii) Suppose that B and C are strongly reversible S-dcpo's, A is a strongly reversible S-dcpo with zero top element $\top$, $f : B \to C$ is an sc-embedding in **SR-Dcpo**-S and $g : B \to A$ is an S-dcpo map. Define $h : C \to A$ as:

$$h(x) = \begin{cases} g(x) & \text{if } x \in B, \\ \top & \text{otherwise.} \end{cases}$$

We show that h is an S-dcpo map. The proof of the fact that h is continuous, is similar to the proof of Theorem 3.2. The only part which needs to be changed, is to show that h preserves the action. To see this, let $x \in C$ and $s \in S$. Then, two cases may occur. If $xs \in B$ then, since B is a strongly reversible S-dcpo, $x = xs^2 \in B$. Therefore, $h(xs) = g(xs) = g(x)s = h(x)s$. Otherwise, if $xs \notin B$ then, $x \notin B$. The latter is because, on the contrary, if $x \in B$ then, since B is an S-dcpo, we have $xs \in B$, which is a contradiction. Therefore, $h(xs) = \top = \top s = h(x)s$.

(ii)⇒(iii) is clear, and the proof of (iii)⇒(i) is similar to the proof of Lemma 3.1. ■

Corollary 3.9. **SR-Dcpo**-S *has enough injective objects with respect to sc-embeddings.*

Let $\mathcal{C}'$ be the full subcategory of **Dcpo**-S whose objects are S-dcpo's A with the property that for every $a \in A$ and $s \in S$, $a \leq as$. In the following we see a similar result to the categories of reversible and strongly reversible S-dcpo's for $\mathcal{C}'$.

Theorem 3.10. *For any object A in $\mathcal{C}'$, the following conditions are equivalent:*

(i) *A has a zero top element.*

(ii) *A is sc-injective.*

(iii) *A is sc-absolute retract.*

Proof. (i)$\Rightarrow$(ii) Suppose that B and C are objects of $\mathcal{C}'$, A is an object of $\mathcal{C}'$ with the zero top element $\top$, $f : B \to C$ is an *sc*-embedding in $\mathcal{C}'$ and $g : B \to A$ is an S-dcpo map. Define $h : C \to A$ as:

$$h(x) = \begin{cases} g(x) & \text{if } x \in B, \\ \top & \text{otherwise.} \end{cases}$$

We show that h is an S-dcpo map. The proof of the fact that h is continuous, is similar to the proof of Theorem 3.2. The only part which needs to be changed, is showing that h preserves the action. To see this, let $x \in C$ and $s \in S$. Then, two cases may occur. If $xs \in B$ then, since for every $s \in S$, $x \leq xs$ and B is Scott closed in C, we get $x \in B$ and $h(xs) = g(xs) = g(x)s = h(x)s$. Otherwise, if $xs \notin B$ then, $x \notin B$. The latter is because, on the contrary, if $x \in B$ then, since B is an S-dcpo, we have $xs \in B$, which is a contradiction. Therefore, $h(xs) = \top = \top s = h(x)s$.

(ii)$\Rightarrow$(iii) is clear, and the proof of (iii)$\Rightarrow$(i) is similar to the proof of Lemma 3.1. ∎

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A MONGE-AMPÈRE TYPE OPERATOR IN 2-DIMENSIONAL SPECIAL LAGRANGIAN GEOMETRY

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Abstract. In this paper, we construct a Monge-Ampère type operator in 2-dimensional special Lagrangian geometry based on the calibrated geometry developed by Harvey and Lawson. We give a special Lagrangian version of the Chern-Levine-Nirenberg estimate for complex Monge-Ampère operator, which enables us to define the Monge-Ampère type operator on continuous ϕ-plurisubharmonic functions on a domain in $\mathbb{C}^2$.

Keywords: Monge-Ampère type operator, special Lagrangian n-plane, ϕ-plurisubharmonic, Radon transform.

MR Subject Classification: 53C38, 32U05, 44A12.

1. Introduction

In [8]-[11], Harvey and Lawson introduce several concrete calibrations, and study operators, which define the notion of plurisubharmonic functions in calibrated geometry. These functions generalize the classical plurisubharmonic functions from complex geometry and enjoy their important properties. Based on that fact, we present a operator in 2-dimensional special Lagrangian geometry which has similar properties as complex Monge-Ampère operator.

Recall the definitions in [8], [10]. A *calibration* ϕ of degree p is a closed p-form on a Riemannian manifold X with the property that $\phi(\xi) \leq 1$ for all unit simple tangent p-vectors ξ on X. A unit simple tangent p-vectors ξ on X satisfying

$$\phi(\xi) = 1 \tag{1}$$

is called ϕ-plane. We denote by $G(\phi)$ the set of all ϕ-planes on X. If the covariant derivative of a calibration is zero, then it is called parallel. In [9], Harvey and Lawson give the definition of ϕ-plurisubharmonic function for general calibrations.

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Here we only need the definition for parallel calibrations. Let's recall it. For any smooth function f on X, the d^{ϕ}-operator is defined as

$$d^{\phi} f := \nabla f \rfloor \phi,$$

where $\rfloor$ is the interior product of a differential form and ∇f is the gradient of f on X. For a parallel calibration ϕ, a function $f \in C^{\infty}(X)$ is called *ϕ-plurisubharmonic* if

$$dd^{\phi} f(\xi) \ \geq \ 0, \quad \text{for each } \xi \in G(\phi).$$

Let $D'(X)$ be the dual of the space of smooth functions on X. A distribution $f \in D'(X)$ is called *ϕ-plurisubharmonic* if

$$dd^{\phi} f(\xi)(\lambda) \ \geq 0$$

for every smooth section $\xi \in G(\phi)$ and every smooth compactly supported non-negative multiple λ of the volume form on X. It is easy to see this definition is compatible with the definition of $f \in C^{\infty}(X) \subset D'(X)$. Denote $PSH(X, \phi)$ both the smooth functions and the distributions which are ϕ-plurisubharmonic on X. In n-dimensional complex Euclidean space $\mathbb{C}^n$ with coordinates $z = (z_1, z_2, ..., z_n)$, the closed n-form $\phi = RedZ = Re(dz_1 \wedge dz_2 \wedge ... \wedge dz_n)$ is a parallel calibration, called *special Lagrangian calibration.* Consider the grassmannian $G(n, 2n)$ of oriented real n-planes in $\mathbb{C}^n$. $\xi \in G(n, 2n)$ is called *Lagrangian* if $Ju \perp \xi$ for all $u \in \xi$, where J is the complex structure on $\mathbb{R}^{2n}$. Let SU_n be the special unitary group. If Lagrangian n-plane ξ satisfies

$$\xi = A\xi_0,$$

where $A \in SU_n$ and $\xi_0 \equiv span_{\mathbb{R}}\{e_1, e_3, ..., e_{2n-1}\} \cong \mathbb{R}^n$, then ξ is called *special Lagrangian.* Here $e_1, ..., e_{2n}$ are orthonormal basis for $\mathbb{R}^{2n}$, e_{2j-1} is a vector with the jth position 1 and others 0, e_{2j} is a vector with the $(n+j)$th position 1 and others 0. We know that a unit simple tangent vector ξ is a ϕ-plane with $\phi = RedZ$ if and only if ξ is special Lagrangian, see Proposition 2.2, or Theorem 1.10 in Section 3 of [8].

For the calibration $\phi = Redz_1 \wedge dz_2$ on special Lagrangian geometry $\mathbb{C}^2$, let f_1, f_2 be two smooth ϕ-plurisubharmonic functions on $\mathbb{C}^2$. We define the Monge-Ampère type operator on f_1 and f_2 as

$$dd^{\phi} f_1 \wedge dd^{\phi} f_2.$$

That operator has similar properties as complex Monge-Ampère operator. Especially, it has an estimate, which is similar to the Chern-Levine-Nirenberg estimate for complex Monge-Ampère operator established in [3]-[5], [14] etc. Now we give that estimate for $dd^{\phi} f_1 \wedge dd^{\phi} f_2$ on special Lagrangian geometry $\mathbb{C}^2$.

Theorem 1.1. *Let Ω be an open neighborhood of a compact set $K \subseteq \mathbb{C}^2$ and U be a compact neighborhood of $K \subset \Omega$. For any $\psi \in C_0^\infty(\Omega)$, there exist a constant $C > 0$, which depends on U, Ω and $\|\psi\|_{C^2}$, such that for smooth functions $f_1, f_2 \in PSH(\Omega, \phi)$ with $\phi = Re(dz_1 \wedge dz_2)$, the following estimate holds,*

$$\left| \int_K \psi dd^\phi f_1 \wedge dd^\phi f_2 \right| \leq C \|f_1\|_K \|f_2\|_U. \tag{2}$$

Based on that estimate, we can define the Monge-Ampère type operator on continuous ϕ-plurisubharmonic functions.

Theorem 1.2. *Let f_1, f_2 be continuous ϕ-plurisubharmonic functions on a domain $\Omega \subseteq \mathbb{C}^2$. Let $f_{1,N}, f_{2,N}$ be two sequences of twice continuously differentiable ϕ-plurisubharmonic functions converging to f_1 and f_2 uniformly on compact subsets of Ω respectively. Then $dd^\phi f_{1,N} \wedge dd^\phi f_{2,N}$ weakly converges to a distribution on Ω. This distribution depends only on f_1 and f_2, not on the choice of approximating sequences $f_{1,N}$ and $f_{2,N}$.*

We denote by $dd^\phi f_1 \wedge dd^\phi f_2$ the limit in Theorem 1.2. So the Monge-Ampère type operator is well defined on continuous ϕ-plurisubharmonic functions f_1 and f_2, though the currents $dd^\phi f_1$ and $dd^\phi f_2$ can't do exterior product on the form.

The complex Monge-Ampère operator is a positive distribution, so it is a measure. But we only know that the special Lagrangian version Monge-Ampère operator is a distribution. Hence, perhaps it doesn't enjoy some deeper results on complex Monge-Ampère operator. For example, Alesker [1], [2] deals with boundary value problem for Monge-Ampère equation. In [1], he first proves that Monge-Ampère operator is well-defined as a measure, and then he proves the uniqueness for the boundary value problem. In addition, in [2], he also proves the existence of the solution. So, next step, we need to investigate the special Lagrangian version Monge-Ampère operator from some other points of view.

The proof of Theorem 1.2 roughly follows the lines of the classical proof. In the process of proving, we need a fact that the linear combinations of delta-functions of special Lagrangian n-planes in $\mathbb{C}^n$ are dense in the space of all generalized functions, which can be induced by the fact that Radon transform over special Lagrangian n-planes is injective, see Proposition 3.1. Hence, we show the Radon transform over the special Lagrangian n-planes in section **??**.

2. Monge-Ampère type operator on $\mathbb{C}^2$

In this section, we give the representation of $dd^\phi f$ for special Lagrangian calibration ϕ and a symmetrical property on the operator $dd^\phi f_1 \wedge dd^\phi f_2$ in $\mathbb{C}^2$, which enables us to get the estimate in Theorem 1.1.

Consider the special Lagrangian calibration $\phi = \; RedZ$ on $\mathbb{C}^n$. Let Z_{ij} be the form obtained from $dz_1 \wedge dz_2 \wedge ... \wedge dz_n$ by replacing dz_i with $d\bar{z}_j$ (in the ith position). For a smooth function f, we have the following proposition.

Proposition 2.1.

$$dd^{\phi}f = 2Re\left\{\sum_{k,j=1}^{n}\frac{\partial^2 f}{\partial\bar{z}_k\partial\bar{z}_j}Z_{kj}\right\} + \frac{1}{2}(\Delta f)Re(dZ). \tag{3}$$

Proof. Given a smooth ϕ-plurisubharmonic function f. Since

$$d^{\phi}f=\nabla f\rfloor\phi=\sum_{k=1}^{n}(-1)^{k-1}\left(\frac{\partial f}{\partial\bar{z}_k}dz_1\wedge\ldots\wedge\widehat{dz_k}\wedge\ldots\wedge dz_n+\frac{\partial f}{\partial z_k}d\bar{z}_1\wedge\ldots\wedge\widehat{d\bar{z}_k}\wedge\ldots\wedge d\bar{z}_n\right),$$

where $dz_1\wedge\ldots\wedge\widehat{dz_k}\wedge\ldots\wedge dz_n$ denotes the form obtained from $dz_1\wedge dz_2\wedge\ldots\wedge dz_n$ by removing dz_k. We have,

$$\begin{aligned} dd^{\phi}f = d(\nabla f\rfloor\phi) &= \sum_{k=1}^{n}\frac{\partial^2 f}{\partial\bar{z}_k\partial z_k}dz_1\wedge\ldots dz_n + \sum_{k,j=1}^{n}\frac{\partial^2 f}{\partial\bar{z}_k\partial\bar{z}_j}Z_{kj} \\ &+ \frac{\partial^2 f}{\partial z_k\partial z_j}\overline{Z}_{kj} + \frac{\partial^2 f}{\partial\bar{z}_k\partial z_k}d\bar{z}_1\wedge\ldots d\bar{z}_n \\ &= 2Re\left\{\sum_{k,j=1}^{n}\frac{\partial^2 f}{\partial\bar{z}_k\partial\bar{z}_j}Z_{kj}\right\} + \frac{1}{2}(\Delta f)Re(dZ). \end{aligned}$$

Particularly, for $n=2$, we have

$$\begin{aligned} dd^{\phi}f &= \frac{1}{2}\left(\frac{\partial^2 f}{\partial x_1^2}-\frac{\partial^2 f}{\partial y_1^2}\right)(dx_1\wedge dx_2+dy_1\wedge dy_2) \\ &-\frac{\partial^2 f}{\partial x_1\partial y_1}(dx_1\wedge dy_2-dy_1\wedge dx_2) \\ &+\frac{1}{2}\left(\frac{\partial^2 f}{\partial x_2^2}-\frac{\partial^2 f}{\partial y_2^2}\right)(dx_1\wedge dx_2+dy_1\wedge dy_2) \\ &+\frac{\partial^2 f}{\partial x_2\partial y_2}(dx_1\wedge dy_2-dy_1\wedge dx_2) \\ &-\left(\frac{\partial^2 f}{\partial x_1\partial y_2}+\frac{\partial^2 f}{\partial x_2\partial y_1}\right)dx_2\wedge dy_2+\left(\frac{\partial^2 f}{\partial x_2\partial y_1}+\frac{\partial^2 f}{\partial y_2\partial x_1}\right)dx_1\wedge dy_1 \\ &+\frac{1}{2}\sum_{k=1}^{2}\left(\frac{\partial^2 f}{\partial x_k^2}+\frac{\partial^2 f}{\partial y_k^2}\right)(dx_1\wedge dx_2-dy_1\wedge dy_2) \\ &=\left(\frac{\partial^2 f}{\partial x_1^2}+\frac{\partial^2 f_2}{\partial x_2^2}\right)dx_1\wedge dx_2-\left(\frac{\partial^2 f}{\partial y_1^2}+\frac{\partial^2 f}{\partial y_2^2}\right)dy_1\wedge dy_2 \\ &+\left(\frac{\partial^2 f}{\partial x_2\partial y_2}-\frac{\partial^2 f}{\partial x_1\partial y_1}\right)(dx_1\wedge dy_2-dy_1\wedge dx_2) \\ &+\left(\frac{\partial^2 f}{\partial x_1\partial y_2}+\frac{\partial^2 f}{\partial x_2\partial y_1}\right)(dx_1\wedge dy_1-dx_2\wedge dy_2). \end{aligned} \tag{4}$$

∎

Proposition 2.2. *For the calibration $\phi = RedZ$ in $\mathbb{C}^n$, with $dZ = dz_1 \wedge dz_2 \wedge ... \wedge dz_n$, the unit simple n-plane ξ is a ϕ-plane if and only if ξ is special Lagrangian.*

Before proving the proposition, we need two lemmas. The proofs of those lemmas have appeared in [8], so we cite them without proofs.

Lemma 2.1. (Theorem 1.7 in [8]) *For any $\xi \in G(n, 2n)$,*

$$\mid dZ(\xi) \mid^2 = \left(RedZ(\xi)\right)^2 + \left(ImdZ(\xi)\right)^2 = \mid \xi \wedge J\xi \mid.$$

Lemma 2.2. (Lemma 1.9 in [8]) $|\xi \wedge J\xi| \leq |\xi|^2$, *for any $\xi \in G(n, 2n)$, with equality if and only if ξ is Lagrangian.*

Proof of Proposition 2.2. Let ξ be a ϕ-plane. Then $|\xi| = 1$ and

$$\phi(\xi) = RedZ(\xi) = 1. \tag{5}$$

Denote $\varepsilon_1, ..., \varepsilon_n$ an oriented orthonormal basis of ξ. Then by Lemma 2.1 and Lemma 2.2, $\left(RedZ(\xi)\right)^2 + \left(ImdZ(\xi)\right)^2 = |\xi \wedge J\xi| \leq |\xi|^2 = 1$. So $ImdZ(\xi) = 0$ and $|\xi \wedge J\xi| = |\xi|^2$ by (5). Hence, ξ is special Lagrangian.

For the inverse, let ξ be a unit simple vector, i.e., $|\xi| = 1$. Suppose ξ is special Lagrangian and $\varepsilon_1, ..., \varepsilon_n$ is an oriented basis for $\xi \in G(n, 2n)$. Denote $e_1, ..., e_n, Je_1, ..., Je_n$ the standard basis for $\mathbb{R}^n \oplus \mathbb{R}^n = \mathbb{C}^n$, and A the linear map sending e_j to ε_j and Je_j to $J\varepsilon_j$. Then $det_{\mathbb{C}} A = 1$ and so $ImdZ(\xi) = Im(det_{\mathbb{C}} A) = 0$. Then, by Lemma 2.1 and Lemma 2.2 again,

$$\left(RedZ(\xi)\right)^2 = \left(RedZ(\xi)\right)^2 + \left(ImdZ(\xi)\right)^2 = |\xi \wedge J\xi| = 1.$$

So ξ is a ϕ-plane. ∎

Let $z = (z_1, z_2)$ be the coordinates of $\mathbb{C}^2$, where $z = x + iy$ with $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Now we give a proposition of $dd^\phi f_1 \wedge dd^\phi f_2$ in $\mathbb{C}^2$. We follow the approach to quaternionic Monge-Ampère operator of S. Alesker in [1].

Proposition 2.3. *Let f_0, f_1, f_2 be real valued compactly supported smooth functions on $\mathbb{C}^2$. Then the 3-linear functional*

$$L(f_0, f_1, f_2) = \int_{\mathbb{C}^2} f_0 \; dd^\phi f_1 \wedge dd^\phi f_2, \tag{6}$$

is symmetric with respect to f_0, f_1, f_2.

Proof. Since $dd^\phi f_1$ is a 2-form, we have $dd^\phi f_1 \wedge dd^\phi f_2 = dd^\phi f_2 \wedge dd^\phi f_1$, i.e., L is symmetric with respect to f_1 and f_2. Thus it is sufficient to check that

$$L(f_0, f_1, f_2) = L(f_1, f_0, f_2), \tag{7}$$

for any smooth compactly supported functions f_0, f_1, f_2. Both sides of (7) make sense if f_0 is a generalized function. Since linear combinations of delta-functions of points δ_z are dense in the space of all the generalized functions, it is sufficient to prove (7) for $f_0 = \delta_0$, namely

$$dd^\phi f_1 \wedge dd^\phi f_2|_{z=0} = \int_{\mathbb{C}^2} f_1 dd^\phi \delta_0 \wedge dd^\phi f_2. \tag{8}$$

Clearly the right hand side of (8) depends only on derivatives at 0 of f_1, f_2, up to order 2. Consider the terms of the Taylor series of f_1 at 0:

$$\begin{aligned} f_1(z) &= f_1(0) + \sum_{i=1}^{2} f'_{1x_i}(0)x_i + f'_{1y_i}(0)y_i \\ &+ \frac{1}{2!}\sum_{i,j=1}^{2}\left(f''_{1x_ix_j}(0)x_ix_j + f''_{1y_iy_j}(0)y_iy_j + 2f''_{1x_iy_j}(0)x_iy_j\right) + O(|z|^3) \\ &= g(z) + h(z) + O(|z|^3), \end{aligned}$$

where g is a polynomial of degree one and h is a quadratic term. So it is sufficient to prove the following two statements:

Case 1. $f_0 = \delta_0$, $f_1 = h$ is a smooth compactly supported function which is a homogeneous polynomial of degree 2 in a neighborhood of 0.

Case 2. $f_0 = \delta_0$, $f_1 = g$ is a smooth compactly supported function which is a polynomial of degree 1 in a neighborhood of 0.

For *Case* 1. Write down $L(h, \delta_0, f_2) = \int_{\mathbb{C}^2} h(z) dd^\phi \delta_0 \wedge dd^\phi f_2$ as a polynomial in $\frac{\partial^2 f_2}{\partial x_i \partial x_j}, \frac{\partial^2 f_2}{\partial x_i \partial y_j}$ etc. and in $\frac{\partial^2 \delta_0}{\partial x_i \partial x_j}, \frac{\partial^2 \delta_0}{\partial x_i \partial y_j}$ etc. Then we see that the derivatives of δ_0 enter at each monomial only once because of linearity of L with respect to each arguments. For example, consider a monomial containing $\frac{\partial^2 \delta_0}{\partial x_i \partial x_j}$,

$$\begin{aligned} \int_{\mathbb{C}^2} h(z)\frac{\partial^2 \delta_0}{\partial x_i \partial x_j} \cdot \partial^2 f_2 &= \frac{\partial^2}{\partial x_i \partial x_j}(h(z)\cdot \partial^2 f_2)|_{z=0} = \frac{\partial^2 h(z)}{\partial x_i \partial x_j}(0)\cdot \partial^2 f_2(0) \\ &= \frac{\partial^2 h(z)}{\partial x_i \partial x_j}\cdot \partial^2 f_2|_{z=0}. \end{aligned}$$

The second identity holds since h and the first derivatives of h at 0 vanish. Thus in each monomial the term $h \cdot \frac{\partial^2 \delta_0}{\partial x_i \partial x_j}\partial^2 f_2$ is just replaced by $\frac{\partial^2 h(z)}{\partial x_i \partial x_j}(0)\partial^2 f_2(0)$. Hence the final expression is

$$\int_{\mathbb{C}^2} h(z) dd^\phi \delta_0 \wedge dd^\phi f_2 = dd^\phi h(z) \wedge dd^\phi f_2|_{z=0}.$$

This proves *Case* 1.

Before proving *Case* 2, we claim that for any smooth compactly supported function g which is equal to a polynomial of degree 1 inside a fixed neighborhood U of origin and a generalized function f_3 with support contained in U, we have

$$\int_{\mathbb{C}^n} g(z) dd^\phi f_3 \wedge dd^\phi f_2 = 0, \tag{9}$$

where f_2 is a smooth function. Since g is a polynomial of degree 1, we have $dd^\phi g=0$. Let f_3 be δ_0 in equation (9). By using that claim, we have $\int_{\mathbb{C}^2} gdd^\phi\delta_0\wedge dd^\phi f_2=0$. Hence,

$$0=dd^\phi g\wedge dd^\phi f_2|_{z=0}=\int_{\mathbb{C}^2} gdd^\phi\delta_0\wedge dd^\phi f_2.$$

This proves *Case* 2.

Let's prove the claim. By the following Proposition 3.1, we know that the linear combinations of delta-functions of special Lagrangian 2-planes are dense in the space of all generalized functions in $\mathbb{C}^2$. Hence it is sufficient to prove the claim for $f_3=\delta_{\xi_0}$, where ξ_0 is a certain special Lagrangian 2-plane through 0 satisfying $y_1=0, y_2=0$. Since δ_{ξ_0} is invariant with respect to translations in directions $x_1=0, x_2=0$, we have $\frac{\partial\delta_{\xi_0}}{\partial x_i}=0$, $\frac{\partial^2\delta_{\xi_0}}{\partial x_i\partial x_j}=0$, $\frac{\partial^2\delta_{\xi_0}}{\partial x_i\partial y_j}=0$ for $i,j=1,2$. By (4) in Proposition 2.1, we have $dd^\phi\delta_{\xi_0}=-\left(\frac{\partial^2\delta_{\xi_0}}{\partial y_1^2}+\frac{\partial^2\delta_{\xi_0}}{\partial y_2^2}\right)dy_1\wedge dy_2$. Hence, by Proposition 2.1, we have

$$\int_{\mathbb{C}^2} g(z)dd^\phi\delta_{\xi_0}\wedge dd^\phi f_2=\int_{\mathbb{C}^2} g(z)(\frac{\partial^2\delta_{\xi_0}}{\partial y_1^2}+\frac{\partial^2\delta_{\xi_0}}{\partial y_2^2})(\frac{\partial^2 f_2}{\partial x_1^2}+\frac{\partial^2 f_2}{\partial x_2^2})dV, \tag{10}$$

where $dV=dx_1\wedge dy_1\wedge dx_2\wedge dy_2$ is the volume element of $\mathbb{C}^2$. Without loss of generality, we can assume $g(z)$ is a polynomial of degree one on z_1.

Denote

$$A_i=\int_{\mathbb{C}^2} g(z_1)\frac{\partial^2\delta_{\xi_0}}{\partial y_1^2}\cdot\frac{\partial^2 f_2}{\partial x_i^2},\quad B_i=\int_{\mathbb{C}^2} g(z_1)\frac{\partial^2\delta_{\xi_0}}{\partial y_2^2}\cdot\frac{\partial^2 f_2}{\partial x_i^2}, i=1,2.$$

Let us consider monomials in the right side of (10). For $i=1,2$,

$$\begin{aligned}
A_i&=\int_{\mathbb{C}^2}\delta_{\xi_0}\frac{\partial^2}{\partial y_1^2}\left(g(z_1)\frac{\partial^2 f_2}{\partial x_i^2}\right)\\
&=\int_{\mathbb{C}^2}\delta_{\xi_0}\left(g(z_1)\frac{\partial^4 f_2}{\partial x_i^2\partial y_1^2}+2\frac{\partial g(z_1)}{\partial y_1}\cdot\frac{\partial^3 f_2}{\partial x_i^2\partial y_1}\right)\\
&=\int_{\mathbb{C}^2}\frac{\partial\delta_{\xi_0}}{\partial x_i}\left(g(z_1)\cdot\frac{\partial^3 f_2}{\partial x_i\partial y_1^2}\right)-\int_{\mathbb{C}^2}\delta_{\xi_0}\frac{\partial g(z_1)}{\partial x_i}\cdot\frac{\partial^3 f_2}{\partial x_i\partial y_1^2}+2C_1\int_{\mathbb{C}^2}\delta_{\xi_0}\cdot\frac{\partial^3 f_2}{\partial x_i^2\partial y_1}\\
&=0-\int_{\mathbb{C}^2}\frac{\partial\delta_{\xi_0}}{\partial x_i}\left(\frac{\partial g(z_1)}{\partial x_i}\cdot\frac{\partial^2 f_2}{\partial y_1^2}\right)+\int_{\mathbb{C}^2}\delta_{\xi_0}\left(\frac{\partial^2 g(z_1)}{\partial x_i^2}\cdot\frac{\partial^2 f_2}{\partial y_1^2}\right)+2C_1\int_{\mathbb{C}^2}\delta_{\xi_0}\frac{\partial^3 f_2}{\partial x_i^2\partial y_1}\\
&=2C_1\int_{\mathbb{C}^2}\frac{\partial\delta_{\xi_0}}{\partial x_i}\frac{\partial^2 f_2}{\partial x_i\partial y_1}=0.
\end{aligned}$$

Here $C_1=\frac{\partial g(z_1)}{\partial y_1}|_{\xi_0}$. The first identity is by the definition of generalized function δ_{ξ_0}. The second one holds since $\frac{\partial^2 g(z_1)}{\partial^2 y_1}=0$.

Similarly,

$$\begin{aligned}
B_i &= \int_{\mathbb{C}^2} \delta_{\xi_0} \frac{\partial^2}{\partial y_2^2}\left(g(z_1)\frac{\partial^2 f_2}{\partial x_i^2}\right) = \int_{\mathbb{C}^2} \delta_{\xi_0} g(z_1)\frac{\partial^4 f_2}{\partial y_2^2 \partial x_i^2} \\
&= \int_{\mathbb{C}^2} \frac{\partial \delta_{\xi_0}}{\partial x_i}\left(g(z_1)\cdot \frac{\partial^3 f_2}{\partial y_2^2 \partial x_i}\right) - \int_{\mathbb{C}^2} \delta_{\xi_0} \frac{\partial g(z_1)}{\partial x_i}\cdot \frac{\partial^3 f_2}{\partial y_2^2 \partial x_i} \\
&= 0 - \int_{\mathbb{C}^2} \frac{\partial \delta_{\xi_0}}{\partial x_i}\left(\frac{\partial g(z_1)}{\partial x_i}\cdot \frac{\partial^2 f_2}{\partial y_2^2}\right) + \int_{\mathbb{C}^2} \delta_{\xi_0}\left(\frac{\partial^2 g(z_1)}{\partial x_i^2}\cdot \frac{\partial^2 f_2}{\partial y_2^2}\right) = 0.
\end{aligned}$$

Hence $\int_{\mathbb{C}^2} g(z) dd^\phi \delta_{\xi_0} \wedge dd^\phi f_2 = A_1 + A_2 + B_1 + B_2 = 0.$ ∎

Before proving Theorem 1.1 and Theorem1.2, we need to show that a function f is ϕ-plurisubharmonic if and only if $dd^\phi f$ is a ϕ-positive current for special Lagrangian calibration ϕ.

Let $\Lambda_p T_x X$ be the vector space of p-vectors at x in a Riemannian manifold X. The corresponding bundle is denoted by $\Lambda_p TX$. Denote $\Lambda^p T^* X$ the dual of $\Lambda_p TX$. Recall [6] that a current T is *representable by integration* if T has measure coefficients when expressed as a generalized differential form. Equivalently, the mass norm $M_K(T)$ of T on each compact set K is finite. Associated with such a current T is a Radon measure $\|T\|$ and a generalized tangent space $\overrightarrow{T}_x \in \wedge_p T_x X$ defined for $\|T\|$ almost every point x. Recall that each $\overrightarrow{T}_x$ has mass norm one. For any p-form α with compact support, we have $T(\alpha) = \int \alpha(\overrightarrow{T}) d\|T\|$. Let $\wedge(\phi)$ be the span of $G(\phi) \subset \Lambda_p TX$ and $\wedge_+(\phi) \subset \wedge(\phi)$ be the convex cone on $G(\phi)$ with vertex the origin. Note that $\wedge_+(\phi)$ is just the cone on $\mathrm{ch}G(\phi)$. The following lemma is needed for a robust understanding of the definition of ϕ-positive current.

Lemma 2.3. (Lemma 5.4 in [9]) *The following conditions are equivalent:*

(1) $\overrightarrow{T} \in \wedge_+(\phi), \ \|T\| - a.e.,$

(2) $\overrightarrow{T} \in chG(\phi), \ \|T\| - a.e.,$

(3) $\phi(\overrightarrow{T}) = 1, \ \|T\| - a.e.$

A *ϕ-positive current* is a p-dimensional current T which is representable by integration and for which the equivalent conditions of Lemma 2.3 are satisfied.

Let $\wedge^+(\phi) \subset \Lambda^p T^* X$ be the polar cone of $\wedge_+(\phi) \subset \Lambda_p TX$. By definition, this is the set of $\alpha \in \Lambda^p T^* X$ such that $\alpha(\xi) \geq 0$ for all $\xi \in \wedge_+(\phi)$, or equivalently,

$$\wedge^+(\phi) := \{\alpha \in \Lambda^p T^* X : \alpha(\xi) \geq 0 \quad \text{for all} \quad \xi \in \wedge_+(\phi)\}.$$

A *$\wedge_+(\phi)$-positive current* is a p-dimension current T satisfying $T(\alpha) \geq 0$, for all p-forms $\alpha \in \wedge^+(\phi)$ with compact support.

Theorem 2.1. (Theorem 5.13 in [9]) *A current T is $\wedge_+(\phi)$-positive if and only if it is ϕ-positive.*

Proposition 5.19 in [9], the appendix: the reduced ϕ-Hessian, is on the relation of ϕ-plurisubharmonic function and ϕ-positive current. Here we only need the version for parallel calibration ϕ. We can rewrite it as follows.

Proposition 2.4. *For a parallel calibration ϕ in Riemannian manifold X, a function $f \in D'(X)$ is ϕ-plurisubharmonic if and only if $dd^\phi f$ is a ϕ-positive current.*

Now, based on Propositions 2.3 and 2.4, we can give the proof of Theorem 1.1.

Proof of Theorem 1.1. Given smooth 2-forms $\omega, \eta \in \Lambda^2 T^*\mathbb{C}^2$. Let $\langle \omega, \eta \rangle_{\Lambda_2}$ be the inner product of ω and η. Let $*$ be the Hodge $*$-operator, (cf. P155 in [15]). Define $\| * \overline{\omega} \|_\Omega := \sup_{z \in \Omega, \xi \in \wedge_+ \phi} | * \overline{\omega}(z)(\xi)|$. Note that there exists a constant C satisfying $C \geq \| * \overline{\omega} \|_\Omega$, such that $C\phi(\xi) - *\overline{\omega}(\xi) \geq 0$ for any ξ in the convex hull of $G(\phi)$, i.e., $C\phi - *\overline{\omega} \in \wedge^+(\phi)$. Since $*\phi = \phi$, we have

$$C\phi - *\overline{\omega} = *(C(*\phi) - \overline{\omega}) \in \wedge^+(\phi).$$

$dd^\phi f_2$ is a ϕ-positive current since f_2 is ϕ-plurisubharmonic by Proposition 2.4. Then we have

$$\begin{aligned}\int_\Omega dd^\phi f_2 \wedge (C(*\phi) - \omega) &= \int_\Omega dd^\phi f_2 \wedge *(\overline{*(C(*\phi) - \overline{\omega})}) \\ &= \langle\ dd^\phi f_2, \quad * (C(*\phi) - \overline{\omega})\ \rangle_{\Lambda_2} \\ &= dd^\phi f_2 \Big(* (C(*\phi) - \overline{\omega}) \Big) \geq 0.\end{aligned}$$

The first identity holds since ω is a 2-form, $*(*\overline{\omega}) = \overline{\omega}$, and the fact $*\phi = \phi$, $\overline{\phi} = \phi$. The second identity is by the fact $\langle \alpha, \beta \rangle_{\Lambda_2} = \alpha \wedge *\overline{\beta}$ vol, for $\alpha, \beta \in \Lambda^2 T^*\mathbb{C}^2$, cf. P156 in [15]. The third one is by the definition of ϕ-positive current. Hence, for $\omega \in \Lambda^2 T^*\mathbb{C}^2$, there exists a constant $C > 0$ depending on ω, such that

$$\int_\Omega dd^\phi f_2 \wedge \omega \leq C \int_\Omega dd^\phi f_2 \wedge (*\phi).$$

Similarly, there exists a constant $C > 0$ depending on ω, such that $-\int_\Omega dd^\phi f_2 \wedge \omega \leq C \int_\Omega dd^\phi f_2 \wedge (*\phi)$. Then

$$| \int_\Omega dd^\phi f_2 \wedge \omega | \leq C \int_\Omega dd^\phi f_2 \wedge (*\phi). \tag{11}$$

Since $\phi \in \wedge^+(\phi)$ and $dd^\phi f_2$ is ϕ-positive current, we have

$$dd^\phi f_2(\phi) = \int_\Omega dd^\phi f_2 \wedge (*\phi) > 0.$$

A direct calculation shows that

$$*\phi = \frac{1}{4} dd^\phi \|z\|^2.$$

Choose a non-negative function $\psi_0 \in C_0^\infty(\Omega)$, satisfying $\psi_0 \mid_K \equiv 1$ and vanishing on $\Omega \setminus U$.

For any $\psi \in C_0^\infty(\Omega)$, we have

$$\begin{aligned}
&\left|\int_K \psi dd^\phi f_1 \wedge dd^\phi f_2\right| = \left|\int_K f_1 dd^\phi \psi \wedge dd^\phi f_2\right| \leq \|f_1\|_K \int_K C_1(*\phi) \wedge dd^\phi f_2 \\
&= \frac{1}{4}C_1\|f_1\|_K \int_K dd^\phi \|z\|^2 \wedge dd^\phi f_2 \leq \frac{1}{4}C_1\|f_1\|_K \int_\Omega \psi_0 dd^\phi \|z\|^2 \wedge dd^\phi f_2 \\
&= \frac{1}{4}C_1\|f_1\|_K \int_\Omega f_2 dd^\phi \|z\|^2 \wedge dd^\phi \psi_0 \leq \frac{1}{4}C_1 C_2 \|f_1\|_K \|f_2\|_U \int_\Omega dd^\phi \|z\|^2 \wedge *\phi \\
&= C_1 C_2 \|f_1\|_K \|f_2\|_U \int_\Omega *\phi \wedge *\phi = C_1 C_2 \|f_1\|_K \|f_2\|_U \int_\Omega 2dV \\
&= C\|f_1\|_K \|f_2\|_U,
\end{aligned}$$

where C_1 and C_2 are chosen to satisfy $C_1 \geq \|dd^\phi \psi\|_K$ and $C_2 \geq \|dd^\phi \psi_0\|_K$, $C = C_1 C_2 \int_\Omega 2dV$. The first and third equations are due to Proposition 2.3. The first and third inequalities are by equation (11). ∎

Proof of Theorem 1.2. By Theorem 1.1, we see that for any compact subset $K \subseteq \Omega$, the sequence of $dd^\phi f_{1,N} \wedge dd^\phi f_{2,N}|_K$ is bounded. Thus it is sufficient to show that for any continuous compactly supported function φ the sequence $\int_\Omega \varphi dd^\phi f_{1,N} \wedge dd^\phi f_{2,N}$ is a Cauchy sequence. Fix $\varepsilon > 0$, and a function $\psi \in C_0^\infty(\Omega)$ such that $\|\varphi - \psi\|_\Omega < \varepsilon$. Fix an arbitrary compact subset $K \subseteq \Omega$ and a compact neighborhood U of K in Ω. We have

$$\begin{aligned}
&\left|\int_K (\psi - \varphi)(dd^\phi f_{1,M} \wedge dd^\phi f_{2,M} - dd^\phi f_{1,N} \wedge dd^\phi f_{2,N})\right| \\
&= \left|\int_K \frac{1}{2}(\psi - \varphi)(dd^\phi(f_{1,M} - f_{1,N}) \wedge dd^\phi(f_{2,M} + f_{2,N}) + dd^\phi(f_{1,M} + f_{1,N}) \wedge dd^\phi(f_{2,M} - f_{2,N}))\right| \\
&\leq C\|\psi - \varphi\|_\Omega (\|f_{1,M} - f_{1,N}\|_K \|f_{2,M} + f_{2,N}\|_U + \|f_{1,M} + f_{1,N}\|_K \|f_{2,M} - f_{2,N}\|_U) \\
&\leq C\varepsilon(\|f_{1,M} - f_{1,N}\|_K \|f_{2,M} + f_{2,N}\|_U + \|f_{1,M} + f_{1,N}\|_K \|f_{2,M} - f_{2,N}\|_U).
\end{aligned}$$

For sufficient large M and N, the last expression can be estimated by $4C\varepsilon\|f_1\|_K\|f_2\|_U$. Hence it is sufficient to prove that for any function $\psi \in C_0^\infty(\Omega)$, the sequence $\int_\Omega \psi dd^\phi f_{1,N} \wedge dd^\phi f_{2,N}$ is a Cauchy sequence. By using Theorem 1.1 again, we get

$$\begin{aligned}
&\left|\int_\Omega \psi(dd^\phi f_{1,M} \wedge dd^\phi f_{2,M} - dd^\phi f_{1,N} \wedge dd^\phi f_{2,N})\right| \\
&= \left|\int_{K'} \frac{1}{2}\psi(dd^\phi(f_{1,M} - f_{1,N}) \wedge dd^\phi(f_{2,M} + f_{2,N}) + dd^\phi(f_{1,M} + f_{1,N}) \wedge dd^\phi(f_{2,M} - f_{2,N}))\right| \\
&\leq C(\|f_{1,M} - f_{1,N}\|_{K'} \|f_{2,M} + f_{2,N}\|_{U'} + \|f_{1,M} + f_{1,N}\|_{K'} \|f_{2,M} - f_{2,N}\|_{U'}),
\end{aligned}$$

where $K' = supp\psi \subseteq \Omega$ and U' is a compact neighborhood of K' in Ω. Hence, $dd^\phi f_{1,N} \wedge dd^\phi f_{2,N}$ has weak limit. Choose another two sequences $g_{1,M}$ and $g_{2,M}$

which are also two twice continuous differentiable ϕ-plurisubharmonic functions converging to f_1 and f_2 uniformly on compact subsets of Ω respectively. We have

$$\begin{aligned}&\left|\int_{\Omega}\psi(dd^{\phi}f_{1,M}\wedge dd^{\phi}f_{2,M}-dd^{\phi}g_{1,M}\wedge dd^{\phi}g_{2,M})\right|\\&=\left|\int_{K'}\frac{1}{2}\psi(dd^{\phi}(f_{1,M}-g_{1,M})\wedge dd^{\phi}(f_{2,M}+g_{2,M})+dd^{\phi}(f_{1,M}+g_{1,M})\wedge dd^{\phi}(f_{2,M}-g_{2,M}))\right|\\&\leq C'(\|f_{1,M}-g_{1,M}\|_{K'}\|f_{2,M}+g_{2,M}\|_{U'}+\|f_{1,M}+g_{1,M}\|_{K'}\|f_{2,M}-g_{2,M}\|_{U'}).\end{aligned}$$

When M tends to $+\infty$, the right of the inequality tends to 0. Hence they have the same limit. We denote this limit by $dd^{\phi}f_1\wedge dd^{\phi}f_2$. The theorem is proved. ∎

3. Radon Transform over special Lagrangian n-planes in $\mathbb{C}^n$

In this section, we give the definition of Radon transform over the special Lagrangian n-planes in $\mathbb{C}^n$. We prove that Radon transform is injective, which enables us to prove the Theorem 1.2.

The theory of Radon transforms associated to a double fibration

$$\begin{array}{ccc} & G/(H_X\cap H_\Xi) & \\ & \swarrow \qquad \searrow & \\ X=G/H_X & & \Xi=G/H_\Xi,\end{array} \tag{12}$$

is introduced in Helgason [12,13], where X and Ξ are two left coset spaces of G, H_X and H_Ξ are closed subgroups of G. Two elements $x\in X$, $\xi\in\Xi$ are said to be *incident* if as cosets in G they intersect. Let

$$\begin{aligned}\check{x}&=\{\xi\in\Xi: x \text{ and } \xi \text{ incident}\},\\ \hat{\xi}&=\{x\in X: x \text{ and } \xi \text{ incident}\}.\end{aligned} \tag{13}$$

The *Radon transform* $f\to\hat{f}$ associated to the double fibration (12) is defined as

$$\hat{f}(\xi)=\int_{\hat{\xi}}f(x)d\mu_{\hat{\xi}}(x),\quad \text{for any } \xi\in\Xi, \tag{14}$$

where f is a rapidly decreasing function on Ξ, $d\mu_{\hat{\xi}}$ is the normalized H_Ξ-invariant measure on $\hat{\xi}$. The *dual transform* $\psi\to\check{\psi}$ for rapidly decreasing function ψ on Ξ is

$$\check{\psi}(x)=\int_{\check{x}}\psi(\xi)d\mu_{\check{x}}(\xi), \tag{15}$$

where $d\mu_{\check{x}}$ is the normalized H_X-invariant measure on $\check{x}$. The integrals (14) and (15) are well-defined since f and ψ are rapidly decreasing.

Let $SLAG_0\subseteq G(n,2n)$ be the set of all special Lagrangian n-planes through 0. By definition, SU_n acts transitively on $SLAG_0$, and the isotropic subgroup of SU_n

at the point $\xi_0 = span_{\mathbb{R}}\{e_1, e_3, ..., e_{2n-1}\}$ is SO_n acting diagonally on $\mathbb{R}^n \oplus \mathbb{R}^n$. Thus $SLAG_0 \cong SU_n/SO_n$.

Let $SLAG \equiv \{(\xi, v) \mid \xi \in SLAG_0,\ v \in \mathbb{R}^{2n}\}/SO_n \rtimes \mathbb{R}^n$, it is the set of all planes of the form $\xi + v$, $\xi \in SLAG_0$, $v \in \mathbb{R}^{2n}$. Namely, $SLAG \cong SU_n \rtimes \mathbb{R}^{2n}/SO_n \rtimes \mathbb{R}^n$. Here $\mathbb{R}^n = span_{\mathbb{R}}\{e_1, e_3, ..., e_{2n-1}\}$ and the production on Lie group $SU_n \rtimes \mathbb{R}^{2n}$ is defined as, for $(A_1, v_1), (A_2, v_2) \in SU_n \rtimes \mathbb{R}^{2n}$, $(A_1, v_1)(A_2, v_2) = (A_1A_2, A_1v_2 + v_1)$. Thus we have the following double fibration,

$$\begin{array}{ccc} & SU_n \rtimes \mathbb{R}^{2n}/SO_n & \\ & \swarrow \qquad \searrow & \\ \mathbb{R}^{2n} \cong SU_n \rtimes \mathbb{R}^{2n}/SU_n & & SLAG \cong SU_n \rtimes \mathbb{R}^{2n}/SO_n \rtimes \mathbb{R}^n. \end{array} \tag{16}$$

Now we can define the Radon transform and its dual associated to the double fibration (16). The Radon transform $\hat{f}$ of a rapidly decreasing function f on $\mathbb{R}^{2n}$ is

$$\hat{f}(\xi) = \int_{\hat{\xi}} f(x) d\mu_{\hat{\xi}}(x), \quad \text{for any } \xi \in SLAG, \tag{17}$$

where $d\mu_{\hat{\xi}}$ is the normalized $SO_n \rtimes \mathbb{R}^n$-invariant measure on $\hat{\xi}$. The dual transform $\psi \to \check{\psi}$ for rapidly decreasing function ψ on $SLAG$ is

$$\check{\psi}(x) = \int_{\check{x}} \psi(\xi) d\mu_{\check{x}}(\xi), \tag{18}$$

where $d\mu_{\check{x}}$ is the normalized SU_n-invariant measure on $\check{x}$. The integrals (17) and (18) are well-defined since f and ψ are rapidly decreasing.

In our case, we know $\hat{\xi}$ are the points in $\mathbb{R}^{2n}$ that lie in ξ by the second equation of (13), i.e.

$$\hat{\xi} = \xi.$$

The $SO_n \rtimes \mathbb{R}^n$-invariant measure $d\mu_{\hat{\xi}}$ on $\hat{\xi}$ is the Lebesgue measure on ξ up to a constant factor. So the definition of Radon transform (14) can be written as

$$\hat{f}(\xi) = \int_{\xi} f(x) dm(x), \quad \text{for any } \xi \in SLAG, \tag{19}$$

where dm is the Lebesgue measure on ξ.

The following inversion formula (20) has appeared in Grinberg [7], section 8, we cite this theorem without proof.

Theorem 3.1. *Let G be a subgroup of the group of isometries of $M = \mathbb{R}^n$. Assume that G acts transitively on M and that M is still a two-point homogeneous space of G. Let X be a fixed k-plane in M and let R be the k-plane transform restricted to the set of planes GX. Then R is invertible with inversion formula:*

$$c_{k,n} \Delta^{\frac{k}{2}} R^t R = I. \tag{20}$$

Here $\Delta^{\frac{1}{2}}$ is the pseudodifferential operator on $\mathbb{R}^n$, R^t is the dual of this Radon transform R, and $c_{k,n}$ is a constant.

Let G in Theorem 3.1 be the group SU_n, M in Theorem 3.1 be S^{2n-1} and X in Theorem 3.1 be $\mathbb{R}^n$, a fixed n-plane in S^{2n-1}. By using the fact that SU_n acts transitively on the unit sphere S^{2n-1}, and Theorem 3.1, we can get the Radon transform over special Lagrangian n-planes is injective.

Similar to [1], we have the following proposition.

Proposition 3.1. *The linear combinations of delta-functions of special Lagrangian n-planes in $\mathbb{C}^n$ are dense in the space of distributions.*

Proof. Let $\mathcal{S}(\mathbb{C}^n)$ be the rapidly decreasing functions on $\mathbb{C}^n$ and $\mathcal{S}'(\mathbb{C}^n)$ be continuous dual of $\mathcal{S}(\mathbb{C}^n)$ with the weak topology. Denote Z the closure of all linear combinations of delta-functions of special Lagrangian n-planes in $\mathbb{C}^n$ in the weak topology. Thus $Z \subset \mathcal{S}'(\mathbb{C}^n)$. Assume $Z \neq \mathcal{S}'(\mathbb{C}^n)$, then there exists $u \in \mathcal{S}'(\mathbb{C}^n) \setminus Z$. By the Hahn-Banach theorem, there exists a continuous linear functional l on $\mathcal{S}'(\mathbb{C}^n)$ such that

$$l(u) \neq 0, \tag{21}$$

$$l(Z) = 0. \tag{22}$$

But any continuous linear functional a on $\mathcal{S}'(\mathbb{C}^n)$ is given by an element of $\mathcal{S}(\mathbb{C}^n)$, namely there exists an element $f \in \mathcal{S}(\mathbb{C}^n)$ such that

$$a(\psi) = \psi(f), \quad \text{for any } \psi \in \mathcal{S}'(\mathbb{C}^n).$$

Let us apply this fact for our functional l. That means there exists $g \in \mathcal{S}(\mathbb{C}^n)$ such that $l(\psi) = \psi(g)$ for any $\psi \in \mathcal{S}'(\mathbb{C}^n)$. (21) implies that

$$l(\psi) = \psi(g) \neq 0,$$

so g is not identically 0. (22) means that for any $\delta_\xi \in Z$, ξ is a special Lagrangian n-planes, we have

$$l(\delta_\xi) = \delta_\xi(g) = 0.$$

This means that the Radon transform over special Lagrangian n-planes of g vanishes. By the injectivity of this Radon transform, we know $g = 0$. This is a contradiction. ∎

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THE FRACTIONAL $(D_\xi^\alpha G/G)$-EXPANSION METHOD AND ITS APPLICATIONS FOR SOLVING FOUR NONLINEAR SPACE-TIME FRACTIONAL PDES IN MATHEMATICAL PHYSICS

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Abstract. The fractional $(D_\xi^\alpha G/G)$-expansion method is applied in this article to find the exact traveling wave solutions with parameters for four nonlinear space-time fractional partial differential equations (PDEs), namely the space-time fractional Potential Kadomtsev-Petviashvili (PKP) equation, the space-time fractional symmetric regularized long wave (SRLW) equation, the space-time fractional Sharma-Tasso Olver (STO) equation and the space-time fractional Kolmogorov-Petrovskii-Piskunov (KPP) equation. When these parameters are taken special values, we obtain three types of solutions via the solitary, trigonometric and rational solutions. Comparison between our recent results and the well-known results is given. The solutions of these equations with numerical simulations are presented.

Keywords: fractional $(D_\xi^\alpha G/G)$-expansion method; nonlinear space-time fractional PDEs; exact traveling wave solutions; modified Riemann-Liouville derivative.

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1. Introduction

Exact traveling wave solutions for nonlinear fractional partial differential equations (NFPDEs) are of fundamental and important in applied science because they are widely employed to explain some of the nonlinear fractional phenomena and dynamical processes existed in nature world. Fractional partial differential equations have been studied due to their special appearance in different fields, such as physics, biology, engineering, signal processing control theory, the finance and fractal dynamics, see for example the articles [11], [15], [20], [22], [25], [28], [29]. For better realizing the mechanisms of the complicated nonlinear physical phenomena as well as further applications in practical life, the exact solutions of such equations obtained in the articles [2], [16], [19], [27], [36], [37]. In the past several decades, new exact solutions may help to find new phenomena. A variety of powerful methods, such as the finite difference method [17], the finite element method [7], the differential transform method [3], [21], the Adomian decomposition method [4], [5], [12], [23], the variational iteration method [13], [24], [35], the homotopy perturbation method [8], the (G'/G)-expansion method [6], [9], [26], [31], [34], [38], the fractional $(D_\xi^\alpha G/G)$-expansion method [34], [39]-[42], the Jacobi elliptic equation method [39], the fractional sub-equation method [1], [10], [34], [37]-[41], the modified simple equation method [32], the homogeneous balance method [33], the variation of parameters method [30] and so on.

The objective of this paper is to apply the fractional $(D_\xi^\alpha G/G)$-expansion method [34], [39]-[42] for solving the nonlinear fractional NFPDEs, namely the space-time fractional Potential Kadomtsev-Petviashvili (PKP) equation, the space-time fractional symmetric regularized long wave (SRLW) equation, the space-time fractional Sharma-Tasso Olver (STO) equation and the space-time fractional Kolmogorov-Petrovskii-Piskunov (KPP) equation in the sense of the modified Riemann-Liouville derivative obtained in [14], [18]. All these equations have been discussed in [31] using a different technique, namely the fractional complex transformation technique combined with the improved (G'/G)-expansion method.

The modified Riemann-Liouville derivative of order α [14], [18] is defined by the following expression:

$$D_t^\alpha f(t) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)} \dfrac{d}{dt} \displaystyle\int_0^t (t-\eta)^{-\alpha} \left[f(\eta) - f(0)\right] d\eta, & 0 < \alpha \leq 1, \\ \left[f^{(n)}(t)\right]^{(\alpha-n)}, & n \leq \alpha < n+1,\ n \geq 1. \end{cases} \tag{1.1}$$

We list some important properties for the modified Riemann-Liouville derivative as follows:

$$D_t^\alpha t^r = \frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)} t^{r-\alpha},\ r > 0 \tag{1.2}$$

$$D_t^\alpha \left[f(t)g(t)\right] = f(t)D_t^\alpha g(t) + g(t)D_t^\alpha f(t) \tag{1.3}$$

$$D_t^\alpha \left[f\left(g(t)\right)\right] = f'_g\left(g(t)\right) D_t^\alpha g(t) = D_g^\alpha f(g(t)) \left[g'(t)\right]^\alpha \tag{1.4}$$

This paper is organized as follows: In Section 2, we give the description of the fractional $(D_\xi^\alpha G/G)$-expansion method. In Section 3, we apply this method to find many exact solutions for the space-time nonlinear fractional PKP equation, the space-time nonlinear fractional SRLW equation, the space-time nonlinear fractional STO equation and the space-time nonlinear fractional KPP equation. In Section 4, conclusions and discussions are obtained.

2. Description the fractional $(D_\xi^\alpha G/G)$-expansion method

Suppose that we have the following nonlinear fractional PDE in the form:

$$F(u, D_t^\alpha u, D_x^\alpha u, D_t^{2\alpha} u, D_x^{2\alpha} u, ...) = 0, \quad 0 < \alpha \leq 1, \tag{2.1}$$

where $D_t^\alpha u, D_x^\alpha u, D_t^{2\alpha} u, D_x^{2\alpha} u, ...$ are the modified Riemann Liouville derivatives and F is a polynomial in $u(x,t)$ and its partial fractional derivatives, in which the highest order fractional derivatives and the nonlinear terms are involved. In the following, we give the main steps of this method:

Step 1: using the wave transformation

$$u(x,t) = u(\xi), \quad \xi = kx + ct, \tag{2.2}$$

where k, c are nonzero constants, to reduce equation (2.1) to the following nonlinear fractional ODE:

$$P(u, c^\alpha D_\xi^\alpha u, k^\alpha D_\xi^\alpha u, c^{2\alpha} D_\xi^{2\alpha} u, k^{2\alpha} D_\xi^{2\alpha} u, ...) = 0, \tag{2.3}$$

where P is a polynomial in $u(\xi)$ and its total fractional derivatives.

Step 2: Assume that equation (2.3) has the formal solution:

$$u(\xi) = \sum_{i=-N}^{N} a_i \left[\frac{D_\xi^\alpha G(\xi)}{G(\xi)} \right]^i, \tag{2.4}$$

where a_i $(i = 0, \pm 1, \pm 2, ..., \pm N)$ are constants to be determined later, such that $a_N \neq 0$ or $a_{-N} \neq 0$, while the function $G(\xi)$ satisfies the following fractional ordinary differential equation (ODE):

$$D_\xi^{2\alpha} G(\xi) + \lambda D_\xi^\alpha G(\xi) + \mu G(\xi) = 0, \tag{2.5}$$

where λ, μ are arbitrary constants.

Step 3: Determining the positive integer N in (2.4) by using the homogeneous balance between the highest order fractional derivatives and the nonlinear terms in equation (2.3).

Step 4: Substituting (2.4) along with equation (2.5) into equation (2.3), we have a polynomial in $\left(\frac{D_\xi^\alpha G(\xi)}{G(\xi)}\right)$. Equating each coefficient of this polynomial to be zero yields a system of algebraic equations which can be solved by using the Maple or Mathematica to find the values a_i $(i = 0, \pm 1, \pm 2, ...)$ and k, c.

Step5: It is well-known [34], [39]-[42] that $\left(\frac{D_\xi^\alpha G(\xi)}{G(\xi)}\right)$ has the following forms:

$$(2.6)\quad \frac{\sqrt{\lambda^2-4\mu}}{2}\left[\frac{c_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\eta\right)+c_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\eta\right)}{c_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\eta\right)+c_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\eta\right)}\right]-\frac{\lambda}{2},\ \lambda^2-4\mu>0,$$

$$(2.7)\quad \frac{\sqrt{4\mu-\lambda^2}}{2}\left[\frac{-c_1\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\eta\right)+c_2\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\eta\right)}{c_1\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\eta\right)+c_2\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\eta\right)}\right]-\frac{\lambda}{2},\ \lambda^2-4\mu<0,$$

$$(2.8)\quad \frac{c_2}{c_1+c_2\eta}-\frac{\lambda}{2},\qquad \lambda^2-4\mu=0,$$

where $\eta=\frac{\xi^\alpha}{\Gamma(1+\alpha)}$, while c_1, c_2 are arbitrary constants.

Step 6: Substituting the values a_i, k, c as well as the values (2.6)-(2.8) into (2.4), we have the exact traveling wave solutions of equation (2.1).

3. Applications

In this section, we apply the above method described in Section 2 to find the exact traveling wave solutions of the following four nonlinear fractional PDEs:

Example 1. The space-time nonlinear fractional PKP equation. This equation is well-known [1], [31] and has the form:

$$(3.1)\qquad \frac{1}{4}D_x^{4\alpha}u+\frac{3}{2}D_x^\alpha uD_x^{2\alpha}u+\frac{3}{4}D_y^{2\alpha}u+D_t^\alpha(D_x^\alpha u)=0.$$

This equation has been discussed in [31] using a different technique, namely the fractional complex transformation technique combined with the improved (G'/G)-expansion method. Let us now solve equation (3.1) using the method of Section 2. To this end, we use the following wave transformation:

$$(3.2)\qquad u(x,y,t)=u(\xi),\quad \xi=k_1x+k_2y+ct,$$

where k_1, k_2 and c are constants, to reduce equation (3.1) to the following fractional ODE

$$(3.3)\qquad k_1^{4\alpha}D_\xi^{3\alpha}u+3k_1^{3\alpha}\left(D_\xi^\alpha u\right)^2+(3k_2^{2\alpha}+4k_1^\alpha c^\alpha)D_\xi^\alpha u=0,$$

By balancing $D_\xi^{3\alpha}u$ with $\left(D_\xi^\alpha u\right)^2$, we have $N=1$. Consequently, equation (3.3) has the formal solution:

$$(3.4)\qquad u(\xi)=a_1\left(\frac{D_\xi^\alpha G}{G}\right)+a_0+a_{-1}\left(\frac{D_\xi^\alpha G}{G}\right)^{-1},$$

where a_1, a_0, a_{-1} are constants to be determined later, such that $a_1 \neq 0$ or $a_{-1} \neq 0$. Substituting (3.4) along with equation (2.5) into equation (3.3), collecting all the terms of the same orders $\left(\frac{D_\xi^\alpha G}{G}\right)^i$, $(i = 0, \pm 1, \pm 2, ...)$ and setting each coefficient to zero, we have the following set of algebraic equations:

$$
\begin{aligned}
\left(\tfrac{D_\xi^\alpha G}{G}\right)^4 \quad &: \quad -3a_1 k_1^{3\alpha}(2k_1^\alpha - a_1) = 0,\\
\left(\tfrac{D_\xi^\alpha G}{G}\right)^3 \quad &: \quad -6a_1 \lambda k_1^{3\alpha}(2k_1^\alpha - a_1) = 0,\\
\left(\tfrac{D_\xi^\alpha G}{G}\right)^2 \quad &: \quad -k_1^{4\alpha}(8a_1\mu + 7a_1\lambda^2) + 3k_1^{3\alpha}(a_1^2\lambda^2 - 2a_1a_{-1} + 2a_1^2\mu)\\
&\qquad -a_1(3k_2^{2\alpha} + 4k_1^\alpha c^\alpha) = 0,\\
\left(\tfrac{D_\xi^\alpha G}{G}\right) \quad &: \quad -k_1^{4\alpha}(8a_1\mu\lambda + a_1\lambda^3) + 3k_1^{3\alpha}(2a_1^2\mu\lambda - 4a_1a_{-1}\lambda)\\
&\qquad -a_1\lambda(3k_2^{2\alpha} + 4k_1^\alpha c^\alpha) = 0,\\
\left(\tfrac{D_\xi^\alpha G}{G}\right)^0 \quad &: \quad k_1^{4\alpha}(-2a_1\mu^2 - a_1\mu\lambda^2 + a_{-1}\lambda^2 + 2a_{-1}\mu) + 3k_1^{3\alpha}(a_{-1}^2\\
&\qquad -4a_1a_{-1}\mu + a_1^2\mu^2 - 2a_1a_{-1}\lambda^2) + (a_{-1} - a_1\mu)(3k_2^{2\alpha} + 4k_1^\alpha c^\alpha) = 0,\\
\left(\tfrac{D_\xi^\alpha G}{G}\right)^{-1} \quad &: \quad a_{-1}\lambda\left[k_1^{4\alpha}(8\mu + \lambda^2) + 3k_1^{3\alpha}(2a_{-1} - 4a_1\mu) + (3k_2^{2\alpha} + 4k_1^\alpha c^\alpha)\right] = 0,\\
\left(\tfrac{D_\xi^\alpha G}{G}\right)^{-2} \quad &: \quad k_1^{4\alpha}(8a_{-1}\mu^2 + 7a_{-1}\mu\lambda^2) + 3k_1^{3\alpha}(a_{-1}^2\lambda^2 + 2a_{-1}^2\mu - 2a_1a_{-1}\mu^2)\\
&\qquad +a_{-1}\mu(3k_2^{2\alpha} + 4k_1^\alpha c^\alpha) = 0,\\
\left(\tfrac{D_\xi^\alpha G}{G}\right)^{-3} \quad &: \quad 6a_{-1}\mu\lambda k_1^{3\alpha}(2\mu k_1^\alpha + a_{-1}) = 0,\\
\left(\tfrac{D_\xi^\alpha G}{G}\right)^{-4} \quad &: \quad 3a_{-1}\mu^2 k_1^{3\alpha}(2\mu k_1^\alpha + a_{-1}) = 0.
\end{aligned}
$$

On solving the above algebraic equations with the aid of Maple or Mathematica, we have the following cases:

Case1.

$$
\begin{aligned}
&\lambda = 0,\ \mu = \mu,\ k_1^\alpha = k_1^\alpha,\ k_2^\alpha = k_2^\alpha,\ c^\alpha = \frac{1}{4k_1^\alpha}(16k_1^{4\alpha}\mu - 3k_2^{2\alpha}),\\
&a_{-1} = -2k_1^\alpha\mu,\ a_1 = 2k_1^\alpha.
\end{aligned} \tag{3.5}
$$

Case2.

$$
\begin{aligned}
&\lambda = 0,\ \mu = \mu,\ k_1^\alpha = k_1^\alpha,\ k_2^\alpha = k_2^\alpha,\ c^\alpha = \frac{1}{4k_1^\alpha}(4k_1^{4\alpha}\mu - 3k_2^{2\alpha}),\\
&a_{-1} = 0,\ a_1 = 2k_1^\alpha.
\end{aligned} \tag{3.6}
$$

Case3.

$$
\begin{aligned}
&\mu = \mu,\ k_1^\alpha = k_1^\alpha,\ k_2^\alpha = k_2^\alpha,\ c^\alpha = \frac{-1}{4k_1^\alpha}(k_1^{4\alpha}(\lambda^2 - 4\mu) + 3k_2^{2\alpha}),\\
&a_{-1} = -2k_1^\alpha\mu,\ a_1 = 0.
\end{aligned} \tag{3.7}
$$

Let us now write down the following exact solutions of the space-time fractional PKP equation (3.1) for case 1 (similarly for cases 2 and 3 which are omitted here for simplicity):

(i) If $\mu < 0$ (Hyperbolic function solutions)

In this case, we have the exact wave solution:

$$\begin{aligned} u(x,y,t) &= 2k_1^\alpha\sqrt{-\mu}\left[\frac{c_1\cosh\left(\sqrt{-\mu}\eta\right)+c_2\sinh\left(\sqrt{-\mu}\eta\right)}{c_1\sinh\left(\sqrt{-\mu}\eta\right)+c_2\cosh\left(\sqrt{-\mu}\eta\right)}\right]+a_0 \\ &\quad +2k_1^\alpha\sqrt{-\mu}\left[\frac{c_1\cosh\left(\sqrt{-\mu}\eta\right)+c_2\sinh\left(\sqrt{-\mu}\eta\right)}{c_1\sinh\left(\sqrt{-\mu}\eta\right)+c_2\cosh\left(\sqrt{-\mu}\eta\right)}\right]^{-1}. \end{aligned} \tag{3.8}$$

If we set $c_1 = 0$ and $c_2 \neq 0$ in (3.8) we have the solitary wave solution:

$$u_1(x,y,t) = 2k_1^\alpha\sqrt{-\mu}\left[\coth\left(\sqrt{-\mu}\eta\right)+\tanh\left(\sqrt{-\mu}\eta\right)\right]+a_0, \tag{3.9}$$

while if we set $c_2 = 0$ and $c_1 \neq 0$ in (3.8) we have the same solitary wave solution (3.9).

(ii) If $\mu > 0$ (Trigonometric function solutions)

In this case, we have the exact wave solution:

$$\begin{aligned} u(x,y,t) &= 2k_1^\alpha\sqrt{\mu}\left[\frac{-c_1\sin\left(\sqrt{\mu}\eta\right)+c_2\cos\left(\sqrt{\mu}\eta\right)}{c_1\cos\left(\sqrt{\mu}\eta\right)+c_2\sin\left(\sqrt{\mu}\eta\right)}\right]+a_0 \\ &\quad -2k_1^\alpha\sqrt{\mu}\left[\frac{-c_1\sin\left(\sqrt{\mu}\eta\right)+c_2\cos\left(\sqrt{\mu}\eta\right)}{c_1\cos\left(\sqrt{\mu}\eta\right)+c_2\sin\left(\sqrt{\mu}\eta\right)}\right]^{-1}. \end{aligned} \tag{3.10}$$

If we set in $c_1 = 0$ and $c_2 \neq 0$ (3.10) we have the periodic wave solution:

$$u_3(x,y,t) = 2k_1^\alpha\sqrt{\mu}\left[\cot\left(\sqrt{\mu}\eta\right)-\tan\left(\sqrt{\mu}\eta\right)\right]+a_0, \tag{3.11}$$

while if we set $c_2 = 0$ and $c_1 \neq 0$ in (3.10) we have the same periodic wave solution (3.11)

(iii) If $\mu = 0$ (Rational function solutions)

In this case we have the rational solution

$$u(x,y,t) = 2k_1^\alpha\left[\frac{c_2}{c_1+c_2\eta}\right]+a_0, \tag{3.12}$$

where $\eta = \dfrac{\xi^\alpha}{\Gamma(1+\alpha)}$.

Example 2. The space-time nonlinear fractional SRLW equation. This equation is well-known [1], [31] and has the form:

$$D_t^{2\alpha}u + D_x^{2\alpha}u + uD_t^{\alpha}(D_x^{\alpha}u) + D_t^{\alpha}uD_x^{\alpha}u + D_t^{2\alpha}\left(D_x^{2\alpha}u\right) = 0, \tag{3.13}$$

This equation has been discussed in [31] using a different technique, namely the fractional complex transformation technique combined with the improved $(G\prime/G)$-expansion method. Let us now solve equation (3.13) using the method of Section 2. To this end, we use the following wave transformation:

$$u(x,t) = u(\xi), \quad \xi = kx + ct, \tag{3.14}$$

where k and c are constants, to reduce equation (3.13) to the following fractional ODE

$$(k^{2\alpha} + c^{2\alpha})u + \frac{k^\alpha c^\alpha}{2}u^2 + k^{2\alpha}c^{2\alpha}D_\xi^{2\alpha}u = 0. \tag{3.15}$$

By balancing $D_\xi^{2\alpha}u$ with u^2, we have $N = 2$. Consequently, equation (3.15) has the formal solutions

$$u(\xi) = a_2\left(\frac{D_\xi^\alpha G}{G}\right)^2 + a_1\left(\frac{D_\xi^\alpha G}{G}\right) + a_0 + a_{-1}\left(\frac{D_\xi^\alpha G}{G}\right)^{-1} + a_{-2}\left(\frac{D_\xi^\alpha G}{G}\right)^{-2}, \tag{3.16}$$

where a_2, a_1, a_0, a_{-1}, a_{-2} are constants to be determined later , such that $a_2 \neq 0$ or $a_{-2} \neq 0$. Substituting (3.16) along with equation (2.5) into equation (3.15), collecting all the terms of the same orders $\left(\frac{D_\xi^\alpha G}{G}\right)^i$, $(i = 0, \pm1, \pm2, ...)$ and setting each coefficient to be zero, we have the following set of algebraic equations:

$$\left(\frac{D_\xi^\alpha G}{G}\right)^4 : \frac{k^\alpha c^\alpha}{2}a_2^2 + 6a_2k^{2\alpha}c^{2\alpha} = 0,$$

$$\left(\frac{D_\xi^\alpha G}{G}\right)^3 : k^\alpha c^\alpha a_1a_2 + k^{2\alpha}c^{2\alpha}(2a_1 + 10a_2\lambda) = 0,$$

$$\left(\frac{D_\xi^\alpha G}{G}\right)^2 : a_2(k^{2\alpha} + c^{2\alpha}) + \frac{k^\alpha c^\alpha}{2}(a_1^2 + 2a_0a_2) + k^{2\alpha}c^{2\alpha}(8a_2\mu + 4a_2\lambda^2 + 3a_1\lambda) = 0,$$

$$\left(\frac{D_\xi^\alpha G}{G}\right) : a_1(k^{2\alpha} + c^{2\alpha}) + k^\alpha c^\alpha(a_2a_{-1} + a_0a_1) + k^{2\alpha}c^{2\alpha}(6a_2\mu\lambda + 2a_1\mu + a_1\lambda^2) = 0,$$

$$\begin{aligned}\left(\frac{D_\xi^\alpha G}{G}\right)^0 : a_0(k^{2\alpha} + c^{2\alpha}) + \frac{k^\alpha c^\alpha}{2}(a_0^2 + 2a_2a_{-2} + 2a_1a_{-1}) + k^{2\alpha}c^{2\alpha}(2a_2\mu^2 + a_1\mu\lambda \\ + a_{-1}\lambda + 2a_{-2}) = 0,\end{aligned}$$

$$\left(\frac{D_\xi^\alpha G}{G}\right)^{-1} : a_{-1}(k^{2\alpha}+c^{2\alpha})+k^\alpha c^\alpha(a_1a_{-2}+a_0a_{-1})+k^{2\alpha}c^{2\alpha}(6a_{-2}\lambda+2a_{-1}\mu+a_{-1}\lambda^2) = 0,$$

$$\left(\frac{D_\xi^\alpha G}{G}\right)^{-2} : a_{-2}(k^{2\alpha}+c^{2\alpha})+\frac{k^\alpha c^\alpha}{2}(a_{-1}^2+2a_0a_{-2})+k^{2\alpha}c^{2\alpha}(8a_{-2}\mu+4a_{-2}\lambda^2+3a_{-1}\mu\lambda) = 0,$$

$$\left(\frac{D_\xi^\alpha G}{G}\right)^{-3} : k^\alpha c^\alpha a_{-1}a_{-2} + k^{2\alpha}c^{2\alpha}(2a_{-1}\mu^2 + 10a_{-2}\mu\lambda) = 0,$$

$$\left(\frac{D_\xi^\alpha G}{G}\right)^{-4}: \frac{k^\alpha c^\alpha}{2}a_{-2}^2 + 6a_{-2}\mu^2 k^{2\alpha}c^{2\alpha} = 0.$$

On solving the above algebraic equations with the aid of Maple or Mathematica, we have the following cases:

Case1.

$$\begin{aligned}
&\lambda = \lambda,\ c^\alpha = c^\alpha,\ k^\alpha = k^\alpha,\ \mu = \frac{1}{4k^{2\alpha}c^{2\alpha}}\left(\lambda^2 k^{2\alpha}c^{2\alpha} - \left(k^{2\alpha} + c^{2\alpha}\right)\right),\\
&a_{-1} = \frac{-3\lambda}{k^\alpha c^\alpha}\left(\lambda^2 k^{2\alpha}c^{2\alpha} - \left(k^{2\alpha} + c^{2\alpha}\right)\right),\ a_1 = a_2 = 0,\\
&a_{-2} = \frac{-3}{4k^{3\alpha}c^{3\alpha}}\left(\lambda^2 k^{2\alpha}c^{2\alpha} - \left(k^{2\alpha} + c^{2\alpha}\right)\right)^2,\\
&a_0 = \frac{-1}{k^\alpha c^\alpha}\left(3\lambda^2 k^{2\alpha}c^{2\alpha} - \left(k^{2\alpha} + c^{2\alpha}\right)\right).
\end{aligned} \tag{3.17}$$

Case2.

$$\begin{aligned}
&c^\alpha = c^\alpha,\ k^\alpha = k^\alpha,\ \mu = \frac{-\left(144\left(k^{2\alpha} + c^{2\alpha}\right) - a_1^2\right)}{576k^{2\alpha}c^{2\alpha}},\ \lambda = \frac{-a_1}{12k^\alpha c^\alpha}\\
&a_0 = \frac{1}{48k^\alpha c^\alpha}\left(48\left(k^{2\alpha} + c^{2\alpha}\right) - a_1^2\right),\ a_{-1} = a_{-2} = 0,\\
&a_1 = a_1, a_2 = -12k^\alpha c^\alpha.
\end{aligned} \tag{3.18}$$

Let us now write down the following exact solutions of the space-time fractional SRLW equation (3.13) for case 1 (similarly for case 2 which is omitted here for simplicity):

(i) If $\lambda^2 - 4\mu > 0$ (Hyperbolic function solutions)

In this case, we have the exact wave solution:

$$\begin{aligned}
u(x,t) = &\frac{-\left[3\lambda^2 k^{2\alpha}c^{2\alpha} - (k^{2\alpha} + c^{2\alpha})\right]}{k^\alpha c^\alpha} - \frac{3\lambda\left[\lambda^2 k^{2\alpha}c^{2\alpha} - (k^{2\alpha} + c^{2\alpha})\right]}{k^\alpha c^\alpha}\\
&\times\left[-\frac{\lambda}{2} + \frac{1}{2}\sqrt{\frac{k^{2\alpha} + c^{2\alpha}}{k^{2\alpha}c^{2\alpha}}}\left(\frac{c_1\cosh\left(\frac{1}{2}\sqrt{\frac{k^{2\alpha}+c^{2\alpha}}{k^{2\alpha}c^{2\alpha}}}\eta\right) + c_2\sinh\left(\frac{1}{2}\sqrt{\frac{k^{2\alpha}+c^{2\alpha}}{k^{2\alpha}c^{2\alpha}}}\eta\right)}{c_1\sinh\left(\frac{1}{2}\sqrt{\frac{k^{2\alpha}+c^{2\alpha}}{k^{2\alpha}c^{2\alpha}}}\eta\right) + c_2\cosh\left(\frac{1}{2}\sqrt{\frac{k^{2\alpha}+c^{2\alpha}}{k^{2\alpha}c^{2\alpha}}}\eta\right)}\right)\right]^{-1}\\
&-\frac{3\left[\lambda^2 k^{2\alpha}c^{2\alpha} - (k^{2\alpha} + c^{2\alpha})\right]^2}{4k^{3\alpha}c^{3\alpha}}\\
&\times\left[-\frac{\lambda}{2} + \frac{1}{2}\sqrt{\frac{k^{2\alpha} + c^{2\alpha}}{k^{2\alpha}c^{2\alpha}}}\left(\frac{c_1\cosh\left(\frac{1}{2}\sqrt{\frac{k^{2\alpha}+c^{2\alpha}}{k^{2\alpha}c^{2\alpha}}}\eta\right) + c_2\sinh\left(\frac{1}{2}\sqrt{\frac{k^{2\alpha}+c^{2\alpha}}{k^{2\alpha}c^{2\alpha}}}\eta\right)}{c_1\sinh\left(\frac{1}{2}\sqrt{\frac{k^{2\alpha}+c^{2\alpha}}{k^{2\alpha}c^{2\alpha}}}\eta\right) + c_2\cosh\left(\frac{1}{2}\sqrt{\frac{k^{2\alpha}+c^{2\alpha}}{k^{2\alpha}c^{2\alpha}}}\eta\right)}\right)\right]^{-2}
\end{aligned} \tag{3.19}$$

If we set $c_1 = 0$ and $c_2 \neq 0$ in (3.19) we have the solitary wave solution:

$$\begin{aligned} u_1(x,t) &= \frac{-\left[3\lambda^2 k^{2\alpha} c^{2\alpha} - (k^{2\alpha}+c^{2\alpha})\right]}{k^\alpha c^\alpha} - \frac{3\lambda\left[\lambda^2 k^{2\alpha} c^{2\alpha} - (k^{2\alpha}+c^{2\alpha})\right]}{k^\alpha c^\alpha} \\ &\times \left[-\frac{\lambda}{2} + \frac{1}{2}\sqrt{\frac{k^{2\alpha}+c^{2\alpha}}{k^{2\alpha}c^{2\alpha}}}\tanh\left(\frac{1}{2}\sqrt{\frac{k^{2\alpha}+c^{2\alpha}}{k^{2\alpha}c^{2\alpha}}}\eta\right)\right]^{-1} \\ &- \frac{3\left[\lambda^2 k^{2\alpha} c^{2\alpha} - (k^{2\alpha}+c^{2\alpha})\right]^2}{4k^{3\alpha}c^{3\alpha}}\left[-\frac{\lambda}{2} + \frac{1}{2}\sqrt{\frac{k^{2\alpha}+c^{2\alpha}}{k^{2\alpha}c^{2\alpha}}}\tanh\left(\frac{1}{2}\sqrt{\frac{k^{2\alpha}+c^{2\alpha}}{k^{2\alpha}c^{2\alpha}}}\eta\right)\right]^{-2} \end{aligned} \tag{3.20}$$

while if we set $c_2 = 0$ and $c_1 \neq 0$ in (3.19) we have the solitary wave solution:

$$\begin{aligned} u_2(x,t) &= \frac{-\left[3\lambda^2 k^{2\alpha} c^{2\alpha} - (k^{2\alpha}+c^{2\alpha})\right]}{k^\alpha c^\alpha} - \frac{3\lambda\left[\lambda^2 k^{2\alpha} c^{2\alpha} - (k^{2\alpha}+c^{2\alpha})\right]}{k^\alpha c^\alpha} \\ &\times \left[-\frac{\lambda}{2} + \frac{1}{2}\sqrt{\frac{k^{2\alpha}+c^{2\alpha}}{k^{2\alpha}c^{2\alpha}}}\coth\left(\frac{1}{2}\sqrt{\frac{k^{2\alpha}+c^{2\alpha}}{k^{2\alpha}c^{2\alpha}}}\eta\right)\right]^{-1} \\ &- \frac{3\left[\lambda^2 k^{2\alpha} c^{2\alpha} - (k^{2\alpha}+c^{2\alpha})\right]^2}{4k^{3\alpha}c^{3\alpha}}\left[-\frac{\lambda}{2} + \frac{1}{2}\sqrt{\frac{k^{2\alpha}+c^{2\alpha}}{k^{2\alpha}c^{2\alpha}}}\coth\left(\frac{1}{2}\sqrt{\frac{k^{2\alpha}+c^{2\alpha}}{k^{2\alpha}c^{2\alpha}}}\eta\right)\right]^{-2} \end{aligned} \tag{3.21}$$

If $c_2 \neq 0$ and $c_1^2 < c_2^2$, then we have the solitary wave solution:

$$\begin{aligned} u_3(x,t) &= \frac{-\left[3\lambda^2 k^{2\alpha} c^{2\alpha} - (k^{2\alpha}+c^{2\alpha})\right]}{k^\alpha c^\alpha} - \frac{3\lambda\left[\lambda^2 k^{2\alpha} c^{2\alpha} - (k^{2\alpha}+c^{2\alpha})\right]}{k^\alpha c^\alpha} \\ &\times \left[-\frac{\lambda}{2} + \frac{1}{2}\sqrt{\frac{k^{2\alpha}+c^{2\alpha}}{k^{2\alpha}c^{2\alpha}}}\coth\left(\xi_1 + \frac{1}{2}\sqrt{\frac{k^{2\alpha}+c^{2\alpha}}{k^{2\alpha}c^{2\alpha}}}\eta\right)\right]^{-1} \\ &- \frac{3\left[\lambda^2 k^{2\alpha} c^{2\alpha} - (k^{2\alpha}+c^{2\alpha})\right]^2}{4k^{3\alpha}c^{3\alpha}} \\ &\times \left[-\frac{\lambda}{2} + \frac{1}{2}\sqrt{\frac{k^{2\alpha}+c^{2\alpha}}{k^{2\alpha}c^{2\alpha}}}\coth\left(\xi_1 + \frac{1}{2}\sqrt{\frac{k^{2\alpha}+c^{2\alpha}}{k^{2\alpha}c^{2\alpha}}}\eta\right)\right]^{-2} \end{aligned} \tag{3.22}$$

where $\xi_1 = \tanh^{-1}\left(\frac{c_2}{c_1}\right)$, while if $c_1 \neq 0$ and $c_2^2 < c_1^2$, then we have the solitary wave solution:

$$\begin{aligned} u_4(x,t) &= \frac{-\left[3\lambda^2 k^{2\alpha} c^{2\alpha} - (k^{2\alpha}+c^{2\alpha})\right]}{k^\alpha c^\alpha} - \frac{3\lambda\left[\lambda^2 k^{2\alpha} c^{2\alpha} - (k^{2\alpha}+c^{2\alpha})\right]}{k^\alpha c^\alpha} \\ &\times \left[-\frac{\lambda}{2} + \frac{1}{2}\sqrt{\frac{k^{2\alpha}+c^{2\alpha}}{k^{2\alpha}c^{2\alpha}}}\tanh\left(\xi_1 + \frac{1}{2}\sqrt{\frac{k^{2\alpha}+c^{2\alpha}}{k^{2\alpha}c^{2\alpha}}}\eta\right)\right]^{-1} \\ &- \frac{3\left[\lambda^2 k^{2\alpha} c^{2\alpha} - (k^{2\alpha}+c^{2\alpha})\right]^2}{4k^{3\alpha}c^{3\alpha}} \\ &\left[-\frac{\lambda}{2} + \frac{1}{2}\sqrt{\frac{k^{2\alpha}+c^{2\alpha}}{k^{2\alpha}c^{2\alpha}}}\tanh\left(\xi_1 + \frac{1}{2}\sqrt{\frac{k^{2\alpha}+c^{2\alpha}}{k^{2\alpha}c^{2\alpha}}}\eta\right)\right]^{-2} \end{aligned} \tag{3.23}$$

where $\xi_1 = \coth^{-1}\left(\frac{c_2}{c_1}\right)$ and $\eta = \dfrac{\xi^\alpha}{\Gamma(1+\alpha)}$.

Example 3. The space-time nonlinear fractional STO equation. This equation is well-known [31], [40] and has the form:

$$D_t^{\alpha}u + 3\beta\left(D_x^{\alpha}u\right)^2 + 3\beta u^2 D_x^{\alpha}u + 3\beta u D_x^{2\alpha}u + \beta D_x^{3\alpha}u = 0, \tag{3.24}$$

where $0 < \alpha \leq 1$. equation (3.24) has been investigated in [40] using the fractional sub-equation method. It is also discussed in [31] using a different technique, namely the fractional complex transformation technique combined with the improved (G'/G)-expansion method. Let us now solve equation (3.24) using the method of Section 2. To this end, we use the wave transformation (3.14) to reduce equation (3.24) to the following fractional ODE:

$$c^{\alpha}u + 3\beta k^{2\alpha}uD_{\xi}^{\alpha}u + \beta k^{\alpha}u^3 + \beta k^{3\alpha}D_{\xi}^{2\alpha}u = 0. \tag{3.25}$$

By balancing $D_{\xi}^{2\alpha}u$ with u^3, we have $N = 1$. Consequently, equation (3.25) has the formal solution:

$$u(\xi) = a_1\left(\frac{D_{\xi}^{\alpha}G}{G}\right) + a_0 + a_{-1}\left(\frac{D_{\xi}^{\alpha}G}{G}\right)^{-1}, \tag{3.26}$$

where are a_1, a_0, a_{-1} constants to be determined later, such that $a_1 \neq 0$ or $a_{-1} \neq 0$. Substituting (3.26) along with equation (2.5) into equation (3.25), collecting all the terms of the same orders $\left(\frac{D_{\xi}^{\alpha}G}{G}\right)^i$, $(i = 0, \pm 1, \pm 2, ...)$ and setting each coefficient to zero, we have the following set of algebraic equations:

$$\left(\frac{D_{\xi}^{\alpha}G}{G}\right)^3 : a_1\beta k^{\alpha}(-3a_1k^{\alpha} + a_1^2 + 2k^{2\alpha}) = 0,$$

$$\left(\frac{D_{\xi}^{\alpha}G}{G}\right)^2 : -3\beta k^{2\alpha}(a_1^2\lambda + a_0a_1) + 3a_0a_1^2\beta k^{\alpha} + 3a_1\lambda\beta k^{3\alpha} = 0,$$

$$\left(\frac{D_{\xi}^{\alpha}G}{G}\right) : a_1c^{\alpha} - 3\beta k^{2\alpha}(a_1^2\mu + a_0a_1\lambda) + \beta k^{\alpha}\left(3a_0^2a_1 + 3a_1^2a_{-1}\right) + \beta k^{3\alpha}\left(a_1\lambda^2 + 2a_1\mu\right) = 0,$$

$$\left(\frac{D_{\xi}^{\alpha}G}{G}\right)^0 : a_0c^{\alpha} + 3\beta k^{2\alpha}(a_0a_{-1} - a_0a_1\mu) + \beta k^{\alpha}\left(a_0^3 + 6a_0a_1a_{-1}\right) + \beta k^{3\alpha}\left(a_1\mu\lambda + a_{-1}\lambda\right) = 0,$$

$$\begin{aligned}\left(\frac{D_{\xi}^{\alpha}G}{G}\right)^{-1} &: a_{-1}c^{\alpha} + 3\beta k^{2\alpha}(a_{-1}^2 + a_0a_{-1}\lambda) + \beta k^{\alpha}\left(3a_0^2a_{-1} + 3a_{-1}^2a_1\right) \\ &+ \beta k^{3\alpha}\left(a_{-1}\lambda^2 + 2a_{-1}\mu\right) = 0,\end{aligned}$$

$$\left(\frac{D_{\xi}^{\alpha}G}{G}\right)^{-2} : 3\beta k^{2\alpha}(a_{-1}^2\lambda + a_0a_{-1}\mu) + 3a_0a_{-1}^2\beta k^{\alpha} + 3a_{-1}\mu\lambda\beta k^{3\alpha} = 0,$$

$$\left(\frac{D_{\xi}^{\alpha}G}{G}\right)^{-3} : a_{-1}\beta k^{\alpha}(-3a_{-1}\mu k^{\alpha} + a_{-1}^2 + 2\mu^2k^{2\alpha}) = 0.$$

On solving the above algebraic equations with the aid of Maple or Mathematica, we have the following cases:

Case1.

$$\begin{aligned}&\lambda = \lambda,\ \mu = \mu,\ \beta = \beta,\ k^{\alpha} = k^{\alpha},\ c^{\alpha} = -\beta k^{3\alpha}(\lambda^2 - 4\mu), \\ &a_1 = 2k^{\alpha}, a_0 = \lambda k^{\alpha},\ a_{-1} = 0.\end{aligned} \tag{3.27}$$

Case2.

$$\begin{aligned}&\lambda=\lambda,\ \mu=\mu,\ \beta=\beta,\ k^\alpha=k^\alpha,\ c^\alpha=-\beta k^{3\alpha}(\lambda^2-4\mu),\\&a_1=0, a_0=-\lambda k^\alpha,\ a_{-1}=-2\mu k^\alpha.\end{aligned}\tag{3.28}$$

Let us now write down the following exact wave solutions of the space-time fractional STO equation (3.24) for case 1 (similarly for case 2 which is omitted here for simplicity):

(i) If $\lambda^2-4\mu>0$ (Hyperbolic function solutions)

In this case, we have the exact wave solution:

$$u(x,t)=k^\alpha\sqrt{\lambda^2-4\mu}\left[\frac{c_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\eta\right)+c_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\eta\right)}{c_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\eta\right)+c_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\eta\right)}\right].\tag{3.29}$$

If we set $c_1=0$ and $c_2\neq0$ in (3.29) we have the solitary wave solution:

$$u_1(x,t)=k^\alpha\sqrt{\lambda^2-4\mu}\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\eta\right).\tag{3.30}$$

while if we set $c_2=0$ and $c_1\neq0$ in (3.29) we have the solitary wave solution:

$$u_2(x,t)=k^\alpha\sqrt{\lambda^2-4\mu}\coth\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\eta\right).\tag{3.31}$$

If $c_2\neq0$ and $c_1^2<c_2^2$, then we have the solitary wave solution:

$$u_3(x,t)=k^\alpha\sqrt{\lambda^2-4\mu}\coth\left(\xi_1+\frac{\sqrt{\lambda^2-4\mu}}{2}\eta\right),\tag{3.32}$$

where $\xi_1=\tanh^{-1}\left(\frac{c_2}{c_1}\right)$, while if $c_1\neq0$ and $c_2^2<c_1^2$, then we have the solitary wave solution:

$$u_4(x,t)=k^\alpha\sqrt{\lambda^2-4\mu}\tanh\left(\xi_1+\frac{\sqrt{\lambda^2-4\mu}}{2}\eta\right),\tag{3.33}$$

where $\xi_1=\coth^{-1}\left(\frac{c_2}{c_1}\right)$.

(ii) If $\lambda^2-4\mu<0$ (Trigonometric function solutions)

In this case, we have the exact wave solution:

$$u(x,t)=k^\alpha\sqrt{4\mu-\lambda^2}\left[\frac{-c_1\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\eta\right)+c_2\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\eta\right)}{c_1\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\eta\right)+c_2\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\eta\right)}\right],\tag{3.34}$$

If we set $c_1 = 0$ and $c_2 \neq 0$ in (3.34) we have the periodic wave solution:

$$u_1(x,t) = k^\alpha \sqrt{4\mu - \lambda^2} \cot\left(\frac{\sqrt{4\mu - \lambda^2}}{2}\eta\right), \tag{3.35}$$

while if we set $c_2 = 0$ and $c_1 \neq 0$ in (3.34) we have the periodic wave solution:

$$u_2(x,t) = -k^\alpha \sqrt{4\mu - \lambda^2} \tan\left(\frac{\sqrt{4\mu - \lambda^2}}{2}\eta\right). \tag{3.36}$$

If $c_2 \neq 0$ and $c_1^2 < c_2^2$, then we have the periodic wave solution:

$$u_3(x,t) = k^\alpha \sqrt{4\mu - \lambda^2} \cot\left(\xi_1 + \frac{\sqrt{4\mu - \lambda^2}}{2}\eta\right), \tag{3.37}$$

where $\xi_1 = \tan^{-1}\left(\frac{c_1}{c_2}\right)$, while if $c_1 \neq 0$ and $c_2^2 < c_1^2$, then we have the periodic wave solution:

$$u_4(x,t) = k^\alpha \sqrt{4\mu - \lambda^2} \tan\left(\xi_1 + \frac{\sqrt{4\mu - \lambda^2}}{2}\eta\right), \tag{3.38}$$

where $\xi_1 = \cot^{-1}\left(\frac{c_1}{c_2}\right)$ and $\eta = \frac{\xi^\alpha}{\Gamma(1+\alpha)}$.

Example 4. The space-time nonlinear fractional KPP equation. This equation is well-known [6], [8], [31], [32] and has the form:

$$D_t^\alpha u - D_x^{2\alpha} u + \mu_1 u + \gamma u^2 + \delta u^3 = 0, \tag{3.39}$$

where $0 < \alpha \leq 1$ and μ_1, γ, δ are nonzero constants. This equation is important in the physical fields and it includes the fisher equation. Huxlay equation, Burgers equation, Chaffee-Infanfe eqution and Fitzhugh-Nagumo equation. When $\alpha = 1$ equation (3.39) has been discussed in [6] by using the -expansion method and in [32] using the modified simple equation method. Equation (3.39) has been studied in [8] by using the homotopy perturbation method. This equation has been discussed in [31] by using a different technique, namely the fractional complex transformation technique combined with the improved (G'/G)-expansion method. Let us now solve equation (3.39) by using the method of Section 2. To this end, we use the wave transformation (3.14) to reduce equation (3.39) to the following fractional ODE:

$$c^\alpha D_\xi^\alpha u - k^{2\alpha} D_\xi^{2\alpha} u + \mu_1 u + \gamma u^2 + \delta u^3 = 0. \tag{3.40}$$

By balancing u^3 with $D_\xi^{2\alpha} u$, we have $N = 1$. Consequently, equation (3.40) has the formal solutions:

$$u(\xi) = a_1 \left(\frac{D_\xi^\alpha G}{G}\right) + a_0 + a_{-1}\left(\frac{D_\xi^\alpha G}{G}\right)^{-1}, \tag{3.41}$$

where are a_1, a_0, a_{-1} constants to be determined later, such that $a_1 \neq 0$ or $a_{-1} \neq 0$. Substituting (3.41) along with equation (2.5) into equation (3.40), collecting all the terms of the same orders $\left(\frac{D_\xi^\alpha G}{G}\right)^i$, $(i = 0, \pm 1, \pm 2, ...)$ and setting each coefficient to zero, we have the following set of algebraic equations:

$\left(\frac{D_\xi^\alpha G}{G}\right)^3 : -2a_1 k^{2\alpha} + \delta a_1^3 = 0,$

$\left(\frac{D_\xi^\alpha G}{G}\right)^2 : -a_1 c^\alpha - 3a_1 \lambda k^{2\alpha} + a_1^2 \gamma + 3a_0 a_1^2 \delta = 0,$

$\left(\frac{D_\xi^\alpha G}{G}\right) : -a_1 \lambda c^\alpha - k^{2\alpha}(a_1 \lambda^2 + 2a_1 \mu) + a_1 \mu_1 + 2a_0 a_1 \gamma + \delta\left(3a_0^2 a_1 + 3a_1^2 a_{-1}\right) = 0,$

$\left(\frac{D_\xi^\alpha G}{G}\right)^0 : c^\alpha(a_{-1} - a_1 \mu) - k^{2\alpha}(a_1 \mu \lambda + a_{-1}\lambda) + a_0 \mu_1 + \gamma\left(a_0^2 + 2a_1 a_{-1}\right) + \delta\left(a_0^3 + 6a_0 a_1 a_{-1}\right) = 0,$

$\left(\frac{D_\xi^\alpha G}{G}\right)^{-1} : a_{-1}\lambda c^\alpha - k^{2\alpha}(a_{-1}\lambda^2 + 2a_{-1}\mu) + a_{-1}\mu_1 + 2a_0 a_{-1}\gamma + \delta\left(3a_1 a_{-1}^2 + 3a_0^2 a_{-1}\right) = 0,$

$\left(\frac{D_\xi^\alpha G}{G}\right)^{-2} : a_{-1}\mu c^\alpha - 3a_{-1}\mu\lambda k^{2\alpha} + a_{-1}^2 \gamma + 3a_0 a_{-1}^2 \delta = 0,$

$\left(\frac{D_\xi^\alpha G}{G}\right)^{-3} : -2a_{-1}\mu^2 k^{2\alpha} + \delta a_{-1}^3 = 0.$

On solving the above algebraic equations with the aid of Maple or Mathematica, we have the following cases:

Case1.

$$\begin{aligned} &\lambda = 0,\ \gamma = \gamma,\ k^\alpha = k^\alpha,\ \delta = \delta,\ \mu = \frac{-1}{32\delta k^{2\alpha}}(\gamma^2 - 4\delta\mu_1), \\ &a_{-1} = \frac{-(\gamma^2 - 4\delta\mu_1)}{32\delta k^\alpha}\sqrt{\frac{2}{\delta}},\ a_1 = -k^\alpha\sqrt{\frac{2}{\delta}},\ c^\alpha = \frac{k^\alpha \gamma}{\sqrt{2\delta}},\ a_0 = \frac{-\gamma}{2\delta}. \end{aligned} \tag{3.42}$$

Case2.

$$\begin{aligned} &\lambda = \lambda,\ \mu = \mu,\ k^\alpha = k^\alpha,\ \delta = \delta,\ \mu_1 = \frac{1}{4\delta}\left[-2\delta k^{2\alpha}(\lambda^2 - 4\mu) + \gamma^2\right], \\ &a_{-1} = 0,\ a_0 = \left[\lambda k^\alpha\sqrt{\tfrac{1}{2\delta}} - \tfrac{\gamma}{2\delta}\right],\ a_1 = 2k^\alpha\sqrt{\tfrac{1}{2\delta}},\ c^\alpha = -k^\alpha\gamma\sqrt{\tfrac{1}{2\delta}}. \end{aligned} \tag{3.43}$$

Let us now write down the following exact solutions of the space-time fractional KPP equation (3.39) for case 1 (similarly for case 2 which is omitted here for simplicity):

(i) If $\mu < 0$ (Hyperbolic function solutions)

In this case, we have the exact wave solution:

$$\begin{aligned} u(x,t) &= -k^\alpha\sqrt{\frac{-2\mu}{\delta}}\left[\frac{c_1\cosh(\sqrt{-\mu}\eta) + c_2\sinh(\sqrt{-\mu}\eta)}{c_1\sinh(\sqrt{-\mu}\eta) + c_2\cosh(\sqrt{-\mu}\eta)}\right] - \frac{\gamma}{2\delta} \\ &\quad - \frac{(\gamma^2 - 4\delta\mu_1)}{16\delta k^\alpha}\sqrt{\frac{-1}{2\mu\delta}}\left[\frac{c_1\cosh(\sqrt{-\mu}\eta) + c_2\sinh(\sqrt{-\mu}\eta)}{c_1\sinh(\sqrt{-\mu}\eta) + c_2\cosh(\sqrt{-\mu}\eta)}\right]^{-1}. \end{aligned} \tag{3.44}$$

If we set $c_1 = 0$ and $c_2 \neq 0$ in (3.44) we have the solitary wave solution:

$$\begin{aligned} u_1(x,t) = &-k^\alpha \sqrt{\frac{-2\mu}{\delta}} \tanh\left(\sqrt{-\mu}\eta\right) - \frac{\gamma}{2\delta} \\ &- \frac{(\gamma^2 - 4\delta\mu_1)}{16\delta k^\alpha} \sqrt{\frac{-1}{2\mu\delta}} \coth\left(\sqrt{-\mu}\eta\right), \end{aligned} \tag{3.45}$$

while if we set $c_2 = 0$ and $c_1 \neq 0$ in (3.44) we have the solitary wave solution:

$$\begin{aligned} u_2(x,t) = &-k^\alpha \sqrt{\frac{-2\mu}{\delta}} \coth\left(\sqrt{-\mu}\eta\right) - \frac{\gamma}{2\delta} \\ &- \frac{(\gamma^2 - 4\delta\mu_1)}{16\delta k^\alpha} \sqrt{\frac{-1}{2\mu\delta}} \tanh\left(\sqrt{-\mu}\eta\right), \end{aligned} \tag{3.46}$$

If $c_2 \neq 0$ and $c_1^2 < c_2^2$, then we have the solitary wave solution:

$$\begin{aligned} u_3(x,t) = &-k^\alpha \sqrt{\frac{-2\mu}{\delta}} \tanh\left(\xi_1 + \sqrt{-\mu}\eta\right) - \frac{\gamma}{2\delta} \\ &- \frac{(\gamma^2 - 4\delta\mu_1)}{16\delta k^\alpha} \sqrt{\frac{-1}{2\mu\delta}} \coth\left(\xi_1 + \sqrt{-\mu}\eta\right), \end{aligned} \tag{3.47}$$

where $\xi_1 = \tanh^{-1}\left(\frac{c_1}{c_2}\right)$.

(ii) If $\mu > 0$ (Trigonometric function solutions)

In this case, we have the exact wave solutions:

$$\begin{aligned} u(x,t) \; = \; &-k^\alpha \sqrt{\frac{2\mu}{\delta}} \left[\frac{-c_1 \sin\left(\sqrt{\mu}\eta\right) + c_2 \cos\left(\sqrt{\mu}\eta\right)}{c_1 \cos\left(\sqrt{\mu}\eta\right) + c_2 \sin\left(\sqrt{\mu}\eta\right)}\right] - \frac{\gamma}{2\delta} \\ &- \frac{(\gamma^2 - 4\delta\mu_1)}{16\delta k^\alpha} \sqrt{\frac{1}{2\mu\delta}} \left[\frac{-c_1 \sin\left(\sqrt{\mu}\eta\right) + c_2 \cos\left(\sqrt{\mu}\eta\right)}{c_1 \cos\left(\sqrt{\mu}\eta\right) + c_2 \sin\left(\sqrt{\mu}\eta\right)}\right]^{-1}. \end{aligned} \tag{3.48}$$

If we set $c_1 = 0$ and $c_2 \neq 0$ in (3.48) we have the periodic wave solution:

$$u_1(x,t) = -k^\alpha \sqrt{\frac{2\mu}{\delta}} \cot\left(\sqrt{\mu}\eta\right) - \frac{\gamma}{2\delta} - \frac{(\gamma^2 - 4\delta\mu_1)}{16\delta k^\alpha} \sqrt{\frac{1}{2\mu\delta}} \tan\left(\sqrt{\mu}\eta\right), \tag{3.49}$$

while if we set $c_2 = 0$ and $c_1 \neq 0$ in (3.48) we have the periodic wave solution:

$$u_2(x,t) = k^\alpha \sqrt{\frac{2\mu}{\delta}} \tan\left(\sqrt{\mu}\eta\right) - \frac{\gamma}{2\delta} + \frac{(\gamma^2 - 4\delta\mu_1)}{16\delta k^\alpha} \sqrt{\frac{1}{2\mu\delta}} \cot\left(\sqrt{\mu}\eta\right). \tag{3.50}$$

If $c_2 \neq 0$ and $c_1^2 < c_2^2$, then we have the periodic wave solution:

$$\begin{aligned} u_3(x,t) = &-k^\alpha \sqrt{\frac{2\mu}{\delta}} \cot\left(\xi_1 + \sqrt{\mu}\eta\right) - \frac{\gamma}{2\delta} \\ &- \frac{(\gamma^2 - 4\delta\mu_1)}{16\delta k^\alpha} \sqrt{\frac{1}{2\mu\delta}} \tan\left(\xi_1 + \sqrt{\mu}\eta\right), \end{aligned} \tag{3.51}$$

where $\xi_1 = \tan^{-1}\left(\frac{c_1}{c_2}\right)$, while if $c_1 \neq 0$ and $c_2^2 < c_1^2$, then we have the periodic wave solution:

$$\begin{aligned} u_4(x,t) = &- k^\alpha \sqrt{\frac{2\mu}{\delta}} \tan\left(\xi_1 + \sqrt{\mu}\eta\right) - \frac{\gamma}{2\delta} \\ &- \frac{(\gamma^2 - 4\delta\mu_1)}{16\delta k^\alpha} \sqrt{\frac{1}{2\mu\delta}} \cot\left(\xi_1 + \sqrt{\mu}\eta\right). \end{aligned} \tag{3.52}$$

where $\xi_1 = \cot^{-1}\left(\frac{c_1}{c_2}\right)$ and $\eta = \dfrac{\xi^\alpha}{\Gamma(1+\alpha)}$.

(iii) If $\mu = 0$ (Rational function solution)

In this case we have the rational solution

$$u(x,t) = -k^\alpha \sqrt{\frac{2}{\delta}} \left[\frac{c_2}{c_1 + c_2\eta}\right] - \frac{\gamma}{2\delta} - \frac{1}{32\delta k^\alpha} \sqrt{\frac{2}{\delta}} \left[\frac{c_2}{c_1 + c_2\eta}\right]^{-1}. \tag{3.53}$$

where $\eta = \dfrac{\xi^\alpha}{\Gamma(1+\alpha)}$.

4. Physical explanations of our obtained solutions

Solitary, periodic and rational wave solutions can be obtained from the exact solutions by setting particular values in its unknown parameters. In this section, we have presented some graphs for solitary and periodic wave solutions constructed by taking suitable values of involved unknown parameters to visualize the underlying mechanism of the original equations (3.1), (3.13), (3.24) and (3.39). By using the mathematical software Maple, the plots of some solutions have been shown in Figs.1-5 as follows:

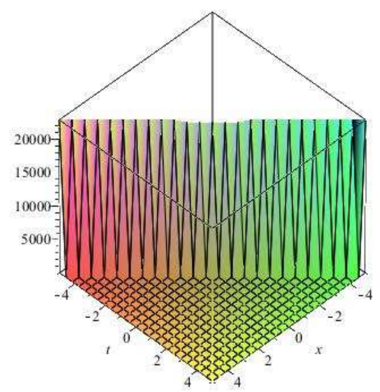

Fig.1. The plot of $u_1(x,0,t)$ of (3.9) with $k_1 = 1,\ \mu = -1,\ a_0 = 0,\ c = 1,\ \alpha = \frac{1}{2}$

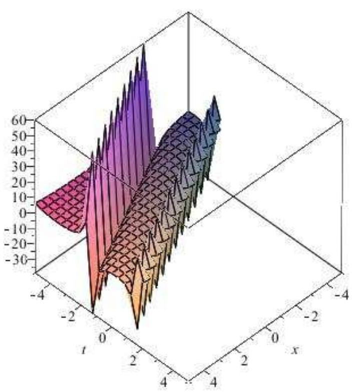

Fig.2. The plot of $u_3(x,0,t)$ of (3.11) with $k_1 = 1,\ \mu = 1,\ a_0 = 0,\ c = -1,\ \alpha = \frac{1}{2}$

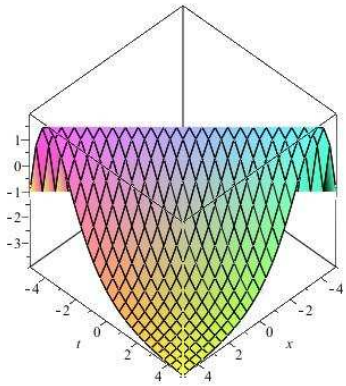

Fig.3. The plot of (3.20) with $k = 1$, $c = 1$, $\lambda = 2$, $\alpha_0 = \frac{2}{3}$

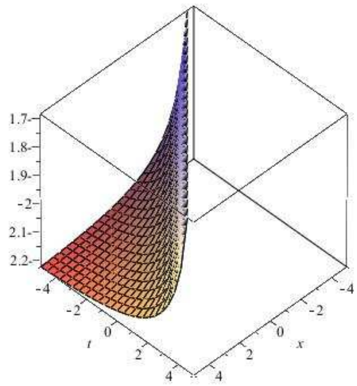

Fig.4. The plot of (3.45) with $k = 1$, $c = -1$, $\mu = -1$, $\mu_1 = 3$, $\delta = 1$, $\gamma = 2$, $\alpha = \frac{3}{8}$

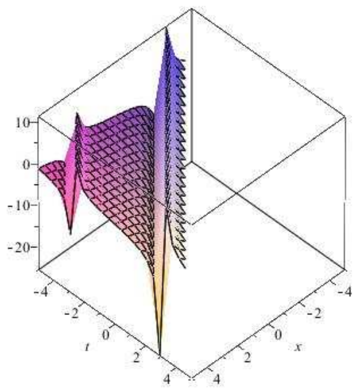

Fig.5. The plot of (3.50) with $k = 1$, $c = -1$, $\mu = 1$, $\mu_1 = -1$, $\delta = 1$, $\gamma = 2$, $\alpha = \frac{1}{2}$

5. Conclusions and discussions

The fractional $(D^{\alpha}_{\xi}G/G)$-expansion method was applied in this paper to construct new exact traveling wave solutions for four nonlinear space- time fractional partial differential equations (PDEs) namely, the space-time fractional Potential Kadomtsev Petviashvili (PKP) equation, the space-time fractional symmetric regularized long wave (SRLW) equation, the space-time fractional Sharma-Tasso Olver (STO) equation and the space-time fractional Kolmogorov-Petrovskii-Piskunov (KPP) equation. The graphical representations of some solutions of these equations have been presented. In [31] these equations have been discussed by using the fractional complex transformation technique combined with the improved (G'/G)-expansion method for finding the exact solutions of these equations. On comparing our results in this paper with those obtained in [31] we deduce that our results are new and different. This method can be applied to many other nonlinear fractional partial differential equations (NFPDEs) in the mathematical physics. Finally, with help of Maple or Mathematica, we have made sure that our new solutions obtained in this article satisfy the original four space- time fractional PDEs.

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ROUGH FUZZY (FUZZY ROUGH) STRONG h-IDEALS OF HEMIRINGS

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Abstract. By means of Dubois and Prade's idea, we apply rough fuzzy sets and fuzzy rough sets to algebraic structures. The concepts of rough fuzzy strong h-ideals (rough fuzzy prime ideals) and fuzzy rough strong h-ideals (fuzzy rough prime ideals) of hemirings are introduced, respectively. The relationships between them are investigated. Some characterizations of these two kinds of rough set theory of hemirings are explored.

Keyword: rough set; strong h-ideal; rough fuzzy strong h-ideal; fuzzy rough strong h-ideal; rough fuzzy prime ideal; fuzzy rough prime ideal; hemiring.

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1. Introduction

Rough set theory, a new mathematical approach to deal with inexact and uncertain knowledge, was originally proposed by Pawlak [11]. There are at least two methods for the development of this theory, the constructive and axiomatic approaches. In constructive methods, lower and upper approximations are constructed by basic concepts, such as equivalence relations on a universal set and neighborhood systems [12]. The Pawlak approximation operators are defined by an equivalence relation. However, an equivalence relation in Pawlak rough set models seem to be a very restrictive condition. Hence, some more general models have been put forward, such as [10], [15], [20], [22]. In 1990, Dubois and Prade [4] introduced the concepts of fuzzy rough sets and rough fuzzy sets. After that, some researchers put forward some generalized fuzzy rough sets, such as [21]. Nowadays, this theory has been applied successfully to many areas, such as knowledge discovery, machine learning, information sciences, and intelligent systems, and so on.

It naturally leads to a question of what happen if we substitute an algebraic system instead of a universe set. In 1997, Kuroki [7] studied the rough sets in semigroups and put forward two new algebraic structures, called rough semigroups and rough ideals. Further, Davvaz [2], [3] introduced the concept of rough ideals with respect to an ideal of rings. After that, many researches investigated rough set theory in different algebraic structures, for example, see [1], [6].

We know that the ideals of semirings play a crucial role in the structure theory, but ideals in semirings do not in general coincide with the usage of ideals in semirings is somewhat limited. To overcome this difficulty, Torre [13] studied h-ideals and k-ideals of hemirings. In 2004, Jun [5] applied fuzzy set theory to hemirings. Zhan and Dudek [18] introduced the concept of h-hemiregular hemirings and investigated this kind of hemirings by fuzzy h-ideals. Further, some characterization of h-hemiregular and h-intra-henmiregular henmirings were investigated by Yin [16], [17]. In particular, some generalized fuzzy h-ideals of hemirings were studied by Ma [8], [9].

The paper is organized as follows. In Section 2, we recall some concepts and results on rough sets and hemirings. In Section 3, we introduce the concept of rough fuzzy strong h-ideals and rough fuzzy prime ideals of hemirings and investigate some related properties. Finally, some characterizations of fuzzy rough strong h-ideals and fuzzy rough prime ideals of hemiring are investigated in Section 4.

2. Preliminaries

In this section, we divide into three parts. Some basic concepts and results on hemirings, strong h-ideals, congruence relations and rough sets are pointed out in these three subsections.

2.1. (Prime) strong h-ideals

By zero of a semiring $(S,+,\cdot)$ we mean an element $0 \in S$ such that $0 \cdot x = x \cdot 0 = 0$ and $0 + x = x + 0 = x$ for all $x \in S$. A semiring with a zero and a commutative semiring $(S,+)$ is called a hemiring[13]. In this paper, S always denotes a hemiring.

A non-empty subset A of S is called a subhemiring of S if A is closed under addition and multiplication. A non-empty subset A of S is a left(right) ideal of S if A closed under addition and $SA \subseteq A (AS \subseteq A)$. An ideal A of S is prime ideal such that $xy \in A$ for some $x, y \in S$ implies $x \in A$ or $y \in A$. A subhemiring

A of S is called an h-subhemiring if $x, z \in S, a, b \in A$ and $x + a + z = b + z$, implies $x \in A$. Left(right) h-ideals are defined similarly. A subhemiring A of S is called a strong h-subhemring if $x, y, z \in S, a, b \in A$ and $x + a + z = y + b + z$ implies $x \in y + A$. Strong h-ideals are defined similarly. Clearly, every strong h-subhemiring (h-ideal) is an h-subhemiring (h-ideal).

2.2. Congruence relations

Recall that an equivalence relation θ on S is a reflexive, symmetric and transitive binary relation on S. If θ is an equivalence relation on S, then the equivalence class of $x \in S$ is the set $\{y \in S|(x, y) \in \theta\}$, denoted by $[x]_\theta$. An equivalence relation θ on S is called a congruence relation if $(a, b) \in \theta$ implies $(a+x, b+x) \in \theta$ and $(ax, bx) \in \theta$ and $(xa, xb) \in \theta$ for all $x \in S$.

A congruence relation θ on S is called complete if $[ab]_\theta = \{xy|x \in [a]_\theta, y \in [b]_\theta\}$ for all $a, b \in S$.

Let I be a strong h-ideal of S, $x, y \in S$. We say x is congruent to y modulo I, denoted by $x \equiv y(\text{mod } I)$, if and only if there exist $a, b \in I$ and $z \in S$ such that $x + a + z = y + b + z$. Clearly, $x \equiv y(\text{mod } I)$ is a congruence relation on S.

Lemma 2.1 [17] *Let I be a strong h-ideal of S. If $x, y \in S$, then*

(1) $x \in [y]_I$ *if and only if* $x \in y + I$,

(2) $[x]_I + [y]_I = [x + y]_I$,

(3) $\{ab|a \in [x]_I, b \in [y]_I\} \subseteq [xy]_I$.

Remark 2.2 If I is an h-ideal of S, then the assertions (1) and (2) in above lemma may not be true as shown in the following.

Example 2.3 The set $\mathbb{N}_0$ of all non-negative integers with usual addition and multiplication operations is a hemiring, let $I =< 3 >$, then I is an h-ideal, but it is not a strong h-ideal of S, since $1+6+1 = 4+3+1$, but $1 \notin 4+I$. By calculations, $[4]_I = \{1, 4, 7, 10, ...\}$, $[5]_I = \{2, 5, 8, 11, ...\}$ and $[9]_I = \{0, 3, 6, 9, 12, ...\}$. Thus, we have

(1) $1 \in [4]_I$, but $1 \notin 4 + I$.

(2) $[4]_I + [5] \neq [9]_I$.

Proposition 2.4 *Let I be an idempotent ($II = I$) strong h-ideal of S. If $x, y \in S$, then*

$$\{ab|a \in [x]_I, b \in [y]_I\} = [xy]_I.$$

Proof. Let $c \in [xy]_I$, by Lemma 2.1 (1), we have $c \in xy + I = xy + II \subseteq xy + xI + Iy + II = (x + I)(y + I) = [x]_I[y]_I$, that is, $[xy]_I \subseteq [x]_I[y]_I$. Combing Lemma 2.1(3), we have $\{ab|a \in [x]_I, b \in [y]_I\} = [xy]_I$. ■

2.3. Rough strong h-ideals

Definition 2.5 [11] For an approximation space(U,θ), by a rough approximation in (U,θ), we mean a mapping $Apr : P(U) \to P(U) \times P(U)$ defined by for any $X \in P(U)$, $Apr(X) = (\underline{Apr}(X), \overline{Apr}(X))$, where $\underline{Apr}(X) = \{x \in U | [x]_\theta \subseteq X\}$ and $\overline{Apr}(X) = \{x \in U | [x]_\theta \sqcap X \neq \phi\}$. $\underline{Apr}(X)(\overline{Apr}(X))$ is called a lower (upper)-rough approximation of X in (U,θ). $Apr(X) = (\underline{Apr}(X), \overline{Apr}(X))$ is called a rough set if $\underline{Apr}(X) \neq \overline{Apr}(X)$.

Let I be a strong h-ideal of S and $A \subseteq S$. Then the sets
$\underline{Apr}_I(A) = \{x \in S | [x]_I \subseteq A\} = \{x \in S | x + I \subseteq A\}$ and
$\overline{Apr}_I(A) = \{x \in S | [x]_I \cap A \neq \emptyset\} = \{x \in S | x + I \cap A \neq \emptyset\}$
are called, resp., lower and upper approximations of A with respect to (briefly, w.r.t.) I.

Clearly, $\underline{Apr}_I(A)$ and $\overline{Apr}_I(A)$ are subsets of S and $\underline{Apr}_I(A) \subseteq A \subseteq \overline{Apr}_I(A)$. We call $Apr_I(A) = (\underline{Apr}_I(A), \overline{Apr}_I(A))$ is a rough set on S if $\underline{Apr}_I(A) \neq \overline{Apr}_I(A)$.

Definition 2.6 Let I be a strong h-ideal of S and $A \subseteq S$. Then A is called a lower(upper) rough strong h-ideal w.r.t I of S if $\underline{Apr}_I(A)(\overline{Apr}_I(A))$ is strong h-ideal of S. Moreover, $Apr_I(A)$ is called a rough strong h-ideal w.r.t. I of S if both $\underline{Apr}_I(A)$ and $\overline{Apr}_I(A)$ are strong h-ideals of S.

Example 2.7 Let $S = \mathbb{Z}_8$, $I = \{0, 2, 4, 6\}$ and $A = \{0, 1, 2, 4, 6\}$. By calculations, $\overline{Apr}_I(A) = S$ and $\underline{Apr}_I(A) = \{0, 1, 2, 4, 6\}$. Hence $Apr_I(A)$ is a rough strong h-ideal w.r.t. I of S.

Lemma 2.8 [17] *Let I be a strong h-ideal of S and A any non-empty subset in S. Then $\overline{Apr}_I(A) = I + A$.*

Lemma 2.9 [17] *Let I and A be any two strong h-ideals of S. Then $\overline{Apr}_I(A)$ is a strong h-ideal of S.*

Lemma 2.10 [17] *Let I and A be any two strong h-ideals of S. If $\underline{Apr}_I(A) \neq \emptyset$, then $\underline{Apr}_I(A) = A$ and $\underline{Apr}_I(A)$ is a strong h-ideal of S.*

Lemma 2.11 *Let I be any idempotent strong h-ideal of S and A be any prime ideal of S. Then $\overline{Apr}_I(A)$ is a prime ideal of S.*

Proof. Assume that I is a strong h-ideal of S and A is a prime ideal of S. Then we have

(1) $\overline{Apr}_I(A) + \overline{Apr}_I(A) = I + A + I + A = I + A = \overline{Apr}_I(A)$.

(2) $S\,\overline{Apr}_I(A) = S(I + A) = SI + SA \subseteq I + A = \overline{Apr}_I(A)$ and $\overline{Apr}_I(A)\,S = (I + A)S = IS + AS \subseteq I + A = \overline{Apr}_I(A)$.

(3) Let $xy \in \overline{Apr}_I(A)$ for some $x, y \in S$, then $[xy]_I \cap A = [x]_I[x]_I \cap A \neq \emptyset$. So there exist $x' \in [x]_I$ and $y' \in [y]_I$ such that $x'y' \in A$. Thus $[x]_I \cap A \neq \emptyset$ or $[y]_I \cap A \neq \emptyset$, and so $x \in \overline{Apr}_I(A)$ or $y \in \overline{Apr}_I(A)$. Therefore $\overline{Apr}_I(A)$ is a prime ideal of S. ∎

Lemma 2.12 *Let I be an idempotent strong h-ideal of S and A a prime strong h-ideal of S. Then $\overline{Apr}_I(A)$ is a prime strong h-ideal of S.*

Proof. Combining Lemmas 2.9 and 2.11, we obtain easily. ∎

Lemma 2.13 *Let I be any idempotent strong h-ideal of S and A be any prime ideal of S. If $\underline{Apr}_I(A) \neq \emptyset$, then $\underline{Apr}_I(A)$ is a prime ideal of S.*

Proof. (1) Let A be any prime ideal of S and $x, y \in \underline{Apr}_I(A)$. Then $[x]_I \subseteq A$ and $[y]_I \subseteq A$. Since I is an h-ideal of S, so $[x+y]_I = [x]_I + [y]_I \subseteq A + A \subseteq A$. That is $x + y \in \underline{Apr}_I(A)$.

(2) Let $s \in S$ and $x \in \overline{Apr}_I(A)$, then $[x]_I \subseteq A$. Since I is an idempotent h-ideal of S, then $[sx]_I = [s]_I[x]_I \subseteq [s]_I A \subseteq A$, this implies $sx \in \underline{Apr}_I(A)$. Similarly, we can get $xs \in \underline{Apr}_I(A)$.

(3) Suppose that $\underline{Apr}_I(A)$ is not a prime ideal, then there exist $x, y \in S$ such that $xy \in \underline{Apr}_I(A)$, but $x \notin \underline{Apr}_I(A)$ and $y \notin \underline{Apr}_I(A)$, that is, $[x]_I[y]_I = [xy]_I \subseteq A$ but $[x]_I \nsubseteq A$ and $[y]_I \nsubseteq A$, then there exist $x' \in [x]_I$ and $y'[y]_I$ such that $x' \notin A$, $y' \notin A$ and $x'y' \in [x]_I[y]_I \subseteq A$. Since A is a prime ideal, we have $x' \in A$ or $y' \in A$, this contradiction the assumption. Hence $\underline{Apr}_I(A)$ is a prime ideal of S. ∎

The following lemma follows from Lemma 2.10.

Lemma 2.14 *Let I be a strong h-ideals of S and A a prime strong h-ideal of S. If $\underline{Apr}_I(A) \neq \emptyset$, then $\underline{Apr}_I(A)$ is a prime strong h-ideal of S.*

3. Rough fuzzy strong h-ideals

In this section, we introduce the concept of rough fuzzy strong h-ideals of hemirings and investigate some related properties.

A fuzzy set μ in S of the form

$$\mu(y) = \begin{cases} r & \text{if } y = x, \\ 0 & \text{if } y \neq x. \end{cases}$$

is called a fuzzy point with support x and r, and is denoted by x_r. In particular, if $r = 1$, we denote x_1.

Definition 3.1 [17] Let μ and ν be any fuzzy sets of S, we define the sum, denoted by $\mu + \nu$, of μ and ν by

$$(\mu + \nu)(x) = \bigvee_{x=a+b} \mu(a) \wedge \nu(b)$$

for all $x \in S$.

In particular, for any $x \in S$, define $x + \mu$ by

$$(x + \mu)(y) = \begin{cases} \bigvee\limits_{y=x+a} \mu(a) & \text{if } \exists a \in S. \text{ s.t. } y = x + a, \\ 0 & \text{otherwise.} \end{cases}$$

For all $x \in S$, we have $x + \mu = x_1 + \mu$.

Definition 3.2 A fuzzy set μ of S is called a fuzzy ideal of S if for all $x, y \in S$, if it satisfies:
(1) $\mu(x+y) \geq \mu(x) \wedge \mu(y)$,
(2) $\mu(xy) \geq \mu(x) \vee \mu(y)$.

Definition 3.3 A fuzzy ideal μ of S is called a fuzzy prime ideal of S if $\mu(xy) = \mu(x)$ or $\mu(xy) = \mu(y)$ for all $x, y \in S$.

Definition 3.4 [17] A fuzzy ideal μ of S is called a fuzzy strong h-ideal of S if $x + a + z = y + b + z \rightarrow (y_1 + \mu)(x) \geq \mu(a) \wedge \mu(b)$ for all $x, y, z, a, b \in S$.

A fuzzy strong h-ideal μ of S is called normal if $\mu(0) = 1$.

Example 3.5 Let $R[x]$ be a polynomial ring in a real number field R. Define $A = \{x^2 f(x) | f(x) \in R[x]\}$ and $B = \{g(x) = a_1 + a_2 x + ... \in R[x] \mid a_1 > 0, a_i \in R, i = 1, 2...\}$. Let $S = A \cup B$. Then it is a hemiring. One can check that A is a strong h-ideal of S. Define a fuzzy set μ by

$$\mu(x) = \begin{cases} 0.8 & \text{if } x \in A, \\ 0.4 & \text{if } x \in B. \end{cases}$$

One can check that μ is a fuzzy strong h-ideal of S.

Let μ be a fuzzy set of S and $r \in [0, 1]$. Then the sets $\mu_r = \{x \in S \mid \mu(x) \geq r\}$ and $\mu_r^s = \{x \in S \mid \mu(x) > r\}$ are called r-level subset and r-strong level of μ, respectively.

Definition 3.6 A fuzzy ideal μ of S is called a fuzzy prime strong h-ideal of S if μ is both a fuzzy prime ideal and a fuzzy strong h-ideal of S.

Theorem 3.7 [17] *A fuzzy set μ of S is a fuzzy strong h-ideal of S if and only if non-empty subset μ_r (μ_r^s) is a strong h-ideal of S for all $r \in [0, 1]$.*

Theorem 3.8 *A fuzzy set μ of S is a fuzzy prime ideal of S if and only if non-empty subset μ_r (μ_r^s) is a prime ideal of S for all $r \in [0, 1]$.*

Proof. We only prove the case for μ_r. The proof of μ_r^s is similar.

Let μ be a fuzzy prime ideal of S, $x, y \in \mu_r$ and $a \in S$. Then $\mu(x + y) \geq \mu(x) \wedge \mu(y) \geq r, \mu(ax) \geq \mu(a) \vee \mu(x) \geq r$, and so $x + y, ax \in \mu_r$. Similarly, we get $xa \in \mu_r$, hence μ_r is an ideal of S.

Now, let $x, y \in \mu_r$ for some $x, y \in S$, then $\mu(xy) \geq r$. Since μ is a fuzzy prime ideal of S, we have $\mu(xy) = \mu(x) \geq r$ or $\mu(xy) = \mu(y) \geq r$. Thus $x \in \mu_r$ or $y \in \mu_r$. Therefore, μ_r is a prime ideal of S.

Conversely, assume that the given conditions hold. Let $x', y' \in S$. If possible, let $\mu(x' + y') < \mu(x') \wedge \mu(y')$. Choose r such that $\mu(x' + y') < r < \mu(x') \wedge \mu(y')$. Then $x', y' \in \mu_r$, but $x' + y' \notin \mu_r$, a contradiction. Hence $\mu(x + y) \geq \mu(x) \wedge \mu(y)$ for all $x, y \in S$. Similarly, we have $\mu(xy) \geq \mu(x) \vee \mu(y)$ for all $x, y \in S$.

We can make clear from the above discussion that $\mu(xy) \geq \mu(x)$ and $\mu(xy) \geq \mu(y)$. Now assume there exist $x', y' \in S$ such that $\mu(x'y') \neq \mu(x')$ and $\mu(x'y') \neq \mu(y')$, then we have $\mu(x'y') > \mu(x')$ and $\mu(x'y') > \mu(y')$, choose r such that $\mu(x'y') > r > \mu(x')$ and $\mu(x'y') > r > \mu(y')$. Then $x'y' \in \mu_r$ but $x' \notin \mu_r$ and $y' \notin \mu_r$, a contradiction. Hence $\mu(xy) = \mu(x)$ or $\mu(xy) = \mu(y)$ for all $x, y \in S$. This implies that μ is a fuzzy prime ideal of S. ■

Now, we introduce the concepts of rough fuzzy strong h-ideals (rough fuzzy prime ideals and rough fuzzy prime strong h-ideals) of hemirings.

Definition 3.9 Let I be a strong h-ideal of S and μ a fuzzy set of S. Then we define the two fuzzy sets $\underline{Apr}_I(\mu)$ and $\overline{Apr}_I(\mu)$ as follows:

$$\underline{Apr}_I(\mu)(x) = \bigwedge_{y \in [x]_I} \mu(y)$$

and

$$\overline{Apr}_I(\mu)(x) = \bigvee_{y \in [x]_I} \mu(y),$$

for all $x \in S$.

The fuzzy sets $\underline{Apr}_I(\mu)$ and $\overline{Apr}_I(\mu)$ are called, resp., the lower and upper approximations of μ w.r.t. I of S. Moreover, $Apr_I(\mu) = (\underline{Apr}_I(\mu), \overline{Apr}_I(\mu))$ is called a rough fuzzy set w.r.t. I of S if $\underline{Apr}_I(\mu) \neq \overline{Apr}_I(\mu)$.

Definition 3.10 Let I be a strong h-ideal of S and μ a fuzzy set of S. Then μ is called a lower(upper) rough fuzzy strong h-ideal (rough fuzzy prime ideal, rough fuzzy prime strong h-ideal) w.r.t. I of S if $\underline{Apr}_I(\mu)$ ($\overline{Apr}_I(\mu)$) is a fuzzy strong h-ideal (fuzzy prime ideal, fuzzy prime strong h-ideal) of S. Moreover, $Apr_I(\mu) = (\underline{Apr}_I(\mu), \overline{Apr}_I(\mu))$ is called a rough fuzzy strong h-ideal (rough fuzzy prime ideal, rough fuzzy prime strong h-ideal) w.r.t. I of S if both $\underline{Apr}_I(\mu)$ and $\overline{Apr}_I(\mu)$ are fuzzy strong h-ideals (fuzzy prime ideals, fuzzy prime strong h-ideals) of S.

Example 3.11 Consider a hemiring $S = \{0, a, b, c\}$ is the Klein's four group with the multiplication $xy = c$ if $x, y \in \{b, c\}$ and $xy = 0$ otherwise.

Let $I = \{0, a\}$, it is a strong h-ideal of S. Moreover, $[0]_I = \{0, a\}$ and $[b]_I = \{b, c\}$.

Define a fuzzy set μ of S by $\mu(0) = 0.8$ and $\mu(a) = \mu(b) = \mu(b) = 0.6$.

By calculations, we have

$$\underline{Apr}_I(\mu) = \frac{0.6}{0} + \frac{0.6}{a} + \frac{0.6}{b} + \frac{0.6}{c}$$

and

$$\overline{Apr}_I(\mu) = \frac{0.8}{0} + \frac{0.8}{a} + \frac{0.6}{b} + \frac{0.6}{c}.$$

This implies that $Apr_I(\mu)$ is a rough fuzzy prime strong h-ideal w.r.t. I of S.

Now, we give the (strong) level subset of lower and upper rough approximations of a fuzzy set μ w.r.t. I of hemirings.

Theorem 3.12 *Let I be a strong h-ideal of S. If μ is a fuzzy set of S and $r \in [0,1]$. Then*

(1) $(\underline{Apr}_I(\mu))_r = \underline{Apr}_I(\mu_r)$,

(2) $(\overline{Apr}_I(\mu))_r^s = \overline{Apr}_I(\mu_r^s)$.

Proof. For any $x \in S$, we have

$$\text{(i)} \quad x \in (\underline{Apr}_I(\mu))_r \Leftrightarrow \underline{Apr}_I(\mu)(x) \geq r \Leftrightarrow \bigwedge_{y \in [x]_I} \mu(y) \geq r \Leftrightarrow \forall y \in [x]_I, \mu(y) \geq r \Leftrightarrow [x]_I \subseteq \mu_r \Leftrightarrow x \in \underline{Apr}_I(\mu_r).$$

$$\text{(ii)} \quad x \in (\overline{Apr}_I(\mu))_r^s \Leftrightarrow \overline{Apr}_I(\mu)(x) > r \Leftrightarrow \bigvee_{y \in [x]_I} \mu(y) > r \Leftrightarrow \exists y \in [x]_I, \mu(y) > r \Leftrightarrow [x]_I \cap \mu_r^s \neq \emptyset \Leftrightarrow x \in \overline{Apr}_I(\mu_r^s).$$

This completes the proof. ∎

Theorem 3.13 *Let I be a strong h-ideal of S. If μ is a fuzzy strong h-ideal of S, then $\overline{Apr}_I(\mu)$ is a fuzzy strong h-ideal of S.*

Proof. Let μ be a fuzzy strong h-ideal of S. For any $r \in [0,1]$, by Theorem 3.12(2), $(\overline{Apr}_I(\mu))_r^s = \overline{Apr}_I(\mu_r^s)$. By Theorem 3.7, we know that μ_r^s is a strong h-ideal of S. Hence, by Lemma 2.9, $\overline{Apr}_I(\mu_r^s)$ is a strong h-ideal of S, and so $(\overline{Apr}_I(\mu))_r^s$ is also a strong h-ideal of S. Then, by Theorem 3.7, $\overline{Apr}_I(\mu)$ is a fuzzy strong h-ideal of S. ∎

Theorem 3.14 *Let I be a strong h-ideal of S. If μ is a fuzzy strong h-ideal of S and $\underline{Apr}_I(\mu) \neq \emptyset$, then $\underline{Apr}_I(\mu)$ is a fuzzy strong h-ideal of S.*

Proof. Let μ be a fuzzy strong h-ideal of S. Since $\underline{Apr}_I(\mu) \neq \emptyset$, there exists $r \in [0,1]$ such that $(\underline{Apr}_I(\mu))_r = \underline{Apr}_I(\mu_r) \neq \emptyset$ by Theorem 3.12(1). Let r be any value that fulfills the above property. Then it is clear that $\mu_r \neq \emptyset$, and we know from Theorem 3.7 that μ_r is a strong h-ideal of S. Hence, by Lemma 2.10, $\underline{Apr}_I(\mu_r)$ is a strong h-ideal of S, and so $(\underline{Apr}_I(\mu))_r$ is also a strong h-ideal of S. Then, by Theorem 3.7, $\underline{Apr}_I(\mu)$ is a fuzzy strong h-ideal of S. ∎

Corollary 3.15 *Let I be a strong h-ideal of S. If μ is a fuzzy strong h-ideal of S and $\underline{Apr}_I(\mu) \neq \emptyset$, then $Apr_I(\mu)$ is a rough fuzzy strong h-ideal w.r.t. I of S.*

Remark 3.16 The converse of Corollary 3.15 may not be true as shown in the following :

Example 3.17 Let $S = \mathbb{Z}_8$ and $I = \{0, 2, 4, 6\}$. Then I is a strong h-ideal of S. Clearly, $[0]_I = \{0, 2, 4, 6\}$ and $[1]_I = \{1, 3, 5, 7\}$.

Define a fuzzy set μ of S by $\mu(0) = 1, \mu(1) = \mu(2) = \mu(4) = \mu(6) = 0.8$ and $\mu(3) = \mu(5) = \mu(7) = 0.6$. Then we can check that μ is not a fuzzy strong h-ideal of S.

By calculations, we have

$$\overline{Apr}_I(\mu) = \frac{1}{0} + \frac{0.8}{1} + \frac{1}{2} + \frac{0.8}{3} + \frac{1}{4} + \frac{0.8}{5} + \frac{1}{6} + \frac{0.8}{7}$$

and

$$\underline{Apr}_I(\mu) = \frac{0.8}{0} + \frac{0.6}{1} + \frac{0.8}{2} + \frac{0.6}{3} + \frac{0.8}{4} + \frac{0.6}{5} + \frac{0.8}{6} + \frac{0.6}{7}.$$

This means that $\overline{Apr}_I(\mu)$ and $\underline{Apr}_I(\mu)$ are both fuzzy strong h-ideals of S, and so $Apr_I(\mu) = (\underline{Apr}_I(\mu), \overline{Apr}_I(\mu))$ is a rough fuzzy strong h-ideal w.r.t. I of S, but μ is not a fuzzy strong h-ideal of S.

Theorem 3.18 *Let I be an idempotent strong h-ideal of S. If μ is a fuzzy prime ideal of S, then $\overline{Apr}_I(\mu)$ is a fuzzy prime ideal of S.*

Proof. Let μ be a fuzzy prime ideal of S. For any $r \in [0, 1]$, by Theorem 3.12(2), $(\overline{Apr}_I(\mu))_r^s = \overline{Apr}_I(\mu_r^s)$. By Theorem 3.8, we know that μ_r^s is a prime ideal of S. Then by Lemma 2.11, $\overline{Apr}_I(\mu_r^s)$ is a prime ideal of S, and so $(\overline{Apr}_I(\mu))_r^s$ is also a prime ideal of S. Then, by Theorem 3.8, $\overline{Apr}_I(\mu)$ is a fuzzy prime ideal of S. ■

Theorem 3.19 *Let I be an idempotent strong h-ideal of S. If μ is a fuzzy prime ideal of S and $\underline{Apr}_I(\mu) \neq \emptyset$, then $\underline{Apr}_I(\mu)$ is a fuzzy prime ideal of S.*

Proof. Let μ be a fuzzy prime ideal of S. Since $\underline{Apr}_I(\mu) \neq \emptyset$, there exists $r \in [0, 1]$ such that $(\underline{Apr}_I(\mu))_r = \underline{Apr}_I(\mu_r) \neq \emptyset$ by Theorem 3.12(1). Let r be any value that fulfills the above property. Then it is clear that $\mu_r \neq \emptyset$, and we know from Theorem 3.8 that μ_r is a prime ideal of S. And then by Lemma 2.13, $\underline{Apr}_I(\mu_r)$ is a prime ideal of S, and so $(\underline{Apr}_I(\mu))_r$ is also a prime ideal of S. Then, by Theorem 3.8, $\underline{Apr}_I(\mu)$ is a fuzzy prime ideal of S. ■

Combining Theorems 3.18 and 3.19, we can obtain the following result:

Corollary 3.20 *Let I be a strong h-ideal of S. If μ is a fuzzy prime ideal of S and $\underline{Apr}_I(\mu) \neq \emptyset$, then $Apr_I(\mu)$ is a rough fuzzy prime ideal w.r.t. I of S.*

4. Fuzzy rough strong h-ideals

In this section, we investigate fuzzy rough strong h-ideals (fuzzy rough prime ideals) of hemirings and show that every rough fuzzy strong h-ideal (rough fuzzy prime ideal) is a fuzzy rough strong h-ideal (fuzzy rough prime ideal) of hemirings.

Definition 4.1 Let μ be a normal fuzzy strong h-ideal of S and ν a fuzzy set of S. Define two fuzzy sets $\underline{Apr}_\mu(\nu)$ and $\overline{Apr}_\mu(\nu)$ in S by

$$\underline{Apr}_\mu(\nu)(x) = \bigwedge_{y\in S}(1 - \bigvee_{x+a+z=y+b+z} \mu(a) \wedge \mu(b)) \vee \nu(y)$$

and

$$\overline{Apr}_\mu(\nu)(x) = \bigvee_{y\in S}(\bigvee_{x+a+z=y+b+z} \mu(a) \wedge \mu(b)) \wedge \nu(y)$$

for all $x \in S$.

For any fuzzy set ν of S, $Apr_\mu(\nu) = (\underline{Apr}_\mu(\nu), \overline{Apr}_\mu(\nu))$ is called a fuzzy rough set w.r.t. μ of S if $\underline{Apr}_\mu(\nu) \neq \overline{Apr}_\mu(\nu)$.

Remark 4.2

(1) If μ is a crisp strong h-ideal of S and ν is a fuzzy set of S, then $\underline{Apr}_\mu(\nu)(x) = \bigwedge_{y\in[x]_\mu} \nu(y)$ and $\overline{Apr}_\mu(\nu)(x) = \bigvee_{y\in[x]_\mu} \nu(y)$, for all $x \in S$.
This means that every rough fuzzy set is a fuzzy rough set.

(2) If μ is a crisp strong h-ideal of S and ν is a non-empty subset of S, then $\underline{Apr}_\mu(\nu) = \{x \in S | x + \mu \subseteq \nu\}$ and $\overline{Apr}_\mu(\nu) = \{x \in S | (x + \mu) \cap \nu \neq \emptyset\}$.
This means that $(\underline{Apr}_\mu(\nu), \overline{Apr}_\mu(\nu))$ is a Pawlak rough set.

Definition 4.3 Let μ be a normal fuzzy strong h-ideal of S and ν a fuzzy set of S. Then ν is called a lower (upper) fuzzy rough strong h-ideal (fuzzy rough prime ideal, fuzzy rough prime strong h-ideal) w.r.t μ of S if $\underline{Apr}_\mu(\nu)(\overline{Apr}_\mu(\nu))$ is a fuzzy strong h-ideal (fuzzy prime ideal, fuzzy prime strong h-ideal) of S. Moreover, $Apr_\mu(\nu) = (\underline{Apr}_\mu(\nu), \overline{Apr}_\mu(\nu))$ is called a fuzzy rough strong h-ideal (fuzzy rough prime ideal, fuzzy rough prime strong h-ideal) w.r.t μ of S if both $\underline{Apr}_\mu(\nu)$ and $\overline{Apr}_\mu(\nu)$ are fuzzy strong h-ideals (fuzzy prime ideals, fuzzy prime strong h-ideals) of S.

Example 4.4 Let $S = Z_8$. Define two fuzzy sets μ and ν of S by

$\mu(0) = 1, \mu(2) = \mu(4) = \mu(6) = 0.8$ and $\mu(1) = \mu(3) = \mu(5) = \mu(7) = 0.4$.
$\nu(0) = 1, \nu(1) = \nu(2) = \nu(4) = \nu(6) = 0.7$ and $\nu(3) = \nu(5) = \nu(7) = 0.3$.

By calculations, $\underline{Apr}_\mu(\nu)$ and $\overline{Apr}_\mu(\nu)$ are fuzzy strong h-ideals of S, and so $Apr_\mu(\nu) = (\underline{Apr}_\mu(\nu), \overline{Apr}_\mu(\nu))$ is a fuzzy rough strong h-ideal w.r.t. μ of S.

Now, we give strong level subsets of lower and upper rough approximations of a fuzzy set w.r.t. μ of hemirings.

Theorem 4.5 *Let μ be a normal fuzzy strong h-ideal and ν a fuzzy set of S and $r \in [0,1]$. Then*

(1) $(\overline{Apr}_\mu(\nu))_r^s = \overline{Apr}_{\mu_r^s}(\nu_r^s)$,

(2) $(\underline{Apr}_\mu(\nu))_r^s = \underline{Apr}_{\mu_{1-r}}(\nu_r^s)$.

Proof. For any $x \in S$, we have

(1) $x \in \overline{Apr}_\mu(\nu)_r^s$

$$\Leftrightarrow \bigvee_{y\in S}\left(\bigvee_{x+a+z=y+b+z} \mu(a)\wedge\mu(b)\right)\wedge \nu(y) > r$$

$\Leftrightarrow \exists y,z,a,b \in S$ with $x+a+z=y+b+z$ s.t $\mu(a)\wedge\mu(b)\wedge\mu(y) > r$

$\Leftrightarrow \exists y,z,a,b \in S$ with $x+a+z=y+b+z$ s.t $a,b \in \mu_r^s$ and $y \in \nu_r^s$

$\Leftrightarrow x \in \mu_r^s + y$ and $y \in \nu_r^s$

$\Leftrightarrow x \in \mu_r^s + \nu_r^s$

$\Leftrightarrow x \in \overline{Apr}_{\mu_r^s}(\nu)_r^s$.

(2) $x \in (\underline{Apr}_\mu(\nu))_r^s$

$$\Leftrightarrow \bigwedge_{y\in S}\left(1-\bigvee_{x+a+z=y+b+z} \mu(a)\wedge\mu(b)\right)\vee \nu(y) > r$$

$$\Leftrightarrow \forall y \in S,\ \left(1-\bigvee_{x+a+z=y+b+z} \mu(a)\wedge\mu(b)\right)\vee \nu(y) > r$$

$$\Leftrightarrow \forall y \in S,\ \left(1-\bigvee_{x+a+z=y+b+z} \mu(a)\wedge\mu(b)\right)\leq r \Rightarrow \nu(y) > r$$

$$\Leftrightarrow \forall y \in S,\ \bigvee_{x+a+z=y+b+z} \mu(a)\wedge\mu(b) \geq 1-r \Rightarrow \nu(y) > r$$

$\Leftrightarrow \forall y \in S$, if there exist $a,b,z \in S$ such that $x+a+z=y+b+z$ and $\mu(a)\wedge\mu(b)\geq 1-r$, then $\nu(y) > r$,

$\Leftrightarrow \forall y \in S, y \in [x]_{\mu_{1-r}} \Rightarrow y \in \nu_r^s$

$\Leftrightarrow [x]_{\mu_{1-r}} \subseteq \nu_r^s \Leftrightarrow x \in \underline{Apr}_{\mu_{1-r}}(\nu_r^s)$. ∎

Now, we investigate the properties of fuzzy rough strong h-ideals of hemirings.

Theorem 4.6 *Let μ be a normal fuzzy strong h-ideals of S and ν a fuzzy strong h-ideal of S, then ν is an upper fuzzy rough strong h-ideal w.r.t. μ of S.*

Proof. Let ν be a fuzzy strong h-ideal of S. For any $r \in [0,1]$, by Lemma 4.5, $(\overline{Apr}_\mu(\nu))_r^s=\overline{Apr}_{\mu_r^s}(\nu_r^s)$. By Theorem 3.7, we know that μ_r^s and ν_r^s are both strong h-ideals of S. Then by Lemma 2.9, $\overline{Apr}_{\mu_r^s}(\nu_r^s)$ is a strong h-ideal of S, and so

$(\overline{Apr}_\mu(\nu))_r^s$ is also a strong h-ideal of S. Then, by Theorem 3.7, $\overline{Apr}_\mu(\nu)$ is a fuzzy strong h-ideal of S. Thus, ν is an upper fuzzy rough strong h-ideal w.r.t. μ of S. ■

Similarly, we can obtain the following:

Theorem 4.7 *Let μ be a normal fuzzy strong h-ideal of S and ν a fuzzy strong h-ideal of S and $\underline{Apr}_\mu(\nu) \neq \emptyset$, then ν is a lower fuzzy rough strong h-ideal w.r.t. μ of S.*

Combining Theorems 4.6 and 4.7, we can obtain the following result:

Corollary 4.8 *Let μ be a normal fuzzy strong h-ideal of S, if ν is a fuzzy strong h-ideal of S and $\underline{Apr}_\mu(\nu) \neq \emptyset$, then $Apr_\mu(\nu) = (\underline{Apr}_\mu(\nu), \overline{Apr}_\mu(\nu))$ is a fuzzy rough strong h-ideal w.r.t μ of S.*

Remark 4.9 The converse of the above theorem may not be true as shown in the following example.

Example 4.10 Consider Example 4.4. We know that $Apr_\mu(\nu)$ is a fuzzy rough strong h-ideal w.r.t. μ of S, but ν is not a fuzzy strong h-ideal of S.

Theorem 4.11 *Let μ be a normal fuzzy strong h-ideals of S and $\mu_r\mu_r=\mu_r$ for all $r \in (0,1]$, ν a fuzzy prime ideal (fuzzy prime strong h-ideal) of S, then ν is an upper fuzzy rough prime ideal (upper fuzzy rough prime strong h-ideal) w.r.t. μ of S.*

Proof. Let μ be a fuzzy strong h-ideal of S. For any $r \in [0,1]$, by Theorem 4.5(1), $(\overline{Apr}_\mu(\nu))_r^s = \overline{Apr}_{\mu_r^s}(\nu_r^s)$. By Theorems 3.7 and 3.8, we know that μ_r^s is a strong h-ideal of S and ν_r^s is a prime ideal of S. Since $\mu_r\mu_r = \mu_r$, then by Lemma 2.11, $\overline{Apr}_{\mu_r^s}(\nu_r^s)$ is a prime ideal of S, and so $(\overline{Apr}_\mu(\nu))_r^s$ is also a prime ideal of S. And then, by Theorem 3.8, $\overline{Apr}_\mu(\nu)$ is a fuzzy prime ideal of S. Thus, ν is an upper fuzzy rough strong h-ideal w.r.t. μ of S. Similarly, we can prove the upper fuzzy rough prime strong h-ideal. ■

Theorem 4.12 *Let μ be a normal fuzzy strong h-ideal of S and $\mu_r\mu_r = \mu_r$ for all $r \in (0,1]$, ν a fuzzy prime ideal (fuzzy prime strong h-ideal) of S and $\underline{Apr}_\mu(\nu) \neq \emptyset$, then ν is a lower fuzzy rough prime ideal (fuzzy rough prime strong h-ideal) w.r.t. μ of S.*

Proof. Let μ be a normal fuzzy strong h-ideal of S. For any $r \in [0,1]$, by Theorem 4.5 (2), $(\underline{Apr}_\mu(\nu))_r^s = \underline{Apr}_{\mu_{1-r}}(\nu_r^s)$. Let r be any value that fulfills the above property, then it is clear that $\mu_r \neq \emptyset$. By Theorems 3.7 and 3.8, we know that μ_{1-r} is a strong h-ideal of S and ν_r^s is a prime ideal of S. Since $\mu_r\mu_r = \mu_r$, then by Lemma 2.13, $\underline{Apr}_{\mu_{1-r}}(\nu_r^s)$ is a prime ideal of S, and so $(\underline{Apr}_\mu(\nu))_r^s$ is also a prime ideal of S. Then, by Theorem 3.8, $\underline{Apr}_\mu(\nu)$ is a fuzzy prime ideal of S. Thus, ν is an lower fuzzy rough prime ideal w.r.t. μ of S. Similarly, we can prove the lower fuzzy rough prime strong h-ideal. ■

From the above discussion, we get the following result:

Corollary 4.13 *Let μ be a normal fuzzy strong h-ideal of S, if ν is a fuzzy prime ideal (fuzzy prime strong h-ideal) of S and $\underline{Apr}_{\mu}(\nu) \neq \emptyset$, then $Apr_{\mu}(\nu) = (\underline{Apr}_{\mu}(\nu), \overline{Apr}_{\mu}(\nu))$ is a fuzzy rough prime ideal (fuzzy rough prime strong h-ideal) w.r.t μ of S.*

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ON FUZZY SOFT GRAPHS

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Abstract. Fuzzy sets and soft sets are two different soft computing models for representing vagueness and uncertainty. We apply these soft computing models in combination to study vagueness and uncertainty in graphs. We introduce the notions of fuzzy soft graphs, strong fuzzy soft graphs, complete fuzzy soft graphs, regular fuzzy soft graphs, and investigate some of their properties.

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1. Introduction

Molodtsov [25] initiated the concept of soft set theory as a new mathematical tool for dealing with uncertainties. It has been demonstrated that soft sets have potential applications in various fields such as the smoothness of functions, game theory, operations research, Riemann integration, Perron integration, probability theory, and measurement theory [25], [27]. Since then research on soft sets has been very active and received much attention from researchers worldwide. Feng et al. [16], [18] combined soft sets with rough sets and fuzzy sets, obtaining three types of hybrid models: rough soft sets, soft rough sets, and soft-rough fuzzy sets. Ali et al. [8] discussed the fuzzy sets and fuzzy soft sets induced by soft sets. To extend the expressive power of soft sets, Jiang et al. [20] presented ontology-based soft sets, which extended soft sets with description logics. Ali et al. [9] proposed several new operations in soft set theory. Gong et al. [19] initiated the concept of bijective soft sets. Babitha and Sunil [10] extend the concepts of relations and functions in the context of soft set theory. Moreover, Maji et al. [24] presented the definition of fuzzy soft sets and Roy et al. [30] presented some applications of this notion to decision making problems.

Fuzzy graph theory is finding an increasing number of applications in modeling real time systems where the level of information inherent in the system varies with different levels of precision. Fuzzy models are becoming useful because of their aim in reducing the differences between the traditional numerical models used in engineering and sciences and the symbolic models used in expert systems. Kaufmann's initial definition of a fuzzy graph [21] was based on Zadeh's fuzzy relations [32]. Bhattacharya [11] gave some remarks on fuzzy graphs. Mordeson and Peng [26] defined the concept of complement of fuzzy graph and studied some operations on fuzzy graphs. Akram *et al.* [1]-[6] introduced many new concepts, including bipolar fuzzy graphs, strong intuitionistic fuzzy graphs, intuitionistic fuzzy hypergraphs, and intuitionistic fuzzy trees. Thumbakara and George [31] discussed the concept of soft graphs in the specific way. On the other hand, Akram and Nawaz [7] have introduced the concepts of soft graphs and vertex-induced soft graphs in broad spectrum. In this paper, we introduce the notions of fuzzy soft graphs, strong fuzzy soft graphs, complete fuzzy soft graphs, regular fuzzy soft graphs, and investigate some of their properties.

2. Preliminaries

First we review some definitions which can be found in [32, 24, 28, 31]. By a *graph*, we mean a pair $G^* = (V, E)$, where V is the set and E is a relation on V. The elements of V are vertices of G^* and the elements of E are edges of G^*. We call $V(G^*)$ the vertex set and $E(G^*)$ the edge set of G^*. A *fuzzy set* A on a set V is characterized by its membership function $\mu_A : V \to [0,1]$, where $\mu_A(u)$ is degree of membership of element u in fuzzy set A for each $u \in V$. A *fuzzy relation* on V is a fuzzy subset of $V \times V$. A *fuzzy relation* ν on V is a fuzzy relation on μ if $\nu(u,v) \leq \mu(u) \wedge \mu(v)$ for all u, v in V. A *fuzzy graph* $G' = (\mu, \nu)$ is a pair of functions $\mu : V \to [0,1]$ and $\nu : V \times V \to [0,1]$, where for all $u, v \in V$, we have $\nu(u,v) \leq \mu(u) \wedge \mu(v)$. The underlying crisp graph of a fuzzy graph $G' = (\mu, \nu)$ is denoted by $G'^* = (\mu^*, \nu^*)$, where $\mu^* = \{u \in V : \mu(u) > 0\}$ and $\nu^* = \{(u,v) \in V \times V : \nu(u,v) > 0\}$. The *strength of connectedness* between two nodes u, v is defined as the maximum of strengths of all paths between u and v and is denoted by $\mathrm{CONNG}(u,v)$. A fuzzy graph G' is *connected* if $\mathrm{CONNG}(u,v) > 0$ for all $u, v \in V$. The fuzzy graph $H = (\tau, \rho)$ is called a *fuzzy subgraph* of $G' = (\mu, \nu)$ if $\tau(u) \leq \mu(u)$ for all $u \in V$ and $\rho(u,v) \leq \nu(u,v)$ for all $u, v \in V$. A fuzzy graph $G' = (\mu, \nu)$ is a *strong* if $\nu(u,v) = \mu(u) \wedge \mu(v)$ for all $(u,v) \in E$ and is a *complete fuzzy graph* if $\nu(u,v) = \mu(u) \wedge \mu(v)$ for all $u, v \in V$. The *order* of fuzzy graph G is $O(G') = \sum\limits_{u \in V} \mu(u)$. The *size* of fuzzy graph G is $S(G') = \sum\limits_{(u,v) \in E} \nu(u,v)$. The *complement* of a fuzzy graph $G' = (\mu, \nu)$ is a fuzzy graph $\bar{G}' = (\bar{\mu}, \bar{\nu})$ where $\bar{\mu} = \mu$ and $\bar{\nu}(u,v) = \mu(u) \wedge \mu(v) - \nu(u,v)$ for all $u, v \in V$. The degree of a vertex u in fuzzy graph $G' = (\mu, \nu)$ is $\deg_{G'}(u) = \deg(u) = \sum\limits_{u \neq v} \mu(u, v) = \sum\limits_{uv \in E} \mu(u, v)$. A fuzzy graph $G' = (\mu, \nu)$ is said to be a *regular* if every vertex which is adjacent to vertices having same degrees.

Definition 2.1 [25] A pair $\mathfrak{S} = (F, A)$ is called a *soft set* over U, where $A \subseteq P$ is a parameter set and $F : A \to \mathscr{P}(U)$ is a set-valued mapping, called the *approximate function* of the soft set $\mathfrak{S}$. In other words, a *soft set* over U is a parameterized family of subsets of U. For any $\epsilon \in A$, $F(\epsilon)$ may be considered as set of ϵ-approximate elements of soft set (F, A).

Maji et al. [24] defined the fuzzy soft set in the following way.

Definition 2.2 Let U be an initial universe, P the set of all parameters, $A \subset P$ and $\mathcal{P}(U)$ the collection of all fuzzy subsets of U. Then $(\widetilde{F}, A)$ is called *fuzzy soft set*, where $\widetilde{F} : A \to \mathcal{P}(U)$ is a mapping, called *fuzzy approximate function* of the fuzzy soft set $(\widetilde{F}, A)$.

Definition 2.3 [14] Let $(\widetilde{F}_1, A)$ and $(\widetilde{F}_2, B)$ be two fuzzy soft sets over a common universal set U. Then a relation R of $(\widetilde{F}_1, A)$ to $(\widetilde{F}_2, B)$ can be defined as a fuzzy approximate function $R : A \times B \to P(U^2)$ such that $e_i \in A$, $e_j \in B$ and for all $u_p \in F_1(e_i)$, $u_q \in F_2(e_j)$, the relation R is characterized by the following membership function,

$$\nu_R(u_1, u_k) = \mu_{F_1(e_i)}(u_1) \times \mu_{F_2(e_j)}(u_k),$$

where $u_1 \in F_1(e_i)$, $u_k \in F_2(e_j)$.

3. Fuzzy soft graphs

Definition 2.1 A *fuzzy soft graph* $\widetilde{G} = (G^*, \widetilde{F}, \widetilde{K}, A)$ is a 4-tuple such that

(a) $G^* = (V, E)$ is a simple graph,

(b) A is a nonempty set of parameters,

(c) $(\widetilde{F}, A)$ is a fuzzy soft set over V,

(d) $(\widetilde{K}, A)$ is a fuzzy soft set over E,

(e) $(\widetilde{F}(a), \widetilde{K}(a))$ is a fuzzy (sub)graph of G^* for all $a \in A$. That is,

$$\widetilde{K}(a)(xy) \leq \min\{\widetilde{F}(a)(x), \widetilde{F}(a)(y)\}$$

for all $a \in A$ and $x, y \in V$. The fuzzy graph $(\widetilde{F}(a), \widetilde{K}(a))$ is denoted by $\widetilde{H}(a)$ for convenience.

On the other hand, a fuzzy soft graph is a parameterized family of fuzzy graphs.

The class of all fuzzy soft graphs of G^* is denoted by $\mathcal{F}(G^*)$.

Example 2.2 Consider a simple graph $G^* = (V, E)$ such that

$$V = \{a_1, a_2, a_3\} \text{ and } E = \{a_1a_2, a_2a_3, a_3a_1\}.$$

Let $A = \{e_1, e_2, e_3\}$ be a parameter set and $(\widetilde{F}, A)$ be a fuzzy soft set over V with its fuzzy approximate function $\widetilde{F} : A \to \mathcal{P}(V)$ defined by

$$\widetilde{F}(e_1) = \{a_1|0.2, a_2|0.6, a_3|0.8\},$$
$$\widetilde{F}(e_2) = \{a_1|0.1, a_2|0.3, a_3|0.7\},$$
$$\widetilde{F}(e_3) = \{a_1|0.4, a_2|0.5, a_3|0.9\}.$$

Let $(\widetilde{K}, A)$ be a fuzzy soft set over E with its fuzzy approximate function $\widetilde{K} : A \to \mathcal{P}(E)$ defined by

$$\widetilde{K}(e_1) = \{a_1a_2|0.1, a_2a_3|0.2, a_3a_1|0.1\},$$
$$\widetilde{K}(e_2) = \{a_1a_2|0.1, a_2a_3|0.2, a_3a_1|0.1\},$$
$$\widetilde{K}(e_3) = \{a_1a_2|0.4, a_2a_3|0.4, a_3a_1|0.3\}.$$

Thus, $\widetilde{H}(e_1) = (\widetilde{F}(e_1), \widetilde{K}(e_1))$, $\widetilde{H}(e_2) = (\widetilde{F}(e_2), \widetilde{K}(e_2))$ and $\widetilde{H}(e_3) = (\widetilde{F}(e_3), \widetilde{K}(e_3))$ are fuzzy graphs of G^* as shown in Figure 1.

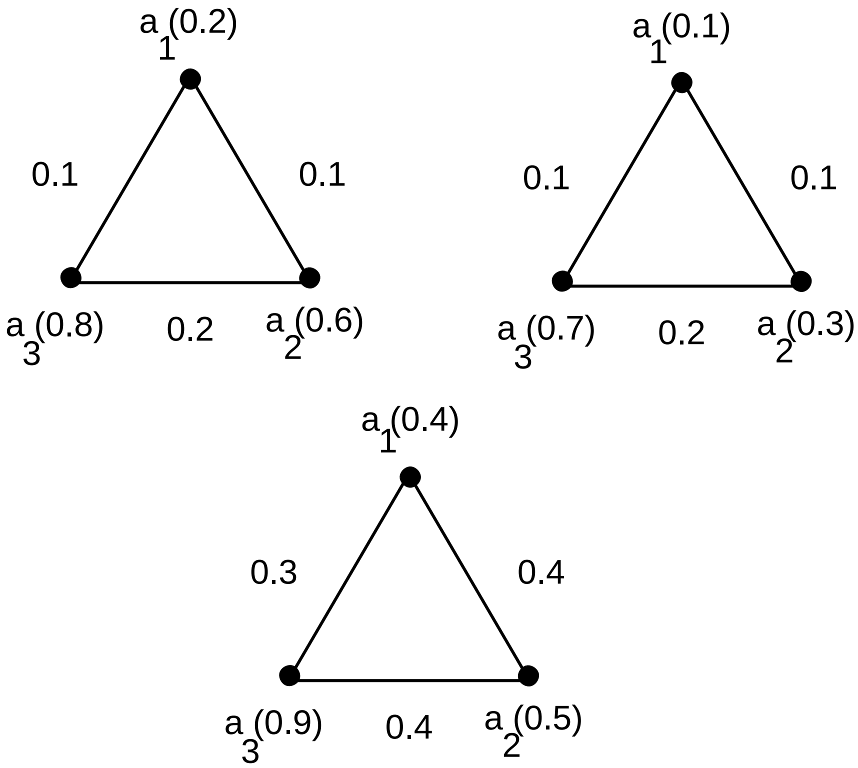

Figure 1: Fuzzy subgraphs

It is easy to verify that $\widetilde{G} = (G^*, \widetilde{F}, \widetilde{K}, A)$ is a fuzzy soft graph.

Example 2.3 Consider a crisp graph $G^* = (V, E)$ such that

$$V = \{a_1, a_2, a_3, a_4, a_5\} \text{ and } E = \{a_1a_2, a_2a_3, a_3a_4, a_4a_5, a_5a_1, a_2a_5\}.$$

Let $A = \{e_1, e_3, e_5\}$ be a parameter set and $(\widetilde{F}, A)$ be a fuzzy soft set over V with its approximate function $\widetilde{F} : A \to \mathcal{P}(V)$ defined by

$$\begin{aligned}
\tilde{F}(e_1) &= \{a_1|0.5, a_2|0.7, a_3|0.0, a_4|0.0, a_5|0.4\},\\
\widetilde{F}(e_3) &= \{a_1|0.0, a_2|0.9, a_3|0.8, a_4|0.6, a_4|0.0\},\\
\widetilde{F}(e_5) &= \{a_1|0.1, a_2|0.5, a_3|0.0, a_4|0.7, a_5|0.8\}.
\end{aligned}$$

Let $(\widetilde{K}, A)$ be a fuzzy soft set over E with its fuzzy approximate function $\widetilde{K} : A \to \mathcal{P}(E)$ defined by

$$\begin{aligned}
\widetilde{K}(e_1) &= \{a_1a_2|0.4, a_2a_3|0.0, a_3a_4|0.0, a_4a_5|0.0, a_1a_5|0.2, a_2a_5|0.3\},\\
\widetilde{K}(e_3) &= \{a_1a_2|0.0, a_2a_3|0.5, a_3a_4|0.6, a_4a_5|0.0, a_5a_1|0.0, a_2a_5|0.0\},\\
\widetilde{K}(e_5) &= \{a_1a_2|0.1, a_2a_3|0.0, a_3a_4|0.0, a_4a_5|0.6, a_1a_5|0.1, a_2a_5|0.4\}.
\end{aligned}$$

Thus, the fuzzy subgraphs are,

$$\begin{aligned}
\widetilde{H}(e_1) &= (\tilde{F}(e_1), \tilde{K}(e_1)),\\
\widetilde{H}(e_3) &= (\widetilde{F}(e_3), \widetilde{K}(e_3)),\\
\widetilde{H}(e_5) &= (\widetilde{F}(e_5), \widetilde{K}(e_5)).
\end{aligned}$$

It is clear that $\widetilde{H}(e_1)$, $\widetilde{H}(e_3)$ and $\widetilde{H}(e_5)$ are connected fuzzy graphs corresponding to the parameters e_1, e_3, e_5, respectively, as shown in Figure 2.

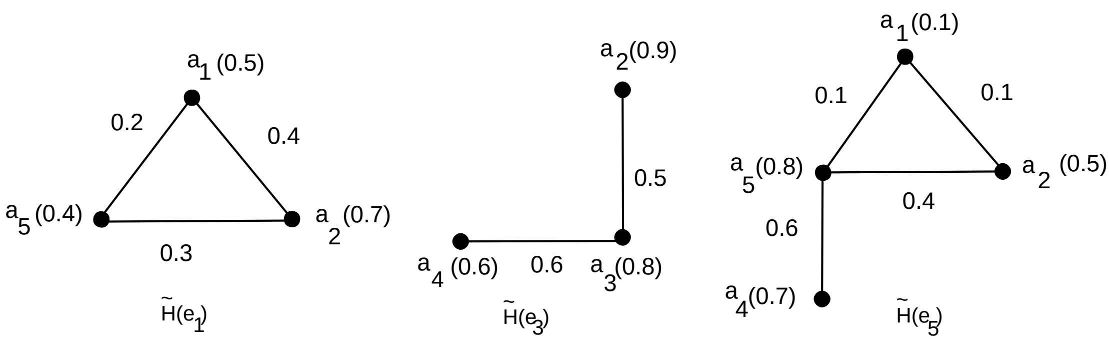

Figure 2: Fuzzy subgraphs $\widetilde{H}(e_1)$,$\widetilde{H}(e_3)$,$\widetilde{H}(e_5)$

Hence, $\widetilde{G} = \{\widetilde{H}(e_1), \widetilde{H}(e_3), \widetilde{H}(e_5)\}$ is a fuzzy soft graph of G^*.

Definition 2.4 The *order* of a fuzzy soft graph is $Ord(\widetilde{G}) = \sum_{e_i \in A} (\sum_{a \in V} \tilde{F}(e_i)(a))$.

Definition 2.5 *The size of a fuzzy soft graph is* $Siz(\widetilde{G}) = \sum_{e_i \in A} (\sum_{ab \in E} \tilde{K}(e_i)(ab))$.

In Example 2.3, the order of fuzzy soft graph is $= \sum_{e_i \in A} (\sum_{a \in V} \tilde{F}(e_i)(a))$ $= \sum_{e_i \in A} (\tilde{F}(e_i)(a_1) + \tilde{F}(e_i)(a_2) + \tilde{F}(e_i)(a_3) + \tilde{F}(e_i)(a_4) + \tilde{F}(e_i)(a_5) = (0.5 + 0.7 + 0.4) + (0.6 + 0.9 + 0.8) + (0.1 + 0.5 + 0.7 + 0.8) = 1.6 + 2.3 + 2.1 = 6.0$

The size of fuzzy soft graph is $= \sum_{e_i \in A} (\sum_{ab \in E} \tilde{K}(e_i)(ab)) = \sum_{e_i \in A} (\tilde{K}(e_i)(a_1a_2) + \tilde{K}(e_i)(a_2a_3) + \tilde{K}(e_i)(a_3a_4) + \tilde{K}(e_i)(a_4a_5) + \tilde{K}(e_i)(a_5a_1) + \tilde{K}(e_i)(a_2a_5)) = (0.4 + 0.3 + 0.2) + (0.5 + 0.6) + (0.1 + 0.4 + 0.1 + 0.6) = 0.9 + 1.1 + 1.2 = 3.2.$

Definition 2.6 A fuzzy soft graph $\widetilde{G}$ is a *strong fuzzy soft graph* if $\tilde{H}(e)$ is a strong fuzzy graph for all $e \in A$, i.e.,

$$\tilde{K}(e)(ab) = \min\{\tilde{F}(e)(a), \tilde{F}(e)(b)\}$$

for all $ab \in E$.

A fuzzy soft graph $\widetilde{G}$ is a *complete fuzzy soft graph* if $\tilde{H}(e)$ is a complete fuzzy graph for all $e \in A$. That is,

$$\tilde{K}(e)(ab) = \min\{\tilde{F}(e)(a), \tilde{F}(e)(b)\}$$

for all $a, b \in V$.

Example 2.7 Consider the crisp graph $G^* = (V, E)$ where

$$V = \{a_1, a_2, a_3, a_4\} \text{ and } E = \{a_1a_2, a_2a_3, a_3a_4, a_4a_1\}.$$

Let $A = \{e_1, e_2\}$ be a parameter set. Let $(\widetilde{F}, A)$ be a fuzzy soft set over V with its approximate function $\tilde{F} : A \to \mathcal{P}(V)$ given by

$$\tilde{F}(e_1) = \{a_1|0.5, a_2|0.3, a_3|0.2, a_4|0.9\},$$
$$\tilde{F}(e_2) = \{a_1|0.7, a_2|0.5, a_3|0.1, a_4|0.8\}.$$

Let $(\widetilde{K}, A)$ be a fuzzy soft set over E with its approximate function $\tilde{K} : A \to \mathcal{P}(E)$ given by

$$\tilde{K}(e_1) = \{a_1a_2|0.3, a_2a_3|0.2, a_3a_4|0.2, a_4a_1|0.5\},$$
$$\tilde{K}(e_2) = \{a_1a_2|0.5, a_2a_3|0.1, a_3a_4|0.1, a_4a_1|0.7\}.$$

It is easy to see that $\tilde{H}(e_1) = (\tilde{F}(e_1), \tilde{K}(e_1))$ and $\tilde{H}(e_2) = (\tilde{F}(e_2), \tilde{K}(e_2))$ are strong fuzzy graphs. Hence $\widetilde{G}$ is a strong fuzzy soft graph of G^* as shown in Figure 3.

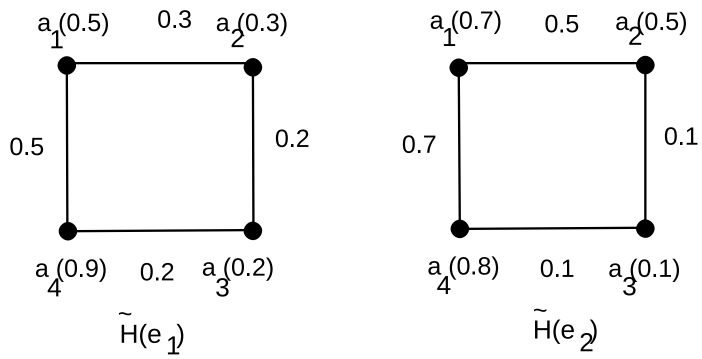

Figure 3: Strong fuzzy soft graph

Example 2.8 Consider the simple graph $G^* = (V, E)$ where

$$V = \{a_1, a_2, a_3, a_4\} \text{ and } E = \{a_1a_2, a_2a_3, a_3a_4, a_4a_1, a_1a_3, a_2a_4\}.$$

Let $A = \{e_1, e_2\}$. Let $(\widetilde{F}, A)$ be a fuzzy soft set over V with its approximate function $\widetilde{F} : A \to \mathcal{P}(V)$ defined by

$$\widetilde{F}(e_1) = \{a_1|0.5, a_2|0.3, a_3|0.2, a_4|0.9\},$$
$$\widetilde{F}(e_2) = \{a_1|0.4, a_2|0.3, a_3|0.2, a_4|0.7\}.$$

Let $(\widetilde{K}, A)$ be a fuzzy soft set over E with its approximate function $\widetilde{K} : A \to \mathcal{P}(E)$ defined by

$$\widetilde{K}(e_1) = \{a_1a_2|0.3, a_2a_3|0.2, a_3a_4|0.2, a_4a_1|0.5, a_1a_3|0.2, a_2a_4|0.3\},$$
$$\widetilde{K}(e_2) = \{a_1a_2|0.3, a_2a_3|0.2, a_3a_4|0.2, a_4a_1|0.4, a_1a_3|0.2, a_2a_4|0.3\}.$$

It is clear that $\widetilde{H}(e_1) = (\widetilde{F}(e_1), \widetilde{K}(e_1))$ and $\widetilde{H}(e_2) = (\widetilde{F}(e_2), \widetilde{K}(e_2))$ are complete fuzzy graphs. Hence $\widetilde{G}$ is a complete fuzzy soft graph as shown in Figure 4.

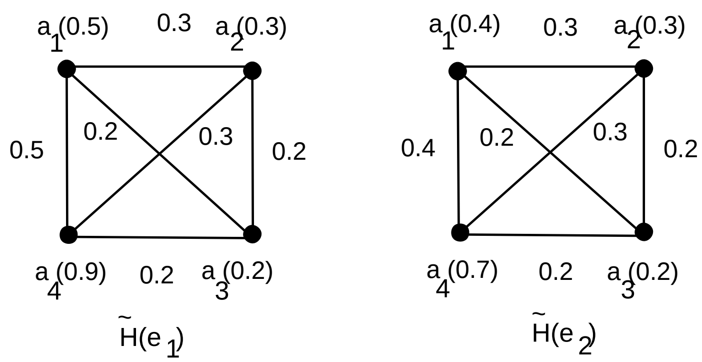

Figure 4: Complete fuzzy soft graph

Definition 2.9 Let $G^* = (V, E)$ be a crisp graph and $\widetilde{G}$ be a fuzzy soft graph of G^*. Then $\widetilde{G}$ is said to be a *regular fuzzy soft graph* if $\widetilde{H}(e)$ is a regular fuzzy graph for all $e \in A$. If $\widetilde{H}(e)$ is a regular fuzzy graph of degree r for all $e \in A$, then $\widetilde{G}$ is a r-regular fuzzy soft graph.

Example 2.10 Consider a crisp graph such that

$$V = \{a_1, a_2, a_3, a_4\} \text{ and } E = \{a_1a_2, a_2a_3, a_3a_4, a_4a_1\}.$$

Let $A = \{e_1, e_2, e_3, e_4\}$ be a parameter set and let $(\widetilde{F}, A)$ be a fuzzy soft set over V with its approximate function $\widetilde{F} : A \to \mathcal{P}(V)$ given by

$$\begin{aligned}
\widetilde{F}(e_1) &= \{a_1|0.3, a_2|0.4, a_3|0.5, a_4|0.2\},\\
\widetilde{F}(e_2) &= \{a_1|0.5, a_2|0.4, a_3|0.6, a_4|0.7\},\\
\widetilde{F}(e_3) &= \{a_1|0.3, a_2|0.5, a_3|0.3, a_4|0.7\},\\
\widetilde{F}(e_4) &= \{a_1|0.5, a_2|0.6, a_3|0.7, a_4|0.8\}.
\end{aligned}$$

Let $(\widetilde{K}, A)$ be a fuzzy soft set over E with its approximate function $\widetilde{K} : A \to \mathcal{P}(E)$ given by

$$\begin{aligned}
\widetilde{K}(e_1) &= \{a_1a_2|0.1, a_2a_3|0.2, a_3a_4|0.1, a_4a_1|0.2\},\\
\widetilde{K}(e_2) &= \{a_1a_2|0.2, a_2a_3|0.4, a_3a_4|0.2, a_4a_1|0.4\},\\
\widetilde{K}(e_3) &= \{a_1a_2|0.2, a_2a_3|0.3, a_3a_4|0.2, a_4a_1|0.3\},\\
\widetilde{K}(e_4) &= \{a_1a_2|0.5, a_2a_3|0.4, a_3a_4|0.5, a_4a_1|0.4\}.
\end{aligned}$$

By routine computations, it is easy to see that fuzzy graphs

$$\begin{aligned}
\widetilde{H}(e_1) &= (\widetilde{F}(e_1), \widetilde{K}(e_1)),\\
\widetilde{H}(e_2) &= (\widetilde{F}(e_2), \widetilde{K}(e_2)),\\
\widetilde{H}(e_3) &= (\widetilde{F}(e_3), \widetilde{K}(e_3)),\\
\widetilde{H}(e_4) &= (\widetilde{F}(e_4), \widetilde{K}(e_4)),
\end{aligned}$$

are regular are shown in Figure 5. Hence $\widetilde{G}$ is a regular fuzzy soft graph of G^*.

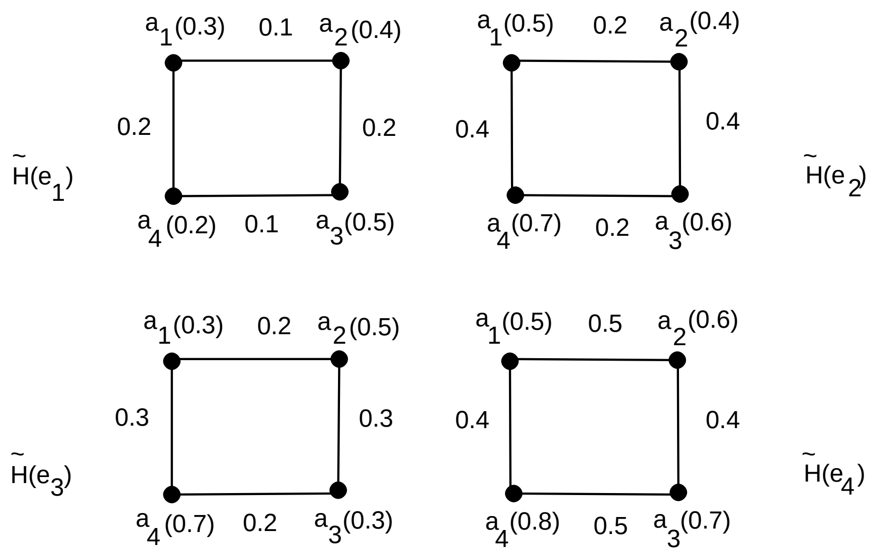

Figure 5: Regular fuzzy soft graph

Definition 2.11 Let $G^* = (V, E)$ be a simple graph and $\widetilde{G}$ be a fuzzy soft graph of G^*. Then $\widetilde{G}$ is said to be a *totally regular fuzzy soft graph* if $\widetilde{H}(e)$ is a totally regular fuzzy graph for all $e \in A$. If $\widetilde{H}(e)$ is a totally regular fuzzy graph of total degree r for all $e \in A$, then $\widetilde{G}$ is called r-totally regular fuzzy soft graph.

Example 2.12 Consider a simple graph $G^* = (V, E)$ where

$$V = \{a_1, a_2, a_3, a_4\} \text{ and } E = \{a_1a_2, a_2a_3, a_3a_4\}.$$

Let $A = \{e_1, e_2\}$ be a parameter set. Let $(\widetilde{F}, A)$ be a fuzzy soft set over V with its approximate function $\widetilde{F} : A \to \mathcal{P}(V)$ given by

$$\widetilde{F}(e_1) = \{a_1|0.3, a_2|0.2, a_3|0.2, a_4|0.3\},$$
$$\widetilde{F}(e_2) = \{a_1|0.5, a_2|0.4, a_3|0.5, a_4|0.6\}.$$

Let $(\widetilde{K}, A)$ be a fuzzy soft set over E with its approximate function $\widetilde{K} : A \to \mathcal{P}(E)$ given by

$$\widetilde{K}(e_1) = \{a_1a_2|0.1, a_2a_3|0.1, a_3a_4|0.1\},$$
$$\widetilde{K}(e_2) = \{a_1a_2|0.2, a_2a_3|0.1, a_3a_4|0.1\}.$$

Fuzzy graphs are $\widetilde{H}(e_1) = (\widetilde{F}(e_1), \widetilde{K}(e_1))$ and $\widetilde{H}(e_2) = (\widetilde{F}(e_2), \widetilde{K}(e_2))$ as shown in Figure 6. By routine computations, we have

$$\text{tdeg}(a_1) = 0.4, \quad \text{tdeg}(a_2) = 0.4,$$
$$\text{tdeg}(a_3) = 0.4, \quad \text{tdeg}(a_4) = 0.4,$$

in fuzzy graph $\widetilde{H}(e_1)$, so $\widetilde{H}(e_1)$ is a totally regular fuzzy graph.

Also,

$$\text{tdeg}(a_1) = 0.7, \quad \text{tdeg}(a_2) = 0.7,$$
$$\text{tdeg}(a_3) = 0.7, \quad \text{tdeg}(a_4) = 0.7,$$

in fuzzy graph $\widetilde{H}(e_2)$, so $\widetilde{H}(e_2)$ is a totally regular fuzzy graph. Hence $\widetilde{G}$ is totally regular fuzzy soft graph. But $\deg(a_1) = 0.1$, $\deg(a_2) = 0.2$ in fuzzy subgraph $\widetilde{H}(e_1)$. Since $\deg(a_1) \neq \deg(a_2)$, so $\widetilde{H}(e_1)$ is not regular fuzzy graph. Hence $\widetilde{G}$ is not regular fuzzy soft graph.

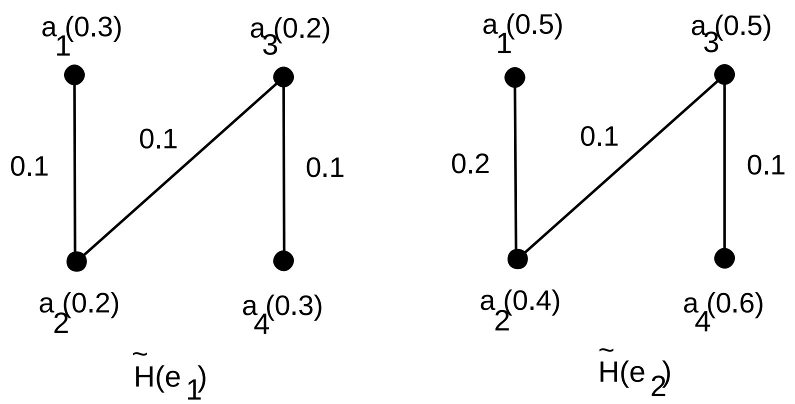

Figure 6: Fuzzy subgraphs

Example 2.13 Consider a simple graph $G^* = (V, E)$ as taken in Example 2.2.

Let $A = \{e_1, e_2, e_3\}$. Let $(\widetilde{F}, A)$ be a fuzzy soft set over V with its fuzzy approximate function $\widetilde{F} : A \to \mathcal{P}(V)$ given by

$$\begin{aligned}
\widetilde{F}(e_1) &= \{a_1|0.5, a_2|0.6, a_3|0.2\},\\
\widetilde{F}(e_2) &= \{a_1|0.2, a_2|0.1, a_3|0.4\},\\
\widetilde{F}(e_3) &= \{a_1|0.5, a_2|0.6, a_3|0.7\}.
\end{aligned}$$

Let $(\widetilde{K}, A)$ be a fuzzy soft set over E with its approximate function $\widetilde{K} : A \to \mathcal{P}(E)$ given by

$$\begin{aligned}
\widetilde{K}(e_1) &= \{a_1a_2|0.2, a_2a_3|0.2, a_3a_1|0.2\},\\
\widetilde{K}(e_2) &= \{a_1a_2|0.1, a_2a_3|0.1, a_3a_1|0.1\},\\
\widetilde{K}(e_3) &= \{a_1a_2|0.5, a_2a_3|0.5, a_3a_1|0.5\}.
\end{aligned}$$

By routine calculations, it is easy to see that $\widetilde{H}(e_1)$, $\widetilde{H}(e_2)$ and $\widetilde{H}(e_3)$ are regular fuzzy graphs of G^*. Hence $\widetilde{G}$ is a regular fuzzy soft graph. But $\widetilde{H}(e)$ is not a totally regular fuzzy graph for all $e \in A$. Hence $\widetilde{G}$ is not a totally regular fuzzy soft graph.

Example 2.14 Consider a simple graph $G^* = (V, E)$ where

$$V = \{a_1, a_2, a_3, a_4, a_5\} \text{ and } E = \{a_1a_2, a_2a_3, a_3a_4, a_4a_5, a_5a_1\}.$$

Let $A = \{e_1, e_2\}$ and $(\widetilde{F}, A)$ be a fuzzy soft set over V with its fuzzy approximate function $\widetilde{F} : A \to \mathcal{P}(V)$ given by

$$\begin{aligned}
\widetilde{F}(e_1) &= \{a_1|0.5, a_2|0.5, a_3|0.5, a_4|0.5, a_5|0.5\},\\
\widetilde{F}(e_2) &= \{a_1|0.3, a_2|0.3, a_3|0.3, a_4|0.3, a_5|0.3\}.
\end{aligned}$$

Let $(\widetilde{K}, A)$ be a fuzzy soft set over E with its fuzzy approximate function $\widetilde{K} : A \to \mathcal{P}(E)$ given by

$$\begin{aligned}
\widetilde{K}(e_1) &= \{a_1a_2|0.4, a_2a_3|0.4, a_3a_4|0.4, a_4a_5|0.4, a_5a_1|0.4\},\\
\widetilde{K}(e_2) &= \{a_1a_2|0.2, a_2a_3|0.2, a_3a_4|0.2, a_4a_5|0.2, a_5a_1|0.2\}.
\end{aligned}$$

Clearly, $\deg(a_i) = 0.8$ in fuzzy graph $\widetilde{H}(e_1)$ and $\deg(a_i) = 0.4$ in fuzzy graph $\widetilde{H}(e_2)$ for $i = 1, 2, 3, 4, 5$. So $\widetilde{G}$ is a regular fuzzy soft graph. Also $\mathrm{tdeg}(a_i) = 1.3$ in $\widetilde{H}(e_1)$ and $\mathrm{tdeg}(a_i) = 0.7$ in $\widetilde{H}(e_2)$ for $i = 1, 2, 3, 4, 5$. Hence $\widetilde{G}$ is totally regular fuzzy soft graph.

We have seen, in the above examples, there is no relationship between regular and totally regular fuzzy soft graph. So we proof the following theorems.

Theorem 2.15 *Let $G^* = (V, E)$ be a simple graph and $\widetilde{G}$ be a fuzzy soft graph of G^*. If $\widetilde{G}$ is a regular fuzzy soft graph and $\widetilde{F}$ is a constant function in fuzzy graph $\widetilde{H}(e_i)$ of G^* for all $e_i \in A$ for $i = 1, 2, 3, ..., n$. Then $\widetilde{G}$ is a totally regular fuzzy soft graph.*

Proof. Suppose that $\widetilde{G}$ is a regular fuzzy soft graph and $\widetilde{F}$ is a constant function. Then $\widetilde{F}(e_i)(a) = c_i$, c_i is a constant, $c_i \in [0,1], \forall\, a \in V, \forall\, e_i \in A$ for $i = 1, 2, 3, ..., n$ and $\deg(a) = r_i$ in fuzzy graphs $\widetilde{H}(e_i)$, $\forall\, e_i \in A$ for $i = 1, 2, 3, ..., n$ and $\forall\, a \in V$. Since $\mathrm{tdeg}(a) = \deg(a) + \widetilde{F}(e_i)(a)$. This implies $\mathrm{tdeg}(a) = r_i + c_i$ in fuzzy graphs $\widetilde{H}(e_i)$, $\forall\, e_i \in A$ for $i = 1, 2, 3, ..., n$ and for all $a \in V$. Hence $\widetilde{G}$ is a totally regular fuzzy soft graph. ∎

Theorem 2.16 *Let $G^* = (V, E)$ be a simple graph and $\widetilde{G}$ be a fuzzy soft graph of G^*. If $\widetilde{G}$ is a totally regular fuzzy soft graph and $\widetilde{F}$ is a constant function in fuzzy graph $\widetilde{H}(e_i)$ for all $e_i \in A$ for $i = 1, 2, 3..., n$. Then $\widetilde{G}$ is a regular fuzzy soft graph.*

Proof. Suppose that $\widetilde{G}$ is a totally regular fuzzy soft graph and $\widetilde{F}$ is a constant function. Then $\widetilde{F}(e_i)(a) = c_i$, c_i is a constant, $c_i \in [0,1]$, $\forall\, a \in V$, $\forall\, e_i \in A$ for $i = 1, 2, 3, ..., n$ and $\mathrm{tdeg}(a) = r_i$ in $\widetilde{H}(e_i)$, $\forall\, e_i \in A$ for $i = 1, 2, 3, ..., n$ and for all $a \in V$. As $\mathrm{tdeg}(a) = \deg(a) + \widetilde{F}(e_i)(a)$ in $\widetilde{H}(e_i)$, $\forall\, e_i \in A$ for $i = 1, 2, 3, ..., n$ and for all $a \in V$. This implies $\deg(a) = \mathrm{tdeg}(a) - \widetilde{F}(e_i)(a)$ in $\widetilde{H}(e_i)$, $\forall\, e_i \in A$ for $i = 1, 2, 3, ..., n$ and for all $a \in V$. This implies $\deg(a) = r_i - c_i$ in $\widetilde{H}(e_i)$, $\forall\, e_i \in A$ for $i = 1, 2, 3, ..., n$ and for all $a \in V$. Hence $\widetilde{G}$ is a regular fuzzy soft graph. ∎

Theorem 2.17 *If $\widetilde{G}$ is both regular and totally regular fuzzy soft graph. Then $\widetilde{F}$ is a constant function in $\widetilde{H}(e_i)$ of G^* for all $e_i \in A$ for $i = 1, 2, 3, ..., n$.*

Proof. Let $\widetilde{G}$ be both regular and totally regular fuzzy soft graph. Then $\deg(a) = r_i$ and $\mathrm{tdeg}(a) = s_i$ in fuzzy subgraphs $\widetilde{H}(e_i)$ for all $e_i \in A$ for $i = 1, 2, 3, ..., n$ and for all $a \in V$. This implies $\deg(a) + \widetilde{F}(e_i)(a) = s_i$ in $\widetilde{H}(e_i)$ for all $e_i \in A$ for $i = 1, 2, 3, ..., n$ and for all $a \in V$. This implies $r_i + \widetilde{F}(e_i)(a) = s_i$ in $\widetilde{H}(e_i)$ for all $e_i \in A$ for $i = 1, 2, 3, ..., n$ and for all $a \in V$. This implies $\widetilde{F}(e_i)(a) = s_i - r_i$ in $\widetilde{H}(e_i)$ for all $e_i \in A$ for $i = 1, 2, 3, ..., n$ and for all $a \in V$. Hence $\widetilde{F}$ is a constant function in $\widetilde{H}(e_i)$ of G^* for all $e_i \in A$ for $i = 1, 2, 3, ..., n$. ∎

The converse of the above theorem is not true in general, that is, if $\widetilde{F}(e)$ is a constant function then $\widetilde{G}$ need not be both regular and totally regular fuzzy soft graph.

Example 2.18 Consider a simple graph $G^* = (V, E)$ as taken in Example 2.7. Let $A = \{e_1, e_2\}$. Let $(\widetilde{F}, A)$ be a fuzzy soft set over V with its approximate function $\widetilde{F} : A \to \mathcal{P}(V)$ given by

$$\widetilde{F}(e_1) = \{a_1|0.6, a_2|0.6, a_3|0.6, a_4|0.6\},$$
$$\widetilde{F}(e_2) = \{a_1|0.4, a_2|0.4, a_3|0.4, a_4|0.4\}.$$

Let $(\widetilde{K}, A)$ be a fuzzy soft set over E with its approximate function $\widetilde{K} : A \to \mathcal{P}(E)$ given by

$$\widetilde{K}(e_1) = \{a_1a_2|0.1, a_2a_3|0.2, a_3a_4|0.5, a_4a_1|0.3\},$$
$$\widetilde{K}(e_2) = \{a_1a_2|0.2, a_2a_3|0.4, a_3a_4|0.3, a_4a_1|0.1\}.$$

Clearly, $\widetilde{F}(e_i)$ is constant in fuzzy graphs $\widetilde{H}(e_i)$ for $i = 1, 2$. But $\widetilde{G}$ is neither regular nor totally regular fuzzy soft graph.

Theorem 2.19 *Let $\widetilde{G}$ be a fuzzy soft graph over an odd cycle $G^* = (V, E)$. Then $\widetilde{G}$ is regular fuzzy soft graph if and only if $\widetilde{K}$ is a constant function in fuzzy subgraph $\widetilde{H}(e_i)$ over $H^*(e_i)$, where $H^*(e_i)$ is an odd cycle for all $e_i \in A$ for $i = 1, 2, 3, ..., n$.*

Proof. Suppose that $\widetilde{K}$ is a constant function. Then $\widetilde{K}(e_i)(ab) = c_i$, a constant, $c_i \in [0, 1]$, for all $e_i \in A$ for $i = 1, 2, 3, ..., n$ in fuzzy graph $\widetilde{H}(e_i)$ and for all $ab \in E$. So $\deg(a) = 2c_i$, in fuzzy graph $\widetilde{H}(e_i)$ for all $e_i \in A$ for $i = 1, 2, 3, ..., n$ and for all $a \in V$. Hence $\widetilde{G}$ is regular fuzzy soft graph.

Conversely, assume that $\widetilde{G}$ is a regular fuzzy soft graph of G^*. Let $d_1, d_2, d_3, ...,$ d_{2n+1} be the edges of G^* in that order. Let $\widetilde{K}(e_i)(d_1) = r_i$ in $\widetilde{H}(e_i)$ for all $e_i \in A$ for $i = 1, 2, 3, ..., n$. Since $\widetilde{H}(e_i)$ is s_i- regular fuzzy graphs for all $e_i \in A$ for $i = 1, 2, 3, ..., n$. Then $\widetilde{K}(e_i)(d_2) = s_i - r_i$ for all $e_i \in A$ for $i = 1, 2, 3, ..., n$. $\widetilde{K}(e_i)(d_3) = s_i - (s_i - r_i) = r_i$ and so on. Therefore,

$$\widetilde{K}(e_i)(d_j) = \begin{cases} r_i, & \text{if } j \text{ is odd} \\ s_i - r_i, & \text{if } j \text{ is even} \end{cases}$$

So $\widetilde{K}(e_i)(d_1) = \widetilde{K}(e_i)(d_{2n+1}) = r_i$ for all $e_i \in A$ for $i = 1, 2, 3, ..., n$. Thus, if d_1 and d_{2n+1} incident at vertex v, then $\deg(v) = s_i$ $\widetilde{H}(e_i)$ for all $e_i \in A$ for $i = 1, 2, 3, ..., n$. Then $\widetilde{K}(e_i)(d_1) + \widetilde{K}(e_i)(d_{2n+1}) = s_i$ for all $e_i \in A$ for $i = 1, 2, ..., n$.

$$r_i + r_i = s_i, \qquad 2r_i = s_i, \qquad r_i = \frac{s_i}{2}$$

So

$$s_i - r_i = s_i - \frac{s_i}{2} = \frac{s_i}{2}.$$

Therefore, $\widetilde{K}(e_i)(d_j) = \frac{s_i}{2}$ in fuzzy graphs $\widetilde{H}(e_i)$ for all j and for $i = 1, 2, ..., n$. Hence $\widetilde{K}$ is a constant function. ∎

Theorem 2.20 *Let $\widetilde{G}$ be a fuzzy soft graph over an even cycle $G^* = (V, E)$. Then $\widetilde{G}$ is regular fuzzy soft graph if and only if $\widetilde{K}$ is a constant function or alternate edges have same membership degrees in fuzzy subgraph $\widetilde{H}(e_i)$ over $H^*(e_i)$, where $H^*(e_i)$ is an even cycle for all $e_i \in A$ for $i = 1, 2, ..., n$.*

Proof. If either $\widetilde{K}$ is a constant function or alternate edges have same membership degrees. Then $\widetilde{G}$ is regular fuzzy soft graph.

Conversely, assume that $\widetilde{G}$ is a regular fuzzy soft graph of G^*. Let $d_1, d_2, d_3, ...,$ d_{2n} be the edges of G^* in that order. Let $\widetilde{K}(e_i)(d_1) = r_i$ in $\widetilde{H}(e_i)$ for all $e_i \in A$ for $i = 1, 2, 3, ..., n$. Since $\widetilde{H}(e_i)$ is s_i- regular fuzzy graphs, for $i = 1, 2, 3, ..., n$. Then $\widetilde{K}(e_i)(d_2) = s_i - r_i$, for $i = 1, 2, 3, ..., n$. $\widetilde{K}(e_i)(d_3) = s_i - (s_i - r_i) = r_i$ and so on. Therefore,

$$\widetilde{K}(e_i)(d_j) = \begin{cases} r_i, & \text{if } j \text{ is odd} \\ s_i - r_i, & \text{if } j \text{ is even} \end{cases}$$

Proceeding as theorem 2.19, if $r_i = s_i - r_i$, then $\widetilde{K}$ is a constant function. If $r_i \neq s_i - r_i$, then alternate edges have same membership degrees. ∎

Note that the above theorems do not hold for totally regular fuzzy soft graphs. To illustrate we consider the following examples.

For example, consider an odd cycle $G^* = (V, E)$, where $V = \{a_1, a_2, a_3, a_4, a_5\}$ and $E = \{a_1a_2, a_2a_3, a_3a_4, a_4a_5, a_5a_1\}$. Let $A = \{e_1, e_2\}$ and $(\widetilde{F}, A)$ be a fuzzy soft set over V with its approximate function $\widetilde{F} : A \to \mathcal{P}(V)$ defined by

$$\begin{aligned}\widetilde{F}(e_1) &= \{a_1|0.5, a_2|0.4, a_3|0.3, a_4|0.3, a_5|0.5\},\\ \widetilde{F}(e_2) &= \{a_1|0.5, a_2|0.4, a_3|0.4, a_4|0.5, a_5|0.4\}.\end{aligned}$$

Let $(\widetilde{K}, A)$ be a fuzzy soft set over E with its approximate function $\widetilde{K} : A \to \mathcal{P}(E)$ by

$$\begin{aligned}\widetilde{K}(e_1) &= \{a_1a_2|0.2, a_2a_3|0.2, a_3a_4|0.3, a_4a_5|0.2, a_5a_1|0.1\},\\ \widetilde{K}(e_2) &= \{a_1a_2|0.1, a_2a_3|0.3, a_3a_4|0.1, a_4a_5|0.2, a_5a_1|0.2\}.\end{aligned}$$

By routine calculations, it is easy to see that $\widetilde{H}(e_1) = (\widetilde{F}(e_1), \widetilde{K}(e_1))$ and $\widetilde{H}(e_2) = (\widetilde{F}(e_2), \widetilde{K}(e_2))$ are totally regular fuzzy graphs of G^*. Hence $\widetilde{G}$ is totally regular fuzzy soft graph. But $\widetilde{K}$ is not a constant function.

Now, we take an even cycle $G^* = (V, E)$ as taken in Example 2.7.

Let $A = \{e_1, e_2\}$. Let $(\widetilde{F}, A)$ be a fuzzy soft set over V with its approximate function $\widetilde{F} : A \to \mathcal{P}(V)$ given by

$$\begin{aligned}\widetilde{F}(e_1) &= \{a_1|0.7, a_2|0.8, a_3|0.6, a_4|0.5\},\\ \widetilde{F}(e_2) &= \{a_1|0.8, a_2|0.6, a_3|0.4, a_4|0.6\}.\end{aligned}$$

Let $(\widetilde{K}, A)$ be a fuzzy soft set over E with its fuzzy approximate function $\widetilde{K} : A \to \mathcal{P}(E)$ given by

$$\begin{aligned}\widetilde{K}(e_1) &= \{a_1a_2|0.2, a_2a_3|0.2, a_3a_4|0.4, a_4a_1|0.3\},\\ \widetilde{K}(e_2) &= \{a_1a_2|0.1, a_2a_3|0.4, a_3a_4|0.3, a_4a_1|0.2\}.\end{aligned}$$

$\widetilde{H}(e_1)$ and $\widetilde{H}(e_2)$ are totally regular fuzzy graphs of G^*. Hence $\widetilde{G}$ is totally regular fuzzy soft graph. But in $\widetilde{H}(e)$, neither $\widetilde{K}$ is a constant function nor alternate edges have same membership degrees for all $e \in A$.

Theorem 2.21 *If $\widetilde{G}$ is a regular fuzzy soft graph and $\widetilde{F}$ is a constant function, then $\widetilde{G}^c$ is a regular fuzzy soft graph.*

Theorem 2.22 *If $\widetilde{G}$ is a totally regular fuzzy soft graph and $\widetilde{F}$ is a constant function, then $\widetilde{G}^c$ is a totally regular fuzzy soft graph.*

Theorem 2.23 *A regular fuzzy soft graph $\widetilde{G}$ on G^* with $|V| \geq 3$ and $\widetilde{H}(e_i)$ is regular fuzzy graph of degree $s_i > 0$, $i = 1, 2, \cdots, n$ have no end node.*

Proof. Since $\widetilde{H}(e_i)$ is regular fuzzy graph of degree s_i, so $\deg_{\widetilde{H}(e_i)}(a) = s_i$ for all $a \in V$, for all $e_i \in A$ for $i = 1, 2, ..., n$. As $s_i > 0$, $\deg_{\widetilde{H}(e_i)}(a) > 0$ for all $a \in V$. That is, every node is adjacent to at least one other node. On contrary, suppose that b is an end node, then $\deg_{\widetilde{H}(e_i)}(b) = s_i = \widetilde{K}_{\widetilde{H}(e_i)}(ab)$. Since $\widetilde{H}(e_i)$ is regular fuzzy graph with $|V| \geq 3$ for $i = 1, 2, ..., n$ then a must be adjacent to an other node $c \neq b$. Then $\deg_{\widetilde{H}(e_i)}(a) = \widetilde{K}_{\widetilde{H}(e_i)}(ab) + \widetilde{K}_{\widetilde{H}(e_i)}(ac) > \widetilde{K}_{\widetilde{H}(e_i)}(ab)$ for $i = 1, 2, ..., n$. $\Rightarrow \deg_{\widetilde{H}(e_i)}(a) > s_i$, which is a contradiction to the fact that $\widetilde{H}(e_i)$ is regular fuzzy graph of degree s_i for $i = 1, 2, ..., n$. Hence $\widetilde{G}$ have no end node. ■

Definition 2.24 Let $\widetilde{G}$ be a fuzzy soft graph on G^*. Then $\widetilde{G}$ is called a *partially regular fuzzy soft graph* if $\widetilde{H}(e)$ is partially regular fuzzy graph for all $e \in A$.

If $\widetilde{G}$ is both regular and partially regular fuzzy soft graph, then $\widetilde{G}$ is called a *full regular fuzzy soft graph.*

Example 2.25 Consider a simple graph $G^* = (V, E)$ as taken in Example 2.7. Let $A = \{e_1, e_2\}$ and let $(\widetilde{F}, A)$ be a fuzzy soft set over V with its approximate function $\widetilde{F} : A \to \mathcal{P}(V)$ given by

$$\widetilde{F}(e_1) = \{a_1|0.4, a_2|0.5, a_3|0.7, a_4|0.3\},$$
$$\widetilde{F}(e_2) = \{a_1|0.9, a_2|0.6, a_3|0.8, a_4|0.4\}.$$

Let $(\widetilde{K}, A)$ be a fuzzy soft set over E with its approximate function $\widetilde{K} : A \to \mathcal{P}(E)$ by

$$\widetilde{K}(e_1) = \{a_1a_2|0.3, a_2a_3|0.4, a_3a_4|0.1, a_4a_1|0.2\},$$
$$\widetilde{K}(e_2) = \{a_1a_2|0.5, a_2a_3|0.4, a_3a_4|0.2, a_4a_1|0.3\}.$$

Fuzzy subgraphs are $\widetilde{H}(e_1) = (\widetilde{F}(e_1), \widetilde{K}(e_1))$ and $\widetilde{H}(e_2) = (\widetilde{F}(e_2), \widetilde{K}(e_2))$. Since the underlying graphs of $\widetilde{H}(e_1)$ and $\widetilde{H}(e_2)$ are regular so $\widetilde{H}(e_1)$ and $\widetilde{H}(e_2)$ are partially regular fuzzy graphs as shown in Figure 7.

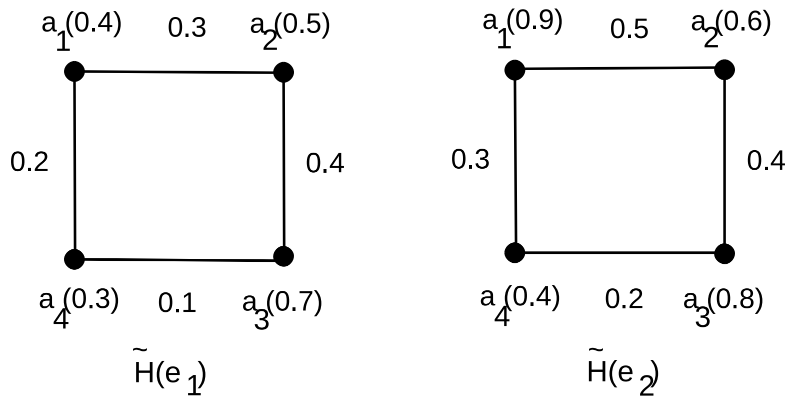

Figure 7: Fuzzy subgraphs

Hence $\widetilde{G}$ is a partially regular fuzzy soft graph of G^*.

Remark 1 Every regular fuzzy soft graph may not be a partially regular fuzzy soft graph.

Remark 2 Every partially regular fuzzy soft graph may not be a regular fuzzy soft graph.

Theorem 2.26 *Let $\widetilde{G}$ be a fuzzy soft graph such that $\widetilde{K}$ is constant in $\widetilde{H}(e_i)$ for all $e_i \in A$ for $i = 1, 2, ..., n$. Then $\widetilde{G}$ is a regular fuzzy soft graph if and only if $\widetilde{G}$ is a partially regular fuzzy soft graph.*

Proof. Suppose that $\widetilde{K}(e_i)(ab) = c_i$, a constant for all $ab \in E$ and for all $e_i \in A$ for $i = 1, 2, .., n$. Now, $\deg_{\widetilde{H}(e_i)}(a) = \sum_{ab\in E} \widetilde{K}(e_i)(ab) = \sum_{ab\in E} c_i = c_i\deg_{H^*(e_i)}(a)$ for all $a \in V$, $e_i \in A$ for $i = 1, 2, ..., n$. Let $\widetilde{G}$ be a regular fuzzy soft graph. Then $\deg_{\widetilde{H}(e_i)}(a) = c_i\deg_{H^*(e_i)}(a) = t_i$ for all $a \in V$ and for all $e_i \in A$ for $i = 1, 2, ..., n$. $\Rightarrow \deg_{H^*(e_i)}(a) = \frac{t_i}{c_i}$ for all $a \in V$, $e_i \in A$ for $i = 1, 2, ..., n$. $\Rightarrow H^*(e_i)$ is regular graph for all $e_i \in A$ for $i = 1, 2, ..., n$. So $\widetilde{H}(e_i)$ is a partially regular fuzzy graph and hence $\widetilde{G}$ is a partially regular fuzzy soft graph.

Conversely, suppose that $\widetilde{G}$ is a partially regular fuzzy soft graph. Assume that $H^*(e_i)$ is regular of degree $'s_i'$ for all $e_i \in A$ for $i = 1, 2, ..., n$. Then $\deg_{\widetilde{H}(e_i)}(a) = c_i\deg_{H^*(e_i)}(a) = c_i s_i$ for all $a \in V$ and for all $e_i \in A$ for $i = 1, 2, ..., n$. Hence $\widetilde{G}$ is a regular fuzzy soft graph. ∎

Remark 3 A regular or partially regular fuzzy soft graph need not be a full regular fuzzy soft graph.

Theorem 2.27 *Let $\widetilde{G}$ be a strong fuzzy soft graph such that $\widetilde{F}$ is a constant function. Then $\widetilde{G}$ is a regular fuzzy soft graph if and only if $\widetilde{G}$ is a partially regular fuzzy soft graph.*

Proof. Suppose that $\widetilde{F}(e_i)(a) = c_i$, where c_i is a constant for all $e_i \in A$ and for all $a \in V$ for $i = 1, 2, ..., n$. Since $\widetilde{G}$ is a strong fuzzy soft graph, then $\widetilde{H}(e_i)$ is a strong fuzzy graph for all $e_i \in A$ for $i = 1, 2, ..., n$. This implies $\widetilde{K}(e_i)(ab) = \min(\widetilde{F}(e_i)(a), \widetilde{F}(e_i)(b)) = c_i$ for all $ab \in E$. Thus $\widetilde{K}$ is a constant function. Proceeding in the same way as in Theorem 2.26 we proof the theorem. ∎

4. Conclusions

Fuzzy graph has numerous applications in modern sciences and technology, especially in research areas of computer science including database theory, data mining, neural networks, expert systems, cluster analysis, control theory, and image capturing. Fuzzy sets and soft sets are two different soft computing models for representing vagueness and uncertainty. We have applied these soft computing models in combination to study vagueness and uncertainty in graphs. We have investigated some properties of regular fuzzy soft graphs. We plan to extend our research of fuzzification to (1) Interval-valued fuzzy soft graphs; (2) Bipolar fuzzy soft regular graphs.

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A NOTE ON THE TRIPARTITE RAMSEY NUMBERS $r_t(C_4; 2)$ AND $r_t(C_4; 3)$

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Abstract. The k-colored tripartite Ramsey numbers $r_t(G; k)$ is the smallest positive integer n such that any k-coloring of lines of a complete tripartite graph $K_{n,n,n}$ there always exists a monochromatic subgraph isomorphic to G. The values of $r_t(C_4; 2) = 3$, and $r_t(C_4; 3) = 7$ are discussed in the article *The tripartite Ramsey numbers* $r_t(C_4; 2)$ *and* $r_t(C_4; 3)$ of the Italian Journal of Pure and Applied Mathematics, n. 33-2014. However, there are our technical mistakes on three figures of the article. In this note we correct these mistakes.

Keywords and phrases: tripartite Ramsey numbers, bipartite Ramsey numbers, Ramsey numbers, tripartite graphs.

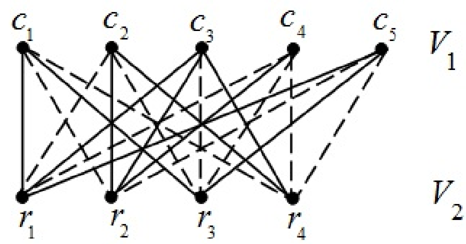
AMS Subject Classification: 05C55; 05D10.

Mistakes and Corrections

The details of the discussions on $r_t(C_4;2) = 3$ and $r_t(C_4;3) = 7$ are shown on pages 383-400 of [1]. There are mistakes, due to our technical problems, on Figure 2.1, Figure 2.2, and Figure 2.3, i.e. lines of the graphs are not shown as intended. The corrections are given below.

(1) Figure 2.1 on page 385 of [1] should be replaced by

(a)



(b)

$$\begin{array}{cc} & \begin{array}{ccccc} c_1 & c_2 & c_3 & c_4 & c_5 \end{array} \\ \begin{array}{c} r_1 \\ r_2 \\ r_3 \\ r_4 \end{array} & \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Figure 2.1

(2) Figure 2.2 on page 386 of [1] should be replaced by

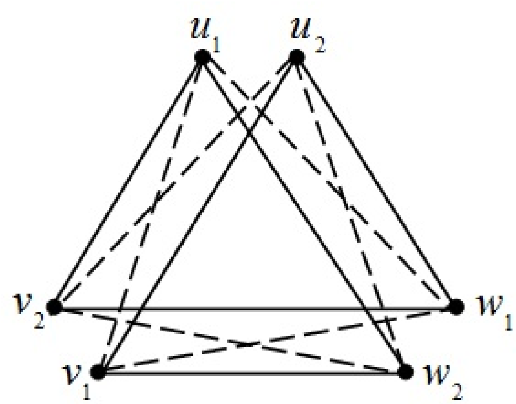


Figure 2.2.

(3) Figure 2.3 on page 388 of [1] should be replaced by

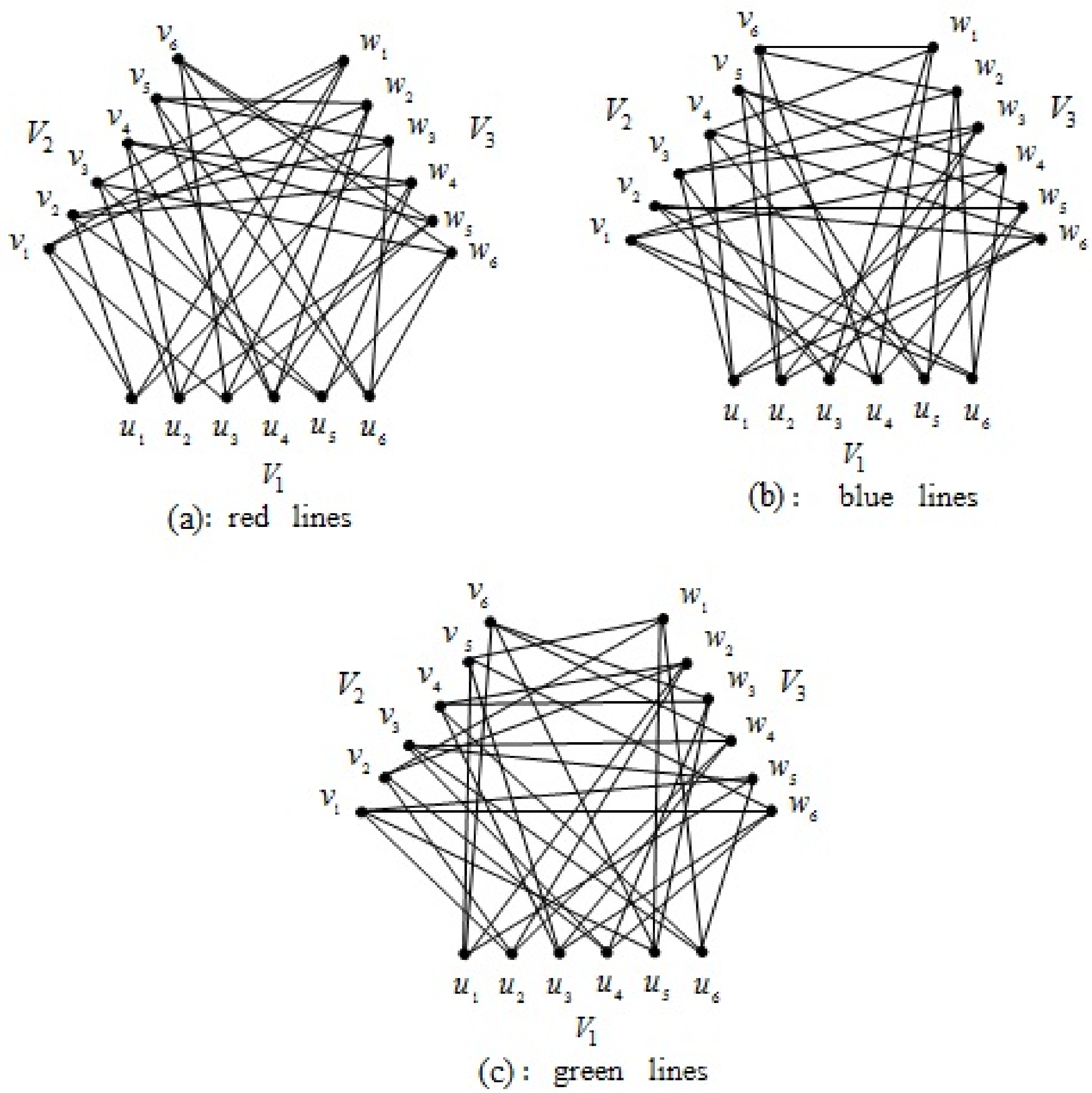


Figure 2.3.

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[1] Buada, S., Samana, D., Longani, V., *The Tripartite Ramsey Numbers $r_t(C_4;2)$ and $r_t(C_4;3)$*, Italian Journal of Pure and Applied Mathematics, 33 (2014), 383-400.

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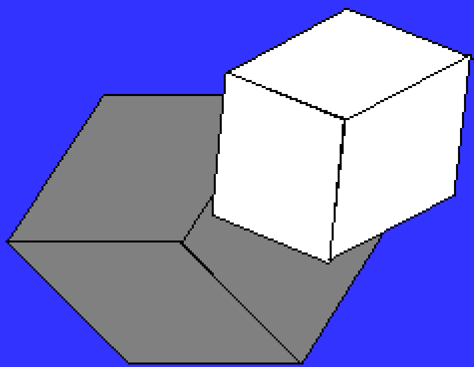

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