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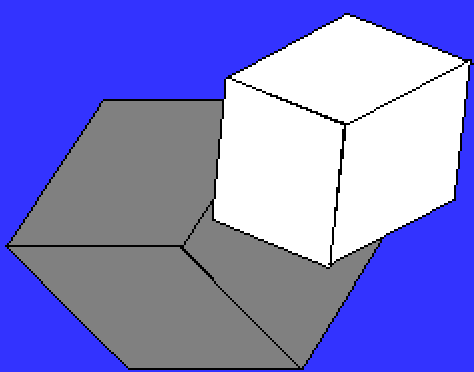

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A NOTE ON DIMENSION OF WEAK HYPERVECTOR SPACES

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Abstract. In this paper, we study the dimension of weak hypervector spaces. First, we define linearly dependence and independence of vectors and also basis of weak hypervector spaces and then prove some results in this field. Finally, we consider weak subhypervector spaces of such spaces.

Keywords: hypervector space, linearly dependent and independent, basis, coordinate, dimension, normal hypervector space.

2000 AMS Subject Classifications: 46J10, 47B48.

1. Introduction

The concept of hyperstructure was first introduced by Marty [4] in 1934 and has attracted attention of many authors in last decades and has constructed some other structures such as hyperrings, hypergroups, hypermodules, hyperfields, and hypervector spaces. These constructions have been applied to many disciplines such as geometry, hypergraphs, binary relations, combinatorics, codes, cryptography, probability, and etc. A wealth of applications of this concepts is given in [2], [3], [10] and [11].

In 1988, the concept of hypervector space was first introduced by Scafati-Tallini. She studied more properties of this new structure in [7] and [8].

In the present paper, we want to state the concepts of linearly dependence and independence, basis, dimension of weak hypervector spaces and prove some results in this field. In [1], the definitions of these concepts are given and using them some results for anti hypervector spaces are proved. In the mentioned paper, some hypervector spaces have been introduced *dimensionless* that means that the hypervector spaces don't have any collection of linearly independent vectors. Since, in [9], it was proved that a normed weak right or left antidistributive hypervector space is a normed classical vector space, we would like to introduce different definitions of mentioned concepts for weak hypervector spaces that are more universal-such that don't create the dimensionless spaces. This paper is arranged as follows. In Section 3, we identify a certain element of weak hypervector spaces and then introduce a certain class of such spaces that we call them by *normal* and show the defined coordinate of any element in finite dimensional normal hypervector spaces is unique. In Section 4, we consider weak subhypervector spaces and prove some results.

2. Linear dependence and independence of vectors

Definition 2.1. [9] Let $(X, +)$ be an abelian group and F be a field. Then a weak hypervector space is a quadruple $(X, +, o, F)$, where o is a mapping

$$o : F \times X \longrightarrow P_*(X)$$

such that the following conditions are satisfied:

(i) $\forall \mathrm{a} \in \mathrm{F}, \forall \mathrm{x}, \mathrm{y} \in \mathrm{X},\ [\mathrm{ao}(\mathrm{x}+\mathrm{y})] \cap [\mathrm{aox}+\mathrm{aoy}] \neq \emptyset$,

(ii) $\forall \mathrm{a}, \mathrm{b} \in \mathrm{F}, \forall \mathrm{x} \in \mathrm{X},\ [(\mathrm{a}+\mathrm{b})\mathrm{ox}] \cap [\mathrm{aox}+\mathrm{box}] \neq \emptyset$,

(iii) $\forall \mathrm{a}, \mathrm{b} \in \mathrm{F}, \forall \mathrm{x} \in \mathrm{X},\ \mathrm{ao}(\mathrm{box}) = (\mathrm{ab})\mathrm{ox}$,

(iv) $\forall \mathrm{a} \in \mathrm{F}, \forall \mathrm{x} \in \mathrm{X},\ \mathrm{ao}(-\mathrm{x}) = (-\mathrm{a})\mathrm{ox} = -(\mathrm{aox})$,

(v) $\forall \mathrm{x} \in \mathrm{X},\ \mathrm{x} \in 1\mathrm{ox}$.

We call (i) and (ii) *weak right* and *left distributive* laws, respectively. Note that the set $ao(box)$ in (3) is of the form $\bigcup_{y \in box} aoy$.

In the following we give some examples of weak hypervector spaces.

Example 2.2. The set $\mathbb{R}^2$ with usual sum and the following scalar product is a weak hypervector space on $\mathbb{R}$.

$$aox = \begin{cases} \overline{Ox} & x \neq 0, \\ \{0\} & x = 0, \end{cases}$$

where $\overline{Ox}$ is the line passing through the origin and the point x.

Example 2.3. The set $\mathbb{R}^2$ with usual sum and the following scalar product is a weak hypervector space on $\mathbb{R}$.

$$aox = \begin{cases} \text{segment } -Ox & x \neq 0 \wedge a < 0 \\ \text{segment } Ox & x \neq 0 \wedge a > 0 \\ \{0\} & x = 0 \vee a = 0 \end{cases},$$

where the segment Ox and the segment $-Ox$ are the closed segments connecting the origin to the point x and $-x$, respectively.

Example 2.4. The set $\mathbb{C}$ with usual sum and the following scalar product is a weak hypervector space on $\mathbb{R}$.

$$aox = \begin{cases} \{re^{i\theta} : 0 \leq r \leq |a||x|, \theta = \arg(x)\} & x \neq 0, \\ \{0\} & x = 0. \end{cases}$$

Example 2.5. The set $\mathbb{C}$ with usual sum and the following scalar product is a weak hypervector space on $\mathbb{R}$.

$$aox = \begin{cases} \{re^{i\theta} : 0 \leq r \leq |a||x|, 0 \leq \theta \leq 2\pi\} & x \neq 0, \\ \{0\} & x = 0. \end{cases}$$

Lemma 2.6. *If X is a weak hypervector space over F, $0 \neq a \in F$ and $x \in X$, then there exists a z in aox such that we have $x \in a^{-1}oz$.*

Proof. Since a is nonzero, from $x \in 1ox$ we obtain $x \in a^{-1}o(aox)$. So there exists a z in aox such that $x \in a^{-1}oz$. ∎

By the above lemma we have the following definition.

Definition 2.7. Let X is a weak hypervector space over F, $a \in F$ and $x \in X$. Essential point of aox, that we denote it by e_{aox}, for $a \neq 0$ is the element of aox such that $x \in a^{-1}oe_{aox}$. For $a = 0$, we define $e_{aox} = 0$.

Remark 2.8. Note that e_{aox} is not unique, necessarily. Hence we denote the set of all these elements by E_{aox}. When in this note we use e_{aox} in an equation, we intend is any element of E_{aox}. See the examples listed above. In Examples 2.3 and 2.4, E_{aox} is equals to the singleton sets

$$\begin{cases} \{-x\} & x \neq 0 \wedge a < 0, \\ \{x\} & x \neq 0 \wedge a > 0, \\ \{0\} & x = 0 \vee a = 0, \end{cases} \quad \text{and} \quad \begin{cases} \{|a|x\} & x \neq 0, \\ \{0\} & x = 0, \end{cases}$$

respectively, but in Examples 2.2 and 2.5, E_{aox} is equals to the sets

$$\begin{cases} \overline{Ox} & x \neq 0, \\ \{0\} & x = 0, \end{cases} \quad \text{and} \quad \begin{cases} \{re^{i\theta} : r = |a||x|, 0 \leq \theta \leq 2\pi\} & x \neq 0, \\ \{0\} & x = 0, \end{cases}$$

respectively.

Definition 2.9. A subset $M = \{x_1, ..., x_n\}$ of X is said to be linearly independent if the equation $0 = \sum_{i=1}^{n} e_{\alpha_i o x_i}$ implies that $\alpha_1 = \alpha_2 = ... = \alpha_n = 0$, where $\alpha_1, ..., \alpha_n$ are scalars. M is said to be linearly dependent if M isn't linearly independent.

An arbitrary subset M of X is linearly independent if every nonempty finite subset of M is linearly independent.

Definition 2.10. A subset M of X is said to be a basis of X if M is linearly independent and spans the elements of X. It means that for any x of X there exists scalars $\alpha_1, ..., \alpha_n$ such that $x = \sum_{i=1}^{n} e_{\alpha_i o x_i}$, where $\{x_1, ..., x_n\}$ is a subset of M. If there exists a finite basis for X, then X is said to be a finite dimensional weak hypervector space.

Definition 2.11. If X is a finite dimensional weak hypervector space, then an ordered basis of X is a finite sequence of vectors that is a basis of X.

Definition 2.12. Let $\{x_1, ..., x_n\}$ be an ordered basis of X and $x \in X$ such that $x = \sum_{i=1}^{n} e_{\alpha_i o x_i}$, where $\alpha_1, ..., \alpha_n$ are scalars. Then $(\alpha_1, ..., \alpha_n)$ is said to be the coordinate of x.

Remark 2.13. By Remark 2.8, it is clear that the coordinate of an element isn't unique, necessarily.

In the follow we introduce a special class of weak hypervector spaces.

Definition 2.14. Let X be a weak hypervector space over F with the following properties

(i) $(\mathrm{E}_{\mathrm{a_1ox}} + \mathrm{E}_{\mathrm{a_2ox}}) \cap \mathrm{E}_{\mathrm{(a_1+a_2)ox}} \neq \emptyset, \ \forall \mathrm{x} \in \mathrm{X}, \ \forall \mathrm{a_1, a_2} \in \mathrm{F}$,

(ii) $(\mathrm{E}_{\mathrm{aox_1}} + \mathrm{E}_{\mathrm{aox_2}}) \cap \mathrm{E}_{\mathrm{ao(x_1+x_2)}} \neq \emptyset, \ \forall \mathrm{x_1, x_2} \in \mathrm{X}, \ \forall \mathrm{a} \in \mathrm{F}$.

Then X is called normal weak hypervector space.

Lemma 2.15. *Let X is a weak hypervector space over F, $a, b \in F$ and $x \in X$. Then the following properties hold.*

(i) $x \in E_{1ox}$

(ii) *If $b \neq 0$, then $aoe_{box} = abox$*

(iii) $E_{-aox} = -E_{aox}$

(iv) *If $a \neq 0$, then there exists an $y \in X$ such that $x \in E_{aoy}$.*

(v) *If X is normal, then E_{aox} is singleton.*

Proof. (i) This is obtained from $x \in 1ox$, immediately.

(ii) Since $b \neq 0$, by Definition 2.7 we have $e_{box} \in box$ and $x \in b^{-1}oe_{box}$. The relation $e_{box} \in box$ together with

$$abox = ao(box) = \bigcup_{y \in box} aoy$$

yields that $aoe_{box} \subseteq abox$. Also the relation $x \in b^{-1}oe_{box}$ together with

$$aoe_{box} = abo(b^{-1}oe_{box}) = \bigcup_{y \in b^{-1}oe_{box}} aboy$$

yields that $abox \subseteq aoe_{box}$. Therefore we obtain $aoe_{box} = abox$.

(iii) By part (iv) of Definition 2.1 we have

$$\begin{aligned} E_{-aox} &= \{z; z \in -aox, x \in (-a)^{-1}oz\} \\ &= \{z; -z \in aox, x \in a^{-1}o(-z)\} \\ &= \{-z; z \in aox, x \in a^{-1}oz\} \\ &= -\{z; z \in aox, x \in a^{-1}oz\} = -E_{aox}. \end{aligned}$$

(iv) By setting $y = e_{a^{-1}ox}$, the assertion follows from the parts (i) and (ii).

(v) By normality of X we have $(E_{-aox} + E_{aox}) \cap E_{(-a+a)ox} \neq \emptyset$. Since by Definition 2.7, $E_{(-a+a)ox} = E_{0ox}$ is equal to zero, so we obtain $E_{-aox} + E_{aox} = 0$. Thus the assertion follows from this and part (iii). ∎

Remark 2.16. By part (v) of preceding lemma we can replace E by e in Definition 2.14.

Lemma 2.17. *Let X be a weak hypervector space over F. X is normal if and only if*

$$e_{a_1ox} + e_{a_2ox} = e_{(a_1+a_2)ox}, \ \forall x \in X, \ \forall a_1, a_2 \in F,$$

$$e_{aox_1} + e_{aox_2} = e_{ao(x_1+x_2)}, \ \forall x_1, x_2 \in X, \ \forall a \in F.$$

Proof. It is clear by part (v) of Lemma 2.10 and Remark 2.11. ∎

Remark 2.18. In general, the reverse of part (v) of Lemma 2.15 is not true. As mentioned in Remark 2.8, for any $x \in \mathbb{R}^2$ and $a \in \mathbb{R}$, E_{aox} in Examples 2.3 is singleton, but $\mathbb{R}^2$ is not normal, because for any positive real number a and b we have $E_{box} = E_{aox} = \{x\}$ and so $e_{aox} + e_{box} = \{x\} + \{x\} = \{2x\}$ and $e_{(a+b)ox} = \{x\}$. Therefore,

$$(e_{aox} + e_{box}) \cap e_{(a+b)ox} = \emptyset.$$

Theorem 2.19. *Let X be a normal weak hypervector space. If $\{x_1, ..., x_m\}$ is a basis for X, then every linear independent set of X has at most m elements.*

Proof. Let $\{x_1, ..., x_m\}$ be a basis for X and $\{y_1, ..., y_n\}$ be a linear independent set of X. Thus for any $1 \leq j \leq n$ there exist $\alpha_{1j}, \alpha_{2j}, ..., \alpha_{mj} \in F$ such that $y_j = \sum\limits_{i=1}^{m} e_{\alpha_{ij}ox_i}$. Let $c_1, ..., c_n \in F$. By Lemmas 2.15 and 2.17, we have

$$\begin{aligned} e_{c_1oy_1} + ... + e_{c_noy_n} &= e_{c_1o\sum_{i=1}^m e_{\alpha_{i1}ox_i}} + ... + e_{c_no\sum_{i=1}^m e_{\alpha_{in}ox_i}} \\ &= \sum_{i=1}^{m} e_{c_1o(e_{\alpha_{i1}ox_i})} + ... + \sum_{i=1}^{m} e_{c_no(e_{\alpha_{in}ox_i})} \\ &= \sum_{i=1}^{m} e_{c_1\alpha_{i1}ox_i} + ... + \sum_{i=1}^{m} e_{c_1\alpha_{in}ox_i} \\ &= \sum_{j=1}^{n} e_{c_j\alpha_{1j}ox_1} + ... + \sum_{j=1}^{n} e_{c_j\alpha_{mj}ox_m} \\ &= e_{(\sum_{j=1}^n c_j\alpha_{1j})ox_1} + ... + e_{(\sum_{j=1}^n c_j\alpha_{mj})ox_m}. \end{aligned}$$

If $e_{c_1oy_1} + ... + e_{c_noy_n} = 0$, then $c_1 = ... = c_n = 0$. Since $\{x_1, ..., x_m\}$ is linear independent, by above relation we obtain

$$\sum_{j=1}^{n} c_j\alpha_{1j} = ... = \sum_{j=1}^{n} c_j\alpha_{mj} = 0.$$

Assume, on the contrary, that $n > m$. The remain of proof is the same proof of this lemma in the classical vector space. With the same reason, we can conclude that there exist at least a nonzero c_j and this contradiction completes the proof. ■

Corollary 2.20. *Let X be a normal and finite dimensional weak hypervector space. Then any two basis of X have equal numbers of elements.*

Proof. It is clear by Theorem 2.19. ■

Definition 2.21. Let X be a normal and finite dimensional weak hypervector space. The dimension of X is defined the numbers of the elements of the basis of X.

Lemma 2.22. *If X is a finite dimensional normal weak hypervector space, then the coordinate of any element of X is unique.*

Proof. It is clear by part (v) of Lemma 2.15. ■

Now, we are able to identify the dimension of defined weak hypervector spaces in Examples 2.4 and 2.5.

Example 2.23. We show the dimension of defined weak hypervector space in Example 2.4 is 2. Let i and j be the unit vectors in direction of x-axis and y-axis, respectively. We show $\{i, j\}$ is linearly independent. So let a and b be real such that $e_{aoi} + e_{boj} = 0$. This implies $|a| + |b| = 0$ and then $a = b = 0$. Now, let $x = re^{i\theta}$ be an arbitrary element of C. So, we can write $x = e_{rcos\theta oi} + e_{rsin\theta oj}$. We proved that C is spanned by a linearly independent set that has 2 elements, so its dimension is 2.

Example 2.24. In this example we show the dimension of defined weak hypervector space in Example 2.5 is 1. We show $\{i\}$ is linearly independent. So, let a be real such that $e_{aoi} = 0$. It means that the elements on the circle with radius $|a|$ and center origin are zero. So we obtain $a = 0$.

If $x = re^{i\theta}$ be an arbitrary element of C, then $x = e_{roi}$. Therefore, C is spanned by $\{i\}$, hence its dimension is 1.

3. Weak subhypervector spaces

Definition 3.1. Let X be a weak hypervector space over F. A nonempty subset M of X is called a weak subhypervector space of X, when M satisfies the following properties:

(i) $x + y \in M, \ \forall x, y \in M$,

(ii) $e_{aox} \in M, \ \forall a \in F, \ \forall x \in M$.

Theorem 3.2. *Let X be a weak hypervector space over F and M be a nonempty subset of X. M is a weak subhypervector space of X if and only if for all $a \in F$ and $x, y \in M$ we have $e_{aox} + y \in M$.*

Proof. The necessity part is obvious. For the converse, let $x, y \in M$ and $a \in F$. Since, by assumption, $E_{1ox} + y \subseteq M$ and, by part (i) of Lemma 3.15, $x \in E_{1ox}$ we obtain $x + y \in M$. By part (iii) of Lemma 3.15 since $E_{-1ox} = -E_{1ox}$ so $-x \in E_{-1ox}$ and hence by $E_{-1ox} + x \subseteq M$ we obtain $0 \in M$ and thus $e_{aox} + 0 \in M$. So the proof is completed. ∎

Theorem 3.3. *Let X be a normal weak hypervector space over F and $\emptyset \neq S \subseteq X$. Then the following set is the smallest weak subhypervector space of X containing S:*

$$[S] = \left\{ \sum_{i=1}^{n} e_{a_i o s_i}; a_i \in F, s_i \in S, n \in \mathbb{N} \right\}.$$

Proof. For any $s \in S$ by part (i) of Lemma 3.15, $s \in E_{1oS}$. So $S \subseteq [S]$. Let $x, y \in [S]$. So, $x = \sum\limits_{i=1}^{n} e_{a_i o s_i}$ and $y = \sum\limits_{j=1}^{m} e_{b_j o t_j}$, for $a_i, b_j \in F$ and $s_i, t_j \in S$. Let $n < m$. We obtain $x + y = \sum\limits_{i=1}^{n+m} e_{c_i o u_i}$ such that, for $1 \leq i \leq n$, $c_i = a_i$, $u_i = s_i$ and, for $n+1 \leq i \leq m+n$, $c_i = b_{i-n}$, $u_i = t_{i-n}$. So, $x + y \in [S]$. Since X is normal, we have $e_{bo\sum_{i=1}^{n} e_{a_i o s_i}} = \sum\limits_{i=1}^{n} e_{boz_{a_i o s_i}}$. By part (ii) of Lemma 3.15, this implies $e_{bo\sum_{i=1}^{n} e_{a_i o s_i}} = \sum\limits_{i=1}^{n} e_{ba_i o s_i}$. Hence, we obtain $e_{bo\sum_{i=1}^{n} e_{a_i o s_i}} \in S$ and then $[S]$ is a weak subhypervector space of X containing S. Now, let M be a weak subhypervector space of X containing S and $x \in S$. So, $x = \sum\limits_{i=1}^{n} e_{a_i o s_i}$, for $a_i \in F$ and $s_i \in S$. Since $S \subseteq M$, we have $e_{a_i o s_i} \in M$ and then $x = \sum\limits_{i=1}^{n} e_{a_i o s_i} \in M$. Thus $[S] \subseteq M$ and so the proof is completed. ∎

Remark 3.4. If $\{M_i\}_{i \in I}$ be a collection of weak subhypervector spaces of X, then $M = \bigcap\limits_{i \in I} M_i$ is a weak subhypervector space, clearly.

It is clear that $[S]$ is equal to intersection of all weak subhypervector spaces of X containing S. We say that $[S]$ is the spanned weak subhypervector space by S.

Proposition 3.5. *Let X be a normal weak hypervector space over F. Then X with the same defined sum and the following scalar product is a classical vector space:*

$$ax = e_{aox},$$

for all $a \in F$ and $x \in X$.

Proof. By part (v) of Lemma 3.115, this defined scalar product is well-defined. Checking the properties of a vector space for X is easy. ∎

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GENERALIZED EXPONENTIAL OPERATORS AND DIFFERENCE EQUATIONS

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Abstract. The present paper deals with the generalization of exponential operators used by Dattoli and Levi for translation and diffusive operator which were utilized to establish analytical solutions of difference and integral equations. The generalization of their technique is expected to cover wide range of such utilization.

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1. Introduction

In 2000, Dattoli and Levi [1] discussed general methods for the solution of difference equations, arising in physical and biological problems.Their technique play crucial role in unifying the generalized families of the difference equations. The present paper deals with the generalization of exponential operators used in [1] to operators of the type $a^{\lambda q(x)\frac{d}{dx}}$, where base a $(a > 0,\ a \neq 1)$ is a real number. In particular when $a = e$, the operator reduces to the operators used by Dattoli et al. [1].

The action of the generalized exponential operator on a generic function $f(x)$ is defined as

$$a^{\lambda q(x)\dfrac{d}{dx}} f(x) = e^{(\lambda \ln(a))q(x)\frac{d}{dx}} f(x) = f(F^{-1}(\lambda \ln(a) + F(x))). \tag{1.1}$$

where $F(x)$ (called the Similarity Factor (SF)) denotes the function

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$$F(x) = \int^x \frac{d\xi}{q(\xi)},$$

and $F^{-1}(\sigma)$ is its inverse. For $q(x) = 1$, the SF is given by

$$F(x) = \int^x d\xi = x, \tag{1.2}$$

therefore $F^{-1}(x) = x$, then operator (1.1) reduces to the ordinary translation or shift operator as follows:

$$a^{\lambda \frac{d}{dx}} f(x) = f(F^{-1}(\lambda \ln(a) + x)) = f(\lambda \ln(a) + x). \tag{1.3}$$

Another example of application of operator (1.1), for $q(x) = x$, the SF is given by

$$F(x) = \int^x \frac{d\xi}{\xi} = \ln(x), \tag{1.4}$$

so that $F^{-1}(x) = e^x$, and hence operator (1.1) reduces to the dilatation operator

$$a^{\lambda x \frac{d}{dx}} f(x) = f(F^{-1}(\lambda \ln(a) + \ln(x))) = f(e^{\lambda \ln(a) + \ln(x)}) = f(a^\lambda x). \tag{1.5}$$

The ordinary shift operators and their properties play a central role within the context of the theory of difference equations [3]. One can, therefore, suspect that the above generalized exponential operators and the wealth of their properties can be exploited to develop tools which allow the solution of different forms of difference equations.

1(a). Particular case: The substitution of $a = e$, into equations (1.1), (1.3) and (1.5) reduce to equations (1), (2′) and (3) of Dattoli et al. [1].

A simple example of how the exponential operators can help us to solve difference equations may be illuminating. Let us consider the linear dilatation difference equation of the type

$$b_1 f(a^2 x) + b_2 f(ax) + b_3 f(x) = 0, \tag{1.6}$$

which, according to equation (1.5), equation (1.6) can be written in the following form

$$\left[b_1 \; a^{2x \frac{d}{dx}} + b_2 a^{x \frac{d}{dx}} + b_3\right] f(x) = 0. \tag{1.7}$$

Suppose $f(x) = R^{\ln(x)}$, we have

$$a^{\lambda x \frac{d}{dx}} R^{\ln(x)} = e^{\lambda \ln(a) x \frac{d}{dx}} R^{\ln(x)},$$

where $q(x) = x$, so that $F(x) = \ln(x)$ and $F^{-1}(x) = e^x$ or $F^{-1}(\lambda \ln(a) + \ln(x)) = e^{\lambda \ln(a) + \ln(x)} = xa^\lambda$. Therefore,

$$a^{\lambda x \frac{d}{dx}} R^{\ln x} = R^{\ln(xa^\lambda)} = R^{\lambda \ln(a)} R^{\ln x}. \tag{1.8}$$

Hence we can associate with equation (1.7) the characteristic equation

$$[b_1 R^{2\ln(a)} + b_2 R^{\ln(a)} + b_3] R^{\ln(x)} = 0, \quad \text{or} \quad b_1 R^{2\ln(a)} + b_2 R^{\ln(a)} + b_3 = 0, \tag{1.9}$$

whose roots $R_1^{\ln(a)}$ and $R_2^{\ln(a)}$ allow to write $f(x)$ in terms of the following linear combination of independent solutions:

$$f(x) = c_1 R_1^{\ln(x)} + c_2 R_2^{\ln(x)} = \sum_{\alpha=1}^{2} c_\alpha R_\alpha^{\ln(x)}. \tag{1.10}$$

The above example indicates that we can extend well-established methods of solutions of difference equations to other types of equations reducible to ordinary difference equations, after a proper change of variable implicit in equations (1.1), (1.3).

1(b). Particular case: The replacement of a with e in equations (1.6), (1.7), (1.8) and (1.9) give raise to equations (5), (6), (7), and (8) of Dattoli et al. [1].

To give a further example of the flexibility of the formalism associated with exponential operators, let us consider the generalized Heat Equation of the following type

$$\begin{cases} \dfrac{\partial}{\partial\lambda} Q(x, \lambda\ln(a)) = \ln(a)\left[q(x)\dfrac{\partial}{\partial x}\right]^2 Q(x, \lambda\ln(a)), \\ Q(x, 0) = g(x), \end{cases} \tag{1.11}$$

which can formally be solved by rewriting equation (1.11) as

$$\frac{\partial}{\partial\lambda} Q(x, \lambda\ln(a)) - \ln(a)\left[q(x)\frac{\partial}{\partial x}\right]^2 Q(x, \lambda\ln(a)) = 0,$$

which can formally be solved by considering this as ordinary linear differential equation of order one, whose I.F. is determined as

$$e^{-\int \ln(a)\left[q(x)\frac{\partial}{\partial x}\right]^2 d\lambda} = e^{-\ln(a)\left[q(x)\frac{\partial}{\partial x}\right]^2 \lambda} = a^{-\lambda\left[q(x)\frac{\partial}{\partial x}\right]^2},$$

we can, therefore, find its general solution as

$$Q(x, \lambda\ln(a)) a^{-\lambda\left[q(x)\frac{\partial}{\partial x}\right]^2} = C,$$

where C in any constant and using the given initial condition, we get

$$Q(x, 0) = g(x) = C,$$

and, finally, we obtain the solution of the Heat equation (1.11) as

$$Q(x, \lambda\ln(a)) = a^{\lambda[q(x)\frac{\partial}{\partial x}]^2} g(x). \tag{1.12}$$

Using the identity

$$e^{b^2} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\xi^2+2b\xi} d\xi,$$

and replacing b^2 by $\lambda \ln(a)[q(x)\frac{\partial}{\partial x}]^2$, we have

$$e^{\lambda \ln(a)[q(x)\frac{\partial}{\partial x}]^2} = a^{\lambda [q(x)\frac{\partial}{\partial x}]^2} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\xi^2+2\sqrt{\lambda \ln(a)}\xi} d\xi. \tag{1.13}$$

Using equation (1.1), finally yields the solution of equation (1.11) in the form of an integral transform, which can be viewed as a generalized Gauss transform

$$a^{\lambda [q(x)\frac{\partial}{\partial x}]^2} g(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\xi^2} g(F^{-1}(2\xi\sqrt{\lambda \ln(a)} + F(x))) d\xi. \tag{1.14}$$

or, in other words, we have

$$Q(x, \lambda \ln(a)) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\xi^2} g(F^{-1}(2\xi\sqrt{\lambda \ln(a)} + F(x))) d\xi. \tag{1.14$'$}$$

It is evident that the formalism associated with generalized exponential operators can be exploited in many flexible ways in finding the general solution of a large number of problems. This paper is devoted to the discussion of methods which provide the solution of the classes of "difference" and generalized "Heat" equations and we shall see that the techniques we propose offer reliable analytical tools and efficient numerical algorithms.

1(c). Particular case: To put $a = e$, in the equations (1.11), (1.12), and (1.14) give raise the same forms of the equations (10), (11) and (13) respectively of Dattoli et al. [1].

2. Generalized difference equations

Before discussing the problem in its generality, let us consider the equation of the following type, as a further example, which reduces to ([1]; p. 655 (14)) when we consider $a = e$, we have

$$\sum_{\alpha=0}^{N} b_\alpha f(x \cos(\alpha \ln(a)) + \sqrt{1-x^2} \sin(\alpha \ln(a))) = 0, \tag{2.1}$$

which belongs to the families of generalized difference equation. This equation can be obtained by the action of the generalized exponential operator on the function $f(x)$.

$$\sum_{\alpha=0}^{N} b_\alpha f(\sin(\sin^{-1}x)\cos(\alpha\ln(a)) + \cos(\cos^{-1}\sqrt{1-x^2})\sin(\alpha\ln(a))) = 0,$$

$$\sum_{\alpha=0}^{N} b_\alpha f(\sin(\sin^{-1}x)\cos(\alpha\ln(a)) + \cos(\sin^{-1}x)\sin(\alpha\ln(a))) = 0,$$

$$\sum_{\alpha=0}^{N} b_\alpha f(\sin(\sin^{-1}x) + \alpha\ln(a)) = 0, \ \sum_{\alpha=0}^{N} b_\alpha a^{\alpha\sqrt{1-x^2}} f(x) = 0.$$

According to the discussion of the previous section, the use of the exponential operator

$$\sum_{\alpha=0}^{N} b_\alpha \widehat{A}^\alpha f(x) = 0,$$

where

$$\widehat{A} = a^{\sqrt{1-x^2}\frac{d}{dx}} \tag{2.2}$$

allows to cast (2.1) in the operator form

$$\Psi(\widehat{A})f(x) = 0,$$

where

$$\Psi(\widehat{A}) = \sum_{\alpha=0}^{N} b_\alpha \widehat{A}^\alpha. \tag{2.3}$$

In this case, the SF associated with (2.2) is

$$F(x) = \sin^{-1}(x). \tag{2.4}$$

Independent solutions of (2.1) can, therefore, be constructed in terms of the function $R^{\sin^{-1}(x)}$, which satisfies the identity

$$\widehat{A}^\alpha R^{\sin^{-1}(x)} = R^{\alpha\ln(a)} R^{\sin^{-1}(x)}, \tag{2.5}$$

the general solution of (2.1) can finally be written as

$$f(x) = \sum_{\alpha=0}^{N} c_\alpha R_\alpha^{\sin^{-1}(x)}. \tag{2.6}$$

Similarly, if we consider the following example, we have

$$\sum_{\alpha=0}^{N} b_\alpha f(x\cos(\alpha\ln(a)) - \sqrt{1-x^2}\sin(\alpha\ln(a))) = 0, \tag{2.1$'$}$$

which belongs to the families of generalized difference equation. According to the discussion of the previous section, the use of the exponential operator

$$\widehat{A} = a^{-\sqrt{1-x^2}\frac{d}{dx}} \tag{2.2$'$}$$

allows to cast (2.1)′ in the operator form (2.3). In this case the SF associated with (2.2)′ is

$$F(x) = \cos^{-1}(x). \tag{2.4$'$}$$

Independent solutions of (2.1)′ can be therefore constructed in terms of the function $R^{\cos^{-1}(x)}$, which satisfies the identity

$$\widehat{A}^{\alpha} R^{\cos^{-1}(x)} = R^{\alpha \ln(a)} R^{\cos^{-1}(x)}. \tag{2.5$'$}$$

The general solution of (2.1)′ can finally be written as

$$f(x) = \sum_{\alpha=0}^{N} c_\alpha R_\alpha^{\cos^{-1}(x)}, \tag{2.6$'$}$$

where $R_\alpha^{\ln(a)}$ are the roots of the characteristic equation

$$\Psi(R^{\ln(a)}) = 0. \tag{2.7}$$

From the above discussion it is now clear that, whenever one deals with equations of the type

$$\sum_{\alpha=0}^{N} b_\alpha f(F^{-1}(\alpha \ln(a) + F(x))) = 0, \tag{2.8}$$

one can associate it with the generalized exponential operator

$$\widehat{A} = a^{q(x)\frac{d}{dx}}, \tag{2.9}$$

which allows to cast (2.8) in the operator form (2.3) and we get the relevant solution in the form

$$f(x) = \sum_{\alpha=0}^{N} c_\alpha R_\alpha^{\int^x \frac{d\xi}{q(\xi)}}. \tag{2.10}$$

2(a). Particular case: when we substitute $a = e$ in equations (2.1), (2.2), (2.3), (2.5), (2.7), (2.8) and (2.9) then these equation lead to equations (14), (15), (16), (18), (20), (21) and (22) respectively due to Dattoli et al. [1].

A useful example is given by the equation

$$\sum_{\alpha=0}^{N} b_\alpha f\left(\frac{x}{1-\alpha\ln(a)x}\right)=0, \tag{2.11}$$

by making use of the shift operator $a^{x^2\frac{d}{dx}}$, which allows to cast (2.11) in the operator form (2.3), i.e.,

$$\sum_{\alpha=0}^{N} b_\alpha f\left(-\frac{1}{\alpha\ln(a)-\frac{1}{x}}\right)=0,$$

$$\sum_{\alpha=0}^{N} b_\alpha a^{\alpha x^2\frac{d}{dx}} f(x)=0,$$

$$\sum_{\alpha=0}^{N} b_\alpha \widehat{A}^\alpha f(x)=0,$$

where

$$\widehat{A}=a^{x^2\frac{d}{dx}},\quad \Psi(\widehat{A})f(x) \;=\; 0 \text{ and } \quad \Psi(A)=\sum_{\alpha=0}^{N} b_\alpha \widehat{(A)^\alpha}.$$

In this case, the SF associated with (2.11) is $F(x)=-\dfrac{1}{x}$. Its solution can thus be written as

$$f(x)=\sum_{\alpha=1}^{N} c_\alpha R_\alpha^{-\frac{1}{x}}. \tag{2.12}$$

The validity of the above solutions is limited to the case in which $R_\alpha^{\ln(a)}$ is not a multiple root of the characteristic equation; this point will be discussed in the concluding section.

2(b). Particular case: Replacing a with e in equation (2.11) reduce to Dattoli et al. ([1]; p.656(24)).

In the tunes of Dattoli et al. ([1]; p. 656(26)), let us introduce the following operational identities:

$$\begin{cases} \widehat{A}^{\pm\alpha} b^{\int^x \frac{d\xi}{q(\xi)}} = b^{\pm\alpha\ln(a)} b^{\int^x \frac{d\xi}{q(\xi)}}, \\ \widehat{A}^{\pm\alpha}(b^{\int^x \frac{d\xi}{q(\xi)}}\phi(x)) = b^{\int^x \frac{d\xi}{q(\xi)}}(b^{\ln(a)}\widehat{A})^{\pm\alpha}\phi(x). \end{cases} \tag{2.13}$$

valid for exponential operators of the form (2.9).

We note that, according to the first of (2.13), the non-homogeneous equation

$$\Psi(\widehat{A})f(x)=Cb^{\int^x \frac{d\xi}{q(\xi)}}, \tag{2.14}$$

where C is a constant and $b^{\ln(a)}$ is not a root of the characteristic equation, admits the particular solution

$$f(x) = \frac{Cb^{\int^x \frac{d\xi}{q(\xi)}}}{\Psi(b^{\ln(a)})}. \tag{2.15}$$

In the slightly more complicated case

$$\Psi(\widehat{A})f(x) = Cb^{\int^x \frac{d\xi}{q(\xi)}}\phi(x), \tag{2.16}$$

the second of (2.13) yields

$$f(x) = Cb^{\int^x \frac{d\xi}{q(\xi)}} \frac{1}{\Psi(b^{\ln(a)}\widehat{A})}\phi(x). \tag{2.17}$$

Further comments shall be discussed in the concluding section.

2(c). Particular case: When $a = e$, equations (2.14), (2.15), (2.16) and (2.17) convert into equations (27), (28), (29) and (30) of Dattoli et al. [1].

3. Generalized shift operators and Jackson derivatives

In the previous section, we have considered linear equations involving discrete power of the generalized exponential operator. Here, we shall discuss examples in which the exponents are not necessarily integers. The introductory example is

$$\frac{f(a^\lambda x) - f(x)}{\lambda \ln(a)} = g(x), \tag{3.1}$$

where $f(x)$ is unknown, $\lambda \in C$, and $g(x)$ is an analytical function. The use of the dilatation operator allows to cast equation (2.17) in the form of a the Jackson derivative [4], namely

$$\frac{a^{\lambda \frac{d}{d\xi}} - 1}{\lambda \ln(a)} f(x) = g(x). \tag{3.2}$$

The operator on the left hand side can formally be inverted and by writing the differentiation variable in terms of the inverse of the SF we find

$$f(e^\xi) = \frac{\lambda \ln(a)}{a^{\lambda \frac{d}{d\xi}} - 1} g(e^\xi). \tag{3.3}$$

The operator on the r.h.s. of (3.3) can be expanded as

$$\frac{\lambda\ln(a)}{a^{\lambda\frac{d}{d\xi}}-1}=\frac{\lambda\ln(a)}{\lambda\ln(a)\frac{d}{d\xi}+\frac{1}{2!}(\lambda\ln(a)\frac{d}{d\xi})^2+\frac{1}{3!}(\lambda\ln(a)\frac{d}{d\xi})^3+\cdots}$$

$$=\frac{1}{\frac{d}{d\xi}\left[1+\frac{1}{2!}\left(\lambda\ln(a)\frac{d}{d\xi}\right)+\frac{1}{3!}\left(\lambda\ln(a)\frac{d}{d\xi}\right)^2+\cdots\right]}$$

$$=D_\xi^{-1}\left[1+\left(\frac{1}{2!}\lambda\ln(a)\frac{d}{d\xi}+\frac{1}{3!}\left(\lambda\ln(a)\frac{d}{d\xi}\right)^2+\cdots\right)\right]^{-1}$$

$$=D_\xi^{-1}\left[1-\left(\frac{1}{2!}\lambda\ln(a)\frac{d}{d\xi}+\frac{1}{3!}\left(\lambda\ln(a)\frac{d}{d\xi}\right)^2+\cdots\right)\right.$$

$$\left.+\left(\frac{1}{2!}\lambda\ln(a)\frac{d}{d\xi}+\frac{1}{3!}\left(\lambda\ln(a)\frac{d}{d\xi}\right)^2+\cdots\right)^2+\cdots\right]$$

$$=D_\xi^{-1}\left[1-\frac{1}{2}\lambda\ln(a)\frac{d}{d\xi}+\left(\frac{1}{4}-\frac{1}{6}\right)\left(\lambda\ln(a)\frac{d}{d\xi}\right)^2-\left(\frac{1}{24}+\frac{1}{8}\right)\left(\lambda\ln(a)\frac{d}{d\xi}\right)^3+\cdots\right]$$

$$=D_\xi^{-1}\left[1-\frac{1}{2}\lambda\ln(a)\frac{d}{d\xi}+\frac{1}{12}\left(\lambda\ln(a)\frac{d}{d\xi}\right)^2-\frac{1}{6}\left(\lambda\ln(a)\frac{d}{d\xi}\right)^3+\cdots\right]$$

$$=D_\xi^{-1}\left[B_0+\frac{B_1}{1!}\lambda\ln(a)\frac{d}{d\xi}+\frac{B_2}{2!}\left(\lambda\ln(a)\frac{d}{d\xi}\right)^2+\frac{B_3}{3!}\left(\lambda\ln(a)\frac{d}{d\xi}\right)^3+\cdots\right]$$

or

$$\frac{\lambda\ln(a)}{a^{\lambda\frac{d}{d\xi}}-1}=D_\xi^{-1}\sum_{n=0}^{\infty}\frac{B_n}{n!}(\lambda\ln(a))^n\left(\frac{d}{d\xi}\right)^n, \tag{3.4}$$

where

$$B_0=1,\ B_1=-\frac{1}{2},\ B_2=\frac{1}{6},\ B_3=-1,... \tag{3.5}$$

are Bernoulli numbers (see [9]; p.300(9)) and D_ξ^{-1} is the inverse of the derivative operator. Since $g(x)$ has a Taylor expansion ($g(x)=\sum_{m=0}^{\infty}b_m x^m$), we get from equations (3.3), (3.4)

$$
\begin{aligned}
f(e^{\xi}) &= D_{\xi}^{-1}\sum_{n=0}^{\infty}\frac{B_n}{n!}(\lambda\ln(a))^n\left(\frac{d}{d\xi}\right)^n\left(\sum_{m=0}^{\infty}b_m e^{m\xi}\right)\\
&= D_{\xi}^{-1}\left[B_0+\frac{B_1}{1!}\lambda\ln(a)\frac{d}{d\xi}+\frac{B_2}{2!}\left(\lambda\ln(a)\frac{d}{d\xi}\right)^2+\frac{B_3}{3!}\left(\lambda\ln(a)\frac{d}{d\xi}\right)^3+\cdots\right]\sum_{m=0}^{\infty}b_m e^{m\xi}\\
&= D_{\xi}^{-1}\left[1-\frac{1}{2}\lambda\ln(a)\frac{d}{d\xi}+\frac{1}{12}\left(\lambda\ln(a)\frac{d}{d\xi}\right)^2-\frac{1}{6}\left(\lambda\ln(a)\frac{d}{d\xi}\right)^3+\cdots\right]\sum_{m=0}^{\infty}b_m e^{m\xi}\\
&= D_{\xi}^{-1}\Bigg[b_0+\sum_{m=1}^{\infty}b_m e^{m\xi}-\frac{1}{2}\lambda\ln(a)\sum_{m=0}^{\infty}b_m m e^{m\xi}\\
&\qquad +\frac{1}{12}(\lambda\ln(a))^2\sum_{m=0}^{\infty}b_m(m)^2e^{m\xi}-\frac{1}{6}(\lambda\ln(a))^3\sum_{m=0}^{\infty}b_m(m)^3e^{m\xi}+\cdots\Bigg]\\
&= \Bigg[b_0\xi+\sum_{m=1}^{\infty}\frac{b_m}{m}e^{m\xi}-\frac{1}{2}\lambda\ln(a)\sum_{m=0}^{\infty}b_m e^{m\xi}\\
&\qquad +\frac{1}{12}(\lambda\ln(a))^2\sum_{m=0}^{\infty}b_m(m)e^{m\xi}-\frac{1}{6}(\lambda\ln(a))^3\sum_{m=0}^{\infty}b_m(m)^2e^{m\xi}+\cdots\Bigg]\\
&= b_0\xi+\left(1-\frac{\lambda\ln(a)}{2}+\frac{(\lambda\ln(a))^2}{12}-\frac{(\lambda\ln(a))^3}{6}+\cdots\right)b_1e^{\xi}\\
&\qquad +\left(1-\frac{2\lambda\ln(a)}{2}+\frac{(2\lambda\ln(a))^2}{12}-\frac{(2\lambda\ln(a))^3}{6}+\cdots\right)\frac{b_2}{2}e^{2\xi}+\cdots\\
&= b_0\xi+b_1e^{\xi}\left[1+\left(\frac{\lambda\ln(a)}{2!}+\frac{(\lambda\ln(a))^2}{3!}+\cdots\right)\right]^{-1}\\
&\qquad +b_2e^{\xi}\left[1+\left(\frac{2\lambda\ln(a)}{2!}+\frac{(2\lambda\ln(a))^2}{3!}+\cdots\right)\right]^{-1}+\cdots\\
&= b_0\xi+\frac{b_1e^{\xi}}{(1+\frac{\lambda\ln(a)}{2!}+\frac{(\lambda\ln(a))^2}{3!}+\cdots)}+\frac{b_2e^{\xi}}{2[1+(\frac{2\lambda\ln(a)}{2!}+\frac{(2\lambda\ln(a))^2}{3!}+\cdots)]}+\cdots\\
&= b_0\xi+\frac{\lambda\ln(a)b_1e^{\xi}}{\frac{(\lambda\ln(a)}{1!}+\frac{(\lambda\ln(a))^2}{2!}+\frac{(\lambda\ln(a))^3}{3!}+\cdots}+\frac{\lambda\ln(a)b_2e^{\xi}}{\frac{2\lambda\ln(a)}{1!}+\frac{2(\lambda\ln(a))^2}{2!}+\frac{(2\lambda\ln(a))^3}{3!}+\cdots}+\cdots\\
&= b_0\xi+\frac{\lambda\ln(a)b_1e^{\xi}}{e^{\lambda\ln(a)}-1}+\frac{\lambda\ln(a)b_2e^{2\xi}}{e^{2\lambda\ln(a)}-1}+\frac{\lambda\ln(a)b_3e^{3\xi}}{e^{3\lambda\ln(a)}-1}+\cdots
\end{aligned}
$$

or

$$f(e^{\xi})=\sum_{m=1}^{\infty}b_m\frac{\lambda\ln(a)e^{m\xi}}{e^{\lambda m\ln(a)}-1}+b_0\xi,$$

or

$$f(e^{\xi}) = \sum_{m=1}^{\infty} b_m \frac{\lambda \ln(a) e^{m\xi}}{a^{\lambda m} - 1} + b_0 \xi. \tag{3.6}$$

Going back to the original variable, we get

$$f(x) = \sum_{m=1}^{\infty} b_m \frac{\lambda \ln(a) x^m}{a^{\lambda m} - 1} + b_0 \ln(x). \tag{3.7}$$

The series on the right hand side of equation (3.6) provides the solution of our problem.

Taking another example, $g(x) = \sin(x) = \displaystyle\sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!}$, we find

$$
\begin{aligned}
f(e^{\xi}) &= D_{\xi}^{-1} \sum_{n=0}^{\infty} \frac{B_n (\lambda \ln(a))^n}{n!} \left(\frac{d}{d\xi}\right)^n \sum_{m=0}^{\infty} \frac{(-1)^m e^{(2m+1)\xi}}{(2m+1)!} \\
&= D_{\xi}^{-1} \left[B_0 + \frac{B_1}{1!} \lambda \ln(a) \frac{d}{d\xi} + \frac{B_2}{2!} \left(\lambda \ln(a) \frac{d}{d\xi}\right)^2 \right. \\
&\qquad \left. + \frac{B_3}{3!} \left(\lambda \ln(a) \frac{d}{d\xi}\right)^3 + \cdots \right] \cdot \sum_{m=0}^{\infty} \frac{(-1)^m e^{(2m+1)\xi}}{(2m+1)!} \\
&= D_{\xi}^{-1} \left[1 - \frac{1}{2} \lambda \ln(a) \frac{d}{d\xi} + \frac{1}{12} \left(\lambda \ln(a) \frac{d}{d\xi}\right)^2 \right. \\
&\qquad \left. - \frac{1}{6} \left(\lambda \ln(a) \frac{d}{d\xi}\right)^3 + \cdots \right] \cdot \sum_{m=0}^{\infty} \frac{(-1)^m e^{(2m+1)\xi}}{(2m+1)!} \\
&= D_{\xi}^{-1} \left[\sum_{m=0}^{\infty} \frac{(-1)^m e^{(2m+1)\xi}}{(2m+1)!} - \frac{1}{2} \left(\lambda \ln(a) \frac{d}{d\xi}\right) \sum_{m=0}^{\infty} \frac{(-1)^m e^{(2m+1)\xi}}{(2m+1)!} \right. \\
&\quad \left. + \frac{1}{12} \left(\lambda \ln(a) \frac{d}{d\xi}\right)^2 \sum_{m=0}^{\infty} \frac{(-1)^m e^{(2m+1)\xi}}{(2m+1)!} - \frac{1}{6} \left(\lambda \ln(a) \frac{d}{d\xi}\right)^3 \sum_{m=0}^{\infty} \frac{(-1)^m e^{(2m+1)\xi}}{(2m+1)!} \cdots \right] \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)(2m+1)!} e^{(2m+1)\xi} - \frac{1}{2} \lambda \ln(a) \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} e^{(2m+1)\xi} \\
&\quad + \frac{1}{12} (\lambda \ln(a))^2 \sum_{m=0}^{\infty} \frac{(-1)^m (2m+1)}{(2m+1)!} \; e^{(2m+1)\xi} \\
&\qquad - \frac{1}{6} (\lambda \ln(a))^3 \cdot \sum_{m=0}^{\infty} \frac{(-1)^m (2m+1)^2}{(2m+1)!} e^{(2m+1)\xi} \cdots
\end{aligned}
$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m e^{(2m+1)\xi}}{(2m+1)(2m+1)!} \left[1 - \frac{\lambda \ln(a)(2m+1)}{2} \right.$$
$$\left. + \frac{(\lambda \ln(a))^2 (2m+1)^2}{12} - \frac{(\lambda \ln(a))^3 (2m+1)^3}{6} + \cdots \right]$$
$$= \sum_{m=0}^{\infty} \frac{(-1)^m e^{(2m+1)\xi}}{(2m+1)(2m+1)!} \left[1 + \frac{\lambda \ln(a)(2m+1)}{2!} + \frac{(\lambda \ln(a))^2 (2m+1)^2}{3!} + \cdots \right]^{-1}$$

or

$$f(e^{\xi}) = \sum \frac{\lambda \ln(a)}{a^{\lambda(2m+1)} - 1} \frac{(-1)^m}{(2m+1)!} e^{(2m+1)\xi}$$

or

$$f(x) = \sum \frac{\lambda \ln(a)}{a^{\lambda(2m+1)} - 1} \frac{(-1)^m}{(2m+1)!} x^{2m+1}. \tag{3.8}$$

It is essentially the series defining $g(x)$, provided that b_m is replaced by

$$\frac{b_m \lambda \ln a}{a^{\lambda m} - 1}.$$

If, e.g., we take $g(x) = \cos(x)$, we find

$$f(x) = \sum \frac{\lambda \ln(a)}{a^{\lambda(2m)} - 1} \frac{(-1)^m}{(2m)!} x^{2m}, \tag{3.8$'$}$$

and for $g(x) = e^{x^q}$, we get

$$f(x) = \sum_{m=1}^{\infty} \frac{\lambda \ln(a)}{a^{\lambda q m} - 1} \frac{x^{qm}}{m!} + \ln(x). \tag{3.9}$$

We can, therefore, conclude that the primitive of a Jackson derivative can be constructed according to the above-quoted recipe.

This method can also be generalized and the concept of Jackson derivative extended to other forms of exponential operators. In this case we consider equation of the type

$$\frac{f(x)\cos(\lambda \ln(a)) + \sqrt{1-x^2}\sin(\lambda \ln(a))}{\lambda \ln(a)} = g(x), \tag{3.10}$$

with the assistance of equation (2.2), we write equation (3.10) as follows:

$$\frac{a^{\lambda\sqrt{1-x^2}} f(x) - f(x)}{\lambda \ln(a)} = g(x) \quad \text{or} \quad \frac{(a^{\lambda\sqrt{1-x^2}} - 1)}{\lambda \ln(a)} f(x) = g(x), \tag{3.11}$$

by assuming $g(x)$ is an odd function there taking $x = \sin\xi$, we have

$$\frac{d}{d\xi} = \frac{d}{dx}\frac{dx}{d\xi} = \cos\xi\frac{d}{dx},$$

$$\frac{(a^{\lambda\frac{d}{d\xi}} - 1)}{\lambda\ln(a)} f(\sin\xi) = g(\sin\xi), \quad \text{or} \tag{3.12}$$

$$f(\sin\xi) = \frac{\lambda\ln(a)}{a^{\lambda\frac{d}{d\xi}} - 1} g(\sin\xi),$$

let us find out the expansion of the first factor of the r.h.s. of equation (3.12) with the help of equation (3.4), we have

$$f(\sin\xi) = D_\xi^{-1}\sum_{n=0}^{\infty}\frac{B_n}{n!}(\lambda\ln(a))^n\left(\frac{d}{d\xi}\right)^n g(\sin\xi), \tag{3.13}$$

since $g(x)$ in an odd and analytic function, then $g(\sin\xi)$, can be expanded by Taylor expansion such as $g(\sin\xi) = \sum_{m=0}^{\infty} b_{2m+1}(\sin\xi)^{2m+1}$, we have from (3.13),

$$\begin{aligned}
&f(\sin\xi) = D_\xi^{-1}\sum_{n=0}^{\infty}\frac{B_n}{n!}(\lambda\ln(a))^n\left(\frac{d}{d\xi}\right)^n\sum_{m=0}^{\infty} b_{2m+1}(\sin\xi)^{2m+1}\\
&= D_\xi^{-1}\sum_{n=0}^{\infty}\frac{B_n}{n!}(\lambda\ln(a))^n\left(\frac{d}{d\xi}\right)^n\sum_{m=0}^{\infty} b_{2m+1}\left[\frac{e^{i\xi} - e^{-i\xi}}{2i}\right]^{2m+1}\\
&= D_\xi^{-1}\sum_{n=0}^{\infty}\frac{B_n}{n!}(\lambda\ln(a))^n\left(\frac{d}{d\xi}\right)^n\sum_{m=0}^{\infty}\frac{b_{2m+1}(-i)^{2m+1}}{2^{2m+1}}(-e^{-i\xi})^{2m+1}[1 - e^{2i\xi}]^{2m+1}\\
&= D_\xi^{-1}\left[B_0 + \frac{B_1(\lambda\ln(a))}{1!}\left(\frac{d}{d\xi}\right) + \frac{B_2(\lambda\ln(a))^2}{2!}\left(\frac{d}{d\xi}\right)^2 + \frac{B_3(\lambda\ln(a))^3}{3!}\left(\frac{d}{d\xi}\right)^3\cdots\right]\\
&\cdot\sum_{m=0}^{\infty}\frac{b_{2m+1}(-i)^{2m+1}}{2^{2m+1}}(-e^{-i\xi})^{2m+1}\sum_{s=0}^{2m+1}\binom{2m+1}{s}(-1)^{2m+1-s}e^{[2(2m+1-s)]i\xi}.
\end{aligned}$$

Substituting the values of Bernoulli's numbers from equation (3.5), we have

$$\begin{aligned}
(3.14) = D_\xi^{-1}&\left[1 - \frac{(\lambda\ln(a))}{2}\left(\frac{d}{d\xi}\right) + \frac{(\lambda\ln(a))^2}{12}\left(\frac{d}{d\xi}\right)^2 - \frac{(\lambda\ln(a))^3}{6}\left(\frac{d}{d\xi}\right)^3\cdots\right]\\
&\cdot\sum_{m=1}^{\infty}\frac{b_{2m+1}(-i)^{2m+1}}{2^{2m+1}}\sum_{s=0}^{2m+1}\binom{2m+1}{s}(-1)^s e^{i[2(m-s)+1]\xi}\\
&= \sum_{m=1}^{\infty}\frac{b_{2m+1}(-i)^{2m+1}}{2^{2m+1}}D_\xi^{-1}\left[\sum_{s=0}^{2m+1}\binom{2m+1}{s}(-1)^s e^{i[2(m-s)+1]\xi} - \frac{(\lambda\ln(a))}{2}\left(\frac{d}{d\xi}\right)\right.
\end{aligned}$$

$$\cdot \sum_{s=0}^{2m+1} \binom{2m+1}{s} (-1)^s e^{i[2(m-s)+1]\xi} + \frac{(\lambda \ln(a))^2}{12} \left(\frac{d}{d\xi}\right)^2$$

$$\left. \cdot \sum_{s=0}^{2m+1} \binom{2m+1}{s} (-1)^s e^{i[2(m-s)+1]\xi} - \cdots \right]$$

$$= \sum_{m=1}^{\infty} \frac{b_{2m+1}(-i)^{2m+1}}{2^{2m+1}} D_{\xi}^{-1} \left[\sum_{s=0}^{2m+1} \binom{2m+1}{s} (-1)^s e^{i[2(m-s)+1]\xi} \right.$$

$$- \frac{(\lambda \ln(a))}{2} \sum_{s=0}^{2m+1} \binom{2m+1}{s} (-1)^s [i\{2(m-s)+1\}\xi] e^{i[2(m-s)+1]\xi}$$

$$\left. + \frac{(\lambda \ln(a))^2}{12} \sum_{s=0}^{2m+1} \binom{2m+1}{s} (-1)^s [i\{2(m-s)+1\}\xi]^2 e^{i[2(m-s)+1]\xi} - \cdots \right]$$

$$= \sum_{m=1}^{\infty} \frac{b_{2m+1}(-i)^{2m+1}}{2^{2m+1}} \left[\sum_{s=0}^{2m+1} \binom{2m+1}{s} \frac{(-1)^s e^{i[2(m-s)+1]\xi}}{i[2(m-s)+1]} \right.$$

$$- \frac{(\lambda \ln(a))}{2} \sum_{s=0}^{2m+1} \binom{2m+1}{s} (-1)^s e^{i[2(m-s)+1]\xi}$$

$$\left. + \frac{(\lambda \ln(a))^2}{12} \sum_{s=0}^{2m+1} \binom{2m+1}{s} (-1)^s [i\{2(m-s)+1\}\xi] e^{i[2(m-s)+1]\xi} - \cdots \right]$$

$$= \sum_{m=1}^{\infty} \frac{b_{2m+1}(-i)^{2m+1}}{2^{2m+1}} \left[\sum_{s=0}^{2m+1} \binom{2m+1}{s} \frac{(-1)^s e^{i[2(m-s)+1]\xi}}{i[2(m-s)+1]} \right.$$

$$\left\{ 1 - \frac{(\lambda \ln(a))}{2} [i\{2(m-s)+1\}] + \frac{(\lambda \ln(a))^2}{12} [i\{2(m-s)+1\}]^2 \right.$$

$$\left. \left. - \frac{(\lambda \ln(a))^3}{6} [i\{2(m-s)+1\}]^3 + \cdots \right\} \right]$$

$$= \sum_{m=1}^{\infty} \frac{b_{2m+1}(-i)^{2m+1}}{2^{2m+1}} \left[\sum_{s=0}^{2m+1} \binom{2m+1}{s} \right.$$

$$\left. \cdot \frac{(-1)^s e^{i[2(m-s)+1]\xi}}{i[2(m-s)+1] \left\{ 1 + \frac{1}{2!}[i\{2(m-s)+1\}\lambda \ln(a)] + \frac{1}{3!}[i\{2(m-s)+1\}\lambda \ln(a)]^2 + \cdots \right\}} \right]$$

$$= \sum_{m=1}^{\infty} \frac{b_{2m+1}(-i)^{2m+1}}{2^{2m+1}} \left[\sum_{s=0}^{2m+1} \binom{2m+1}{s} \right.$$

$$\left. \cdot \frac{(-1)^s \lambda \ln(a) (e^{i\xi})^{2(m-s)+1}}{[i\{2(m-s)+1\}\lambda \ln(a)] + \frac{1}{2!}[i\{2(m-s)+1\}\lambda \ln(a)]^2 + \frac{1}{3!}[i\{2(m-s)+1\}\lambda \ln(a)]^3 + \cdots} \right]$$

$$= \sum_{m=1}^{\infty} \frac{b_{2m+1}(-i)^{2m+1}}{2^{2m+1}} \left[\sum_{s=0}^{2m+1} \binom{2m+1}{s} \frac{(-1)^s \lambda \ln(a) (\cos\xi + i \sin\xi)^{2(m-s)+1}}{e^{i[2(m-s)+1]\lambda \ln(a)} - 1} \right]$$

or

$$f(\sin\xi) = \sum_{m=1}^{\infty} \frac{b_{2m+1}\lambda\ln(a)}{2^{2m+1}}(-i)^{2m+1} \cdot \left[\sum_{s=0}^{2m+1} \binom{2m+1}{s} \frac{(-1)^s(\sqrt{1-\sin\xi^2}+i\sin\xi)^{2(m-s)+1}}{e^{i[2(m-s)+1]\lambda\ln(a)}-1}\right].$$

Finally,

$$f(x) = \sum_{m=1}^{\infty} \frac{b_{2m+1}\lambda\ln(a)}{2^{2m+1}}(-i)^{2m+1} \cdot \left[\sum_{s=0}^{2m+1} \binom{2m+1}{s} \frac{(-1)^s(\sqrt{1-x^2}+ix)^{2(m-s)+1}}{a^{i[2(m-s)+1]\lambda}-1}\right] \tag{3.15}$$

It is interesting to note that, in this case too, the criterion to evaluate the primitive of the Jackson derivative, associated with the operator (2.2), can easily be inferred.

Let us note that the procedure we have discussed can also be extended to the cases involving the generalized Gauss transform. In fact the solution of

$$\frac{a^{\lambda(x\frac{d}{dx})^2}-1}{\lambda\ln(a)}f(x) = g(x) \tag{3.16}$$

or in other words, we have

$$\frac{\lambda\ln(a)}{a^{\lambda(x\frac{d}{dx})^2}-1}g(x) = f(x), \tag{3.17}$$

let us suppose $x = e^{\xi}$, then

$$\frac{d}{d\xi} = \frac{d}{dx}\frac{dx}{d\xi} = e^{\xi}\frac{d}{dx} = x\frac{d}{dx}\cdot$$

Now, from equation (3.17), we have

$$\frac{\lambda\ln(a)}{a^{\lambda(\frac{d}{d\xi})^2}-1}g(e^{\xi}) = f(e^{\xi}). \tag{3.18}$$

Further, after following the steps as we followed in getting the result (3.4), we obtain the expansion of first factor of l.h.s. as

$$\frac{\lambda\ln(a)}{a^{\lambda(\frac{d}{d\xi})^2}-1} = D_{\xi}^{-2}\sum_{n=0}^{\infty}\frac{B_n}{n!}(\lambda\ln(a))^n\left(\frac{d}{d\xi}\right)^{2n}. \tag{3.19}$$

Now by the virtue of the analyticity of $g(x)$, we expand $g(x)$, by Taylor series, i.e.,

$$f(e^{\xi}) = D_{\xi}^{-2} \sum_{n=0}^{\infty} \frac{B_n}{n!} (\lambda \ln(a))^n \left(\frac{d}{d\xi} \right)^{2n} \sum_{m=0}^{\infty} b_m e^{mx},$$

proceeding of the steps as proceeded in finding the result (3.7), we obtain

$$f(e^{\xi}) = \sum_{m=1}^{\infty} b_m \frac{\lambda \ln(a)}{e^{\lambda \ln(a) m^2} - 1} e^{m\xi} + b_0 \xi$$

or in other words, if we take $b_0 = 0$, then

$$f(x) = \sum_{m=1}^{\infty} b_m \frac{\lambda \ln(a)}{a^{\lambda m^2} - 1} x^m. \tag{3.20}$$

Further comments on the results of this section will be discussed in the forthcoming concluding section.

3(a). Particular case: If we substitute $a = e$, in equations (3.1), (3.2), (3.3), (3.4), (3.7), (3.8), (3.9), (3.10), (3.15), (3.16) and (3.20), then we obtain the equations (31), (32), (33), (34), (36), (37), (38), (39), (40), (41) and (42) on page numbers 657-658 due to Dattoli et al. [1].

4. Remarks

In the previous section we have considered linear difference equations, a (trivial) non-linear example, similar to "Riccati" equation, is given blow

$$f(ax) - f(x) + b^{\ln(x)} f(ax) f(x) = 0, \tag{4.1}$$

which can be solved using the auxiliary function $g(x) = \dfrac{1}{f(x)}$ and thus getting

$$\frac{1}{f(x)} - \frac{1}{f(ax)} + b^{\ln(x)} = 0, \qquad \text{or}$$
$$g(ax) - g(x) = b^{\ln(x)}. \tag{4.2}$$

From operator (1.5), we have

$$a^{x \frac{d}{dx}} g(x) - g(x) = b^{\ln(x)}, \tag{4.3}$$

or, in other words, we write the above equation (4.3) as

$$\begin{aligned} g(x) &= \frac{b^{\ln(x)}}{a^{x\frac{d}{dx}}-1} \\ &= -[1-a^{x\frac{d}{dx}}]^{-1}b^{\ln(x)} \\ &= -[1+a^{x\frac{d}{dx}}+a^{2x\frac{d}{dx}}+a^{3x\frac{d}{dx}}+\cdots]b^{\ln(x)} \\ &= -[b^{\ln(x)}+a^{x\frac{d}{dx}}b^{\ln(x)}+a^{2x\frac{d}{dx}}b^{\ln(x)}+a^{3x\frac{d}{dx}}b^{\ln(x)}+\cdots] \\ &= -[b^{\ln(x)}+b^{\ln(a)}b^{\ln(x)}+b^{2\ln(a)}b^{\ln(x)}+b^{3\ln(a)}b^{\ln(x)}+\cdots] \\ &= -[1+b^{\ln(a)}+b^{2\ln(a)}+b^{3\ln(a)}+\cdots]b^{\ln(x)} \\ &= -[\frac{1}{1-b^{\ln(a)}}]b^{\ln(x)} \\ &= \frac{b^{\ln(x)}}{b^{\ln(a)}-1} \end{aligned}$$

thus finding as a particular solution

$$f(x)=\frac{1}{g(x)}=\frac{b^{\ln(a)}-1}{b^{\ln(x)}}. \tag{4.4}$$

Moreover, in general, equations of the type

$$\sum_{\alpha=0}^{N} b_\alpha f(F^{-1}(\alpha\ln(a)+F(x)))=\epsilon[f(x)]^n, \tag{4.5}$$

standard perturbation methods can be used. At the lowest order in $\epsilon(f\cong f_0+\epsilon f_1)$, we find

$$\sum_{\alpha=0}^{N} b_\alpha f_0(F^{-1}(\alpha\ln(a)+F(x)))=0 \tag{4.6}$$

and

$$\sum_{\alpha=0}^{N} b_\alpha f_1(F^{-1}(\alpha\ln(a)+F(x)))=R^{n\int^x \frac{d\xi}{q(\xi)}}, \tag{4.7}$$

where $R^{\ln(a)}$ is one of the roots of the characteristic equation associated with (4.5). The first-order contribution f_1 can therefore be evaluated by using equation (2.15), which should be modified as follows:

$$\begin{aligned} f(x) &= \frac{C(\int^x \frac{d\xi}{q(\xi)})b^{\int^x \frac{d\xi}{q(\xi)}-1}}{\Psi'(b^{\ln(a)})}, \\ \Psi'(b^{\ln(a)}) &= \left[\frac{d}{dR}\Psi(R^{\ln(a)})\right]_{R=b}, \end{aligned} \tag{4.8}$$

if $b^{\ln(a)}$ is a simple root of the characteristic equation.

4(a). Particular case: The replacement of a with e in the equations (4.1), (4.4) and (4.8) reduce to the equations (43), (45) and (49) of Dattoli et al. [1].

Let us now go back to the problem of treating exponential operators of the type

$$\widehat{A}_{m,\lambda} = a^{\lambda(q(x)\frac{d}{dx})^m}. \tag{4.9}$$

We have seen that, for $m = 2$ and $\lambda > 0$, they can be viewed as generalized Gauss transform. Before discussing the problem more deeply, we recall the following important relation [2]:

$$\begin{cases} a^{\lambda(\frac{d}{dx})^m} x^n = H_n^{(m)}(x, \lambda \ln(a)), \\ \\ H_n^{(m)}(x, \lambda \ln(a)) = n! \displaystyle\sum_{r=0}^{[\frac{n}{m}]} \frac{(\lambda \ln(a))^r x^{n-mr}}{r!(n-mr)!}. \end{cases} \tag{4.10}$$

which holds for negative or positive λ and $H_n^{(m)}(x, \lambda \ln(a))$ are Kampé de Feriét polynomials, and satisfy the identity

$$\frac{\partial}{\partial \lambda} H_n^{(m)}(x, \lambda \ln(a)) = \ln(a) \left(\frac{\partial}{\partial x}\right)^m H_n^{(m)}(x, \lambda \ln(a)). \tag{4.11}$$

According to equation (4.8) we also find

$$\begin{aligned} a^{\lambda(\frac{d}{dx})^m} g(x) &= a^{\lambda(\frac{d}{dx})^m} \sum_{n=0}^{\infty} b_n x^n \\ &= \sum_{n=0}^{\infty} b_n H_n^{(m)}(x, \lambda \ln(a)). \end{aligned} \tag{4.12}$$

Therefore, it is easy to realize that

$$A_{m,\lambda} x = \sum_{n=0}^{\infty} \phi_n H_n^{(m)}(F(x), \lambda \ln(a)), \tag{4.13}$$

where we have assumed that the function F^{-1} can be expanded in power series

$$F^{-1}(\zeta) = \sum_{n=0}^{\infty} \phi_n \zeta^n. \tag{4.14}$$

It is clear that equation (4.12) can be further handled to extend the action of operators (4.8) to any function $g(x)$. It is worth considering the possibility of extending the definition of operators (4.8) to the case of not necessarily integer m. In the case of $m = \frac{1}{2}$, equation (4.9) should be replaced by

$$\begin{cases} a^{\lambda(\frac{d}{dx})^{\frac{1}{2}}} x^n = H_n^{(\frac{1}{2})}(x, \lambda \ln(a)), \\ \\ H_n^{(\frac{1}{2})}(x, \lambda \ln(a)) = n! \displaystyle\sum_{r=0}^{2n} \frac{(\lambda \ln(a))^r x^{n-\frac{r}{2}}}{r!\Gamma(n-\frac{r}{2}+1)}. \end{cases} \tag{4.15}$$

It is evident that in this case $H_n^{(\frac{1}{2})}(x,\lambda\ln(a))$ is a relation analogous to (4.10) holds, namely

$$\frac{\partial^2}{\partial\lambda^2}H_n^{(\frac{1}{2})}(x,\lambda\ln(a)) = (\ln(a))^2\left(\frac{\partial}{\partial x}\right)H_n^{(\frac{1}{2})}(x,\lambda\ln(a)). \tag{4.16}$$

or involving semi derivatives [8]

$$\frac{\partial}{\partial\lambda}H_n^{(\frac{1}{2})}(x,\lambda\ln(a)) = \ln(a)\left(\frac{\partial}{\partial x}\right)^{\frac{1}{2}}H_n^{(\frac{1}{2})}(x,\lambda\ln(a)). \tag{4.16$'$}$$

This definition can be extended to any $m = \dfrac{1}{p}$ (p, integer).

4(b). Particular case: Equations (4.9), (4.10), (4.11), (4.12), (4.13), (4.15) and (4.16) lead to Dattoli's et al. [1] equations (50), (51), (52), (53), (54), (56) and (57).

Concluding remark. It is hope that for the values other than e some more use of the generalized exponential operators can be obtained.

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ANALYSIS OF BLOOD FLOW THROUGH AN ARTERY WITH MILD STENOSIS: A TWO-LAYERED MODEL

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Abstract. In this paper, a model of blood flow through a constricted arterial segment has been considered. We have proposed a trapezium shaped geometry of mild axisymmetric stenosis. The flow of blood with artery has been represented by a two-layered model consisting of a core layer and a peripheral layer. It has been observed that the resistance to flow and wall shear stress increase as the peripheral layer viscosity increases. The results are compared graphically with those of previous investigators.

Keywords. arterial wall, blood flow, peripheral layer viscosity, stenosis.

AMS Classification: 92C35.

1. Introduction

Stenosis is an arterial disease in which the internal lumen of blood vessel is affected by some abnormal growth. This leads to blockage of the artery and then myocardial infarction. Keeping in mind the consequences of this type of blockages various researchers analyzed the problem experimentally and theoretically. The study of blood flow through stenotic arteries plays an important role in the diagnostic and fundamental understanding of cardiovascular diseases. The fluid mechanical behaviour of an artery having stenosis in its lumen had been given considerable attention by Lee and Fung [5], Rodbard [11], Young and Tsai [18]. The fluid behaviour of biological systems was thoroughly discussed by Leyton [6] and McDonald [7]. To understand the effects of mild stenosis several researchers Haldar [2], Misra and Chakravarty [8], Young [17] investigated the flow of blood through constricted tube treating blood as a Newtonian fluid. Nakamura and Sawada [9], Shukla et al. [12] extended the model by assuming that blood behaves like a non-Newtonian fluid. In all above models, the blood flow was represented by a single-layered model. Bugliarello and Sevilla [1] have shown experimentally that the blood flowing through narrow tubes can be well represented by a two-layered model instead of one. In this type of models there is a peripheral layer of plasma and a core region of suspension of red blood cells. Shukla et al. [13] have taken two-layered model to analyze the peripheral layer viscosity. Singh et al. [15] discussed a model of blood flow through an artery formulated for generalized geometry of multiple, mild and radially non-symmetric stenosis. Ponalagusamy [10] focused on slip velocity, thickness of peripheral layer and core layer viscosity at the vessel wall. Srivastava [16] studied analytically and numerically the effects of mild stenosis on blood flow characteristics in a two-fluid model. Recently, Joshi et al. [3] analyzed the flow of blood by taking a two-layered model with composite shaped geometry of constriction. In this paper the flow of blood has been analyzed with an artery having trapezium shaped mild stenosis. The results obtained are compared with previous investigators.

2. Formulation of the Model

In this paper, the flow of blood in a cylindrical tube having axisymmetric mild stenosis have been presented by a two-layered model. The external layer shows peripheral layer of plasma and the internal core layer describes the suspension of red blood cells. The schematic diagram is as follows:

The geometry of the stenotic tube without peripheral layer is described as follows,

$$(1)\ R(z)=\begin{cases} R_0-\dfrac{2\delta_s}{(L_0-\alpha}(z-d), & d\le z\le d+\dfrac{L_0-\alpha}{2}\\ R_0-\delta_s, & d+\dfrac{L_0-\alpha}{2}\le z\le d+\dfrac{L_0+\alpha}{2}\\ R_0-\delta_s+\frac{2\delta_s}{(L_0-\alpha)}\left(z-d-\dfrac{L_0+\alpha}{2}\right), & d+\dfrac{L_0+\alpha}{2}\le z\le d+L_0\\ R_0, & \text{elsewhere.}\end{cases}$$

where the symbols stand for

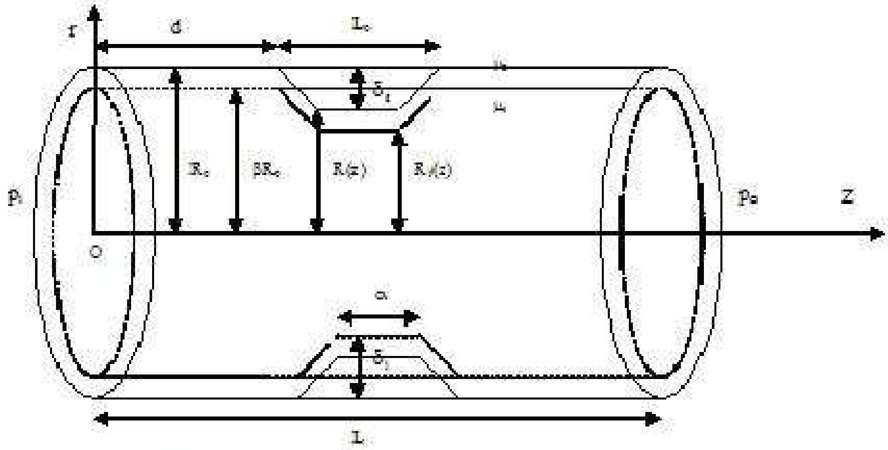

Figure 1: Geometry of Stenosed Artery

R_0 : Radius of the non-stenotic region
$R(z)$: Radius of the stenotic region
$R_1(z)$: Radius of the central core layer in stenotic region
L : The length of the artery
L_0 : The length of the stenosis
d : Location of stenosis
p_i : Inlet fluid pressure
p_0 : Instantaneous outlet fluid pressure
δ_s : Instantaneous maximum height of the stenosis
δ_i : Maximum bulging of interface
μ_1 : Viscosity of fluid in central core layer
μ_2 : Viscosity of fluid in peripheral layer
α : A parameter ($\alpha \geq 0$)
β : Ratio of central core radius to the tube radius

The governing equation of blood flow is given by Kapur [4],

$$0 = -\frac{dp}{dz} + \frac{1}{r}\frac{\partial}{\partial r}\left\{\mu(r)\, r\,\frac{\partial w}{\partial r}\right\}, \tag{2}$$

where w is axial velocity, p is fluid pressure and $\mu(r)$ is viscosity of fluid.

The boundary conditions are,

$$w = 0 \quad at \quad r = R(z) \tag{3}$$

and

$$\frac{\partial w}{\partial r} = 0 \quad at \quad r = 0. \tag{4}$$

Solving equation (2) under boundary conditions (3) and (4), we get

$$w = \left(-\frac{1}{2}\frac{dp}{dz}\right)\int_r^R \frac{r}{\mu(r)}\, dr. \tag{5}$$

The volumetric flow rate is given by

$$Q = \int_0^R 2\,\pi\,r\,w\,dr, \tag{6}$$

which on using equation (5) gives,

$$Q = \left(-\frac{\pi}{2}\frac{dp}{dz}\right)\int_0^R \frac{r^3 dr}{\mu\,(r)}. \tag{7}$$

Thus, the pressure gradient can be obtained as,

$$\frac{dp}{dz} = -\frac{2Q}{\pi I(z)}, \tag{8}$$

where

$$I(z) = \int_0^R \frac{r^3 dr}{\mu\,(r)}. \tag{9}$$

Integrating equation (8) using conditions $p = p_i$ at $z = 0$ and $p = p_0$ at $z = L$, we have

$$p_i - p_0 = \frac{2Q}{L}\int_0^L \frac{dz}{I(z)}. \tag{10}$$

The resistance to flow is defined by,

$$\lambda = \frac{p_i - p_0}{Q} \tag{11}$$

From equations (1), (10) and (11), we can find

$$\lambda = \frac{2}{\pi}\left[\frac{L - L_0}{I_0} + \int_d^{d+L_0} \frac{dz}{I(z)}\right], \tag{12}$$

where

$$I_0 = \int_0^{R_0} \frac{r^3}{\mu\,(r)}\,dr. \tag{13}$$

Now, the shear stress at wall is given by,

$$\tau_R = \left[-\mu\,(r)\,\frac{\partial\,w}{\partial\,r}\right]_{r=R(z)}. \tag{14}$$

By using equations (5) and (8) in (14), we can find shear stress at maximum height of stenosis as follows,

$$\tau_s = \left[\frac{R(z)Q}{\pi I(z)}\right]. \tag{15}$$

For finding the effects of peripheral layer viscosity, the viscosity function $\mu(r)$ can be defined as,

$$\mu(r) = \begin{cases} \mu_1; & 0 \le r \le R_1(z), \\ \mu_2; & R_1(z) \le r \le R(z), \end{cases} \tag{16}$$

where μ_1 and μ_2 are the viscosities of the central and the peripheral layers respectively. The function $R_1(z)$ represents the shape of the central layer with stenosis. The mathematical representation of this model can be described as,

$$R_1(z)=\begin{cases} \beta R_0 - \dfrac{2\delta_i}{(L_0-\alpha)}(z-d), & d\leq z\leq d+\dfrac{L_0-\alpha}{2} \\ \beta R_0-\delta_i, & d+\dfrac{L_0-\alpha}{2}\leq z\leq d+\dfrac{L_0+\alpha}{2} \\ \beta R_0-\delta_i+\dfrac{2\delta_i}{(L_0-\alpha)}\left(z-d-\dfrac{L_0+\alpha}{2}\right), & d+\dfrac{L_0+\alpha}{2}\leq z\leq d+L_0 \\ \beta R_0, & \text{elsewhere.} \end{cases} \tag{17}$$

By using equation (16) in equation (5), velocities w_c,w_p and then the corresponding volumetric flow rates Q_c, Q_p can be obtained as follows,

$$Q_c = \int_0^{R_1} 2\pi r w_c\, dr = \left(-\frac{\pi}{8\mu_2}\frac{dp}{dz}\right) 2{R_1}^2\left[R^2-\left(1-\frac{\bar{\mu}_2}{2}\right)R_1^2\right] \tag{18}$$

$$Q_p = \int_{R_1}^{R} 2\pi r w_p\, dr = \left(-\frac{\pi}{8\mu_2}\frac{dp}{dz}\right)\left(R^2-R_1^2\right)^2 \tag{19}$$

where $\bar{\mu}_2 = \mu_2/\mu_1$.

Thus, the total volumetric flow rate Q is defined as,

$$Q = Q_c + Q_p = \left(-\frac{\pi}{8\mu_2}\frac{dp}{dz}\right)\left(R^4-(1-\overline{\mu_2})R_1^4\right). \tag{20}$$

Equation (20) can also be obtained by using equation (16) in equation (7) which shows that Q is a constant.

Integrating equation (18), (19) and (20) by assuming that the pressure drop is same in each case across the length of artery. We obtain,

$$Q_c = \frac{(p_i-p_0)\ \pi R_0^4 M_1}{4\mu_2 L\left(1-\frac{L_0}{L}+M_1G_1\right)}, \tag{21}$$

where

$$M_1 = \beta^2\left[1-\left(1-\frac{\overline{\mu_2}}{2}\right)\beta^2\right. \tag{22}$$

and

$$G_1 = g_1+g_2+g_3, \tag{23}$$

where

$$g_1 = \frac{1}{L}\int_d^{d+\frac{L_0-\alpha}{2}} \frac{dz}{\left(\frac{R_1}{R_0}\right)^2\left[\left(\frac{R}{R_0}\right)^2-\left(1-\frac{\overline{\mu_2}}{2}\right)\left(\frac{R_1}{R_0}\right)^2\right]} \tag{24}$$

$$g_2 = \frac{1}{L}\int_{d+\frac{L_0-\alpha}{2}}^{d+\frac{L_0+\alpha}{2}} \frac{dz}{\left(\frac{R_1}{R_0}\right)^2\left[\left(\frac{R}{R_0}\right)^2-\left(1-\frac{\overline{\mu_2}}{2}\right)\left(\frac{R_1}{R_0}\right)^2\right]} \tag{25}$$

(26)
$$g_3 = \frac{1}{L}\int_{d+\frac{L_0+\alpha}{2}}^{d+L_0} \frac{dz}{\left(\frac{R_1}{R_0}\right)^2\left[\left(\frac{R}{R_0}\right)^2 - \left(1-\frac{\overline{\mu_2}}{2}\right)\left(\frac{R_1}{R_0}\right)^2\right]}$$

and

(27)
$$Q_p = \frac{(p_i - p_0)\ \pi R_0^4 M_2}{8\mu_2 L\left(1 - \frac{L_0}{L} + M_2 G_2\right)},$$

where

(28)
$$M_2 = \left(1-\beta^2\right)^2$$

(29)
$$G_2 = g_4 + g_5 + g_6,$$

where

(30)
$$g_4 = \frac{1}{L}\int_{d}^{d+\frac{L_0-\alpha}{2}} \frac{dz}{\left[\left(\frac{R}{R_0}\right)^2 - \left(\frac{R_1}{R_0}\right)^2\right]^2}$$

(31)
$$g_5 = \frac{1}{L}\int_{d+\frac{L_0-\alpha}{2}}^{d+\frac{L_0+\alpha}{2}} \frac{dz}{\left[\left(\frac{R}{R_0}\right)^2 - \left(\frac{R_1}{R_0}\right)^2\right]^2}$$

(32)
$$g_6 = \frac{1}{L}\int_{d+\frac{L_0+\alpha}{2}}^{d+L_0} \frac{dz}{\left[\left(\frac{R}{R_0}\right)^2 - \left(\frac{R_1}{R_0}\right)^2\right]^2}.$$

Now,

(33)
$$Q = \frac{(p_i - p_0)\ \pi R_0^4 M}{8\mu_2 L\left(1 - \frac{L_0}{L} + MG\right)},$$

where

(34)
$$M = 1 - (1-\overline{\mu_2})\beta^4$$

(35)
$$G = g_7 + g_8 + g_9,$$

where

(36)
$$g_7 = \frac{1}{L}\int_{d}^{d+\frac{L_0-\alpha}{2}} \frac{dz}{\left[\left(\frac{R}{R_0}\right)^4 - (1-\overline{\mu_2})\left(\frac{R_1}{R_0}\right)^4\right]}$$

(37)
$$g_8 = \frac{1}{L}\int_{d+\frac{L_0-\alpha}{2}}^{d+\frac{L_0+\alpha}{2}} \frac{dz}{\left[\left(\frac{R}{R_0}\right)^4 - (1-\overline{\mu_2})\left(\frac{R_1}{R_0}\right)^4\right]}$$

(38)
$$g_9 = \frac{1}{L}\int_{d+\frac{L_0+\alpha}{2}}^{d+L_0} \frac{dz}{\left[\left(\frac{R}{R_0}\right)^4 - (1-\overline{\mu_2})\left(\frac{R_1}{R_0}\right)^4\right]}.$$

From equations (21), (27) and (33) and using $Q = Q_c + Q_p$, we can find

(39)
$$\frac{M}{\left(1 - \frac{L_0}{L} + MG\right)} = \frac{2M_1}{\left(1 - \frac{L_0}{L} + M_1 G_1\right)} + \frac{M_2}{\left(1 - \frac{L_0}{L} + M_2 G_2\right)}.$$

Now, using $R_1 = \beta R$ in equation (17), we get

$$(40)\quad R(z) = \begin{cases} R_0 - \dfrac{2\delta_i}{\beta(L_0-\alpha)}(z-d), & d \le z \le d + \dfrac{L_0-\alpha}{2} \\ R_0 - \dfrac{\delta_i}{\beta}, & d + \dfrac{L_0-\alpha}{2} \le z \le d + \dfrac{L_0+\alpha}{2} \\ R_0 - \dfrac{\delta_i}{\beta} + \dfrac{2\delta_i}{\beta(L_0-\alpha)}\left(z-d-\dfrac{L_0+\alpha}{2}\right), & d + \dfrac{L_0+\alpha}{2} \le z \le d+L_0 \\ R_0, & \text{elsewhere.} \end{cases}$$

On comparing equation (1) and (40), we can observe that

$$(41)\qquad \delta_i = \beta\delta_s.$$

Now by keeping in mind equation (16), the dimensionless resistance to flow $\overline{\lambda}$ and the dimensionless shear stress $\overline{\tau}_s$ can be obtained by using equation (33) in equations (11) and (15) respectively,

$$(42)\qquad \overline{\lambda} = \frac{\lambda}{\lambda_0} = \frac{\overline{\mu_2}}{M}\left(1 - \frac{L_0}{L} + MG\right),$$

where $\overline{\mu_2} = \frac{\mu_2}{\mu_1}$ and $\lambda_0 = \frac{8\mu_1 L}{\pi R_0^4}$, and

$$(43)\qquad \overline{\tau_s} = \frac{\tau_s}{\tau_0} = \frac{\overline{\mu_2}\left(1 - \frac{\delta_s}{R_0}\right)}{\left[\left(1 - \frac{\delta_s}{R_0}\right)^4 - (1 - \overline{\mu_2})\left(\beta - \frac{\delta_i}{R_0}\right)^4\right]},$$

where $\tau_0 = \frac{4\mu_1 Q}{\pi R_0^3}$, and λ_0, τ_0 are the resistance to flow and wall shear stress for the case of no stenosis respectively, with $\overline{\mu_2} = 1$.

Evaluating the integrals (36), (37) and (38) after using equation (41) and rewriting the expressions for $\overline{\lambda}$ and $\overline{\tau_s}$ which are as follows,

$$(44)\qquad \begin{aligned} \overline{\lambda} = \tfrac{\overline{\mu_2}}{M}\Big[1 - \tfrac{L_0}{L} + \tfrac{1}{L}\Big\{&(L_0-\alpha)\left(1 + 2\left(\tfrac{\delta_s}{R_0}\right) + \tfrac{10}{3}\left(\tfrac{\delta_s}{R_0}\right)^2 + \cdots\right) \\ &+\alpha\left(1 + 4\left(\tfrac{\delta_s}{R_0}\right) + 10\left(\tfrac{\delta_s}{R_0}\right)^2 + \cdots\right)\Big\}\Big] \end{aligned}$$

and

$$(45)\qquad \overline{\tau_s} = \frac{\overline{\mu_2}}{\left[\left(1 - \frac{\delta_s}{R_0}\right)^3 M\right]},$$

here $\overline{\tau_s}$ obtained is same as in Shukla et al. [13]. If $\overline{\mu_2} = 1$ in equations (44) and (45), we get

$$(46)\qquad \begin{aligned} \overline{\lambda} = 1 - \tfrac{L_0}{L} + \tfrac{1}{L}\Big\{&(L_0-\alpha)\left(1 + 2\left(\tfrac{\delta_s}{R_0}\right) + \tfrac{10}{3}\left(\tfrac{\delta_s}{R_0}\right)^2 + \cdots\right) \\ &+\alpha\left(1 + 4\left(\tfrac{\delta_s}{R_0}\right) + 10\left(\tfrac{\delta_s}{R_0}\right)^2 + \cdots\right)\Big\}, \end{aligned}$$

which is same as the ratio obtained by Singh et al. [14], and

$$\overline{\tau_s} = \left(1 - \frac{\delta_s}{R_0}\right)^{-3}, \tag{47}$$

which is same as obtained by Young [17].

3. Conclusion

In this paper, we consider a two-layered model of blood flow through a stenosed artery. It is assumed that when blood flows in a cylindrical tube there exists two layers. The central core layer consists of erythrocytes surrounded by a peripheral plasma layer. Both fluids have different viscosities. The expressions for dimensionless resistance to flow $\overline{\lambda}$ and dimensionless shear stress $\overline{\tau_s}$ have been plotted by using MATLAB software for different values of parameters. Graphs in appendix represent the variations of $\overline{\lambda}$and $\overline{\tau_s}$ with $\frac{\delta_s}{R_0}$ for different values of $\overline{\mu_2}$ and $\frac{L_0}{L}$. It has been observed that $\overline{\lambda}$ and $\overline{\tau_s}$ increase with the increase in the height of stenosis, this increase have also been noted when $\overline{\mu_2}$ is increased. It is further noted that resistance to flow $\overline{\lambda}$(denoted by broken lines) is lower in the present model as compared to previous investigation of Shukla et al. [13]. Also, using the data $\overline{\mu_2} = 0.3$, $\frac{L_0}{L} = 1.0$, $\alpha = 0.25$, $\beta = 0.95$ and $\frac{\delta_s}{R_0} = 0.1$ in equations (44) and (45), it can be noted that $\overline{\lambda}$ and $\overline{\tau_s}$ are decreased by 13% and 4% respectively when compared with the case of no stenosis with $\overline{\mu_2} = 1$. Again, in the absence of peripheral layer these characteristics are increased by 24% and 37% respectively for the same stenosis size and $\overline{\mu_2} = 1$. These values are almost same to previous ones thus it seems that the results of present analysis of two-layered model can be useful to explain the flow behaviour of stenotic arteries.

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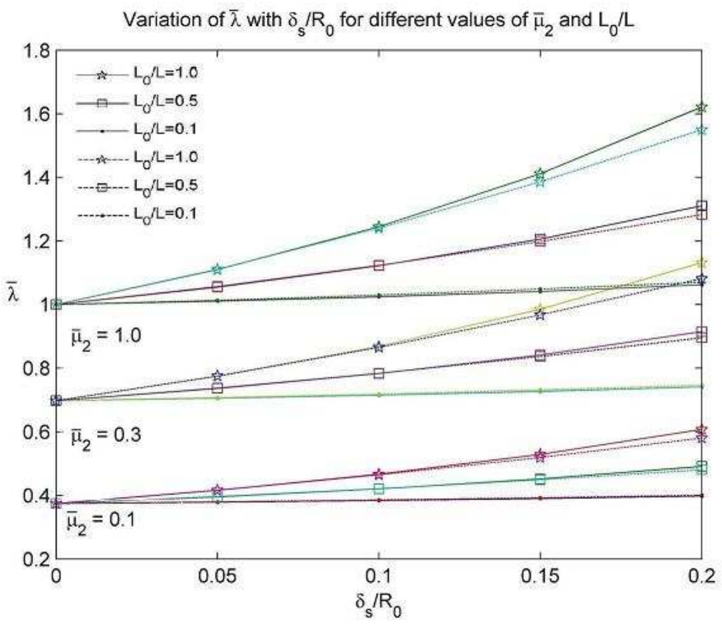
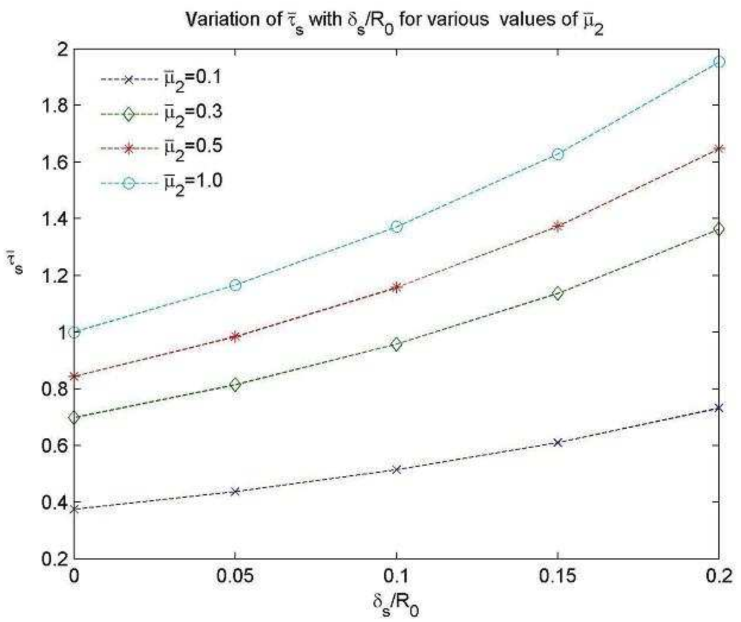
Appendix


Variation of $\bar{\lambda}$ with δ_s/R_0 for different values of $\bar{\mu}_2$ and L_0/L
$L_0/L=1.0$
$L_0/L=0.5$
$L_0/L=0.1$
$L_0/L=1.0$
$L_0/L=0.5$
$L_0/L=0.1$
$\bar{\mu}_2 = 1.0$
$\bar{\mu}_2 = 0.3$
$\bar{\mu}_2 = 0.1$
$\bar{\lambda}$
δ_s/R_0


Variation of $\bar{\tau}_s$ with δ_s/R_0 for various values of $\bar{\mu}_2$
$\bar{\mu}_2=0.1$
$\bar{\mu}_2=0.3$
$\bar{\mu}_2=0.5$
$\bar{\mu}_2=1.0$
$\bar{\tau}_s$
δ_s/R_0

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THE EXPLICIT EXPRESSION OF THE DRAZIN INVERSE OF SUMS OF TWO MATRICES AND ITS APPLICATION

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Abstract. In this paper, we give explicit expressions of $(P \pm Q)_d$ of two matrices P and Q, in terms of P, Q, P_d and Q_d, under the condition that $PQ = P^2$, and apply the result to finding an explicit representation for the Drazin inverse of a 2×2 block matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ under some conditions.

Keywords: Drazin inverse, group inverse, representation, block matrix, idempotent matrix.
AMS Classification: 15A09.

1. Introduction

Recently, the representations of the Drazin inverse of matrices have been widely investigated (see, for example, [2], [10], [11], [15], [16], [17] and the literature mentioned below). In [12], Meyer and Rose presented a representation for the Drazin inverse of $\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$ in terms of sub-blocks A, B, D and their Drazin inverses. And in [1] later, Campbell and Meyer suggested to find an explicit representation for the Drazin inverse of a 2×2 block matrix in relation to its sub-blocks. Since then, a lot of special cases of this problem have been studied (see, for example, [3],

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[4], [5], [6], [7], [8], [13] and references therein). But no one can solve the general problem up to the present.

These investigations motivate us to deal with a representation for the Drazin inverse of a 2×2 block matrix by exploiting an explicit expression of the Drazin inverse of sums of two matrices. The paper is organized as follows. In this section, we will introduce some notions and lemmas. In Section 2, we will present these explicit expressions of differences and sums of two matrices P and Q under the conditions $PQ = P^2$. In Section 3, we will deduce an explicit representation for the Drazin inverse of the 2×2 block matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ under the conditions $AB = 0$ and $D^2 = \frac{1}{2}CB$ in terms of its sub-blocks and A_d and D_d.

Throughout this paper the symbol $\mathbb{C}^{m\times n}$ stands for the set of $m \times n$ complex matrices and $I \in \mathbb{C}^{n\times n}$ stands for the unit matrix. Let $A \in \mathbb{C}^{n\times n}$, the Drazin inverse, denoted by A_d, of matrix A is defined as the unique matrix satisfying

$$A^{k+1}A_d = A^k, \quad A_dAA_d = A_d, \quad AA_d = A_dA$$

where $k = \mathrm{Ind}(A)$ is the index of A. In particular, if $\mathrm{Ind}(A) = 1$, then A_d is called the group inverse, denoted by A_g, of A (see [1], [9], [14]). Apparently, if A is nonsingular, then $\mathrm{Ind}(A) = 0$, otherwise $\mathrm{Ind}(A) \geq 1$, especially $\mathrm{Ind}(0) = 1$. If A is nilpotent, then $A_d = 0$. If X is nonsingular and $B = XAX^{-1}$, then $B_d = XA_dX^{-1}$. For convenience, we write $A^\pi = I - AA_d$ and

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{1.1}$$

where $A \in \mathbb{C}^{n\times n}, B \in \mathbb{C}^{n\times m}, C \in \mathbb{C}^{m\times n}$ and $D \in \mathbb{C}^{m\times m}$.

Also, we need two functions: the ceiling function $\lceil x \rceil$, the smallest integer greater than or equal to x, and the floor function $\lfloor x \rfloor$, the largest integer less than or equal to x.

Now, we list and deduce some lemmas.

Lemma 1.1 [12, Theorem 3.2] *Let M be given by* (1.1) *with $C = 0$, $t = \mathrm{Ind}(A)$ and $l = \mathrm{Ind}(D)$. Then*

$$M_d = \begin{bmatrix} A_d & S \\ 0 & D_d \end{bmatrix}$$

where

$$S = A^\pi \sum_{n=0}^{t-1} A^n B D_d^{n+2} + \sum_{n=0}^{l-1} A_d^{n+2} B D^n D^\pi - A_d B D_d.$$

Lemma 1.2 [8, Theorem 3.1] *Let M be given by* (1.1) *with $D = 0$, $t = \mathrm{Ind}(A)$ and $r = \mathrm{Ind}(BC)$. If $AB = 0$, then*

$$M_d = \begin{bmatrix} XA & (BC)_dB \\ CX & 0 \end{bmatrix}$$

where

$$X = (BC)^{\pi} \sum_{n=0}^{r-1} (BC)^n A_d^{2n+2} + \sum_{n=0}^{\lceil \frac{t}{2} \rceil - 1} (BC)_d^{n+1} A^{2n} A^{\pi}.$$

Lemma 1.3 *Let $P, Q \in \mathbb{C}^{n \times n}$ with $s = \mathrm{Ind}(P)$ and $h = \mathrm{Ind}(Q)$. If $PQ = 0$, then* (i) *for any positive integer $n \geq 2$,*

$$(P+Q)^n = P^n + Q^n + \sum_{i=1}^{n-1} Q^i P^{n-i}. \tag{1.2}$$

(ii)[10, Theorem 2.1]

$$(P+Q)_d = \sum_{n=0}^{s-1} Q_d^{n+1} P^n P^{\pi} + Q^{\pi} \sum_{n=0}^{h-1} Q^n P_d^{n+1}.$$

Proof. (i) When $n = 2$, obviously, $(P+Q)^2 = P^2 + Q^2 + QP$ holds. Assume that (1.2) holds for $k \geq 2$, namely, $(P+Q)^k = P^k + Q^k + \sum\limits_{i=1}^{k-1} Q^i P^{k-i}$, $k \geq 2$. Then for $k+1$,

$$(P+Q)^{k+1} = (P^k + Q^k + \sum_{i=1}^{k-1} Q^i P^{k-i})(P+Q) = P^{k+1} + Q^{k+1} + \sum_{i=1}^{k} Q^i P^{k+1-i}.$$

Hence, by induction, (1.2) holds for any positive integer n.

Remark 1: If $P^r = 0$ in Lemma 1.3(ii), then $(P+Q)_d = \sum\limits_{n=0}^{r-1} Q_d^{n+1} P^n$. ∎

The following lemma generalizes [1, Theorem 7.8.4(iii)].

Lemma 1.4 *Let $P \in \mathbb{C}^{n \times m}$, $Q \in \mathbb{C}^{m \times n}$, then $(PQ)_d = P(QP)_d^2 Q$.*

Lemma 1.5 *Let $P, Q \in \mathbb{C}^{n \times n}$. If $PQ = P^2$, then for any positive integer n,*

$$(Q-P)^n = Q^{n-1}(Q-P). \tag{1.3}$$

Proof. When $n = 1$, obviously, (1.3) holds. Assume that (1.3) holds for k, namely, $(Q-P)^k = Q^{k-1}(Q-P)$. Then for $k+1$,

$$(Q-P)^{k+1} = Q^{k-1}(Q-P)(Q-P) = Q^k(Q-P).$$

Hence, by induction, (1.3) holds for any positive integer n. ∎

2. The Drazin inverse of differences and sums of two matrices

In this section, we will investigate how to express $[I - Q + (WP)^k]_d$ for any positive integer k, and $(P \pm Q)_d$ under some conditions. We begin with the following theorem.

Theorem 2.1 *Let $P, Q \in \mathbb{C}^{n\times n}$ with $\mathrm{Ind}(P) = s$. If $PQ = P$ and Q is idempotent, then*
(i)

$$(2.1)\qquad (I-Q+P^k)_d = \begin{cases} (I-Q)\displaystyle\sum_{n=0}^{\lceil \frac{s}{k}\rceil -1} P^{kn}P^{\pi} + QP_d^k, & \text{if } k < s; \\ (I-Q)P^{\pi} + QP_d^k, & \text{if } k \geq s. \end{cases}$$

(ii)

$$(2.2)\qquad [I-Q+(QP)^k]_d \;=\; I-Q+QP_d^k.$$

Proof. (i) Since $Q = Q^2$, $h = \mathrm{Ind}(I-Q) = 1$. From $PQ = P$, we have $P^k(I-Q) = 0$ for any positive integer k, and $(P^k)^{\pi} = P^{\pi}$. Thus, by Lemma 1.3(ii), we get

$$(2.3)\qquad [I-Q+P^k]_d = (I-Q)\sum_{n=0}^{s_0-1} P^{kn}P^{\pi} + QP_d^k$$

where $s_0 = \mathrm{Ind}P^k$.

Note that $n \geq \lceil \frac{s}{k}\rceil$ implies $kn \geq s$ and that when $kn \geq s$, $P^{kn}P^{\pi} = 0$. So (2.3) becomes

$$[I-Q+P^k]_d = (I-Q)\sum_{n=0}^{\lceil \frac{s}{k}\rceil -1} P^{kn}P^{\pi} + QP_d^k.$$

Consequently, (2.1) holds.

(ii) Replacing P with QP in the proof of (i) yields

$$(2.4)\qquad [I-Q+(QP)^k]_d = (I-Q)\sum_{n=0}^{\lceil \frac{l}{k}\rceil -1} [(QP)^{kn} - (QP)^{kn+1}(QP)_d] + Q(QP)_d^k$$

where $l = \mathrm{Ind}(QP)^k$.

Since $PQ = P$, by Lemma 1.4,

$$(QP)_d = Q(PQ)_d^2P = QP_d^2P = QP_d.$$

In general, by induction, we can easily show that for any positive integer k,

$$(2.5)\qquad (QP)_d^k = QP_d^k.$$

By the above equation and Lemma 1.4,

$$P_d = (PQ)_d = P(QP)_d^2Q = PQP_d^2Q = P_dQ.$$

Since $(I-Q)(QP)^i = 0$ for any positive integer i, by (2.4) and (2.5), we reach (2.2). ∎

Now, we present the expression of $(P \pm Q)_d$.

Theorem 2.2 *Let* $P, Q \in \mathbb{C}^{n\times n}$ *with* $\mathrm{Ind}(P) = s \geq 1$ *and* $\mathrm{Ind}(Q) = r$. *If* $PQ = P^2$ *and* $h = \mathrm{Ind}(P-Q) \geq 1$, *then* $h \leq r+1$ *and*
(i)

$$(P-Q)_d = Q_d^2 P - Q_d = Q_d^2(P-Q). \tag{2.6}$$

(ii)

$$(P-Q)(P-Q)_d = Q_d(Q-P). \tag{2.7}$$

(iii)

$$\begin{aligned}(P+Q)_d \;=\; & \frac{1}{2}Q_d + \sum_{n=0}^{s-1} 2^{n-1} Q_d^{n+1} P^n P^{\pi} + Q^{\pi} \sum_{n=0}^{h-1} 2^{-(n+2)} Q^n P_d^{n+1} \\ & + 2^{-(h+1)} Q^{\pi} Q^{h-1} P_d^h.\end{aligned} \tag{2.8}$$

In particular, when $h = r+1$,

$$(P+Q)_d \;=\; \frac{1}{2}Q_d + \sum_{n=0}^{s-1} 2^{n-1} Q_d^{n+1} P^n P^{\pi} + Q^{\pi} \sum_{n=0}^{r-1} 2^{-(n+2)} Q^n P_d^{n+1}. \tag{2.9}$$

Proof. Since $\mathrm{Ind}(P) = s \geq 1$, there exists a nonsingular matrix W_1 such that

$$P = W_1 \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} W_1^{-1} \quad \textit{and} \quad P_d = W_1 \begin{bmatrix} P_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} W_1^{-1} \tag{2.10}$$

where P_1 is nonsingular and P_2 is nilpotent with $P_2^s = 0$. Partitioning $W_1^{-1} Q W_1$ conformably with $W_1^{-1} P W_1$, we have

$$Q = W_1 \begin{bmatrix} Q_1 & Q_4 \\ Q_3 & Q_2 \end{bmatrix} W_1^{-1}.$$

Since $PQ = P^2$, we can deduce $P_1 = Q_1$, $Q_4 = 0$ and $P_2 Q_2 = P_2^2$. Thus

$$P - Q = W_1 \begin{bmatrix} 0 & 0 \\ -Q_3 & P_2 - Q_2 \end{bmatrix} W_1^{-1} \text{ and } Q_d = W_1 \begin{bmatrix} P_1^{-1} & 0 \\ H & (Q_2)_d \end{bmatrix} W_1^{-1} \tag{2.11}$$

where H is some matrix obtained by using Lemma 1.1.

Since $P_2 Q_2^{k-1} = P_2^k = 0, k \geq \max\{s, 2\}$, we have

$$\begin{aligned} (Q_2^s P_2)^2 &= Q_2^s (P_2 Q_2^s) P_2 = 0, \\ P_2 (Q_2)_d &= P_2 Q_2^s (Q_2)_d^{s+1} = 0, \\ (Q_2^s P_2)_d &= 0. \end{aligned}$$

By (1.3) and Remark 1, therefore, we have

$$(P_2 - Q_2)_d^{s+1} = (-1)^s (Q_2^s P_2 - Q_2^{s+1})_d = (-1)^s[-(Q_2)_d^{s+1} + (Q_2)_d^{s+2} P_2]$$

and then, by Lemma 1.5,

$$\begin{aligned}
(P_2-Q_2)_d &= (P_2-Q_2)_d^{s+1}(P_2-Q_2)^s \\
&= (-1)^{2s-1}[-(Q_2)_d^{s+1}+(Q_2)_d^{s+2}P_2](Q_2^{s-1}P_2-Q_2^s) \\
&= (Q_2)_d^2P_2-(Q_2)_d, \\
(P_2-Q_2)_d^2 &= [(Q_2)_d^2P_2]^2-(Q_2)_d^2P_2(Q_2)_d-(Q_2)_d(Q_2)_d^2P_2+(Q_2)_d^2 \\
&= (Q_2)_d^2-(Q_2)_d^3P_2.
\end{aligned}$$

By Lemma 1.1, we get that

$$\begin{aligned}
(P-Q)_d &= W_1\begin{bmatrix} 0 & 0 \\ -(P_2-Q_2)_d^2Q_3 & (P_2-Q_2)_d \end{bmatrix}W_1^{-1} \\
&= W_1\begin{bmatrix} 0 & 0 \\ -[(Q_2)_d^2P_2-(Q_2)_d]^2Q_3 & (Q_2)_d^2P_2-(Q_2)_d \end{bmatrix}W_1^{-1} \\
&= (Q_d^2P-Q_d)P^\pi-(Q_d^2-Q_d^3P)P^\pi QPP_d \\
&= Q_d^2P-Q_d-(Q_d^2P-Q_d)PP_d-(Q_d^2-Q_d^3P)(QPP_d-P_dPQPP_d) \\
&= Q_d^2P-Q_d-(Q_d^2P-Q_d)PP_d-Q_d^2(QPP_d-P^2P_d) \\
&= Q_d^2P-Q_d \;=\; Q_d^2(P-Q).
\end{aligned}\tag{2.12}$$

Since $\mathrm{Ind}(Q)=r$, $Q_dQ^{r+1}=Q^r$. By (1.3) and (2.12),

$$\begin{aligned}
(P-Q)_d(P-Q)^{r+2} &= Q_d^2(P-Q)^{r+3} &&= (-1)^{r+2}Q_d^2Q^{r+2}(P-Q) \\
&= (-1)^rQ^r(P-Q) &&= (P-Q)^{r+1}.
\end{aligned}$$

Hence $\mathrm{Ind}(P-Q)=\mathrm{Ind}(Q-P)\le r+1$.

(ii) By (i) and Lemma 1.5,

$$(P-Q)(P-Q)_d=(P-Q)_d(P-Q)=Q_d^2(P-Q)^2=Q_d(Q-P).$$

(iii) By (i) and (ii),

$$(Q-P)_d^2=Q_d^2(Q-P)(Q-P)_d=Q_d^3(Q-P).$$

Generally, for any positive integer n, by induction,

$$(Q-P)_d^n \;=\; Q_d^{n+1}(Q-P).\tag{2.13}$$

From $PQ=P^2$, it follows easily that $P(P-Q)=0$ and $PQ^k=P^{k+1}$ for

$k \geq 1$. Thus, by Lemma 1.2 and (1.3), (2.7) and (2.13), we have

$$\begin{aligned}
(P+Q)_d &= [2P+(Q-P)]_d \\
&= \sum_{n=0}^{s-1}(Q-P)_d^{n+1}(2P)^nP^\pi + (Q-P)^\pi\sum_{n=0}^{h-1}(Q-P)^n(2P)_d^{n+1} \\
&= \sum_{n=0}^{s-1}2^nQ_d^{n+2}(Q-P)P^nP^\pi + (Q^\pi+Q_dP)\left[\frac{1}{2}P_d + \sum_{n=1}^{h-1}2^{-(n+1)}Q^{n-1}(Q-P)P_d^{n+1}\right] \\
&= \sum_{n=0}^{s-1}2^n(Q_d^{n+1}P^n - Q_d^{n+2}P^{n+1})P^\pi + \frac{1}{2}(Q^\pi+Q_dP)P_d \\
&\qquad +Q^\pi\sum_{n=1}^{h-1}2^{-(n+1)}Q^{n-1}(Q-P)P_d^{n+1} \\
&= Q_dP^\pi + \sum_{n=1}^{s-1}2^nQ_d^{n+1}P^nP^\pi - \sum_{n=1}^{s-1}2^{n-1}Q_d^{n+1}P^nP^\pi + \frac{1}{2}(Q^\pi+Q_dP)P_d \\
&\qquad +Q^\pi\sum_{n=1}^{h-1}2^{-(n+1)}Q^nP_d^{n+1} - Q^\pi\sum_{n=1}^{h-2}2^{-(n+2)}Q^nP_d^{n+1} - 2^{-2}Q^\pi P_d \\
&= \frac{1}{2}Q_d + \frac{1}{2}Q_dP^\pi + \sum_{n=1}^{s-1}2^{n-1}Q_d^{n+1}P^nP^\pi \\
&\qquad +2^{-h}Q^\pi Q^{h-1}P_d^h + Q^\pi\sum_{n=1}^{h-2}2^{-(n+2)}Q^nP_d^{n+1} + 2^{-2}Q^\pi P_d \\
&= \frac{1}{2}Q_d + \sum_{n=0}^{s-1}2^{n-1}Q_d^{n+1}P^nP^\pi + Q^\pi\sum_{n=0}^{h-1}2^{-(n+2)}Q^nP_d^{n+1} + 2^{-(h+1)}Q^\pi Q^{h-1}P_d^h.
\end{aligned}$$

When $h = r+1$, (2.9) follows from (2.8). ∎

Remark 2. If $h = 0$ in Theorem 2.2, then $P - Q$ is nonsingular and therefore $P = 0$ since $PQ = P^2$. Similarly, if $s = 0$, then P is nonsingular and so $P = Q$. Thus Theorem 2.2 is trivial for the two special cases.

If P is idempotent in Theorem 2.2, then $P^kP^\pi = 0, k \geq 0$. So we have the following result.

Corollary 2.1 *Let $P, Q \in \mathbb{C}^{n\times n}$ with $r = \mathrm{Ind}(Q)$ and $h = \mathrm{Ind}(P-Q) \geq 1$. If $PQ = P$ and P is idempotent, then $h \leq r+1$ and*

$$(P+Q)_d = Q_d - \frac{1}{2}Q_dP + Q^\pi\sum_{n=0}^{h-1}2^{-(n+2)}Q^nP + 2^{-(h+1)}Q^\pi Q^{h-1}P$$

In particular, when $h = r+1$,

$$(P+Q)_d = Q_d - \frac{1}{2}Q_dP + Q^\pi\sum_{n=0}^{r-1}2^{-(n+2)}Q^nP$$

If P and Q are both idempotent in Theorem 2.2, then

$$(P+Q)_d = Q + \frac{1}{4}P - \frac{3}{4}QP.$$

Since

$$(P+Q)^2(P+Q)_d = (P+Q)\left(\frac{1}{2}P + Q - \frac{1}{2}QP\right) = P+Q,$$

we have the result below.

Corollary 2.2 *Let $P, Q \in \mathbb{C}^{n\times n}$ be idempotent. If $PQ = P$, then*

$$(P-Q)_d = QP - Q \quad \textit{and} \quad (P+Q)_g = Q + \frac{1}{4}P - \frac{3}{4}QP.$$

Theorem 2.3 *Let $P, Q \in \mathbb{C}^{n\times n}$. If $PQ = P^2$ and $QP = Q^2$, then*

$$(P+Q)_d = \frac{1}{4}(P_d + Q_d) \quad \textit{and} \quad (P-Q)_d = 0.$$

Proof. Premultiplying $QP = Q^2$ by Q_d^2 yields $Q_dP = QQ_d$ and then $Q_dPP_d = QQ_dP_d$. So, by Lemma 1.4, we get

$$QP_d = QPP_d^2 = Q^2(PQ)_d = Q^2P(QP)_d^2Q = Q^3Q_d^4Q = QQ_d = Q_dP,$$

and then $Q_dQP_d = Q_d$. Thus for $n \geq 1$,

$$\begin{aligned} Q_dP^{n-1}(I - PP_d) &= (Q_d - Q_dPP_d)P^{n-1} = 0, && (2.14)\\ (I - QQ_d)Q^nP_d &= (I - QQ_d)Q^{n-1}Q_dP = 0. && (2.15) \end{aligned}$$

Since $PQ = P^2$, by Theorem 2.2(iii), we have (2.8) and put (2.14) and (2.15) in (2.8). As a result,

$$(P+Q)_d = \frac{1}{2}Q_d + \frac{1}{4}(I - Q_dQ)P_d = \frac{1}{2}Q_d + \frac{1}{4}(P_d - Q_d) = \frac{1}{4}(P_d + Q_d).$$

Since $(P-Q)^2 = P^2 - PQ - QP + Q^2 = 0$, $(P-Q)_d = 0$. ∎

3. The Drazin inverse of 2×2 block matrices

In this section, we turn our attention to the representation for the Drazin inverse of a 2-by-2 block matrix M given by (1.1), in terms of its sub-blocks and A_d and D_d, by Theorem 2.2.

Theorem 3.1 *Let M be given by (1.1) with $t = \mathrm{Ind}(A)$, $s = \mathrm{Ind}\left(\begin{bmatrix} A & \frac{1}{2}B \\ C & 0 \end{bmatrix}\right)$, $q = \mathrm{Ind}\left(\begin{bmatrix} 0 & \frac{1}{2}B \\ 0 & D \end{bmatrix}\right)$, $h = \mathrm{Ind}\left(\begin{bmatrix} A & 0 \\ C & -D \end{bmatrix}\right)$ and $r = \mathrm{Ind}(BC)$. If $D^2 = \frac{1}{2}CB$*

and $AB = 0$, then $h \leq s+1$ and

$$\begin{aligned} M_d &= \frac{1}{2}\begin{bmatrix} L & BD_d^2 \\ 2K(1,r)+S(t) & 0 \end{bmatrix} + \sum_{n=1}^{q-1} 2^{n-1}\begin{bmatrix} \frac{1}{2}BK(n,r)A_d & 0 \\ K(n+1,r) & 0 \end{bmatrix} \\ &\quad -\frac{1}{3}(1-4^{-\lceil \frac{h}{2}\rceil})G(0,t) - \frac{1}{6}(1-4^{-\lfloor \frac{h}{2}\rfloor})G(1,t) + \sum_{n=2}^{h-1} 2^{-(n+2)}G(n,n) \\ &\quad +2^{-(h+1)}[G(h-1,h-1)-G(h-1,t)]. \end{aligned} \tag{3.1}$$

In particular, when $h = s+1$,

$$\begin{aligned} M_d &= \frac{1}{2}\begin{bmatrix} L & BD_d^2 \\ 2K(1,r)+S(t) & 0 \end{bmatrix} + \sum_{n=1}^{q-1} 2^{n-1}\begin{bmatrix} \frac{1}{2}BK(n,r)A_d & 0 \\ K(n+1,r) & 0 \end{bmatrix} \\ &\quad -\frac{1}{3}(1-4^{-\lceil \frac{s}{2}\rceil})G(0,t) - \frac{1}{6}(1-4^{-\lfloor \frac{s}{2}\rfloor})G(1,t) + \sum_{n=2}^{s-1} 2^{-(n+2)}G(n,n) \end{aligned} \tag{3.2}$$

where $\sum\limits_{n=k}^{m} = 0$ whenever $m < k$ and

$$L = (2I - BD_d^2C)A_d + BK(1,r-1)A_d + BD_d^2S(t)A - \frac{1}{2}BD_dS(t), \tag{3.3}$$

$$S(n) = \sum_{k=0}^{\lceil \frac{n}{2}\rceil -1} D_d^{2k+2}CA^{2k}A^{\pi}, \tag{3.4}$$

$$G(n,m) = \begin{cases} \begin{bmatrix} \frac{1}{2}BD_d^2S(m)A & 0 \\ D_dS(m)A & 0 \end{bmatrix}, & n \text{ is even;} \\ \\ \begin{bmatrix} \frac{1}{2}BD_d^3S(m-2)A^2 & 0 \\ D_d^2S(m-2)A^2 & 0 \end{bmatrix}, & n \text{ is odd,} \end{cases} \tag{3.5}$$

$$K(n,m) = D^{\pi}\sum_{k=0}^{m-1} D^{n+2k-1}CA_d^{n+2k+1}. \tag{3.6}$$

Proof. Assume that

$$W = \begin{bmatrix} Y_0 & Y_1 \\ Y_2 & 0 \end{bmatrix} = \begin{bmatrix} Y_0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & Y_1 \\ Y_2 & 0 \end{bmatrix} := W_1 + W_2$$

where $Y_0Y_1 = 0$. So, for $k \geq 1$,

$$\begin{aligned} W_1^k &= \begin{bmatrix} Y_0^k & 0 \\ 0 & 0 \end{bmatrix}, \qquad W_2^{2k} = \begin{bmatrix} (Y_1Y_2)^k & 0 \\ 0 & (Y_2Y_1)^k \end{bmatrix}, \\ W_2^{2k-1} &= \begin{bmatrix} 0 & (Y_1Y_2)^{k-1}Y_1 \\ (Y_2Y_1)^{k-1}Y_2 & 0 \end{bmatrix}. \end{aligned}$$

and then

$$W_2^{2k}W_1^{n-2k} = \begin{bmatrix} (Y_1Y_2)^kY_0^{n-2k} & 0 \\ 0 & 0 \end{bmatrix},$$

$$W_2^{2k-1}W_1^{n-2k+1} = \begin{bmatrix} 0 & 0 \\ (Y_2Y_1)^{k-1}Y_2Y_0^{n-2k+1} & 0 \end{bmatrix},$$

$$HW_2^{2j}W_1^k = 0, \quad \text{for } j > 1, \tag{3.7}$$

where $H = \begin{bmatrix} 0 & * \\ 0 & * \end{bmatrix}$ partitioned conformably to W. For $n \geq 2$, by Lemma 1.3(i) and (3.7),

$$\begin{aligned} HW^n &= H(W_1^n + W_2^n + \sum_{i=1}^{n-1} W_2^iW_1^{n-i}) = HW_2^n + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} HW_2^{2k-1}W_1^{n-2k+1} \\ &= HW_2^n + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} H \begin{bmatrix} 0 & 0 \\ (Y_2Y_1)^{k-1}Y_2Y_0^{n-2k+1} & 0 \end{bmatrix}. \end{aligned} \tag{3.8}$$

Rewrite M as

$$M = \begin{bmatrix} A & \frac{1}{2}B \\ C & 0 \end{bmatrix} + \begin{bmatrix} 0 & \frac{1}{2}B \\ 0 & D \end{bmatrix} := P + Q.$$

From the conditions that $2D^2 = CB$ and $AB = 0$, it follows $PQ = Q^2$, namely $Q^TP^T = (Q^T)^2$ where the symbol F^T stands for the transpose of a matrix F. Note that $\mathrm{Ind}\left(\begin{bmatrix} A_1 & 0 \\ A_3 & A_2 \end{bmatrix}\right) \geq 1$ whenever square matrix A_1 or A_2 is singular by [12, Theorem 2.1]. So $q \geq 1$. If A is nonsingular, then $B = 0$ since $AB = 0$. Thus $D^2 = 0$ and then D is singular. So $h \geq 1$. Therefore, applying Theorem 2.2 to M^T, we have $h = \mathrm{Ind}(P - Q) \leq \mathrm{Ind}(P) + 1 = s + 1$ and

$$\begin{aligned} M_d &= \frac{1}{2}P_d + Q^\pi \sum_{n=0}^{q-1} 2^{n-1}Q^nP_d^{n+1} + \sum_{n=0}^{h-1} 2^{-(n+2)}Q_d^{n+1}P^nP^\pi \\ &\quad + 2^{-(h+1)}Q_d^hP^{h-1}P^\pi. \end{aligned} \tag{3.9}$$

Obviously, for $n \geq 1$,

$$Q^n = \begin{bmatrix} 0 & \frac{1}{2}BD^{n-1} \\ 0 & D^n \end{bmatrix}, \quad Q_d^n = \begin{bmatrix} 0 & \frac{1}{2}BD_d^{n+1} \\ 0 & D_d^n \end{bmatrix}, \quad Q^\pi = \begin{bmatrix} I & -\frac{1}{2}BD_d \\ 0 & D^\pi \end{bmatrix}.$$

By Lemma 1.4,

$$C\left(\frac{1}{2}BC\right)_d = C\left(\frac{1}{2}BC\right)_d^2\left(\frac{1}{2}BC\right) = \left(\frac{1}{2}CB\right)_d C = D_d^2C, \tag{3.10}$$

$$C(BC)_dB = \frac{1}{2}D_d^2CB = D_dD, \tag{3.11}$$

$$D_dC(BC)^\pi = D_dC(I - (BC)_dBC) = D_dC - D_d^2DC = 0 \tag{3.12}$$

So, by Lemma 1.2 and (3.11),

$$P_d = \begin{bmatrix} XA & (BC)_d B \\ CX & 0 \end{bmatrix}, \qquad P^\pi = \begin{bmatrix} (BC)^\pi - XA^2 & 0 \\ -CXA & D^\pi \end{bmatrix}$$

where

$$X = \left(\frac{1}{2}BC\right)^\pi \sum_{m=0}^{r-1} \left(\frac{1}{2}BC\right)^m A_d^{2m+2} + \sum_{m=0}^{\lceil \frac{t}{2} \rceil - 1} \left(\frac{1}{2}BC\right)_d^{m+1} A^{2m} A^\pi$$

and $AX = A_d$.

Now, taking $W = P$, we have $W_1 = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$ and $W_2 = \begin{bmatrix} 0 & \frac{1}{2}B \\ C & 0 \end{bmatrix}$, and, accordingly, $Y_0 = A, Y_1 = \frac{1}{2}B$, and $Y_2 = C$. Thus,

$$(Y_2 Y_1)^{k-1} = \left(\frac{1}{2}CB\right)^{k-1} = D^{2k-2}.$$

Taking $H = Q_d^{n+1}$ in (3.8) for $n \geq 2$ yields

$$\begin{aligned} Q_d^{n+1} P^n P^\pi &= Q_d^{n+1} W_2^n P^\pi + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \begin{bmatrix} 0 & \frac{1}{2}BD_d^{n+2} \\ 0 & D_d^{n+1} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ D^{2k-2}CA^{n-2k+1} & 0 \end{bmatrix} P^\pi \\ &= Q_d^{n+1} W_2^n P^\pi + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \begin{bmatrix} \frac{1}{2}BD_d^{n-2k+4}CA^{n-2k+1}[(BC)^\pi - XA^2] & 0 \\ D_d^{n-2k+3}CA^{n-2k+1}[(BC)^\pi - XA^2] & 0 \end{bmatrix} \\ &= Q_d^{n+1} W_2^n P^\pi + \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor - 1} \begin{bmatrix} \frac{1}{2}BD_d^{n-2k+2}CA^{n-2k-1}A^\pi & 0 \\ D_d^{n-2k+1}CA^{n-2k-1}A^\pi & 0 \end{bmatrix}. \end{aligned} \tag{3.13}$$

since $A[(BC)^\pi - XA^2] = AA^\pi$ and $k \leq n/2$.

In order to continue conveniently the above computation, we write

$$V(n) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor - 1} D_d^{n-2k+1} CA^{n-2k-1} A^\pi, \quad n \geq 2,$$

and consider it in terms of the parity of n as follows. If $n = 2w+1, w \geq 1$, then since

$$S((2w+1)-2) = \sum_{k=0}^{\lceil \frac{2w-1}{2} \rceil - 1} D_d^{2k+2} CA^{2k} A^\pi = \sum_{k=0}^{w-1} D_d^{2k+2} CA^{2k} A^\pi,$$

we have

$$\begin{aligned} V(2w+1) &= \sum_{k=0}^{w-1} D_d^{2w-2k+2} CA^{2w-2k} A^\pi = \sum_{i=0}^{w-1} D_d^{2i+4} CA^{2i+2} A^\pi \\ &= D_d^2 S((2w+1)-2) A^2. \end{aligned}$$

If $n = 2w, w \geq 1$, then

$$\begin{aligned} V(2w) &= \sum_{k=0}^{w-1} D_d^{2w-2k+1} C A^{2w-2k-1} A^\pi = \sum_{i=0}^{w-1} D_d^{2i+3} C A^{2i+1} A^\pi \\ &= D_d S(2w) A. \end{aligned}$$

Hence,

$$V(n) = \begin{cases} D_d S(n) A, & n \text{ is even;} \\ D_d^2 S(n-2) A^2, & n \text{ is odd} \end{cases}$$

and then for $n \geq 2$,

$$G(n,n) = \begin{bmatrix} \frac{1}{2} B D_d V(n) & 0 \\ V(n) & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} B D_d^{n-2k+2} C A^{n-2k-1} A^\pi & 0 \\ D_d^{n-2k+1} C A^{n-2k-1} A^\pi & 0 \end{bmatrix} \tag{3.14}$$

We also need to deal with $Q_d^{n+1} W_2^n P^\pi$. Likewise, we consider it in terms of the parity of n as follows. When $n = 2k$,

$$\begin{aligned} Q_d^{2k+1} W_2^{2k} P^\pi &= \begin{bmatrix} 0 & \frac{1}{2} B D_d^{2k+2} \\ 0 & D_d^{2k+1} \end{bmatrix} \begin{bmatrix} (\frac{1}{2} BC)^k & 0 \\ 0 & D^{2k} \end{bmatrix} \begin{bmatrix} (BC)^\pi - X A^2 & 0 \\ -CXA & D^\pi \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{2} B D_d^2 C X A & 0 \\ -D_d C X A & 0 \end{bmatrix}, \end{aligned}$$

and when $n = 2k-1$, by (3.12),

$$\begin{aligned} Q_d^{2k} W_2^{2k-1} P^\pi &= \begin{bmatrix} 0 & \frac{1}{2} B D_d^{2k+1} \\ 0 & D_d^{2k} \end{bmatrix} \begin{bmatrix} 0 & (\frac{1}{2} BC)^{k-1} B \\ D^{2k-2} C & 0 \end{bmatrix} \begin{bmatrix} (BC)^\pi - X A^2 & 0 \\ -CXA & D^\pi \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{2} B D_d^3 C X A^2 & 0 \\ -D_d^2 C X A^2 & 0 \end{bmatrix}. \end{aligned}$$

By (3.10) and (3.12),

$$\begin{aligned} D_d C X A &= D_d C \sum_{k=0}^{\lceil \frac{t}{2} \rceil - 1} \left(\frac{1}{2} BC \right)_d^{k+1} A^{2k+1} A^\pi = D_d \sum_{k=0}^{\lceil \frac{t}{2} \rceil - 1} D_d^{2k+2} C A^{2k+1} A^\pi \\ &= D_d S(t) A, \\ D_d^2 C X A^2 &= D_d^2 \sum_{k=0}^{\lceil \frac{t}{2} \rceil - 1} D_d^{2k+2} C A^{2k+2} A^\pi = D_d^2 \sum_{k=0}^{\lceil \frac{t}{2} \rceil - 2} D_d^{2k+2} C A^{2k+2} A^\pi \\ &= D_d^2 \sum_{k=0}^{\lceil \frac{t-2}{2} \rceil - 1} D_d^{2k+2} C A^{2k+2} A^\pi = D_d^2 S(t-2) A^2. \end{aligned}$$

Thus

$$Q_d^{n+1} W_2^n P^\pi = -G(n,t). \tag{3.15}$$

Hence, putting (3.14) and (3.15) in (3.13) yields

$$Q_d^{n+1}P^nP^\pi \;=\; G(n,n)-G(n,t) \tag{3.16}$$

where $n \geq 2$. Especially,

$$Q_d^hP^{h-1}P^\pi \;=\; G(h-1,h-1)-G(h-1,t). \tag{3.17}$$

By (3.12),

$$\begin{aligned}\frac{1}{4}Q_dP^\pi+\frac{1}{8}Q_d^2PP^\pi &= \frac{1}{4}\begin{bmatrix} 0 & \frac{1}{2}BD_d^2 \\ 0 & D_d \end{bmatrix}\begin{bmatrix} (BC)^\pi - XA^2 & 0 \\ -CXA & D^\pi \end{bmatrix} \\ &\quad +\frac{1}{8}\begin{bmatrix} 0 & \frac{1}{2}BD_d^3 \\ 0 & D_d^2 \end{bmatrix}\begin{bmatrix} A & \frac{1}{2}B \\ C & 0 \end{bmatrix}\begin{bmatrix} (BC)^\pi - XA^2 & 0 \\ -CXA & D^\pi \end{bmatrix} \\ &= \frac{1}{4}\begin{bmatrix} -\frac{1}{2}BD_d^2CXA & 0 \\ -D_dCXA & 0 \end{bmatrix}+\frac{1}{8}\begin{bmatrix} -\frac{1}{2}BD_d^3CXA^2 & 0 \\ -D_d^2CXA^2 & 0 \end{bmatrix} \\ &= -2^{-2}G(0,t)-2^{-3}G(1,t). \end{aligned} \tag{3.18}$$

Hence, we can obtain, by (3.16) and (3.18),

$$\begin{aligned}\sum_{n=0}^{h-1}2^{-(n+2)}Q_d^{n+1}P^nP^\pi &= \frac{1}{4}Q_dP^\pi+\frac{1}{8}Q_d^2PP^\pi+\sum_{n=2}^{h-1}2^{-(n+2)}[G(n,n)-G(n,t)] \\ &= \sum_{n=2}^{h-1}2^{-(n+2)}G(n,n)-\sum_{n=0}^{h-1}2^{-(n+2)}G(n,t). \end{aligned} \tag{3.19}$$

In particular, when $h<3$, the first sum $\sum\limits_{n=2}^{h-1}=0$ in (3.19).

Note that by (3.5), $G(1,t)=G(3,t)=\cdots=G(2k-1,t)=\cdots$ and $G(0,t)=G(2,t)=\cdots=G(2k,t)=\cdots$. Thus

$$\begin{aligned}\sum_{n=0}^{h-1}2^{-(n+2)}G(n,t) &= \sum_{k=0}^{\lceil\frac{h}{2}\rceil-1}2^{-(2k+2)}G(2k,t)+\sum_{k=0}^{\lfloor\frac{h}{2}\rfloor-1}2^{-(2k+3)}G(2k+1,t) \\ &= \frac{1}{3}(1-4^{-\lceil\frac{h}{2}\rceil})G(0,t)+\frac{1}{6}(1-4^{-\lfloor\frac{h}{2}\rfloor})G(1,t). \end{aligned} \tag{3.20}$$

Next, taking $W=P_d$, we have $W_1=\begin{bmatrix} XA & 0 \\ 0 & 0 \end{bmatrix}$ and $W_2=\begin{bmatrix} 0 & (BC)_dB \\ CX & 0 \end{bmatrix}$, and, accordingly, $Y_0=XA, Y_1=(BC)_dB$, and $Y_2=CX$. Thus

$$(XA)^i=X(AX)^{i-1}A=XA_d^{i-1}A, \text{ for } i\geq 1,$$

and, by $AB=0$ and Lemma 1.4,

$$Y_2Y_1=CX(BC)_dB=C\left(\frac{1}{2}BC\right)_d(BC)_dB=\left(\frac{1}{2}CB\right)_d=D_d^2.$$

Note that

$$Q^\pi Q^n = \begin{bmatrix} I & -\frac{1}{2}BD_d \\ 0 & I-DD_d \end{bmatrix}\begin{bmatrix} 0 & \frac{1}{2}BD^{n-1} \\ 0 & D^n \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2}BD^{n-1}D^\pi \\ 0 & D^nD^\pi \end{bmatrix}.$$

Therefore, taking $H = Q^n$ in (3.8) for $n \geq 1$ yields

$$\begin{aligned} Q^\pi Q^n P_d^{n+1} &= Q^\pi Q^n W_2^{n+1} \\ &+ \sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} \begin{bmatrix} 0 & \frac{1}{2}BD^{n-1}D^\pi \\ 0 & D^nD^\pi \end{bmatrix}\begin{bmatrix} 0 & 0 \\ D_d^{2k-2}CX(XA)^{n-2k+2} & 0 \end{bmatrix} \\ &= Q^\pi Q^n W_2^{n+1} + \begin{bmatrix} \frac{1}{2}BD^{n-1}D^\pi CX^2AA_d^{n-1} & 0 \\ D^nD^\pi CX^2AA_d^{n-1} & 0 \end{bmatrix}. \end{aligned} \tag{3.21}$$

Consider $Q^\pi Q^n W_2^{n+1}$ as follows. When $n = 2k$,

$$Q^\pi Q^n W_2^{2k+1} = \begin{bmatrix} 0 & \frac{1}{2}BD^{n-1}D^\pi \\ 0 & D^nD^\pi \end{bmatrix}\begin{bmatrix} 0 & (Y_1Y_2)^kY_1 \\ D_d^{2k}Y_2 & 0 \end{bmatrix} = 0.$$

And when $n = 2k-1$,

$$Q^\pi Q^n W_2^{2k} = \begin{bmatrix} 0 & \frac{1}{2}BD^{n-1}D^\pi \\ 0 & D^nD^\pi \end{bmatrix}\begin{bmatrix} (Y_1Y_2)^k & 0 \\ 0 & D_d^{2k} \end{bmatrix} = 0.$$

Hence

$$Q^\pi Q^n W_2^{n+1} = 0. \tag{3.22}$$

By (3.10) and (3.11),

$$\begin{aligned} X &= (BC)^\pi A_d^2 + (BC)^\pi \sum_{k=1}^{r-1} \frac{1}{2}BD^{2k-2}CA_d^{2k+2} + \sum_{k=0}^{\lceil \frac{t}{2} \rceil - 1} \frac{1}{2}BC(\frac{1}{2}BC)_d^{m+2}A^{2k}A^\pi \\ &= (BC)^\pi A_d^2 + \frac{1}{2}BD^\pi \sum_{k=1}^{r-1} D^{2k-2}CA_d^{2k+2} + \frac{1}{2}B\sum_{k=0}^{\lceil \frac{t}{2} \rceil - 1} D_d^{2k+4}CA^{2k}A^\pi \\ &= (BC)^\pi A_d^2 + \frac{1}{2}BD^\pi \sum_{k=1}^{r-1} D^{2k-2}CA_d^{2k+2} + \frac{1}{2}BD_d^2S(t) \end{aligned}$$

and then

$$\begin{aligned} CX &= D^\pi CA_d^2 + D^\pi D^2 \sum_{k=1}^{r-1} D^{2k-2}CA_d^{2k+2} + D^2D_d^2S(t) \\ &= D^\pi \sum_{k=0}^{r-1} D^{2k}CA_d^{2k+2} + S(t) \\ &= K(1,r) + S(t), \end{aligned} \tag{3.23}$$

$$D^{\pi}CX = K(1,r), \tag{3.24}$$

$$\begin{aligned} XA &= (BC)^{\pi}A_d + \tfrac{1}{2}BD^{\pi}\textstyle\sum_{k=1}^{r-1} D^{2k-2}CA_d^{2k+1} + \tfrac{1}{2}BD_d^2S(t)A \\ &= \left(I - \tfrac{1}{2}BD_d^2C\right)A_d + \tfrac{1}{2}BD^{\pi}\textstyle\sum_{k=0}^{r-2} D^{2k}CA_d^{2k+3} + \tfrac{1}{2}BD_d^2S(t)A \\ &= \left(I - \tfrac{1}{2}BD_d^2C\right)A_d + \tfrac{1}{2}BK(1,r-1)A_d + \tfrac{1}{2}BD_d^2S(t)A, \end{aligned} \tag{3.25}$$

$$D^{\pi}CX^2A = K(1,r)XA = K(1,r)A_d. \tag{3.26}$$

By Lemma 1.4, (3.23) and (3.24),

$$\begin{aligned} Q^{\pi}P_d &= \begin{bmatrix} I & -\tfrac{1}{2}BD_d \\ 0 & D^{\pi} \end{bmatrix}\begin{bmatrix} XA & (BC)_dB \\ CX & 0 \end{bmatrix} \\ &= \begin{bmatrix} XA - \tfrac{1}{2}BD_dCX & (BC)_dB \\ D^{\pi}CX & 0 \end{bmatrix} \\ &= \begin{bmatrix} XA - \tfrac{1}{2}BD_dS(t) & \dfrac{1}{2}BD_d^2 \\ K(1,r) & 0 \end{bmatrix}, \end{aligned} \tag{3.27}$$

and, by (3.21), (3.22) and (3.26)

$$Q^{\pi}Q^nP_d^{n+1} = \begin{bmatrix} \tfrac{1}{2}BD^{n-1}K(1,r)A_dA_d^{n-1} & 0 \\ D^nK(1,r)A_dA_d^{n-1} & 0 \end{bmatrix} = \begin{bmatrix} \tfrac{1}{2}BK(n,r)A_d & 0 \\ K(n+1,r) & 0 \end{bmatrix}. \tag{3.28}$$

Consequently, by (3.27), (3.28),(3.23) and (3.25),

$$\begin{aligned} &\frac{1}{2}P_d + Q^{\pi}\sum_{n=0}^{q-1} 2^{n-1}Q^nP_d^{n+1} \\ &= \frac{1}{2}\begin{bmatrix} XA & \dfrac{1}{2}BD_d^2 \\ CX & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} XA - \dfrac{1}{2}BD_dS(t) & \dfrac{1}{2}BD_d^2 \\ K(1,r) & 0 \end{bmatrix} \\ &+ \sum_{n=1}^{q-1} 2^{n-1}\begin{bmatrix} \dfrac{1}{2}BK(n,r)A_d & 0 \\ K(n+1,r) & 0 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} L & BD_d^2 \\ 2K(1,r)+S(t) & 0 \end{bmatrix} + \sum_{n=1}^{q-1} 2^{n-1}\begin{bmatrix} \tfrac{1}{2}BK(n,r)A_d & 0 \\ K(n+1,r) & 0 \end{bmatrix} \end{aligned} \tag{3.29}$$

where

$$L = (2I - BD_d^2C)A_d + BK(1,r-1)A_d + BD_d^2S(t)A - \frac{1}{2}BD_dS(t).$$

Especially, when $q = 1$, the sum $\sum\limits_{n=1}^{q-1} = 0$ in (3.29).

Putting (3.17), (3.19), (3.20), (3.22) and (3.29) in (3.9) yields (3.1).

If $h = s + 1$, then (3.9) becomes

$$M_d = \frac{1}{2}P_d + Q^{\pi}\sum_{n=0}^{q-1} 2^{n-1}Q^nP_d^{n+1} + \sum_{n=0}^{s-1} 2^{-(n+2)}Q_d^{n+1}P^nP^{\pi}.$$

So, putting (3.19), (3.20), (3.22) and (3.29) in the above equation yields (3.2). ■

Adding some restrictions to D, we have the following results.

Corollary 3.3 *Let M be given by (1.1). If $D^2 = CB = 0$ and $AB = 0$, then*

$$M_d = \begin{bmatrix} A_d + BCA_d^3 + BDCA_d^4 & 0 \\ CA_d^2 + DCA_d^3 & 0 \end{bmatrix} \tag{3.30}$$

Proof. Assume that indices of matrices mentioned are the same as those in Theorem 3.1. Since $D^2 = 0$, then $D_d = 0$. Putting $D_d = 0$ in (3.3) $\sim$ (3.6), we have

$$\begin{aligned} S(n) &= 0, & K(1,m) &= CA_d^2, \\ G(n,m) &= 0, & K(2,m) &= DCA_d^3, \\ L &= 2A_d + BCA_d^3, & K(n,m) &= 0 \ (n \geq 3). \end{aligned}$$

Substituting the above equations in (3.1) yields (3.30). ■

Corollary 3.4 *Let M be given by (1.1) with $t = \mathrm{Ind}(A)$, $s = \mathrm{Ind}\left(\begin{bmatrix} A & \frac{1}{2}B \\ C & 0 \end{bmatrix}\right)$, $q = \mathrm{Ind}\left(\begin{bmatrix} 0 & \frac{1}{2}B \\ 0 & D \end{bmatrix}\right)$ and $h = \mathrm{Ind}\left(\begin{bmatrix} A & 0 \\ C & -D \end{bmatrix}\right)$. If $D^2 = D = \frac{1}{2}CB$ and $AB = 0$, then $h \leq s + 1$ and*

$$\begin{aligned} M_d =& \frac{1}{2}\begin{bmatrix} H & BD \\ 2(I-D)CA_d^2 + S(t) & 0 \end{bmatrix} + 2^{-(h+1)}[G(h-1,h-1) - G(h-1,t)] \\ & -\frac{1}{3}(1 - 4^{-\lceil \frac{h}{2} \rceil})G(0,t) - \frac{1}{6}(1 - 4^{-\lfloor \frac{h}{2} \rfloor})G(1,t) + \sum_{n=2}^{h-1} 2^{-(n+2)}G(n,n). \end{aligned} \tag{3.31}$$

In particular, when $h = s + 1$,

$$\begin{aligned} M_d =& \frac{1}{2}\begin{bmatrix} H & BD \\ 2(I-D)CA_d^2 + S(t) & 0 \end{bmatrix} - \frac{1}{3}(1 - 4^{-\lceil \frac{s}{2} \rceil})G(0,t) \\ & -\frac{1}{6}(1 - 4^{-\lfloor \frac{s}{2} \rfloor})G(1,t) + \sum_{n=2}^{s-1} 2^{-(n+2)}G(n,n) \end{aligned} \tag{3.32}$$

where

$$H = (2I - BDC)A_d + qB(I - D)CA_d^3 + BS(t)A - \frac{1}{2}BS(t),$$

$$S(n) = \sum_{k=0}^{\lceil \frac{n}{2} \rceil - 1} DCA^{2k}A^{\pi},$$

$$G(n,m) = \begin{cases} \begin{bmatrix} \frac{1}{2}BS(m)A & 0 \\ S(m)A & 0 \end{bmatrix}, & n \text{ is even;} \\ \\ \begin{bmatrix} \frac{1}{2}BS(m-2)A^2 & 0 \\ S(m-2)A^2 & 0 \end{bmatrix}, & n \text{ is odd.} \end{cases}$$

Proof. Since $D = D^2$, then $\mathrm{Ind}(D) = 1$ and $D_d = D$ and then $K(n,m) = 0$, $n \geq 2$, and $K(1,m) = (I - D)CA_d^2$. Clearly,

$$S(n) = \sum_{k=0}^{\lceil \frac{n}{2} \rceil - 1} DCA^{2k}A^{\pi}, \tag{3.33}$$

$$G(2k,m) = \begin{bmatrix} \frac{1}{2}BS(m)A & 0 \\ S(m)A & 0 \end{bmatrix}; \; G(2k-1,m) = \begin{bmatrix} \frac{1}{2}BS(m-2)A^2 & 0 \\ S(m-2)A^2 & 0 \end{bmatrix}, \tag{3.34}$$

$$L = (2I - BDC)A_d + B(I - D)CA_d^3 + BS(t)A - \frac{1}{2}BS(t). \tag{3.35}$$

By [12, Theorem 2.1], $1 \leq q \leq 2$. Thus when $q = 2$,

$$\sum_{n=1}^{q-1} 2^{n-1} \begin{bmatrix} \frac{1}{2}BK(n,r)A_d & 0 \\ K(n+1,r) & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}B(I - D)CA_d^3 & 0 \\ 0 & 0 \end{bmatrix}.$$

However when $q = 1$, $\sum\limits_{n=1}^{q-1} = 0$. Therefore, Define

$$\begin{aligned} H &= L + (q-1)B(I - D)CA_d^3 \\ &= (2I - BDC)A_d + qB(I - D)CA_d^3 + BS(t)A - \frac{1}{2}BS(t). \end{aligned} \tag{3.36}$$

Putting (3.33) $\sim$ (3.36) in (3.1) and (3.2), respectively, yields (3.31) and (3.32). ∎

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SUPER PRINCIPAL FIBER BUNDLE WITH SUPER ACTION

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Abstract. We introduce super action for supermanifolds to devoted to principal fiber bundle with structural generalized Lie groups. We present a product super fiber bundle, also we extend the coordinate bundle in the sense of Steenrod to show that super coordinate bundles are equivalent if and only if their super actions agree.

Keywords: principal fiber bundle, Lie group, supermanifold.

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1. Introduction

In physics and mathematics, supermanifolds are generalizations of the manifold concept based on ideas coming from supersymmetry, also a super Lie group is a group object in the category of supermanifolds. An affine super algebraic group is a group object in the category of affine supervarieties.

In this paper we introduce super action for manifolds and supermanifolds to devoted to principal fiber bundle with structural generalized Lie groups, also we extend coordinate bundle in the sense of **Steenrod** ([7]), for the generalized Lie group T. We show that our definition of super fiber bundle is an equivalence class of super coordinate bundles, which is another way of saying that super coordinate bundles are equivalent if and only if their super actions agree. Therefore, we have a set of super transition functions defined for an open covering of a manifold M, which determine a super principal bundle whose transition maps relative to the covering are old super transitions. Note that this generalization is a different structure from fiber bundle of top spaces introduced in ([6,2]). The concept of generalized Lie groups is a different structure from super Lie group, but results may be useful for researchers who working in $Superspace\ time$ and $Quantum\ field\ theory$. Moreover, we present a useful theorem of super product fiber bundle.

Definition 1.1. Let M be a real manifold and * be an involution over the fiber turning it into a * algebra. Then the resulting algebra is called a real supermanifold.

Note that, in physics the points of this algebra isn't a point set space and so doesn't "really" exist. A supermanifold is a concept in noncommutative geometry.

Example 1.2. Each supermanifold is a real C^∞ manifold.

Now, we present the concept of generalized Lie group(top space), which is both a generalized group and a manifold, such that the generalized group operations are C^∞. The several authors studied various aspects and concepts of generalized group, well known from standard groups, [4,5].

Definition 1.3. ([6],[1]) A non-empty Hausdorff smooth d-dimensional manifold T is called a top space if there is an associative action "$\cdot$" on T such that $t \cdot s \in T$, for every $t, s \in T$, and satisfies in the following conditions (Note that $t \cdot s$ showed that by ts.):

(i) For each $t \in T$, there is a unique $e(t) \in T$ such that $te(t) = e(t)t = t$;

(ii) For each $t \in T$, there is $s \in T$ such that $ts = st = e(t)$;

(iii) For all $t, s \in T$, $e(ts) = e(t)e(s)$;

(iv) The mappings

$$\begin{array}{rcl} m_1 : T \times T & \longrightarrow & T \\ (t,s) & \longmapsto & ts \end{array} \quad \text{and} \quad \begin{array}{rcl} m_2 : T & \longrightarrow & T \\ t & \longmapsto & t^{-1} \end{array}$$

are smooth mappings.

Note that, $e(t)$ is called the identity of t. Moreover, s is called the inverse of t, and denoted by t^{-1}.

Example 1.4. ([6]) Each Lie group is top space.

There is a Lie groups which is not top space, for example:

Example 1.5. ([6]) The $n-$torus $\mathbb{T}^n = \frac{\mathbb{R}}{\mathbb{Z}}$ with the product:

$$((r^1, ..., r^n) + \mathbb{Z}, ((s^1, ..., s^n) + \mathbb{Z}) \mapsto ((r^1 + s^1, ..., r^{n-1} + s^{n-1}, r^n) + \mathbb{Z})$$

is a top space, which is not a Lie group.

2. Super action of top spaces

All of us knew how important actions of Lie groups on manifolds are, in this section at first we introduce a super action for supermanifolds and generalized Lie groups as a generalization of usual action for Lie groups, and then we shall introduce the notion of a super principal fiber bundle, also we shall present on the most important class of super fiber bundle, moreover we shall make a super coordinate bundle, in the sense of Steenrod, to find a balance with respect to the super actions.

Definition 2.1. ([2]) A super action(or generalized action) of top space T on supermanifold (or manifold) M is differentiable map $\lambda : M \times T \longrightarrow M$ which satisfies the conditions:

(i) For any $m \in M$, there is $e(t)$ in T such that $\lambda(m, e(t)) = m$;

(ii) $\lambda(\lambda(m, t_1), t_2) = \lambda(m, t_1 t_2)$ for all $t_1, t_2 \in T$ and $m \in M$.

In this case, we called T *superacts* on M.

Remark. We often use the notion tm instead of $\lambda(m, t)$, so the second condition in this notation is $(mt_1)t_2 = m(t_1 t_2)$.

Note that, if T is a Lie group, then a super action is an action (C^∞ action).

Example 2.2. $T = R \times R - \{0\}$ with product $(a, b).(e, f) = (be, bf)$ is top space [Remark: $e((a, b)) = (a/b, 1)$ and $(a, b)^{-1} = (a/b^2, 1/b)$]. The map $\lambda : R \times T \longrightarrow R$ defined by $\lambda(c, (a, b)) = ac/b$ is a super action.

T is said to super acts $effectively$, if $m \cdot t = m$, for all $t \in T$, implies that $t \in e(T)$. Also, T supper acts *transitivity* to the right if for every $m, n \in M$, there is $t \in T$ such that $mt = n$.

T super acts $freely$ if the only elements of T having fixed point on M are belong to the identity set $e(T)$, where $e(T)$ the set of all identity elements in T, also the set $O(m) = \{mt | t \in T\}$ is called the *orbit* of m.

Now, we introduce a product super fiber bundle:

Definition 2.3. A super principal fiber bundle is a set (P, T, M), where P, M are differentiable manifolds, T is a top space such that:

(i) T super acts freely to the right on P;

(ii) M is the quotient space of P, by equivalence under T, and projection $\pi : P \to M$ is differentiable;

(iii) For every $m \in M$, there is an open set U of m and differentiable map $F_U : \pi^{-1}(U) \to T$ such that $F_U(pt) = e(t)F_U(p)t$, for all $t \in T$ and $p \in U$.

Also, the map of $\pi^{-1}(U) \to U \times T$ given by $p \mapsto (\pi(p), F_U(p))$ is a diffeomorphism. In this case, P is called the super bundle space, M the base space, and T the structural top space.

Note that, we can easily extend this definition for spurmanifolds P and M, moreover, if T is a Lie group, then the concept of super principal fiber bundle is the concept of principal fiber bundle, it clear that, there is a super principal fiber bundle which is not a principal fiber bundle (see Example 1.5). Now, by Definition 2.3 we make a product super bundle, which is very important case in the sense of Steenrod ([6]):

Theorem 2.4. *Let T be a top space, M a manifold (or supermanifold), and $P = T \times M$, also P provided with the right super action of T on itself in the second factor, then (P, T, M) is a super principal fiber bundle.*

Proof. At first, we define a suitable super action:

Define $P \times T \to P$ by $((m,t),t^{'}) \mapsto (m, e(t^{'})tt^{'})$.

We show that this C^{∞}-map satisfies in the conditions of Definition 2.3.

Let $(m,t) \in P$, since $t \in T$, then $e(t) \in T$.

Moreover, $(m,t)e(t) = (m, e(e(t))te(t))$.

By uniqueness of identity in Definition 1.3, we have $e(e(t)) = e(t)$.

Then $(m, e(e(t))te(t)) = (m,t)$.

Moreover, $(((m,t),t_1),t_2) = ((m, e(t_1)tt_1),t_2) = (m, e(t_2)e(t_1)tt_1t_2)$.

By definition of top spaces, we knew that $e(t_1)e(t_2) = e(t_1t_2)$, so

$$(m, e(t_2)e(t_1)tt_1t_2) = (m, e(t_1t_2)tt_1t_2) = ((m,t),t_1t_2).$$

Therefore, T super acts on P.

Step I. Let $(m,t) \in M \times T$, and there exist $t^{'} \in T$ such that

$$(m,t)t^{'} = (m, e(t^{'})tt^{'}) = (m,t).$$

Then $e(t^{'})tt^{'} = t$, then $e(e(t^{'})tt^{'}) = e(t)$.

By uniqueness of identity, we have $e(e(t^{'})tt^{'}) = e(t)$.

Then $e(t^{'}) = e(t))$, and so $e^{-1}(e(t^{'})) = e^{-1}(e(t))$.

Therefore, $e(t^{'})tt^{'} = e(t)tt^{'} = tt^{'}$.

Since $e(t^{'})tt^{'} = t$, then $tt^{'} = t$, and then, by multiplying $t^{'}$ from the left side, we have $t^{-1}tt^{'} = t^{-1}t$. Then $e(t)t^{'} = e(t)$.

Since $e(t) = e(t^{'})$, then $e(t)t^{'} = e(t^{'})t^{'} = t^{'}$. Hence $t^{'} = e(t)$.

Therefore, $t^{'} \in e(T)$, thus T super acts freely on P.

Step II. Let $\sim$ be a relation on P, defined by:

$$(m_1,t_1) \sim (m_2,t_2) \text{ iff there is } t \in T \text{ such that } (m_1, e(t)t_1t) = (m_2,t_2),$$

this relation is an equivalence relation, because:

Let (m,t) belong to P. Since $e(t) \in T$ and $(m,t)e(t) = (m, e(t)te(t)) = (m,t)$, then $\sim$ is a reflexive relation.

Let (m_1,t_1), (m_2,t_2) belong to P, and $(m_1,t_1) \sim (m_2,t_2)$. Then there is $t \in T$ such that $(m_1, e(t)t_1t) = (m_2,t_2)$. Therefore,

$$m_1 = m_2 \text{ and } e(t)t_1t = t_2. \qquad (*)$$

By multiplying $e(t_1)$ from the left side, we have $e(t_1)e(t)t_1e(t) = e(t_1)t_2$. Since

$$\begin{aligned} e(t_1)e(t)t_1t &= e(t_1)e(t)e(t_1)t_1t; && \text{(because: } e(t_1)t_1 = t_1) \\ &= e(t_1)t_1t && \text{(because: } e(t_1)e(t)e(t_1) = e(t_1)) \\ &= t_1t \end{aligned}$$

and

$$t_1t = e(t_1)t_2 \qquad \text{(because: } e(t_1)e(t)t_1t = e(t_1)t_2,$$

then $t_1 t = e(t_1)t_2$).

By multiplying t^{-1} from the right side, we have: $t_1 e(t) = e(t_1)t_2 t^{-1}$.

By multiplying $e(t_1)$ from the right side, we have:

$$(**) \qquad t_1 e(t)e(t_1) = e(t_1)t_2 t^{-1} e(t_1).$$

We knew that

$$t_1 e(t)e(t_1) = (t_1 e(t_1))e(t)e(t_1) = t_1(e(t_1)e(t)e(t_1)) = t_1 e(t_1) = t_1.$$

Then, by $(**)$ we have $t_1 = e(t_1)t_2 t^{-1} e(t_1)$.
By $(*)$, we knew that $e(t) = e(t_2)$(because: $e(e(t)t_1 t) = e(t)$) and then

$$\begin{array}{rcll} e(t_1)t_2 t^{-1} e(t_1) = & & e(t_1)e(t_2)t_2 t^{-1} e(t_1) & \\ & = & e(t_1)e(t)t_2 t^{-1} e(t_1) & \text{(because: } e(t_2) = e(t)) \\ & = & e(t_1)e(t^{-1})t_2 t^{-1} e(t_1) & \text{(because: } e(t^{-1}) = e(t)) \\ & = & e((t^{-1}e(t_1))^{-1})t_2 t^{-1} e(t_1) & \text{(because: } (t^{-1}e(t_1))^{-1} = e(t_1)t) \\ & = & e((t^{-1}e(t_1))t_2 t^{-1} e(t_1)). & \end{array}$$

Then, there is $t^{-1}e(t_1) \in T$ such that:

$$t_1 = e((t^{-1}e(t_1))^{-1})t_2 t^{-1} e(t_1),$$

and so $(m_2, t_2) \sim (m_1, t_1)$, therefore $\sim$ is a symmetric relation.

Let (m_1, t_1), (m_2, t_2) and (m_3, t_3) belong to P, and $(m_1, t_1) \sim (m_2, t_2)$, $(m_2, t_2) \sim (m_3, t_3)$, then $m_1 = m_2$ and $m_2 = m_3$, respectively.

Also, there are $t', t'' \in T$ such that: $e(t')t_1 t' = t_2$ and $e(t'')t_2 t'' = t_3$.

Therefore, $e(t'')e(t')t_1 t' t'' = t_3$. Then $e(t'' t')t_1 t' t'' = t_3$, and so

$$e((t'' t')^{-1})t_1 t' t'' \quad = \quad t_3 \quad \text{(because: } e(s) = e(s^{-1}), \text{ for all } s \in T.)$$

Then $e((t')^{-1}(t'')^{-1})t_1 t' t'' = t_3$. Hence $e(t')e(t'')t_1 t' t'' = e((t')^{-1}(t'')^{-1})t_1 t' t'' = t_3$. Therefore, $e(t' t'')t_1 t' t'' = e(t')e(t'')t_1 t' t'' = t_3$. Then $(m_1, t_1) \sim (m_3, t_3)$, therefore $\sim$ is a transitive relation.

Step III. Now, we show that M is the quotient space of P by this equivalence relation under T.

Let $(m, t_1), (m, t_2)$ belong to P, since $e(t_2), (t_1)^{-1}t_2$ belong to T, and

$$\begin{array}{rcl} e((t_1)^{-1}t_2)t_1(t_1)^{-1}t_2 & = & = e(((t_1)^{-1}t_2)^{-1})e(t_1)t_2 \\ & = & e((t_2)^{-1})e((t_1)^{-1})^{-1}e(t_1)t_2 \\ & = & e(t_2)e(t_1)e(t_2)t_2 \end{array}$$

and, since $e(t_2)t_2 = t_2$ and $e(t^{-1})e(t_1) = e(t_1)$), we have

$$e(t_2)e(t_1)e(t_2)t_2 = e(t_2)t_2 = t_2.$$

Then

$$(m, t_1) \sim (m, t_2).$$

Also, by Definition 1.3, it is clear that the projection $\pi : P \to M$ is a C^∞ map.

Now, let $m \in M$ be given, it is clear that for each open set U of m in M, we have F_U commutes with right super action, i.e., $F_U(pt^{'}) = F_U(p)t^{'}$ for every $t^{'} \in T$ and $p \in P$, because:

$$\begin{aligned} F_U((m,t),t^{'}) &= F_U(m, e(t^{'})tt^{'}) \\ &= e(t^{'})tt^{'} \\ &= e(t^{'})F_U((m,t))t^{'}. \end{aligned}$$

Also, the map $p \mapsto (\pi(p), F_U(p))$ from $\pi^{-1}(U)$ into $U \times T$, which is the identity map, is a diffeomorphism.

Then (P, T, M) is a super fiber bundle, which is called product super fiber bundle. ■

Now, we present the structure of super coordinate bundle with structural top space T, where T is a top space with the finite identity elements, i.e., the cardinality of $e(T)$ is finite, note that we knew that $e^{-1}(e(t))$ is Lie group, for all $t \in T$, and also $e^{-1}(e(t))$ is diffeomorphism to $e^{-1}(e(t^{'}))$, for all $t, t^{'} \in T$ [2, Corollaries 3.2 and 3.3].

Let (P, T, M) be a super principal fiber bundle, and let $\{U_i\}$ be an open covering of M such that $\pi^{-1}(U_i)$ can be represented as a product space via the function $F_i : \pi^{-1}(U_i) \to T$. We define a map $G_{ij}^{e(t)} : U_i \cap U_j \to T$ as follows:

If $m \in U_i \cap U_j$, let $p \in \pi^{-1}(m) \cap \{p_0 t_0 | t_0 \in e^{-1}(e(t)), p_0 \in \pi^{-1}(m)\}$, and put

$$(G_{ij})^{e(t)}(m) = e(t)F_j(p)e(t)(F_i(p))^{-1}.$$

Now, we want to show that the definition of $(G_{ij})^{e(t)}$ is independent of the choice of p.

If $p^{'} \in \pi^{-1}(m) \cap \{p_0 t_0 | t_0 \in e^{-1}(e(t)), p_0 \in \pi^{-1}(m)\}$, then there is $t_1 \in e^{-1}(e(t))$ such that $p^{'} = pt_1$.

Now, we have:

$$\begin{aligned} F_j(p^{'})(F_i(p^{'}))^{-1} &= F_j(pt_1)(F_i(pt_1))^{-1} \\ &= e(t_1)F_j(p)t_1 e(t_1)(F_i(p))^{-1}(t_1)^{-1} \\ &= e(t_1)F_j(p)t_1(F_i(p))^{-1}(t_1)^{-1} \\ &= e(t_1)F_j(p)t_1(t_1)^{-1}(F_i(p))^{-1} \\ &= e(t_1)F_j(p)e(t_1)(F_i(p))^{-1}. \end{aligned}$$

Since $e^{-1}(e(t_1)) = e^{-1}(e(t))$ and $e(t_1) = e(t)$, then

$$e(t_1)F_j(p)e(t_1)(F_i(p))^{-1} = e(t_1)F_j(p)e(t)(F_i(p))^{-1}.$$

Moreover, it is easy to show that:

$$(G_{ik})^{e(t)}(m)(G_{kj})^{e(t)}(m) = (G_{ij})^{e(t)}(m),$$

for all $m \in U_i \cap U_j \cap U_k \cap e^{-1}(e(t))$ and $t \in T$.

The functions $(G_{ij})^{e(t)}$ are called the *super transition functions* corresponding to the covering $\{U_i\}$.

By this covering we extend a coordinate principal bundle in the sense of Steenrod ([6]), for all top spaces T.

Since T is a top space with the finite identity element $e(T)$, then

$$T = \overset{\circ}{\bigcup_{e(t)\in e(T)}} (e^{-1}(e(t))), \qquad [2, \text{Corollary } 3.3].$$

Now, suppose M is covered by domains of coordinate systems(i.e., $\{U_i\}$), let

$$P = \overset{\circ}{\bigcup_{e(t)\in e(T)}} \bigcup_i U_i \times e^{-1}(e(t))$$

and

$$\bigcup_i (U_i \times e^{-1}(e(t))) \times e^{-1}(e(t)) \to P$$

defined by $((u,t),t^{'}) \mapsto (u,tt^{'})$.

For all $t \in T$, by the usual manner, we can induce a C^∞ structure on $\bigcup_i (U_i \times e^{-1}(e(t))$.

Since $e^{-1}(e(t))$ is diffeomorphism to $e^{-1}(e(t^{'}))$ [2, Corollary 3.2], for all $t,t^{'} \in T$, then all of the C^∞ structures are diffeomorphic.

Then, by the projection map and the usual manner, we have a differentiable structure on

$$\overset{\circ}{\bigcup_{e(t)\in e(T)}} \bigcup_i \bigcup (U_i \times e^{-1}(e(t))),$$

where $\pi_{e(t)} : \bigcup_i (U_i \times e^{-1}(e(t))) \to M$ is a projection map, for all $t \in T$.

Hence, in the sense of Steenrod, our definition of super fiber bundle is an equivalence class of super coordinate bundle.

Example 2.5. The Euclidean subspace $R^* = R - \{0\}$, with the product $(a,b) \mapsto a|b|$, is a top space with the identity element $e(T) = \{+1,-1\}$, then $\lambda : R^* \times R \longrightarrow R$ defined by $\lambda(a,m) = am$ is a super action of top space R^* on Euclidean manifold R.

Let

$$U_1 = (-2,\infty) \quad \text{and} \quad U_2 = (-\infty, 2).$$

Now, let

$$P = \overset{\circ}{\bigcup_{e(t)\in e(T)}} \bigcup_i (U_i \times e^{-1}(e(t)))$$

and

$$\overset{\circ}{\bigcup_{e(t)\in e(T)}} \bigcup_i ((U_i \times e^{-1}(e(t))) \times e^{-1}(e(t))) \to R$$

defined by

$$((u,t),t^{'}) \mapsto u.$$

It is clear that, by the usual manner, we can induce a differentiable structure by the projection maps:

$$\pi_{e(1)} : ((-2,\infty) \times R^{+}) \bigcup ((-\infty,2) \times R^{+}) \to R$$

and

$$\pi_{e(-1)} : ((-2,\infty) \times R^{-}) \bigcup ((-\infty,2) \times R^{-}) \to R.$$

Since $((-2,\infty) \times R^{+}) \bigcup ((-\infty,2) \times R^{+})$ and $((-2,\infty) \times R^{-}) \bigcup ((-\infty,2) \times R^{-})$ are disjoint diffeomorphic manifolds, then we have a C^{∞} structure on P such that the projection maps be C^{∞} maps, it is clear that (P, R^{*}, R) is super fiber bundle.

Conclusion

In the extension mode of coordinate bundle, we found the important following condition:

- All of the super coordinate bundles, in the sense of Steenrod, are equivalent if and only if their super actions agree.

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INTUITIONISTIC FUZZY α-IRRESOLUTE FUNCTIONS

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Abstract. In this paper the concept of intuitionistic fuzzy α-irresolute functions are introduced and studied. Besides giving characterizations of these functions, several interesting properties of these functions are also given. We also study relationship between this function with other existing functions.

Keywords: intuitionistic fuzzy α-open set, intuitionistic fuzzy α-continuous, intuitionistic fuzzy α-irresolute.

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1. Introduction

Ever since the introduction of fuzzy sets by L.A. Zadeh [10], the fuzzy concept has invaded almost all branches of mathematics. The concept of fuzzy topological spaces was introduced and developed by C.L. Chang [2]. Atanassov [1] introduced the notion of intuitionistic fuzzy sets, Coker [3] introduced the intuitionistic fuzzy topological spaces. In this paper, we have introduced the concept of intuitionistic fuzzy α-irresolute functions and studied their properties. Also, we have given characterizations of intuitionistic fuzzy α-irresolute functions. We also study the relationship between this function with other existing functions.

2. Preliminaries

Definition 2.1. [1] Let X be a nonempty fixed set and I the closed interval $[0,1]$. An intuitionistic fuzzy set (IFS) A is an object of the following form

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\},$$

where the mappings $\mu_A : X \to I$ and $\nu_A : X \to I$ denote the degree of membership (namely) $\mu_A(x)$ and the degree of nonmembership (namely) $\nu_A(x)$ for each element $x \in X$ to the set A respectively, and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for each $x \in X$.

Definition 2.2. [1] Let A and B are IFSs of the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$ and $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle \mid x \in X\}$. Then

(i) $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$;

(ii) $\overline{A} = \{\langle x, \nu_A(x), \mu_A(x)\rangle \mid x \in X\}$;

(iii) $A \cap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x)\rangle \mid x \in X\}$;

(iv) $A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x)\rangle \mid x \in X\}$.

We will use the notation $A = \{\langle x, \mu_A, \nu_A\rangle \mid x \in X\}$ instead of $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$.

Definition 2.3. [3] $0_\sim = \{\langle x, 0, 1\rangle \mid x \in X\}$ and $1_\sim = \{\langle x, 1, 0\rangle \mid x \in X\}$.

Let $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta \leq 1$. An intuitionistic fuzzy point (IFP) $p_{(\alpha,\beta)}$ is an intuitionistic fuzzy set defined by

$$p_{(\alpha,\beta)}(x) = \begin{cases} (\alpha, \beta) & \text{if } x = p, \\ (0, 1) & \text{otherwise.} \end{cases}$$

Let X and Y are two non-empty sets and $f : (X, \tau) \to (Y, \sigma)$ be a function.

If $B = \{\langle y, \mu_B(y), \nu_B(y)\rangle \mid y \in Y\}$ is an IFS in Y, then the pre-image of B under f is denoted and defined by

$$f^{-1}(B) = \{\big\langle x, f^{-1}(\mu_B(x)), f^{-1}(\nu_B(x))\big\rangle \mid x \in X\}.$$

Since μ_B, ν_B are fuzzy sets, we explain that $f^{-1}(\mu_B(x)) = \mu_B(f(x))$.

Definition 2.4. [3] An intuitionistic fuzzy topology(IFT) in Coker's sense on a nonempty set X is a family τ of intuitionistic fuzzy sets in X satisfying the following axioms:

(i) $0_\sim, 1_\sim \in \tau$;

(ii) $G_1 \cap G_2 \in \tau$, for any $G_1, G_2 \in \tau$;

(iii) $\cup G_i \in \tau$ for any arbitrary family $\{G_i \mid i \in J\} \subseteq \tau$.

In this paper, by (X, τ) or, simply, by X we will denote the intuitionistic fuzzy topological space (IFTS). Each IFS which belongs to τ is called an intuitionistic fuzzy open set (IFOS) in X. The complement $\overline{A}$ of an IFOS A in X is called an intuitionistic fuzzy closed set(IFCS) in X.

Definition 2.5. [3] Let (X, τ) be an IFTS and $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$ be an IFS in X. Then, the intuitionistic fuzzy closure and intuitionistic fuzzy interior of A are defined by

(i) $\mathrm{cl}(A) = \cap\{C : C$ is an IFCS in X and $C \supseteq A\}$;

(ii) $\mathrm{int}(A) = \cup\{D : D$ is an IFOS in X and $D \subseteq A\}$.

It can be also shown that $\mathrm{cl}(A)$ is an IFCS, $\mathrm{int}(A)$ is an IFOS in X and A is an IFCS in X if and only if $\mathrm{cl}(A) = A$; A is an IFOS in X if and only if $\mathrm{int}(A) = A$.

Proposition 2.1. [3] *Let (X, τ) be an IFTS and A, B be IFSs in X. Then, the following properties hold:*

(i) $\mathrm{cl}\overline{A} = \overline{(\mathrm{int}(A))}, \mathrm{int}\overline{(A)} = \overline{(\mathrm{cl}(A))}$;

(ii) $\mathrm{int}(A) \subseteq A \subseteq \mathrm{cl}(A)$.

Definition 2.6. [5] An IFS A in an IFTS X is called an intuitionistic fuzzy pre open set (IFPOS) if $A \subseteq \mathrm{int}(\mathrm{cl}\, A)$. The complement of an IFPOS A in IFTS X is called an intuitionistic fuzzy preclosed (IFPCS) in X.

Definition 2.7. [5] An IFS A in an IFTS X is called an intuitionistic fuzzy α-open set (IFαOS) if and only if $A \subseteq \mathrm{int}(\mathrm{cl}(\mathrm{int}\, A))$. The complement of an IFαOS A in X is called intuitionistic fuzzy α-closed (IFαCS) in X.

Definition 2.8. [5] An IFS A in an IFTS X is called an intuitionistic fuzzy semi open set (IFSOS) if and only if $A \subseteq \mathrm{cl}(\mathrm{int}(A))$. The complement of an IFSOS A in X is called intuitionistic fuzzy semi closed(IFSCS) in X.

Definition 2.9. Let f be a mapping from an IFTS X into an IFTS Y. The mapping f is called:

(i) intuitionistic fuzzy continuous if and only if $f^{-1}(B)$ is an IFOS in X, for each IFOS B in Y [5];

(ii) intuitionistic fuzzy α-continuous if and only if $f^{-1}(B)$ is an IFαOS in X, for each IFOS B in Y [5];

(iii) intuitionistic fuzzy pre continuous if and only if $f^{-1}(B)$ is an IFPOS in X, for each IFOS B in Y [5];

(iv) intuitionistic fuzzy semi continuous if and only if $f^{-1}(B)$ is an IFSOS in X, for each IFOS B in Y [5].

Definition 2.10. [9] Let (X, τ) be an IFTS and $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$ be an IFS in X. Then the intuitionistic fuzzy α-closure and intuitionistic fuzzy α-interior of A are defined by

(i) $\alpha\mathrm{cl}(A) = \bigcap\{C : C$ is an IFαCS in X and $C \supseteq A\}$;

(ii) $\alpha\mathrm{int}(A) = \bigcup\{D : D$ is an IFαOS in X and $D \subseteq A\}$.

Definition 2.11. Let f be a mapping from an IFTS of X into an IFTS of Y. The mapping f is called intuitionistic fuzzy strongly α-continuous if and only if $f^{-1}(B)$ is an IFαOS in X, for each IFSOS B in Y.

3. Intuitionistic fuzzy α-irresolute functions

Definition 3.1. A function $f : (X, \tau) \to (Y, \sigma)$ from an intuitionistic fuzzy topological space (X,τ) to another intuitionistic fuzzy topological space (Y,σ) is said to be intuitionistic fuzzy α-irresolute(IF α-irresolute) if $f^{-1}(B)$ is an IFαOS in (X,τ) for each IFαOS B in (Y,σ).

$$\begin{tikzcd}
\text{IF strongly } \alpha\text{-continuous} \arrow[rr] \arrow[dr] & & \text{IF } \alpha\text{-continuous} \\
& \text{IF } \alpha\text{-irresolute} \arrow[ur] \arrow[dl] \arrow[dr] & \\
\text{IF semi continuous} & & \text{IF pre continuous}
\end{tikzcd}$$

Proposition 3.1. *Every intuitionistic fuzzy α-irresolute is an intuitionistic fuzzy pre continuous.*

Proof. Follows from the definitions. ∎

However, the converse of the above Proposition 3.1 need not to be true, as shown by the following example.

Example 3.1. Let $X = \{a, b\}$, $Y = \{c, d\}$, $\tau = \{0_\sim, 1_\sim, A\}$, $\sigma = \{0_\sim, 1_\sim, B\}$, where

$$\begin{aligned}
A &= \left\{\left\langle x, \left(\frac{a}{0.6}, \frac{b}{0.5}\right), \left(\frac{a}{0.4}, \frac{b}{0.4}\right)\right\rangle;\ x \in X\right\}, \\
B &= \left\{\left\langle y, \left(\frac{c}{0.2}, \frac{d}{0.4}\right), \left(\frac{c}{0.6}, \frac{d}{0.5}\right)\right\rangle\ \ y \in Y\right\}.
\end{aligned}$$

Define an intuitionistic fuzzy mapping $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = d$, $f(b) = c$. B is an IFOS in (Y, σ). $f^{-1}(B) = \left\{\left\langle x, \left(\frac{a}{0.4}, \frac{b}{0.2}\right), \left(\frac{a}{0.5}, \frac{b}{0.6}\right)\right\rangle;\ x \in X\right\}$ is an IFPOS in (X, τ), since $\mathrm{int}(\mathrm{cl}(f^{-1}(B))) = 1_\sim$, and $f^{-1}(B) \subseteq \mathrm{int}(\mathrm{cl}(f^{-1}(B)))$. Hence, f is an IF pre continuous. B is an IFαOS in (Y, σ) and $\mathrm{int}(\mathrm{cl}(\mathrm{int}(f^{-1}(B)))) = 0_\sim$, $f^{-1}(B) \nsubseteq \mathrm{int}(\mathrm{cl}(\mathrm{int}(f^{-1}(B))))$. Hence, $f^{-1}(B)$ is not IFαOS in (X, τ) which implies f is not IF α-irresolute function.

Proposition 3.2. *Every intuitionistic fuzzy α-irresolute is an intuitionistic fuzzy α-continuous.*

Proof. Follows from the definitions. ∎

However, the converse of the above Proposition 3.2 need not to be true, as shown by the following example.

Example 3.2. Let $X = \{a, b, c\} = Y$, $\tau = \{0_\sim, AB, A \cup B, A \cap B, 1_\sim\}$, $\sigma = \{0_\sim, 1_\sim, C\}$, where

$$\begin{aligned} A &= \left\{\left\langle x, \left(\frac{a}{0.5}, \frac{b}{0.3}, \frac{c}{0.6}\right), \left(\frac{a}{0.5}, \frac{b}{0.7}, \frac{c}{0.4}\right)\right\rangle;\ x \in X\right\}, \\ B &= \left\{\left\langle x, \left(\frac{a}{0.2}, \frac{b}{0.4}, \frac{c}{0.3}\right) \left(\frac{a}{0.7}, \frac{b}{0.6}, \frac{c}{0.7}\right)\right\rangle;\ x \in X\right\} \\ C &= \left\{\left\langle y, \left(\frac{a}{0.5}, \frac{b}{0.4}, \frac{c}{0.6}\right), \left(\frac{a}{0.5}, \frac{b}{0.6}, \frac{c}{0.4}\right)\right\rangle;\ y \in Y\right\} \\ D = &= \left\{\left\langle y, \left(\frac{a}{0.5}, \frac{b}{0.4}, \frac{c}{0.6}\right), \left(\frac{a}{0.4}, \frac{b}{0.5}, \frac{c}{0.4}\right)\right\rangle;\ y \in Y\right\} \end{aligned}$$

Define an intuitionistic fuzzy mapping $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = a$, $f(b) = b$, $f(c) = c$. C is an IFOS in (Y, σ). $f^{-1}(C) = \big\{\big\langle x, \big(\frac{a}{0.5}, \frac{b}{0.4}, \frac{c}{0.6}\big), \big(\frac{a}{0.5}, \frac{b}{0.6}, \frac{c}{0.4}\big)\big\rangle;$ $x \in X\big\}$ and $\text{int}(\text{cl}(\text{int} f^{-1}(C))) = A \cup B$. Thus $f^{-1}(C) \subseteq \text{int}(\text{cl}(\text{int} f^{-1}(C)))$. Hence $f^{-1}(C)$ is IFαOS in X, which implies f is IF α-continuous. D is an IFS in Y and $D \subseteq \text{int}(\text{cl}(\text{int}\, D)) = 1_\sim$. Hence, D is IFαOS in Y. $f^{-1}(D) = \big\{\big\langle x, \big(\frac{a}{0.5}, \frac{b}{0.4}, \frac{c}{0.6}\big), \big(\frac{a}{0.4}, \frac{b}{0.5}, \frac{c}{0.4}\big)\big\rangle;\ x \in X\big\}$ and $\text{int}(\text{cl}(\text{int} f^{-1}(D))) = A \cup B$. Thus, $f^{-1}(D) \nsubseteq \text{int}(\text{cl}(\text{int} f^{-1}(D)))$. Hence, $f^{-1}(D)$ is not IFαOS in X. So, f is not IF α-irresolute function.

Proposition 3.3. *Every intuitionistic fuzzy α-continuous is an intuitionistic fuzzy pre continuous.*

Proof. Follows from the definitions. ∎

However, the converse of the above Proposition 3.3 need not to be true, as shown by the following example.

Example 3.3. Let $X = \{a, b\}$, $Y = \{c, d\}$, $\tau = \{0_\sim, 1_\sim, A\}$, $\sigma = \{0_\sim, 1_\sim, B\}$, where

$$\begin{aligned} A &= \left\{\left\langle x, \left(\frac{a}{0.6}, \frac{b}{0.5}\right), \left(\frac{a}{0.4}, \frac{b}{0.4}\right)\right\rangle;\ x \in X\right\}, \\ B &= \left\{\left\langle y, \left(\frac{c}{0.2}, \frac{d}{0.4}\right), \left(\frac{c}{0.6}, \frac{d}{0.5}\right)\right\rangle;\ y \in Y\right\}. \end{aligned}$$

Define an intuitionistic fuzzy mapping $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = d$, $f(b) = c$. B is an IFOS in (Y, σ). $f^{-1}(B) = \big\{\big\langle x, \big(\frac{a}{0.4}, \frac{b}{0.2}\big), \big(\frac{a}{0.5}, \frac{b}{0.6}\big)\big\rangle;\ x \in X\big\}$ is an IFPOS in X since $\text{int}(\text{cl}(f^{-1}(B))) = 1_\sim$, and $f^{-1}(B) \subseteq \text{int}(\text{cl}(f^{-1}(B)))$. Hence, f is an IF pre continuous. $f^{-1}(B)$ is not IFαOS in X since $f^{-1}(B) \nsubseteq \text{int}(\text{cl}(\text{int}(f^{-1}(B)))) = 0_\sim$. Hence, f is not IFα-continuous function.

Proposition 3.4. *Every intuitionistic fuzzy α-irresolute is an intuitionistic fuzzy semi continuous.*

Proof. Follows from the definitions. ■

However, the converse of the above Proposition 3.4 need not to be true, as shown by the following example.

Example 3.4. Let $X = \{a, b\}$, $Y = \{c, d\}$, $\tau = \{0_\sim, 1_\sim, A\}$, $\sigma = \{0_\sim, 1_\sim, B\}$, where

$$\begin{aligned} A &= \left\{\left\langle x, \left(\frac{a}{0.3}, \frac{b}{0.4}\right), \left(\frac{a}{0.6}, \frac{b}{0.5}\right)\right\rangle;\ x \in X\right\}, \\ B &= \left\{\left\langle y, \left(\frac{c}{0.4}, \frac{d}{0.5}\right), \left(\frac{c}{0.5}, \frac{d}{0.5}\right)\right\rangle;\ y \in Y\right\}. \end{aligned}$$

Define an intuitionistic fuzzy mapping $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = d$, $f(b) = c$. B is an IFOS in (Y, σ). $f^{-1}(B) = \left\{\left\langle x, \left(\frac{a}{0.5}, \frac{b}{0.4}\right), \left(\frac{a}{0.5}, \frac{b}{0.5}\right)\right\rangle;\ x \in X\right\}$ is an IFSOS in (X, τ) since $\mathrm{cl}(\mathrm{int}(f^{-1}(B))) = \overline{A}$. Hence $f^{-1}(B) \subseteq \mathrm{cl}(\mathrm{int}(f^{-1}(B)))$, which implies f is an IF Semi continuous. B is an IFOS in Y and B is an IFαOS in Y since $B \subseteq \mathrm{int}(\mathrm{cl}(\mathrm{int}(B))) = B$. $f^{-1}(B) = \left\{\left\langle x, \left(\frac{a}{0.5}, \frac{b}{0.4}\right), \left(\frac{a}{0.5}, \frac{b}{0.5}\right)\right\rangle;\ x \in X\right\}$ $\mathrm{int}(\mathrm{cl}(\mathrm{int}(f^{-1}(B)))) = A$ and $f^{-1}(B) \nsubseteq \mathrm{int}(\mathrm{cl}(\mathrm{int}(f^{-1}(B))))$ which implies $f^{-1}(B)$ is not IFαOS in Y. Hence f is not IF α-irresolute function.

Proposition 3.5. *Every intuitionistic fuzzy strongly α-continuous is an intuitionistic fuzzy α-irresolute.*

Proof. Follows from the definitions. ■

However, the converse of the above Proposition 3.5 is need not be true, in general as shown by the following example.

Example 3.5. Let $X = \{a, b\}$, $Y = \{c, d\}$, $\tau = \{0_\sim, 1_\sim, A\}$, $\sigma = \{0_\sim, 1_\sim, B\}$ where

$$\begin{aligned} A &= \left\{\left\langle x, \left(\frac{a}{0.2}, \frac{b}{0.4}\right), \left(\frac{a}{0.3}, \frac{b}{0.4}\right)\right\rangle;\ x \in X\right\}, \\ B &= \left\{\left\langle y, \left(\frac{c}{0.4}, \frac{d}{0.2}\right), \left(\frac{c}{0.4}, \frac{d}{0.3}\right)\right\rangle;\ y \in Y\right\}, \\ C &= \left\{\left\langle y, \left(\frac{c}{0.4}, \frac{d}{0.3}\right), \left(\frac{c}{0.4}, \frac{d}{0.2}\right)\right\rangle;\ y \in Y\right\}. \end{aligned}$$

Define an intuitionistic fuzzy mapping $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = d$, $f(b) = c$. B is an IFOS in (Y,σ) and $\mathrm{int}(\mathrm{cl}(\mathrm{int}(B))) = B$. Hence $B \subseteq \mathrm{int}(\mathrm{cl}(\mathrm{int}(B)))$. Thus B is an IFαOS in (Y,σ). $f^{-1}(B) = \left\{\left\langle x, \left(\frac{a}{0.2}, \frac{b}{0.4}\right), \left(\frac{a}{0.3}, \frac{b}{0.4}\right)\right\rangle; x \in X\right\}$ is an IFαOS in (X,τ) since $\mathrm{int}(\mathrm{cl}(\mathrm{int}(f^{-1}(B)))) = A$. Hence $f^{-1}(B) \subseteq \mathrm{int}(\mathrm{cl}(\mathrm{int}(f^{-1}(B))))$ which implies f is an IF α-irresolute. C is an IFS in Y. Also C is an IFSOS in Y since $C \subseteq \mathrm{cl}(\mathrm{int}(C)) = \overline{B}$. $f^{-1}(C) = \left\{\left\langle x, \left(\frac{a}{0.3}, \frac{b}{0.4}\right), \left(\frac{a}{0.2}, \frac{b}{0.4}\right)\right\rangle;\ x \in X\right\}$ $\mathrm{int}(\mathrm{cl}(\mathrm{int}(f^{-1}(C)))) = A$ and $f^{-1}(C) \nsubseteq \mathrm{int}(\mathrm{cl}(\mathrm{int}(f^{-1}(C))))$ which implies $f^{-1}(C)$ is not IFαOS in Y. Hence f is not IF strongly α-continuous.

Proposition 3.6. *Every intuitionistic fuzzy strongly α-continuous is an intuitionistic fuzzy α-continuous.*

Proof. Follows from the definitions. ∎

However the converse of the above Proposition 3.6 is need not be true, as shown by the following example.

Example 3.6. Let $X = \{a, b\}$, $Y = \{c, d\}$, $\tau = \{0_\sim, 1_\sim, A\}$, $\sigma = \{0_\sim, 1_\sim, B\}$, where

$$\begin{aligned} A &= \left\{\left\langle x, \left(\frac{a}{0.2}, \frac{b}{0.4}\right), \left(\frac{a}{0.3}, \frac{b}{0.4}\right)\right\rangle;\ x \in X\right\}, \\ B &= \left\{\left\langle y, \left(\frac{c}{0.4}, \frac{d}{0.2}\right), \left(\frac{c}{0.4}, \frac{d}{0.3}\right)\right\rangle;\ y \in Y\right\}, \\ C &= \left\{\left\langle y, \left(\frac{c}{0.4}, \frac{d}{0.3}\right), \left(\frac{c}{0.4}, \frac{d}{0.2}\right)\right\rangle;\ y \in Y\right\}. \end{aligned}$$

Define an intuitionistic fuzzy mapping $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = d$, $f(b) = c$. B is an IFOS in (Y,σ). $f^{-1}(B) = \{\langle x, (\frac{a}{0.2}, \frac{b}{0.4}), (\frac{a}{0.3}, \frac{b}{0.4})\rangle;\ x \in X\}$ is an IFαOS in (X,τ) since int(cl(int($f^{-1}(B)$))) $= A$. Hence $f^{-1}(B) \subseteq$ int(cl(int($f^{-1}(B)$))) which implies f is an IFα-continuous. C is an IFS in Y. Also C is an IFSOS in Y since $C \subseteq$ cl(int(C)) $= \overline{B}$. $f^{-1}(C) = \{\langle x, (\frac{a}{0.3}, \frac{b}{0.4}), (\frac{a}{0.2}, \frac{b}{0.4})\rangle;\ x \in X\}$ int(cl(int($f^{-1}(C)$))) $= A$ and $f^{-1}(C) \nsubseteq$ int(cl(int($f^{-1}(C)$))) which implies $f^{-1}(C)$ is not IFαOS in Y. Hence f is not IF strongly α-continuous.

Theorem 3.1. *If $f : (X, \tau) \to (Y, \sigma)$ be a mapping from an IFTS X into an IFTS Y. Then the following are equivalent.*

(a) *f is IF α-irresolute.*

(b) *$f^{-1}(B)$ is IFαCS in X for each IFαCS B in Y.*

(c) *$f(\alpha \mathrm{cl}\, A) \subseteq \alpha\, \mathrm{cl}\, f(A)$ for each IFS A in X.*

(d) *$\alpha\, \mathrm{cl}\, f^{-1}(B) \subseteq f^{-1}(\alpha\, \mathrm{cl}\, B)$ for each IFS B in Y.*

(e) $f^{-1}(\alpha \mathrm{int}\, B) \subseteq \alpha\, \mathrm{int}\, f^{-1}(B)$ for each IFS B in Y.

Proof. (a)⇒(b): It can be proved by using the complement and the definition of IFα-irresolute. Let B be IFαCS in Y, then 1-B is IFαOS in Y. Since f is IFα-irresolute, $f^{-1}(1 - B) =$ 1-$f^{-1}(B)$ is IFαOS in X. Hence $f^{-1}(B)$ is IFαCS in X. Thus (a)⇒(b) is proved.

(b)⇒(c): Let A be IFS in X. Then A$\subseteq f^{-1}(f(A)) \subseteq f^{-1}(\alpha cl f(A))$.
$\alpha cl f(A)$ is IFαCS in Y, by (b) $f^{-1}(\alpha cl f(A))$ is an IFαCS in X.
αcl(A) $\subseteq f^{-1}(\alpha cl f(A))$ and $f(\alpha cl(A)) \subseteq f(f^{-1}(\alpha cl f(A))) = \alpha cl f(A)$.
Thus $f(\alpha cl(A)) \subseteq \alpha cl f(A)$. Hence (b)⇒(c) is proved.

(c)⇒(d): For any IFS B in Y, let $f^{-1}(B)=$ A; by (c), $f(\alpha\mathrm{cl}f^{-1}(B))\subseteq \alpha cl f(f^{-1}(B)) \subseteq \alpha cl(B)$ and $(\alpha\mathrm{cl}f^{-1}(B))\subseteq f^{-1}(f(\alpha cl f^{-1}(B)))\subseteq f^{-1}(\alpha cl B)$. Thus $(\alpha cl f^{-1}(B)) \subseteq f^{-1}(\alpha cl B)$. Hence (c)⇒(d) is proved.

(d)⇒(e): We know that αint(B) $=\overline{\alpha cl(\overline{B})}$
$f^{-1}(\alpha\mathrm{intB}) = f^{-1}\overline{(\alpha cl(\overline{B}))} = \overline{(f^{-1}(\alpha cl(\overline{B})))} \subseteq \overline{(\alpha cl(f^{-1}(\overline{B}))} = \overline{\alpha int\overline{(f^{-1}(B))}} \subseteq$ αint(f^{-1}(B)) Hence (d)⇒(e) is proved.

(e)⇒(a):Let B be any IFαOS in Y. Then B = αint(B).
$f^{-1}(\alpha$intB$)= f^{-1}$(B) $\subseteq \alpha$int(f^{-1}(B)).
By definition, f^{-1}(B) $\supseteq \alpha$int(f^{-1}(B)). So f^{-1}(B) $=\alpha$int(f^{-1}(B)). Thus f^{-1}(B) is an IFαOS in X which implies f is IF α-irresolute. Thus (e)⇒(a) is proved. ■

4. Properties of intuitionistic fuzzy α-irresolute functions

Lemma 4.1. [3] *Let f:X→Y be a mapping, and A_α be a family of IF sets of Y. Then*

(a) $f^{-1}(\bigcup A_\alpha) = \cup f^{-1}(A_\alpha)$

(b) $f^{-1}(\cap A_\alpha) = \cap f^{-1}(A_\alpha)$.

Lemma 4.2. [6] *Let $f : X_i \to Y_i$ be a mapping and A,B are IFS's of Y_1 and Y_2 respectively then $(f_1 \times f_2)^{-1}(A\times B) = {f_1}^{-1}(A)\times {f_2}^{-1}(B)$.*

Lemma 4.3. [6] *Let $g : X \to X\times Y$ be a graph of a mapping $f : (X, \tau) \to (Y, \sigma)$. If A and B are IFS's of X and Y respectively, then $g^{-1}(1_\sim \times B) = (1_\sim \cap f^{-1}(B))$.*

Lemma 4.4. [6] *Let X and Y be intuitionistic fuzzy topological spaces, then (X, τ) is product related to (Y, σ) if for any IFS C in X, D in Y whenever $\overline{A} \not\supseteq C$, $\overline{B} \not\supseteq D$ implies $\overline{A}\times 1_\sim \cup 1_\sim \times \overline{B} \supseteq C\times D$ there exists $A_1 \in \tau$, $B_1 \in \sigma$ such that $\overline{A_1} \supseteq C$ and $\overline{B_1} \supseteq D$ and $\overline{A_1}\times 1_\sim \cup 1_\sim \times \overline{B_1} = \overline{A}\times 1_\sim \cup 1_\sim \times \overline{B}$.*

Lemma 4.5. *Let X and Y be intuitionistic fuzzy topological spaces such that X is product related to Y. Then the product A×B of an IFαOS A in X and an IFαOS B in Y is an IFαOS in fuzzy product spaces X×Y.*

Theorem 4.1. *Let $f : X \to Y$ be a function and assume that X is product related to Y. If the graph $g : X \to X\times Y$ of f is IF α-irresolute then so is f.*

Proof. Let B be IFαOS in Y. Then by Lemma 4.3 f^{-1}(B) $= 1_\sim \cap f^{-1}$(B) $= g^{-1}$ ($1_\sim \times$ B). Now, $1_\sim \times B$ is IFαOS in $X\times Y$. Since g is IF α-irresolute, $g^{-1}(1_\sim \times B)$ is IFαOS in X. Hence $f^{-1}(B)$ is IFαOS in X. Thus f is IF α-irresolute function. ■

Theorem 4.2. *If a function $f : X \to \Pi Y_i$ is IF α-irresolute, then $P_i \circ f : X \to Y_i$ is IF α-irresolute, where P_i is the projection of ΠY_i onto Y.*

Proof. Let B_i be any IFαOS of Y_i. Since P_i is IF continuous and IFOS, it is IFαOS. Now P_i: $\Pi Y_i \rightarrow Y_i$; ${P_i}^{-1}(B_i)$ is IFαOS in Π Y_i. Therefore, P_i is IF α-irresolute function. Now $(P_i \circ f)^{-1}(B_i) = f^{-1}$ $({P_i}^{-1}(B_i))$, since f is IF α-irresolute and ${P_i}^{-1}(B_i)$ is IFαOS, f^{-1} $({P_i}^{-1}(B_i))$ is IFαOS. Hence $(P_i \circ f)$ is IF α-irresolute.

Theorem 4.3. *If $f_i : X_i \to Y_i$, $(i = 1, 2)$ are IF α-irresolute and X_1 is product related to X_2, then $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is IF α-irresolute.*

Proof. Let $C = \cup(A_i \times B_i)$ where A_i and B_i, $i = 1, 2$ are IFα-open sets of Y_1 and Y_2 respectively. Since Y_1 is product related to Y_2, by Lemma 4.5, that C $= \cup(A_i \times B_i)$ is IFα-open of $Y_1 \times Y_2$. Using Lemmas 4.1 and 4.2 we obtain $(f_1 \times f_2)^{-1}(C) = (f_1 \times f_2)^{-1} \cup(A_i \times B_i) = \cup$ $({f_1}^{-1}(A_i) \times {f_2}^{-1}(B_i))$. Since f_1 and f_2 are IF α-irresolute, we conclude that $(f_1 \times f_2)^{-1}(C)$ is an IFαOS in X_1 $\times$ X_2 and hence $f_1 \times f_2$ is IF α-irresolute function. ∎

Theorem 4.4. *A mapping $f : X \to Y$ from an IFTS X into an IFTS Y is IF α-irresolute if and only if for each IFP $p_{(\alpha,\beta)}$ in X and IFαOS B in Y such that $f(p_{(\alpha,\beta)}) \in B$, there exists an IF$\alpha$OS A in X such that $p_{(\alpha,\beta)} \in A$ and $f(A) \subseteq B$.*

Proof. Let f be any IF α-irresolute mapping, $p_{(\alpha,\beta)}$ be an IFP in X and B be any IFαOS in Y such that $f(p_{(\alpha,\beta)})\in B$. Then $p_{(\alpha,\beta)} \in f^{-1}$(B) $= \alpha$intf^{-1}(B). Let A $= \alpha$intf^{-1}(B). Then A is an IFαOS in X which containing IFP $p_{(\alpha,\beta)}$ and $f(A) = f(\alpha int f^{-1}(B)) \subseteq f(f^{-1}(B)) = B$.

Conversely, let B be an IFαOS in Y and $p_{(\alpha,\beta)}$ be IFP in X such that $p_{(\alpha,\beta)} \in$ f^{-1}(B). According to assumption there exists IFαOS A in X such that $p_{(\alpha,\beta)}$ $\in$A and $f(A) \subseteq B$. Hence $p_{(\alpha,\beta)}$ $\in$A $\subseteq f^{-1}$(B) and $p_{(\alpha,\beta)} \in A = \alpha$int$A \subseteq \alpha$int$f^{-1}(B)$. Since $p_{(\alpha,\beta)}$ be an arbitrary IFP and f^{-1}(B) is union of all IFP containing in f^{-1}(B) we obtain that $f^{-1}(B) = \alpha$int$f^{-1}(B)$. So, f is an IF α-irresolute mapping. ∎

Definition 4.1. [4] Let (X,τ) be an IFTS and let A be any IFS in X. Then A is called IFdense set if clA $= 1_\sim$ and A is called nowhere IFdense set if int(cl(A))$= 0_\sim$.

Theorem 4.5. *If a function $f : (X, \tau) \to (Y, \sigma)$ is IF α-irresolute, then $f^{-1}(A)$ is IFα-closed in X for any nowhere IFdense set A of Y.*

Proof. Let A be any nowhere IF dense set in Y. Then int(clA)$= 0_\sim$. Now, 1-int(clA) $= 1_\sim$.

$\Longrightarrow$ cl(1-cl(A)) $= 1_\sim$ which implies cl(int(1-A)) $= 1_\sim$. Since int$1_\sim$ $= 1_\sim$, int(cl(int(1-A))) $=$ int$1_\sim$ $= 1_\sim$. Hence 1-A $\subseteq$ int(cl(int(1-A))) $= 1_\sim$. Then 1-A is IFαOS in Y. Since f is IF α-irresolute, f^{-1}(1-A) is IFαOS in X. Hence $f^{-1}(A)$ is IFαCS in X. ∎

Theorem 4.6. *The following hold for functions $f : X \to Y$ and $g : Y \to Z$.*

(i) *If f is IF α-irresolute and g is IF α-irresolute $g \circ f$ is IF α-irresolute.*

(ii) *If f is IF α-irresolute and g is IF strongly α-continuous then $g \circ f$ is IF strongly α-continuous.*

(iii) *If f is IF α-irresolute and g is IFα-continuous then $g \circ f$ is IF α-irresolute.*

Proof. (i) Let B be an IFαOS in Z. Since g is IF α-irresolute, g^{-1}(B) is an IFαOS in Y. Now $(g \circ f)^{-1}(B) = f^{-1}(g^{-1}(B))$. Since f is IF α-irresolute, $f^{-1}(g^{-1}(B))$ is IFαOS in X. Hence $g \circ f$ is IF α-irresolute.

(ii) Let B be an IFSOS in Z. Since g is IF strongly α-continuous, g^{-1}(B) is an IFαOS in Y. Now $(g \circ f)^{-1}(B) = f^{-1}(g^{-1}(B))$. Since f is IF α-irresolute, $f^{-1}(g^{-1}(B))$ is IFαOS in X. Hence $g \circ f$ is IF strongly α-continuous.

(iii) Let B be an IFOS in Z. Since g is IFα-continuous, g^{-1}(B) is an IFαOS in Y. Now, $(g \circ f)^{-1}(B) = f^{-1}(g^{-1}(B))$. Since f is IF α-irresolute, $f^{-1}(g^{-1}(B))$ is IFαOS in X. Hence $g \circ f$ is IF α-irresolute. ∎

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INVERTIBLE ELEMENTS IN BCK-ALGEBRAS

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Abstract. In this article, we introduce the notion of cyclic BCK-algebra and study some of its main properties. In addition, we obtain a structure theorem for bounded commutative of finite order and use it to prove a Lagrange-like theorem for the above class of algebras. Finally, we use the notion of invertible elements to obtain a new characterization of implicative BCK-algebras and study the intersection of all maximal ideals of bounded BCK-algebras.

Keywords: cyclic BCK-algebras, simple BCK-algebras, implicative BCK-algebras, J-semisimple.

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1. Introduction

Introduced in the mid 60's by Iseki and Imai, the theory of BCK-algebras has been the object of intense development for the past four decades. BCK-algebras are in many respect similar to rings, and most of the theory has been developed in perfect agreement with the classical ring theory. For instance, there is a notion of ideal in BCK-algebras, and they have been studied extensively ([1], [5], [7]). The notion of zero-divisor was introduced in [1] and used to characterize prime ideals. In the same line of ideas, we introduced the notion of invertible elements a BCK-algebra and used it to characterize maximal ideals [2].

In the present work, we introduced Cyclic BCK-algebras, which seems to be structurally the simplest, as they are representable using only one element. We explore some of the applications of invertible elements. We prove that for every order, there exists a unique (up to isomorphism) cyclic BCK-algebra of that order. This approach gives a new description and characterization of bounded commutative BCK-chains of finite order as treated in [4]. We also prove that sub-algebras of a cyclic BCK-algebra are also cyclic. Using the newly introduced notion and some results from [4], we obtain a structure theorem for bounded commutative BCK-algebras. As a consequence of this theorem, we obtain a Lagrange-like theorem for finite bounded commutative BCK-algebras. A new characterization of implicative algebras among bounded commutative algebras in terms of invertible elements is also established. In the final section, we revisit the notion of J-semisimple for bounded BCK-algebras as introduced in 1978 by J. Ahsan and E. Deeba (for details on J-semisimple see [3, §5.3]). The notion J-semisimplicity for BCK-algebras corresponds to the well known semisimplicity in classical ring theory. Having introduced the notion of invertibility, we are able to use this to offer a fresh treatment of J-simisimplicity in BCK-algebras. Among other things, we prove that every finite bounded commutative BCK-algebra is J-semisimple.

2. Preliminaries and notations

A *BCK-algebra* is a set X with a binary operation $\star$ and a constant 0 satisfying the following axioms:

(i) $(x \star y) \star (x \star z) \leq z \star y$

(ii) $x \star (x \star y) \leq y$

(iii) $x \leq x$

(iv) $0 \leq x$

(v) $x \leq y$ and $y \leq x$ implies that $x = y$

where $x \leq y$ if and only if $x \star y = 0$. A key property of BCK-algebras as proved in [[6], Eq.3] is the following:

$$(x \star y) \star z = (x \star z) \star y \tag{3}$$

In addition, if there is an element $1 \in X$ such that $x \leq 1$ for all $x \in X$, then X is said to be *bounded* and we set $Nx = 1 \star x$. If we denote $x \wedge y$ by $y \star (y \star x)$, then X is said to be *commutative* if $x \wedge y = y \wedge x$ for all $x, y \in X$. A BCK-algebra is *implicative* if $x \star (y \star x) = x$ for all $x, y \in X$.

A subset I of a BCK-algebra X is an *ideal*, if $0 \in I$ and if $x, y \star x \in I$, then $y \in I$. Let A be a subset of a BCK-algebra X, the ideal generated by A, denoted by $< A >$ is the set of all $x \in X$ such that $(\dots(x \star a_1) \star a_2) \star \dots a_n) = 0$ for some $a_1, a_2, \dots a_n \in A$ [7, Theorem 3]. If $A = \{a\}$, we denote $< A >$ by $< a >$ the ideal generated by a. We denote the expression $(\dots(x \star a) \star a) \star \dots a)$ by $x \star a^n$, where n is the number of times a appears in the expression, in particular $x \star a^0 = x$. In which case $< a >= \{x \in X : \exists n > 0, x \star a^n = 0\}$. Let I be an ideal of X. The relation $\sim$ defined on X by $x \sim y$ if and only if $x \star y, y \star x \in I$ is an equivalence relation. Let C_x denote the class of $x \in X$ and X/I the set of equivalence classes C_x, $x \in X$. It is clear that $C_0 = I$. Define on X/I a binary operation $\star$ given by $C_x \star C_y = C_{x \star y}$ and $C_x \leq C_y$ if and only if $x \star y \in I$. Then X/I together with $\star$ and its constant I is a BCK-algebra called the quotient BCK-algebra of X determined by I. The maximal ideals of BCK-algebras have the usual meaning. A proper ideal P of a commutative BCK-algebra X is called *prime* if $a \wedge b \in P$ implies $a \in P$ or $b \in P$.

If X and Y are BCK-algebras, a homomorphism from X to Y is a map $f : X \to Y$ such that $f(x \star y) = f(x) \star f(y)$.

If X and Y are bounded, we require that homomorphisms between X and Y map 1 to 1. By isomorphism between two BCK-algebras, we mean a bijective homomorphism. We recall the following definitions and results which are mainly from [2].

Definition 2.1 Let X be a BCK-algebra. An $u \in X$ is *invertible* if $< u >= X$.

Example 2.2

1. It is clear that 1 is always an invertible element of any bounded BCK-algebra as $\langle 1 \rangle = X$.

2. Examples of nontrivial invertible elements of a BCK-algebra can be found under this section (see Example 2.5).

We denote by $\mathcal{U}_X$ the set of units of a bounded BCK-algebra X or simply $\mathcal{U}$ when there is no risk of confusion.

Proposition 2.3 *If X is a bounded BCK-algebra, then u is invertible in X if and only if there exists $n > 0$ such that $1 \star u^n = 0$.*

Proof. If u is an invertible element of X, then $1 \in < u >= X$. So there exists $n > 0$ such that $1 \star u^n = 0$. Conversely, if there exists $n > 0$ such that $1 \star u^n = 0$, then $1 \in < u >$ and hence $< u >= X$. ∎

In [7], Iseki and Tanaka introduced *simple* BCK-algebras as BCK-algebras having only two ideals, $\{0\}$ and X. A trivial example of simple bounded BCK-algebra is $X = \{0, 1\}$. We will provide more examples later. We have the following easy characterization of simple algebras using invertible elements.

Lemma 2.4 *A BCK-algebra X is simple if and only if all non-zero elements of X are invertible.*

Proof. Suppose X is simple. Then $< x >= X$ for all $x \in X \setminus \{0\}$. So x is invertible. Conversely, if all $x \in X \setminus \{0\}$ are invertible, it is clear that X is simple. ∎

Example 2.5

1. **Example of a simple non-commutative bounded BCK-algebra**
For $n > 1$, consider $X = \{a_1, a_2 \ldots, a_n\}$ and order X by $a_i \leq a_{i+1}$ for all $i \geq 1$. Define $\star$ on X by:

$$a_i \star a_j = \begin{cases} 0 & : \ i \leq j \\ a_2 & : \ 1 < j < i \\ a_i & : \ j = 1 \end{cases}$$

Then X is a bounded non-commutative simple BCK-algebra. For the simplicity, note that for every $i > 1$, $a_n \star a_i^2 = 0$, hence a_i is invertible. Therefore, every non-zero element of X is invertible, and by Lemma 2.4, X is simple.

2. **Example of a simple commutative bounded BCK-algebra**
For $n > 1$, consider $\mathbb{X}_n = \{a_0, a_1, a_2 \ldots, a_{n-1}\}$ and order X by $a_i \leq a_{i+1}$ for all $i \geq 0$. Define $\star$ on X by: $a_i \star a_j = 0$ if $i \leq j$ and $a_i \star a_j = a_k$ if $i - j = k > 0$. Then $\mathbb{X}_n$ is a bounded commutative simple BCK-algebra.

3. Cyclic BCK-algebras

Recall that the order of a BCK-algebra X is the cardinality of X and is often denoted by $o(X)$ or $|X|$. We start by introducing the order of elements in bounded BCK-algebras.

Definition 3.1 Let X be a bounded BCK-algebra with unit 1. The *order* of $x \in X$, denoted by $o(x)$, is the smallest positive integer n such that $1 \star x^n = 0$. If such n does not exist, we say that x has infinite order.

It is obvious that 1 is the only invertible element of order 1. It is also clear from the definition that an $x \in X$ has finite order if and only if x is invertible. If X is a bounded implicative different than $\{0, 1\}$, then each $x \in X \setminus \{1\}$ has infinite order. This is because, in a bounded implicative BCK-algebra, we have $1 \star x^n = 1 \star x$ for all $n \geq 1$.

Lemma 3.2 *Let u be an invertible element of X of order n. Then the following elements $0, 1, 1 \star u^i$, $i = 1, \ldots, n-1$, of X are pairwise distinct.*

Proof. By contradiction, suppose that $1 \star u^i = 1 \star u^j$ for some $0 \leq i < j \leq n$. Then

$$\begin{aligned} 0 &= 1 \star u^n \\ &= (1 \star u^j) \star u^{n-j} \\ &= (1 \star u^i) \star u^{n-j} \qquad \text{since } 1 \star u^i = 1 \star u^j \\ &= 1 \star u^{n-j+i} \end{aligned}$$

But $n-j+i<n$ which is a contradiction with the order of u being n. So, $1\star u^i\neq 1\star u^j$ for $1\leq i<j\leq n$, as required. ∎

Next, we give a result that provides an upper bound for the orders in finite BCK-algebras.

Proposition 3.3 *Let X be a finite bounded BCK-algebra. Then for every invertible element u of X, $o(u)<o(X)$.*

Proof. This is immediate from Lemma 3.2. ∎

Remark 3.4 The upper bound given in Proposition 3.3 is the sharpest possible. In fact consider the BCK-algebra of Example 2.5 2, then it is easy to see that a_1 has order n.

We introduce the following definition.

Definition 3.5 A *cyclic* BCK-algebra X is a finite bounded BCK-algebra with an invertible element $u\in X$ such that $o(u)=o(X)-1$.

The above definition is motivated by the following result.

Proposition 3.6 *Let X be cyclic BCK-algebra of order n. Then, there exists $u\in X$ invertible such that*

$$X=\{1\star u^i : i=0,1,2,\dots,n-1\}$$

where $1\star u^0=1$.

Proof. Since X is cyclic of order n, there exists an invertible element $u\in X$ of order $n-1$. It follows from Lemma 3.2 that the elements $1\star u^i$'s are pairwise distinct. Therefore, $|\{1\star u^i : i=0,1,2,\dots,n-1\}|=n$. Thus as $|X|=n$, we obtain that $X=\{1\star u^i : i=0,1,2,\dots,n-1\}$ as needed. ∎

Recall that if X is a BCK-algebra and $a\in X\setminus\{0\}$, then a is called an *atom* of X if for every $x\in X$, if $x\leq a$, then $x=0$ or $x=a$. We start with the following easy lemma.

Lemma 3.7 *Every Cyclic BCK-algebra X is a BCK-chain. Furthermore, the atom of X is the generator for X, in particular the generator is unique.*

Proof. Let X be a cyclic algebra of order n, by Proposition 3.6, there exists $u\in X$ such that $X=\{1\star u^i : i=0,1,2,\dots,n-1\}$. Note that $i\geq j$ implies $1\star u^i\leq 1\star u^j$, therefore,

$$0=1\star u^{n-1}\leq 1\star u^{n-2}\leq\cdots\leq 1\star u\leq 1\star u^0=1$$

Hence X is a chain.

It remains to show that $1\star u^{n-2}=u$. Note that, since $u\neq 0$, and $X=\{1\star u^i : i=0,1,2,\dots,n-1\}$, then $u=1\star u^i$ for some $i=0,1,\dots,n-2$. But since $u\star u=0$, then $1\star u^{i+1}=0$, thus by the definition of order, we get $n-1\leq i+1$. Hence $i\geq n-2$, thus $i=n-2$ and $u=1\star u^{n-2}$ as needed. Note that $1\star u^{n-2}$ is the atom of X. ∎

Remark 3.8 It is straightforward to see that properties such as cyclicity, commutativity, and simplicity are preserved by BCK-isomorphisms. It is also clear that orders of elements are preserved by BCK-isomorphisms as well.

Theorem 3.9 *Every Cyclic BCK-algebra of order n is isomorphic to the BCK-algebra $\mathbb{X}_n$ of Example 2.5 2. In particular, every cyclic BCK-algebra is simple and commutative.*

Proof. From Lemma 3.7, we have $X = \{0 \leq u \leq 1 \star u^{n-3} \cdots \leq 1 \star u \leq 1\}$. For simplicity, we denote $1 \star u^{n-i-1}$ by x_i, so $x_0 = 0$, $x_1 = u$, and so on.

We need to show that $x_i \star x_j = 0$ if $i \leq j$ and $x_i \star x_j = x_k$ if $i - j = k > 0$.

It is clear that $x_i \star x_j = 0$ if $i \leq j$.

It remains to show that $x_i \star x_j = x_k$ if $i - j = k > 0$.

For this, we first prove the case $k = 1$, that is $i = j + 1$.

We have

$$\begin{aligned}(x_i \star x_j) \star x_1 &= (x_i \star x_j) \star u \\ &= (x_i \star u) \star x_j \\ &= x_j \star x_j \qquad\qquad \text{since } x_j = x_i \star u \\ &= 0\end{aligned}$$

Hence $x_i \star x_j \leq u$, so $x_i \star x_j = 0$ or $x_i \star x_j = u$. But $x_i \star x_j = 0$ implies $x_i = x_j$ which is impossible. Therefore, $x_i \star x_j = u = x_1$ as needed.

For the general step, assume $k > 1$ and $i = j + k$. For simplicity, name $x_i \star x_j$ by a. Since $X = \{1 \star u^i : i = 0, 1, 2, \ldots, n-1\}$, we know that $a = 1 \star u^r$ for some $r = 0, 1, \ldots, n-1$.

On the other hand, we have

$$\begin{aligned}a \star u^{k-1} &= (x_i \star x_j) \star u^{k-1} \\ &= (x_i \star u^{k-1}) \star x_j \\ &= x_{j+1} \star x_j \qquad \text{since } x_{j+1} = x_i \star u^{k-1} \\ &= u \qquad\qquad\quad \text{by the step above}\end{aligned}$$

Thus $u = a \star u^{k-1} = 1 \star u^{r+k-1}$, therefore $1 \star u^{r+k} = u \star u = 0$, thus $n - 1 \leq r + k$. Hence $r \geq n - k - 1(1)$. On the other hand, since $u \neq 0$ and $u = 1 \star u^{r+k-1}$, then $r + k - 1 \leq n - 2$ as $1 \star u^s = 0$ for all $s \geq n - 1$. Therefore, $r \leq n - k - 1$, which coupled with (1) imply that $r = n - k - 1$. Thus, $x_i \star x_j = a = 1 \star u^{n-k-1} = x_k$ as required.

It is now clear that $x_i \mapsto a_i$ is an isomorphism between X and $\mathbb{X}_n$, so $X \cong \mathbb{X}_n$ as needed.

Now, since $\mathbb{X}_n$ is commutative and simple, it follows from Remark 3.8 that X is commutative and simple. ∎

Remark 3.10 As stated in Theorem 3.9, cyclic BCK-algebras are commutative and simple. Note however that a finite bounded simple BCK-algebra needs not be cyclic [e.g., Example 2.5(1)]. Neither is every finite bounded commutative BCK-algebra cyclic [e.g., B_{4-2-3} from [3]], but every finite bounded commutative and simple BCK-algebra is cyclic as we now prove.

Proposition 3.11 *A finite bounded BCK-algebra is cyclic if and only if it is commutative and simple.*

Proof. Note that the necessity is from Theorem 3.9 and the sufficiency is an immediate consequence of [Cor. 2.3.2, Thm. 2.3.3] from [3]. ∎

A combination of [4, Theorem 7], [4, Theorem 8] and Proposition 3.11 yields the following structure result for bounded commutative BCK-algebras.

Theorem 3.12 *Every finite non-zero bounded commutative BCK-algebra is a product of cyclic BCK-algebras.*

It follows from Theorem 3.12 that every bounded commutative BCK-algebra of prime order is cyclic. This result is for bounded commutative BCK-algebras what the fundamental theorem of finite Abelian groups is for Abelian groups.

Proposition 3.13 *Every non-zero sub-algebra of a cyclic BCK-algebra is cyclic.*

Proof. Let X be a cyclic BCK-algebra algebra and S a non-zero sub-algebra of X. Then, by Theorem 3.9, X is commutative and simple. Therefore, by Theorem 2.3.8 of [3], S is simple. But being a sub-algebra of a commutative algebra, S is also commutative. On the other hand by Lemma 3.7, X is a finite BCK-chain, thus S is bounded. Therefore, S is bounded, commutative and simple, hence cyclic by Proposition 3.11. ∎

We have the following result that can rightfully be called the Lagrange's theorem for bounded commutative BCK-algebras.

Proposition 3.14 *Let X be a bounded commutative BCK-algebra of finite order. Then, for every ideal I of X,*

$$|X| = |I||X/I|$$

In particular, the order $|I|$ of I divides $|X|$.

Proof. First, note that from Theorem 3.12, there exist simple algebras X_i, i=1, ..., n, such that $X = \prod_{i=1}^{n} X_i$. For each k=1, ..., n, let $I_k := \{a \in X_k : \iota_k(a) \in I\}$, where $\iota_k : X_k \to X$ is the natural inclusion.

Claim. $I = I_1 \times \cdots \times I_n$.
Let $x = (x_1, \ldots, x_n) \in I$. Then for each $k = 1, \ldots, n$, $x \star \widehat{x} = \iota_k(x_k)$, where $\widehat{x}$ is obtained from x by keeping every coordinate, but replacing the k^{th}-coordinate by 0. Since $x \in I$, then $x \star \widehat{x} \in I$ and therefore $\iota_k(x_k) \in I$ for all $k = 1, \ldots, n$. Thus, $x_k \in I_k$ for all $k = 1, \ldots, n$ and hence $x \in I_1 \times \cdots \times I_n$.

Conversely, if $x = (x_1, \ldots, x_n) \in I_1 \times \cdots \times I_n$, then

$$(\ldots((x \star \iota_n(x)) \star \iota_{n-1}(x)) \ldots \star \iota_1(x)) = 0 \in I.$$

Since $\iota_k(x) \in I$ for all $k = 1, \ldots, n$ and I is an ideal of X, one obtains that $x \in I$. In addition, since each X_k is simple, then $I = I_1 \times \cdots \times I_n$ with $I_k = \{0\}$ or X_k.

Now, let $A = \{k \in \{1, \ldots, n\} | I_k = \{0\}\}$, so $I_k = X_k$ for all $k \notin A$. For every $x = (x_k) \in X$, it is easy to see that $C_x = \{(a_k) \in X | a_k = x_k \text{ for all } k \in A\}$. It follows that the map $(a_k) \mapsto (b_k)$ (where $b_k = 0$ for $k \in A$ and $b_k = a_k$ otherwise) is a bijection between C_x and I. Therefore, for every $x \in X$, $|C_x| = |I|$. Since $X = \biguplus_{X/I} C_x$, then

$$|X| = \sum_{X/I} |C_x| = \sum_{X/I} |I| = |I||X/I|.$$

Hence $|X| = |I||X/I|$, in particular $|I|$ divides $|X|$ as required. ∎

Remark 3.15 It is important to point out that to prove Proposition 3.14, we have just proved that any ideal I of X is the product of ideals coming from each component X_i, but this is not true in general for infinite product. In fact if we consider the direct sum of algebras, it is an ideal of the direct product that is not the direct product of ideals.

Remark 3.16 Note that Proposition 3.14 holds for any BCK-algebra that is a product of finite simple algebras as this was the only aspect of finite bounded commutative BCK-algebras used in the proof.

Proposition 3.17 *Let $X_1, X_2, \ldots, X_n$ be bounded BCK-algebras and $X = \prod_{i=1}^{n} X_i$. Then*

$$\mathcal{U}_X = \times_{i=1}^{n} \mathcal{U}_{X_i}$$

Proof. Let $x = (x_1, \ldots, x_n) \in X$ such that each x_i is invertible in X_i. Then, for each i, there exists n_i such that $1 \star x_i^{n_i} = 0$. If $m = \max\{n_i\}$, then $1 \star x^m = 0$, so x is invertible. The reverse inclusion is even simpler. ∎

Remark 3.18 Proposition 3.17 is not true in general for infinite product. For instance, consider the real interval $X_i = [0, 1]$ with $\star$ defined as $x \star y = \max\{0, x - y\}$ for each $i \in \mathbb{N}$. Now, let $X = \prod_{i \in \mathbb{N}} X_i$ and $x = (1/i)_{i \in \mathbb{N}} \in X$. Now, for each i, $1/i$ is invertible in X_i as X_i is simple, but we claim that x is not invertible. Suppose that x is invertible. Then, there is a positive integer n such that $1 \star x^n = 0$, i.e., for all $i \in \mathbb{N}$, $1 \star (1/i)^n = 0$ $(*)$. But there exists an $i_0 > n$ such that $(k+1)/i_0 < 1$ for each $k = 1, ..., n$. Thus, $1 - k/i_0 > 1/i_0$ for each $k = 1, \ldots, n$. So then $1 \star (1/i_0)^n = 1 - n/i_0 \neq 0$, which is a contradiction with $(*)$.

Most of the theory of BCK-algebras studied so far in this article is in perfect agreement with the classical ring theory. But this is a major difference between the two theories because the units of direct product of rings is the product of units.

We obtain the following characterization of implicative BCK-algebras using invertible elements.

Proposition 3.19 *Let X be a non-zero bounded commutative BCK-algebra. Then X is implicative if and only if $\mathcal{U}_X = \{1\}$.*

Proof. Note that if X is a non-zero bounded implicative, and $x \in \mathcal{U}_X$, then there exists $n \geq 1$ such that $1 \star x^n = 0$. But since $1 \star x^n = 1 \star x$ [6, Prop. 6], hence $x = 1$ and $\mathcal{U}_X = \{1\}$.

Conversely, suppose that X is a non-zero bounded commutative BCK-algebra such that $\mathcal{U}_X = \{1\}$. Since X is commutative and bounded, in order to show that X is implicative, it is enough by [6, Prop. 6] to show that $x \vee Nx = 1$ for all $x \in X$. Let $x \in X$, then since $Nx \leq x \vee Nx$, we have $1 \star (x \vee Nx) \leq 1 \star Nx = 1 \wedge x = x$. Hence, $1 \star (x \vee Nx) \leq x$, which implies $1 \star (x \vee Nx)^2 \leq x \star (x \vee Nx) = 0$ as $x \leq x \vee Nx$. Therefore, $1 \star (x \vee Nx)^2 = 0$, hence $x \vee Nx \in \mathcal{U}_X = \{1\}$ and $x \vee Nx = 1$ as needed. ■

Remark 3.20 Proposition 3.19 is not true in general for bounded non-commutative algebras. In fact there are bounded positive implicative algebras X with $\mathcal{U}_X=\{1\}$ that are not implicative, see for example the BCK-algebra B_{5-2-12} from [3].

4. J-Semisimple BCK-algebras

Given a bounded BCK-algebra X, we denote by $J(X)$ the intersection of all maximal ideals of X.

Definition 3.1 A bounded BCK-algebra X is *J-semisimple* if $J(X) = 0$.

Example 3.2

1. For every $n \geq 1$, $J(\mathbb{X}_n) = 0$, because $\mathbb{X}_n$ is simple. So, for all n, $\mathbb{X}_n$ is J-semisimple.

2. Let X is an unbounded BCK-algebra, and $\widehat{X}$ be the Iséki's extension i.e., the BCK-algebra obtained by adjunction of a unit [see [5], Section II]. Then, since X is the unique maximal ideal of $\widehat{X}$, then $J(\widehat{X}) = X$. Thus, $\widehat{X}$ is not J-semisimple.

3. [3, Ex.1.2.2] In this example, $\mathbb{Z}^+$ denotes the set of all non-negative integers. Let $A := \{a_n : n \in \mathbb{Z}^+\}$ such that $A \cap \mathbb{Z}^+ = \emptyset$ and let $X = A \cup \mathbb{Z}^+$. Define a binary operation $\star$ on X as follows: For every $m, n \in \mathbb{Z}^+$,

$$\begin{aligned} m \star n &= \max\{0, m - n\} \\ m \star a_n &= 0 \\ a_m \star n &= a_{m+n} \\ a_m \star a_n &= n \star m \end{aligned}$$

Then, it is easy to see that X is a bounded commutative BCK-algebra with unit a_0. In fact, the BCK-order on X is given by the chain

$$0 \leq 1 \leq \cdots a_2 \leq a_1 \leq a_0.$$

One can verify that $\mathcal{U}_X = A$ and that $\mathbb{Z}^+$ is the ideal of X generated by 1. Therefore, by [2, Thm. 4.3], X is a local BCK-algebra with unique maximal ideal $\mathbb{Z}^+$. In fact, it is quite straightforward to see that X has only three ideals: $0, \mathbb{Z}^+$ and X. In either case, $J(X) = \mathbb{Z}^+$ and X is not J-semisimple.

4. Let $X = \prod_{n\in\mathbb{N}} \mathbb{X}_n$. Since each $\mathbb{X}_n$ is simple, then by Proposition 3.7 below, X is an infinite J-semisimple BCK-algebra.

Recall that in ring theory the Jacobson radical of a ring is the intersection of all maximal ideals of the ring. So the notion of Jacobson radical coincides with $J(X)$ in the context of bounded BCK-algebras. Motivated by the fact that the Jacobson radical of a ring is characterized in terms of units of the ring, we give a characterization of $J(X)$ in terms of invertible elements of any bounded BCK-algebra X.

Proposition 3.3 *Let X be a bounded BCK algebra with unit 1. Then*

$$J(X) = \{x \in X : 1 \star x^n \in \mathcal{U}_X \text{ for all } n \in \mathbb{N}\}$$

Proof. First note that for any $x \in X$, if there exists $n \geq 1$ so that $1 \star x^n \notin \mathcal{U}_X$, then there exists a maximal ideal M of X such that $1 \star x^n \in M$. Therefore, since $1 \notin M$, it follows that $x \notin M$. Hence $x \notin J(X)$.

Conversely, suppose that $1 \star x^n$ is invertible for all n. Suppose that there is a maximal ideal M such that $x \notin M$. Then since X/M is simple [2, Prop. 3.6], it follows by [2, Lemma 3.5] that $C_x \in X/M$ is invertible. So there is an m such that $C_{1\star x^m} = C_0$. Thus $1 \star x^m \in M$ a contradiction. ■

Corollary 3.4 *Let X be a bounded BCK-algebra that is J-semisimple and A be a subalgebra of X. If $1 \in A$, then A is J-semisimple.*

Proof. Since $1 \in A$, it follows from Proposition 3.3 that $J(A) \subseteq J(X) = 0$, hence $J(A) = 0$. ■

Recall that a bounded BCK-algebra X is *multiply implicative* if, for all $x, y \in X$, there is a positive integer $n = n(x, y)$ such that $x \star (y \star x^n) = x$. In particular, any bounded implicative BCK-algebra is multiply implicative.

Corollary 3.5 *Every bounded multiply implicative BCK-algebra is J-semisimple.*

Proof. Let X be a bounded multiply implicative BCK-algebra and let $x \in J(X)$. Then there is an $n = n(x, 1)$ such that $x \star (1 \star x^n) = x$. Letting $u = 1 \star x^n$, we have $x \star u = x$ and so $u \in I := \{y \in X : x \star y = x\}$. It is not hard to see that I is an ideal of X. In fact, $x \star 0 = x$ so $0 \in I$. Now let $y \star z, z \in I$. On one hand, we know that $x \star y \leq x$, on the other hand, $x = x \star (y \star z) = (x \star z) \star (y \star z) \leq x \star y$ [3, Cor.1.1.6]. Thus $x = x \star y$ and $y \in I$ as needed.

But then, by Proposition 3.3, $u = 1 \star x^n$ is invertible and it then follows that $I = X$. Thus $1 \in I$ and $x = x \star 1 = 0$. Hence $J(X) = 0$ and X is J-semisimple. ■

Remark 3.6 It is well known that the converse of Corollary 3.5 is not true (see Example 5.3.1 [3]).

Proposition 3.7 *Let $(X_i)_{i\in I}$ be a family of bounded simple BCK-algebras and $X = \prod_{i\in I} X_i$ the direct product of the family $(X_i)_{i\in I}$. Then X is J-semisimple.*

Proof. We claim that $J(X) \subseteq \prod_{i\in I} J(X_i)$ in which case $J(X) = 0$ as each $J(X_i) = 0$ since, for each $i \in I$, X_i is a bounded simple BCK-algebra. To prove the claim, let $x = (x_i)_{i\in I} \in J(X)$. Then, by Proposition 3.3, $1 \star x^n = (1 \star x_i^n)_{i\in I}$ is invertible for all positive integer n. By definition of invertible, it is clear that if $y = (y_i)_{i\in I} \in X$ is invertible, then for each $i \in I$ y_i is invertible in X_i. So then, for each $i \in I$, $1 \star x_i^n$ is invertible for all positive integer n in X_i. So $x_i \in J(X_i)$. ■

Remark 3.8

1. The same argument in the proof of proposition 3.7 will show that a direct product of J-semisimple BCK-algebras is again J-semisimple.

2. To prove Proposition 3.7, we used the fact that $J(\prod_{i\in I} X_i) \subseteq \prod_{i\in I} J(X_i)$. In fact, for finite product, one can easily, combining Proposition 3.17 and Proposition 3.3 , see that

$$J\left(\prod_{i=1}^{n} X_i\right) = \prod_{i=1}^{n} J(X_i).$$

   An open question is whether equality still holds for infinite product.

Since every bounded commutative BCK-algebra of finite order is a product of simple BCK-algebras [Theorem 3.12] we obtain as a consequence of Proposition 3.7 the following:

Corollary 3.9 *Every bounded commutative BCK-algebra of finite order is J-semisimple.*

Remark 3.10

1. The Iséki's extension $\widehat{X}$ in Example 3.2.2. is bounded non commutative and may be chosen to be finite, but $J(\widehat{X}) = X \neq 0$.

2. If a finite bounded local BCK-algebra is not simple, then it is non-commutative.

3. It is worth pointing out that Corollary 3.9 is not true in general for infinite algebras. In fact, there exists infinite bounded commutative BCK-algebras that are not J-semisimple. For instance, the BCK-algebra X of Example 3.2.3. is bounded and commutative, but with $J(X) \neq 0$.

4. Also there are bounded BCK-algebras that are J-semisimple but not commutative nor simple. For instance picking $X = B_{5-4-1} \times B_{5-4-2}$ it is clear that $J(X) = 0$ and that X is not commutative and not simple. In fact, taking any product of non commutative simple BCK-algebras would provide such examples.

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A NEW CHARACTERIZATION OF SIMPLETIC GROUP $S_8(2)$

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Abstract. It is a well-known fact that characters of a finite group can give important information of the group's structure. Also it was proved by chen [1] that a finite simple group can be uniquely determined by its character table. In this paper, the authors attempt to investigate how to characterize a finite almost simple group by using less information of its character table, and successfully characterize the simpletic group $S_8(2)$ by its order and at most two irreducible character degrees of its character table.

Keywords: finite group, character degree, simple group, order.

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1. Introduction

In this paper, G represents a finite group. We use $Irr(G)$ to denote the set of all irreducible complex characters of G, and $cd(G) = \{\nu(1)|\nu \in Irr(G)\}$ the set of all irreducible character degrees of G. Moreover, $cd^*(G)$ denotes the Multi-set of degrees of irreducible characters, i.e., each element of the set $cd^*(G)$ can occur many times upon the number of characters of the same degree. In particular, $|cd^*(G)| = |Irr(G)|$. $Syl_p(G)$ denotes the set of all Sylow p-subgroups of G, where $p \in \pi(G)$, and G_p denotes a Sylow p-subgroup of G. $H \cdot M$ denotes the

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non-split extension of H by M and $H : M$ the split extension of H by M. For any finite group G, $L_i(G)$ denotes the ith largest irreducible character degree of G. Particularly, $L_1(G)$ and $L_2(G)$ are the largest and the second largest irreducible character degree of G, respectively. All further unexplained symbols and notations are standard and can be found, for instance, in [2].

A finite group is called a K_n-group if $|\pi(G)| = n$, where n is a positive integer. K_3-simple groups were classified many years ago, which are A_5, A_6, $L_2(7)$, $L_2(8)$, $L_2(17)$, $L_3(3)$, $U_3(3)$ and $U_4(2)$ (cf. [3]). Classifying finite groups by the properties of their characters is an interesting and difficult topic in group theory. It is a well-known fact that characters of a finite group can give important information about the group's structure. For example, in [1], Chen proved that a finite simple group can be uniquely determined by its character table. In 2000, Huppert [4] put forward the following conjecture.

Huppert's Conjecture. *Let H be any nonabelian simple group, and G a group such that $cd(G) = cd(H)$, then $G \cong H \times A$, where A is an abelian group.*

Huppert conjectured that each finite non-abelian simple group G is characterized by $cd(G)$, the set of degrees of its complex irreducible characters. In [4, 5, 6], he confirmed that the conjecture holds for simple groups such as $L_2(q)$ and $S_z(q)$. Moreover, he also proved this conjecture holds for 19 out of 26 sporadic simple groups, and a few others (cf. [4], [5], [6]). In [7], Daneshkhah showed that the conjecture holds for another three sporadic simple groups Co_1, Co_2 and Co_3. Xu, et al. attempt to characterize the finite simple groups by less information of its characters, and for the first time successfully characterize the simple K_3-groups (cf. [8]) and sporadic simple groups by their orders and one or both of its largest and second largest irreducible character degrees (cf. [9], [10], [11]). For convenience, we summarize some results of these articles which will be used later in the following propositions:

Proposition. (cf. [10]) *Let G be a finite group and M a non-Abelian simple group, then the following assertions hold:*

(i) *If M is one of the simple groups: A_5, $L_2(7)$, $L_2(17)$, $L_3(3)$, $U_4(2)$, M_{11}, M_{23}, J_1, J_3 and J_4, then $G \cong M$ if and only if $|G| = |M|$ and $L_1(G) = L_1(M)$;*

(ii) *If M is isomorphic to one of the simple groups: simple K_3-groups, Mathieu simple groups and Janko simple groups, then $G \cong M$ if and only if $|G|=|M|$, $L_1(G) = L_1(M)$ and $L_2(G) = L_2(M)$;*

(iii) *If $M = A_6$, then G is isomorphic to one of the groups: $G \cong A_6$, $G \cong S_3 \times A_5$ and $G \cong Z_3 \rtimes S_5$, if and only if $|G| = |M|$ and $L_1(G) = L_1(M)$;*

(iv) *If $M = M_{22}$, then $G \cong M_{22}$ or $H \times M_{11}$, where H is a Frobenius group with elementary kernel of order 8 and a cyclic complement of order 7, if and only if $|G| = |M_{22}|$ and $L_1(G) = L_1(M_{22})$.*

In the following, we continue this investigation, and show that the simpletic simple group $S_8(2)$ by its order and at most two irreducible character degrees of its character tables.

Our main results are the following:

Theorem. *Let G be a finite group and $|G| = |S_8(2)|$, then $G \cong S_8(2)$ if and only if $L_1(G) = L_1(S_8(2))$ and $L_2(G) = L_2(S_8(2))$.*

2. Preliminaries

In this section, we consider some results which will be applied for our further investigations.

Lemma 2.1. *Let G be a finite solvable group of order $q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_s^{\alpha_s}$, where $q_1, q_2, \cdots, q_s$ are distinct primes, and if $(kq_s + 1) \nmid q_i^{\alpha_i}$ for each $i \leq s-1$ and $k > 0$, then the Sylow q_s-subgroup is normal in G.*

Proof. Let N be a minimal normal subgroup of G. Since G is solvable, then we have $|N| = q^m$. If $q = q_s$, by induction on G/N, it is easy to see the Sylow q_s-subgroup is normal in G. Now, assume that $q = q_i$ for some $i < s$. By induction of the factor group G/N, we obtain that the Sylow q_s-subgroup Q/N of G/N is normal in G/N. Thus $Q \trianglelefteq G$. Let P be a Sylow q_s-subgroup of Q. Then $Q = NP$. By Sylow's Theorem, $|Q : N_Q(P)| = q_i^l (l \leq m \leq \alpha_i)$ and $q_s \big| q_i^l - 1$. But this means that $(kq_s + 1) \big| q^{\alpha_i}$, and then $k = 0$ by assumption. Hence $P \trianglelefteq Q$. Since $Q \trianglelefteq G$, we have $P \trianglelefteq G$, as required.

Lemma 2.2. *Let G be a non-solvable group, then G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that K/H is a direct product of isomorphic non-abelian simple groups and $|G/K| \big| |Out(K/H)|$.*

Proof. Let G be a non-solvable group and M/N a minimal normal subgroup of G/N. Then M/N is a direct product of isomorphic non-abelian simple groups. Hence, $C_{G/N}(M/N) \bigcap M/N = Z(M/N) = 1$, and so

$$M/N \cong \frac{M/N \times C_{G/N}(M/N)}{C_{G/N}(M/N)} \leq \frac{G/N}{C_{G/N}(M/N)}.$$

Thus $\dfrac{G/N}{C_{G/N}(M/N) \times M/N}$ is a subgroup of $Out(M/N)$.

Let $K/N = C_{G/N}(M/N) \times M/N$ and $H/N = C_{G/N}(M/N)$, then $G/K \leq Out(M/N)$ and $K/H \cong M/N$, and the normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, as desired.

Lemma 2.3. (cf. [2]) *Let S be a finite non-abelian simple group with $\pi(S) \subseteq \{2, 3, 5, 7, 17\}$, then S is isomorphic to one of the following simple groups listed in Table 1. In particular, if $|\pi(Out(S))| \neq 1$, then $\pi(Out(S)) \subseteq \{2, 3\}$.*

Table 1. Finite nonabelian simple groups with $\pi(S) \subseteq \{2, 3, 5, 7, 17\}$

G	$\|G\|$	$Out(G)$	G	$\|G\|$	$Out(G)$
A_5	$2^2 \cdot 3 \cdot 5$	2	$U_3(5)$	$2^4 \cdot 3^2 \cdot 5^3 \cdot 7$	6
$L_2(7)$	$2^3 \cdot 3 \cdot 7$	2	$S_4(7)$	$2^8 \cdot 3^2 \cdot 5^2 \cdot 7^4$	2
A_6	$2^3 \cdot 3^2 \cdot 5$	4	A_9	$2^6 \cdot 3^4 \cdot 5 \cdot 7$	2
$L_2(8)$	$2^3 \cdot 3^2 \cdot 7$	3	$O_8^+(2)$	$2^{12} \cdot 3^5 \cdot 5^2 \cdot 7$	6
$L_2(17)$	$2^4 \cdot 3^2 \cdot 17$	2	$S_4(4)$	$2^8 \cdot 3^2 \cdot 5^2 \cdot 17$	4
$L_2(16)$	$2^4 \cdot 3 \cdot 5 \cdot 17$	4	$S_6(2)$	$2^9 \cdot 3^4 \cdot 5 \cdot 7$	1
A_7	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	2	$L_2(49)$	$2^4 \cdot 3 \cdot 5^2 \cdot 7^2$	4
$U_3(3)$	$2^5 \cdot 3^3 \cdot 7$	2	J_2	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$	2
He	$2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$	2	A_{10}	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7$	2
$O_8^-(2)$	$2^{12} \cdot 3^4 \cdot 5 \cdot 7 \cdot 17$	2	$L_2(31)$	$2^5 \cdot 3 \cdot 5 \cdot 31$	2
$L_4(4)$	$2^{12} \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 17$	4	$U_4(3)$	$2^7 \cdot 3^6 \cdot 5 \cdot 7$	8
A_8	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	2	$U_4(2)$	$2^6 \cdot 3^4 \cdot 5$	2
$L_3(4)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	12	$S_8(2)$	$2^{16} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17$	1

3. Proof of Theorem

In this section, we will prove the following theorem:

Theorem. *Let G be a finite group and $|G| = |S_8(2)|$, then $G \cong S_8(2)$ if and only if $L_1(G) = L_1(S_8(2))$ and $L_2(G) = L_2(S_8(2))$.*

Proof. By hypothesis, we have that $|G| = 2^{16} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17$, $L_1(G) = 2 \cdot 3^4 \cdot 5^2 \cdot 17$ and $L_2(G) = 2^{16}$. Let $\chi, \beta \in Irr(G)$ such that $\chi(1) = L_1(G) = 2 \cdot 3^4 \cdot 5^2 \cdot 17$ and $\beta(1) = L_2(G) = 2^{16}$.

We first prove that G is nonsolvable. Otherwise, G is a solvable group. Let T be a Hall-subgroup of G with order $|T| = 2^{16} \cdot 3^5 \cdot 5^2 \cdot 17$. Considering the permutation representation of G on the right cosets of T with kernel T_G, we have that $G/T_G \lesssim S_7$. Since orders of solvable subgroups of S_7 divided by 7 are 7, 14 and 21, then $|T_G|$ can be equal to one of $2^{16} \cdot 3^5 \cdot 5^2 \cdot 17$, $2^{15} \cdot 3^5 \cdot 5^2 \cdot 17$ and $2^{16} \cdot 3^4 \cdot 5^2 \cdot 17$.

If $|T_G| = 2^{16} \cdot 3^5 \cdot 5^2 \cdot 17$, let $\Delta \in Irr(T_G)$ such that $[\beta_{T_G}, \Delta] \neq 0$, then we have $\beta(1)/\Delta(1) \big| |G : T_G| = 7$, and so $\Delta(1) = 2^{16}$. Let $\Omega \in Irr(T_G)$ such that $[\chi_{T_G}, \Omega] \neq 0$, then $\chi(1)/\Omega(1) \big| |G : T_G| = 7$. Thus $\Omega(1) = 2 \cdot 3^4 \cdot 5^2 \cdot 17$. Let Q be a minimal normal subgroup of T_G, then we assert that $|Q| = 3$.

If $|Q| = 2$, then $\Delta(1) \big| |T_G : Q| = 2^{15} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17$, a contradiction.

Similarly, we can prove that $|Q| \neq 2^i$, where $2 \leq i \leq 16$.

If $|Q| = 3^2$, then $\Omega(1) \big| |T_G : Q| = 2^{16} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 17$, a contradiction.

By the similar arguments as above, we can prove that $|Q| \neq 3^m$, 5, 5^2 or 17, where $3 \leq m \leq 5$. Hence, we have that $|Q| = 3$, and so the assertion holds. Let $\lambda \in Irr(Q)$ such that $[\Delta_Q, \lambda] \neq 0$. Set $e = [\Delta_Q, \lambda]$, $t = |T_G : I_{T_G}(\lambda)|$, we have

$\Delta(1) = et\lambda(1)$. Since Q is abelian and $Aut(Q) \cong Z_2$, we have that $\lambda(1) = 1$, and hence $t \leq 2$. Thus $e \geq 2^{15}$. But $[\Delta_Q, \Delta_Q] = e^2 t \geq 2^{31} > |T_G : Q| = 2^{16} \cdot 3^4 \cdot 5^2 \cdot 17$, a contradiction to [13, 2.29].

If $|T_G| = 2^{15} \cdot 3^5 \cdot 5^2 \cdot 17$, let $\theta \in Irr(T_G)$ such that $[\chi_{T_G}, \theta] \neq 0$, then $\chi(1)/\theta(1) \big| |G : T_G| = 14$, and so $\theta(1) = 2 \cdot 3^4 \cdot 5^2 \cdot 17$. Hence $\theta(1)^2 > |T_G|$, a contradiction.

If $|T_G| = 2^{16} \cdot 3^4 \cdot 5^2 \cdot 17$, let $\vartheta \in Irr(T_G)$ such that $[\beta_{T_G}, \vartheta] \neq 0$, then $\beta(1)/\vartheta(1) \big| |G : T_G| = 21$. Hence $\vartheta(1) = 2^{16}$. But $\vartheta(1)^2 > |T_G|$, a contradiction.

Therefore, G is nonsolvable. By Lemma 2.2, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic non-abelian simple groups and $|G/K| \big| |Out(K/H)|$. Since $|G| = 2^{16} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17$, then by Table 1 we have that K/H can be isomorphic to one of the groups: A_5, A_6, $U_4(2)$, $L_2(7)$, $L_2(8)$, $U_3(3)$, A_7, $L_3(4)$, A_8, A_9, J_2, A_{10}, $S_6(2)$, $O_8^+(2)$, $L_2(17)$, $L_2(16)$, $S_4(4)$, $O_8^-(2)$, $L_4(4)$, $A_5 \times A_5$, $A_6 \times A_6$ and $S_8(2)$.

Case 1. $K/H \cong A_5$

By Lemma 2.2, we have $|G : K| = 1$ or 2. We will only discuss the case $|G : K| = 1$ since by the same reason we can prove that $|G : K| \neq 2$.

If $|G : K| = 1$, then $|H| = 2^{14} \cdot 3^4 \cdot 5 \cdot 7 \cdot 17$. If H is solvable, then H has a Hall-subgroup D with index 7. Considering the permutation representation of H on the right cosets of D with kernel D_H, we have that $H/D_H \lesssim S_7$. Since orders of solvable subgroups of S_7 divided by 7 are 7, 14 and 21. Hence $|D_H| = 2^{14} \cdot 3^4 \cdot 5 \cdot 17$, $2^{13} \cdot 3^4 \cdot 5 \cdot 17$ and $2^{14} \cdot 3^3 \cdot 5 \cdot 17$.

If $|D_H| = 2^{13} \cdot 3^4 \cdot 5 \cdot 17$, let $\alpha \in Irr(D_H)$ such that $[\beta_{D_H}, \alpha] \neq 0$, then $\beta(1)/\alpha(1) \big| |G : D_H| = 2^{13} \cdot 3 \cdot 5 \cdot 7$, and so $\alpha(1) = 2^{13}$. But $\alpha(1)^2 > |D_H|$, a contradiction.

Similarly, we can prove that $|D_H| \neq 2^{14} \cdot 3^4 \cdot 5 \cdot 17$ or $2^{14} \cdot 3^3 \cdot 5 \cdot 17$.

Therefore, H is nonsolvable. By Lemma 2.2, H has a normal series $1 \trianglelefteq N \trianglelefteq M \trianglelefteq H$ such that M/N is a direct product of isomorphic non-abelian simple groups and $|H/M| \big| |Out(M/N)|$. Since $|H| = 2^{14} \cdot 3^4 \cdot 5 \cdot 7 \cdot 17$, then by Table 1 M/N can be isomorphic to one of A_5, A_6, $U_4(2)$, $L_2(7)$, $L_2(8)$, $U_3(3)$, A_7, $L_3(4)$, A_8, A_9, $S_6(2)$, $L_2(17)$, $L_2(16)$ and $O_8^-(2)$.

Subcase 1.1. $M/N \cong A_5$

By Table 1, we have $|H/M| \big| |Out(A_5)| = 2$. Hence $|H : M| = 1$ or 2.

If $|H : M| = 1$, then $|N| = 2^{12} \cdot 3^3 \cdot 7 \cdot 17$. Let $\Delta \in Irr(N)$ such that $[\beta_N, \Delta] \neq 0$, we have that $\beta(1)/\Delta(1) \big| |G : N| = 2^4 \cdot 3^2 \cdot 5^2$. Therefore, $\Delta(1) = 2^{12}$. But $\Delta(1)^2 > |N|$, a contradiction.

If $|H : M| = 2$, then $|N| = 2^{11} \cdot 3^3 \cdot 7 \cdot 17$. By the same reason as above, N is nonsolvable. By Lemma 2.2, N has a normal series: $1 \trianglelefteq B \trianglelefteq A \trianglelefteq N$ such that A/B is a direct product of isomorphic non-abelian simple groups and $|N/A| \big| |Out(A/B)|$. Since $|N| = 2^{12} \cdot 3^3 \cdot 7 \cdot 17$, then we have that $A/B \cong L_2(7)$, $L_2(8)$, $U_3(3)$ or $L_2(17)$.

If $A/B \cong L_2(7)$, then $|N : A| = 1$ or 2. Assume that $|B| = 2^v \cdot 3^2 \cdot 17$, where $8 \leq v \leq 9$. Let $\Omega \in Irr(B)$ such that $[\beta_B, \Omega] \neq 0$, we have $\beta(1)/\Omega(1) \big| |G : B| = 2^{16-v} \cdot 3^3 \cdot 5^2 \cdot 7$. Hence $\Omega(1) = 2^v$. But $\Omega(1)^2 > |B|$, a contradiction.

Similarly, we can prove that $A/B \ncong L_2(8)$, $U_3(3)$ or $L_2(17)$.

By the same reason as Subcase 1.1, M/N cannot be isomorphic to A_5, A_6 or $U_4(2)$.

Subcase 1.2. $M/N \cong L_2(7)$

In this case, $|H/M|\big||Out(L_2(7))| = 2$. Hence $|H : M| = 1$ or 2. Since the method of proof for $|H : M| = 2$ is the same as $|H : M| = 1$, we only discuss the case $|H : M| = 1$.

If $|H : M| = 1$, we have $|N| = 2^{10} \cdot 3^3 \cdot 5 \cdot 17$. If N is solvable, then N has a Hall-subgroup R with index 5. Considering the permutation representation of N on the right cosets of R with kernel R_N, we have that $N/R_N \lesssim S_5$. Since orders of solvable subgroups of S_5 divided by 5 are 5, 10 and 20, then $|R_N|$ can be equal to one of $2^{10} \cdot 3^3 \cdot 17$, $2^9 \cdot 3^3 \cdot 17$ and $2^8 \cdot 3^3 \cdot 17$.

If $|R_N| = 2^{10} \cdot 3^3 \cdot 17$, let $\Lambda \in Irr(R_N)$ such that $[\beta_{R_N}, \Lambda] \neq 0$, one has that $\beta(1)/\Lambda(1)\big||G : R_N| = 2^6 \cdot 3^2 \cdot 5^2 \cdot 7$, and then $2^{10}\big|\Lambda(1)$. But $\Lambda(1)^2 > |R_N|$, a contradiction. Similarly, we can prove that $|R_N| \neq 2^9 \cdot 3^3 \cdot 17$.

If $|R_N| = 2^8 \cdot 3^3 \cdot 17$, let U be a minimal normal subgroup of G and $U \leq R_N$. We assert that $|U| = 3$.

If $|U| = 2$, then $\beta(1)\big||G : U| = 2^{15} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17$, a contradiction. Similarly, we have $|U| \neq 2^i$, where $2 \leq i \leq 10$. If $|U| = 3^2$, then $\chi(1)\big||G : U| = 2^{16} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 17$, a contradiction. Also, we can prove that $|U| \neq 3^3$ or 17. Therefore, $|U| = 3$, and so the assertion holds. In this case, $|R_N/U| = 2^8 \cdot 3^2 \cdot 17$. Since R_N is solvable, then there exists a Hall-subgroup V of R_N with index 9. Let $V_{R_N} = \bigcap_{x \in R_N} V^x \leq V$. Then $R_N/V_{R_N} \lesssim S_9$. Since orders of solvable subgroups of S_9 divided by 9 are 9, 18, 36, 72, 144, 288, 576 and 1152, then we have that the order of V_{R_N} may be one of $2^8 \cdot 3 \cdot 17$, $2^7 \cdot 3 \cdot 17$, $2^6 \cdot 3 \cdot 17$, $2^5 \cdot 3 \cdot 17$, $2^4 \cdot 3 \cdot 17$, $2^3 \cdot 3 \cdot 17$, $2^2 \cdot 3 \cdot 17$ and $2 \cdot 3 \cdot 17$.

If $|V_{R_N}| = 2^8 \cdot 3 \cdot 17$, let $\theta \in Irr(V_{R_N})$ such that $[\beta_{V_{R_N}}, \theta] \neq 0$, then $\beta(1)/\theta(1)\big||G : V_{R_N}| = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7$, and so $\theta(1) = 2^8$. But $\theta(1)^2 > |V_{R_N}|$, a contradiction. Similarly, we have that $|V_{R_N}| \neq 2^7 \cdot 3 \cdot 17$, $2^6 \cdot 3 \cdot 17$, $2^2 \cdot 3 \cdot 17$ or $2 \cdot 3 \cdot 17$.

If $|V_{R_N}| = 2^5 \cdot 3 \cdot 17$, Considering the permutation representation of V_{R_N} on the right cosets of P with kernel $P_{V_{R_N}}$, where $|V_{R_N} : P| = 3$, we have that $V_{R_N}/P_{V_{R_N}} \lesssim S_3$. Hence $|P_{V_{R_N}}| = 2^5 \cdot 17$ or $2^4 \cdot 17$.

If $|P_{V_{R_N}}| = 2^5 \cdot 17$, let $\mu \in Irr(P_{V_{R_N}})$ such that $[\beta_{P_{V_{R_N}}}, \mu] \neq 0$, then $\beta(1)/\mu(1)\big||G : P_{V_{R_N}}| = 2^{11} \cdot 3^4 \cdot 5^2 \cdot 7$, and so $\mu(1) = 2^5$. Thus $\mu(1)^2 > |P_{V_{R_N}}|$, which is a contradiction. If $|P_{V_{R_N}}| = 2^4 \cdot 17$, let $F \in Irr(P_{V_{R_N}})$ such that $[\chi_{P_{V_{R_N}}}, F] \neq 0$, then $\chi(1)/F(1)\big||G : P_{V_{R_N}}| = 2^{12} \cdot 3^4 \cdot 5^2 \cdot 7$, and hence $F(1) = 17$. In the case that $F(1)^2 > |P_{V_{R_N}}|$, a contradiction.

Therefore, N is nonsolvable, and so the assertion is true. By Lemma 2.2, N has a normal series $1 \trianglelefteq B \trianglelefteq A \trianglelefteq N$ such that A/B is a direct product of isomorphic non-abelian simple groups and $|N/A|\big||Out(A/B)|$. By Table 1, we have $A/B \cong A_5$, A_6, $L_2(17)$ or $L_2(16)$.

If $A/B \cong A_5$, then $|N : A| = 1$ or 2. If $|N : A| = 1$, we have $|B| = 2^8 \cdot 3^2 \cdot 17$.

Let $\zeta \in Irr(B)$ such that $[\beta_B, \zeta] \neq 0$, then $\beta(1)/\zeta(1) \big| |G : B| = 2^8 \cdot 3^3 \cdot 5^2 \cdot 7$, and hence $\zeta(1) = 2^8$. But $\zeta(1)^2 > |B|$, a contradiction. If $|N : A| = 2$, we have $|B| = 2^7 \cdot 3^2 \cdot 17$. By Lemma 2.1, the Sylow-17 subgroup B_{17} is normal in B, and so B_{17}, *char*, B. Since $H \lhd \lhd G$, then $B_{17} \trianglelefteq G$. Hence $\chi(1) \big| |G : B_{17}| = 2^{16} \cdot 3^5 \cdot 5^2 \cdot 7$, a contradicion.

Similarly, we can prove that $A/B \ncong A_6$, $L_2(17)$ or $L_2(16)$.

By the similar arguments as Subcase 1.2, we can prove that M/N cannot be isomorphic to one of $L_2(8)$, $U_3(3)$, A_7, $L_3(4)$, A_8, A_9, $S_6(2)$, $L_2(17)$ and $L_2(16)$.

Subcase 1.3. $M/N \cong O_8^-(2)$

In this case, we have $|H/M| \big| |Out(O_8^-(2))| = 2$. Thus $|H : M| = 1$ or 2.

If $|H : M| = 1$, then we have $|N| = 4$, and so $N \leq Z(H)$. In this case, $H/N \cong O_8^-(2)$. Therefore, H is a central extension of N by $O_8^-(2)$. Since the Shur Multiplier is 1, then H is a splitting central extension of N by $O_8^-(2)$, and so $H \cong Z_4 \times O_8^-(2)$. Let $\Phi \in Irr(H)$ such that $[\beta_H, \Phi] \neq 0$. Then $\beta(1)/\Phi(1) \big| |G : H| = 2^2 \cdot 3 \cdot 5$, and we have $\Phi(1) = 2^{14}$. However, by the structure of H, we see that H has no irreducible character of degree 2^{14}, which leads to a contradiction.

If $|H : M| = 2$, then $|N| = 2$, and hence $N \leq Z(H)$. In this case, M is a splitting central extension of N by $O_8^-(2)$, and so $M \cong Z_4 \times O_8^-(2)$. By [13, Clifford 6.2], M has an irreducible character of degree 2^{13}, a contradiction to the structure of M.

By the same reason as Case 1, we can prove that K/H cannot be isomorphic to one of the groups: A_6, $U_4(2)$, $L_2(7)$, $L_2(8)$, $U_3(3)$, A_7, $L_3(4)$, A_8, $S_6(2)$, $L_2(17)$, $O_8^-(2)$, $A_5 \times A_5$ and $A_6 \times A_6$.

Case 2. $K/H \cong A_7$

In this case, $|G : K| = 1$ or 2. We suppose that $|H| = 2^a \cdot 3^3 \cdot 5 \cdot 17$, where $12 \leq a \leq 13$. Let $\Omega \in Irr(H)$ such that $[\beta_H, \Omega] \neq 0$, then we have $\beta(1)/\Omega(1) \big| |G : H| = 2^{16-a} \cdot 3^2 \cdot 5 \cdot 7$, and so $\Omega(1) = 2^a$. But $\Omega(1)^2 > |H|$, a contradiction.

Similarly, we can show that $K/H \ncong A_9$, J_2, A_{10}, $O_8^+(2)$, $L_2(16)$, $S_4(4)$ or $L_4(4)$.

Case 3. $K/H \cong S_8(2)$

In this case, by order comparison we have $H = 1$. Therefore, $G \cong S_8(2)$. Thus we complete the proof of Theorem. ∎

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ON CHARACTERISTIC SUBGRAPH OF A GRAPH

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Abstract. A subgraph H of a graph G is called characteristic if $\varphi(H) = H$, for each automorphism $\varphi \in Aut(G)$. In this paper, the main properties of this new concept in algebraic graph theory are presented.

Keywords: characteristic subgraph, Boolean algebra, automorphism group.

AMS Subject Classifications: 05C10, 20E45.

1. Introduction

Throughout this paper graph means simple finite graph and we follow the terminology and notation of [1, 3] for graphs. An trivial graph on n vertices consists of n isolated vertices with no edges. This graph is denoted by $\emptyset_n$. We refer to [2] for general properties of lattices and Boolean algebras.

We assume that G is a graph and u, v are vertices of G. The edge connecting u and v is denoted by uv and the distance $d_G(u, v)$ is defined as the length of a shortest path connecting u and v in G. The eccentricity $\varepsilon(u)$ is the largest distance between u and any other vertex v of G. The maximum eccentricity over all vertices of G is called the diameter of G and denoted by $d(G)$. The minimum eccentricity is said to be radius and denoted by $r(G)$. The center of G is the set of all vertices u such that $\varepsilon(u) = r(G)$. A self-centered graph is one that the center is the same as vertex set. Consider the graph G whose vertices are the n-tuple $(b_1, b_2, ..., b_n)$

with $b_i \in \{0,1\}$ and two vertices are adjacent if the corresponding tuples differ in precisely one place. Such a graph is called a $n-$dimentional hypercube and denote it by Q_n.

Suppose G and H are graphs. H is said to be a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ and it is a spanning subgraph of G if $V(H) = V(G)$ and $E(H) \subseteq E(G)$. A subgraph H of G is called characteristic if for every automorphism $\beta \in Aut(G)$, $\beta(H) = H$. We use the notation $H \leq_{ch} G$ to show that H is a characteristic subgraph of G. It is easily seen that the graph G itself and its trivial spanning subgraph are characteristic in G.

It is easy to see that, if P_n is a path of length n with vertex set $V(P_n) = \{v_0, v_1, ..., v_n\}$, then, for each i, $1 \leq i \leq n/2$, $v_i v_{i+1} \cdots v_{n-i}$ is a characteristic subgraph of P_n.

2. Main results

In this section some basic properties of characteristic subgraphs are investigated. We first introduce an important class of characteristic subgraphs of a given graph G. Define $G[i] = \{x \in V(G) \mid \deg_G(x) = i\}$. Then it is easy to see that for each i, $1 \leq i \leq \Delta(G)$, where Δ is maximum degree of vertices, $\langle G[i] \rangle$ is a characteristic subgraph of G. We start by considering the center of a graph.

Theorem 2.1. $\langle C(G) \rangle \leq_{ch} G$.

Proof. Let $a \in C(G)$, then $\varepsilon(a) = r(G)$ and there exists $b \in V(G)$, such that $\varepsilon(a) = d(a,b) = m$, for a positive integer m. Thus, there exists a path $P : a = a_0, a_1, ..., a_m = b$ connecting a and b. Let $\varphi \in Aut(G)$ be arbitrary. Then, $\varphi(a) = \varphi(a_0), \varphi(a_1), ..., \varphi(a_m) = \varphi(b)$ is a path. Hence $\varepsilon(\varphi(a)) \geq \varepsilon(a)$. Now, suppose $\varepsilon(\varphi(a)) = k$, then there exists $d \in V(G)$, such that $d(\varphi(a), d) = k$ and $\varphi(a) = d_0, d_1, ..., d_k = d$ is a path. Since φ is an automorphism, for each $1 \leq i \leq k$ there exists $a_i \in V(G)$ such that $d_i = \varphi(a_i)$ and $\varphi(a) = \varphi(a_0), \varphi(a_1), ..., \varphi(a_k) = d$ is a path. Hence the following path exists in G, $a = a_0, a_1, ..., a_k$. Therefore, $\varepsilon(a) \geq k = \varepsilon(\varphi(a))$ and then $\varepsilon(a) = \varepsilon(\varphi(a))$. This shows that $\varphi(\langle C(G) \rangle) = \langle C(G) \rangle$, as desired. ∎

Corollary 2.2. *If T is a tree and $\varphi \in Aut(T)$ then φ has at least a fixed vertex or a fixed edge.*

Proof. By a well-known fact in graph theory the center of a tree T is a single vertex or two adjacent vertices of T. So, every automorphism of T fix a vertex or an edge of T. ∎

Theorem 2.3. *The number of characteristic spanning subgraphs of a non-trivial graph is always even.*

Proof. Suppose H is a characteristic spanning subgraph of G. For each characteristic spanning subgraph H, we define a spanning subgraph H' with edge set

$E(H') = E(G)\backslash E(H)$. Since H is a characteristic subgraph, for each $\varphi \in Aut(G)$, $\varphi(H) = H$, it means that $E(\varphi(H)) = E(H)$ and so

$$E(G)\backslash E(\varphi(H)) = E(G)\backslash E(H).$$

By the definition of $E(H')$, we have $E(\varphi(H')) = E(G)\backslash E(\varphi(H))$, and then $E(\varphi(H')) = E(H')$. This implies that $\varphi(H') = H'$ and so H' is a characteristic spanning subgraph of G. Since H and H' are different, the number of characteristic spanning subgraphs of a non-trivial graph is always even. ∎

Remark. Notice that trivial graphs have exactly one characteristic spanning subgraph.

Corollary 2.4. *Suppose G is not regular or self-centered then G has at least four characteristic subgraphs.*

Proof. Our main proof is separated into two different cases as follows:

Case 1: *G is self-centered.* In this case, there are positive integers i, j such that $\langle G[i]\rangle \leq_{ch} G$ and $\langle G[j]\rangle \leq_{ch} G$. Define H_i and H_j to be spanning subgraphs of G such that $E(H_i) = E(\langle G[i]\rangle)$ and $E(H_j) = E(\langle G[j]\rangle)$. Obviously, H_i and H_j are characteristic spanning subgraphs of G. Let $D = \{deg_G(u)|u \in V(G)\}$, if $D = \{i, j\}$, then $H_i' = H_j$ and also $H_j' = H_i$. By considering G and trivial spanning subgraph, we obtain at least four characteristic. Now if $D \supsetneqq \{i, j\}$, by Theorem 2.3, H_i' and H_j' are different characteristic subgraphs of G. By considering G and trivial spanning subgraph, we obtain at least six characteristic subgraphs.

Case 2: *G is not self-centered.* In this case $\langle C(G)\rangle \lneqq G$. Let H be a spanning subgraph of G, with edge set $E(H) = E\big(\langle C(G)\rangle\big)$. By Theorem 2.1, $H \leq_{ch} G$ and by Theorem 2.3 and considering G itself and trivial spanning subgraph of G, we obtain at least four characteristic subgraphs, as desired. ∎

Definition 2.5. A subgraph K of G is minimal characteristic in G if no proper nontrivial subgraph of K is characteristic in G.

It is easy to see that every characteristic subgraph of G contains a minimal characteristic subgraph of G.

Theorem 2.6. *Let G be a graph and H be a minimal characteristic subgraph of G. Then there exists positive integer i such that $H \leq \langle G[i]\rangle$.*

Proof. Suppose H has a vertex of degree i and $K = \{v \in V(H)|\ \deg_G(v) = i\}$. It is clear that $\langle K\rangle$ is a characteristic subgraph of G. Since $\langle K\rangle \leq H$ and H is a minimal characteristic subgraph, $\langle K\rangle = H$. It means that for each $v \in V(H)$, $\deg_G(v) = i$, proving the result. ∎

Corollary 2.7. *Let G be a graph and for a positive integer i, $G[i]$ is singleton, then $\langle G[i]\rangle$ is a characteristic subgraph of G.*

The converse of Corollary 2.7 does not hold. To do this, we assume that G is a cycle of length n. Then $G[2] = V(G)$ and $\langle G[2]\rangle$ is minimal in G.

Theorem 2.8. *A graph G is vertex transitive graph if and only if except G and its trivial spanning subgraph, it doesn't have any spanning characteristic subgraph.*

Proof. Suppose that G is a vertex transitive graph having a non-trivial characteristic subgraph H, $H \neq G$. Let $v \in V(H)$ and $u \in V(G)/V(H)$. Since G is vertex transitive, there exists $\alpha \in Aut(G)$ such that $\alpha(v) = u$, which is impossible. Conversely, we assume that G is a graph with exactly two spanning characteristic subgraph. If G is not vertex transitive, then there exist vertices u and v such that for each $\alpha \in Aut(G)$, $\alpha(v) \neq u$. Define H to be the orbit of u under the natural action of $Aut(G)$ on $V(G)$. A simple calculation shows that $\langle H\rangle$ is a characteristic subgraph of G, a contradiction. ∎

Suppose $S(G)$ and $CSS(G)$ denote the set of all spanning and characteristic spanning subgraphs of G, respectively. In the following theorem, we prove that $S(G)$ has a Boolean algebra structure.

Theorem 2.9. *Let G be a graph on n vertices, then $S(G)$ is closed under taking intersection and union and $(S(G), \cap, \cup, ')$ is a Boolean algebra in which for each element $H \in S(G)$, H' is a spanning subgraph such that $E(H') = E(G) - E(H)$. Moreover, $|S(G)| = 2^{|E(G)|}$.*

Proof. Let $H, K \in S(G)$, then $H \cap K$ and $H \cup K$ are spanning subgraphs of G. One can see that $(S(G), \cap, \cup)$ is a bounded distributive lattice in which $0_{S(G)} = \emptyset_n$ and $1_{S(G)} = G$. Moreover, $H \cap H' = \emptyset_n = 0_{S(G)}$ and $H \cup H' = G = 1_{S(G)}$ and so $S(G)$ is complemented. Therefore, $(S(G), \cap, \cup, ')$ is a Boolean algebra.

Notice that spanning subgraphs of G with exactly one edge are atoms of this Boolean algebra and conversely, each atom of this Boolean algebra has this form. Since the number of atoms of this Boolean algebra is $|E(G)|$ and every finite Boolean algebra is atomic, by a well-known theorem in Boolean algebras, $|S(G)| = 2^{|E(G)|}$. ∎

Theorem 2.10. *$CSS(G)$ is closed under taking intersection and union and $(CSS(G), \cap, \cup, ')$ is a sub-Boolean algebra of $(S(G), \cap, \cup, ')$.*

Proof. Let $H, K \in CSS(G)$. Obviously, for each $\varphi \in Aut(G)$,

$$\varphi(H \cap K) = \varphi(H) \cap \varphi(K) = H \cap K \text{ and } \varphi(H \cup K) = \varphi(H) \cup \varphi(K) = H \cup K.$$

By a similar argument as the proof of Theorem 2.9, one can show that $(CSS(G), \cap, \cup, ')$ is a Boolean algebra with $0_{CSS(G)} = \emptyset_n$ and $1_{CSS(G)} = G$. Since $CSS(G) \subseteq S(G)$, then $(CSS(G), \cap, \cup, ')$ is sub-Boolean algebra of $(S(G), \cap, \cup, ')$. ∎

Corollary 2.11. *$H \in CSS(G)$ is an atom of Boolean algebra $(CSS(G), \cap, \cup, ')$ if and only if H is a minimal characteristic subgraph of G. In particular, if G is a graph with exactly m minimal characteristic subgraph then $|CSS(G)| = 2^m$.*

Corollary 2.12. *Every characteristic spanning subgraph of G can be represented as the union of some minimal characteristic subgraphs of G.*

Proof. By Theorem 2.9 every element of a Boolean algebra is a supremum of some atoms, as desired. ∎

In the end of this paper, we construct a graph $S'(G)$ with $V(S'(G)) = S(G)$ and two spanning subgraph H and K are adjacent if and only if $E(H) \subseteq E(K)$ and $|E(H)| = |E(K)| + 1$. In the following theorem, it is proved that $S'(G)$ is a hypercube.

Theorem 2.13. *$S'(G) \cong Q_m$, where $m = |E(G)|$ and Q_m is $m-$dimentional hypercube.*

Proof. Suppose $E(G) = \{e_1, e_2, ..., e_m\}$. We can represent each spanning subgraph $H \in S(G)$, by an $m-$array $(e'_1, e'_2, ..., e'_m)$ as follows:

$$e'_i = \begin{cases} e_i & e_i \in E(H), \\ 0 & e_i \notin E(H). \end{cases}$$

Define $\psi : Q_m \longrightarrow S(G)$ by $\psi\big((a_1, a_2, ...a_m)\big) = (b_1, b_2, ...b_m)$ such that

$$b_i = \begin{cases} e_i & a_i = 1, \\ 0 & a_i = 0. \end{cases}$$

Suppose that $\alpha, \beta \in V(Q_m)$ such that $\alpha\beta \in E(Q_m)$. Then, by the definition of the hypercube, $\alpha = (t_1, t_2, ..., t_m)$ and $\beta = (s_1, s_2, ..., s_m)$ such that there exists a positive integer $k, 1 \leq k \leq m$, $t_k \neq s_k$ and for all $l \neq k$, $t_l = s_l$. Obviously, $(t_k = 0 \textit{ and } s_k = 1)$ or $(t_k = 1 \textit{ and } s_k = 0)$. Without losing generality, we assume that $t_k = 0 \textit{ and } s_k = 1$. Then, by definition of ψ:

$$\psi(\alpha) = \psi\big((t_1, t_2, ..., t_k = 0, ..., t_m)\big) = (b_1, b_2, ..., b_k = 0, ..., b_m),$$

$$\psi(\beta) = \psi\big((s_1, s_2, ..., s_k = 1, ..., s_m)\big) = (b_1, b_2, ..., b_k = e_k, ..., b_m),$$

and, by adjacency in $S'(G)$, it can easily seen that the subgraph by representing vector $\psi(\alpha) = (b_1, b_2, ..., 0, ..., b_m)$ is adjacent with subgraph by representing vector $\psi(\beta) = (b_1, b_2, ..., e_k, ..., b_m)$. Hence $\psi(\alpha)\psi(\beta) \in E(S(G))$, and so ψ is a homomorphism of graphs. Since ψ is bijective, it is isomorphism and so $S'(G) \cong Q_m$. ∎

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MINIMUM COMPLEXITY AND LOW-WEIGHT NORMAL POLYNOMIALS OVER FINITE FIELDS

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Abstract. In this paper, by using some algorithms, the distribution of the complexity of normal polynomials over finite fields of characteristic three with degree extensions up to 16 is provided. Also, the current results on the smallest known complexity for the remaining degree extensions up to 300 by using a combination of theorems and known exact values are given. In what follows, by using some algorithms, a table of normal trinomials and pentanomials with minimum complexity among all normal trinomials and pentanomials, respectively over $\mathbb{F}_3$, with their complexities for each degree n with $3^n \leq 10^{50}$ is presented. Also, either normal trinomials or pentanomials with minimum weight over $\mathbb{F}_3$, for each n, $106 \leq n \leq 300$ are listed.

Keywords: Complexity, finite fields, normal polynomial, trinomial, pentanomial.

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1. Introduction

For a prime power $q = p^s (s \in \mathbb{N})$ and a positive integer $n \geqslant 2$, let $\mathbb{F}_q$ be the finite fields with q elements and $\mathbb{F}_{q^n}$ be its extension of degree n. A normal basis of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$ is a basis of the form $N = \{\alpha, \alpha^q, \alpha^{q^2}, \ldots, \alpha^{q^{n-1}}\}$, where $\alpha \in \mathbb{F}_{q^n}$. In this case, α is called a normal element of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$, or that α generates the normal basis N. Let $\alpha_i = \alpha^{q^i}$,$0 \leq i \leq n-1$, and $T = (t_{ij})$ be the $n \times n$ matrix given by

$$\alpha \cdot \alpha_i = \sum_{j=0}^{n-1} t_{ij}\alpha_j \quad 0 \leq i \leq n-1,\ t_{ij} \in \mathbb{F}_q. \tag{1}$$

The matrix T in (1) is called the *multiplication table* of the normal basis N. If α is a normal element, the multiplication table of the normal basis generated by α is also referred as the multiplication table of α. The number of non-zero entries in T is called the *complexity* of the normal basis N, denoted by c_N. The following theorem gives a lower bound for c_N.

Theorem 1.1 (Mullin et al. [18]) *For any normal basis N of F_{q^n} over F_q, $c_N \geq 2n - 1$.*

A normal basis N is *called optimal normal basis* (ONB) if $c_N = 2n - 1$. It is well known that, when using normal bases, the speed of multiplications over $\mathbb{F}_q$ depends directly on c_N (see [6], Section 11.2.2), and it is important to use a normal basis in $\mathbb{F}_q$, with the lowest possible complexity. When no optimal normal basis exists, the problem of classifying the low complexity normal bases stays open.

A monic irreducible polynomial $F(x) \in \mathbb{F}_q[x]$ is called *normal polynomial* or *N-polynomial* if its roots form a normal basis or, equivalently, if they are linearly independent over $\mathbb{F}_q$. The elements in a normal basis are exactly the roots of some N-polynomial. Hence an N-polynomial is just another way of describing a normal basis. On the other side, the polynomials of low weight (with minimum number of nonzero coefficients) can lead to more efficient implementation of the arithmetic of $\mathbb{F}_q$. In [23] Seroussi presents a table of either irreducible trinomials or pentanomials (polynomials with three or five number of nonzero coefficients) over the finite field $\mathbb{F}_2$, For each degree n in the range $2 \leq n \leq 10,000$. In general the problem of existence of normal trinomials or pentonomials of each fixed degree n stays open. Peterson and Weldon [20] list a set of normal polynomial of degree $n \leq 34$ over $\mathbb{F}_2$. For a better understanding of the behavior of the complexities of normal elements in $\mathbb{F}_q$, tables summarizing the complexity distribution are important tools. In ([12], Section 3.3), Jungnickel provides a table with minimum and maximum complexities of normal elements in $\mathbb{F}_{2^n}$, for each n smaller than or equal to 30. In ([12], Section 5.4), for $31 \leq n \leq 60$, he provides a table due to Geiselmann [8] with the lowest complexities found via free polynomials. Masuda, Moura, Panario and Thomson [15] provide the distribution of the complexity of normal elements for binary fields with degree extensions up to 39, using some algorithms that test field elements. They also provide the current results on the smallest known complexity for the remaining degree extensions up to 528, using a combination of constructive theorems and known exact values.

Morgan and Mullen [17] published tables of primitive normal polynomials with three, four or five number of nonzero coefficients of degree n over $\mathbb{F}_p$ for each $p^n \leq 10^{50}$ with $p \leq 97$.

The software and hardware implementation of arithmetic in the fields with characteristic three has been intensively studied in recent years. Some results in the hardware or software implementations in the finite fields with characteristic three can be found in [9], [10], [11], [13], [19]. The elements of the finite fields $\mathbb{F}_{3^s}$ can be represented using a normal basis.

In this paper, using some algorithms that efficiently test normality of polynomials and compute the complexities of them, we provide the distribution of the complexity of normal polynomials for finite fields of characteristic three with degree extensions up to 16. We also provide current results on the lowest known complexity for the remaining degree extensions up to 300 by using a combination of theorems and known exact values. In the continue, using some algorithms, a table of normal trinomials and pentanomials with minimum complexity among all normal trinomials and pentanomials, respectively over $\mathbb{F}_3$, with their complexities for each degree n with $3^n \leq 10^{50}$ is presented. Also, either normal trinomials or pentanomials with minimum weight over $\mathbb{F}_3$, for each n, $106 \leq n \leq 300$ are listed.

2. Preliminaries

We need the following results for next study. The following proposition determine that whether an irreducible polynomial is normal or not.

Proposition 2.1 (Gao, see [16], Theorem 4.6) *Let $f(x)$ be an irreducible polynomial of degree n over $\mathbb{F}_q$. Let $\alpha_i = \alpha^{q^i}$, and $t_i = Tr_{q^n|q}(\alpha_0\alpha_i)$, $0 \leq i \leq n-1$, where $\alpha \in \mathbb{F}_{q^n}$ is a root of $f(x)$. Then $f(x)$ is a normal polynomial over $\mathbb{F}_q$ if and only if the polynomial $N(x) = \sum_{i=0}^{n-1} t_i x^i \in \mathbb{F}_q[x]$ is relatively prime to $x^n - 1$.*

Notation 2.2 *Let $f(x) = \sum_{i=0}^{n} c_i x^i$ be an N-polynomial over $\mathbb{F}_q$. Then $c_{n-1} \neq 0$.*

Proposition 2.3 ([16], Corollary 4.19) *Let $n{=}p^e$ for some e, and $f(x){=}\sum_{i=0}^{n} c_i x^i$ be an irreducible polynomial over $\mathbb{F}_q$. Then $f(x)$ is an N-polynomial if and only if $c_{n-1} \neq 0$.*

The number of normal polynomials of degree n over F_q will be compute by the following proposition.

Proposition 2.4 (Ore, [12], Theorem 3.1.5) *Let q be a power of the prime p, let n be a positive integer and write $n = p^a m$, where p does not divide m. Then the number of normal bases of F_{q^n} over F_q equals*

$$\frac{1}{n}\ \phi_q(x^n - 1) = (q^n/n) \prod_{d|m} (1 - q^{-o_d(q)})^{\phi(d)/o_d(q)},$$

where $\phi_q(f)$ is the number of polynomials of degree smaller than the degree of f which are relatively prime to f, $o_n(a)$ is the order of a modulo n, and $\phi(d)$ is the number of positive integers smaller than d that are relatively prime to d.

Proposition 2.5 ([16], Theorem 5.2) *Let $n+1$ is prime and q is primitive in Z_{n+1}. Then $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$ has an optimal normal basis.*

Proposition 2.6 ([12], Theorem 3.3.13) *Let α and β generate normal bases A and B for $\mathbb{F}_{q^m}$ and $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$, respectively. Assume that m and n are coprime and put $\gamma = \alpha \cdot \beta$. Then*

1. *γ generates a normal bases N for $\mathbb{F}_{q^{mn}}$ over $\mathbb{F}_q$.*
2. *$c_N = c_A \cdot c_B$.*

Let n, k be integers such that $r = nk+1$ is a prime, and let q be a prime power with $gcd(q, r) = 1$. Let β be a primitive rth root of unity in $\mathbb{F}_{q^{nk}}$. Furthermore, let G be the unique subgroup of order k in Z_r^*. The element $\alpha = \sum_{\gamma \in G} \beta^\gamma$ is called a Gauss period of type (n, k) over $\mathbb{F}_q$. In fact $\alpha \in \mathbb{F}_{q^n}$ and is a normal element of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$ if and only if $gcd(e, n) = 1$, where e is the index of q modulo r.

In this case, a normal basis generated by a normal Gauss periods is denoted by *GNB*. We will use of the normal Gauss periods over $\mathbb{F}_3$, listed in the Table 3 in [7], for construction of our tables.

Proposition 2.7 (see [1] and [5]) *Let α be a normal Gauss period of type (n, k) over $\mathbb{F}_3$ and N is the normal basis generated by α. Then*

1. $c_N \leq (k+1)n - k$.
2. *if* $k \equiv 0 \ (mod\ 3)$ *then* $c_N \leq nk - 1$.
3. *if* $k = 2$ *or 3 then* $c_N = 3n - 2$.
4. *if* $k = 4$ *then* $c_N \leq 5n - 7$.
5. *if* $k = 5$ *or 6 then* $c_N \leq 6n - 11$.

We give two classes of theorems which allows us to construct even more low complexity normal bases in subfields of finite fields containing optimal normal bases and normal basis generated by a Gauss period.

Proposition 2.8 ([5], Theorem 4.2) *Let α be a type $(n, 3)$ Gauss period generating a normal basis N of $\mathbb{F}_{3^n}$ over $\mathbb{F}_3$. Further, suppose $n = ml$ be integers, $\beta = Tr_{3^n|3^m}(\alpha)$ and c_N is the complexity of the normal basis generated by β. Then*

1. *If m and l are odd, then* $c_N \leq 3lm - 2$.
2. *Otherwise,* $c_N \leq 3lm - 1$.

Proposition 2.9 [4] *Let $\alpha \in \mathbb{F}_{q^n}$ generate an optimal normal basis of Type I of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$, $n > 2$, and $\beta = Tr_{q^n|q^m}(\alpha)$ with $m = \frac{n}{k}$ and $k \leq m$. Suppose c_N is the complexity of the normal basis generated by β. Then*

1. *If m is even and k is odd, then* $c_N \leq (k+1)m - 3k + 2$.
2. *Otherwise* $c_N \leq (k+1)m - k$.

The normal basis generated by Propositions 2.8 and 2.9 are denoted by *Trace GNB* and *Trace ONB*, respectively.

3. Algorithms

We assume that the arithmetic in $\mathbb{F}_q$ is given. The cost measure of an algorithm will be the number of operations in $\mathbb{F}_q$. The algorithms in this paper use basic polynomial operations like products, divisions, gcd's and to solve linear equation systems. We consider in this paper exclusively FFT (Fast Fourier Transform) based arithmetic, similar results hold for classical arithmetic. Let $M(n) = n \log(n) \log\log(n)$. The cost of multiplying and dividing two polynomials of degree at most n using fast arithmetic ([3], [21], [22]) can be taken as $O(M(n))$ and $O(\log(n)M(n))$, respectively, and the cost of gcd's between two polynomials of degree at most n can be taken as $O(\log(n)M(n))$ operations in $\mathbb{F}_q$.

The cost of computing $h^q \pmod{f}$ by using the classical *repeated squaring* method (see [14], pp. 441-442), where h is a polynomial over $\mathbb{F}_q$ of degree less than n, is $O(\log(q)M(n))$ in $\mathbb{F}_q$ using FFT based methods. The cost of solving a system of n linear equations with n variables is $O(n^2)$.

The following algorithm is a variant of Algorithm 2.1 in [2]. This algorithm efficiently tests normality of a given irreducible polynomial over $\mathbb{F}_q$.

Algorithm 3.1

Input: *An irreducible polynomial* $f(x) = \sum_{i=0}^{n} c_i x^i \in \mathbb{F}_q[x]$.

output: *Either 'f(x) is normal' or 'f(x) is not normal'.*

1) *if* $c_{n-1} \neq 0$
2) $\quad n := deg(f(x))$;
3) $\quad$ *if* $n = p^e$, *for some* $e \in \mathbb{N}$, *then 'f(x) is a normal polynomial'*
4) $\quad$ *else*
5) $\quad\quad$ *for* $j := 0$ *to* $n-1$
6) $\quad\quad$ $t_j := \sum_{i=0}^{n-1} ((x^{q^j+1} \pmod{f(x)})^{q^i} \pmod{f(x)})$;
7) $\quad\quad$ *end for*
8) $\quad$ $g := gcd\left(\sum_{j=0}^{n-1} t_j x^j, x^n - 1\right)$;
9) $\quad$ *if* $g = 1$, *then 'f(x) is a normal polynomial'*
10) $\quad$ *else 'f(x) is not a normal polynomial';*
11) $\quad$ *end if*
12) $\quad$ *end if*
13) *else 'f(x) is not a normal polynomial';*
14) *end if*

Obviously, by Notation 2.2, the irreducible polynomial $f(x) = \sum_{i=0}^{n} c_i x^i \in \mathbb{F}_q[x]$ is not an N-polynomial over F_q if $c_{n-1} = 0$. Also according to Proposition 2.3, if $n = p^e$, for some $e \in \mathbb{N}$, then $f(x) = \sum_{i=0}^{n} c_i x^i \in \mathbb{F}_q[x]$ is a normal polynomial over $\mathbb{F}_q$ if and only if $c_{n-1} \neq 0$. So the correctness of the above algorithm is based on Proposition 2.1.

Theorem 3.2 *The above algorithm correctly tests normality of irreducible polynomials, and uses* $O(nM(n)\ (n\log(q) + \log(n)))$ *operations in* $\mathbb{F}_q$.

Proof. The basic idea of this algorithm is to compute $x^{q^j+1} \pmod{f(x)}$ and $(x^{q^j+1})^{q^i} \pmod{f(x)}$ for each $0 \leq i, j \leq n-1$ by repeated squaring method, and then to take the correspondent *gcd*.

Obviously, line 6 of the algorithm computes $t_j = Tr_{q^n|q}(\alpha^{q^j+1})$, for each $0 \leq j \leq n-1$, where $\alpha \in \mathbb{F}_{q^n}$ is a root of $f(x)$. Hence, according to Proposition 2.1, the above algorithm correctly tests normality of irreducible polynomials. Our algorithm computes $x^{q^j} \pmod{f(x)}$, $x^{q^j} \cdot x \pmod{f(x)}$ and $(x^{q^j} \cdot x)^{q^i} \pmod{f(x)}$ for each $0 \leq i, j \leq n-1$. The number of polynomial multiplications in this algorithm, to compute all powers using repeated squaring is $(n-1)\log q + (n-1)^2 \log q$. So the cost of all exponentiations is $O(n^2 M(n)\log(q))$. The number of polynomial multiplications in this algorithm, to compute all $x^{q^j} \cdot x \pmod{f(x)}$, $0 \leq j \leq n-1$

is $n+n\log n$, and so the cost of compute them is $O(nM(n)\log(n))$. Therefore, in the worst case, the total cost of our algorithm, with consideration the cost of a gcd in the algorithm, is $O(nM(n)(n\log(q)+\log(n)))$ using FFT based multiplication algorithms. ∎

Since the elements in a normal basis are exactly the roots of some N-polynomials, there is a canonical one-to-one correspondence between N-polynomials and normal basis. So we may denote the complexity of the normal polynomial $f(x)$ by c_N, when the elements in the normal basis N are exactly the roots of $f(x)$. The following algorithm which is given in [2], computes the complexity of normal polynomials over $\mathbb{F}_q$.

Algorithm 3.3
Input: *An N-polynomial $f(x)$ of degree n over $\mathbb{F}_q$.*
Out put: *The complexity c_N of normal polynomial $f(x)$.*

1) $c_N := 0$;
2) *for* $i := 1$ *to* n
3) *find solution* $T_i = (t_{i1}, t_{i2}, ..., t_{in})$ *of the following linear equation system:*

$$k_i(x) = \sum_{j=1}^{n} t_{ij} r_j(x), \tag{2}$$

where $r_j(x) = x^{q^{j-1}} \pmod{f(x)}$ *and* $k_i(x) = x^{q^{i-1}+1} \pmod{f(x)}$;
4) *for* $j := 1$ *to* n
5) *if* $t_{ij} \neq 0$ *then* $c_N = c_N + 1$;
6) *end for;*
7) *end for;*
8) *return* c_N.

Theorem 3.4 *The above algorithm computes the complexity of a normal polynomial, and uses $O(n(n^2 + M(n)\log(qn)))$ operations in $\mathbb{F}_q$.*

Proof. The basic idea of this algorithm is to compute $x^{q^{i-1}} \pmod{f(x)}$ for each $1 \leq i \leq n$ by repeated squaring, and then to solve the correspondent linear equation system. Clearly, line 3 of the algorithm solves the linear equation system

$$\alpha^{q^{i-1}+1} = \sum_{j=1}^{n} t_{ij} \alpha^{q^{j-1}} \quad 1 \leq i \leq n, \; t_{ij} \in \mathbb{F}_q,$$

where $\alpha \in \mathbb{F}_{q^n}$ is a root of $f(x)$. Hence, by (2), the above algorithm correctly computes the complexity of a normal polynomial.

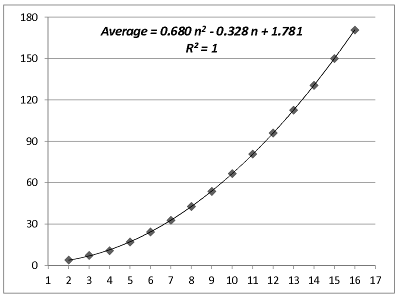
This algorithm computes $x^{q^j} \pmod{f(x)}$ and $x^{q^j} \cdot x \pmod{f(x)}$ for each $0 \leq j \leq n-1$. The number of polynomial multiplications in this algorithm, to compute all powers using repeated squaring is $(n-1)\log q$. So the cost of all exponentiations is $O(nM(n)\log q)$. The number of polynomial multiplications in this algorithm to compute all $x^{q^j} \cdot x \pmod{f(x)}$, $0 \leq j \leq n-1$ is $n + n\log n$, and so the cost of to compute them is $O(nM(n)\log(n))$. The cost of solving n times a system of n linear equations and n variables is $O(n^3)$. Therefore, the total cost of this algorithm is $O(n(n^2 + M(n)\log(qn)))$ using FFT based multiplication algorithms. ∎

4. Tables

By using the Algorithms 3.1 and 3.3, in Table 1, a statistical table consist the number of normal polynomials found ($No = \frac{1}{n}\,\phi_3(x^n - 1)$), minimum and maximum complexities (Min, Max), average, variance and standard deviation of their complexities ($Avg, Var, Std.\ Dev$) is given. Using Table 1 the distribution of the complexity of normal polynomials for finite fields of characteristic three with degree extensions up to 16 is provided. The Tables 3 and 4 give us the current results on the smallest known complexity for the degree extensions from 17 up to 300 by using a combination of theorems and known exact values. For convenience, in the Table 2, using the Proposition 2.5, we list all the values of $n \leq 10000$ for which there is an optimal normal polynomial of degree n over $\mathbb{F}_3$. We also provide a table, in the Table 5, of normal trinomials and pentanomials with minimum complexity among all normal trinomials and pentanomials, respectively over $\mathbb{F}_3$, with their complexities for each degree n with $3^n \leq 10^{50}$. In the continue, either normal trinomials or pentanomials with minimum weight over $\mathbb{F}_3$, for each n, $106 \leq n \leq 300$ are presented. For this, in our search procedure, for a given n, $106 \leq n \leq 300$, we first tried to locate a normal trinomial of degree n over $\mathbb{F}_3$. Failing this, a search was conducted among pentanomials. Among those polynomials, the polynomial $f(x)$ listed is such that the degree of $f(x) - x^n - a_{n-1}x^{n-1}$ for each $a_{n-1} \in \mathbb{F}_3^*$ is lowest. The results of this search procedure in a table in the Table 6 are listed.

The choice of n with $3^n \leq 10^{50}$, and $106 \leq n \leq 300$ as the stopping point for the Tables 5 and 6 respectively, are quite arbitrary, and it should not be particularly difficult to extend the tables to larger values of n. In the Tables 5 and 6 only the nonzero terms are represented. For example the polynomial $x^{35} + x^{34} + 2x^{23} + 2x^{10} + 1$ over $\mathbb{F}_3$ is represented as $35:1, 34:1, 23:2, 10:2, 0:1$. In the Table 5 the first column, headed by $f(x)$, contains the minimum complexity trinomials and pentanomials, the second column headed by c_N, contains the corresponding value of the complexity of $f(x)$, and the third column, headed by No, contains the number of all normal trinomials and pentanomials, which are founded by Algorithm 3.1.

n	No	Min	Max	Avg	Var	Std.Dev	Notes
2	2	4	4	4	0	0	
3	6	7	8	7.3333	0.2222	0.4714	
4	8	7	13	10.75	6.1875	2.4875	Optimal
5	32	13	20	17.0625	7.8086	2.7944	
6	54	11	32	24.3333	22.6667	4.761	Optimal
7	208	24	41	32.7596	15.6249	3.9528	
8	256	22	55	42.8203	29.7099	5.4507	
9	1458	25	69	53.7407	35.7723	5.981	
10	2560	28	85	66.6336	45.0837	6.7144	
11	10648	31	109	80.7938	51.2433	7.1584	
12	17496	49	127	96.071	62.7923	7.9242	
13	70304	58	145	112.6281	73.6445	8.5816	
14	151424	40	168	130.6227	85.6254	9.2534	
15	629856	43	199	150.0063	97.3465	9.8664	
16	819200	31	220	170.6615	110.8023	10.5263	Optimal



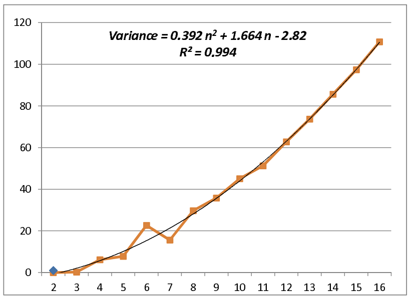


Table 1: The statistics on the complexity of normal polynomials of degree $n \leq 16$ over F_3

In the Table 6, the columns, headed by $f(x)$, contains either normal trinomials or pentanomials with minimum weight. All results in the tables were generated with some matlab programs based on the algorithms, which were stated in the Section . All tables and programs are available in electronic form from the authors.

4	6	16	18	28	30	42	52	78	88
100	112	126	136	138	148	162	172	196	198
210	222	232	256	268	280	282	292	316	330
352	378	388	400	448	460	462	486	508	520
556	568	570	592	606	616	630	640	652	676
690	700	738	750	772	796	808	810	820	822
856	858	880	906	928	940	952	976	1012	1038
1048	1060	1062	1086	1096	1108	1122	1192	1216	1228
1230	1276	1278	1290	1300	1326	1360	1372	1408	1422
1432	1446	1458	1480	1482	1492	1552	1566	1578	1600
1612	1626	1636	1662	1696	1698	1708	1720	1722	1732
1746	1830	1888	1900	1912	1948	1950	1972	1986	1996
1998	2010	2068	2080	2082	2128	2140	2142	2152	2212
2236	2238	2272	2308	2310	2332	2346	2356	2370	2380
2392	2416	2466	2476	2502	2538	2548	2608	2632	2646
2656	2658	2682	2692	2706	2718	2728	2730	2740	2752
2766	2776	2788	2800	2836	2860	2896	2908	2956	2968
3040	3088	3136	3162	3208	3256	3258	3270	3306	3328
3330	3388	3390	3412	3448	3460	3462	3532	3546	3556
3558	3570	3580	3582	3592	3616	3642	3676	3700	3726
3760	3796	3820	3822	3832	3916	3918	3928	3930	3942
3988	4000	4002	4012	4026	4048	4072	4132	4156	4158
4216	4218	4228	4230	4240	4242	4252	4288	4326	4336
4348	4362	4372	4396	4408	4420	4422	4446	4456	4480
4492	4506	4516	4518	4566	4602	4636	4638	4648	4650
4662	4672	4722	4758	4792	4816	4830	4876	4888	4902
4936	4972	4986	4998	5008	5020	5022	5080	5118	5188
5236	5260	5272	5296	5308	5332	5346	5380	5392	5406
5416	5418	5430	5440	5442	5476	5478	5502	5562	5572
5646	5656	5668	5682	5692	5716	5740	5778	5800	5812
5826	5848	5860	5896	5922	5980	6006	6028	6052	6088
6112	6150	6162	6172	6196	6198	6220	6256	6268	6316
6328	6342	6352	6366	6378	6388	6426	6448	6450	6472
6546	6568	6570	6606	6618	6652	6688	6690	6736	6760
6762	6822	6832	6856	6868	6870	6906	6916	6976	7000
7012	7038	7108	7120	7158	7192	7206	7228	7242	7252
7348	7410	7432	7456	7458	7516	7528	7540	7576	7602
7648	7672	7698	7722	7758	7792	7816	7828	7866	7876
7878	7900	7926	7936	7948	8008	8058	8068	8080	8092
8116	8166	8178	8236	8262	8272	8286	8296	8310	8368
8418	8428	8430	8442	8466	8536	8538	8562	8572	8596
8598	8608	8622	8646	8668	8692	8718	8730	8740	8752
8836	8838	8848	8860	8862	8886	8922	8968	8970	9006
9028	9040	9042	9066	9090	9126	9136	9150	9160	9172
9186	9198	9208	9256	9280	9282	9292	9318	9376	9390
9402	9412	9460	9462	9472	9496	9510	9520	9532	9546
9628	9630	9642	9676	9678	9688	9738	9748	9786	9810
9832	9870	9882	9906	9928	9966				

Table 2: All values of $n \leq 10000$ for which there is an optimal normal polynomial of degree n over $\mathbb{F}_3$

n	min	property	Method
17	91	Random	
18	35	Optimal	Pro 2.5
19	176	Random	
20	91	a.b(4,5)	pro 2.6
21	61	Trace ONB(k=2)	pro 2.9
22	64	GNB(k=3)	pro 2.7
23	137	Trace GNB (46, 3)	pro 2.8
24	154	a.b(3,8)	pro 2.6
25	121	Trace ONB(k=4)	pro 2.9
26	76	Trace ONB(k=2)	pro 2.9
27	183	Trace ONB(k=6)	pro 2.9
28	55	Optimal	pro 2.5
29	253	Trace ONB(k=8)	pro 2.9
30	59	Optimal	pro 2.5
31	537	Random	
32	280	GNB(k=8)	pro 2.7
33	187	GNB(k=6)	pro 2.7
34	166	Trace ONB(k=4)	pro 2.9
35	209	Trace GNB (70, 3)	pro 2.8
36	175	a.b(9,4)	pro 2.6
37	181	Trace ONB(k=4)	pro 2.9
38	569	GNB(k=15)	pro 2.7
39	115	Trace ONB(k=2)	pro 2.9
40	286	a.b(8,5)	pro 2.6
41	738	Trace GNB (246, 3)	pro 2.8
42	83	Optimal	pro 2.5
43	211	Trace ONB(k=4)	pro 2.9
44	130	Trace ONB(k=2)	pro 2.9
45	325	a.b(9,5)	pro 2.6
46	136	GNB(k=3)	pro 2.7
47	323	Trace ONB(k=6)	pro 2.9
48	217	a.b(3,16)	pro 2.6
49	241	Trace ONB(k=4)	pro 2.9
50	148	Trace ONB(k=2)	pro 2.9
51	637	a.b(3,17)	pro 2.6
52	103	Optimal	pro 2.5
53	1093	Trace ONB(k=20)	pro 2.9
54	209	Trace ONB(k=3)	pro 2.9
55	319	GNB(k=6)	pro 2.7
56	166	Trace ONB(k=2)	pro 2.9
57	617	Trace ONB(k=10)	pro 2.9
58	283	GNB(k=4)	pro 2.7
59	1103	Trace ONB(k=10)	pro 2.9
60	637	a.b(3,20)	pro 2.6
61	1021	Trace ONB(k=16)	pro 2.9
62	1301	GNB(k=21)	pro 2.7
63	187	Trace ONB(k=2)	pro 2.9
64	313	GNB(k=4)	pro 2.7
65	577	Trace ONB(k=8)	pro 2.9
66	196	GNB(k=3)	pro 2.7
67	331	Trace ONB(k=4)	pro 2.9
68	202	Trace ONB(k=2)	pro 2.9
69	205	Trace ONB(k=2)	pro 2.9
70	208	GNB(k=3)	pro 2.7
71	631	Trace ONB(k=8)	pro 2.9
72	550	a.b(9,8)	pro 2.6
73	361	Trace ONB(k=4)	pro 2.9
74	220	Trace ONB(k=2)	pro 2.9
75	847	a.b(3,25)	pro 2.6
76	1232	a.b(4,19)	pro 2.6
77	744	a.b(7,11)	pro 2.6
78	155	Optimal	pro 2.5
79	391	Trace ONB(k=4)	pro 2.9
80	467	Trace ONB(k=5)	pro 2.9
81	241	Trace ONB(k=2)	pro 2.9
82	737	GNB(k=9)	pro 2.7
83	497	Trace GNB (166, 3)	pro 2.8
84	385	a.b(3,28)	pro 2.6
85	1183	a.b(5,17)	pro 2.6
86	256	Trace ONB(k=2)	pro 2.9
87	1771	a.b(3,29)	pro 2.6

n	min	property	Method
88	175	Optimal	pro 2.5
89	3785	Trace ONB(k=42)	pro 2.9
90	713	GNB(k=7)	pro 2.7
91	1392	a.b(7,13)	pro 2.6
92	541	GNB(k=5)	pro 2.7
93	3759	a.b(3,31)	pro 2.6
94	369	Trace ONB(k=3)	pro 2.9
95	695	Trace ONB(k=6)	pro 2.9
96	1960	a.b(3,32)	pro 2.6
97	1633	Trace ONB(k=16)	pro 2.9
98	292	Trace ONB(k=2)	pro 2.9
99	295	Trace ONB(k=2)	pro 2.9
100	199	Optimal	Pro 2.5
101	701	Trace ONB(k=6)	pro 2.9
102	1213	GNB(k=11)	pro 2.7
103	4387	Trace ONB(k=42)	pro 2.9
104	611	Trace ONB(k=5)	pro 2.9
105	313	Trace ONB(k=2)	pro 2.9
106	1156	GNB(k=10)	pro 2.7
107	955	Trace ONB(k=8)	pro 2.9
108	1281	a.b(4,27)	pro 2.6
109	3349	Trace ONB(k=30)	pro 2.9
110	433	Trace ONB(k=3)	pro 2.9
111	1267	a.b(3,37)	pro 2.6
112	223	Optimal	Pro 2.5
113	3473	Trace ONB(k=30)	pro 2.9
114	673	GNB(k=5)	pro 2.7
115	571	Trace ONB(k=4)	pro 2.9
116	346	Trace ONB(k=2)	pro 2.9
117	5685	Trace ONB(k=48)	pro 2.9
118	1155	Trace ONB(k=9)	pro 2.9
119	1063	Trace ONB(k=8)	pro 2.9
120	2002	a.b(3,40)	pro 2.6
121	361	Trace ONB(k=2)	pro 2.9
122	1090	Trace ONB(k=8)	pro 2.9
123	727	GNB(k=6)	pro 2.7
124	1723	GNB(k=13)	pro 2.7
125	749	Trace GNB (250, 3)	pro 2.8
126	251	Optimal	Pro 2.5
127	631	Trace ONB(k=4)	pro 2.9
128	382	Trace ONB(k=2)	pro 2.9
129	1409	Trace ONB(k=10)	pro 2.9
130	641	GNB(k=4)	pro 2.7
131	785	Trace GNB (262, 3)	pro 2.8
132	910	a.b(3,44)	pro 2.6
133	4224	a.b(7,19)	pro 2.6
134	400	Trace ONB(k=2)	pro 2.9
135	939	Trace ONB(k=6)	pro 2.9
136	271	Optimal	Pro 2.5
137	953	Trace ONB(k=6)	pro 2.9
138	275	Optimal	Pro 2.5
139	691	Trace ONB(k=4)	pro 2.9
140	418	Trace ONB(k=2)	pro 2.9
141	2261	a.b(3,47)	pro 2.6
142	706	Trace ONB(k=4)	pro 2.9
143	1798	a.b(11,13)	pro 2.6
144	775	a.b(9,16)	pro 2.6
145	1585	GNB(k=10)	pro 2.7
146	436	Trace ONB(k=2)	pro 2.9
147	1607	GNB(k=10)	pro 2.7
148	295	Optimal	Pro 2.5
149	1333	Trace ONB(k=8)	pro 2.9
150	889	GNB(k=5)	pro 2.7
151	1051	Trace ONB(k=6)	pro 2.9
152	1360	Trace ONB(k=8)	pro 2.9
153	2275	a.b(9,17)	pro 2.6
154	609	Trace ONB(k=3)	pro 2.9
155	4775	Trace ONB(k=30)	pro 2.9
156	721	a.b(3, 52)	pro 2.6
157	4525	Trace ONB(k=28)	pro 2.9
158	472	Trace ONB(k=2)	pro 2.9

Table 3: The smallest known complexities of normal polynomials of degree $17 \leq n \leq 300$ over $\mathbb{F}_3$

n	*min*	*property*	*Method*
159	5531	GNB(k=34)	pro 2.7
160	796	Trace ONB(k=4)	pro 2.9
161	4961	Trace ONB(k=30)	pro 2.9
162	323	Optimal	Pro 2.5
163	811	Trace ONB(k=4)	pro 2.9
164	973	GNB(k=5)	pro 2.7
165	493	Trace ONB(k=2)	pro 2.9
166	496	GNB(k=3)	pro 2.7
167	7471	Trace ONB(k=44)	pro 2.9
168	1162	a.b(3,56)	pro 2.6
169	841	Trace ONB(k=4)	pro 2.9
170	1523	GNB(k=8)	pro 2.7
171	2051	GNB(k=12)	pro 2.7
172	343	Optimal	pro 2.5
173	1037	Trace GNB (346, 3)	pro 2.8
174	1565	GNB(k=9)	pro 2.7
175	871	Trace ONB(k=4)	pro 2.9
176	526	Trace ONB(k=2)	pro 2.9
177	1234	Trace ONB(k=6)	pro 2.9
178	2669	GNB(k=15)	pro 2.7
179	1603	Trace ONB(k=8)	pro 2.9
180	2275	a.b(9, 20)	pro 2.6
181	1261	Trace ONB(k=6)	pro 2.9
182	2716	GNB(k=14)	pro 2.7
183	2003	Trace ONB(k=10)	pro 2.9
184	1465	GNB(k=7)	pro 2.7
185	1657	Trace ONB(k=8)	pro 2.9
186	2789	GNB(k=15)	pro 2.7
187	1303	Trace ONB(k=6)	pro 2.9
188	1115	Trace ONB(k=5)	pro 2.9
189	565	Trace ONB(k=2)	pro 2.9
190	563	GNB(k=3)	pro 2.7
191	3991	Trace ONB(k=20)	pro 2.9
192	2191	a.b(3,64)	pro 2.6
193	961	Trace ONB(k=4)	pro 2.9
194	580	Trace ONB(k=2)	pro 2.9
195	2135	GNB(k=10)	pro 2.7
196	391	Optimal	pro 2.5
197	3725	Trace ONB(k=18)	pro 2.9
198	395	Optimal	pro 2.5
199	991	Trace ONB(k=4)	pro 2.9
200	598	Trace ONB(k=2)	pro 2.9
201	2201	Trace ONB(k=10)	pro 2.9
202	801	Trace ONB(k=3)	pro 2.9
203	2435	GNB(k=12)	pro 2.7
204	1444	a.b(3,68)	pro 2.6
205	1021	Trace ONB(k=4)	pro 2.9
206	4471	Trace ONB(k=21)	pro 2.9
207	3425	a.b(9,23)	pro 2.6
208	2278	GNB(k=10)	pro 2.7
209	5456	a.b(11,19)	pro 2.6
210	419	Optimal	pro 2.5
211	9032	Trace ONB(k=42)	pro 2.9
212	1261	GNB(k=5)	pro 2.7
213	1485	Trace ONB(k=6)	pro 2.9
214	2344	Trace ONB(k=10)	pro 2.9
215	1499	Trace ONB(k=6)	pro 2.9
216	4026	a.b(8,27)	pro 2.6
217	12888	a.b(7,31)	pro 2.6
218	3224	GNB(k=15)	pro 2.7
219	2527	a.b(3,73)	pro 2.6
220	1093	GNB(k=4)	pro 2.7
221	1541	Trace ONB(k=6)	pro 2.9
222	443	Optimal	pro 2.5
223	6883	Trace ONB(k=30)	pro 2.9
224	670	Trace ONB(k=2)	pro 2.9
225	3025	a.b(9,25)	pro 2.6
226	3389	GNB(k=15)	pro 2.7
227	N/A		
228	4319	a.b(4,57)	pro 2.6
229	7069	Trace ONB(k=30)	pro 2.9

n	*min*	*property*	*Method*
230	688	Trace ONB(k=2)	pro 2.9
231	784	Trace ONB(k=2)	pro 2.9
232	463	Optimal	pro 2.5
233	7657	Trace ONB(k=32)	pro 2.9
234	1393	GNB(k=5)	pro 2.7
235	1171	Trace ONB(k=4)	pro 2.9
236	2116	Trace ONB(k=8)	pro 2.9
237	1653	Trace ONB(k=6)	pro 2.9
238	1183	GNB(k=4)	pro 2.7
239	2143	Trace ONB(k=8)	pro 2.9
240	1333	a.b(15,16)	pro 2.6
241	1681	Trace ONB(k=6)	pro 2.9
242	1444	a.b(2,121)	pro 2.6
243	727	Trace ONB(k=2)	pro 2.9
244	1216	Trace ONB(k=4)	pro 2.9
245	3133	a.b(5,49)	pro 2.6
246	736	GNB(k=3)	pro 2.7
247	1723	Trace ONB(k=6)	pro 2.9
248	2965	Trace ONB(k=11)	pro 2.9
249	2240	GNB(k=9)	pro 2.7
250	748	GNB(k=3)	pro 2.7
251	4751	Trace ONB(k=18)	pro 2.9
252	1309	a.b(4,63)	pro 2.6
253	1258	GNB(k=4)	pro 2.7
254	760	Trace ONB(k=2)	pro 2.9
255	3811	Optimal	pro 2.5
256	511	Optimal	pro 2.5
257	N/A		
258	1573	GNB(k=5)	pro 2.7
259	4344	a.b(7,37)	pro 2.6
260	778	Trace ONB(k=2)	pro 2.9
261	1551	GNB(k=6)	pro 2.7
262	785	GNB(k=3)	pro 2.7
263	1835	Trace ONB(k=6)	pro 2.9
264	1225	a.b(3,88)	pro 2.6
265	1321	Trace ONB(k=4)	pro 2.9
266	2386	Trace ONB(k=8)	pro 2.9
267	26495	a.b(3,89)	pro 2.6
268	535	Optimal	pro 2.5
269	2413	Trace ONB(k=8)	pro 2.9
270	1077	Trace ONB(k=3)	pro 2.9
271	1891	Trace ONB(k=6)	pro 2.9
272	1621	GNB(k=5)	pro 2.7
273	2760	a.b(7,39)	pro 2.6
274	1089	Trace ONB(k=3)	pro 2.9
275	3299	GNB(k=12)	pro 2.7
276	1435	a.b(4,69)	pro 2.6
277	1381	Trace ONB(k=4)	pro 2.9
278	832	Trace ONB(k=2)	pro 2.9
279	5285	Trace ONB(k=18)	pro 2.9
280	559	Optimal	pro 2.5
281	8681	Trace ONB(k=30)	pro 2.9
282	563	Optimal	pro 2.5
283	1975	Trace ONB(k=6)	pro 2.9
284	850	Trace ONB(k=2)	pro 2.9
285	853	Trace ONB(k=2)	pro 2.9
286	1137	Trace ONB(k=3)	pro 2.9
287	2003	Trace ONB(k=6)	pro 2.9
288	7000	a.b(9,32)	pro 2.6
289	8353	Trace ONB(k=28)	pro 2.9
290	3679	a.b(5,58)	pro 2.6
291	2031	Trace ONB(k=6)	pro 2.9
292	583	Optimal	pro 2.5
293	6133	Trace ONB(k=20)	pro 2.9
294	1753	GNB(k=5)	pro 2.7
295	14399	a.b(5,59)	pro 2.6
296	886	Trace ONB(k=2)	pro 2.9
297	4441	Trace ONB(k=14)	pro 2.9
298	1486	Trace ONB(k=4)	pro 2.9
299	2683	Trace ONB(k=8)	pro 2.9
300	1393	a.b(3,100)	pro 2.6

Table 4: Continue of Table 3.

f(x)	C_N	No	f(x)	C_N	No
2:1,1:1,0:2	4	2	52:1,51:1,37:2,14:1,0:2	1685	300
3:1,2:1,0:2	7	2	53:1,52:1,34:2,19:2,0:2	1758	642
4:1,3:1,0:2	13	2	54:1,53:1,35:1,33:1,0:1	1857	458
4:1,3:1,2:1,1:1,0:1	7	4	55:1,54:1,49:2,15:2,0:2	1923	508
5:1,4:1,0:2	19	2	56:1,55:1,10:1,7:2,0:2	1951	276
5:1,4:1,2:1,1:2,0:2	13	14	57:1,56:1,26:2,4:2,0:1	2072	590
6:1,5:1,3:2,2:2,0:1	16	16	58:1,57:1,55:1,7:1,0:1	2152	420
7:1,6:1,3:2,2:2,0:2	24	38	59:1,58:1,42:2,33:2,0:2	2207	652
8:1,7:1,6:1,2:2,0:2	39	14	60:1,59:1,27:2,23:1,0:2	2310	318
9:1,8:1,3:2,1:2,0:2	43	62	61:1,60:1,52:2,37:2,0:2	2355	844
10:1,9:1,8:2,2:2,0:1	57	46	62:1,61:1,51:1,15:2,0:2	2448	502
11:1,10:1,7:1,3:2,0:2	66	86	63:1,62:1,45:2,27:1,0:2	2537	772
12:1,11:1,3:2,1:2,0:2	81	40	64:1,63:1,27:1,20:1,0:1	2607	368
13:1,12:1,0:2	121	2	65:1,64:1,20:2,7:2,0:1	2692	676
13:1,12:1,11:2,10:1,0:2	95	110	66:1,65:1,15:1,13:2,0:2	2785	604
14:1,13:1,8:1,3:1,0:1	112	78	67:1,66:1,60:2,10:2,0:1	2872	676
15:1,14:1,8:1,3:1,0:1	126	152	68:1,67:1,44:2,43:2,0:2	2962	398
16:1,15:1,11:1,2:1,0:1	149	80	69:1,68:1,64:1,36:2,0:2	3048	804
17:1,16:1,0:2	189	2	70:1,69:1,34:2,32:2,0:1	3118	584
17:1,16:1,7:2,2:2,0:2	160	150	71:1,70:1,61:2,50:1,0:2	3226	768
18:1,17:1,9:2,8:2,0:1	52	130	72:1,71:1,68:1,8:2,0:2	3348	296
19:1,18:1,8:2,3:2,0:1	218	176	73:1,72:1,0:2	3390	2
20:1,19:1,6:2,5:2,0:2	239	108	73:1,72:1,69:2,15:1,0:2	3410	980
21:1,20:1,15:2,4:2,0:1	258	206	74:1,73:1,66:2,35:2,0:1	3497	640
22:1,21:1,14:1,9:1,0:1	275	150	75:1,74:1,42:2,11:2,0:2	3600	950
23:1,22:1,19:2,14:2,0:1	317	226	76:1,75:1,23:1,11:2,0:2	3690	610
24:1,23:1,10:1,3:1,0:1	356	92	77:1,76:1,57:2,36:1,0:2	3800	990
25:1,24:1,8:2,3:2,0:1	377	282	78:1,77:1,66:2,38:1,0:2	3900	596
26:1,25:1,15:2,13:2,0:2	409	156	79:1,78:1,68:1,63:1,0:1	3998	750
27:1,26:1,15:1,3:2,0:2	435	274	80:1,79:1,35:1,4:1,0:1	4120	392
28:1,27:1,26:1,1:2,0:2	82	210	81:1,80:1,48:1,16:2,0:2	4192	1002
29:1,28:1,15:2,14:1,0:2	511	302	82:1,81:1,48:1,11:1,0:1	4293	672
30:1,29:1,5:1,3:1,0:1	551	258	83:1,82:1,67:1,1:1,0:1	4387	1124
31:1,30:1,21:2,11:2,0:2	588	260	84:1,83:1,53:2,10:2,0:1	4543	528
32:1,31:1,22:1,17:2,0:2	629	158	85:1,84:1,54:1,48:2,0:2	4619	1278
33:1,32:1,8:2,4:2,0:1	679	316	86:1,85:1,46:1,15:2,0:2	4771	708
34:1,33:1,24:2,20:1,0:2	707	230	87:1,86:1,83:2,55:1,0:2	4881	1144
35:1,34:1,23:2,10:2,0:1	754	402	88:1,87:1,66:1,35:1,0:1	5000	494
36:1,35:1,33:1,17:1,0:1	805	194	89:1,88:1,52:2,48:2,0:1	5111	1114
37:1,36:1,35:2,3:1,0:2	856	458	90:1,89:1,77:2,35:1,0:2	5201	888
38:1,37:1,15:1,13:1,0:1	891	272	91:1,90:1,71:2,6:2,0:1	5335	782
39:1,38:1,15:2,13:2,0:2	963	340	92:1,91:1,80:1,9:2,0:2	5475	668
40:1,39:1,28:2,23:2,0:2	1008	232	93:1,92:1,81:2,55:1,0:2	5577	1098
41:1,40:1,0:2	1116	2	94:1,93:1,53:2,8:1,0:2	5656	802
41:1,40:1,17:2,11:1,0:2	1038	520	95:1,94:1,9:2,3:2,0:2	5801	1168
42:1,41:1,26:1,14:2,0:2	1104	382	96:1,95:1,41:2,22:2,0:1	5958	452
43:1,42:1,26:1,25:1,0:1	1138	374	97:1,96:1,18:1,11:1,0:1	6089	1270
44:1,43:1,18:2,17:2,0:2	1226	270	98:1,97:1,43:2,33:1,0:2	6186	850
45:1,44:1,22:1,4:2,0:2	1275	486	99:1,98:1,57:2,18:2,0:2	6340	1270
46:1,45:1,19:2,15:1,0:2	1330	380	100:1,99:1,94:2,33:2,0:2	6449	748
47:1,46:1,44:1,39:2,0:2	1391	522	101:1,100:1,66:1,26:2,0:2	6581	1260
48:1,47:1,44:2,13:2,0:1	1475	198	102:1,101:1,54:2,52:2,0:1	6731	1018
49:1,48:1,26:2,14:1,0:2	1488	702	103:1,102:1,75:2,29:1,0:2	6864	1054
50:1,49:1,40:2,10:2,0:1	1549	366	104:1,103:1,27:1,18:1,0:1	6995	454
51:1,50:1,0:2	1705	2	105:1,104:1,33:2,28:2,0:2	7106	1146
51:1,50:1,42:2,33:2,0:2	1658	576			

Table 5: Minimum complexity trinomials and pentanomials of degree $n \leq 105$ over F_3 with their complexities

106:1,105:1,10:2,1:2,0:2	156:1,155:1,27:2,1:1,0:2	206:1,205:1,25:1,1:2,0:2	256:1,255:1,15:2,1:1,0:2
107:1,106:1,14:2,1:2,0:1	157:1,156:1,6:1,1:1,0:1	207:1,206:1,8:1,1:2,0:2	257:1,256:1,9:1,1:2,0:2
108:1,107:1,16:2,1:2,0:2	158:1,157:1,28:1,1:1,0:1	208:1,207:1,26:1,1:2,0:2	258:1,257:1,12:1,1:1,0:1
109:1,108:1,17:2,1:2,0:2	159:1,158:1,24:2,1:2,0:1	209:1,208:1,12:1,1:2,0:2	259:1,258:1,28:1,1:2,0:2
110:1,109:1,49:2,1:1,0:2	160:1,159:1,2:2,1:2,0:1	210:1,209:1,43:1,1:1,0:1	260:1,259:1,10:2,1:2,0:2
111:1,110:1,16:1,1:1,0:1	161:1,160:1,17:2,1:2,0:2	211:1,210:1,9:1,1:2,0:2	261:1,260:1,19:2,1:2,0:2
112:1,111:1,14:1,1:1,0:1	162:1,161:1,15:2,1:2,0:2	212:1,211:1,28:2,1:2,0:1	262:1,261:1,6:1,1:1,0:1
113:1,112:1,10:2,1:2,0:2	163:1,162:1,11:1,1:2,0:2	213:1,212:1,8:1,1:1,0:1	263:1,262:1,12:1,1:2,0:2
114:1,113:1,28:1,1:2,0:2	164:1,163:1,17:2,1:1,0:2	214:1,213:1,24:1,1:1,0:1	264:1,263:1,83:2,1:1,0:2
115:1,114:1,2:2,1:2,0:2	165:1,164:1,4:1,1:2,0:2	215:1,214:1,13:1,1:2,0:2	265:1,264:1,12:1,1:1,0:1
116:1,115:1,17:1,1:2,0:2	166:1,165:1,6:1,1:1,0:1	216:1,215:1,18:1,1:2,0:2	266:1,265:1,11:1,1:2,0:2
117:1,116:1,12:1,1:2,0:2	167:1,166:1,18:1,1:1,0:1	217:1,216:1,30:2,1:2,0:2	267:1,266:1,2:2,1:2,0:2
118:1,117:1,22:2,1:2,0:1	168:1,167:1,6:2,1:2,0:1	218:1,217:1,8:1,1:2,0:2	268:1,267:1,110:1,1:1,0:1
119:1,118:1,8:2,1:2,0:2	169:1,168:1,6:1,1:2,0:2	219:1,218:1,2:2,1:2,0:2	269:1,268:1,5:2,1:1,0:2
120:1,119:1,14:2,1:2,0:1	170:1,169:1,3:2,1:1,0:2	220:1,219:1,3:2,1:1,0:2	270:1,269:1,16:1,1:1,0:1
121:1,120:1,0:2	171:1,170:1,23:2,1:1,0:2	221:1,220:1,4:2,1:2,0:1	271:1,270:1,3:2,1:2,0:2
122:1,121:1,11:2,1:2,0:2	172:1,171:1,7:1,1:1,0:1	222:1,221:1,16:2,1:2,0:1	272:1,271:1,94:1,1:2,0:2
123:1,122:1,4:1,1:2,0:2	173:1,172:1,20:1,1:1,0:1	223:1,222:1,15:2,1:1,0:2	273:1,272:1,29:2,1:2,0:2
124:1,123:1,38:1,1:1,0:1	174:1,173:1,46:2,1:2,0:1	224:1,223:1,28:2,1:2,0:1	274:1,273:1,53:2,1:1,0:2
125:1,124:1,11:1,1:1,0:1	175:1,174:1,28:2,1:2,0:1	225:1,224:1,11:1,1:2,0:2	275:1,274:1,10:2,1:2,0:2
126:1,125:1,6:2,1:2,0:2	176:1,175:1,163:1,1:1,0:1	226:1,225:1,22:2,1:2,0:2	276:1,275:1,15:1,1:1,0:1
127:1,126:1,3:1,1:2,0:2	177:1,176:1,16:1,1:1,0:1	227:1,226:1,13:1,1:2,0:2	277:1,276:1,8:1,1:2,0:2
128:1,127:1,10:1,1:1,0:1	178:1,177:1,22:2,1:2,0:2	228:1,227:1,16:2,1:2,0:2	278:1,277:1,9:1,1:2,0:2
129:1,128:1,2:2,1:2,0:2	179:1,178:1,9:1,1:2,0:2	229:1,228:1,3:1,1:1,0:1	279:1,278:1,15:1,1:2,0:2
130:1,129:1,51:1,1:2,0:2	180:1,179:1,115:2,1:1,0:2	230:1,229:1,22:2,1:2,0:1	280:1,279:1,14:1,1:2,0:2
131:1,130:1,4:2,1:2,0:1	181:1,180:1,7:1,1:1,0:1	231:1,230:1,42:2,1:2,0:1	281:1,280:1,3:1,1:2,0:2
132:1,131:1,18:1,1:1,0:1	182:1,181:1,13:1,1:2,0:2	232:1,231:1,90:2,1:2,0:2	282:1,281:1,15:2,1:1,0:2
133:1,132:1,11:2,1:2,0:2	183:1,182:1,42:2,1:2,0:2	233:1,232:1,29:2,1:2,0:2	283:1,282:1,3:1,1:2,0:2
134:1,133:1,10:1,1:1,0:1	184:1,183:1,13:2,1:1,0:2	234:1,233:1,11:2,1:2,0:2	284:1,283:1,19:2,1:1,0:2
135:1,134:1,16:1,1:2,0:2	185:1,184:1,6:1,1:1,0:1	235:1,234:1,2:2,1:2,0:1	285:1,284:1,18:1,1:1,0:1
136:1,135:1,20:2,1:2,0:1	186:1,185:1,8:1,1:2,0:2	236:1,235:1,4:2,1:2,0:1	286:1,285:1,33:1,1:2,0:2
137:1,136:1,0:2	187:1,186:1,15:2,1:1,0:2	237:1,236:1,14:1,1:1,0:1	287:1,286:1,2:2,1:2,0:1
138:1,137:1,46:2,1:2,0:1	188:1,187:1,3:1,1:1,0:1	238:1,237:1,3:1,1:2,0:2	288:1,287:1,34:2,1:2,0:1
139:1,138:1,2:2,1:2,0:1	189:1,188:1,9:1,1:2,0:2	239:1,238:1,12:2,1:2,0:1	289:1,288:1,8:1,1:1,0:1
140:1,139:1,23:1,1:1,0:1	190:1,189:1,40:1,1:1,0:1	240:1,239:1,17:1,1:2,0:2	290:1,289:1,8:2,1:2,0:1
141:1,140:1,5:1,1:1,0:1	191:1,190:1,13:2,1:1,0:2	241:1,240:1,2:1,1:2,0:2	291:1,290:1,25:2,1:2,0:2
142:1,141:1,4:1,1:1,0:1	192:1,191:1,14:2,1:2,0:1	242:1,241:1,91:1,1:2,0:2	292:1,291:1,3:2,1:1,0:2
143:1,142:1,5:1,1:2,0:2	193:1,192:1,5:1,1:1,0:1	243:1,242:1,45:1,1:2,0:2	293:1,292:1,10:1,1:1,0:1
144:1,143:1,18:2,1:2,0:1	194:1,193:1,40:2,1:2,0:1	244:1,243:1,31:1,1:1,0:1	294:1,293:1,40:2,1:2,0:1
145:1,144:1,2:1,1:2,0:2	195:1,194:1,11:1,1:1,0:1	245:1,244:1,24:2,1:2,0:1	295:1,294:1,56:2,1:2,0:2
146:1,145:1,28:2,1:2,0:1	196:1,195:1,3:2,1:1,0:2	246:1,245:1,88:1,1:1,0:1	296:1,295:1,102:1,1:2,0:2
147:1,146:1,3:1,1:2,0:2	197:1,196:1,45:1,1:2,0:2	247:1,246:1,21:1,1:2,0:2	297:1,296:1,78:1,1:1,0:1
148:1,147:1,26:1,1:2,0:2	198:1,197:1,22:1,1:1,0:1	248:1,247:1,20:2,1:2,0:1	298:1,297:1,34:2,1:2,0:2
149:1,148:1,10:1,1:2,0:2	199:1,198:1,15:1,1:2,0:2	249:1,248:1,11:2,1:2,0:2	299:1,298:1,10:1,1:1,0:1
150:1,149:1,21:1,1:2,0:2	200:1,199:1,19:2,1:2,0:2	250:1,249:1,107:2,1:1,0:2	300:1,299:1,3:2,1:1,0:2
151:1,150:1,39:2,1:1,0:2	201:1,200:1,20:1,1:2,0:2	251:1,250:1,53:1,1:2,0:2	
152:1,151:1,7:2,1:1,0:2	202:1,201:1,53:2,1:1,0:2	252:1,251:1,33:1,1:1,0:1	
153:1,152:1,7:1,1:2,0:2	203:1,202:1,10:2,1:2,0:1	253:1,252:1,40:1,1:1,0:1	
154:1,153:1,35:1,1:1,0:1	204:1,203:1,12:1,1:2,0:2	254:1,253:1,26:1,1:1,0:1	
155:1,154:1,2:1,1:2,0:2	205:1,204:1,18:1,1:1,0:1	255:1,254:1,4:2,1:2,0:2	

Table 6: All normal trinomials or pentanomials with minimum weight of degree $106 \leq n \leq 300$ over F_3

5. Conjectures

By using the data in the Table 1, the statements of conjectures, which have been stated in [15] for finite fields with characteristic two, are studied for finite fields of characteristic three. the results of our studying procedure are described in the following conjectures.

Conjecture 5.1 *An upper bound for average of the complexities of normal polynomials of degree n over $\mathbb{F}_3$ is* $\dfrac{n^2-n+2}{3}$.

Conjecture 5.2 *An upper bound for variance of the complexities of normal polynomials of degree n over $\mathbb{F}_3$ is* $\dfrac{n^2+3n-5}{2}$.

Conjecture 5.3 *Since the probability of finding a normal polynomial of complexity $c_N \leq kn$ is similar to finding the density under the normalized curve as $P(c_N \leq kn) = P\left[Z \leq \dfrac{(kn-\mu)}{\sigma}\right]$, where μ and σ^2 denote the average and the variance of the complexities of normal polynomials of degree n over $\mathbb{F}_3$, respectively. So, according to the above conjectures, by giving $\mu = \dfrac{n^2-n+2}{3}$ and $\sigma^2 = \dfrac{n^2+3n-5}{2}$, computing the Z-score of kn gives $z_p = \dfrac{kn - \frac{n^2-n+2}{3}}{\sqrt{\frac{n^2+3n-5}{2}}}$. Hence, it is obvious that if k is a constant, then the Z-score becomes infinitely small. Relating this to the complexity distribution, this implies that the upper bound on the minimum complexity vanishes, which is a contradiction since the minimum possible complexity is $2n-1$. So there is no constant k such that the complexity c_N of $\mathbb{F}_{3^n}$ is bounded above by kn for all n.*

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CYCLIC HYPERGROUPS WHICH ARE INDUCED BY THE CHARACTER OF SOME FINITE GROUPS

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Abstract. Let G be a finite group and $\hat{G}$ be the set of all irreducible characters of G. In this paper, the hypergroups obtained from the character table $\hat{G}$ are considered. Moreover, we show that $\hat{S}_n$ for $n \geq 3$ and $\hat{A}_n$ for $n \geq 4$ are single-power cyclic hypergroups and $\hat{D_{2n}}$ is cyclic with finite period.

Keywords: polygroup, cyclic hypergroup, character.

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1. Introduction

Hypergroups have been studied by many researchers in various fields for a long time; for examples, see [2], [12] and [4]. Cyclic hypergroups already considered at the beginning of the theory's history by [13] have been later on studied in depth by Vougiouklis [11] and afterwards by Leoreanu [8]. The hypergroup H will be called cyclic with finite period with respect to $h \in H$, if there exists a positive integer $s \in \mathbb{Z}^+$, such that

$$H = h^1 \cup h^2 \cup ... \cup h^s,$$

where $h^t = \underbrace{h.h...h}_{t\ times}$.

The minimum such a s will be called period of the generator h. If there exists $h \in H$ and $s \in \mathbb{Z}^+$, the minimum one, such that $H = h^s$, then H will be called single -power cyclic and h is a generator with single- power period s. Quasicanonical hypergroups were introduced by P. Corsini and later were studied by P. Bonansinga and Ch. Massouros. They satisfy all the conditions of canonical hypergroups, except the commutativity. Later, S. D. Comer in [1] introduced this class of hypergroups independently, using the name of polygroups. A polygroup is a system $\mathcal{P} =< P, ., e, ^{-1} >$, where $e \in P$, $^{-1}$ is a unitary operation on P, maps $P \times P$ into the non-empty subset of P, and the following axioms hold for all $x, y, z \in P$:

1. $(x.y).z = x.(y.z)$;
2. $e.x = x.e = x$;
3. $x \in y.z$ implies $y \in x.z^{-1}$ and $z \in y^{-1}.x$.

Roth in [10] showed that for a finite group G, there exists a polygroup system $\left\langle \hat{G}, *, \chi_1, ^{-1} \right\rangle$ where $\hat{G}$ is the set of all irreducible characters [6] of G. Later on, $\hat{G}$ have been studied in various fields. McMullen in [9] proved that $\mathbb{C}\hat{G}$ is semisimple and Comer in [1] showed that a natural hypergroup is associated with every character algebra and also showed certain edge coloring of graphs give raise to hypergroups with special properties. Symmetry groups have been widely applied in chemistry [3] and crystallography [7]. Many of these applications, have involved coset decomposition, decompositions into conjugacy classes and group characters. Let S_n be symmetric group on n letters and A_n be alternating group on n letters and D_{2n} be dihedral group. In this paper, we will show that the hypergroups which obtained from character tables of S_n for $n \geq 3$ and A_n for $n \geq 4$ are single-power cyclic hypergroup. In continue, we will show the hypergroup $\hat{D_{2n}}$ is cyclic hypergroup with finite period.

2. Preliminaries

In this section, we mention some fundamental notions and facts of character of finite groups and character hypergroups, referring to Issacs's book [6] and Roth's paper [10].

Let G be a finite group and F be a field. Also, let V be a finite dimensional vector space on F. A representation of G over V is a homomorphism

$$T : G \longrightarrow GL(V), T(xy) = T(x)T(y); \quad \forall x, y \in G.$$

A representation T of G is called irreducible, if V is an irreducible $FG - mod$. Let the dimension of V over F be n. Then $GL(V) \cong GL(n, F)$ where $GL(n, F)$ is the set of all square invertible matrixes. Let T be a representation of G. Then the character χ of G afforded by T is the function given by $\chi(g) = trT(g)$ and χ

is an irreducible character if the representation T is irreducible. For a character χ, the kernel of χ is defined by

$$\ker \chi = \{g \in G : \chi(g) = \chi(e)\}.$$

If $\ker \chi = \{e\}$, then χ is called a faithful character. We assume that the field F is equal to complex number. If χ and ψ are any two complex characters of G, then (χ, ψ) denotes the usual inner product:

$$(\chi, \psi) = \frac{1}{|G|} \sum_{g \in G} \chi(g)\psi(g^{-1}).$$

Let $Irr(G) = \{\chi_1, \chi_2, ..., \chi_k\}$, where χ_i for $1 \leq i \leq k$ are irreducible characters of G. Since we need some well known results relate to character theory we bring them in follow:

Theorem 2.1. [6](Orthogonality Relations) *Let χ_i, χ_j and χ be complex characters of G and $g, h \in G$. Then*

$$\begin{aligned} \frac{1}{|G|} \sum_{g \in G} \chi_i(g)\chi_j(g^{-1}) &= \delta_{ij}, \\ \frac{1}{|G|} \sum_{\chi \in Irr(G)} \chi(g)\chi(h) &= 0. \end{aligned}$$

Theorem 2.2. [6] *The character table of D_{2n} for even integer $n = 2m$, $\epsilon = e^{\frac{2\pi i}{n}}$ and $1 \leq j \leq m-1$ is as follow:*

Table I

D_{2n}	1	a^m	$a^r (1 \leq r \leq m-1)$	b	ab
$\lvert c(g_i) \rvert$	$2n$	$2n$	n	4	4
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	$(-1)^m$	$(-1)^r$	1	-1
χ_4	1	$(-1)^m$	$(-1)^r$	-1	1
ψ_j	2	$2(-1)^j$	$\epsilon^{jr} + \epsilon^{-jr}$	0	0

Theorem 2.3. [6] *The character table of D_{2n} for odd integer n, $\epsilon = e^{\frac{2\pi i}{n}}$ and $1 \leq j \leq \frac{n-1}{2}$ is as follow:*

Table II

D_{2n}	1	$a^r (1 \leq r \leq \frac{n-1}{2})$	b
$\lvert c(g_i) \rvert$	$2n$	n	2
χ_1	1	1	1
χ_2	1	1	-1
ψ_j	2	$\epsilon^{jr} + \epsilon^{-jr}$	0

Suppose that the group G acts on a set Ω and $g \in G$. Then we define the set of fixed points of g by

$$fix(g) = \{\alpha \in \Omega | \alpha^g = \alpha\}.$$

Theorem 2.4. [6] *In symmetric group S_n, $\chi(g) = |fix(g)| - 1$ is a faithful irreducible character.*

Theorem 2.5. [6] *In alternating group A_n, $(\chi(g)) \downarrow_{A_n}$ is a faithful irreducible character.*

Theorem 2.6. (Cauchy-Frobenius Lemma) [5] *Let G be a finite group acting on a finite set Ω. Then G has m orbits on Ω where*

$$m|G| = \sum_{g \in G} |fix(g)|.$$

Let G be a finite group with $\hat{G} = \{\chi_1, \chi_2, ..., \chi_k\}$. Roth in [10], introduced the character polygroup $< \hat{G}, *, \chi_1, ^{-1} >$ where the product $\chi_i * \chi_j$ is the set of those irreducible components which appear in the element wise product $\chi_i \chi_j$. Further, $\bar{\chi}$, the complex conjugate of χ, is the inverse of χ. If $\theta \in \chi * \psi$, then $(\theta, \chi\psi) > 0$, hence $(\theta\bar{\chi}, \psi) > 0$ and $\psi \in \theta\bar{\chi}$.

Lemma 2.7. [10] *Let G be a finite abelian group. Then $\hat{G}$ is isomorphic to G.*

A key theorem in the study of the character hypergroup $\hat{G}$ is the classical theorem of Burnside:

Theorem 2.8. (Burnside) [6] *Let χ be a faithful character of G and suppose $\chi(g)$ takes on exactly m different values for $g \in G$. Then every $\psi \in Irr(G)$ is a constituent of one of the characters $(\chi)^j$ for $0 \leq j < m$.*

3. Main results

In this section, we will obtain the main results related to the character table of symmetric group S_n, A_n and D_{2n}. In fact, we give Theorems 3.4, 3.5, 3.8 and 3.11 as the main results. For an irreducible character χ_i, we let $\chi_i^t = \underbrace{\chi_i * \chi_i * ... * \chi_i}_{t \ times}$, where the hyper operation $*$ is as above.

Lemma 3.1. *In the symmetric group S_n for $n \geq 3$, χ takes on exactly n different values for any $g \in S_n$.*

Proof. We know that S_n has conjugacy classes of the forme $1^{n-i}i$ where $0 \leq i \leq n$ and $i \neq 1$. Moreover, in each of classes we have $\chi(g) = n - i - 1$. ∎

Lemma 3.2. *In the alternating group A_n for $n \geq 4$, $\chi \downarrow_{A_n}$ takes on exactly $n-1$ different values for $g \in A_n$.*

Proof. A_n has conjugacy classes of the forme $1^{n-i}i$ where i is an odd integer such that $0 \leq i \leq n$ and $i \neq 1$ and also has conjugacy classes of the forme $1^{n-i^2}i^2$ and $1^{n-ij}ij$ for even integer i, j. ∎

Let Ω be a finite set and for any positive integer t, $\Omega^t = \underbrace{\Omega \times \Omega \times ... \times \Omega}_{t\ times}$. Then we give a corollary of Cauchy-Frobenius Lemma as follow:

Corollary 3.3. *Let Ω be a finite set and G be a finite group acting on Ω^t. Then G has m orbits on Ω^t where*

$$\frac{1}{G}\sum_{g\in G}|fix(g)|^t = m.$$

Proof. Consider the set

$$\mathcal{F} = \{(\omega_1, \omega_2, ..., \omega_t, g) \in \Omega \times \Omega \times ... \times \Omega \times G \mid (\omega_1, \omega_2, ..., \omega_t)^g = (\omega_1, \omega_2, ..., \omega_t)\}.$$

we shall count the number of $\mathcal{F}$ in two ways. First, suppose that the orbits of G are $\Omega_1', \Omega_2', ..., \Omega_m'$. Then, using the orbit-stabilizer property, we have

$$|\mathcal{F}| = \sum_{i=1}^{m}\sum_{(\omega_1,\omega_2,...,\omega_t)\in\Omega_i'}\frac{|G|}{|\Omega_i'|} = \sum_{i=1}^{m}|G| = m|G|.$$

Second,

$$|\mathcal{F}| = \sum_{g\in G}|fix(g)|.$$

The result follows. ∎

Theorem 3.4. *For $n \geq 3$, $\hat{S_n}$ is a single-power cyclic polygroup with respect to generator $\chi(g) =\mid fix(g) \mid -1$. In fact $(\hat{S_n}) = \chi^{n-1}$.*

Proof. By the Burnside theorem 2.8, we have $\hat{S_n} = \chi^0 \cup \chi^1 \cup ... \cup \chi^{n-1}$. We must prove that $\chi \in \chi^2$. We know that for $\Omega = \{1, 2, ..., n\}$, S_n acts on Ω, Ω^2 and Ω^3. The action on Ω^2 has two orbits $A_1 = \{(i,i) \mid i \in \Omega\}$ and $A_2 = \{(i,j) \mid i, j \in \Omega, i \neq j\}$. Similarly the action on Ω^3 has five orbits.

Now, put $F(g) = |fix(g)|$. Then, by the Cauchy-Frobenius Lemma 2.6 and the previous lemma:

$$\frac{1}{|G|}\sum_{g\in G}F(g) = 1, \qquad \frac{1}{|G|}\sum_{g\in G}F(g)^2 = 2, \qquad \frac{1}{|G|}\sum_{g\in G}F(g)^3 = 5.$$

Hence,

$$(\chi, \chi^2) = \frac{1}{|G|}\sum_{g\in G}\chi(g)^3 = \frac{1}{G}\sum_{g\in G}(F(g)-1)^3 = 1.$$

Therefore, $\chi \in \chi^2$. Consequently,

$$\chi^2 \subseteq \chi^3 \subseteq ... \subseteq \chi^{n-1}.$$

Hence,

$$\hat{S_n} = \chi^0 \cup \chi^1 \cup ... \cup \chi^{n-1} = \chi^{n-1}. \qquad \blacksquare$$

Since the alternating group A_n is an important simple group, we would like to give some results as above on it.

Theorem 3.5. *For $n \geq 4$, $\hat{A_n}$ is a single-power cyclic polygroup with respect to generator $\chi \downarrow_{A_n}$. In fact, $\hat{A_n} = (\chi \downarrow_{A_n})^{n-2}$.*

Proof. By the Burnside theorem 2.8, we have

$$\hat{A_n} = (\chi \downarrow_{A_n})^0 \cup (\chi \downarrow_{A_n})^1 \cup ... \cup (\chi \downarrow_{A_n})^{n-2}.$$

Since $(\chi, \chi^2) \neq 0$, we have $(\chi \downarrow_{A_n}, (\chi \downarrow_{A_n})^2) \neq 0$. Hence, $\chi \downarrow_{A_n} \in (\chi \downarrow_{A_n})^2$. Therefore,

$$(\chi \downarrow_{A_n})^2 \subseteq (\chi \downarrow_{A_n})^3 \subseteq ... \subseteq (\chi \downarrow_{A_n})^{n-2}. \qquad \blacksquare$$

Since dihedral groups are famous between of all non-abelian groups, we show that $\hat{D_{2n}}$ has a cyclic hypergroup structure.

Lemma 3.6. *Consider the dihedral group D_{2n} for even integer $n = 2m$. Let m be an even integer. Then:*
Case (a). *For an even integer j, $\psi_j \in \psi_1^m$. Also the multiplicities of ψ_j in ψ_1^m is $\begin{pmatrix} m \\ \frac{m-j}{2} \end{pmatrix}$.*
Case (b). *For an odd integer j, $\psi_j \in \psi_1^{m-1}$. Also the multiplicities of ψ_j in ψ_1^{m-1} is $\begin{pmatrix} m-1 \\ \frac{m-j-1}{2} \end{pmatrix}$.*

Proof. Case (a). Let j be an even integer. Then, we have

$$(\psi_1^m, \psi_j) = \frac{2^{m+1} + 2^{m+1}}{2n} + \frac{(\epsilon + \epsilon^{-1})^m (\epsilon^j + \epsilon^{-j}) + ...}{n}$$

$$= \frac{2^{m+1} + \sum_{k=0}^{m} \epsilon^{-m+2k+j} + \sum_{k=0}^{m} \epsilon^{-m+2k-j} + \sum_{k=0}^{m} \epsilon^{-2m+4k+2j} + ...}{n}$$

$$\begin{array}{llllll}
= \frac{1}{n}\Big(2^{m+1} & +\epsilon^{-m+j} & +\begin{pmatrix} m \\ 1 \end{pmatrix}\epsilon^{-m+2+j} & +... & +\begin{pmatrix} m \\ m-1 \end{pmatrix}\epsilon^{m-2+j} & +\epsilon^{m+j} \\
 & +\epsilon^{-2m+2j} & +\begin{pmatrix} m \\ 1 \end{pmatrix}\epsilon^{-2m+4+2j} & +... & +\begin{pmatrix} m \\ m-1 \end{pmatrix}\epsilon^{2m-4+2j} & +\epsilon^{2m+2j} \\
\vdots & & & & & \\
 & +\epsilon^{-m^2+m(1+j)-j} & +\begin{pmatrix} m \\ 1 \end{pmatrix}\epsilon^{-m^2+m(3+j)-2-j} & +... & +\begin{pmatrix} m \\ m-1 \end{pmatrix}\epsilon^{m^2+m(j-3)+2-j} & +\epsilon^{m^2+m(j-1)-j} \\
 & +\epsilon^{-m-j} & +\begin{pmatrix} m \\ 1 \end{pmatrix}\epsilon^{-m+2-j} & +... & +\begin{pmatrix} m \\ m-1 \end{pmatrix}\epsilon^{m-2-j} & +\epsilon^{m-j} \\
 & +\epsilon^{-2m-2j} & +\begin{pmatrix} m \\ 1 \end{pmatrix}\epsilon^{-2m+4-2j} & +... & +\begin{pmatrix} m \\ m-1 \end{pmatrix}\epsilon^{2m-4-2j} & +\epsilon^{2m-2j} \\
\vdots & & & & & \\
 & +\epsilon^{-m^2+m(1-j)+j} & +\begin{pmatrix} m \\ 1 \end{pmatrix}\epsilon^{-m^2+m(3-j)-2+j} & +... & +\begin{pmatrix} m \\ m-1 \end{pmatrix}\epsilon^{m^2-m(3+j)+2+j} & +\epsilon^{m^2-m(1+j)+j}\Big).
\end{array}$$

In this equality, we consider the column $A_k, A_k^{'}, B_k$ and $B_k^{'}$ as follows:
for all $0 \leq k \leq \frac{m-j}{2}$, A_k is equal to the column

$$\begin{array}{l}\epsilon^{-m+2k+j}\\ +\epsilon^{-2m+4k+2j}\\ \vdots\\ +\epsilon^{-m^2+2mk+m-2k+mj-j},\end{array}$$

B_k is equal to the column

$$\begin{array}{l}\epsilon^{-m+2k-j}\\ +\epsilon^{-2m+4k-2j}\\ \vdots\\ +\epsilon^{-m^2+2mk+m-2k-mj+j}.\end{array}$$

and for all $\frac{m-j}{2} < k \leq m$, $A_k^{'}$ is equal to the column

$$\begin{array}{l}\epsilon^{-m+2k+j}\\ +\epsilon^{-2m+4k+2j}\\ \vdots\\ +\epsilon^{-m^2+2mk+m-2k+mj-j},\end{array}$$

and $B_k^{'}$ is equal to column

$$\begin{array}{l}\epsilon^{-m+2k-j}\\ +\epsilon^{-2m+4k-2j}\\ \vdots\\ +\epsilon^{-m^2+2mk+m-2k-mj+j}.\end{array}$$

Now by using the orthogonality relations 2.1 for $A_k, A_k^{'}, B_k$ and $B_k^{'}$ and by some manipulations we get that

$$A_k + B_k^{'} = -2\left(\begin{array}{c} m \\ k \end{array}\right), A_k^{'} + B_k = -2\left(\begin{array}{c} m \\ k \end{array}\right), k \neq \frac{m-j}{2}.$$

And, for $k = \frac{m-j}{2}$, we have for each component of A_k and B_k is equal to one.

Hence,

$$(\psi_1^m, \psi_j) = \frac{1}{2m}(2^{m+1} + 2(m-1)\left(\begin{array}{c} m \\ \frac{m-j}{2} \end{array}\right) - 2\sum_{k=0}^{m}\left(\begin{array}{c} m \\ k \end{array}\right) + 2\left(\begin{array}{c} m \\ \frac{m-j}{2} \end{array}\right)).$$

but

$$-2\sum_{k=0}^{m}\left(\begin{array}{c} m \\ k \end{array}\right) = -2(1+1)^m = -2. - 2^m = -2^{m+1}.$$

So

$$(\psi_1^m, \psi_j) = \frac{1}{2m}.2m\left(\begin{array}{c} m \\ \frac{m-j}{2} \end{array}\right) = \left(\begin{array}{c} m \\ \frac{m-j}{2} \end{array}\right).$$

and hence the proof of Case (a) is completed.

Case (b). The proof is similar to Case (a). ∎

Lemma 3.7. *Consider the dihedral group D_{2n} for even integer $n = 2m$. Let m be an odd integer. Then:*

Case (a). *For an even integer j, $\psi_j \in \psi_1^{m-1}$ and the multiplicities of ψ_j in ψ_1^{m-1} is $\begin{pmatrix} m-1 \\ \frac{m-j-1}{2} \end{pmatrix}$.*

Case (b). *For an odd integer j, $\psi_j \in \psi_1^{m}$ and the multiplicities of ψ_j in ψ_1^{m} is $\begin{pmatrix} m \\ \frac{m-j}{2} \end{pmatrix}$.*

Proof. The proof is similar to Lemma 3.6. ∎

Theorem 3.8. *Consider the dihedral group D_{2n} and $n = 2m$. Then $\hat{D_{2n}}$ is a cyclic hypergroup with generator ψ_1. In fact, $\hat{D_{2n}} = \psi_1^{m-1} \cup \psi_1^{m}$.*

Proof. First let m be an even integer. By Lemma 3.6 it is enough to show that for $1 \leq i \leq 4$, χ_i are in ψ_1^m. Since m is an even integer and by the character value number of χ_1, χ_2 in the character table of D_{2n}, we have:

$$\chi_1, \chi_2 \in \psi_1^m.$$

Now, for χ_3 we have:

$$(\psi_1^m, \chi_3) = \frac{2^m + 2^m}{2n} + \frac{-(\epsilon + \epsilon^{-1})^m + (\epsilon^2 + \epsilon^{-2})^m - ...}{n}$$

$$= \frac{2^m - \sum_{k=0}^{m} \epsilon^{-m+2k} + \sum_{k=0}^{m} \epsilon^{-2m+4k} - ...}{n}$$

$$\begin{array}{llllll} = \frac{1}{n}\Big(2^m & -\epsilon^{-m} & -\begin{pmatrix} m \\ 1 \end{pmatrix} \epsilon^{-m+2} & -... & -\begin{pmatrix} m \\ m-1 \end{pmatrix} \epsilon^{m-2} & -\epsilon^{m} \\ & +\epsilon^{-2m} & +\begin{pmatrix} m \\ 1 \end{pmatrix} \epsilon^{-2m+4} & +... & +\begin{pmatrix} m \\ m-1 \end{pmatrix} \epsilon^{2m-4} & +\epsilon^{2m} \\ \vdots & & & & & \\ & +\epsilon^{-m^2+m} & +\begin{pmatrix} m \\ 1 \end{pmatrix} \epsilon^{-m^2+3m-2} & +... & +\begin{pmatrix} m \\ m-1 \end{pmatrix} \epsilon^{m^2-3m+2} & +\epsilon^{m^2-m}\Big). \end{array}$$

In this equality, we consider the column A_k and B_k as follows: for all $0 \leq k \leq \frac{m}{2}$, A_k is equal to the column

$$\begin{array}{l} \epsilon^{-m+2k} \\ +\epsilon^{-2m+4k} \\ \vdots \\ +\epsilon^{-m^2+2mk+m-2k}. \end{array}$$

and, for all $\frac{m}{2} < k \leq m$, B_k its equal to column

$$\begin{aligned}&\epsilon^{m-2k}\\&+\epsilon^{2m-4k}\\&\vdots\\&+\epsilon^{m^2-2mk-m+2k}.\end{aligned}$$

Now, by using the orthogonality relations for A_k and B_k and some manipulations, we get that

$$A_k + B_k = -2\begin{pmatrix} m \\ k \end{pmatrix}.$$

for $k = \dfrac{m}{2}$, we have that each component of A_k is equal to one. Hence,

$$(\psi_1^m, \chi_3) = \frac{1}{2m}\Big(2^m + (m-1)\begin{pmatrix} m \\ \frac{m}{2} \end{pmatrix} - 2\sum_{k=1}^{\frac{m}{2}-1}\begin{pmatrix} m \\ k \end{pmatrix} + 2m - 2\Big).$$

But

$$-2\sum_{k=1}^{\frac{m}{2}-1}\begin{pmatrix} m \\ k \end{pmatrix} = -2^m + \begin{pmatrix} m \\ \frac{m}{2} \end{pmatrix} + 2.$$

So

$$(\psi_1^m, \chi_3) = \frac{2m + m\begin{pmatrix} m \\ \frac{m}{2} \end{pmatrix}}{2m} \neq 0.$$

Therefore, $\chi_3 \in \psi_1^m$, and similarly $\chi_4 \in \psi_1^m$. Hence the proof is completed.

Now, let m be an odd integer. The proof in this case is similar to the above. ∎

Lemma 3.9. *Consider the dihedral group D_{2n} for odd integer n. Put $m = \frac{n-1}{2}$ and let m be an even integer. Then:*

Case (a). *For an even integer j, $\psi_j \in \psi_1^m$ and the multiplicities of ψ_j in ψ_1^m is $\begin{pmatrix} m \\ \frac{m-j}{2} \end{pmatrix}$.*

Case (b). *For an odd integer j, $\psi_j \in \psi_1^{m-1}$ and the multiplicities of ψ_j in ψ_1^{m-1} is $\begin{pmatrix} m-1 \\ \frac{m-j-1}{2} \end{pmatrix}$.*

Proof. The proof is similar to Lemma 3.6. ∎

Lemma 3.10. *Consider the dihedral group D_{2n} for odd integer n. Put $m = \frac{n-1}{2}$ and let m be an odd integer. Then:*

Case (a). *For an even integer j, $\psi_j \in \psi_1^{m-1}$ and the multiplicities of ψ_j in ψ_1^{m-1} is $\begin{pmatrix} m-1 \\ \frac{m-j-1}{2} \end{pmatrix}$.*

Case (b). *For an odd integer j, $\psi_j \in \psi_1^m$ and the multiplicities of ψ_j in ψ_1^m is $\begin{pmatrix} m \\ \frac{m-j}{2} \end{pmatrix}$.*

Proof. The proof is similar to Lemma 3.6. ∎

Theorem 3.11. *Consider the dihedral group D_{2n} for odd integer $n > 3$. Then $\hat{D_{2n}}$ is a cyclic with finite period hypergroup with generator ψ_1 where $m = \frac{n-1}{2}$. In fact, $\hat{D_{2n}} = \psi_1^{m-1} \cup \psi_1^m$.*

Proof. By Lemmas 3.9 and 3.10, it is enough to show that χ_1 and χ_2 are in ${\psi_1}^{m-1} \cup {\psi_1}^m$. By the character value number of χ_1 and χ_2 in the character table of D_{2n}, we have $\chi_1, \chi_2 \in {\psi_1}^m$ if m is an even integer and $\chi_1, \chi_2 \in {\psi_1}^{m-1}$ if m is an odd integer. This completes the proof. ∎

4. Conclusion

In this paper, a relation between character theory and polygroup theory has obtained. In fact, we could give a structure of hypergroup by character tables and using a special hyperaction on them. Now there is a question, can we extended this idea to an arbitrary finite group in which its character tables is known?

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A NOTE ON NON-FRAGMENTABLE SUBSPACE OF $\ell_\infty^c(\Gamma)$

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Abstract. In this paper we consider $\ell_\infty^c(\Gamma)$ where Γ is uncountable and introduce subspaces $\{A_{\mathcal{P}}\}_{\mathcal{P}\in\Sigma}$ of $\ell_\infty^c(\Gamma)$ which are fragmented by a metric that generates the discrete topology but $A = \bigcup_{\mathcal{P}\in\Sigma} A_{\mathcal{P}}$ is not countable unions of fragmentable subspaces.

Keywords: Discrete topology; fragmentability of topological space ; topological game.

2010 Mathematics Subject Classification: 91A44, 54A05.

1. Introduction

A topological space X is fragmentable if there exists a metric $d(.,.)$ on X such that for every $\varepsilon > 0$ and every nonempty set $A \subseteq X$ there exists a nonempty subset $B \subseteq A$ which is relatively open in A and d-$diam(B) = sup\{d(x,y) : x, y \in B\} < \varepsilon$. In such a case we say that the metric d fragments X. Obviously subspaces of fragmentable space are fragmentable, metric spaces are fragmentable and if τ_1 and τ_2 are two topology on set X such that τ_1 is stronger than τ_2 and (X, τ_2) is fragmentable then (X, τ_1) is fragmentable.

If X is countable union of fragmentable closed subspaces then X is fragmentable [1, Theorem 5.1.10]. This is not true when we replace countable by uncountable. Let Γ be an uncountable set and $Y = \ell_\infty^c(\Gamma)$ be the space of all bounded real-valued functions with countable support defined on Γ. This space by supremum norm is closed subspace of $\ell_\infty^c(\Gamma)$. In next section we introduce a subspace of Y which is uncountable unions of subspaces which (by weak topology) are fragmentable by discrete metric but the space is not even countable unions of fragmentable spaces.

In [6] the following topological game was used to characterize the fragmentability of the space X. Two player $\mathcal{A}$ and $\mathcal{B}$ alternatively select subset of X. The player $\mathcal{A}$ starts the game by choosing some nonempty subset A_1 of X, then the player $\mathcal{B}-$ chooses some nonempty relatively open subset B_1 of A_1. Then again $\mathcal{A}$ selects an arbitrary nonempty subset $A_2 \subseteq B_1$ and $\mathcal{B}$ responds by choosing some nonempty relatively open subset B_2 of A_2. Continuing this alternative selection of sets the two players generate a sequence of sets

$$A_1 \supseteq B_1 \supseteq A_2 \supseteq B_2 \supseteq \cdots$$

which we call a play and denote by $p = (A_i, B_i)_{i \geq 1}$. We say that the player $\mathcal{B}$ is winner whenever the set $\bigcap\limits_{i \geq 1} A_i = \bigcap\limits_{i \geq 1} B_i$ contains at most one point, otherwise the player $\mathcal{A}$ is winner. A strategy w for the player $\mathcal{B}$ is a mapping which assigns to each partial play, $A_1 \supseteq B_1 \supseteq A_2 \supseteq B_2 \supseteq \cdots \supseteq A_k$, some nonempty set $B_k = w(A_1, B_1, ..., A_k)$ which is relatively open subset of A_k.

We call the play $p = (A_i, B_i)_{i \geq 1}$, a w-play if, $B_i = w(A_1, B_1, ...s, A_i)$ for every $i \geq 1$. The strategy w is a winning strategy for $\mathcal{B}$ if, the player $\mathcal{B}$ wins every w-play. We denote such a game by G_f.

The following theorem determines the relation between fragmentability and topological game:

Theorem 1.1 [6, Theorem 1.1] *The topological space X is fragmentable if and only if the player $\mathcal{B}$ has a winning strategy for the game G.*

The following theorem is proved in [7, Lemma 3] about spaces which are countable unions of fragmentable spaces:

Theorem 1.2 *If X is countable unions of fragmentable subspaces then there exists a strategy w for the player $\mathcal{B}$ in the game G such that for every w-play $p = (A_i, B_i)_{i \geq 1}$ the set $\bigcap\limits_{i \geq 1} B_i$ contains at most countable point.*

Let τ_1, τ_2 be two (not necessarily distinct) topologies on the set X. We say that (X, τ_1) is fragmented by a metric d which majorizes the topology τ_2 if the topology generated by d is stronger than or equal to the topology τ_2.

Theorem 1.3 [5, Theorem 1.2] *Let τ_1, τ_2 be two (not necessarily distinct) topologies on the set X. The space (X, τ_1) is fragmented by a metric d which majorizes τ_2 if and only if there exists a strategy w for the player $\mathcal{B}$ in the game G in (X, τ_1) such that, for every w-play $p = (A_i, B_i)_{i \geq 1}$ either $\bigcap\limits_{i \geq 1} A_i = \bigcap\limits_{i \geq 1} B_i = \emptyset$ or $\bigcap\limits_{i \geq 1} B_i = \{x\}$ for some $x \in X$, and for every τ_2-open set U that contains x, there exists some integer $k > 0$ with $B_k \subseteq U$.*

Let (X, τ) be a topological space which fragmented by metric d. By use of recent Theorem we can determine that d generates the topology τ or not.

In general, d does not generate the topology τ. For example $(X = \ell_\infty, weak)$ is fragmentable since $(B_X, weak^*)$ is metrizable and $X = \bigcup_{n\in\mathbb{N}} nB_X$ but it is proved in [4, Example 3.2] that this space is not fragmented by a metric which majorizes the weak topology.

If the topology on X is discrete then obviously X is fragmented by each metric on it and then X is fragmented by a metric which generates the discrete topology.

2. Results

Let $x \in Y = \ell^c_\infty(\Gamma)$, $supp(x) = \{\alpha \in \Gamma : x(\alpha) \neq 0\}$, $A = \{x \in \ell^c_\infty(\Gamma) : x(\alpha) = 1, \alpha \in supp(x)\}$.

Let Σ be the collection of all partitions of Γ such that, for each partition $\mathcal{P} \in \Sigma$, if $I \in \mathcal{P}$, then I is countable.

Let $\mathcal{P} \in \Sigma$, define $A_\mathcal{P} = \{x \in A : supp(x) = I$, for some $I \in \mathcal{P}\}$, obviously $A = \bigcup_{\mathcal{P}\in\Sigma} A_\mathcal{P}$.

Theorem 2.1 *If $\mathcal{P} \in \Sigma$ then $(A_\mathcal{P}, weak)$ is discrete.*

Proof. Let $x_0 \in A_\mathcal{P}$. We show that $(\{x_0\}, weak)$ is open in $A_\mathcal{P}$.

If $\alpha \in supp(x_0)$, then $x_0(\alpha) = 1$ and $x(\alpha) = 0$ for other $x \in A_\mathcal{P}$, that implies $x_0 \notin \overline{A_\mathcal{P} \setminus \{x_0\}}$. Therefore, there exists $f \in Y^*$ such that $f(x_0) = 1$ and $f(x) = 0$ for other $x \in A_\mathcal{P}$. Put $B = \{x \in A_\mathcal{P} : |f(x - x_0| < \frac{1}{2}\}$. B is open in $A_\mathcal{P}$ by weak topology and contains just x_0. ∎

For every $\mathcal{P} \in \Sigma$, the set $A_\mathcal{P}$ by weak topology is closed in A. Theorem 2.1 implies the following theorem:

Theorem 2.2 *If $\mathcal{P} \in \Sigma$ then $(A_\mathcal{P}, weak)$ is fragmented by a metric which generates the discrete topology.*

It is proved in [2, Theorem 3.1] that $(A, weak)$ is not fragmentable. By use of property of Y^*, we show that $(A, weak)$ is not countable unions of fragmentable subspaces.

Lemma 2.3 *Let Γ_1 be an uncountable subset of Γ and $y \in Y^*$, then there exists an uncountable subset $J_y(\Gamma_1)$ of Γ_1 such that $y(x) = 0$ for each $x \in A$ where $supp(x) \subseteq J_y(\Gamma_1)$.*

Proof. It is proved in [3] that for $y \in Y^*$, there exists a countable subset I_y of Γ such that $y(x) = 0$ for $x \in A$ where $supp(x) \subseteq {I_y}^c$. If $J_y(\Gamma_1) = \Gamma_1 \cap I^c_y$, then $y(x) = 0$, for $x \in A$ where $supp(x) \subseteq J_y(\Gamma_1)$. ∎

Corollary 2.4 *Let $\Gamma_1 \subseteq \Gamma$ be uncountable and $y_1, y_2, \ldots, y_n \in Y^*$, then there exists an uncountable subset $J_{y_1 y_2 \cdots y_n}(\Gamma_1)$ of Γ_1 such that*

$$y_1(x) = y_2(x) = \cdots = y_n(x) = 0 \text{ for each } x \in A,$$

where

$$supp(x) \subseteq J_{y_1 y_2 \cdots y_n}(\Gamma_1).$$

Proof. We get $J_1 = J_{y_1}(\Gamma_1)$ and $J_2 = J_{y_2}(J_1)$ and continue this process to get

$$J_n = J_{y_n}(J_{n-1}).$$

Put $J_{y_1 y_2 \cdots y_n}(\Gamma_1) = J_n$, then

$$y_1(x) = y_2(x) = \cdots = y_n(x), \text{ for } x \in A,$$

where

$$supp(x) \subseteq J_{y_1 y_2 \cdots y_n}(\Gamma_1).$$

■

Theorem 2.5 $(A, weak)$ *is not countable unions of fragmentable subspaces.*

Proof. By Theorem 1.2 it is enough to show that there exists a play $p = (A_i, B_i)_{i \geq 1}$ in the game G such that $\bigcap_{i \geq 1} B_i$ has uncountable point. Let player $\mathcal{A}$ select $A_1 = A$ and player $\mathcal{B}$ select non-empty and relatively open subset $B_1 \subseteq A_1$. Let $x_1 \in B_1$, then there are $y_{11}, y_{12}, \ldots, y_{1n_1} \in Y^*$ and $\varepsilon_1 > 0$ such that $B_1' \subseteq B_1$, where $B_1' = \{x \in A_1 : |y_{11}(x - x_1)| < \varepsilon_1, \ldots, |y_{1n_1}(x - x_1)| < \varepsilon_1\}$. Put $I_1 = supp(x_1)$ and $J_1 = J_{y_{11} y_{12} \cdots y_{1n_1}}({I_1}^c)$. Let

$$A_2 = \{x \in B_1' : I_1 \subseteq supp(x) \subseteq I_1 \cup J_1\}.$$

Let $B_2 \subseteq A_2$ (non-empty and relatively open) be selected. Let $x_2 \in B_2$, then there are $y_{21}, y_{22}, \ldots, y_{2n_2} \in Y^*$ and $\varepsilon_2 > 0$ such that $B_2' \subseteq B_2$ where

$$B_2' = \{x \in A_2 : |y_{21}(x - x_2)| < \varepsilon_2, \ldots, |y_{2n_2}(x - x_2)| < \varepsilon_2\}.$$

Put $I_2 = supp(x_2)$ and $J_2 = J_{y_{21} y_{22} \cdots y_{2n_2}}({I_2}^c \cap J_1)$. Let

$$A_3 = \{x \in B_2' : I_2 \subseteq supp(x) \subseteq I_2 \cup J_2\}.$$

We get B_3' similarly. Following this process, in m'th stage we have $I_m = supp(x_m)$ and $J_m = J_{y_{m1} y_{m2} \cdots y_{mn_m}}({I_m}^c \cap J_{m-1})$ and

$$A_m = \{x \in B_{m-1}' : I_m \subseteq supp(x) \subseteq I_m \cup J_m\}.$$

We have

$$I_1 \subseteq I_2 \subseteq \cdots \subseteq I_m \subseteq \cdots, J_1 \supseteq J_2 \supseteq \cdots \supseteq J_m \supseteq \cdots.$$

Since I_n and $J_n \setminus J_{n+1}$ are countable for each $n \in \mathbb{N}$, $\left(\bigcup_{n\in\mathbb{N}} I_n\right)$ is countable and J is uncountable where $J = \bigcap_{n\in\mathbb{N}} J_n$. For each $n \in \mathbb{N}$, we have $I_n \cap J_n = \varnothing$, then

$$\left(\bigcup_{n\in\mathbb{N}} I_n\right) \cap \left(\bigcap_{n\in\mathbb{N}} J_n\right) = \varnothing.$$

Let $x \in A$ such that $x(\alpha) = 1$, for every $\alpha \in \bigcup_{n\in\mathbb{N}} I_n$ and for one $\alpha \in J$ and $x(\alpha) = 0$, for other α. We have $x = x_1 + x_1'$ where $x_1' \in A$ and $supp(x_1') \subseteq J_1$. Then $y_{1i}(x - x_1) = y_{1i}(x_1') = 0$, for each $1 \leq i \leq n_1$, that follows $x \in B_1'$ and $x \in A_2$. Also we have $x = x_2 + x_2'$, where $x_2' \in A$ and $supp(x_2') \subseteq J_2$, then $y_{2i}(x - x_2) = y_{2i}(x_2') = 0$, for each $1 \leq i \leq n_2$, that follows $x \in B_2'$ and $x \in A_3$. By continuing this process we have $x \in A_n$ for each $n \in \mathbb{N}$, then

$$x \in \bigcap_{n\in\mathbb{N}} A_n.$$

Since J is uncountable, there are uncountable choices for x, then

$$\bigcap_{n\in\mathbb{N}} A_n = \bigcap_{n\in\mathbb{N}} B_n$$

contains uncountable point. ∎

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ON NEW INEQUALITIES OF HERMITE-HADAMARD TYPE FOR GENERALIZED CONVEX FUNCTIONS

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Abstract. In this article, we obtain some inequalities of Hermite-Hadamard type for functions whose third derivatives absolute values are ϕ-convex, log ϕ-convex and quasi-ϕ-convex.

Keywords: Hermite-Hadamard inequality, ϕ-convex functions, log-ϕ-convex, quasi-ϕ-convex function, Holder's integral inequality.

Subject Classification: MSC (2010) 26D07; 26D10; 26D99.

1. Introduction

One of the cornerstones of analysis is the Hadamard inequality, if $[a, b] \quad (a < b)$ is a real interval and $f : [a, b] \to R$ a convex function, then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2} \tag{1.1}$$

Over the last decade this has been extended in a number of ways. An important question is the estimating the difference between the middle and rightmost term in the (1.1). The following identity is a useful building block.

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For several results which generalize, improve and extend the inequalities (1.1), we refer the interested reader to [1,2], [10-16].

We recall that the notion of quasi-convex functions generalized the notion of convex functions. More precisely, a function $f : [a, b] \to R$ is said to be quasi-convex on $[a, b]$ if

$$f(\lambda x + (1-\lambda) y) \leq \max\{f(x), f(y)\}, \quad \forall x, y \in [a, b].$$

Any convex function is a quasi-convex function but the reverse are not true, because there exist quasi-convex functions which are not convex, (see, e.g., [2]).

Recently, D.A. Ion [3] obtained two inequalities of the right hand side of Hermite-Hadamard's type functions whose derivatives in absolute values are quasi-convex functions, as follows:

Theorem 1. *Let $f : I^0 \subseteq R \to R$ be a differentiable function on I^0, $a, b \in I^0$, with $a < b$, and, if $|f'|$ is quasi-convex on $[a, b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)}{4}\max\{|f'(a)|, |f'(b)|\}.$$

Theorem 2. *Let $f : I^0 \subseteq R \to R$ be a differentiable function on I^0, $a, b \in I^0$, with $a < b$, and, if $|f'|^{p/(p-1)}$ is quasi-convex on $[a, b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{b-a}{2(p+1)^{1/p}}\left(\max\left\{|f'(a)|^{p/(p-1)} + |f'(b)|^{p/(p-1)}\right\}\right)^{(p-1)/p}.$$

Alomari, Darus and Dragomir in [4] introduced the following theorems for twice differentiable quasi-convex functions:

Theorem 3. *Let $f : I^0 \subseteq R \to R$ be a twice differentiable function on I^0, $a, b \in I^0$, with $a < b$, and, if $|f''|$ is quasi-convex on $[a, b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)^2}{12}\max\{|f''(a)|, |f''(b)|\}.$$

Theorem 4. *Let $f : I^0 \subseteq R \to R$ be a twice differentiable function on I^0, $a, b \in I^0$, with $a < b$, and, if $|f''|^{p/(p-1)}$ is quasi-convex on $[a, b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{(b-a)^2}{8}\left(\frac{\sqrt{\pi}}{2}\right)^{1/p}\left(\frac{\Gamma(1+p)}{\Gamma\left(\frac{3}{2}+p\right)}\right)^{1/p}\left(\max\left\{|f''(a)|^q+|f''(b)|^q\right\}\right)^{1/q}.$$

Theorem 5. *Let $f : I^0 \subseteq R \to R$ be a twice differentiable function on I^0 $a, b \in I^0$, with $a < b$, and, if $|f''|^q$ is quasi-convex on $[a, b]$, $q \geq 1$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)^2}{12}\left(\max\left\{|f''(a)|^q,|f''(b)|^q\right\}\right)^{1/q}.$$

This paper is in the direction of the results discussed in [5] but here we use ϕ-convex, log ϕ-convex and quasi-ϕ-convex functions instead of s-convex function. After this introduction, in section 2 we found some new integral inequalities of the type of Hermite Hadamard's for generalized convex functions.

2. Main results

To establish our principal results, we first obtain the following definitions.

Let K be a closed set R^n and let $f, \phi : K \to R$ and $\phi : K \times K \to R$ be continuous functions . we recall the following results, which are due to Noor [6], [7], Noor [8], [9] as follows:

Definition 2.1. Let $x \in K$. Then the set K is said to be ϕ-convex at x with respect to ϕ, if

$$x+\lambda e^{i\phi}(y-x) \in K,\ \forall x, y \in K,\ \lambda \in [0,1].$$

Observation 2.2. We would like to mention that the Definition 2.1 of a ϕ-convex set has a clear geometric interpretation. This definition essentially says that there is a path starting from a point x which is contained in K. We don't require that the point y should be one of the end points of the path. This observation plays an important role in our analysis. Note that, if we demand that y should be an end point of the path for every pair of points, $x, y \in K$, then $e^{i\phi}(y-x) = y-x$ if and only if $\phi = 0$, and consequently $\phi-$convexity reduces to convexity. Thus, it is true that every convex set is also an $\phi-$convex set, but the converse is not necessarily true.

Definition 2.3. The function f on the $\phi-$convex set $K$is said to be ϕ-convex with respect to ϕ, if

$$f\left(x+\lambda e^{i\phi}(y-x)\right) \leq +(1-\lambda)f(x)+\lambda f(y),\ \forall x, y \in K,\ \lambda \in [0,1].$$

The function f is said to be ϕ-concave if and only if $-f$ is ϕ-convex.

It is to be noted that every convex function is ϕ-convex function, but the converse is not true.

Definition 2.4. The function f on the quasi $\phi-$convex set $K$is said to be quasi ϕ-convex with respect to ϕ, if

$$f\left(x+\lambda e^{i\phi}\left(y-x\right)\right)\leq \max\left\{f\left(x\right),f\left(y\right)\right\}.$$

Definition 2.5. The function f on the quasi $\phi-$convex set $K$is said to be logarithmic ϕ-convex with respect to ϕ, if

$$f\left(x+\lambda e^{i\phi}\left(y-x\right)\right)\leq \left(f\left(x\right)\right)^{1-\lambda}\left(f\left(y\right)\right)^{\lambda},\ x,y\in K,\ \lambda\in[0,1],$$

where $f\left(\cdot\right)>0$.

From the above definitions, we have

$$\begin{aligned} f\left(x+\lambda e^{i\phi}\left(y-x\right)\right) &\leq \left(f\left(x\right)\right)^{1-\lambda}\left(f\left(y\right)\right)^{\lambda} \\ &\leq \left(1-\lambda\right)f\left(x\right)+\lambda f\left(y\right) \\ &\leq \max\left\{f\left(x\right),f\left(y\right)\right\}. \end{aligned}$$

Lemma 2.6. *Suppose $f:K=\left[a,a+e^{i\phi}\left(b-a\right)\right]\to(0,\infty)$ be a ϕ-convex function on the interval of real numbers K^0 (the interior of K) and $a,b\in K^0$ with $a<a+e^{i\phi}\left(b-a\right)$ and $0\leq\phi\leq\frac{\pi}{2}$. Then the following inequality holds:*

$$\begin{aligned} &\tfrac{f(a)+f(a+e^{i\phi}(b-a))}{2}-\tfrac{1}{e^{i\phi}(b-a)}\int\limits_a^{a+e^{i\phi}(b-a)}f(x)dx-\tfrac{e^{i\phi}(b-a)}{12}\left[f'\left(a+e^{i\phi}\left(b-a\right)\right)-f'\left(a\right)\right] \\ &=\tfrac{\left(e^{i\phi}(b-a)\right)^3}{12}\int\limits_0^1\psi\left(1-\psi\right)\left(2\psi-1\right)f'''\left(a+\psi e^{i\phi}\left(b-a\right)d\psi\right) \end{aligned}$$

A simple proof of this inequality can be done by integrating by parts on the right hand side. The details are left to the interested reader. The next theorem gives a new result of the Hermite-Hadamard inequality for$\phi-$convex function.

Theorem 2.7. *Let $K\subset R$ be an open interval, $a,a+e^{i\phi}\left(b-a\right)\in K$ with $a<a+e^{i\phi}\left(b-a\right)$. Suppose $f:K=\left[a,a+e^{i\phi}\left(b-a\right)\right]\to(0,\infty)$ be a three times differentiable mapping such that f''' is integrable and $0\leq\phi\leq\frac{\pi}{2}$. If $\left|f'''\right|$ is ϕ-convex function on $\left[a,a+e^{i\phi}\left(b-a\right)\right]$, then following inequality holds:*

$$\begin{aligned} &\left|\tfrac{f(a)+f(a+e^{i\phi}(b-a))}{2}-\tfrac{1}{e^{i\phi}(b-a)}\int\limits_a^{a+e^{i\phi}(b-a)}f(x)dx-\tfrac{e^{i\phi}(b-a)}{12}\left[f'\left(a+e^{i\phi}\left(b-a\right)\right)-f'\left(a\right)\right]\right| \\ &\leq\tfrac{\left(e^{i\phi}(b-a)\right)^3}{384}\max\left\{\left|f'''\left(a\right)\right|,\left|f'''\left(b\right)\right|\right\}. \end{aligned}$$

Proof. From Lemma 2.6 and using the ϕ-convexity of $|f'''|$, we get

$$\left|\frac{f(a)+f(a+e^{i\phi}(b-a))}{2}-\frac{1}{e^{i\phi}(b-a)}\int_a^{a+e^{i\phi}(b-a)} f(x)dx-\frac{e^{i\phi}(b-a)}{12}\left[f'\left(a+e^{i\phi}(b-a)\right)-f'(a)\right]\right|$$
$$\leq \frac{\left(e^{i\phi}(b-a)\right)^3}{12}\left|\int_0^1 \psi(1-\psi)\left|(2\psi-1)\right| f'''\left(a+\psi e^{i\phi}(b-a)\,d\psi\right)\right|$$
$$\leq \frac{\left(e^{i\phi}(b-a)\right)^3}{12}\int_0^1 \psi(1-\psi)\left|(2\psi-1)\right|\left[(1-\psi)\left|f'''(a)\right|+\psi\left|f'''(b)\right|\right]d\psi$$
$$\leq \frac{\left(e^{i\phi}(b-a)\right)^3}{384}\left[\left|f'''(a)\right|+\left|f'''(b)\right|\right],$$

which completes the proof. ∎

Observation 2.8. If we take $e^{i\phi}(b-a)=b-a$ in Theorem 2.7, then inequality reduces to the [Corollary 3.1.1(2), 5].

Theorem 2.9. *Suppose $f: K=\left[a,a+e^{i\phi}(b-a)\right]\to(0,\infty)$ be a three times differentiable mapping on K^0 and f''' is integrable on $\left[a,a+e^{i\phi}(b-a)\right]$. Assume $p\in R$ with $p>1$. If $\left|f'''\right|^{p/(p-1)}$ is ϕ-convex function on the interval of real numbers K^0 (the interior of K) and $a,b\in K^0$ with $a<a+e^{i\phi}(b-a)$ and $0\leq\phi\leq\frac{\pi}{2}$. Then following inequality holds:*

$$\left|\frac{f(a)+f(a+e^{i\phi}(b-a))}{2}-\frac{1}{e^{i\phi}(b-a)}\int_a^{a+e^{i\phi}(b-a)} f(x)dx-\frac{e^{i\phi}(b-a)}{12}\left[f'\left(a+e^{i\phi}(b-a)\right)-f'(a)\right]\right|$$
$$\leq \frac{\left(e^{i\phi}(b-a)\right)^3}{96}\left(\frac{1}{p+1}\right)^{1/p}\left(\frac{|f'''(a)|^{\frac{p}{p-1}}+|f'''(b)|^{\frac{p}{p-1}}}{2}\right)^{\frac{p-1}{p}}.$$

Proof. Suppose that $a,a+e^{i\phi}(b-a)\in K$. By assumption, Holder's inequality, then we have

$$\left|\frac{f(a)+f(a+e^{i\phi}(b-a))}{2}-\frac{1}{e^{i\phi}(b-a)}\int_a^{a+e^{i\phi}(b-a)} f(x)dx-\frac{e^{i\phi}(b-a)}{12}\left[f'\left(a+e^{i\phi}(b-a)\right)-f'(a)\right]\right|$$
$$\leq \frac{\left(e^{i\phi}(b-a)\right)^3}{12}\int_0^1 \psi(1-\psi)\left|(2\psi-1)\right|\left|f'''\left(a+\psi e^{i\phi}(b-a)\right)\right|d\psi$$
$$\leq \frac{\left(e^{i\phi}(b-a)\right)^3}{12}\left(\int_0^1 \psi^p(1-\psi)^p\left|(2\psi-1)\right|d\psi\right)^{1/p}\left(\int_0^1\left|f'''\left(a+\psi e^{i\phi}(b-a)\right)\right|^{\frac{p-1}{p}}d\psi\right)^{\frac{p-1}{p}}$$
$$\leq \frac{\left(e^{i\phi}(b-a)\right)^3}{12}\left(\int_0^1 \psi^p(1-\psi)^p\left|(2\psi-1)\right|d\psi\right)^{1/p}$$
$$\cdot\left(\int_0^1\left[(1-\psi)\left|f'''(a)\right|^{\frac{p}{p-1}}+\psi\left|f'''(b)\right|^{\frac{p}{p-1}}\right]d\psi\right)^{\frac{p-1}{p}}$$
$$=\frac{\left(e^{i\phi}(b-a)\right)^3}{96}\left(\frac{1}{p+1}\right)^{1/p}\left(\frac{|f'''(a)|^{\frac{p}{p-1}}+|f'''(b)|^{\frac{p}{p-1}}}{2}\right)^{\frac{p-1}{p}},$$

where we use the fact that $\int_0^1 \psi^p(1-\psi)^p\left|2\psi-1\right|d\psi=\frac{1}{2^{2p+1}(p+1)}$,, which completes the proof. ∎

Observation 2.10. If we take $e^{i\phi}(b-a)=b-a$ in Theorem 2.9, then inequality reduces to the [Corollary 3.2.1, 5].

Theorem 2.11. *Let $K\subset R$ be an open interval, $a, a+e^{i\phi}(b-a)\in K$ with $a<a+e^{i\phi}(b-a)$. Suppose $f:K=\left[a,a+e^{i\phi}(b-a)\right]\to(0,\infty)$ be a three times differentiable mapping such that f''' is integrable and $0\leq\phi\leq\frac{\pi}{2}$. If $|f'''|$ is log ϕ-convex function on $\left[a,a+e^{i\phi}(b-a)\right]$. Then, the following inequality holds:*

$$\left|\frac{f(a)+f(a+e^{i\phi}(b-a))}{2}-\frac{1}{e^{i\phi}(b-a)}\int\limits_a^{a+e^{i\phi}(b-a)}f(x)dx-\frac{e^{i\phi}(b-a)}{12}\left[f'\left(a+e^{i\phi}(b-a)\right)-f'(a)\right]\right|$$
$$\leq\frac{\left(e^{i\phi}(b-a)\right)^3}{(\log|f'''(a)|-\log|f'''(b)|)^3}\left[A\left(|f'''(b)|,|f'''(a)|\right)-L\left(|f'''(b)|,|f'''(a)|\right)\right].$$

Proof. From Lemma 2.6 and using the log-ϕ-convexity of $|f'''|$, we get

$$\left|\frac{f(a)+f(a+e^{i\phi}(b-a))}{2}-\frac{1}{e^{i\phi}(b-a)}\int\limits_a^{a+e^{i\phi}(b-a)}f(x)dx-\frac{e^{i\phi}(b-a)}{12}\left[f'\left(a+e^{i\phi}(b-a)\right)-f'(a)\right]\right|$$
$$\leq\frac{\left(e^{i\phi}(b-a)\right)^3}{12}\int\limits_0^1\psi(1-\psi)\left|(2\psi-1)\right|\left|f'''\left(a+\psi e^{i\phi}(b-a)\right)\right|d\psi$$
$$\leq\frac{\left(e^{i\phi}(b-a)\right)^3}{12}\int\limits_0^1\psi(1-\psi)\left|(2\psi-1)\right|\left(\left|f'''(a)\right|^{1-\psi}.\left|f'''(b)\right|^{\psi}\right)$$
$$=\frac{\left(e^{i\phi}(b-a)\right)^3}{12}.\left[\frac{6(|f'''(b)|+|f'''(a)|)}{(\log|f'''(b)|-\log|f'''(a)|)^3}-\frac{12(|f'''(b)|-|f'''(a)|)}{(\log|f'''(b)|-\log|f'''(a)|)^4}\right]$$
$$=\frac{\left(e^{i\phi}(b-a)\right)^3}{(\log|f'''(b)|-\log|f'''(a)|)^3}\left[A\left(|f'''(b)|,|f'''(a)|\right)-L\left(|f'''(b)|,|f'''(a)|\right)\right],$$

which completes the proof. ∎

Theorem 2.12. *Suppose $f:K=\left[a,a+e^{i\phi}(b-a)\right]\to(0,\infty)$ be a three times differentiable mapping on K^0 and f''' is integrable on $\left[a,a+e^{i\phi}(b-a)\right]$. Assume $p\in R$ with $p>1$. If $|f'''|^{p/(p-1)}$ is log ϕ-convex function on the interval of real numbers K^0 (the interior of K) and $a,b\in K^0$ with $a<a+e^{i\phi}(b-a)$ and $0\leq\phi\leq\frac{\pi}{2}$. Then, the following inequality holds:*

$$\left|\frac{f(a)+f(a+e^{i\phi}(b-a))}{2}-\frac{1}{e^{i\phi}(b-a)}\int\limits_a^{a+e^{i\phi}(b-a)}f(x)dx-\frac{e^{i\phi}(b-a)}{12}\left[f'\left(a+e^{i\phi}(b-a)\right)-f'(a)\right]\right|$$
$$\leq\frac{\left(e^{i\phi}(b-a)\right)^3}{96}\left(\frac{1}{p+1}\right)^{1/p}\left(\frac{p-1}{p}\right)^{\frac{p-1}{p}}\left(\frac{|f'''(a)|^{\frac{p}{p-1}}+|f'''(b)|^{\frac{p}{p-1}}}{\log|f'''(b)|+\log|f'''(a)|}\right)^{\frac{p-1}{p}}.$$

Proof. From Lemma 2.6, and using the well known Holder integral inequality, we have

$$
\begin{aligned}
&\left| \frac{f(a)+f(a+e^{i\phi}(b-a))}{2} - \frac{1}{e^{i\phi}(b-a)} \int_{a}^{a+e^{i\phi}(b-a)} f(x)dx - \frac{e^{i\phi}(b-a)}{12}\left[f'\left(a+e^{i\phi}(b-a)\right) - f'(a)\right] \right| \\
&\le \frac{\left(e^{i\phi}(b-a)\right)^3}{12} \int_0^1 \psi(1-\psi)\left|(2\psi-1)\right| \left|f'''\left(a+\psi e^{i\phi}(b-a)\right)\right| d\psi \\
&\le \frac{\left(e^{i\phi}(b-a)\right)^3}{12} \left(\int_0^1 \psi^p (1-\psi)^p \left|(2\psi-1)\right| d\psi \right)^{1/p} \left(\int_0^1 \left|f'''\left(a+\psi e^{i\phi}(b-a)\right)\right|^{\frac{p}{p-1}} d\psi \right)^{\frac{p-1}{p}} \\
&\le \frac{\left(e^{i\phi}(b-a)\right)^3}{12} \left(\int_0^1 \psi^p (1-\psi)^p \left|(2\psi-1)\right| d\psi \right)^{1/p} \\
&\cdot \left(\int_0^1 \left|f'''(a)\right|^{\frac{p}{p-1}(1-\psi)} + \left|f'''(b)\right|^{\frac{p}{p-1}\psi} d\psi \right)^{\frac{p-1}{p}} \\
&= \frac{\left(e^{i\phi}(b-a)\right)^3}{96} \left(\frac{1}{p+1}\right)^{1/p} \left(\frac{p-1}{p}\right)^{\frac{p-1}{p}} \left(\frac{\left|f'''(a)\right|^{\frac{p}{p-1}} + \left|f'''(b)\right|^{\frac{p}{p-1}}}{\log\left|f'''(b)\right| + \log\left|f'''(a)\right|} \right)^{\frac{p-1}{p}},
\end{aligned}
$$

where we use the fact that

$$
\int_0^1 \psi^p (1-\psi)^p \left|2\psi - 1\right| d\psi = \frac{1}{2^{2p+1}(p+1)},
$$

which completes the proof. ∎

Theorem 2.13. *Suppose $f : K = \left[a, a+e^{i\phi}(b-a)\right] \to (0,\infty)$ be a three times differentiable mapping on K^0 and f''' is integrable on $\left[a, a+e^{i\phi}(b-a)\right]$. If $|f'''|$ is quasi ϕ-convex function on the interval of real numbers K^0 (the interior of K) and $a, b \in K^0$ with $a < a+e^{i\phi}(b-a)$ and $0 \le \phi \le \frac{\pi}{2}$. Then, the following inequality holds:*

$$
\begin{aligned}
&\left| \frac{f(a)+f(a+e^{i\phi}(b-a))}{2} - \frac{1}{e^{i\phi}(b-a)} \int_{a}^{a+e^{i\phi}(b-a)} f(x)dx - \frac{e^{i\phi}(b-a)}{12}\left[f'\left(a+e^{i\phi}(b-a)\right) - f'(a)\right] \right| \\
&\le \frac{\left(e^{i\phi}(b-a)\right)^3}{192} \max\left\{\left|f'''(a)\right|, \left|f'''(b)\right|\right\}.
\end{aligned}
$$

Proof. From Lemma 2.6 and using the quasi-ϕ-convexity of $|f'''|$, we get

$$
\begin{aligned}
&\left| \frac{f(a)+f(a+e^{i\phi}(b-a))}{2} - \frac{1}{e^{i\phi}(b-a)} \int_{a}^{a+e^{i\phi}(b-a)} f(x)dx - \frac{e^{i\phi}(b-a)}{12}\left[f'\left(a+e^{i\phi}(b-a)\right) - f'(a)\right] \right| \\
&\le \frac{\left(e^{i\phi}(b-a)\right)^3}{12} \int_0^1 \psi(1-\psi)\left|(2\psi-1)\right| \left|f'''\left(a+\psi e^{i\phi}(b-a)\right)\right| d\psi \\
&\le \frac{\left(e^{i\phi}(b-a)\right)^3}{12} \max\left\{\left|f'''(a)\right|, \left|f'''(b)\right|\right\} . \int_0^1 \psi(1-\psi)\left|(2\psi-1)\right| d\psi \\
&\le \frac{\left(e^{i\phi}(b-a)\right)^3}{192} \max\left\{\left|f'''(a)\right|, \left|f'''(b)\right|\right\},
\end{aligned}
$$

which completes the proof. ∎

Theorem 2.14. *Suppose $f : K = \left[a, a + e^{i\phi}(b-a)\right] \to (0,\infty)$ be a three times differentiable mapping on K^0 and f''' is integrable on $\left[a, a + e^{i\phi}(b-a)\right]$. Assume $p \in R$ with $p > 1$. If $|f'''|^{p/(p-1)}$ is quasi ϕ-convex function on the interval of real numbers K^0 (the interior of K) and $a, b \in K^0$ with $a < a + e^{i\phi}(b-a)$ and $0 \le \phi \le \frac{\pi}{2}$. Then, the following inequality holds:*

$$\left|\frac{f(a)+f(a+e^{i\phi}(b-a))}{2} - \frac{1}{e^{i\phi}(b-a)} \int_a^{a+e^{i\phi}(b-a)} f(x)dx - \frac{e^{i\phi}(b-a)}{12}\left[f'\left(a+e^{i\phi}(b-a)\right) - f'(a)\right]\right|$$
$$\le \frac{\left(e^{i\phi}(b-a)\right)^3}{96}\left(\frac{1}{p+1}\right)^{1/p}\left[\max\left\{|f'''(a)|^{\frac{p}{p-1}}, |f'''(b)|^{\frac{p}{p-1}}\right\}\right]^{\frac{p-1}{p}}.$$

Proof. From Lemma 2.6, and using the well known Holder integral inequality, we have

$$\left|\frac{f(a)+f(a+e^{i\phi}(b-a))}{2} - \frac{1}{e^{i\phi}(b-a)} \int_a^{a+e^{i\phi}(b-a)} f(x)dx - \frac{e^{i\phi}(b-a)}{12}\left[f'\left(a+e^{i\phi}(b-a)\right) - f'(a)\right]\right|$$
$$\le \frac{\left(e^{i\phi}(b-a)\right)^3}{12}\int_0^1 \psi(1-\psi)\left|(2\psi-1)\right|\left|f'''\left(a+\psi e^{i\phi}(b-a)\right)\right|d\psi$$
$$\le \frac{\left(e^{i\phi}(b-a)\right)^3}{12}\left(\int_0^1 \psi^p(1-\psi)^p\left|(2\psi-1)\right|d\psi\right)^{1/p}\left(\int_0^1 \left|f'''\left(a+\psi e^{i\phi}(b-a)\right)\right|^{\frac{p}{p-1}}d\psi\right)^{\frac{p}{p-1}}.$$

Since $|f'''|^q$ is quasi-ϕ-convex, we have

$$\int_0^1 \left|f'''\left(a+\psi e^{i\phi}(b-a)\right)\right|^{\frac{p}{p-1}}d\psi \le \left\{\max|f'''(a)|^{\frac{p}{p-1}}, |f'''(b)|^{\frac{p}{p-1}}\right\}.$$

Hence

$$\le \frac{\left(e^{i\phi}(b-a)\right)^3}{96}\left(\frac{1}{(p+1)}\right)^{\frac{1}{p}}\left\{\max|f'''(a)|^{\frac{p}{p-1}}, |f'''(b)|^{\frac{p}{p-1}}\right\}^{\frac{1}{q}},$$

where we use the fact that

$$\int_0^1 \psi^p(1-\psi)^p\left|2\psi-1\right|d\psi = \frac{1}{2^{2p+1}(p+1)},$$

which completes the proof. ∎

Theorem 2.15. *Suppose $f : K = \left[a, a + e^{i\phi}(b-a)\right] \to (0,\infty)$ be a three times differentiable mapping on K^0 and f''' is integrable on $\left[a, a + e^{i\phi}(b-a)\right]$. Assume $q \in R$ with $q \ge 1$.If $|f'''|^q$ is quasi-ϕ-convex function on the interval of real numbers K^0 (the interior of K) and $a, b \in K^0$ with $a < a + e^{i\phi}(b-a)$ and $0 \le \phi \le \frac{\pi}{2}$. Then, the following inequality holds*

$$\left|\frac{f(a)+f(a+e^{i\phi}(b-a))}{2} - \frac{1}{e^{i\phi}(b-a)} \int_a^{a+e^{i\phi}(b-a)} f(x)dx - \frac{e^{i\phi}(b-a)}{12}\left[f'\left(a+e^{i\phi}(b-a)\right) - f'(a)\right]\right|$$
$$= \frac{\left(e^{i\phi}(b-a)\right)^3}{192}\left(\max\left\{|f'''(a)|^q, |f'''(b)|^q\right\}\right)^{\frac{1}{q}}.$$

Proof. Suppose that $q \geq 1$. From Lemma 2.6 and using the well known power mean inequality, we have

$$\left|\tfrac{f(a)+f(a+e^{i\phi}(b-a))}{2} - \tfrac{1}{e^{i\phi}(b-a)} \int\limits_a^{a+e^{i\phi}(b-a)} f(x)dx - \tfrac{e^{i\phi}(b-a)}{12}\left[f'\left(a+e^{i\phi}(b-a)\right) - f'(a)\right]\right|$$
$$\leq \tfrac{\left(e^{i\phi}(b-a)\right)^3}{12} \int\limits_0^1 \psi(1-\psi)\left|(2\psi-1)\right| \left|f'''\left(a+\psi e^{i\phi}(b-a)\right)\right| d\psi$$
$$\leq \tfrac{\left(e^{i\phi}(b-a)\right)^3}{12} \left(\int\limits_0^1 \psi(1-\psi)\left|(2\psi-1)\right| d\psi\right)^{1-1/q}$$
$$\cdot \left(\int\limits_0^1 \psi(1-\psi)\left|(2\psi-1)\right| \left|f'''\left(a+\psi e^{i\phi}(b-a)\right)\right|^q d\psi\right)^{1/q}$$
$$\leq \tfrac{\left(e^{i\phi}(b-a)\right)^3}{12} \left(\tfrac{1}{16}\right)^{1-1/q} \cdot \left(\tfrac{1}{16}\max\left\{\left|f'''(a)\right|^q, \left|f'''(b)\right|^q\right\}\right)^{1/q}$$
$$= \tfrac{\left(e^{i\phi}(b-a)\right)^3}{192} \left(\max\left\{\left|f'''(a)\right|^q, \left|f'''(b)\right|^q\right\}\right)^{1/q},$$

where we use the fact

$$\int\limits_0^1 \psi(1-\psi)\left|(2\psi-1)\right| d\psi = \frac{1}{16}.$$

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REGULAR SUB-SEQUENTIALLY DENSE INJECTIVE IN THE CATEGORY OF S-POSETS

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Abstract. Sequentially dense monomorphisms were first introduced and studied by Giuli for projection algebras and followed by Ebrahimi, Mahmoudi, Moghaddasi and Shahbaz for S-acts. In this paper we use the notion of sub-Cauchy sequences and introduce the class of regular sub-sequentially dense monomorphisms for S-posets, denoted by $\mathcal{M}_s$. We investigate the properties of the class $\mathcal{M}_s$ and study $\mathcal{M}_s$-injectivity of S-posets. Further, we find $\mathcal{M}_s$-injective hull of S-posets over some kind of semigroups.

1. Introduction and preliminary

In [5], M.M. Ebrahimi and M. Mahmoudi introduced the interesting concept of a Cauchy sequence in an S-set, a set A together with an action of a semigroup S on A, see [11]. Then the concept of the notion of sequentially dense monomorphism and injectivity with respect to this kind of monomorphism have been investigated, see [6].

Introducing the category of S-posets, a poset A together with an action of a posemigroup on A, and algebraic study on this category was initiated by S. Fakhruddin, see [9, 10]. Recently, there are more people interested on studying on this category [3], [2], [13], [8]. In this paper, we are going to study sub-sequentially dense injective objects and the behavior of this type of injectivity in the category of S-posets. To do so, we first introduce the notions of sub-Cauchy sequences and convergent sub-Cauchy sequences. We then, in Section 2, use these concepts to define a categorical closure operator C_s, as introduced by Tholen and

Dikranjan in [14], on the category of S-posets. We also consider the regular and dense monomorphisms with respect to the closure C_s, that is $\mathcal{M}_s$-monomorphisms, and briefly study some categorical properties of this class of monomorphism which is needed subsequently. In Section 3, we study essential monomorphisms and injectivity relative to this class of monomorphisms in the category of S-posets. And finally, in Section 4, we give an example about what we did through the paper and study these concepts in the left zero posemigroups.

Now we briefly recall some concepts and we then give the preliminaries. A semigroup S is said to be *partially ordered* (or simply, a *posemigroup*) if it is also a poset whose partial order is compatible with the binary operation. An *S-poset* is a (possibly empty) poset A together with a monotone map $\lambda : A \times S \to A$, called the action of S on A, such that, for all $a \in A$ and $s, t \in S$, we have $a(st) = (as)t$, where $A \times S$ is considered as a poset with componentwise order and we denote $\lambda(a, s)$ by as. By an S-poset morphism (or morphism), we mean a monotone map between S-posets which is equivariant (or preserves the action).

Definition 1.1. Let S be a posemigroup and A be an S-poset. Then

(i) by a *sub-Cauchy sequence* over an S-poset A we mean a family $(a_s)_{s\in S}$ of ascending elements of A, that is $a_s \leq a_t$ if $s \leq t$, with $a_s t \geq a_{st}$ for all $s, t \in S$.

(ii) by a *sub-Cauchy sequence induced by* $a \in A$ we mean the sub-Cauchy sequence $(a_s = as)_{s\in S}$ and we show it by λ_a.

(iii) by a *limit* of a sub-Cauchy sequence $(a_s)_{s\in S}$ over A in some extension B of A we mean an element $b \in B$ such that $bs = a_s$, for all $s \in S$, and denote it by $lim(a_s)_{s\in S} = b$.

(iv) An S-poset A is said to be *sub-sequentially complete* (or simply *sub-s-complete*) if any sub-Cauchy sequence over A has a limit in A.

Separated S-posets, in which any two distinct points a and b in A can be separated by at least one $s \in S$, by $as \neq bs$, are an important class of s-posets. Here we should note that limits are not necessarily unique, unless A is separated.

In the next definition we generalize the concept of separated S-poset to the sub-separated one.

Definition 1.2.

(i) Let S be a posemigroup. An S-poset A is called *sub-separated* if $a \leq b$ in A whenever $as \leq bs$ for all $s \in S$.

(ii) For an S-poset A, we define $\mathcal{S}(A)$ to be the poset of all sub-Cauchy sequences over A with point-wise order. More explicitly, $\mathcal{S}(A) = \{(a_s)_{s\in S} \mid a_s t \geq a_{st}\}$ and $(a_s)_{s\in S} \leq (a'_s)_{s\in S}$ whenever $a_s \leq a'_s$, for all $s \in S$.

Remark 1.3. Every Sub-separated S-poset is a separated one. Indeed, $as = bs$, for all $s \in S$ implies $as \leq bs$ and $bs \leq as$, for all $s \in S$. Hence $a \leq b$ and $b \leq a$, by sub-separateness, and hence $a = b$.

Theorem 1.4.

(i) *The poset $\mathcal{S}(A)$ is an S-poset.*

(ii) *For every separated S-poset A, the S-poset $\mathcal{S}(A)$ is an extension of A. Also if A is sub-separated then $\mathcal{S}(A)$ is a regular extension of A.*

(iii) *The S-poset $\mathcal{S}(A)$ is sub-separated if $S^2 = S$.*

Proof. (i) To prove, first for a given S-poset A we define an action of S over $\mathcal{S}(A)$ as follows:

$$(a_s)_{s\in S} \cdot t = (a_{ts})_{s\in S},$$

for every $(a_s)_{s\in S} \in \mathcal{S}(A)$.

We note that, $\mathcal{S}(A)$ is closed under the defined action. Indeed, for every sub-Cauchy sequence $(a_s)_{s\in S}$ and $t \in S$, the sequence $(a_s)_{s\in S} \cdot t = (a_{ts})_{s\in S}$ is also a sub-Cauchy sequence, since for all $s \in S$ we have $a_{ts}s' \geq a_{(ts)s'}$.

Now we check the properties of being an S-posets:

(1) For every $t, t' \in S$, and a sub-Cauchy sequence $(a_s)_{s\in S}$, we have $((a_s)_{s\in S} \cdot t) \cdot t' = (a_{ts})_{s\in S} \cdot t'$. Take $(a_{ts})_{s\in S}$ to be $(b_s)_{s\in S}$, so $(a_{ts})_{s\in S} \cdot t' = (b_s)_{s\in S} \cdot t' = (b_{t's})_{s\in S} = (a_{t(t's)})_{s\in S} = (a_{(tt')s)})_{s\in S} = (a_s)_{s\in S} \cdot (tt')$.

(2) For sub-Cauchy sequences $(a_s)_{s\in S}$, $(a'_s)_{s\in S}$ and $t \in S$, if $(a_s)_{s\in S} \leq (a'_s)_{s\in S}$ then $(a_{ts})_{s\in S} \leq (a'_{ts})s \in S$, for every $s, t \in S$. Therefore $(a_s)_{s\in S} \cdot t \leq (a'_s)_{s\in S} \cdot t$.

(3) And, finally, for $t, t' \in S$ and sub-Cauchy sequence $(a)_{s\in S}$, if $t \leq t'$ then $ts \leq t's$, and hence $a_{ts} \leq a_{t's}$, by definition of sub-Cauchy sequence. Therefore, $(a_{ts})_{s\in S} \leq (a_{t's})_{s\in S}$, that is, $(a_s)_{s\in S}t \leq (a_s)_{s\in S}t'$.

(ii) To prove this part we consider the map $f : A \to \mathcal{S}(A)$ defined by $f(a) = \lambda_a$, see Definition 1.1 (ii). The defined f is one-one, since A is separated. Also since A is an S-poset, $as \leq bs$, for every $s \in S$, if $a \leq b$. This means that $\lambda_a \leq \lambda_b$, if $a \leq b$. That is f is order preserving. And finally f is equivariant, because $f(at) = (ats = a_{ts})_{s\in S} = (a_s)_{s\in S} \cdot t$.

Now suppose that A is, moreover, sub-separated. We show that the above S-poset embedding is an order embedding, too. Because if $\lambda_a \leq \lambda_b$ then $as \leq bs$, for every $s \in S$, and hence $a \leq b$, by sub-separateness of A.

(iii) To see that $\mathcal{S}(A)$ is sub-separated, take sub-Cauchy sequences $\gamma = (a_s)_{s\in S}$ and $\gamma' = (a'_s)_{s\in S}$ with $\gamma \cdot t \leq \gamma' \cdot t$ for all $t \in S$. This means that $a_{ts} \leq a'_{ts}$ for all $t, s \in S$. Now since $S^2 = S$, each $r \in S$ is in the form of st for some $t, s \in S$, then we have $a_r = a_{st} \leq a'_{st} = a'_r$ which means $\gamma \leq \gamma'$. ∎

2. Sub-sequential closure operator and dense regular monomorphisms

In this section, we are going to introduce a categorical closure operator C_s, in the sense of Tholen and Dikranjan in [14]. This is a weakening of C^d closure operator in [12], on the category of S-posets. We also consider the class of regular and dense monomorphisms with respect to the closure operator C_s and briefly study some categorical properties of this class of monomorphism which is needed in the sequel.

Definition 2.1. For an S-poset B, and a sub S-poset A of B, by the *s-closure* of A in B we mean $C_s(A) = \{b \in B | \forall s \in S, \exists a \in A; bs \leq a\}$.

We say that A is sub-*s-dense* in B if $C_s(A) = B$.

An S-poset morphism $f : A \to B$ is said to be *sub-sequentially dense* (or, simply, *sub-s-dense*) if $f(A)$ is a sub-s-dense sub S-poset of B.

Note 2.2. We show that the above introduced C_s is a categorical closure operator, in the sense of [14].

(Extensive) To prove $A \subseteq C_s(A)$, let $x \in A$. Then, since $xs \in A$ for all $s \in S$ and $xs \leq xs$ for $xs \in A$, $A \subseteq C_s(A)$.

(Monotonicity) To prove $A_1 \subseteq A_2$ implies $C_s(A_1) \subseteq C_s(A_2)$, let $x \in C_s(A_1)$. Then, for each $s \in S$ there exists $a \in A_1$ such that $xs \leq a$. But, since $A_1 \subseteq A_2$, there exists $a \in A_2$ such that $xs \leq a$. Therefore $C_s(A_1) \subseteq C_s(A_2)$.

(Continuity) We show that $f(C_s(A)) \leq C_s(f(A))$, for all S-poset morphism f from A. Let $y \in f(C_s(A))$. Then, there exists $x \in C_s(A)$ such that $y = f(x)$. Now for each $s \in S$ there exist $a \in A$ such that $xs \leq a$. Therefore, $ys = f(x)s = f(xs) \leq f(a)$, that is, $y \in C_s(A)$.

Theorem 2.3. *The sub-sequential closure operator is idempotent, that is,* $C_s(C_s(A)) = C_s(A)$, *if* $S^2 = S$.

Proof. By extensive, we have $C_s(A) \subseteq C_s(C_s(A))$. For the converse, let $x \in C_s(C_s(A))$. Then, for each $s \in S$, there exists $a_s \in C_s(A)$ such that $xs \leq a_s$. But, since $a \in C_s(A)$, there exist $a' \in A$ such that $as \leq a'_s$, for every $s \in S$. Now, using $S = S^2$, we have $xs = xs_1s_2 = (xs_1)s_2 \leq a_{s_1}s_2 \leq a'_{s_2}$, hence $x \in C_s(A)$. Therefore, $C_s(C_s(A)) \subseteq C_s(A)$ and hence $C_s(C_s(A)) = C_s(A)$. ■

Theorem 2.4. *Let B be an S-poset and A be a sub S-poset of B in which every subset has a least element. Then A in B is sub-s-dense if and only if for every sub-Cauchy sequence λ_b, induced by $b \in B$, there exists a sub-Cauchy sequence $(a_s)_{s\in S}$ in A such that $\lambda_b \leq (a_s)_{s\in S}$.*

Proof. ($\Rightarrow$) Let A be sub-*s-dense* in B and λ_b be the sub-Cauchy sequence induced by $b \in B$. Then for each $s \in S$ consider the subset $A_s = \{a \in A \mid bs \leq a\}$ of A. Then A_s has a least element such as a_s. Now we consider $(a_s)_{s\in S}$ which is a sub-Cauchy sequence. Because $b_{st} = bst = b_st \leq a_st \in A_{st}$, hence $a_{st} \leq a_st$.

($\Leftarrow$) The converse is trivial. ■

Theorem 2.5. *Let A be an S-poset in which every subset has a least element. Then the S-poset morphism $f : A \to \mathcal{S}(A)$ which maps every $a \in A$ to λ_a is sub-s-dense.*

Proof. We show that $C_s(f(A)) = \mathcal{S}(A)$. For each $\gamma = (a_s)_{s\in S} \in \mathcal{S}(A)$, since every subset of A has a least element, the set of upper bounds of $\gamma = (a_s)_{s\in S}$ has a least element, say $a \in A$. Therefore $\gamma \cdot t \leq \lambda_a$, for every $t \in S$, because $\gamma \cdot t = (a_{ts})s \in S$ and $a_{ts} \leq a_t s \leq as$. ∎

In Theorem 1.4, part (ii) it has been shown that every separated S-poset A is embedded in $\mathcal{S}(A)$. Now, by Theorem 2.5, we get the following corollary:

Corollary 2.6. *Every sub-separated S-poset A in which every subset has a least element is regular sub-s-dense sub-S-poset of $\mathcal{S}(A)$.*

Notation 2.7. In the rest of this paper we are going to study injectivity with respect to the class of regular sub-sequentially dense monomorphisms in the category of S-posets. So from now on we denote the class of regular sub-sequentially dense monomorphisms by $\mathcal{M}_s$.

In the following we see that the class of sub-s-dense S-poset monomorphisms has "good" properties with respect to composition.

Theorem 2.8. *Let $\mathcal{M}_s$ be the class of all sub-s-dense S-poset monomorphisms. Then,*

(i) *the class $\mathcal{M}_s$ is composition closed if $S^2 = S$.*

(ii) *the class $\mathcal{M}_s$ is isomorphism closed; that is, contains all isomorphisms and is closed under composition with isomorphisms.*

(iii) *the class $\mathcal{M}_s$ is left cancelable; that is, $gf : A \to B \to C, g : B \to C \in \mathcal{M}_s$ imply $f \in \mathcal{M}_s$.*

Proof. (i) It is easy to see that the composition of regular monomorphisms is a regular monomorphism. Now let $f : A \to B$ and $g : B \to C$ be $\mathcal{M}_s$-monomorphisms between S-poset. Then for every $c \in C$ and $s \in S$, since $S^2 = S$, the inequality $cs = cs_1s_2 \geq g(b_{s_1})s_2 = g(b_{s_1}s_2)$, for some b_{s_1}, is given by the fact that g is sub-s-dense. Now since f is sub-s-dense, $b_{s_1}s_2 \geq f(a_s)$, for some $a_s \in A$. So $cs = cs_1s_2 \geq g(b_{s_1})s_2 = g(b_{s_1}s_2) \geq g(f(a_s))$.

(ii) It is trivial.

(iii) Suppose that $f(a_1) \leq f(a_2)$. Then $gf(a_1) \leq gf(a_2)$ and hence, by regularity of gf, we have $a_1 \leq a_2$. That is f is regular. Also for each $b \in B$ and $s \in S$, there exists $a_s \in A$ such that $g(bs) = g(b)s \geq gf(a_s)$. Now, using regularity of g, we get $bs \geq f(a_s)$. That is $f \in \mathcal{M}_s$. ∎

Theorem 2.9. *In the category of S-posets, pushouts transfer $\mathcal{M}_s$-morphisms.*

Proof. As mentioned in [3], the pushout of S-poset maps $f : A \to B$ and $g : A \to C$ is the quotient of the coproduct $B\dot{\cup}C$ by the congruence $\Theta(H)$

generated by $H = \{(f(a), g(a)) \mid a \in A\}$. Now consider the pushout diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "\overline{g}"] \\ C \arrow[r, "\overline{f}"'] & (B\dot{\cup}C)/\Theta(H) \end{tikzcd}$$

in which f is an $\mathcal{M}_s$-morphisms. If $\overline{f}(c_1) = [c_1]_\Theta \leq [c_2]_\Theta = \overline{f}(c_2)$, then there exist $a_1, ..., a_n \in A$ such that

$$c_1 = g(a_1)\Theta(H)f(a_1) \leq f(a_2)\Theta(H)g(a_2) \leq \cdots \leq f(a_n)\Theta(H)g(a_n) = c_2.$$

Now, by regularity of f, we have

$$a_1 \leq a_2, a_3 \leq a_4, ..., a_{n-1} \leq a_n,$$

hence, since g is order preserving, we have

$$c_1 = g(a_1) \leq g(a_2) \leq g(a_3) \leq g(a_4) \leq \cdots \leq g(a_{n-1}) \leq g(a_n) = c_2.$$

That is, $\overline{f}$ is regular. Also, $\overline{f}$ is sub-sequentially dense, because, for each $[x]_\Theta \in (B\dot{\cup}C)/\Theta(H)$, $[x]_\Theta = [c]_\Theta$ or $[x]_\Theta = [b]_\Theta$, for some $c \in C$ and $b \in B$. In the first case $[x]_\Theta s = [c]_\Theta s = [cs]_\Theta = \overline{f}(cs)$, and in the second case, since f is sub-sequentially dense, there exists $a_s \in A$ with $bs \leq f(a_s)$. Hence $[x]_\Theta s = [b]_\Theta s = [bs]_\Theta \leq [f(a_s)]_\Theta = [g(a_s)]_\Theta = \overline{f}(g(a_s))$. ■

3. Regular sub-sequential dense injectivity

This section is devoted to the investigation of the behavior of $\mathcal{M}_s$-injectivity in the category of S-posets. First we give some definitions.

Definition 3.1. An S-poset A is called:

(1) *regular sub-sequentially dense injective* or briefly $\mathcal{M}_s$*-injective* if it is injective with respect to $\mathcal{M}_s$-monomorphisms. That is every commutative diagram

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d, "g"'] & A \arrow[dl, dotted, "\overline{g}"] \\ A & \end{tikzcd}$$

with $f \in \mathcal{M}_s$, can be completed by some S-poset morphism $\overline{g}$.

(2) *regular sub-sequentially absolute retract* or briefly $\mathcal{M}_s$*-absolute retract* if it is a retract of each of its regular sub-s-dense extensions.

Theorem 3.2. *An S-poset A is $\mathcal{M}_s$-injective if and only if for every $\mathcal{M}_s$-monomorphism $h : B \to B \sqcup cS$ to a singly generated extension of B, every S-poset morphism $f : B \to A$ can be extended to $g : B \sqcup cS \to A$.*

Proof. One direction is obvious. For the converse, let $h : B \to C$ be a sub-dense regular monomorphism and $f : B \to A$ be an S-poset morphism. Applying Zorn's Lemma on the poset of all sub S-poset D of C with $h(B) \subseteq D$ and such that there exists an S-poset morphism $g : D \to A$ such that $gh = f$, we get a maximal such S-poset, say D. If $D = C$ then the proof is complete, otherwise there exists $c \in C - D$. Now $h : B \to D \sqcup cS$ is an sub-dense regular monomorphism and by hypothesis there is an S-poset morphism $\overline{g}$ which extends g. This contradicts the maximality of D, so $C = D$. ∎

Theorem 3.3. *If a sub-separated S-poset A is sub-complete then it is sub-sequentially absolute retract.*

Proof. Let $f : A \to B$ be an $\mathcal{M}_s$-morphism. Then, by Theorem 2.4, for each $b \in B$, there exists a sub-Cauchy sequence $(a_S)_{s\in S}$ in A such that $\lambda_b \leq (a_s)_{s\in S}$. Since A is sub-complete, $(a_s)_{s\in S}$ has a limit such as $a_b \in A$. Now we define $g : B \to A$ by $g|_A = id_A$ and for $b \in B - A$, $g(b) = a_b$. The defined g is order preserving, because if $b \leq b'$ in B three following cases may occur:

Case 1. Both $b, b' \in B - A$. In this case $bt \leq b't$, for every $t \in S$. Hence $(a_s)_{s\in S} \leq (a'_s)_{s\in S}$, see how we get $(a_s)_{s\in S}$ and $(a'_s)_{s\in S}$ in the proof of Theorem 2.4. Therefore $a_s = a_b s \leq a_{b'} s = a'_s$ and, since A is sub-separated $a_b \leq a_{b'}$, $g(b) \leq g(b')$.

Case 2. Now let $b \in B - A$, $b' \in A$. Then if $b \leq b'$ then, for all $t \in S$, $bt \leq b't$ but, since $b't \in A$ and a_t is a least element in A such that $bt \leq a_t$, we have $a_t \leq b't$, for all $t \in S$. Therefore, $a_b t = a_t \leq b't$ for all $t \in S$. Now $a_b \leq b'$ is a clear result of sub-separateness of A, thus $g(b) \leq g(b')$.

Case 3. If $b, b' \in A$ there is nothing to prove.

Now, we show that g is equivariant. The only thing we have to check is $g(bs) = g(b)s$ when $b \in B - A$, for every $s \in S$. But, two cases may happen:

Case 1. $bs \in A$, for some $s \in S$. Then $g(bs) = bs$. So, by definition of a_s, $a_s = bs$. Since $a_b = \lim(a_s)_{s\in S}$, $a_b s = bs$. That is $g(b)s = g(bs)$.

Case 2. $bs \in B - A$, for some $s \in S$. Then take $g(bs)$ to be $a_{bs} = \lim(a'_{(t)})_{t\in S}$ and $g(b)$ to be $a_b = \lim(a_t)_{t\in S}$. By definition of $(a'_{(t)})_{t\in S}$ and $(a_t)_{t\in S}$, we have $a_{st} = a'_t$, for every $t \in S$. Hence $a_b st = a_{st} = a'_t$, for every $t \in S$. So

$$g(bs) = \lim(a'_{(t)})_{t\in S} = a_b s = g(b)s,$$

and we are done. ∎

Theorem 3.4. *An S-poset A is $\mathcal{M}_s$-injective if and only if it is an $\mathcal{M}$-absolute retract.*

Proof. Since in the category of S-posets, pushouts transfer $\mathcal{M}_s$- morphisms, by Theorem 2.2 in [4] we are done. ∎

Corollary 3.5. *Every sub-separated and sub-complete S-poset A is $\mathcal{M}_s$-injective.*

Proof. By Theorems 3.3 and 3.4 there is nothing to prove. ∎

Now, we study $\mathcal{M}_s$-essential extensions to get the $\mathcal{M}_s$-injective hulls.

Definition 3.6. An $\mathcal{M}_s$-monomorphism $f : A \to B$ is called *$\mathcal{M}_s$-essential* if it is essential with respect to $\mathcal{M}_s$-monomorphisms. That is if $g \circ f : A \to \ B \to C$ is a $\mathcal{M}_s$-monomorphism then g is an $\mathcal{M}_s$-monomorphism.

Also, by a *sub-s-regular injective hull* or, briefly, *$\mathcal{M}_s$-injective hull* of an S-poset A, we mean an $\mathcal{M}_s$-essential extension $f : A \to B$ in which B is an $\mathcal{M}_s$-injective.

Notice that an $\mathcal{M}_s$-injective hull is unique up to isomorphism.

Lemma 3.7. *A regular monomorphism f is $\mathcal{M}_s$-essential if and only if it is essential and s-dense.*

Proof. Let f be sub-s-essential. Then, by definition, f is s-dense. Also it is essential, for if $g : B \to C$ is a morphism such that $g \circ f$ is an S-monomorphism then since $g \circ f : A \to g(B)$ is a sub-dense monomorphism (this is because f is sub-dense and $g : B \to g(B)$) and f is sub-dense essential we get that $g : B \to g(B)$ is a sub-dense monomorphism and hence g is monomorphism. The converse is easy to show. Just notice that if the composition $g \circ f$ of two S-maps is an s-dense monomorphism then g is an s-dense monomorphism, too. ∎

We are going to find an $\mathcal{M}_s$-essential extension for an S-poset A, Banaschewski's condition, and Theorems of well behavior of injectivity for $\mathcal{M}_s$-injectivity, see [1]. To do so, for each S-poset A, we take the poset $\mathcal{C}(A)$ to be the class of all Cauchy sequences; that is $\mathcal{C}(A) = \{(a_s)_{s\in S} \mid a_{st} = a_s t$, for every $s, t \in S$ and $a_s \leq a_t$ if $s \leq t\}$ in which $(a_s)_{s\in S} \leq (a'_s)_{s\in S}$ whenever $a_s \leq a'_s$, for every $s \in S$.

Now, we have the following theorem:

Theorem 3.8. *Given an S-poset A,*

(i) *the poset $\mathcal{C}(A)$ is an S-poset.*

(ii) *if A is sub-separated then the S-poset $\mathcal{C}(A)$ is an $\mathcal{M}_s$-essential extension of A.*

Proof. (i) To make $\mathcal{C}(A)$ an S-poset, we define an action of S on $\mathcal{C}(A)$ as follows:

$$(a_s)_{s\in S}\cdot t=\lambda_{a_t},\ \text{for every } (a_s)_{s\in S}\in\mathcal{C}(A) \text{ and for every } t\in S.$$

Obviously, $\mathcal{C}(A)$ is closed under the defined action. Now note that $\mathcal{C}(A)$ with the defined action is an S-poset, because:

(1) for every $t,t'\in S$, and each Cauchy sequence $(a_s)_{s\in S}$, we have

$$((a_s)_{s\in S}\cdot t)\cdot t'=\lambda_{a_t}\cdot t'=\lambda_{a_t t'}=\lambda_{a_{tt'}}=(a_s)_{s\in S}\cdot(tt').$$

The third equality is given by the definition of the Cauchy sequence $(a_s)_{s\in S}$.

(2) for each pair of the Cauchy sequences $(a_s)_{s\in S}$, $(a'_s)_{s\in S}$ and $t\in S$, if $(a_s)_{s\in S}\leq(a'_s)_{s\in S}$ then $a_t\leq a'_t$, and hence $\lambda_{a_t}\leq\lambda_{a_{t'}}$. That is, $(a_s)_{s\in S}{\cdot}t\leq(a'_s)_{s\in S}{\cdot}t$.

(3) and finally for $t,t'\in S$ and a Cauchy sequence $(a)_{s\in S}$, if $t\leq t'$ then $a_t\leq a_{t'}$, by Definition 1.1, therefore $\lambda_a\leq\lambda_{a_{t'}}$.

(ii) To prove this part, we consider the map $f:A\to\mathcal{C}(A)$ defined by $f(a)=\lambda_a$. First we show that f is an S-poset monomorphism. Indeed, f is one-one since A is separated. Also since A is an S-poset, $a\leq b$ implies $as\leq bs$, for every $s\in S$. So, if $a\leq b$ then $\lambda_a\leq\lambda_b$. That is, f is order preserving. And, finally, f is equivariant, because $f(at)=(ats=at_s=a_{ts})_{s\in S}=(a_s)_{s\in S}\cdot t$.

Regularity of f is a clear consequence of the fact that A is sub-separated. Also, f is sub-sequentially dense. Because for every Cauchy sequence γ and every $s\in S$, we have $\gamma\cdot s\leq\lambda_{a_s}\in f(A)$.

Now, we are going to show that f is $\mathcal{M}_s$-essential. Let $gf:A\to\mathcal{C}(A)\to C$ be an $\mathcal{M}_s$-morphism, for some S-poset morphism g, and $g(\gamma)=g(\gamma')$. Then

$$g(\gamma)s=g(\gamma\cdot s)=g(\lambda_{a_s})=g(\lambda_{a'_s})=g(\gamma'\cdot s)=g(\gamma')s,$$

that is $gf(a_s)=g(f(a'_s))$. Since gf is one-one, we have $a_s=a'_s$.

Now, suppose that $g(\gamma)\leq g(\gamma')$, then

$$g(\gamma)s=g(\gamma\cdot s)=g(\lambda_{a_s})=gf(a_s)\leq gf(a'_s)=g(\lambda_{a'_s})=g(\gamma'\cdot s)=g(\gamma)s,$$

for each $s\in S$. Using the regularity of gf, we have $a_s\leq a'_s$, for every $s\in S$. So $\gamma\leq\gamma'$. Finally, we show that g is sub-sequentially dense. For each $c\in C$, since gf is sub-sequentially dense, for every $s\in S$, there exists $a_s\in A$ such that $cs\leq gf(a_s)$. That is $cs\leq g(\lambda_{(a_s)})$ and we are done. ∎

Theorem 3.9. *Let $S^2=S$. Then for each S-poset A, the S-poset $\mathcal{C}(A)$ is sub-s-complete.*

Proof. Let $\gamma=(\gamma_s)_{s\in S}$ be a sub-Cauchy sequence in $\mathcal{C}(A)$. That is $\gamma_s=(a^s_t)_{t\in S}$ is a Cauchy sequence, for every $s\in S$. So we have $a^s_t t'=a^s_{tt'}$, for every $s,t,t'\in S$. And hence, for each $t\in S$,

$$\gamma\cdot t=\lambda_{\gamma_t}=(\gamma_t\cdot s)_{s\in S}=(\lambda_{a^t_s})_{s\in S}=(a^t_s t')_{s,t'\in S}=(a^t_{st'})_{s,t\in S}=(a^t_k)_{k\in S}=\gamma_t,$$

in which the sixth equality is given by $S^2=S$. So γ is a limit for the sub-Cauchy sequence $(\gamma_s)_{s\in S}$. ∎

Theorem 3.10. *Let $S^2 = S$ and A be a sub-separated S-poset. Then $\mathcal{C}(A)$ is the $\mathcal{M}_s$-injective hull of A.*

Proof. By Theorems 3.3 and 3.9, $\mathcal{C}(A)$ is $\mathcal{M}_s$-absolute retract. And since we have Banaschewki's condition for sub-separated S-posets, so $\mathcal{C}(A)$ is injective, see Corollary 3.5. Now, by Theorem 3.8, we are done. ∎

4. Left zero posemigroup

In this short section, we examine the above results for a special case of ordered semigroups. The semigroup we have chosen here is the left zero one. We will see that the $\mathcal{M}_s$-injective hulls in this special and important class of S-posets, have a simpler form.

To do so, we will also use the morphism notation A^S of Cauchy sequences and imitate the above proofs in this notation. We will also see that the fixed elements of an S-poset A in this class play an important role to simplify the $\mathcal{M}_s$-injective hulls.

An element a of an S-poset A is called a *fixed element* if $as = a$ for all $s \in S$. The set of all fixed elements of an S-poset A is denoted by $FixA$, which is in fact a sub S-poset of A.

Note also that, a partial order on a left zero semigroup is automatically compatible with the operation of S. For, if $s \leq t$ and $r \in S$, then we have $sr = s \leq t = tr$ and $rs = r = rt$.

Remark 4.1. Analogously to the proof of Theorem 3.8 part (i), one can easily see that $(FixA)^S$ is an S-poset, except that we should note that the action on $(FixA)^S$ is of the form $f \cdot t : S \to A$ which maps every $s \in S$ to $f(t)s = f(ts) = f(t)$, for every $f \in (FixA)^S$ and every $t \in S$.

Theorem 4.2. *Let S be a left zero posemigroup and A be an S-poset. Then, the S-poset $(FixA)^S$ is sub-complete.*

Proof. Let $\gamma = (f_s)_{s\in S}$ be a sub-Cauchy sequence over $(FixA)^S$. Hence $f_s \cdot r \geq f_{sr} = f_s$ for all $f_s \in (FixA)^S$ and $r \in S$. That is $f_s \cdot r(t) = f_s(rt) = f_s(r) \geq f_s(t)$ for all $r, t \in S$ and this implies that f_s is a constant sequence. So $f_s(t) = f_s(s)$. Now, consider $f : S \to FixA$ which maps each $s \in S$ to $f_s(s)$. Then, for every $s \in S$, we have

$$f \cdot s(t) = f(st) = f(s) = f_s(s) = f_s(t).$$

That is, f is a limit of the sub-Cauchy sequence $(f_s)_{s\in S}$ in $(FixA)^S$. ∎

According to the above theorem and Theorems 3.3 and 3.4, we have the following.

Theorem 4.3. *Let S be a left zero posemigroup and A be a sub-separated S-poset. Then, the S-poset $(FixA)^S$ is sub-sequential regular injective.*

Theorem 4.4. *Let S be a left zero posemigroup and A be a sub-separated S-poset. Then, the S-poset $(FixA)^S$ is the sub-sequential regular injective hull of A.*

Proof. First, we show that $(FixA)^S$ is sub-sequential dense regular essential extension of A. To do so, we define the map $\varphi : A \to (FixA)^S$ by

$$\varphi(a) = \lambda_a : S \to FixA,$$

in which $\lambda_a(s) = as$. We note that $\varphi(A)$ is s-dense in $(FixA)^S$. Because, for each $f \in (FixA)^S$ and $t \in S$ consider $\lambda_{f(t)}$. Then we have

$$f \cdot t(s) = f(ts) = f(t) = f(t)s = \lambda_{f(t)}(s), \tag{4.1}$$

so $(FixA)^S \subseteq C_s(\varphi(A))$. It is clear that φ is a regular monomorphism.

Now, we show that φ is essential. Let $\alpha : (FixA)^S \to B$ be an S-poset morphism such that $\alpha \circ \varphi$ is one-one and $\alpha(f) = \alpha(f')$ for some $f, f' \in (FixA)^S$. Hence, for each $t \in S$ we have $\alpha(f)t = \alpha(f')t$. Thus, by (4.1),

$$\alpha \circ \varphi(f(t)) = \alpha(\lambda_{f(t)}) = \alpha(\lambda_{f'(t)}) = \alpha \circ \varphi(f'(t)),$$

for each $t \in S$, and therefore $f(t) = f'(t)$, for each $t \in S$. Now, Lemma 3.7 ensures that φ is sub s-essential and by the former theorem we are done. ∎

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STRONG CONVERGENCE THEOREMS FOR FIXED POINT PROBLEMS AND EQUILIBRIUM PROBLEMS WITH APPLICATIONS

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Abstract. In this paper, we present a new iterative algorithm for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of an equilibrium problem, and we prove the strong convergence theorems in Hilbert spaces. We also apply our results to the convex minimization and variational inequality problems. Our results extend and improve some recent results of Cai, Tang and Liu [Cai, Y., Tang, Y. and Liu, L.: *Iterative algorithms for minimum-norm fixed point of nonexpansive mapping in Hilbert space*. Fixed Point Theory Appl., 2012:49 (2012)] and others.

1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H. A mapping $T : C \to C$ is called nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. The set of fixed points of T is denoted by $\mathrm{Fix}(T)$. It is well known that $\mathrm{Fix}(T)$ is closed and convex. Halpern [1] introduced an iterative method for approximation of fixed points of a nonexpansive mapping as follows: $u \in C,\ x_1 \in C$ and

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n,$$

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where $\{\alpha_n\}$ is a sequence in $[0,1]$. Wittmann [2] proved that $\{x_n\}$ converges strongly to $P_{\mathrm{Fix}(T)}u$ if $\{\alpha_n\}$ satisfies the conditions:

$$\lim_{n\to\infty}\alpha_n = 0, \sum_{n=1}^{\infty}\alpha_n = \infty \text{ and } \sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty,$$

where $P_{\mathrm{Fix}(T)}$ is the metric projection onto $\mathrm{Fix}(T)$. Especially, letting $0 = u \in C$, we see that the sequence $\{x_n\}$ converges strongly to $P_{\mathrm{Fix}(T)}0$, i.e., the minimum-norm fixed point of T. Cai, Tang and Liu [3] presented an iterative method for finding the minimum-norm fixed point of T.

Let f be a bifunction from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for f is to find $\overline{x} \in C$ such that $f(\overline{x}, y) \geq 0$ for all $y \in C$. The set of such solutions is denoted by $EP(f)$. Numerous problems in physics, optimization and economics can be reduced to find a solution of the equilibrium problem (for instance, see [4]). For solving equilibrium problem, we assume that the bifunction f satisfies the following conditions:

(A1) $f(x,x) = 0$ for all $x \in C$;

(A2) f is monotone, i.e., $f(x,y) + f(y,x) \leq 0$ for all $x, y \in C$;

(A3) For every $x, y, z \in C$, $\limsup_{t\downarrow 0} f(tz + (1-t)x, y) \leq f(x,y)$;

(A4) $f(x,\cdot)$ is convex and lower semicontinuous for each $x \in C$.

Some methods have been proposed to solve the equilibrium problem (see [4]–[12], [15]).

The methods for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of an equilibrium problem have received much attention in recent years. Tada and Takahashi [5] introduced an iterative scheme as follows.

$$\begin{cases} x_0 = x \in H, \\ u_n \in C, \\ \text{such that } f(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0 \text{ for all } y \in C, \\ \omega_n = (1-\alpha_n)x_n + \alpha_n T u_n, \\ C_n = \{z \in H : \|\omega_n - z\| \leq \|x_n - z\|\}, \\ D_n = \{z \in H : \langle x_n - z, x - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n\cap D_n}(x_0), \end{cases}$$

where $\{\alpha_n\} \subset [a,1]$ for some $a \in (0,1)$ and $\{r_n\} \subset (0,\infty)$ satisfies $\liminf_{n\to\infty} r_n > 0$. They proved that the sequence $\{x_n\}$ converges to $P_{\mathrm{Fix}(T)\cap EP(f)}x$. Takahashi and Takahashi [6] obtained the following iterative scheme

$$\begin{cases} x_1 \in H, \\ u_n \in C, \\ \text{such that } f(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0 \text{ for all } y \in C, \\ x_{n+1} = \alpha_n g(x_n) + (1-\alpha_n)Tu_n, \end{cases}$$

where g is a contraction on H. Under appropriate conditions they showed that the sequence $\{x_n\}$ converges strongly to an element of the set $\mathrm{Fix}(T) \cap EP(f)$.

Motivated by the above results, in this paper, we present a new iterative algorithm for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of an equilibrium problem, and prove the strong convergence theorems in Hilbert spaces. Finally, we give the applications to the convex minimization and variational inequality problems. Our results extend and improve some recent results of Cai, Tang and Liu [3] and others.

2. Preliminaries

Throughout this paper, let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, and let C be a nonempty closed convex subset of H. We write $x_n \to x$ to indicate that the sequence $\{x_n\}$ converges strongly to x. Similarly, the notation $x_n \rightharpoonup x$ means weak convergence. It is well known that H satisfies Opial's condition, that is, for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, we have

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\| \text{ for all } y \neq x. \tag{2.1}$$

For any $x \in H$, there exists a unique point $P_C x \in C$ such that

$$\|x - P_C x\| \leq \|x - y\|, \text{ for all } y \in C.$$

P_C is called the metric projection of H onto C. Note that P_C is a nonexpansive mapping of H onto C. For $x \in H$ and $z \in C$, we have

$$z = P_C x \iff \langle z - y, x - z\rangle \geq 0 \text{ for every } y \in C. \tag{2.2}$$

We need the following lemmas.

Lemma 2.1. [4] *Let C be a nonempty closed convex subset of H, and let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1) $-$ (A4). *If $r > 0$ and $x \in H$, then there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \textit{ for all } y \in C.$$

Lemma 2.2. [11] *Let C be a nonempty closed convex subset of H, and let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1) – (A4). *For $r > 0$, define a mapping $T_r : H \to 2^C$ as follows:*

$$T_r(x) = \{z \in C : f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \text{ for all } y \in C\}.$$

Then the following hold:

(i) *T_r is single-valued;*

(ii) *T_r is firmly nonexpansive, i.e., for any $x, y \in H$,*

$$\langle x - y, T_r x - T_r y\rangle \geq \|T_r x - T_r y\|^2;$$

(iii) $\mathrm{Fix}(T_r) = EP(f)$*;*

(iv) *$EP(f)$ is closed and convex.*

Lemma 2.3. [12] *Suppose that (A1) – (A4) hold. If $x, y \in H$ and $r_1, r_2 > 0$, then*

$$\|T_{r_2}y - T_{r_1}x\| \leq \|y - x\| + \frac{|r_2 - r_1|}{r_2}\|T_{r_2}y - y\|.$$

Lemma 2.4. [13] *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with*

$$0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1.$$

Suppose $x_{n+1} = (1 - \beta_n)x_n + \beta_n y_n$ for all integers $n \geq 1$ and

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then

$$\lim_{n\to\infty} \|y_n - x_n\| = 0.$$

The following lemma is an immediate consequence of the inner product on H.

Lemma 2.5. *For all $x, y \in H$, the inequality $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$ holds.*

Lemma 2.6. [14] *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying $a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n\beta_n$, where*

(i) $\{\alpha_n\} \subset (0, 1)$, $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$*;*

(ii) $\limsup\limits_{n\to\infty} \beta_n \leq 0$.

Then $\lim\limits_{n\to\infty} a_n = 0$.

3. Strong convergence theorems

In this section, we introduce a new iterative algorithm for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of an equilibrium problem, and we prove the strong convergence theorems in Hilbert spaces.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H, and let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1) – (A4). *Suppose that $T : C \to C$ is a nonexpansive mapping such that* $\mathrm{Fix}(T) \cap EP(f) \neq \emptyset$. *For $\lambda \in (0, 1)$ and $\omega \in C$, let $\{x_n\}$ and $\{u_n\}$ be sequences generated by*

$$\tag{3.1} \begin{cases} x_1 \in C \textit{ chosen arbitrarily}, \\ u_n \in C, \\ \textit{such that } f(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0 \textit{ for all } y \in C, \\ x_{n+1} = (1 - \alpha_n)[\lambda T u_n + (1 - \lambda)x_n] + \alpha_n \omega, \end{cases}$$

where the sequences $\{\alpha_n\} \subset (0, 1)$ and $\{r_n\} \subset (0, \infty)$ satisfy the following conditions:

(1) $\lim\limits_{n\to\infty} \alpha_n = 0$ *and* $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$;

(2) $\liminf\limits_{n\to\infty} r_n > 0$ *and* $\lim\limits_{n\to\infty} |r_{n+1} - r_n| = 0$.

Then the sequence $\{x_n\}$ converges strongly to $P_{\mathrm{Fix}(T)\cap EP(f)}\omega$.

Proof. Note that $\mathrm{Fix}(T) \cap EP(f)$ is a closed convex subset of H since $\mathrm{Fix}(T)$ and $EP(f)$ are closed and convex. For simplicity, we write $\Omega := \mathrm{Fix}(T) \cap EP(f)$.

From Lemmas 2.1 and 2.2, we have $u_n = T_{r_n} x_n$, and for any $z \in \Omega$,

$$\tag{3.2} \|u_n - z\| = \|T_{r_n} x_n - T_{r_n} z\| \leq \|x_n - z\|.$$

It follows that

$$\tag{3.3} \begin{aligned} \|x_{n+1} - z\| &= \|(1 - \alpha_n)[\lambda T u_n + (1 - \lambda)x_n] + \alpha_n \omega - z\| \\ &\leq (1 - \alpha_n)\|\lambda T u_n + (1 - \lambda)x_n - z\| + \alpha_n\|\omega - z\| \\ &\leq (1 - \alpha_n)[\lambda\|T u_n - z\| + (1 - \lambda)\|x_n - z\|] + \alpha_n\|\omega - z\| \\ &\leq (1 - \alpha_n)[\lambda\|u_n - z\| + (1 - \lambda)\|x_n - z\|] + \alpha_n\|\omega - z\| \\ &\leq (1 - \alpha_n)\|x_n - z\| + \alpha_n\|\omega - z\| \\ &\leq \max\{\|x_n - z\|, \|\omega - z\|\}. \end{aligned}$$

From a simple inductive process, we get

$$\|x_{n+1} - z\| \leq \max\{\|x_1 - z\|, \|\omega - z\|\},$$

which yields that $\{x_n\}$ is bounded. So is the sequence $\{u_n\}$.

Setting

$$y_n = \frac{(1-\alpha_n)\lambda T u_n + \alpha_n \omega}{\alpha_n + (1-\alpha_n)\lambda},$$

one has

$$\begin{aligned}
&\|y_{n+1} - y_n\| \\
&\quad = \Big\| \frac{(1-\alpha_{n+1})\lambda T u_{n+1} + \alpha_{n+1}\omega}{\alpha_{n+1} + (1-\alpha_{n+1})\lambda} - \frac{(1-\alpha_n)\lambda T u_n + \alpha_n \omega}{\alpha_{n+1} + (1-\alpha_{n+1})\lambda} \\
&\quad + \frac{(1-\alpha_n)\lambda T u_n + \alpha_n \omega}{\alpha_{n+1} + (1-\alpha_{n+1})\lambda} - \frac{(1-\alpha_n)\lambda T u_n + \alpha_n \omega}{\alpha_n + (1-\alpha_n)\lambda} \Big\| \\
&\quad \le \Big\| \frac{(\alpha_{n+1}-\alpha_n)\omega + (1-\alpha_{n+1})\lambda T u_{n+1} - (1-\alpha_n)\lambda T u_n}{\alpha_{n+1} + (1-\alpha_{n+1})\lambda} \Big\| \\
&\quad + \Big| \frac{1}{\alpha_{n+1} + (1-\alpha_{n+1})\lambda} - \frac{1}{\alpha_n + (1-\alpha_n)\lambda} \Big| \cdot \big\|(1-\alpha_n)\lambda T u_n + \alpha_n \omega\big\| \\
&\quad \le \frac{1}{\alpha_{n+1} + (1-\alpha_{n+1})\lambda} \big[\big\|(\alpha_{n+1}-\alpha_n)\omega\big\| \\
&\qquad + \big\|(1-\alpha_{n+1})\lambda T u_{n+1} - (1-\alpha_n)\lambda T u_n\big\|\big] \\
&\quad + \Big| \frac{1}{\alpha_{n+1} + (1-\alpha_{n+1})\lambda} - \frac{1}{\alpha_n + (1-\alpha_n)\lambda} \Big| \cdot \big\|(1-\alpha_n)\lambda T u_n + \alpha_n \omega\big\| \\
&\quad \le \frac{1}{\alpha_{n+1} + (1-\alpha_{n+1})\lambda} \big[\big\|(\alpha_{n+1}-\alpha_n)\omega\big\| \\
&\qquad + \big\|(1-\alpha_{n+1})\lambda T u_{n+1} - (1-\alpha_{n+1})\lambda T u_n\big\| \\
&\qquad + \big\|(1-\alpha_{n+1})\lambda T u_n - (1-\alpha_n)\lambda T u_n\big\|\big] \\
&\qquad + \Big| \frac{1}{\alpha_{n+1} + (1-\alpha_{n+1})\lambda} - \frac{1}{\alpha_n + (1-\alpha_n)\lambda} \Big| \cdot \big\|(1-\alpha_n)\lambda T u_n + \alpha_n \omega\big\| \\
&\quad \le \frac{1}{\alpha_{n+1} + (1-\alpha_{n+1})\lambda} \big[\big\|(\alpha_{n+1}-\alpha_n)\omega\big\| + (1-\alpha_{n+1})\lambda\big\|u_{n+1} - u_n\big\| \\
&\qquad + \big|\alpha_{n+1} - \alpha_n\big|\lambda\big\|T u_n\big\|\big] \\
&\quad + \Big| \frac{1}{\alpha_{n+1} + (1-\alpha_{n+1})\lambda} - \frac{1}{\alpha_n + (1-\alpha_n)\lambda} \Big| \cdot \big\|(1-\alpha_n)\lambda T u_n + \alpha_n \omega\big\|.
\end{aligned}$$

Since $\liminf_{n\to\infty} r_n > 0$, without loss of generality, we may assume that there exists a real number c such that $r_n > c > 0$ for all $n \ge 1$. According to Lemma 2.3, it follows that

$$
\begin{aligned}
&\|y_{n+1}-y_n\|\\
&\quad\leq \frac{1}{\alpha_{n+1}+(1-\alpha_{n+1})\lambda}\big[\big\|(\alpha_{n+1}-\alpha_n)\omega\big\|+[(1-\alpha_{n+1})\lambda+\alpha_{n+1}]\big\|u_{n+1}-u_n\big\|\\
&\quad+\big|\alpha_{n+1}-\alpha_n\big|\lambda\big\|Tu_n\big\|\big]\\
&\quad+\Big|\frac{1}{\alpha_{n+1}+(1-\alpha_{n+1})\lambda}-\frac{1}{\alpha_n+(1-\alpha_n)\lambda}\Big|\cdot\big\|(1-\alpha_n)\lambda Tu_n+\alpha_n\omega\big\|\\
&\quad\leq \frac{1}{\alpha_{n+1}+(1-\alpha_{n+1})\lambda}\big[\big\|(\alpha_{n+1}-\alpha_n)\omega\big\|+[(1-\alpha_{n+1})\lambda+\alpha_{n+1}]\big(\big\|x_{n+1}-x_n\big\|\\
&\qquad+\frac{|r_{n+1}-r_n|}{c}\big\|u_{n+1}-x_{n+1}\big\|\big)+|\alpha_{n+1}-\alpha_n|\lambda\big\|Tu_n\big\|\big]\\
&\qquad+\Big|\frac{1}{\alpha_{n+1}+(1-\alpha_{n+1})\lambda}-\frac{1}{\alpha_n+(1-\alpha_n)\lambda}\Big|\cdot\big\|(1-\alpha_n)\lambda Tu_n+\alpha_n\omega\big\|\\
&\quad\leq \big\|x_{n+1}-x_n\big\|+\frac{1}{\alpha_{n+1}+(1-\alpha_{n+1})\lambda}\big[\big\|(\alpha_{n+1}-\alpha_n)\omega\big\|\\
&\qquad+[(1-\alpha_{n+1})\lambda+\alpha_{n+1}]\frac{|r_{n+1}-r_n|}{c}\big\|u_{n+1}-x_{n+1}\big\|+|\alpha_{n+1}-\alpha_n|\lambda\big\|Tu_n\big\|\big]\\
&\qquad+\Big|\frac{1}{\alpha_{n+1}+(1-\alpha_{n+1})\lambda}-\frac{1}{\alpha_n+(1-\alpha_n)\lambda}\Big|\cdot\big\|(1-\alpha_n)\lambda Tu_n+\alpha_n\omega\big\|.
\end{aligned}
$$

Therefore, we get

$$\limsup_{n\to\infty}(\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\|)\leq 0.$$

It follows from Lemma 2.4 that $\lim\limits_{n\to\infty}\|y_n-x_n\|=0$. Thus,

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}[\alpha_n+(1-\alpha_n)\lambda]\|y_n-x_n\|=0. \tag{3.4}$$

For any $z\in\Omega$, we have

$$
\begin{aligned}
\|u_n-z\|^2&=\|T_{r_n}x_n-T_{r_n}z\|^2\\
&\leq\langle x_n-z,u_n-z\rangle\\
&=\frac{1}{2}\left[\|x_n-z\|^2+\|u_n-z\|^2-\|x_n-u_n\|^2\right],
\end{aligned}
$$

which yields

$$\|u_n-z\|^2\leq\|x_n-z\|^2-\|x_n-u_n\|^2. \tag{3.5}$$

By (3.1), one has

$$\begin{aligned}
\|x_{n+1}-z\|^2 &= \|(1-\alpha_n)[\lambda Tu_n+(1-\lambda)x_n]+\alpha_n\omega-z\|^2\\
&= \|(1-\alpha_n)[\lambda Tu_n+(1-\lambda)x_n-z]+\alpha_n(\omega-z)\|^2\\
&\le (1-\alpha_n)\|\lambda Tu_n+(1-\lambda)x_n-z\|^2+\alpha_n\|\omega-z\|^2\\
&\le (1-\alpha_n)\|\lambda(Tu_n-z)+(1-\lambda)(x_n-z)\|^2+\alpha_n\|\omega-z\|^2\\
&\le (1-\alpha_n)[\lambda\|u_n-z\|^2+(1-\lambda)\|x_n-z\|^2]+\alpha_n\|\omega-z\|^2\\
&\le (1-\alpha_n)[\lambda\|x_n-z\|^2-\lambda\|x_n-u_n\|^2+(1-\lambda)\|x_n-z\|^2]\\
&\quad+\alpha_n\|\omega-z\|^2\\
&\le \|x_n-z\|^2-(1-\alpha_n)\lambda\|x_n-u_n\|^2+\alpha_n\|\omega-z\|^2,
\end{aligned}$$

which implies

$$\begin{aligned}
(1-\alpha_n)\lambda\|x_n-u_n\|^2 &\le \|x_n-z\|^2-\|x_{n+1}-z\|^2+\alpha_n\|\omega-z\|^2\\
&\le \|x_n-x_{n+1}\|(\|x_n-z\|+\|x_{n+1}-z\|)+\alpha_n\|\omega-z\|^2.
\end{aligned}$$

Consequently, equality (3.4) and $\lim_{n\to\infty}\alpha_n=0$ imply

$$\lim_{n\to\infty}\|x_n-u_n\|=0. \tag{3.6}$$

Observe that

$$\begin{aligned}
&\|x_n-Tx_n\|\\
&\le \|x_n-x_{n+1}\|+\|x_{n+1}-Tx_n\|\\
&\le \|x_n-x_{n+1}\|+\|(1-\alpha_n)[\lambda Tu_n+(1-\lambda)x_n]+\alpha_n\omega-Tx_n\|\\
&\le \|x_n-x_{n+1}\|+\|(1-\alpha_n)[\lambda Tu_n+(1-\lambda)x_n-Tx_n]+\alpha_n(\omega-Tx_n)\|\\
&\le \|x_n-x_{n+1}\|+\|(1-\alpha_n)[\lambda(Tu_n-Tx_n)+(1-\lambda)(x_n-Tx_n)]\\
&\quad+\alpha_n(\omega-Tx_n)\|\\
&\le \|x_n-x_{n+1}\|+(1-\alpha_n)\lambda\|u_n-x_n\|+(1-\alpha_n)(1-\lambda)\|x_n-Tx_n\|\\
&\quad+\alpha_n\|\omega-Tx_n\|,
\end{aligned}$$

which deduces

$$\begin{aligned}
&[1-(1-\alpha_n)(1-\lambda)]\|x_n-Tx_n\|\\
&\qquad\le \|x_n-x_{n+1}\|+(1-\alpha_n)\lambda\|u_n-x_n\|+\alpha_n\|\omega-Tx_n\|.
\end{aligned}$$

It follows from (3.4), (3.6) and $\lim_{n\to\infty}\alpha_n=0$ that

$$\lim_{n\to\infty}\|x_n-Tx_n\|=0. \tag{3.7}$$

We shall prove that

$$\limsup_{n\to\infty}\langle \omega - z_0, x_n - z_0\rangle \le 0,$$

where $z_0 = P_\Omega\,\omega$. In order to show this inequality, we can choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle \omega - z_0, x_n - z_0\rangle = \lim_{i\to\infty}\langle \omega - z_0, x_{n_i} - z_0\rangle. \tag{3.8}$$

Based on the boundedness of $\{x_{n_i}\}$, without loss of generality, we assume that $x_{n_i} \rightharpoonup p$. Now we show that $p \in \Omega$. As $\{x_n\} \subset C$ and C is a closed convex set, one has $p \in C$. We firstly prove that $p \in EP(f)$. According to (3.1), we have

$$f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0 \text{ for all } y \in C.$$

The monotonicity of f yields

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge f(y, u_n) \text{ for all } y \in C.$$

Replacing n by n_i, we obtain

$$\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle \ge f(y, u_{n_i}) \text{ for all } y \in C.$$

It follows from (3.6) and (A4) that

$$f(y, p) \le 0 \text{ for all } y \in C.$$

For $0 < t \le 1$, $y \in C$, set $y_t = ty + (1-t)p$. Then $y_t \in C$ and $f(y_t, p) \le 0$. Hence

$$0 = f(y_t, y_t) \le tf(y_t, y) + (1-t)f(y_t, p) \le tf(y_t, y).$$

Dividing by t, we see that

$$f(y_t, y) \ge 0.$$

Letting $t \downarrow 0$, we get

$$f(p, y) \ge 0 \text{ for all } y \in C.$$

That is, $p \in EP(f)$.

Now we prove that $p \in \mathrm{Fix}(T)$. Otherwise, assume that $p \notin \mathrm{Fix}(T)$, that is, $p \ne Tp$. The Opial's condition and (3.7) imply

$$\begin{aligned}
\liminf_{i\to\infty}\|x_{n_i} - p\| &< \liminf_{i\to\infty}\|x_{n_i} - Tp\| \\
&= \liminf_{i\to\infty}\|x_{n_i} - Tx_{n_i} + Tx_{n_i} - Tp\| \\
&= \liminf_{i\to\infty}\|Tx_{n_i} - Tp\| \\
&\le \liminf_{i\to\infty}\|x_{n_i} - p\|.
\end{aligned}$$

This is a contradiction. Thus $p \in \mathrm{Fix}(T)$. By (3.8) and the property of metric projection, we have

$$\begin{aligned}\limsup_{n\to\infty}\langle \omega - z_0, x_n - z_0\rangle &= \lim_{i\to\infty}\langle \omega - z_0, x_{n_i} - z_0\rangle \\ &= \langle \omega - z_0, p - z_0\rangle \leq 0.\end{aligned} \tag{3.9}$$

Using (3.1) again, we obtain

$$\begin{aligned}\|x_{n+1} - z_0\|^2 &= \|(1-\alpha_n)[\lambda T u_n + (1-\lambda)x_n - z_0] + \alpha_n(\omega - z_0)\|^2 \\ &\leq (1-\alpha_n)\|\lambda(Tu_n - z_0) + (1-\lambda)(x_n - z_0)\|^2 \\ &\quad + 2\langle \alpha_n(\omega - z_0), x_{n+1} - z_0\rangle \\ &\leq (1-\alpha_n)\|x_n - z_0\|^2 + 2\alpha_n\langle \omega - z_0, x_{n+1} - z_0\rangle.\end{aligned}$$

It follows from (3.9) and Lemma 2.6 that $\{x_n\}$ converges strongly to z_0. ∎

Remark 1. The iterative algorithm in Theorem 3.1 weakens the conditions in Theorem 3.1 of Wang et al. [15] and Theorem 3.2 of Takahashi and Takahashi [6] in the following aspects:

(i) the condition $\sum_{n=1}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty$ is removed;

(ii) the condition $\sum_{n=1}^{\infty}|r_{n+1} - r_n| < \infty$ is weakened by the condition

$$\lim_{n\to\infty}|r_{n+1} - r_n| = 0.$$

We immediately obtain the following corollaries by Theorem 3.1.

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Suppose that $T : C \to C$ is a nonexpansive mapping such that* $\mathrm{Fix}(T) \neq \emptyset$. *For $\lambda \in (0,1)$ and $\omega \in C$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 \in C \text{ chosen arbitrarily,} \\ x_{n+1} = (1-\alpha_n)[\lambda T x_n + (1-\lambda)x_n] + \alpha_n\omega, \end{cases} \tag{3.10}$$

where the sequence $\{\alpha_n\} \subset (0,1)$ satisfies the following conditions:

(1) $\lim_{n\to\infty}\alpha_n = 0$;

(2) $\sum_{n=1}^{\infty}\alpha_n = \infty$.

Then the sequence $\{x_n\}$ converges strongly to $P_{\mathrm{Fix}(T)}\omega$.

Proof. Letting $f(x,y) \equiv 0$ for all $x, y \in C$ and $r_n = 1$ in Theorem 3.1, we obtain the desired result. ∎

Remark 2. Corollary 3.2 includes, as a special case, Theorem 3.2 of Cai, Tang and Liu [3].

Corollary 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H, and let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1) − (A4) *such that $EP(f) \neq \emptyset$. For $\lambda \in (0,1)$ and $\omega \in C$, let $\{x_n\}$ and $\{u_n\}$ be sequences generated by*

$$(3.11)\quad \begin{cases} x_1 \in C \ \textit{chosen arbitrarily}, \\ u_n \in C, \ \textit{such that } f(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0 \ \textit{for all } y \in C, \\ x_{n+1} = (1-\alpha_n)[\lambda u_n + (1-\lambda)x_n] + \alpha_n \omega, \end{cases}$$

where the sequences $\{\alpha_n\} \subset (0,1)$ and $\{r_n\} \subset (0,\infty)$ satisfy the following conditions:

(1) $\lim\limits_{n\to\infty} \alpha_n = 0$ *and* $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$;

(2) $\liminf\limits_{n\to\infty} r_n > 0$ *and* $\lim\limits_{n\to\infty} |r_{n+1} - r_n| = 0$.

Then the sequence $\{x_n\}$ converges strongly to $P_{EP(f)}\omega$.

Proof. Putting $T = I$ in Theorem 3.1, we get the result. ∎

4. Applications

In this section, we apply the results in the preceding section to convex minimization problem and variational inequality problem.

Now, we consider the convex minimization problem

$$(4.1)\quad \begin{cases} \min \phi(x) \\ x \in C, \end{cases}$$

where $\phi(x)$ is a proper lower semicontinuous convex function of H into $(-\infty, +\infty]$ such that C is included in $\text{dom}\phi = \{x \in H : \phi(x) < +\infty\}$. We denote by $\text{Sol}(\phi, C)$ the set of solutions of (4.1). Let $f : C \times C \to \mathbb{R}$ be a bifunction defined by

$$f(x,y) = \phi(y) - \phi(x).$$

It is clear that $f(x,y)$ satisfies $(A1) - (A4)$ and $EP(f) = \text{Sol}(\phi, C)$. Therefore the following result is obtained.

Theorem 4.1. *Let $\phi(x)$ be a proper lower semicontinuous convex function of H into $(-\infty, +\infty]$ and C a nonempty closed convex subset of H such that C is included in* $\mathrm{dom}\phi$. *Suppose that* $\mathrm{Sol}(\phi, C) \neq \emptyset$. *For $\lambda \in (0,1)$ and $\omega \in C$, let $\{x_n\}$ and $\{u_n\}$ be sequences generated by*

$$\begin{cases} x_1 \in C \text{ chosen arbitrarily}, \\ u_n = \arg\min_{y\in C}\{\phi(y) + \frac{1}{2r_n}\|y - x_n\|^2\}, \\ x_{n+1} = (1-\alpha_n)[\lambda u_n + (1-\lambda)x_n] + \alpha_n\omega, \end{cases} \tag{4.2}$$

where the sequences $\{\alpha_n\} \subset (0,1)$ and $\{r_n\} \subset (0,\infty)$ satisfy the following conditions:

(1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(2) $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$.

Then the sequence $\{x_n\}$ converges strongly to an element of $\mathrm{Sol}(\phi, C)$.

Proof. Letting $f(x,y) = \phi(y) - \phi(x)$ in Corollary 3.3, we get the conclusion. ∎

Next, we study the variational inequality problem. Let $A : C \to C$ be a mapping. The variational inequality problem for A is to find $z \in C$ such that

$$\langle Az, y - z\rangle \geq 0, \text{ for all } y \in C. \tag{4.3}$$

The set of its solutions is denoted by $VI(C, A)$. For $\lambda > 0$, it is easy to see that a point z is a solution of variational inequality (4.3) if and only if $z \in \mathrm{Fix}(P_C(I - \lambda A))$. Given a positive constant α, a mapping $A : C \to C$ is said to be $\alpha-$inverse strongly monotone if

$$\langle x - y, Ax - Ay\rangle \geq \alpha\|Ax - Ay\|^2 \text{ for all } x, y \in C.$$

Let $A : C \to C$ be $\alpha-$inverse strongly monotone and $0 < \lambda \leq 2\alpha$. Then mapping $I - \lambda A$ is nonexpansive (see [16]). Using Corollary 3.2, we obtain the strong convergence theorem for variational inequality problem.

Theorem 4.2. *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $A : C \to C$ be an $\alpha-$inverse strongly monotone mapping with $\alpha > 0$. Suppose that $VI(C, A) \neq \emptyset$. For $\lambda \in (0,1)$ with $\lambda \leq 2\alpha$ and $\omega \in C$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 \in C \text{ chosen arbitrarily}, \\ x_{n+1} = (1-\alpha_n)[\lambda P_C(I - \lambda A)x_n + (1-\lambda)x_n] + \alpha_n\omega, \end{cases} \tag{4.4}$$

where the sequence $\{\alpha_n\} \subset (0,1)$ satisfies the following conditions:

(1) $\lim_{n\to\infty} \alpha_n = 0$;

(2) $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Then the sequence $\{x_n\}$ converges strongly to an element of $VI(C, A)$.

Proof. Since $0 < \lambda \leq 2\alpha$, the mapping $I - \lambda A$ is nonexpansive. So is $P_C(I - \lambda A)$. Corollary 3.2 yields the result. ∎

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NEW EXTENDED (G'/G)-EXPANSION METHOD FOR TRAVELING WAVE SOLUTIONS OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS (NPDEs) IN MATHEMATICAL PHYSICS

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Abstract. The new extended (G'/G)-expansion method is proposed to construct abundant exact traveling wave solutions involving free parameters to the nonlinear partial differential equations (NPDEs) in mathematical physics. We highlight the power of the new extended (G'/G)-expansion method in providing generalized solitary wave solutions of different physical structures applying it in the right-handed noncommutative burgers and the $(1+1)$-dimensional compound KdVB equations. By this application, we enhanced new traveling wave solutions of the equations which can be used to exploit some practical physical and mechanical phenomena. Moreover,when the parameters are replaced by special values, the well-known solitary wave solutions of the equation rediscovered from the traveling waves that may imply some physical meaningful results in fluid mechanics, gas dynamics, traffic flow, nonlinear dispersion and dissipation effects.

Keywords: the new extended (G'/G)-expansion method; the right-handed noncommutative burgers equation; the $(1+1)$-dimensional compound KdVB equation; solitons wave solutions; traveling wave solutions.

Mathematics Subject Classification : 35C07, 35C08, 35P99.

1. Introduction

The world around us is inherently nonlinear, and nonlinear evolution equations are widely used to describe the complex physical phenomena that come out in a broad range of scientific applications, such as the fluid dynamics, nuclear physics, high energy physics, plasma physics, solid state physics, condensed matter physics, elastic media, optical fibers, biology, chemical kinematics, chemical physics, and geochemistry, etc. The exact solutions of nonlinear partial differential equations (NPDEs) play a significant role in nonlinear science and engineering. Recently, a number of prominent mathematicians and physicists who are interested in the nonlinear physical phenomena have investigated exact solutions of NPDEs to understand the physical mechanism of the phenomena using symbolical computer programs such as Maple, Matlab, Mathematica that facilitate complex and tedious algebraical computations.

For example, the wave phenomena observed in fluid dynamics [4], [14], plasma and elastic media [5], [12] and optical fibers [11], [19] etc. Some of the existing powerful methods for deriving exact solutions of NLEEs are Backlund transformation method [10], Darboux Transformations [8], tanh-function method [18], Exp-function method [7] and so on. Wang et al. [17] firstly proposed the (G'/G)-expansion method, then many diverse group of researchers extended this method by different names like an improved (G'/G)-expansion method [3], improved (G'/G)-expansion method [24],extended (G'/G)-expansion method [2], [15], generalized (G'/G)-expansion method [13], modified simple equation method [6] with different auxiliary equations. Zayed [20] established extended (G'/G)-expansion method for solving the $(3+1)$-dimensional NLEEs in mathematical physics. We (Roshid et. al.) [15] also used this method to find new exact traveling wave solutions of nonlinear Klein-Gordon equation. Recently, Khan et al. [25]found traveling and soliton wave solutions of GZK-BBM and right-handed non-commutative burgers equations by Modified Simple Equations method.

In this article, our motivation is to add new more general traveling wave solutions of right-handed non-commutative burgers and the $(1+1)$-dimensional compound KdVB equations via new extended (G'/G)-expansion. The performances of the method will encourage other researchers to apply it in other nonlinear evolution equations.

2. Materials and method

For given nonlinear evolution equations with independent variables x and t, we consider the following form

$$F(u, u_t, u_x, u_{xt}, u_{tt}, \ldots.) = 0. \tag{1}$$

By using the traveling wave transformation

$$u(x,t) = u(\xi), \xi = x - Vt, \tag{2}$$

where u is an unknown function depending on x and t, and is a polynomial F in $u(\xi) = u(x,t)$ and its partial derivatives and V is a constant to be determined later. The existing steps of method are as follows:

Step 1: Using equation (2) in equation (1), we can convert equation (1) to an ordinary differential equation

$$Q(u, -Vu\prime, u\prime, -Vu\prime\prime, V^2u\prime\prime, ...) = 0 \tag{3}$$

Step 2: Assume the solutions of equation (3) can be expressed in the form

$$u(\xi) = \sum_{i=-n}^{n} \left\{ \frac{a_i(G'/G)^i}{[1+\lambda G'/G]^i} + b_i(G'/G)^{i-1}\sqrt{\sigma\left[1+\frac{(G'/G)^2}{\mu}\right]} \right\} \tag{4}$$

with $G = G(\xi)$ satisfying the differential equation

$$G'' + \mu G = 0 \tag{5}$$

in which the value of σ must be ± 1,$\mu \neq 0$, a_i, b_i $(i = -n, ..., n)$, and λ are constants to be determined later. We can evaluate n by balancing the highest-order derivative term with the nonlinear term in the reduced equation (3).

Step 3: Inserting equation (4) into equation (3) and making use of equation (5) and then, extracting all terms of like powers of $(G'/G)^j$ and $(G'/G)^j\sqrt{\sigma[1+\frac{(G'/G)^2]}{\mu}}$ together, set each coefficient of them to zero yield an over-determined system of algebraic equations and then solving this system of algebraic equations for a_i, b_i $(i = -n, ..., n)$ and V, we obtain several sets of solutions.

Step 4: For the general solutions of Eq.(5), we have

$$\mu < 0,\ \frac{G'}{G} = \sqrt{-\mu}\left(\frac{Asinh(\sqrt{-\mu}\xi) + Bcosh(\sqrt{-\mu}\xi)}{Acosh(\sqrt{-\mu}\xi) + Bsinh(\sqrt{-\mu}\xi)}\right) = f_2(\xi) \tag{6}$$

$$\mu > 0,\ \frac{G'}{G} = \sqrt{\mu}\left(\frac{Acos(\sqrt{\mu}\xi) - Bsin(\sqrt{\mu}\xi)}{Asin(\sqrt{\mu}\xi) + Bcos(\sqrt{\mu}\xi)}\right) = f_1(\xi) \tag{7}$$

where A, B are arbitrary constants. At last, inserting the values of a_i, b_i $(i = -n, ..., n)$, V and (6), (7) into equation (4) and obtain required traveling wave solutions of equation (1).

Remark 1. It is noteworthy to observe that if we put $\lambda = 0$ in the equation (5), then the proposed new extended (G'/G)-expansion coincide with the Guo and Zhou's extended (G'/G)-expansion [2]. On the other hand if we put $b_i = 0$ and $\lambda = 0$ in the equation (5), then the proposed method is identical to the improved (G'/G)-expansion method presented by Zhang et al. [24]. Again if we set $b_i = 0$ and $\lambda = 0$ and negative the exponents of (G'/G) are zero in equation (5), then the proposed method turn into the basic (G'/G)-expansion method introduced by Wang et al. [17]. Thus the methods presented in the Ref. [2], [17], [24] are only special cases of the proposed new extended (G'/G)-expansion method.

3. Application of our method

To test the validity of our method, let us consider two important equations of mathematical physics to construct exact traveling wave solutions:

Example 3.1. In this subsection, we will bring to bear the new extended (G'/G)-expansion method to find the traveling wave solutions to the right-handed nc-Burgers equation:

$$u_t = u_{xx} + 2uu_x \tag{8}$$

Using the traveling wave transformation (2), (8) is reduced to the following ODE:

$$u'' + 2uu' + Vu' = 0 \tag{9}$$

Integrating (9) with respect to ξ and setting the constant of integration to zero, we obtain

$$u' + u^2 + Vu = 0 \tag{10}$$

Balancing the highest order derivative and nonlinear term, we obtain$N = 1$. Now, the solutions of equation (10), according to equation (4) is

$$\begin{aligned} u(\xi) &= a_0 + \frac{a_1(G'/G)}{1+\lambda(G'/G)} + \frac{a_{-1}(1+\lambda(G'/G))}{(G'/G)} \\ &\quad + (b_0(G'/G)^{-1} + b_1 + b_{-1}(G'/G)^{-2})\sqrt{\sigma[1+(G'/G)^2/\mu]} \end{aligned} \tag{11}$$

where $G = G(\xi)$ satisfies equation (5). Substituting equations (11) and (5) into equation (10), collecting all terms with the like powers of $(G'/G)^j$ and $(G'/G)^j$ $\sqrt{\sigma[1+(G'/G)^2/\mu]}$, and setting them to zero, we obtain a over-determined system that consists of eighteen algebraic equations (which are omitted for convenience).

Solving this over-determined system with the assist of Maple and inserting in equation (11), we have the following results.

Set 1: $V = \pm\sqrt{-\mu}, a_0 = \mp\frac{1}{2}\sqrt{-\mu}, a_1 = \frac{1}{2}, b_1 = \pm\sqrt{\left(\frac{\mu}{4\sigma}\right)}, \lambda = a_{-1} = b_0 = b_{-1} = 0.$
Now, when $\mu > 0$, then using (7) and (11), we have

$$u_1(\xi) = -\frac{1}{2}\sqrt{-\mu} + \frac{1}{2}f_1(\xi) \pm \sqrt{\left(\frac{\mu}{4\sigma}\right)}\sqrt{\sigma(1+f_1^2(\xi)/\mu)}, \tag{12}$$

where $\xi = x - \sqrt{-\mu}t$

$$u_2(\xi) = \frac{1}{2}\sqrt{-\mu} + \frac{1}{2}f_1(\xi) \pm \sqrt{\left(\frac{\mu}{4\sigma}\right)}\sqrt{\sigma(1+f_1^2(\xi)/\mu)}, \tag{13}$$

where $\xi = x + \sqrt{-\mu}t$ and when $\mu < 0$, then using (6) and (11), we have

$$u_3(\xi) = -\frac{1}{2}\sqrt{-\mu} + \frac{1}{2}f_2(\xi) \pm \sqrt{\left(\frac{\mu}{4\sigma}\right)}\sqrt{\sigma(1+f_2^2(\xi)/\mu)}, \tag{14}$$

where $\xi = x - \sqrt{-\mu}t$

$$u_4(\xi) = \frac{1}{2}\sqrt{-\mu} + \frac{1}{2}f_2(\xi) \pm \sqrt{\left(\frac{\mu}{4\sigma}\right)}\sqrt{\sigma(1 + f_2^2(\xi)/\mu)}, \tag{15}$$

where $\xi = x + \sqrt{-\mu}t$

Set 2: $V = \pm 2\sqrt{-\mu},\ a_0 = a_0,\ a_1 = \dfrac{a_0^2 \pm 2a_0\sqrt{-\mu}}{\mu},\ \lambda = \dfrac{-a_0 \mp \sqrt{-\mu}}{\mu}$,
$a_1 = b_{-1} = b_0 = b_1 = 0$. Now, when $\mu > 0$, then using (7) and (11), we have

$$u_5(\xi) = a_0 + \frac{a_0^2 + 2a_0\sqrt{-\mu}}{\mu} \times \frac{f_1(\xi)}{1 + \frac{-a_0 - \sqrt{-\mu}}{\mu} f_1(\xi)}, \tag{16}$$

where $\xi = x - 2\sqrt{-\mu}t$

$$u_6(\xi) = a_0 + \frac{a_0^2 - 2a_0\sqrt{-\mu}}{\mu} \times \frac{f_1(\xi)}{1 + \frac{-a_0 + \sqrt{-\mu}}{\mu} f_1(\xi)}, \tag{17}$$

where $\xi = x + 2\sqrt{-\mu}t$ and when $\mu < 0$, then using (6) and (11), we have

$$u_7(\xi) = a_0 + \frac{a_0^2 + 2a_0\sqrt{-\mu}}{\mu} \times \frac{f_2(\xi)}{1 + \frac{-a_0 - \sqrt{-\mu}}{\mu} f_2(\xi)}, \tag{18}$$

where $\xi = x - 2\sqrt{-\mu}t$

$$u_8(\xi) = a_0 + \frac{a_0^2 - 2a_0\sqrt{-\mu}}{\mu} \times \frac{f_2(\xi)}{1 + \frac{-a_0 + \sqrt{-\mu}}{\mu} f_2(\xi)}, \tag{19}$$

where $\xi = x + 2\sqrt{-\mu}t$

Set 3: $V = \pm 2\sqrt{-\mu}, a_{-1} = -\mu, a_0 = \pm\sqrt{-\mu}, \lambda = a_1 = b_1 = b_{-1} = b_0 = 0$. Now, when $\mu > 0$, then using (7) and (11), we have

$$u_9(\xi) = -\sqrt{-\mu} - \mu f_1^{-1}(\xi), \tag{20}$$

where $\xi = x - 2\sqrt{-\mu}t$

$$u_{10}(\xi) = \sqrt{-\mu} - \mu f_1^{-1}(\xi), \tag{21}$$

where $\xi = x + 2\sqrt{-\mu}t$ and when $\mu < 0$, then using (6) and (11), we have

$$u_{11}(\xi) = -\sqrt{-\mu} - \mu f_2^{-1}(\xi), \tag{22}$$

where $\xi = x - 2\sqrt{-\mu}t$

$$u_{12}(\xi) = \sqrt{-\mu} - \mu f_2^{-1}(\xi), \tag{23}$$

where $\xi = x + 2\sqrt{-\mu}t$

Set 4: $V = \pm 4\sqrt{-\mu}, a_{-1} = -\mu, a_0 = \pm 2\sqrt{-\mu}, a_1 = 1, \lambda = b_1 = b_{-1} = b_0 = 0$. Now, when $\mu > 0$, then using (7) and (11), we have

$$u_{13}(\xi) = 2\sqrt{-\mu} + f_1(\xi) - \mu f_1^{-1}(\xi), \tag{24}$$

where $\xi = x - 4\sqrt{-\mu}t$

$$u_{14}(\xi) = -2\sqrt{-\mu} + f_1(\xi) - \mu f_1^{-1}(\xi), \tag{25}$$

where $\xi = x + 4\sqrt{-\mu}t$ and when $\mu < 0$, then using (6) and (11), we have

$$u_{15}(\xi) = 2\sqrt{-\mu} + f_2(\xi) - \mu f_2^{-1}(\xi), \tag{26}$$

where $\xi = x - 4\sqrt{-\mu}t$

$$u_{16}(\xi) = -2\sqrt{-\mu} + f_2(\xi) - \mu f_2^{-1}(\xi), \tag{27}$$

where $\xi = x + 4\sqrt{-\mu}t$

Set 5: $V = \pm 2\sqrt{-\mu}, a_{-1} = -\mu, a_0 = a_0, \lambda = \dfrac{a_0 \pm \sqrt{-\mu}}{\mu}, b_{-1} = a_1 = b_1 = b_0 = 0$. Now, when $\mu > 0$, then using (7) and (11), we have

$$u_{17}(\xi) = a_0 - \mu(f_1(\xi))^{-1}\left[1 + \frac{a_0 + \sqrt{-\mu}}{\mu} f_1(\xi)\right], \tag{28}$$

where $\xi = x - 2\sqrt{-\mu}t$

$$u_{18}(\xi) = a_0 - \mu(f_1(\xi))^{-1}\left[1 + \frac{a_0 - \sqrt{-\mu}}{\mu} f_1(\xi)\right], \tag{29}$$

where $\xi = x + 2\sqrt{-\mu}t$ and when $\mu < 0$, then using (6) and (11), we have

$$u_{19}(\xi) = a_0 - \mu(f_2(\xi))^{-1}\left[1 + \frac{a_0 + \sqrt{-\mu}}{\mu} f_2(\xi)\right], \tag{30}$$

where $\xi = x - 2\sqrt{-\mu}t$

$$u_{20}(\xi) = a_0 - \mu(f_2(\xi))^{-1}\left[1 + \frac{a_0 - \sqrt{-\mu}}{\mu} f_2(\xi)\right], \tag{31}$$

where $\xi = x + 2\sqrt{-\mu}t$

Set-6: $V = \pm\sqrt{-\mu}, a_{-1} = -\mu/2, a_0 = a_0, \lambda = \dfrac{2a_0 \pm \sqrt{-\mu}}{\mu}, b_0 = \pm\mu\sqrt{(}1/4\sigma)$, $a_1 = b_1 = b_{-1} = 0$. Now, when $\mu > 0$, then using (7) and (11), we have

$$u_{21}(\xi) = a_0 - \frac{\mu + (2a_0 + \sqrt{-\mu})f_1(\xi)}{2f_1(\xi)} + \mu f_1^{-1}(\xi)\sqrt{\frac{1}{4}(1 + f_1^2(\xi)/\mu)}, \tag{32}$$

where $\xi = x - \sqrt{-\mu}t$

$$u_{22}(\xi) = a_0 - \frac{\mu + (2a_0 - \sqrt{-\mu})f_1(\xi)}{2f_1(\xi)} - \mu f_1^{-1}(\xi)\sqrt{\frac{1}{4}(1 + f_1^2(\xi)/\mu)}, \tag{33}$$

where $\xi = x + \sqrt{-\mu}t$

$$u_{23}(\xi) = a_0 - \frac{\mu + (2a_0 + \sqrt{-\mu})f_1(\xi)}{2f_1(\xi)} - \mu f_1^{-1}(\xi)\sqrt{\frac{1}{4}(1 + f_1^2(\xi)/\mu)}, \tag{34}$$

where $\xi = x - \sqrt{-\mu}t$

$$u_{24}(\xi) = a_0 - \frac{\mu + (2a_0 - \sqrt{-\mu})f_1(\xi)}{2f_1(\xi)} + \mu f_1^{-1}(\xi)\sqrt{\frac{1}{4}(1 + f_1^2(\xi)/\mu)}, \tag{35}$$

where $\xi = x + \sqrt{-\mu}t$ and when $\mu < 0$, then using (6) and (11), we have

$$u_{25}(\xi) = a_0 - \frac{\mu + (2a_0 + \sqrt{-\mu})f_2(\xi)}{2f_2(\xi)} + \mu f_2^{-1}(\xi)\sqrt{\frac{1}{4}(1 + f_2^2(\xi)/\mu)}, \tag{36}$$

where $\xi = x - \sqrt{-\mu}t$

$$u_{26}(\xi) = a_0 - \frac{\mu + (2a_0 - \sqrt{-\mu})f_2(\xi)}{2f_2(\xi)} - \mu f_2^{-1}(\xi)\sqrt{\frac{1}{4}(1 + f_2^2(\xi)/\mu)}, \tag{37}$$

where $\xi = x + \sqrt{-\mu}t$

$$u_{27}(\xi) = a_0 - \frac{\mu + (2a_0 + \sqrt{-\mu})f_2(\xi)}{2f_2(\xi)} - \mu f_2^{-1}(\xi)\sqrt{\frac{1}{4}(1 + f_2^2(\xi)/\mu)}, \tag{38}$$

where $\xi = x - \sqrt{-\mu}t$

$$u_{28}(\xi) = a_0 - \frac{\mu + (2a_0 - \sqrt{-\mu})f_2(\xi)}{2f_2(\xi)} + \mu f_2^{-1}(\xi)\sqrt{\frac{1}{4}(1 + f_2^2(\xi)/\mu)}, \tag{39}$$

where $\xi = x + \sqrt{-\mu}t$

Remark 2. Comparison between Khan et al. [25] solutions and new solutions:

(i) If $C_1 = v, C_2 = 1$ in the paper of Khan et al. [25], from sub section 3.2 in example 2, the exact solution (35) turns into the solitary wave solution (36) $-\dfrac{v}{2}\{1 - coth(v(x - vt)/2)\}$ but if we put $v/2 = \sqrt{-\mu}, A = 1, B = 0$ in our solution u_{11} becomes $-\frac{v}{2}\{1 - coth(v(x - vt)/2)\}$

(ii) If $C_1 = -v, C_2 = 1$ in the paper of Khan et al. [25], from Section 3.2 in Example 2, the exact solution (35) turns into the solitary wave solution (37) $-\dfrac{v}{2}\{1 - tanh(v(x - vt)/2)\}$ but if we put $v/2 = \sqrt{-\mu}, A = 0, B = 1$ in our solution u_{11} becomes $-\dfrac{v}{2}\{1 - tanh(v(x - vt)/2)\}$

Example 3.2. In this subsection, we will bring to bear the new extended (G'/G)-expansion method to find the traveling wave solutions of the (1+1)-dimensional compound KdVB equation in the form:

$$u_t + \alpha u u_x + \beta u^2 u_x + \gamma u_{xx} - \delta u_{xxx} = 0, \tag{40}$$

where α, β, γ and δ are constants.

This equation can be thought of as a generalization of KdV-mKdV and Burgers equations involving nonlinear dispersion and dissipation effects. The traveling wave solutions of equation (40) have been found in [23] using (G'/G)-expansion method. To this end, we are going to find new traveling solution of the equation by our proposed method. Using traveling wave transformation (2), equation (40) is reduced to the following ODE:

$$C - Vu + \tfrac{1}{2}\alpha u^2 + \tfrac{1}{3}\beta u^3 + \gamma u' - \delta u'' = 0, \tag{41}$$

where C is an integration constant. Considering the homogeneous balance between the highest order derivative and nonlinear term, we obtain$N = 1$. Now, the solutions of equation (40), according to equation (4) is same of the equation (11). Substituting equation (11) and equation (5) into equation (41), collecting all terms with the like powers of $(G'/G)^j$and $(G'/G)^j\sqrt{\sigma[1 + (G'/G)^2/\mu]}$, and setting them to zero, we obtain a over-determined system that consists of eighteen algebraic equations (which are omitted for convenience).

Solving this over-determined system with the assist of Maple and inserting in equation (11), we have the following results.

Set 1. $C = \dfrac{1}{72\delta\beta}\{(288\alpha\mu\delta^2\beta + 3\alpha^3\delta - 6\alpha\beta\gamma^2) \mp (8\beta\gamma^3 + 1152\beta\gamma\mu\delta^2)/\sqrt{6\delta/\beta}\}$, $V = -\dfrac{96\delta^2\mu\beta + 3\delta\alpha^2 - 2\beta\gamma^2}{12\delta\beta}$, $\lambda = 0$, $a_1 = \mp\sqrt{\dfrac{6\delta}{\beta}}$, $a_{-1} = \pm\mu\sqrt{\dfrac{6\delta}{\beta}}$, $a_0 = -\dfrac{\alpha}{2\beta} \mp \dfrac{\gamma}{\sqrt{6\delta\beta}}$, $b_1 = b_0 = b_{-1} = 0$.

Now, when $\mu > 0$, then using (7) and (11), we have

$$u^{\pm}_{2,1}(\xi) = -\frac{\alpha}{2\beta} \pm \frac{\gamma}{\sqrt{6\delta\beta}} \mp \mu\sqrt{\frac{6\delta}{\beta}} f_1^{-1}(\xi) \pm \sqrt{\frac{6\delta}{\beta}} f_1(\xi), \tag{42}$$

and when $\mu < 0$, then using (6) and (11), we have

$$u^{\pm}_{2,2}(\xi) = -\frac{\alpha}{2\beta} \pm \frac{\gamma}{\sqrt{6\delta\beta}} \mp \mu\sqrt{\frac{6\delta}{\beta}} f_2^{-1}(\xi) \pm \sqrt{\frac{6\delta}{\beta}} f_2(\xi), \tag{43}$$

where $\xi = x + \dfrac{96\delta^2\mu\beta + 3\delta\alpha^2 - 2\beta\gamma^2}{12\delta\beta} t$

Set 2. $C = \dfrac{1}{72\delta\beta^2}\{(72\alpha\mu\delta^2\beta + 3\alpha^3\delta - 6\alpha\beta\gamma^2) \pm (8\beta\gamma^2 + 288\beta\gamma\mu\delta^2)/\sqrt{6\delta/\beta}\}$,

$V = -\frac{24\delta^2\mu\beta + 3\delta\alpha^2 - 2\beta\gamma^2}{12\delta\beta}$, $\lambda = 0$, $a_1 = \pm\sqrt{\frac{6\delta}{\beta}}$, $a_0 = -\frac{\alpha}{2\beta} \pm \frac{\gamma}{\sqrt{6\delta\beta}}$, $a_{-1} = b_1 = b_0 = b_{-1} = 0$. Now, when $\mu > 0$, then using (7) and (11), we have

$$u^{\pm}_{2,3}(\xi) = -\frac{\alpha}{2\beta} \pm \frac{\gamma}{\sqrt{6\delta\beta}} \pm \sqrt{\frac{6\delta}{\beta}} f_1(\xi), \tag{44}$$

and when $\mu < 0$, then using (6) and (11), we have

$$u^{\pm}_{2,4}(\xi) = -\frac{\alpha}{2\beta} \pm \frac{\gamma}{\sqrt{6\delta\beta}} \pm \sqrt{\frac{6\delta}{\beta}} f_2(\xi), \tag{45}$$

where $\xi = x + \frac{24\delta^2\mu\beta + 3\delta\alpha^2 - 2\beta\gamma^2}{12\delta\beta}t$

Set 3. $C = \frac{1}{216\delta^2\beta^2}\{\pm(72\gamma\mu\beta^2\delta^2 + 8\beta^2\gamma^3)\sqrt{3\delta/2\beta} + 54\alpha\beta\mu\delta^3 - 18\alpha\beta\gamma^2\delta\}$, $V = -\frac{6\delta^2\mu\beta + 3\delta\alpha^2 - 2\beta\gamma^2}{12\delta\beta}$, $\lambda = 0$, $a_1 = \pm\sqrt{\frac{3\delta}{2\beta}}$, $a_0 = -\frac{\alpha}{2\beta} \pm \frac{\gamma}{\sqrt{6\delta\beta}}$, $b_1 = \pm\sqrt{\frac{3\delta\mu}{2\beta\sigma}}$, $a_{-1} = b_{-1} = b_0 = 0$. Now, when $\mu > 0$, then using (7) and (11), we have

$$u^{\pm}_{2,5}(\xi) = -\frac{\alpha}{2\beta} \pm \frac{\gamma}{\sqrt{6\delta\beta}} \pm \sqrt{\frac{3\delta}{2\beta}} f_1(\xi) \mp \sqrt{\frac{3\delta\mu}{2\beta\sigma}}\sqrt{\sigma(1 + f_1^2(\xi)/\mu)}, \tag{46}$$

and when $\mu < 0$, then using (6) and (11), we have

$$u^{\pm}_{2,6}(\xi) = -\frac{\alpha}{2\beta} \pm \frac{\gamma}{\sqrt{6\delta\beta}} \pm \sqrt{\frac{3\delta}{2\beta}} f_2(\xi) \mp \sqrt{\frac{3\delta\mu}{2\beta\sigma}}\sqrt{\sigma(1 + f_2^2(\xi)/\mu)}, \tag{47}$$

where $\xi = x + \frac{6\delta^2\mu\beta + 3\delta\alpha^2 - 2\beta\gamma^2}{12\delta\beta}t$

Set 4. $C = \frac{1}{72\delta\beta^2}\{(72\alpha\mu\delta^2\beta + 3\alpha^3\delta - 6\alpha\beta\gamma^2) \mp (8\beta\gamma^2 + 288\beta\gamma\mu\delta^2)/\sqrt{6\delta/\beta}\}$, $V = -\frac{24\delta^2\mu\beta + 3\delta\alpha^2 - 2\beta\gamma^2}{12\delta\beta}$, $\lambda = \lambda$, $a_{-1} = \pm\mu\sqrt{\frac{6\delta}{\beta}}$, $a_0 = -\frac{\alpha}{2\beta} \pm \frac{\gamma + 6\delta\mu\lambda}{\sqrt{6\delta\beta}}$, $a_1 = b_{-1} = b_0 = b_1 = 0$. Now, when $\mu > 0$, then using (7) and (11), we have

$$u^{\pm}_{2,7}(\xi) = -\frac{\alpha}{2\beta} \pm \frac{\gamma + 6\delta\mu\lambda}{\sqrt{6\delta\beta}} \mp \mu\sqrt{\frac{6\delta}{\beta}}\frac{1 + \lambda f_1(\xi)}{f_1(\xi)}, \tag{48}$$

and when $\mu < 0$, then using (6) and (11), we have

$$u^{\pm}_{2,8}(\xi) = -\frac{\alpha}{2\beta} \pm \frac{\gamma + 6\delta\mu\lambda}{\sqrt{6\delta\beta}} \mp \mu\sqrt{\frac{6\delta}{\beta}}\frac{1 + \lambda f_2(\xi)}{f_2(\xi)}, \tag{49}$$

where $\xi = x + \dfrac{24\delta^2\mu\beta + 3\delta\alpha^2 - 2\beta\gamma^2}{12\delta\beta}t$

Set 5. $C = \dfrac{1}{72\delta\beta^2}\{(72\alpha\mu\delta^2\beta + 3\alpha^3\delta - 6\alpha\beta\gamma^2) \pm (8\beta\gamma^2 + 288\beta\gamma\mu\delta^2)/\sqrt{6\delta/\beta}\}$, $V = -\dfrac{24\delta^2\mu\beta + 3\delta\alpha^2 - 2\beta\gamma^2}{12\delta\beta}$, $\lambda = \lambda$, $a_1 = \pm(\lambda^2\mu+1)\sqrt{\dfrac{6\delta}{\beta}}$, $a_0 = -\dfrac{\alpha}{2} \pm \dfrac{\gamma - 6\delta\mu\lambda}{\sqrt{6\delta\beta}}$, $a_{-1} = b_{-1} = b_0 = b_1 = 0$. Now, when $\mu > 0$, then using (7) and (11), we have

$$u^{\pm}_{2,9}(\xi) = -\frac{\alpha}{2} \pm \frac{\gamma - 6\delta\mu\lambda}{\sqrt{6\delta/\beta}} \pm (\lambda^2\mu + 1)\sqrt{\frac{6\delta}{\beta}}\frac{f_1(\xi)}{1 + \lambda f_1(\xi)}, \tag{50}$$

and when $\mu < 0$, then using (6) and (11), we have

$$u^{\pm}_{2,10}(\xi) = -\frac{\alpha}{2} \pm \frac{\gamma - 6\delta\mu\lambda}{\sqrt{6\delta/\beta}} \pm (\lambda^2\mu + 1)\sqrt{\frac{6\delta}{\beta}}\frac{f_2(\xi)}{1 + \lambda f_2(\xi)}, \tag{51}$$

where $\xi = x + \dfrac{24\delta^2\mu\beta + 3\delta\alpha^2 - 2\beta\gamma^2}{12\delta\beta}t$

Set 6. $C = \dfrac{1}{72\delta\beta^2}\{(18\alpha\mu\delta^2\beta + 3\alpha^3\delta - 6\alpha\beta\gamma^2) \mp (2\gamma^3 + 18\gamma\mu\delta^2)/\sqrt{6\delta\beta}\}$, $V = -\dfrac{6\delta^2\mu\beta + 3\delta\alpha^2 - 2\beta\gamma^2}{12\delta\beta}$, $\lambda = \lambda$, $a_{-1} = \pm\mu\sqrt{\dfrac{3\delta}{2\beta}}$, $a_0 = -\dfrac{\alpha}{2\beta} \mp \dfrac{\gamma + 3\delta\mu\lambda}{\sqrt{6\delta\beta}}$, $b_0 = \pm\mu\sqrt{\dfrac{3\delta}{2\beta\sigma}}$, $a_1 = b_{-1} = b_1 = 0$. Now, when $\mu > 0$, then using (7) and (11), we have

$$\begin{aligned} u_{2,11}(\xi) &= -\frac{\alpha}{2\beta} - \frac{\gamma+3\delta\mu\lambda}{\sqrt{6\delta\beta}} + \mu\sqrt{\frac{3\delta}{2\beta}}\frac{1+\lambda f_1(\xi)}{f_1(\xi)} \\ &\mp \mu\sqrt{\frac{3\delta}{2\beta\sigma}}\sqrt{\sigma(1+f_1^2(\xi)/\mu)}, \end{aligned} \tag{52}$$

$$\begin{aligned} u_{2,12}(\xi) &= -\frac{\alpha}{2\beta} + \frac{\gamma+3\delta\mu\lambda}{\sqrt{6\delta\beta}} - \mu\sqrt{\frac{3\delta}{2\beta}}\frac{1+\lambda f_1(\xi)}{f_1(\xi)} \\ &\mp \mu\sqrt{\frac{3\delta}{2\beta\sigma}}\sqrt{\sigma(1+f_1^2(\xi)/\mu)}, \end{aligned} \tag{53}$$

and when $\mu < 0$, then using (6) and (11), we have

$$\begin{aligned} u_{2,13}(\xi) &= -\frac{\alpha}{2\beta} - \frac{\gamma+3\delta\mu\lambda}{\sqrt{6\delta\beta}} + \mu\sqrt{\frac{3\delta}{2\beta}}\frac{1+\lambda f_2(\xi)}{f_2(\xi)} \\ &\mp \mu\sqrt{\frac{3\delta}{2\beta\sigma}}\sqrt{\sigma(1+f_2^2(\xi)/\mu)}, \end{aligned} \tag{54}$$

$$
(55) \quad \begin{aligned} u_{2,14}(\xi) &= -\frac{\alpha}{2\beta} + \frac{\gamma + 3\delta\mu\lambda}{\sqrt{6\delta\beta}} - \mu\sqrt{\frac{3\delta}{2\beta}}\frac{1+\lambda f_2(\xi)}{f_2(\xi)} \\ &\mp \mu\sqrt{\frac{3\delta}{2\beta\sigma}}\sqrt{\sigma(1+f_2^2(\xi)/\mu)}, \end{aligned}
$$

where $\xi = x + \dfrac{6\delta^2\mu\beta + 3\delta\alpha^2 - 2\beta\gamma^2}{12\delta\beta}t$

Set 7. $C = -\dfrac{1}{216\delta^2\beta^2}\{\pm(72\gamma\mu\beta^2\delta^2 + 8\beta^2\gamma^3)\sqrt{3\delta/2\beta} - 9\delta^2\alpha^3 - 54\alpha\beta\mu\delta^3 + 18\alpha\beta\gamma^2\delta\}$, $V = -\dfrac{6\delta^2\mu\beta + 3\delta\alpha^2 - 2\beta\gamma^2}{12\delta\beta}$, $\lambda = -\dfrac{\gamma}{3\mu\delta}$, $a_{-1} = \pm\mu\sqrt{\dfrac{3\delta}{2\beta}}$, $a_0 = -\dfrac{\alpha}{2\beta}$, $b_0 = \pm\mu\sqrt{\dfrac{3\delta}{2\beta\sigma}}$, $a_1 = b_{-1} = b_1 = 0$. Now, when $\mu > 0$, then using (7) and (11), we have

$$
(56) \quad u_{2,15}(\xi) = -\frac{\alpha}{2\beta} \pm \mu\sqrt{\frac{3\delta}{2\beta}}\frac{1 - \frac{\gamma}{3\mu\delta}f_1(\xi)}{f_1(\xi)} \pm \mu\sqrt{\frac{3\delta}{2\beta\sigma}}\sqrt{\sigma(1+f_1^2(\xi)/\mu)},
$$

and when $\mu < 0$, then using (6) and (11), we have

$$
(57) \quad u_{2,16}(\xi) = -\frac{\alpha}{2\beta} \pm \mu\sqrt{\frac{3\delta}{2\beta}}\frac{1 - \frac{\gamma}{3\mu\delta}f_2(\xi)}{f_2(\xi)} \pm \mu\sqrt{\frac{3\delta}{2\beta\sigma}}\sqrt{\sigma(1+f_2^2(\xi)/\mu)},
$$

where $\xi = x + \dfrac{6\delta^2\mu\beta + 3\delta\alpha^2 - 2\beta\gamma^2}{12\delta\beta}t$

Note. For correct solutions, we have one solution taking upper signs and another solution taking lower sign but there is no restriction on the signs of b_1, b_0, b_{-1}.

Remark 3. Comparison between Zayed [23] solutions and new solutions:

In the paper of Zayed [23], from Section 3 in Example 2, the exact solution (23) turns our solution (45) when $\gamma = 0$ and $AB = -\mu/2$. After then, if we use the same conditions on A, B like [23], we can get all the solitary solutions of [23].

Remark 4. We have verified all the achieved solutions by putting them back into the original equation (8) of Example 3.1 and into the original equation (40) of Example 3.2 with the aid of Maple 13.

4. Results and discussion

We have constructed twenty eight exact traveling wave solutions for the nc-Burger equations and thirty two solutions for the (1 + 1)-dimensional compound KdVB equation including solitons, periodic solutions via the new extended (G'/G)-expansion method. It is important to state that one (for each equation) of our obtained solutions is in good agreement with the existing results which are shown in Remarks 2, 3. Beyond Remarks 2, 3, we have constructed new exact traveling wave solutions u_1 to u_{28} for ncBurgers equation and solutions $u^{\pm}_{1,2}$ to $u^{\pm}_{2,16}$ for (1+1)-dimensional compound KdVB equation which have not been reported in the previous literature. In addition, the graphical representations of some obtained traveling wave solutions are shown in Figure 1 to Figure 13.

Graphical representations of the solutions: The graphical illustrations of the solutions are depicted in the figures with the aid of Maple. Solutions u_3, u_4, u_7, u_8, $u_{25}, u_{28}, u^{\pm}_{2,4}, u^{\pm}_{2,6}, u^{\pm}_{2,8}, u^{\pm}_{2,10}, u_{2,13}, u_{2,14}$ and $u^{\pm}_{2,16}$ describes the kink wave. Kink waves are traveling waves which arise from one asymptotic state to another. The kink solutions are approach to a constant at infinity. Fig. 1 and Fig. 2 below shows the shape of the exact Kink-type solution of u_3 and u_{28} the right-handed noncommutative burgers equation (8). The shape of figures of solutions $u_4, u_7, u_8, u_{25}, u^{\pm}_{2,8}, u^{\pm}_{2,10}, u_{2,13}, u_{2,14}$ are similar to the figure of solution of u_3 and $u^{\pm}_4, u^{\pm}_6, u^{\pm}_{16}$ are similar to Fig. 2. So, the figures of these solutions are omitted for convenience.

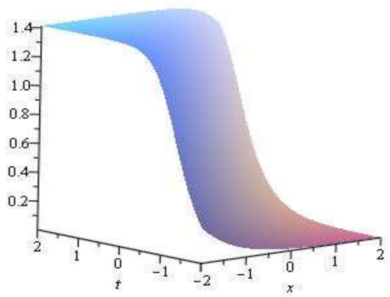

Fig-1: Kink solution for Eq (14) with μ= -2, σ = -1, B=1, A=2

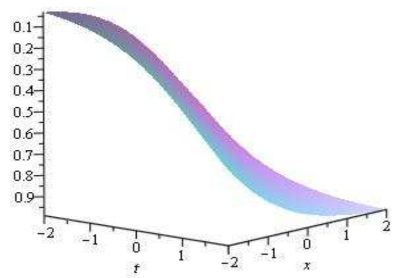

Fig-2: Kink solution for Eq (39) with μ= -1, σ =B=1, A=2, a0= -2

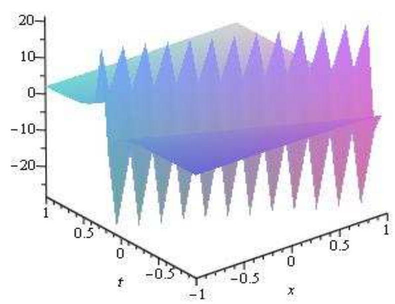

Fig-3: Kink solution for Eq (31) with μ= -1, σ =B=1, A=2, a0= -1

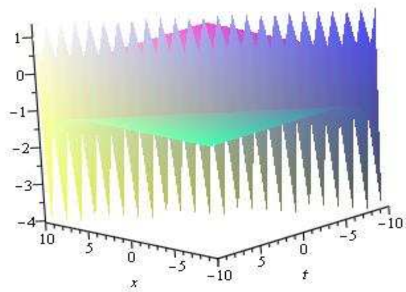

Fig-4: Singular Kink solution for Eq (38) with μ= -1, σ =B=1, A=2, a0= -

Solutions $u_{11}, u_{12}, u_{15}, u_{16}, u_{19}, u_{20}$ and u_{27} comes infinity as in hyperbolic function, are singular Kink solution. Fig. 3 and Fig. 4 shows the shape of the exact singular Kink-type solution of u_{20} and u_{27} respectively. The shape of figures of solutions $u_{11}, u_{12}, u_{15}, u_{16}, u_{19}$ are similar to the figure of solution u_{20}, and so, the figures of these solutions are omitted for convenience. Another type of singular kink type figure is expressed by the solution u_{26} whose figure is described by Fig. 5 and singular kink type figure of the of KdVB equation is expressed by solutions $u^{\pm}_{2,2}$ whose figure is described by Fig. 10.

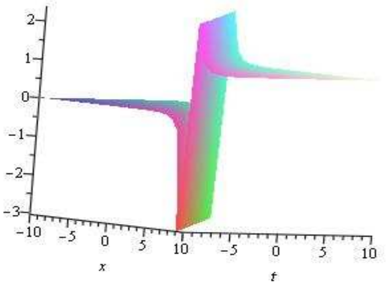

Fig-5: Singular Kink solution for Eq (37) with μ= -1, σ =B=1, A=2, a0= -

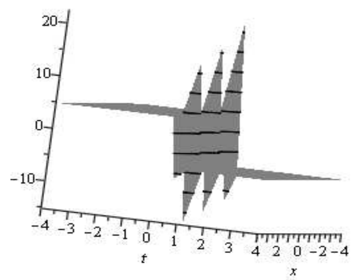

Fig-10: singular kink solution to Eq(47) for α=β=γ=δ=σ =1, μ= -1, A=2, B=3

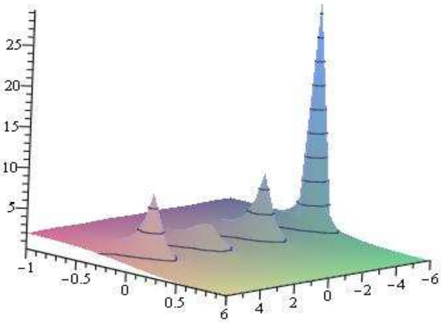

Fig-6: Singular Kink solution for (21) with μ=σ =B=1, A=3

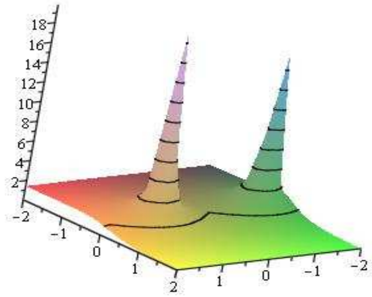

Fig-7: Singular Kink solution for (33) with A=μ=2, σ =B=1, a0=-2

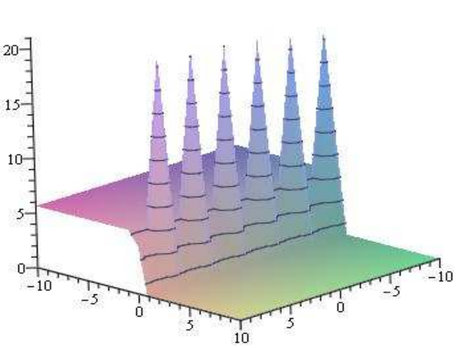

Fig-8: Singular Kink solution for (25) with μ=2, σ =B=1, A=3

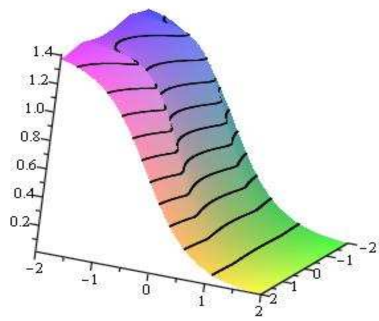

Fig-9: Singular Kink solution for (32) with A=μ=2, σ =B=1, a0=-2

Solutions $u_1, u_2, u_5, u_6, u_9, u_{10}, u_{13}, u_{14}, u_{17}, u_{18}, u_{21}, u_{22}, u_{23}$ and u_{24} comes infinity as in trigonometric function, are singular Kink solution. Fig. 6, Fig. 7, Fig. 8 and Fig. 9 shows the shape of the exact singular Kink-type solution of u_{10}, u_{22}, u_{14}, and u_{21}, respectively. The shape of figures of solutions $u_1, u_2, u_5, u_6, u_9, u_{18}$ are similar to the figure of solution u_{10}, the shape of figure of solution u_{13} is similar to the figure of solution u_{25} and the shape of figure of solution u_{24} is similar to the figure of solution u_{21}, and so, the figures of these solutions are omitted for convenience.

Solutions $u^{\pm}_{2,1}, u^{\pm}_{2,3}, u^{\pm}_{2,5}, u^{\pm}_{2,7}, u^{\pm}_{2,9}, u_{2,11}, u_{2,12}$ and $u^{\pm}_{2,15}$ of KdVB equation (trigonometric functions as $cos(x-t)$) are periodic solutions. Fig. 11 Fig. 12 and Fig. 13 shows the shape of the periodic solution of $u^{-}_{2,1}$, $u^{-}_{2,5}$ and $u_{2,15}$ (when b_0 take $-$, a_{-1} take $+$ sign), respectively. The shape of figures of solutions $u^{+}_{2,1}$, $u^{\pm}_{2,3}$, $u^{\pm}_{2,7}$, $u^{\pm}_{2,9}$, $u_{2,11}$, $u_{2,12}$ are similar to the figure of solution $u^{-}_{2,1}$, the shape of figure of solution $u_{2,15}$ (b_0 take $+$, a_{-1} take $-$), $u_{2,11}$ (when sign of a_0 take $-$, a_{-1} take $+$, b_1 take $-$) and $u_{2,12}$ (when sign of a_0 take $+$, a_{-1} take $-$, b_1 take $+$) is similar to the Fig. 13. Others are similar to the Fig. 11, and so, the figures of these solutions are omitted for convenience.

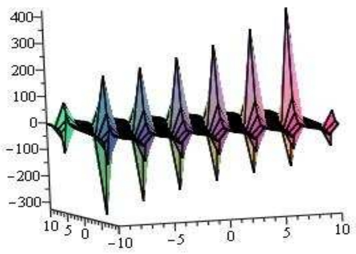

Fig-11: Periodic solution of Eq.(43) for
α=β=γ=δ=σ =1, A=μ=2, B=3

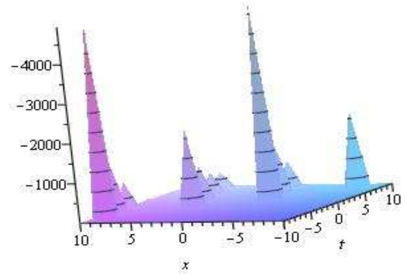

Fig-12: Periodic solution of Eq.(46) for
α=β=γ=δ=1, σ = -1, A=μ=2, B=3

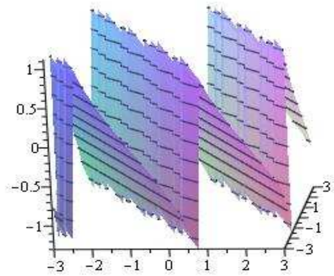

Fig-13: Periodic solution of Eq.(56) for
α=β=γ=δ=σ =μ=1, A=2, B=3

5. Conclusion

The new extended (G'/G)-expansion method is presented to search exact traveling wave solutions for NPDEs. In addition, this method is also computerizable, which allows us to perform complicated and tedious algebraic calculation using a computer by the help of symbolic programs such as Maple, Mathematica, Matlab, and so on. We apply it to the right-handed non-commutative burgers and the $(1+1)$-dimensional compound KdVB equations. As a results, many plentiful new hyperbolic functions and periodic solutions with free parameters including soliton solutions are obtained. Overall, the results reveal that the presented method is effective, productive and more powerful than the original (G'/G)-expansion method and it can be applied for other kind of nonlinear evolution equations in mathematical physics.

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REVERSE MAGIC STRENGTH OF FESTOON TREES

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Abstract. In this paper, we prove that the reverse super edge-magic strength of some different festoon trees.

Keywords. graph labeling, reverse super edge-magic labeling, Festoon trees.

1. Introduction

Consider a family of finite number of stars. Arrange them in an array and join one centre of a star to that of the next one. The tree so obtained is called a $f$estoon tree. In ([9]), we defined a reverse magic labeling of a graph $G(V, E)$ as a bijection $f : V \cup E \to \{1, 2, 3, ..., v + \epsilon\}$ such that for all edges $xy, f(xy) - \{f(x) + f(y)\}$ are the same where v and ϵ denote the order and size of the graph G. A graph is said to be A reverse edge-magic labeling f is called reverse super edge-magic if $f(V) = \{1, 2, 3, ..., \}$ and $f(E) = \{v + 1, v + 2, v + 3, ..., v + \epsilon\}$. A graph G is called reverse super edge-magic if there exists a reverse super edge-magic labeling of G. In ([5]), Hugand introduced the concept of reverse super edge-magic strength of a graph. A reverse edge-magic labeling of a graph $G(V, E)$ is a bijection $f : V \cup E \to \{1, 2, 3, ..., v + \epsilon\}$ such that for all edges $xy, f(xy) - \{f(x) + f(y)\}$ is a constant which is denoted by $c(f)$. The reverse edge-magic strength of a graph $G, rsm(G)$, is defined as the minimum of all $c(f)$ where the minimum is taken over all reverse edge-magic labelings f of G. A reverse magic labeling of a graph $G(V, E)$ is called reverse super edge-magic labeling of G if $f(V) = \{1, 2, ..., v\}$ and $f(E) = \{v + 1, v + 2, ..., v + \epsilon\}$.

In ([5]), the following results have been proved.

1. $\text{rsm}(P_{2n}) = n - 1$, $\text{rsm}(P_{2n+1}) = n$
2. $\text{rsm}(K_{1,n}) = n - 1$
3. $\text{rsm}(B_{n,n}) = n$
4. $\text{rsm}(P_n^2) = n - 2$
5. $\text{rsm}(C_{2n+1}) = n$
6. $\text{rsm}(< K_{1,n} : 2 >) = 2n$
7. $\text{rsm}[(2n + 1)P_2] = n$

Note 1. Let f be a reverse super edge-magic labeling of G with the constant $c(f)$. Then, adding all the constants obtained at each edge, we get

$$\varepsilon\ c(f)\ = \sum_{e\in E} f(e) - \sum_{v\in V} d(v)f(v)$$

In this paper, we determined the reverse super edge-magic strength of some different festoon trees.

2. Reverse super edge-magic strength of Festoon trees

In this paper, we obtain the reverse super edge-magic strength of festoon tree $T_\alpha, n+\alpha$ vertices of the stars $K_{1,n}, n_i \geq 1,\ 1 \leq i \leq \alpha,\ \sum_{i=1}^{\alpha} n_i = n.$

Theorem 2.1. $rsm(T_\alpha) = n + \alpha - n_1 - n_3 - \ldots - n_{\alpha-1} - \frac{\alpha}{2} - 1.$

Proof. We prove this theorem by assigning reverse super edge-magic labeling to T_α, α is even.

Let u_i be the center and v_i be the set of pendent vertices of i^{th} star. Each v_i has n_i vertices, $i = 1, 2, ..., \alpha$. Thus $n_1 + n_2 + n_3 + \cdots + n_\alpha = n$, $v = n + \alpha$, $\varepsilon = n+\alpha-1$. Then the following labeling f is a reverse super edge-magic labeling of T_α.

$$\begin{aligned}
f(u_{2i}) &= n_1 + n_3 + n_5 plus n_{2i-1} + i, \quad i = 1, 2, 3, ..., \frac{\alpha}{2} \\
f(u_{2i-1}) &= f(u_\alpha) + n_2 + n_4 + n_6 + \cdots + n_{2i-2} + i, \quad i = 1, 2, 3, ..., \frac{\alpha}{2} \\
f(v_{2i-1}) &= \{n_1 + n_3 + \cdots + n_{2i-3} + i, ..., n_1 + n_3 + \cdots + n_{2i-1} + i - 1\}, \\
&\qquad\qquad\qquad\qquad\qquad\qquad i = 1, 2, 3, ..., \frac{\alpha}{2} \\
f(v_{2i}) &= \{f(u_1) + n_2 + n_4 + \ldots + n_{2i-2} + i, ..., f(u_1) + n_2 + n_4 \\
&\qquad\qquad + \cdots + n_{2i} + i - 1\}, \quad i = 1, 2, 3, ..., \frac{\alpha}{2}
\end{aligned}$$

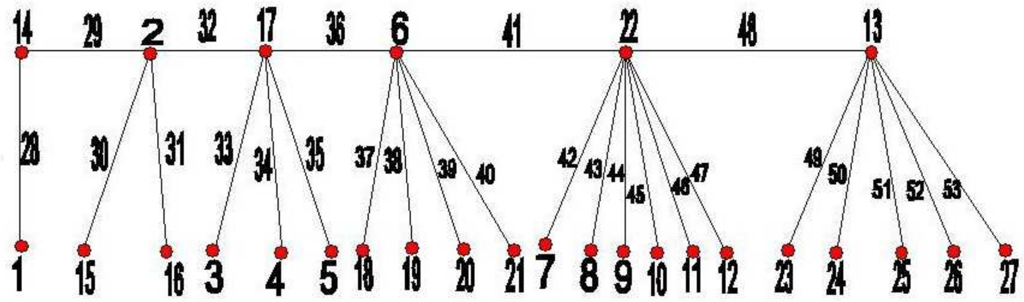

Figure 1:

This labeling of vertices gives $f(x)+f(y)$ for all edges $xy \in E$ to vary from $f(u_\alpha)+2$ to $f(u_\alpha)+n+\alpha$. Thus, for each edge $e=xy$, if $f(x)+f(y)=f(u_\alpha)+i$, $i=2,3,...,n+\alpha$, then $f(e)=n+\alpha+i$. By Note 1, if f is a reverse super edge-magic labeling of T_α with constant $c(f)$.

$$\varepsilon\ c(f)=\sum_{e\in E} f(e)-\sum_{v\in V} d(v)f(v)$$

$$\begin{aligned}
(n+\alpha-1)\ c(f) &= \sum_{e\in E} f(e)-\Big\{\sum_{i=1,v\in V_i}^{\alpha} f(v)+\sum_{i=1}^{\alpha} d(u_i)f(u_i)\Big\}\\
&= \sum_{e\in E} f(e)-\Big\{\sum_{i=1,v\in V_i}^{\alpha} f(v)+(n_1+1)f(u_1)\\
&\quad +(n_2+2)f(u_2)+\ldots+(n_{\alpha-1}+2)f(u_{\alpha-1})+(n_\alpha+1)f(u_\alpha)\Big\}\\
&= \sum_{e\in E} f(e)-\Big\{\Big[\sum_{i=1,v\in V_i}^{\alpha} f(v)+f(u_1)+f(u_2)+\ldots+f(u_\alpha)\Big]\\
&\quad +n_1f(u_1)+(n_2+1)f(u_2)+\ldots+(n_{\alpha-1}+1)f(u_{\alpha-1})+n_\alpha f(u_\alpha)\Big\}\\
&= \sum_{e\in E} f(e)-\Big\{\sum_{i=1_i}^{n+\alpha-1} i+n_1[f(u_\alpha)+1]+(n_2+1)(n_1+1)\\
&\quad +(n_3+1)[f(u_\alpha)+n_2+2]+(n_4+1)[n_1+n_3+2]\\
&\quad +\ldots+(n_{\alpha-2}+1)\big[n_1+n_3+\ldots+n_{\alpha-3}+\big(\frac{\alpha}{2}-1\big)\big]\\
&\quad +(n_{\alpha-1}+1)\big[f(u_\alpha)+n_2+n_4+\ldots+n_{\alpha-2}+\frac{\alpha}{2}\big]\\
&\quad +n_\alpha\big[n_1+n_3+\ldots+n_{\alpha-1}+\frac{\alpha}{2}\big]\Big\}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(n+\alpha-1)[n+\alpha+1+2n+2\alpha-1]}{2} \\
&\quad - \Big\{ \frac{(n+\alpha-1)(n+\alpha)}{2} + f(u_\alpha)\big[n_1+n_3+\ldots+n_{\alpha-1}+\frac{\alpha-2}{2}\big] \\
&\quad + n_1 + (n_2+1)(n_1+1) + (n_3+1)(n_2+2) + (n_4+1)(n_1+n_3+2) \\
&\quad + (n_5+1)(n_2+n_4+3) + \ldots + (n_{\alpha-2}+1)\big[n_1+n_3+\ldots+n_{\alpha-3}+\big(\frac{\alpha}{2}-1\big)\big] \\
&\quad + (n_{\alpha-1}+1)\big[n_2+n_4+\ldots+n_{\alpha-2}+\frac{\alpha}{2}\big] + n_\alpha\big[n_1+n_3+\ldots+n_{\alpha-1}+\frac{\alpha}{2}\big]\Big\} \\
&= \frac{(n+\alpha-1)[3n+3\alpha]}{2} - \frac{(n+\alpha-1)(n+\alpha)}{2} \\
&\quad - \Big\{\big(n_1+n_3+\ldots+n_{\alpha-1}+\frac{\alpha}{2}\big)\big(n_1+n_3+\ldots+n_{\alpha-1}+\frac{\alpha-2}{2}\big) \\
&\quad + n_1 + (n_3+1)(n_2+2) + (n_5+1)(n_2+n_4+3) + \ldots \\
&\quad + (n_{\alpha-1}+1)\big[n_2+n_4+\ldots+n_{\alpha-2}+\frac{\alpha}{2}\big] \\
&\quad + (n_2+1)(n_1+1) + (n_4+1)(n_1+n_3+2) + \ldots \\
&\quad + (n_{\alpha-2}+1)\big[n_1+n_3+\ldots+n_{\alpha-3}+\big(\frac{\alpha}{2}-1\big)\big] \\
&\quad + n_\alpha\big[n_1+n_3+\ldots+n_{\alpha-1}+\frac{\alpha}{2}\big]\Big\} \\
&= (n+\alpha-1)[n+\alpha] - \Big\{\big(n_1+n_3+\ldots+n_{\alpha-1}+\frac{\alpha}{2}\big)\big(n_1+n_3+\ldots+n_{\alpha-1}+\frac{\alpha-2}{2}\big) \\
&\quad + n_1(n_1+n_2+n_3+\ldots+n_\alpha) + n_3(n_1+n_2+n_3+\ldots+n_\alpha) + \ldots \\
&\quad + \frac{\alpha}{2}(n_1+n_2+n_3+\ldots+n_\alpha) + \big(\frac{\alpha}{2}+\frac{\alpha}{2}-1\big)(n_1+n_3+\ldots+n_{\alpha-1}) \\
&\quad + \big(2+3+4+\ldots+\frac{\alpha}{2}\big) + \big(1+2+3+\ldots+\frac{\alpha}{2}-1\big)\Big\} \\
&= (n+\alpha-1)[n+\alpha] \\
&\quad - \Big\{\big(n_1+n_3+\ldots+n_{\alpha-1}+\frac{\alpha}{2}\big)(n_1+n_3+\ldots+n_{\alpha-1}+\alpha-1) \\
&\quad + \frac{\alpha}{2}\big(\frac{\alpha}{2}-1\big) + \frac{\alpha}{2}(\alpha-1) + \frac{\frac{\alpha}{2}(\frac{\alpha}{2}+1)}{2} + \frac{\frac{\alpha}{2}(\frac{\alpha}{2}-1)}{2} - 1\Big\} \\
&= (n+\alpha-1)[n+\alpha] - \Big\{\big(n_1+n_3+\ldots+n_{\alpha-1}+\frac{\alpha}{2}\big) + (n+\alpha-1)\Big\}
\end{aligned}
$$

$$
\begin{aligned}
(n+\alpha-1)c(f) &= (n+\alpha-1)(n+\alpha) \\
&\quad - \Big\{\big(n_1+n_3+\ldots+n_{\alpha-1}+\frac{\alpha}{2}\big) + (n+\alpha-1)\Big\} \\
c(f) &= n+\alpha-\big(n_1+n_3+\ldots+n_{\alpha-1}+\frac{\alpha}{2}\big) \\
\therefore rsm(T_\alpha) &= n+\alpha-(n_1+n_3+\ldots+n_{\alpha-1}+\frac{\alpha}{2})
\end{aligned}
$$

■

Note 2. $rsm(T_\alpha)$ can be increased if the number of vertices in a star at any odd position is less that the no. of vertices in a star at any even position.

For example, we have $rsm(T_6) = 17$ in Fig. 2.

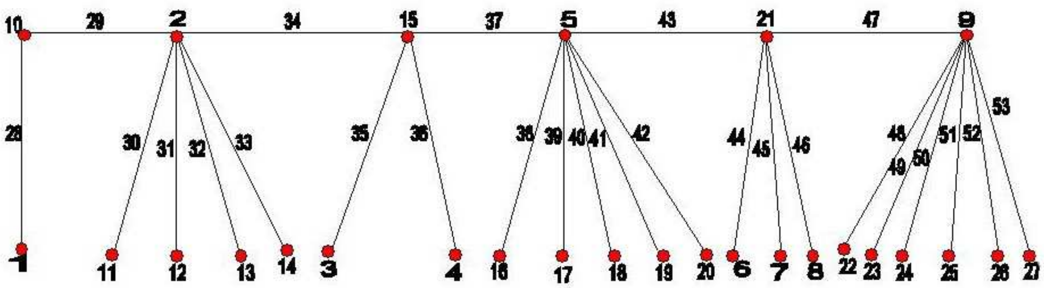

Figure 2: $\alpha = 6,\ n_1 = 1,\ n_2 = 4,\ n_3 = 2,\ n_4 = 5,\ n_5 = 3,\ n_6 = 6,\ rsm(T_6) = 17$

Note 3. $rsm(T_\alpha)$ can also be reduced if the number of vertices in the star at any odd position is greater than the number of vertices in the star at any even position. For example, we have $rsm(T_6) = 8$ in Fig. 3

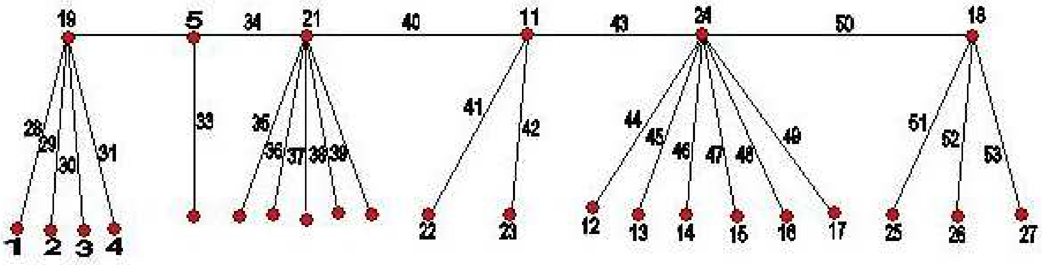

Figure 3: $\alpha = 6,\ n_1 = 4,\ n_2 = 1,\ n_3 = 5,\ n_4 = 2,\ n_5 = 6,\ n_6 = 3,\ rsm(T_6) = 8$

Theorem 2.2. $rsm(T_\alpha) = n + \alpha - n_2 - n_4 - \ldots - n_{\alpha-1} - \left(\dfrac{\alpha+3}{2}\right)$

Proof. We prove this theorem by assigning reverse super edge-magic labeling to T_α, α is odd.

Let u_i be the center and v_i be the set of pendent vertices of the i^{th} star.

Each v_i has n_i vertices, $i = 1, 2, ..., \alpha$. Thus

$$n_1 + n_2 + n_3 + \cdots + n_\alpha = n,\ v = n + \alpha,\ \varepsilon = n + \alpha - 1.$$

Then, the following labeling f is a reverse super edge-magic labeling of T_α.

$$
\begin{aligned}
f(u_1) &= 1 \\
f(u_{2i}) &= f(u_\alpha) + n_1 + n_3 + n_5 + \cdots + n_{2i-1} + i, && i = 1, 2, 3, ...\frac{(\alpha-1)}{2} \\
f(u_{2i-1}) &= f(u_1) + n_2 + n_4 + n_6 + \cdots + n_{2i-2} + i - 1, && i = 2, 3, ...\frac{(\alpha-1)}{2} \\
f(v_{2i-1}) &= \{f(u_\alpha) + n_1 + n_3 + \cdots + n_{2i-3} + i, ..., \\
&\quad f(u_\alpha) + n_1 + n_3 + \cdots + n_{2i-1} + i - 1\}, && i = 1, 2, 3, ..., \frac{\alpha+1}{2} \\
f(v_{2i}) &= \{f(u_1) + n_2 + n_4 + \cdots + n_{2i-2} + i, ..., \\
&\quad f(u_1) + n_2 + n_4 + \cdots + n_{2i} + i - 1\}, && i = 1, 2, 3, ..., \frac{\alpha-1}{2}.
\end{aligned}
$$

This labeling of vertices gives $f(x) + f(y)$ for all edges $xy \in E$ to vary from $f(u_\alpha) + 2$ to $f(u_\alpha) + n + \alpha$. Thus for each edge $e = xy$ if $f(x) + f(y) = f(u_\alpha) + i$, $i = 2, 3, ..., n + \alpha$ then $f(e) = n + \alpha - 1$.

For example, the reverse super edge-magic labeling of $T_\alpha, \alpha = 7$ is shown in Fig. 4.

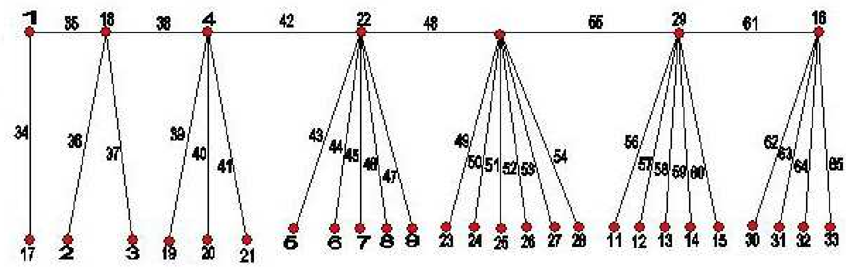

Figure 4: α=7, n_1=1, $n_2 = 2$, n_3=3, n_4=5, n_5=6, n_6=5, $rsm(T_7)$=16

By Note 1, if f is a reverse super edge-magic labeling of T_α with constant $c(f)$.

$$
\begin{aligned}
\varepsilon\, c(f) &= \sum_{e\in E} f(e) - \sum_{v\in V} d(v)f(v) \\
(n+\alpha-1)\, c(f) &= \sum_{e\in E} f(e) - \left\{ \sum_{i=1, v\in V_i}^{\alpha} f(v) + \sum_{i=1}^{\alpha} d(u_i)f(u_i) \right\} \\
&= \sum_{e\in E} f(e) - \Big\{ \sum_{i=1, v\in V_i}^{\alpha} f(v) + (n_1+1)f(u_1) + (n_2+2)f(u_2) + ... \\
&\quad + (n_{\alpha-1}+2)f(u_{\alpha-1}) + (n_\alpha+1)f(u_\alpha) \Big\} \\
&= \sum_{e\in E} f(e) - \Big\{ \Big[\sum_{i=1, v\in V_i}^{\alpha} f(v) + f(u_1) + f(u_2) + \ldots + f(u_\alpha) \Big] \\
&\quad + n_1 f(u_1) + (n_2+1)f(u_2) + \cdots + (n_{\alpha-1}+1)f(u_{\alpha-1}) + n_\alpha f(u_\alpha) \Big\}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{e\in E} f(e) - \Big\{ \sum_{i=1_i}^{n+\alpha-1} i + n_1 f(u_1) + (n_2+1)[f(u_\alpha)+n_2+1] \\
&\quad + (n_3+1)[f(u_\alpha)+n_2+1] \\
&\quad + (n_4+1)[f(u_\alpha)+n_1+n_3+2] + (n_5+1)[f(u_1)+n_2+n_4+2] + \cdots \\
&\quad + (n_{\alpha-1}+1)\big[f(u_\alpha)+n_1+n_3+\ldots+n_{\alpha-2}+\big(\frac{\alpha-1}{2}\big)\big] \\
&\quad + n_\alpha\big[f(u_1)+n_2+n_4+\ldots+n_{\alpha-1}+\frac{\alpha-1}{2}\big]\Big\} \\
&= \sum_{e\in E} f(e) - \Big\{\frac{(n+\alpha-1)(n+\alpha)}{2} + f(u_1)\big[n_1+n_3+\ldots+n_\alpha+\frac{\alpha-3}{2}\big] \\
&\quad + f(u_\alpha)\big[n_2+n_2+\ldots+n_{\alpha-1}+\frac{\alpha-1}{2}\big] \\
&\quad + (n_2+1)(n_1+1) + (n_3+1)(n_2+1) \\
&\quad + (n_4+1)(n_1+n_3+2) + (n_5+1)(n_2+n_4+2) + \ldots \\
&\quad + (n_{\alpha-1}+1)\big[n_1+n_3+\ldots+n_{\alpha-2}+\big(\frac{\alpha-1}{2}\big)\big] \\
&\quad + n_\alpha\big[n_2+n_4+\ldots+n_{\alpha-1}+\frac{\alpha-1}{2}\big]\Big\} \\
&= \{(n+\alpha+1)+(n+\alpha+2)+\ldots+(2n+2\alpha-1)\} \\
&\quad - \Big\{\frac{(n+\alpha-1)(n+\alpha)}{2} + \big(n_1+n_3+\ldots+n_\alpha+\frac{(\alpha-3)}{2}\big) \\
&\quad + \big(n_2+n_4+\ldots+n_{\alpha-1}+\frac{\alpha+1}{2}\big)\big(n_2+n_4+\ldots+n_{\alpha-1}+\frac{\alpha-1}{2}\big) \\
&\quad + (n_2+1)(n_1+1) + (n_4+1)(n_1+n_3+2) + \ldots \\
&\quad + (n_{\alpha-1}+1)\big[n_1+n_3+\ldots+n_{\alpha-2}+\frac{\alpha-1}{2}\big] \\
&\quad + (n_3+1)(n_2+1) + (n_5+1)(n_2+n_4+2) + \ldots \\
&\quad + n_\alpha\big[n_2+n_4+\ldots+n_{\alpha-1}+\frac{(\alpha-1)}{2}\big]\Big\} \\
&= \frac{n+\alpha-1}{2}[n+\alpha+1+2n+2\alpha-1] \\
&\quad - \Big\{\frac{(n+\alpha-1)(n+\alpha)}{2} - \big(n_1+n_3+\ldots+n_\alpha+\frac{(\alpha-3)}{2}\big) \\
&\quad + (n_2+n_4+\ldots+n_{\alpha-1})(n_2+n_4+\ldots+n_{\alpha-1}+\alpha) \\
&\quad + \big(\frac{\alpha+1}{2}\big)\big(\frac{\alpha-1}{2}\big) + \big(\frac{\alpha-1}{2}\big)[n_1+n_2+\ldots+n_\alpha] \\
&\quad + \big(1+2+3+\ldots+\frac{\alpha-1}{2}\big) + \big(1+2+3+\ldots+\frac{\alpha-3}{2}\big)\Big\} \\
&= \frac{n+\alpha-1}{2}[3n+3\alpha-n-\alpha] - \Big\{\big(n_1+n_3+\ldots+n_\alpha+\frac{(\alpha-3)}{2}\big) \\
&\quad + (n_2+n_4+\ldots+n_{\alpha-1})(n+\alpha) + \big(\frac{\alpha+1}{2}\big)(n+\alpha) + \big(\frac{\alpha+1}{2}\big)\big(\frac{\alpha-1}{2}\big) \\
&\quad + \alpha\big(\frac{\alpha+1}{2}\big) + \big(\frac{(\frac{\alpha-1}{2})(\frac{\alpha+1}{2})}{2} + \frac{(\frac{\alpha-3}{2})(\frac{\alpha-1}{2})}{2}\big)\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{n+\alpha-1}{2}[2n+2\alpha] - \Big\{ \big(n_1+n_3+\ldots+n_\alpha+\frac{(\alpha-3)}{2}\big) \\
&\quad + (n_2+n_4+\ldots+n_{\alpha-1})(n+\alpha) \\
&\quad + \big(\frac{\alpha^2-1}{2}\big) + \alpha\big(\frac{\alpha-1}{2}\big) + \frac{\alpha^2-1}{8} + \frac{\alpha^2-4\alpha+3}{2} + \big(\frac{\alpha+1}{2}\big)(n+\alpha)\Big\} \\
&= (n+\alpha)\big[n+\alpha-1-n_2-n_4-\ldots-n_{\alpha-1}-\frac{\alpha+1}{2}\big] \\
&\quad - \Big\{ \big(n_1+n_3+\ldots+n_\alpha+\frac{(\alpha-3)}{2}\big)+ \\
&\quad + \frac{1}{2}\,[2\alpha^2-2-4\alpha^2+4\alpha^2+\alpha^2-1+\alpha^2-4\alpha+3]\Big\} \\
&= (n+\alpha)\big[n+\alpha-1-n_2-n_4-\ldots-n_{\alpha-1}-\frac{\alpha+1}{2}\big] \\
&\quad - \big[n_1+n_3+\ldots+n_\alpha+\frac{(\alpha-3)}{2}\big] \\
&= (n+\alpha-1)\big[n+\alpha-1-n_2-n_4-\ldots-n_{\alpha-1}-\big(\frac{\alpha+1}{2}\big)\big] \\
&\quad + \big[n+\alpha-1-n_2-n_4-\ldots-n_{\alpha-1}-\big(\frac{\alpha+1}{2}\big)\big] \\
&\quad - \big[n_1+n_3+\ldots+n_\alpha+\frac{(\alpha-3)}{2}\big]
\end{aligned}
$$

$$
\begin{aligned}
(n+\alpha-1)c(f) &= (n+\alpha-1)\big[n+\alpha-1-n_2-n_4-\ldots-n_{\alpha-1}-\big(\frac{\alpha+1}{2}\big)\big] \\
c(f) &= n+\alpha-n_2-n_4-\ldots-n_{\alpha-1}-\big(\frac{\alpha+3}{2}\big) \\
rsm[T_\alpha] &= n+\alpha-n_2-n_4-\ldots-n_{\alpha-1}-\big(\frac{\alpha+3}{2}\big)
\end{aligned}
$$

■

Note 4. $rsm(T_\alpha)$ is odd, will also be increased if the number of vertices in the star at any even position is less than the number of vertices in the star at any odd position as shown in Fig. 5.

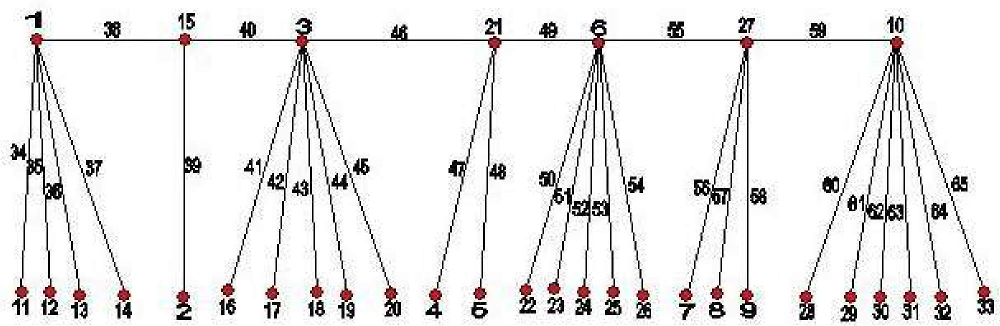

Figure 5: α=7, n_1=4, n_2=1, n_3=5, n_4=2, n_5=5, n_6=3, n_7=6, $rsm(T_7)$=22

Note 5. $rsm(T_\alpha)$ is odd, will also be decreased if the number of vertices in the star at any even position is greater than the number of vertices in the star at any odd position as shown in Fig. 6.

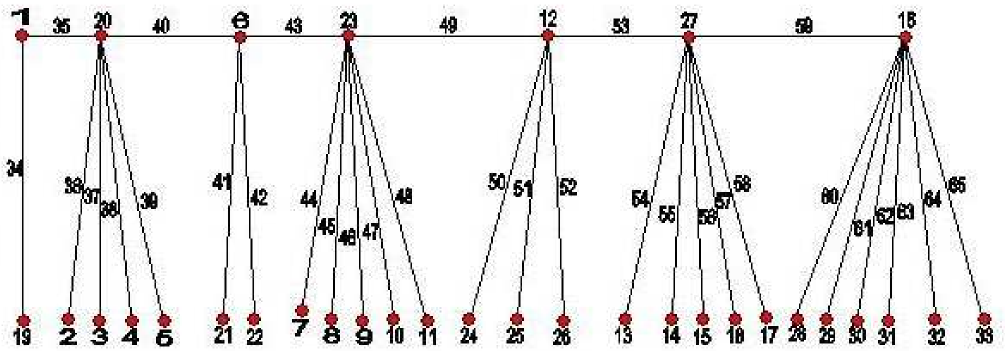

Figure 6: α=7, n_1=1, n_2=4, n_3=2, n_4=5, n_5=3, n_6=5, n_7=6, $rsm(T_6)$=14

3. Observation

Theorem 3.1. $rsm\left(\langle K_{1,1}, K_{1,2}, K_{1,r}\rangle\right) = r - 2$.

Proof. This is obtained by joining centres u, v, w of $K_{1,1}, K_{1,2}$, and $K_{1,r}$ to a new X. The pendent vertex of the star $K_{1,1}$ is denoted by u_1; the pendent vertices of $K_{1,2}$ are denoted by v_1, v_2 and the pendent vertices of $K_{1,r}$ are denoted by $w_1, w_2, ..., w_r$. The reverse super edge-magic labeling f of the graph is given below: $f(u_1) = 1$, $f(v) = 2$, $f(w_1) = 3$, $f(w) = 4$, $f(u) = 5$, $f(v_1) = 6$, $f(x) = 7$, $f(v_2) = 8$ and $f(w_i) = i + 7$, $i = 2, 3, 4, ..., r$, $f(wu_1) = r + 8$, $f(ww_1) = r + 9$, $f(vv_1) = r + 10$, $f(xv) = r + 11$, $f(vv_2) = r + 12$, $f(xw) = r + 13$, $f(xu) = r + 14$, $f(ww_i) = f(w_r) + 3r + 1 + i$, $i = 2, 3, 4, ...$ Thus, $f(xu) - \{f(x) + f(u)\} = r + 14 - \{7 + 5\} = r + 2$. ∎

This observation is illustrated in Fig. 7 for $r = 4$.

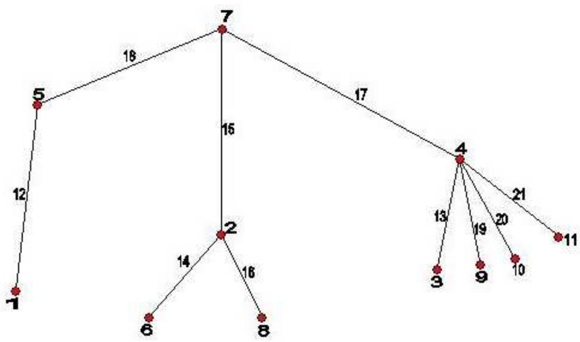

Figure 7:

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ON HYPER PSEUDO MV-ALGEBRAS

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Abstract. In this paper, we introduce the notion of hyper pseudo MV-algebra as a generalization of pseudo MV-algebra and hyper MV-algebra. Then we investigate some properties of this structure and attempt to construct a hyper pseudo MV-algebra from a l-group. Finally, we proved some theorems about filters, ideals, congruence relations, in hyper pseudo MV-algebras.

Keywords: hyper (Pseudo) MV-algebra, l-group, filter, ideal, congruence.

Mathematics Subject Classification: 03G12, 06F35, 06D35, 03G25.

1. Introduction

MV-algebras were introduced by C.C. Chang [2] in 1958, as an algebraic model of infinite valued logic. In [10], Mundici showed that any MV-algebra is an interval of an Abelian lattice ordered group with a strong unit. Georgescu and Iorgulescu [4] introduced a new non-commutative algebraic structures, which were called pseudo MV-algebras. It can be obtained by dropping commutative axioms in MV-algebras, which are a generalization of MV-algebras.

The hyper algebraic structure theory was introduced in 1934 [9] by Marty at 8th Congress of Scandinavian Mathematicians. Recently in [6], Sh. Ghorbani et al. applied the hyper structure to MV-algebras and introduced the concept of hyper MV-algebra which is a generalization of MV-algebra and investigated some related results. Since then many researchers have worked on this structure (see [5], [7], [8], [11]).

In this paper, the concept of hyper pseudo MV-algebra was introduced. We show that any pseudo MV-algebra (hyper MV-algebra) is a hyper pseudo MV-algebra and verify some of the properties of this algebra as mentioned in the abstract.

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2. Preliminaries

Definition 2.1. [4] A *Pseudo MV-algebra* is an algebra $(M, +, -, \sim, 0, 1)$ of type $(2, 1, 1, 0, 0)$ such that the following axioms hold for all $x, y, z \in M$,

(A1) $x + (y + z) = (x + y) + z$,
(A2) $x + 0 = 0 + x = x$,
(A3) $x + 1 = 1 + x = 1$,
(A4) $1^- = 1^\sim = 0$,
(A5) $(x^- + y^-)^\sim = (x^\sim + y^\sim)^-$,
(A6) $x + (x^\sim.y) = y + (y^\sim.x) = (x.y^-) + y = (y.x^-) + x$,
(A7) $x.(x^- + y) = (x + y^\sim).y$,
(A8) $(x^-)^\sim = x$.

where $x.y = (x^- + y^-)^\sim$.

Definition 2.2. [1] A *lattice-ordered group (l-group)* is an algebra $(G, \vee, \wedge, +, -, 0)$ such that $(G, \vee, \wedge)$ is a lattice, $(G, +, -, 0)$ is a group and $+$ is an order preserving map. If $(G, \vee, \wedge, +, -, 0)$ is a lattice-ordered group, then we use $[0, u]$ to denote $\{x \in G | 0 \le x \le u\}$, for any $0 \le u$.

Proposition 2.3. *[4] Let $(G, \vee, \wedge, +, -, 0)$ be an l-group and $0 \le u$, for some $u \in G$. Define $x * y = (x + y) \wedge u$, $x^- = u - x$ and $x^\sim = -x + u$, for any $x, y \in G$. Then $([0, u], *, -, \sim, 0, u)$ is a pseudo MV-algebra.*

Definition 2.4. [6] A *hyper MV-algebra* is a non-empty set M endowed with a binary hyper operation $\oplus$, a unary operation $*$ and a constant 0 satisfying the following conditions: for all $x, y, z \in M$

$(hMV1)$ $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,
$(hMV2)$ $x \oplus y = y \oplus x$,
$(hMV3)$ $(x^*)^* = x$,
$(hMV4)$ $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$,
$(hMV5)$ $0^* \in x \oplus 0^*$,
$(hMV6)$ $0^* \in x \oplus x^*$
$(hMV7)$ if $x \ll y$ and $y \ll x$, then $x = y$, where $x \ll y$ is defined by $0^* \in x^* \oplus y$.

For every $A, B \subseteq M$, we define $A \ll B$ if and only if there exist $a \in A$ and $b \in B$ such that $a \ll b$ and $A \oplus B = \cup\{a \oplus b | a \in A, b \in B\}$. Also, we define $0^* = 1$ and $A^* = \{a^* | a \in A\}$.

3. Hyper pseudo MV-algebras

Definition 3.1. A *hyper pseudo MV-algebra* is a non-empty set M with a binary hyperoperation $+$, two unary operations $'$, $\sim$ and two constants $0, 1$ satisfying the following conditions: for all $x, y, z \in M$,

$(HSMV1)$ $x + (y + z) = (x + y) + z$,
$(HSMV2)$ $1 \in (x + 1) \cap (1 + x)$,

$(HSMV3)$ $1^{\sim} = 1' = 0$,
$(HSMV4)$ $(x' + y')^{\sim} = (x^{\sim} + y^{\sim})'$,
$(HSMV5)$ $x + (x^{\sim} \odot y) = y + (y^{\sim} \odot x) = (x \odot y') + y = (y \odot x') + x$,
$(HSMV6)$ $x \odot (x' + y) = (x + y^{\sim}) \odot y$,
$(HSMV7)$ $(x')^{\sim} = x$,
$(HSMV8)$ $1 \in (x + x^{\sim}) \cap (x' + x)$,
$(HSMV9)$ $1 \in (x' + y) \cap (y' + x)$ implies $x = y$,
$(HSMV10)$ $1 \in x' + y$ if and only if $1 \in y + x^{\sim}$.

where $y \odot x = (x' + y')^{\sim}$, $A' = \{a' | a \in A\}$, $A^{\sim} = \{a^{\sim} | a \in A\}$, $A \odot B = \cup\{a \odot b | a \in A, b \in B\}$ and $A + B = \cup\{a + b | a \in A, b \in B\}$, for any $A, B \subseteq M$. A hyper pseudo MV-algebra M is called *proper* if there exists $a, b \in M$ such that $2 \leq |a + b|$. Let $(M, +, ', \sim, 0, 1)$ be a hyper pseudo MV-algebra, which is not proper. Then for any $x, y \in M$, there exists $a_{x,y} \in M$ such that $x + y = \{a_{x,y}\}$. Define the operation $* : M \to M$ by $x * y = a_{x,y}$. It can be easily obtained that $(M, *, ', \sim, 0, 1)$ is a pseudo MV-algebra and we say that $(M, +, ', \sim, 0, 1)$ is a pseudo MV-algebra.

Example 3.2. Let $(M, +, -, \sim, 0, 1)$ be a pseudo MV-algebra. For any $x, y \in M$, we define $x \oplus y = \{x + y\}$. Then $(M, \oplus, -, \sim, 0, 1)$ is a hyper pseudo MV-algebra.

Example 3.3. Let $(M, \oplus, *, 0, 1)$ be a hyper MV-algebra. Then $(M, \oplus, *, *, 0, 0^*)$ is a hyper pseudo MV-algebra.

Example 3.4. (i) Let $M = \{0, a, b, c, 1\}$. Consider the following tables:

Table 1

+	0	a	b	c	1
0	{0}	{0,a}	{0,b}	{0,c}	M
a	{0,a}	{0,a}	{a,b}	M	M
b	{b,0}	M	{0,b}	{c,b}	M
c	{c,0}	{c,a}	M	{0,c}	M
1	M	M	M	M	M

Table 2

	0	a	b	c	1
′	1	b	c	a	0
∼	1	c	a	b	0

Then we get

Table 3

⊙	0	a	b	c	1
0	M	M	M	M	M
a	M	{1,a}	M	{a,c}	{1,a}
b	M	{a,b}	{1,b}	M	{1,b}
c	M	M	{b,c}	{1,c}	{1,c}
1	M	{a,1}	{b,1}	{c,1}	{1}

It is not difficult to show that $(M, +, ', \sim, 0, 1)$ is a hyper pseudo MV-algebra. Moreover, $+$ is not commutative and so M is not a hyper MV-algebra.

(ii) Let $N = \{0, a, b, c, d, 1\}$ and $M = \{0, a, b, c, 1\}$. Consider the following tables:

Table 4

+	0	a	b	c	d	1
0	{0}	{0,a}	{0,b}	{0,c}	{0,d}	M
a	{0,a}	{0,a}	{a,b}	M	{0,a,d}	M
b	{b,0}	M	{0,b}	{c,b}	{0,b,d}	M
c	{c,0}	{c,a}	M	{0,c}	{0,c,d}	M
d	{d,0}	{d,a}	{d,b}	{d,c}	N	N
1	M	M	M	M	N	M

Table 5

	0	a	b	c	d	1
$'$	1	b	c	a	d	0
$\sim$	1	c	a	b	d	0

Easy calculations show that $(N, +,', \sim, 0, 1)$ is a hyper pseudo MV-algebra.

Remark 3.5.

(i) It is easy, to show that if $(M, +,', \sim, 0, 1)$ is a hyper pseudo MV-algebra such that $x + y = y + x$ and $x' = x^{\sim}$, for any $x, y \in M$, then $(M, +,', 0)$ is a hyper MV-algebra.

(ii) Let $(M, +,', \sim, 0, 1)$ be a hyper pseudo MV-algebra. Define a relation $\leq$ on M by $x \leq y$ if and only if $1 \in x' + y$, for any $x, y \in M$. Then by $(HSMV8)$ and $(HSMV9)$, we conclude that $\leq$ is a reflexive and antisymmetric relation on M.

Proposition 3.6. *For any integer* $5 \leq m$*, there is at least one proper hyper pseudo* MV*-algebra of order* m*.*

Proof. Let $5 \leq m$, $n = m - 2$, $M = \{a_1, a_2, ..., a_n\}$ and $N = M \cup \{0, 1\}$, where $0, 1 \notin M$. Define $a_i' = a_{i+1}$ and $a_j^{\sim} = a_{j-1}$, for any $i, j \in \{1, 2, ..., n\}$, where $a_0 = a_n$ and $a_{n+1} = a_1$. Consider the following hyper operation on M.

$$x + y = \begin{cases} \{0\} & \text{if } x = y = 0, \\ \{0, y\} & \text{if } x = 0,\ y = a_i \text{ or } y = 0,\ x = a_i, \\ N & \text{if} x = 1 \text{ or } y = 1, \\ \{0, a_i\} & \text{if } x = y = a_i, \\ N & \text{if } x = a_i,\ y = a_j,\ j = i - 1, \\ \{a_i, a_j\} & \text{otherwise.} \end{cases}$$

It is easy to show that $(N, +,', \sim, 0, 1)$ is a proper hyper pseudo MV-algebra of order m. ∎

From now on, in this paper, $(M, +,', \sim, 0, 1)$ or simply M will denote a hyper pseudo MV-algebra, unless otherwise stated. If $A, B \subseteq M$, $A \ll B$ $(A \leq B)$ means that $a \leq b$, for some $a \in A$ and $b \in B$ (for any $a \in A$, there exists $b \in B$ such that $a \leq b$). Moreover, if $A = \{a\}$, then we write $a \ll B$ instead of $A \ll B$.

Proposition 3.7. *Suppose that $(G, \vee, \wedge, +, -, 0)$ is a l-group and $0 \leq u$, for some $u \in G$. Let $\downarrow A = \{x \in [0,u] | x \leq a$, for all $a \in A\}$ and $\uparrow A = \{x \in [0,u] | a \leq x$, for all $a \in A\}$, for any $A \subseteq G$. Define $x \oplus y = \downarrow \{x + y, u\} = \downarrow \{x * y\}$, $x^{-} = u - x$ and $x^{\sim} = -x + u$, for any $x, y \in G$, where $*$ is the operation which is defined in Proposition 2.3. Then $([0,u], \oplus, ', \sim, 0, u)$ is a hyper pseudo MV-algebra.*

Proof. Let $x, y, z \in [0,u]$. By Proposition 2.3, the operations $'$ and $\sim$ are one to one and onto maps. Moreover, $x \leq y \Leftrightarrow y' \leq x' \Leftrightarrow y^{\sim} \leq x^{\sim}$, for all $x, y \in [0,u]$.

(1) Let $x \in [0,u]$. Then $0 \leq x$ and so $u = 0 + u \leq x + u$. Hence $u \in \downarrow \{x + u, u\} = x \oplus u$. By the similar way, $u \in u \oplus x$. Therefore, $u \in (x \oplus u) \cap (u \oplus x)$.

(2) By Proposition 2.3, $u' = u^{\sim} = 0$ and $(x')^{\sim} = x$, for all $x \in [0,u]$.

(3) Let $a \in [0,u]$. Then $0 \leq a \leq u$ and so $0 = a - a \leq u - a = u'$, $-a \leq 0$. Hence $u - a \leq u + 0 = u$. Therefore, $a' \in [0,u]$. By the similar way, we can show that $a^{\sim} \in [0,u]$. On the other hand, let $b' \in [0,u]$, for some $b \in G$. Then $0 \leq u - b \leq u$, so $b = 0 + b \leq (u - b) + b = u$ and $-b = -u + (u - b) \leq -u + u = 0$. Hence $0 \leq b$. Therefore, $b \in [0,u]$ if and only if $b' \in [0,u]$, for any $b \in G$. Similarly, we can show that $b \in [0,u]$ if and only if $b^{\sim} \in [0,u]$. Hence

$$
\begin{aligned}
(x' \oplus y')^{\sim} &= \{a^{\sim} \in [0,u] | a \leq x' + y', a \in [0,u]\} \\
&= \{a^{\sim} | a \in [0,u], (x' + y')^{\sim} \leq a^{\sim}\} \\
&= \{a^{\sim} | a \in [0,u], (x^{\sim} + y^{\sim})' \leq a^{\sim}\}, \text{ by Proposition 2.3} \\
&= \{a^{\sim} | (a^{\sim})' \in [0,u], (x^{\sim} + y^{\sim})' \leq a^{\sim}\} \\
&= \{t | t' \in [0,u], (x^{\sim} + y^{\sim})' \leq t\} \\
&= \{t | t' \in [0,u], t^{\sim} \leq (x^{\sim} + y^{\sim})\} \\
&= \{t | t' \in [0,u], t^{\sim} \in (x^{\sim} \oplus y^{\sim})\} \\
&= \{t | t' \in [0,u], t \in (x^{\sim} \oplus y^{\sim})'\} \\
&= \{t | t \in [0,u], t \in (x^{\sim} \oplus y^{\sim})'\} \\
&= (x^{\sim} \oplus y^{\sim})'.
\end{aligned}
$$

(4) By Proposition 2.3, $x + x^{\sim} = u = x' + x$. Hence $u \in \downarrow \{x + x^{\sim}, u\}$ and $u \in \downarrow \{x' + x, u\}$. Therefore, $u \in (x \oplus x^{\sim}) \cap (x' \oplus x)$.

(5) Let $u \in (x' \oplus y) \cap (y' \oplus x)$. Then by $u \in x' \oplus y$, we get $u \leq x' + y$ and so $u + (-y) \leq (x' + y) + (-y) = x'$. Similarly, $u \in y' \oplus x$ implies $x' \leq y'$. Hence $x' = y'$ and so $x = y$.

(6) Since $([0,u], +, ', \sim, 0, u)$ is a pseudo MV-algebra, then we have

$$
u \in x' \oplus y \Leftrightarrow u \leq x' + y \Leftrightarrow y' \leq x' \Leftrightarrow y^{\sim} \leq x^{\sim} \Leftrightarrow -y + u \leq x^{\sim} \Leftrightarrow u \leq y + x^{\sim}
$$

(7) We want to show that $x \oplus (y \oplus z) = (x \oplus y) \oplus z$.

$$
x \oplus (y \oplus z) = \cup \{x \oplus a | a \in y \oplus z\} = \{t \in [0,u] | t \leq x * a, a \in y \oplus z\}
$$

For any $a \in y \oplus z$, we have $a \leq y * z$, $a \in [0,u]$ and so $x + a \leq x + (y * z)$. Hence $x * a \leq x * (y * z)$, it follows that $x \oplus (y \oplus z) \subseteq \downarrow \{x * (y * z)\}$. Moreover, $y * z \leq y * z$

and so $\downarrow \{x * (y * z)\} \subseteq x \oplus (y \oplus z)$. Therefore, $x \oplus (y \oplus z) =\downarrow \{x * (y * z), u\}$. By the similar way, $(x \oplus y) \oplus z =\downarrow \{(x * y) * z, u\}$. Now, by Proposition 2.3, we get $x \oplus (y \oplus z) = (x \oplus y) \oplus z$.

(8) Let $a \odot b = (b^{\sim} \oplus a^{\sim})'$, for all $a, b \in [0, u]$. Then

$$\begin{aligned} x \odot y = (y^{\sim} \oplus x^{\sim})' &= \{a' | a \in [0, u], a \in y^{\sim} \oplus x^{\sim}\} \\ &= \{a' \in [0, u] | a \leq y^{\sim} * x^{\sim}\} \\ &= \{a' \in [0, u] | (y^{\sim} * x^{\sim})' \leq a'\} \\ &= \{t \in [0, u] | (y^{\sim} * x^{\sim})' \leq t\}, \text{ since } ' \text{ is one to one and onto} \\ &= \uparrow \{x.y\}. \end{aligned}$$

Now, we prove that, $(HSMV5)$ holds.

$$x \oplus (x^{\sim} \odot y) = \cup\{x \oplus a | a \in x^{\sim} \odot y\} = \{t \in [0, u] | t \leq x * a, a \in \uparrow \{x^{\sim}.y\}\}$$

By Proposition 2.3, $x^{\sim}.y \in [0, u]$ and so $u \in \uparrow \{x^{\sim}.y\}$. Hence $x \oplus (x^{\sim} \odot y) = [0, u]$. By the similar way, we can show that $y \oplus (y^{\sim} \odot x) = (x \odot y') \oplus y = (y \odot x') \oplus x = [0, u]$. Therefore, $(HSMV5)$ holds.

(9) By definition of $\oplus$, it is obvious that $0 \in x' \oplus y$. Hence by Proposition 2.3, we obtain

$$[0, u] =\uparrow \{x * 0\} = x \odot 0 \subseteq x \odot x \odot (x' \oplus y) \subseteq [0, u]$$

By the similar way, we obtain $(x \oplus y^{\sim}) \odot y = [0, u]$. Therefore, $(HSMV6)$ holds.

From (1)-(9), it follows that $([0, u], \oplus, ', \sim, 0, u)$ is a hyper pseudo MV-algebra. ∎

Lemma 3.8. *The following properties hold: for any $x, y, z \in M$.*

(i) $(x^{\sim})' = x$, $0^{\sim} = 0' = 1$ *and* $1'' = 1 = (1^{\sim})^{\sim}$.
Moreover, $x = y \Leftrightarrow x' = y' \Leftrightarrow x^{\sim} = y^{\sim}$,

(ii) $0 \in (x \odot 0) \cap (0 \odot x)$,

(iii) $x \odot y = (y^{\sim} + x^{\sim})'$,

(iv) $0 \in (x \odot x') \cap (x^{\sim} \odot x)$,

(v) $x \ll x + y$ *and* $y \leq x + y$,

(vi) $(x \odot y) \odot z = x \odot (y \odot z)$,

(vii) $x \leq y \Leftrightarrow 0 \in y^{\sim} \odot x \Leftrightarrow 0 \in x \odot y' \Leftrightarrow 1 \in y + x^{\sim}$,

(viii) $x \odot y \ll z \Leftrightarrow x \ll z + y^{\sim} \Leftrightarrow y \ll x' + z$,

(ix) $(x + y)' = y' \odot x'$, $(x + y)^{\sim} = y^{\sim} \odot x^{\sim}$ *and* $x + y = (y' \odot x')^{\sim} = (y^{\sim} \odot x^{\sim})'$,

(x) $(x \odot y)' = y' + x'$ *and* $(x \odot y)^{\sim} = y^{\sim} + x^{\sim}$,

(xi) $(x^{\sim} \odot y) + y^{\sim} = (y^{\sim} \odot x) + x^{\sim}$,

(xii) $x \odot (x' + y) = y \odot (y' + x)$,

(xiii) $x \leq 1$ *and* $0 \leq x$.

Proof. (i) Since $1 \in x + x^{\sim}$, then by $(HSMV7)$, $1 \in (x + ((x^{\sim})')^{\sim})$. On the other hand, $1 \in ((x^{\sim})' + x^{\sim})$ and so $1 \in (x + u^{\sim}) \cap (u + x^{\sim})$, where $u = (x^{\sim})'$. Hence by $(HSMV9)$, $x = (x^{\sim})'$.
The proof of the other parts follows from $(HSMV7)$ and the first part of (i).

(ii) By $(HSMV2)$, $1 \in 1 + x'$ and so $1^{\sim} \in (1 + x')^{\sim}$. Hence by $(HSMV3)$, $0 \in x \odot 0$. By the similar way, we can show that $0 \in (0 \odot x)$.

(iii) The proof is Straightforward by $(HSMV4)$.

(iv) By $(HSMV8)$, $1 \in (x + x^{\sim}) \cap (x' + x)$, and so by $(HSMV7)$ and (iii), $0 \in (x + x^{\sim})' = ((x')^{\sim} + x^{\sim})' = x \odot x'$. On the other hand by (i), $0 \in (x' + x)^{\sim} = (x' + (x^{\sim})')^{\sim} = x^{\sim} \odot x$.

(v) By $(HSMV2)$ and $(HSMV8)$, we get $1 \in 1 + y \subseteq (x' + x) + y = x' + (x + y)$ and so $1 \in x' + c$, for some $c \in x + y$. Hence $x \leq c$, so $x \ll x + y$. Moreover, $1 \in x + 1 \subseteq x + (y + y^{\sim}) = (x + y) + y^{\sim}$. Hence $1 \in d + y^{\sim}$, for some $d \in x + y$. Therefore, $y \leq d$ and so $y \ll x + y$.

(iv)
$$\begin{aligned}
(x \odot y) \odot z &= (y' + x')^{\sim} \odot z \\
&= \cup\{a^{\sim} \odot z | a \in y' + x'\} \\
&= \cup\{(z' + (a^{\sim})')^{\sim} | a \in y' + x'\} \\
&= \cup\{(z' + a)^{\sim} | a \in y' + x'\}, \text{ by (i)} \\
&= (z' + (y' + x'))^{\sim} \\
&= ((z' + y') + x')^{\sim} \\
&= (((z' + y')^{\sim})' + x')^{\sim}, \text{ by (i)} \\
&= (y \odot z)' + x')^{\sim} \\
&= \cup\{(a' + x')^{\sim} | a \in y \odot z\} \\
&= \cup\{x \odot a | a \in y \odot z\} \\
&= x \odot (y \odot z).
\end{aligned}$$

(vii) Let $x, y \in M$. Then

$$x \leq y \Leftrightarrow 1 \in x' + y \Leftrightarrow 0 \in (x' + y)^{\sim} \Leftrightarrow 0 \in y^{\sim} \odot x \tag{3.1}$$

Moreover, by $(HSMV10)$, we have

$$1 \in x' + y \Leftrightarrow 1 \in y + x^{\sim} \Leftrightarrow 0 \in ((y')^{\sim} + x^{\sim})' \Leftrightarrow 0 \in x \odot y' \tag{3.2}$$

By using of (3.1) and (3.2), the proof is completed.

(viii)
$$\begin{aligned}
x \odot y \ll z &\Leftrightarrow 1 \in (x \odot y)' + z \Leftrightarrow 1 \in (y' + x') + z \\
&\Leftrightarrow 1 \in y' + (x' + z) \Leftrightarrow y \ll x' + z \\
x \odot y \ll z &\Leftrightarrow 1 \in z + (x \odot y)^{\sim} \Leftrightarrow 1 \in z + (y^{\sim} + x^{\sim}) \\
&\Leftrightarrow 1 \in (z + y^{\sim}) + x^{\sim} \Leftrightarrow x \ll z + y^{\sim}
\end{aligned}$$

(ix) By $(HSMV7)$, $(x + y)' = ((x')^{\sim} + (y')^{\sim})' = y' \odot x'$. Similarly, by using of (i), we can show that $(x + y)^{\sim} = y^{\sim} \odot x^{\sim}$.

(x) By $(HSMV7)$ and (iii), we get $(x \odot y)^{\sim} = ((y^{\sim} + x^{\sim})')^{\sim} = y^{\sim} + x^{\sim}$. The proof of the other part is similar.

(xi) By $(HSMV5)$, we obtain $(x^{\sim} \odot (y^{\sim})') + y^{\sim} = (y^{\sim} \odot (x^{\sim})') + x^{\sim}$. Therefore,

$$(x^{\sim} \odot y) + y^{\sim} = (y^{\sim} \odot x) + x^{\sim}$$

(xii)
$$\begin{aligned} x \odot (x' + y) &= ((x' + y)^{\sim} + (x')^{\sim})' \\ &= (y^{\sim} \odot x) + x^{\sim} \\ &= ((x^{\sim} \odot y) + y^{\sim})', \text{ by (xi)} \\ &= y \odot (x^{\sim} \odot y)' \\ &= y \odot (y' + x) \end{aligned}$$

(xiii) Let $x \in M$. By $(HSMV2)$ and (i), $0 \leq x$. Moreover, $1 \in 1 + x^{\sim}$ and so $x \leq 1$. ∎

Theorem 3.9. *Let $(M, +, ', \sim, 0, 1)$ be a hyper pseudo MV-algebra. Then*

(i) *$(M, \odot, \sim, ' 1, 0)$ is a hyper pseudo MV-algebra, too. It is called the dual hyper pseudo MV-algebra of $(M, +, ', \sim, 0, 1)$.*

(ii) *If $x + y = y + x$, for any $x, y \in M$, then M is a hyper MV-algebra.*

Proof. (i) It follows from Lemma 3.8.

(ii) Suppose that $x + y = y + x$, for any $x, y \in M$. Let $x \in M$. Then by $(HSMV8)$ and Lemma 3.8(i), we get $1 \in x' + x = x' + (x^{\sim})' = (x^{\sim})' + x'$ and so Lemma 3.8(vii) implies $x^{\sim} \leq x'$. Moreover, by $1 \in x + x^{\sim} = (x')^{\sim} + x^{\sim} = x^{\sim} + (x')^{\sim}$, we conclude that $x^{\sim} \leq x'$. Hence, $x' = x^{\sim}$, for any $x \in M$. Now, it is clear that $(M, +, ', 0)$ is a hyper MV-algebra. ∎

Proposition 3.10. *Let $x, y \in M$. Then the following properties hold:*

(i) $x \leq y \Leftrightarrow y' \leq x' \Leftrightarrow y^{\sim} \leq x^{\sim}$,

(ii) *$x \leq y$ implies $x + a \ll y + a$ and $a + x \ll a + y$, for all $a \in M$,*

(iii) *$x \leq y$ implies $a \odot x \ll a \odot y$ and $x \odot a \ll y \odot a$, for all $a \in M$,*

(iv) $x \odot y \ll x, y$,

(v) *$0 + 0 = \{0\}$ and $1 \odot 1 = \{1\}$,*

(vi) *$y \in x + 0$ ($y \in 0 + x$) implies $y \leq x$. Moreover, if $y \in x \odot 1$ ($y \in 1 \odot x$), then $x \leq y$,*

(vii) $x \in (0 + x) \cap (x + 0)$,

(viii) $x \in (1 \odot x) \cap (x \odot 1)$,

(ix) *$x + 0 = y + 0$ (0+x=0+y) implies $x = y$.*

Proof. (i)

$y' \leq x' \Leftrightarrow 1 \in x' + (y')^{\sim} = x' + y \Leftrightarrow x \leq y \Leftrightarrow 1 \in y + x^{\sim} = (y^{\sim})' + x^{\sim} \Leftrightarrow y^{\sim} \leq x^{\sim}$

(ii) Let $x \leq y$ and $a \in M$. Then we have

$$\begin{aligned}(y+a)+(x+a)^{\sim} &= (y+a)+(a^{\sim}\odot x^{\sim})\\ &= y+(a+(a^{\sim}\odot x^{\sim}))\\ &= y+(x^{\sim}+(((x^{\sim})^{\sim}\odot a)),\ \text{by } (HSMV5)\\ &= (y+x^{\sim})+((x^{\sim})^{\sim}\odot a)\\ &\supseteq 1+((x^{\sim})^{\sim}\odot a),\ \text{since } x\leq y.\end{aligned}$$

Hence by $(HSMV2)$, $1 \in (y+a)+(x+a)^{\sim}$ and so $1 \in u+v^{\sim}$, for some $u \in y+a$ and $v \in x+a$. Thus, by Lemma 3.8(vii), $v \leq u$. It follows that, $x+a \ll y+a$. On the other hand,

$$\begin{aligned}(a+x)'+(a+y) &= (x'\odot a')+(a+y),\ \text{by Lemma 3.8(ix)}\\ &= ((x'\odot a')+a)+y\\ &= ((a\odot x'')+x')+y\\ &= (a\odot x'')+(x'+y)\\ &\supseteq (a\odot x'')+1,\ \text{since } x\leq y\end{aligned}$$

Hence by $(HSMV2)$, $1 \in (a+x)'+(a+y)$ and so there exists $u \in a+x$ and $v \in a+y$ such that $1 \in u'+v$, whence $u \leq v$. Therefore, $a+y \ll a+y$.

(iii) Let $x \leq y$ and $a \in M$. Then by (i), $y' \leq x'$ and so by (ii), $a'+y' \ll a'+x'$. Hence $u \leq v$, for some $u \in a'+x'$ and $v \in a'+v'$, whence by (i), $v^{\sim} \leq u^{\sim}$. Therefore, $x \odot a = (a'+x')^{\sim} \ll (a'+y')^{\sim} = y \odot a$. By the similar way, we can show that $a \odot x \ll a \odot y$.

(iv) By Lemma 3.8(v), we know that $x', y' \leq x'+y'$ and so by (i), $(x'+y')^{\sim} \ll (x')^{\sim}, (y')^{\sim}$. Hence $y \odot x \ll x, y$.

(v) Let $b \in 0+0$. Then by $(HSMV8)$, we get $1 \in b+b^{\sim} \subseteq (0+0)+b^{\sim} = 0+(0+b^{\sim})$ and so $1 \in 0+x$, for some $x \in 0+b^{\sim} = 1'+b^{\sim}$. Hence by $(HSMV2,3,10)$, $x=1$, and so $1 \in 0+b^{\sim}$. Therefore, $b \leq 0$ and so by Lemma 3.8(xiii) and $(HSMV8)$, $b=0$. The proof of the other part follows from Theorem 3.9.

(vi) Let $x \in M$ and $y \in 0+x$. Then by (v), $0+x = (0+0)+x = 0+(0+x) \supseteq 0+y$ and so

$$1 \in 0+1 \subseteq 0+(y+y^{\sim}) = (0+y)+y^{\sim} \subseteq (0+x)+y^{\sim} = 0+(x+y^{\sim})$$

Hence there exists $u \in x+y^{\sim}$ such that $1 \in 0+u = 1'+u$. Thus $u=1$ and so $1 \in x+y^{\sim}$. Therefore, $y \leq x$. The proof of other part is similar. Now, let $y \in x \odot 1$. Then $y \in (0+x')^{\sim}$ and so $y' \in 0+x'$. It follows that $y' \leq x'$ and so by (i), $x \leq y$. The proof of the other part is similar.

(vii) Let $x \in M$. Then $1 \in 0+1 \subseteq 0+(x+x^{\sim}) = (0+x)+x^{\sim}$ and so $1 \in u+x^{\sim}$, for some $u \in 0+x$. Hence $x \leq u$. Now, by (vi), we get $x=u$.

Therefore, $x \in x+0$. On the other hand, $1 \in 1+0 \subseteq (x'+x)+0 = x'+(x+0)$. Thus $1 \in x'+u$, for some $u \in x+0$ and so $x \leq u$. Hence by (vi), we get $x = u$ and so $x \in x+0$.

(viii) Let $x \in M$. By (vii), $x' \in 0+x'$, and so $x = (x')^{\sim} \in (1'+x')^{\sim} = x \odot 1$. By the similar way, we can show that $x \in 1 \odot x$.

(ix) Let $x+0 = y+0$, for some $x, y \in M$. Since $1 \in x'+x$, then we get

$$1 \in (x'+x)+0 = x'+(x+0) = x'+(y+0) = (x'+y)+0.$$

Hence there exists $u \in x'+1$ such that $1 \in u+0 = u+1^{\sim}$. By $(HSMV2)$ and $(HSMV9)$, we get $u = 1$ and so $1 \in x'+y$. By the similar way, we can show that $1 \in y+x'$. Therefore, $(HSMV9)$ implies $x = y$. The proof of the other part is similar. ∎

4. Some hyper operations in hyper pseudo MV-algebra

In this section we define the concept of hyper operations, $\vee$, $\wedge$, $/$ and $\backslash$ on a hyper pseudo MV-algebras which we will use for definitions of ideals and filters in hyper pseudo MV-algebras. Then we obtain some of theirs properties and relation between $\leq$, $+$, $\odot$ and these hyper operations.

Proposition 4.1. *Consider the hyper operations $/$ and $\backslash$ on M, were defined by $a/b = a+b^{\sim}$ and $a\backslash b = a'+b$. Then, for any $x, y, z \in M$, we have:*

(i) $z \ll z/y$ *and* $z \ll x\backslash z$,

(ii) $x \leq y$ *implies* $x/z \ll y/z$ *and* $z\backslash x \ll z\backslash y$ *(*$z/y \ll z/x$ *and* $y\backslash z \ll x\backslash z$*)*,

(iii) $z \vee y^{\sim} \ll z/y$ *and* $y' \vee z \ll y\backslash z$,

(iv) $y\backslash z, z/y \subseteq \{u | u \odot y \ll z\}$ *and* $\{u | u \odot y \ll z\} \leq z/y, y\backslash z$,

(v) $x\backslash(y/z) = (x\backslash y)/z$,

(vi) $y\backslash(x\backslash z) = (x \odot y)\backslash z$,

(vii) $(z/x)/y = z/(x \odot y)$,

(viii) $(x+y)/z = x+(y/z)$,

(ix) $x/(z\backslash x) = z/(x\backslash z)$.

Proof. (i) It follows from Lemma 3.8(v).

(ii) Let $x \leq y$. Then by Proposition 3.10(ii), we have $x+z^{\sim} \ll y+z^{\sim}$ and $z'+x \ll z'+y$. By the similar way, we can prove the other part.

(iii) By Proposition 3.10(iv), $z^{\sim} \odot y^{\sim} \ll y^{\sim}$ and so there exists $a \in z^{\sim} \odot y^{\sim}$ such that $a \leq y^{\sim}$. Now, by 3.10(ii), we get $z \vee y^{\sim} = z+(z^{\sim} \odot y^{\sim}) \supseteq z+a \ll z+y^{\sim} = z/y$. By the similar way, we can prove the other part.

(iv) Let $a \in z/y$. Then $a \ll z/y = z + y^{\sim}$. Hence by Lemma 3.8(viii), $a \odot y \ll z$ and so $a \in \{u|u \odot y \ll z\}$. Now, let $u \in \{u|u \odot y \ll z\}$. Then by Lemma 3.8(viii), $u \ll z/y$ and so $u \leq b$, for some $b \in z/y$. Therefore, $\{u|u \odot y \ll z\} \leq z/y$. The proof of the other part is similar.

(v) $x\backslash(y/z) = x\backslash(y+z^{\sim}) = x'+(y+z^{\sim}) = (x'+y)+z^{\sim} = (x'+y)/z = (x\backslash y)/z$.

(vi) By Lemma 3.8(x), we get

$$y\backslash(x\backslash z) = y\backslash(x' + z) = y' + (x' + z) = (y' + x') + z = (x \odot y)' + z = (x \odot y)\backslash z.$$

(vii) $(z/x)/y = (z+x^{\sim})/y = (z+x^{\sim})+y^{\sim} = z+(x^{\sim}+y^{\sim}) = z+(y\odot x)^{\sim} = z/(y \odot x)$.

(viii) $(x + y)/z = (x + y) + z^{\sim} = x + (y + z^{\sim}) = x + (y/z)$.

(ix) $x/(z\backslash x) = x/(z' + x) = x + (z' + x)^{\sim} = x + (z' + (x^{\sim})')^{\sim} = x + (x^{\sim} \odot z)$. Now, by $(HSMV5)$, we obtain $x/(z\backslash x) = z+(z^{\sim}\odot x) = z+(x'+z)^{\sim} = z/(x'+z) = z/(x\backslash z)$. ∎

Proposition 4.2. *Let $\vee$ and $\wedge$ be two hyper operations on M, were defined by $x \vee y = x + (x^{\sim} \odot y)$ and $x \wedge y = x \odot (x' + y)$. Then the following hold: for all $x, y, z \in M$,*

(i) $x \in (x \wedge x) \cap (x \vee x)$,

(ii) $x \vee y = y \vee x$ *and* $x \wedge y = y \wedge x$,

(iii) $x \in (x \wedge (x \vee y)) \cap (x \vee (x \wedge y))$,

(iv) $x \in x \vee 0$, $0 \in x \wedge 0$, $x \in x \wedge 1$ *and* $x \in x \vee 1$,

(v) $x \leq y$ *implies* $x \in x \wedge y$ *and* $y \in x \vee y$,

(vi) $x \odot y \ll x \wedge y \ll x, y$,

(vii) $x, y \ll x \vee y$,

(viii) $x \leq y$ *implies* $x \wedge z \ll y \wedge z$ *and* $x \vee z \ll y \vee z$,

(ix) $x \wedge y = (x' \vee y')^{\sim} = (x^{\sim} \vee y^{\sim})'$,

(x) $x \vee y = (x' \wedge y')^{\sim} = (x^{\sim} \wedge y^{\sim})'$,

(xi) $0 \vee 1 = 0 + 1$ *and* $0 \wedge 1 = 1 \odot 0$,

(xii) $\vee$ *is associative if and only if* $\wedge$ *is associative.*

Proof. (i) By Lemma 3.8(iv), $0 \in x^{\sim} \odot x$ and so by Proposition 3.10(vii), $x \in x + 0 \subseteq x + (x^{\sim} + x) = x \vee x$. On the other hand, by Proposition 3.10(viii) and $(HSMV8)$, we have $x \in x \odot 1 \subseteq x \odot (x' + x) = x \wedge x$.

(ii) It follows from $(HSMV7, 8)$.

(iii)

$$\begin{aligned} x \wedge (x \vee y) = x \wedge (x + (x^{\sim} \odot y)) &= x \odot (x' + (x + (x^{\sim} \odot y))) \\ &= x \odot ((x' + x) + (x^{\sim} \odot y)) \\ &\supseteq x \odot 1, \text{ by } (HSMV8) \text{ and } (HSMV2) \\ &\ni x, \text{ by Proposition 3.10(viii)} \end{aligned}$$

By the similar way, we can show that $x \in (x \vee (x \wedge y))$.

(iv) By Lemma 3.8(iv) and 3.10(vii), we get

$$\begin{aligned} &0 \in x \odot x' \subseteq x \odot (x' + 0) = x \wedge 0, \quad \text{and} \\ &x \in x + 0 \subseteq x + (x^{\sim} \odot 0) = x \vee 0. \end{aligned}$$

Moreover, by Proposition 3.10(viii) and $(HSMV8)$, we obtain

$$\begin{aligned} &1 \in x + x^{\sim} \subseteq x + (x^{\sim} \odot 1) = x \vee 1, \quad \text{and} \\ &x \in x \odot 1 \subseteq x \odot (x' + 1) = x \wedge 1. \end{aligned}$$

(v) Let $x \leq y$. Then by Lemma 3.8(vii), $1 \in x' + y$ and $0 \in y^{\sim} \odot x$. Hence by Proposition 3.10 (vii) and (viii), we have

$$\begin{gathered} y \in y + 0 \subseteq y + (y^{\sim} \odot x) = (x \odot y') + y = x \vee y, \\ x \in x \odot 1 \subseteq x \odot (x' + y) = x \wedge y. \end{gathered}$$

(vi) Since by Lemma 3.8(v), $y \leq x' + y$, then Proposition 3.10(iii) implies

$$x \odot y \ll x \odot (x' + y) = x \wedge y.$$

Moreover, by Lemma 3.8(iv) and $(HSMV6)$, $x \wedge y \ll x, y$.

(vii) It is straight consequent of Lemma 3.8(v) and $(HSMV5)$.

(viii) Let $x \leq y$ and $z \in M$. Then by Proposition 3.10(ii), $x + z^{\sim} \ll y + z^{\sim}$ and so by Proposition 3.10(iii), $x \wedge z = (x + z^{\sim}) \odot z \ll (y + z^{\sim}) \odot z = y \wedge z$. Similarly, we can show that $x \vee z \ll y \vee z$.

(ix) We will show that $x \wedge y = (x' \vee y')^{\sim}$. The proof of other part is similar.

$$(x' \vee y')^{\sim} = (x' + ((x')^{\sim} \odot y'))^{\sim} = (x' + (x \odot y'))^{\sim} = (x \odot y')^{\sim} \odot x = (y + x^{\sim}) \odot x = x \wedge y$$

(x) By (ix) and Lemma 3.8(i), we get $(x^{\sim} \wedge y^{\sim})' = (((x^{\sim})' \vee (y^{\sim})')^{\sim})' = x \vee y$. By the similar way, we can show that $x \vee y = (x' \wedge y')^{\sim}$.

(xi) By Proposition 3.10(v), we have $0 + 0 = \{0\}$ and $1 \odot 1 = \{1\}$. Therefore, $0 \vee 1 = 0 + (0^{\sim} \odot 1) = 0 + (1 \odot 1) = 0 + 1$ and $0 \wedge 1 = 1 \odot (1' + 0) = 1 \odot (0 + 0) = 1 \odot 0$.

(xii) Let $\vee$ be a associative hyper operation A be a non-empty subset of M and $x, y, z \in M$. Then by Lemma 3.8(i), we get $\{a' \in M | a' \in A\} = A$ and so

$$\begin{aligned} x \wedge (y \wedge z) &= \cup\{x \wedge a | a \in y \wedge z\} \\ &= \cup\{x \wedge a | a' \in y' \vee z'\}, \text{ by (iv)} \\ &= \cup\{(x' \vee a')^{\sim} | a' \in y' \vee z'\}, \text{ by (iv)} \\ &= \cup\{(x' \vee a')^{\sim} | a' \in y' \vee z'\}, \text{ by (iv)} \\ &= \cup\{(x' \vee a)^{\sim} | a \in y' \vee z'\}, \text{ by (iv)} \\ &= (x' \vee (y' \vee z'))^{\sim} \\ &= ((x' \vee y') \vee z')^{\sim} \end{aligned}$$

By the similar way, we can show that $(x \wedge y) \wedge z = ((x' \vee y') \vee z')^{\sim}$. Hence $\wedge$ is associative. The proof of the converse is similar. ■

Remark 4.3. Let $x, y \in M$.

(i) If $x + y = \{0\}$, then $x = y = 0$ (by Lemma 3.8(v)).

(ii) If $x \vee y = \{0\}$, then $x = y = 0$ (by Proposition 4.2(vii)).

(iii) If $x \wedge y = \{1\}$ ($x \odot y = \{1\}$), then $x = y = 1$ (by Proposition 4.2(vi)).

Proposition 4.4. *Let $x, y \in M$. Then*

(i) $x \ll x \odot y$ *implies* $1 \in x' \vee y$,

(ii) $y \ll x \odot y$ *implies* $1 \in x \vee y^{\sim}$,

(iii) $x + y \ll y$ *implies* $0 \in x^{\sim} \wedge y$,

(iv) $x + y \ll x$ *implies* $0 \in x \wedge y'$.

Proof. (i) By Lemma 3.8(i), we get

$$1 \in x' \vee y \Leftrightarrow 1 \in x' + ((x^{\sim})' \odot y) \Leftrightarrow 1 \in x' + u, (\exists u \in x \odot y) \Leftrightarrow x \ll x \odot y$$

(ii) By Proposition 3.5(vii), we obtain

$$1 \in x \vee y^{\sim} \Leftrightarrow 1 \in (x \odot (y^{\sim})') + y^{\sim} \Leftrightarrow 1 \in v + y^{\sim} (\exists v \in x \odot y) \Leftrightarrow y \ll x \odot y.$$

(iii) Proposition 3.5(vii) implies

$$0 \in x^{\sim} \wedge y \Leftrightarrow 0 \in x^{\sim} \odot ((x^{\sim})' + y) \Leftrightarrow 0 \in x^{\sim} \odot u (\exists u \in x + y) \Leftrightarrow x \ll x + y.$$

(iv) $0 \in x \wedge y' \Leftrightarrow 0 \in (x + (y')^{\sim}) \odot y' \Leftrightarrow 0 \in v \odot y' (\exists v \in x + y) \Leftrightarrow y \ll x + y$. ■

5. Filters on hyper pseudo MV-algebras

In this section, we attempt to verify filter and ideal theories in hyper pseudo MV-algebras. Then the relation between them was obtained. We show that I is an ideal of M if and only if I is a filter of dual pseudo MV-algebra of M.

Definition 5.1. Let I be a non-empty subset of M. Then I is called

- a *weak ideal* of M if

  $(x+y)\cap I \neq \emptyset$, for all $x, y \in I$,

  $y \in I$ and $x \leq y$ imply $x \in I$, for all $x \in M$.

- an *ideal* of M if

  $(y/x)' \ll I$ and $y \in I$ imply $x \in I$, for all $x \in M$,

  $(x+y) \subseteq I$, for all $x, y \in I$.

Clearly, any ideal of M is a weak ideal. An ideal (a weak ideal) I of M is called proper, if $I \subsetneqq M$. It is easy to show that if I is an ideal of M, then I is proper if and only if $x \in I$ implies $x', x^{\sim} \in M' - I$, for all $x \in M$. Moreover, if I is an ideal, $x \leq y$ and $y \in I$, then $0 \in x \odot y'$ and so $(y/x)' \ll I$. Hence $x \in I$.

Example 5.2. Consider the hyper pseudo MV-algebra in Example 3.4(i). Then $\{0\}$, $\{0, a\}$, $\{0, b\}$ and $\{0, c\}$ are ideals of M. Moreover, $\{0, a, b\}$, $\{0, b, c\}$ and $\{0, a, c\}$ are weak ideals of M. But, they are not ideal.

Definition 5.3. Let F be a non-empty subset of M. Then F is called

- a *weak filter* of M if

  $x \in F$ and $x \leq y$ imply $y \in F$, for any $y \in M$

  $(x \odot y) \cap F \neq \emptyset$, for all $x, y \in F$,

- a *filter* of M if

  $F \ll x\backslash y$ and $x \in F$ imply $y \in F$, for any $y \in M$,

  $(x \odot y) \subseteq F$, for all $x, y \in F$.

Clearly, any filter of M is a weak filter.

Proposition 5.4. *Let F be a non-empty subset of M such that $(x \odot y) \subseteq F$, for all $x, y \in F$. Then*

(i) *F is a filter if and only if $x \in F$ and $x \leq y$ imply $y \in F$, for any $x, y \in M$.*

(ii) *F is a filter if and only if $F \ll y/x$ and $x \in F$ imply $y \in F$, for any $x, y \in M$.*

Proof. (i) Let F be a filter of M, $x \in F$ and $x \leq y$. Then by $1 \in x\backslash y$, we have $F \ll x\backslash y$ and so by assumption $y \in F$. Conversely, let $x \in F$ and $F \ll x\backslash y$, for some $y \in M$. Then there exists $a \in F$ such that $a \ll x\backslash y$, so $1 \in a\backslash(x\backslash y)$. Hence by Proposition 4.1(vi), $1 \in (x \odot a)\backslash y$. Since $x, y \in F$, then by assumption $x \odot a \subseteq F$ and so there exists $u \in F$ such that $1 \in u\backslash y$. Now, by assumption, we get $y \in F$. Therefore, F is a filter of L.

(ii) Suppose that F be a filter of M, $F \ll y/x$ and $x \in F$, for some $y \in M$. Then there exists $a \in F$ such that $a \ll y + x^{\sim}$ and so $1 \in (y/x)/a$. By Proposition 4.1(vii), we get $1 \in y/(x \odot a)$. Since F is a filter, then $x \odot a \subseteq F$ and so there exists $b \in F$ such that $1 \in y/b$. Hence $b \leq y$ and $b \in F$, whence by (i), $y \in F$. The proof of the converse is straightforward by $(HSMV10)$ and (i). ∎

Example 5.5. Let M be a hyper pseudo MV-algebra in Example 3.4(i). Then

(i) $\{1, a\}$, $\{1, b\}$ and $\{1, c\}$ are three filters of M.

(ii) $\{1, a, b\}$ and $\{1, a, b\}$ are weak filters of M. But, they are not filters.

Proposition 5.6. *Let F be a filter, I be an ideal of M and $x, y \in F$. Then $(x \wedge y) \cap I \neq \emptyset$, $(x \vee y) \cap F \neq \emptyset$, $(x \vee y) \cap I \neq \emptyset$ and $(x \wedge y) \cap F \neq \emptyset$.*

Proof. From Lemma 3.8(v) and (vii), it follows that $(x \wedge y) \cap I \neq \emptyset$ and $(x \vee y) \cap F \neq \emptyset$. By Proposition 4.2(vi), there exists $a \in x^{\sim} \odot y$ such that $a \leq y$ and so $x + a \ll x + y \subseteq I$. Hence $(x \vee y) \cap I = x + (x^{\sim} \odot y) \cap I \neq \emptyset$. On the other hand, by Proposition 4.2(vi), there exits $b \in (x' + y)$ such that $y \leq a$ and so $F \supseteq x \odot y \ll x \odot a$. Hence $(x \wedge y) \cap F = x \odot (x' + y) \cap F \neq \emptyset$. ∎

By Proposition 3.10(v), in any hyper pseudo MV-algebra, $\{1\}$ is a filter and $\{0\}$ is an ideal.

Theorem 5.7. *Let I be a non-empty subset of M. Then the following are equivalent:*

(i) *I is an ideal of M,*

(ii) *$I^{\sim}$ is a filter of M,*

(iii) *I' is a filter of M.*

Proof. (i) $\Rightarrow$ (ii) Suppose that I is an ideal of M and $a, b \in I^{\sim}$. Then there exist $x, y \in I$ such that $a = x^{\sim}$ and $b = y^{\sim}$. By Lemma 3.8(ix), we get $a \odot b = x^{\sim} \odot y^{\sim} = (y + x)^{\sim}$. Since I is an ideal of M and $x, y \in I$, then $(y + x)^{\sim} \subseteq I^{\sim}$. Now, let $I^{\sim} \ll a \backslash b$, for some $a \in I^{\sim}$ and $b \in M$. Then there exist $x, y \in I$ such that $a = x^{\sim}$ and $y^{\sim} \ll x^{\sim} \backslash b$. By Proposition 3.10(i), we conclude that $(x^{\sim} \backslash b)' \ll y$, so $= (x/b')' = (x + (b')^{\sim})' = (x + b)' \ll y$. Since $x \in I$ and I is an ideal of M, then we get $b' \in I$. Therefore, $b \in I^{\sim}$ and so $I^{\sim}$ is a filter of M.

(ii) $\Rightarrow$ (i) Let $I^{\sim}$ be a filter of M and $x, y \in I$. Then $x^{\sim}, y^{\sim} \in I^{\sim}$ and $(x + y)^{\sim} = (y^{\sim} \odot x^{\sim}) \subseteq I^{\sim}$. Hence $x + y \subseteq I$. Now, let $(y/x)' \ll I$, for some $y \in I$ and $x \in M$. Then By Proposition 3.10(i), $I^{\sim} \ll y/x = y + x^{\sim} = (y^{\sim})' + x^{\sim} = y^{\sim} \backslash x^{\sim}$ and $y^{\sim} \in I^{\sim}$. Since $I^{\sim}$ is a filter, then we get $x^{\sim} \in I^{\sim}$ and so $x \in I$. Therefore, I is an ideal of M.

(i) $\Rightarrow$ (iii) Assume that I is an ideal of M and $x, y \in I'$. Then $x = a'$ and $y = b'$, for some $a, b \in I$. By Lemma 3.8(ix), $x \odot y = a' \odot b' = (b + a)' \subseteq I'$. Now, let $I' \ll (x \backslash y)$, $x \in I$ and $y \in M$. Then $(x^{\sim}/y^{\sim})' = y^{\sim} \odot (x^{\sim})' = y^{\sim} \odot x = (x' + y)^{\sim} \ll I$. By $x \in I$, we get $y^{\sim} \in I$ and so $y \in I'$. Therefore, I' is a filter of M.

(iii) $\Rightarrow$ (i) It is easy to show that $x + y \subseteq I$, for all $x, y \in I$. Let $(y/x)' \ll I$, for some $y \in I$ and $x \in M$. Then by Proposition 3.10(i), $I' \ll (y/x)'' = (x \odot y')' = (y')' + x' = y' \backslash x'$. Since I' is a filter of M and $y' \in I'$, then we get $x' \in I'$ and so $x = (x')^{\sim} \in I$. Therefore, I is an ideal of M. ∎

Corollary 5.8. *Let I be a non-empty subset of M such that $x + y \subseteq I$, for any $x, y \in I$. Then I is an ideal if and only if $y \in I$ and $x \leq y$ imply $x \in I$, for any $x \in M$.*

Proof. Suppose that $y \in I$ and $x \leq y$ imply $x \in I$, for any $x \in M$. We show that $I^{\sim}$ is a filter. Let $x, y \in I^{\sim}$. Then $x', y' \in I$ and so $y' + x' \subseteq I$. It follows that $x \odot y \subseteq I^{\sim}$. Now, let $x \leq y$ and $x \in I^{\sim}$. Then $y' \leq x'$ and $x' \in I$, so by assumption, $y' \in I$. Hence $y \in I^{\sim}$, whence by Proposition 5.4, $I^{\sim}$ is a filter of M. Therefore, by Theorem 5.7, I is an ideal of M. The proof of the converse is clear. ∎

Definition 5.9. Let $(M', +, ', \sim, 0, 1)$ be a hyper pseudo MV-algebra and $f : M \to M'$ be a map such that $f(0) = 0$ and $f(x') = f(x)'$, for all $x \in M$. Then f is called

(i) a *weak homomorphism* if $f(x + y) \subseteq f(x) + f(y)$, for all $x, y \in M$,

(ii) a *homomorphism* if $f(x + y) = f(x) + f(y)$, for all $x, y \in M$.

For any weak homomorphism $f : M \to M'$, we use $ker(f)$ to denote $f^{-1}(0) = \{x \in M | f(x) = 0\}$. A homomorphism $f : M \to M'$ is called *isomorphism* if f is a one to one and onto map. We use $M \cong M'$ to denote, there exists a isomorphism from M to M'.

Example 5.10. Let M be a hyper pseudo MV-algebra in Example 3.4(i).

(i) Let $M' = \{0, a, b, c, d, 1\}$. Consider the following tables:

Table 6

+	0	a	b	c	d	1
0	{0}	{0,a}	{0,b}	{0,c}	{0,d}	M'
a	{0,a}	{0,a}	{a,b}	M'	{a,d}	M'
b	{b,0}	M'	{0,b}	{c,b}	{b,d}	M'
c	{c,0}	{c,a}	M'	{0,c}	{c,d}	M'
d	{d,0}	{d,a}	{d,b}	{d,c}	M'	M'
1	M'	M'	M'	M'	M'	M'

Table 7

	0	a	b	c	d	1
$'$	1	b	c	a	d	0
$\sim$	1	c	a	b	d	0

Easy calculations show that $(M', +, ', \sim, 0, 1)$ is a hyper pseudo MV-algebra. Let $f : M \to M'$ be a map was defined by $f(x) = x$, for all $x \in M$. Then f is a weak homomorphism.

(ii) Define $f : M \to M$ by $f(0) = 0$, $f(1) = 1$, $f(a) = b$, $f(b) = c$ and $f(c) = a$. It is not difficult to check that $f : M \to M$ is a homomorphism.

Theorem 5.11. *Let $f : M \to M'$ be a weak homomorphism and I be an ideal of M'. Then*

(i) *$f(x^{\sim}) = f(x)^{\sim}$. Moreover, $x \leq y$ implies $f(x) \leq f(y)$, for any $x, y \in M$,*

(ii) *$ker(f)$ is an ideal of M,*

(iii) $f^{-1}(I)$ *is an ideal of* M,

(iv) *If* F *is a filter of* M', *then* $f^{-1}(F)$ *is a filter of* M,

(iv) *If* f *is a homomorphism, then* f *is one to one if and only if* $ker(f) = \{0\}$,

(v) F *is a filter of* M *if and only if* F *is an ideal of dual pseudo* MV*-algebra of* M,

(vi) *If* f *is onto and* I *is an ideal of* M *containing* $ker(f)$, *then* $f(I)$ *is an ideal of* M'.

Proof. (i) Let $x \in M$. Then by $f(x) = f((x^{\sim})') = f(x^{\sim})'$, we get $f(x)^{\sim} = (f(x^{\sim})')^{\sim} = f(x^{\sim})$. Now, let $x \leq y$. then $1 \in x' + y$ and so $1 = f(0)' = f(1) \in f(x' + y) = f(x)' + f(y)$. Hence $f(x) \leq f(y)$.

(ii) Let $x, y \in ker(f)$. Then $f(x) = f(y) = 0$ and so by Proposition 3.10(v), $f(x + y) \subseteq f(x) + f(y) = 0 + 0 = 0$. Hence $x + y \subseteq ker(f)$. Now, let $(y/x)' \ll ker(f)$ and $y \in ker(f)$. Then $x \odot y' \leq a$, for some $a \in ker(f)$. By Lemma 3.8(viii), we get $x \leq a + y$. Since $a, y \in ker(f)$, then $a + y \subseteq ker(f)$ and so $f(x) = 0$. Therefore, $ker(f)$ is an ideal of M.

(iii) Assume that I is an ideal of M'. If $x \odot y' \ll f^{-1}(I)$ and $y \in f^{-1}(I)$, for some $x, y \in M$, then by (i), $f(x) \odot f(y)' \ll I$ and $f(y) \in I$. Since I is an ideal of M', then we have $f(x) \in I$. Hence $x \in f^{-1}(I)$. Now, let $a, b \in f^{-1}(I)$. Then $f(x), f(y) \in I$. Hence $f(x + y) \subseteq f(x) + f(y) \subseteq I$ and so $x + y \subseteq f^{-1}(I)$. Therefore, $f^{-1}(I)$ is an ideal of M.

(iv) Let F be a filter of M'. Then by Theorem 5.7, F' is a filter of M, so by (iii), $f^{-1}(F')$ is an ideal of M. It suffices to show that $f^{-1}(F') = f^{-1}(F)'$. Let $x \in M$. Hence

$$x \in f^{-1}(F') \Leftrightarrow f(x) \in F' \Leftrightarrow f(x^{\sim}) = f(x)^{\sim} \in F \Leftrightarrow x^{\sim} \in f^{-1}(F) \Leftrightarrow x \in f^{-1}(F)'$$

(iv) Clearly, if f is one to one, then $ker(f) = \{0\}$. Suppose that $ker(f) = \{0\}$ and $f(x) = f(y)$, for some $x, y \in M$. Then $1 \in (f(x)' + f(y)) \cap (f(y)' + f(x))$. Since f is a homomorphism, then by (i), we get $0 \in f((x' + y)^{\sim}) \cap f((y' + x)^{\sim})$. Hence by assumption, $0 \in (x' + y)^{\sim} = y^{\sim} \odot x$ and $0 \in (y' + x)^{\sim} = x^{\sim} \odot y$. Now, by Lemma 3.8(vii), we conclude that $x = y$. Therefore, f is a one to one map.

(v) Straightforward.

(vi) Let I be an ideal of M containing $ker(f)$ and $x, y \in f(I)$. Then there exist $a, b \in I$ such that $x = f(a)$ and $y = f(b)$. Hence $x + y = f(a) + f(b) = f(a + b) \subseteq f(I)$. Now, let $(y/x)' \ll f(I)$ and $y \in f(I)$, for some $x, y \in M'$. Since f is onto, then there are $a \in M$ and $b \in I$ such that $x = f(a)$ and $y = f(b)$. By $f((b/a)') = (f(b)/f(a))' \ll f(I)$, we get, there are $u \in I$ and $c \in (b/a)'$ such that $f(c) \leq f(u)$ and so $0 \in (f(u)/f(c))' = f((u/c)')$. Hence $(u/c)' \ll ker(f) \subseteq I$. Since $u \in I$ and I is an ideal of M, then $c \in I$ and so $(b/a)' \ll I$. Therefore, $a \in I$ (since $b \in I$ and I is an ideal) and so $x \in f(I)$. That is, $f(I)$ is an ideal of M'. ∎

Corollary 5.12. *If* M' *be a hyper pseudo* MV*-algebra and* $f : M \to M'$ *be a homomorphism, then* $f^{-1}(1)$ *is a filter of* M.

Proof. Clearly, $f : (M, \odot, \sim, ', 1, 0) \rightarrow (M', \odot, ', 1, 0)$ is a homomorphism, where $(M, \odot, \sim, ', 1, 0)$ and $(M', \odot, ', 1, 0)$ are dual hyper pseudo MV-algebra of M and M', respectively. Hence by Theorem 5.11(ii), $f^{-1}(1) = ker(f)$ is an ideal of $(M, \odot, \sim, ', 1, 0)$. Therefore, by Theorem 5.11(v), $f^{-1}(1)$ is a filter of M. ∎

6. Congruence relation on hyper pseudo MV-algebras

Definition 6.1. Let θ be an equivalence relation on M and A, B be two subsets of M. Then

- $A\theta B$ means that $a\theta b$, for some $a \in A$ and $b \in B$,
- $A\overline{\theta}B$ means that for any $a \in A$, there exists $b \in B$ such that $a\theta b$ and for any $b \in B$ there exists $a \in A$ such that $b\theta a$. Moreover, if $A = \{a\}$, then we write $a\theta B$, for $A\theta B$,
- $A\overline{\overline{\theta}}B$ means that $a\theta b$, for any $a \in A$ and $b \in B$.

It can be easily obtained that, $A\overline{\theta}B$ and $B\overline{\theta}C$ imply $A\overline{\theta}C$, for any subset A, B, C of M.

Definition 6.2. Let θ be an equivalence relation on M. Then θ is called the *congruence relation*, if it satisfies the following conditions:

(C1) $x\theta y$ and $a\theta b$ imply $(x + a)\overline{\theta}(y + b)$,for any $x, y, a, b \in M$,

(C2) $x\theta y$ implies $x'\theta y'$ and $x^{\sim}\theta y^{\sim}$, for any $x, y \in M$,

(C3) $x\theta y$ if and only if $1\theta(x' + y)$ and $1\theta(y' + x)$, for any $x, y \in M$,

(C4) $1\theta(x' + y)$ if and only if $1\theta(y + x^{\sim})$, for any $x, y \in M$.

We use $Con(M)$ to denote the set of all congruence relations on M.

Note 6.3. Let θ be an equivalence relation on M.

(i) If θ is an equivalence relation on M satisfies on (C1), (C2), then $x\theta y$ implies that $1 \in x' + x\overline{\theta}x' + y$ and $1 \in y' + y\overline{\theta}y' + x$.

(ii) If $[1] = \{x \in M | x\theta 1\} = \{1\}$, then by (C3) and $(HSMV9)$ we get $\theta = \{(x, x) | x \in M\}$.

Example 6.4. Let $M = \{0, a, b, c, d, e, f, 1\}$. Consider the following tables:

Table 8

+	0	a	b	c	d	e	f	1
0	{0}	{0,a}	{0,b}	{0,c}	{0,d}	{0,e}	{0,f}	M
a	{0,a}	{0,a}	{0,a,b}	{0,a,c}	{0,a,d}	{0,a,e}	M	M
b	{b,0}	{0,a,b}	{0,b}	{0,b,c}	{0,b,d}	M	M	M
c	{c,0}	{0,c,a}	M	{0,c}	{0,c,d}	{0,c,e}	{0,c,f}	M
d	{d,0}	M	M	{0,d,c}	{d,0}	{0,d,e}	{0,d,f}	M
e	{e,0}	{0,a,e}	{0,b,e}	{0,c,e}	M	{0,e}	{0,e,f}	M
f	{f,0}	{0,a,f}	{0,b,f}	M	M	{0,f,e}	{0,f}	M
1	M	M	M	M	M	M	M	M

Table 9

	0	a	b	c	d	e	f	1
$'$	1	d	c	f	e	b	a	0
$\sim$	1	f	e	b	a	d	c	0

Then $(M, +,', \sim, 0, 1)$ is a hyper pseudo MV-algebra.

Let $\theta = \{(x, y)|x, y \in \{0, a, c, e\}\} \cup \{(x, y)|x, y \in \{1, b, d, f\}\}$. Easy calculations show that θ is a congruence relation on M.

Proposition 6.5. *Let θ be a congruence relation on M and $[x] = \{a \in M|x\theta a\}$, for any $x \in M$. Then*

(i) *θ is a congruence relation on dual hyper pseudo MV-algebra of $(M, +,', \sim, 0, 1)$,*

(ii) *$[1]$ is a filter of M,*

(iii) *$[0]$ is an ideal of M.*

(iv) *$[x]$ is convex, for any $x \in M$, that is $a \leq b \leq c$ and $a, b \in [x]$ imply $b \in [x]$, for any $a, b, c \in M$,*

(v) *If $x \in M$ and $a, b \in [x]$, then $a \wedge b \cap [x] \neq \emptyset$ and $a \vee b \cap [x] \neq \emptyset$.*

Proof. (i) Let M^* be the dual hyper pseudo MV-algebra of M. Clearly, (C2) holds. Let $x\theta y$ and $a\theta b$. Then by (C2), $x'\theta y'$ and $a'\theta b'$ and so by (C1), we have $a' + x'\overline{\theta}b' + y'$. Hence by (C2), $x \odot a\overline{\theta}y \odot b$, whence (C2) hold in M^*. Now, let $0\theta(x^\sim \odot y)$ and $0\theta(y^\sim \odot x)$, then $0\theta(y' + x)^\sim$ and so $0\theta a^\sim$, for some $a \in y' + x$. Hence $1\theta a$, so $1\theta y' + x$. By the similar way, we can show that $1\theta x' + y$, which implies that $x\theta y$. By Note 6.3(i), (C3), hold. Let $x, y \in M$. Then by (C4), we get

$$0\theta(x^\sim \odot y) \Leftrightarrow 0\theta(y' + x)^\sim \Leftrightarrow 1\theta(y' + x) \Leftrightarrow 1\theta(y' + x) \Leftrightarrow 0\theta(x' + y)^\sim = (y^\sim \odot x)$$

Therefore, θ is a congruence relation on M^*.

(ii) Let $x, y \in [1]$. Then by Proposition 3.10(v), we get $(x \odot y)\overline{\theta}(1 \odot 1) = 1$. Hence $x \odot y \subseteq [1]$. Now, let $a \in [1]$ and $a \leq b$, for some $b \in M$. Then $1 \in a' + b$. Moreover, $(b' + a)\overline{\theta}(b' + 1)$ and $1 \in b' + 1$, so $(b' + a)\theta 1$. It follows from (C3), $a\theta b$. Therefore, by Proposition 5.4, $[1]$ is a filter of M.

(iii) By (i) $[0]$ is a filter of M^*. Since $[0] = [1]'$, then by Theorem 5.7, $[0]$ is an ideal of M.

(iv) Let $b, x \in M$ and $a, c \in [x]$ such that $a \leq b \leq c$. Then $1 \in a' + b$, $1 \in b' + c$. Since $a\theta c$, then $(a' + b)\overline{\theta}(c' + b)$, so by $1 \in a' + b$, we get $1\theta(c' + b)$. By (C3), we conclude that $a\theta c$. Therefore, $[x]$ is convex.

(v) It is easy to show that, $a\theta b$ and $x\theta y$, imply $(a\vee x)\overline{\theta}(b\vee y)$ and $(a\wedge x)\overline{\theta}(b\wedge y)$, for any $x, y, a, b \in M$. Now, let $x \in M$ and $a, b \in [x]$. Then $(a \vee b)\overline{\theta}(x \vee x)$. Hence by Proposition 4.2(i), $(a \vee b)\theta x$. By the similar way, we can show that $(a \wedge b)\theta x$. ∎

Theorem 6.6. *Let θ be a congruence relation on M and M/θ be the set of all congruence classes of M with respect to θ. Then $(M/\theta, \oplus, -, *, [0], [1])$ is a hype pseudo MV-algebra, where $[x] \oplus [y] = [x \oplus y]$, $[x]^- = [x']$ and $[x]^* = [x^\sim]$, for any $x, y \in X$.*

Proof. Since θ is a congruence relation on M, then clearly, $\oplus$ is a hyper operation on M/θ and $-$, $*$ are well define. Moreover, by definition of $\oplus$, $-$ and $*$, it can be easily obtained that $(HSMV1)$-$(HSMV8)$ hold in M/θ. Let $x, y \in M$ and $[1] \in ([x]^- \oplus [y]) \cap ([y]^- \oplus [y])$, then $[1] \in [x' + y] \cap [y' + x]$ and so $1\theta(x' + y)$, $1\theta(y' + x)$. Hence by (C3), $x\theta y$. That is $[x] = [y]$. Moreover, by (C4), we have

$$[1] \in [x]^- \oplus [y] \Leftrightarrow [1] \in [x' + y] \Leftrightarrow 1\theta x' + y \Leftrightarrow 1\theta y + x^{\sim} \Leftrightarrow [1] \in [y] \oplus [x]^*$$

Therefore, $(M/\theta, \oplus, -, *, [0], [1])$ is a hype pseudo MV-algebra. ∎

Definition 6.7. A filter F of M is called *normal filter* or simply N-filter if the relation θ_F by the following definition

$$\theta_F = \{(x, y) \in M \times M | (x' + y) \cap F \neq \emptyset, \quad (y' + x) \cap F \neq \emptyset\}$$

is a congruence relation on M. If F is a normal filter, the we use M/F to denote the set of all equivalence relation of M with respect to θ. We use $NF(M)$ to denote the set of all N-filters of M.

Proposition 6.8. *Let F be a normal filter of M. Then $x \in F$ if and only if $x'' \in F$ if and only if $(x^{\sim})^{\sim} \in F$, for any $x \in M$.*

Proof. Let $x \in M$ and F be a normal filter of M and $*$, $-$ be the hyper operations in Theorem 6.6. Then there exists a congruence relation θ on M such that $F = [1]_\theta$. If $x \in F$, then $[x] = [1]$ and so $[x''] = ([x]^-)^- = ([1]^-)^- = [1]$. Hence $x'' \in F$. Conversely, let $x'' \in F$. Then $[x''] = [1]$ and so $[x]^-)^- = [1]$. Hence $[x] = [1]$, whence $x \in F$. By the similar way, we can show that $x \in F$ if and only if $x'' \in F$. ∎

In the next theorem, we try to find relation between N-filters and congruence relations on M.

Theorem 6.9.

(i) *F is a N-filter of M if and only if there exists a congruence relation θ on M such that $F = [1]$.*

(ii) *There is a bijection map between the set of all N-filters of M and the set of all congruence relations on M.*

Proof. (i) Suppose that F is a N-filter of M. Then the relation

$$\theta_F = \{(x, y) \in M \times M | (x' + y) \cap F \neq \emptyset, \quad (y' + x) \cap F \neq \emptyset\}$$

is a congruence relation on M. We show that $F = [1]$. Let $x \in F$. Then by Proposition 3.10(vii), and $(HSMV2)$ we have $1 \in x' + 1$ and $x \leq 0 + x = 1' + x$ and so by (C3), $x\theta_F 1$. Now, let $x \in [1]$, then by (C3), $1' + x \cap F \neq \emptyset$. Since by Proposition 3.10(vi), $1' + x \leq x$, then by Proposition 5.4, $x \in F$. Therefore, $F = [1]$. Conversely, let there exists a congruence relation ϕ on M such that $F = [1]_\phi$ and

$$\theta = \{(x, y) \in M \times M | (x' + y) \cap F \neq \emptyset, \quad (y' + x) \cap F \neq \emptyset\}.$$

We show that $\theta = \phi$. Let $x\theta y$, for some $x, y \in M$. Then $(x' + y) \cap F \neq \emptyset$ and $(y' + x) \cap F \neq \emptyset$ and so $1\phi(x' + y)$ and $1\phi(y' + x)$. Since ϕ is a congruence relation, then by (C3), $x\phi y$. Now, let $x\phi y$, then by (C3), $1\phi(x' + y)$ and $1\phi(y' + x)$ and so $(x' + y) \cap F \neq \emptyset$ and $(y' + x) \cap F \neq \emptyset$. Hence $x\theta y$. Therefore, θ is a congruence relation on M and so F is a N-filter.

(ii) Define the map $f : Con(M) \to NF(M)$ by $f(\theta) = [1]_\theta$, for any $\theta \in Con(M)$. It follows from (i), f is an onto map. Moreover, by the proof of (i), we can show that f is a bijection map. ∎

Note 6.10. Let F be a normal filter of M. Then there exists a congruence relation θ on M such that $F = [1]$. In the proof of Theorem 6.9, we shown that $\theta_F = \theta$.

Theorem 6.11. *Let M' be a hyper pseudo MV-algebra and $f : M \to M'$ be a homomorphism. Then $\theta = \{(x, y) \in M \times M | f(x) = f(y)\}$ is a congruence relation on M.*

Proof. Clearly, θ is an equivalence relation on M. Since f is a homomorphism, then (C1) and (C2) hold. Now, let $1\theta(x' + y)$ and $1\theta(y' + x)$, for some $x, y \in M$. Then $1 = f(1) \in f(x' + y) = f(x)' + f(y)$. By the similar way, $1 \in f(y)' + f(x)$. Since M' is a hyper pseudo MV-algebra, then $f(x) = f(y)$ and so $(x, y) \in \theta$. Hence by Note 6.3(i), (C3) holds. Since M' is a hyper pseudo MV-algebra, then

$$\begin{aligned}(x' + y)\theta 1 &\Leftrightarrow 1 = f(1) \in f(x)' + f(y) \Leftrightarrow 1 \in f(y) + f(x)^{\sim} \\ &\Leftrightarrow f(1) \in f(y + x^{\sim}) \Leftrightarrow (y + x^{\sim})\theta 1.\end{aligned}$$

Therefore, θ is a congruence relation on M. ∎

Corollary 6.12. *Let M' be a hyper pseudo MV-algebra and $f : M \to M'$ be a homomorphism. Then $f^{-1}(1)$ is a N-filter of M.*

Proof. Since $\{1\}$ is a filter of M', then by Theorem 5.11(iv), $f^{-1}(1)$ is a filter of L. Hence by Theorem 6.9 and 6.11 and $f^{-1}(1) = [1]$, we conclude that $f^{-1}(1)$ is a N-filter of M. ∎

Theorem 6.13. *Let G be a N-filter of M. Then there is a one to one corresponding between the set of all filters of M/G, that is $F(M/G)$ and the set of all filters of M containing F, that is $F(M, G)$.*

Proof. Let $f : F(M/G) \to F(M, G)$ be a map was defined by $f(H) = \{x \in M \mid x \in [a]$, for some $[a] \in H\,\}$. Let H be a filter of $M/[F]$. Then $[1] \in H$ and so by Theorem 6.9, $F = [1] \subseteq f(H)$. If $x, y \in F(H)$, then $[x], [y] \in H$ and so $[y \odot x] = ([x]^- \oplus [y]^-)^* \subseteq H$. Hence $y \odot x \subseteq f(H)$. Now, let $x \in f(H)$ and $x \leq y$, for some $y \in M$. Then $1 \in x' + y$, so $[1] \in [x]^- \oplus [y]$. Hence $[x] \leq [y]$ in hyper pseudo MV-algebra M/F. Since H is a filter of $M/[F]$, then $[y] \in H$ and so $y \in f(H)$. Therefore, $f(H)$ is a filter of M containing F. Clearly, f is one to one. Now, we show that f is onto. Let G be a filter of M containing F. Then we define $H = \{[x] \in M/[F] | x \in G\}$. Clearly, $[1] \in H$. Let $[x], [y] \in H$. Then there exist $a, b \in G$ such that $[x] = [a]$ and $[y] = [b]$. Since G is a filter, then

$(a'+b')^{\sim} \subseteq G$ and so $([x]^- \oplus [y]^-)^* = ([a]^- \oplus [b]^-)^* = [(a'+b')^{\sim}] \subseteq H$. Let $[x] \in H$ and $[x] \leq [y]$, for some $[y] \in M/F$. Then there exists $a \in G$ such that $[x] = [a]$ and $[1] \in [x]^- + [y] = [x' + y]$. Hence $1\theta(x'+y)\overline{\theta}(a'+y)$, where θ is a congruence relation in Definition 6.7. Since $[1] \subseteq G$ and $1\theta a' + y$, then we get $a' + y \cap G \neq \emptyset$ and so by $a \in G$, we conclude that $y \in G$. Hence $[y] \in H$ and so H is a filter of $M/[F]$. It is easy to show that $f(H) = G$. Therefore, f is one to one and onto. ∎

Theorem 6.14. *Let M' be a hyper pseudo MV-algebra, $f : M \to M'$ be a homomorphism and $F = f^{-1}(1)$. Then M/F is a hyper pseudo MV-algebra and $g : M/F \to M'$, was defined by $g([x]) = f(x)$, for any $[x] \in M/F$ is a one to one homomorphism.*

Proof. Since f is a homomorphism, then by Corollary 6.12, F is a N-filter of M and so M/F is a hyper pseudo MV-algebra. If θ be a congruence relation in Theorem 6.11, then by Note 6.10, $\theta = \theta_F$. Clearly, g is a homomorphism. We will show that g is one to one. Let $g([x]) = g([y])$, for some $x, y \in M$. Then by definition of g, we get $f(x) = f(y)$ and so $[x] = [y]$. Hence g is one to one. ∎

From now on in this section, for convenience, we use the same notations for the hyper operation and operations of M and M/θ, for any congruence relation θ on M.

Theorem 6.15. *Let F and G be two normal filters of M such that $F \subseteq G$. Then G/F is a normal filter of M/F and $\dfrac{M/F}{G/F} \cong M/G$.*

Proof. Since F and G are normal filters of M, then there exist two congruence relations θ and φ on M such that $F = [1]_\varphi$ and $G = [1]_\theta$. Consider the following relation on M/F.

$$\Theta = \{([x]_F, [y]_F) \in M/F \times M/F | (x, y) \in \theta\}$$

Clearly, Θ is an equivalence relation on M/F. We show that it is a congruence relation. Let $([x]_F, [y]_F) \in \Theta$. Then $(x, y) \in \theta$ and so $(x', y') \in \theta$. It follows that $([x']_F, [y']_F) = ([x]'_F, [y]'_F) \in \Theta$. By the similar way, we can show that $([x]^{\sim}_F, [y]^{\sim}_F) \in \Theta$. Let $([a]_F, [b]_F) \in \Theta$. Then $(a, b) \in \theta$. Since θ is a congruence relation on M, then we get $(x + a)\overline{\theta}(y + b)$. By definition of Θ, we conclude that $([x]_F + [a]_F)\overline{\Theta}([y]_F + [a]_F)$. Now, let $a, x \in M$ and $[1]_F\Theta([x]'_F + [a]_F)$ and $[1]_F\Theta([a]'_F + [x]_F)$. Then $[1]_F\Theta[x' + a]_F$, so there is $u \in x' + a$ such that $1\theta u$ and so $1\theta(x' + a)$. By the similar way, $1\theta(a' + x)$. Hence by (C3), $x\theta a$ and so $[x]_F = [a]_F$. Finally, we show that (C4) hold. Let $x, y \in M$ such that $[1]_F\Theta([x]'_F + [y]_F)$, then similar to the above arguments $1\theta(x' + y)$. Since θ is a congruence relation, then by (C4) $1\theta(y + x^{\sim})$ and so $[1]_F\Theta([y]_F + [x]^{\sim}_F)$. Hence Θ is a congruence relation on M/F. By the proof of Theorem 6.13, G/F is a filter of M/F. Suppose that $[x]_F \in M/F$. Then, by

$$[x]_F \in [[1]_F]_\Theta \Leftrightarrow [x]_F\Theta[1]_F \Leftrightarrow x\theta 1 \Leftrightarrow x \in [1]_\theta = [1]_F \Leftrightarrow x \in G \Leftrightarrow [x]_F \in G/F,$$

we obtain $[[1]_F] = G/F$, and so, by Theorem 6.13(i), G/F is a normal filter of M/F. Define $f : M/F \to M/G$, by $f([1]_F) = [1]_G$, for any $x \in M$. Clearly, f is an onto homomorphism. Moreover, $f^{-1}([1]_G) = \{[x]_F | x \in G\} = G/F$. Therefore, by Theorem 6.14, we have $\dfrac{M/F}{G/F} \cong M/G$. ∎

Definition 6.16. A congruence relation θ on M is called *strong congruence relation* if we replace (C1) with $(C'1)$ in Definition 6.2.

$(C'1)$ $x\theta y$ and $a\theta b$ imply $(x + a)\overline{\overline{\theta}}(y + b)$, for any $x, y, a, b \in M$.

Theorem 6.17. *If θ is a strong congruence relation on M, then M/θ is a pseudo MV-algebra.*

Proof. Let θ be a strong congruence relation on M. It suffices to show that $[x]+[y]$ has only one element, for any $x, y \in M$. Let $x, y \in M$ and $[a], [b] \in [x] + [y]$. Then by definition of $+$, there exist $u, v \in x + y$ such that $a\theta u$ and $b\theta v$. Since θ is a strong congruence relation, then $(x + y)\overline{\overline{\theta}}(x + y)$ and so $u\theta v$. Hence $a\theta b$, so $[a] = [b]$. Therefore, M/θ is a pseudo MV-algebra. ∎

7. Conclusions

In this paper, we defined the concept of hyper pseudo MV-algebras and investigate some properties of this structure. Then we introduced the concept of ideal and filter in hyper pseudo MV-algebras and obtained the relation between them. Finally, we used congruence relation in hyper pseudo MV-algebra and construct quotient hyper MV-algebras. For future research, we can work on category of hyper pseudo MV-algebras, relation between this category and category of hyper MV-algebras, relation between hyper pseudo MV-algebras and hyper pseudo K-algebras and fundamental relations on these structures.

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MERGING STATES IN DETERMINISTIC FUZZY FINITE TREE AUTOMATA BASED ON FUZZY SIMILARITY MEASURES

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Abstract. This paper presents a contribution to the problem of measuring fuzzy similarity of states and merging them in a Deterministic Fuzzy Finite Tree Automaton (DFFTA). The main question is: how to merge some states of a complete and reduced DFFTA such that the languages of original automaton and minimized one be similar but not necessarily equal? In order to solving this problem, we generalize the concepts of distance and similarity measures between fuzzy sets to distance and similarity measures between states of DFFTA. Then, we define the notions of normal DFFTA and introduce an efficient algorithm (polynomial order of time complexity) for discovering similar state sets of a DFFTA.

Keywords: deterministic fuzzy tree automata, state reduction, fuzzy similarity measure.

1. Introduction

Fuzzy sets were introduced by Zadeh [29] as an extension of the classical notion of set. This theory can be used in a wide range of domains. One such domain is fuzzy automata theory first introduced by Wee [26]. A fuzzy automaton is a device which accepts a fuzzy set of words called fuzzy language. Automata have a long history both in theory and application [3], [2], [20], [19], [9], [1].

Finite Tree Automata (FTA) was introduced by Doner [7], [8] and Thatcher and Wright [24], [23]. Their goal was to prove the decidability of the weak second order theory of multiple successors. An FTA accepts a set of trees called recognizable tree language [5]. Trees appears in mathematics, computer science and other areas as formal terms, algebraic expressions, parse and derivation trees, computation trees, and generally as representations of hierarchically organized structures. A fuzzy set of trees, called recognizable fuzzy tree language if can be accepted by some Fuzzy FTA (FFTA) [18], [4], [10], [19]. The membership grade of each tree in language of FFTA is known as behavior of automaton. When the membership grade of trees takes values in a lattice (rather than in the unit interval of real numbers), the language called L-fuzzy tree language [12], [14], [13].

From a practical perspective, it is important that the considered automata are as small as possible (minimal). As well, some decision problems such as equivalence and intersection non-emptiness are closely related to minimization problem [5]. Current studies on minimizing DFFTA and deterministic weighted tree automata, focus on the two main strategies; language preserving minimization [12], [17], [15], [16], [20] and behavior preserving minimization [17].

Similarity plays an essential role in taxonomy, recognition, case based reasoning and many other fields [27], [25], [11]. We use the concept of similarity or approximate equality [22], [28] modeled on classes of fuzzy sets to define similarity between some states of DFFTA. We use this idea for similarity based merging states (state reduction) of DFFTA. We show how to find set of states similar to a given state and prove that merging a set of states which are pair wise similar, makes a DFFTA which its language is similar to original DFFTA.

In addition, we introduce the problem of similarity based state merging in DFFTA and show that this problem is not partitioning states by an equivalence relation, and it is different from language preserving and behavior preserving minimizing DFFTA.

This paper is organized as follows. Section 2 presents some mathematical preliminaries about L-Fuzzy sets and fuzzy finite tree automata. In Section 3, firstly, the concept of similarity and distance between L-fuzzy sets and its generalization to states of automata are presented. Then, we define the concept of normal DFFTA and introduce DFFTA normalizing algorithm. Also, we show how to find the set of similar states in a normal DFFTA.

2. Preliminaries

2.1. L-Fuzzy sets

We will present our results in the context of L-fuzzy sets, i.e., all the results presented below hold when membership takes values in a lattic. Basic concepts of the theory of ordered sets and lattices will be used as usual, see e.g., [6], [21]. Given a set L we can equip it with an order relationship $\leq$ and thus obtain a partially order set $(L, \leq)$. If, for every pair $x, y \in L$, $\inf(x, y)$ and $\sup(x, y)$ exist, we say that $(L, \leq)$ is a lattice. We denote $\inf(x, y)$ by $x \wedge y$ and $\sup(x, y)$ by $x \vee y$; then $\wedge, \vee$ are binary operations on L, and we say that $\ell = (L, \leq, \wedge, \vee)$ is a lattice. Also, denote by 0 and 1 the minimum and maximum elements of lattice L, respectively. A lattice $(L, \leq, \wedge, \vee)$ is complete if the least upper bound $\bigvee S$ and the greatest lower bound $\bigwedge S$ exist for all $S \subseteq L$. A completely distributive lattice is a complete lattice in which arbitrary meets ($\wedge$) distribute over arbitrary joins ($\vee$) and vice versa.

Definition 1. ([13],[21]) Given a nonempty set X and a lattice $\ell = (L, \leq, \wedge, \vee)$, an *L-fuzzy* set is characterized by its membership function $\mu_A : X \to L$, and $\mu_A(x)$ is interpreted as the grade of membership of element x in L-fuzzy set A for each $x \in X$. We denote by $\mathcal{F}(X, \ell)$ the set of all L-fuzzy sets on universal set X with membership grades in L.

Definition 2. ([13],[21]) Let X be a nonempty set and $\ell = (L, \leq, \wedge, \vee)$ be a lattice. For every $A, B \in \mathcal{F}(X, \ell)$ we have $A \subseteq B \Leftrightarrow \forall x \in X; \mu_A(x) \leq \mu_B(x)$.

2.2. Fuzzy finite tree automata

Our definitions in this section, are different from that in [4], [10], [17], [18], [20] only in some minor details.

The set of natural numbers is denoted by $\mathbb{N}$, and the set of finite strings over $\mathbb{N}$ is $\mathbb{N}^*$. The empty string is denoted by ε. A $\Sigma-$ *alphabet* is a finite and nonempty set of symbols. A *ranked alphabet* is a couple $(\Sigma, Arity : \Sigma \to \mathbb{N} \cup \{0\})$, which is the disjoint union of sets of $n-ary$ symbols $\Sigma_n = \{\sigma | Arity(\sigma) = n\}$ for all $n \geq 0$. The set $T_\Sigma(Q)$ of $\Sigma -$ *trees* indexed by Q is inductively defined to be the smallest set such that $Q \subseteq T_\Sigma(Q)$ and $\sigma(t_1, \ldots, t_n) \subseteq T_\Sigma(Q)$ for every $\sigma \in \Sigma_n$ and $t_1, \ldots, t_n \in T_\Sigma(Q)$. We write T_Σ for $T_\Sigma(\phi)$.

Definition 3. Let $\ell = (L, \leq, \wedge, \vee)$ be a completely distributive lattice. A *fuzzy finite tree automaton* over ℓ is a system $M = (\Sigma, Q, \Gamma, \delta, \ell, \rho, \beta)$, where:

1. Σ is a finite set of ranked alphabets called input symbols.
2. Q is a finite set of symbols called states.
3. $\Gamma : Q \to L$ is an L-fuzzy set on Q and called the set of final states.
4. $\delta = \{\delta_\sigma : Q^n \times Q \times \Sigma_n \to L \,|\, \sigma \in \Sigma_n,\, n \geq 0\}$ is a finite set called transition rules.
5. $\rho : T_\Sigma(Q) \times Q \to L$ is called the run map of FFTA M, and defined by induction on structure of $t \in T_\Sigma(Q)$:
   (a) If $t = \sigma \in \Sigma_0$, then $\rho(t)(q) = \delta(q, \sigma)$, for all $q \in Q$.
   (b) If $t = \sigma(t_1, \ldots, t_n)$ for some $\sigma \in \Sigma_n$ and $t_1, \ldots, t_n \in T_\Sigma$, then
   $$\rho(t)(q) = \bigvee_{q_1,\ldots,q_n \in Q} \left(\delta(q_1, \ldots, q_n, q, \sigma) \wedge \bigwedge_{i=1}^{n} \rho(t_i)(q_i) \right).$$

6. $\beta : T_\Sigma \to L$ is an L-fuzzy set on a set of trees $t \in T_\Sigma$, called behavior of FFTA M, and defined by:
$$\beta(t) = \bigvee_{q \in Q} \rho(t)(q) \wedge \Gamma(q).$$

An FFTA $M = (\Sigma, Q, \Gamma, \delta, \ell, \rho, \beta)$ *accepts* a tree $t \in T_\Sigma$ iff $\beta(t) > 0$. Also, the set of all trees accepted by M is known as fuzzy tree language $L(M)$ *recognized* by FFTA M. As well, $\mu_{L(M)}(t) = \beta(t)$. In other words, a recognizable fuzzy tree language is a fuzzy tree language recognized by some FFTA.

An FFTA $M = (\Sigma, Q, \Gamma, \delta, \ell, \rho, \beta)$ is called *deterministic* if for every $\sigma \in \Sigma_n$ and $q_1, \ldots, q_n \in Q$, where $n \geq 0$, there exist at most one $q \in Q$, such that $\delta(q_1, \ldots q_n, q, \sigma) > 0$.

An FFTA $M = (\Sigma, Q, \Gamma, \delta, \ell, \rho, \beta)$ is called *complete* if for every $\sigma \in \Sigma_n$ and $q_1, \ldots, q_n \in Q$, where $n \geq 0$, there exist at least one $q \in Q$, such that $\delta(q_1, \ldots q_n, q, \sigma) > 0$.

An FFTA $M = (\Sigma, Q, \Gamma, \delta, \ell, \rho, \beta)$ is called *reduced* if for every $q \in Q$, there exists at least one $t \in T_\Sigma$ such that $\rho(t)(q) > 0$.

3. Similarity based merging states of DFFTA

3.1. Distance and similarity measure

Definition 4. ([21],[28]) Let X be a nonempty set and ℓ be a lattice. A real function $\mathcal{D} : \mathcal{F}(X, \ell)^2 \to [0, 1]$ is called a *distance measure* on $\mathcal{F}(X, \ell)$, if $\mathcal{D}$ satisfies the following properties:

1. $\mathcal{D}(A, B) = \mathcal{D}(B, A)$; $\forall A, B \in \mathcal{F}(X, \ell)$,
2. $\mathcal{D}(A, A) = 0$; $\forall A \in \mathcal{F}(X, \ell)$,
3. $\forall A, B, C \in \mathcal{F}(X, \ell)$, if $A \subseteq B \subseteq C$, then $\mathcal{D}(A, B) \leq \mathcal{D}(A, C)$ and $\mathcal{D}(B, C) \leq \mathcal{D}(A, C)$;

Definition 5. ([21],[28]) Let X be a nonempty set and ℓ be a lattice. A real function $\mathcal{S} : \mathcal{F}(X, \ell)^2 \to [0, 1]$ is called a *similarity measure* on $\mathcal{F}(X, \ell)$, if $\mathcal{S}$ satisfies the following properties:

1. $\mathcal{S}(A, B) = \mathcal{S}(B, A)$; $\forall A, B \in \mathcal{F}(X, \ell)$,
2. $\mathcal{S}(A, A) = \bigvee \{\mathcal{S}(B, C) \mid B, C \in \mathcal{F}(X, \ell)\}$; $\forall A \in \mathcal{F}(X, \ell)$,
3. $\forall A, B, C \in \mathcal{F}(X, \ell)$, if $A \subseteq B \subseteq C$, then $\mathcal{S}(A, C) \leq \mathcal{S}(A, B)$ and $\mathcal{S}(A, C) \leq \mathcal{S}(B, C)$.

Proposition 6. ([28]) *There exists a one-to-one correlation between all distance measures and all similarity measures, where a distance measure $\mathcal{D}$ and its corresponding similarity measure $\mathcal{S}$ satisfy $\mathcal{D} + \mathcal{S} = 1$.*

The similarity measure generated by the distance measure $\mathcal{D}$ is denoted by $\mathcal{S}\langle\mathcal{D}\rangle = 1 - \mathcal{D}$, and the distance measure generated by similarity measure $\mathcal{S}$ is denoted by $\mathcal{D}\langle\mathcal{S}\rangle = 1 - \mathcal{S}$.

Lemma 7. *Let $M = (\Sigma, Q, \Gamma, \delta, \ell, \rho, \beta)$ be a DFFTA and $L^{\gamma}(q)$ be an L-fuzzy set with membership function $\mu_{L^{\gamma}(q)}(t) = \rho(t)(q) \wedge \gamma$, where $q \in Q$ and $\gamma \in \ell$. Then,*

$$L(M) = \bigcup_{q \in Q} L^{\Gamma(q)}(q).$$

Proof. Since M is a DFFTA, for every $t \in T_{\Sigma}$ with $\mu_{L(M)}(t) > 0$, so there exists exactly one $q \in Q$ such that $\rho(t)(q) \wedge \Gamma(q) > 0$. Therefore,

$$L(M) = \bigcup_{q \in Q} \{t \mid t \in T_{\Sigma}, \rho(t)(q) \wedge \Gamma(q) > 0\} = \bigcup_{q \in Q} L^{\Gamma(q)}(q). \qquad \blacksquare$$

Definition 8. Let $M = (\Sigma, Q, \Gamma, \delta, \ell, \rho, \beta)$ be a DFFTA. An operator $\Pi : \mathcal{P}(\ell) \to \ell$ is called centroid operation on δ, iff for every $\overline{\delta} \subseteq \delta$, $M' = (\Sigma, Q, \Gamma, \delta', \ell, \rho', \beta')$, $M'' = (\Sigma, Q, \Gamma, \delta'', \ell, \rho'', \beta'')$ and $\gamma, \gamma', \gamma'' \in \ell$ with

$$\forall q_1, \ldots, q_n, q \in Q, \sigma \in \Sigma_n, n \geq 0;$$

$$\gamma = \delta(q_1, \ldots, q_n, q, \sigma),$$

$$\gamma'' = \prod_{\substack{n \geq 0,\ \sigma \in \Sigma_n, \\ q_1, \ldots, q_n,\ q \in Q, \\ \overline{\delta}(q_1, \ldots, q_n, q, \sigma) > 0}} \overline{\delta}(q_1, \ldots, q_n, q, \sigma),$$

$$\overline{\delta}\,(q_1, \ldots, q_n, q, \sigma) > 0 \Rightarrow \begin{cases} \delta'(q_1, \ldots, q_n, q, \sigma) = \gamma', \\ \delta''(q_1, \ldots, q_n, q, \sigma) = \gamma'', \end{cases}$$

$$\overline{\delta}(q_1, \ldots, q_n, q, \sigma) = 0 \Rightarrow \begin{cases} \delta'(q_1, \ldots, q_n, q, \sigma) = \gamma, \\ \delta''(q_1, \ldots, q_n, q, \sigma) = \gamma, \end{cases}$$

where, ρ', β' are corresponding to δ', and ρ'', β'' are corresponding to δ''; it holds $\mathcal{D}(L(M), L(M'')) \leq \mathcal{D}(L(M), L(M'))$.

Definition 9. Let $M = (\Sigma, Q, \Gamma, \delta, \ell, \rho, \beta)$ be a DFFTA, Π be a centroid operation on δ, $q, q' \in Q$ and $t \in T_{\Sigma}(Q)$. Then, $\rho(t)\left(q \xrightarrow{\Pi} q'\right)$ is defined by:

1. $\rho(t)\left(q \xrightarrow{\Pi} q\right) = \rho(t)(q)$,
2. $\rho(t)\left(q \xrightarrow{\Pi} q'\right)$

$$= \bigvee_{q_1, \ldots, q_n \in Q} \left(\left(\prod_{\substack{q_i \notin \{q, q'\} \Rightarrow q'_i = q_i, \\ q_i \in \{q, q'\} \Rightarrow q'_i \in \{q, q'\}, \\ i \in \{1, \ldots, n\},\ p \in \{q, q'\}}} \delta(q'_1, \ldots, q'_n, p, \sigma)\right) \wedge \left(\bigvee_{\substack{q_i \notin \{q, q'\} \Rightarrow q'_i = q_i, \\ q_i \in \{q, q'\} \Rightarrow q'_i \in \{q, q'\}, \\ i \in \{1, \ldots, n\},\ p \in \{q, q'\}}} \bigwedge_{i=1}^{n} \rho\,(t_i)\,(q'_i \xrightarrow{\Pi} q_i)\right)\right)$$

Definition 10. Let $M = (\Sigma, Q, \Gamma, \delta, \ell, \rho, \beta)$ be an FFTA, $\mathcal{D}$ be a distance measure on $L(M)$, $\mu_{L^{\gamma}(q)} = \rho(t)(q) \wedge \gamma$ and $\mu_{L^{\gamma}(q \xrightarrow{\Pi} q')} = \rho(t)\left(q \xrightarrow{\Pi} q'\right) \wedge \gamma$; where Π is a centroid operation on δ, $\gamma \in \ell$, $q, q' \in Q$ and $t \in T_{\Sigma}(Q)$. Then, we develope the distance measure $\mathcal{D}$ on Q by:

$$\mathcal{D}^{\Pi}(q, q') = \bigwedge_{\gamma \in \ell} \mathcal{D}\left(L^{\Gamma(q)}(q) \cup L^{\Gamma(q')}(q'), L^{\gamma}\left(q \xrightarrow{\Pi} q'\right)\right); \ \forall q, q' \in Q.$$

Theorem 11. *Let $M = (\Sigma, Q, \Gamma, \delta, \ell, \rho, \beta)$ be a DFFTA, Π be a centroid operation on δ, $\mathcal{D}$ be a generalized distance measures on $L(M)$ and Q, $x \in [0, 1]$ and Q' be a subset of Q such that for every $q, q' \in Q'$ it holds $\mathcal{D}^{\Pi}(q, q') \leq x$. Then, there exists a $\gamma \in \ell$ such that for every $q \in Q'$ it holds:*

$$\mathcal{D}\left(\bigcup_{q' \in Q'} L^{\Gamma(q')}(q'), \ \bigcup_{q' \in Q'} L^{\gamma}\left(q' \xrightarrow{\Pi} q\right)\right) \leq x.$$

Proof. Let

$$\gamma_{q,q'} = \bigwedge \left\{\gamma \in \ell \mid \mathcal{D}^{\Pi}(q, q') = \mathcal{D}\left(L^{\Gamma(q)}(q) \bigcup L^{\Gamma(q')}(q'), \ L^{\gamma}\left(q \xrightarrow{\Pi} q'\right)\right)\right\},$$

$$M_q = \bigvee_{q' \in Q'} \gamma_{q,q'} \ , \ m_q = \bigwedge_{q' \in Q'} \gamma_{q,q'} \ ; \ \forall q \in Q',$$

$$M = \bigwedge_{q \in Q'} M_q \ \textit{and} \ m = \bigvee_{q \in Q'} m_q.$$

Now, we prove that the above inequality holds for all $\gamma \in [m, M]$. To show this, on the contrary, let there exists a $\gamma' \in [m, M]$ such that

$$\mathcal{D}\left(\bigcup_{q' \in Q'} L^{\Gamma(q')}(q'), \ \bigcup_{q' \in Q'} L^{\gamma'}\left(q' \xrightarrow{\Pi} q\right)\right) > x.$$

Since M is deterministic and from Lemma 7, $L^{\Gamma(q)}(q) \cap L^{\Gamma(q')}(q') = \phi$ for every $q, q' \in Q$. So, there exists a $q \in Q'$ with $\mathcal{D}\left(L^{\Gamma(q)}(q), L^{\gamma'}(q)\right) > x$. Furthermore, we have $[m, M] \subseteq [m_q, M_q]$ which implies that $L^{m_q}(q) \subseteq L^{\gamma'}(q) \subseteq L^{M_q}(q)$. Let $M_q, m_q \in Q'$ be two states such that $M_q = \gamma_{q,q_M}$ and $m_q = \gamma_{q,q_m}$, where $q_m, q_M \in Q'$. Now, if $\Gamma(q) \leq \gamma'$, then $L^{\Gamma(q)}(q) \subseteq L^{\gamma'}(q) \subseteq L^{M_q}(q)$, and $\mathcal{D}^{\Pi}(q, q_M) > x$, which is a contradiction. On the other hand, if $\gamma' \leq \Gamma(q)$, and $L^{m_q}(q) \subseteq L^{\gamma'}(q) \subseteq L^{\Gamma(q)}(q)$, thus $\mathcal{D}^{\Pi}(q, q_m) > x$, which also is a contradiction. ∎

Theorem 12. *Let $M = (\Sigma, Q, \Gamma, \delta, \ell, \rho, \beta)$ be a DFFTA, Π be a centroid operation on δ, and $\mathcal{D}$ be a generalized distance measures on $L(M)$ and Q; $x \in [0, 1]$, and Q' be a subset of Q. If there exists a $\gamma \in \ell$ such that for every $q \in Q'$ it holds:*

$$\mathcal{D}\left(\bigcup_{q' \in Q'} L^{\Gamma(q')}(q'), \ \bigcup_{q' \in Q'} L^{\gamma}\left(q \xrightarrow{\Pi} q'\right)\right) \leq x.$$

Then, for any FFTA $\tilde{M} = (\Sigma, \tilde{Q}, \tilde{\Gamma}, \tilde{\delta}, \ell, \tilde{\rho}, \tilde{\beta})$ with

1. $\tilde{Q} = \{\tilde{q}\} \cup Q \backslash Q'$, *where* $\tilde{q} \notin Q$,
2. $\tilde{\Gamma}(\tilde{q}) = \gamma$ *and* $\tilde{\Gamma}(q) = \Gamma(q); \forall q \in \tilde{Q} \backslash \{\tilde{q}\}$,
3. $\forall \tilde{q}_1, \ldots, \tilde{q}_n \in \tilde{Q}, \forall q \in \tilde{Q} \backslash \{\tilde{q}\}$;

$$\tilde{\delta}(\tilde{q}_1, \ldots, \tilde{q}_n, q, \sigma) = \prod_{\substack{\tilde{q}_i = \tilde{q} \Rightarrow q'_i \in Q', \\ \tilde{q}_i \neq \tilde{q} \Rightarrow q'_i = \tilde{q}_i, \\ 1 \leq i \leq n}} \delta(q'_1, \ldots, q'_n, q, \sigma),$$

$$\tilde{\delta}(\tilde{q}_1, \ldots, \tilde{q}_n, \tilde{q}, \sigma) = \prod_{\substack{\tilde{q}_i = \tilde{q} \Rightarrow q'_i \in Q', \\ \tilde{q}_i \neq \tilde{q} \Rightarrow q'_i = \tilde{q}_i, \\ 1 \leq i \leq n, q' \in Q'}} \delta(q'_1, \ldots, q'_n, q', \sigma),$$

4. $\tilde{\rho}$ *and* $\tilde{\beta}$ *are corresponding to* $\tilde{\delta}$,

it holds $\mathcal{D}\left(L(M), L(\tilde{M})\right) \leq x$.

Proof. Let

$$L_1 = \bigcup_{q \in Q'} L^{\Gamma(q)}(q) \quad \text{and} \quad L_2 = \bigcup_{q \in Q \backslash Q'} L^{\Gamma(q)}(q).$$

Then, from Lemma 7, we have

a) $L(M) = \bigcup_{q \in Q} L^{\Gamma(q)}(q) = L_1 \cup L_2$.

b) $L(\tilde{M}) = \bigcup_{q \in \tilde{Q}} L^{\tilde{\Gamma}(q)}(q) = L^{\gamma}(\tilde{q}) \cup L_2$.

Now, from (a) and (b), and by the assumption of the theorem,

$$\mathcal{D}\left(L(M), L(\tilde{M})\right) = \mathcal{D}\left(L_1, L^{\gamma}(\tilde{q})\right) \leq x. \qquad \blacksquare$$

3.2. Normalizing DFFTA

Remark 13. In this manuscript, without lose of generality, we assume that Σ and Q are ordered sets.

Definition 14. Let $M = (\Sigma, Q, \Gamma, \delta, \ell, \rho, \beta)$ be an FFTA. An ordering on set δ is defined as follows:

1. If $\sigma < \sigma'$ then $\delta(q_1, ..., q_n, q, \sigma) < \delta(q'_1, ..., q'_m, q', \sigma')$.
2. If there exists $i \in \{1, ..., n\}$ such that $q_i < q'_i$, and $q_j = q'_j$ for $j \in \{1, ..., i-1\}$, then $\delta(q_1, ..., q_n, q, \sigma) < \delta(q'_1, ..., q'_n, q', \sigma)$.
3. If $q < q'$ then $\delta(q_1, ..., q_n, q, \sigma) < \delta(q_1, ..., q_n, q', \sigma)$.

where $q_1, ..., q_n, q, q'_1, ..., q'_m, q' \in Q$, $0 \leq n \leq m$, $\sigma \in \Sigma_n$ and $\sigma' \in \Sigma_m$.

Definition 15. Let S be an ordered set. We define the function $f^* : S \to \mathbb{N}$ called offset of $x \in S$ by

$$f^*(x) = |\{x' \in S | x' < x\}|.$$

Lemma 16. *Let $M = (\Sigma, Q, \Gamma, \delta, \ell, \rho, \beta)$ be a complete and reduced DFFTA. Then, for every $r, r' \in \delta$ with*

$$r : \delta(q_1, ..., q'_i, ..., q_n, q', \sigma),$$
$$r' : \delta(q_1, ..., q_i, ..., q_n, q, \sigma),$$

it holds

$$f^*(r) = f^*(r') + (f^*(q'_i) - f^*(q_i)) \times |Q|^{n-i},$$

where, $q_1, ..., q_i, ..., q_n, q, q', q'_i \in Q$, $\sigma \in \Sigma_n$, $q_i < q'_i$ and $1 \leq i \leq n$.

Proof. Since M is complete, reduced and deterministic; order of rules $r \in \delta$ related to each $\sigma \in \Sigma_n$ is like the sequence of $(n+1)$–*digit* numbers in base $|Q|$, where the value of each digit $q \in Q$ is $f^*(q)$. Now, the proof is straightforward. ∎

Corollary 17. *Let M be a complete and reduced DFFTA. The time complexity of accessing the membership grade of each fuzzy transition rule is $O(l)$, where l is the maximum rank of Σ–alphabet.*

Example 18. Let $M = (\Sigma, Q, \Gamma, \delta, \ell, \rho, \beta)$ be a complete and reduced DFFTA defined by

$$\begin{aligned}
&\Sigma_0 = \{\alpha\}, \Sigma_1 = \{\lambda\}, \Sigma_2 = \{\sigma\}, \Sigma = \{\alpha, \lambda, \sigma\}, \ell = ([0,1], \geq, \wedge, \vee),\\
&Q = \{q_1, q_2, q_3, q_4\}, \Gamma = \{(q_1, 0.7), (q_2, 0.5), (q_3, 0.5)\} \text{ and}\\
&\delta = \{r_1 : \delta(q_1, \alpha) = 0.9, r_2 : \delta(q_1, q_4, \lambda) = 0.8, r_3 : \delta(q_2, q_4, \lambda) = 0.8,\\
&\quad r_4 : \delta(q_3, q_4, \lambda) = 0.8, r_5 : \delta(q_4, q_2, \lambda) = 0.8, r_6 : \delta(q_1, q_1, q_3, \sigma) = 0.7,\\
&\quad r_7 : \delta(q_1, q_2, q_3, \sigma) = 0.7, r_8 : \delta(q_1, q_3, q_2, \sigma) = 0.6, r_9 : \delta(q_1, q_4, q_4, \sigma) = 0.6,\\
&\quad r_{10} : \delta(q_2, q_1, q_3, \sigma) = 0.6, r_{11} : \delta(q_2, q_2, q_3, \sigma) = 0.6, r_{12} : \delta(q_2, q_3, q_2, \sigma) = 0.6,\\
&\quad r_{13} : \delta(q_2, q_4, q_4, \sigma) = 0.2, r_{14} : \delta(q_3, q_1, q_2, \sigma) = 0.3, r_{15} : \delta(q_3, q_2, q_2, \sigma) = 0.3,\\
&\quad r_{16} : \delta(q_3, q_3, q_3, \sigma) = 0.3, r_{17} : \delta(q_3, q_4, q_4, \sigma) = 0.2, r_{18} : \delta(q_4, q_1, q_4, \sigma) = 0.1,\\
&\quad r_{19} : \delta(q_4, q_2, q_4, \sigma) = 0.1, r_{20} : \delta(q_4, q_3, q_4, \sigma) = 0.1, r_{21} : \delta(q_4, q_4, q_4, \sigma) = 1\}.
\end{aligned}$$

Now, it holds

$$f^*(r_{16}) = f^*(r_{14}) + (f^*(q_3) - f^*(q_1))|Q|^{Arity(\sigma)-2} = 13 + (2-0) \times 4^{2-2} = 15.$$

Definition 19. Let $M = (\Sigma, Q, \Gamma, \delta, \ell, \rho, \beta)$ be an FFTA. For any $q \in Q$ we define the *accessibility grade set* by

$$\mathcal{A}(q) = \{\rho(q)(t) | t \in T_\Sigma, \rho(q)(t) > 0\}.$$

Lemma 20. *Let $M = (\Sigma, Q, \Gamma, \delta, \ell, \rho, \beta)$ be an FFTA. Then, the set of maximum accessibility grade $(\vee \mathcal{A}(q))$ of all $q \in Q$ can be calculated in $O(l|Q|^l)$, where l is the maximum rank of $\Sigma -$ alphabet.*

Proof. We define the Algorithm 1 for computing $\bigvee \mathcal{A}(q)$ for all $q \in Q$:

Algorithm 1. Computing maximum accessibility grade of all states.

---

0 Input: $M = (\Sigma, Q, \Gamma, \delta, \ell, \rho, \beta)$

1 $\forall q \in Q; V(q) = 0$

2 $\forall \sigma \in \Sigma_0, q \in Q; V(q) = V(q) \vee \delta(q, \sigma)$

3 $A_{\max} = \bigvee_{q \in Q} V(q)$

4 $Q_{acc} = \phi$

5 Repeat

6 $\quad Q_{new} = \{q | q \in Q,\ V(q) = A_{\max}\}$

7 $\quad \forall q_1, \ldots, q_n \in Q_{acc} \cup Q_{new}, \{q_1, \ldots, q_n\} \cap Q_{new} \neq \phi, q \in Q,\ \sigma \in \Sigma_{n>0};$

\- $\qquad V(q) = V(q) \vee (\delta_\sigma(q_1, \ldots, q_n, q, \sigma) \wedge A_{\max})$

8 $\quad A_{\max} = \bigvee_{q \in Q, V(q) < A_{\max}} V(q)$

9 $\quad Q_{acc} = Q_{acc} \cup Q_{new}$

10 Until $Q = Q_{acc}$

11 Output: $V(q); \forall q \in Q$.

Lines 5 to 10 consist a loop that in each iteration, processes at least one transition rule. Since each transition rule will be processed only one time, the number of repetitions of this loop is $O(|\delta|)$. We note that the number of repeating lines 7 to 9 is not more than $|\delta|$ times (see $\{q_1, \ldots, q_n\} \cap Q_{new} \neq \phi$). Furthermore, making each transition rule requires combining l states. In the other hand, according to Lemma 7 and Corollary 17 accessing the membership grade of each rule is $O(l)$, which can be merged with the process of transition rule making. Therefore this loop can be done in $O(l|\delta|)$. ∎

Definition 21. Let $M = (\Sigma, Q, \Gamma, \delta, \ell, \rho, \beta)$ be an FFTA. For any $q \in Q$ we define the *behavior grade set* by

$$\mathcal{B}(q) = \{\beta(t) | t \in T_\Sigma, \rho(q)(t) \wedge \Gamma(q) > 0\}.$$

Lemma 22. *Let $M = (\Sigma, Q, \Gamma, \delta, \ell, \rho, \beta)$ be a DFFTA. Then, for every $q \in Q$, it holds*

$$\mathcal{B}(q) = \{\gamma \wedge \Gamma(q) | \gamma \in \mathcal{A}(q)\}.$$

Proof. Since M is deterministic, the condition $\rho(q)(t) \wedge \Gamma(q) > 0$ implies that $\beta(t) = \rho(q)(t) \wedge \Gamma(q)$. Hence,

$$\mathcal{B}(q) = \{\rho(t)(q) \wedge \Gamma(q) | t \in T_\Sigma, q \in Q, \rho(q)(t) > 0\} = \{\gamma \wedge \Gamma(q) | \gamma \in \mathcal{A}(q)\}. \quad ∎$$

Corollary 23. *Let $M = (\Sigma, Q, \Gamma, \delta, \ell, \rho, \beta)$ be a DFFTA. Then, for all $q \in Q$, it holds*

$$\vee\mathcal{B}(q) = \vee\mathcal{A}(q) \wedge \Gamma(q).$$

Proof. Using Lemma 22 we have

$$\mathcal{B}(q) = \{\gamma \wedge \Gamma(q) | \gamma \in \mathcal{A}(q)\} = \vee\{\gamma | \gamma \in \mathcal{A}(q)\} \wedge \Gamma(q) = \vee\mathcal{A}(q) \wedge \Gamma(q). \qquad \blacksquare$$

Corollary 24. *Let $M = (\Sigma, Q, \Gamma, \delta, \ell, \rho, \beta)$ be a DFFTA. Then, the $\vee\mathcal{B}(q)$ for all $q \in Q$ can be calculated in $O(l|Q|^l)$, where l is the maximum rank of $\Sigma-$alphabet.*

Proof. It is and immediate consequence of the Lemma 16 and Corollary 23. $\blacksquare$

Definition 25. Let $M = (\Sigma, Q, \Gamma, \delta, \ell, \rho, \beta)$ be an FFTA. The *next* set of every $q \in Q$ is defined by:

$$\begin{aligned} next(q) = \{q' | 1 \leq i \leq n, \sigma \in \Sigma_n, q_1, \ldots, q_{i-1}, q_{i+1}, \ldots, q_n, q' \in Q, \\ \delta(q_1, \ldots, q_{i-1}, q, q_{i+1}, \ldots, q_n, q', \sigma) \geq 0\} \end{aligned}$$

Lemma 26. *Let $M = (\Sigma, Q, \Gamma, \delta, \ell, \rho, \beta)$ be a DFFTA. Then, the order of time complexity for calculating next set of all $q \in Q$ is $O(l|Q|^l)$, where l is the maximum rank of $\Sigma-$ alphabet.*

Proof. It is sufficient that for all $\delta(q_1, \ldots, q_n, q, \sigma) \geq 0$, with $q_1, \ldots, q_n, q \in Q$, $1 \leq n$ and $\sigma \in \Sigma_n$, add q to sets $next(q_1)$,..., $next(q_n)$. $\blacksquare$

Definition 27. Let $M = (\Sigma, Q, \Gamma, \delta, \ell, \rho, \beta)$ be an FFTA. The *follow* set of each $q \in Q$ is the smallest set with the following properties:

1. $next(q) \subseteq follow(q)$,
2. If $q' \in follow(q)$ then, $next(q') \subseteq follow(q)$.

Lemma 28. *Let $M = (\Sigma, Q, \Gamma, \delta, \ell, \rho, \beta)$ be a DFFTA. Then, the order of time complexity for calculating the follow set of all $q \in Q$ is $O(l|Q|^{\max(l,2)})$, where l is the maximum rank of $\Sigma-$ alphabet.*

Proof. According to Lemma 26, the time complexity of calculating *next* set of all $q \in Q$ is $O(l|Q|^l)$. Now, according to properties of *follow* set, computing it for each state can be done by a simple recursive process with the time complexity $O(|Q|^2)$. We mention that the total time complexity, when $l = 1$ is $O(|Q|^2)$ and otherwise, is $O(l|Q|^l)$. $\blacksquare$

Definition 29. Let $M = (\Sigma, Q, \Gamma, \delta, \ell, \rho, \beta)$ be a DFFTA. The *maximum follow grade* of each $q \in Q$ is defined by:

$$\mathcal{G}(q) = \bigvee_{q' \in follow(q)} \mathcal{B}(q').$$

Lemma 30. *Let $M = (\Sigma, Q, \Gamma, \delta, \ell, \rho, \beta)$ be a DFFTA. The time complexity for calculating $\mathcal{G}(q)$ for all $q \in Q$ is $O(l|Q|^{\max(l,2)})$, where l is the maximum rank of $\Sigma-$alphabet.*

Proof. It is similar to Lemma 28. ∎

Definition 31. Let $M = (\Sigma, Q, \Gamma, \delta, \ell, \rho, \beta)$ be a DFFTA, $\sigma \in \Sigma_n$, $q_1, \ldots, q_n, q \in Q$ and $\gamma \in \ell$. A transition rule $\delta(q_1, \ldots, q_n, q, \sigma) = \gamma$ is called *normal* iff

$$\gamma = \Big(\Gamma(q) \vee \mathcal{G}(q)\Big) \wedge \bigwedge_{i=1}^{n} \big(\vee \mathcal{A}(q_i)\big).$$

Also, M is a *normal FFTA* iff all transition rules in δ be *normal* and for each $q \in Q$ it holds $\Gamma(q) \leq \vee\mathcal{A}(q)$.

Theorem 32. *Let $M = (\Sigma, Q, \Gamma, \delta, \ell, \rho, \beta)$ be a DFFTA. Then, normalizing M can be done in order of time complexity $O(l|Q|^l)$, where l is the maximum rank of $\Sigma-$alphabet.*

Proof. We define Algorithm 2 for *normalizing* M:

Algorithm 2. Normalizing a DFFTA.

---

0 Input: $M = (\Sigma, Q, \Gamma, \delta, \ell, \rho, \beta)$
1 $\forall q \in Q; V(q) = \vee\mathcal{A}(q)$
2 $A_{\max} = \bigvee_{q\in Q} V(q)$
3 $Q_{acc} = \phi$
4 $\delta^{\Delta} = \phi$
5 Repeat
6 $\quad Q_{new} = \{q | q \in Q, V(q) = A_{\max}\}$
7 $\quad \forall q \in Q, \sigma \in \Sigma_n, n \geq 0, q_1, \ldots, q_n \in Q_{acc} \cup Q_{new}$,
- $\quad\quad \{q_1, \ldots, q_n\} \cap Q_{new} = \phi \Leftrightarrow n = 0$;
- $\quad\quad\quad \delta^{\Delta}(q_1, \ldots, q_n, q, \sigma) = \delta(q_1, \ldots, q_n, q, \sigma) \wedge A_{\max} \wedge \Big(\Gamma(q) \vee \mathcal{G}(q)\Big)$
8 $\quad A_{\max} = \bigvee_{q\in Q, V(q) < A_{\max}} V(q)$
9 $\quad Q_{acc} = Q_{acc} \cup Q_{new}$
10 Until $Q = Q_{acc}$
11 $\forall q \in Q; \Gamma^{\Delta}(q) = \Gamma(q) \wedge V(q)$
12 Output: $M = (\Sigma, Q, \Gamma^{\Delta}, \delta^{\Delta}, \ell, \rho, \beta)$.

According to Lemma 30, calculating $\mathcal{G}(q)$ for all $q \in Q$ can be done in $O(l|\delta|)$. Lines 5 to 10 are a loop that in each repetition, adds at least one transition rule to δ^{Δ}. Thus making all transition rules must be repeats $|\delta|$ times and making each rule, involves combining n states. Therefore, this loop can be done in $O(n|\delta|)$. Calculating the complexity of other lines is straightforward. ∎

3.3. Merging states of DFFTA

Definition 33. Let $M = (\Sigma, Q, \Gamma, \delta, \ell, \rho, \beta)$ be a DFFTA, Π be a centroid operation on δ, $\mathcal{S}\langle\mathcal{D}\rangle$ be a similarity measure corresponding to distance measure $\mathcal{D}$ on $L(M)$ and Q, and $\tau \in [0,1]$. The similarity relation $\overset{\mathcal{S},\Pi,\tau}{\longleftrightarrow}$ on Q, is defined by:

$$q \overset{\mathcal{S},\Pi,\tau}{\longleftrightarrow} q' \Longleftrightarrow 1 - \mathcal{D}^{\Pi}(q, q') \geq \tau \ ; \ \forall q, q' \in Q.$$

Lemma 34. *Let $M = (\Sigma, Q, \Gamma, \delta, \ell, \rho, \beta)$ be a normal DFFTA, Π be a centroid operation on $\delta, \mathcal{S}$ be a similarity measures on $L(M)$ and Q, $\tau \in [0,1]$, and $\overset{\mathcal{S},\Pi,\tau}{\longleftrightarrow}$ be a similarity relation on Q. Also, let Q' be a subset of Q such that for every $q, q' \in Q'$ we have $q \overset{\mathcal{S},\Pi,\tau}{\longleftrightarrow} q'$, $\delta_{Q'} = \{r : \delta(q_1, ..., q_n, q, \sigma) = c | c > 0, \ \sigma \in \Sigma_n, q_1, ..., q_n \in Q'\}$, and let $d = \prod_{r \in \delta_{Q'}} r$ and $\delta'_{Q'} = \{r : \delta(q_1, ..., q_n, q, \sigma) = d \mid \sigma \in \Sigma_n, q_1, ..., q_n \in Q'\}$. Then, it holds $\mathcal{S}(\delta_{Q'}, \delta'_{Q'}) \geq \tau$.*

Proof. Let $x = 1 - \tau$. for every $q, q' \in Q'$ it holds $q \overset{\mathcal{S},\Pi,\tau}{\longleftrightarrow} q'$; thus, $\mathcal{D}^{\Pi}(q, q') \leq x$. Then, from Theorem 11, there exists a $\gamma \in \ell$ such that for every $q \in Q'$ it holds:

$$\mathcal{D}\left(\bigcup_{q' \in Q'} L^{\Gamma(q')}(q'), \ \bigcup_{q' \in Q'} L^{\gamma}\left(q' \overset{\Pi}{\longrightarrow} q\right)\right) \leq x.$$

From Theorem 12 and since M is normal, we have $\mathcal{D}(\delta_{Q'}, \delta'_{Q'}) \leq x$. ∎

Theorem 35. *Let $M = (\Sigma, Q, \Gamma, \delta, \ell, \rho, \beta)$ be a complete and reduced DFFTA and $\mathcal{S}$ be a similarity measure corresponding to distance measures $\mathcal{D}$ on $L(M)$ and Q. Then, the similarity relation $\overset{\mathcal{S},\Pi,\tau}{\longleftrightarrow}$ is not transitive for some $\tau \in [0,1]$ and centroid operation Π on δ.*

Proof. Let $M = (\Sigma, Q, \Gamma, \delta, \ell, \rho, \beta)$ be DFFTA defined by:

$$\begin{aligned}
&\Sigma_0 = \{\alpha\}, \ \Sigma_2 = \{\sigma\}, \ \Sigma = \{\alpha, \sigma\}, \ Q = \{q_1, q_2, q_3\},\\
&\Gamma = \{(q_1, 0.2), (q_2, 0.3), (q_3, 0.6)\}, \text{ and}\\
&\delta = \{r_1 : \delta(q_1, \alpha) = 0.6, r_2 : \delta(q_1, q_1, q_2, \sigma) = 0.6,\\
&\qquad r_3 : \delta(q_1, q_2, q_3, \sigma) = 0.6, r_4 : \delta(q_1, q_3, q_3, \sigma) = 0.6,\\
&\qquad r_5 : \delta(q_2, q_1, q_1, \sigma) = 0.6, r_6 : \delta(q_2, q_2, q_1, \sigma) = 0.6,\\
&\qquad r_7 : \delta(q_2, q_3, q_2, \sigma) = 0.6, r_8 : \delta(q_3, q_1, q_2, \sigma) = 0.6,\\
&\qquad r_9 : \delta(q_3, q_1, q_1, \sigma) = 0.6, r_{10} : \delta(q_3, q_2, q_1, \sigma) = 0.6\}.
\end{aligned}$$

Also, let $\ell = ([0,1], \geq, \wedge, \vee)$ be a lattice and $\overset{\mathcal{S},\text{mid},0.8}{\longleftrightarrow}$ be a similarity relation, where $\mathcal{S}(A, B) = 1 - \bigvee_{a \in A, b \in B} |a - b|; \forall A, B \in \mathcal{F}(T_{\Sigma}, \ell)$ and $mid(C) = \frac{\vee C + \wedge C}{2}$; $\forall C \subseteq [0,1]$. (In this example, $[0,1]$ is the unit interval of real numbers.) Now, we have $q_1 \overset{\mathcal{S},\text{mid},0.8}{\longleftrightarrow} q_2$ and $q_2 \overset{\mathcal{S},\text{mid},0.8}{\longleftrightarrow} q_3$, where $q_1 \overset{\mathcal{S},\text{mid},0.8}{\longleftrightarrow} q_3$ does not hold. ∎

Corollary 36. *Let $M = (\Sigma, Q, \Gamma, \delta, \ell, \rho, \beta)$ be a DFFTA, $\prod$ be a centroid operation on δ, $\mathcal{S}$ be a similarity measure on $L(M)$ and on Q, $\tau \in [0,1]$, and $\xleftrightarrow{\mathcal{S},\Pi,\tau}$ be a similarity relation on Q. Then, for some $P = \bigcup\limits_{q \in Q} \{q' \xleftrightarrow{\mathcal{S},\Pi,\tau} q\}$, there exists $p, p' \in P$ such that $p \cap p' \neq \phi$.*

Definition 37. Let $M = (\Sigma, Q, \Gamma, \delta, \ell, \rho, \beta)$ be a reduced and complete DFFTA. A *Merging Dependency Graph* (MDG) $G = (V, E)$ on M is a directed graph with the following properties:

1. $V = \Big\{\{q, q'\} |\, q, q' \in Q\Big\}$,
2. $E \subseteq V^2$,
3. $\big(\{p, p'\}, \{q, q'\}\big) \in E$ if and only if there exist $\sigma \in \Sigma_n$, $1 \leq i \leq n$ and $q_1, ..., q_n, q, q', p, p' \in Q$ such that $\delta(q_1, ..., q_{i-1}, q, q_{i+1}, ..., q_n, p, \sigma) > 0$ and $\delta(q_1, ..., q_{i-1}, q', q_{i+1}, ..., q_n, p', \sigma) > 0$.

Lemma 38. *Let $M = (\Sigma, Q, \Gamma, \delta, \ell, \rho, \beta)$ be a complete and reduced DFFTA. The order of time complexity of making MDG $G = (V, E)$ for M is $O(l\,|Q|^{l+1})$, where l is the maximum arity of $\Sigma -$ alphabet.*

Proof. All rules in δ must be processed for constructing set E. As well, the set of rules related to each $\sigma \in \Sigma_n$, denoted by δ_σ, compare with each other. Therefore, it is sufficient to prove that the order of time complexity for processing all rules in δ_σ is $O(n|Q|^{n+1})$. Now, we introduce Algorithm 3 to process all rules in δ_σ for constructing the edges in set E:

Algorithm 3. Making MDG for a DFFTA.

---

0 Input: $M = (\Sigma, Q, \Gamma, \delta, \ell, \rho, \beta)$
1 $E = \phi$
2 $\forall r : \delta(q_1, \ldots, q_n, q, \sigma) > 0;$
3    $\forall i \in \{1, \ldots, n\};$
4       $\forall r' : \delta(q_1, \ldots, q'_i, \ldots, q_n, q', \sigma) > 0, q'_i \in Q, q'_i > q_i;$
5          $E = E \cup \Big\{\big(\{q, q'\}, \{q_i, q'_i\}\big)\Big\}$
6 Output: E.

The proof of correctness of this algorithm is straightforward; thus, we analyze its order of time complexity. Lines 2 to 5 are three nested loops which cause lines 4 and 5 process $\frac{n|Q|^{n+1}}{2}$ times. According to Lemma 16, the order of time complexity for finding offset of r' by r is $O(1)$. Furthermore, we assume that a two dimension array (adjacency matrix) is used for holding set E. Thus, the order of time complexity for adding an edge to E is $O(1)$. Therefore, the order of time complexity for processing all rules in $\delta_\sigma$is $O(n|Q|^{n+1})$. ∎

Lemma 39. *Let $M = (\Sigma, Q, \Gamma, \delta, \ell, \rho, \beta)$ be a DFFTA, Π be a centroid operation on δ, $\mathcal{S}$ be a similarity measures on $L(M)$ and Q, and $\tau \in [0, 1]$. Then, making graph $G = (V, E)$ such that $V = Q$ and E is corresponding to similarity relation $\overset{\mathcal{S},\Pi,\tau}{\longleftrightarrow}$ on Q can be done in order of complexity $O(l|Q|^{\max(2l,4)})$, where l is the maximum arity of $\Sigma-$alphabet.*

Proof. Without lose of generality, we can assume that M is normal. Now, we introduce Algorithm 4 for processing all rules in δ_σ and constructing graph G.

Algorithm 4. Making MDG corresponding to similarity relation $\overset{\mathcal{S},\Pi,\tau}{\longleftrightarrow}$.

---

0 Input: $M = (\Sigma, Q, \Gamma, \delta, \ell, \rho, \beta)$

1 $E = \left\{(q, q')|q, q' \in Q, \mathcal{S}^{\frac{1}{2}}\left(\mathcal{B}(q), \mathcal{B}(q')\right) \geq \tau\right\}$

2 $\forall\ r : \delta(q_1, \ldots, q_n, q, \sigma) > 0\ ,\ r' : \delta(q'_1, \ldots, q'_n, q', \sigma) > 0;$

3 $\quad R_{rr'} = \phi$

4 $\quad \mathcal{S}^{\frac{1}{2}}(r, r') < \tau\ ,\ \forall i \in \{1, \ldots, n\}; (q_i, q'_i) \in E\ \Rightarrow$

5 $\qquad R_{rr'} = \left\{(q_i, q'_i)|i \in \{1, \ldots, n\}\right\}$

6 $R = \bigcup\limits_{r,r' \in \delta} R_{rr'}$

7 Repeat

8 $\quad (q, q') = \bigvee R$

9 $\quad E = E - \{(q, q')\}$

10 $\quad \forall r, r' \in \delta; (q, q') \in R_{rr'} \Rightarrow R_{rr'} = \phi$

11 $\quad R = \bigcup\limits_{r,r' \in \delta} R_{rr'}$

12 Until $R = \phi$

13 Output: $G = (Q, E)$.

This algorithm clusters transition rules that cannot merge together (because of their grade of transition) by clustering the set Q. Also, line 1 makes an initial clustering based on the similarity of states. According to Corollary 24, the time complexity of line 1 is $O(l|Q|^l)$, where l is the maximum arity of Σ. Lines 2 to 5 is a loop that repeats $|Q|^{2n}$ times, where n is the arity of σ. The order of lines 4 and 5 is $O(l)$; therefore, the order of time complexity for lines 2 to 5 is $O(l|Q|^{2l})$. As well, the order of time complexity of line 6 is $O(l|Q|^{2l})$. Lines 7 to 12 is a loop that repeats $O(|Q|^2)$ times and its order of time complexity is $O(l|Q|^{\max(2l,4)})$. Therefore, the total order of time complexity of algorithm $O(l|Q|^{\max(2l,4)})$. ∎

4. Conclusion

We contribute the problem of similarity based merging states of DFFTA. Firstly, the concept of similarity and distance measure of fuzzy sets is generalized for states of FFTA and a similarity relation $\overset{\mathcal{S},\Pi,\tau}{\longleftrightarrow}$ is defined on Q. We prove that this relation is not transitive; therefore, minimizing DFFTA by $\overset{\mathcal{S},\Pi,\tau}{\longleftrightarrow}$, is not similar to traditional minimization algorithms. As well, the concept of normalizing DFFTA with a polynomial time algorithm is introduced. Then, normal DFFTA is used for obtaining MDG and defining an ordering on Q^2. Furthermore, we present Algorithm 4 and show that the order of time complexity for making similarity relation graph on Q corresponding to $\xleftrightarrow{\mathcal{S},\Pi,\tau}$, is $O(l|Q|^{max(4,2l)})$, where l is the maximum arity of Σ-*alphabet*.

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ON AN INERTIA FACTOR GROUP OF $2^8{:}O_8^+(2)$

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Abstract. The group $\overline{G} = 2^6{:}A_8$ is an inertia factor group of $2^8{:}O_8^+(2)$. As an inertia factor group, our group $\overline{G}$ plays an essential role in the construction of the character table of $2^8{:}O_8^+(2)$. In this paper we look at two ways of constructing $\overline{G}$. In the first method, we use combinatorics and the natural action of A_8 on $W \cong 2^6$. In the second method, we use a computational method to construct $\overline{G} = N{:}A_8$ inside $O_8^+(2)$. We show that A_8 acts irreducibly on both W and N and we prove that the two groups are indeed isomorphic.

Keywords: split extension, maximal subgroup, inertia factor groups, orthogonal group
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1. Introduction

The group $\overline{G} = 2^6{:}A_8$ is an inertia factor group of $2^8{:}O_8^+(2)$. This group is also a maximal subgroup of $O_8^+(2)$ of index 135 and order 1290240. As an inertia factor it plays an essential role in the construction of the character table of $2^8{:}O_8^+(2)$ as there is a block of irreducible characters in this table that corresponds to $\overline{G}$. In the construction of $\overline{G}$, A_8 acts on the elementary abelian group 2^6. The action on 2^6 is multiplication on the right of the six dimensional row vector space $N = 2^6$. This requires A_8 to be represented by 6×6 matrices. It then becomes necessary to reconstruct A_8 from a 8×8 representation to a 6×6 representation. In this paper we look at two ways to do this.

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Although it is much simpler and natural to consider the embedding of $2^6{:}A_8$ into $O_8^+(2)$ (see Section 3), but it is interesting to construct this group combinatorially and this is our main reason for discussing the first method. In our first method (Section 2), we first take an 8-dimensional module V on which S_8 acts naturally by permuting its basis elements. We then obtain two submodules of V, namely M_1 and M_2 of dimensions 1 and 7 respectively. Let $W = M_2/M_1$, then $dim(W) = 6$ and W is a G-invariant where $G = S_6$ or A_8 (see Theorem 2.2). Let α and β be two permutation cycles of orders 7 and 3 respectively where $A_8 =< \alpha, \beta >$. Then, by the action of α and β on the generators of W, we get a matrix representation of both α and β. These are 6×6 matrix representations. We are then able to represent A_8 by 6×6 matrices. We show that (in Section 2) A_8 acts irreducibly on W and this action produces three orbits of lengths $1, 28$ and 35 respectively. These have corresponding point stabilizers which we obtain from the ATLAS [4] . We are then able to construct $\overline{G}$.

For the second method (Section 3) we use GAP [6]. We first construct $O_8^+(2)$ from the general orthogonal group $GO_8^+(2)$. We then construct, $\overline{G} = 2^6{:}A_8$, inside $O_8^+(2)$. This has only one proper normal subgroup, namely $N \cong 2^6$, which we can always obtain from $\overline{G}$. We then obtain the 6 generators of N which are 8×8 matrices. From the generators of $\overline{G}$, we are able to get two, 8×8 matrix generators of A_8 namely, a and b each of order 4. We then let a and b act on the generators of N by conjugation. Since $N \trianglelefteq \overline{G}$ the result of these actions are elements of N. We get a 6×6 matrix representation of both. This leads us to a 6×6 representation of A_8. We then let this A_8, using GAP, to act on N and in Theorem 3.1 we show that N is irreducible under this action. The two groups constructed in Sections 2 and 3 are indeed isomorphic (see Corollary 3.2).

This work is taken from the dissertation of the second author [18], for more of his work, one can also go to [14], [15], [16]. For further references one can also read [1], [8], [9], [12], [17], [19]. For the general theory of ordinary characters of finite groups, readers are referred to [7].

Our notation will be consistent with that used in the ATLAS [4].

2. The Combinatorics Method

The combinatorics method can also be found in [1] and [17] and is used extensively in [13] and [20]. The group S_8 acts naturally on a module of dimension 8 by permuting the basis elements which generate the module. Let V be the 8-dimensional natural module of A_8 over $GF(2)$, where $V = \langle e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8 \rangle$, and $e_i^2 = 1$ for $i \in \{1, 2, 3, 4, 5, 6, 7, 8\}$. We regard V as a multiplicative elementary abelian 2-group of order 2^8.

Theorem 2.1 *Let V be the natural module of S_8 over $GF(2)$. Then there exist S_8 submodules M_1 and M_2 of V such that $V \supset M_2 \supset M_1 \supset 0$ and that*

$$dim(M_2) = 7 \quad and \quad dim(M_1) = 1.$$

Proof. Let $V = \langle e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\rangle$, and $e_i^2 = 1$ for $i \in \{1, 2, 3, 4, 5, 6, 7, 8\}$. Then S_8 acts naturally on V and this natural action results in the following orbits:

1. $O_0 = \{1_V\}$ and $|O_0| = 1$.
2. $O_1 = \{e_i | 1 \le i \le 8\}$ and $|O_1| = 8$.
3. $O_2 = \{e_ie_j | 1 \le i, j \le 8, i \ne j\}$ and $|O_2| = \left(\begin{array}{c} 8 \\ 2 \end{array}\right) = 28$.
4. $O_3 = \{e_ie_je_k | 1 \le i, j, k \le 8,$ distinct $i, j, k\}$ and $|O_3| = \left(\begin{array}{c} 8 \\ 3 \end{array}\right) = 56$.
5. $O_4 = \{e_ie_je_ke_l | 1 \le i, j, k, l \le 8,$ distinct $i, j, k, l\}$ and $|O_4| = \left(\begin{array}{c} 8 \\ 4 \end{array}\right) = 70$.
6. $O_5 = \{e_ie_je_ke_le_m | 1 \le i, j, k, l, m \le 8,$ distinct $i, j, k, l, m\}$ and $|O_5| = \left(\begin{array}{c} 8 \\ 5 \end{array}\right) = 56$.
7. $O_6 = \{e_ie_je_ke_le_me_n | 1 \le i, j, k, l, m, n \le 8,$ distinct $i, j, k, l, m, n\}$ and $|O_6| = \left(\begin{array}{c} 8 \\ 6 \end{array}\right) = 28$.
8. $O_7 = \{e_ie_je_ke_le_me_ne_o | 1 \le i, j, k, l, m, n, o \le 8,$ distinct $i, j, k, l, m, n, o\}$ and $|O_7| = \left(\begin{array}{c} 8 \\ 7 \end{array}\right) = 8$.
9. $O_8 = \{e_ie_je_ke_le_me_ne_oe_p | 1 \le i, j, k, l, m, n, o, p \le 8,$ distinct $i, j, k, l, m, n, o, p\}$ and $|O_8| = \left(\begin{array}{c} 8 \\ 8 \end{array}\right) = 1$.

Thus S_8 produces 9 orbits on V.

Set $M_1 = \langle e_1e_2e_3e_4e_5e_6e_7e_8\rangle$. Then M_1 is an S_8-invariant submodule of V with $dim(M_1) = 1$.

Now, set $M_2 = O_0 \cup O_2 \cup O_4 \cup O_6 \cup O_8$. Then $|M_2| = 128$, so we have $dim(M_2) = 7$.

Since $M_1 = O_0 \cup O_8$, we obtain that $V \supset M_2 \supset M_1 \supset 0$. This implies that M_2 is a reducible S_8-invariant submodule of V. ∎

Since S_8 is 8-transitive, A_8 is 6-transitive on $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$. It is clear that $O_0, O_1, O_2, O_4, O_5, O_6$ are also orbits under the action of A_8. Now since A_8 does not have a proper subgroup of index less than 8, O_7 remains as an orbit of length 8. Obviously O_8 also remains as an orbit of length 1.

Theorem 2.2 *Let $W = M_2/M_1$, then $dim(W) = 6$. Also W is a G-invariant module where $G = S_8$ or A_8.*

Proof. It is clear that $dim(W) = 6$, since $dim(M_1) = 1$ and $dim(M_2) = 7$. If $g \in G$ and $\alpha \in M_2$, then since M_2 is G invariant, $g(\alpha M_1) = g(\alpha)M_1 \in M_2/M_1$ $\forall\ g \in G$ and $\alpha \in M_2$. So W is S_8 (A_8) invariant. ■

Let
$$W = \langle e_1e_2M_1, e_1e_3M_1, e_1e_4M_1, e_1e_5M_1, e_1e_6M_1, e_1e_7M_1\rangle .$$
The set $B = \{e_1e_2, e_1e_3, e_1e_4, e_1e_5, e_1e_6, e_1e_7\}$ is a linearly independent set.

Let
$$\begin{aligned} \gamma_1 &= e_1e_2M_1, & \gamma_2 &= e_1e_3M_1,\\ \gamma_3 &= e_1e_4M_1, & \gamma_4 &= e_1e_5M_1,\\ \gamma_5 &= e_1e_6M_1, & \gamma_6 &= e_1e_7M_1. \end{aligned}$$

Also, if $\alpha = (1\ 2\ 3\ 4\ 5\ 6\ 7)$ and $\beta = (6\ 7\ 8)$, then
$$A_8 = \langle \alpha, \beta\rangle .$$

We obtain
$$\alpha : \gamma_1 \to \gamma_1\gamma_2,\ \gamma_2 \to \gamma_1\gamma_3,\ \gamma_3 \to \gamma_1\gamma_4,\ \gamma_4 \to \gamma_1\gamma_5,\ \gamma_5 \to \gamma_1\gamma_6,\ \text{and} \quad \gamma_6 \to \gamma_1.$$

We give two examples for the action of α. Under the action of α we have
$$\gamma_2 = e_1e_3M_1 \to e_2e_4M_1 = e_1e_2e_1e_4M_1 = \gamma_1\gamma_3$$

That is $\alpha(\gamma_2) = \gamma_1\gamma_3$. Also
$$\gamma_6 = e_1e_7M_1 \to e_2e_1M_1 = \gamma_1.$$

That is $\alpha(\gamma_6) = \gamma_1$. Hence α can be represented by the following matrix
$$\alpha = \begin{pmatrix} 1&1&0&0&0&0\\ 1&0&1&0&0&0\\ 1&0&0&1&0&0\\ 1&0&0&0&1&0\\ 1&0&0&0&0&1\\ 1&0&0&0&0&0 \end{pmatrix},$$
with $o(\alpha) = 7$.

Similarly, for β we have
$$\beta : \gamma_1 \to \gamma_1,\ \gamma_2 \to \gamma_2,\ \gamma_3 \to \gamma_3,\ \gamma_4 \to \gamma_4,\ \gamma_5 \to \gamma_6,\ \gamma_6 \to \gamma_1\gamma_2\gamma_3\gamma_4\gamma_5\gamma_6$$

As an example, we see that
$$\gamma_6 = e_1e_7M_1 \to e_1e_8M_1 = e_2e_3e_4e_5e_6e_7M_1 = \gamma_1\gamma_2\gamma_3\gamma_4\gamma_5\gamma_6.$$

That is $\beta(\gamma_6) = \gamma_1\gamma_2\gamma_3\gamma_4\gamma_5\gamma_6$. Here we obtain

$$\beta = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix},$$

with $o(\beta) = 3$. We are now able to write all the elements of A_8 as 6×6 matrices.

By acting A_8 directly on W, using the orbits of A_8 on M_2 and the fact that $M_1 = \{1, e_1e_2e_3e_4e_5e_6e_7e_8\}$, we can see that A_8 has 3 orbits namely

$$\begin{aligned} \Delta_0 &= \{O_0M_1\} = \{O_8M_1\} = \{M_1\}, \\ \Delta_1 &= \{O_2M_1\} = \{O_6M_1\} = \{e_ie_jM_1 \mid \text{distinct } \ e_i, e_j\}, \\ \Delta_2 &= \{O_4M_1\} = \{e_ie_je_ke_lM_1 | \text{distinct } \ e_i, e_j, e_k, e_l\}. \end{aligned}$$

Clearly, $|\Delta_0| = 1,\ |\Delta_1| = 28,\ \ |\Delta_2| = \frac{70}{2} = 35$ and $W = \Delta_0 \cup \Delta_1 \cup \Delta_2$.

Theorem 2.3 *A_8 acts irreducibly on W.*

Proof. Let $U \leq W$ be such that $U \neq 0$ and U is A_8 invariant. Since $U \neq 0$ $\exists\ x \in U$ such that $x \neq 0$. Since $x \neq 0$ and $U \leq W$ we have two cases.

Case 1. Suppose $x \in \Delta_1$, $x = e_ie_jM_1$ for distinct i, j. Hence $g(x) \in U \ \ \forall \ g \in A_8$. However

$$\{g(x)|g \in A_8\} = \Delta_1 \quad \Rightarrow \Delta_1 \subseteq U \quad \Rightarrow e_ie_jM_1 \in U \ \ \forall\ i, j.$$

Hence we have $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6 \in U$.

Case 2. Suppose $x \in \Delta_2$, then $x = e_ie_je_ke_lM_1$ for some distinct e_i, e_j, e_k, e_l. Hence $g(x) \in U \quad \forall \ \ g \in A_8$. Now

$$\{g(x)|g \in A_8\} = \Delta_2 \quad \Rightarrow \Delta_2 \subseteq U.$$

Since $\Delta_2 \subseteq U$, $e_ke_le_me_rM_1$ and $e_ke_le_me_iM_1$ are in U for distinct k, l, m, r, i. Since U is closed we get

$$(e_ke_le_me_rM_1)(e_ke_le_me_iM_1) = e_ie_r \in U \ \ \forall \ \text{ distinct } \ \ k, l, m, r, i.$$

This shows that $U \subseteq W$. So similar to case 1, we have $U = W$.

Hence W is a unique 6-dimensional $GF(2)$ module that A_8 acts irreducibly on. ∎

Our group $\bar{G}$ can now be realized by the split extension $W{:}A_8$, where $W \cong 2^6$. In here the multiplication is defined by

$$(w_1, g_1).(w_2, g_2) = (w_1(w_2)^{g_1}, g_1g_2)$$

for $w_1, w_2 \in W$ and $g_1, g_2 \in A_8$.

Remark 2.4 In V, the set of vectors of even weight, that is our M_2, forms a 7-dimensional S_8-submodule (this is true in general for S_n, these vectors form an $(n-1)$-dimensional S_n-submodule). Our module M_2 under the standard dot product on V becomes a symplectic space.

We can define an invariant quadratic form Q associated with this symplectic form as follows:

$$Q(\alpha) = \begin{cases} 0, & \alpha \in O_0 \cup O_4 \cup O_8 \\ \\ 1, & \alpha \in O_2 \cup O_6 \ . \end{cases}$$

The quotient space $W = M_2/M_1$ still admits an invariant quadratic form $\overline{Q}$ given by

$$\overline{Q}(\alpha) = \begin{cases} 0, & \alpha \in \Delta_0 \cup \Delta_2 \\ \\ 1, & \alpha \in \Delta_1 \ , \end{cases}$$

where $W = \Delta_0 \cup \Delta_1 \cup \Delta_2$. Hence $S_8 \leq GO_6^{\epsilon}(2)$ and since $|GO_6^-(2)| = 2^6 \times 3^4 \times 5$, we can easily deduce that $S_8 = GO_6^+(2)$ since these two groups have same order. Therefore

$$A_8 = O_6^+(2) \cong GL(4,2).$$

So our group $\overline{G} = 2^6{:}A_8$ can be identified as the extension of $O_6^+(2)$ by its natural module.

Since our group $\overline{G} = 2^6{:}A_8$ is an inertia factor group of $2^8{:}O_8^+(2)$, in the next section we aim to construct $\overline{G}$ as a subgroup of $O_8^+(2)$.

3. The Computational Method

In this section for all our computations we use GAP. We first construct $O_8^+(2)$ inside the general orthogonal group $GO_8^+(2)$. This we do by getting the maximal normal subgroup of $GO_8^+(2)$ and this is a group of 8×8 matrices of size 174182400 over $GF(2)$. We then construct $\overline{G} = 2^6{:}A_8$ inside $O_8^+(2)$ by first constructing an 8-dimensional row vector space U, over $GF(2)$. We then let $O_8^+(2)$ to act on U and we get three orbits of lengths $1, 120$ (non-isotropic points) and 135 (isotropic points). Using the ATLAS [4] and Programme C [18], the maximal subgroup of index 135 is $2^6{:}A_8$ which corresponds to the third orbit. We then get the stabilizer of a representative of this orbit in $O_8^+(2)$, which gives us a group of 8×8 matrices of size 1290240 which is our $2^6{:}A_8$. We are now ready to construct 2^6 and A_8 inside our $\overline{G}$. We use GAP for our computations.

The group $N = 2^6$ is the only proper normal subgroup of $\overline{G}$ and we use GAP to obtain this normal subgroup. We then obtain its generators, which are given below.

$$\gamma_1 = \begin{bmatrix} 1&0&0&0&0&0&0&1\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&1&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{bmatrix}, \quad \gamma_2 = \begin{bmatrix} 1&0&0&0&0&0&0&1\\ 0&1&0&0&0&0&0&1\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&1\\ 0&0&0&0&0&1&0&1\\ 1&1&0&0&1&1&1&0\\ 0&0&0&0&0&0&0&1 \end{bmatrix},$$

$$\gamma_3 = \begin{bmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&1\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&1\\ 1&0&0&0&1&0&1&0\\ 0&0&0&0&0&0&0&1 \end{bmatrix}, \quad \gamma_4 = \begin{bmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&1\\ 0&0&1&0&0&0&0&1\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&1\\ 0&0&0&0&0&1&0&0\\ 1&0&0&1&0&1&1&0\\ 0&0&0&0&0&0&0&1 \end{bmatrix},$$

$$\gamma_5 = \begin{bmatrix} 1&0&0&0&0&0&0&1\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&1\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&1\\ 0&0&0&0&0&1&0&0\\ 0&1&0&1&0&1&1&0\\ 0&0&0&0&0&0&0&1 \end{bmatrix}, \quad \gamma_6 = \begin{bmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&1\\ 0&0&0&1&0&0&0&1\\ 0&0&0&0&1&0&0&1\\ 0&0&0&0&0&1&0&1\\ 0&0&1&1&1&1&1&0\\ 0&0&0&0&0&0&0&1 \end{bmatrix}.$$

We then turn our attention to A_8. We first obtain the generators of $\overline{G}$. We use GAP [6] to get two generators, a and b, of A_8 . We give a, b and their inverses below. Note that $o(a) = o(b) = 4$.

$$a = \begin{bmatrix} 1&0&0&0&0&1&0&0\\ 0&1&0&1&0&1&0&1\\ 1&0&1&0&0&1&0&0\\ 0&0&0&1&0&0&0&0\\ 1&1&0&0&1&1&0&0\\ 0&0&0&0&0&1&0&0\\ 1&0&0&0&0&1&1&0\\ 0&0&0&0&0&0&0&1 \end{bmatrix}, \quad b = \begin{bmatrix} 0&0&0&0&0&1&0&0\\ 0&0&1&1&1&1&1&0\\ 1&0&0&1&0&0&0&1\\ 1&0&1&0&0&0&0&1\\ 1&1&0&0&0&1&1&1\\ 1&1&1&1&0&1&0&1\\ 0&1&0&0&0&0&1&0\\ 1&1&1&1&0&0&0&0 \end{bmatrix}$$

$$a^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad b^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

Noting that the generators of 2^6 and A_8 are both 8×8 matrices, thus we have $\bar{G} = \langle \gamma_i, a, b : 1 \leq i \leq 6 \rangle$, as a maximal subgroup of $O_8^+(2)$.

Computing the conjugate of each γ_i with respect to a, that is $a\gamma_i a^{-1}$ and noting that 2^6 is normal in $2^6{:}A_8$, we get that

$$a\gamma_i a^{-1} = \gamma_{j_1}\gamma_{j_2}\cdots\gamma_{j_k},$$

where $\gamma_{j_r} = \gamma_j$ or 1 for some $j_r = 1, ..., 6$. We denote this as

$$\gamma_i \longrightarrow \gamma_{j_1}\gamma_{j_2}\cdots\gamma_{j_k}.$$

We then get

$$\gamma_1 \longrightarrow \gamma_1,\ \gamma_2 \longrightarrow \gamma_2\gamma_5,\ \gamma_3 \longrightarrow \gamma_2\gamma_4\gamma_5\gamma_6,\ \gamma_4 \longrightarrow \gamma_4,\ \gamma_5 \longrightarrow \gamma_1\gamma_4\gamma_5,\ \gamma_6 \longrightarrow \gamma_1\gamma_2\gamma_3\gamma_4.$$

Similarly with b we get

$$\gamma_1 \longrightarrow \gamma_1\gamma_2\gamma_3\gamma_4\gamma_6,\ \gamma_2 \longrightarrow \gamma_2\gamma_3\gamma_4\gamma_5\gamma_6,\ \gamma_3 \longrightarrow \gamma_1\gamma_2\gamma_3\gamma_5\gamma_6,\ \gamma_4 \longrightarrow \gamma_1\gamma_3\gamma_4,$$

$$\gamma_5 \longrightarrow \gamma_4,\ \gamma_6 \longrightarrow \gamma_2\gamma_3\gamma_4.$$

Representing this information in matrix form, where the i-th row will correspond to the i-th conjugate, we get a 6×6 matrix representation of $G = A_8$. Hence we have $A_8 = \langle a', b' \rangle$, where

$$a' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}, \quad b' = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

By methods of coset analysis which can also be found in [10], [11], when $G = A_8$ acts on N we obtain three orbits of lengths $1, 28$ and 35 respectively. These have corresponding point stabilizers K_1, K_2 and K_3 of indices $1, 28$ and 35 respectively. One can immediately see that $K_1 = G$ and K_2, K_3 must each sit in a maximal subgroup of G. However any maximal subgroup of G which contains

K_i must have an order divisible by $|K_i|$ and its index in G must divide 28 and 35 respectively. From the ATLAS [4], we get that up to isomorphism and conjugacy there is only one maximal subgroup of G, in each case, that would contain K_2 and the other K_3 and these are the symmetric group S_6 and the group $2^4{:}(S_3 \times S_3)$ respectively. However, since $|K_2| = |S_6|$ we have $K_2 \cong S_6$. Similarly we have $K_3 \cong 2^4{:}(S_3 \times S_3)$. For each $g \in G$, the number of fixed points $g \in G$ in N is equal to $k = |C_N(g)|$. Since the zero vector of N is fixed by every $g \in G$ we have

$$k = 1 + \chi(G|K_2)(g) + \chi(G|K_3)(g) = 1 + (\chi(G|K_2) + \chi(G|K_3))(g).$$

From this, we determine $\chi = \chi(A_8|2^6)$, the permutation character of A_8 on 2^6. We have

$$\chi = 1a + I_{S_6}^{A_8} + I_{2^4:(S_3\times S_3)}^{A_8} = 3 \times 1a + 7a + 14a + 2 \times 20a,$$

where $I_{S_6}^{A_8} = 1a + 7a + 20a$ and $I_{2^4:(S_3\times S_3)}^{A_8} = 1a + 14a + 20a$ are the characters of A_8 induced from the identity characters of S_6 and $2^4{:}(S_3 \times S_3)$ respectively. Since $C_N(g) \leq N$, we must have $k = 2^n$ where $n \in \{1, 2, 3, 4, 5, 6\}$. Hence we obtain the values of the $k's$ in Table 1.

Table 1

$[g]_{A_8}$	1a	2a	2b	3a	3b	4a	4b	5a	6a	6b	7a	7b	15a	15b
$\chi(A_8\|K_2)$	28	4	8	10	1	0	2	3	1	2	0	0	0	0
$\chi(A_8\|K_3)$	35	11	7	5	2	3	1	0	2	1	0	0	0	0
k	64	16	16	16	4	4	4	4	4	4	1	1	1	1

Theorem 3.1 *A_8 acts irreducibly on N.*

Proof. As we seen above, the action of A_8 on N produces three orbits of lengths 1, 28 and 35. If H is an A_8-invariant subgroup of N, then H is a union of these orbits and hence $|H| = 1+28a+35b$, where $a, b \in \{0, 1\}$. Now since $H = 2^i$, where $0 \leq i \leq 6$, we must have $a = b = 0$ or $a = b = 1$. This implies that $H = \{1_N\}$ or $H = N$. ∎

Corollary 3.2 *The two groups constructed in Sections 2 and 3 are isomorphic.*

Proof. Since by [2] A_8 has a unique (up to isomorphism) modular representation of degree 2 over $GF(2)$, proof follows from Theorems 2.3 and 3.1. ∎

Since the two $2^6{:}A_8$ constructed are isomorphic, from now on, we use one of them in the rest of our discussion. So we consider $\bar{G} = N{:}G$, where $N \cong 2^6$ and $G \cong A_8$ as in this section (Section 3).

4. Actions of A_8 on N and $Irr(N)$

We determined the action of A_8 on N in Section 3. We notice that A_8 produces three orbits of lengths 1, 28 and 35 on N. Hence it produces three orbits on $Irr(N)$. Here we use Programme C [18] to act G on $Irr(N)$. To be able to do this we need to rewrite N as a row vector space V of dimension 6 over $GF(2)$, that is V:= FullRowSpace(GF(2),6). We have two procedures at our disposal. First we can act G on V from right and this action gives us the orbits of G acting as a permutation group on the conjugacy classes of N. Secondly, we act G^T, that is the set consists of transpose of elements of G, on V from right. This action is equivalent to multiplying the column vectors of V on the left by G. This action gives the orbits of G acting as a permutation group on the irreducible characters of N.

From the above, the action of G on $Irr(N)$ produces three orbits of lengths $1, 28$ and 35 respectively. We then take representatives of the orbits of lengths 28 and 35. For each of the orbit representative we find its stabilizer in G. For the representative of the orbit of length 28, the corresponding stabilizer, that is H_2, is a group of 6×6 matrices of size 720 isomorphic to S_6. For the orbit of length 35 the corresponding stabilizer, that is H_3, is a group of 6×6 matrices of size 576 isomorphic to $2^4 : (S_3 \times S_3)$.

Remark 4.1 Since $A_8 = O_6^+(2)$ acts naturally on $N = 2^6$, the three orbits mentioned above are the zero vector, non-isotropic points and isotropic points respectively.

5. The Conjugacy Classes of $2^6{:}A_8$

We first give the representatives of the conjugacy classes of A_8, in terms of 6×6 matrices in Table 2.

From the methods of coset analysis, which can also be found in [1],[10], [8], [11], [12], [17], [19] and by Programmes A and B [18], we are able to compute the conjugacy classes of $2^6{:}A_8$ which are given in Table 3. We give a very brief summary of coset analysis. We look at the action of $\overline{G}$ on $N\overline{g}$, for the split extension it is suffices to look at the coset $Ng, g \in G$. First N acts on Ng and we get k orbits. Then we act $C_G(g)$ on these orbits and f_j of these orbits, fuse to form one orbit with $\sum f_j = k$, and d_j a representative of these fused orbits. For this we use Programme A [18].

Table 2. Conjugacy classes of $H_1 = G = A_8$

$[g]_G$	6×6 matrix	$\lvert[g]_G\rvert$	$[g]_G$	6×6 matrix	$\lvert[g]_G\rvert$
1A	$\begin{pmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{pmatrix}$	1	2A	$\begin{pmatrix}0&0&0&1&0&0\\1&0&1&0&0&1\\0&1&0&1&0&1\\1&0&0&0&0&0\\0&1&1&1&1&1\\0&0&0&0&0&1\end{pmatrix}$	105
2B	$\begin{pmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&1\\1&0&1&0&1&0\end{pmatrix}$	210	3A	$\begin{pmatrix}0&1&1&1&1&0\\0&1&0&0&0&0\\1&1&0&0&0&0\\1&1&1&1&0&0\\0&0&0&1&0&0\\0&0&0&1&1&1\end{pmatrix}$	112
3B	$\begin{pmatrix}1&1&1&0&0&0\\1&0&1&1&0&1\\1&0&1&0&0&1\\0&1&1&1&0&0\\0&1&0&0&0&0\\0&0&1&1&1&1\end{pmatrix}$	1 120	4B	$\begin{pmatrix}0&0&1&1&1&1\\0&0&1&1&0&0\\0&0&0&0&1&0\\1&0&1&0&1&1\\0&1&0&1&0&1\\1&1&1&0&0&0\end{pmatrix}$	1 260
4A	$\begin{pmatrix}0&0&1&1&1&0\\0&0&1&1&0&0\\0&0&0&1&0&0\\1&0&1&1&0&1\\0&1&0&1&0&1\\1&1&1&1&1&1\end{pmatrix}$	2 520	5a	$\begin{pmatrix}0&1&0&1&0&1\\1&0&1&0&0&1\\0&0&0&1&0&0\\0&0&0&1&1&1\\0&0&0&0&0&1\\0&1&1&1&1&0\end{pmatrix}$	1 344
6A	$\begin{pmatrix}0&0&0&1&0&0\\1&0&1&0&0&1\\1&0&1&0&1&1\\0&1&1&1&1&0\\1&0&0&0&0&0\\1&1&1&1&1&1\end{pmatrix}$	1 680	6B	$\begin{pmatrix}0&0&1&1&1&1\\0&0&1&0&0&0\\1&0&1&1&0&1\\1&0&1&0&1&1\\0&1&1&1&1&0\\1&0&0&0&0&0\end{pmatrix}$	3 360
7A	$\begin{pmatrix}0&0&0&0&1&0\\0&0&1&1&0&0\\1&0&0&1&0&1\\1&0&1&1&0&1\\0&1&1&0&0&0\\1&0&0&0&0&0\end{pmatrix}$	2 880	7B	$\begin{pmatrix}1&0&1&0&0&1\\1&0&0&1&0&1\\1&1&0&0&0&0\\1&1&1&1&1&1\\0&0&0&1&1&1\\0&1&1&0&0&0\end{pmatrix}$	2 880
15A	$\begin{pmatrix}1&0&1&0&1&1\\1&0&1&0&0&1\\1&1&1&1&0&0\\1&1&1&1&1&1\\0&0&0&1&1&1\\0&1&1&0&0&0\end{pmatrix}$	1 344	15B	$\begin{pmatrix}0&0&1&1&1&0\\0&0&1&1&1&1\\1&1&1&1&1&1\\1&0&0&0&0&0\\0&1&1&0&0&0\\1&0&1&1&0&1\end{pmatrix}$	1 344

Table 3. Conjugacy classes of $\bar{G} = 2^6{:}A_8$

$g \in A_8$	k	f_j	d_j	w	$[x]_{2^6:A_8}$	$\lvert C_{2^6:A_8}(x)\rvert$
$1A$	2^6	1	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$1A$	1 290 240
		28	$(0,0,0,0,0,1)$	$(0,0,0,0,0,1)$	$2A$	46 080
		35	$(0,0,0,1,0,1)$	$(0,0,0,1,0,1)$	$2B$	36 864
$2A$	2^4	1	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$2C$	3 072
		1	$(0,0,1,0,0,1)$	$(0,0,0,0,0,0)$	2D	3 072
		1	$(0,0,1,1,1,1)$	$(0,0,0,0,0,0)$	2E	3 072
		1	$(0,0,1,1,1,1)$	$(0,0,0,0,0,0)$	2F	3 072
		12	$(0,0,0,0,0,1)$	$(1,1,1,0,0,1)$	4A	256
$2B$	2^4	1	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$2G$	1 536
		1	$(0,0,1,1,1,1)$	$(0,0,1,0,0,0)$	4B	1 536
		3	$(1,0,1,0,1,0)$	$(0,0,0,0,0,0)$	2H	512
		3	$(0,0,1,1,0,1)$	$(0,0,1,0,1,0)$	4C	512
		8	$(0,0,0,0,0,1)$	$(0,0,0,1,1,0)$	4D	192
$3A$	2^4	1	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$3A$	2 880
		5	$(1,1,0,0,1,0)$	$(0,0,1,1,1,1)$	6A	576
		10	$(0,1,0,0,1,1)$	$(0,0,1,1,0,0)$	6B	288
$3B$	2^2	1	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$3B$	72
		1	$(1,1,0,1,0,0)$	$(1,0,0,0,0,1)$	$6B$	72
		1	$(1,0,1,0,1,0)$	$(1,0,1,0,1,0)$	$6C$	72
		1	$(0,0,0,1,0,0)$	$(1,1,1,1,1,1)$	$6D$	72
$4A$	2^2	1	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$4E$	64
		1	$(1,1,0,1,0,0)$	$(0,0,0,0,0,0)$	$4F$	64
		1	$(1,0,1,0,1,0)$	$(0,0,0,0,0,0)$	$4G$	64
		1	$(0,0,0,1,0,0)$	$(0,0,0,0,0,0)$	$4H$	64
$4B$	2^2	1	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$4I$	32
		1	$(1,1,0,1,0,0)$	$(0,0,1,1,1,1)$	$8A$	32
		1	$(1,0,1,0,1,0)$	$(0,0,0,0,0,0)$	$4J$	32
		1	$(0,0,0,1,0,0)$	$(1,1,1,0,1,0)$	$8B$	32
$5A$	2^2	1	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$5A$	60
		3	$(0,1,0,1,0,0)$	$(0,0,0,1,0,1)$	$10A$	20
$6A$	2^2	1	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$6F$	48
		1	$(1,0,1,0,1,0)$	$(0,1,1,0,1,0)$	$12A$	48
		2	$(1,0,1,0,1,0)$	$(1,0,1,0,1,0)$	$12B$	24
$6B$	2^2	1	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$6G$	24
		1	$(0,0,0,0,0,0)$	$(0,0,0,0,0,1)$	$6H$	24
		1	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$6I$	24
		1	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$6J$	24
$7A$	1	1	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$7A$	7
7B	1	1	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$7B$	7
15A	1	1	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$15A$	15
15B	1	1	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$15B$	15

Let $o(dg) = k$ and $o(g) = m$. If $w = (dg)^m$, then if $w = (0,0,0,0,0,0)$, $k = m$. On the other hand if $w \neq (0,0,0,0,0,0)$, then $k = 2m$. To get w we use Programme B [18]. We obtain that $2^6{:}A_8$ has altogether 41 conjugacy classes which are given in Table 3 above.

Remark 5.1 The group $O_8^+(2)$ has three conjugacy classes of maximal subgroups isomorphic to $2^6{:}A_8$. Our group $\overline{G}$, which is the stabilizer of a non-isotropic point in $O_8^+(2)$, is first of these groups as listed in the ATLAS. Table 3 shows that $N = [1A] \cup [2A]_{\overline{G}} \cup [2B]_{\overline{G}}$, with $|[2A]_{\overline{G}}| = 28$ and $|[2B]_{\overline{G}}| = 35$. The fusion of $\overline{G}$ into $O_8^+(2)$ implies $[2A]_{\overline{G}} \to [2B]_{O_8^+(2)}$ and $[2B]_{\overline{G}} \to [2A]_{O_8^+(2)}$. This confirms that $\overline{G} = N_{O_8^+(2)}(2A_{35}B_{28})$ as it is stated in the ATLAS.

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CONVERGENCE OF LAGRANGE-HERMITE INTERPOLATION

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Abstract. In this paper, we consider explicit representations and convergence of Lagrange–Hermite Interpolation on two disjoint set of nodes, which are obtained by projecting vertically the zeros of $\left(1-x^2\right) P_n^{(\alpha,\beta)}(x)$ and $\left(1-x^2\right) P_n^{(\alpha,\beta)\prime}(x)$ respectively on the unit circle, where $P_n^{(\alpha,\beta)}(x)$ stands for Jacobi polynomials.

Keywords: Jacobi polynomial, Lagrange interpolation, Hermite interpolation, explicit representation, convergence.

2000 Mathematics Subject Classification: 41A05, 30E10.

1. Introduction

In 1975, L.G. Pál [10] introduced a new type of interpolation on the zeros of two different polynomials. He considered two systems of real numbers $\{x_n\}_{k=0}^{n}$ and $\{x_n^*\}_{k=0}^{n-1}$, which are the zeros of $W_n(x)$ and $W_n'(x)$ respectively, then there exists a unique polynomials $P(x)$ of degree at most $2n-1$ satisfying the interpolatory properties:

$$\begin{aligned} P(x_k) &= y_k, \quad k = 1\,(1)\,n, \\ P'(x_k^*) &= y_k, \quad k = 1\,(1)\,n-1, \end{aligned}$$

and gave the explicit formulae of this polynomial. In another paper, L.G. Pál [11] considered the $(0;0,1)$-Interpolation and obtained the convergence for the same. In 2003, H.P. Dikshit [7] also considered the Pál-type interpolation on non-uniformly distributed nodes on the unit circle. Later on P. Mathur [9] considered $(0,1;0)$ interpolation on infinite interval. P. Mathur and his associates [13] considered $(0;0,1)$ interpolation.

In a paper, S. Xie [15] considered the regularity of $(0, 1, ..., r-2, r)$ and $(0, 1, ..., r-2, r)^*$ −interpolation on the sets obtained by projecting vertically the zeros of $(1-x^2)\, P_n^{(\alpha,\beta)}(x)$ on the unit circle, where $P_n^{(\alpha,\beta)}(x)$ stands for the Jacobi polynomials. After that S. Bahadur [3, 5] considered $(0,1;0)$ and $(0;0,1)$ interpolation on the unit circle and establish the convergence theorem for the same. In another paper, the authors [6] have considered Hermite–Lagrange interpolation and established a convergence theorem on the unit circle.

In this paper, we consider Lagrange–Hermite interpolation on the unit circle. Here, we consider two pairwise disjoint sets $\{z_k\}_{k=0}^{2n+1}$ and $\{t_k\}_{k=0}^{2n-1}$, which are the vertically projected zeros of $(1-x^2)\, P_n^{(\alpha,\beta)}(x)$ and $(1-x^2)\, P_n^{(\alpha,\beta)\prime}(x)$ on the unit circle, respectively.

Let Z_n and T_n be two sets satisfying:

$$Z_n = \begin{cases} z_k : k = 0(1)2n+1 : \\ z_0 = 1,\ z_{2n+1} = -1 \\ z_k = \cos\theta_k + i\sin\theta_k,\ z_{n+k} = -z_k,\ k = 1\,(1)\,n \end{cases}$$

(1.1) and

$$T_n = \begin{cases} t_k : k = 0(1)2n-1 : \\ t_0 = 1,\ t_{2n-1} = -1 \\ t_k = \cos\phi_k + i\sin\phi_k,\ t_{n+k} = -t_k, k = 1\,(1)\,n-1 \end{cases},$$

which are the zeros of $(1-x^2)\, P_n^{(\alpha,\beta)}(x)$ and $P_n^{(\alpha,\beta)\prime}(x)$, respectively.

In Section 2, we give some preliminaries and in Section 3, we describe the problem and obtained the regularity of the same. In Section 4, we give the explicit formulae of the interpolatory polynomials. In Sections 5 and 6, estimation of interpolatory polynomials and convergence are given, respectively.

2. Preliminaries

In this section, we shall give some well known results, which we shall use.

The differential equation satisfied by $P_n^{(\alpha,\beta)}(x)$ is:

$$\begin{aligned}(2.1)\qquad &\left(1-x^2\right) P_n^{(\alpha,\beta)\prime\prime}(x) + \left[\beta-\alpha-(\alpha+\beta+2)\,x\right] P_n^{(\alpha,\beta)\prime}(x) \\ &\qquad + n\,(n+\alpha+\beta+1)\, P_n^{(\alpha,\beta)}(x) = 0\end{aligned}$$

$$(2.2)\qquad W(z) = \prod_{k=1}^{2n}(z-z_k) = K_n P_n^{(\alpha,\beta)}\left(\frac{1+z^2}{2z}\right) z^n$$

$$(2.3)\qquad H(z) = \prod_{k=1}^{2n-2}(z-t_k) = K_n^* P_n^{(\alpha,\beta)\prime}\left(\frac{1+z^2}{2z}\right) z^{n-1}.$$

We shall require the fundamental polynomials of Lagrange interpolation based on Z_n and T_n

$$L_k\ (z) = \frac{R(z)}{R'\,(z_k)\,(z - z_k)},\ k = 0(1)2n+1, \tag{2.4}$$

where $R(z) = (z^2 - 1)\,W(z)$,

$$l_k\ (z) = \frac{H(z)}{H'\,(t_k)\,(z - t_k)},\ k\ = 1(1)2n-2. \tag{2.5}$$

We will also use the following results

$$(-1)^n\,W'\,(z_{n+k})\ = W'\,(z_k) = -\frac{1}{2}K_n P_n^{(\alpha,\beta)\prime}\,(x_k)\left(1 - z_k^2\right) z_k^{n-2}, \qquad k = 1\,(1)\,n. \tag{2.6}$$

We will also use the following well known inequalities (see [8])

$$(1-x^2)^{\frac{1}{2}}\,P_n^{(\alpha,\beta)}\,(x) = o\left(n^{\alpha-1}\right), \tag{2.7}$$

$$(1-x_k^2)^{-1} \sim \left(\frac{k}{n}\right)^{-2} \tag{2.8}$$

$$\left|\ P_n^{(\alpha,\beta)\prime}\,(x_k)\right| \sim k^{-\alpha-\frac{3}{2}}\ n^{\alpha+2} \tag{2.9}$$

$$\left|P_n^{(\alpha,\beta)}\,(x)\right| = o\left(n^{\alpha}\right), \tag{2.10}$$

$$(1-x^2)\left|P_n^{(\alpha,\beta)\prime}\,(x)\right| \le cn^{\alpha+1}, \tag{2.11}$$

$$\left|\ P_n^{(\alpha,\beta)}\,(x_k)\right| \sim\ k^{-\alpha-\frac{1}{2}}\ n^{\alpha}, \tag{2.12}$$

$$\left|\ P_n^{(\alpha,\beta)\prime}\,(x)\right| \sim o\left(n^{\alpha+2}\right). \tag{2.13}$$

3. The problem

Let $\{z_k\}_{k=0}^{2n+1}$ and $\{t_k\}_{k=0}^{2n-1}$ be the two disjoint sets of nodes obtained by projecting vertically the zeros of $(1-x^2)\,P_n^{(\alpha,\beta)}\,(x)$ and $(1-x^2)P_n^{(\alpha,\beta)\prime}\,(x)$ on the unit circle respectively, we seek to determine the interpolatory polynomial $R_n\,(z)$ of degree $\le 6n-1$ satisfying the conditions:

$$\begin{cases} R_n\,(z_k) = \alpha_k, & k = 0\,(1)\,2n+1, \\ R_n\,(t_k) = \beta_k, & k = 1\,(1)\,2n-2, \\ R_n'\,(t_k) = \gamma_k, & k = 0\,(1)\,2n-1, \end{cases} \tag{3.1}$$

where α_k, β_k and γ_k are arbitrary complex numbers. We are also interested in establishing a convergence theorem for the same.

Regularity

Theorem 1. *The Lagrange–Hermite interpolation is regular on* Z_n *and* T_n.

Proof. It is sufficient, if we show the unique solution of (3.1) is $R_n(z) \equiv 0$, when all data $\alpha_k = \beta_k = \gamma_k = 0$. Clearly, in this case we have $R_n(z) = R(z)\, H(z)\, q(z)$, where $q(z)$ is a polynomial of degree $\leq 2n - 1$.

Let $R_n(z) = R(z)\, H(z)\, q(z)$. Obviously, $R_n(z_k) = 0$ and $R_n(t_k) = 0$. Then, from $R_n'(t_k) \equiv 0$, we have $q(t_k) = 0$. Therefore, $q(z) = (az + b)\, H(z)$, where a and b are arbitrary constants.

As $q(\pm 1) = 0$, we get $a = b = 0$. It indicates

$$R_n(z) \equiv q(z) \equiv 0.$$

Hence the theorem follows.

4. Explicit representation of interpolatory polynomials

We shall write $R_n(z)$ satisfying (3.1) as

$$R_n(z) = \sum_{k=0}^{2n+1} \alpha_k A_k(z) + \sum_{k=1}^{2n-2} \beta_k B_k(z) + \sum_{k=0}^{2n-1} \gamma_k C_k(z), \tag{4.1}$$

where $A_k(z)$, $B_k(z)$ and $C_k(z)$ are unique polynomials, each of degree atmost $6n - 1$ satisfying the following conditions:

For $k = 0\,(1)\,2n + 1$

$$\begin{cases} A_k(z_j) = \delta_{jk}; & j = 0\,(1)\,2n+1, \\ A_k(t_j) = 0; & j = 1\,(1)\,2n-2, \\ A_k'(t_j) = 0; & j = 0\,(1)\,2n-1. \end{cases} \tag{4.2}$$

For $k = 1\,(1)\,2n - 2$

$$\begin{cases} B_k(z_j) = 0; & j = 0\,(1)\,2n+1, \\ B_k(t_j) = \delta_{jk}; & j = 1\,(1)\,2n-2, \\ B_k'(t_j) = 0; & j = 0\,(1)\,2n-1. \end{cases} \tag{4.3}$$

For $k = 0\,(1)\,2n - 1$

$$\begin{cases} C_k(z_j) = 0; & j = 0\,(1)\,2n+1, \\ C_k(t_j) = 0; & j = 1\,(1)\,2n-2 \\ C_k'(t_j) = \delta_{jk}; & j = 0\,(1)\,2n-1 \end{cases} \tag{4.4}$$

Theorem 2. *For* $k = 1(1)2n-2$, *we have,*

$$C_k(z) = \frac{(z^2-1)\,R(z)\,H(z)\,l_k(z)}{(t_k^2-1)\,R(t_k)\,H'(t_k)}. \tag{4.5}$$

For $k = 0, 2n-1$

$$C_k(z) = -\frac{(1+t_k z)\,H^2(z)\,R(z)}{2H^2(t_k)\,R'(t_k)}. \tag{4.6}$$

Theorem 3. *For* $k = 1(1)2n-2$, *we have,*

$$B_k(z) = \frac{(z^2-1)\,R(z)\,l_k^2(z)}{(t_k^2-1)\,R(t_k)} - \left\{\frac{2t_k}{(t_k^2-1)} + \frac{R'(t_k)}{R(t_k)} + \frac{H''(t_k)}{H'(t_k)}\right\} C_k(z), \tag{4.7}$$

where $C_k(z)$ *is given by* (4.5).

Theorem 4. *For* $k = 0(1)2n+1$, *we have,*

$$A_k(z) = \frac{(z^2-1)\,H^2(z)\,L_k(z)}{(z_k^2-1)\,H^2(z_k)}. \tag{4.8}$$

One can establish Theorems 2, 3 and 4 owing to conditions (4.2), (4.3) and (4.4), respectively.

5. Estimation of fundamental polynomials

Lemma 1. [2] *Let* $L_k(z)$ *be given by* (2.4). *Then*

$$\sum_{k=0}^{2n+1} |\, L_k(z)\, | \le c \sum_{k=0}^{2n+1} \frac{1}{k^{-\alpha+\frac{3}{2}}}, \tag{5.1}$$

where c *is a constant independent of* n *and* z.

Lemma 2. [6] *Let* $l_k(z)$ *be given by* (2.5). *Then*

$$\sum_{k=1}^{2n-2} |\, l_k(z)\, | \le c \sum_{k=1}^{2n-2} \frac{1}{k^{-\alpha-\frac{1}{2}}}, \tag{5.2}$$

where c *is a constant independent of* n *and* z.

Lemma 3. *Let* $C_k(z)$ *be given by* (4.5),*we have*

$$\sum_{k=0}^{2n-1} |C_k(z)| \le c \log n, \qquad -1 < \alpha \le -\frac{1}{2},\ |z| \le 1 \tag{5.3}$$

and c *is a constant independent of* n *and* z.

Proof. From (4.5) and (4.6) using (2.8) , (2.10) , (2.12) and Lemma 2, we get the required result.

Lemma 4. *Let* $B_k(z)$ *be given by* (4.7)*, we have*

$$\sum_{k=1}^{2n-2} |B_k(z)| \leq cn \log n, \quad -1 < \alpha \leq -\frac{1}{2}, \ |z| \leq 1 \tag{5.4}$$

where c *is a constant independent of* n *and* z*.*

Proof. From (4.6), using (2.7) , (2.8) , (2.12) and Lemmas 2 and 3, we get the required result.

Lemma 5. *Let* $A_k(z)$ *be given by* (4.8)*, we have*

$$\sum_{k=0}^{2n+1} |A_k(z)| \leq cn \log n, \ -1 < \alpha \leq -\frac{1}{2}, \ |z| \leq 1, \tag{5.7}$$

where c *is a constant independent of* n *and* z*.*

Proof. Proof is similar to Lemma 3.

6. Convergence

Let $f(z)$ be analytic for $|z| < 1$ and continuous for $|z| \leq 1$ and $\omega(f, \delta)$ be the modulus of continuity of $f(e^{ix})$.

Theorem 5. *Let* $f(z)$ *be continuous in* $|z| \leq 1$ *and analytic in* $|z| < 1$*. Let the arbitrary numbers* β_k*'s and* γ_k'*s be such that:*

$$\begin{cases} |\beta_k| = o\left(\omega_2(f, n^{-1})\right), & k = 1\,(1)\,2n-2, \\ |\gamma_k| \ = o\left(n\omega_2(f, n^{-1})\right), & k = 1\,(1)\,2n-2. \end{cases} \tag{6.1}$$

Then, $\{R_n\}$ *defined by*

$$R_n(z) = \sum_{k=0}^{2n+1} f(z_k) A_k(z) + \sum_{k=1}^{2n-2} \beta_k B_k(z) + \sum_{k=0}^{2n-1} \gamma_k C_k(z) \tag{6.2}$$

satisfies the relation

$$|R_n(z) - f(z)| = o\left(n\omega_2(f, n^{-1}) \log n\right) \ \text{for} \ -1 < \alpha \leq -\frac{1}{2} \tag{6.3}$$

where $\omega_2(f, n^{-1})$ *is the modulus of continuity of* $f(z)$ *.*

Remark. To prove Theorem 5, we shall need the following:

Let $f(z)$ be continuous in $|z| \leq 1$ and $f' \in Lip\,\nu$, $\nu > 0$. Then the sequence $\{R_n\}$ converges uniformly to $f(z)$ in $|z| \leq 1$, follows from (6.3) provided

$$\omega_2(f, n^{-1}) = o\left(n^{-1-\nu}\right). \tag{6.4}$$

Let $f(z)$ be continuous in $|z| \leq 1$ and analytic in $|z| < 1$. Then there exists a polynomial $F_n(z)$ of degree $6n-1$ satisfying Jackson's inequality

$$|F_n(z) - f(z)| \leq c\omega_2\left(f, n^{-1}\right), \quad z = e^{i\theta}\,(0 < \theta \leq 2\pi) \tag{6.5}$$

and also an inequality due to O. Kiš [8]

$$\left|F_n^{(m)}(z)\right| \leq cn^m \omega_2\left(f, n^{-1}\right), \quad \text{for } m \in I^+. \tag{6.6}$$

Proof. Since $R_n(z)$ be given by (6.2) is a uniquely determined polynomial of degree $\leq 6n-1$, the polynomial $F_n(z)$ satisfying (6.5) and (6.6) can be expressed as

$$R_n(z) = \sum_{k=0}^{2n+1} F_n(z_k) A_k(z) + \sum_{k=1}^{2n-2} F_n(t_k) B_k(z) + \sum_{k=0}^{2n-1} F_n'(t_k) C_k(z).$$

Then,

$$\begin{aligned} |R_n(z) - f(z)| &\leq |R_n(z) - F_n(z)| + |F_n(z) - f(z)| \\ &\leq \sum_{k=0}^{2n+1} |f(z_k) - F_n(z_k)|\,|A_k(z)| + \sum_{k=1}^{2n-2} \{|\beta_k| + |F_n(t_k)|\}\,|B_k(z)| \\ &+ \sum_{k=0}^{2n-1} \{|\gamma_k| + |F_n'(t_k)|\}\,|C_k(z)| + |F_n(z) - f(z)|. \end{aligned}$$

Using $z = e^{i\theta}$ $(0 < \theta \leq 2\pi)$, (6.1), (6.4), (6.5), (6.6) and Lemmas 3, 4 and 5, we get (6.3).

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ASYMMETRIC CLOPEN SETS IN THE BITOPOLOGICAL SPACES

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Abstract. In the paper the behavior of clopen sets in bitopological spaces and some properties of generalized objects (e.g., (i, j)-quasi components and (i, j)-clopen compact subsets) are investigated. By using asymmetric clopen sets we introduce new classes of (i, j)-clopen irresolute and (i, j)-weakly clopen-continuous maps. Also, some their relations to p-ultra-Hausdorff bitopological structures are established. Characterizations and a preserving theorem of pairwise connected spaces are obtained.

Keywords: bitopological space, clopen, zero dimension, quasi-component.

2010 Mathematics Subject Classification: Primary 54E55.

1. Introduction

The clopen sets (i.e., sets that are both closed and open) play an important role in characterizations of the objects which define fundamental constructions of classical topology (see, e.g., [4], [6], [7], [10] etc.). It is well known that such sets are actually used in mathematical analysis, logics and theoretical computer sciences. In bitopological spaces, considerations of so called $(1, 2)$-clopen and $(2, 1)$-clopen sets seem to be not applied widely, although there are few interesting articles in this direction (see, e.g., [1], [12]). Motivated by this gap in the bitopological case we try to develop some theoretical constructions for asymmetric quasi-components, ultra-Hausdorff separation and continuous-like mappings by using asymmetric clopens. We obtain new characterizations and a preservation theorem of pairwise connected spaces due to Pervin [11].

Throughout the paper, for a bitopological space (X, τ_1, τ_2) we use the following notations: the interior and the closure of a subset A of X with respect to the topology τ_i are denoted by $\tau_i intA$ and $\tau_i clA$, respectively, where $i \in \{1, 2\}$. If O is open in τ_i, then we write $O \in \tau_i$, while, for the τ_i-closed set F, we use the notation $F \in co\tau_i$ (in this case, for brevity, O and F are meant also as an i-open and an i-closed set, respectively). We denote by $\tau_i^A = \{A \cap U | U \in \tau_i\}$ the topology induced on the set A from the τ_i. Next, in several results, we apply few important notions on bitopological structures, which are completely concerned in [5], but for classical topological ones see, e.g., [7]. The family of all τ_i-open neighborhoods of a subset M of X is denoted by $\sum_i^X(M)$. The bitopological space (X, τ_1, τ_2) is briefly denoted by BS (X, τ_1, τ_2).

2. (i, j)-Clopen sets

Definition 2.1. A subset A of a BS (X, τ_1, τ_2) is called an (i, j)-clopen set if $A \in \tau_i \cap co\tau_j$, where $i, j \in \{1, 2\}, i \neq j$.

Below the class of all (i, j)-clopen subsets of (X, τ_1, τ_2) will be denoted by $(i, j) - Clp(X)$. If $i = j$, we get the well known notion of general topology –the clopen set. Therefore, the class of $i - Clp(X)$ will denote the collection of all τ_i-clopen subsets of (X, τ_1, τ_2).

The following three propositions might be easily verified and we omit the proofs.

Proposition 2.1. *Let A and B be subsets of a BS (X, τ_1, τ_2).*

(1) *$A \in (i, j) - Clp(X)$ if and only if $X \setminus A \in (j, i) - Clp(X)$.*

(2) *If $A \in (i, j) - Clp(X)$ and $B \in (j, i) - Clp(X)$, then $A \setminus B \in (i, j) - Clp(X)$.*

(3) *The following equation holds:*

$$(1, 2) - Clp(X) \cap (2, 1) - Clp(X) = 1 - Clp(X) \cap 2 - Clp(X).$$

Proposition 2.2. *Let $A_\alpha \in (i, j) - Clp(X)$ for each $\alpha \in \Lambda$, $A = \bigcap_{\alpha \in \Lambda} A_\alpha$ and $B = \bigcup_{\alpha \in \Lambda} A_\alpha$. Then the following hold:*

(1) *$A \in co\tau_j$ and $B \in \tau_i$,*

(2) *$A, B \in (i, j) - Clp(X)$ if Λ is finite,*

(3) *$A \in (i, j) - Clp(X)$ (resp. $B \in (i, j) - Clp(X)$) if (X, τ_i) (resp. (X, τ_j)) is an Alexandorff space.*

Proposition 2.3. *If A is a subset of a BS (X, τ_1, τ_2) and $B \in (i,j) - Clp(X)$, then $A \cap B$ is (i,j)-clopen in the subspace (A, τ_1^A, τ_2^A).*

In [3], Dochviri introduced the notion of p-open (resp. p-closed) sets to obtain various characterizations of bitopological objects. Recall that a nonempty set A of a BS (X, τ_1, τ_2) is said to be p-open (resp. p-closed) if there exist $G_1 \in \tau_1$ and $G_2 \in \tau_2$ (resp. $F_1 \in co\tau_1$ and $F_2 \in co\tau_2$) such that $A = G_1 \cap G_2$ (resp. $A = F_1 \cup F_2$). The classes of p-open and p-closed sets of a given BS (X, τ_1, τ_2) are denoted by $p - O(X)$ and $p - C(X)$, respectively. The conjugate classes of sets were introduced in [2]. According to [2], a set A of a BS (X, τ_1, τ_2) is said to be p-quasi-open (resp. p-quasi-closed) if there exist $G_1 \in \tau_1$ and $G_2 \in \tau_2$ (resp. $F_1 \in co\tau_1$ and $F_2 \in co\tau_2$) such that $A = G_1 \cup G_2$ (resp. $A = F_1 \cap F_2$). The classes of p-quasi-open and p-quasi closed sets are denoted by $p - qO(X)$ and $p - qC(X)$, respectively. It is obvious that the complement of a p-open (resp. p-quasi-open) set is p-closed (resp. p-quasi-closed), and vice versa. By applying the above mentioned classes of sets we conclude: if $A \in (i,j) - Clp(X)$ and $B \in (j,i) - Clp(X)$ then $A \cap B \in p - O(X) \cap p - qC(X)$ and $A \cup B \in p - C(X) \cap p - qO(X)$.

Definition 2.2. A map $f : (X, \tau_1, \tau_2) \to (Y, \gamma_1, \gamma_2)$ is said to be

(1) i-open (resp. i-continuous) if $f : (X, \tau_i) \to (Y, \gamma_i)$ is an open (resp. continuous) map.

(2) j-closed if $f : (X, \tau_j) \to (Y, \gamma_j)$ is a closed map.

(3) p-continuous if both $f : (X, \tau_1) \to (Y, \gamma_1)$ and $f : (X, \tau_2) \to (Y, \gamma_2)$ are continuous [8].

(4) p-homeomorphism if f is bijective and both f and f^{-1} are p-continuous, where f^{-1} denotes the inverse to f.

The proof of the following proposition is obvious.

Proposition 2.4. *If a map $f : (X, \tau_1, \tau_2) \to (Y, \gamma_1, \gamma_2)$ is i-open and j-closed and $A \in (i,j) - Clp(X)$, then $f(A) \in (i,j) - Clp(Y)$.*

Below, we obtain another conditions under which (i,j)-clopen sets are preserved.

Definition 2.3. A BS (X, τ_1, τ_2) is said to be (i,j)-stable [Ko] if any $A \in co\tau_i$ implies j-compactenss of A.

Note that if (X, τ_1, τ_2) is (j,i)-stable and i-Hausdorff then $(i,j) - Clp(X) \subset i - Clp(X)$.

Definition 2.4. A map $f : (X, \tau_1, \tau_2) \to (Y, \gamma_1, \gamma_2)$ is said to be $(i,j) - \Delta$ continuous if $f : (X, \tau_i) \to (Y, \gamma_j)$ is continuous.

Proposition 2.5. *Let (X, τ_1, τ_2) be a (j,i)-stable BS and a map $f : (X, \tau_1, \tau_2) \to (Y, \gamma_1, \gamma_2)$ be both i-open and $(i,j)-\Delta$-continuous. If a BS (Y, γ_1, γ_2) is j-T_2, then $f(A) \in (i,j) - Clp(Y)$ for each $A \in (i,j) - Clp(X)$.*

Proof. Note that, in the (j,i)-stable BS (X, τ_1, τ_2), for each $A \in (i,j) - Clp(X)$ A is a τ_i-compact subset. By $(i,j) - \Delta$-continuity of f, $f(A)$ is a τ_j-compact subset of (Y, γ_1, γ_2). Hence $f(A) \in co\gamma_j$ and combining this fact with the i-openness of f the proof is done. ■

3. p-Connected spaces

Definition 3.1. A BS (X, τ_1, τ_2) is said to be pairwise connected (briefly p-connected) if X could not be represented as the union of the disjoint sets $A \in \tau_1 \setminus \{\emptyset\}$ and $B \in \tau_2 \setminus \{\emptyset\}$ [11].

In another case (X, τ_1, τ_2) is called a p-disconnected BS.

Theorem 3.1. *For a BS (X, τ_1, τ_2), the following properties are equivalent:*

(1) *(X, τ_1, τ_2) is p-connected;*

(2) *X cannot be represented as the union of nonempty disjoint $A \in (i,j) - Clp(X)$ and $B \in (j,i) - Clp(X)$;*

(3) *There exists no nonempty proper (i,j)-clopen set.*

Proof. (1)⇒(2): Suppose that there exist $A \in (i,j) - Clp(X)$, $B \in (j,i) - Clp(X)$ such that $\emptyset \neq A$, $\emptyset \neq B$, $A \cap B = \emptyset$ and $A \cup B = X$. Then $\emptyset \neq A \in \tau_i$, $\emptyset \neq B \in \tau_j$, $A \cap B = \emptyset$ and $A \cup B = X$. This shows that (X, τ_1, τ_2) is not p-connected.

(2)⇒(3): Suppose that there exists $A \in (i,j) - Clp(X)$ such that $\emptyset \neq A \subset X$ and $A \neq X$. Then, by Proposition 2.1 $X \setminus A \in (j,i) - Clp(X)$. Therefore, X is represented as the union of nonempty disjoint sets $A \in (i,j) - Clp(X)$ and $X \setminus A \in (j,i) - Clp(X)$. This contradicts (2).

(3)⇒(1): Suppose that X is not p-connected. Then there exist $A \in \tau_1$ and $B \in \tau_2$ such that $\emptyset \neq A$, $\emptyset \neq B$, $A \cap B = \emptyset$ and $A \cup B = X$. Therefore, there exists a nonempty proper set A such that $A \in \tau_1 \cap co\tau_2$. This contradicts (3). ■

Definition 3.2. A map $f : (X, \tau_1, \tau_2) \to (Y, \gamma_1, \gamma_2)$ is said to be (i,j)-clopen-irresolute if $f^{-1}(V) \in (i,j) - Clp(X)$ for each $V \in (i,j) - Clp(Y)$, where $i \neq j$, $i,j \in \{1,2\}$.

If a map is both $(1,2)$-clopen-irresolute and $(2,1)$-clopen-irresolute then it is called to be p-clopen-irresolute. Every p-continuous map is p-clopen-irresolute but the converse is not always true as shown by the following example.

Example 3.1. Let us consider a set $X = \{m, n, p, q, k\}$, together with topologies $\tau_1 = \{\emptyset, X\} \cup \{\{m\}, \{n, p\}, \{m, n, p\}\}$ and $\tau_2 = \{\emptyset, X\} \cup \{\{q, k\}\}$. Then, we observe that $(1,2) - Clp(X) = \{\emptyset, X, \{m, n, p\}\}$ and $(2,1) - Clp(X) = \tau_2$. Moreover, let $Y = \{a, b, c, d\}$ be endowed with the following topologies $\gamma_1 = \{\emptyset, Y\} \cup \{\{a, b\}, \{c\}, \{a, b, c\}\}$ and $\gamma_2 = \{\emptyset, Y\} \cup \{\{c, d\}\}$, then $(1,2) - Clp(Y) = \{\emptyset, Y, \{a, b\}\}$ and $(2,1) - Clp(Y) = \gamma_2$. If we define a map $f : (X, \tau_1, \tau_2) \to (Y, \gamma_1, \gamma_2)$ via the equations: $f(m) = f(n) = a, f(p) = b, f(q) = c, f(k) = d$, then it is p-clopen-irresolute. But $f : (X, \tau_1) \to (Y, \gamma_1)$ is not continuous and f is not p-continuous.

It is known that the p-connectedness is preserved under p-continuous surjections [11]. The following proposition is an improvement of this result.

Proposition 3.1. *The p-connectedness is preserved by (i,j)-clopen-irresolute surjections.*

Proof. Let $f : (X, \tau_1, \tau_2) \to (Y, \gamma_1, \gamma_2)$ be an (i,j)-clopen-irresolute surjection and (X, τ_1, τ_2) be p-connected. Suppose that (Y, γ_1, γ_2) is not p-connected. By Theorem 3.1, there exists a nonempty proper (i,j)-clopen set V of Y. Since f is an (i,j)-clopen-irresolute surjection, then $f^{-1}(V)$ is a nonempty proper (i,j)-clopen subset of X. By Theorem 3.1, (X, τ_1, τ_2) is not p-connected. ∎

Recall that a BS (X, τ_1, τ_2) is said to be (i,j)-zero dimensional if a basis $\mathbf{B}(\tau_i)$ for the topology τ_i is formed with $co\tau_j$, i.e. $\mathbf{B}(\tau_i) = co\tau_j$ [12]. It is obvious that a BS (X, τ_1, τ_2) is (i,j)-zero dimensional if and only if $\mathbf{B}(\tau_i) = (i,j) - Clp(X)$.

Theorem 3.2. *If (X, τ_1, τ_2) is an (i,j)-zero dimensional and i-T_1 BS, then $card(A) \leq 1$ for every p-connected subspace (A, τ_1^A, τ_2^A).*

Proof. Assume there exists a p-connected subset A of X with $card(A) \geq 2$. Then, there exists $U \in (i,j) - Clp(X)$ such that $a \in U \subset X \setminus \{b\}$ for any pair of distinct points $a, b \in A$. Note that $A \setminus U \in \tau_j^A \setminus \{\emptyset\}$, $A \cap U \in \tau_i^A \setminus \{\emptyset\}$ and A is the disjoint union of $(A \setminus U)$ and $(A \cap U)$. Hence we get a contradiction to our assumption. ∎

Definition 3.3. The subset $\bigcap\{U(x) | x \in U(x) \in (i,j) - Clp(X)\}$ of a BS (X, τ_1, τ_2) is called the (i,j)-quasi-component of a point $x \in X$ and is denoted by $(i,j) - Q_x$.

Proposition 3.2. *Let x be a point in a BS (X, τ_1, τ_2). Then the following hold:*

(1) $(i,j) - Q_x \in co\tau_j \setminus \{\emptyset\}$.

(2) *If $y \in (i,j) - Q_x$, then $(i,j) - Q_y \subset (i,j) - Q_x$.*

Proof. (1) This follows immediately from Definition 3.3.

(2) Suppose that $a \notin (i,j) - Q_x$. Then there exists $V_a \in (i,j) - Clp(X)$ such that $x \in V_a$ and $a \notin V_a$. Since $y \in (i,j) - Q_x, y \in U(x)$ for every

$U(x) \in (i,j) - Clp(X)$ containing x. Therefore, $y \in V_a \in (i,j) - Clp(X)$ and $a \notin V_a$. Hence $a \notin (i,j) - Q_y$. Consequently, $(i,j) - Q_y \subset (i,j) - Q_x$. ■

Proposition 3.3. *If $f : (X, \tau_1, \tau_2) \to (Y, \gamma_1, \gamma_2)$ is (i,j)-clopen-irresolute, then $f((i,j) - Q_x) \subset (i,j) - Q_{f(x)}$ for each $x \in X$.*

Proof. Since $(i,j) - Q_{f(x)} = \cap\{V | f(x) \in V \in (i,j) - Clp(Y)\}$, we have

$$\begin{aligned} f^{-1}((i,j) - Q_{f(x)}) &= f^{-1}(\cap\{V|f(x) \in V \in (i,j) - Clp(Y)\}) \\ &= \cap f^{-1}(\{V|f(x) \in V \in (i,j) - Clp(Y)\}) \\ &\supset \cap\{U|x \in U \in (i,j) - Clp(X)\} \\ &= (i,j) - Q_x. \end{aligned}$$

Therefore, $f((i,j) - Q_x) \subset f(f^{-1}((i,j) - Q_{f(x)})) \subset (i,j) - Q_{f(x)}$. ■

Corollary 3.1. *Let $f : (X, \tau_1, \tau_2) \to (Y, \gamma_1, \gamma_2)$ be a map.*

(1) *If f is p-continuous, then $f((i,j) - Q_x) \subset (i,j) - Q_{f(x)}$.*

(2) *If f is a p-homeomorphism, then $f((i,j) - Q_x) = (i,j) - Q_{f(x)}$.*

Proof. (1) Since every p-continuous function is (i,j)-clopen-irresolute, the proof follows immediately from Proposition 3.3.

(2) Since a p-homeomorphism is a bijection such that f and f^{-1} are p-continuous, the proof follows immediately from (1). ■

The greatest p-connected subset containing a point $x \in X$ is called the p-component of x in a BS (X, τ_1, τ_2) and is denoted by $p - C_x$.

Theorem 3.3. *In a BS (X, τ_1, τ_2), the following implication holds:*

$$p - C_x \subset (1,2) - Q_x \cap (2,1) - Q_x.$$

Proof. Consider a set $G_x \in (1,2) - Clp(X)$ containing a point $x \in X$. Then from $G_x \cap (X \setminus G_x) = \emptyset$ it follows that $(p - C_x \cap G_x) \cap (p - C_x \setminus G_x) = \emptyset$. Moreover, we have $p - C_x \cap G_x \neq \emptyset$. Hence $p - C_x \setminus G_x = \emptyset$, or equivalently $p - C_x \subset G_x$. It is obvious that $p - C_x \subset \bigcap G_x = (1,2) - Q_x$. Similarly we obtain that $p - C_x \subset (2,1) - Q_x$. Thereby, the implication $p - C_x \subset (1,2) - Q_x \cap (2,1) - Q_x$ is valid. ■

4. p-ultra-Hausdorff spaces

Definition 4.1. A BS (X, τ_1, τ_2) is said to be

(1) p-Urysohn if for any pair of distinct points x_1, $x_2 \in X$ there exist the neighborhoods $U \in \sum_i^X(x_1)$ and $V \in \sum_j^X(x_2)$ such that $\tau_j clU \cap \tau_i clV = \emptyset$ (see [5]).

(2) p-ultra-Hausdorff if for any pair of distinct points x_1,$x_2 \in X$ there exist $U_{x_1} \in (i,j)-Clp(X)$ and $V_{x_2} \in (j,i)-Clp(X)$ such that $x_1 \in U_{x_1}$, $x_2 \in V_{x_2}$ and $U_{x_1} \cap V_{x_2} = \emptyset$.

It should be especially noticed that if $i = j$ then the notion of p-ultra-Hausdorff coincides with the notion of ultra-Hausdorff for topological spaces, given in [13].

Example 4.1. Let X be a set with $card(X) \geq \aleph_0$, τ_d-discrete and $\tau_{cof.}$-cofinite (i.e., all finite subsets of X are closed, and vice versa) topologies on X, respectively. Then it is obvious that $(X, \tau_d, \tau_{cof.})$ is p-ultra-Hausdorff BS.

Remark 4.1. It should be mentioned that p-ultra-Hausdorff implies p-Urysohn but the converse does not hold. Find example which show that the BS Space is p-Urysohn but not p-ultra-Hausdorff. Under which conditions the converse is true?

Proposition 4.1. *A BS (X, τ_1, τ_2) is p-ultra-Hausdorff if and only if for any distinct points x_1, x_2, there exist $U \in (i,j) - Clp(X)$ such that $x_1 \in U$, $x_2 \notin U$ and $V \in (i,j) - Clp(X)$ such that $x_2 \in V, x_1 \notin V$.*

Definition 4.2. A subset K of a BS (X, τ_1, τ_2) is said to be (i,j)-clopen-compact relative to X if every cover of K by (i,j)-clopen sets of X has a finite subcover.

It should be noticed that every i-compact subset of (X, τ_1, τ_2) is (i,j)-clopen-compact relative to X.

Proposition 4.2. *If a BS (X, τ_1, τ_2) is (i,j)-zero dimensional and a subset K of X is (i,j)-clopen-compact relative to X, then K is a τ_i-compact subset of X.*

Proof. Consider a family $\mathcal{U} = \{U_\alpha | U_\alpha \in \tau_i\}_{\alpha\in\Lambda}$ such that $K \subset \bigcup\limits_{\alpha\in\Lambda} U_\alpha$. Then, since (X, τ_1, τ_2) is (i,j)-zero dimensional, we can write $U_\alpha = \bigcup\limits_{\alpha_\beta\in\Omega_\alpha} V_{\alpha_\beta}$ for each $\alpha \in \Lambda$, where $V_{\alpha_\beta} \in (i,j) - Clp(X)$. Denote $\Omega \equiv \bigcup\limits_{\alpha\in\Lambda} \Omega_\alpha$, then the family $\mathcal{V} = \{V_\beta\}_{\beta\in\Omega}$ is a cover of K by (i,j)-clopen sets of X. Therefore, one can extract from the family $\mathcal{V}$ a finite collection $\{V_{\beta_1}, V_{\beta_2}, ..., V_{\beta_n}\}$ such that $K \subset \bigcup\limits_{p=1}^{n} V_{\beta_p}$. Now we can choose finite collection $\{U_{\alpha(\beta_1)}, U_{\alpha(\beta_2)}, ..., U_{\alpha(\beta_n)}\}$ from the covering $\mathcal{U}$ such that $K \subset \bigcup\limits_{p=1}^{n} U_{\alpha(\beta_p)}$. ∎

Theorem 4.1. *For a BS (X, τ_1, τ_2), the following are equivalent:*

(1) *X is p-ultra-Hausdorff;*

(2) *For each set K of X which is (i,j)-clopen compact relative to X, $K = \cap\{V | K \subset V \in (i,j) - Clp(X)\}$;*

(3) *For each $x \in X$, $x = \cap\{V | x \in V \in (i,j) - Clp(X)\}$.*

Proof. (1) $\Rightarrow$ (2): Let K be any set of X which is (i,j)-clopen-compact relative to X. It is obvious that $K \subset \cap\{V_\alpha | K \subset V_\alpha \in (i,j)-Clp(X)\}$. Suppose that $x \notin K$. Then, by Proposition 4.1, for each $k \in K$ there exists $V_k \in (i,j)-Clp(X)$ such that $x \notin V_k$ and $k \in V_k$. Since $\{V_k | k \in K\}$ is a cover of K by (i,j)-clopen sets of X, there exist a finite points $k_1, k_2, ..., k_n \in K$ such that $K \subset \bigcup\limits_{m=1}^{n} V_{k_m}$. Now, set $V = \bigcup\limits_{m=1}^{n} V_{k_m}$. Then $K \subset V \in (i,j)-Clp(X)$ and $x \notin V$. Therefore, we have $x \notin \cap\{V_\alpha | K \subset V_\alpha \in (i,j)-Clp(X)\}$. Hence $K \supset \cap\{V_\alpha | K \subset V_\alpha \in (i,j)-Clp(X)\}$. Consequently, we obtain the assertion (2).

(2) $\Rightarrow$ (3): This is obvious since every singleton is (i,j)-clopen-compact relative to X.

(3) $\Rightarrow$ (1): For any distinct points $x_1, x_2 \in X$, by (3) we have $x_1 \notin \cap\{V_\alpha \mid x_2 \in V_\alpha \in (i,j)-Clp(X)\}$. Therefore, there exists $V \in (i,j)-Clp(X)$ such that $x_2 \in V$ and $x_1 \notin V$. Similarly, we have $x_2 \notin \cap\{V_\alpha | x_1 \in V_\alpha \in (i,j)-Clp(X)\}$. Therefore, there exists $U \in (i,j)-Clp(X)$ such that $x_1 \in U$ and $x_2 \notin U$. By Proposition 4.1, X is p-ultra-Hausdorff. ∎

Corollary 4.1. *If a BS (X,τ_1,τ_2) is p-ultra-Hausdorff and K is (i,j)-clopen-compact relative to X, then $K \in co\tau_j$.*

Proof. This follows from Proposition 2.2 and Theorem 4.1. ∎

Definition 4.3. A map $f : (X,\tau_1,\tau_2) \to (Y,\gamma_1,\gamma_2)$ is said to be (i,j)-weakly clopen-continuous (resp. (i,j)-clopen-continuous) if, for each $x \in X$ and each $V \in \sum_i^Y(f(x))$, there exists a set $U \in (i,j)-Clp(X)$ containing x such that $f(U) \subset \gamma_j cl(V)$ (resp. $f(U) \subset V$), where $i \neq j$, $i,j \in \{1,2\}$.

Example 4.2. Let us consider the set $X = \{m,n,p,q,k\}$, together with topologies $\tau_1 = \{\emptyset, X\} \cup \{\{m\},\{n,p\},\{m,n,p\}\}$ and $\tau_2 = \{\emptyset, X\} \cup \{\{q,k\}\}$. Then, we observe that $(1,2)-Clp(X) = \{\emptyset, X, \{m,n,p\}\}$ and $(2,1)-Clp(X) = \{\emptyset, X, \{q,k\}\}$. Moreover, let $Y = \{a,b,c\}$ be endowed with the following topologies $\gamma_1 = \{\emptyset, Y\} \cup \{\{a\}\}$ and $\gamma_2 = \{\emptyset, Y\} \cup \{\{c\}\}$. If we define a map $f : (X,\tau_1,\tau_2) \to (Y,\gamma_1,\gamma_2)$ via the equations: $f(m) = f(n) = a, f(p) = b, f(q) = f(k) = c$, then f is $(1,2)$-weakly clopen-continuous. But it is not $(1,2)$-clopen-continuous.

Theorem 4.2. *If $f : (X,\tau_1,\tau_2) \to (Y,\gamma_1,\gamma_2)$ is (i,j)-weakly clopen-continuous (resp. (i,j)-clopen-continuous) and A is a subset of X, then $f|_A : (A, \tau_1^A, \tau_2^A) \to (Y,\gamma_1,\gamma_2)$ is (i,j)-weakly clopen-continuous (resp. (i,j)-clopen-continuous).*

Proof. We prove only the case of (i,j)-weakly clopen-continuous. Let $x \in A$ and $V \in \sum_i^Y(f(x))$. Since f is (i,j)-weakly clopen-continuous, then there exists an (i,j)-clopen set U in X containing x such that $f(U) \subset W = \gamma_j cl(V)$. Because $U \subset f^{-1}(W)$, we have the implication $U \cap A \subset f^{-1}(W) \cap A = (f|_A)^{-1}(W)$ and $x \in U \cap A \in (i,j)-Clp(A)$. Consequently, $f|_A$ is (i,j)-weakly clopen-continuous. ∎

Theorem 4.3. *If $f : (X, \tau_1, \tau_2) \to (Y, \gamma_1, \gamma_2)$ is a p-weakly clopen-continuous injective map and a BS (Y, γ_1, γ_2) is p-Urysohn, then (X, τ_1, τ_2) is p-ultra-Hausdorff.*

Proof. For any pair of distinct points $x, y \in X$, there exist $U \in \sum_i^Y(f(x))$ and $V \in \sum_j^Y(f(y))$ such that $\gamma_j cl(U) \cap \gamma_i cl(V) = \emptyset$. Since f is p-weakly clopen-continuous, there exist $G_x \in (i,j) - Clp(X)$ and $H_y \in (j,i) - Clp(X)$ containing x and y, respectively, such that $f(G_x) \subset \gamma_j cl(U)$ and $f(H_y) \subset \gamma_i cl(V)$. Since $\gamma_j cl(U) \cap \gamma_i cl(V) = \emptyset$, then $G_x \cap H_y = \emptyset$. This shows that (X, τ_1, τ_2) is p-ultra-Hausdorff. ∎

Theorem 4.4. *If a map $f : (X, \tau_1, \tau_2) \to (Y, \gamma_1, \gamma_2)$ is (i,j)-weakly clopen-continuous and (X, τ_j) is Alexandorff, then f is (i,j)-clopen-irresolute.*

Proof. Let V be any (i,j)-clopen set of Y and $x \in f^{-1}(V)$. Then $V \in \sum_i^Y(f(x))$. Since f is (i,j)-weakly clopen-continuous, there exists $U_x \in (i,j) - Clp(X)$ such that $x \in U_x$ and $f(U_x) \subset \gamma_j cl(V) = V$. For each $x \in f^{-1}(V)$, we have $x \in U_x \subset f^{-1}(V)$ and hence $\bigcup_{x \in f^{-1}(V)} U_x = f^{-1}(V)$. Since (X, τ_j) is Alexandorff, by Proposition 2.2 $f^{-1}(V) \in (i,j) - Clp(X)$. Therefore, f is (i,j)-clopen-irresolute. ∎

Theorem 4.5. *If a map $f : (X, \tau_1, \tau_2) \to (Y, \gamma_1, \gamma_2)$ is (i,j)-clopen irresolute and (Y, γ_1, γ_2) is (i,j)-zero dimensional, then f is (i,j)-clopen-continuous.*

Proof. Let $x \in X$ and $V \in \sum_i^Y(f(x))$. Since Y is (i,j)-zero dimensional, there exists $W \in (i,j) - Clp(Y)$ containing $f(x)$ such that $W \subset V$. Since f is (i,j)-clopen irresolute, we have $f^{-1}(W) \in (i,j) - Clp(X)$. Set $U = f^{-1}(W)$, then $x \in U$ and $f(U) \subset W \subset V$. This shows f is (i,j)-clopen-continuous. ∎

Corollary 4.6. *Let (X, τ_j) be an Alexandroff topological space and (Y, γ_1, γ_2) is an (i,j)-zero dimensional BS. Then a map $f : (X, \tau_1, \tau_2) \to (Y, \gamma_1, \gamma_2)$ is (i,j)-clopen-continuous if and only if it is (i,j)-clopen irresolute.*
Proof. This follows immediately from Theorems 4.4 and 4.5. ∎

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A REFINEMENT ON THE GROWTH FACTOR IN GAUSSIAN ELIMINATION FOR ACCRETIVE-DISSIPATIVE MATRICES

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Abstract. In this note, we give a refinement of the growth factor in Gaussian elimination for accretive-dissipative matrix A which is due to Lin [Calcolo, 2013, DOI 10.1007/s10092-013-0089-1].

Keywords: accretive-dissipative matrix, growth factor, Gaussian elimination.
MSC (2010): 47A63.

1. Introduction

Let $M_n(\mathbb{C})$ be the set of $n \times n$ complex matrices and A be a non-singular matrix in $M_n(\mathbb{C})$. Consider the linear system

$$Ax = b \tag{1.1}$$

and let $A^{(k)} = (a_{ij}^{(k)})$ be the matrix resulted from applying the first $k(1 \leq k \leq n-1)$ steps of Gaussian elimination to A; in particular, $A^{(n-1)}$ is the upper triangular matrix obtained from the LU factorization of A.

The quantity

$$\rho_n(A) = \frac{\max\limits_{i,j,k} |a_{ij}^{(k)}|}{\max\limits_{i,j} |a_{ij}|} \tag{1.2}$$

is called the growth factor (in Gaussian elimination) of A.

For any $A = (a_{ij}) \in M_n(\mathbb{C})$, A^* stands for the conjugate transpose of A. Similarly, x^* means the conjugate transpose of x for any $x \in \mathbb{C}$. $A \in M_n(\mathbb{C})$ is accretive-dissipative if it can be written as

$$A = B + iC, \tag{1.3}$$

where B and C are both (Hermitian) positive definite.

If B, C are real symmetric positive definite in (1.3), then A is called a Higham matrix.

For nonsingular matrix A, its condition number is denoted by

$$\kappa(A) := \sqrt{\frac{\lambda_{\max}(A^*A)}{\lambda_{\min}(A^*A)}}$$

which is the ratio of largest and smallest singular values of A.

It is conjectured in [1] that

$$\rho_n(A) \leq 2 \tag{1.4}$$

for any Higham matrix A.

It is proved in [1] that if A in (1.1) is a Higham matrix, then no pivoting is needed in Gaussian elimination.

George et al. obtained the following result in [2]:

Theorem 1 *Let $A \in \mathrm{M}_n(\mathbb{C})$ be accretive-dissipative. Then*

$$\rho_n(A) < 3\sqrt{2}. \tag{1.5}$$

Furthermore, if A is a Higham matrix, then

$$\rho_n(A) < 3. \tag{1.6}$$

They proved Theorem 1 by the Theorem 2 in [2] below:

Theorem 2 *Let $A \in \mathrm{M}_n(\mathbb{C})$ be accretive-dissipative. Then*

$$\frac{|a_{ij}^{(k)}|}{|a_{ij}|} < 3, \qquad j = 1, \ldots, n; \quad k = 1, \ldots, n-1. \tag{1.7}$$

Lin [3] got a stronger result as follows:

Theorem 3 *Let $A \in \mathrm{M}_n(\mathbb{C})$ be accretive-dissipative. Then*

$$\frac{|a_{ij}^{(k)}|}{|a_{ij}|} < 2\sqrt{2}, \qquad j = 1, \ldots, n; \quad k = 1, \ldots, n-1. \tag{1.8}$$

Consequently,

$$\rho_n(A) < 4. \tag{1.9}$$

If A is a Higham matrix, then

$$\rho_n(A) < 2\sqrt{2}. \tag{1.10}$$

2. The main theorem

In this paper, we show a refinement of (1.8) which is a main result. Moreover, we get the refinements of (1.9) and (1.10):

Theorem 4 *Let $A \in \mathrm{M}_n(\mathbb{C})$ be accretive-dissipative. Then*

$$\frac{|a_{ij}^{(k)}|}{|a_{ij}|} < \left[1 + \left(\frac{1-\kappa}{1+\kappa}\right)^2\right]\sqrt{2}, \ j = 1, ..., n; \ k = 1, ..., n-1, \tag{2.1}$$

Consequently, $\rho_n(A) < 2\left[1 + \left(\frac{1-\kappa}{1+\kappa}\right)^2\right]$. *If A is a Higham matrix, then*

$$\rho_n(A) < \left[1 + \left(\frac{1-\kappa}{1+\kappa}\right)^2\right]\sqrt{2},$$

where $\kappa \in [1, +\infty)$ is the maximum of the condition numbers of B and C.

Proof. We fix numbers $k \in 1, 2, ..., n-1$ and j, where $j \geq k+1$. Denote by A_k, B_k and C_k, respectively, the leading principal order k submatrices in A, B and C. Consider the $(k+1) \times (k+1)$ matrix

$$A_{kj} = \begin{pmatrix} A_k & u \\ v^T & a_{jj} \end{pmatrix}$$

where

$$u^T = (a_{1j}, a_{2j}, ..., a_{kj})$$

and

$$v^T = (a_{j1}, a_{j2}, ..., a_{jk}).$$

Note that A_{kj} is a principal order $k+1$ submatrix in A.

Defining the vectors

$$b^T = (b_{1j}, b_{2j}, ..., b_{kj})$$

and

$$c^T = (c_{1j}, c_{2j}, ..., c_{kj}),$$

we can rewrite A_{kj} as

$$A_{kj} = \begin{pmatrix} B_k + iC_k & b + ic \\ b^* + ic^* & b_{jj} + ic_{jj} \end{pmatrix}$$

It is easy to see that $a_{jj}^{(k)}$ can be obtained by performing block Gaussian eliminations in A_{kj}; namely,

$$a_{jj}^{(k)} = a_{jj} - v^T A_k^{-1} u = b_{jj} + ic_{jj} - (b^* + ic^*)(B_k + iC_k)^{-1}(b + ic).$$

Setting

$$a_{jj}^{(k)} = \beta + i\gamma, \quad \beta, \gamma \in \mathbf{R},$$

we have by [2, Theorem 2.1]

$$\beta = b_{jj} - b^* X_k b + c^* X_k c - b^* Y_k c - c^* Y_k b$$

and

$$\gamma = c_{jj} + b^* Y_k b - c^* Y_k c - b^* X_k c - c^* X_k b,$$

where

$$X_k = (B_k + C_k B_k^{-1} C_k)^{-1} \tag{2.2}$$

$$Y_k = (C_k + B_k C_k^{-1} B_k)^{-1} \tag{2.3}$$

with

$$\begin{pmatrix} B_k & b \\ b^* & b_{jj} \end{pmatrix} \text{ and } \begin{pmatrix} C_k & c \\ c^* & c_{jj} \end{pmatrix} \tag{2.4}$$

positive definite. It is known that $\beta, \gamma > 0$.

By simple computation, we have

$$\pm(b^* Y_k c + c^* Y_k b) \le b^* Y_k b + c^* Y_k c; \tag{2.5}$$

$$\pm(b^* X_k c + c^* X_k b) \le b^* X_k b + c^* X_k c. \tag{2.6}$$

From (2.2) and (2.3) we have [2, Lemma 2.3]

$$X_k \le \frac{1}{2} C_k^{-1} \text{ and } Y_k \le \frac{1}{2} B_k^{-1}. \tag{2.7}$$

From (2.4) and [4, (6)], we get

$$(\frac{\lambda_1 - \lambda_n}{\lambda_1 + \lambda_n})^2 b_{jj} \ge b^* B_k^{-1} b \text{ and } (\frac{\lambda_1' - \lambda_n'}{\lambda_1' + \lambda_n'})^2 c_{jj} \ge c^* C_k^{-1} c \tag{2.8}$$

In (2.8), λ_1 and λ_n (λ_1' and λ_n') are the largest and the smallest eigenvalues of B (C), respectively.

Note that $f(x) = (\frac{x-1}{x+1})^m (m \geq 1)$ is increasing for $x \in [1, \infty)$. Then we have

$$\begin{aligned}
|a_{jj}^{(k)}| &= |\beta + i\gamma| \\
&\leq \beta + \gamma \\
&= b_{jj} - b^* X_k b + c^* X_k c - b^* Y_k c - c^* Y_k b \\
&\quad + c_{jj} + b^* Y_k b - c^* Y_k c - b^* X_k c - c^* X_k b \\
&\leq b_{jj} - b^* X_k b + c^* X_k c + (b^* Y_k b + c^* Y_k c) \qquad \text{(by (2.5))} \\
&+ \; c_{jj} + b^* Y_k b - c^* Y_k c + (b^* X_k b + c^* X_k c) \qquad \text{(by (2.6))} \\
&= b_{jj} + 2b^* Y_k b + c_{jj} + 2c^* X_k c \\
&\leq b_{jj} + b^* B_k^{-1} b + c_{jj} + c^* C_k^{-1} c \qquad \text{(by (2.7))} \\
&\leq b_{jj} + \left(\frac{\lambda_n - \lambda_1}{\lambda_n + \lambda_1}\right)^2 b_{jj} + c_{jj} + \left(\frac{\lambda_n^{'} - \lambda_1^{'}}{\lambda_n^{'} + \lambda_1^{'}}\right)^2 c_{jj} \quad \text{(by (2.8))} \\
&= \left[1 + \left(\frac{\lambda_n - \lambda_1}{\lambda_n + \lambda_1}\right)^2\right] b_{jj} + \left[1 + \left(\frac{\lambda_n^{'} - \lambda_1^{'}}{\lambda_n^{'} + \lambda_1^{'}}\right)^2\right] c_{jj} \\
&\leq \left[1 + \left(\frac{1-\kappa}{1+\kappa}\right)^2\right] (b_{jj} + c_{jj}) \\
&\leq \left[1 + \left(\frac{1-\kappa}{1+\kappa}\right)^2\right] \sqrt{2} |b_{jj} + i c_{jj}| \\
&= \left[1 + \left(\frac{1-\kappa}{1+\kappa}\right)^2\right] \sqrt{2} |a_{jj}|.
\end{aligned}$$

where κ =max$(\frac{\lambda_1}{\lambda_n}, \frac{\lambda_1^{'}}{\lambda_n^{'}}) \geq 1$, i.e., the maximum of the condition numbers of B and C. This completes the proofs of (2.1).

It is easy to know that $\left[1 + \left(\frac{1-\kappa}{1+\kappa}\right)^2\right] \sqrt{2} < 2\sqrt{2}$ for $\kappa \in [1, +\infty)$. So (2.1) is a refinement of (1.8).

To show the remaining claims, we need the following facts:

Fact1. [2, Corollary 2.3] The property of being an accretive-dissipative matrix is hereditary under Gaussian elimination.

Fact2. [2, Lemma 2.1, 2.2] If $A = (a_{ij}) \in M_n(\mathbb{C})$ is accretive-dissipative, then $\sqrt{2} \max\limits_{l} |a_{ll}| \geq \max\limits_{l \neq j} |a_{lj}|$. If A is a Higham matrix, then $\max\limits_{l} |a_{ll}| \geq \max\limits_{l,j} |a_{lj}|$.

Suppose $\max_{j,k} |a_{jj}^{(k)}| = |a_{j_0 j_0}^{(k_0)}|$ for some j_0, k_0, then by (2.1) the result below holds:

$$\rho_n(A) = \frac{\max_{i,j,k} |a_{ij}^{(k)}|}{\max_{i,j} |a_{ij}|} \leq \frac{\sqrt{2} \max_{j,k} |a_{ij}^{(k)}|}{\max_{i,j} |a_{ij}|} \leq \frac{\sqrt{2} |a_{j_0 j_0}^{(k_0)}|}{|a_{j_0 j_0}|} < 2 \left[1 + \left(\frac{1-\kappa}{1+\kappa}\right)^2\right].$$

Similarly, if A is a Higham matrix, then $\rho_n(A) < \left[1 + \left(\frac{1-\kappa}{1+\kappa}\right)^2\right] \sqrt{2}$. The proof is thus complete. ∎

3. Conclusion

Our results in Theorem 4 are refinements of the results in Lin [3, Theorem 3]. Although it is a minor improvement, the result is much closer to the final solution of Higham's conjecture.

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ON A SPECIAL CLASS OF FINITE p-GROUPS OF MAXIMAL CLASS

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Abstract. In this paper, we study the finite $p-$group G of maximal class in which every nonabelian subgroup H satisfies $C_G(H) = Z(H)$. We prove here that a finite $p-$group G of maximal class is metabelian if every nonabelian subgroup H satisfies $C_G(H) = Z(H)$, furthermore, if $p \neq 3$, $|G| \geq p^{2p}$, then there is an abelian subgroup of index p in G.

Keywords and phrases: a $CGZ-$group; a $p-$group of maximal class; an abelian $p-$group.

AMS Subject Classification: 20D15.

1. Introduction

Finite $p-$groups of maximal class is an important class of finite $p-$groups. It is well known that a finite $p-$group of maximal class can be determined by centralizers of some subgroups. For example, M. Suzuki [1, Proposition 1.8] proved that a finite $p-$group G is of maximal class if and only if there is a subgroup N of order p^2 such that $C_G(N) = N$; in [1, Proposition 10.17], it is showed that if G is a finite $p-$group, $B \leq G$ is a nonabelian subgroup of order p^3 such that $C_G(B) = Z(B)$, then G is of maximal class.

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In [1], it is posed the following problem: Classify every the finite $p-$ group G all of whose every nonabelian subgroup H satisfies $C_G(H) = Z(H)$. In this paper, we will study this problem. For convenience, we give the following definition.

Definition 1. A nonabelian $p-$group G is called a $CGZ-$group if and only if every nonabelian subgroup H of G satisfies $C_G(H) = Z(H)$.

In this paper, we mainly study the finite $CGZ-$groups of maximal class. We get that a finite $p-$group G of maximal class is metabelian if G is a $CGZ-$group. Furthermore, if $p \neq 3$ and $|G| \geq p^{2p}$, then there is an abelian subgroup of index p in G.

2. Main results and proofs

Lemma 2.1 *Let G be a nonabelian $p-$group and let H be a nonabelian subgroup of G. If G is a $CGZ-$group, then H is also a $CGZ-$group.*

Proof. Let H_1 be a nonabelian subgroup of H. Since $C_H(H_1) = H \cap C_G(H_1)$ and $C_G(H_1) \leq H_1$, we have $C_H(H_1) \leq H_1$. So H is also a $CGZ-$group. ∎

Proposition 2.2 *Let G be a nonabelian $p-$group with $exp(G) = p$, where $p \geq 3$. Then G is a $CGZ-$group if and only if G is a finite $p-$group of maximal class, and there is an abelian subgroup of index p.*

Proof. Since G is nonabelian, there is an element $a \in Z_2(G) - Z(G)$ and $b \in G$ such that $ab \neq ba$. We have that the subgroup $H = \langle a, b\rangle$ is a nonabelian subgroup of order p^3 with $|Z(H)| = p$. Since $Z(G) \leq C_G(H) = Z(H)$, then $Z(G)$ has order p. By [1, Proposition 10.17], G is a $p-$group of maximal class.

Let N be a normal subgroup of order p^2 and let $K = C_G(N)$. Then $|G : K| \leq p$. If K is not abelian, by Lemma 2.1, K is also a $CGZ-$group, similarly, we have $|Z(K)| = p$, which contradicts to $N \leq Z(K)$. So K is an abelian subgroup.

Conversely, if G is a finite $p-$group of maximal class, and G has a maximal abelian subgroup A with $|G : A| = p$. Then $|Z(G)| = p$. Let H be a nonabelian subgroup of G and let $H_1 = A \cap H$. Then $|H : H_1| = p$ and there is an element $x \in G - A$ such that $H = \langle x, H_1\rangle$, $G = \langle A, x\rangle$. It follows that $H_1 \trianglelefteq G$ and $Z(G) \leq H_1$. We see that

$$A \geq C_G(H) = C_G(x) = C_A(x) = Z(G).$$

Clearly $C_H(x) = Z(H) \leq Z(G) = C_G(x)$. Hence we have $C_G(H) = Z(H) = Z(G)$ and G is a $CGZ-$group. ∎

Noticing that a $p-$group G of maximal class with $|G| \geq p^{p+1}$ has an element g of order p^2, we have by Proposition 2.2, that if a nonabelian $p-$group G with $\exp(G) = p$ and G is a $CGZ-$group, then $|G| \leq p^p$.

Lemma 2.3 *Let G be a nonabelian $p-$group. If G is a $CGZ-$group and $\Omega_1(G) \cap Z(G)$ has an elementary abelian subgroup of order $\geq p^3$, then $\Omega_1(G) \leq Z(G)$ and $|\Omega_1(G)| = p^3$.*

Proof. Let $N = \Omega_1(G) \cap Z(G)$. If there is an element x of order p such that $x \notin Z(G)$, then there is an element $y \in Z_2(G) - Z(G)$ of order p. We can get an element $y_1 \in G$ such that $[y, y_1] \neq 1$. Let $H = \langle y, y_1 \rangle$. Then $|\Omega_1(H)| \leq p^3$ and $y \in \Omega_1(H)$. We have that the subgroup N is not contained in H, which contradicts to $N \leq C_G(H)$. It follows that $\Omega_1(G) \leq Z(G)$.

Let K be a minimal nonabelian subgroup of G. Then $|\Omega_1(K)| \leq p^3$. Since $\Omega_1(G) \leq C_G(K)$, we have $|\Omega_1(G)| = p^3$. ■

For convenience, we give the following two lemmas.

Lemma 2.4 [3, lemma 14.14] *Let G be a finite $p-$group of maximal class. If $|G| \leq p^{p+1}$, then $exp(G^{'}) \leq p$.*

Lemma 2.5 [4] *Let G be a finite $p-$group of maximal class and order p^m, where $p > 2$, $m > p+1$. Then G is irregular and*

(a) *Let G_1 be a maximal subgroup of G. Then G_1 is either of maximal class or regular with $|G_1 : \mho_1(G_1)| = p^{p-1}$.*

(b) *If $N \trianglelefteq G$ is of order p^{p-1}, then $exp(N) = p$.*

(c) $[G_i, G_{j+1}] \leq G_{i+j+1}$.

Proposition 2.6 *Let G be a finite $p-$group of maximal class. If G is a CGZ-group, then $G^{'}$ is abelian.*

Proof. If $|G| \leq p^5$, it is easy to see that $G^{'}$ is abelian. If $p = 2$, by [5], G has a cyclic subgroup of index 2. By [2, Theorem 6], for a $3-$group G of maximal class, $G^{'}$ is also abelian. Hence we only need to consider the cases $|G| \geq p^6$ and $p \geq 5$.

Suppose that $|G| \leq p^{p+1}$. By Lemma 2.4, we have $\exp(G^{'}) = p$. Let N be a normal subgroup of order p^2. Then $|G : C_G(N)| \leq p$. Note that G is a $p-$group of maximal class, we have $N \leq G^{'}$ and $|Z(G^{'})| \geq p^2$. By Proposition 2.2 we get that $G^{'}$ is abelian.

Suppose that $|G| \geq p^{p+2}$. Let M be a normal subgroup of order p^3. Then $M = G_{n-3} \leq G^{'} = G_2$. By Lemma 2.5, $[G_2, G_{n-3}] \leq G_n = 1$, we have that $M \leq Z(G^{'})$. It follows from Lemma 2.5 that $|\Omega_1(G^{'})| = p^{p-1}$. By Lemma 2.3, we have that $G^{'}$ is abelian. ■

By [3, Theorem 14.11], if G is a finite $p-$group of maximal class and of order p^n in which $G^{'}$ is abelian, then $C_G(G_i/G_{i+2}) = C_G(G_j/G_{j+2})$ for positive numbers $1 \leq i, j \leq n-2$. Hence we have the following corollary.

Corollary 2.7 *Let G be a finite $p-$group of maximal class of order p^n. If G is a $CGZ-$group, then $C_G(G_i/G_{i+2}) = C_G(G_j/G_{j+2})$ for every positive numbers $1 \leq i, j \leq n-2$.*

Proposition 2.8 *Let G be a finite $3-$group of maximal class. Then G is a $CGZ-$group.*

Proof. Suppose $|G| \leq 3^4$. Then G has an abelian subgroup of index 3 in G. It is easy to get that G is a $CGZ-$group.

Now, suppose $|G| \geq 3^5$. Let $H < G$ be a nonabelian subgroup. Then $H \leq M$, where M is a maximal subgroup of G. By Lemma 2.5, M is either of maximal class or regular with $|M : \mho_1(M)| = 3^2$. By [2, Theorem 6], G is metabelian, thus $G^{'}$ is abelian and $|M : G^{'}| = 3$. If M is regular, then M is metacyclic subgroup since $|M : \mho_1(M)| = 3^2$. It follows that M is a minimal nonabelian subgroup. Hence $H = M$, we have that $C_G(H) = Z(H)$. Suppose that M is a $3-$group of maximal class. Let $H_1 = H \cap G^{'}$. Then there is an element $x \in M$ such that $M = \langle x, G^{'} \rangle$ and $H = \langle x, H_1 \rangle$, where $|H : H_1| = 3$. We have that

$$G^{'} \geq C_M(H) = C_M(x) = Z(H).$$

If $C_G(H) \neq Z(H)$, there is an element $y \in G - M$ such that $y \in C_G(H)$. Note that

$$G = \langle y, M \rangle = \langle x, y, G^{'} \rangle = \langle x, y \rangle,$$

we have that G is abelian, a contradiction. Hence $C_G(H) = Z(H)$ and G is a $CGZ-$group. ■

Since the finite $2-$groups of maximal class are classified and every finite $2-$group of maximal class has a cyclic subgroup of index 2 (see [5]). From the proof of Proposition 2.2, we have that all finite $2-$groups of maximal class are $CGZ-$groups. Proposition 2.8 shows that all finite $3-$group of maximal class are $CGZ-$groups too. So we assume in following theorem that $p \geq 5$.

Theorem 2.9 *Let G be finite $p-$group of maximal class, where $p \geq 5$. If G is a $CGZ-$group of order $p^n > p^{2p}$, then G has an abelian subgroup of index p.*

Proof. Let M be a normal subgroup of order p^3. Note that $|G| = p^n \geq p^{2p}$ and $p \geq 5$. By Lemma 2.5, G has a normal subgroup T of order p^{p-1} and $\exp(T) = p$. It is easy to see that $M \leq T$ and then M is an elementary abelian subgroup. Suppose that

$$M = \langle x \rangle \times \langle y \rangle \times \langle z \rangle,$$

where $x \in Z_3(G) - Z_2(G), y \in Z_2(G) - Z(G), z \in Z(G)$. Then $Z(G) = G_{n-1} = \langle z \rangle$. Let $N = \langle y \rangle \times \langle z \rangle$. We have $N = Z_2(G) \trianglelefteq G$.

By Corollary 2.7, $C_G(G_i/G_{i+i}) = C_G(G_j/G_{j+2})$ for positive numbers $1 \leq i$, $j \leq n-2$. It follows that

$$C_G(M/Z(G)) = C_G(N).$$

Let $G_1 = C_G(N)$, then $|G : G_1| = p$. Hence there are two elements $a, b \in G$ such that $G = \langle a, b \rangle$ and $G_1 = \langle b, G^{'} \rangle$. Since $N \leq Z(G_1)$ is of order $\geq p^2$, then G_1 is not of maximal class. By Lemma 2.5, G_1 is a regular $p-$ subgroup.

Let $H = \langle x, b\rangle$. Notice that $G_1 = C_G(M/Z(G))$, we have $[x, b] \in Z(G)$. Suppose that $[x, b] = 1$. Since $|G_1 : G'| = p$ and G' is abelian, one has that $x \in Z(G_1)$, it follows that $M \leq Z(G_1)$. By Lemma 2.3, G_1 is abelian.

Suppose that $[x, b] \neq 1$. Now $\mathrm{cl}(H) = 2$. It follows from $C_G(M/Z(G)) = C_G(N) = G_1$ that $[x, b] \in Z(G)$. We have that $H' = Z(G)$ has order p. Hence H is a minimal nonabelian $p-$subgroup, and we have $Z(H) = \langle b^p, z\rangle$. Assume that $z \in \langle b^p\rangle$, then $Z(H) = \langle b^p\rangle$. But

$$N \leq Z(G_1) \leq C_{G_1}(H) = Z(H),$$

a contradiction. Hence $z \notin \langle b^p\rangle$, and then $Z(H) = \langle b^p\rangle \times \langle z\rangle$. Since $|G_1 : G'| = p$ and G' is abelian. We have $b^p \in G'$, and then $b^p \in Z(G_1)$. Similarly, $(b^g)^p = (b^p)^g \in Z(G_1)$ for every element $g \in G$. Since $Z(G_1) \leq C_{G_1}(H) \leq Z(H)$, it follows that $(b^p)^g \in Z(H)$. Assume that $|b^p| \geq p^2$. Then $(b^p)^g = (b^p)^{l_1} z^{l_2}$, where $(l_1, p) = 1$. It follows that $\langle b^{p^2}\rangle = \langle (b^{p^2})^g\rangle$, so $\langle b^{p^2}\rangle \trianglelefteq G$. By the hypothesis, G is a $p-$group of maximal class, surely $|Z(G)| = p$. Notice that we have proved that $\langle b^p\rangle \cap Z(G) = 1$, we get $b^{p^2} = 1$. Since $G_1 = \langle b\rangle G'$ and $G' = \langle [b, a]\rangle^G$, we have by Lemma 2.5, that G_1 is regular and $|\Omega_1(G_1)| \leq p^{p-1}$. Hence $\exp(G_1) \leq p^2$ and $|G_1 : \mho_1(G_1)| = |\Omega_1(G_1)|$, consequently $\mho_1(G_1) \leq \Omega_1(G_1)$. It follows that

$$|G_1| = |\mho_1(G_1)||\Omega_1(G_1)| \leq p^{2p-2}.$$

We have $|G| \leq p^{2p-1}$, which contradicts to the hypothesis that $|G| \geq p^{2p}$. ∎

Remark. There exists a finite $p-$group G of maximal class which is also a $CGZ-$group, but G has no abelian subgroup of index p in G and $|G| < p^{2p}$. For example,

$$G = \langle a, b, c, d | a^{p^2} = b^p = c^p = d^p = 1, [b, a] = c, [c, a] = d, [c, b] = a^p, [d, b] = a^p, [d, a] = 1\rangle,$$

where $p \geq 5$. We have G is a $p-$group of maximal class with $|G| = p^5$, and G is a $CGZ-$group without any abelian subgroup of index p in G.

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CONJUGACY CLASS SIZES OF SUBGROUPS AND THE STRUCTURE OF FINITE GROUPS

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Abstract. The authors investigate the influences of conjugacy class sizes of subgroups of a finite groups G on the structure of G. Some sufficient conditions for a finite group to be p-nilpotent, p- solvable and supersolvable are obtained.

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1. Introduction

One of the questions that were studied extensively is what can be said about the structure of the group G if some information is known about the arithmetical structure of $Con(G)$, the set of the conjugacy classes of G. Answers in many cases were given. On the other hand, few studies about the conjugacy classes of subgroups of a group G were done. In [12], the author proved that a finite group G is p-nilpotent for some prime p if and only if $(p, |G : N_G(Q)|) = 1$ for any Sylow subgroups Q of G. In the same paper, the author showed also that if $|G : N_G(Q)|$ is square-free for any Sylow subgroups Q of G, then either G is supersolvable or $G = HK$, where H is normal in G and $H = PSL(2, p)$ or $SL(2, p)$ for some prime $p = 8k + 5$, K is a supersolvable subgroup of G. Guo Wenbin proved in [6] that if $|G : N_G(Q)|$ is prime power numbers for any Sylow subgroups Q of G, then G is solvable. Further, if $|G : N_G(Q)|$ is prime power numbers or odd numbers, then G is a solvable group. Recently, in [2], Berkovich and Kazarin showed that G is solvable with $nl(G) \leq 2$ if $|G : N_G(H)|$ is a power of a prime for all primary subgroup $H \leq G$. In this paper, we consider the conjugacy class sizes of subgroups of a finite groups G and investigate the influences of conjugacy class sizes of subgroups of G on the structure of G.

In what follows, G is a finite group of order $|G|$; $\pi(G)$ denotes the set of all prime divisors of $|G|$; $nl(G)$ denotes the nilpotent length of $|G|$ and $cl(G)$ denotes the nilpotent class of $|G|$. The p-length of $|G|$ is denoted by $l_p(G)$. All further unexplained notation and terminologies are standard can be found in [5].

2. Preliminaries

In this section, we give some lemmas which are useful in the sequel.

Lemma 2.1 *Let $N \trianglelefteq G$, and $H \leq G$. Then*

(1) $|N : N_N(H)|$ *and* $|\overline{G} : N_{\overline{G}}(\overline{H})|$ *divide* $|G : N_G(H)|$, *if N is contained in H, where $\overline{G} = G/N$.*

(2) $|G : N_G(NH)|$ *divides* $|G : N_G(H)|$.

Proof. (1) Clearly,

$$|N : N_N(H)| = |NN_G(H) : N_G(H)|,$$

which divides $|G : N_G(H)|$.

Also

$$|\overline{G} : N_{\overline{G}}(\overline{H})| \,|\, |\overline{G} : \overline{N_G(H)N}| = |G : N_G(H)N|.$$

On the other hand,

$$|G : N_G(H)| = |G : N_G(H)N||N_G(H)N : N_G(H)|.$$

Hence

$$|\overline{G} : N_{\overline{G}}(\overline{H})| \,|\, |G : N_G(H)|.$$

(2) follows by the fact that $N_G(H) \leq N_G(NH)$. ∎

Lemma 2.2 *Suppose that $\pi \subseteq \pi(G)$ and $x \in H$, where H is a π-Hall subgroup of group G. If $|G : N_G(\langle x \rangle)|$ is a π-number. Then $\langle x \rangle \leq O_\pi(G)$.*

Proof. Indeed, $G = N_G(\langle x \rangle)H$. So $\langle x \rangle^G = \langle x \rangle^{N_G(\langle x \rangle)H} = \langle x \rangle^H \leq H$. ∎

Lemma 2.3 *Let G be a group and p be the smallest odd prime divisor of $|G|$. Suppose that there exits an element x of order p such that $|G : N_G(\langle x \rangle)|$ is a power of two. Then G is not a non-abelian simple group.*

Proof. Assume that G is a non-abelian simple group, $H = N_G(\langle x\rangle)$ is a proper subgroup of G with index of 2^α, where α is a natural number. Then G is one of the groups list in theorem 1 of [7].

If $G = A_{2^n}$, then $H = A_{2^n-1}$. As $2^n > 6$, we obtain that A_{2^n-1} is also a simple group. This is impossible.

If $G = PSL(n,q)$, then $|G : N_G(\langle x\rangle)| = \frac{q^n-1}{q-1}$. It is easy to check that $n = 2$ and $q = 2^\alpha - 1$ is a prime in this case. Hence, $|N_G(\langle x\rangle)| = q(q-1)$. Suppose that $o(x) = q$, then $q-1 = 2^\alpha - 2$ is a divisor of $|G|$. Hence $2^\alpha - 2 = 2^\beta$ by the choice of x, where α, β are both natural numbers. Therefore, $\alpha = 2$, and $q = 3$, a contradiction. Hence, we may assume that $o(x)|q-1$. Since $N_G(\langle x\rangle)/C_G(\langle x\rangle)$ is isomorphic to some subgroups of $Aut(\langle x\rangle)$, we have $|N_G(\langle x\rangle)| = q(q-1)$ is a divisor of $|C_G(\langle x\rangle)||Aut(\langle x\rangle)|$. On the other hand, by the structure of $PSL(2,q)$ we know that $q \not| |C_G(\langle x\rangle)|$, which implies that $(|C_G(\langle x\rangle)||Aut(\langle x\rangle)|, q) = 1$. Thus, we obtain a contradiction.

Assume that $G = PSL(2,11)$. Then, $N_G(\langle x\rangle) = H = A_5$. Hence, we obtain $\langle x\rangle \trianglelefteq A_5$, a contradiction.

If $G = M_{23}$, then $H = M_{22}$. Since in this case $|G : N_G(\langle x\rangle)| = 23 \neq 2^\alpha$, we get a contradiction. Similarly, we have that $G \neq M_{11}$ and $PSU(4,2)$, a final contradiction. The proof is complete. ∎

Lemma 2.4 [4] *Every group of odd order is solvable.*

Lemma 2.5 [3, Theorem 1] *If the subgroup H of the group G is quasinormal in G, then H/H_G is nilpotent.*

Lemma 2.6 [1, Theorem 3] *Let the group $G = HK$ be the m-permutable product of the subgroups H and K. Assume that H is supersolvable and K is nilpotent. If K permutes with every Sylow subgroup of H, then G is supersolvable.*

3. Main results

We first prove the following:

Theorem 3.1 *Let p be the minimal odd divisor of $|G|$. Suppose that $|G : N_G(\langle x\rangle)|$ is a power of two for any element $x \in G$ of order p. Then G is p-solvable.*

Proof. Assume that the theorem is false and let G be a counterexample of minimal order. Then we have:

(1) Any nontrivial normal subgroup of G is p-solvable and $O_{p'}(G) = 1$.

This follows by Lemma 2.1 and the choice of G.

(2) $O_p(G) > 1$.

It follows by Lemma 2.3 that there exists a nontrivial minimal normal subgroup N of G. By Lemma 2.1, N is p-solvable. Since $O_{p'}(N) \leq O_{p'}(G) = 1$, we obtain $O_{p'}(N) = 1$. Hence $1 < O_p(N) \leq O_p(G)$.

(3) Every subgroup of order p is contained in $O_p(G)$.

Suppose that there is a subgroup $\langle x\rangle$ of order p such that $\langle x\rangle$ is not contained in $O_p(G)$. It follows that $|G : N_G(\langle x\rangle)|$ is a power of two by hypothesis. Hence $N_G(\langle x\rangle)$ contains a Sylow p-subgroup of G, giving $O_p(G) \le N_G(\langle x\rangle)$. Let $C = C_G(O_p(G))$. Then $\langle x\rangle \le C \trianglelefteq G$, and $O_p(C) = O_p(G)$ and hence $O_{pp'}(C) = O_p(C) \times H$, where, H is a nontrivial p'-group. Therefore, $1 < H < O_{p'}(G) = 1$, a contradiction. Hence $C = G$.

Let $\overline{G} = G/O_p(G)$, and $\overline{x} = xO_p(G)$ be an element of order p. Then

$$|\overline{G} : N_{\overline{G}}(\langle\overline{x}\rangle)|||G : N_G(\langle x\rangle)|$$

is a power of two. Again by Lemma 2.3, $\overline{G}$ is not a non-abelian simple group. Hence there exists a normal subgroup N of G such that $O_p(G) < N < G$. Now, $O_p(N) = O_p(G) \le Z(G)$. This implies that $O_{p'}(G) > 1$, which contradicts to (1).

(4) G/M is a non-abelian simple group and $p||G/M|$, where M be a maximal normal subgroup of G which is p-solvable.

This follows immediately by (1).

(5) The final contradiction.

Let S_0 be a Sylow 2-subgroup of M and $H = N_G(S_0)$. Then, by Frattini argument, we obtain that $G = MH$. It follows by (4) that $H/H \cap M \cong G/M$ is a non-abelian simple group and $p||G/H/H \cap M|$. Let $D = C_H(O_p(H))$. Then D is p-nilpotent by Ito' theorem. Since $(H \cap M)D/H \cap M$ is normal in $H/H \cap M$, we have $D \le H \cap M$.

By Lemma 2.4, there is a Sylow 2-subgroup S of G such that $S_0 < S$ and $S \le H$. Let S_1/S_0 be a subgroup of order two of $Z(S/S_0)$. Then $S_1 = S_0\langle u\rangle$, where $u^2 \in S_0$ and $S_0 \trianglelefteq S_1$. Let $K = O_p(H)S_1$. If S_1 is normal in K, then $S_0 \le C_H(O_p(H)) = D \le H \cap M$, which contradicts to the fact that S_1 be a Sylow 2-subgroup of M. Therefore, K has no normal Sylow 2-subgroup, of course K is not nilpotent. Suppose that W is a minimal non-nilpotent group of K. Then $W = X\langle v\rangle$, where $o(v) = 2^\alpha$, $\Phi(\langle v\rangle) = \langle v^2\rangle \le Z(W)$, the center of W, and X is a normal p-group of W of exponent p. Now, $\langle v\rangle$ acts irreducible on $X/\Phi(X)$ and v induces an automorphism of order two of $X/\Phi(X)$. Hence, $|X/\Phi(X)| = p$ and $X = \langle x\rangle$ is a cyclic group of order p. Further, $S_0 \le C_G(\langle x\rangle) < N_G(\langle x\rangle)$. So $N_G(\langle x\rangle)$ contains $S_0\langle v\rangle$, a Sylow 2-subgroup of K. Without lose of generality, we may assume $S_1 = S_0\langle v\rangle$. Now, $|G : N_G(\langle x\rangle)|$ is a power of two by assumption. Thus $G = N_G(\langle x\rangle)S$. If $G = N_G(\langle x\rangle)$, then $C_G(\langle x\rangle) \trianglelefteq G$. By (1), $C_G(\langle x\rangle)$ is p-solvable, of course we have G is p-solvable, a contradiction. If $G > N_G(\langle x\rangle)$, we have

$$S_1^G = S_1^{SN_G(\langle x\rangle)} = S_1^{N_G(\langle x\rangle)} \le N_G(\langle x\rangle)$$

since $S_1/S_0 \le Z(S/S_0)$. Hence S_1^G is p-solvable, and we have $S_1^G \le M$. Therefore, $|M|_2 = |S_0| < |S_1| \le |M|_2$, the final contradiction. This completes our proof. ∎

Theorem 3.2 *Let G be a group. If $|G : N_G(\langle x\rangle)|$ is a power of a prime for all $x \in G$ of prime power order. Then G is solvable with $nl(G) \le 2$ and $l_p(G) \le 1$ for any prime divisor p of $|G|$.*

Proof. First of all, G is not a non-abelian simple group. Suppose G is simple. Then $|\pi(G)| \geq 3$. Let $P \in Syl_p(G)$, and $x \in Z(P)$, the center of P, where $p \in \pi(G)$. Then, by hypothesis $|G : N_G(\langle x\rangle)|$ is power of a prime r, which is different from p. Let R be a Sylow r-subgroup of G and let $y \in Z(R)$. Then $|G : N_G(\langle y\rangle)|$ is also a power of a prime. It is well known that G must be the simple group $PSL(2,7)$ (see [7, p.304, Note]). Obviously, $PSL(2,7)$ can not satisfy the hypothesis of our Theorem. This is a contradiction. Hence G is not a non-abelian simple group and there exists a proper normal subgroup N in G. By Lemma 2.1, G/N and N are all solvable groups, therefor G is solvable.

In order to prove that $l_p(G) \leq 1$, we suppose G is a counterexample of minimal order. Then by [8, Lemma 6.9, VI], we know $\Phi(G) = 1$ and $F(G) = O_p(G)$ is the unique minimal normal subgroup of G. Hence there exists a proper subgroup M of G such that $G = F(G) \rtimes M$, the semiproduct of $F(G)$ and M. Let $M_p \in Syl_p(M)$ and $x \in Z(M_p)$. Then $F(G)$ is not contained in $N_G(\langle x\rangle)$. Otherwise we have that $x \in C_G(F(G))$, a contradiction since $C_G(F(G)) \leq F(G)$. Thus $|G : N_G(\langle x\rangle)|$ is a power of p. By Lemma 2.2, $x \in O_p(G) = F(G)$, a contradiction. Then we obtain that $F(G)$ must be a Sylow subgroup of G. Therefore, $l_p(G) = 1$, and G is not a counterexample, a contradiction too.

Now, we will show that $nl(G) \leq 2$. Assume that G is a counterexample of minimal order. Since $F(G/\Phi(G)) = F(G)/\Phi(G)$, we have $\Phi(G) = 1$ by induction. Moreover, $F(G) = O_p(G)$ is the unique minimal normal subgroup of G. Hence, there exists a proper subgroup M of G such that $G = F(G) \rtimes M$. By using the same argument as the above, we get M is a p'-group and $|G : N_G(\langle x\rangle)|$ is a power of p for any $x \in M$ of prime power order. It follows by Lemma 2.1 that $|G : N_G(F(G)\langle x\rangle)|$ divides $|G : N_G(\langle x\rangle)|$. However, $|G : N_G(F(G)\langle x\rangle)|$ is coprime to p since $F(G)$ is the Sylow p-subgroup of G. Hence $F(G)\langle x\rangle$ is normal in G, which implies that all cyclic subgroup of $G/F(G)$ is normal in $G/F(G)$. Hence $G/F(G)$ is a Dedekind group, and $nl(G) \leq 2$, which contradicts to the choice of G. The proof is complete. ∎

Moreover, for a group G, let $Norm(G) = \cap\{N_G(\langle a\rangle) | \forall a \in G\}$. Then, we have $Norm(G) \leq Z_2(G)$ by [11].

The following theorem gives a sufficient condition for a group to be p-nilpotent.

Theorem 3.3 *Let G be a solvable group and p a prime divisor of $|G|$ such that q does not divide $p-1$ for any prime divisor q of $|G|$. Suppose that $|G : N_G(\langle x\rangle)|$ is not divided by p^2 for any $x \in G$ of prime power order. Then G is a p-nilpotent group. Furthermore, if P is a Sylow p-subgroup of G, then $cl(P) \leq 3$.*

Proof. It follows by Lemma 2.1 that the conclusion holds for proper quotient groups of G. Hence we may assume that G has a unique minimal normal subgroup N, since the class of p-nilpotent groups forms a saturated formation. Clearly, we may assume also that N is an elementary abelian group of order r^n for a prime r and a natural number n. Obviously $r = p$. Let M be a maximal normal subgroup of G. Then $|G/M| = q$ is a prime. By Lemma 2.1, M satisfies the assumptions of the theorem and therefore it is p-nilpotent. If $q = p$, then it follows that the normal p-complement of M is also the normal p-complement of G, a contradiction. So we have $q \neq p$. Since G/N is p-nilpotent and it has no quotient group of order p, we have G/N is a p'-group. If $M \neq N$, then $O_{p'}(M) \neq 1$ since M is p-nilpotent. Hence $N \leq O_{p'}(M)$ by the uniqueness of N, a contradiction. Therefore $M = N$ and $|G/N| = q$. If $n = 1$, then we have $q|p-1$, since $|G/N| = |N_G(N)/C_G(N)|$ divides $|Aut(N)|$. This is a contradiction. Hence $n \geq 2$. By the Schur-Zassenhaus Theorem, $G = N\langle x\rangle$, where $o(x) = q$. If there is an element $u(\neq 1) \in N$ such that $u \in N_G(\langle x\rangle)$, then $u \in Z(G)$ since N is abelian. Thus, $N = \langle u\rangle \leq Z(G)$, a contradiction. Hence, we obtain $N_G(\langle x\rangle) = \langle x\rangle$. This implies that $p^2||G : N_G(\langle x\rangle)|$, a contradiction too.

It remains to prove that $cl(P) \leq 3$. It follows by the first part of the proof that $P \cong G/O_{p'}(G)$. Thus P satisfies the assumptions of the theorem. Let $x \in P$, by the hypothesis, $|P : N_P(\langle x\rangle)| \leq p$, and so $\Phi(P) \leq N_P(\langle x\rangle)$ for all $x \in P$, giving $\Phi(P) \leq Norm(P) \leq Z_2(P)$. Hence $cl(P) \leq 3$. Our proof is complete now. ∎

The following two theorems give some sufficient conditions for a group to be supersolvable.

Theorem 3.4 *Let G be a solvable group. Suppose that $|G : N_G(\langle x\rangle)|$ is a square-free number for all $x \in G$ of prime power order. Then G is supersolvable.*

Proof. Assume that the result is false and G be a counterexample of minimal order. Since G is solvable, we have that G has a minimal normal subgroup N of order p^n, where p is a prime and n is a natural number. Since the class of supersolvable groups forms a saturated formation, we may suppose that N is a unique minimal normal subgroup of G and $\Phi(G) = 1$. If $n = 1$, then G is supersolvable since G/N is supersolvable, which contradicts to the choice of G. Hence $n > 1$. Since $N \nleq \Phi(G)$, there exists a maximal subgroup M of G such that $G = MN$, $M \cap N = 1$, and $M \cong G/N$ is supersolvable. Let Q be a minimal normal subgroup of M. Then $Q = \langle x\rangle$, and $N_G(\langle x\rangle) \geq M$, where $x \in M$ is of prime order. Assume that $N_G(\langle x\rangle) \cap N \neq 1$, then $N_G(\langle x\rangle) = G$. Hence $Q \trianglelefteq G$. This implies that G is supersolvable since G/Q is supersolvable, a contradiction. Now,

$$p < p^n||NN_G(\langle x\rangle) : N_G(\langle x\rangle)| = |G : N_G(\langle x\rangle)|,$$

contrary to the hypothesis. The proof is hence completed. ∎

Recall that a subgroup K of a group G is said to be quasi-normal in G if $KH = HK$ for any subgroup H of G.

Theorem 3.5 *Let A and B be quasi-normal subgroups of a solvable group G such that $G = AB$. Suppose that $|G : N_G(\langle x\rangle)|$ is a square-free number for every $x \in A \cup B$ of prime power order. Then G is supersolvable.*

Proof. Assume that the theorem is not true and G a counterexample of minimal order. Because supersolvable groups form a saturated formation, we may suppose that G has a unique minimal normal subgroup N and $\Phi(G) = 1$. Let $|N| = p^n$ for a prime p and a natural number n, then $n > 1$. Obviously, $F(G) = N = C_G(N)$. If either $A_G = 1$ or $B_G = 1$, then either A or B is nilpotent by Lemma 2.5. Therefore, by Lemma 2.6 we obtain G is supersolvable, a contradiction. Now, we have $N \leq A_G$ and $N \leq B_G$ by the uniqueness of N. Since A and B are quasi-normal in G, $F(A)$ and $F(B)$ are contained in $F(G) = N$. Therefore, $F(A) = F(B) = N$.

Let q be the largest prime divisor of $|A|$, and $S_q \in Syl_q(A)$. Then $S_q \trianglelefteq A$ since A is supersolvable and $S_q \leq N$. This implies that $p = q$ and N is the Sylow p- subgroup of A. By the same reason, we know that p is the largest prime divisor of $|B|$ and N is a Sylow subgroup of B. Therefore, N is a Sylow subgroup of G and p is the largest prime divisor of $|G|$. Let K/N be a minimal normal subgroup of G/N. By a result of [9], we may assume that $K \leq A$ or $K \leq B$. Since G/N is supersolvable, we have $|K/N| = q$ is a prime, which is not equal to p. By the Schur-Zassenhaus Theorem, $K = N\langle v\rangle$, where $o(v) = q$. If there is an element $u(\neq 1) \in N$ such that $u \in N_K(\langle v\rangle)$, then $u \in Z(K)$ since N is abelian. Thus, we have either $Z(K) = K$ or $Z(K) < N$, a contradiction. Hence we obtain $N_K(\langle v\rangle) = \langle v\rangle$. This implies that $|N| = |K : N_K(\langle v\rangle)|$, that is, $p^2 \| K : N_K(\langle v\rangle)|$, contrary to the hypothesis. The proof is complete. ∎

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AVERAGE D-DISTANCE BETWEEN VERTICES OF A GRAPH

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Abstract. The D-distance between vertices of a graph G is obtained by considering the path lengths and as well as the degrees of vertices present on the path. The average D-distance between the vertices of a connected graph is the average of the D-distances between all pairs of vertices of the graph. In this article we study the average D-distance between the vertices of a graph.

Keywords: D-distance, average D-distance, diameter.

2000 Mathematics Subject Classification: 05C12.

1. Introduction

The concept of distance is one of the important concepts in study of graphs. It is used in isomorphism testing, graph operations, hamiltonicity problems, extremal problems on connectivity and diameter, convexity in graphs etc. Distance is the basis of many concepts of symmetry in graphs.

In addition to the usual distance, $d(u, v)$ between two vertices $u, v \in V(G)$ we have detour distance (introduced by Chartrand et al, see [2]), superior distance (introduced by Kathiresan and Marimuthu, see [6]), signal distance (introduced by Kathiresan and Sumathi, see [7]), degree distance etc.

In an earlier article [9], the authors introduced the concept of D-distance between vertices of a graph G by considering not only path length between vertices, but also the degrees of all vertices present in a path while defining the D-distance. In a natural way we can extend this concept to D-distance between edges.

The article is arranged as follows. In §2, we collect some definitions and results for easy reference. In §3, we study some properties of average D-distance and in §4, we calculate the average D-distance between vertices for some classes of graphs.

2. Preliminaries

Throughout this article, by a graph $G(V, E)$ or simply G, we mean a non-trivial, finite, undirected graph without multiple edges and loops. Further all graphs we consider are *connected.*

In this section we given some definitions and state some results for later use. We begin with D-distance in graphs.

Definition 1 If u, v are vertices of a connected graph G the D-length of a connected $u-v$ path s is defined as $l^D(s) = l(s) + deg(v) + deg(u) + \sum deg(w)$ where sum runs over all intermediate vertices w of s and $l(s)$ is the length of the path.

Definition 2 The D-distance, $d^D(u, v)$, between two vertices u, v of a connected graph G is defined as $d^D(u, v) = min\left\{l^D(s)\right\}$ where the minimum is taken over all $u - v$ paths s in G. In other words,

$$d^D(u, v) = min\left\{l(s) + deg(u) + deg(v) + \sum deg(w)\right\}$$

where the sum runs over all intermediate vertices w in s and minimum is taken over all $u - v$ paths s in G.

Definition 3 Let G be a connected graph of order n. The average distance of G denoted by $\mu(G)$, is defined as

$$\mu(G) = ({}^n c_2)^{-1} \sum d(u, v),$$

where $d(u, v)$ denotes the distance between the vertices u and v. See [3, 4, 5].

Similarly, we can define the average D-distance of a graph as follows:

Definition 4 Let G be a connected graph of order n. The average D-distance between vertices of G denoted by μ^D, is defined as

$$\mu^D(G) = ({}^n c_2)^{-1} \sum d^D(u, v),$$

where $d^D(u, v)$ denotes the D-distance between the vertices u and v.

Definition 5 Let G be a connected graph of order n having m edges with $V(G) = \{v_1, v_2, v_3, \cdots, v_n\}$. The D-distance matrix of G denoted as $D^D(G)$, is defined as

$$D^D(G) = \left[d^D_{i,j}\right]_{n\times n},$$

where $d^D_{i,j} = d^D(v_i, v_j)$ is the D-distance between the vertices v_i and v_j.

Obviously, $D^D(G)$ is an $n \times n$ symmetric matrix with all diagonal entries being zero.

Definition 6 Let G be a graph, then the average degree of G, denoted as $d(G)$, is given by

$$d(G) = \frac{1}{|V|}\sum d(v),$$

where $d(v)$ is the degree of the vertex v .

Definition 7 The total D-distance $[TDD]$ of graph G is the number given by

$$\frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{n} d^D(v_i, v_j),$$

where n is the number of vertices.

3. Average D-distance

In this section, we prove some results on average D-distance between vertices. We begin with a theorem which connects the number of vertices and average D-distance. This leads to some more results.

Theorem 1 *Let G_1 and G_2 be two connected graphs having same number edges and same diameters. If the number of vertices in G_1 is more than the number of vertices in G_2 then average D-distance of G_1 is more than average D-distance of G_2.*

Proof. Since the diameters of these two graphs are the same, the largest entries in the D-distance matrix of these graphs are the same. The number of the pairs of vertices is more in G_1 and hence total D-distance value of G_1 is more. Since number of edges in G_1 and G_2 are same. This implies the average D-distance of G_1 is more than average D-distance of G_2. ■

Theorem 2 *Let G_1 and G_2 be two connected graphs of same number of edges and $diam(G_1) < diam(G_2)$. Then $\mu^D(G_1) > \mu^D(G_2)$.*

Proof. Since G_1, G_2 have same number of edges and $diam(G_1) < diam(G_2)$, it is clear that $|V(G_1)| > |V(G_2)|$. Then by Theorem 1, we have $\mu^D(G_1) > \mu^D(G_2)$. ■

Theorem 3 *Let G_1 and G_2 be two connected graphs having same number of edges and diameters. If $\delta(G_1) < \delta(G_2)$ then $\mu^D(G_1) > \mu^D(G_2)$.*

Proof. $\delta(G_1) < \delta(G_2)$ implies $|V(G_1)| > |V(G_2)|$. Then, by Theorem 1, $\mu^D(G_1) > \mu^D(G_2)$. ∎

Theorem 4 *Let G_1 and G_2 be two connected graphs having same number of edges and same diameters. If $\delta^1(G_1) < \delta^1(G_2)$ then $\mu^D(G_1) > \mu^D(G_2)$.*

Proof. Since $\delta^1(G_1) < \delta^1(G_2)$ we have $|V(G_1)| > |v(G_2)|$. Then, by Theorem 1, $\mu^D(G_1) > \mu^D(G_2)$. ∎

4. Results on some classes of graphs

Here we calculate the average D-distance for some classes of graphs.

Theorem 1 *The average D-distance of complete graph K_n is $\mu^D(K_n) = 2n - 1$.*

Proof. Every vertex taken from K_n has $n-1$ vertex neighbors. The D-distance between any vertex and its neighbors is $2n-1$. Thus the total D-distance $TDD = ({}^nc_2)(2n-1)$ and hence

$$\mu^D(K_n) = \frac{TDD}{{}^nc_2} = 2n - 1.$$

∎

Theorem 2 *The average D-distance of complete bipartite graph $K_{m,n}$ is*

$$\mu^D(K_{m,n}) = \frac{2mn(m+n+1) + n(n-1)(2m+n+2) + m(m-1)(2n+m+2)}{(m+n)(m+n-1)}.$$

Proof. Let $V(K_{m,n})$=$A \cup B$, where A= $\{v_1, v_2, v_3, ..., v_m\}$, B= $\{w_1, w_2, w_3, ..., w_n\}$. Then $d^D(v_i, v_j) = 2m+n+2, d^D(w_i, w_j) = 2n+m+2$ and $d^D(v_i, w_j) = m+n+1$. Thus, the total D-distance

$$TDD = ({}^mc_2)\,(2m+n+2) + ({}^nC_2)\,(2n+m+2) + mn(m+n+1).$$

Hence the average D-distance

$$\begin{aligned}\mu^D(K_{m,n}) &= \frac{TDD}{{}^{(m+n)}c_2}\\ &= \frac{2mn(m+n+1) + n(n-1)(2m+n+2) + m(m-1)(2n+m+2)}{(m+n)(m+n-1)}\end{aligned}$$

∎

Theorem 3 *The average D-distance of star graph $St_{n,1}$ is given by*

$$\mu^D(St_{n,1}) = \frac{2(n+2)+(n-1)(n+4)}{n-1}.$$

Proof. In Star graph there are $n+1$ vertices. For the central vertex there are n neighbor vertices and all other vertices has only one neighbor vertex, namely the central vertex. For the central vertex there are no distinct vertices, where as all others have $n-1$ distinct vertices. The D-distance between any vertex and its distinct vertex is $(n+2)(n+4)$. Finally, $TDD = n(n+2) + ({}^n c_2)\,(n+4)$, then

$$\mu^D(St_{n,1}) = \frac{TDD}{{}^n c_2} = \frac{2(n+2)+(n-1)(n+4)}{n-1}.$$

■

Theorem 4 *The average D-distance of the path graph P_n is*

$$\mu^D(P_n) = \frac{2a_n}{n},$$

where $a_n = a_{n-1} + n + 1$ with $a_1 = 0$.

Proof. For P_n, the D-distance matrix is the $n \times n$ symmetric matrix

$$\begin{bmatrix} 0 & 4 & 7 & 10 & \cdots & 3n-5 & 3n-2 \\ & 0 & 5 & 8 & \cdots & 3n-7 & 3n-4 \\ & & 0 & 5 & \cdots & 3n-12 & 3n-9 \\ & & & \cdots & \cdots & \cdots & \cdots \\ & & & & 0 & 5 & 7 \\ & & & & & 0 & 4 \\ & & & & & & 0 \end{bmatrix}$$

By adding all entries in the upper triangular or lower triangular matrix we get the total D-distance, which is $a_n(n-1)$ in this case, where $a_n (n \geq 2)$ is a constant given by $\{0, 3, 7, 12, 18, 25, \cdots\}$ or, recursively, $a_n = a_{n-1} + n + 1$ with $a_1 = 0$. Then,

$$\mu^D(P_n) = \frac{TDD}{({}^n C_2)} = \frac{2a_n}{n}.$$ ■

Theorem 5 *The average D-distance of the cyclic graph C_n is*

$$\mu^D(C_{2n}) = \frac{a_n}{2n-1},$$

where $a_n = a_{n-1} + 6n + 1$ with $a_2 = 18$ and $\mu^D(C_{2n-1}) = \frac{3}{2}n + 2$, $\forall n \geq 2$.

Proof. Case (i). First, we consider the case of cyclic graphs of odd order, (C_{2n-1}) $(n \geq 2)$. As $\mu^D(C_{2n-1})$ is regular, the elements of any row in the DDM, except the diagonal element are $0, 5, 8, 11, ..., 3n-1, 3n-1, ..., 11, 8, 5$. Then, the total D-distance, $TDD = 2(5+8+11+\cdots+(3n-1) = \frac{1}{4}(3n+4)(2n-1)(2n-2)$. Thus the average D-distance, $\mu^D(C_{2n-1}) = \frac{3}{2}n+2$.

Case (ii). Here, we consider the case of even order cyclic graphs $(C_{2n})(n \geq 2)$. In this case, the elements of any row in DDM except the diagonal element, are $0, 5, 8, 11, ..., 3n+2, ..., 11, 8, 5$ Like above we can show that $\mu^D(C_{2n}) = \dfrac{a_n}{2n-1}$ where $a_n = a_{n-1} + 6n + 1$ with $a_2 = 18$. ∎

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A NOTE ON HERMITE-HADAMARD INEQUALITIES FOR PRODUCTS OF CONVEX FUNCTIONS VIA RIEMANN-LIOUVILLE FRACTIONAL INTEGRALS[1]

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Abstract. In this paper, we obtain some new Hermite-Hadamard type inequalities for products of convex functions via Riemann-Liouville integrals. We conclude that Our methods considered here may be a stimulant for further investigations concerning Hermite-Hadamard type inequalities for products of various kinds of convex functions involving Riemann-Liouville fractional integrals.

Keywords: Hermite-Hadamard inequality; convex function; Riemann-Liouville fractional integrals.

2000 Mathematics Subject Classification: 26D15; 26D10.

1. Introduction

If $f : I \to R$ is a convex function on the interval I, then for any $a, b \in I$ with $a \neq b$ we have the following double inequality

$$f(\frac{a+b}{2}) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

This remarkable result is well known in the literature as the Hermite-Hadamard inequality.

Since then, some refinements of the Hermite-Hadamard inequality on convex functions have been extensively investigated by a number of authors (e.g., [1], [2], [3], [4], [5], [6], [7] and [8]).

In [9], B.G. Pachpatte established two new Hermite-Hadamard type inequalities for products of convex functions as follows.

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Theorem 1.1. *Let f and g be real-valued, nonnegative and convex functions on $[a,b]$. Then*

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \le \frac{1}{3}M(a,b)+\frac{1}{6}N(a,b), \tag{1.2}$$

$$2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)g(x)dx+\frac{1}{6}M(a,b)+\frac{1}{3}N(a,b), \tag{1.3}$$

where $M(a,b)=f(a)g(a)+f(b)g(b)$ and $N(a,b)=f(a)g(b)+f(b)g(a)$.

Some new integral inequalities involving two nonnegative and integrable functions that are related to the Hermite-Hadamard type are also obtained by many authors. In [10], B.G. Pachpatte proposed some Hermite-Hadamard type inequalities involving two log-convex functions. An analogous result for s-convex functions is established by Kirmaci et. al. in [8]. In [12], M.Z. Sarikaya presented some integral inequalities for two h-convex functions. For recent results and generalizations concerning Hermite-Hadamard type inequality for product of two functions see [13] and the references given therein.

It is remarkable that M.Z. Sarikaya et al. [11] proved the following interesting inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

Theorem 1.2. *Let $f:[a,b]\to R$ be a positive function with $a<b$ and $f\in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J_{a^+}^\alpha f(b)+J_{b^-}^\alpha f(a)] \le \frac{f(a)+f(b)}{2}, \tag{1.4}$$

with $\alpha>0$.

We remark that the symbol $J_{a^+}^\alpha$ and $J_{b^-}^\alpha f$ denote the left-sided and right-sided Riemann-Liouville fractional integrals of the order $\alpha \ge 0$ with $a \ge 0$ which are defined by

$$J_{a^+}^\alpha f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt, \qquad x>a,$$

and

$$J_{b^-}^\alpha f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt, \qquad x<b,$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function defined by

$$\Gamma(\alpha)=\int_0^\infty e^{-t}t^{\alpha-1}dt.$$

In this paper, we obtain some new Hermite-Hadamard type inequalities for products of convex functions via Riemann-Liouville integrals.

2. Main results

Theorem 2.1. *Let f and g be real-valued, nonnegative and convex functions on $[a,b]$. Then*

$$\begin{aligned}\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[J_{a+}^{\alpha}f(b)g(b)+J_{b-}^{\alpha}f(a)g(a)] &\leq \Big(\frac{\alpha}{\alpha+2}-\frac{\alpha}{\alpha+1}+\frac{1}{2}\Big)M(a,b)\\ &\quad+\frac{\alpha}{(\alpha+1)(\alpha+2)}N(a,b),\end{aligned}$$

where $M(a,b)=f(a)g(a)+f(b)g(b)$, $N(a,b)=f(a)g(b)+f(b)g(a)$.

Proof. Since f and g are convex on $[a,b]$, then for $t\in[0,1]$ we get

$$f(ta+(1-t)b)\leq tf(a)+(1-t)f(b), \tag{2.1}$$

and

$$g(ta+(1-t)b)\leq tg(a)+(1-t)g(b). \tag{2.2}$$

From (2.1) and (2.2), we get

$$\begin{aligned}f(ta+(1-t)b)g(ta+(1-t)b) &\leq t^2f(a)g(a)+(1-t)^2f(b)g(b)\\ &\quad+t(1-t)[f(a)g(b)+f(b)g(a)].\end{aligned}$$

Similarly, we have

$$\begin{aligned}f((1-t)a+tb)g((1-t)a+tb) &\leq (1-t)^2f(a)g(a)+t^2f(b)g(b)\\ &\quad+t(1-t)[f(a)g(b)+f(b)g(a)].\end{aligned}$$

So

$$\begin{aligned}&f(ta+(1-t)b)g(ta+(1-t)b)+f((1-t)a+tb)g((1-t)a+tb)\\ &\leq(2t^2-2t+1)[f(a)g(a)+f(b)g(b)]+2t(1-t)[f(a)g(b)+f(b)g(a)].\end{aligned}$$

Multiplying both sides of above inequality by $t^{\alpha-1}$, then integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\begin{aligned}&\int_0^1 t^{\alpha-1}f(ta+(1-t)b)g(ta+(1-t)b)dt\\ &\quad+\int_0^1 t^{\alpha-1}f((1-t)a+tb)g((1-t)a+tb)dt\\ &\quad=\int_b^a\Big(\frac{b-u}{b-a}\Big)^{\alpha-1}f(u)g(u)\frac{du}{a-b}+\int_a^b\Big(\frac{v-a}{b-a}\Big)^{\alpha-1}f(v)g(v)\frac{dv}{b-a}\\ &\quad=\frac{\Gamma(\alpha)}{(b-a)^{\alpha}}[J_{a+}^{\alpha}f(b)g(b)+J_{b-}^{\alpha}f(a)g(a)]\end{aligned}$$

$$
\begin{aligned}
&\leq [f(a)g(a)+f(b)g(b)]\int_0^1 t^{\alpha-1}(2t^2-2t+1)dt \\
&+2[f(a)g(b)+f(b)g(a)]\int_0^1 t^{\alpha-1}t(1-t)dt \\
&=\Big(\frac{2}{\alpha+2}-\frac{2}{\alpha+1}+\frac{1}{\alpha}\Big)[f(a)g(a)+f(b)g(b)] \\
&+\frac{2}{(\alpha+1)(\alpha+2)}[f(a)g(b)+f(b)g(a)] \\
&=\Big(\frac{2}{\alpha+2}-\frac{2}{\alpha+1}+\frac{1}{\alpha}\Big)M(a,b)+\frac{2}{(\alpha+1)(\alpha+2)}N(a,b).
\end{aligned}
$$

So

$$
\begin{aligned}
\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J_{a^+}^\alpha f(b)g(b)+J_{b^-}^\alpha f(a)g(a)] \quad \leq \quad & \Big(\frac{\alpha}{\alpha+2}-\frac{\alpha}{\alpha+1}+\frac{1}{2}\Big)M(a,b) \\
&+\frac{\alpha}{(\alpha+1)(\alpha+2)}N(a,b),
\end{aligned}
$$

which completes the proof. ∎

Corollary 2.2. *With the notations in Theorem* 2.1*, if we choose* $g:[a,b]\to R$ *as* $g(x)=1$ *for all* $x\in[a,b]$*, we obtain*

$$\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J_{a^+}^\alpha f(b)+J_{b^-}^\alpha f(a)]\leq\frac{f(a)+f(b)}{2},$$

which is the right hand side of (1.4).

Proof. Since $g(x)=1$ for all $x\in[a,b]$, from $M(a,b)=N(a,b)=f(a)+f(b)$, we can get the desire result. ∎

Corollary 2.3. *With the notations in Theorem* 2.1*, if* $\alpha=1$*, then*

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx\leq\frac{1}{3}M(a,b)+\frac{1}{6}N(a,b),$$

which is the right hand side of (1.2).

Theorem 2.4. *Let* f *and* g *be real-valued, nonnegative and convex functions on* $[a,b]$*. Then*

$$
\begin{aligned}
2f\Big(\frac{a+b}{2}\Big)g\Big(\frac{a+b}{2}\Big) \quad \leq \quad & \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J_{a^+}^\alpha f(b)g(b)+J_{b^-}^\alpha f(a)g(a)] \\
+ \quad & M(a,b)\frac{\alpha}{(\alpha+1)(\alpha+2)} \\
+ \quad & N(a,b)\Big(\frac{\alpha}{\alpha+2}-\frac{\alpha}{\alpha+1}+\frac{1}{2}\Big),
\end{aligned}
$$

where $M(a,b)=f(a)g(a)+f(b)g(b)$*,* $N(a,b)=f(a)g(b)+f(a)g(b)$*.*

Proof. We can write

$$\frac{a+b}{2}=\frac{(1-t)a+tb}{2}+\frac{ta+(1-t)b}{2},$$

so

$$\begin{aligned}
&f\Big(\frac{a+b}{2}\Big)g\Big(\frac{a+b}{2}\Big)\\
&\quad=f\Big(\frac{ta+(1-t)b}{2}+\frac{(1-t)a+tb}{2}\Big)g\Big(\frac{ta+(1-t)b}{2}+\frac{(1-t)a+tb}{2}\Big)\\
&\quad\le\frac{1}{4}\Big[f(ta+(1-t)b)+f((1-t)a+tb)\Big]\Big[g(ta+(1-t)b)+g((1-t)a+tb)\Big]\\
&\quad=\frac{1}{4}\Big[f(ta+(1-t)b)g(ta+(1-t)b)+f((1-t)a+tb)g((1-t)a+tb)\Big]\\
&\quad+\frac{1}{4}\Big[f(ta+(1-t)b)g((1-t)a+tb)+f((1-t)a+tb)g(ta+(1-t)b)\Big]\\
&\quad\le\frac{1}{4}\Big[f(ta+(1-t)b)g(ta+(1-t)b)+f((1-t)a+tb)g((1-t)a+tb)\Big]\\
&\quad+\frac{1}{4}\Bigg\{\Big[tf(a)+(1-t)f(b)\Big]\Big[(1-t)g(a)+tg(b)\Big]\\
&\quad+\Big[(1-t)f(a)+tf(b)\Big]\Big[tg(a)+(1-t)g(b)\Big]\Bigg\}\\
&\quad=\frac{1}{4}\Big[f(ta+(1-t)b)g(ta+(1-t)b)+f((1-t)a+tb)g((1-t)a+tb)\Big]\\
&\quad+\frac{1}{4}\Bigg\{2t(1-t)\Big[f(a)g(a)+f(b)g(b)\Big]+\Big[(1-t)^2+t^2\Big]\Big[f(a)g(b)+f(b)g(a)\Big]\Bigg\}.
\end{aligned}$$

Multiplying both sides of above inequality by $t^{\alpha-1}$, then integrating the resulting inequality with respect to t over $[0,1]$, we get

$$\begin{aligned}
&f\Big(\frac{a+b}{2}\Big)g\Big(\frac{a+b}{2}\Big)\int_0^1 t^{\alpha-1}dt\\
&\quad\le\frac{1}{4}\Big[\int_0^1 t^{\alpha-1}f(ta+(1-t)b)g(ta+(1-t)b)dt\\
&\quad+\int_0^1 t^{\alpha-1}f((1-t)a+tb)g((1-t)a+tb)dt\Big]\\
&\quad+\frac{1}{4}\Bigg\{\Big[f(a)g(a)+f(b)g(b)\Big]\int_0^1 t^{\alpha-1}2t(1-t)dt\\
&\quad+\Big[f(a)g(b)+f(b)g(a)\Big]\int_0^1 t^{\alpha-1}[(1-t)^2+t^2]dt\Bigg\}.
\end{aligned}$$

That is,

$$\begin{aligned}\frac{1}{\alpha}f\Big(\frac{a+b}{2}\Big)g\Big(\frac{a+b}{2}\Big) &\leq \frac{1}{4}\Big[\frac{\Gamma(\alpha)}{(b-a)^{\alpha}}[J^{\alpha}_{a^+}f(b)g(b)+J^{\alpha}_{b^-}f(a)g(a)]\Big]\\ &+ \frac{1}{4}\Big\{M(a,b)\int_0^1 t^{\alpha-1}2t(1-t)dt\\ &+ N(a,b)\int_0^1 t^{\alpha-1}[(1-t)^2+t^2]dt\Big\}.\end{aligned}$$

From

$$\int_0^1 t^{\alpha-1}2t(1-t)dt=\frac{2}{(\alpha+1)(\alpha+2)}$$

and

$$\int_0^1 t^{\alpha-1}[(1-t)^2+t^2)]dt=\Big(\frac{2}{\alpha+2}-\frac{2}{\alpha+1}+\frac{1}{\alpha}\Big),$$

we get

$$\begin{aligned}2f\Big(\frac{a+b}{2}\Big)g\Big(\frac{a+b}{2}\Big) &\leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[J^{\alpha}_{a^+}f(b)g(b)+J^{\alpha}_{b^-}f(a)g(a)]\\ &+ M(a,b)\frac{\alpha}{(\alpha+1)(\alpha+2)}\\ &+ N(a,b)\Big(\frac{\alpha}{\alpha+2}-\frac{\alpha}{\alpha+1}+\frac{1}{2}\Big),\end{aligned}$$

which completes the proof. ∎

Corollary 2.5. *With the notations in Theorem* 2.4*, if* $\alpha=1$*, then*

$$2f\Big(\frac{a+b}{2}\Big)g\Big(\frac{a+b}{2}\Big)\leq\frac{1}{b-a}\int_a^b f(x)g(x)dx+\frac{1}{6}M(a,b)+\frac{1}{3}N(a,b),$$

which is the right hand side of (1.3).

Corollary 2.6. *With the notations in Theorem* 2.4*, if we choose* $g:[a,b]\to R$ *as* $g(x)=1$ *for all* $x\in[a,b]$*, we have*

$$2f\Big(\frac{a+b}{2}\Big)\leq\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[J^{\alpha}_{a^+}f(b)+J^{\alpha}_{b^-}f(a)]+\frac{f(a)+f(b)}{2}.$$

3. Conclusion

In this paper, we establish some new Hermite-Hadamard type inequalities for products of convex functions via Riemann-Liouville integrals. An interesting topic is whether we can use the methods in this paper to establish the Hermite-Hadamard inequality for other kinds of convex functions.

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THE HOMOGENEOUS BALANCE METHOD AND ITS APPLICATIONS FOR FINDING THE EXACT SOLUTIONS FOR NONLINEAR EVOLUTION EQUATIONS

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Abstract. In this article, we apply the homogeneous balance method to find the exact solutions of some nonlinear evolution equations in mathematical physics, namely, the Kaup-Kupershmidt equation, the Ito equation, the Caudrey-Dodd-Gibbon equation, the Lax equation and the Sawada-Kotera equation. These equations have wide applications in quantum mechanics and non linear optics. The efficiency of this method for constructing these exact solutions is demonstrated.

Keywords: the homogeneous balance method, nonlinear evolution equations, exact solutions.

AMS Subject Classifications: 35K99, 35P05, 35P99.

1. Introduction

When a nonlinear evolution equation is analyzed, one of the most important question is the construction of the exact solutions of that equation. Searching for exact solutions of that equation plays an important role in the study of nonlinear physical phenomena. Nonlinear wave phenomena appears in various scientific and engineering fields, such as fluid mechanics, plasma physics, optical fibers, biology, solid state physics, chemical kinematics, chemical physics, geochemistry and so on. In the past several decades, exact solutions may help to find new phenomena. Many powerful methods for obtaining these exact solutions are presented, such as the inverse scattering transform [1], the Hirota method [3], the truncated Painleve expansion [9], the Backlund transform [1], [15], the exp-function method [20], [21], the simplest equation method [10], the Weierstrass elliptic function method [7], the Jacobi elliptic function method [13], [14], the tanh-function method [18], the (G'/G)-expansion method [22], the modified simple equation method [5], [25], the homogeneous balance method [8], [16], [17], [26] and so on. To realize these

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methods, one applies some special functions, then exact solutions read as a finite series in these special functions.

The objective of this article is to demonstrate efficiency of the homogeneous balance method for finding exact solutions of some nonlinear evolution equations in the mathematical physics, namely, the Kaup-Kupershmidt equation, the Ito equation, the Caudrey-Dodd-Gibbon equation, the Lax equation and the Sawada-Kotera equation. This article is organized as follows. In Section 2, we give the description of the homogeneous balance method. In Section 3, we apply this method to five nonlinear evolution equations indicated above. In Section 4, physical explanations of our obtained solutions are given. In Section 5, conclusions are given.

2. Description of the homogeneous balance method

Suppose we have a nonlinear evolution equation in the form

$$F(u, u_t, u_x, u_{xx}, ...) = 0, \tag{2.1}$$

where F is a polynomial in $u(x,t)$ and its partial derivatives in which the highest order derivatives and nonlinear terms are involved. In the following, we give the main steps of this method [8], [16], [17], [26]:

Step 1. Using the wave transformation

$$u(x,t) = u(\xi),\ \xi = kx + \omega t, \tag{2.2}$$

to reduce equation (2.1) to the following ODE:

$$P(u, u', u'', ...) = 0, \tag{2.3}$$

where P is a polynomial in $u(\xi)$ and its total derivatives, while k, ω are constants and $' = d/d\xi$

Step 2. We suppose that equation (2.3) has the formal solution

$$u(\xi) = \sum_{n=0}^{N} a_n Q(\xi)^n, \tag{2.4}$$

where $a_n (n = 0, 1, ...N)$ are constants to be determined, such that $a_N \neq 0$, and $Q(\xi)$ is the solution of the equation

$$Q'(\xi) = Q^2(\xi) - Q(\xi). \tag{2.5}$$

equation (2.5) has the solution

$$Q(\xi) = \frac{1}{1 \pm e^{\xi}}. \tag{2.6}$$

Step 3. We determine the positive integer N in equation (2.4) by considering the homogeneous balance between the highest order derivatives and the nonlinear terms in equation (2.3).

Step 4. Substitute equation (2.4) into equation (2.3), we calculate all the necessary derivatives $u', u'', ...$ of the function $u(\xi)$. As a result of this substitution, we get a polynomial of $Q^i, (i = 0, 1, 2, ...)$. In this polynomial we gather all terms of same powers and equating them to zero, we obtain a system of algebraic equations which can be solved by the Maple or Mathematica to get the unknown parameters a_n $(n = 0, 1, ..., N)$, k and ω. Consequently, we obtain the exact solutions of equation (2.1).

3. Applications

In this section, we apply the homogeneous balance method to find the exact solutions of the following nonlinear partial differential equations:

3.1. Example 1. The Kaup-Kupershmidt (KK) equation

This equation is well known [6], [11], [23], [24] and has the form

$$u_t + 20u^2 u_x + 25u_x u_{xx} + 10uu_{3x} + u_{5x} = 0. \tag{3.1}$$

Let us now solve equation (3.1) by using the homogeneous balance method. To this end, we use the wave transformation (2.2) to reduce equation (3.1) to the following ODE:

$$\omega u' + 20ku^2 u' + 25k^3 u' u'' + 10k^3 uu^{(3)} + k^5 u^{(5)} = 0. \tag{3.2}$$

Balancing $u^{(5)}$ with $u^2 u'$ yields $N = 2$. Consequently, equation (3.2) has the formal solution

$$u = a_0 + a_1 Q + a_2 Q^2, \tag{3.3}$$

where a_0, a_1 and a_2 are constants to be determined such that $a_2 \neq 0$. From equation (3.3), we get

$$u' = (Q-1)Q\left(a_1 + 2Qa_2\right), \tag{3.4}$$

$$u'' = (Q-1)Q\left[(-1+2Q)a_1 + 2Q(-2+3Q)a_2\right], \tag{3.5}$$

$$u^{(3)} = (Q-1)Q\left[\left(1-6Q+6Q^2\right)a_1 + 2Q\left(4-15Q+12Q^2\right)a_2\right], \tag{3.6}$$

$$u^{(4)} = (Q-1)Q[\left(-1+14Q-36Q^2+24Q^3\right)a_1 \\ +2Q\left(-8+57Q-108Q^2+60Q^3\right)a_2], \tag{3.7}$$

$$u^{(5)} = (Q-1)Q[\left(1-30Q+150Q^2-240Q^3+120Q^4\right)a_1 \tag{3.8}$$

$$+2Q\left(16-195Q+660Q^2-840Q^3+360Q^4\right)a_2]. \tag{3.9}$$

Substituting (3.3)-(3.8) into (3.2) and equating all the coefficients of powers of $Q(\xi)$ to zero, we obtain

$$-k^5a_1 - \omega a_1 - 10k^3a_0a_1 - 20ka_0^2a_1 = 0, \tag{3.10}$$

$$\begin{aligned}&31k^5a_1 + \omega a_1 + 70k^3a_0a_1 + 20ka_0^2a_1 - 35k^3a_1^2 - \\ &- 40ka_0a_1^2 - 32k^5a_2 - 2\omega a_2 - 80k^3a_0a_2 - 40ka_0^2a_2 = 0,\end{aligned} \tag{3.11}$$

$$\begin{aligned}&- 180k^5a_1 - 120k^3a_0a_1 + 170k^3a_1^2 + 40ka_0a_1^2 - 20ka_1^3 + 422k^5a_2 \\ &+ 2\omega a_2 + 380k^3a_0a_2 + 40ka_0^2a_2 - 240k^3a_1a_2 - 120ka_0a_1a_2 = 0,\end{aligned} \tag{3.12}$$

$$\begin{aligned}&390k^5a_1 + 60k^3a_0a_1 - 245k^3a_1^2 + 20ka_1^3 - 1710k^5a_2 - 540k^3a_0a_2 \\ &+ 1000k^3a_1a_2 + 120ka_0a_1a_2 - 80ka_1^2a_2 - 280k^3a_2^2 - 80ka_0a_2^2 = 0,\end{aligned} \tag{3.13}$$

$$\begin{aligned}&- 360k^5a_1 + 110k^3a_1^2 + 3000k^5a_2 + 240k^3a_0a_2 - 1310k^3a_1a_2 + 80ka_1^2a_2 \\ &+ 1080k^3a_2^2 + 80ka_0a_2^2 - 100ka_1a_2^2 = 0,\end{aligned} \tag{3.14}$$

$$120k^5a_1 - 2400k^5a_2 + 550k^3a_1a_2 - 1340k^3a_2^2 + 100ka_1a_2^2 - 40ka_2^3 = 0, \tag{3.15}$$

$$720k^5a_2 + 540k^3a_2^2 + 40ka_2^3 = 0. \tag{3.16}$$

Solving the system of equations (3.9)-(3.15) by using the Maple or Mathematica, we obtain

Case 1.

$$\omega = -\frac{k^5}{16}, \quad a_0 = -\frac{k^2}{8}, \quad a_1 = \frac{3k^2}{2}, \quad a_2 = -a_1. \tag{3.17}$$

The solution of equation (3.1) corresponding to (3.16) is

$$u_1(x,t) = \frac{-k^2}{8} + \frac{3k^2}{8}\sec h^2\left[\frac{k}{2}x - \frac{k^5}{32}t\right], \tag{3.18}$$

$$u_2(x,t) = \frac{-k^2}{8} - \frac{3k^2}{8}\csc h^2\left[\frac{k}{2}x - \frac{k^5}{32}t\right], \tag{3.19}$$

Case 2.

$$\omega = -11k^5, \quad a_0 = -k^2, \quad a_1 = 12k^2, \quad a_2 = -a_1. \tag{3.20}$$

The solution of equation (3.1) corresponding to (3.19) is

$$u_3(x,t) = -k^2 + 3k^2\sec h^2\left[\frac{k}{2}x - \frac{11k^5}{2}t\right], \tag{3.21}$$

$$u_4(x,t) = -k^2 - 3k^2\csc h^2\left[\frac{k}{2}x - \frac{11k^5}{2}t\right]. \tag{3.22}$$

3.2. Example 2. The Ito equation

This equation is well known [4], [23], [24] and has the form:

$$u_t + 2u^2 u_x + 6u_x u_{xx} + 3uu_{3x} + u_{5x} = 0. \tag{3.23}$$

Let us solve equation (3.22) by using the homogeneous balance method. To this end, we use the wave transformation (2.2) to reduce equation (3.22) to the following ODE:

$$\omega u' + 2ku^2 u' + 6k^3\ u'u'' + 3k^3 uu^{(3)} + k^5 u^{(5)} = 0. \tag{3.24}$$

Balancing $u^{(5)}$ with u^2u' yields $N = 2$. Consequently, equation (3.23) has the formal solution (3.3). Substituting (3.3)-(3.8) into (3.23) and equating all the coefficients of powers of $Q(\xi)$ to zero, we obtain

$$-k^5 a_1 - \omega a_1 - 3k^3 a_0 a_1 - 2k a_0^2 a_1 = 0, \tag{3.25}$$

$$\begin{aligned} &31k^5 a_1 + \omega a_1 + 21k^3 a_0 a_1 + 2k a_0^2 a_1 - 9k^3 a_1^2 - 4k a_0 a_1^2 - 32k^5 a_2 \\ &\quad -2\omega a_2 - 24k^3 a_0 a_2 - 4k a_0^2 a_2 = 0, \end{aligned} \tag{3.26}$$

$$\begin{aligned} &-180k^5 a_1 - 36k^3 a_0 a_1 + 45k^3 a_1^2 + 4k a_0 a_1^2 - 2k a_1^3 + 422k^5 a_2 \\ &\quad +2\omega a_2 + 114k^3 a_0 a_2 + 4k a_0^2 a_2 - 63k^3 a_1 a_2 - 12k a_0 a_1 a_2 = 0, \end{aligned} \tag{3.27}$$

$$\begin{aligned} &390k^5 a_1 + 18k^3 a_0 a_1 - 66k^3 a_1^2 + 2k a_1^3 - 1710k^5 a_2 - 162k^3 a_0 a_2 \\ &\quad +267k^3 a_1 a_2 + 12k a_0 a_1 a_2 - 8k a_1^2 a_2 - 72k^3 a_2^2 - 8k a_0 a_2^2 = 0, \end{aligned} \tag{3.28}$$

$$\begin{aligned} &-360k^5 a_1 + 30k^3 a_1^2 + 3000k^5 a_2 + 72k^3 a_0 a_2 - 354k^3 a_1 a_2 \\ &\quad +8k a_1^2 a_2 + 282k^3 a_2^2 + 8k a_0 a_2^2 - 10k a_1 a_2^2 = 0, \end{aligned} \tag{3.29}$$

$$120k^5 a_1 - 2400k^5 a_2 + 150k^3 a_1 a_2 - 354k^3 a_2^2 + 10k a_1 a_2^2 - 4k a_2^3 = 0, \tag{3.30}$$

$$720k^5 a_2 + 144k^3 a_2^2 + 4k a_2^3 = 0. \tag{3.31}$$

Solving the system of equations (3.24)-(3.30), using the Maple or Mathematica we obtain

$$\omega = -6k^5, \quad a_0 = -\frac{5k^2}{2}, \quad a_1 = 30k^2, \quad a_2 = -a_1. \tag{3.32}$$

The solution of equation (3.22) corresponding to (3.31) is

$$u_1(x,t) \;=\; \frac{-5k^2}{2} + \frac{15k^2}{2} \sec h^2 \left[\frac{k}{2}x - \frac{6k^5}{2}t\right], \tag{3.33}$$

$$u_2(x,t) \;=\; \frac{-5k^2}{2} - \frac{15k^2}{2} \csc h^2 \left[\frac{k}{2}x - \frac{6k^5}{2}t\right]. \tag{3.34}$$

3.3. Example 3. The Caudrey-Dodd-Gibbon equation (CDG)

This equation is well known [2], [23], [24] and has the form:

$$u_t + 180u^2u_x + 30u_xu_{xx} + 30uu_{3x} + u_{5x} = 0. \tag{3.35}$$

Let us solve equation (3.34) by using the homogeneous balance method. To this end. we use the wave transformation (2.2) to reduce equation (3.34) to the following ODE :

$$\omega u' + 180ku^2u' + 30k^3\ u'u'' + 30\ k^3uu^{(3)}\ + k^5u^{(5)} = 0. \tag{3.36}$$

Balancing $u^{(5)}$ with u^2u' yields $N = 2$. Consequently, equation (3.35) has the formal solution (3.3). Substituting (3.3)-(3.8) into (3.35) and equating all the coefficients of powers of $Q(\xi)$ to zero, we obtain

$$-k^5a_1 - \omega a_1 - 30k^3a_0a_1 - 180ka_0^2a_1 = 0, \tag{3.37}$$

$$\begin{aligned} &31k^5a_1 + \omega a_1 + 210k^3a_0a_1 + 180ka_0^2a_1 - 60k^3a_1^2 \\ &\quad -360ka_0a_1^2 - 32k^5a_2 - 2\omega a_2 - 240k^3a_0a_2 - 360ka_0^2a_2 = 0, \end{aligned} \tag{3.38}$$

$$\begin{aligned} &-180k^5a_1 - 360k^3a_0a_1 + 330k^3a_1^2 + 360ka_0a_1^2 - 180ka_1^3 \\ &\quad +422k^5a_2 + 2\omega a_2 + 1140k^3a_0a_2 + 360ka_0^2a_2 - 450k^3a_1a_2 \\ &\quad -1080ka_0a_1a_2 = 0, \end{aligned} \tag{3.39}$$

$$\begin{aligned} &390k^5a_1 + 180k^3a_0a_1 - 510k^3a_1^2 + 180ka_1^3 - 1710k^5a_2 \\ &\quad -1620k^3a_0a_2 + 2010k^3a_1a_2 + 1080ka_0a_1a_2 - 720ka_1^2a_2 \\ &\quad -480k^3a_2^2 - 720ka_0a_2^2 = 0, \end{aligned}$$

$$\begin{aligned} &-360k^5a_1 + 240k^3a_1^2 + 3000k^5a_2 + 720k^3a_0a_2 - 2760k^3a_1a_2 \\ &\quad +720ka_1^2a_2 + 1980k^3a_2^2 + 720ka_0a_2^2 - 900ka_1a_2^2 = 0, \end{aligned} \tag{3.40}$$

$$\begin{aligned} &120k^5a_1 - 2400k^5a_2 + 1200k^3a_1a_2 - 2580k^3a_2^2 \\ &\quad +900ka_1a_2^2 - 360ka_2^3 = 0, \end{aligned} \tag{3.41}$$

$$720k^5a_2 + 1080k^3a_2^2 + 360ka_2^3 = 0. \tag{3.42}$$

Solving the system of equations (3.36)-(3.42) using the Maple or Mathematica, we obtain

Case 1.

$$\omega = -k^5 - 30k^3a_0 - 180ka_0{}^2, \quad a_1 = k^2, \quad a_2 = -a_1. \tag{3.43}$$

The solution of equation (3.34) corresponding to (3.43) is

$$u_1(x,t) \;=\; a_0 + \frac{k^2}{4}\sec h^2\left[\frac{k}{2}\ x - \frac{(k^5 + 30k^3a_0 + 180ka_0{}^2)}{2}\ t\right], \tag{3.44}$$

$$u_2(x,t) \;=\; a_0 - \frac{k^2}{4}\csc h^2\left[\frac{k}{2}\ x - \frac{(k^5 + 30k^3a_0 + 180ka_0{}^2)}{2}\ t\right], \tag{3.45}$$

where a_0 is an arbitrary constant.

Case 2.

$$\omega = -k^5, \quad a_0 = -\frac{k^2}{6}, \quad a_1 = 2k^2, \quad a_2 = -a_1. \tag{3.46}$$

The solution of equation (3.34) corresponding to (3.46) is

$$u_1(x,t) = \frac{-k^2}{6} + \frac{k^2}{2}\sec h^2\left[\frac{k}{2}\,x - \frac{k^5}{2}\,t\right]. \tag{3.47}$$

$$u_2(x,t) = \frac{-k^2}{6} - \frac{k^2}{2}\csc h^2\left[\frac{k}{2}\,x - \frac{k^5}{2}\,t\right]. \tag{3.48}$$

3.4. Example 4. The Lax equation

This equation is well known [12], [23], [24] and has the form:

$$u_t + 30u^2u_x + 20u_xu_{xx} + 10uu_{3x} + u_{5x} = 0. \tag{3.49}$$

Let us now solve equation (3.49) by using the homogeneous balance method. To this end, we use the wave transformation (2.2) to reduce equation (3.49) to the following ODE:

$$\omega u' + 30ku^2u' + 20k^3\ u'u'' + 10k^3uu^{(3)} + k^5u^{(5)} = 0. \tag{3.50}$$

Balancing $u^{(5)}$ with u^2u' yields $N = 2$. Consequently, equation (3.50) has the formal solution (3.3). Substituting (3.3)-(3.8) into (3.50) and equating all the coefficients of powers of $Q(\xi)$ to zero, we obtain

$$-k^5a_1 - \omega a_1 - 10k^3a_0a_1 - 30ka_0^2a_1 = 0, \tag{3.51}$$

$$\begin{aligned} &31k^5a_1 + \omega a_1 + 70k^3a_0a_1 + 30ka_0^2a_1 - 30k^3a_1^2 - 60ka_0a_1^2 \\ &\quad -32k^5a_2 - 2\omega a_2 - 80k^3a_0a_2 - 60ka_0^2a_2 = 0, \end{aligned} \tag{3.52}$$

$$\begin{aligned} &-180k^5a_1 - 120k^3a_0a_1 + 150k^3a_1^2 + 60ka_0a_1^2 - 30ka_1^3 \\ &\quad +422k^5a_2 + 2\omega a_2 + 380k^3a_0a_2 + 60ka_0^2a_2 \\ &\quad -210k^3a_1a_2 - 180ka_0a_1a_2 = 0, \end{aligned} \tag{3.53}$$

$$\begin{aligned} &390k^5a_1 + 60k^3a_0a_1 - 220k^3a_1^2 + 30ka_1^3 - 1710k^5a_2 \\ &\quad -540k^3a_0a_2 + 890k^3a_1a_2 + 180ka_0a_1a_2 \end{aligned} \tag{3.54}$$

$$\quad -120ka_1^2a_2 - 240k^3a_2^2 - 120ka_0a_2^2 = 0, \tag{3.55}$$

$$\begin{aligned} &-360k^5a_1 + 100k^3a_1^2 + 3000k^5a_2 + 240k^3a_0a_2 - 1180k^3a_1a_2 \\ &\quad +120ka_1^2a_2 + 940k^3a_2^2 + 120ka_0a_2^2 - 150ka_1a_2^2 = 0, \end{aligned} \tag{3.56}$$

$$\begin{aligned} &120k^5a_1 - 2400k^5a_2 + 500k^3a_1a_2 - 1180k^3a_2^2 \\ &\quad +150ka_1a_2^2 - 60ka_2^3 = 0, \end{aligned} \tag{3.57}$$

$$720k^5a_2 + 480k^3a_2^2 + 60ka_2^3 = 0. \tag{3.58}$$

Solving the system of equations (3.51)-(3.57)using the Maple or Mathematica, we obtain

Case 1.

$$\omega = -k^5 - 10k^3 a_0 - 30ka_0^2, \quad a_1 = 2k^2, \quad a_2 = -a_1. \tag{3.59}$$

The solution of equation (3.49) corresponding (3.58) is

$$u_1(x,t) = a_0 + \frac{k^2}{2} \sec h^2 \left[\frac{k}{2}\, x - \frac{(k^5 + 10k^3 a_0 + 30ka_0^2)}{2}\, t\right], \tag{3.60}$$

$$u_2(x,t) = a_0 - \frac{k^2}{2} \csc h^2 \left[\frac{k}{2}\, x - \frac{(k^5 + 10k^3 a_0 + 30ka_0^2)}{2}\, t\right], \tag{3.61}$$

where a_0 is an arbitrary constant.

Case 2.

$$\omega = -\frac{7k^5}{2}, \quad a_0 = -\frac{k^2}{2}, \quad a_1 = 6k^2, \quad a_2 = -a_1. \tag{3.62}$$

The solution of equation (3.49) corresponding (3.61) is

$$u_3(x,t) = \frac{-k^2}{2} + \frac{3k^2}{2} \sec h^2 \left[\frac{k}{2} x - \frac{7k^5}{4} t\right], \tag{3.63}$$

$$u_4(x,t) = \frac{-k^2}{2} - \frac{3k^2}{2} \csc h^2 \left[\frac{k}{2} x - \frac{7k^5}{4} t\right]. \tag{3.64}$$

3.5. Example 5. The Sawada-Kotera (SK) equation

This equation is well known [19], [23], [24] and has the form:

$$u_t + 5u^2 u_x + 5u_x u_{xx} + 5uu_{3x} + u_{5x} = 0. \tag{3.65}$$

Let us solve equation (3.64)using the homogeneous balance method. To this end, we use the wave transformation (2.2) to reduce equation (3.64) to the following ODE:

$$\omega u' + 5ku^2 u' + 5k^3\, u' u'' + 5k^3 uu^{(3)} + k^5 u^{(5)} = 0. \tag{3.66}$$

Balancing $u^{(5)}$ with $u^2 u'$ yields $N = 2$. Consequently, equation (3.65) has the formal solution (3.3). Substituting (3.3)-(3.8) into (3.65) and equating all the coefficients of powers of $Q(\xi)$ to zero, we obtain

$$-k^5a_1 - \omega a_1 - 5k^3a_0a_1 - 5ka_0^2a_1 = 0, \tag{3.67}$$

$$31k^5a_1 + \omega a_1 + 35k^3a_0a_1 + 5ka_0^2a_1 - 10k^3a_1^2 - 10ka_0a_1^2 - 32k^5a_2 = 0$$
$$-2\omega a_2 - 40k^3a_0a_2 - 10ka_0^2a_2 = 0, \tag{3.68}$$

$$\begin{aligned} &-180k^5a_1 - 60k^3a_0a_1 + 55k^3a_1^2 + 10ka_0a_1^2 - 5ka_1^3 \\ &\quad +422k^5a_2 + 2\omega a_2 + 190k^3a_0a_2 + 10ka_0^2a_2 \\ &\quad -75k^3a_1a_2 - 30ka_0a_1a_2 = 0, \end{aligned} \tag{3.69}$$

$$\begin{aligned} &390k^5a_1 + 30k^3a_0a_1 - 85k^3a_1^2 + 5ka_1^3 - 1710k^5a_2 \\ &\quad -270k^3a_0a_2 + 335k^3a_1a_2 + 30ka_0a_1a_2 \end{aligned} \tag{3.70}$$

$$-20ka_1^2a_2 - 80k^3a_2^2 - 20ka_0a_2^2 = 0, \tag{3.71}$$

$$\begin{aligned} &-360k^5a_1 + 40k^3a_1^2 + 3000k^5a_2 + 120k^3a_0a_2 - 460k^3a_1a_2 \\ &\quad +20ka_1^2a_2 + 330k^3a_2^2 + 20ka_0a_2^2 - 25ka_1a_2^2 = 0, \end{aligned} \tag{3.72}$$

$$\begin{aligned} &120k^5a_1 - 2400k^5a_2 + 200k^3a_1a_2 - 430k^3a_2^2 \\ &\quad +25ka_1a_2^2 - 10ka_2^3 = 0, \end{aligned} \tag{3.73}$$

$$720k^5a_2 + 180k^3a_2^2 + 10ka_2^3 = 0. \tag{3.74}$$

Solving the system of equations (3.66)-(3.72) using the Maple or Mathematica, we obtain

Case 1.

$$\omega = -k^5 - 5k^3a_0 - 5ka_0^2, \quad a_1 = 6k^2, \quad a_2 = -a_1. \tag{3.75}$$

The solution of equation (3.64) corresponding (3.73) is

$$u_1(x,t) = a_0 + \frac{3k^2}{2}\sec h^2\left[\frac{k}{2}\,x - \frac{(k^5 + 5k^3a_0 + 5ka_0^2)}{2}\,t\right], \tag{3.76}$$

$$u_2(x,t) = a_0 - \frac{3k^2}{2}\csc h^2\left[\frac{k}{2}\,x - \frac{(k^5 + 5k^3a_0 + 5ka_0^2)}{2}\,t\right], \tag{3.77}$$

where a_0 is an arbitrary constant.

Case 2.

$$\omega = -k^5, \quad a_0 = -k^2, \quad a_1 = 12k^2, \quad a_2 = -a_1. \tag{3.78}$$

The solution of equation (3.64) corresponding (3.76) is

$$u_3(x,t) = -k^2 + 3k^2\sec h^2\left[\frac{k}{2}\,x - \frac{k^5}{2}\,t\right], \tag{3.79}$$

$$u_4(x,t) = -k^2 - 3k^2\csc h^2\left[\frac{k}{2}\,x - \frac{k^5}{2}\,t\right]. \tag{3.80}$$

4. Physical explanations of our obtained solutions

Solitary bell-type waves have been obtained. In this section we have presented some graphs of these solutions by taking suitable values of involved unknown parameters to visualize the underlying mechanism of the original equations. Using mathematical software Maple or Mathematica, the plots of some obtained solutions of equations (3.1), (3.22) and (3.34) have been shown in Figs. 1-3.

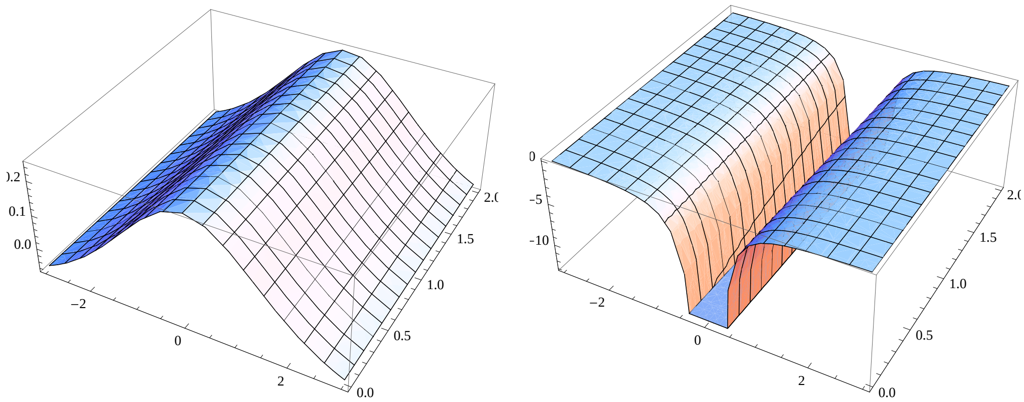

Figure 1: The plot of the solutions (3.17) and (3.18), when $k = 1$

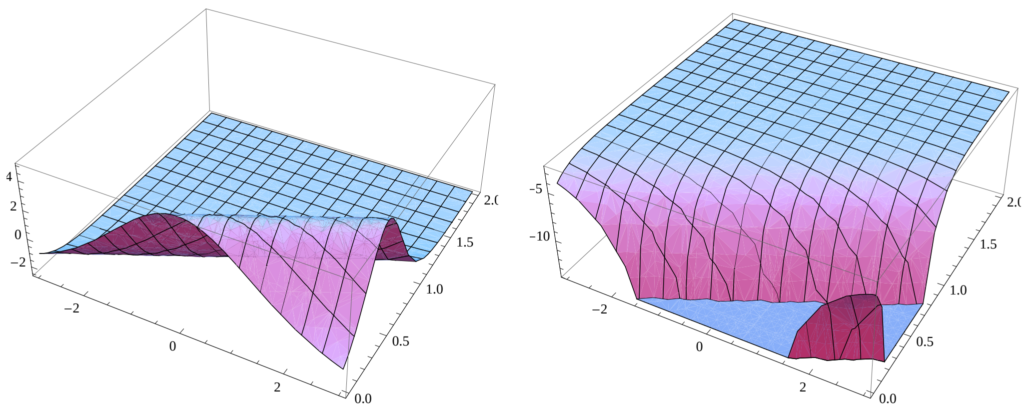

Figure 2: The plot of the solutions (3.32) and (3.33), when $k = 1$

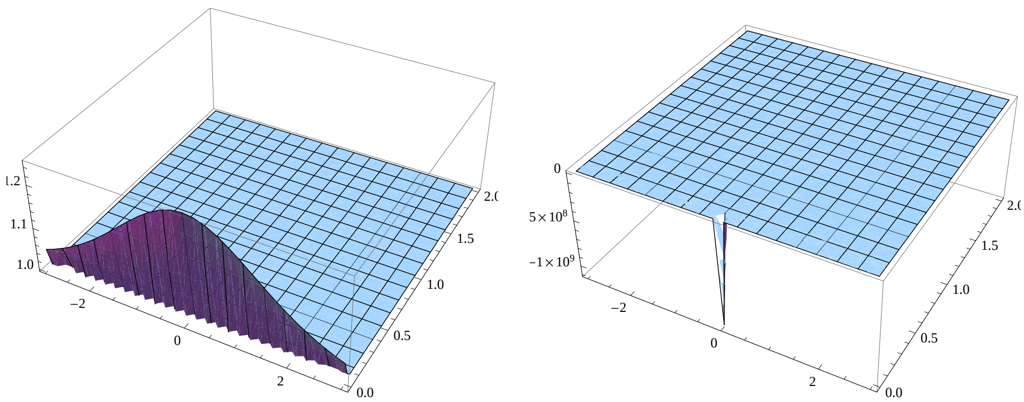

Figure 3: The plot of the solutions (3.44) and (3.45), when $a_0 = 1$, $k = 1$

5. Conclusions

The homogeneous balance method presented in this article has been applied to the nonlinear Kaup-Kupershmidt equation, the nonlinear Ito equation, the nonlinear Caudrey-Dodd-Gibbon equation, the nonlinear Lax equation and the Sawada-Kotera equation for finding the exact solutions of these equations which attract the attention of many authors. On comparing this method with the other methods, we see that the homogeneous balance method is much more simpler than these methods. Also we deduce that the homogeneous balance method is direct, effective and can be applied to many other nonlinear evolution equations.

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SOME STRUCTURAL PROPERTIES OF HYPER KS-SEMIGROUPS

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Abstract. This study introduces a new class of algebra related to hyper BCK-algebras and semihypergroups, called hyper KS-semigroups. It presents some characterizations of a hyper KS-semigroup with respect to its hyper subKS-semigroups, hyper KS-ideals, and reflexive hyper KS-ideals and their relationships. A quotient structure is constructed from a hyper KS-semigroup via a reflexive hyper KS-ideal and some properties are established. This paper also shows some properties of hyper KS-semigroups homomorphism and specifically, the three isomorphism theorems for hyper KS-semigroups are proved. Moreover, this paper shows that the hyper product of any nonempty finite family of hyper KS-semigroups is also a hyper KS-semigroup and investigates some related properties.

Keywords: hyper KS-semigroup, hyper subKS-semigroup, hyper KS-ideal, reflexive hyper KS-ideal, hyper P-ideal.

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1. Introduction

In 1966, Y. Imai and K. Iseki introduced a class of algebra with one binary operation called BCK-algebra in their paper entitled "On Axiom systems of Propositional Calculi XIV" [7]. This algebra is a generalization of the concept of set-theoretic difference and propositional calculi. On the other hand, K. H. Kim [11]

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introduced a new class of algebra called KS-semigroups, which is a combination of BCK-algebra and semigroup. He characterized the KS-semigroups from its ideal up to the first isomorphism theorem. Cawi, in her masteral thesis [3], proved the second and third isomorphism theorems for KS-semigroups and gave other characterizations parallel to ring theory.

Hyperstructure theory (also called multivalued algebras) was introduced by F. Marty at the 8th congress of Scandinavian Mathematicians in 1934. Recall that in a classical algebraic structure, the image of two elements of a set is an element of the set, while in an algebraic hyperstructure, the image of two elements is a set. This theory has been studied by many mathematicians and several books have been written in this topic, for example see [4], [5], [6]. It is considered as a generalization of classical algebraic structures. In this paper, we apply the theory of hyperstructures to KS-semigroups and provide some characterizations.

2. Preliminaries

Let H be a non-empty set endowed with a *hyper operation* " $*$ " that is, " $*$ " is a function from $H \times H$ to $P^*(H) = P(H) \setminus \{\varnothing\}$. For two nonempty subsets A and B of H, $A * B = \bigcup\limits_{a \in A, b \in B} a * b$. We shall use $x * y$ instead of $x * \{y\}, \{x\} * y$, or $\{x\} * \{y\}$. When A is a nonempty subset of H and $x \in H$, we agree to write $A * x$ instead of $A * \{x\}$. Similarly, we write $x * A$ for $\{x\} * A$. In effect, $A * x = \bigcup\limits_{a \in A} a * x$ and $x * A = \bigcup\limits_{a \in A} x * a$.

Definition 2.1 [8] A *semihypergroup* is a hypergroupoid $(H, \cdot)$ such that for all $x, y, z \in H$, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$, that is, $\bigcup\limits_{u \in x \cdot y} u \cdot z = \bigcup\limits_{v \in y \cdot z} x \cdot v$.

Definition 2.2 [9] A *hyper BCK-algebra* is a nonempty set H endowed with a hyperoperation " $*$ " and a constant 0 satisfying the following axioms: for all $x, y, z \in H$,

(H1) $(x * z) * (y * z) < x * y$,

(H2) $(x * y) * z = (x * z) * y$,

(H3) $x * H < x$,

(H4) $x < y$ and $y < x$ imply $x = y$,

where (a) $x < y$ is defined by $0 \in x * y$, and (b) for every $A, B \subseteq H$, $A < B$ is defined as follows: for all $a \in A$, there exists $b \in B$ such that $a < b$. In such case, we call " $<$ " the *hyper order* in H.

Theorem 2.3 [9] *In any hyper BCK-algebra* $(H, *, 0)$, *the following hold: for any* $x, y, z \in H$ *and for any nonempty subsets* A, B *of* H, (a1) $x < x$ *and* (a2) $A \subseteq B$ *implies* $A < B$.

Definition 2.4 [9] Let I be a nonempty subset of a hyper BCK-algebra $(H, *, 0)$. Then I is said to be a *hyper BCK-ideal* of H if $0 \in I$ and $x * y < I$ and $y \in I$ imply $x \in I$ for all $x, y \in H$. If additionally $x * x \subseteq I$ for all $x \in H$, then I is said to be a *reflexive hyper BCK-ideal* of H.

Lemma 2.5 [10] *If* I *is a reflexive hyper BCK-ideal of a hyper BCK-algebra* H, *then* $(x * y) \cap I \neq \varnothing$ *implies* $x * y \subseteq I$ *for all* $x, y \in H$.

Theorem 2.6 [12] *Let* $f : (H_1, *_1, 0_1) \to (H_2, *_2, 0_2)$ *be a homomorphism of hyper BCK-algebras. Then* $H_1/Kerf \cong Imf$.

Theorem 2.7 [12] *Let* I *be a reflexive hyper BCK-ideal and* J *be a hyper BCK-ideal of* H *such that* $I \subseteq J$. *Then* $(H/I)/(J/I) \cong H/J$.

3. Hyper SubKS-semigroups, Hyper KS-Ideals and Hyper P-ideals

In this section, we introduce the concept of hyper KS-semigroup and we study the notions of hyper subKS-semigroups, hyper KS-ideals and hyper P-ideals of a hyper KS-semigroup and investigate some of their properties.

Definition 3.1 A *hyper KS-semigroup* is a nonempty set H together with two hyperoperations "$*$" and "$\cdot$" and a constant 0 satisfying the following conditions:

(i) $(H, *, 0)$ is a hyper BCK-algebra.

(ii) $(H, \cdot)$ is a semihypergroup having zero as a bilaterally absorbing element, that is, $x \cdot 0 = 0 \cdot x = \{0\}$ for all $x \in H$; and

(iii) "$\cdot$" is left and right distributive over "$*$", that is, for any $x, y, z \in H$

$$x \cdot (y * z) = (x \cdot y) * (x \cdot z) \text{ and } (x * y) \cdot z = (x \cdot z) * (y \cdot z).$$

From now on, a hyper KS-semigroup $(H, *, \cdot, 0)$ shall be denoted by H and for all $x, y \in H$, we agree to write $x \cdot y$ as xy.

Example 3.2 Let $H = \{0, 1, 2\}$. Define the operation "$*$" and "$\cdot$" by the Cayley's table shown below. Then by routine calculations, H is a hyper KS-semigroup.

$*$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{0,1\}$
2	$\{2\}$	$\{1,2\}$	$\{0,1,2\}$

$\cdot$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{0\}$	$\{1\}$	$\{0,1\}$
2	$\{0\}$	$\{0,1\}$	$\{0,1,2\}$

Lemma 3.3 *Let A, B and C be nonempty subsets of a hyper KS-semigroup H. Then $A < B$ and $B \subseteq C$ imply $A < C$.*

Definition 3.4 Let I be a nonempty subset of a hyper KS-semigroup H. Then I is said to be a *hyper subKS-semigroup* of H if for all $x, y \in I$, $x * y \subseteq I$ and $xy \subseteq I$. I is a *hyper left* (resp. *hyper right*) *stable* if $xa \subseteq I$ (resp. $ax \subseteq I$) for all $x \in H$ and for all $a \in I$. I is a *hyper stable* if I is both hyper left and right stable. I is a *hyper left* (resp. *hyper right*) *KS-ideal* if I is a hyper left (resp. hyper right) stable and for any $x, y \in H$, $x * y < I$ and $y \in I$ imply that $x \in I$. I is a *hyper KS-ideal* if I is both a hyper left and a hyper right KS-ideal. I is said to be *reflexive* if for all $x \in H, x * x \subseteq I$ and $xx \subseteq I$.

Example 3.5 Consider the hyper KS-semigroup $H = \{0, 1, 2\}$ in Example 3.2 and $I = \{0, 1\}$. By routine calculations, for all $x, y \in I$ such that $x * y < I$ and $y \in I$ imply $x \in I$ and for all $a \in I$, $xa, ax \subseteq I$. Hence, I is a hyper KS-ideal of H.

The following remark follows from Definition 3.4.

Remark 3.6 For any hyper KS-semigroup H, the following hold.

(i) $\{0\}$ and H are hyper KS-ideals of H.

(ii) $0 \in I$ for any hyper KS-ideal I of H.

(iii) Every hyper KS-ideal is a hyper BCK-ideal.

Proposition 3.7 *Every hyper KS-ideal of a hyper KS-semigroup is a hyper subKS-semigroup.*

The converse of Proposition 3.7 (ii) may not be true in general. Consider the following example.

Example 3.8 Let $H = \{0, 1, 2\}$. Define the operation "$*$" and "$\cdot$" by the Cayley's table shown below. Then by routine calculations, H is a hyper KS-semigroup.

$*$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{0,1\}$
2	$\{2\}$	$\{1,2\}$	$\{0,1,2\}$

$\cdot$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{0\}$	$\{1\}$	$\{0,1,2\}$
2	$\{0\}$	$\{0,1\}$	$\{0,2\}$

Consider $I = \{0, 1\}$. Observe that I is a hyper subKS-semigroup but I is not hyper right stable since $1 \cdot 2 = \{0, 1, 2\} \nsubseteq I$.

Lemma 3.9 *Let A and B be nonempty subsets of a hyper KS-semigroup H and I be a hyper KS-ideal of H. Then $B \subseteq I$ implies $A \cdot B \subseteq I$ and $B \cdot A \subseteq I$.*

Theorem 3.10 *Let $\{A_i : i \in \mathscr{I}\}$ be a nonempty collection of subsets of a hyper KS-semigroup H.*

(i) *If* A_i *is a hyper KS-ideal of* H *for all* $i \in \mathscr{I}$*, then so is* $\bigcap_{i\in\mathscr{I}} A_i$.

(ii) *If* A_j *is a hyper subKS-semigroup where* $j \in \mathscr{I}$ *and* A_{i_0} *is a hyper KS-ideal of* H *for some* $i_0 \in \mathscr{I}$*, then* $\bigcap_{i\in\mathscr{I}} A_i$ *is a hyper KS-ideal of* A_j, $j \neq i_0$*. Moreover, if* A_{i_0} *is reflexive in* H*, then* $\bigcap_{i\in\mathscr{I}} A_i$ *is reflexive in* A_j.

Proof. Let $\{A_i : i \in \mathscr{I}\}$ be a nonempty collection of subsets of H.

(i) Suppose that A_i is a hyper KS-ideal of H for all $i \in \mathscr{I}$. Then $0 \in A_i$ for all $i \in \mathscr{I}$ and so $0 \in \bigcap_{i\in\mathscr{I}} A_i$. Thus, $\bigcap_{i\in\mathscr{I}} A_i \neq \varnothing$. Clearly, $\bigcap_{i\in\mathscr{I}} A_i \subseteq A_i$ for all $i \in \mathscr{I}$. Let $a \in \bigcap_{i\in\mathscr{I}} A_i$ and $x \in H$. Then $a \in A_i$ for all $i \in \mathscr{I}$. Since A_i is hyper stable in H for all for all $i \in \mathscr{I}$, it follows that $xa, ax \subseteq A_i$ for all $i \in \mathscr{I}$. Thus, $xa, ax \subseteq \bigcap_{i\in\mathscr{I}} A_i$ and so $\bigcap_{i\in\mathscr{I}} A_i$ is hyper stable in H. Suppose that $x, y \in H$ such that $x * y < \bigcap_{i\in\mathscr{I}} A_i$ and $y \in \bigcap_{i\in\mathscr{I}} A_i$. Since $\bigcap_{i\in\mathscr{I}} A_i \subseteq A_i$ for all $i \in \mathscr{I}$, it follows by Lemma 3.3 that $x * y < A_i$ for all $i \in \mathscr{I}$. Also, $y \in A_i$ for all $i \in \mathscr{I}$. Hence, A_i hyper KS-ideals for all $i \in \mathscr{I}$ imply $x \in A_i$ for all $i \in \mathscr{I}$. Therefore, $\bigcap_{i\in\mathscr{I}} A_i$ is a hyper KS-ideal of H.

(ii) Suppose that A_j is a hyper subKS-semigroup where $j \in \mathscr{I}$ and A_{i_0} is a hyper KS-ideal of H for some $i_0 \in \mathscr{I}$. Since $0 \in A_i$ for all $i \in \mathscr{I}$, $0 \in \bigcap_{i\in\mathscr{I}} A_i$ and so $\bigcap_{i\in\mathscr{I}} A_i \neq \varnothing$. Clearly, $\bigcap_{i\in\mathscr{I}} A_i \subseteq A_i$ for all $i \in \mathscr{I}$. Let $x \in A_j$ where $j \neq i_0$ and $a \in \bigcap_{i\in\mathscr{I}} A_i$. Then $a \in A_i$ for all $i \in \mathscr{I}$. In particular, $a \in A_j$. Since A_j is a hyper subKS-semigroup, $xa, ax \subseteq A_j$. Also, since A_{i_0} is hyper stable in H, it follows that $xa, ax \subseteq A_{i_0}$. Thus, $xa, ax \subseteq A_i$ for all $i \in \mathscr{I}$ and so $xa, ax \subseteq \bigcap_{i\in\mathscr{I}} A_i$. Hence, $\bigcap_{i\in\mathscr{I}} A_i$ is hyper stable in A_j. Suppose that $x, y \in A_j$ where $j \neq i_0$ such that $x * y < \bigcap_{i\in\mathscr{I}} A_i$ and $y \in \bigcap_{i\in\mathscr{I}} A_i$. Then $y \in A_i$ for all $i \in \mathscr{I}$. Since $\bigcap_{i\in\mathscr{I}} A_i \subseteq A_i$ for all $i \in \mathscr{I}$, it follows by Lemma 3.3 that $x * y < A_i$ for all $i \in \mathscr{I}$. If $i = i_0$, then $x \in A_{i_0}$ since A_{i_0} hyper KS-ideal of H. Note that by assumption, $x \in A_j$ and so $x \in A_i$ for all $i \in \mathscr{I}$. Hence, $x \in \bigcap_{i\in\mathscr{I}} A_i$. Thus, $\bigcap_{i\in\mathscr{I}} A_i$ is a hyper KS-ideal of A_j. Moreover, if A_{i_0} is reflexive in H, then for all $x \in A_j$, $x * x, xx \subseteq A_{i_0}$ and since A_j is a hyper subKS-semigroup, $x * x, xx \subseteq A_j$ and so $x * x, xx \subseteq A_i$ for all $i \in \mathscr{I}$. Therefore, $x * x, xx \subseteq \bigcap_{i \in \mathscr{I}} A_i$ implying that $\bigcap_{i \in \mathscr{I}} A_i$ is reflexive in A_j. ■

Definition 3.11 Let θ be an equivalence relation on a hyper KS-semigroup H and A, B be nonempty subsets of H.

(i) $A\overline{\theta}B$ means that, for all $a \in A$ there exists $b \in B$ such that $a\theta b$ and for all $b \in B$ there exists $a \in A$ such that $a\theta b$,

(ii) θ is said to be a *congruence relation* on H if for all $x, y, u, v \in H$, $x\theta y$ and $u\theta v$ imply that $(x * u)\overline{\theta}(y * v)$ and $(xu)\overline{\theta}(yv)$.

From [13], the relation " $\sim_I$ " on H is defined by $x \sim_I y$ if and only if $x*y \subseteq I$ and $y * x \subseteq I$ for all $x, y \in H$. Then, $\sim_I$is an equivalence relation on H. Also, the relation " $\sim_I$ " on $P^*(H)$ is defined by $A \approx_I B$ if and only if for all $a \in A$ there exists $b \in B$ such that $a \sim_I b$, and for all $b \in B$ there exists $a \in A$ such that $a \sim_I b$, for all $A, B \in P^*(H)$. Then, $\approx_I$ is an equivalence relation on $P^*(H)$, where I is a reflexive hyper KS-ideal of H. In [12], it was proved that $x\theta y$ and $u\theta v$ imply $(x * u)\overline{\theta}(y * v)$ for all $x, y, u, v \in H$, (i) $I = I_0$ and (ii) $I_x = I_y$ if and only if $x \sim_I y$.

Theorem 3.12 *The relation $\sim_I$ is a congruence relation on a hyper KS-semigroup H.*

Proof. Let $x, y, u, v \in H$ such that $x \sim_I y$ and $u \sim_I v$. Then $x * y, y * x$, $u * v, v * u \subseteq I$. We only need to show that $xu \approx_I yv$. Let $a \in xu$ and $b \in yv$. Then

$$a * b \subseteq xu * xv = x(u * v) \subseteq I \text{ and } b * a \subseteq xv * xu = x(v * u) \subseteq I.$$

Thus, $a \sim_I b$ and so $xu \approx_I xv$. Also, let $c \in xv$ and $d \in yv$. Then $c*d \subseteq xv*yv = (x * y)v \subseteq I$ and $d * c \subseteq yv * xv = (y * x)v \subseteq I$. Thus, $c \sim_I d$ and so $xv \approx_I yv$. Since $\approx_I$ is transitive, $xu \approx_I yv$.

Therefore, $\sim_I$ is a congruence relation on H. ■

In [12], if $(H, *, 0)$ is a hyper BCK-algebra and I is a reflexive hyper BCK-ideal, then $(H/I, \circledast, I_0)$ is a hyper BCK-algebra under the hyperoperation " $\circledast$ " and hyper order " $\ll$ " which is defined as $I_x \circledast I_y = \{I_z : z \in x * y\}$ and $I_x \ll I_y \Leftrightarrow I_0 \in I_x \circledast I_y$ where $I_x = \{y \in H : x \sim_I y\}$ and $H/I = \{I_x : x \in H\}$.

Theorem 3.13 *Let I be a reflexive hyper KS-ideal of a hyper KS-semigroup H. Define the hyperoperation "$\odot$" on H by $I_x \odot I_y = \{I_z : z \in xy\}$ for all $I_x, I_y \in H/I$. Then $(H/I, \circledast, \odot, I_0)$ is a hyper KS-semigroup.*

Proof. We only need to show that $(H/I, \odot)$ is a semihypergroup such that I_0 is a bilaterally absorbing element and that " $\odot$ " is left and right distributive over " $\circledast$ ". First, we show that the hyperoperation " $\odot$ " on H/I is well-defined. Let

$x, x', y, y' \in H$ such that $I_x = I_{x'}$ and $I_y = I_{y'}$. Then $x \sim_I x'$ and $y \sim_I y'$. Since $\sim_I$ is a congruence relation, $xx' \overline{\sim}_I yy'$, it follows that for all $z \in xx'$, there exists $w \in yy'$ such that $z \sim_I w$ and for all $t \in yy'$, there exists $s \in xx'$ such that $t \sim_I s$. Thus, $I_z = I_w \in I_y \odot I_{y'}$ implying that $z \in yy'$ and also $I_t = I_s \in I_x \odot I_{x'}$ implying that $t \in xx'$. Hence, $I_x \odot I_{x'} \subseteq I_y \odot I_{y'}$ and $I_y \odot I_{y'} \subseteq I_x \odot I_{x'}$. Therefore, $I_x \odot I_{x'} = I_y \odot I_{y'}$ and so "$\odot$" is well-defined. Now, we show that H/I is associative.

Let $I_x, I_y, I_z \in H/I$ and let $I_w \in (I_x \odot I_y) \odot I_z$. Then $I_w \in I_u \odot I_z$ for some $u \in xy$. Thus, $w \in uz \subseteq (xy)z = x(yz)$ and so $I_w \in I_x \odot (I_y \odot I_z)$. Hence, $(I_x \odot I_y) \odot I_z \subseteq I_x \odot (I_y \odot I_z)$. Similarly, let $I_s \in I_x \odot (I_y \odot I_z)$. Then $I_s \in I_x \odot I_t$ for some $t \in yz$. Thus, $s \in xt \subseteq x(yz) = (xy)z$ and so $I_s \in (I_x \odot I_y) \odot I_z$. Hence, $I_x \odot (I_y \odot I_z) \subseteq (I_x \odot I_y) \odot I_z$. Therefore, $(I_x \odot I_y) \odot I_z = I_x \odot (I_y \odot I_z)$ and so $(H/I, \odot)$ is a semihypergroup. Moreover, for all $I_x \in H/I$,

$$I_x \odot I_0 = \{I_z : z \in x0 = \{0\}\} = \{I_0\} = \{I_z : z \in 0x = \{0\}\} = I_0 \odot I_x.$$

Furthermore, let $I_w \in I_x \odot (I_y \circledast I_z)$. Then $I_w \in I_x \odot I_u$ for some $u \in y * z$. By Definition 3.1 (iii), $w \in xu \subseteq x(y * z) = xy * xz$. This implies that $I_w \in (I_x \odot I_y) \circledast (I_x \odot I_z)$ and so $I_x \odot (I_y \circledast I_z) \subseteq (I_x \odot I_y) \circledast (I_x \odot I_z)$. On the other hand, let $I_{w'} \in (I_x \odot I_y) \circledast (I_x \odot I_z)$. Then $I_{w'} \in I_{u'} \circledast I_{v'}$ for some $u' \in xy$ and for some $v' \in xz$. By Definition 3.1 (iii), $w' \in u' * v' \subseteq xy * xz = x(y * z)$. Hence, $I_{w'} \in I_x \odot (I_y \circledast I_z)$ and so $(I_x \odot I_y) \circledast (I_x \odot I_z) \subseteq I_x \odot (I_y \circledast I_z)$. Therefore, $(I_x \odot I_y) \circledast (I_x \odot I_z) = I_x \odot (I_y \circledast I_z)$ and so " $\odot$ " is left distributive over " $\circledast$ ".

Finally, let $I_w \in (I_y \circledast I_z) \odot I_x$. Then there exists $u \in y * z$ such that $I_w \in I_u \odot I_x$. By Definition 3.1 (iii), $w \in ux \subseteq (y * z)x = yx * zx$. Hence, $I_w \in (I_y \odot I_x) \circledast (I_z \odot I_x)$ and so $(I_y \circledast I_z) \odot I_x \subseteq (I_y \odot I_x) \circledast (I_z \odot I_x)$. On the other hand, let $I_{w'} \in (I_y \odot I_x) \circledast (I_z \odot I_x)$. Then $I_{w'} \in I_{u'} \circledast I_{v'}$ for some $u' \in yx$ and for some $v' \in zx$. By Definition 3.1 (iii), $w' \in u' * v' \subseteq yx * zx = (y * z)x$. Hence, $I_{w'} \in (I_y \circledast I_z) \odot I_x$ and so $(I_y \odot I_x) \circledast (I_z \odot I_x) \subseteq (I_y \circledast I_z) \odot I_x$. Therefore, $(I_y \odot I_x) \circledast (I_z \odot I_x) = (I_y \circledast I_z) \odot I_x$ and so " $\odot$ " is right distributive over " $\circledast$ ".

Therefore, $(H/I, \circledast, \odot, I_0)$ is a hyper KS-semigroup. ■

4. Hyper KS-semigroup Homomorphism

In this section, we study the concept of hyper KS-semigroup homomorphism. A hyper KS-semigroup homomorphism is a function between two hyper KS-semigroups which respects the hyperoperations. More precisely:

Definition 4.1 Let $(H_1, *_1, \cdot_1, 0_1)$ and $(H_2, *_2, \cdot_2, 0_2)$ be hyper KS-semigroups and $f : H_1 \to H_2$ be a map. Then f is called a *hyper KS-semigroup homomorphism* if $f(x *_1 y) = f(x) *_2 f(y)$ and $f(x \cdot_1 y) = f(x) \cdot_2 f(y)$ for all $x, y \in H_1$.

Theorem 4.2 *Let $f : G \to H$ be a hyper KS-semigroup homomorphism. Then $Ker\, f$ is a hyper KS-ideal of G.*

Theorem 4.3 *Let $f : G \to H$ be a hyper KS-semigroup epimorphism and I be a hyper KS-ideal of G containing $Kerf$ such that $Kerf$ is reflexive. Then $f(I)$ is a reflexive hyper KS-ideal of H.*

Proof. Let $x \in H$ and $a \in f(I)$. Since f is onto, there exists $y \in G$ such that $x = f(y)$. Also, $a = f(b)$ for some $b \in I$. Since I is hyper stable in G, $yb, by \subseteq I$ and so $xa = f(y)f(b) = f(yb) \subseteq f(I)$ and $ax = f(b)f(y) = f(by) \subseteq f(I)$. Thus, $f(I)$ is hyper left and hyper right stable in H.

Let $a, b \in H$ such that $a*b < f(I)$ and $b \in f(I)$. Then $b = f(y)$ for some $y \in I$ and $a = f(x)$ for some $x \in G$ since f is onto. Also, since f is a homomorphism, $f(x*y) = f(x)*f(y) = a*b < f(I)$. Let $z \in x*y$. Then $f(z) \in f(x*y) < f(I)$ and so $f(z) < f(w)$ for some $w \in I$. This means that $0 \in f(z)*f(w) = f(z*w)$ since f is a homomorphism. This implies that $0 = f(t)$ for some $t \in z*w$. Since $f(t) = 0$, $t \in Kerf$ and so $(z*w) \cap Kerf \neq \varnothing$. Since $Kerf$ is reflexive, by Lemma 2.5, $z*w \subseteq Kerf \subseteq I$. By Theorem 2.3(a2), $z*w < I$. Since I is a hyper KS-ideal and $w \in I$, it follows that $z \in I$ and so $x*y \subseteq I$. By Theorem 2.3(a2), $x*y < I$. Since I is a hyper KS-ideal and $y \in I$, $x \in I$. Hence, $a = f(x) \in f(I)$ and so $f(I)$ is a hyper KS-ideal of H.

Let $y \in H$. Then $f(x) = y$ for some $x \in G$ since f is onto and $x*x, xx \subseteq Kerf \subseteq I$ since $Kerf$ is reflexive. Now, f a homomorphism implies $y*y = f(x)*f(x) = f(x*x) \subseteq f(Kerf) \subseteq f(I)$ and $yy = f(x)f(x) = f(xx) \subseteq f(Kerf) \subseteq f(I)$. Hence, $f(I)$ is reflexive. ∎

Theorem 4.4 *Let $f : G \to H$ be a hyper KS-semigroup homomorphism and I be a hyper KS-ideal of H. Then $f^{-1}(I)$ is a hyper KS-ideal of G. Moreover, if I is reflexive in H, then $f^{-1}(I)$ is reflexive in G.*

In [12], if I is a reflexive hyper BCK-ideal of a hyper BCK-algebra H, then there exists a canonical surjective homomorphism $\varphi : H \to H/I$ given by $\varphi(x) = I_x$ with kernel I.

Theorem 4.5 *If I is a reflexive hyper KS-ideal of H, then the mapping $\varphi : H \to H/I$ given by $\varphi(x) = I_x$ is an epimorphism with kernel I.*

The proof follows by showing that $\varphi(x \cdot_1 y) = \varphi(x) \cdot_2 \varphi(y)$ for all $x, y \in H$.

Theorem 4.6 (First Isomorphism Theorem) *Let $f : G \to H$ be a hyper KS-semigroup homomorphism and $Kerf$ be a reflexive hyper KS-ideal of G. Then $G/Kerf \cong Imf$.*

Proof. Since $Kerf$ is a reflexive hyper KS-ideal and by Theorem 3.13, $G/Kerf$ is a hyper KS-semigroup. Define a mapping $\varphi : G/Kerf \to Imf$ by $\varphi(J_x) = f(x)$ for all $J_x \in G/Kerf$.

Let $J_x, J_y \in G/Kerf$ such that $J_x = J_y$. Then $x \sim_{Kerf} y$ and so $x*y, y*x \subseteq Kerf$. Now, f homomorphism implies $f(x)*f(y) = f(x*y) \subseteq f(Kerf) = \{0\}$ and $f(y)*f(x) = f(y*x) \subseteq f(Kerf) = \{0\}$. Since $x*y, y*x \neq \varnothing$, it follows that $f(x)*f(y) = \{0\}$ and $f(y)*f(x) = \{0\}$, that is, $0 \in f(x)*f(y)$ and $0 \in f(y)*f(x)$. So, $f(x) < f(y)$ and $f(y) < f(x)$. By Definition 2.2(H4), $f(x) = f(y)$ and so $\varphi(J_x) = \varphi(J_y)$. Hence, φ is well-defined.

Let $J_x, J_y \in G/Kerf$ such that $\varphi(J_x) = \varphi(J_y)$. Then $f(x) = f(y)$. By Theorem 2.3(a1), $f(x) < f(y)$ and $f(y) < f(x)$. Thus, by Definition 2.2 and since f is a homomorphism, we have $0 \in f(x) * f(y) = f(x * y)$ and $0 \in f(y) * f(x) = f(y * x)$. This means that there exists $t \in x * y$ and $s \in y * x$ such that $f(t) = 0$ and $f(s) = 0$. This implies that $t, s \in Kerf$ and so $(x * y) \cap Kerf \neq \varnothing$ and $(y * x) \cap Kerf \neq \varnothing$. By Lemma 2.5, $x * y, y * x \subseteq Kerf$. Hence, $x \sim_{Kerf} y$ and so $J_x = J_y$. Therefore, φ is one-to-one. Clearly, φ is onto. Now, for all $J_x, J_y \in G/Kerf$, $\varphi(J_x \circledast J_y) = \{\varphi(J_t) : J_t \in J_x \circledast J_y\} = \{f(t) : t \in x * y\} = f(x * y) = f(x) * f(y) = \varphi(J_x) * \varphi(J_y)$ and $\varphi(J_x \odot J_y) = \{\varphi(J_t) : J_t \in J_x \odot J_y\} = \{f(t) : t \in xy\} = f(xy) = f(x)f(y) = \varphi(J_x)\varphi(J_y)$. Hence, φ is an isomorphism. Therefore, $G/Kerf \cong Imf$. ∎

Definition 4.7 Let M be a nonempty subset of H and N be a reflexive hyper KS-ideal of H. Define the subset MN of H by

$$MN = \bigcup_{m \in M} N_m,$$

where $N_m = \{x \in H : m \sim_N x\}$.

From the preceding definition and since $\sim_N$ is reflexive, for all $m \in M$, $m \sim_N m$. Hence, $m \in N_m \subseteq \bigcup_{m \in M} N_m = MN$. Thus, we $M \subseteq MN$.

Proposition 4.8 *Let M be a nonempty subset of H and N be a reflexive hyper KS-ideal of H. If $M \eqsim_N N$, then $M \subseteq N$ and $MN = N$.*

Proof. The result follows from Definition 4.7 ∎

Theorem 4.9 *Let M be a hyper subKS-semigroup and N be a reflexive hyper KS-ideal of a hyper KS-semigroup H. Then MN is a hyper subKS-semigroup of H.*

Proof. Let $x, y \in MN$. Then $x \in N_a$ and $y \in N_b$ for some $a, b \in M$. This implies that $x \sim_N a$ and $y \sim_N b$. By Theorem 3.12, we have $x * y \eqsim_N a * b$ and $xy \eqsim_N ab$. Thus, for all $w \in x * y$, there exists $z \in a * b \subseteq M$ such that $w \sim_N z$ and for all $u \in xy$, there exists $v \in ab \subseteq M$ such that $u \sim_N v$. This means that $w \in N_z \subseteq MN$ and $u \in N_v \subseteq MN$ and so $w, u \in MN$. Hence, $x * y, xy \subseteq MN$. Therefore, MN is a hyper subKS-semigroup of H. ∎

Theorem 4.10 (Second Isomorphism Theorem) *Let M be a hyper subKS-semigroup and N be a reflexive hyper KS-ideal of H. Then $M/(M \cap N) \cong MN/N$.*

Proof. Let M be a hyper subKS-semigroup and N be a reflexive hyper KS-ideal of H. Define $\pi : H \to H/N$ by $\pi(x) = N_x$ for all $x \in H$. Define $f = \pi_{|M}$. By

Theorem 4.5, π is a well-defined homomorphism. It follows that f is also a well-defined homomorphism. By Theorem 4.6, $M/Kerf \cong Imf$. Now, by Theorem 2.3(a2) and Definition 3.4 (iv), we have

$$\begin{aligned} Kerf &= \{x \in M \mid f(x) = N_0\} = \{x \in M \mid N_x = N_0\} \\ &= \{x \in M \mid x \sim_N 0\} = \{x \in M \mid x \in N\} = M \cap N. \end{aligned}$$

Furthermore, we show that $Imf = MN/N$. Let $N_x \in Imf$. Then there exists $m \in M$ such that $f(m) = N_x$. This implies that $\pi(m) = N_m = N_x$ and so $m \sim_N x$. This means that $x \in N_m \subseteq MN$. Hence, $N_x \in MN/N$ and so $Imf \subseteq MN/N$. On the other hand, let $N_y \in MN/N$. Then $y \in MN$, that is, $y \in N_x$ for some $x \in M$. Thus, $y \sim_N x$ and so $N_y = N_x = f(x) \in Imf$. Hence, $MN/N \subseteq Imf$ and so $Imf = MN/N$. Therefore, $M/(M \cap N) \cong MN/N$. ∎

Lemma 4.11 *Let I and J be hyper KS-ideals of H such that I is reflexive and $I \subseteq J$. Then*

(i) *I is a reflexive hyper KS-ideal of the hyper subKS-semigroup J, and*

(ii) *the quotient hyper KS-semigroup J/I is a reflexive hyper KS-ideal of H/I.*

Proof. The result follows from Definition 3.4 (v) and Proposition 3.7.

Theorem 4.12 (Third Isomorphism Theorem) *Let I and J be hyper KS-ideals of H such that I is reflexive and $I \subseteq J$. Then $(H/I)/(J/I) \cong H/J$.*

Proof. Let I and J be hyper KS-ideals of H such that I is reflexive and $I \subseteq J$. Observe that for all $x \in H$, $x * x, xx \subseteq I \subseteq J$ and so it follows that J is also reflexive. By Lemma 4.11, J/I and $(H/I)/(J/I)$ are defined. Define the function $f : H/I \to H/J$ by $f(I_x) = J_x$. Note that by Theorem 2.7, f is a well-defined homomorphism with respect to the hyperoperation "$*$", $Ker\ f = J/I$ and $Im\ f = H/J$. Thus, we only need to show that f is a homomorphism with respect to the hyperoperation "$\cdot$". Let $I_x, I_y \in H/I$. Then

$$f(I_x \odot I_y) = \{f(I_t) : I_t \in I_x \odot I_y\} = \{f(I_t) : t \in xy\} = \{J_t : t \in xy\} = J_x \odot J_y.$$

Hence, $(H/I)/(J/I) \cong H/J$. ∎

5. Hyper Product of Hyper KS-semigroups

In this section, we develop some more of the abstract theory of hyper KS-semigroups. In particular, we show that the hyper product of any nonempty finite family of hyper KS-semigroups is also a hyper KS-semigroup.

Let $(H_1, *_1, \cdot_1, 0_1)$ and $(H_2, *_2, \cdot_2, 0_2)$ be hyper KS-semigroups with hyper orders denoted by $<_1$ and $<_2$, respectively. Define the structure $H_1 \times H_2$ by $H_1 \times H_2 = \{(a, b) : a \in H_1, b \in H_2\}$. In [2], $(H_1 \times H_2, \circledast, (0_1, 0_2))$ is a hyper

BCK-algebra where $(a_1, b_1) \circledast (a_2, b_2) = (a_1 *_1 a_2, b_1 *_2 b_2)$ and $(a_1, b_1) \ll (a_2, b_2) \Leftrightarrow a_1 <_1 a_2$ and $b_1 <_2 b_2$ for all $(a_1, b_1), (a_2, b_2) \in H_1 \times H_2$. For any sets $A \in P^*(H_1)$ and $B \in P^*(H_2)$, $(A, B) = \{(a, b) : a \in A, b \in B\}$. H_1 and H_2 shall mean the hyper KS-semigroups $(H_1, *_1, \cdot_1, 0_1)$ and $(H_2, *_2, \cdot_2, 0_2)$ with hyper orders $<_1$ and $<_2$, respectively.

Definition 5.1 Define the hyperoperation "$\odot$" on $H_1 \times H_2$ by $(a_1, b_1) \odot (a_2, b_2) = (a_1 \cdot_1 a_2, b_1 \cdot_2 b_2)$ for all $(a_1, b_1), (a_2, b_2) \in H_1 \times H_2$. Then $(H_1 \times H_2, \circledast, \odot, (0_1, 0_2))$ is called the *hyper product* of H_1 and H_2.

The following theorem holds since the hyperoperation $\odot$ is componentwise.

Theorem 5.2 *The hyper product of two hyper KS-semigroups is also a hyper KS-semigroup.*

Now, we extend the hyper product $H_1 \times H_2$ of H_1 and H_2 to any finite family of hyper KS-semigroups and obtain the following result.

Theorem 5.3 *Let $\{H_i | i = 1, 2, \ldots, n\}$ be a nonempty family of hyper KS-semigroups. Then $\left(\prod_{i=1}^{n} H_i, \circledast, \odot, 0\right)$ is a hyper KS-semigroup.*

Theorem 5.4 *Let H_1 and H_2 be hyper KS-semigroups.*

(i) *If I_1 and I_2 are hyper subKS-semigroups of H_1 and H_2, respectively, then $I_1 \times I_2$ is a hyper subKS-semigroup of $H_1 \times H_2$.*

(ii) *If I_1 and I_2 are hyper KS-ideals of H_1 and H_2, respectively, then $I_1 \times I_2$ is a hyper KS-ideal of $H_1 \times H_2$.*

(iii) *If I_1 and I_2 are reflexive in H_1 and H_2, respectively, then $I_1 \times I_2$ is reflexive in $H_1 \times H_2$.*

Proof. Let $(x, y), (p, q) \in I_1 \times I_2$. Then $x, p \in I_1$ and $y, q \in I_2$.

(i) Suppose that I_1 and I_2 are hyper subKS-semigroups of H_1 and H_2, respectively. Then $x * p, xp \subseteq I_1$ and $y * q, yq \subseteq I_2$ implying that $(x, y) \circledast (p, q) = (x * p, y * q) \subseteq I_1 \times I_2$ and $(x, y) \odot (p, q) = (xp, yq) \subseteq I_1 \times I_2$. Thus, $I_1 \times I_2$ is a hyper subKS-semigroup of $H_1 \times H_2$.

(ii) Suppose that I_1 and I_2 are hyper KS-ideals of H_1 and H_2, respectively. Let $(r, s) \in H_1 \times H_2$. Then $r \in H_1$ and $s \in H_2$. Since I_1 and I_2 are hyper stable in H_1 and H_2, respectively, $rx, xr \subseteq I_1$ and $sy, ys \subseteq I_2$ where $x \in H_1, y \in H_2$. It follows that $(r, s) \odot (x, y) = (rx, sy) \subseteq I_1 \times I_2$ and $(x, y) \odot (r, s) = (xr, ys) \subseteq I_1 \times I_2$ and so $I_1 \times I_2$ is hyper stable in $H_1 \times H_2$. Suppose that $(m, n), (r, s) \in H_1 \times H_2$ such that $(m, n) \circledast (r, s) \ll I_1 \times I_2$ and $(r, s) \in I_1 \times I_2$. Then $(m * r, n * s) \ll I_1 \times I_2$. This means that for all $(a, b) \in (m * r, n * s)$, there exists $(c, d) \in I_1 \times I_2$ such that $(a, b) \ll (c, d)$, that is, $a < c$ and $b < d$. Note that $a \in m * r$ and $c \in I_1$ imply $m * r < I_1$. Since $r \in I_1$, we have $m \in I_1$. Also, $b \in n * s$ and $d \in I_2$ imply $n * s < I_2$. Since $s \in I_2$, it follows that $n \in I_2$. Hence, $(m, n) \in I_1 \times I_2$. Therefore, $I_1 \times I_2$ is a hyper KS-ideal of $H_1 \times H_2$.

(iii) Let $(r, s) \in H_1 \times H_2$. Then $r \in H_1$ and $s \in H_2$. Since I_1 and I_2 are reflexive in H_1 and H_2, respectively, it follows that $r * r, rr \subseteq I_1$ and $s * s, ss \subseteq I_2$. Thus, $(r*r, s*s), (rr, ss) \subseteq I_1 \times I_2$. Therefore, $I_1 \times I_2$ is reflexive in $H_1 \times H_2$. ∎

Theorem 5.5 *Let $\varphi_1 : G_1 \to H_1$ and $\varphi_2 : G_2 \to H_2$ be hyper KS-semigroup homomorphisms. Define $\varphi : G_1 \times G_2 \to H_1 \times H_2$ by $\varphi((a_1, a_2)) = (\varphi_1(a_1), \varphi_2(a_2))$ for all $(a_1, a_2) \in G_1 \times G_2$. Then*

(i) *φ is a hyper KS-homomorphism;*

(ii) *$Ker\varphi = Ker\varphi_1 \times Ker\varphi_2$;*

(iii) *$Im\varphi = Im\varphi_1 \times Im\varphi_2$; and*

(iv) *φ is a monomorphism (resp. epimorphism) if and only if φ_i is a monomorphism (resp. epimorphism) for each $i = 1, 2$.*

Proof. Define $\varphi : G_1 \times G_2 \to H_1 \times H_2$ by $\varphi((a_1, a_2)) = (\varphi_1(a_1), \varphi_2(a_2))$ for all $(a_1, a_2) \in G_1 \times G_2$. Since φ_1 and φ_2 are well-defined, it also follows that φ is well-defined.

Let $(a_1, a_2), (b_1, b_2) \in G_1 \times G_2$. Then

$$\begin{aligned}
\varphi((a_1, a_2) \circledast (b_1, b_2)) &= \varphi((a_1 * b_1, a_2 * b_2)) \\
&= \{\varphi((s, t)) \mid s \in a_1 * b_1, t \in a_2 * b_2\} \\
&= \{(\varphi_1(s), \varphi_2(t)) \mid s \in a_1 * b_1, t \in a_2 * b_2\} \\
&= (\varphi_1(a_1 * b_1), \varphi_2(a_2 * b_2)) \\
&= (\varphi_1(a_1) * \varphi_1(b_1), \varphi_2(a_2) * \varphi_2(b_2)) \\
&= (\varphi_1(a_1), \varphi_2(a_2)) \circledast (\varphi_1(b_1), \varphi_2(b_2)) \\
&= \varphi(a_1, a_2) \circledast \varphi(b_1, b_2).
\end{aligned}$$

and

$$\begin{aligned}
\varphi((a_1, a_2) \odot (b_1, b_2)) &= \varphi((a_1 b_1, a_2 b_2)) \\
&= \{\varphi((s, t)) \mid s \in a_1 b_1, t \in a_2 b_2\} \\
&= \{(\varphi_1(s), \varphi_2(t)) \mid s \in a_1 b_1, t \in a_2 b_2\} \\
&= (\varphi_1(a_1 b_1), \varphi_2(a_2 b_2)) \\
&= (\varphi_1(a_1)\varphi_1(b_1), \varphi_2(a_2)\varphi_2(b_2)) \\
&= (\varphi_1(a_1), \varphi_2(a_2)) \odot (\varphi_1(b_1), \varphi_2(b_2)) \\
&= \varphi(a_1, a_2) \odot \varphi(b_1, b_2).
\end{aligned}$$

Thus, φ is a homomorphism. Furthermore,

$$\begin{aligned}
Ker\varphi &= \{(a_1, a_2) \mid \varphi((a_1, a_2)) = (0_1, 0_2) \in H_1 \times H_2\} \\
&= \{(a_1, a_2) | (\varphi_1(a_1), \varphi_2(a_2)) = (0_1, 0_2)\} \\
&= \{(a_1, a_2) \mid \varphi_1(a_1) = 0_1, \varphi_2(a_2) = 0_2\} \\
&= \{(a_1, a_2) \mid a_1 \in Ker\varphi_1, a_2 \in Ker\varphi_2\} \\
&= Ker\varphi_1 \times Ker\varphi_2
\end{aligned}$$

and

$$\begin{aligned} Im\varphi &= \{\varphi((a_1, a_2)) \mid (a_1, a_2) \in G_1 \times G_2\} \\ &= \{(\varphi_1(a_1), \varphi_2(a_2)) \mid a_1 \in G_1, a_2 \in G_2\} \\ &= \{(\varphi_1(a_1), \varphi_2(a_2)) \mid \varphi_1(a_1) \in Im\varphi_1, \varphi_2(a_2) \in Im\varphi_2\} \\ &= Im\varphi_1 \times Im\varphi_2. \end{aligned}$$

Moreover, suppose that φ is one-to-one. Let $a_1, b_1 \in G_1$ and $a_2, b_2 \in G_2$ such that $\varphi_1(a_1) = \varphi_1(b_1)$ and $\varphi_2(a_2) = \varphi_2(b_2)$. Then $\varphi((a_1, a_2)) = (\varphi_1(a_1), \varphi_2(a_2)) = (\varphi_1(b_1), \varphi_2(b_2)) = \varphi((b_1, b_2))$. Since φ is one-to-one, $(a_1, a_2) = (b_1, b_2)$ and so $a_1 = b_1$ and $a_2 = b_2$. Thus, φ_1 and φ_2 are one-to-one. Conversely, suppose that φ_1 and φ_2 are one-to-one. Let $(a_1, a_2), (b_1, b_2) \in G_1 \times G_2$ such that $\varphi((a_1, a_2)) = \varphi((b_1, b_2))$. Then $(\varphi_1(a_1), \varphi_2(a_2)) = \varphi((a_1, a_2)) = \varphi((b_1, b_2)) = (\varphi_1(b_1), \varphi_2(b_2))$. This implies that $\varphi_1(a_1) = \varphi_1(b_1)$ and $\varphi_2(a_2) = \varphi_2(b_2)$. Since φ_1 and φ_2 are one-to-one, it follows that $a_1 = b_1$ and $a_2 = b_2$ and so $(a_1, a_2) = (b_1, b_2)$. Hence, φ is one-to-one.

Finally, assume that φ is onto. Let $x_1 \in H_1$ and $x_2 \in H_2$. Then $(x_1, x_2) \in H_1 \times H_2$. Since φ is onto, there exists $(a_1, a_2) \in G_1 \times G_2$ such that $(\varphi_1(a_1), \varphi_2(a_2)) = \varphi((a_1, a_2)) = (x_1, x_2)$. It follows that $\varphi_1(a_1) = x_1$ and $\varphi_2(a_2) = x_2$. Hence, φ_1 and φ_2 are onto. Conversely, assume that φ_1 and φ_2 are onto. Let $(x_1, x_2) \in H_1 \times H_2$. Then $x_1 \in H_1$ and $x_2 \in H_2$. Since φ_1 and φ_2 are onto, there exists $a_1 \in G_1$ and $a_2 \in G_2$ such that $\varphi_1(a_1) = x_1$ and $\varphi_2(a_2) = x_2$ and so $\varphi((a_1, a_2)) = (\varphi_1(a_1), \varphi_2(a_2)) = (x_1, x_2)$. Therefore, φ is onto. ■

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WEAK OPEN SETS ON SIMPLE EXTENSION IDEAL TOPOLOGICAL SPACE

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Abstract. In this paper we intend to introduce a new class of sets known as e-$\mathcal{I}^+$-open sets, defined in the light of simple extension topology and ideal topology. This set is investigated and found to be a weaker form of e-$\mathcal{I}$-open sets. We have also generalized this concept and studied its properties.

Keywords: ideal topological space, e-open, e-$\mathcal{I}$-open sets, simple extension to topology, e-$\mathcal{I}^+$-open.

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1. Introduction

Levine [9] in 1964, defined one topology,τ^+, to be *simple extension* of another topology, τ, on the same set X by $\tau^+(B) = \{O \cup (\acute{O} \cap B) | O, \acute{O} \in \tau\}$ for some $B \notin \tau$. He investigated the question of whether (X, τ^+) has certain properties possesses by (X, τ), the properties included regularity, complement, and normality. By the definition of simple expansion we infer that all topologies are simple expansion topologies.

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An ideal $\mathcal{I}$ on a topological space $(X, \mathcal{I})$ is a nonempty collection of subsets of X which satisfies the following conditions:

$$A \in \mathcal{I} \text{ and } B \subset A \text{ implies } B \in \mathcal{I};\ A \in \mathcal{I} \text{ and } B \in \mathcal{I} \text{ implies } A \cup B \in \mathcal{I}.$$

Applications to various fields were further investigated by Jankovic and Hamlett [7] Dontchev et al. [4]; Mukherjee et al. [10]; Arenas et al. [3]; et al. Nasef and Mahmoud [11] etc. Given a topological space $(X, \tau, \mathcal{I})$ with an ideal $\mathcal{I}$ on X and if $\wp(X)$ is the set of all subsets of X. Then operator $(.)^* : \wp(X) \to \wp(X)$, called a local function [13, 7] of A with respect to τ and $\mathcal{I}$ is defined as follows: for $A \subseteq X$,

$$A^*(\mathcal{I}, \tau) = \{x \in X \mid U \cap A \notin \mathcal{I} \text{ for every } U \in \tau(x)\},$$

where $\tau(x) = \{U \in \tau \mid x \in U\}$.

A Kuratowski closure operator $Cl^*(x) = A \cup A^*(\mathcal{I}, \tau)$.

When there is no chance for confusion, we will simply write A^* for $A^*(\mathcal{I}, \tau)$. X^* is often a proper subset of X.

A subset A of an ideal space $(X, \tau, \mathcal{I})$ is said to be *R-I*-open (resp. *R-I*-closed) [15] if $A = Int(Cl^*(A))$ $(resp.\, A = Cl^*(Int(A))$. A point $x \in X$ is called $\delta - I$-cluster point of A if $Int(Cl^*(U)) \cap A \neq \emptyset$ for each open set V containing x. The family of all δ-$\mathcal{I}$-cluster points of A is called the δ-$\mathcal{I}$-closure of A and is denoted by $\delta Cl_I(A)$. The set δ-$\mathcal{I}$-interior of A is the union of all *R-I*-open sets of X contained in A and its denoted by $\delta Int_I(A)$. A is said to be δ-$\mathcal{I}$-closed if $\delta Cl_I(A) = A$ [15].

The subject of ideals in topological spaces has been studied by Kuratowski [8] and Vaidyanathaswamy [14]. Jankovic and Hamlett [7] introduced the notation of $\mathcal{I}$-open sets in ideal topological space, and investigated further properties of ideal space. Further Abd El-Monsef et al. [2] investigated $\mathcal{I}$-open sets and $\mathcal{I}$-continuous functions. Hatir [6] introduced the notion of *semi**-$\mathcal{I}$-open sets and obtained a decomposition of $\mathcal{I}$-continuity. The notion of *pre**-$\mathcal{I}$-open sets to obtain decomposition of continuity was introduced by E. Ekici and T. Noiri [5]. In addition to this, the concept of e-$\mathcal{I}$-open sets and e-$\mathcal{I}$-continuous functions have been introduced by [1].

Definition 1.1. A subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is called

1. *semi**-$\mathcal{I}$-open [6] if $A \subset Cl(\delta Int_I(A))$.

2. *pre**-$\mathcal{I}$-open [5] if $A \subseteq Int(\delta Cl_I(A))$.

3. $\delta\alpha$-$\mathcal{I}$-open [6] if $A \subset Int(Cl(\delta Int_I(A)))$.

4. $\delta\beta_I$-open [6] if $A \subset Int(Cl(\delta Int_I(A)))$.

5. e-$\mathcal{I}$-open [1] if $A \subset Cl(\delta Int_I(A)) \cup Int(\delta Cl_I(A))$.

In this paper we have made an attempt to extend these concept of $\mathcal{I}$-openness, *semi**-$\mathcal{I}$-openness, *pre**-$\mathcal{I}$-openness, e-$\mathcal{I}$-openness, $\delta\alpha$-$\mathcal{I}$-openness, $\delta\beta_I$-openness in simple extension topology.

2. e-$\mathcal{I}$-Open In Simple Extension

In all below definitions the interior $Int(A)$ refers to the interior in usual topology, δCl_I^+ denote the family of all δ-$\mathcal{I}^+$-cluster points of A, where. A point $x \in X$ is called δ-$\mathcal{I}^+$-cluster point of A if $Int(Cl^{+*}(U)) \cap A \neq \emptyset$ for each open set V containing x, $Cl^{*+}(A)$ is denoted the closure with respect to the ideal topological space under simple extension. And δInt_I^+ is the union of all R-I^+-open sets of X contained in A. Here a new local function is defined on the simple ideal topological space (SEITS) and its denoted as $A^{+*} = \{x \in X \mid U \cap A \notin \mathcal{I} \text{ for every } U \in \tau^+(B)\}$ as known as extend local functions with respect to τ^+ and $\mathcal{I}$. Also we defined a closure operator as $Cl^{+*}(A) = A \cup A^{+*}$. A subset A of $(X, \tau^+, \mathcal{I})$ is called $*+$ perfect if $A = A^{*+}$. The family of all e-$\mathcal{I}^+$-open defined by EI^+O.

Definition 2.1. Let A be a subset of simple extension ideal topological space (SEITS), then A is said to be

(1) $\mathcal{I}^+$ open set [12] if $A \subset Int(A^{*+})$.

(2) e^+-open if $A \subset Int(\delta Cl^+(A)) \cup Cl(\delta Int^+(A))$.

(3) R-I^+-open if $A = Int(Cl^{+*}(A))$.

(4) *semi**-$\mathcal{I}^+$-open if $A \subset Cl(\delta Int_I^+(A))$.

(5) *pre**-$\mathcal{I}^+$-open if $A \subseteq Int(\delta Cl_I^+(A))$.

(6) $\delta\alpha$-$\mathcal{I}^+$-open if $A \subset Int(Cl(\delta Int_I^+(A)))$.

(7) $\delta\beta_I^+$-open if $A \subset Int(Cl(\delta Int_I^+(A)))$.

(8) e-$\mathcal{I}^+$-open if $A \subset Cl(\delta Int_I^+(A)) \cup Int(\delta Cl_I^+(A))$.

Theorem 2.2. *Let $(X, \tau^+, \mathcal{I})$ be an simple extension ideal topological space (SEITS) the following hold:*

(1) *Every open is e-$\mathcal{I}^+$-open,*

(2) *Every e-$\mathcal{I}^+$-open is e-$\mathcal{I}$-open,*

(3) *Every $\mathcal{I}^+$-open is e-$\mathcal{I}^+$-open.*

Proof. (1) Let A be any subset of $(X, \tau^+, \mathcal{I})$ if A is open in τ we have:

$$\begin{aligned} A &= Int(A) \\ &\subset Int(\delta Cl_I^{+*}(A)) \\ &\subset Int(\delta Cl_I^{+*}(A)) \cup Cl(\delta Int_I^+(A))) \end{aligned}$$

Then A is e-$\mathcal{I}^+$-open.

(2) By the definition of e-$\mathcal{I}^+$-open and e-$\mathcal{I}$-open and since $Cl^{+*}(A) \subset Cl^*(A)$, then $\delta Cl_I^{+*}(A) \subset \delta Cl_I^*(A)$, under theses conditions every e-$\mathcal{I}^+$-open is e-$\mathcal{I}$-open.

(3) Obvious. ∎

Remark 2.3. From the above Theorem we know the class of e-$\mathcal{I}^+$-open sets is properly placed between an open set and e-$\mathcal{I}$-open set. But the converse no need to be true.

Example 2.4. Let $X = \{a, b, c\}$ with a topology $\tau = \{\emptyset, X, \{a\}, \{a, b\}\}$ and an ideal $\mathcal{I} = \{\varnothing, \{b\}\}$,$B = \{b\}$, $\tau^+(B) = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Then the set $A = \{a, c\}$ is e-$\mathcal{I}^+$-open, but it is not open in the topology τ and τ^+.

Example 2.5. Let $X = \{a, b, c\}$ with a topology $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and an ideal $\mathcal{I} = \{\varnothing, \{c\}\}$,$B = \{b, c\}$, $\tau^+(B) = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$. Here $\{a, c\}$ is e-$\mathcal{I}$-open, but it is not e-$\mathcal{I}^+$-open.

Proportion 2.6. *For any simple extension ideal topological space (SEITS) $(X, \tau^+, \mathcal{I})$ and $A \subset X$ we have:*

(1) *If $I = \emptyset$, then A is e-$\mathcal{I}^+$-open if and only if A is e^+-open.*

(2) *If $I = \wp(X)$, then A is e-$\mathcal{I}^+$-open if and only if $A \in \tau$.*

(3) *If $I = N$, then A is e-$\mathcal{I}^+$-open if and only if A is e^+-open, where N the ideal of nowhere dense.*

Proof. (1) Let $I = \emptyset$ and $A \subset X$. We have $\delta Cl_I^+(A)) = \delta Cl^+(A))$, $\delta Int_I^+(A)) = \delta Int^+(A))$ and $A^{+*} = Cl^+(A)$. on other hand, $Cl^{+*}(A) = A^{+*} \cup A = Cl^+(A)$. Hence $A^{+*} = Cl^+(A) = Cl^{+*}(A)$. Thus (1) follows immediately.

(2) Let $I = P(X)$ then $A^{+*} = \emptyset$, for any $A \subset X$. Since A is e-$\mathcal{I}^+$-open, we have

$$\begin{aligned} A &\subset Cl(\delta Int_I^+(A)) \cup Int(\delta Cl_I^+(A)) \\ &= Int[Int(Cl(\delta Int_I^+(A))) \cup \delta Cl_I^+(A)] \\ &\subset Int[Cl(\delta Int_I^+(A)) \cup \delta Cl_I^+(A)] \\ &\subset Int[\delta Cl_I^+(\delta Int_I^+(A \cup A))] \\ &\subset Int[\delta Cl_I^+(\delta Int_I^+(A))] \\ &\subset Int[Cl(Int(A))] \end{aligned}$$

This show $A \in \tau$.

(3)$\Leftarrow$ Every e-$\mathcal{I}^+$-open is e^+-open.

Let A be e-$\mathcal{I}^+$-open then, $A \subset Cl(\delta Int_I^+(A)) \cup Int(\delta Cl_I^+(A))$. By using this fact when $I = \emptyset$ part (1), $A^{+*} = Cl^+(A) = Cl^{+*}(A)$, we have $\delta Cl_I^+(A) = \delta Cl^+(A)$, $\delta Int_I^+(A) = \delta Int^+(A)$, since $\delta Cl_I^+(A)$ is the family of all δ-$\mathcal{I}^+$-cluster point of A, and $\delta Int_I^+(A)$ the union of all R-$\mathcal{I}^+$-open set of X we have respectively,

$$\begin{aligned}\emptyset \neq Int(Cl^{*+}(U)) \cap A &= Int(U^{*+} \cup U) \cap A = Int(Cl^+(U) \cup U) \cap A\\ &= Int(Cl^+(U)) \cap A \neq \emptyset\end{aligned}$$

From this we get $\delta Cl_I^+(A) = \delta Cl^+(A)$, and

$$\begin{aligned}A = Int(Cl^{*+}(A)) &= Int(A^{*+} \cup A) = Int[Cl^+(A) \cup A]\\ &= Int(Cl^+(A)) = A\end{aligned}$$

From this we get $\delta Int_I^+(A) = \delta Int^+(A)$. This show that

$$A \subset Cl(\delta Int_I^+(A)) \cup Int(\delta Cl_I^+(A)) \subset Cl(\delta Int^+(A)) \cup Int(\delta Cl^+(A)).$$

Now, let us consider $I = N$ and A is e^+-open.

$\Rightarrow$ If $I = N$ then $A^{+*} = Cl^{+*}(Int(Cl^{+*}A))$.

Since A is e^+-open then $A \subset Cl(\delta Int^+(A)) \cup Int(\delta Cl^+(A))$. Then

$$\begin{aligned}&\emptyset \neq Int(Cl^+(U)) \cap A = Int(U^+ \cup U) \cap A\\ &\quad = Int(Cl^+(Int(Cl^+(U)) \cup U) \cap A \subset Int(Cl^{+*}(Int(Cl^{+*}(U))) \cup U) \cap A\\ &\quad = Int(U^{*+} \cup U) \cap A = Int(Cl^{+*}(U)) \cap A \neq \emptyset\end{aligned}$$

From this we get $\delta Cl^+(A) \subset \delta Cl_I^+(A)$, and

$$\begin{aligned}A &= Int(Cl^+(A)) = Int(A^+ \cup A) = Int[Cl^+(Int(Cl^+(A))) \cup A]\\ &\subset Int[Cl^{+*}(Int(Cl^{+*}(A))) \cup A] = Int(A^{*+} \cup A) = Int(Cl^{+*}(A)) = A\end{aligned}$$

From this we get $\delta Int^+(A) \subset \delta Int_I^+(A)$.
A is e-$\mathcal{I}^+$-open. Hence the proof. ∎

Proposition 2.7. *Let A be a subset of (SITES) $(X, \tau^+, \mathcal{I})$ then the following properties hold:*

(1) *Every semi*-$\mathcal{I}^+$-open is e-$\mathcal{I}^+$-open,*

(2) *Every pre*-$\mathcal{I}^+$-open is e-$\mathcal{I}^+$-open,*

(3) *Every e-$\mathcal{I}^+$-open is $\delta\beta_I^+$-open.*

(4) *Every $\delta\alpha$-$\mathcal{I}^+$-open is $\delta\beta_I^+$-open.*

Proof. (1) and (2) are obvious from the definition of e-$\mathcal{I}^{+}$-open set.

(3) Let A be e-$\mathcal{I}^{+}$-open. Then we have,

$$\begin{aligned} A &\subset Cl(\delta Int_I^{+}(A)) \cup Int(\delta Cl_I^{+}(A)) \\ &\subset Cl(Int(\delta Int_I^{+}(A))) \cup Int(Int(\delta Cl_I^{+}(A))) \\ &\subset Cl(Int(\delta Int_I^{+}(A)) \cup Int(\delta Cl_I^{+}(A))) \\ &\subset Cl[Int(\delta Int_I^{+}(A)) \cup \delta Cl_I^{+}(A)] \\ &\subset Cl[Int(\delta Cl_I^{+}(A \cup A))] \\ &= Cl(Int(\delta Cl_I^{+}(A))). \end{aligned}$$

This show that A is an $\delta\beta_I^{+}$-open set.

(4) proof is obvious. ■

Remark 2.8. From above the following implication,

$$\begin{tikzcd}
\delta_I^{+}\text{ open} \arrow[r] \arrow[d] & \delta\alpha\text{-}\mathcal{I}^{+}\text{-open} \arrow[r] & semi^{*}\text{-}\mathcal{I}^{+}\text{-open} \arrow[dd] \\
\text{open} \arrow[d] & & \\
pre^{*}\text{-}\mathcal{I}^{+}\text{-open} \arrow[rr] \arrow[dr] & & \text{e-}\mathcal{I}^{+}\text{-open} \arrow[dl] \\
& \delta\beta_{\mathcal{I}}^{+}\text{-open} &
\end{tikzcd}$$

A is called δ_I^{+} open if for each $x \in A$, there exist a R-$\mathcal{I}^{+}$-open set G such that $x \in G \subset A$. None of these implications is reversible as shown by examples given below.

Example 2.9. Let $X = \{a, b, c\}$ with a topology $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$, $\mathcal{I} = \{\emptyset, \{b\}\}$, $B = \{b, c\}$, $\tau^{+}(B) = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$. Then the set $A = \{a, c\}$ is e-$\mathcal{I}^{+}$-open, but it is not pre^{*}-$\mathcal{I}^{+}$-open.

Example 2.10. Let $X = \{a, b, c\}$ with a topology $\tau = \{\emptyset, X, \{c\}\}$ and an ideal $\mathcal{I}$=$\{\emptyset, \{b\}\}$. Let B=$\{a\}$, then $\tau^{+}(B) = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}\}$. Here the set $A = \{b, d\}$ is e-$\mathcal{I}^{+}$-open, but it is not $semi^{*}$-$\mathcal{I}^{+}$-open. Because $Cl(\delta Int_I(A)) \cup Int(\delta Cl_I(A)) = Cl(\{a\}) \cup Int(X) = \{a, b\} \cup X = X \supset A$ and hence A is e-$\mathcal{I}^{+}$-open. Since $Cl(\delta Int_I(A)) = Cl(\{a\}) = \{a, b\} \not\supseteq A$. So A is not $semi^{*}$-$\mathcal{I}^{+}$-open.

Example 2.11. Let $X = \{a, b, c\}$ with a topology $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and an ideal $\mathcal{I} = \{\emptyset, \{b\}\}$. Let $B = \{b, c\}$, $\tau^{+}(B) = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$. Here $A = \{a, c\}$ is e-$\mathcal{I}^{+}$-open, but it is not $\delta\alpha_{\mathcal{I}}^{+}$-open. Because $Cl(\delta Int_I(A)) \cup Int(\delta Cl_I(A)) = Cl(\{a\}) \cup Int(X) = \{a\} \cup X = X \supset A$ and hence A is e-$\mathcal{I}^{+}$-open. Since $Int(Cl(\delta Int_I(A))) = Int(Cl(\{a\})) = \{a\} \not\supseteq A$. So A is not $\delta\alpha_{\mathcal{I}}^{+}$-open.

Theorem 2.12. *Let $(X, \tau, \mathcal{I})$ an ideal in topological space and A, B subsets of X. Then, for local functions the following properties hold:*

(1) *If* $A \subset B$, *then* $A^{*+} \subset B^{*+}$,

(2) *For another ideal* $J \supset I$ *on* X, $A^{*+}(J) \subset A^{*+}(\mathcal{I})$,

(3) $A^{*+} \subset Cl(A)$,

(4) $A^{*+}(\mathcal{I}) = Cl(A^{*+}) \subset Cl(A)$ *(i.e* A^{*+}

(5) $(A^{*+})^{*+} \subset A^{*+}$,

(6) $(A \cup B)^{*+} = A^{*+} \cup B^{*+}$,

(7) $A^{*+}\text{-}B^{*+} = (A - B)^{*+} - B^{*+} \subset (A - B)^{*+}$,

(8) *If* $U \in \tau$, *then* $U \cap A^{*+} = U \cap (U \cap A)^{*+} \subset (U \cap A)^{*+}$,

(9) *If* $I \in \mathcal{I}$, *then* $(A\text{-}I)^{*+} \subset A^{*+} = (A \cup I)^{*+}$,

Proof. Obvious using the Definition of A^{*+}. ∎

Proposition 2.13. *Let* $(X, \tau^+, \mathcal{I})$ *be SEITS and let* A, $U \subseteq X$. *If* A *is* e-$\mathcal{I}^+$*-open set and* $U \in \tau$. *Then* $A \cap U$ *is an* e-I^+*-open.*

Proof. By assumption $A \subset Cl(\delta Int_I(A)) \cup Int(\delta Cl_I(A))$ and $U \subseteq Int(U)$. By Theorem 2.12 (8) we have,

$$\begin{aligned}
A \cap U \subset & (Cl(\delta Int_I^+(A)) \cup Int(\delta Cl_I^+(A))) \cap Int(U) \\
& \subset (Cl(\delta Int_I^+(A)) \cap Int(U)) \cup (Int(\delta Cl_I^+(A)) \cap Int(U)) \\
& \subset (Cl(\delta Int_I^+(A)) \cap Cl(Int(U))) \cup (Int(\delta Cl_I^+(A)) \cap Cl(Int(U))) \\
& \subset (Cl(\delta Int_I^+(A)) \cap Int(U)) \cup (Int(Cl(\delta Cl_I^+(A)) \cap Cl(Cl(Int(U))))) \\
& \subset Cl(\delta Int_I^+(A \cap U) \cup (Int(Cl(\delta Cl_I^+(A)) \cap Cl(Int(U)))) \\
& \subset Cl(\delta Int_I^+(A \cap U)) \cup (Int(Cl(\delta Cl_I^+(A)) \cap Int(U))) \\
& \subset Cl(\delta Int_I^+(A \cap U)) \cup (Int(\delta Cl_I^+(A \cap U))).
\end{aligned}$$

Thus $A \cap U$ is e-$\mathcal{I}^+$-open. ∎

Proposition 2.14. *Let* $(X, \tau^+, \mathcal{I})$ *be SEITS then the following hold.*

(1) *The union of any family of* e-$\mathcal{I}^+$*-open sets is an* e-$\mathcal{I}^+$*-open set.*

(2) *The intersection of arbitrary family of* e-$\mathcal{I}^+$*-closed sets is* e-$\mathcal{I}^+$*-closed.*

(3) *If* $A \in EI^+O(X, \tau^+, \mathcal{I})$ *and* $B \in \tau$, *then* $A \cap B \in EI^+O(X, \tau^+, \mathcal{I})$.

Proof. (1) Let $\{A_\alpha | \alpha \in \Delta\}$ be a family of e-$\mathcal{I}^+$-open set, $A_\alpha \subset Cl(\delta Int_I^+(A_\alpha)) \cup Int(\delta Cl_I^+(A_\alpha))$. Hence

$$\begin{aligned}
\cup_\alpha A_\alpha & \subset \cup_\alpha [Cl(\delta Int_I^+(A_\alpha)) \cup Int(\delta Cl_I^+(A_\alpha))] \\
& \subset \cup_\alpha [Cl(\delta Int_I^+(A_\alpha))] \cup \cup_\alpha [Int(\delta Cl_I^+(A_\alpha))] \\
& \subset [Cl(\cup_\alpha (\delta Int_I^+(A_\alpha))] \cup [Int(\cup_\alpha (\delta Cl_I^+(A_\alpha))] \\
& \subset [Cl(\cup_\alpha (\delta Int_I^+(A_\alpha))] \cup [Int(\cup_\alpha (\delta Cl_I^+(A_\alpha))] \\
& \subset [Cl(\delta Int_I^+(\cup_\alpha A_\alpha))] \cup [Int(\delta Cl_I^+(\cup_\alpha A_\alpha))].
\end{aligned}$$

$U_\alpha A_\alpha$ is e-$\mathcal{I}^+$-open.

(2) Let $\{B_\alpha/\alpha \in \Delta\}$ be a family of e-$\mathcal{I}^+$-closed set. Then $\{B^c_\alpha/\alpha \in \Delta\}$ be a family of e-$\mathcal{I}^+$-open set. By (1) $\cup^c_\alpha A_\alpha$ is e-$\mathcal{I}^+$-open. Hence $(\cap_\alpha A_\alpha)^c = \cup^c_\alpha A_\alpha$ is e-$\mathcal{I}^+$-open $(\cap_\alpha A_\alpha)$ is e-$\mathcal{I}^+$-closed set. Hence the proof.

(3) Let $A \in EI^+O(X, \tau^+, \mathcal{I})$ and $B \in \tau$ then $A \subset Cl(\delta Int^+_I(A)) \cup Int(\delta Cl^+_I(A))$ and

$$\begin{aligned} A \cap B &\subset [Cl(\delta Int^+_I(A)) \cup Int(\delta Cl^+_I(A))] \cap B \\ &\subset [Cl(\delta Int^+_I(A)) \cap B] \cup [Int(\delta Cl^+_I(A)) \cap B] \\ &\subset [Cl(\delta Int^+_I(A \cap B))] \cup [Int(\delta Cl^+_I(A \cap B))]. \end{aligned}$$

This proof come from the fact $\delta Int^+_I(A)$ is the union of all R-$\mathcal{I}^+$-open of X contend in A. Then

$$\begin{aligned} A = Int(Cl^{*+}(A)) \Rightarrow A \cap B &= Int(Cl^{*+}(A)) \cap B \\ &= Int(A^{*+} \cup A) \cap B \\ &= Int[(A \cap B) \cup (A^{*+} \cap B)] \\ &\subset Int[Cl^{*+}(A \cap B)] = A \cap B \end{aligned}$$

Hence $Cl(\delta Int^+_I(A)) \cap B \subset Cl(\delta Int^+_I(A \cap B))$, and other part is obvious. ■

Let $(X, \tau^+, \mathcal{I})$ be a SEITS and A be a subset of X, we denoted the relative topology [12] on A by τ^+/A and $\mathcal{I}/A = \{A \cap I : I \in \mathcal{I}\}$ is clearly ideal on A.

Lemma 2.15. *Let $(X, \tau^+, \mathcal{I})$ be a SEITS and A, B subset of X such that $B \subset A$. Then $B^{+*}(\tau^+|_A, \mathcal{I}|_A) = B^{+*}(\tau^+, \mathcal{I}) \cap A$.*

Proposition 2.16. *Let $(X, \tau^+, \mathcal{I})$ be a SEITS and let A, $U \subseteq X$.*
If $V \in EI^+O(X, \tau^+, \mathcal{I})$ set and $U \in \tau$. Then $U \cap V \in EIO(U, \tau^+|_U, \mathcal{I}|_U)$.

Proof. Since U is open, we have $Int_U(A) = Int(A)$ for any subset A of U. By using this fact and Theorem (2.12). We get the proof. ■

Definition 2.17. [12] A point $x \in X$ is said to be $\mathcal{I}^+$ limit point of A if for every $\mathcal{I}^+$ open set U in X, $U \cap (A \backslash x) \neq \emptyset$. The set of all $\mathcal{I}^+$ limit point of A is called the $\mathcal{I}^+$ derived set of A denoted by $D^+_I(A)$.

Definition 2.18. Let A be a subset of X.

(1) The intersection of all e-$\mathcal{I}^+$-closed containing A is called the e-I^+-closure of A and its denoted by $Cl^{I+}_e(A)$,

(2) The e-$\mathcal{I}^+$-interior of A, denoted by $Int^{I+}_e(A)$, is defined by the union of all e-I^+-open sets contained in A.

Definition 2.19. Let A be a subset of $(X, \tau^+, \mathcal{I})$. A point $x \in X$ is said to be $\mathcal{I}^+$ limit point of A if for every e-$\mathcal{I}^+$ open set U in X, $U \cap (A \backslash x) \neq \emptyset$. The set of all $e - \mathcal{I}^+$ limit point of A is called the $e - \mathcal{I}^+$ derived set of A denoted by $D^+_{eI}(A)$.

Since every open is pre^*-$\mathcal{I}^+$ open set and every pre^*-$\mathcal{I}^+$ open set is e-$\mathcal{I}^+$ open set we have. $D^+_{eI}(A) \subset D(A)$ for any $A \subset X$. Moreover, since every closed set is e-$\mathcal{I}^+$-closed set we have $A \subset Cl^{I+}_e(A) \subset Cl(A)$.

Lemma 2.20. *If $D^+_{eI}(A) = D(A)$, then we have $Cl^{I+}_e(A) = Cl(A)$*

Proof. Straightforward. ∎

Corollary 2.21. *If $D(A) \subset D^+_{eI}(A)$ for every subset $A \subset X$. Then for any subset C and B of X, we have $Cl^{I+}_e(B \cup C) = Cl^{I+}_e(B) \cup Cl^{I+}_e(C)$.*

Theorem 2.22. *If A be a subset of $(X, \tau^+, \mathcal{I})$, then $x \in Cl^{I+}_e(A)$ if and only if every e-$\mathcal{I}^+$ open set U containing x intersect A.*

Proof. Let us prove that $x \in Cl^{I+}_e(A)$ if and only if there exists e-$\mathcal{I}^+$ open set U containing x which does not intersect A, hence $x \notin Cl^{I+}_e(A) \Rightarrow x \notin X \backslash Cl^{I+}_e(A)$ which does not intersect A.

Conversely, let U be e-$\mathcal{I}^+$-open set U containing x which does not intersect A. Then $(X \backslash U)$ is e-$\mathcal{I}^+$-open set U containing A and $x \in (X \backslash U)$ but $Cl^{I+}_e(A) \subset X \backslash U$. ∎

Theorem 2.23. $Cl^{I+}_e(A) = A \cup D^+_{eI}(A)$.

Proof. If $x \in D^+_{eI}(A)$. Then, for every e-$\mathcal{I}^+$ open set U containing x, we have $U \cap (A \backslash x) \neq \emptyset$. Therefore $x \in Cl^{I+}_e(A)$, i.e.,

$$A \cup D^+_{eI}(A) \subseteq Cl^{I+}_e(A) \qquad (*)$$

Conversely, let $x \in Cl^{I+}_e(A)$. If $x \in A$, then $x \in A \cup D^+_{eI}(A)$. Let $x \notin A$, since $x \in Cl^{I+}_e(A)$ every e-$\mathcal{I}^+$-open set U containing x intersects A. But $x \notin A \Rightarrow U \cap (A \backslash x) \neq \emptyset$. Therefore $x \in Cl^{I+}_e(A)$, i.e.,

$$Cl^{I+}_e(A) \subseteq A \cup D^+_{eI}(A) \qquad (**)$$

From $(*)$ and $(**)$, we get $Cl^{I+}_e(A) = A \cup D^+_{eI}(A)$. ∎

3. Generalized e-$\mathcal{I}^+$-Closed Sets

Definition 3.1. A subset A of a SEITS $(X, \tau^+, \mathcal{I})$ is said to be gEI^+-closed if $Cl^{I+}_e(A) \subset U$ whenever $A \subset U$ and $U \in \tau^+$.

The set of all gEI^+ closed sets of X is denoted as $GEI^+C(X)$.

Example 3.2. *Let $X = \{a, b, c\}$ with a topology $\tau = \{\emptyset, X, \{a\}, \{a, b\}\}$ and an ideal $\mathcal{I} = \{\varnothing, \{b\}\}$, $B = \{b\}$, $\tau^+(B) = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Then the sets $\{\emptyset, X, \{a\}, \{a, c\}, \{a, b\}\}$ are e-$\mathcal{I}^+$-open, and gEI^+-closed sets are $\{\emptyset, X, \{b\}, \{c\}, \{b, c\}, \{a, c\}\}$.*

Since every $\mathcal{I}^+$-closed set is e-$\mathcal{I}^+$-closed we have $Cl^{I+}_e(A) \subseteq I^+Cl(A)$.

Theorem 3.3. Let $(X, \tau^+, \mathcal{I})$ be an simple extension ideal topological space (SEITS) then the following hold:

(1) *Every $\mathcal{I}^+$-closed set is gEI^+-closed.*

(2) *Every e-$\mathcal{I}^+$-closed set is gEI^+-closed.*

Proof. (1) Since A is $\mathcal{I}^+$-closed set we have $A = I^+Cl(A) \subseteq U$ by the above note we have $Cl_e^{I+}(A) \subseteq I^+Cl(A)$, then $Cl_e^{I+}(A) \subseteq U$ whenever $A \subseteq U$ and $U \in \tau^+$. Hence the proof.

(2) Let A be e-$\mathcal{I}^+$-closed set. Then $A = Cl_e^{I+}(A) \subseteq U$. Hence A is gEI^+-closed. But the converse need not be true. ∎

Example 3.4. *Let $X = \{a, b, c\}$ with a topology $\tau = \{\emptyset, X, \{a\}, \{a, b\}\}$ and an ideal $\mathcal{I} = \{\emptyset, \{b\}\}$, $B = \{b\}$, $\tau^+ = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Then the set $\{a, c\}$ is gEI^+-closed sets but not e-$\mathcal{I}^+$-closed.*

Theorem 3.5. *If A be gEI^+-closed of a SEITS $(X, \tau^+, \mathcal{I})$, $Cl_e^{I+}(A) \backslash A$ does not contains any nonempty closed set.*

Proof. (1) Let S be closed set such that $S \subseteq Cl_e^{I+}(A) \backslash A$. Then $(X \backslash S)$ is open and

$$S \subseteq Cl_e^{I+}(A) \cap A^c. \tag{$*$}$$

$S \subseteq A^c \Rightarrow A \subset (X \backslash S)$. Since A is gEI^+-closed we have $Cl_e^{I+}(A) \subseteq (X \backslash S)$. Hence

$$S \subseteq X \backslash Cl_e^{I+}(A). \tag{$**$}$$

From $(*)$ and $(**)$, we have $S \subseteq Cl_e^{I+}(A) \cap X \backslash Cl_e^{I+}(A) = \emptyset$. Hence $Cl_e^{I+}(A) \backslash A$. ∎

Theorem 3.6. *If A be gEI^+-closed set of a SEITS $(X, \tau^+, \mathcal{I})$ and $A \subseteq B \subseteq Cl_e^{I+}(A)$ then B is also gEI^+-closed.*

Proof. Let A be gEI^+-closed set and $A \subseteq B \subseteq Cl_e^{I+}(A)$. Then $Cl_e^{I+}(A) \subseteq Cl_e^{I+}(B) \subseteq Cl_e^{I+}(A)$ which implies that $Cl_e^{I+}(A) = Cl_e^{I+}(B)$ let us now consider U to be open set in $(X, \tau^+, \mathcal{I})$ containing B. Then $A \subseteq U$ and A is gEI^+-closed. $\Rightarrow Cl_e^{I+}(A) \subseteq U \Rightarrow Cl_e^{I+}(B) \subseteq U$. Then B is gEI^+-closed set. ∎

Theorem 3.7. *A gEI^+-closed set A is also e-I^+-closed if and only if $Cl_e^{I+}(A) \backslash A$ is closed.*

Proof. Let A be e-I^+-closed $A = Cl_e^{I+}(A)$. If $Cl_e^{I+}(A) \backslash A = \emptyset$ which is closed.

Conversely, let $Cl_e^{I+}(A) \backslash A$ is closed. By Theorem (3.6) we know that $Cl_e^{I+}(A) \backslash A$ does not contains any nonempty closed set. Therefor $Cl_e^{I+}(A) \backslash A = \emptyset \Rightarrow Cl_e^{I+}(A) = A$. Hence A is e-I^+-closed. ∎

Theorem 3.8. *If A and B are gEI^+-closed sets such that $D(A) \subseteq D_{eI}^+(A)$ and $D(B) \subseteq D_{eI}^+(B)$. Then $A \cup B$ is gEI^+-closed.*

Proof. Let U be an open set such that $A \cup B \subseteq U$. Then since A and B are gEI^+-closed sets we have $Cl_e^{I+}(A) \subseteq U$ $Cl_e^{I+}(B) \subseteq U$. Since $D(A) \subseteq D_{eI}^+(A)$, thus $D(A) = D_{eI}^+(A)$ and by Lemma (2.20) $Cl(A) = Cl_e^{I+}(A)$ and, similarly, $Cl(B) = Cl_e^{I+}(B)$. Thus

$$Cl_e^{I+}(A \cup B) \subseteq Cl(A \cup B) = Cl(A) \cup Cl(B) = Cl_e^{I+}(A) \cup Cl_e^{I+}(B) \subseteq U.$$

This implies $A \cup B$ is gEI^+-closed. ∎

Proposition 3.9. *If A and B are gEI^+-closed sets such that $D(A) \subseteq D^+_{eI}(A)$ and $D(B) \subseteq D^+_{eI}(B)$. Then $A \cup B$ is gEI^+-closed.*

Proof. Let U be an open set such that $A \cup B \subseteq U$. Then since A and B are gEI^+-closed sets we have $Cl^{I+}_e(A) \subseteq U$ $Cl^{I+}_e(B) \subseteq U$. Since $D(A) \subseteq D^+_{eI}(A)$, thus $D(A) = D^+_{eI}(A)$ and by Lemma (2.20) $Cl(A) = Cl^{I+}_e(A)$ and similarly $Cl(B) = Cl^{I+}_e(B)$. Thus

$$Cl^{I+}_e(A \cup B) \subseteq Cl(A \cup B) = Cl(A) \cup Cl(B) = Cl^{I+}_e(A) \cup Cl^{I+}_e(B) \subseteq U.$$

This implies $A \cup B$ is gEI^+-closed. ∎

Definition 3.10. Let $B \subseteq A \subseteq X$. The set B is said to be gEI^+-closed relative to A if ${}_ACl^{I+}_e(B) \subseteq U$ whenever $B \subseteq U$ and U is open in A, where ${}_ACl^{I+}_e(B) = A \cap Cl^{I+}_e(B)$.

Theorem 3.11. *If $B \subseteq A \subseteq X$ and A is gEI^+-closed and open, then B is gEI^+-closed relative to A if and only if B is gEI^+-closed in X.*

Proof. Let A be a gEI^+-closed and open. Let B is gEI^+-closed relative to A. Since A be an gEI^+-closed and open then $Cl^{I+}_e(A) \subseteq A$. Therefore, $Cl^{I+}_e(B) \subseteq Cl^{I+}_e(A) \subseteq A$. Therefore, ${}_ACl^{I+}_e(B) \subseteq Cl^{I+}_e(B) \cap A = Cl^{I+}_e(B)$. Now, let U be open in X and $B \subseteq U$. Then, $U \cap A$ is open in A and $B \subseteq U \cap A$. Since B is gEI^+-closed relative to A we have ${}_ACl^{I+}_e(B) \subseteq U \cap A$. Hence ${}_ACl^{I+}_e(B) \subseteq U \cap A \subseteq U$. Therefore, B is gEI^+-closed. Conversely, let B is gEI^+-closed in X. Consider U is an open in A and $B \subseteq U$. Then $U = V \cap A$ where V is open in $(X, \tau^+, \mathcal{I})$. Now $B \subseteq V and\ B$ is gEI^+-closed in X. This implies $Cl^{I+}_e(B) \cap A \subset V \cap A = U$, i.e., $Cl^{I+}_e(B) \cap A \subseteq U$. ∎

Definition 3.12. A set A is said to be gEI^+-open if and only if $(X \backslash A)$ is gEI^+-closed. The family of all gEI^+-open subset of X is denoted by $GEI^+O(X)$. The largest gEI^+-open set contained in X is called the gEI^+-interior of A and is denoted by $gEI^+(Int(A))$ also A is gEI^+-open if and only if $gEI^+(Int(A)) = A$.

Proposition 3.13. *Let $(X, \tau^+, \mathcal{I})$ be an simple extension ideal topological space (SEITS) then Statement $Cl^{I+}_e(X \backslash A) = X \backslash Cl^{I+}_e(A)$ hold.*

Proof. Let $x \in Cl^{I+}_e(X \backslash A)$.
⇔ every e-$\mathcal{I}^+$-open set U containing x intersects $(X \backslash A)$.
⇔ there is no e-$\mathcal{I}^+$-open set U containing x and contained in A.
⇔ $x \in X \backslash Cl^{I+}_e(A)$. ∎

Theorem 3.14. *A subset A of a SETIS $(X, \tau^+, \mathcal{I})$ is gEI^+-open if and only if $S \subseteq Int^{I+}_e(A)$ where S is closed and $S \subseteq A$.*

Proof. Let A be gEI^+-open and suppose that S is closed and $S \subseteq A$. Then $(X \backslash A)$ is gEI^+-closed and $(X \backslash A) \subset (X \backslash S)$. Now, $(X \backslash S)$ is open and $(X \backslash A)$ is gEI^+-closed. Therefore, $Cl^{I+}_e(X \backslash A) \subseteq (X \backslash S)$. By Proportion (3.13) $Cl^{I+}_e(X \backslash A) = X \backslash Int^{I+}_e(A)$. Hence $X \backslash Int^{I+}_e(A) \subseteq (X \backslash S)$. i.e., $S \subseteq Int^{I+}_e(A)$.

Conversely, let $S \subseteq Int^{I+}_e(A)$ where S is closed and $S \subseteq A$. Now, to prove A is gEI^+-open is the same as proving $(X \backslash A)$ gEI^+-closed. Let G be an open set containing $(X \backslash A)$ then $S = (X \backslash G)$ is closed set such that $S \subseteq A$. Therefore, $S \subseteq Int^{I+}_e(A)$. i.e., $Cl^{I+}_e(X \backslash A) = (X \backslash Cl^{I+}_e(X \backslash A)) \subset (X \backslash S)$ $Cl^{I+}_e(X \backslash A) \subseteq G$. Therefore, $(X \backslash A)$ gEI^+-closed. i.e., A is gEI^+-open. Hence the proof. ∎

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EXISTENCE AND UNIQUENESS THEOREM FOR A SOLUTION OF FUZZY IMPULSIVE DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we prove the existence and uniqueness of a solution of the fuzzy impulsive differential equation $x^{'}(t) = f(t, x(t)), x(t_0) = x_0, \Delta x(t_k) = I_k(x(t_k))$ by using the method of successive approximation. We also consider the ϵ-approximate solution for the above fuzzy impulsive differential equation.

Keywords: Fuzzy differential equation, Levelwise continuous, Fuzzy impulsive equation, Fuzzy integral, Fuzzy ϵ-approximate solution.

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1. Introduction

Knowledge about differential equations is often incomplete or vague. For example, initial conditions or the values of functional relationships may not be known precisely. In such a situation, the usage of fuzzy differential equations (FDEs) is a natural way to model dynamical systems under possibilistic uncertainty. FDEs is a very important topic from the theoretical point of view (see e.g. [8] and references therein) as well as of their applications, for example, in modelling hydraulic [2], in population models [3, 12], in modelling of a three-phase induction

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motor [26]. The study of fuzzy differential equations forms a suitable setting for modelling dynamical systems.

Some authors have studied fuzzy differential equations. FDEs were first formulated by Kaleva [7]. He discussed the properties of differentiable fuzzy set value mappings and give the existence and uniqueness theorem for a solution of the fuzzy differential equation. Seikkala [23] defined the fuzzy derivatives which is generalization of the Hukuhara derivative in [19]. Since then there appeared a lot of papers concerning different approaches to the theory of FDEs. A rich collection of results from the theory of FDEs is contained in the monographs of Lakshmikantham and Mohapatra [8]. Park [17] studied the existence and uniqueness theorem for fuzzy differential equations. For the cauchy problem $x^{'} = f(t,x), x(t_0) = x_0$, the local existence theorems are proved in [25], and the existence theorems under compactness-type conditions are investigated in [24] when the fuzzy valued mapping f satisfies the generalized Lipschitz condition. There appeared a lot of papers concerning different approaches to the theory of FDEs (see, e.g., [14], [15], [16]).

On the other hand, the theory of impulsive differential equations or implicit impulsive integro-differential equations has been emerging as an important area of investigation in recent years and has been developed very rapidly due to the fact that such equations find a wide range of applications modeling adequately many real processes observed in physics, chemistry, biology and engineering. Correspondingly, applications of the theory of impulsive differential equations to different areas were considered by many authors (see, e.g, [5], [11], [13]). There are not too many papers on impulsive fuzzy differential equations, but some basic results on impulsive fuzzy differential equations can be found in [4], [6], [9], [20], [21]. For the monographs of the theory of impulsive diffferential equations, we can refer the books of Bainov and Simenov [1], Lakshmikantham et.al [10], Samoilenko and Perestyuk [22].

Motivated and inspired by the above works, In this paper, we prove the existence and uniqueness theorem of a solution to the fuzzy impulsive differential equation,

$$\begin{aligned} x'(t) &= f(t, x(t)), \\ x(t_0) &= x_0, \\ \Delta x(t_k) &= I_k(x(t_k)), \quad k = 1, 2..., m \end{aligned} \tag{1.1}$$

where $f : I \times E^d \to E^d$ is levelwise continuous and satisfies a generalized Lipschitz condition, $x_0 \in E^d$, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, where $x(t_k^-)$ and $x(t_k^+)$ represent the left and right limits of $x(t)$ at $t = t_k$ respectively. Under some hypotheses, we also consider the ϵ-approximate solution for the above fuzzy differential equation.

The paper is organized as follows. In Section 2, we collect the fundamental notions and facts which will be used in the rest of the article. In Section 3, we prove the existence and uniqueness theorem of a solution to the fuzzy impulsive differential equation (1.1).

2. Preliminaries

In this section, our aim is to give a background of the fuzzy set space, and an overview of properties used by us, of integration and differentiation of fuzzy set-valued mappings.

Let A, B be nonempty compact subsets of $\mathbb{R}^d$. The Hausdorff metric is defined as follows

$$d_H(A,B) = \max\{d_H^*(A,B), d_H^*(B,A)\},$$

where

$$d_H^*(A,B) = \sup_{x\in A}\inf_{y\in B} \|x-y\|$$

and $\|.\|$ denotes usual Euclidean norm in $\mathbb{R}^d$.

We have $d_H^*(A,B) = 0$ if and only if $A \subset B$ and $d_H^*(A,B) \leq d^*(A,C) + d_H^*(C,B)$ for nonempty compact subsets A, B, C of $\mathbb{R}^d$.

Let $\mathcal{K}(\mathbb{R}^d)$ denote a family of all nonempty compact convex subsets of $\mathbb{R}^d$ and define addition and scalar multiplication in $\mathcal{K}(\mathbb{R}^d)$ as usual, i.e, for $A, B \in \mathcal{K}(\mathbb{R}^d)$ and $\lambda \in \mathbb{R}$.

$$A + B = \{a + b | a \in A, b \in B\}, \qquad \lambda A = \{\lambda a | a \in A\}.$$

Denote

$$E^d = \{u : \mathbb{R}^d \to [0,1] \mid \quad u \text{ satisfies (i)-(iv) below}\},$$

(i) u is normal, i.e., there exists an $x_0 \in \mathbb{R}^d$ such that $u(x_0) = 1$,

(ii) u is fuzzy convex, that is, $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}$ for any $x, y \in \mathbb{R}^d$ and $0 \leq \lambda \leq 1$,

(iii) u is upper semicontinuous,

(iv) $[u]^0 = \mathrm{cl}\{x \in \mathbb{R}^d : u(x) > 0\}$ is compact, where cl denotes the closure in $(\mathbb{R}^d, \|.\|)$.

For $\alpha \in (0,1]$, denote $[u]^\alpha = \{x \in \mathbb{R}^d | u(x) \geq \alpha\}$. We will call this set an α-cut (α-level set) of u. For $u \in E^d$ one has that $[u]^\alpha \in \mathcal{K}(\mathbb{R}^d)$ for every $\alpha \in (0,1]$.

If $g : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is a function then according to Zadeh's extension principle we can extend g to $E^d \times E^d \to E^d$ by the formula

$$g(u,v)(z) = \sup_{z=g(x,y)} min\{u(x), v(y)\}.$$

It is well known that if g is continuous then $[g(u,v)]^\alpha = g([u]^\alpha, [v]^\alpha)$ for all $u, v \in E^d$, $\alpha \in [0,1]$. Especially, for addition and a scalar multiplication in fuzzy number space E^d, we have:

$$[u+v]^\alpha = [u]^\alpha + [v]^\alpha, \qquad [\lambda u]^\alpha = \lambda [u]^\alpha,$$

where $u, v \in E^d$, $\lambda \in \mathbb{R}$ and $\alpha \in [0,1]$.

Define $D : E^d \times E^d \to [0, \infty)$ by the expression

$$D(u,v) = \sup_{0 \le \alpha \le 1} d_H([u]^\alpha, [v]^\alpha),$$

where d_H is the Hausdorff metric in $\mathcal{K}(\mathbb{R}^d)$. It is easy to see that D is a metric in E^d. In fact (E^d, D) is a complete metric space, and for every $u, v, w, z \in E^d$, $\lambda \in \mathbb{R}$, one has $D(u+w, v+w) = D(u,v), D(u+v, w+z) \le D(u,w) + D(v,z)$, $D(\lambda u, \lambda v) = |\lambda| D(u,v)$ (see, e.g., [18]).

We define $\widehat{\theta} \in E^d$ as $\widehat{\theta} = \chi_{\{0\}}$, where for $x \in \mathbb{R}^d$ we have

$$\chi_{\{x\}}(y) = 1 \quad \text{if } y = x \text{ and}$$
$$\chi_{\{x\}}(y) = 0 \quad \text{if } y \neq x.$$

Let $[a,b] \subset \mathbb{R}$ be a compact interval, $-\infty < a, b < +\infty$. A fuzzy valued mapping $F : [a,b] \to E^d$ is strongly measurable if for all $\alpha \in [0,1]$ the set-valued mapping $[F(.)]^\alpha : [a,b] \to \mathcal{K}(\mathbb{R}^d)$ is measurable, i.e., the set $\{t \in [a,b] \mid [F(t)]^\alpha \cap C \neq \emptyset\}$ for each closed set $C \subset \mathbb{R}^d$ is Lebesgue measurable. A fuzzy mapping $F : [a,b] \to E^d$ is called integrably bounded if there exists an integrable function $h : [a,b] \to \mathbb{R}$ such that $\|x\| \le h(t)$ for all $x \in [F(t)]^0$.

Definition 2.1. (Puri and Ralescu [18]). Let $F : [a,b] \to E^d$. The integral of F over $[a,b]$, denoted by $\displaystyle\int_a^b F(t)dt$, is defined levelwise by the expression

$$\left[\int_a^b F(t)dt\right]^\alpha = \int_a^b [F(t)]^\alpha dt$$
$$= \left\{\int_a^b f(t)dt \mid f : [a,b] \to \mathbb{R}^d \text{ is measurable selection for } [F(.)]^\alpha\right\},$$

for all $\alpha \in (0,1]$.

By virtue of Remark 4.1 in [7], we have that

$$\left[\int_a^b F(t)dt\right]^0 = \int_a^b [F(t)]^0 dt.$$

We recall (see [7]) some properties of integrability for fuzzy mappings.

1. Let $F, G : [a,b] \to E^d$ be integrable and $\lambda \in \mathbb{R}$. Then

   (i) $\displaystyle\int_a^b \big(F(t) + G(t)\big)dt = \int_a^b F(t)dt + \int_a^b G(t)dt,$

   (ii) $\displaystyle\int_a^b \lambda F(t)dt = \lambda \int_a^b F(t)dt,$

   (iii) $D(F, G)$ is integrable,

(iv) $D\Big(\int_a^b F(t)dt, \int_a^b G(t)dt\Big) \leq \int_a^b D\big(F(t), G(t)\big)dt.$

2. If $F : [a, b] \to E^d$ is continuous then it is integrable.

3. If $F : [a, b] \to E^d$ is integrable and $c \in [a, b]$, then
$\int_a^b F(t)dt = \int_a^c F(t)dt + \int_c^b F(t)dt.$

Let $u, v \in E^d$. If there exists $w \in E^d$ such that $u = v + w$, then we call w the H-difference of u and v and we denote it by $u \ominus v$. Note that $u \ominus v \neq u + (-1)v$.

Definition 2.2. (Puri and Ralescu [19]) A mapping $F : [a, b] \to E^d$ is differentiable at $t_0 \in [a, b]$ if there exists $F^{'}(t_0) \in E^d$ such that the limits

$$\lim_{h \to 0^+} \frac{1}{h}(F(t_0 + h) \ominus F(t_0)),$$
$$\lim_{h \to 0^+} \frac{1}{h}(F(t_0) \ominus F(t_0 - h))$$

exist and equal to $F^{'}(t_0)$. The limits are taken in the metric space (E^d, D), and at the boundary points we consider only the one-sided derivatives.

Definition 2.3. The integral of a fuzzy mapping $F : [a, b] \to E^d$ is defined levelwise by

$$\left[\int_a^b F(t)dt\right]^\alpha = \int_a^b F_\alpha(t)dt,$$

i.e., the set of all $\int_a^b f(t)dt$ such that $f : [a, b] \to \mathbb{R}^d$ is a measurable selection for F_α for all $\alpha \in [0, 1]$.

Definition 2.4. A strongly measurable and integrably bounded mapping $F : [a, b] \to E^d$ is said to be integrable over $[a, b]$ if $\displaystyle\int_a^b F(t)dt \in E^d$.

Note that if $F : [a, b] \to E^d$ is strongly measurable and integrably bounded, then F is integrable. Further if $F : [a, b] \to E^d$ is continuous, then it is integrable.

Definition 2.5. A mapping $F : [a, b] \to E^d$ is called differentiable at $t_0 \in [a, b]$ if for any $\alpha \in [0, 1]$, the set-valued mapping $F_\alpha(t) = [F(t)]^\alpha$ is Hukuhara differentiable at point t_0 with $DF_\alpha(t_0)$ and the family $\{DF_\alpha(t_0) : \alpha \in [0, 1]\}$ define a fuzzy number $F(t_0) \in E^d$. If $F : [a, b] \to E^d$ is differentiable at $t_0 \in [a, b]$, then we say that $F'(t_0)$ is the fuzzy derivative of $F(t)$ at the point t_0.

Theorem 2.1. *Let $F : [a, b] \to E^d$ be differentiable. Denote $F_\alpha(t) = [f_\alpha(t), g_\alpha(t)]$. Then f_α and g_α are differentiable and*

$$[F'(t)]^\alpha = [f'_\alpha(t), g'_\alpha(t)].$$

Theorem 2.2. *Let $F : [a, b] \to E^d$ be differentiable and assume that the derivative F' is integrable over $[a, b]$. Then, for each $s \in [a, b]$, we have*

$$F(s) = F(a) + \int_a^s F'(t)dt.$$

Definition 2.6. A mapping $f : [a, b] \times E^d \to E^d$ is called levelwise continuous at a point $(t_0, x_0) \in [a, b] \times E^d$ provided for any fixed $\alpha \in [0, 1]$ and arbitrary $\epsilon > 0$, there exists $\delta(\epsilon, \alpha) > 0$ such that

$$d([f(t, x)]^\alpha, [f(t_0, x_0)]^\alpha) < \epsilon$$

whenever $|t - t_0| < \delta(\epsilon, \alpha)$ and $d([x]^\alpha, [x_0]^\alpha) < \delta(\epsilon, \alpha)$ for all $t \in [a, b], x \in E^d$.

Definition 2.7. A mapping $f : [a, b] \times E^d \to E^d$ is called levelwise continuous at a point $(t_0, x_0) \in [a, b] \times E^d$ provided for any fixed $\alpha \in [0, 1]$ and arbitrary $\epsilon > 0$, there exists $\delta(\epsilon, \alpha) > 0$ such that

$$d([f(t, x)]^\alpha, [f(t_0, x_0)]^\alpha) < \epsilon$$

whenever $|t - t_0| < \delta(\epsilon, \alpha)$ and $d([x]^\alpha, [x_0]^\alpha) < \delta(\epsilon, \alpha)$ for all $t \in [a, b], x \in E^d$.

3. Existence and uniqueness results

Assume that $f : I \times E^d \to E^d$ is levelwise continuous, where the interval $I = \{t : |t - t_0| \leq \delta \leq a\}$.

Consider the fuzzy differential equation (1.1). We denote $J_0 = I \times B(x_0, b)$, where $a > 0$, $b > 0$, $x_0 \in E^d$,

$$B(x_0, b) = \{x \in E^d \mid D(x, x_0) \leq b\}. \tag{3.1}$$

Definition 3.1. A mapping $x : I \to E^d$ is a solution to the problem (1.1) if it is levelwise continuous and satisfies the integral equation

$$x(t) = x_0 + \int_{t_0}^t f(s, x(s))ds + \sum_{0<t<t_k} I_k(x(t_k)),\ k = 1, 2, ..., m,\ \forall t \in I. \tag{3.2}$$

According to the method of successive approximation, let us consider the sequence $\{x_n(t)\}$ such that

$$x_n(t) = x_0 + \int_{t_0}^t f(s, x_{n-1}(s))ds + \sum_{0<t<t_k} I_k(x(t_k)), n = 1, 2, ..., \tag{3.3}$$

where $x_0(t) = x_0, t \in I$.

Theorem 3.1. *Assume that*

(i) *a mapping $f : J_0 \to E^d$ is levelwise continuous,*

(ii) *for any pair $(t, x), (t, y) \in J_0$ we have*

$$d([f(t,x)]^\alpha, [f(t,y)]^\alpha) \leq Ld([x]^\alpha, [y]^\alpha), \tag{3.4}$$

where $L > 0$ is a given constant and for any $\alpha \in [0, 1]$.

(iii) *There exists a constant κ and χ such that*

(a) $d\Big([I_k(x(t_k))]^\alpha, [I_k(y(t_k))]\Big)^\alpha \leq \kappa,$

(b) $d\big([I_k(x(t_k))]^\alpha, 0\big) \leq \chi.$

Then there exists a unique solution $x = x(t)$ of (1.1) defined on the interval

$$|t - t_0| \leq \delta = min\left\{a, \frac{b}{M}\right\}, \tag{3.5}$$

where $M = D(f(t,x), \hat{0}), \hat{0} \in E^d$ such that $\hat{0}(t) = 1$ for $t = 0$ and 0 otherwise and for any $(t, x) \in J_0$.

Moreover, there exists a fuzzy set valued mapping $x : I \to E^d$ such that $D(x_n(t), x(t)) \to 0$ on $|t - t_0| \leq \delta$ as $n \to \infty$.

Proof. Let $t \in I$, from (3.3), it follows that, for $n = 1$,

$$x_1(t) = x_0 + \int_{t_0}^{t} f(s, x_0)ds + \sum_{0<t<t_k} I_k(x(t_k)) \tag{3.6}$$

which proves that $x(t)$ is levelwise continuous on $|t - t_0| \leq a$ and, hence on $|t - t_0| \leq \delta$.

Moreover, for any $\alpha \in [0, 1]$ we have

$$\begin{aligned} d([x_1(t)]^\alpha, [x_0]^\alpha) &= d\left(\left[\int_{t_0}^{t} f(s, x_0)ds\right]^\alpha, 0\right) + d\big([I_k(x(t_k))]^\alpha, 0\big) \\ &\leq \int_{t_0}^{t} d([f(s, x_0)]^\alpha, 0)ds + d\left([I_k(x(t_k))]^\alpha, 0\right) \end{aligned} \tag{3.7}$$

and by the definition of D, we get

$$D(x_1(t), x_0) \leq M|t - t_0| + \chi \leq M\delta + \chi = b + \chi \tag{3.8}$$

Now, assume that $x_{n-1}(t)$ is levelwise continuous on $|t - t_0| \leq \delta$, and that

$$D(x_{n-1}(t), x_0) \leq M|t - t_0| + \chi \leq M\delta + \chi = b + \chi \tag{3.9}$$

From (3.3), we deduce that $x_n(t)$ is levelwise continuous on $|t - t_0| \leq \delta$ and that

$$D(x_n(t), x_0) \leq M|t - t_0| + \chi \leq M\delta + \chi = b + \chi \tag{3.10}$$

Consequently, we conclude that $\{x_n(t)\}$ consists of levelwise continuous mappings on $|t-t_0| \le \delta$, and

$$(t, x_n(t)) \in J_0, \quad |t-t_0| \le \delta,\ n = 1, 2, 3... \tag{3.11}$$

Let us prove that there exists a fuzzy set valued mapping $x : I \to E^d$ such that $D(x_n(t), x(t)) \to 0$ uniformly on $|t-t_0| \le \delta$ as $n \to \infty$.

For n=2, from (3.3),

$$x_2(t) = x_0 + \int_{t_0}^{t} f(s, x_1(s))ds + \sum_{0<t<t_k} I_k(x(t_k)). \tag{3.12}$$

From (3.6) and (3.12) , we have

$$\begin{aligned} d([x_2(t)]^\alpha, [x_1(t)]^\alpha) &= d\left(\left[\int_{t_0}^{t} f(s, x_1(s))ds\right]^\alpha, \left[\int_{t_0}^{t} f(s, x_0(s))ds\right]^\alpha\right) \\ &\quad + d([I_k(x_2(t_k))]^\alpha, [I_k(x_1(t_k))]^\alpha) \\ &\le \int_{t_0}^{t} d([f(s, x_1(s))]^\alpha, [f(s, x_0)]^\alpha)ds \\ &\quad + d([I_k(x_2(t_k))]^\alpha, [I_k(x_1(t_k))]^\alpha) \end{aligned} \tag{3.13}$$

for any $\alpha \in [0,1]$. According to condition (3.4), we obtain

$$\begin{aligned} d([x_2(t)]^\alpha, [x_1(t)]^\alpha) &\le \int_{t_0}^{t} Ld([x_1(s)]^\alpha, [x_0]^\alpha)ds \\ &\quad + d([I_k(x_2(t_k))]^\alpha, [I_k(x_1(t_k))]^\alpha) \end{aligned} \tag{3.14}$$

and by the definition of D, we obtain

$$D(x_2(t), x_1(t)) \le L \int_{t_0}^{t} D(x_1(s), x_0(s))ds + \kappa. \tag{3.15}$$

Now, we can apply the first inequality (3.8) in the right hand side of (3.15) to get

$$D(x_2(t), x_1(t)) \le ML\frac{|t-t_0|^2}{2!} + \kappa \le ML\frac{\delta^2}{2!} + \kappa. \tag{3.16}$$

Starting from (3.8) and (3.16), assume that

$$D(x_n(t), x_{n-1}(t)) \le ML^{n-1}\frac{|t-t_0|^n}{n!} + \kappa \le ML^{n-1}\frac{\delta^n}{n!} + \kappa, \tag{3.17}$$

and let us prove that such an inequality holds for $D(x_{n+1}(t), x_n(t))$.

Indeed, from (3.3) and condition (3.4), it follows that

$$
\begin{aligned}
d([x_{n+1}(t)]^\alpha,[x_n(t)]^\alpha) &= d\left(\left[\int_{t_0}^t f(s,x_n(s))ds\right]^\alpha,\left[\int_{t_0}^t f(s,x_{n-1}(s))ds\right]^\alpha\right)\\
&\quad +d([I_k(x(t_k))]^\alpha,[I_k(y(t_k))]^\alpha)\\
&\le \int_{t_0}^t d([f(s,x_n(s))]^\alpha,[f(s,x_{n-1}(s))]^\alpha)ds\\
&\quad + d([I_k(x_2(t_k))]^\alpha,[I_k(x_1(t_k))]^\alpha)\\
&\le \int_{t_0}^t Ld([x_n(s)]^\alpha,[x_{n-1}(s)]^\alpha)ds\\
&\quad + d([I_k(x_2(t_k))]^\alpha,[I_k(x_1(t_k))]^\alpha)
\end{aligned}
\tag{3.18}
$$

for any $\alpha \in [0,1]$ and from the definition of D, we have

$$
D(x_{n+1}(t),x_n(t)) \le L\int_{t_0}^t D([x_n(s)]^\alpha,[x_{n-1}(s)]^\alpha)ds+\kappa. \tag{3.19}
$$

According to (3.17), we get

$$
\begin{aligned}
D(x_{n+1}(t),x_n(t)) &\le ML^n\int_{t_0}^t \frac{|s-t_0|^n}{n!}ds+\kappa\\
&= ML^n\frac{|t-t_0|^{n+1}}{(n+1)!}+\kappa \le ML^n\frac{\delta^{n+1}}{(n+1)!}+\kappa.
\end{aligned}
\tag{3.20}
$$

Consequently, inequality (3.17) holds for $n = 1, 2....$ We can also write

$$
D(x_n(t),x_{n-1}(t)) \le \frac{M}{L}.\frac{(L\delta)^n}{n!}+\kappa \tag{3.21}
$$

for $n = 1, 2...$, and $|t-t_0| \le \delta$.

Let us mention now that

$$
x_n(t) = x_0 + [x_1(t)-x_0] + ... + [x_n(t)-x_{n-1}(t)], \tag{3.22}
$$

which implies that the sequence $\{x_n(t)\}$ and the series

$$
x_0 + \sum_{n-1}^{\infty}[x_n(t)-x_{n-1}(t)] \tag{3.23}
$$

have the same convergence properties.

From (3.21), according to the convergence criterion of Weierstrass, it follows that the series having the general term $x_n(t)-x_{n-1}(t)$, so $D(x_n(t),x_{n-1}(t)) \to 0$ uniformly on $|t-t_0| \le \delta$ as $n \to \infty$.

Hence, there exists a fuzzy set -valued mapping $x : I \to E^d$ such that $D(x_n(t),x(t)) \to 0$ uniformly on $|t-t_0| \le \delta$ as $n \to \infty$.

From (3.4), we get

$$d\Big([f(t,x_n(t))]^\alpha,[f(t,x(t))]^\alpha\Big) \le Ld\Big([x_n(t)]^\alpha,[x(t)]^\alpha\Big) \tag{3.24}$$

for any $\alpha \in [0,1]$. By the definition of D,

$$D\Big(f(t,x_n(t)),f(t,x(t))\Big) \le LD(x_n(t)),x(t)) \to 0 \tag{3.25}$$

uniformly on $|t-t_0| \le \delta$ as $n \to \infty$.

Taking (3.25) in to account, from (3.3), we obtain, for $n \to \infty$,

$$x(t) = x_0 + \int_{t_0}^{t} f(s,x(s))ds + \sum_{0<t<t_k} I_k(x(t_k)). \tag{3.26}$$

Consequently, there is at least one levelwise continuous solution of (1.1).

We want to prove now that this solution is unique, that is, from

$$y(t) = x_0 + \int_{t_0}^{t} f(s,y(s))ds + \sum_{0<t<t_k} I_k(y(t_k)) \tag{3.27}$$

on $|t-t_0| \le \delta$, it follows that $D\Big(x(t),y(t)\Big) \equiv 0$. Indeed, from (3.3) and (3.27), we obtain

$$\begin{aligned} d([y(t)]^\alpha,[x_n(t)]^\alpha) &= d\left(\left[\int_{t_0}^{t} f(s,y(s))ds\right]^\alpha, \left[\int_{t_0}^{t} f(s,x_{n-1}(s))ds\right]^\alpha\right) \\ &\quad + d([I_k(y(t_k))]^\alpha,[I_k(x_n(t_k))]^\alpha) \\ &\le \int_{t_0}^{t} d([f(s,y(s))ds]^\alpha,[f(s,x_{n-1}(s))ds]^\alpha)ds \\ &\quad + d([I_k(y(t_k))]^\alpha,[I_k(x_n(t_k))]^\alpha) \\ &\le \int_{t_0}^{t} Ld([y(s)]^\alpha,[x_{n-1}(s)]^\alpha)ds \\ &\quad + d([I_k(y(t_k))]^\alpha,[I_k(x_n(t_k))]^\alpha) \end{aligned} \tag{3.28}$$

for any $\alpha \in [0,1], n = 1,2....$

By the definition of D, we obtain

$$D\Big(y(t),x_n(t)\Big) \le L\int_{t_0}^{t} D\Big(y(s),x_{n-1}(s)\Big)ds + \kappa, \quad n = 1,2..., \tag{3.29}$$

But $D(y(t),x_0) \le b$ on $|t-t_0| \le \delta$, $y(t)$ being a solution of (3.27). It follows from (3.29) that

$$D\Big(y(t),x_1(t)\Big) \le bL|t-t_0| + \kappa \tag{3.30}$$

on $|t - t_0| \leq \delta$. Now, assume that

$$D(y(t), x_n(t)) \leq bL^n \frac{|t - t_0|^n}{n!} + \kappa \tag{3.31}$$

on the interval $|t - t_0| \leq \delta$. From

$$D(y(t), x_{n+1}(t)) \leq L \int_{t_0}^{t} D(y(s), x_n(s)) ds + \kappa \tag{3.32}$$

and (3.31), one obtains

$$D\big(y(t), x_{n+1}(t)\big) \leq bL^{n+1} \frac{|t - t_0|^{n+1}}{(n+1)!} + \kappa \tag{3.33}$$

Consequently, (3.31) holds for any n, which leads to the conclusion

$$D(y(t), x_n(t)) = D(x(t), x_n(t)) \to 0 \tag{3.34}$$

on the interval $|t - t_0| \leq \delta$ as $n \to \infty$. This proves the uniqueness of the solution for (1.1). ∎

Definition 3.2. A mapping $x : L \to E^d$ is an ϵ-approximate solution of (1.1) if the following properties hold

(a) $x(t)$ is levelwise continuous on $|t - t_0| \leq \delta$,

(b) the derivative $x'(t)$ exists and it is levelwise continuous,

(c) for all t for which $x'(t)$ is defined,we have

$$D\Big(x'(t), f(t, x(t))\Big) < \epsilon. \tag{3.35}$$

Theorem 3.2. *A mapping $f : J_0 \to E^d$ is levelwise continuous, and let $\epsilon > 0$ be arbitrary.Then there exists at least one ϵ-approximate solution of* (1.1)*, defined on $|t - t_0| \leq \delta = min\{a, b/M\}$, where $M = D(f(t, x), \hat{0}), \hat{0} \in E^d$ and for any $(t, x) \in J_0$.*

Proof. In as much as a mapping $f : J_0 \to E^d$ is a levelwise continuous on a compact set J_0, it follows that $f(t, x)$ is uniformly levelwise continuous.

Consequently, for any $\alpha \in [0, 1]$, we can find $\delta > 0$ such that

$$d([f(t, x)]^\alpha, [f(s, y)]^\alpha) < \epsilon.$$

Now, we construct the approximate solution for $t \in [t_0, t_0+\delta]$, the construction being completely similar for $t \in [t_0 - \delta, t_0]$.

Let us consider a division

$$t_0 < t_1 < ... < t_n = t_0 + \delta \tag{3.36}$$

of $[t_0, t_0 + \delta]$ such that

$$\max_k (t_k - t_{k-1}) < \lambda = min\left\{\delta, \frac{\delta}{M}\right\}. \tag{3.37}$$

We define a mapping $x : I \to E^d$ as follows

$$x(t_0) = x_0, \tag{3.38}$$

$$x(t) = x(t_k) + f(t_k, x(t_k))(t - t_k) \tag{3.39}$$

on $t_k < t < t_{k+1}, k = 0, 1, ...n - 1$. It is obvious that a mapping $x : I \to E^d$ satisfies the first two properties from the definition of an ϵ-approximate solution.

Now, we want to prove that the last property is also fulfilled.

Indeed, $x'(t) = f(t_k, x(t_k))$ on (t_k, t_{k+1}) and for any $\alpha \in [0, 1]$,

$$d\Big([x'(t)]^\alpha, [f(t, x(t))]^\alpha\Big) = d\Big(\big[f\big(t_k, x(t_k)\big)\big]^\alpha, \big[f(t, x(t))\big]^\alpha\Big) < \epsilon \tag{3.40}$$

since $|t - t_k| < \lambda \le \delta$,

$$d\Big([x(t)]^\alpha, [x(t_k)]^\alpha\Big) \le d\Big([f\big(t_k, x(t_k)\big)]^\alpha, 0\Big)|t - t_k| < M\lambda \le \delta. \tag{3.41}$$

Thus, by the definition of D, we have

$$D\Big(x'(t), f(t, x(t))\Big) < \epsilon \tag{3.42}$$

on $|t - t_0| < \delta$ and $(t, x) \in J_0$. Since I_k is a bounded function, we know that the theorem (3.2) holds. ■

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ON L–FUZZY TOPOLOGICAL TM–SUBSYSTEM

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Abstract. Recently, in 2010, Tamilarasi and Megalai introduced a new class of algebras known as TM-algebras. In this paper, we discuss the notion of an L-fuzzy topological TM-subsystem.

Keywords: BCK/BCI algebra, TM-algebra, fuzzy set, L-fuzzy set, L-fuzzy TM-algebra, fuzzy topology.

AMS Classification: 54A40, 03E72, 06F35.

Introduction

To evaluate the modern concept of uncertainty in real physical world, L.A. Zadeh [8] introduced the notion of fuzzy sets, in which the boundaries are not crisp or sharp but flexible. In [4], Goguen generalized the notion of fuzzy sets into L-fuzzy sets where L- can be a complete lattice.

Recently, in 2010, Tamilarasi and Megalai introduced a new class of algebras, called TM-algebras [6]. In their paper they investigated the relationship between TM-algebras and other algebras. They claimed that the TM-algebra is a generalization of BCH/BCI/BCK and Q algebras. In [1], the authors, while studying L-fuzzy structures on TM-algebras, brought out the fact that the TM-algebra is not a generalization of BCH/BCI/BCK algebras by giving counter examples.

The notion of a fuzzy set provides a natural framework for generalizing many of the concepts of general topology. The theory of fuzzy topological spaces is developed by Chang [3], Wong [7], Lowen [5] and others. In our paper [2], we have studied the notion of Fuzzy Topological subsystem on a TM-algebra. In this paper, we introduce the notion of an L-fuzzy topological TM-subsystem and investigate some simple but elegant results.

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2. Preliminaries

In this section we recall some basic definitions that are required in the sequel.

Definition 2.1 Let X be a non-empty set. A mapping $\mu : X \to L$ is called an L-fuzzy set of X,where L is a complete lattice, with sup 1 and inf 0.

Definition 2.2 Let A and B be any two fuzzy sets in a non-empty set X.

1. The union of A and B, denoted by, $A \cup B$ is defined to be the L-fuzzy set $(A \cup B)(x) = \mu_A(x) \vee \mu_B(x)\ for\ all\ x \in X$.
2. The intersection of A and B, denoted by, $A \cap B$ is defined to be the L-fuzzy set $(A \cap B)(x) = \mu_A(x) \wedge \mu_B(x)\ for\ all\ x \in X$.
3. $A \subset B \ \Rightarrow \ A(x) \leq B(x)\ for\ all\ x \in X$.
4. The complement of A is defined to be $A'(x) = 1 - A(x)\ for\ all\ x \in X$.

Definition 2.3 A fuzzy topology is a family T of fuzzy sets in X which satisfies the following conditions

1. $\phi, X \in T$
2. If $A, B \in T$ then $A \cap B \in T$
3. If $A_i \in T$ for each $i \in I$ then $\cup_I A_i \in T$ where I is an indexing set.

Remark 2.1 If X is a set with a fuzzy topology T then (X, T) is called a fuzzy topological space and any element in T is called a T-open fuzzy set in X.

Definition 2.4 Let f be a function from X to Y. Let σ be a fuzzy set in Y. The inverse image of σ under f is defined as $\sigma_{f^{-1}}(x) = \sigma(f(x))\ \forall x\ \in X$. Let μ be a fuzzy set in X. The image of μ under f is defined as

$$\mu_f(y) = \begin{cases} \sup\limits_{z \in f^{-1}(y)} \mu(z), & f^{-1}(y) \text{ is not empty}, \quad \forall y \in Y. \\ 0 & \text{otherwise,} \end{cases}$$

Definition 2.5 A TM-Algebra $(X, *, 0)$ is a non-empty set X with a constant 0 and a binary operation $*$ satisfying the following axioms:

1. $X * 0 = X$
2. $(X * Y) * (X * Z) = Z * Y\ for\ all\ x, y, z \in X$.

Definition 2.6 L-Fuzzy TM-Subalgebra. Let L be a complete lattice with sup 1 and inf 0. An L-fuzzy subset μ of a TM-Algebra $(X, *, 0)$ is called an L-fuzzy TM-Subalgebra of X if, for all $x, y\ \in X$, $\mu(x * y) \geq \mu_(x) \wedge \mu_(y)$

Definition 2.7 Let $(X, *)$ be a TM-algebra. X is said to be an L-fuzzy topological TM-system if there is a family (X, L_τ) of L-fuzzy subsets in X which satisfies the following conditions

1. $\phi, X \in L_\tau$
2. If $A, B \in L_\tau$ then $A \cap B \in L_\tau$
3. If $A_i \in L_\tau$ for each $i \in I$ then $\cup_I A_i \in L_\tau$ where I is an indexing set.

Remark 2.2 If X is a set with an L-fuzzy topology L_τ then (X, L_τ) is called an L-fuzzy topological TM-system and any element in L_τ is called an L_τ-open fuzzy set in X.

3. On L-Fuzzy Topological TM-Subsystem

Definition 3.8 Let $(X, *)$ be a TM-Algebra. Let (X, L_τ) be an L-Fuzzy Topological TM-System. Let A be an L-fuzzy set in X. Then the induced L-fuzzy topological TM-system is the intersection of the L-fuzzy set A with L_τ-open fuzzy sets of X. The induced L-fuzzy topological TM-system is denoted by L_{τ_A}. (A, L_{τ_A}) is called an L-fuzzy topological TM-subsystem.

Example 3.1 Consider the set $X = \{0, 1, 2, 3, 4, 5\}$ with the following cayley table

$*$	0	1	2	3	4	5
0	0	3	4	1	2	5
1	1	0	2	3	5	4
2	2	4	0	5	1	3
3	3	1	5	0	4	2
4	4	5	3	2	0	1
5	5	2	1	4	3	0

Then $(X, *)$ is a TM-algebra.

Let L be a complete lattice with sup 1 and inf o. Let $t_1, t_2, t_3, t_4, t_5, t_6, t_7 \in L$ such that $0 \leq t_1 \leq t_2 \leq t_3 \leq t_4 \leq t_5 \leq t_6 \leq t_7 \leq 1$. Let the fuzzy subsets $\mu_i : X \to L$, $i = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$ be given by

$$\mu_1(x) = \begin{cases} 1 & \text{if } x = 0 \\ t_5 & \text{if } x = 1, 3 \\ t_3 & \text{if } x = 2, 4, 5 \end{cases} \qquad \mu_2(x) = \begin{cases} t_7 & \text{if } x = 0 \\ 0 & \text{if } x = 1, 2, 3, 4 \\ t_3 & \text{if } x = 5 \end{cases}$$

$$\mu_3(x) = \begin{cases} t_3 & \text{if } x = 0, 5 \\ 0 & \text{if } x = 1, 2, 3, 4 \end{cases} \qquad \mu_4(x) = \begin{cases} 1 & \text{if } x = 0 \\ t_4 & \text{if } x = 1, 3 \\ t_3 & \text{if } x = 2, 4, 5 \end{cases}$$

$$\mu_5(x) = \begin{cases} 1 & \text{if } x = 0 \\ t_2 & \text{if } x = 1, 2, 3, 4 \\ t_3 & \text{if } x = 5 \end{cases} \qquad \mu_6(x) = \begin{cases} 1 & \text{if } x = 0 \\ t_6 & \text{if } x = 1, 2 \\ t_4 & \text{if } x = 3 \end{cases}$$

$$\mu_7(x) = \begin{cases} 1 & \text{if } x = 0 \\ t_1 & \text{if } x = 1,2,3,4 \\ t_3 & \text{if } x = 5 \end{cases} \quad \mu_8(x) = \begin{cases} 1 & \text{if } x = 0 \\ t_5 & \text{if } x = 1,3 \\ t_4 & \text{if } x = 2,4,5 \end{cases}$$

$$\mu_9(x) = \begin{cases} t_6 & \text{if } x = 0 \\ 0 & \text{if } x = 1,2,3,4 \\ t_3 & \text{if } x = 5 \end{cases} \quad \mu_{10}(x) = \begin{cases} 1 & \text{if } x = 0 \\ t_7 & \text{if } x = 1,3 \\ t_4 & \text{if } x = 2,4,5 \end{cases}$$

Then the collection $L_\tau = \{\phi, X, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6, \mu_7, \mu_9, \mu_{10}\}$ is an L-Fuzzy Topology on X. Hence (X, L_τ) is an L-fuzzy topological TM-system. Choose $A{=}\mu_8$. Then $L_{\tau_A} = \{\phi, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_7, \mu_8, \mu_9\}$ and $A = (\mu_8, L_{\tau_A})$ is an L-fuzzy topological TM-subsystem.

Definition 3.9 Let $(X, *)$, $(Y, *)$ be two TM-Algebras. Let (X, L_T),(Y, L_U) be two an L-fuzzy topological TM-systems. A mapping f of (X, L_T) into (Y, L_U) is L-fuzzy continuous iff the inverse image of each L_U-open fuzzy set is an L_T-open fuzzy set.

Example 3.2 Consider the TM-algebra $(X, *)$ as in the 3.1. Let L be a complete lattice with sup 1 and inf 0. Let $t_1, t_2, t_3, t_4, t_5, t_6, t_7 \in L$ such that $0 \le t_1 \le t_2 \le t_3 \le t_4 \le t_5 \le t_6 \le t_7 \le 1$. Consider the L-fuzzy subsets $\mu_i : X \to L$, $i = 1,2,3,4,5,6,7,8$ be given by

$$\mu_1(x) = \begin{cases} 1 & \text{if } x = 0 \\ t_5 & \text{if } x = 1,3 \\ t_3 & \text{if } x = 2,4,5 \end{cases} \quad \mu_2(x) = \begin{cases} 1 & \text{if } x = 0 \\ t_4 & \text{if } x = 1,3 \\ t_3 & \text{if } x = 2,4,5 \end{cases}$$

$$\mu_3(x) = \begin{cases} 1 & \text{if } x = 0 \\ t_2 & \text{if } x = 1,3 \\ 0 & \text{if } x = 2,4,5 \end{cases} \quad \mu_4(x) = \begin{cases} 1 & \text{if } x = 0 \\ t_6 & \text{if } x = 1,3 \\ t_4 & \text{if } x = 2,4,5 \end{cases}$$

$$\mu_5(x) = \begin{cases} 1 & \text{if } x = 0 \\ t_5 & \text{if } x = 1,3 \\ t_4 & \text{if } x = 2,4,5 \end{cases} \quad \mu_6(x) = \begin{cases} 1 & \text{if } x = 0 \\ t_1 & \text{if } x = 1,2 \\ 0 & \text{if } x = 3,4 \end{cases}$$

$$\mu_7(x) = \begin{cases} 1 & \text{if } x = 0 \\ t_7 & \text{if } x = 1,3 \\ t_4 & \text{if } x = 2,4,5 \end{cases} \quad \mu_8(x) = \begin{cases} 1 & \text{if } x = 0 \\ t_3 & \text{if } x = 1,3 \\ 0 & \text{if } x = 2,4,5 \end{cases}$$

Then the collection $L_T = \{\phi, X, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6, \mu_7, \mu_8\}$ is an L-Fuzzy Topology on X. Hence (X, L_T) is an L-fuzzy topological TM-system. Consider the set $Y = \{0, a, b, c, d, e\}$ with the following Cayley table

$*$	0	a	b	c	d	e
0	0	e	d	c	b	a
a	a	0	e	d	c	b
b	b	a	0	e	d	c
c	c	b	a	0	e	d
d	d	c	b	a	0	e
e	e	d	c	b	a	0

Then $(Y, *)$ is a TM-algebra.

Let L be a complete lattice with sup 1 and inf 0. Let $t_1, t_2, t_3, t_4, t_5, t_6 \in L$ such that $0 \leq t_1 \leq t_2 \leq t_3 \leq t_4 \leq t_5 \leq t_6 \leq 1$. Consider the L-fuzzy subsets $\sigma_i : Y \to L, i = 1, 2, 3, 4, 5$ be given by

$$\sigma_1(y) = \begin{cases} 1 & \text{if } y = 0 \\ t_3 & \text{if } y = a, b, d, e \\ t_5 & \text{if } y = c \end{cases} \qquad \sigma_2(y) = \begin{cases} 1 & \text{if } y = 0 \\ t_4 & \text{if } y = a, b, d, e \\ t_6 & \text{if } y = c \end{cases}$$

$$\sigma_3(y) = \begin{cases} 1 & \text{if } y = 0 \\ 0 & \text{if } y = a, b, d, e \\ t_2 & \text{if } y = c \end{cases} \qquad \sigma_4(y) = \begin{cases} 1 & \text{if } y = 0 \\ t_3 & \text{if } y = a, b, d, e \\ t_4 & \text{if } y = c \end{cases}$$

$$\sigma_5(y) = \begin{cases} 1 & \text{if } y = 0 \\ 0 & \text{if } y = a, b, d, e \\ t_1 & \text{if } y = c \end{cases}$$

Then the collection $L_U = \{\phi, Y, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5\}$ is an L-Fuzzy Topology on Y. Hence (Y, L_U) is an L-fuzzy topological TM-system. Here an L-fuzzy topological TM-systems (X, L_T) and (Y, L_U) have the same values of t_i's $i = 1, 2, 3, 4, 5, 6$.

Let $f : X \to Y$ be the function given by, $f(0) = 0$, $f(1) = c$, $f(2) = e$, $f(3) = c$, $f(4) = b$, $f(5) = b$, $\sigma_{f^{-1}}(x) = \sigma(f(x))$ for all x in X for any L-fuzzy set σ in Y,

$$(\sigma_1)_{f^{-1}}(x) = \mu_1(x) \quad (\sigma_2)_{f^{-1}}(x) = \mu_4(x) \quad (\sigma_3)_{f^{-1}}(x) = \mu_3(x)$$
$$(\sigma_4)_{f^{-1}}(x) = \mu_2(x), (\sigma_5)_{f^{-1}}(x) = \mu_6(x), \quad x \in X$$

Hence the inverse image of each L_U-open fuzzy set is L_T-open and hence the function f is an L-fuzzy continuous.

Definition 3.10 Let $(X, *)$, $(Y, *)$ be two TM-Algebras. Let (X, L_T),(Y, L_U) be two an L-fuzzy topological TM-systems . A mapping f of (X, L_T) into (Y, L_U) is an L-fuzzy open iff the image of each L_T-open fuzzy set is L_U-open fuzzy set.

Example 3.3 Consider the TM-algebra $(X, *)$ as in the example 3.1, Let L be a complete lattice with sup 1 and inf 0. Let $t_1, t_2, t_3, t_4, t_5 \in L$ such that $0 \leq t_1 \leq t_2 \leq t_3 \leq t_4 \leq t_5 \leq 1$. Consider the L-fuzzy subsets $\mu_i : X \to L, i = 1, 2, 3, 4$ be given by

$$\mu_1(x) = \begin{cases} t_5 & \text{if } x = 0 \\ 0 & \text{if } x = 1, 2, 3, 4 \\ t_3 & \text{if } x = 5 \end{cases} \qquad \mu_2(x) = \begin{cases} 1 & \text{if } x = 0 \\ t_2 & \text{if } x = 1, 2, 3, 4 \\ t_3 & \text{if } x = 5 \end{cases}$$

$$\mu_3(x) = \begin{cases} t_4 & \text{if } x = 0 \\ 0 & \text{if } x = 1, 2, 3, 4 \\ t_3 & \text{if } x = 5 \end{cases} \qquad \mu_4(x) = \begin{cases} 1 & \text{if } x = 0 \\ t_1 & \text{if } x = 1, 2, 3, 4 \\ t_3 & \text{if } x = 5 \end{cases}$$

Then the collection $L_T = \{\phi, X, \mu_1, \mu_2, \mu_3, \mu_4\}$ is an L-Fuzzy Topology on X. Hence (X, L_T) is an L-fuzzy topological TM-system. Consider the TM-algebra

$(Y, *)$ as in Example 3.2. Let L be a complete lattice with sup 1 and inf 0. Let $t_1, t_2, t_3, t_4, t_5, t_6 \in L$ such that $0 \leq t_1 \leq t_2 \leq t_3 \leq t_4 \leq t_5 \leq t_6 \leq 1$.

Consider the L-fuzzy subsets $\sigma_i : Y \to L, i = 1, 2, 3, 4, 5, 6$ be given by

$$\sigma_1(y) = \begin{cases} t_5 & \text{if } y = 0 \\ 0 & \text{if } y = a, b, d, e \\ t_3 & \text{if } y = c \end{cases} \qquad \sigma_2(y) = \begin{cases} 1 & \text{if } y = 0 \\ t_1 & \text{if } y = a, b, d, e \\ t_4 & \text{if } y = c \end{cases}$$

$$\sigma_3(y) = \begin{cases} t_6 & \text{if } y = 0 \\ 0 & \text{if } y = a, b, d, e \\ t_3 & \text{if } y = c \end{cases} \qquad \sigma_4(y) = \begin{cases} 1 & \text{if } y = 0 \\ t_4 & \text{if } y = a, b, d, e \\ t_6 & \text{if } y = c \end{cases}$$

$$\sigma_5(y) = \begin{cases} 1 & \text{if } y = 0 \\ t_2 & \text{if } y = a, b, d, e \\ t_3 & \text{if } y = c \end{cases} \qquad \sigma_6(y) = \begin{cases} t_4 & \text{if } y = 0 \\ 0 & \text{if } y = a, b, d, e \\ t_3 & \text{if } y = c \end{cases}$$

Then the collection $L_U = \{\phi, Y, \sigma_1, \sigma_2, \sigma_3, \sigma_5, \sigma_6\}$ is an L-Fuzzy Topology on Y. Hence (Y, L_U) is an L-fuzzy topological TM-system. Here an L-fuzzy topological TM-systems (X, L_T) and (Y, L_U) have the same values of t_i's $i = 1, 2, 3, 4, 5$.

Let $f : X \to Y$ be given by, $f(0) = 0$, $f(1) = b$, $f(2) = d$, $f(3) = e$, $f(4) = a$, $f(5) = c$. Then $f^{-1}(0) = 0$, $f^{-1}(b) = 1$, $f^{-1}(d) = 2$, $f^{-1}(e) = 3$, $f^{-1}(a) = 4$, $f^{-1}(c) = 5$.

$$f(\mu_1) = \sigma_1 \quad f(\mu_2) = \sigma_5 \quad f(\mu_3) = \sigma_6 \quad f(\mu_4) = \sigma_2$$

Hence the image of each L_T-open fuzzy set is L_U-open and hence the function f is L-Fuzzy Open.

Definition 3.11 Let $(X, *)$, $(Y, *)$ be two TM-Algebras. Let (X, L_T),(Y, L_U) be two an L-fuzzy topological TM-systems. Let $(A, L_{T_A}), (B, L_{U_B})$ be an L-fuzzy topological subsystems on X and Y. f is said to be a mapping of (A, L_{T_A}) into (B, L_{U_B}) if $f(A) \subset B$.

Definition 3.12 Let $(X, *)$, $(Y, *)$ be two TM-Algebras. Let (X, L_T),(Y, L_U) be two an L-fuzzy topological TM-systems. Let $(A, L_{T_A}), (B, L_{U_B})$ be an L-fuzzy topological subsystems on X and Y. A mapping f from (A, L_{T_A}) into (B, L_{U_B}) is relatively L-fuzzy continuous if and only if $f^{-1}(\sigma_B) \wedge A$ is in L_{T_A} where $\sigma_B \in L_{U_B}$.

Example 3.4 Consider the TM-algebra $(X, *)$ as in Example 3.1. Let L be a complete lattice with sup 1 and inf 0. Let $t_1, t_2, t_3, t_4, t_5, t_6, t_7 \in L$ such that $0 \leq t_1 \leq t_2 \leq t_3 \leq t_4 \leq t_5 \leq t_6 \leq t_7 \leq 1$. Consider the L-fuzzy subsets $\mu_i : X \to L$, $i = 1, 2, 3, 4, 5, 6, 7, 8$ be given by

$$\mu_1(x) = \begin{cases} 1 & \text{if } x = 0 \\ t_5 & \text{if } x = 1, 3 \\ t_3 & \text{if } x = 2, 4, 5 \end{cases} \qquad \mu_2(x) = \begin{cases} t_6 & \text{if } x = 0 \\ 0 & \text{if } x = 1, 2, 3, 4 \\ t_3 & \text{if } x = 5 \end{cases}$$

$$\mu_3(x) = \begin{cases} t_3 & \text{if } x = 0, 5 \\ 0 & \text{if } x = 1, 2, 3, 4 \end{cases} \qquad \mu_4(x) = \begin{cases} 1 & \text{if } x = 0 \\ t_2 & \text{if } x = 1, 2, 3, 4 \\ t_3 & \text{if } x = 5 \end{cases}$$

$$\mu_5(x) = \begin{cases} 1 & \text{if } x = 0 \\ t_3 & \text{if } x = 1,2,3,4 \\ t_4 & \text{if } x = 5 \end{cases} \quad \mu_6(x) = \begin{cases} t_5 & \text{if } x = 0 \\ 0 & \text{if } x = 1,2,3,4 \\ t_3 & \text{if } x = 5 \end{cases}$$

$$\mu_7(x) = \begin{cases} 1 & \text{if } x = 0 \\ t_1 & \text{if } x = 1,2,3,4 \\ t_3 & \text{if } x = 5 \end{cases} \quad \mu_8(x) = \begin{cases} 1 & \text{if } x = 0 \\ t_7 & \text{if } x = 1,3 \\ t_4 & \text{if } x = 2,4,5 \end{cases}$$

Then the collection $L_T = \{\phi, X, \mu_1, \mu_2, \mu_3, \mu_4, \mu_6, \mu_7, \mu_8\}$ is an L-Fuzzy Topology on X. Hence (X, L_T) is an L-fuzzy topological TM-system. Choose $A = \mu_5$. Then $L_{T_A} = \{\phi, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6, \mu_7\}$ and $A = (\mu_5, L_{T_A})$ is an L-fuzzy topological TM-subsystem. Consider the TM-algebra $(Y, *)$ as in Example 3.2.

Let L be a complete lattice with sup 1 and inf 0. Let $t_1, t_2, t_3, t_4, t_5, t_6 \in L$ such that $0 \leq t_1 \leq t_2 \leq t_3 \leq t_4 \leq t_5 \leq t_6 \leq 1$. Consider the L-fuzzy subsets $\sigma_i : Y \to L, i = 1,2,3,4,5$ be given by

$$\sigma_1(y) = \begin{cases} t_6 & \text{if } y = 0 \\ 0 & \text{if } y = a,b,d,e \\ t_3 & \text{if } y = c \end{cases} \quad \sigma_2(y) = \begin{cases} 1 & \text{if } y = 0 \\ t_2 & \text{if } y = a,b,d,e \\ t_3 & \text{if } y = c \end{cases}$$

$$\sigma_3(y) = \begin{cases} 1 & \text{if } y = 0 \\ t_1 & \text{if } y = a,b,d,e \\ t_3 & \text{if } y = c \end{cases} \quad \sigma_4(y) = \begin{cases} t_5 & \text{if } y = 0 \\ 0 & \text{if } y = a,b,d,e \\ t_3 & \text{if } y = c \end{cases}$$

$$\sigma_5(y) = \begin{cases} 1 & \text{if } y = 0 \\ t_4 & \text{if } y = a,b,d,e \\ t_6 & \text{if } y = c \end{cases}$$

Then the collection $L_U = \{\phi, Y, \sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ is an L-Fuzzy Topology on Y. Hence (Y, L_U) is an L-fuzzy topological TM-system. Choose $B = \sigma_5$. Then $L_{U_B} = \{\phi, \sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ and $B = (\sigma_5, L_{U_B})$ is an L-fuzzy topological TM-subsystem. Here an L-fuzzy topological TM-subsystem (A, L_{T_A}) and (B, L_{U_B}) have the same values of t_i's $i = 1,2,3,4,5,6$.

Let $f : X \to Y$ be given by $f(0) = 0,\ f(1) = b,\ f(2) = d,\ f(3) = e,$ $f(4) = a,\ f(5) = c$. Then $f^{-1}(0) = 0,\ f^{-1}(b) = 1,\ f^{-1}(d) = 2,\ f^{-1}(e) = 3,$ $f^{-1}(a) = 4,\ f^{-1}(c) = 5$

$$f(A) = \begin{cases} 1 & \text{if } x = 0 \\ t_3 & \text{if } x = a,b,d,e \\ t_4 & \text{if } x = c \end{cases} \subset B$$

$\sigma_{f^{-1}}(x) = \sigma(f(x))$ for all x in X for any L-fuzzy set σ in Y.

$$(\sigma_1)_{f^{-1}}(x) = \begin{cases} t_6 & \text{if } x = 0 \\ 0 & \text{if } x = 1,2,3,4 \\ t_3 & \text{if } x = 5 \end{cases} \quad (\sigma_1)_{f^{-1}}(x) \wedge A = \mu_2 \in (A, L_{T_A})$$

$$(\sigma_2)_{f^{-1}}(x) = \begin{cases} 1 & \text{if } x = 0 \\ t_2 & \text{if } x = 1,2,3,4 \\ t_3 & \text{if } x = 5 \end{cases} \quad (\sigma_2)_{f^{-1}}(x) \wedge A = \mu_4 \in (A, L_{T_A})$$

$$(\sigma_3)_{f^{-1}}(x) = \begin{cases} 1 & \text{if } x = 0 \\ t_1 & \text{if } x = 1,2,3,4 \\ t_3 & \text{if } x = 5 \end{cases} \quad (\sigma_3)_{f^{-1}}(x) \wedge A = \mu_7 \in (A, L_{T_A})$$

Hence the inverse image of each L_{U_B}-open fuzzy set is L_{T_A}-open. Therefore, the function f is relatively L-fuzzy continuous.

Definition 3.13 Let $(X,*)$, $(Y,*)$ be two TM-Algebras. Let (X, L_T),(Y, L_U) be two an L-fuzzy topological TM-systems. Let $(A, L_{T_A}), (B, L_{U_B})$ be an L-fuzzy topological TM-subsystems. A mapping f from (A, L_{T_A}) into (B, L_{U_B}) is relatively L-fuzzy open if and only iff $f(\mu_A) \in L_{U_B}$ where $\mu_A \in L_{T_A}$.

Example 3.5 Consider the TM-algebra $(X,*)$ as in Example 3.1, Let L be a complete lattice with sup 1 and inf 0. Let $t_1, t_2, t_3, t_4, t_5, t_6 \in L$ such that $0 \leq t_1 \leq t_2 \leq t_3 \leq t_4 \leq t_5 \leq t_6 \leq 1$. Consider the L-fuzzy subsets $\mu_i : X \to L$, $i = 1,2,3,4,5$ be given by

$$\mu_1(x) = \begin{cases} t_6 & \text{if } x = 0 \\ 0 & \text{if } x = 1,2,3,4 \\ t_3 & \text{if } x = 5 \end{cases} \qquad \mu_2(x) = \begin{cases} 1 & \text{if } x = 0 \\ t_2 & \text{if } x = 1,2,3,4 \\ t_3 & \text{if } x = 5 \end{cases}$$

$$\mu_3(x) = \begin{cases} 1 & \text{if } x = 0 \\ t_3 & \text{if } x = 1,2,3,4 \\ t_4 & \text{if } x = 5 \end{cases} \qquad \mu_4(x) = \begin{cases} t_5 & \text{if } x = 0 \\ 0 & \text{if } x = 1,2,3,4 \\ t_3 & \text{if } x = 5 \end{cases}$$

$$\mu_5(x) = \begin{cases} 1 & \text{if } x = 0 \\ t_1 & \text{if } x = 1,2,3,4 \\ t_3 & \text{if } x = 5 \end{cases}$$

Then the collection $L_T = \{\phi, X, \mu_1, \mu_2, \mu_4, \mu_5\}$ is an L-Fuzzy Topology on X. Hence (X, L_T) is an L-fuzzy topological TM-system. Choose $A = \mu_3$.

$L_{T_A} = \{\phi, \mu_1, \mu_2, \mu_4, \mu_5\}$ and $A = (\mu_3, L_{T_A})$ is an L-fuzzy topological TM-subsystem. Consider the TM-algebra $(Y,*)$ as in Example 3.2. Let L be a complete lattice with sup 1 and inf 0. Let $t_1, t_2, t_3, t_4, t_5, t_6 \in L$ such that $0 \leq t_1 \leq t_2 \leq t_3 \leq t_4 \leq t_5 \leq t_6 \leq 1$. Consider the L-fuzzy subsets $\sigma_i : Y \to L$, $i = 1,2,3,4,5,6$ be given by

$$\sigma_1(y) = \begin{cases} t_5 & \text{if } y = 0 \\ 0 & \text{if } y = a,b,d,e \\ t_3 & \text{if } y = c \end{cases} \qquad \sigma_2(y) = \begin{cases} 1 & \text{if } y = 0 \\ t_1 & \text{if } y = a,b,d,e \\ t_3 & \text{if } y = c \end{cases}$$

$$\sigma_3(y) = \begin{cases} t_6 & \text{if } y = 0 \\ 0 & \text{if } y = a,b,d,e \\ t_3 & \text{if } y = c \end{cases} \qquad \sigma_4(y) = \begin{cases} 1 & \text{if } y = 0 \\ t_4 & \text{if } y = a,b,d,e \\ t_6 & \text{if } y = c \end{cases}$$

$$\sigma_5(y) = \begin{cases} 1 & \text{if } y = 0 \\ t_2 & \text{if } y = a,b,d,e \\ t_3 & \text{if } y = c \end{cases} \qquad \sigma_6(y) = \begin{cases} t_4 & \text{if } y = 0 \\ 0 & \text{if } y = a,b,d,e \\ t_3 & \text{if } y = c \end{cases}$$

Then the collection $L_U = \{\phi, Y, \sigma_1, \sigma_2, \sigma_3, \sigma_5, \sigma_6\}$ is an L-Fuzzy Topology on Y. Hence (Y, L_U) is an L-fuzzy topological TM-system. Choose $B = \sigma_4$. Then $L_{U_B} =$

$\{\phi, \sigma_1, \sigma_2, \sigma_3, \sigma_5, \sigma_6\}$ and $B = (\sigma_4, L_{U_B})$ is an L-fuzzy topological TM-subsystem. Here an L-fuzzy topological TM-subsystems (A, L_{T_A}) and (B, L_{U_B}) have the same values of t_i's $i = 1, 2, 3, 4, 5, 6$.

Let $f : X \to Y$ be given by $f(0) = 0$, $f(1) = b$, $f(2) = d$, $f(3) = e$, $f(4) = a$, $f(5) = c$. Then, $f^{-1}(0) = 0$, $f^{-1}(b) = 1$, $f^{-1}(d) = 2$, $f^{-1}(e) = 3$, $f^{-1}(a) = 4$, $f^{-1}(c) = 5$

$$f(A) = \begin{cases} 1 & \text{if } x = 0 \\ t_3 & \text{if } x = a, b, d, e \quad \subset B \\ t_4 & \text{if } x = c \end{cases}$$

$$f(\mu_1) = \sigma_3 \quad f(\mu_2) = \sigma_5 \quad f(\mu_4) = \sigma_1 \quad f(\mu_5) = \sigma_2$$

Hence the image of each L_{T_A}-open fuzzy set is L_{U_B}-open and hence the function f is relatively L-Fuzzy Open.

Theorem 3.1 *Let $(X, *)$, $(Y, *)$ be two TM-Algebras. Let (X, L_T),(Y, L_U) be two an L-fuzzy topological TM-systems. Let (A, L_{T_A}), (B, L_{U_B}) be an L-fuzzy topological TM-subsystems.Let f be an L-fuzzy continuous mapping of (X, L_T) in to (Y, L_U) such that $f(A) \subset B$. Then f is a relatively L-fuzzy continuous mapping of (A, L_{T_A}) into (B, L_{U_B}).*

Proof. Let $\sigma' \in L_{U_B}$. Then there exists $\sigma \in L_U$ such that $\sigma' = \sigma \wedge B$. Hence

$$f^{-1}(\sigma') \wedge A = f^{-1}(\sigma \wedge B) \wedge A = (f^{-1}(\sigma) \wedge f^{-1}(B)) \wedge A = f^{-1}(\sigma) \wedge A \because f(A) \subset B.$$

Since $\sigma \in L_U, f$ is an L-fuzzy continuous and $f^{-1}(\sigma) \in L_T$.

Therefore $f^{-1}(\sigma) \wedge A$ is open in L_{T_A}. Therefore, f is relatively L-fuzzy continuous mapping of (A, L_{T_A}) into (B, L_{U_B}).

Theorem 3.2 *Let f be a L-fuzzy continuous mapping of an L-fuzzy topological TM-system (X, L_T) in to (Y, L_U). Let g be a mapping of (Y, L_U) in to an L-fuzzy topological TM-system (Z, L_V). Then the composition mapping $g \circ f$ is an L-fuzzy continuous mapping of (X, L_T) in to (Z, L_V).*

Proof. Since the functions f, g are L-fuzzy continuous, by Definition 3.9, the inverse image of each L_U-open fuzzy set σ is L_T-open.

The inverse image of each open fuzzy set χ of L_V is an L_U-open fuzzy set.

$$(g \circ f)^{-1}(\chi) = (f^{-1} \circ g^{-1})(\chi) = f^{-1}(g^{-1}(\chi)) = f^{-1}(\sigma)$$

Since $f^{-1}(\sigma)$ is L_T-open fuzzy set, $(g \circ f)^{-1}(\chi)$ is L_T-open fuzzy set.

Hence $(g \circ f)$ is an L-fuzzy continuous.

Theorem 3.3 *Let (A, L_{T_A}),(B, L_{U_B}),(C, L_{V_C}) be an L-fuzzy topological TM-subsystems. Let f, g be the relatively L-fuzzy continuous mappings of (A, L_{T_A}) into (B, L_{U_B}) and (B, L_{U_B}) into (C, L_{V_C}). Then, the composition $g \circ f$ is a relatively L-fuzzy continuous mapping of (A, L_{T_A}) into (C, L_{V_C}).*

Proof. Let $\chi' \in L_{V_C}$. Then, there exists $\chi \in L_V$ such that $\chi' = \chi \wedge C$.

Since g is relatively L-fuzzy continuous from (B, L_{U_B}) into (C, L_{V_C}) by Definition 3.12, $g^{-1}(\chi') \wedge B$ is open in L_{U_B}.

Since f is relatively L-fuzzy continuous from (A, L_{T_A}) into (B, L_{U_B}) by Definition 3.12, $f^{-1}((g^{-1}(\chi') \wedge B)) \wedge A$ is open in L_{T_A}

But $(g \circ f)^{-1}(\chi') \wedge A = f^{-1}(g^{-1}(\chi') \wedge B) \wedge A$ implying that $(g \circ f)^{-1}(\chi') \wedge A$ is open in L_{T_A}.

Hence $f(A) \subset B$, $g \circ f$ is relatively L-fuzzy continuous.

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MIDPOINT DERIVATIVE-BASED TRAPEZOID RULE FOR THE RIEMANN-STIELTJES INTEGRAL

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Abstract. In this paper, the midpoint derivative-based trapezoid rule for the Riemann-Stieltjes integral is presented which uses derivative value at the midpoint. This kind of quadrature rule obtains an increase of two orders of precision over the trapezoid rule for the Riemann-Stieltjes integral and the error term is investigated. At last, the rationality of the generalization of midpoint derivative-based trapezoid rule for Riemann-Stieltjes integral is demonstrated.

Keywords: midpoint derivative; trapezoid rules; Riemann-Stieltjes integral; error term.

2000 Mathematics Subject Classification: 65D30; 65D32.

1. Introduction

In mathematics, the Riemann-Stieltjes integral is a kind of generalization of the Riemann integral, named after Bernhard Riemann and Thomas Stieltjes. It is Stieltjes [1] that first give the definition of this integral in 1894. It serves as an instructive and useful precursor of the Lebesgue integral. It is known that the Riemann-Stieltjes integral has wide applications in the field of probability theory [2-3], stochastic process [4] and functional analysis [5], especially in original

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formulation of F. Riesz's theorem [2], [5] and the spectral theorem for self-adjoint operators in a Hilbert space [2], [5].

Definite integration is one of the most important and basic concepts in mathematics. And it has numerous applications in fields such as physics and engineering. In several practical problems, we need to calculate integrals. As is known to all, as for $I = \int_a^b f(x)dx$, once the primitive function $F(x)$ of integrand $f(x)$ is known, the definite integral of $f(x)$ over the interval $[a, b]$ is given by Newton-Leibniz formula, that is,

$$\int_a^b f(x)dx = F(b) - F(a).$$

However, the explicit primitive function $F(x)$ is not available or its primitive function is not easy to obtain, such as $e^{\pm x^2}$, $\sin x^2$, $\frac{\sin x}{x}$, etc. Moreover, the integrand $f(x)$ is only available at certain points x_i, $i = 1, 2, ...n$. How to get high-precision numerical integration formulas becomes one of the challenges in fields of mathematics [6].

The methods of quadrature are usually based on the interpolation polynomials and can be written in the following form:

$$\int_a^b f(x)dx \approx \sum_{i=0}^{n} w_i f(x_i), \tag{1.1}$$

where there are $n+1$ distinct integration points at $x_0, x_1, ..., x_n$ within the interval $[a, b]$ and $n+1$ weights w_i. If the integration points are uniformly distributed over the interval, so $x_i = x_0 + ih$ in which $h = \frac{b-a}{n}$.

These w_i can be derived in several different ways [7]-[9]. One is interpolate $f(x)$ at the $n + 1$ points $x_0, x_1, ..., x_n$, using the Lagrange polynomials and then integrating the foresaid polynomials to obtain (1.1).

The other is based on the precision of a quadrature formula. Select the w_i, $i = 1, 2, ..., n$, so that the error

$$R_n(f) = \int_a^b f(x)dx - \sum_{i=0}^{n} w_i f(x_i), \tag{1.2}$$

is exactly zero for $f(x) = x^j$, $j = 1, 2, ..., n$. Using the method of undetermined coefficients, this approach generates a system of $n+1$ linear equations for weights w_i. Since the monomials $1, x, ..., x^n$ are linearly independent, the linear system of equations has a unique solution.

The trapezoidal rule is the most well known numerical integration rules of this type. Trapezoidal rule for classical Riemann integral is

$$\int_a^b f(x)dx = \frac{b-a}{2}\left(f(a) + f(b)\right) - \frac{(b-a)^3}{12} f''(\xi), \tag{1.3}$$

where $\xi \in (a, b)$.

In spite of the many accurate and efficient methods for numerical integration being available in [7]-[9], recently Mercer [10] has obtained trapezoid rule for Riemann-Stielsjes integral which engender a generalization of Hadamard's integral inequality. Then he develops Midpoint and Simpson's rules for Riemann-Stielsjes integral [11] by using the concept of relative convexity. Burg [12] has proposed derivative-based closed Newton-Cotes numerical quadrature which uses both the function value and the derivative value on uniformly spaced intervals. Zhao and Li have proposed midpoint derivative-based closed Newton-Cotes quadrature [13] and numerical superiority has been shown. Then, the derivative-based trapezoid rule for the Riemann-Stieltjes integral is presented by Zhao and Zhang [14], which uses derivative values at the endpoints. Recently, Simos and his partners have made a contribution to the Newton-Cotes formula for the Riemann integral and its applications [15]-[17], especially the connection between closed Newton-Cotes, trigonometrically-fitted differential methods, symplectic integrators and efficient solution of the Schrodinger equation [17].

Motivation for the research presented here lies in construction of midpoint derivative-based trapezoid rule for the Riemann-Stieltjes integral, which is generalization of the results in [13]. In this paper, the midpoint derivative-based trapezoid rule for the Riemann-Stieltjes integral is presented. These new scheme is investigated in Section 2. In Section 3, the error term is presented. Finally, conclusions are drawn in Section 4.

2. Midpoint derivative-based trapezoid rule for the Riemann-Stieltjes integral

In this section, by adding the derivative at the midpoint, midpoint derivative-based trapezoid rule for the Riemann-Stieltjes integral is presented.

Theorem 2.1 *Suppose that $f'(t)$ and $g(t)$ are continuous on $[a,b]$ and $g(t)$ is increasing there. The midpoint derivative-based Trapezoid rule for the Riemann-Stieltjes integral is*

$$\begin{aligned}\int_a^b f(t)dg \approx T &= \left(\frac{1}{b-a}\int_a^b g(t)dt - g(a)\right) f(a)\\ &+ \left(g(b) - \frac{1}{b-a}\int_a^b g(t)dt\right) f(b)\\ &+ \left(\int_a^b \int_a^t g(x)dxdt - \frac{b-a}{2}\int_a^b g(t)dt\right) f''(c),\end{aligned} \tag{2.1}$$

where $c = \dfrac{(-2b^2+a^2-ab)\int_a^b g(t)dt + 6b\int_a^b\int_a^t g(x)dxdt - 6\int_a^b\int_a^t\int_a^y g(x)dxdydt}{6\int_a^b\int_a^t g(x)dxdt - 3(b-a)\int_a^b g(t)dt}$.

Proof. Looking for the midpoint derivative-based trapezoid rule for the Riemann-Stieltjes integral, we seek numbers a_0, a_1, c_0, c such that

$$\int_a^b f(t)dg \approx a_0 f(a) + a_1 f(b) + c_0 f''(c)$$

is equality for $f(t) = 1,\ t,\ t^2, t^3$. That is,

$$\begin{cases} \displaystyle\int_a^b 1dg = a_0 + a_1; \\ \displaystyle\int_a^b tdg = a_0 a + a_1 b; \\ \displaystyle\int_a^b t^2 dg = a_0 a^2 + a_1 b^2 + 2c_0; \\ \displaystyle\int_a^b t^3 dg = a_0 a^3 + a_1 b^3 + 6c_0 c. \end{cases}$$

Therefore,

$$\begin{cases} a_0 + a_1 = g(b) - g(a); \\ a_0 a + a_1 b = bg(b) - ag(a) - \displaystyle\int_a^b g(t)dt; \\ a_0 a^2 + a_1 b^2 + 2c_0 = \ b^2 g(b) - a^2 g(a) - 2b \displaystyle\int_a^b g(t)dt \\ \qquad\qquad\qquad\qquad +2 \displaystyle\int_a^b \int_a^t g(x)dxdt; \\ a_0 a^3 + a_1 b^3 + 6c_0 c = b^3 g(b) - a^3 g(a) - 3b^2 \displaystyle\int_a^b g(t)dt \\ \qquad\qquad\qquad\qquad +6b \displaystyle\int_a^b \int_a^t g(x)dxdt - 6 \int_a^b \int_a^t \int_a^y g(x)dxdydt. \end{cases} \tag{2.2}$$

Solving simultaneous equations (2.2) for a_0, a_1, c_0, c, we obtain

$$\begin{cases} a_0 = \dfrac{1}{b-a} \displaystyle\int_a^b g(t)dt - g(a); \\ a_1 = g(b) - \dfrac{1}{b-a} \displaystyle\int_a^b g(t)dt; \\ c_0 = \displaystyle\int_a^b \int_a^t g(x)dx\,dt - \dfrac{b-a}{2} \int_a^b g(t)dt; \\ c = \dfrac{(-2b^2 + a^2 - ab) \int_a^b g(t)dt + 6b \int_a^b \int_a^t g(x)dx\,dt - 6 \int_a^b \int_a^t \int_a^y g(x)dx\,dy\,dt}{6 \int_a^b \int_a^t g(x)dx\,dt - 3\,(b-a) \int_a^b g(t)dt}. \end{cases}$$

∎

So we have the midpoint derivative-based trapezoid rule for the Riemann-Stieltjes integral as desired.

Corollary 2.2 *The precision of the midpoint derivative-based trapezoid rule for the Riemann-Stieltjes integral is* 3.

Proof. From the construction of a_0, a_1, c_0, c, we obtain that the derivative-based trapezoidal rule for the Riemann-Stieltjes integral has degree of precision not less than 3.

In Section 3, by applying Theorem 3.1, we can easily see that the quadratue is not equality for $f(t) = t^4$. So the precision of this method is 3. ∎

3. The error term for Riemann-Stieltjes midpoint derivative-based trapezoid rule

In this section, the error term of the derivative-based trapezoid rule for the Riemann-Stieltjes is investigated. The error term can be found in mainly 3 different ways [8, 9].

Here, we use the concept of precision to calculate the error term, where the error term is related to the difference between the quadrature formula for the monomial $\frac{x^{p+1}}{(p+1)!}$, and the exact value $\frac{1}{(p+1)!}\int_a^b x^{p+1}dx$ where p is the precision of the quadrature formula.

Theorem 3.1 *Suppose that $f'(t)$ and $g(t)$ are continuous on $[a,b]$ and $g(t)$ is increasing there. The derivative-based trapezoid rule for the Riemann-Stieltjes integral with the error term is*

$$\int_a^b f(t)dg \approx T = \left(\frac{1}{b-a}\int_a^b g(t)dt - g(a)\right) f(a) + \left(g(b) - \frac{1}{b-a}\int_a^b g(t)dt\right) f(b)$$

$$+ \left(\int_a^b\int_a^t g(x)dx\,dt - \frac{b-a}{2}\int_a^b g(t)dt\right) f''(c)$$

$$+ \left(\frac{a^3+ab^2+a^2b-3b^3+6(b-a)c^2}{24}\int_a^b g(t)dt + \frac{b^2-c^2}{2}\int_a^b\int_a^t g(x)dx\,dt\right. \tag{3.1}$$

$$\left. - b\int_a^b\int_a^t\int_a^y g(x)dx\,dy\,dt + \int_a^b\int_a^t\int_a^z\int_a^y g(x)dx\,dy\,dz\,dt\right) f^{(4)}(\xi)g'(\eta),$$

where $c = \dfrac{(-2b^2+a^2-ab)\int_a^b g(t)dt + 6b\int_a^b\int_a^t g(x)dx\,dt - 6\int_a^b\int_a^t\int_a^y g(x)dx\,dy\,dt}{6\int_a^b\int_a^t g(x)dx\,dt - 3(b-a)\int_a^b g(t)dt}$,

$\xi, \eta \in (a,b)$.

And the error term $R[f]$ of this method is

$$\begin{aligned}&\left(\frac{a^3+ab^2+a^2b-3b^3+6(b-a)c^2}{24}\int_a^b g(t)dt + \frac{b^2-c^2}{2}\int_a^b\int_a^t g(x)dx\,dt\right.\\ &\left. -b\int_a^b\int_a^t\int_a^y g(x)dx\,dy\,dt + \int_a^b\int_a^t\int_a^z\int_a^y g(x)dx\,dy\,dz\,dt\right) f^{(4)}(\xi)g'(\eta).\end{aligned} \tag{3.2}$$

Proof. Let $f(t) = \dfrac{t^4}{4!}$. So

$$\begin{aligned}\frac{1}{4!}\int_a^b t^4 dg &= \frac{1}{24}\left(b^4 g(b) - a^4 g(a)\right) - \frac{b^3}{6}\int_a^b g(t)dt + \frac{b^2}{2}\int_a^b\int_a^t g(x)dx \\ &\quad -b\int_a^b\int_a^t\int_a^y g(x)dx\,dy\,dt + \int_a^b\int_a^t\int_a^z\int_a^y g(x)dx\,dy\,dz dt.\end{aligned} \tag{3.3}$$

By Theorem 2.1, we have

$$\begin{aligned}\int_a^b f(t)dg \approx T &= \left(\frac{1}{b-a}\int_a^b g(t)dt - g(a)\right) f(a) \\ &\quad + \left(g(b) - \frac{1}{b-a}\int_a^b g(t)dt\right) f(b) \\ &\quad + \left(\int_a^b\int_a^t g(x)dx\,dt - \frac{b-a}{2}\int_a^b g(t)dt\right) f''(c).\end{aligned} \tag{3.4}$$

By (8)-(9), we obtain

$$\begin{aligned}\frac{1}{4!}\int_a^b t^4 dg - T = &\left(\frac{a^3+ab^2+a^2b-3b^3+6(b-a)c^2}{24}\int_a^b g(t)dt\right. \\ &+\frac{b^2-c^2}{2}\int_a^b\int_a^t g(x)dx\,dt - b\int_a^b\int_a^t\int_a^y g(x)dx\,dy\,dt \\ &\left.+\int_a^b\int_a^t\int_a^z\int_a^y g(x)dx\,dy\,dz\,dt\right) f^{(4)}(\xi)g'(\eta).\end{aligned}$$

This implies that

$$\begin{aligned}R[f] = &\left(\frac{a^3+ab^2+a^2b-3b^3+6(b-a)c^2}{24}\int_a^b g(t)dt + \frac{b^2-c^2}{2}\int_a^b\int_a^t g(x)dx\,dt\right. \\ &\left.-b\int_a^b\int_a^t\int_a^y g(x)dx\,dy\,dt + \int_a^b\int_a^t\int_a^z\int_a^y g(x)dx\,dy\,dz\,dt\right) f^{(4)}(\xi)g'(\eta).\end{aligned}$$

∎

Corollary 3.2 *Conditions are the same as in Theorem* 3.1. *When* $g(t) = t$, (3.1) *reduces to the mindpoint derivative-based trapezoid rule* (see [13]) *for the classical Riemann integral.*

Proof. It is easy to obtain

$$\begin{aligned}\int_a^b t\,dt &= \frac{1}{2}b^2 - \frac{1}{2}a^2, \\ \int_a^b\int_a^t x\,dx\,dt &= \frac{1}{6}b^3 - \frac{1}{2}a^2b + \frac{1}{3}a^3, \\ \int_a^b\int_a^t\int_a^y x\,dx\,dy\,dt &= \frac{1}{24}b^4 - \frac{1}{4}a^2b^2 + \frac{1}{3}a^3b - \frac{1}{8}a^4, \\ \int_a^b\int_a^t\int_a^z\int_a^y x\,dx\,dy\,dz\,dt &= \frac{1}{120}b^5 - \frac{1}{12}a^2b^3 + \frac{1}{6}a^3b^2 - \frac{1}{8}a^3b + \frac{1}{30}a^5.\end{aligned}$$

Therefore,

$$c = \frac{(-2b^2+a^2-ab)\int_a^b tdt + 6b\int_a^b\int_a^t xdx\,dt - 6\int_a^b\int_a^t\int_a^y xdx\,dy\,dt}{6\int_a^b\int_a^t xdx\,dt - 3(b-a)\int_a^b tdt} = \frac{a+b}{2}.$$

By Theorem 3.1, we have

$$\begin{aligned}
\int_a^b f(t)dg &= \int_a^b f(t)dt \\
&= \left(\frac{1}{b-a}\int_a^b tdt - g(a)\right) f(a) + \left(g(b) - \frac{1}{b-a}\int_a^b tdt\right) f(b) \\
&\quad + \left(\int_a^b\int_a^t xdx\,dt - \frac{b-a}{2}\int_a^b tdt\right) f''(\frac{a+b}{2}) \\
&\quad + \left(\frac{a^3+ab^2+a^2b-3b^3}{24}\int_a^b tdt + \frac{b^2}{2}\int_a^b\int_a^t xdx\,dt \right. \\
&\quad -b\int_a^b\int_a^t\int_a^y xdx\,dy\,dt + \int_a^b\int_a^t\int_a^z\int_a^y xdx\,dy\,dz\,dt \\
&\quad \left. -\left(\frac{a+b}{2}\right)^2\left(\frac{1}{2}\int_a^b\int_a^t xdx\,dt - \frac{b-a}{4}\int_a^b tdt\right)\right) f^{(4)}(\xi) \\
&= \frac{b-a}{2}(f(a)+f(b)) - \frac{(b-a)^3}{12} f''\left(\frac{a+b}{2}\right) + \frac{(b-a)^5}{480} f^{(4)}(\xi). \quad \blacksquare
\end{aligned}$$

Remark 3.3 From Corollary 3.1, we know that the results in Theorem 3.1 possess the reducibility. When $g(t) = t$, formula (3.1) reduces to the derivative-based trapezoid rule for the classical Riemann integral. So Theorem 3.1 is a reasonable generalization of the results in [13].

4. Conclusions

We briefly summarize our main conclusions in this paper as follows.

1. The midpoint derivative-based trapezoid rule for the Riemann-Stieltjes integral is presented which uses derivative value at the midpoint.

2. This kind of quadrature rule obtains an increase of two orders of precision over the trapezoid rule for the Riemann-Stieltjes integral.

3. The error term for Riemann-Stieltjes midpoint derivative-based trapezoid rule is investigated. And the rationality of the generalization of midpoint derivative-based trapezoid rule for Riemann-Stieltjes integral is demonstrated.

The derivative-based Simpson's rules for the Riemann-Stieltjes integral will be achieved by further research.

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NEW CHARACTERIZATIONS OF SOLUBILITY OF FINITE GROUPS

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Abstract. A subgroup H of a group G is said to be S-supplemented in G if there exists a subgroup T of G such that $G = HT$ and $H \cap T \leq H_{sG}$, where H_{sG} denotes the subgroup of H generated by all those subgroups of H which are S-permutable in G. In this paper, two new characterizations of solubility of finite groups are presented in terms of S-supplemented subgroups of primes power orders, where primes belong to $\{3, 5\}$. In particular, a counterexample is given to show that the conjecture, proposed by Heliel at the end of [A.A. Heliel, *A note on c-supplemented subgroups of finite groups*, Comm. Algebra, 42 (2014), 1650-1656] and related to c-supplemented subgroups of primes power orders, is negative.

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All groups considered are finite.

Following Ballester-Bolinches, Wang and Guo [2], [12], a subgroup H of a group G is said to be c-supplemented in G if G has a subgroup T such that $G = HT$ and $H \cap T \leq H_G$, where H_G denotes the largest normal subgroup of G contained in H. In [1], Asaad and Ramadan prove that a group G is soluble provided that every minimal subgroup of G is c-supplemented in G. Recently, in [6], Heliel has generalized this result and proved the following theorems.

Theorem A. *If each subgroup of prime odd order of a group G is c-supplemented in G, then G is soluble.*

Theorem B. *A group G is soluble if and only if every Sylow subgroup of G of odd order is c-supplemented in G.*

In connection with the above two results, the following conjecture is posed at the end of [6].

Conjecture. *Let G be a group such that every non-cyclic Sylow subgroup P of odd order of G has a subgroup D such that $1 < |D| \leq |P|$ and all subgroups H of P with $|H| = |D|$ are c-supplemented in G. Then, is G soluble?*

In this short note, we first present a counterexample to show that the answer to this conjecture is negative in general and then give a generalization of Theorems A and B.

Example. Let $G = A_5 \times H$, where A_5 is the alternating group of degree 5 and H is an elementary group of order p^n with $p > 5$ and $n \geq 2$. Then G satisfies the condition of the preceding conjecture, but G is insoluble.

Next, we generalize Theorems A and B as the following two results respectively.

Theorem C. *Let G be a group and $\pi = \pi(G) \cap \{3, 5\}$. If every subgroup of G of order p with $p \in \pi$ is c-supplemented in G, then G is soluble.*

Theorem D. *Let G be a group and $\pi = \pi(G) \cap \{3, 5\}$. Then G is soluble if and only if every Sylow p-subgroup of G with $p \in \pi$ is c-supplemented in G and $L_2(8)$ is not involved in G.*

Here, we say that a group K is involved in a group G if K is isomorphic to a homomorphic image of a subgroup H of G. Note that such a homomorphic image is often called a section of G.

Recall that a subgroup H of a group G is said to be S-supplemented in G if G has a subgroup T such that $G = HT$ and $H \cap T \leq H_{sG}$, where H_{sG} denotes the subgroup of H generated by all those subgroups of H which are S-quasinormal (permutable with all Sylow subgroups of G) in G (see Skiba [11] or [10]). By the definition, all c-supplemented subgroups are also S-supplemented subgroups. Hence, Theorems C and D are special cases of the following results.

Theorem E. *Let G be a group and $\pi = \pi(G) \cap \{3, 5\}$. If every subgroup of G of order p with $p \in \pi$ is S-supplemented in G, then G is soluble.*

Theorem F. *Let G be a group and $\pi = \pi(G) \cap \{3, 5\}$. Then G is soluble if and only if every Sylow p-subgroup of G with $p \in \pi$ is S-supplemented in G and $L_2(8)$ is not involved in G.*

In order to prove these two results, we need the following lemmas.

Lemma 1. [3, Theorem 5.4] *Let G be a group such that $(|G|, 15) = 1$. Then G is soluble.*

Lemma 2. [10, Lemma 2.10] *Let G be a group and $H \leq K \leq G$.*

(1) *If H is S-supplemented in G, then H is S-supplemented in K.*

(2) *Suppose that H is normal G. Then K/H is S-supplemented in G/H if and only if K is S-supplemented in G.*

(3) *Suppose that H is normal in G. Then the subgroup EH/H is S-supplemented in G/H for every S-supplemented subgroup E of G satisfying $(|E|, |H|) = 1$.*

Lemma 3. *Let P be a nontrivial normal p-subgroup of a group G with p odd. If all cyclic subgroups of P of order p are S-supplemented in G, then each G-chief factor below P is cyclic.*

Proof. This follows directly from Theorem A in [11]. ∎

Lemma 4. [5, Theorem 1] *Let G be a nonabelian simple group with H a subgroup of G such that $|G : H| = p^a$. Then one of the following holds.*

(1) *$G = A_n$ and $H \simeq A_{n-1}$ with $n = p^a$.*

(2) *$G = L_n(q)$ and H is the stabilizer of a line or hyperplane.*
Then $|G : H| = (q^n - 1)/(q - 1) = p^a$.

(3) *$G = L_2(11)$ and $H \simeq A_5$.*

(4) *$G = M_{23}$ and $H \simeq M_{22}$ or $G = M_{11}$ and $H \simeq M_{10}$.*

(5) *$G = U_4(2)$ and H is the parabolic subgroup of index 27.*

Lemma 5. [9, §5] *Let H be a Hall π-subgroup of the finite simple group G and $3 \notin \pi$. Then either H has a Sylow tower or $H = G = {}^2B_2(q)$.*

Lemma 6. *Let $G = L_n(q)$ and H a subgroup of G such that $|G : H| = 3^a$, where $a \geq 1$. Then $G = L_2(8)$ and $H \simeq 2^3.Z_7$ with index 9.*

Proof. This is a special case of Theorem 1.1 in [8]. ∎

Proof of Theorem E. Suppose the result is false and let G be a counterexample of minimal order. Then

(1) *Every proper subgroup of G is soluble.*

It follows from Lemmas 2 and 1 and the choice of G.

(2) *G is not a nonabelian simple group.*

Assume that G is a nonabelian simple group. Then G is a minimal simple group by (1). Let H be a subgroup of G of order $p \in \pi$. If H is S-quasinormal in G, then $H \leq O_p(G)$, a contradiction. Suppose G has a subgroup T such that $G = HT$ and $H \cap T = 1$. If $p = 3$, then we deduce that G is soluble. If $p = 5$, then G is isomorphic to A_5. However, the subgroups of A_5 of order 3 are not S-supplemented, which contradicts our initial assumption for G. Hence G cannot be a nonabelian simple group.

(3) $\overline{G} = G/\Phi(G)$ *is a minimal simple group.*

By (2), suppose that N is any nontrivial proper normal subgroup of G. Let M be any maximal subgroup of G. By (1), both N and M are soluble. If N is not contained in M, then $G = MN$ and so $G/N \simeq M/M \cap N$ is soluble. It follows that G is soluble, a contradiction. Hence $N \leq \Phi(G)$ and (3) holds.

(4) *Final contradiction.*

By (3), $\overline{G}$ is isomorphic to one of the following simple groups (see Huppert [7, Ch.II, Remark 7.5]):

(i) $L_2(p)$, $p > 3$ is a prime, and 5 does not divide $p^2 - 1$;

(ii) $L_2(3^r)$, r is an odd prime;

(iii) $L_2(2^r)$, r is a prime;

(iv) $Sz(2^r)$, r is an odd prime;

(v) $L_3(3)$.

Suppose first that $\overline{G}$ is isomorphic to one of the simple groups in (i)-(iv). By [7, Ch.II, Theorem 8.27] and [13, p.117, Theorem 4.1], every Sylow p-subgroup of $\overline{G}$ is cyclic, where $p = 3$ or 5. We claim that p does not divide the order of $\Phi(G)$. Otherwise, let P be the Sylow p-subgroup of $\Phi(G)$ and G_p be a Sylow p-subgroup of G. Then G_p/P is cyclic. By Lemma 2 and Lemma 3, every G-chief factor below P is cyclic. It follows that $G/C_G(P)$ is supersoluble (see [4, Corollary 3.2.9]). Thus, $G = C_G(P)$ by (1). Hence $P \leq Z(G)$ and so G_p is abelian. Furthermore, $G = G'$ according to Step (3). Therefore, we have that $P \cap Z(G) \cap G' = P$, a contradiction by [7, Ch.VI, Theorem 14.3]. Thus, the order of $\Phi(G)$ cannot be divisible by p. Let $H/\Phi(G)$ be a subgroup of $\overline{G}$ with order p. Then $H/\Phi(G) = \langle x \rangle \Phi(G)/\Phi(G)$ for some element x of G of order p. By Lemma 2, $H/\Phi(G)$ is S-supplemented in $\overline{G}$. Arguing as in (2), we deduce a contradiction.

Now, suppose that $\overline{G}$ is isomorphic to $PSL_3(3)$. We show that $\Phi(G)$ is a $3'$-group. If not, let P be the Sylow 3-subgroup of $\Phi(G)$. As above, we see that $P \leq Z(G)$. If all subgroups of G of order 3 are contained in P, then by [7, Ch.IV, Theorem 5.5], G is 3-nilpotent, which implies that G is soluble. Thus, for some element x of order 3 in G, $x \notin P$. By the hypothesis, $H = \langle x \rangle$ is S-supplemented in G. Then G has a subgroup T such that $G = HT$ and $H \cap T \leq H_{sG}$. If $H \cap T = 1$, then T is a proper subgroup of G of index 3. It is easy to see that $\Phi(G) \leq T$ and therefore $\overline{G}$ has a subgroup of index 3, a contradiction. Suppose that H is S-quasinormal in G. Then $H \leq O_p(G) \leq \Phi(G)$ and consequently $H \leq P$, a contradiction. Hence 3 does not divide the order of $\Phi(G)$. As in the foregoing paragraph, we derive a contraction, completing the proof.

Proof of Theorem F. If G is soluble, then every Sylow subgroup of G is complemented in G and thereby is S-supplemented in G. In addition, $L_2(8)$ is clearly not involved in G. Hence the necessity holds.

Now, we suppose that all Sylow p-subgroups of G with $p \in \pi$ are S-supplemented in G and G does not involve $L_2(8)$. We proceed by induction on the order of G. Let P be an arbitrary Sylow p-subgroup of G, where $p \in \pi$. Then, by the hypothesis, there exists a subgroup T in G such that $G = PT$ and $P \cap T \leq P_{sG}$. If

$P_{sG} \neq 1$, then $P_{sG} \leq O_p(G)$, where $O_p(G)$ denotes the largest normal p-subgroup of G. Consider the factor group $G/O_p(G)$. Then, it is easy to see that $G/O_p(G)$ satisfies the hypothesis and so $G/O_p(G)$ is soluble by induction. It follows that G is soluble. Hence we may assume that P_{sG} is trivial, which means that P is complemented in G.

Next, we argue that G is not a nonabelian simple group. If not, then G is a nonabelian simple group such that every Sylow p-subgroup of G with $p \in \pi$ is complemented in G by the foregoing discussion. Hence G has a subgroup H such that $|G : H| = p^a$, where $p = 3$ or 5. Thus, G is isomorphic to one of the groups listed in Lemma 4. If G is isomorphic to one of the following:

$$L_2(11),\ M_{23},\ M_{11},\ U_4(2),$$

then $\pi = \{3, 5\}$. By the preceding paragraph, G has two subgroups with two different indices 3^a and 5^b, where $a \geq 1$ and $b \geq 1$, which is impossible (see [5, pp. 304]). If $G = A_n$ and $H \simeq A_{n-1}$, then, by Lemma 5, we have that $n = 5$. But the Sylow 3-subgroups of A_5 are not complemented in A_5, a contradiction. At last, assume that $G = L_n(q)$. Then, by Lemma 6, G must be isomorphic to $L_2(8)$, contrary to our assumption for G. Thus, we have shown that G is not a nonabelian simple group.

Let N be a minimal normal subgroup of G. Then N is nontrivial. If N is an elementary abelian group, then, by Lemma 2, G/N satisfies the hypothesis and so G/N is soluble by induction. Thereby G is soluble. Suppose that N is insoluble. Denote $\pi' = \pi(N) \cap \{3, 5\}$. Obviously, $\pi' \subseteq \pi$. Then, by Lemma 1, $\pi' \neq \emptyset$. Let P be any Sylow p-subgroup of G with $p \in \pi'$ and set $L = PN$. Then $P \cap N$ is complemented in N by the first paragraph and therefore $P \cap N$ is S-supplemented in N. Note that $P \cap N$ is a Sylow p-subgroup of N. Thus, we see that N satisfies the hypothesis and so N is soluble by induction.

This contradiction completes the proof. ∎

Remark.

(1) The converse of Theorem E is not true in general. The alternating group A_4 of degree 4 is such a counterexample because every involution of A_4 is not S-supplemented in A_4 as A_4 has no subgroup of order 6.

(2) In Theorem F, the condition "G does not involve $L_2(8)$" can not be removed. In fact, $L_2(8)$ is a counterexample. In $L_2(8)$, $\pi = \pi(L_2(8)) \cap \{3, 5\} = \{3\}$ and every Sylow 3-subgroup of $L_2(8)$ is complemented in $L_2(8)$. Of course, every Sylow 3-subgroup of $L_2(8)$ is S-supplemented in $L_2(8)$. But $L_2(8)$ is a nonabelian simple group.

With respect to Theorem E and Theorem F, the following problem seems interesting.

Problem. *Let G be a group and $\pi = \pi(G) \cap \{3, 5\}$. Suppose that for every Sylow p-subgroup P of G with $p \in \pi$, G has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with $|H| = |D|$ are S-supplemented in G. What can we say about the structure of G?*

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THE TRIPARTITE RAMSEY NUMBERS $r_t(C_4; 2)$ AND $r_t(C_4; 3)$

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Abstract. The k-colored tripartite Ramsey numbers $r_t(G; k)$ is the smallest positive integer n such that any k-coloring of lines of a complete tripartite graph $K_{n,n,n}$ there always exists a monochromatic subgraph isomorphic to G. When G is C_4 it is known, but unpublished in a journal, that $r_t(C_4; 2) = 3$. In this paper we simplify the proof of $r_t(C_4; 2) = 3$ and show the new result that $r_t(C_4; 3) = 7$.

Keywords and phrases: tripartite Ramsey numbers, bipartite Ramsey numbers, Ramsey numbers, tripartite graphs, bipartite graphs.

AMS Subject Classification: 05C55; 05D10.

1. Introduction

A graph G is *n-partite*, $n \geq 1$, if it is possible to partition $V(G)$ into n subsets $V_1, V_2, \ldots, V_n$ (called partite sets) such that every element of $E(G)$ joins a vertex of V_i to a vertex of $V_j, i \neq j$. For $n = 2$, such graphs are called *bipartite graphs*. For $n = 3$, such graphs are called *tripartite graphs*. A *complete n-partite* G is an *n-partite* graph with partite sets $V_1, V_2, \ldots, V_n$ having the added property that if $u \in V_i$ and $v \in V_j, i \neq j$, then $uv \in E(G)$. When $|V_i| = p_i$, we denote the complete n-partite graph by $K_{p_1,p_2,\ldots,p_n}$.

Consider a complete bipartite graph $K_{s,s}$ of order $p = 2s$. Let each line of $K_{s,s}$ be colored by using either red or blue color. We shall call such a $K_{s,s}$ 2-colored.

Consider a subgraph $K_{m,n}$ of 2-colored $K_{s,s}$. If all lines of $K_{m,n}$ have red(blue) color, we shall say that the $K_{s,s}$ contains a red(blue) $K_{m,n}$. The smallest number s of points such that $K_{s,s}$ always contains red $K_{m,n}$ or blue $K_{m,n}$ is called bipartite Ramsey number and denoted by $r_b(K_{m,n}; 2)$ or $r_b(K_{m,n}, K_{m,n})$.

According to the definition of bipartite Ramsey numbers in this paper, V. Longani [7], has found that

$$\begin{aligned} &r_b(K_{1,n}, K_{1,n}) = 2n - 1 \ (n = 1, 2, 3, ...),\\ &r_b(K_{2,2}, K_{2,2}) = 5,\\ &r_b(K_{2,3}, K_{2,3}) = 9,\end{aligned}$$

and L.W. Beineke and A.J. Schwenk [1], have also found that

$$\begin{aligned} &r_b(K_{2,2}, K_{2,2}) = 5,\\ &r_b(K_{3,3}, K_{3,3}) = 17.\end{aligned}$$

The 3-colored bipartite Ramsey number $r_b(G; 3)$ is the smallest integer n such that any 3-coloring of lines of a complete bipartite graph $K_{n,n}$ there always exists monochromatic subgraph isomorphic to G. In [4], W. Goddard, M.A. Henning, and O.R. Hellermann showed that $r_b(C_4; 3) = 11$.

Consider a complete tripartite graph $K_{s,s,s}$. Let each line of the $K_{s,s,s}$ be colored by using one of k colors. We call such a $K_{s,s,s}$ as k-colored. For 2-coloring, the smallest number s of points such that the $K_{s,s,s}$ always contains monochromatic $K_{m,n}$ is called tripartite Ramsey number $r_t(K_{m,n}; 2)$ or $r_t(K_{m,n}, K_{m,n})$.

In [5], K. Leamyoo and V. Longani, have found that $r_t(K_{2,2}, K_{2,2}) = 3$ (or $r_t(C_4; 2) = 3$). Also in [2], [3], S. Buada and V. Longani, have shown that $r_t(K_{2,3}, K_{2,3}) = 5$, $r_t(K_{2,4}, K_{2,4}) = 7$.

For 3-coloring, the tripartite Ramsey number of the graph G, denoted by $r_t(G; 3)$ (or $r_t(G, G, G)$) is the minimum integer n such that for any 3-coloring of the lines of $K_{n,n,n}$ there always exists monochromatic subgraph G.

2. Main results

The proof that $r_t(C_4; 2) = 3$ in [5] is rather lengthy and has not been published in a journal. In this section we provide a new shorter proof that $r_t(C_4; 2) = 3$ and prove a new result that $r_t(C_4; 3) = 7$.

For a $K_{3,3,3}$, consider all twenty seven lines that are adjacent to all points of a V_i. We shall call the lines that are adjacent to V_i as the lines of V_i. Before proving Theorem 1, we prove Lemma 1 first.

Lemma 1. *Let $K_{3,3,3}$ be a 2-colored complete tripartite graph and each V_1, V_2, and V_3 be the set of three non-adjacent points of the $K_{3,3,3}$. There exists at least one V_i of which the numbers of red lines and blue lines of the V_i are not equal.*

Proof. Consider the three V_i's. Suppose there are nine red lines and nine blue lines of each V_i.

Since there are totally nine red lines of V_1, consider when there are n $(n \geq 0)$ red lines which join points of V_1 and V_2, and so there are $9-n$ red lines which join points of V_1 and V_3. Since for V_2 there are also exactly nine red lines of V_2, therefore there are $9-n$ red lines which join points of V_2 and V_3.

Now we can see that there are $(9-n)+(9-n)$ red lines of V_3. Since there are exactly nine red lines of V_3, therefore

$$\begin{aligned}(9-n)+(9-n) &= 9\\ n &= 4.5.\end{aligned}$$

This is not possible. Therefore, there exist some V_i's of which the numbers of red lines and blue lines of the V_i's are not equal. ∎

In order to prove Theorem 1 and Theorem 2, it would be convenient to represent a 2-colored $K_{m,n}$ by an $m \times n$ matrix $B = [b_{ij}]$.

Given a 2-colored $K_{m,n}$ with V_1 and V_2 as its partite sets of size m and n, respectively. Let $V_1 = \{r_1, r_2, \ldots, r_m\}$ and $V_2 = \{c_1, c_2, \ldots, c_n\}$. For the corresponding $B = [b_{ij}]$ of the $K_{m,n}$, we let $b_{ij} = 1$ if the line r_ic_j is red, and let $b_{ij} = 0$ if the line r_ic_j is blue. For example, the following (a) and (b) in Figure 2.1 illustrate the 2-colored $K_{4,5}$ and its corresponding matrix B. Here, we use dark lines to indicate red lines and dash lines to indicate blue lines.

c_1 c_2 c_3 c_4 c_5 V_1

(a)

V_2

r_1 r_2 r_3 r_4

(b)

$$\begin{array}{c c} & \begin{array}{ccccc} c_1 & c_2 & c_3 & c_4 & c_5 \end{array} \\ \begin{array}{c} r_1 \\ r_2 \\ r_3 \\ r_4 \end{array} & \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Figure 2.1

Theorem 1. $r_t(C_4; 2) = 3$

Proof. Consider the 2-colored $K_{2,2,2}$ graph illustrated in Figure 2.2.

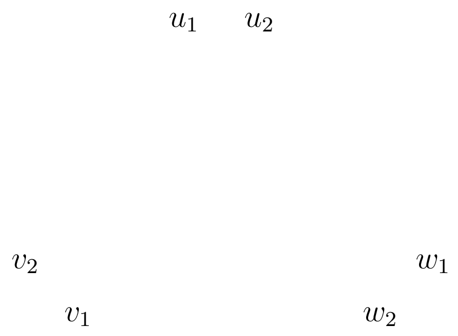

Figure 2.2

It can be seen that the $K_{2,2,2}$ contains neither red C_4 nor blue C_4. Therefore $r_t(C_4; 2) > 2$. That is

$$r_t(C_4; 2) \geq 3 \tag{2.1}$$

Let $K_{3,3,3}$ be a 2-colored complete tripartite graph. Consider the set V_1, V_2 and V_3 of three non-adjacent points of the $K_{3,3,3}$

$$\begin{aligned} V_1 &= \{u_1, u_2, u_3\} \\ V_2 &= \{v_1, v_2, v_3\} \\ V_3 &= \{w_1, w_2, w_3\}. \end{aligned}$$

From Lemma 1, we can assume that from V_1, the numbers of red lines are greater than the numbers of blue lines, that is the numbers of red lines are equal to ten or greater. We only need to consider the case when the numbers of red lines of V_1 is ten and show that in such case the $K_{3,3,3}$ always contain red C_4. For the cases when the number of red lines is greater than ten, the results follow immediately.

Let $V(G_1) = V_1$ and $V(G_2) = V_2 \cup V_3$. For $V(G_1)$, let u_1, u_2, u_3 be respectively replaced by r_1, r_2, r_3. Also for $V(G_2)$, let v_1, v_2, v_3, w_1, w_2, w_3 be respectively replaced by c_1, c_2, c_3, c_4, c_5, c_6. That is,

$$\begin{aligned} V(G_1) &= \{r_1, r_2, r_3\} \\ V(G_2) &= \{c_1, c_2, c_3, c_4, c_5, c_6\}. \end{aligned}$$

By ignoring the lines between V_2 and V_3 and consider the defined $V(G_1)$ and $V(G_2)$. The $K_{3,3,3}$ is now reduced to 2-colored $K_{3,6}$. In order to prove the theorem we only need to show that this $K_{3,6}$ always contains red C_4.

We find the value of $r_t(C_4; 2)$ by considering the 2-colored $K_{3,6}$. If there are $m, n, s, t, u (1 \leq m, n \leq 3$ and $1 \leq s, t, u \leq 6)$ such that some submatrices

$$\begin{bmatrix} b_{ms} & b_{mt} \\ b_{ns} & b_{nt} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \tag{2.2}$$

then the $K_{3,6}$ contains red C_4.

Let d_1, d_2, d_3 be degrees of red lines of r_1, r_2, r_3 respectively. We can choose r_i's such that $d_1 \geq d_2 \geq d_3$. Here we have the conditions that $d_1 + d_2 + d_3 = 10$ and $0 \leq d_i \leq 6$, $i = 1, 2, 3$. Next, we consider two main cases.

Case 1. $d_1 + d_2 \geq 8$.

Here, the possible $d_1 \geq d_2$ are $4 \geq 4$, $5 \geq 3$, $5 \geq 4$, $5 \geq 5$, $6 \geq 3$, $6 \geq 4$, $6 \geq 5$, $6 \geq 6$. It is easy to show that for all of these cases the $K_{3,6}$ always contains red C_4. Consider cases $d_1 = 4$, and $d_2 = 4$, for example. For a case in Table 1, parts of the matrix involving r_1 and r_2 could be

	c_1	c_2	c_3	c_4	c_5	c_6
r_1	1	1	**1**	**1**		
r_2			**1**	**1**	1	1

Table 1:

from which submatrix of the form (2.5) always appears; that is the $K_{3,6}$ contains red C_4.

Case 2. $d_1 + d_2 < 8$.

With the conditions for d_i, there is only one subcase to consider $d_1 = 4$, $d_2 = 3$, $d_3 = 3$. We consider two more possibilities: Subcase 2.1 and Subcase 2.2.

Subcase 2.1. When there are two or more points of c_i's each of which is joined by red lines to both of r_1 and r_2, see Table 2, then we see that the $K_{3,6}$ contains red C_4.

	c_1	c_2	c_3	c_4	c_5	c_6
r_1	1	1	**1**	**1**		
r_2			**1**	**1**	1	

Table 2:

Subcase 2.2. One point of c_i's is joined by red lines to both of r_1 and r_2.

Suppose that r_1 is joined by four red lines to c_1, c_2, c_3, c_4 and r_2 is joined by three red lines to c_4, c_5, c_6. Consider the three red lines joining r_3. Either at least two of three red lines are joined, from r_3, to some points among c_1, c_2, c_3, c_4 or at least two of these three red lines are joined to some points among c_4, c_5, c_6. In either case, we see that red C_4 is contained in the $K_{3,6}$, see Table 3 for example.

	c_1	c_2	c_3	c_4	c_5	c_6
r_1	**1**	**1**	1	1		
r_2				1	1	1
r_3	**1**	**1**				1

Table 3:

Hence the $K_{3,3,3}$ will always contain red C_4. Therefore,

$$r_t(C_4; 2) \leq 3. \tag{2.3}$$

By (2.1) and (2.3), we have $r_t(C_4; 2) = 3$ as required. ∎

Next, we show that $r_t(C_4; 3) = 7$.

Theorem 2. *$r_t(C_4; 3) = 7$.*

Proof. Consider the subgraphs of a $K_{6,6,6}$ illustrated in Figure 2.3. Here, (a), (b), and (c) represent subgraphs of the 3-colored $K_{6,6,6}$ with red, blue, and green lines, respectively.

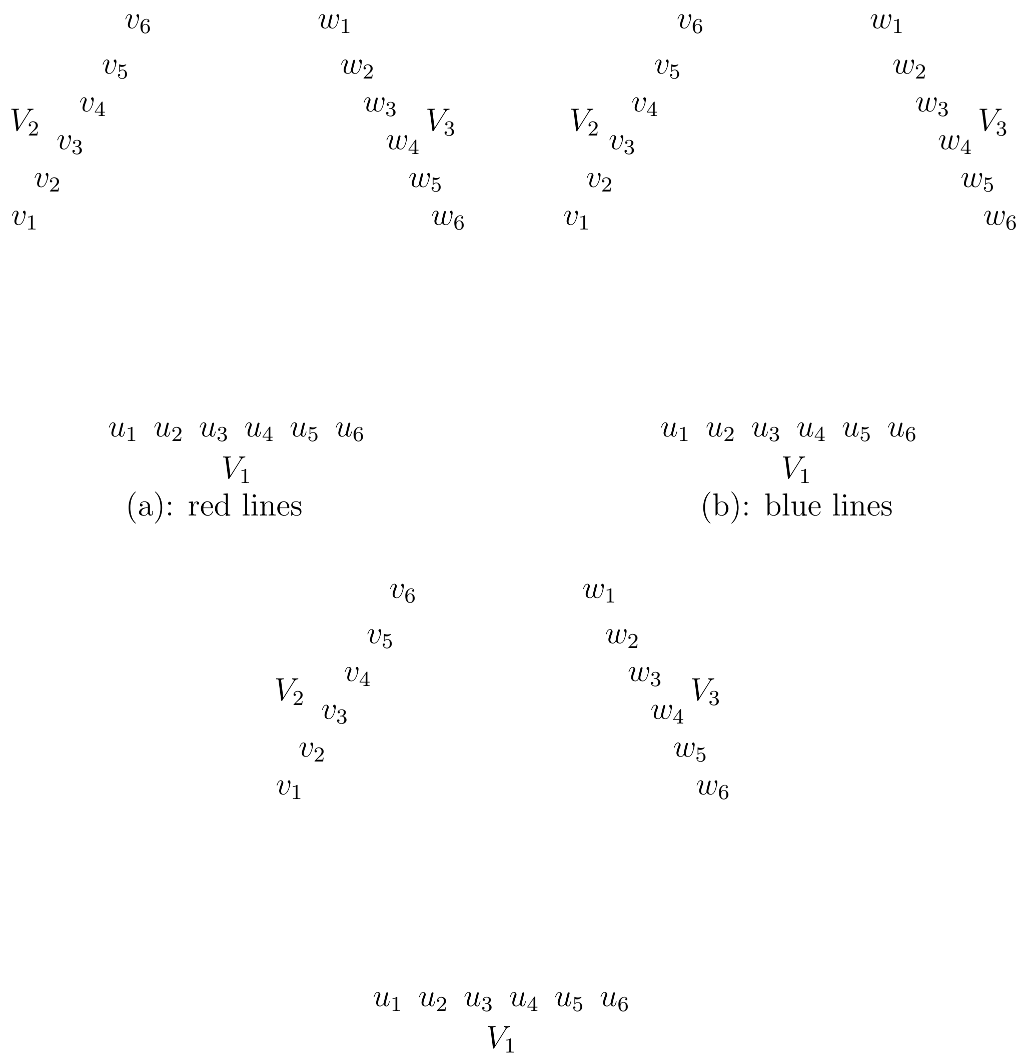

Figure 2.3

It can be verified that the $K_{6,6,6}$ contains no monochromatic C_4. Therefore, $r_t(C_4; 3) > 6$. That is,

$$r_t(C_4; 3) \geq 7. \tag{2.4}$$

Let $K_{7,7,7}$ be a 3-colored complete tripartite graph with $p = 21$. Consider the set V_1, V_2 and V_3 of seven non-adjacent points of the $K_{7,7,7}$:

$$\begin{aligned} V_1 &= \{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}, \\ V_2 &= \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}, \\ V_3 &= \{w_1, w_2, w_3, w_4, w_5, w_6, w_7\}. \end{aligned}$$

We shall call the lines that are adjacent to V_i as the lines of V_i. Since there are totally ninety eight lines of V_1, there are at least thirty three lines of V_1 with the same, say red, color. Consider thirty three of these red lines.

Let $V(G_1) = V_1$ and $V(G_2) = V_2 \cup V_3$. For $V(G_1)$, let u_1, u_2, u_3, u_4, u_5, u_6, u_7 be respectively replaced by r_1, r_2, r_3, r_4, r_5, r_6, r_7. Also for $V(G_2)$, let v_1, v_2, v_3, v_4, v_5, v_6, v_7, w_1, w_2, w_3, w_4, w_5, w_6, w_7 be respectively replaced by c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9, c_{10}, c_{11}, c_{12}, c_{13}, c_{14}. That is,

$$\begin{aligned} V(G_1) &= \{r_1, r_2, r_3, r_4, r_5, r_6, r_7\} \\ V(G_2) &= \{c_1, c_2, \ldots, c_{14}\}. \end{aligned}$$

By ignoring the lines between V_2 and V_3 and consider the defined $V(G_1)$ and $V(G_2)$ the $K_{7,7,7}$ is now reduced to 3-colored $K_{7,14}$. In order to prove the theorem we only need to show that this $K_{7,14}$ always contains red C_4.

We find an upper bound of $r_t(C_4; 3)$ by considering the 3-colored $K_{7,14}$.

If there are $m, n, s, t (1 \leq m, n \leq 7$ and $1 \leq s, t \leq 14)$ such that some submatrices

$$\begin{bmatrix} b_{ms} & b_{mt} \\ b_{ns} & b_{nt} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \tag{2.5}$$

then the $K_{7,14}$ contains red $K_{2,2}$ or C_4.

Let d_1, d_2, d_3, d_4, d_5, d_6, d_7 be degrees of red lines of r_1, r_2, r_3, r_4, r_5, r_6, r_7 respectively. We can choose r_i's such that $d_1 \geq d_2 \geq d_3 \geq d_4 \geq d_5 \geq d_6 \geq d_7$. Here we have the conditions that

$$d_1 + d_2 + d_3 + d_4 + d_5 + d_6 + d_7 = 33$$

and $0 \leq d_i \leq 14$, $i = 1, 2, 3, 4, 5, 6, 7$.

Next, we consider two main cases: Case 1 and Case 2.

Case 1. $d_1 + d_2 + d_3 \geq 18$.

Here, the possible $d_1 \geq d_2 \geq d_3$ are $6 \geq 6 \geq 6$, $7 \geq 6 \geq 5$, $7 \geq 6 \geq 6$, $7 \geq 7 \geq 5$, $7 \geq 7 \geq 6$, $7 \geq 7 \geq 7$, $8 \geq 5 \geq 5$, $8 \geq 6 \geq 5$, $8 \geq 6 \geq 6$, $8 \geq 7 \geq 5$, $8 \geq 7 \geq 6$, $8 \geq 7 \geq 7$, $8 \geq 8 \geq 6$, $8 \geq 8 \geq 6$, $8 \geq 8 \geq 7$, $8 \geq 8 \geq 8$, $9 \geq 5 \geq 5$, $9 \geq 6 \geq 5$, $9 \geq 6 \geq 6$, $9 \geq 7 \geq 5$, $9 \geq 7 \geq 6$, $9 \geq 7 \geq 7$, $9 \geq 8 \geq 5$, $10 \geq 5 \geq 5$, $10 \geq 6 \geq 5$, $10 \geq 6 \geq 6$, $10 \geq 7 \geq 5$, $11 \geq 5 \geq 5$, $11 \geq 6 \geq 5$, $11 \geq 6 \geq 5$, $12 \geq 5 \geq 5$.

It is easy to show that for all of these cases the $K_{7,14}$ always contains red C_4.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	1	1	1	1	1	1								
r_2						1	1	1	1	**1**	**1**			
r_3	1									**1**	**1**	1	1	1

Table 4:

Consider the cases $d_1 = 6$, $d_2 = 6$ and, $d_3 = 6$, for example. For a case in Table 4, parts of the matrix involving r_1, r_2 and, r_3 could be
from which submatrix of the form (2.2) always appears, that is the $K_{7,14}$, and so the $k_{7,7,7}$ contains red C_4.

Case 2. $d_1 + d_2 + d_3 < 18$.

With the conditions for d_i, there are only four subcases to consider.

Subcase 2.1. $d_1 = 7, d_2 = 5, d_3 = 5, d_4 = 4, d_5 = 4, d_6 = 4, d_7 = 4$.

When there are two or more points c_i's each of which is joined to both of r_1 and r_2 by red lines, we can see that the $K_{7,14}$ contains red C_4, see Table 5 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	1	1	1	1	1	**1**	**1**							
r_2						**1**	**1**	1	1	1				

Table 5:

We consider two more possibilities 2.1.1 and 2.1.2.

2.1.1. One c_i is joined by red lines to both of r_1 and r_2.

Suppose that r_1 is joined by seven red lines to $c_1, c_2, c_3, c_4, c_5, c_6, c_7$ and r_2 is joined by five red lines to c_7, c_8, c_9, c_{10}, c_{11}. We now consider four sub-possibilities (1), (2), (3), and (4).

(1) c_{12}, c_{13}, c_{14} are not joined to some r_i's $(i = 3, 4, 5, 6, 7)$ by red lines.

If c_{12}, c_{13}, c_{14} are not joined to r_3 for example, then either at least three red lines from r_3 are joined to points among $c_1, c_2, c_3, c_4, c_5, c_6, c_7$ or at least three red lines from r_3 are joined to points among c_8, c_9, c_{10}, c_{11}. In either case, we can see that red C_4 is contained in the $K_{7,14}$, see Table 6 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	**1**	**1**	1	1	1	1	1							
r_2							1	1	1	1	1			
r_3	**1**	**1**	1					1	1					

Table 6:

(2) One of c_{12}, c_{13}, c_{14} is joined to some r_i's $(i = 3, 4, 5, 6, 7)$ by red lines.

For example, suppose r_3 is joined to c_{12} by red line, then there are four other red lines joining r_3. From these four red lines, either at least two red lines from r_3 are joined to points among $c_1, c_2, c_3, c_4, c_5, c_6, c_7$ or at least two red lines from r_3 are joined to points among c_8, c_9, c_{10}, c_{11}. In either case, we can see that red C_4 is contained in the $K_{7,14}$, see Table 7 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	**1**	**1**	1	1	1	1	1							
r_2							1	1	1	1	1			
r_3	**1**	**1**								1	1	1		

Table 7:

(3) Two of c_{12}, c_{13}, c_{14} are joined to some r_i's $(i = 3, 4, 5, 6, 7)$ by red lines.

For example, suppose r_3 is joined to c_{12}, c_{13} by red lines, then there are three other red lines joining r_3. From these three red lines, then either at least two red lines from r_3 are joined to points among $c_1, c_2, c_3, c_4, c_5, c_6, c_7$ or at least two red lines from r_3 are joined to points among c_8, c_9, c_{10}, c_{11}. In either case, we can see that red C_4 is contained in the $K_{7,14}$, see Table 8 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	**1**	**1**	1	1	1	1	1							
r_2							1	1	1	1	1			
r_3	**1**	**1**									1	1	1	

Table 8:

(4) All c_{12}, c_{13}, c_{14} are joined to some r_i's $(i = 3, 4, 5, 6, 7)$ by red lines.

Suppose r_3 is joined to c_{12}, c_{13}, and c_{14}. Consider r_4 and the four red lines joining r_4. From these four red lines, either at least two red lines are joined to points among $c_1, c_2, c_3, c_4, c_5, c_6, c_7$ or at least two red lines are joined to points among c_8, c_9, c_{10}, c_{11} or at least two red lines are joined to points among c_{12}, c_{13}, c_{14}. In any case, we can see that red C_4 is contained in the $K_{7,14}$, see Table 9 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	1	**1**	**1**	1	1	1	1							
r_2							1	1	1	1	1			
r_3	1										1	1	1	1
r_4		**1**	**1**							1				1

Table 9:

2.1.2. None of c_i's are joined by red lines to both of r_1 and r_2.

Suppose that r_1 is joined by seven red lines to $c_1, c_2, c_3, c_4, c_5, c_6, c_7$ and r_2 is joined by five red lines to c_8, c_9, c_{10}, c_{11}, c_{12}. We now consider three sub-possibilities (1), (2), and (3).

(1) c_{13}, c_{14} are not joined to some r_i's $(i = 3, 4, 5, 6, 7)$ by red lines.

If c_{13}, c_{14} are not joined to r_3 for example, then either at least three red lines from r_3 are joined to points among $c_1, c_2, c_3, c_4, c_5, c_6, c_7$ or at least three red lines from r_3 are joined to points among c_8, c_9, c_{10}, c_{11}, c_{12}. In either case, we can see that red C_4 is contained in the $K_{7,14}$, see Table 10 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	**1**	**1**	**1**	1	1	1	1							
r_2								1	1	1	1	1		
r_3	**1**	**1**	**1**					1	1					

Table 10:

(2) One of c_{13}, c_{14} is joined to some r_i's $(i = 3, 4, 5, 6, 7)$ by red lines.

For example, suppose r_3 is joined to c_{13} by red line, then there are four other red lines joining r_3. From these four red lines, either at least two red lines are joined to points among $c_1, c_2, c_3, c_4, c_5, c_6, c_7$ or at least two red lines are joined to points among c_8, c_9, c_{10}, c_{11}, c_{12}. In either case, we can see that red C_4 is contained in the $K_{7,14}$, see Table 11 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	**1**	**1**	1	1	1	1	1							
r_2								1	1	1	1	1		
r_3	**1**	**1**									1	1	1	

Table 11:

(3) Both of c_{13}, c_{14} are joined to some r_i's $(i = 3, 4, 5, 6, 7)$ by red lines.

Suppose r_3 is joined to c_{13}, and c_{14}, then there are three other red lines joining r_3. From these three red lines, either at least two red lines are joined to points among $c_1, c_2, c_3, c_4, c_5, c_6, c_7$ or at least two red lines are joined to points among c_8, c_9, c_{10}, c_{11}, c_{12}. In either case, we can see that red C_4 is contained in the $K_{7,14}$, see Table 12 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	**1**	**1**	1	1	1	1	1							
r_2								1	1	1	1	1		
r_3	**1**	**1**										1	1	1

Table 12:

Subcase 2.2. $d_1 = 6, d_2 = 6, d_3 = 5, d_4 = 4, d_5 = 4, d_6 = 4, d_7 = 4.$

When there are two or more points c_i's each of which is joined to both of r_1 and r_2 by red lines, then we can see that the $K_{7,14}$ contains red C_4, see Table 13 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	1	1	1	1	**1**	**1**								
r_2					**1**	**1**	1	1	1	1				

Table 13:

We consider two more possibilities 2.2.1 and 2.2.2.

2.2.1. One c_i is joined by red lines to both of r_1 and r_2.

Suppose that r_1 is joined by six red lines to $c_1, c_2, c_3, c_4, c_5, c_6$ and r_2 is joined by six red lines to c_6, c_7, c_8, c_9, c_{10}, c_{11}. Similar to the cases in 2.1.1 of subcase 2.1, we have that the $K_{7,14}$ contains red C_4.

2.2.2. None of c_i's are joined by red lines to both of r_1 and r_2.

Suppose that r_1 is joined by six red lines to $c_1, c_2, c_3, c_4, c_5, c_6$ and r_2 is joined by six red lines to c_7, c_8, c_9, c_{10}, c_{11}, c_{12}. Similar to the cases in 2.1.2 of subcase 2.1, we have that the $K_{7,14}$ contains red C_4.

Subcase 2.3. $d_1 = 6, d_2 = 5, d_3 = 5, d_4 = 5, d_5 = 4, d_6 = 4, d_7 = 4.$

When there are two or more points c_i's each of which is joined to both of r_1 and r_2 by red lines, then we can see that the $K_{7,14}$ contains red C_4, see Table 14 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	1	1	1	1	**1**	**1**								
r_2					**1**	**1**	1	1	1					

Table 14:

We consider two more possibilities 2.3.1 and 2.3.2.

2.3.1. One c_i is joined by red lines to both of r_1 and r_2.

Suppose that r_1 is joined by six red lines to $c_1, c_2, c_3, c_4, c_5, c_6$ and r_2 is joined by five red lines to c_6, c_7, c_8, c_9, c_{10}. We now consider five sub-possibilities (1), (2), (3), (4), and (5).

(1) c_{11}, c_{12}, c_{13}, c_{14} are not joined to some r_i's $(i = 3, 4, 5, 6, 7)$ by red lines.

If c_{11}, c_{12}, c_{13}, c_{14} are not joined to r_3 for example, then either at least three red lines from r_3 are joined to points among c_1, c_2, c_3, c_4, c_5 or at least three red lines are joined to points among c_6, c_7, c_8, c_9, c_{10}. In either case, we can see that red C_4 is contained in the $K_{7,14}$, see Table 15 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	**1**	**1**	1	1	1	1								
r_2						1	1	1	1	1				
r_3	**1**	**1**	1						1	1				

Table 15:

(2) One of c_{11}, c_{12}, c_{13}, c_{14} is joined to some r_i's ($i = 3, 4, 5, 6, 7$) by red lines.

For example, suppose r_3 is joined to c_{11} by red line, then there are four other red lines joining r_3. From these four red lines, then either at least two red lines from r_3 are joined to points among c_1, c_2, c_3, c_4, c_5 or at least two red lines are joined to points among c_6, c_7, c_8, c_9, c_{10}. In either case, we can see that red C_4 is contained in the $K_{7,14}$, see Table 16 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	**1**	**1**	1	1	1	1								
r_2						1	1	1	1	1				
r_3	**1**	**1**							1	1	1			

Table 16:

(3) Two of c_{11}, c_{12}, c_{13}, c_{14} are joined to some r_i's ($i = 3, 4, 5, 6, 7$) by red lines.

For example, suppose r_3 is joined to c_{11}, c_{12} by red lines, then there are three other red lines joining r_3. From these three red lines, then either at least two red lines from r_3 are joined to points among c_1, c_2, c_3, c_4, c_5 or at least two red lines are joined to points among c_6, c_7, c_8, c_9, c_{10}. In either case, we can see that red C_4 is contained in the $K_{7,14}$, see Table 17 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	**1**	**1**	1	1	1	1								
r_2						1	1	1	1	1				
r_3	**1**	**1**								1	1	1		

Table 17:

(4) Three of c_{11}, c_{12}, c_{13}, c_{14} are joined to some r_i's ($i = 3, 4, 5, 6, 7$) by red lines.

Suppose r_3 is joined to c_{11}, c_{12}, c_{13} by red lines. Consider r_4 and the five red lines joining r_4. From these five red lines, if r_4 is joined to c_{14}, then either at least two red lines are joined to points among c_1, c_2, c_3, c_4, c_5 or at least two red lines are joined to points among c_6, c_7, c_8, c_9, c_{10} or at least two red lines are joined to points among c_{11}, c_{12}, c_{13}. In any case, we can see that red C_4 is contained in the $K_{7,14}$, see Table 18 for example. For the case when r_4 is not joined to c_{14}, we have that red C_4 is also formed.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	1	**1**	**1**	1	1	1								
r_2						1	1	1	1	1				
r_3	1									1	1	1	1	
r_4		**1**	**1**				1						1	1

Table 18:

(5) All of c_{11}, c_{12}, c_{13}, c_{14} are joined to some r_i's $(i = 3, 4, 5, 6, 7)$ by red lines.

Suppose r_3 is joined to c_{11}, c_{12}, c_{13}, c_{14} by red lines. Consider r_4 and the five red lines joining r_4. From these five red lines, then either at least two red lines are joined to points among c_1, c_2, c_3, c_4, c_5 or at least two red lines are joined to points among c_6, c_7, c_8, c_9, c_{10} or at least two red lines are joined to points among c_{11}, c_{12}, c_{13}, c_{14}. In any case, we can see that red C_4 is contained in the $K_{7,14}$, see Table 19 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	1	**1**	**1**	1	1	1								
r_2						1	1	1	1	1				
r_3	1										1	1	1	1
r_4		**1**	**1**				1						1	1

Table 19:

2.3.2. None of c_i's are joined by red lines to both of r_1 and r_2.

Suppose that r_1 is joined by six red lines to $c_1, c_2, c_3, c_4, c_5, c_6$ and r_2 is joined by five red lines to c_7, c_8, c_9, c_{10}, c_{11}. Similar to the cases in 2.1.1 of subcase 2.1, we have that the $K_{7,14}$ contains red C_4.

Subcase 2.4. $d_1 = 5, d_2 = 5, d_3 = 5, d_4 = 5, d_5 = 5, d_6 = 4, d_7 = 4$.

When there are two or more points c_i's each of which is joined to both of r_1 and r_2 by red lines, we can see that the $K_{7,14}$ contains red C_4, see Table 20 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	1	1	1	**1**	**1**									
r_2				**1**	**1**	1	1	1						

Table 20:

We consider two more possibilities 2.4.1 and 2.4.2.

2.4.1. One c_i is joined by red lines to both of r_1 and r_2.

Suppose that r_1 is joined by five red lines to c_1, c_2, c_3, c_4, c_5 and r_2 is joined by five red lines to c_5, c_6, c_7, c_8, c_9. We now consider six sub-possibilities (1), (2), (3), (4), (5), and (6).

(1) c_{10}, c_{11}, c_{12}, c_{13}, c_{14} are not joined to some r_i's $(i = 3, 4, 5, 6, 7)$ by red lines.

If c_{10}, c_{11}, c_{12}, c_{13}, c_{14} are not joined to r_3 for example, then either at least three red lines from r_3 are joined to points among c_1, c_2, c_3, c_4 or at least three red lines are joined to points among c_5, c_6, c_7, c_8, c_9. In either case, we can see that red C_4 is contained in the $K_{7,14}$, see Table 21 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	**1**	**1**	1	1	1									
r_2					1	1	1	1	1					
r_3	**1**	**1**	1					1	1					

Table 21:

(2) One of c_{10}, c_{11}, c_{12}, c_{13}, c_{14} is joined to some r_i's $(i = 3, 4, 5, 6, 7)$ by red lines.

For example, suppose r_3 is joined to c_{10} by red line, then there are four other red lines joining r_3. From these four red lines, then either at least two red lines from r_3 are joined to points among c_1, c_2, c_3, c_4 or at least two red lines are joined to points among c_5, c_6, c_7, c_8, c_9. In either case, we can see that red C_4 is contained in the $K_{7,14}$, see Table 22 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	**1**	**1**	1	1	1									
r_2					1	1	1	1	1					
r_3	**1**	**1**						1	1	1				

Table 22:

(3) Two of c_{10}, c_{11}, c_{12}, c_{13}, c_{14} are joined to some r_i's $(i = 3, 4, 5, 6, 7)$ by red lines.

For example, suppose r_3 is joined to c_{10}, c_{11} by red lines, then there are three other red lines joining r_3. From these three red lines, then either at least two red lines from r_3 are joined to points among c_1, c_2, c_3, c_4 or at least two red lines are joined to points among c_5, c_6, c_7, c_8, c_9. In either case, we can see that red C_4 is contained in the $K_{7,14}$, see Table 23 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	**1**	**1**	1	1	1									
r_2					1	1	1	1	1					
r_3	**1**	**1**							1	1	1			

Table 23:

(4) Three of $c_{10}, c_{11}, c_{12}, c_{13}, c_{14}$ are joined to some r_i's $(i = 3, 4, 5, 6, 7)$ by red lines.

Suppose r_3 is joined to c_{10}, c_{11}, c_{12} by red lines. Consider r_4 and the five red lines joining r_4. From these five red lines, if r_4 is not joined to c_{13} and c_{14}, then either at least two red lines are joined to points among c_1, c_2, c_3, c_4 or at least two red lines are joined to points among c_5, c_6, c_7, c_8, c_9 or at least two red lines are joined to points among c_{10}, c_{11}, c_{12}. In any case, we can see that red C_4 is contained in the $K_{7,14}$, see Table 24 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	1	**1**	**1**	1	1									
r_2					1	1	1	1	1					
r_3	1								1	1	1	1		
r_4		**1**	**1**			1	1			1				

Table 24:

If r_4 is joined to one of c_{13} and c_{14}, say c_{13}, then there are four other red lines joining r_4. From these four red lines, either at least two red lines are joined to points among c_1, c_2, c_3, c_4 or at least two red lines are joined to points among c_5, c_6, c_7, c_8, c_9 or at least two red lines are joined to points among c_{10}, c_{11}, c_{12}. In any case, we can see that red C_4 is contained in the $K_{7,14}$, see Table 25 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	1	**1**	**1**	1	1									
r_2					1	1	1	1	1					
r_3	1								1	1	1	1		
r_4		**1**	**1**			1				1			1	

Table 25:

If r_4 is joined to c_{13} and c_{14}, then consider r_5 and the five red lines joining r_5. From these five red lines, then either at least two red lines are joined to points among c_1, c_2, c_3, c_4 or at least two red lines are joined to points among c_5, c_6, c_7, c_8, c_9 or at least two red lines are joined to points among c_{10}, c_{11}, c_{12} or at least two red lines are joined to points c_{13} and c_{14}. In any case, we can see that red C_4 is contained in the $K_{7,14}$, see Table 26 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	1	1	**1**	**1**	1									
r_2					1	1	1	1	1					
r_3	1								1	1	1	1		
r_4		1				1				1			1	1
r_5			**1**	**1**			1				1		1	

Table 26:

(5) Four of c_{10}, c_{11}, c_{12}, c_{13}, c_{14} are joined to some r_i's ($i = 3, 4, 5, 6, 7$) by red lines.

Suppose r_3 is joined to c_{10}, c_{11}, c_{12}, c_{13} by red lines. Consider r_4 and the five red lines joining r_4. From these five red lines, if r_4 is not joined to c_{14}, then either at least two red lines are joined to points among c_1, c_2, c_3, c_4 or at least two red lines are joined to points among c_5, c_6, c_7, c_8, c_9 or at least two red lines are joined to points among c_{10}, c_{11}, c_{12}, c_{13}. In any case, we can see that red C_4 is contained in the $K_{7,14}$, see Table 27 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	1	**1**	**1**	1	1									
r_2					1	1	1	1	1					
r_3	1									1	1	1	1	
r_4		**1**	**1**			1				1	1			

Table 27:

If r_4 is joined to c_{14}, then there are four other red lines joining r_4. From these four red lines,then either at least two red lines are joined to points among c_1, c_2, c_3, c_4 or at least two red lines are joined to points among c_5, c_6, c_7, c_8, c_9 or at least two red lines are joined to points among c_{10}, c_{11}, c_{12}, c_{13}. In any case, we can see that red C_4 is contained in the $K_{7,14}$, see Table 28 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	1	**1**	**1**	1	1									
r_2					1	1	1	1	1					
r_3	1									1	1	1	1	
r_4		**1**	**1**			1				1				1

Table 28:

(6) All of c_{10}, c_{11}, c_{12}, c_{13}, c_{14} are joined to some r_i's ($i = 3, 4, 5, 6, 7$) by red lines.

Suppose r_3 is joined to c_{10}, c_{11}, c_{12}, c_{13}, c_{14} by red lines. Consider r_4 and the five red lines joining r_4. From these five red lines, then either at least two red lines are joined to points among c_1, c_2, c_3, c_4 or at least two red lines are joined to points among c_5, c_6, c_7, c_8, c_9 or at least two red lines are joined to points among c_{10}, c_{11}, c_{12}, c_{13}, c_{14}. In any case, we can see that red C_4 is contained in the $K_{7,14}$, see Table 29 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	**1**	**1**	1	1	1									
r_2					1	1	1	1	1					
r_3										1	1	1	1	1
r_4	**1**	**1**				1				1	1			

Table 29:

2.4.2. None c_i's are joined by red lines to both of r_1 and r_2.

Suppose that r_1 is joined by five red lines to c_1, c_2, c_3, c_4, c_5 and r_2 is joined by five red lines to c_6, c_7, c_8, c_9, c_{10}.

Similar to the cases in 2.3.1 of subcase 2.3, red C_4 is always contained in the $K_{7,14}$.

Therefore,

$$r_t(C_4; 3) \leq 7. \tag{2.6}$$

From the inequalities (2.4) and (2.6), we have $r_t(C_4; 3) = 7$ as required. ∎

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Δ-CONVERGENCE THEOREM FOR TOTAL ASYMPTOTICALLY NONEXPANSIVE MAPPING IN UNIFORMLY CONVEX HYPERBOLIC SPACES

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Abstract. Recently, Chang, et al introduce the concept of total asymptotically nonexpansive mapping which contain the asymptotically nonexpansive mapping. The purpose of the paper is to analyze a three-step iterative scheme for total asymptotically nonexpansive mapping in uniformly convex hyperbolic spaces. Meanwhile, we obtain a Δ-convergence theorem of the three-step iterative scheme for total asymptotically nonexpansive mapping in CAT(0) spaces. Ours results obtained in this paper extend and improve some previous known results.

Keywords and phrases: total asymptotically nonexpansive mapping, three-step iterations, uniformly convex hyperbolic spaces, Δ-convergence theorem, CAT(0) spaces.

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1. Introduction

Throughout this paper, (M, d) will stand for a metric space. For any $x, y \in M$, let $[x, y]$ be an isometric image of the real line interval $[0, d(x, y)]$. Suppose that there exists a family $\mathcal{F}$ of metric segments (or geodesic) such that any two points x, y in M are endpoints of a unique metric segment $[x, y] \in \mathcal{F}$. We shall denote by $(1-\beta)x \oplus \beta y$ the unique point z of $[x, y]$ which satisfies

$$d(x, z) = \beta d(x, y) \quad \text{and} \quad d(z, y) = (1-\beta)d(x, y).$$

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Such metric spaces are usually called convex metric spaces (see [10]). The (M,d) is said to be a geodesic space if every two points of M are joined by a geodesic. For a convex metric spaces (M,d), if

$$d\left(\frac{1}{2}p \oplus \frac{1}{2}x,\ \frac{1}{2}p \oplus \frac{1}{2}y\right) \le \frac{1}{2}d(x,y),$$

for all p, x, y in M, then M is said to be a hyperbolic metric space, for more details about hyperbolic metric space see ([9]). A subset C of a hyperbolic metric space M is convex if $[x,y] \subset C$ whenever x, y are in C. Obviously, normed linear spaces are hyperbolic spaces. One can consider, as nonlinear examples, the Hadamard manifolds ([2]), the Hilbert open unit ball equipped with the hyperbolic metric ([5]), and the CAT(0)spaces. Recall that a geodesic space is a CAT(0) space if and only if it satisfies the (CN) inequality ([13]):

$$d\left(z, \frac{1}{2}x \oplus \frac{1}{2}y\right)^2 \le \frac{1}{2}d(z,x)^2 + \frac{1}{2}d(z,y)^2 - \frac{1}{4}d(x,y)^2.$$

In particular, if x, y, z are points in a CAT(0) space and $t \in [0,1]$, then

$$d((1-t)x \oplus ty, z) \le (1-t)d(x,z) + td(y,z).$$

For more details about CAT(0) spaces, see [1].

In 2011, Khamsi and Khan (see [7]) introduced uniformly convex hyperbolic metric space.

Definition 1.1 Let (M,d) be a hyperbolic metric space, M is said to be uniformly convex, if for any $a \in M$, for every $r > 0$, and for each $\varepsilon > 0$,

$$\delta(r,\varepsilon) = \inf\left\{1 - \frac{1}{r}d\left(\frac{1}{2}x \oplus \frac{1}{2}y, a\right) :\ d(x,a) \le r,\ d(y,a) \le r,\ d(x,y) \ge r\varepsilon\right\} > 0.$$

If (M,d) is uniformly convex hyperbolic spaces, then for every $s \ge 0$, $\varepsilon > 0$, there exists $\eta(s,\varepsilon) > 0$ depending on s and ε such that

$$\delta(r,\varepsilon) > \eta(s,\varepsilon) > 0, \quad \text{for any } r > s.$$

CAT(0) spaces are uniformly convex hyperbolic spaces with $\delta(r,\varepsilon) = 1 - \sqrt{1 - \varepsilon^2/4}$. Uniformly convex hyperbolic spaces can be viewed as 2-uniformly convex (see [7]).

2. Preliminaries

Let us start by making some basic definitions.

Definition 2.1. ([4]) Let C be bounded subset of X, a mapping $T : X \to X$ is called asymptotically nonexpansive, if there exists a sequence $\{k_n\}$ of positive real numbers with $k_n \to 1$ as $n \to \infty$ for which

$$d(T^n x, T^n y) \le k_n d(x,y), \quad \text{for all } x,\ y \in X.$$

Definition 2.2. T is said to be uniformly L-Lipschitzian, if there exists a constant $L > 0$ such that

$$d(T^n x, T^n y) \leq L d(x, y), \quad \forall n \geq 1, x, y \in X.$$

Chang et al. (see [3]) recently introduce the concept of total asymptotically nonexpansive mappings and prove the demiclosed principle for this kind of mappings in CAT(0) spaces.

Definition 2.3. A mapping $T : X \to X$ is said to be $(\{\mu_n\}, \{\nu_n\}, \zeta)-$total asymptotically nonexpansive, if there exist nonnegative sequences $\{\mu_n\}, \{\nu_n\}$ with $\mu_n \to 0$, $\nu_n \to 0$ and a strictly increasing continuous function $\zeta : [0, +\infty) \to [0, +\infty)$ with $\zeta(0) = 0$ such that

$$d(T^n x, T^n y) \leq d(x, y) + \nu_n \zeta(d(x, y)) + \mu_n \quad \forall n \geq 1, x, y \in X.$$

Rermark 2.4. From the above definition, it is to know that, each nonexpansive mapping is a asymptotically nonexpansive mapping with sequence $\{k_n = 1\}$, and each asymptotically nonexpansive mapping is a $(\{\mu_n\}, \{\nu_n\}, \zeta)-$total asymptotically nonexpansive mapping with $\mu_n = 0, \nu_n = k_n - 1, \forall n \geq 1$ and $\zeta(t) = t, t \geq 0$.

Let $\{x_n\}$ be a bounded sequence in M and $C \subset M$ be a nonempty subset of M. The asymptotic radius of $\{x_n\}$ with respect to C is defined by

$$r(C, \{x_n\}) = \inf \left\{ \limsup_{n\to\infty} d(x, x_n) : \ x \in C \right\}.$$

The asymptotic radius of $\{x_n\}$, denoted by $r(\{x_n\})$, is the asymptotic radius of $\{x_n\}$ with respect to M. The asymptotic center of $\{x_n\}$ with respect to C is defined by

$$A(C, \{x_n\}) = \left\{ z \in C : \ \limsup_{n\to\infty} d(z, x_n) = r(\{C, x_n\}) \right\}.$$

When $C = M$, we call the asymptotic center of $\{x_n\}$ and use the notation $A(\{x_n\})$ for $A(C, \{x_n\})$.

Definition 2.5. ([11]) A sequence $\{x_n\}$ in (M, d) is said to Δ-converge to $x \in M$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\Delta - \lim\limits_{n\to\infty} x_n = x$ and call x the Δ-limit of $\{x_n\}$.

Every bounded sequence in a complete CAT(0) space always has a Δ-convergent subsequence ([8]). The proof of following lemma is implicit in the proof of Theorem 3.5 in [3].

Lemma 2.6. *Let C be a closed convex subset of a complete CAT(0) space M, and $T : C \to C$ be a uniformly L-Lipschitzian and $(\{\mu_n\}, \{\nu_n\}, \zeta)-$total asymptotically nonexpansive mapping. Suppose that $\{x_n\}$ is a bounded sequence in C*

such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ *and* $d(x_n, p)$ *converges for each* $p \in F(T)$*, then* $\omega_w(x_n) \subset F(T)$*. Here* $\omega_w(x_n) = \cup A(\{u_n\})$*, the union is taken over all subsequences* $\{u_n\}$ *of* $\{x_n\}$*. Moreover,* $\omega_w(x_n)$ *consists of exactly one point.*

In 2002, Xu and Noor [14] introduced and analyzed a three-step iterative schemes for solving the nonlinear equation $Tx = x$ for asymptotically nonexpansive mappings in Banach space. It has been shown that the three-step iterative scheme gives better numerical results than the two-step and one-step approximate iterations ([6]). Inspired and motivated by these facts, we analyze a three-step iterative scheme for mappings of total asymptotically nonexpansive in uniformly convex hyperbolic spaces.

Algorithm 1. Let C be a nonempty closed subset of a hyperbolic metric space (M, d) and $T : C \to C$ be a uniformly $L-$ Lipschitzian and $(\{\mu_n\}, \{\nu_n\}, \zeta)-$total asymptotically nonexpansive mapping. For a given $x_1 \in C$, compute sequences $\{z_n\}$,$\{y_n\}$,$\{x_n\}$ by the iterative schemes

$$\begin{aligned} z_n &= a_n T^n x_n \oplus (1 - a_n)x_n \\ y_n &= b_n T^n z_n \oplus (1 - b_n)x_n \\ x_{n+1} &= \alpha_n T^n y_n \oplus (1 - \alpha_n)x_n \end{aligned} \tag{2.1}$$

where $\{a_n\}$, $\{b_n\}$, $\{\alpha_n\}$ are real numbers in $[0, 1]$. Let $a_n = 0$ or $a_n = 0$ and $b_n = 0$, we get Ishikawa-type and Krasnoselski-Mann iteration as special cases.

$$\begin{aligned} y_n &= b_n T^n x_n \oplus (1 - b_n)x_n \\ x_{n+1} &= \alpha_n T^n y_n \oplus (1 - \alpha_n)x_n; \end{aligned} \tag{2.2}$$

$$x_{n+1} = \alpha_n T^n x_n \oplus (1 - \alpha_n)x_n; \tag{2.3}$$

For a suitable choice of $\{\alpha_n\}$ and $\{b_n\}$, we obtain a Δ-convergence theorem of the three-step iterative scheme for uniformly $L-$ Lipschitzian and $(\{\mu_n\}, \{\nu_n\}, \zeta)-$total asymptotically nonexpansive mapping in CAT(0) spaces.

3. Main results

The following Lemma is trivial (see [12]).

Lemma 3.1. *Let* $\{a_n\}$*,* $\{\lambda_n\}$ *and* $\{c_n\}$ *be the sequences of nonnegative numbers such that*

$$a_{n+1} \leq (1 + \lambda_n)a_n + c_n, \quad \forall\ n \geq 1.$$

If $\sum_{n=1}^{\infty} \lambda_n < \infty$ *and* $\sum_{n=1}^{\infty} c_n < \infty$*, then* $\lim_{n\to\infty} a_n$ *exists. If there exists a subsequence of* $\{a_n\}$ *which converges to 0, then* $\lim_{n\to\infty} a_n = 0$*.*

Lemma 3.2. *Let (M,d) be uniformly convex hyperbolic spaces and C be a bounded closed nonempty convex subset of M. Let $T : C \to C$ be a uniformly L-Lipschitzian and $(\{\mu_n\},\{\nu_n\},\zeta)-$total asymptotically nonexpansive mapping. If $\{a_n\}$, $\{b_n\}$, $\{\alpha_n\}$ are real numbers in $[0,1]$. Suppose that $x_1 \in C$ and $\{x_n\}$ is given by (2.1).*

(i) *If $\sum_{n=1}^{\infty} \nu_n < \infty$; $\sum_{n=1}^{\infty} \mu_n < \infty$;*

(ii) *there exists a constant $M^* > 0$ such that $\zeta(r) \le M^* r$, $r \ge 0$.*

Then $\lim_{n\to\infty} d(x_n,p)$ exists for each $p \in F(T)$.

Proof. Let $p \in F(T)$. From (2.1), we have

$$\begin{aligned} d(z_n,p) &= d(a_n T^n x_n \oplus (1-a_n)x_n, p) \\ &\le a_n d(T^n x_n, p) + (1-a_n) d(x_n,p) \\ &= a_n d(T^n x_n, T^n p) + (1-a_n) d(x_n,p) \\ &\le a_n(d(x_n,p) + \nu_n \zeta(d(x_n,p)) + \mu_n) + (1-a_n) d(x_n,p) \\ &\le (1+\nu_n M^*) d(x_n,p) + \mu_n \end{aligned} \tag{3.1}$$

$$\begin{aligned} d(y_n,p) &= d(b_n T^n z_n \oplus (1-b_n)x_n, p) \\ &\le b_n d(T^n z_n, p) + (1-b_n) d(x_n,p) \\ &= b_n d(T^n z_n, T^n p) + (1-b_n) d(x_n,p) \\ &\le b_n(d(z_n,p) + \nu_n \zeta(d(z_n,p)) + \mu_n) + (1-b_n) d(x_n,p) \\ &\le b_n((1+\nu_n M^*) d(x_n,p) + \mu_n + \nu_n M^* d(z_n,p) + \mu_n) \\ &\qquad\qquad + (1-b_n) d(x_n,p) \\ &\le (1+2\nu_n M^* + (\nu_n M^*)^2) d(x_n,p) + (\nu_n M^* + 2)\mu_n \end{aligned} \tag{3.2}$$

From (3.2), we get

$$\begin{aligned} d(x_{n+1},p) &= d(\alpha_n T^n y_n \oplus (1-\alpha_n)x_n, p) \\ &\le \alpha_n d(T^n y_n, p) + (1-\alpha_n) d(x_n,p) \\ &\le \alpha_n(d(y_n,p) + \nu_n \zeta(d(y_n,p)) + \mu_n) + (1-\alpha_n) d(x_n,p) \\ &\le (1+3\nu_n M^* + 3(\nu_n M^*)^2 + (\nu_n M^*)^3) d(x_n,p) \\ &\quad + (3+3\nu_n M^* + (\nu_n M^*)^2)\mu_n. \end{aligned} \tag{3.3}$$

In Lemma 3.1, take $a_n = d(x_n,p)$, $\lambda_n = 3\nu_n M^* + 3(\nu_n M^*)^2 + (\nu_n M^*)^3$ and $c_n = (3+3\nu_n M^* + (\nu_n M^*)^2)\mu_n$, then all conditions in Lemma 3.1 are satisfied. The conclusion is obtained from Lemma 3.1 immediately. ∎

Lemma 3.3. *Let (M,d) be uniformly convex hyperbolic spaces and C be a bounded closed nonempty convex subset of M. Let $T : C \to C$ be a uniformly L-Lipschitzian and $(\{\mu_n\},\{\nu_n\},\zeta)-$total asymptotically nonexpansive mapping. If $\{a_n\}$, $\{b_n\}$, $\{\alpha_n\}$ are real numbers in $[0,1]$ satisfying*

(i) $0 < \liminf_n \alpha_n \leq \limsup_n \alpha_n < 1$;

(ii) $0 < \liminf_n b_n \leq \limsup_n b_n < 1$;

(iii) $\sum_{n=1}^{\infty} \nu_n < \infty$; $\sum_{n=1}^{\infty} \mu_n < \infty$;

(iv) *there exists a constant* $M^* > 0$ *such that* $\zeta(r) \leq M^* r$, $r \geq 0$.

Suppose that $x_1 \in C$ *and* $\{x_n\}$ *is given by* (2.1). *Then*

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

Proof. Take a $p \in F(T)$, in view of Lemma 3.2, we can let $\lim_{n\to\infty} d(x_n, p) = r$ for some $r \in \mathbb{R}$. If $r = 0$, then we immediately obtain

$$d(x_n, Tx_n) \leq d(x_n, p) + d(Tx_n, p) = d(x_n, p) + d(Tx_n, Tp) \leq (1 + L)d(x_n, p),$$

by the uniformly L-Lipschitzian of T, then $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

If $r > 0$, then we shall prove that

$$\lim_{n\to\infty} d(T^n y_n, p) = \lim_{n\to\infty} d(T^n z_n, p) = r, \tag{3.4}$$

by showing that for any increasing sequence $\{n_i\}$ of positive integers for which the limits in (3.4) exist, and it follows that the limit is r. From condition (i), we may assume that the corresponding subsequence $\{\alpha_{n_i}\}$ converges to some α; we shall have $\alpha > 0$ because $\{\alpha_{n_i}\}$ is assumed to be bounded away from 0. From conditions (i) and (iii), we have

$$\begin{aligned}
r &= \lim_{n\to\infty} d(x_n, p) = \lim_{i\to\infty} d(x_{n_i+1}, p) \\
&= \lim_{i\to\infty} d\big(\alpha_{n_i} T^{n_i} y_{n_i} \oplus (1 - \alpha_{n_i}) x_{n_i}, p\big) \\
&\leq \lim_{i\to\infty} \big[\alpha_{n_i} d(T^{n_i} y_{n_i}, p) + (1 - \alpha_{n_i}) d(x_{n_i}, p)\big] \\
&\leq \lim_{i\to\infty} \big[\alpha_{n_i} (d(y_{n_i}, p) + \nu_n \zeta(d(y_{n_i}, p)) + \mu_n) + (1 - \alpha_{n_i}) d(x_{n_i}, p)\big] \\
&\leq \alpha \limsup_{i\to\infty} d(x_{n_i}, p) + (1 - \alpha) r \\
&= r.
\end{aligned}$$

It follows that $\lim_{n\to\infty} d(T^n y_n, p) = r$. From condition (ii), we may assume that the corresponding subsequence $\{b_{n_i}\}$ converges to some b; we shall have $b > 0$ because $\{b_{n_i}\}$ is assumed to be bounded away from 0. From (3.1) and conditions (ii), (iii), we have

$$
\begin{aligned}
r &= \lim_{n\to\infty} d(x_n,p) = \lim_{i\to\infty} d(x_{n_i+1},p)\\
&= \lim_{i\to\infty} d\big(\alpha_{n_i}T^{n_i}y_{n_i} \oplus (1-\alpha_{n_i})x_{n_i},p\big)\\
&\le \lim_{i\to\infty} \big[\alpha_{n_i} d(T^{n_i}y_{n_i},T^{n_i}p) + (1-\alpha_{n_i})d(x_{n_i},p)\big]\\
&\le \lim_{i\to\infty} \big[\alpha_{n_i}(d(y_{n_i},p)+\nu_{n_i}\zeta(d(y_{n_i},p))+\mu_{n_i}) + (1-\alpha_{n_i})d(x_{n_i},p)\big]\\
&\le \alpha \lim_{i\to\infty}(b_{n_i}(d(T^{n_i}z_{n_i},p)+(1-b_{n_i})d(x_{n_i},p)+\nu_n M^* d(y_{n_i},p)+\mu_{n_i})\\
&\qquad\qquad +(1-\alpha)d(x_{n_i},p)\\
&= \alpha \lim_{i\to\infty}(b_{n_i}d(T^n z_{n_i},T^{n_i}p)+(1-b_{n_i})d(x_{n_i},p))+(1-\alpha)d(x_{n_i},p)\\
&\le \alpha(b\lim_{i\to\infty}(d(z_{n_i},p)+\nu_{n_i}\zeta(d(z_{n_i},p))+\mu_{n_i})+(1-b)d(x_{n_i},p))\\
&\qquad\qquad +(1-\alpha)d(x_{n_i},p)\\
&\le \alpha(b\lim_{i\to\infty}(d(x_{n_i},p)+(1-b)d(x_{n_i},p))+(1-\alpha)d(x_{n_i},p)\\
&= r.
\end{aligned}
$$

It follows that $\lim_{n\to\infty} d(T^n z_n,p) = r$ holds. In addition, it is easy to see that

$$\lim_{n\to\infty} d\left(\frac{x_n \oplus T^n x_n}{2},p\right) = r.$$

In the sequel, we shall prove

$$\lim_{n\to\infty} d(T^n y_n,x_n) = \lim_{n\to\infty} d(T^n z_n,x_n) = 0.$$

Assume by contradiction that $\{T^n y_n\}$ does not converge $\{x_n\}$, and so we can find $\varepsilon_0 > 0$ and $\{n_k\} \subset \mathbb{N}$ such that

$$d(T^{n_k}y_{n_k},x_{n_k}) \ge \varepsilon_0. \tag{3.5}$$

We can assume $\varepsilon_0 \in (0,2]$, then $\dfrac{\varepsilon_0}{r+1} \in (0,2]$. Since M is uniformly convex, there exits $\theta \in (0,1]$ such that

$$1-\delta\left(r+\theta,\frac{\varepsilon_0}{r+1}\right) \le 1-\eta\left(r,\frac{\varepsilon_0}{r+1}\right) \le \frac{r-\theta}{r+\theta} \tag{3.6}$$

By (3.4), for the above $\theta > 0$, there exists $N_0 \in \mathbb{N}$ such that

$$d(x_{n_k},p) \le r+\theta, d(T^{n_k}y_{n_k},p) \le r+\theta \quad \forall k \ge N_0.$$

For $k \ge N_0$, we also have that

$$d\left(T^{n_k}y_{n_k},x_{n_k}\right) \ge \varepsilon_0 = (r+\theta)\frac{\varepsilon_0}{r+\theta} \ge (r+\theta)\frac{\varepsilon_0}{r+1}.$$

Now, applying the fact that M is uniformly convex and (3.6), we get that

$$\begin{aligned} d\left(\frac{x_{n_k} \oplus T^{n_k} x_{n_k}}{2}, p\right) &\leq \left(1-\delta\left(r+\theta, \frac{\varepsilon_0}{r+1}\right)\right)(r+\theta) \\ &< r-\theta. \end{aligned}$$

Let $n_k \to \infty$, we obtain that

$$r = \lim_{n_k \to \infty} d\left(\frac{x_{n_k} \oplus T^{n_k} x_{n_k}}{2}, p\right) \leq r-\theta.$$

Hence, we get a contradiction, and therefore

$$\lim_{n\to\infty} d(T^n y_n, x_n) = 0. \tag{3.7}$$

This is equivalent to

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{3.8}$$

Using the same way, we can prove that

$$\lim_{n\to\infty} d(T^n z_n, x_n) = 0 \tag{3.9}$$

This is equivalent to $\lim\limits_{n\to\infty} d(y_n, x_n) = 0$ and meanwhile we get

$$\lim_{n\to\infty} d(y_{n+1}, y_n) = 0 \tag{3.10}$$

Thus

$$\begin{aligned} d(x_n, Tx_n) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, T^{n+1} y_{n+1}) \\ &+ d(T^{n+1} y_{n+1}, T^{n+1} y_n) + d(T(T^n y_n), Tx_n) \\ &\leq d(x_n, x_{n+1}) + d(x_{n+1}, T^{n+1} y_{n+1}) \\ &+ Ld(y_{n+1}, y_n) + Ld(T^n y_n, x_n) \end{aligned}$$

By (3.7), (3.8), (3.10) and the uniformly L-Lipschitzian of T, we conclude that

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0. \qquad \blacksquare$$

Theorem 3.4. *Let (M,d) be a completely $CAT(0)$ spaces and C be a bounded closed nonempty convex subset of M. Let $T : C \to C$ be a uniformly L-Lipschitzian and $(\{\mu_n\}, \{\nu_n\}, \zeta)$-total asymptotically nonexpansive mapping. If $F(T) \neq \emptyset$, $\{a_n\}$, $\{b_n\}$, $\{\alpha_n\}$ are real numbers in $[0,1]$ satisfying*

(i) $0 < \liminf\limits_n \alpha_n \leq \limsup\limits_n \alpha_n < 1$;

(ii) $0 < \liminf\limits_n b_n \leq \limsup\limits_n b_n < 1$;

(iii) $\sum_{n=1}^{\infty} \nu_n < \infty$; $\sum_{n=1}^{\infty} \mu_n < \infty$;

(iv) *there exists a constant* $M^* > 0$ *such that* $\zeta(r) \leq M^* r$, $r \geq 0$.

Supposing that $x_1 \in C$ *and* $\{x_n\}$ *is given by* (2.1), Δ*-converges to a fixed point of* T.

Proof. Since CAT(0) spaces are uniformly convex hyperbolic spaces, by Lemma 3.2, $\{d(x_n, p)\}$ is convergent for each $p \in F(T)$. By Lemma 3.3, we have $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. By Lemma 2.6, $\omega_w(x_n)$ consists of exactly one point and is contained in $F(T)$. This shows that $\{x_n\}$ Δ-converges to an element of $F(T)$.■

Remark 3.5.

1. Since normed linear spaces are hyperbolic spaces and total asymptotically nonexpansive mapping which contain the asymptotically nonexpansive mapping. Therefore, Lemma 3.3 extend Lemma 2.2 of Xu and Noor [14].

2. Let $a_n = 0$ or $a_n = 0$ and $b_n = 0$, we get the Ishikawa-type and the Krasnoselski-Mann iteration as special cases, therefore Lemma 3.3 extends Theorem 5.2 of U. Kohlenbach (see [9]) and Lemma 5.4, Theorem 5.7 in [11]. Theorem 3.4 gives some new Δ-convergence theorem different from Theorem 3.5 ([3]) in CAT(0) spaces.

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QUOTIENT RINGS VIA FUZZY CONGRUENCE RELATIONS

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Abstract. This paper aims to introduce fuzzy congruence relations over rings and give constructions of quotient rings induced by fuzzy congruence relations. The Fuzzy First, Second and Third Isomorphism Theorems of rings are established. Finally, we investigate the relationships between fuzzy ideals and fuzzy congruence relations on rings.

Keywords: ring; fuzzy congruence relation; fuzzy ideal; quotient ring.

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1. Introduction

Fuzzy set theory, proposed by L.A. Zadeh [14], has been extensively applied to many scientific fields. In fact, the field grew enormously, and applications were found in areas by many authors (see [1], [13]) as diverse as washing machines to handwriting recognition and other applications.

Following the discovery of fuzzy sets, much attention has been paid to generalize the basic concepts of classical algebra in a fuzzy framework, and thus developing a theory of fuzzy algebras. In recent years, much interest is shown to generalize algebraic structures of groups, rings, modules, etc. The notion of fuzzy ideals of a ring R was put forward and the operations on fuzzy ideals was discussed by several researchers (see, e.g., [4], [5], [6], [7]). Fuzzy congruence relations and fuzzy normal subgroups on groups was shown by N. Kuroki [3]. Later on, L. Filep and I. Maurer [2] and by V. Murali [11] further studied fuzzy congruence relations

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on universal algebras. Fuzzy isomorphism theorems of soft rings were shown by X.P. Liu [8], [9]. General algebraic structure, such as group and ring of congruence relations and ideals to depict the algebraic structure has played a very important role. The various constructions of quotient groups and quotient rings by fuzzy ideals was introduced by Y.L. Liu [7]. Moreover, N. Kuroki has been shown that there exists a one-to-one mapping from all fuzzy normal subgroups and all fuzzy congruence relations of groups. Naturally, the study of the definition and properties about fuzzy congruence relations on rings is a meaningful work.

In this paper, we introduce the notion of fuzzy congruence relations on rings and introduce the notion of quotient rings by fuzzy congruence relations and give the Fuzzy First, Second and Third Isomorphism Theorems of rings based on fuzzy congruence relation. Moreover, we give some properties between fuzzy ideals and fuzzy congruence relations on rings.

2. Preliminaries

From the properties of fuzzy set theory, we know that a fuzzy set defined on a set as follows: let R be a non-empty set, then $\mu : R \to [0,1]$ is called a fuzzy set of R. In this paper, R is always a ring.

Definition 2.1 [10]

(1) A fuzzy set μ of R is called a fuzzy left (resp., right) ideal of R if it satisfies:

  (i) $\mu(x-y) \geq \mu(x) \wedge \mu(y)$ for all x, y of R,

  (ii) $\mu(xy) \geq \mu(y)$ (resp., $\mu(xy) \geq \mu(x)$) for all x, y of R.

(2) A fuzzy set μ of R is called a fuzzy ideal of R if it is both a fuzzy left ideal and a fuzzy right ideal of R.

Clearly, let μ be a fuzzy set of R, if it satisfies $\mu(xy) \geq \mu(x) \vee \mu(y)$, then μ is a fuzzy ideal of R. We denote the set of all fuzzy ideals of R by $FI(R)$.

Definition 2.2 [10] Let μ and ν be two fuzzy sets of R. Then the product $\mu+\nu$ is defined by the following:

$$(\mu+\nu)(z) = \bigvee_{x+y=z} [\mu(x) \wedge \nu(y)],$$

and

$$(\mu+\nu)(z) = 0$$

if z cannot be expressed as $z = x+y$, for all x, y and z of R.

Definition 2.3 [11]

(1) A function α from $R \times R$ to the unit interval [0,1] is called a fuzzy relation on R. Let α and β be two fuzzy relations on R, then the product $\alpha \circ \beta$ is defined by the following:

$$(\alpha \circ \beta)(a, b) = \bigvee_{x \in R} [\alpha(a, x) \wedge \beta(x, b)]$$

for all a, b of R.

(2) Let α and β be two fuzzy relations on R, then the product $\alpha \cap \beta$ is defined by the following way:

$$(\alpha \cap \beta)(x, y) = \alpha(x, y) \wedge \beta(x, y),$$

$$(\alpha \cup \beta)(x, y) = \alpha(x, y) \vee \beta(x, y).$$

Definition 2.4 [12] A relation α on the set R is called left compatible if $(a, b) \in \alpha$ implies $(x + a, x + b) \in \alpha$ and $(xa, xb) \in \alpha$, for all a, b, x of R, and is called right compatible if $(a, b) \in \alpha$ implies $(a + x, b + x) \in \alpha$ and $(ax, bx) \in \alpha$, for all a, b, x of R.

Remark 2.5 For any relation α on the set R

(i) It is called compatible if $(a, b) \in \alpha$ and $(c, d) \in \alpha$ implies $(a + c, b + d) \in \alpha$ and $(ac, bd) \in \alpha$, for all a, b, c, d of R,

(ii) A left (right) compatible equivalence relation on R is called a left (right) congruence relation on R,

(iii) A compatible equivalence relation on R is called a congruence relation on R.

As is well known (see [3]), a relation α on R is a congruence relation if and only if it is both a left and a right congruence relation on R.

3. Fuzzy congruence relations

In this section, we introduce the notion of fuzzy congruence relations on rings and give some properties about fuzzy congruence relations.

Definition 3.1 [3] A fuzzy set α of $R \times R$ is called a fuzzy relation on R. A fuzzy relation α on R is called a fuzzy equivalence relation if it satisfies the following conditions:

(i) $\alpha(x, x) = 1$ for all x of R (fuzzy reflexive),

(ii) $\alpha(x, y) = \alpha(y, x)$ for all of R (fuzzy symmetric),

(iii) $\alpha(x, y) \geq \bigvee_{z \in R} [\alpha(x, z) \wedge \alpha(z, y)]$ for all x, y of R (fuzzy transitive).

We note that α is fuzzy transitive if and only if $\alpha \supset \alpha \circ \alpha$.

Definition 3.2 [12] A fuzzy relation α on R is called a fuzzy left compatible relation if $\alpha(x+a, x+b) \geq \alpha(a,b)$ and $\alpha(xa, xb) \geq \alpha(a,b)$ for all x, a, b of R, and is called a fuzzy right compatible relation if $\alpha(a+x, b+x) \geq \alpha(a,b)$ and $\alpha(ax, bx) \geq \alpha(a,b)$ for all x, a, b of R. It is called a fuzzy compatible relation if $\alpha(a+c, b+d) \geq \alpha(a,b) \wedge \alpha(c,d)$ and $\alpha(ac, bd) \geq \alpha(a,b) \wedge \alpha(c,d)$.

Remark 3.3 A fuzzy relation on R is called a fuzzy compatible relation if and only if it is both a left and a right fuzzy compatible relation on R.

Proposition 3.4 Let α and β be any fuzzy compatible relations on R. Then $\alpha \circ \beta$ is also a fuzzy compatible relation on R.

Proof. For every $a, b, x \in R$. Since α and β are fuzzy compatible relations, we have

$$\begin{aligned}(\alpha \circ \beta)(x+a, x+b) &= \bigvee_{z \in R} [\alpha(x+a, z) \wedge \beta(z, x+b)] \\ &\geq [\alpha(x+a, x+z) \wedge \beta(x+z, x+b)] \\ &\geq [\alpha(a, z) \wedge \beta(z, b)].\end{aligned}$$

Then we have

$$\begin{aligned}(\alpha \circ \beta)(x+a, x+b) &\geq \bigvee_{z \in R} [\alpha(a, z) \wedge \beta(z, b)] \\ &= (\alpha \circ \beta)(a, b),\end{aligned}$$

$$\begin{aligned}(\alpha \circ \beta)(xa, xb) &= \bigvee_{z \in R} [\alpha(xa, z) \wedge \beta(z, xb)] \\ &\geq [\alpha(xa, xz) \wedge \beta(xz, xb)] \\ &\geq [\alpha(a, z) \wedge \beta(z, b)].\end{aligned}$$

Hence

$$\begin{aligned}(\alpha \circ \beta)(xa, xb) &\geq \bigvee_{z \in R} [\alpha(a, z) \wedge \beta(z, b)] \\ &= (\alpha \circ \beta)(a, b).\end{aligned}$$

This means that $\alpha \circ \beta$ is a fuzzy left compatible relation. It can be seen in a similar way that $\alpha \circ \beta$ is a fuzzy right compatible relation. Thus we obtain that $\alpha \circ \beta$ is a fuzzy compatible relation.

Definition 3.5 [12] A fuzzy equivalence relation α on R is called a fuzzy congruence relation if the following conditions are satisfied for all x, y, z, t of R

(i) $\alpha(x+y, z+t) \geq \alpha(x,z) \wedge \alpha(y,t)$,

(ii) $\alpha(xy, zt) \geq \alpha(x,z) \wedge \alpha(y,t)$.

We denote the set of all fuzzy congruence relations on R by $FC(R)$.

Example 3.6 Let $\mathbb{Z}$ be the set of all integers. Then $\mathbb{Z}$ is a ring with respect to the usual addition and multiplication of numbers. The fuzzy relation α on $\mathbb{Z}$ defined by

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x = y, \\ 0.5 & \text{if } x \neq y \text{ and both } x, y \text{ are either or odd,} \\ 0 & \text{otherwise.} \end{cases}$$

is a fuzzy congruence relation on $\mathbb{Z}$.

Proposition 3.7 [12] *Let α be a fuzzy congruence relation on R. Then for all $x, y, z \in R$ we have the following results:*

(i) $\alpha(x,y) \geq \alpha(x+z, y+z) \wedge \alpha(xz, yz) \wedge \alpha(zx, zy)$,

(ii) $\alpha(-x, -y) = \alpha(x,y)$.

Proposition 3.8 [3] *Let α and β be fuzzy congruence relations on R. Then $\alpha \circ \beta$ is a fuzzy congruence relation on R if and only if $\alpha \circ \beta = \beta \circ \alpha$.*

Let α be a fuzzy relation on R. For each $\lambda \in [0,1]$, we put

$$R_\alpha(\lambda) = \{(a,b) : (a,b) \in R \times R, \alpha(a,b) \geq \lambda\}.$$

This set is called the λ-level set of α.

Theorem 3.9 *A fuzzy relation α is a fuzzy congruence relation on R if and only if for each $\lambda \in [0,1]$, the λ-level set $R_\alpha(\lambda)$ is a congruence relation on R.*

Proof. Since α is a fuzzy congruence relation on R, then $\alpha(x,x) = 1$, for every $x \in R$, we have $(x,x) \in R_\alpha(\lambda)$, which means $R_\alpha(\lambda)$ is a reflexive relation.

For all $(x,y) \in R_\alpha(\lambda), \alpha(x,y) = \alpha(y,x) \geq \lambda$, i.e. $(y,x) \in R_\alpha(\lambda)$, $R_\alpha(\lambda)$ is a symmetric relation.

For each $(x,y),(z,y) \in R_\alpha(\lambda)$, $\alpha(x,z) \geq \lambda$, $\alpha(z,y) \geq \lambda$, since $\alpha(x,y) \geq \bigvee_{z \in R} [\alpha(x,z) \wedge \alpha(z,y)]$, then

$$\alpha(x,y) \geq \lambda, \text{ i.e., } (x,y) \in R_\alpha(\lambda).$$

Therefore, $R_\alpha(\lambda)$ is a transitive relation. Consequently, $R_\alpha(\lambda)$ is a equivalence relation on R.

For every $(x,y),(a,b) \in R_\alpha(\lambda)$, since α is a fuzzy congruence relation on R, then we have $\alpha(a+x, b+y) \geq \alpha(a,b) \wedge \alpha(x,y) \geq \lambda$, $\alpha(ax, by) \geq \alpha(a,b) \wedge \alpha(x,y) \geq \lambda$, i.e., $(a+x, b+y) \in R_\alpha(\lambda)$ and $(ax, by) \in R_\alpha(\lambda)$.

Thus $R_\alpha(\lambda)$ is a compatible relation on R, which implies, $R_\alpha(\lambda)$ is a congruence relation on R.

Conversely, let $\lambda \in [0,1]$, since $R_\alpha(\lambda)$ is a congruence relation on R, then for all $x \in R, \alpha(x,x) \geq \lambda$, implies $\alpha(x,x) = 1$, i.e., α is a fuzzy reflexive relation. For each $x, y \in R$, if $\alpha(x, y) \neq \alpha(y, x)$, let $\alpha(x, y) = \lambda_1, \alpha(y, x) = \lambda_2$, if $\lambda_1 > \lambda_2$, then $(y, x) \notin R_\alpha(\lambda_1)$, but $(x, y) \in R_\alpha(\lambda_1)$, since $R_\alpha(\lambda_1)$ is a congruence relation, hence $(y, x) \in R_\alpha(\lambda_1)$, contradiction.

So $\alpha(x, y) = \alpha(y, x)$. When $\lambda_1 < \lambda_2$, the proof of it is similarly. Hence α is fuzzy a symmetric relation. For each $x, y, z \in R$, let $\alpha(x, z) = t_1, \alpha(z, y) = t_2$, if $t_1 \leq t_2$ then $(x, z), (z, y) \in R_\alpha(t_1)$. Since $R_\alpha(t_1)$ is a congruence relation on R, hence $(x, y) \in R_\alpha(t_1)$, and $\alpha(x, y) \geq t_1 = \bigvee_{z \in R} [\alpha(x, z) \wedge \alpha(z, y)]$, thus α is a fuzzy transitive relation. When $t_1 \geq t_2$, the proof of it is similarly, hence α is a fuzzy equivalence relation on R.

For each $a, b, x, y \in R$, let $\alpha(a, b) = s, \alpha(x, y) = t$, if $s \leq t$, then $(a, b) \in R_\alpha(s), (x, y) \in R_\alpha(t) \leq R_\alpha(s)$. Since $R_\alpha(s)$ is a congruence relation on R, hence $\alpha(a + x, b + y) \in R_\alpha(s), (ax, by) \in R_\alpha(s)$, we have $\alpha(a + x, b + y) \geq s = \alpha(a, b) \wedge \alpha(x, y), \alpha(ax, by) \geq s = \alpha(a, b) \wedge \alpha(x, y)$. Hence α is a fuzzy compatible relation on R. When $s \geq t$, the proof is similar. It follows that α is a fuzzy congruence relation on R. This completes the proof.

4. Quotient rings induced by fuzzy congruence relations

In this section, we introduce the notion of quotient rings by fuzzy congruence relations and give the Fuzzy First, Second and Third Isomorphism Theorems of rings by means of fuzzy congruence relations.

Definition 4.1 Let α be a fuzzy congruence relation on R. For every element $x \in R$, we define a subset

$$\alpha_x = \{y \in R | \alpha(x, y) = 1\}$$

of R and $R/\alpha = \{\alpha_x | x \in R\}$.

Theorem 4.2 *If α is a fuzzy congruence relation of R, then R/α is a ring under the binary operations defined by*

$$\alpha_x + \alpha_y = \alpha_{x+y} \text{ and } \alpha_x \alpha_y = \alpha_{xy}$$

for any $x, y \in R$.

Proof. First, we show that the above binary operations are well-defined. In fact, if $\alpha_x = \alpha_{x'}$ and $\alpha_y = \alpha_{y'}$, then we have $\alpha(x, x') = 1$ and $\alpha(y, y') = 1$. Since $\alpha(x, x') \leq \alpha(x + y, x' + y)$ and $\alpha(y, y') \leq \alpha(x' + y, x' + y')$ so

$$\begin{aligned}
\alpha(x + y, x' + y') &\geq \bigvee_{z \in R} [\alpha(x + y, z) \wedge \alpha(z, x' + y')] \\
&\geq \alpha(x + y, x' + y) \wedge \alpha(x' + y, x' + y') \\
&\geq \alpha(x, x') \wedge \alpha(y, y') \\
&= 1.
\end{aligned}$$

Then we have $\alpha(x+y, x'+y') = 1$. This means $\alpha_{x+y} = \alpha_{x'+y'}$.
Again, since $\alpha(x, x') \leq \alpha(xy, x'y)$ and $\alpha(y, y') \leq \alpha(x'y, x'y')$,

$$\begin{aligned} \alpha(xy, x'y') &\geq \bigvee_{z \in R} [\alpha(xy, z) \wedge \alpha(z, x'y')] \\ &\geq \alpha(xy, x'y) \wedge \alpha(x'y, x'y') \\ &\geq \alpha(x, y') \wedge \alpha(y, y') \\ &= 1. \end{aligned}$$

Therefore, $\alpha(xy, x'y') = 1$, this means $\alpha_{xy} = \alpha_{x'y'}$. Hence addition and multiplication are well-defined. Because it is a routine matter to verify that the set R/α is a ring under the binary defined above and we omit its proof.

Notation 4.3 If α is a fuzzy congruence relation on R, then R/α is a ring which has the zero element α_0.

Lemma 4.4 *Let R, R' be rings and f be a homomorphism from R to R'. If α' is a fuzzy congruence relation of R', then the map $f^{-1}(\alpha')$ defined by*

$$f^{-1}(\alpha')(x, y) = \alpha'(f(x), f(y))$$

for all $x, y \in R$ is a fuzzy congruence relation of R.

Proof. For all $x, y, z, \in R$, then

$$f^{-1}(\alpha')(x, x) = 1,\ f^{-1}(\alpha')(x, y) = \alpha'(f(x), f(y)) = \alpha'(f(y), f(x)) = f^{-1}(\alpha')(y, x),$$

which means $f^{-1}(\alpha')$ is a fuzzy reflexive relation and fuzzy symmetric relation of R. Since

$$\begin{aligned} f^{-1}(\alpha')(x, y) &= \alpha'(f(x), f(y)) \\ &\geq \bigvee_{f(z) \in R'} [\alpha'(f(x), f(z)) \wedge \alpha'(f(z), f(y))] \\ &\geq \alpha'(f(x), f(z)) \wedge \alpha'(f(z), f(y)) \\ &= f^{-1}(\alpha')(x, z) \wedge f^{-1}(\alpha')(z, y) \\ &\geq \bigvee_{z \in R} [f^{-1}(\alpha')(x, z) \wedge f^{-1}(\alpha')(z, y)]. \end{aligned}$$

This means $f^{-1}(\alpha')$ is a fuzzy transitive relation of R. So $f^{-1}(\alpha')$ is a fuzzy equivalence relation of R. Again

$$\begin{aligned} f^{-1}(\alpha')(z+x, z+y) &= \alpha'(f(z+x), f(z+y)) \\ &= \alpha'(f(z)+f(x), f(z)+f(y)) \\ &\geq \alpha'(f(x), f(y)) \\ &= f^{-1}(\alpha')(x, y), \end{aligned}$$

$$\begin{aligned} f^{-1}(\alpha')(zx, zy) &= \alpha'(f(zx), f(zy)) \\ &= \alpha'(f(z)f(x), f(z)f(y)) \\ &\geq \alpha'(f(x), f(y)) \\ &= f^{-1}(\alpha')(x, y). \end{aligned}$$

This means that $f^{-1}(\alpha')$ is a fuzzy left compatible relation of R. By the same argument, we can see that $f^{-1}(\alpha')$ a fuzzy right compatible relation of R. Thus we obtain that $f^{-1}(\alpha')$ is a fuzzy congruence relation of R. This completes the proof.

Theorem 4.5 (Fuzzy First Isomorphism Theorem) *Let R, R' be rings, f be an epimorphism from R to R', and α' be a fuzzy congruence relation of R. Then*

$$R/f^{-1}(\alpha') \cong R'/\alpha'.$$

Proof. It follows from Theorem 4.2 and Lemma 4.4, $R/f^{-1}(\alpha')$ and R'/α' are both rings. We define a map h from $R/f^{-1}(\alpha')$ to R'/α' by

$$h(f^{-1}(\alpha')_x) = \alpha'_{f(x)} \text{ for all } x \in R.$$

(i) h is well-defined: If $f^{-1}(\alpha')_x = f^{-1}(\alpha')_y$, then $f^{-1}(\alpha')(x, y) = 1$. By Definition 4.1, we have $\alpha'(f(x), f(y)) = 1$, which implies $\alpha'_{f(x)} = \alpha'_{f(y)}$. Therefore, h is well-defined.

(ii) h is a homomorphism: $h(f^{-1}(\alpha')_x + f^{-1}(\alpha')_y) = h(f^{-1}(\alpha')_{x+y}) = \alpha'_{f(x+y)}$ $= \alpha'_{f(x)+f(y)} = \alpha'_{f(x)} + \alpha'_{f(y)}$; $h(f^{-1}(\alpha')_x f^{-1}(\alpha')_y) = h(f^{-1}(\alpha')_{xy}) = \alpha'_{f(xy)}$ $= \alpha'_{f(x)f(y)} = \alpha'_{f(x)}\alpha'_{f(y)}$. This implies that h is a homomorphism.

(iii) h is an epimorphism: For any $\alpha'_y \in R'/\alpha'$, since f is epimorphic, there exists $x \in R$ such that $f(x) = y$. So $h(f^{-1}(\alpha')_x) = \alpha'_{f(x)} = \alpha'_y$.

(iv) h is a monomorphism: Suppose that $h(f^{-1}(\alpha')_x) = h(f^{-1}(\alpha')_y)$, then $\alpha'_{f(x)} = \alpha'_{f(y)}$, which implies $\alpha'(f(x), f(y)) = 1$. From Lemma 4.4, we have $f^{-1}(\alpha')(x, y) = 1$. Hence $f^{-1}(\alpha')_x = f^{-1}(\alpha')_y$. This means h is a monomorphism.

In conclusion, $R/f^{-1}(\alpha') \cong R'/\alpha'$.

Corollary 4.6 *Let α be a fuzzy congruence relation of R, then the mapping $f : R \to R/\alpha$ defined by $f(x) = \alpha_x$ for all $x \in R$, is a homomorphism.*

Lemma 4.7 *Let α be a fuzzy congruence relation of R, we denote*

$$R_\alpha = \{y \in R | \alpha(0, y) = 1\}.$$

Then R_α is an ideal of R.

Proof. It is clear.

Lemma 4.8 *Let I be an ideal of R, α and β are fuzzy congruence relations of R.*

(i) *If α is restricted to I, then α is a fuzzy congruence relation of I,*

(ii) *$\alpha \cap \beta$ is fuzzy congruence relation of R,*

(iii) *I/α is an ideal of R/α.*

Proof. (i) Obviously.

(ii) For any $x, y \in R$, since $(\alpha \cap \beta)(x, y) = \alpha(x, y) \wedge \beta(x, y)$, $\alpha \cap \beta$ is a fuzzy reflexive relation and a fuzzy symmetric relation, we only show $\alpha \cap \beta$ is a fuzzy transitive relation. Since

$$\begin{aligned}
(\alpha \cap \beta)(x, y) &= \alpha(x, y) \wedge \beta(x, y) \\
&\geq \alpha(x, z) \wedge \alpha(z, y) \wedge \beta(x, z) \wedge \beta(z, y) \\
&= (\alpha \cap \beta)(x, z) \wedge (\alpha \cap \beta)(z, y) \\
&\geq \bigvee_{z \in R} [(\alpha \cap \beta)(x, z) \wedge (\alpha \cap \beta)(z, y)].
\end{aligned}$$

It follows from Definition 3.1 that $\alpha \cap \beta$ is a fuzzy transitive relation. Therefore $\alpha \cap \beta$ is a fuzzy equivalence relation of R.

Again, for every $a \in R$, since

$$\begin{aligned}
(\alpha \cap \beta)(a + x, a + y) &= \alpha(a + x, a + y) \wedge \beta(a + x, a + y) \\
&\geq \alpha(x, y) \wedge \beta(x, y) \\
&= (\alpha \cap \beta)(x, y),
\end{aligned}$$

$$\begin{aligned}
(\alpha \cap \beta)(ax, ay) &= \alpha(ax, ay) \wedge \beta(ax, ay) \\
&\geq \alpha(x, y) \wedge \beta(x, y) \\
&= (\alpha \cap \beta)(x, y).
\end{aligned}$$

This means $\alpha \cap \beta$ is a fuzzy left compatible relation. Similarly $\alpha \cap \beta$ is a fuzzy right compatible relation. Hence $\alpha \cap \beta$ is a fuzzy congruence relation of R.

(iii) First, we show that $\{\alpha_a \mid a \in I\}$ is an ideal of R/α. For any $\alpha_a, \alpha_b \in \{\alpha_a \mid a \in I\}$, where $a, b \in I$. Since I is an ideal, $a - b \in I$, hence $\alpha_a - \alpha_b = \alpha_{a-b} \in \{\alpha_a \mid a \in I\}$. For any $\alpha_a \in \{\alpha_a \mid a \in I\}, \alpha_x \in R/\alpha$, where $a \in I$ and $x \in R$, then $ax, xa \in I$, hence $\alpha_a \alpha_x = \alpha_{ax} \in \{\alpha_a \mid a \in I\}$ and $\alpha_x \alpha_a = \alpha_{xa} \in \{\alpha_a \mid a \in I\}$. Thus $\{\alpha_a \mid a \in I\}$ is an ideal of R/α.

Next, we define $\varphi : I/\alpha \to R/\alpha$ by

$$(\alpha|I)_a \mapsto \alpha_a \text{ for all } (\alpha|I)_a \in I/\alpha.$$

It is easy to verify that $I/\alpha \cong \{\alpha_a \mid a \in I\}$ under the mapping. Hence, we may regard I/α as an ideal of R/α in isomorphic sense, completing the proof.

Theorem 4.9 (Fuzzy Second Isomorphism Theorem) *Let α, β be two fuzzy congruence relations of a ring R with $\alpha_0 \subseteq \beta_0$. Then*

$$(R_\alpha + R_\beta)/\beta \cong R_\alpha/\alpha \cap \beta.$$

Proof. By Lemma 4.8, β is a fuzzy congruence relation of $R_\alpha + R_\beta$ and $\alpha \cap \beta$ is a fuzzy congruence relation of R_α. Thus $(R_\alpha + R_\beta)/\beta$ and $R_\alpha/\alpha \cap \beta$ are both rings. For any $x \in R_\alpha + R_\beta$, then $x = a + b$, where $a \in R_\alpha, b \in R_\beta$, it implies $\alpha(0, a) = 1$ and $\beta(0, b) = 1$.

Define $f : (R_\alpha + R_\beta)/\beta \cong R_\alpha/\alpha \cap \beta$ by

$$f(\beta_x) = (\alpha \cap \beta)_a.$$

If $\beta_x = \beta_{x'}$, where $x' = a' + b', a' \in R_\alpha, b' \in R_\beta$, then we have $\alpha(0, a') = 1$, $\beta(0, b')$=1 and $\beta(x, x') = \beta(a+b, a'+b') = 1$. Since $\alpha(a, a') \geq \alpha(a, 0) \wedge \alpha(0, a') = 1$, so $\alpha(a, a') = 1$. Similarly, $\beta(b, b') = 1$. By Definition 4.1 and Lemma 4.7 and $\alpha_0 \subseteq \beta_0$, we have $R_\alpha \subseteq R_\beta$. Therefore, $a, a' \in R_\beta$, which implies $\beta(0, a) = 1$ and $\beta(0, a') = 1$. Since $\beta(a, a') \geq \beta(a, 0) \wedge \beta(0, a') = 1$, $\beta(a, a') = 1$. From Definition 2.3, $(\alpha \cap \beta)(a, a') = \alpha(a, a') \wedge \beta(a, a') = 1$, which implies $(\alpha \cap \beta)_a = (\alpha \cap \beta)_{a'}$. Hence f is well-defined.

For any $\beta_x, \beta_y \in (R_\alpha + R_\beta)/\beta$, where $x = a + b$, $y = a_1 + b_1$, $a, a_1 \in R_\alpha$ and $b, b_1 \in R_\beta$, $x + y = (a + a_1) + (b + b_1)$, $xy = (a + a_1)(b + b_1) = aa_1 + (ab_1 + ba_1 + bb_1) = aa_1 + b'$, $b' = ab_1 + ba_1 + bb_1 \in R_\beta$. We have $f(\beta_x + \beta_y) = f(\beta_{x+y}) = (\alpha \cap \beta)_{a+a_1} = (\alpha \cap \beta)_a + (\alpha \cap \beta)_{a_1} = f(\beta_x) + f(\beta_y)$, $f(\beta_x\beta_y) = f(\beta_{xy}) = (\alpha \cap \beta)_{aa_1} = (\alpha \cap \beta)_a(\alpha \cap \beta)_{a_1} = f(\beta_x)f(\beta_y)$. Hence f is a homomorphism.

For any $(\alpha \cap \beta)_a \in R_\alpha/(\alpha \cap \beta)$, taking $b \in R_\beta$, $\exists x = a + b \in R_\alpha + R_\beta$, then $\beta_x \in (R_\alpha + R_\beta)/\beta$ and $f(\beta_x) = (\alpha \cap \beta)_a$. Hence f is an epimorphism.

For any $x, y \in R_\alpha + R_\beta$, where $x = a+b, y = a_1+b_1, a, a_1 \in R_\alpha$ and $b, b_1 \in R_\beta$, if $(\alpha \cap \beta)_a = (\alpha \cap \beta)_{a_1}$, then $(\alpha \cap \beta)(a, a_1) = 1$, i.e, $\alpha(a, a_1) \wedge \beta(a, a_1) = 1$, which implies $\alpha(a, a_1) = 1$ and $\beta(a, a_1) = 1$. Since $b, b_1 \in R_\beta$, we have $\beta(0, b) = 1$ and $\beta(0, b_1) = 1$. Hence $\beta(b, b_1) \geq \beta(b, 0) \wedge \beta(0, b') = 1$, and $\beta(b, b_1) = 1$.

Then

$$\begin{aligned} \beta(a + b, a_1 + b_1) &\geq \beta(a + b, a_1 + b) \wedge \beta(a_1 + b, a_1 + b_1) \\ &\geq \beta(a, a_1) \wedge \beta(b, b_1) \\ &= 1. \end{aligned}$$

Thus $\beta(a + b, a_1 + b_1) = 1, \beta(x, y) = 1$, this means $\beta_x = \beta_y$. Therefore f is a monomorphism. Hence, we have shown that$(R_\alpha + R_\beta)/\beta \cong R_\alpha/\alpha \cap \beta$, completing the proof.

Notation 4.10 Let α and β be fuzzy relations of a ring R. We denote $\alpha \leq \beta$, if $\alpha(x, y) \leq \beta(x, y)$ for all $x, y \in R$.

Theorem 4.11 (Fuzzy Third Isomorphism Theorem) *Let α, β be two fuzzy congruence relations of a ring R with $\alpha \leq \beta$. Then*

$$(R/\alpha)/(R_\beta/\alpha) \cong R/\beta.$$

Proof. By Lemma 4.7 and Lemma 4.8, R_β/α is an ideal of R/α.

Define $f : R/\alpha \to R/\beta$ by $f(\alpha_x) = \beta_x$ for all $x \in R$. If $\alpha_x = \alpha_y$, then $\alpha(x, y) = 1$. Since $\alpha \leq \beta$, so $\beta(x, y) \geq \alpha(x, y) = 1$, thus $\beta(x, y) = 1, \beta_x = \beta_y$. Hence f is well-defined.

$f(\alpha_x + \alpha_y) = f(\alpha_{x+y}) = \beta_{x+y} = \beta_x + \beta_y = f(\alpha_x) + f(\alpha_y)$,
$f(\alpha_x\alpha_y) = f(\alpha_{xy}) = \beta_{xy} = \beta_x\beta_y = f(\alpha_x)f(\alpha_y)$,

they mean f is a homomorphism. For any $\beta_x \in R/\beta$, there exist $\alpha_x \in R/\alpha$ such that $f(\alpha_x) = \beta_x$, so f is an epimorphism.

Now, we show $kerf = R_\beta/\alpha$. In fact, $kerf = \{\alpha_x \in R/\alpha \mid f(\alpha_x) = \beta_0\} = \{\alpha_x \in R/\alpha \mid \beta_x = \beta_0\} = \{\alpha_x \in R/\alpha \mid \beta(0,x) = 1\} = \{\alpha_x \in R/\alpha | x \in R_\beta\} = R_\beta/\alpha$. Therefore, $(R/\alpha)/(R_\beta/\alpha) \cong R/\beta$. The proof is complete.

5. Fuzzy ideal and fuzzy congruence relations

In this section, we discuss the relationships between fuzzy ideals and fuzzy congruence relations on R. In particular, we show that there exists a one-to-one mapping from all fuzzy ideals and all fuzzy congruence relations of R.

Theorem 5.1 *Let μ be a fuzzy ideal of R. Then the fuzzy relation α_μ on R, defined by*

$$\alpha_\mu(a,b) = \mu(a-b)$$

, for all (a,b) of $R \times R$, is a fuzzy congruence relation on R.

Proof. For all $a, b \in R$, then we have $\alpha_\mu(a,a) = \mu(0) = 1$, α_μ is a fuzzy reflexive relation; $\alpha_\mu(a,b) = \mu(a-b)$, $\alpha_\mu(b,a) = \mu(b-a)$, so $\alpha_\mu(a,b) = \alpha_\mu(b,a)$, which means α_μ is a fuzzy symmetric relation;

$$\bigvee_{x\in R}[\alpha_\mu(a,x) \wedge \alpha_\mu(x,b)] = \bigvee_{x\in R}[\mu(a-x) \wedge \mu(x-b)] \leq \bigvee_{x\in R}\mu(a-b) = \alpha_\mu(a,b),$$

α_μ is a fuzzy transitive relation. This means that α_μ is a fuzzy equivalence relation on R. For all $a, b, x, y \in R$, since

$$\begin{aligned}
\alpha_\mu(x,a) \wedge \alpha_\mu(y,b) &= \mu(x-a) \wedge \mu(y-b) \\
&\leq \mu[(x+y)-(a+b)] \\
&= \alpha_\mu(x+y, a+b),
\end{aligned}$$

and

$$\begin{aligned}
\alpha_\mu(xy,ab) &= \mu(xy-ab) \\
&= \mu[(x-a)y + a(y-b)] \\
&\geq \mu[(x-a)y] \wedge \mu[a(y-b)] \\
&\geq \mu(x-a) \wedge \mu(y-b) \\
&= \alpha_\mu(x,a) \wedge \alpha_\mu(y,b).
\end{aligned}$$

We obtain that α_μ is a fuzzy congruence relation on R.

Corollary 5.2 *Let μ and ν be two fuzzy ideals on a ring R, then*

(i) $\alpha_{\mu\cap\nu} = \alpha_\mu \cap \alpha_\nu$,

(ii) $\alpha_{\mu+\nu} = \alpha_\mu \circ \alpha_\nu$.

Proof. (i) For each $x, y \in R$,

$$\begin{aligned} \alpha_{(\mu\cap\nu)}(x,y) &= (\mu\cap\nu)(x-y) \\ &= \mu(x-y)\wedge\nu(x-y) \\ &= \alpha_\mu(x,y)\wedge\alpha_\nu(x,y) \\ &= (\alpha_\mu\cap\alpha_\nu)(x,y). \end{aligned}$$

(ii) For every $x, y \in R$,

$$\begin{aligned} (\alpha_\mu\circ\alpha_\nu)(x,y) &= \bigvee_{z\in R}[\alpha_\mu(x,z)\wedge\alpha_\nu(z,y)] \\ &= \bigvee_{z\in R}[\mu(x-z)\wedge\nu(z-y)] \\ &= (\mu+\nu)(x-y) \\ &= \alpha_{(\mu+\nu)}(x,y). \end{aligned}$$

Hence $\alpha_{\mu+\nu} = \alpha_\mu \circ \alpha_\nu$. The proof is complete.

Theorem 5.3 *Let α be any fuzzy congruence relation on a ring R, then the fuzzy set μ_α of R, defined by*

$$\mu_\alpha(a) = \alpha(0, a)$$

is a fuzzy ideal of R for all $a \in R$.

Proof. For all $a, b \in R$, since α is a fuzzy congruence relation, then we have

$$\begin{aligned} \mu_\alpha(a+b) &= \alpha(0,a+b) \\ &= \alpha(0+0,a+b) \\ &\geq \alpha(0,a)\wedge\alpha(0,b) \\ &= \mu_\alpha(a)\wedge\mu_\alpha(b). \end{aligned}$$

Hence $\mu_\alpha(a+b) \geq \mu_\alpha(a)\wedge\mu_\alpha(b)$. Again, $\mu_\alpha(-a) = \alpha(0,-a) = \alpha(0,a) = \mu_\alpha(a)$ and $\mu_\alpha(ab) = \alpha(0,ab) \geq \alpha(0,a)\wedge\alpha(0,b) = \mu_\alpha(a)\wedge\mu_\alpha(b)$. Hence $\mu_\alpha(ab) \geq \mu_\alpha(a)\wedge\mu_\alpha(b)$. Since $\mu_\alpha(0) = \alpha(0,0) = 1$, μ_α is a fuzzy subring of R. Moreover, $\mu_\alpha(a) = \alpha(0,a) = \alpha(0,a)\wedge\alpha(b,b) \leq \alpha(0,ab) = \mu_\alpha(ab)$, hence $\mu_\alpha(ab) \geq \mu_\alpha(a)$. Similarly $\mu_\alpha(ab) \geq \mu_\alpha(b)$. Therefore, we obtain that μ_α is a fuzzy ideal of R. The proof is complete.

Corollary 5.4 *Let α and β be two fuzzy congruence relations on a ring R, then*

(i) $\mu_{\alpha\cap\beta} = \mu_\alpha \cap \mu_\beta$,

(ii) $\mu_{\alpha\circ\beta} = \mu_\alpha + \mu_\beta$.

Proof. (i) For each $x, y \in R$,

$$\begin{aligned}\mu_{(\alpha\cap\beta)}(x) &= (\alpha\cap\beta)(0,x)\\ &= \alpha(0,x)\wedge\beta(0,x)\\ &= \mu_\alpha(x)\wedge\mu_\beta(x)\\ &= (\mu_\alpha\cap\mu_\alpha)(x).\end{aligned}$$

(ii) Since

$$\begin{aligned}\mu_{(\alpha\circ\beta)}(x) &= (\alpha\circ\beta)(0,x)\\ &= \bigvee_{y\in R}[\alpha(0,y)\wedge\beta(y,x)]\\ &= \bigvee_{z\in R}[\mu_\alpha(y)\wedge\mu_\beta(x-y)]\\ &= (\mu_\alpha+\mu_\alpha)(x).\end{aligned}$$

This completes the proof.

Proposition 5.5 *Let R be a ring, then there exists a one-to-one mapping from $FI(R)$ onto $FC(R)$.*

Proof. We define a mapping φ from $FI(R)$ to $FC(R)$ as follows:

$$\varphi(\mu) = \alpha_\mu$$

for every μ of $FI(R)$. By Theorem 5.1, φ is well-defined. If $\alpha_\mu = \alpha_\nu$, then $\alpha_\mu(a,b) = \alpha_\nu(a,b)$, for all $(a,b)\in R\times R$, so $\mu(a-b) = \nu(a-b)$, hence $\mu = \nu$, this means φ is injective. For each $\alpha \in FC(R)$, $a, b \in R$, by Theorem 5.3, $\varphi(\mu_\alpha)(a,b) = \alpha_{\mu_\alpha}(a,b) = \mu_\alpha(a-b) = \alpha(0,a-b) = \alpha(a,b)$. Thus φ is surjective. Consequently, φ is a one-to-one mapping from $FI(R)$ onto $FC(R)$.

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PROPERTIES OF HYPERIDEALS IN ORDERED SEMIHYPERGROUPS

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Abstract. An ordered semihypergroup is a semihypergroup $(S, \circ)$ together with a partial order $\leq$ on S such that the monotone condition holds, i.e., for all $x, y, a \in S$, if $x \leq y$, then for all $u \in x \circ a$ there exists $v \in y \circ a$ such that $u \leq v$, and similarly, for all $u' \in a \circ x$ there exists $v' \in a \circ y$ such that $u' \leq v'$. Indeed, the concept of ordered semihypergroups is a generalization of the concept of ordered semigroups. In this paper, we study some aspects of hyperideals of ordered semihypergroups. We give a necessary and sufficient condition of a subset of Cartesian product of two ordered semihypergroups to be a prime hyperideal. Also, we study right simple element ordered semihypergroups containing right simple elements.

Keywords: algebraic hyperstructure, ordered semigrouip, ordered semihypergroup, hyperideal, prime hyperideal, simple element.
AMS Mathematics Subject Classification: 20N20, 06F05.

1. Introduction and basic definitions

The concept of algebraic hyperstructures was introduced in 1934 by Marty [11] and has been studied in the following decades and nowadays by many mathematicians. Let S be a nonempty set. A mapping $\circ : S \times S \to \mathcal{P}^*(S)$, where $\mathcal{P}^*(S)$ denotes the family of all nonempty subsets of S, is called a *hyperoperation* on S. The couple $(S, \circ)$ is called a *hypergroupoid*. In the above definition, if A and B are two nonempty subsets of S and $x \in S$, then we denote

$$A \circ B = \bigcup_{\substack{a \in A \\ b \in B}} a \circ b, \;\; x \circ A = \{x\} \circ A \;\; \text{and} \;\; A \circ x = A \circ \{x\}.$$

A hypergroupoid $(S, \circ)$ is called a *semihypergroup* if for every $x, y, z \in S$, $x \circ (y \circ z) = (x \circ y) \circ z$, that is

$$\bigcup_{u \in y \circ z} x \circ u = \bigcup_{v \in x \circ y} v \circ z.$$

A nonempty subset A of S is called a *subsemihypergroup* if $x \circ y \subseteq A$ for all x, y in A. Semihypergroups are studied by many authors, for example, Bonansinga and Corsini [2], Davvaz [4], [5], De Salvo et al. [6], Freni [7], Hila et al. [9], Leoreanu [14], and many others. The concept of ordering hypergroups investigated by Chvalina [3] as a special class of hypergroups and studied by him and many others. In [8], Heidari and Davvaz studied a semihypergroup $(S, \circ)$ besides a binary relation $\leq$, where $\leq$ is a partial order relation such that satisfies the monotone condition. Indeed, an *ordered semihypergroup* $(S, \circ, \leq)$ is a semihypergroup $(S, \circ)$ together with a partial order $\leq$ that is *compatible* with the hyperoperation, meaning that for any x, y, z in S,

$$x \leq y \Rightarrow z \circ x \leq z \circ y \text{ and } x \circ z \leq y \circ z.$$

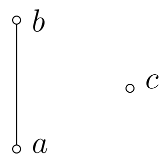
Here, $z \circ x \leq z \circ y$ means for any $a \in z \circ x$ there exists $b \in z \circ y$ such that $a \leq b$. The case $x \circ z \leq y \circ z$ is defined similarly.

Example 1 We have $(S, \circ, \leq)$ is an ordered semihypergroup where the hyperoperation and the order relation are defined by:

$\circ$	a	b	c
a	a	$\{a, b\}$	$\{a, c\}$
b	a	$\{a, b\}$	$\{a, c\}$
c	a	$\{a, b\}$	c

$$\leq \; = \{(a, a), (b, b), (c, c), (a, b)\}.$$

The covering relation and the figure of S are given by: $\prec = \{(a, b)\}$

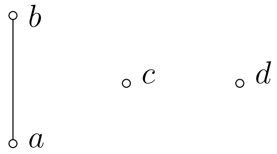


Example 2 We have $(S, \circ, \leq)$ is an ordered semihypergroup where the hyperoperation and the order relation are defined by:

$\circ$	a	b	c	d
a	a	$\{a, b\}$	$\{a, c\}$	$\{a, d\}$
b	a	$\{a, b\}$	$\{a, c\}$	$\{a, d\}$
c	a	b	c	d
d	a	b	c	d

$$\leq = \{(a, a), (b, b), (c, c), (d, d), (a, b)\}.$$

The covering relation and the figure of S are given by: $\prec = \{(a,b)\}$



Note that the concept of ordered semihypergroups is a generalization of the concept of ordered semigroups [1], [10], [13]. Indeed, every ordered semigroup is an ordered semihypergroup.

For a nonempty subset A of an ordered semihypergroup $(S,\circ,\leq)$, we write

$$(A] = \{x \in S \mid x \leq a \text{ for some } a \in A\}.$$

The following is easy to see for nonempty subsets A, B of an ordered semihypergroup $(S,\circ,\leq)$:

(1) $A \subseteq (A]$;

(2) $A \subseteq B \Rightarrow (A] \subseteq (B]$;

(3) $(A] \circ (B] \subseteq (A \circ B]$;

(4) $((A] \circ (B]] = (A \circ B]$;

(5) $(A] \cup (B] = (A \cup B]$.

Let $(S,\circ,\leq)$ be an ordered semihypergroup. A subset A of S is called a *hyperideal* of S if it satisfies the following conditions:

(1) $x \circ y \subseteq A$ and $y \circ x \subseteq A$ for all x in A, y in S;

(2) for $x \in A, y \in S$, $y \leq x$ implies $y \in A$.

Let $(S,\circ,\leq_S)$ and $(T,\diamond,\leq_T)$ be two ordered semihypergroups. Under the coordinatewise multiplication, i.e.,

$$(s_1,t_1) \star (s_2,t_2) = s_1 \circ s_2 \times t_1 \diamond t_2$$

where $(s_1,t_1),(s_2,t_2) \in S \times T$, the Cartesian product $S \times T$ of S and T forms a semihypergroup. Define a partial order $\leq$ on $S \times T$ by

$$(s_1,t_1) \leq (s_2,t_2) \text{ if and only if } s_1 \leq_S s_2 \text{ and } t_1 \leq_T t_2$$

where $(s_1,t_1),(s_2,t_2) \in S \times T$. Then, $(S \times T,\star,\leq)$ is an ordered semihypergroup.

2. Prime ideals of the Cartesian product of two ordered semihypergroups

A hyperideal P of an ordered semihypergroup $(S,\circ,\leq)$ is said to be *prime* if $S \setminus P$ is a subsemihypergroup of S. Note that if a hyperideal P of S is prime, then $P \neq S$. In this section we accept the empty set to be a prime hyperideal. Similar to the method of Petrich [15], we give a necessary and sufficient condition of a subset of Cartesian product of two ordered semihypergroups to be a prime hyperideal.

Theorem 2.1 *Let $(S, \circ, \leq_S)$ and $(T, \diamond, \leq_T)$ be ordered semihypergroups. Then, a subset L of $S \times T$ is a prime hyperideal of $S \times T$ if and only if there exist a prime hyperideal I of S and a prime hyperideal J of T such that $L = (I \times T) \cup (S \times J)$.*

Proof. Assume that there exist a prime hyperideal I of S and a prime hyperideal J of T such that

$$L = (I \times T) \cup (S \times J).$$

If $I = \emptyset$ and $J = \emptyset$, then $L = \emptyset$; hence L is a prime hyperideal of $S \times T$.

Suppose that $I \neq \emptyset$ or $J \neq \emptyset$. Then, $L \neq \emptyset$. We will show that L is a prime hyperideal of $S \times T$. Let $(x, u) \in L$ and $(y, v) \in S \times T$. If $x \in I$, then $x \circ y \subseteq I$ and $y \circ x \subseteq I$; hence

$$(x, u) \star (y, v) = x \circ y \times u \diamond v \subseteq I \times T$$

and

$$(y, v) \star (x, u) = y \circ x \times v \diamond u \subseteq I \times T.$$

Similarly, if $u \in J$, then $(x, u) \star (y, v) \subseteq S \times J$ and $(y, v) \star (x, u) \subseteq S \times J$. Let $(x, u) \in L$ and $(y, v) \in S \times T$ be such that $(y, v) \leq (x, u)$, i.e., $y \leq_S x$, $v \leq_T u$. If $x \in I$, then $y \in I$; hence $(y, v) \in I \times T$. Thus, $(y, v) \in L$. Similarly, if $u \in J$, then $(y, v) \in L$. Therefore, L is a hyperideal of $S \times T$. Next, we assert that $(S \times T) \setminus L$ is a subsemihypergroup of $S \times T$. Since $S \setminus I \neq \emptyset$ and $T \setminus J \neq \emptyset$, it follows that $S \setminus I$ and $T \setminus J$ are semihypergroups of S and of T, respectively. We have

$$(S \times T) \setminus L = (S \setminus I) \times (T \setminus J) \neq \emptyset.$$

Then, $(S \setminus I) \times (T \setminus J)$ is a subsemihypergroup of $S \times T$. Hence, L is a prime hyperideal of $S \times T$.

Conversely, assume that L is a prime hyperideal of $S \times T$. If $L = \emptyset$, then $L = (\emptyset \times T) \cup (S \times \emptyset)$. Assume that $(x, u) \in L$. We assert that $\{x\} \times T \subseteq L$ or $S \times \{u\} \subseteq L$. Suppose that $\{x\} \times T \nsubseteq L$ and $S \times \{u\} \nsubseteq L$. Then, there exist $v \in T$ and $y \in S$ such that $(x, v) \notin L$ and such that $(y, u) \notin L$. We have

$$(x, v) \star (y, u) \star (x, v) \star (y, u) \quad = \quad x \circ y \circ x \circ y \times v \diamond u \diamond v \diamond u$$

and

$$(x \circ y, v) \star (x, u) \star (y, v \diamond u) \quad = \quad x \circ y \circ x \circ y \times v \diamond u \diamond v \diamond u.$$

Since $(x, v) \star (y, u) \star (x, v) \star (y, u) \subseteq (S \times T) \setminus L$, we have

$$(x \circ y, v) \star (x, u) \star (y, v \diamond u) \subseteq (S \times T) \setminus L.$$

But, since $(x, u) \in L$, we have $(x \circ y, v) \star (x, u) \star (y, v \diamond u) \subseteq L$. This is a contradiction. Hence,, $\{x\} \times T \subseteq L$ or $S \times \{u\} \subseteq L$. Let

$$A = \{x \in S \mid \{x\} \times T \subseteq L\} \text{ and } B = \{u \in T \mid S \times \{u\} \subseteq L\},$$

and let

$$I = (A] \text{ and } J = (B].$$

Let $(x, u) \in L$. Then, $\{x\} \times T \subseteq L$ or $S \times \{u\} \subseteq L$. Thus, $x \in I$ or $u \in J$. Hence, $(x, u) \in (I \times T) \cup (S \times J)$. Thus, $L \subseteq (I \times T) \cup (S \times J)$. The reverse inclusion is clear. Hence,

$$L = (I \times T) \cup (S \times J).$$

We will show that I is a prime hyperideal of S. That J is a prime hyperideal of T can be proved similarly. If $I = \emptyset$, then I is a prime hyperideal of S. Assume that $I \neq \emptyset$. If $I = S$, then $L = S \times T$. This is a contradiction since L is a prime hyperideal of $S \times T$. Hence, $S \setminus I \neq \emptyset$. Similarly, $T \setminus J \neq \emptyset$. Let $x, y \in S \setminus I$ and $u \in T \setminus J$. Then,

$$(x, u), (y, u) \in (S \setminus I) \times (T \setminus J).$$

Since L is prime, we have $(S \setminus I) \times (T \setminus J)$ is a subsemihypergroup of $S \times T$. Since

$$x \circ y \times u \diamond u = (x, u) \star (y, u) \subseteq (S \setminus I) \times (T \setminus J)$$

we get $x \circ y \subseteq S \setminus I$. Thus, $S \setminus I$ is a subsemihypergroup of S. Let $x \in I, y \in S$ and $u \in T \setminus J$. Since $x \in I$, $(x, u) \in L$. Since L is a hyperideal of $S \times T$, we have

$$(x, u) \star (y, u) = x \circ y \times u \diamond u \subseteq L$$

and

$$(y, u) \star (x, u) = y \circ x \times u \diamond u \subseteq L.$$

Since $T \setminus J$ is a subsemihypergroup, so $u \diamond u \subseteq T \setminus J$. Since

$$x \circ y \times u \diamond u, y \circ x \times u \diamond u \subseteq L$$

we obtain

$$x \circ y \times u \diamond u, y \circ x \times u \diamond u \subseteq I \times T$$

and hence $x \circ y, y \circ x \subseteq I$. It is clear that if $x \in I$ and $y \in S$ such that $y \leq x$, then $y \in I$. Therefore, I is a prime hyperideal of S. ∎

Suppose that $(S, \circ)$ and $(T, \diamond)$ are semihypergroups. Then, the Cartesian product $S \times T$ is a semihypergroup under the coordinatewise multiplication. Define a partial order $\leq_S$ on S by

$$x \leq_S y \text{ if and only if } x = y \text{ for all } x, y \in S.$$

Then, S forms an ordered semihypergroup. Similarly, T forms an ordered semihypergroup with a partial order $\leq_T$ defined in a similar way. Using Theorem 2.1, we have the following result proved in [15].

Corollary 2.2 *Let $(S, \circ)$ and $(T, \diamond)$ be semihypergroups. Then, a subset L of $S \times T$ is a prime hyperideal of $S \times T$ if and only if $L = (I \times T) \cup (S \times J)$ for some prime hyperideals I and J of S and of T, respectively.*

3. Right simple ordered semihypergroups

Let $(S, \circ, \leq)$ be an ordered semihypergroup. An element a of S is said to be *right simple* if $S = (a \circ S]$. If S contains a right simple element then it is called a *right simple element ordered semihypergroup.* If every element of S is right simple, then S is called a *right simple ordered semihypergroup.*

Theorem 3.1 *If $(S, \circ, \leq_S)$ and $(T, \diamond, \leq_T)$ are two right simple element ordered semihypergroups, then $S \times T$ is a right simple element ordered semihypergroup, too. Moreover, if A and B are the sets of all right simple elements of S and of T, respectively, then $A \times B$ is the set of all right simple elements of $S \times T$.*

Proof. Assume that $(S, \circ, \leq_S)$ and $(T, \diamond, \leq_T)$ are right simple element ordered semihypergroups with the sets of all right simple elements A and B, respectively. If $(a, b) \in A \times B$, then

$$S \times T = (a \circ S] \times (b \diamond T].$$

If $(s, t) \in S \times T$, then $s \in a \circ s'$ for some s' in S, and $t \in b \diamond t'$ for some t' in T. Since $(s, t) \in a \circ s' \times b \diamond t'$, it follows that

$$(s,t) \in \left(\bigcup_{(s,t) \in S \times T} a \circ s \times b \diamond t \right] = ((a, b) \star (S \times T)].$$

Hence, (a, b) is a right simple element of $S \times T$. If (a, b) is a right simple element of $S \times T$, then

$$S \times T = ((a, b) \star (S \times T)] = \left(\bigcup_{(s,t) \in S \times T} a \circ s \times b \diamond t \right].$$

If $(s, t) \in S \times T$, then $(s, t) \leq (u, v)$ for some $(u, v) \in a \circ s' \times b \diamond t'$ where $s' \in S, t' \in T$. Since

$$s \leq u \in a \circ s' \subseteq (a \circ S],$$

we have $s \in (a \circ S]$, and so $S = (a \circ S]$.

Similarly, $T = (b \diamond T]$. Hence, $(a, b) \in A \times B$. ∎

It is well known the following result in semigroup theory. Let S be a right simple element semigroup and let R denote the set of all right simple elements of S. Then, the following conditions holds: (1) R is a subsemigroup of S; (2) If $S \setminus R$ is nonempty, then it is the maximal right ideal of S and is prime, too ([12]). In the following, we extend the above result based on ordered semihypergroups.

Theorem 3.2 *Let $(S, \circ, \leq)$ be right simple element ordered semihypergroup with the set of all right simple elements R. The following statements hold:*

(1) *R is a subsemihypergroup of S.*

(2) *If $S \setminus R$ is nonempty, then it is the maximal right hyperideal of S and is prime, too.*

Proof. If $(S, \circ, \leq)$ is right simple ordered semihypergroup, then it is clear that (1) and (2) hold. Then, we assume that $(S, \circ, \leq)$ is not a right simple ordered semihypergroup.

(1) Let $a, b \in R$. Since $S = (a \circ S]$ and $S = (b \circ S]$, we have

$$S = (a \circ S] = (a \circ (b \circ S]] \subseteq ((a] \circ (b \circ S]] = (a \circ b \circ S],$$

and so $a \circ b \subseteq R$.

(2) Assume that $S \setminus R \neq \emptyset$. Let $x \in S$ and $a \in S \setminus R$. If $a \circ x \subseteq R$, then $S = (a \circ x \circ S] \subseteq (a \circ S]$; hence $a \in R$. This is a contradiction. Thus, $a \circ x \subseteq S \setminus R$. Let $x \in S \setminus R$ and $y \in S$ be such that $y \leq x$. If $y \in R$, then $S = (y \circ S] \subseteq (x \circ S]$; hence $x \in R$. This is a contradiction. Thus, $S \setminus R$ is a right hyperideal of S. Let A be a right hyperideal of S such that $S \setminus R \subset A$. Then, there is an element a in $A \setminus (S \setminus R)$. Since $S = (a \circ S] \subseteq A$, so $A = S$. By (1), it follows directly that $S \setminus R$ is prime. ∎

Theorem 3.3 *If an ordered semihypergroup $(S, \circ, \leq)$ has a unique maximal right hyperideal A such that $S \setminus A \neq (b]$ for all b in $S \setminus A$, then $S \setminus A$ is the set of all right simple elements of S.*

Proof. Let R denote the set of all right simple elements of S. Let $a \in R$. If $a \in A$, then $S = (a \circ S] \subseteq A$. This is a contradiction. Thus, $R \subseteq S \setminus A$. Let $b \in S \setminus A$. We have $(b \circ S]$ is a right hyperideal of S. If $(b \circ S] \subset S$, then by assumption we have $(b \circ S] \subseteq A$; hence $(A \cup \{b\}]$ is a right hyperideal of S. By $A \subset (A \cup \{b\}]$, $S = (A \cup \{b\}]$. Hence, $S \setminus A = (b]$. This is a contradiction. Hence, $(b \circ S] = S$. ∎

Let $(S, \circ, \leq)$ be an ordered semihypergroup. An equivalence relation $\mathcal{R}$ is defined on S by

$$a\mathcal{R}b \text{ if and only if } (a \cup a \circ S] = (b \cup b \circ S]$$

for any a, b in S.

An element a of an ordered semihypersroup $(S, \circ, \leq)$ is said to be *right regular* if $a \in (a^2 \circ S]$.

Theorem 3.4 *Let $(S, \circ, \leq)$ be a right simple element ordered semihypergroup with set of all right simple elements R. Then,*

(1) *R is an $\mathcal{R}$-class of S;*

(2) *every element of R is right regular.*

Proof. (1) If $a, b \in R$, then $S = (a \circ S]$ and $S = (b \circ S]$; hence $(a \cup a \circ S] = (b \cup b \circ S]$. This shows that $a\mathcal{R}b$. Let $x \in S$ be such that $x\mathcal{R}a$ for some a in R. Then, $(x \cup x \circ S] = (a \cup a \circ S] = S$. If $S \setminus R = \emptyset$, then $x \in R$. If $S \setminus R \neq \emptyset$ and $x \in S \setminus R$, then $S = (x \cup x \circ S] \subseteq S \setminus R$; hence $S = S \setminus R$. This is a contradiction. Hence, $x \in R$.

(2) If $a \in R$, then $a \in (a \circ S] \subseteq (a \circ (a \circ S]] \subseteq (a^2 \circ S]$; hence a is right regular. ∎

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PRIME SUBMODULES IN EXTENDED BCK-MODULE

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Abstract. In this paper, by considering the notion of BCK-module, we define the concept of extended BCK-module which is a generalization of BCK-module and we state and prove some related results. Specially, we define the notions of prime submodule and torsion free module and we investigate some important results. Finally, we define the concept of radical of any submodule in extended BCK-modules and we characterize the elements of it.

Keywords: BCK-algebra, BCK-module, extended BCK-module, prime submodule of BCK-module.

Mathematical Subject Classification (2010): 06F35, 06D99.

1. Introduction

The notion of BCK-algebra was formulated first in 1966 by Imai and Iseki. This notion is originated from two different ways. One of the motivations is based on set theory. Another motivation is from classical and non-classical propositional calculus. The notion of BCK-module was introduced in 1994 [2] as an action of a BCK-algebra over a commutative group by M. Aslam, A.B. Thaheem and H.A.S. Abujaabal. The idea was further explored in 1994 by F. Kôpka and F. Chovanec [8]. The concept of BCK-module was extended by R. A. Borzooei, J. Shohani and M. Jafari in 2011 [4]. Now, we introduce a different extended BCK-module that we can obtain some interesting results by it. Since the notion of prime-submodule is fundamental notion in modules theory, in this paper we introduce and investigate it on BCK-modules and we obtain some results as mentioned in the abstract.

2. Preliminaries

Definition 2.1. [9] A BCK-algebra is a structure $X = (X, *, 0)$ of type $(2, 0)$ such that:

$(BCK1)$ $((x * y) * (x * z)) * (z * y) = 0$,

$(BCK2)$ $(x * (x * y)) * y = 0$,

$(BCK3)$ $x * x = 0$,

$(BCK4)$ $0 * x = 0$,

$(BCK5)$ $x * y = y * x = 0$ implies that $x = y$,for all $x, y, z \in X$.

The relation $x \leq y$ which is defined by $x * y = 0$ is a partial order with 0 as least element. In any BCK-algebra X, for all $x, y, z \in X$, we have

$(BCK6)$ $x * y \leq x$, $(x * y) * z = (x * z) * y$.

Definition 2.2. [9] Let $(X, *, 0)$ be a BCK-algebra. Then

(i) $\emptyset \neq X_0 \subseteq X$ is called to be a *subalgebra* of X, if for any $x, y \in X_0$, $x*y \in X_0$,

(ii) $\emptyset \neq I \subseteq X$ is called an *ideal* of X, if $0 \in I$ and for any $x, y \in X$, $x * y \in I$ and $y \in I$, implies that $x \in I$. Specially, generated ideal by x is defined by $(x] = \{y \in X : y * x = 0\}$, for any $x \in X$,

(iii) X is called *bounded*, if there exists $1 \in X$ such that $x \leq 1$, for any $x \in X$. In this case, we set $Nx = 1 * x$,

(iv) X is said to be *commutative*, if $y * (y * x) = x * (x * y)$, for all $x, y \in X$,

(v) proper ideal I of X, is called *prime ideal* if X is commutative and $a \wedge b \in I$ implies that $a \in I$ or $b \in I$, for any $a, b \in X$,

(iv) X is said to be *implicative* if $x * (y * x) = x$, for all $x, y \in X$.

Note. In a BCK-algebra X, we let $x \wedge y = y * (y * x)$ and in a bounded BCK-algebra X, we let $x \vee y = N(Nx \wedge Ny)$, for all $x, y \in X$. Moreover, in bounded commutative BCK-algebra, $x \wedge y$ is the least upper bound and $x \vee y$ is the grate lower bound of x, y, for any $x, y \in X$ and so $(L, \vee, \wedge)$ is a bounded lattice.

Lemma 2.3. [9] *Let X be a bounded implicative BCK-algebra. Then for all $x, y, z \in X$,*

(i) $x \wedge y = x * Ny$,

(ii) $x * (x \wedge y) = x * y$,

(iii) $x \wedge (y * z) = (x \wedge y) * (x \wedge z)$,

(iv) $(x * y) + (y * x) = x + y$, *where* $x + y = (x * y) \vee (y * x)$,

(v) $(x + y) \wedge z = (x \wedge z) + (y \wedge z)$,

(vi) $x + x = 0$ *and so* $x = -x$,

(vii) $x + 0 = 0 + x = x$.

Let A be an ideal of BCK-algebra X. For any $x, y \in X$, we define $x \sim y$ if and only if $x * y \in A$ and $y * x \in A$. So $\sim$ is an equivalence relation on X. Denote the equivalence class containing x by C_x and $\frac{X}{A} = \{C_x : x \in X\}$. Then $\left(\frac{X}{A}, *, C_0\right)$ is a BCK-algebra (quotient BCK-algebra), where $C_x \star C_y = C_{x*y}$, for all $x, y \in X$. Moreover, the relation " $\leq$ " which is defined by, $C_x \leq C_y$ if and only if $x * y \in A$, is a partial order relation. If X is bounded and commutative, then $\frac{X}{A}$ is bounded and commutative, too. Let $(X, *, 0)$ and $(Y, *', 0)$ be two BCK-algebras. A mapping $f : X \to Y$ is called a *homomorphism* if $f(0) = 0$ and $f(x * y) = f(x) *' f(y)$, for any $x, y \in X$ (see [9]).

Definition 2.4. [1] Let X be a BCK-algebra, M be an abelian group under "+" and $(x, m) \to x.m$ be a mapping of $X \times M \longrightarrow M$ such that,

$(XM1)$ $(x \wedge y).m = x.(y.m)$,

$(XM2)$ $x.(m + n) = x.m + x.n$,

$(XM3)$ $0.m = 0$, for all $x, y \in X$ and $m, n \in M$.

Then M is called a *BCK-module* or briefly X-module. If X is bounded and for any $m \in M$, $1.m = m$, then M is called a *unitary X module*.

Definition 2.5. [1] A map $f : M \to N$, where M and N are X-modules, is an X-homomorphism if the following hold:

(i) $f(m + n) = f(m) + f(n)$, for all $m, n \in M$,

(ii) $f(x.m) = x.f(m)$, for all $m \in M$ and $x \in X$.

Proposition 2.6. [3] *Let M and N be two BCK-modules over commutative BCK-algebra X and $Hom(M, N) = \{f : f \text{ is a homomorphism from } M \text{ into } N\}$. Then $(Hom(M, N), +)$ forms an abelian group where $(f + g)(m) = f(m) + g(m)$, for any $f, g \in Hom(M, N)$ and $m \in M$. Moreover by operation $\bullet : X \times Hom(M, N) \longrightarrow Hom(M, N)$, $Hom(M, N)$ is an X-module, where $x \bullet f(m) = x.f(m)$.*

Theorem 2.7. [4] *Let X be a bounded implicative BCK-algebra. Then $(X, +)$, is an abelian group and X is an X-module, where $x + y = (x * y) \vee (y * x)$.*

Note. From now on, in this paper X is a BCK-algebra and M is an abelian group.

3. Extended BCK-Modules

Definition 3.8. Let operation $. : X \times M \longrightarrow M$ satisfies the following axioms:

$(XM1)$ $(x \wedge y).m = x.(y.m)$,

$(XM2)$ $x.(m + n) = x.m + x.n$,

$(XM3)$ $0.m = 0$,

$(XM4)$ $(x * y).m = x.m - y.m$, where $x * y \neq 0$, for $x \neq y$, for all $x, y \in X$ and $m, n \in M$. Then M is called an *extended BCK-module* or briefly X^E-module. If X is bounded and $1.m = m$, for any $m \in M$, then M is called a *unitary X^E-module.*

Example 3.9. Let X be a bounded implicative BCK-algebra such that "$\leq$" is totally ordered and operations "$+,.$":$X \times X \longrightarrow X$ are defined by, $x + y = (x * y) \vee (y * x)$, $x.y = x \wedge y$, for all $x, y \in X$. Then X is an X^E-module. By Theorem 2.7, it is enough to show that $(x * y).z = x.z - y.z$, for any $x, y, z \in X$, where $x * y \neq 0$ for $x \neq y$. If $x = y$, then the proof is clear. Now, let $x * y \neq 0$, for $x \neq y$. Since $x * y \neq 0$, $x \nleqslant y$ and so $y \leqslant x$ and this means that $y * x = 0$. Therefore,

$$\begin{aligned}
(x * y).z &= (x * y) \wedge z, \\
&= (x * y + 0) \wedge z, \quad \text{by Lemma 2.3(vii)} , \\
&= (x * y + y * x) \wedge z, \quad \text{since } y * x = 0, \\
&= (x + y) \wedge z, \quad \text{by Lemma 2.3(iv)}, \\
&= (x \wedge z) + (y \wedge z), \quad \text{by Lemma 2.3(v)}, \\
&= x.z + y.z, \\
&= x.z - y.z, \quad \text{by Lemma 2.3(vi)}.
\end{aligned}$$

Example 3.10. (i) Let X be a bounded commutative BCK-algebra such that $(X, .)$ be an X^E-module and A be an ideal of X. Then $\left(\dfrac{X}{A}, +'\right)$ is an abelian group, where $C_x +' C_y = C_{x+y}$ and $x + y = x * y \vee y * x$, for any $x, y \in X$. Moreover, if operation $\bullet : X \times \dfrac{X}{A} \longrightarrow \dfrac{X}{A}$ is defined by $x \bullet C_y = C_{x.y}$, for any $x, y \in X$, then $\dfrac{X}{A}$ is an X^E-module.

(ii) Let $X = \{0, x\}$ and operation "$*$" on X is defined by $0*x = 0*0 = x*x = 0$ and $x*0 = x$. Then $(X, *, 0)$ is a BCK-algebra. Now, let operation $. : X \times \mathbb{Z} \longrightarrow \mathbb{Z}$ is defined by $x.n = n$ and $0.n = 0$, for any $n \in \mathbb{Z}$. We claim that $\mathbb{Z}$ is an X^E-module. It is clear that $(x \wedge 0).n = 0.n = 0$, $x.(0.n) = x.0 = 0$ and so $(x \wedge 0).n = x.(0.n)$. Similarly $(x \wedge x).n = x.(x.n)$ and $(0 \wedge x).n = 0.(x.n)$. Then $(XM1)$ holds. The proof of $(XM2)$ and $(XM3)$ is clear. Moreover, since $(x * 0).n = n = x.n - 0.n$ and $(x * x).n = 0 = x.n - x.n$, $(XM4)$ holds.

(iii) It is easy to see that BCK-algebra $(X, *, 0)$ in (ii) is bounded with unit x. Moreover, $(X, +)$ is an abelian group, where $a + b = (a * b) \vee (b * a)$, for any $a, b \in X$. Now, let operation $. : X \times X \longrightarrow X$ is defined by $a.b = a \wedge b$, for any $a, b \in X$. Then $(X, +)$ is an X^E-module.

(iv) Let $X = \{0, a, b, 1\}$ and operation "$*$" on X is defined by

$*$	0	a	b	1
0	0	0	0	0
a	a	0	0	0
b	b	b	0	0
1	1	b	a	0

Then $(X, *, 0)$ is a bounded BCK-algebra. Let $M = \{0, a\} \subseteq X$. Then $(M, +)$ is an abelian group, where $x + y = (x * y) \vee (y * x)$, for any $x, y \in M$. We define the operation $. : X \times M \to M$ by

$$x.y = \begin{cases} a, & \text{if } x = b \text{ or } 1 \text{ and } y = a \\ 0, & \text{otherwise} \end{cases}$$

Then M is an X^E-module.

(v) Let $(X, *, 0)$ be a bounded BCK-algebra with unit 1, $1 \neq a \in X$ and $1 * a = 1$ or a. Now, if $Y = \{0, a, 1\}$, then Y is a subalgebra of X and so it is a BCK-algebra. Moreover, let $M = \{0, 1\} \subseteq X$. Then $(M, +)$ is an abelian group, where $x + y = (x * y) \vee (y * x)$, for any $x, y \in M$. Now, let the operation $. : Y \times M \to M$ is defined by $y.m = y \wedge m$ for any $y \in Y$ and $m \in M$. Then M is a Y^E-module.

(vi) Let $X = \{0, 1, 2, 3, 4\}$ and the operation " $*$ " is defined by

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	2	0	2	0
3	3	3	3	0	0
4	4	4	3	2	0

Then $(X, *, 0)$ is a bounded BCK-algebra. Let $Y = \{0, 1, 4\}$ and $M = \{0, 2, 3, 4\}$. It is clear that Y is a subalgebra of X and so is BCK-algebra. It is easy to show that $(M, +)$ is an abelian group, where $x + y = (x * y) \vee (y * x)$, for any $x, y \in M$. Now, we define the operation $. : Y \times M \to M$ by $y.m = y \wedge m$, for any $y \in Y$ and $m \in M$. Then M is a Y^E-module.

(vii) Let $X = \{P, \{2\}, \{1, 2\}\}$ be a subset of BCK-algebra [7, Example 2.8]. Then it is easy to see that $(X, \odot, P)$ is a BCK-algebra. If operation $. : X \times \mathbb{Z} \longrightarrow \mathbb{Z}$ is defined by $\{2\}.n = n$ and $\{1, 2\}.n = P.n = 0$, for any $n \in \mathbb{Z}$, then $\mathbb{Z}$ is an X^E-module.

Theorem 3.11. *Every X^E-module is an X-module.*

Proof. The proof is clear. ■

Example 3.12. Let X be a nonempty set. Then $(\mathcal{P}(X), -)$ is a bounded implicative BCK-algebra and $\mathbb{Z}$ is a $\mathcal{P}(X)$-module with operation $. : \mathcal{P}(X) \times \mathbb{Z} \longrightarrow \mathbb{Z}$ such that $A.n = \mu(A)n$, for any $A \subseteq X$, where for $a \in X$,

$$\mu(A) = \begin{cases} 0, & \text{if } a \notin A \\ 1, & \text{if } a \in A \end{cases}$$

But $\mathbb{Z}$ is not a $\mathcal{P}(X)^E$-module. Since for $A, B \in \mathcal{P}(X)$ such that $a \notin A, a \in B$, we have

$$(A - B).n = \mu(A - B)n = 0 \neq -n = 0 - n = A.n - B.n$$

and so $(XM4)$ is not true.

Definition 3.13. A map $f : M \to N$, where M and N are X^E-modules, is called an X^E-*homomorphism*, if the following hold:

(i) $f(m + n) = f(m) + f(n)$,

(ii) $f(x.m) = x.f(m)$, for all $m, n \in M$ and $x \in X$.

Proposition 3.14. *Let M, N be two X^E-modules and*

$$Hom(M, N) = \{f : f : M \to N \ is \ an \ X^E - homomorphism\}.$$

Then $(Hom(M, N)$ is an X^E-module by the operation which is defined in Proposition 2.6.

Proof. By Proposition 2.6, it is enough to show that $(x * y) \bullet f(m) = x \bullet f(m) - y \bullet f(m)$, for any $x, y \in X$, where $x * y \neq 0$, and $x \neq y$. Now, since N is an X^E-module, we have

$$(x * y) \bullet f(m) = (x * y).f(m) = x.f(m) - y.f(m) = x \bullet f(m) - y \bullet f(m). \quad \blacksquare$$

Theorem 3.15. *Let X be a bounded implicative BCK-algebra. Then, by the assumption of Example* 3.9, $\left(\sum_{i \in I} X, +'\right)$ *is an abelian group, where $\{x_i\}_{i \in I} +' \{y_i\}_{i \in I} = \{x_i + y_i\}_{i \in I}$, for any $\{x_i\}_{i \in I}, \{y_i\}_{i \in I} \in \sum_{i \in I} X$. Moreover, if the operation $. : X \times \sum_{i \in I} X \longrightarrow \sum_{i \in I} X$ is defined by $x.\{x_i\} = \{x \wedge x_i\}$, for any $x, x_i \in X$, $i \in \mathbb{N}$, then $\sum_{i \in I} X$ is an X^E-module.*

Proof. Since by Theorem 2.7, $(X, +)$ is an abelian group, then it is clear that $\left(\sum_{i \in I} X, +'\right)$ is an abelian group.

Now, for any $x, y, x_i, y_i \in X$ and $i \in \mathbb{N}$, we have:

$(XM1)$: $(x \wedge y).\{x_i\} = \{(x \wedge y) \wedge x_i\} = \{x \wedge (y \wedge x_i)\} = x.\{y \wedge x_i\} = x.(y.\{x_i\})$.

$(XM2)$: By Lemma 2.3(v),

$$\begin{aligned} x.(\{x_i\} +' \{y_i\}) &= x.\{x_i + y_i\} = \{x \wedge (x_i + y_i)\} = \{x \wedge x_i + x \wedge y_i\} \\ &= \{x \wedge x_i\} +' \{x \wedge y_i\} = x.\{x_i\} +' x.\{y_i\}. \end{aligned}$$

$(XM3)$: $0.\{x_i\} = \{0 \wedge x_i\} = \{0\}$.
$(XM4)$: Let $x * y \neq 0$ for $x \neq y$. Then by Lemma 2.3(v) and (vi),

$$\begin{aligned} x.\{x_i\} -' y.\{x_i\} &= \{x \wedge x_i\} -' \{y \wedge x_i\} \\ &= \{x \wedge x_i\} +' \{y \wedge x_i\} \\ &= \{x \wedge x_i + y \wedge x_i\} \\ &= \{(x+y) \wedge x_i\} \\ &= (x+y).\{x_i\} \\ &= (x*y + y*x).\{x_i\} \\ &= (x*y+0).\{x_i\} = (x*y).\{x_i\}. \end{aligned}$$ ∎

Theorem 3.16. *Let X be BCK-algebra in Theorem 3.15, and A be an ideal in X. Then $\left(\sum_{i\in I} \frac{X}{A}, \bar{+}\right)$ is an abelian group, where $\{C_{x_i}\}\bar{+}\{C_{y_i}\} = \{C_{x_i+y_i}\}$ and $x_i + y_i = x_i * y_i \vee y_i * x_i$, for any $x_i, y_i \in X$ and $i \in I$. Moreover, if we define $\bullet : X \times \sum_{i\in I} \frac{X}{A} \longrightarrow \sum_{i\in I} \frac{X}{A}$ by $x \bullet \{C_{x_i}\} = \{C_{x\wedge x_i}\}$, then $\left(\sum_{i\in I} \frac{X}{A}\right)$ is an X^E-module.*

Proof. It is easy to show that $\left(\sum_{i\in I} \frac{X}{A}, \bar{+}\right)$ is an abelian group and $\sum_{i\in I} \frac{X}{A}$ is an X^E-module. ∎

Theorem 3.17. *Let $(X, *)$ and $(Y, \star)$ be two BCK-algebras, M be a Y^E-module and $\Phi : X \longrightarrow Y$ be a BCK-homomorphism such that $x \neq 0$ implies that $\phi(x) \neq 0$, for any $x \in X$. If operation $\bullet : X \times M \longrightarrow M$ is defined by $x \bullet m = \phi(x).m$, for any $x \in X$ and $m \in M$, then M is an X^E-module.*

Proof. Let M be a Y^E-module and $\Phi : X \longrightarrow Y$ be a BCK-homomorphism such that $x \neq 0$ implies that $\phi(x) \neq 0$, for any $x \in X$. Then for any $x, y \in X$ and $m, n \in M$, we have:
$(XM1)_X$: By $(XM1)_Y$, we have

$$\begin{aligned} (x \wedge y) \bullet m &= \phi(x \wedge y).m = \phi(y * (y * x)).m = (\phi(y) \star (\phi(y) \star \phi(x))).m, \\ &= (\phi(x) \wedge \phi(y)).m = \phi(x).(\phi(y).m) = x \bullet (y \bullet m). \end{aligned}$$

$(XM2)_X$: By $(XM2)_Y$, we have

$$x \bullet (m+n) = \phi(x).(m+n) = \phi(x).m + \phi(x).n = x \bullet m + x \bullet n$$

$(XM3)_X$: $0 \bullet m = \phi(0).m = 0.m = 0$
$(XM4)_X$: By $(XM4)_Y$, where $x * y \neq 0$, for $x \neq y$ we have

$$(x*y) \bullet m = \phi(x*y).m = (\phi(x) \star \phi(y)).m = \phi(x).m - \phi(y).m = x \bullet m - y \bullet m.$$ ∎

Theorem 3.18. *Let X be a bounded commutative BCK-algebra, $(X,+)$ be an X^E-module and A be an ideal of X. Then $\frac{X}{A}$ is an X^E-module.*

Proof. Let X be a bounded commutative BCK-algebra, $(X,+)$ be an X^E-module and A be an ideal of X. It is easy to show that $\left(\dfrac{X}{A},+'\right)$ is an abelian group, where $C_x +' C_y = C_{x+y}$ and $x+y=(x*y)\vee(y*x)$, for any $x,y\in X$. Let operation $\bullet : X\times \dfrac{X}{A}\longrightarrow \dfrac{X}{A}$ is defined by $x\bullet C_y = C_{x.y}$, for any $x,y\in X$. Then we show that $\dfrac{X}{A}$ is an X^E-module. For $x,x',y\in X$,

$(XM1)_{\frac{X}{A}}$: By $(XM1)$, $(x\wedge x')\bullet C_y = C_{(x\wedge x').y} = C_{x.(x'.y)} = x\bullet(x'\bullet C_y)$

$(XM2)_{\frac{X}{A}}$: By $(XM2)$,

$$x\bullet(C_x+'C_y) = x\bullet C_{y+y'} = C_{x.(y+y')} = C_{x.y+x.y'} = C_{x.y}+'C_{x.y'} = x\bullet C_y +' x\bullet C_{y'}$$

$(XM3)_{\frac{X}{A}}$: By $(XM3)$, $0\bullet C_x = C_0$

$(XM4)_{\frac{X}{A}}$: Let $x*y\neq 0$, for $x\neq y$. By $(XM4)$,

$$(x*y)\bullet C_{y'} = C_{(x*y).y'} = C_{x.y'-y.y'} = C_{x.y'} -' C_{y.y'} = x\bullet C_{y'} -' y\bullet C_{y'}. \qquad \blacksquare$$

3. Prime submodules in X^E-modules

Definition 3.1. A subgroup N of X^E-module M is a *submodule* of M if for any $x\in X$ and any $n\in N$, $x.n\in N$.

Example 3.2. (i) By considering the Example 3.10 (ii), $2\mathbb{Z}$ is a submodule of $\mathbb{Z}$.

(ii) Let X be a bounded implicative BCK-algebra with the assumption of Example 3.9, and $M_r=\{x\in X : x\leq r\}$, where $r\in X$. Then M_r is a submodule of X. First we show that M is a subgroup of X. Let $m,n\in M_r$. By assumption, $m*n=0$ or $n*m=0$. W.L.G, $n*m=0$. Hence, by Lemma 2.3(vi), $m-n=m+n=(m*n)\vee(n*m)=(m*n)\vee 0=m*n$. On the other hand by $(BCK6)$, $m*n\leq m$ and $m\leq r$. Hence $m*n\leq r$ and so $m-n\in M_r$. It means that M_r is a subgroup of X. Now, we will show that $x.m\in M_r$, for any $x\in X$ and $m\in M_r$. By $(BCK4)$ and $(BCK6)$, we have

$$(x.m)*r=(x\wedge m)*r=(m*(m*x))*r=(m*r)*(m*x)=0*(m*x)=0$$

Hence, $x.m\leq r$ and so $x.m\in M_r$. Therefore, M_r is a submodule of X.

Lemma 3.3. *Let M be an X^E-module and N be a submodule of M. Then $\dfrac{M}{N}$ is an X^E-module.*

Proof. Let N be a submodule of M and operation $\bullet : X \times \dfrac{M}{N} \longrightarrow \dfrac{M}{N}$ is defined by $x \bullet (m+N) = x.m+N$, for any $x \in X$ and $m \in M$. Let $x = y$ and $m+N = m'+N$. Then $m - m' \in N$. Since N is a submodule of M, $x.(m - m') = x.m - x.m' \in N$ and so $x \bullet m + N = x \bullet m' + N$. It means that " $\bullet$ " is well defined. For any $x, y \in X$ and $m, m' \in M$,

$(XM1)_{\frac{M}{N}}$: By $(XM1)$,

$$(x \wedge y) \bullet (m+N) = (x \wedge y).m+N = x.(y.m)+N = x \bullet (y.m+N) = x \bullet (y \bullet (m+N))$$

$(XM2)_{\frac{M}{N}}$: By $(XM2)$,

$$\begin{aligned} x \bullet (m + N + m' + N) &= x.(m + m') + N = x.m + x.m' + N \\ &= x.m + N + x.m' + N = x \bullet (m + N) + x \bullet (m' + N) \end{aligned}$$

$(XM3)_{\frac{M}{N}}$: By $(XM3)$, $0 \bullet (m + N) = 0.m + N = N$

$(XM4)_{\frac{M}{N}}$: Let $x * y \neq 0$, for $x \neq y$. By $(XM4)$,

$$\begin{aligned} (x * y) \bullet (m + N) &= (x * y).m + N = (x.m - y.m) + N = x.m + N - y.m + N \\ &= x \bullet (m + N) - y \bullet (m + N). \end{aligned}$$ ∎

Theorem 3.4. *Let M, M' be X^E-modules, $\phi : M \longrightarrow M'$ be an X^E-homomorphism and N be a submodule of M such that $\phi(N) = 0$. Then there exists an X^E-homomorphism from $\dfrac{M}{N}$ to M'.*

Proof. We define $\bar{\phi} : \dfrac{M}{N} \longrightarrow M'$ by $\bar{\phi}(m + N) = \phi(m)$. It is easy to show that $\bar{\phi}$ is well defined and it is an X^E-homomorphism. ∎

Theorem 3.5. *Let M, M' be X^E-modules and $\phi : M \longrightarrow M'$ be an X^E-homomorphism. Then*

(i) *$Ker\phi$ and $Img\phi$ are submodules of M and M', respectively,*

(ii) $\dfrac{M}{Ker\phi} \simeq Img\phi$.

Proof. (i) The proof is clear.

(ii) We know that, $\phi : M \longrightarrow Img\phi$ is an epimorphism. Now, in Theorem 3.4, it is enough to consider $N = Ker\phi$. ∎

Theorem 3.6. *Let M be an X^E-module and N, K are submodules of M. Then*

(i) *$N + K = \{n + k : n \in N, k \in K\}$ and $N \cap K$ are X^E-modules,*

(ii) $\dfrac{K}{N \cap K} \simeq \dfrac{N + K}{N}$,

(iii) $\dfrac{K}{N}$ *is a submodule of* $\frac{M}{N}$ *and* $\dfrac{\frac{M}{N}}{\frac{K}{N}} \simeq \dfrac{M}{K}$, *where* $N \subseteq K$.

Proof. (i) $N + K$ is an X-module (*see*[3]). Now, let $x * y \neq 0$, where $x \neq y$, for any $x, y \in X$ and $n + k \in N + K$. Then,

$$(x*y).(n+k) = (x*y).n+(x*y).k = x.n-y.n+x.k-y.k = x.(n+k)-y.(n+k)$$

and so we have $(XM4)_{N+K}$. Therefore, $N+K$ is an X^E-module. Moreover, $N\cap K$ is an X-module (see [3]) and it is not difficult to verify the condition $(XM4)_{N\cap K}$. Hence $N \cap K$ is an X^E-module, too.

(ii), (iii) The proofs are easy. ∎

Theorem 3.7. *Let X be a bounded commutative BCK-algebra such that $x*y \neq x$, where $x \neq y$ for $x, y \neq 0$, M be an X^E-module and K be a proper submodule of M. Then $(K : M) = \{x \in X : x.M \subseteq K\}$ is a prime ideal of X.*

Proof. First, we show that $(K : M)$ is an ideal of X. If $(K : M) = X$, then $1.M \subseteq K$ and so $M \subseteq K$, which is a contradiction. Since K is a subgroup of M, $0.m = 0 \in K$, for any $m \in M$ and so $0 \in (K : M)$. Now, for any $x, y \in X$, let $x * y \in (K : M) = \{x \in X : x.M \subseteq K\}$ and $y \in (K : M)$. Then $(x * y).m \in K$ and $y.m \in K$, for any $m \in M$. If $x = y$ or $x = 0$, then it is clear that $x.m \in M$. So let $x \neq y$. If $x * y = 0$, then $x * (x * y) = x$ and so $x \wedge y = x$. Since $y.m \in K$, $x.(y.m) \in K$, for any $m \in M$ and K is a submodule of M and so $x.m = (x \wedge y).m \in K$. If $x * y \neq 0$, then by $(XM4)$, $x.m - y.m = (x * y)m \in K$. Since $(K, +)$ is a subgroup of M and $y.m \in K$, we have $x.m \in K$. Therefore, $(K : M)$ is an ideal of X.

Now, we prove that $(K : M)$ is prime. Let $x \wedge y \in (K : M)$, for $x, y \in X$. Then for any $m \in M$, $(x \wedge y).m \in K$ and so $(y * (y * x)).m \in K$. Now if $x = y$, then $x.m = (x * 0).m = (x * (x * x)).m = (x \wedge y).m \in K$. If $x = 0$ or $y = 0$, then it is clear that $x \in (K : M)$ or $y \in (K : M)$. If $x \neq y$, $x, y \neq 0$ and $x * y = 0$, then $x.m = (x * 0).m = (x * (x * y)).m = (y \wedge x).m = (x \wedge y).m \in K$, for any $m \in M$. If $x \neq y$, $x, y \neq 0$, $x * y \neq 0$, $x \neq x * y$ and $x * (x * y) \neq 0$. Then by $(XM4)$, $y.m = x.m-(x.m-y.m) = x.m-(x*y).m = x*(x*y).m = (y\wedge x).m = (x \wedge y).m \in K$, for any $m \in M$. Finally, if $x \neq y$, $x, y \neq 0$, $x * y \neq 0$, $x \neq x * y$ and $x * (x * y) = 0$, then by $(BCK6)$, we have $(x * y) * x = 0$ and so $x = x * y$, which is a contradiction. Therefore, $(K : M)$ is a prime ideal of X. ∎

Proposition 3.8. *Let M be an X^E-module. If for any $x, y \in X$, $x \neq y$ implies that $x * y \neq 0$, then $Ann_X(M) = \{x \in X : x.m = 0, \forall m \in M\}$ is an ideal of X.*

Proof. It is clear that $0 \in Ann_X(M)$. Now, let $x * y, y \in Ann_X(M)$ and $x \neq y$, for any $x, y \in X$. If $x = 0$, then it is clear that $x \in Ann_X(M)$. Now, let $x \neq 0$. Then by $(XM4)$, $x.m = x.m - 0 = x.m - y.m = (x * y).m = 0$, for any $m \in M$, and so $x \in Ann_X(M)$. Therefore, $Ann_X(M)$ is an ideal of X. ∎

Theorem 3.9. *Let M be an X^E-module and I be an ideal of X such that $I \subseteq Ann_X(M)$. If the operation $\bullet : X/I \times M \longrightarrow M$ is defined by $c_x \bullet m = x.m$, for any $x, y \in X$ and $m \in M$, then M is an $(X/I)^E$-module.*

Proof. Let $\bullet : X/I \times M \longrightarrow M$ is defined by $c_x \bullet m = x.m$, for any $x \in X$ and $m \in M$. First we prove that $\bullet$ is well defined. Let $c_x = c_y$, $m = n$ and $x \neq y$, for all $x, y \in X$ and $m, n \in M$. Then $x * y \in I$ and $y * x \in I$. If $x * y = y * x = 0$, then by $(BCK5)$, $x = y$, which is a contradiction. If $x * y \neq 0$ or $y * x \neq 0$, since $I \subseteq Ann_X(M)$, by $(XM4)$, $0 = (x * y).m = x.m - y.m$ or $0 = (y * x).m = y.m - x.m$ and so $x.m = y.m$. Now, since $m = n$, $x.m = y.n$. Hence, $\bullet$ is well defined. Now, we will show that M is an $(X/I)^E$-module.

$(XM1)_{X/I}$: We have $c_x \wedge c_y = c_y \star (c_y \star c_x) = c_{y*(y*x)}$, then by $(XM1)_X$,

$$(c_x \wedge c_y) \bullet m = (y * (y * x).m = (x \wedge y).m = x.(y.m) = c_x \bullet (c_y \bullet m)$$

$(XM2)_{X/I}$: By $(XM2)_X$,

$$c_x \bullet (m + n) = x.(m + n) = x.m + x.n = c_x \bullet m + c_x \bullet n.$$

$(XM3)_{X/I}$: $c_0 \bullet m = 0.m = 0$.

$(XM4)_{X/I}$: Let $c_x \star c_y \neq c_0$, for $c_x \neq c_y$. Hence $c_{x*y} \neq c_0$. Since $0 * (x * y) = 0 \in I$, $(x * y) * 0 = x * y \notin I$ and so $x * y \neq 0$. Therefore, by $(XM4)_X$,

$$(c_x \star c_y) \bullet m = c_{x*y} \bullet m = (x * y).m = x.m - y.m = c_x \bullet m - c_y \bullet m. \qquad \blacksquare$$

Notion. For X^E-module M, $Y \subseteq X$ and submodule N of M, we consider

$$Y.M = YM = \{x.m : x \in Y, m \in M\}.$$

Lemma 3.10. *Let X be a commutative BCK-algebra, M be an X^E-module, N be a submodule of M and I be an ideal of X. Then*

$$I.M + N = \left\{ \sum_{i=1}^{n} t_i.m_i + n : t_i \in I,\ m_i \in M,\ n \in N \right\}$$

is a submodule of M.

Proof. Let N be a submodule of M and I be an ideal of X. It is clear that " + " is an associative operation in $I.M + N$ and $0 \in I.M + N$. Moreover, by $(XM4)$,

$$\sum_{i=1}^{n} t_i.m_i + n - \left(\sum_{i=1}^{n} t_i.m_i + n \right) = \sum_{i=1}^{n} (t_i * t_i).m_i = 0,$$

for any $\sum_{i=1}^{n} t_i.m_i + n \in I.M + N$. Hence, every element in $I.M + N$ has an inverse element and so $I.M + N$ is a subgroup of M. Now, by $(XM1)$ and $(XM2)$,

$$\begin{aligned} x.\left(\sum_{i=1}^{n} t_i.m_i + n \right) &= \sum_{i=1}^{n} x.(t_i.m_i) + x.n = \sum_{i=1}^{n} (x \wedge t_i).m_i + x.n \\ &= \sum_{i=1}^{n} (t_i \wedge x).m_i + x.n = \sum_{i=1}^{n} t_i.(x.m_i) + x.n \in I.M + N, \end{aligned}$$

for any $\sum_{i=1}^{n} t_i.m_i + n \in I.M + N$ and $x \in X$. Therefore, $I.M + N$ is a submodule of M. ■

Theorem 3.11. *Let X be a bounded BCK-algebra, I be a proper ideal of X and M be an X^E-module. Then M/IM is an $(X/I)^E$-module.*

Proof. Let I be a proper ideal of X and M be an X^E-module. By Lemma 3.10, IM is a submodule of M. Now, we define $\bullet : X/I \times M/IM \longrightarrow M/IM$ by $c_x \bullet m + IM = x.m + IM$, for any $x \in X$ and $m \in M$. Since $I \bullet (M/IM) = \{x \bullet (m + IM) : x \in I, m \in M\} = \{x.m + IM : x \in I, m \in M\} = IM$, then $I \subseteq ann_X(M/IM)$. By Lemma 3.9, " $\bullet$ " is well defined. Now, we show that M/IM is an $(X/I)^E$-module, for any $x, y \in X$ and $m, n \in M$.

$(XM1)_{X/I}$: Since $(c_x \wedge c_y) = c_y \star (c_y \star c_x) = c_{y*(y*x)}$, by $(XM1)_X$,

$$\begin{aligned}(c_x \wedge c_y) \bullet (m + IM) &= c_{y*(y*x)} \bullet (m + IM) = (y * (y * x)).m + IM,\\ &= (x \wedge y).m + IM = x.(y.m) + IM,\\ &= c_x \bullet (y.m + IM) = c_x \bullet (c_y \bullet (m + IM))\end{aligned}$$

$(XM2)_{X/I}$: By $(XM2)_X$,

$$\begin{aligned}c_x \bullet ((m + IM) + (n + IM)) &= c_x \bullet (m + n + IM) = x.(m + n) + IM\\ &= (x.m + x.n) + IM, = x.m + IM + x.n + IM\\ &= c_x \bullet (m + IM) + c_x \bullet (n + IM)\end{aligned}$$

$(XM3)_{X/I}$: $c_0 \bullet (m + IM) = 0.m + IM = 0 + IM = IM = 0_{M/IM}$
$(XM4)_{X/I}$: If $c_x = c_y$, then by $(XM4)_X$,

$$\begin{aligned}(c_x \star c_y) \bullet (m + IM) &= c_{x*x} \bullet (m + IM) = c_0 \bullet (m + IM)\\ &= 0.m + IM = (x * x).m + IM,\\ &= x.m + IM - x.m + IM,\\ &= c_x \bullet (m + IM) - c_x \bullet (m + IM)\end{aligned}$$

Now, let $c_x \star c_y \neq 0$ where $c_x \neq c_y$. Then $c_{x*y} \neq c_0$ i.e., $(x * y) * 0 = x * y \notin I$ and so $x * y \neq 0$. Hence, by $(XM4)_X$,

$$\begin{aligned}(c_x \star c_y) \bullet (m + IM) &= c_{x*y} \bullet (m + IM) = (x * y).m + IM = (x.m - y.m) + IM,\\ &= x.m + IM - y.m + IM = c_x \bullet (m + IM) - c_y \bullet (m + IM)\end{aligned}$$

Therefore, M/IM is an $(X/I)^E$-module. ■

Definition 3.12. Let M be an X^E-module and N be a submodule of M. Then N is called a *prime* submodule of M, if $N \neq M$ and for any $x \in X$, $x.m \in N$ implies that $m \in N$ or $x \in (N : M)$.

Example 3.13. By considering the Example 3.10 (ii), $2\mathbb{Z}$ is a prime submodule of $\mathbb{Z}$. It is clear that $2\mathbb{Z}$ is a subgroup of $\mathbb{Z}$. Now, let $x.n \in 2\mathbb{Z}$. If $x \neq 0$, $x.n = n$, then $n \in 2\mathbb{Z}$. If $x = 0$, then $x.n = 0.n = 0$ and so $0 \in (2\mathbb{Z} : \mathbb{Z})$. Hence, $2\mathbb{Z}$ is a prime submodule of $\mathbb{Z}$.

Theorem 3.14. *Let X be a commutative BCK-algebra, M be an X^E-module and $N \neq M$ be a submodule of M. Then N is a prime submodule of M if and only if for any ideal I in X and for any submodule D of M, $ID \subseteq N$ implies that $I \subseteq (N : M)$ or $D \subseteq N$.*

Proof. ($\Rightarrow$) Let N be a prime submodule of M, I be an ideal in X and D be a submodule of M such that $ID \subseteq N$. We show that $I \subseteq (N : M)$ or $D \subseteq N$. Let $I \nsubseteq (N : M)$ and $D \nsubseteq N$. Then there exist $x \in X$ and $d \in D$ such that $x.M \nsubseteq N$ and $d \notin N$. On the other hand, $ID \subseteq N$ implies that $x.d \in N$. Since N is a prime submodule of M, $x.M \subseteq N$, which is a contradiction.

($\Leftarrow$) Let $x \in X$ and $m \in M$ such that $x.m \in N$ and $m \notin N$. Let $I = (x] = \{y \in X : y * x = 0\}$ and $D = \prec m \succ = \{y'.m : y' \in X\}$. For any $y \in I$, we have

$$y.m = (y * 0).m = y * (y * x).m = (x \wedge y).m = (y \wedge x).m = y.(x.m) \in N$$

So $ID = \{y.(y'.m) : y, y' \in X\} = \{y'.(y.m) : y, y' \in X\} \subseteq N$ and so $I \subseteq (N : M)$ or $D \subseteq N$. Since $m \notin N$, $I \subseteq (N : M)$ and this implies that $x.M \subseteq N$. Therefore, N is a prime submodule of M. ∎

Proposition 3.15. *Let M be an X^E-module and N be a submodule of M. Then P is a prime submodule of M if and only if $\frac{P}{N}$ is a prime submodule of $\frac{M}{N}$, where $N \subseteq P$.*

Proof. By Lemma 3.3, the proof is easy. ∎

Definition 3.16. Let M be an X^E-module. M is called *torsion* free if $x.m = 0$ implies that $m = 0$ or $x = 0$, for any $x \in X$ and $m \in M$.

Example 3.17. (i) In Example 3.10(ii), $\mathbb{Z}$ is a torsion free.

(ii) In Example 3.10(iv), M is not a torsion free. Because, $a.a = 0$ but $a \neq 0$.

Theorem 3.18. *Let X be bounded, M be a unitary X^E-module and K be a submodule of M. Then K is a prime submodule of M if and only if $P = (K : M)$ is a prime ideal of X and $\frac{M}{K}$ is a torsion free $\left(\frac{X}{P}\right)^E$-module, where $\left(\frac{X}{P}, \star, P\right)$ is a quotient BCK-algebra.*

Proof. ($\Rightarrow$) Let K be a prime submodule of M. By Theorem 3.7, $P = (K : M)$ is an ideal of X. If $X = (K : M)$, then $1 \in P$ and so $M = K$, which is a contradiction. Now, let $x \wedge y \in P$, for any $x, y \in X$. Then for any $m \in M$, $(x \wedge y).m \in K$ and so by $(XM1)$, $x.(y.m) \in K$. Since K is a prime submodule of M, we have $y.m \in K$ or $x \in (K : M)$. It means that $y \in (K : M)$ or

$x \in (K : M)$. Hence, $(K : M)$ is a prime ideal. Now, we show that $\dfrac{M}{K}$ is a torsion free $\left(\dfrac{X}{P}\right)^E$-module. Let the operation $\bullet : \dfrac{X}{P} \times \dfrac{M}{K} \longrightarrow \dfrac{M}{K}$ is defined by $c_x \bullet (m+K) = x.m+K$, for any $x \in X$, $m \in M$. Similar to the proof of Theorem 3.9, $\bullet$ is well defined. Finally, for any $c_x \in \dfrac{X}{P}$ and $m+K \in \dfrac{M}{K}$, we will show that, $c_x \bullet (m+K) = K$ implies that $c_x = c_0$ or $m+K = K$. Let $c_x \bullet (m+K) = K$, for any $x \in X$ and $m \in M$. Then $x.m+K = K$ and so $x.m \in K$. Since K is a prime submodule of M, $m \in K$ or $x \in (K : M)$. If $m \in K$, then $m + K = K$. Now, if $x \in (K : M) = P$, then $c_x = c_0 = P$ (because $x * 0 = x \in P$ and $0 * x = 0 \in P$). Therefore, $\dfrac{M}{K}$ is a torsion free.

($\Leftarrow$) Let P be a prime ideal in X and $\dfrac{M}{K}$ is a torsion free $\left(\dfrac{X}{P}\right)^E$-module. First we show that $K \subsetneqq M$. Since, if $K = M$, $P = (K : M) = (M : M) = X$, which is a contradiction. Now, let $x.m \in K$, for any $x \in X$, $m \in M$. Hence $x.m + K = K$ and so $c_x \bullet (m + K) = K$. Since $\dfrac{M}{K}$ is torsion free, $c_x = c_0 = P$ or $m + K = K$. This means that $x \in P$ or $m \in K$. Therefore, K is a prime submodule of M. ∎

Theorem 3.19. *Let X be a bounded commutative BCK-algebra, M be a unitary X^E-module, N be a submodule of M and P be a prime ideal of X. Then $K(N, P) = \{m \in M : c.m \in P.M + N, \exists c \in X - P\}$ is a submodule of M and $P.M + N \subseteq K(N, P)$.*

Proof. First, we show that $K(N, P)$ is a subgroup of M. Let $m, n \in K(N, P)$. Then there exists $c, c' \in X - P$ such that $c.m, c'.n \in P.M + N$. Let $t = c \wedge c'$. Then

$$\begin{aligned} t.(m-n) &= (c \wedge c').(m-n), \\ &= c.(c'.(m-n)) \quad \text{by } (XM1), \\ &= c.(c'.m - c'.n), \quad \text{by } (XM2), \\ &= c.(c'.m) - c.(c'.n) \quad \text{by } (XM2), \\ &= (c \wedge c').m - c.(c'.n), \quad \text{by } (XM1), \\ &= (c' \wedge c).m - c.(c'.n), \\ &= c'.(c.m) - c.(c'.n) \in P.M + N, \quad \text{by Lemma 3.10.} \end{aligned}$$

and so $m - n \in K(N, P)$, which means that $K(N, P)$ is a subgroup. Now, let $x \in X$ and $m \in K(N, P)$. Since $m \in K(N, P)$, there exists $c \in X - P$ such that $c.m \in P.M + N$. Now, by Lemma 3.10,

$$c.(x.m) = (c \wedge x).m = (x \wedge c).m = x.(c.m) \in P.M + N$$

Hence, $x.m \in K(N, P)$ and so $K(N, P)$ is a submodule of M. Finally, for any $m \in P.M+N$, if we let $c = 1$ then $c.m = 1.m = m \in P.M+N$, then $m \in K(N, P)$

(note that $1 \in X - P$, otherwise for any $x \in X$, $x * 1 = 0 \in P$ and $1 \in P$ results in $x \in P$ i.e., $X = P$, which is impossible). Therefore, $P.M + N \subseteq K(N, P)$. ■

Theorem 3.20. *Let X be a bounded commutative BCK-algebra, M be a unitary X^E-module, N be a submodule of M and P be a prime ideal of X. Then $K(N, P) = M$ or $K(N, P)$ is a prime submodule of M such that*

$$P = (K(N, P) : M).$$

Proof. Let $K(N, P) \neq M$. We will show that $K(N, P)$ is a prime submodule of M and $P = (K(N, P) : M)$. By Theorem 3.19, $K(N, P)$ is a submodule of M. Let $x.m \in K(N, P)$, for any $x \in X$, $m \in M$. Then there exists $c \in X - P$ such that $c.(x.m) \in P.M + N$. We will show that $m \in K(N, P)$ or $x \in (K(N, P) : M)$. If $x \in P$, then $x.M \subseteq P.M + N \subseteq K(N, P)$ and so $x.M \subseteq K(N, P)$. Hence $x \in (K(N, P) : M)$. If $x \notin P$, then $x \in X - P$. Since P is a prime ideal of X, $c \wedge x \in X - P$. because, if $c \wedge x \in P$, then $c \in P$ or $x \in P$, which is a contradiction. On the other hand, $c.(x.m) \in P.M + N$ and so $(c \wedge x).m \in P.M + N$. Hence, $m \in K(N, P)$. Therefore, $K(N, P)$ is a prime submodule of M. Now, we will prove that $P = (K(N, P) : M)$. Let $p \in P$. Then for any $m \in M$, $p.m \in P.M + N$. Let $c = 1$. Then $c.(p.m) \in P.M + N$ and so $p.m \in K(N, P)$, which implies that $P.M \subseteq K(N, P)$. Hence, $P \subseteq (K(N, P) : M)$. Now, let $q \in (K(N, P) : M)$ such that $q \notin P$. Since $q.M \subseteq K(N, P)$, $q.m \in K(N, P)$, for any $m \in M$. Hence there exists $c \in X - P$ such that $c.(q.t) \in P.M + N$ and so $(c \wedge q).t \in P.M + N$. Now, since P is prime, $c \wedge q \notin P$ i.e., $c \wedge q \in X - P$ and so $t \in K(N, P)$. Hence, $M = K(N, P)$, which is a contradiction. Then $q \in P$ and so $(K(N, P) : M) \subseteq P$. Therefore, $P = (K(N, P) : M)$. ■

Definition 3.21. Let M be an X^E-module and N be a submodule of M. The intersection of all prime submodules of M, including N, is called *radical* of N and it is shown by $rad_M(N)$. If there exists no prime submodule of M consisting of N, then we let $rad_M(N) = M$.

Theorem 3.22. *Let X be a bounded commutative BCK-algebra and M be an X^E-module. Then for any submodule N of M,*

$$rad_M(N) = \bigcap\{K(N, P) : P \textit{ is a prime ideal of } X\}.$$

Proof. Let $T = \bigcap\{K(N, P) : P \textit{ is a prime ideal of } X\}$ and $m \in T$. Let L be a prime submodule of M including of N. Hence, by Theorem 3.7, $Q = (L : M)$ is a prime ideal of X. Since for any prime ideal P of X, $m \in K(N, P)$, $m \in K(N, Q)$ and so there exists $c \in X - Q$ such that $c.m \in Q.M + N = (L : M).M + N \subseteq L + L \subseteq L$. Since L is a prime submodule of M and $c \notin Q = (L : M)$, $m \in L$. Hence $T \subseteq rad_M(N)$. Now, let $m \in rad_M(N)$. Hence, $m \in L$, where L is any prime submodule of M consisting of N and P be a prime ideal of X. If $K(N, P) = M$, then the proof is complete. Let $K(N, P) \neq M$. By Theorem 3.20, $K(N, P)$ is a prime submodule of M and $P = (K(N, P) : M)$. Now, we show that $N \subseteq K(N, P)$. By Theorem 3.19, we have $P.M + N \subseteq K(N, P)$ and so $N \subseteq K(N, P)$. Since $m \in rad_M(N)$, then $m \in K(N, P)$. Hence $m \in T$ and so $rad_M(N) \subseteq T$. Therefore, $rad_M(N) = T$. ■

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OD-CHARACTERIZATION OF ALTERNATING GROUP OF DEGREE $p+3$

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Abstract. Let A_{p+3} be the alternating group of degree $p+3$, where p is a prime, $p+4$ is a composite number, $p+6$ is a prime and $7 \neq p \in \pi(1000!)$. In the present paper, we prove that A_{p+3} is OD-characterizable by using the classification theorem of finite simple groups and Magma soft of computational group theory. This new method is introduced in order to deal with the subtle changes of the prime graph of a group in the discussion of its OD-characterization, which might occur. As a consequence of this theorem not only generalizes the result in [1] (Hoseini, A.A. and Moghaddamfar, A.R., *Frontiers of Mathematics in China*, 5 (3), 2010) but also gives a positive answer to a conjecture in [2] (Shi, W.J., Contemporary Math., 82, 1989).

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1. Introduction

Throughout this paper, groups under consideration are finite, and by a simple group, we always mean a nonabelian simple. For any group G, we denote by $\pi_e(G)$ the set of orders of its elements and by $\pi(G)$ the set of prime divisors of $|G|$. We associate to $\pi(G)$ a graph of G, denoted by $\Gamma(G)$ (cf. [3]). The vertex set of this graph is $\pi(G)$, and two distinct vertices p, q are adjacent by an edge if

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and only if $pq \in \pi_e(G)$, in this case, we write $p \sim q$. We also denote by $\pi(l)$ the set of all primes dividing l, where l is a positive integer.

In this article, we also use the following symbols. Let G be a finite group, then the socle of G is defined as the subgroup generated by minimal normal subgroups of G, denoted as $Soc(G)$. $Syl_p(G)$ denotes the set of all Sylow p-subgroups of G, where $p \in \pi(G)$, and P_r denotes a Sylow r-subgroup of G for $r \in \pi(G)$. Moreover, we use A_n to denote alternating group of degree n. Let q be a prime, we denote by $Exp(n, q)$ the exponent of the largest power of a prime q in the factorization of a positive integer $n(> 1)$. All further unexplained symbols and notations are standard and can be found in [4], for instance.

Definition 1.1 ([5]) Let G be a finite group and $|G| = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where $p_i s$ are primes and $\alpha_i s$ are integers. For $p \in \pi(G)$, let $deg(p) := |\{q \in \pi(G) | p \sim q\}|$, which we call the *degree* of p. We also define $D(G) := (deg(p_1), deg(p_2), ..., deg(p_k))$, where $p_1 < p_2 < \cdots < p_k$. We call $D(G)$ the *degree pattern* of G.

Definition 1.2 ([5]) A group M is called *k-fold OD-characterizable* if there exist exactly k non-isomorphic groups G such that

(1) $|G| = |M|$,

(2) $D(G) = D(M)$.

Moreover, a 1-fold OD-characterizable group is simply called an *OD-characterizable group.*

It is an interesting and difficult topic to determine the structure of finite groups by their orders and degree patterns. This topic is related to following open problem:

Open problem. ([5]) *Let G and M be finite groups satisfying the conditions*

(1) $|G| = |M|$,

(2) $D(G) = D(M)$.

Then

(i) *How far do these conditions effect the structure of G?*

(ii) *Is the number of non-isomorphic groups satisfying* (1) *and* (2) *finite?*

At present, we mention that the problem is still unsolved completely and till now we may not be able to provide a suitable answer for the above questions. This topic was studied in several articles. For example, in a series of articles (Ref. to [1],[6]-[17]), it was shown that many finite almost simple groups are m-fold OD-characterizable, where m is a positive integer and $m \geq 1$. For convenience, we summarize some results of these articles which will be used later in the following propositions:

Proposition 1.3 ([1], [6]-[8]) *A finite group G is OD-characterizable if G is isomorphic to one of the following groups:*

(1) *The alternating groups A_p, A_{p+1} and A_{p+2}, where p is a prime;*
(2) *The alternating groups A_{p+3}, where p is a prime and $7 \neq p \in \pi(100!)$;*
(3) *All finite simple K_4-groups except A_{10};*
(4) *All finite simple groups whose orders are less than 10^8 except for A_{10} and $U_4(2)$.*

Proposition 1.4 ([9]) *A finite group G is 2-fold OD-characterizable if and only if $|G| = |A_{10}|$ and (2) $D(G) = D(A_{10})$.*

2. Main results

According to Proposition 1.3, the alternating groups A_p, A_{p+1} and A_{p+2} are OD-characterizable, and A_{p+3} with $7 \neq p \in \pi(100!)$ is OD-characterizable. Proposition 1.4 says that the alternating group A_{10} is 2-fold OD-characterizable. On the other hand, by [10], we see that all A_n with $10 \neq n \leq 100$ is OD-characterizable. Now, omitting all the above alternating groups except A_{10}, there remains the following groups:

$$A_{10}, A_{106}, A_{112}, A_{116}, ..., A_{126}, A_{134}, A_{135}, A_{136}, A_{142}, ... \tag{2.1}$$

By [3], we know that all the alternating groups in (2.1) have connected prime graph. By these facts above, we see that it is very difficult to investigate how many-fold OD-characterization of alternating groups. In this paper, we continue to investigate this topic and get the following theorem:

Main Theorem. *All alternating groups A_{p+3}, where $p+2$ is a composite number and $p+4$ is a prime and $7 \neq p \in \pi(1000!)$, are OD-characterizable.*

Proposition 1.4 says A_{10} is 2-fold OD-characterizable. It is worth mentioning that A_{10} is the first alternating group which has not been considered for OD-characterizable. Up to now, we do not know whether there exists an alternating group A_n ($n \neq 10$) which isOD-characterizable. Hence, we put forward the following question:

Question. *Are the alternating groups A_n ($n \neq 10$) OD-characterizable?*

In fact, Main Theorem and Proposition 1.3 imply the following corollary.

Corollary 2.1 *Let A_n be an alternating group of degree n. Assume one of the following conditions is fulfilled:*

(1) *$n = p$, $p+1$ or $p+2$, where p is a prime;*
(2) *$n = p+3$, where $p+2$ is a composite number and $p+4$ is a prime and $7 \neq p \in \pi(1000!)$.*

Then A_n is OD-characterizable.

3. Preliminaries

In this section, we give some results which will be applied for our further investigations. We shall utilize the following Lemma 3.1 concerning the set of elements of the alternating and symmetric groups ([18]).

Lemma 3.1 ([18]) *The group S_n (or A_n) has an element of order $m = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_s^{\alpha_s}$, where $p_1, p_2, ..., p_s$ are distinct primes and $\alpha_1, \alpha_2, ..., \alpha_s$ are natural numbers, if and only if $p_1^{\alpha_1} + p_2^{\alpha_2} + \cdots + p_s^{\alpha_s} \leq n$ (or $p_1^{\alpha_1} + p_2^{\alpha_2} + \cdots + p_s^{\alpha_s} \leq n$ for m odd, and $p_1^{\alpha_1} + p_2^{\alpha_2} + \cdots + p_s^{\alpha_s} \leq n-2$ for m even).*

As an immediately corollary of Lemma 3.1, we have

Lemma 3.2 *Let A_n (or S_n) be an alternating group (or a symmetric group) of degree n. Then, the following assertions hold:*

(1) *Let $p, q \in \pi(A_n)$ be odd primes. Then $p \sim q$ if and only if $p + q \leq n$.*
(2) *Let $p \in \pi(A_n)$ be an odd prime. Then $2 \sim p$ if and only if $p + 4 \leq n$.*
(3) *Let $p, q \in \pi(S_n)$. Then $p \sim q$ if and only if $p + q \leq n$.*

Lemma 3.3 ([19]) *Let G be a finite solvable group all of whose elements are of prime power order. Then $|\pi(G)| \leq 2$.*

Lemma 3.4 *Let A_{p+3} be an alternating group of degree $p+3$, where p is a prime and $p+2$ is a composite number. Suppose that $|\pi(A_{p+3})| = d$. Then the following assertions hold.*

(i) *$deg(2) = d-2$. Particularly, $2 \sim r$ for each $r \in \pi(A_{p+3}) \backslash \{p\}$.*

(ii) *$deg(3) = d-1$, i.e., $3 \sim r$ for each $r \in \pi(A_{p+3})$.*

(iii) *$deg(p) = 1$. In other words, $p \sim r$, where $r \in \pi(A_{p+3})$, if and only if $r = 3$.*

(iv) *$Exp(|A_{p+3}|, 2) = \sum_{i=1}^{\infty} \left[\frac{p+3}{2^i}\right] - 1$. In particular, $Exp(|A_{p+3}|, 2) < p+3$.*

(v) *$Exp(|A_{p+3}|, r) = \sum_{i=1}^{\infty} \left[\frac{p+3}{r^i}\right]$ for each $r \in \pi(A_{p+3}) \backslash \{2\}$.*

Furthermore, $Exp(|A_{p+3}|, r) < \frac{p-1}{2}$, where $5 \leq r \in \pi(A_{p+3})$.

Particularly, if $r > \left[\frac{p+3}{2}\right]$, then $Exp(|A_{p+3}|, r) = 1$.

Proof. By Lemma 3.2, one has that $2 \nsim p$. Obviously, $r + 4 \leq p + 3$ for each $r \in \pi(A_{p+3}) \backslash \{p\}$, it follows that $2 \sim r$ and so $deg(2) = d-2$. By the same reason, we have $deg(3) = d-1$. For $r \in \pi(A_{p+3}) \backslash \{2, p\}$, by Lemma 3.2, it is easy to see that $p \sim r$ if and only if $p + r \leq p + 3$. Hence $r = 3$ and $deg(p) = 1$. Therefore, (i), (ii) and (iii) hold.

By the definition of Gauss's integer function, we can get that

$$\begin{aligned} Exp(|A_{p+3}|,2) &= \sum_{i=1}^{\infty}\left[\frac{p+3}{2^i}\right]-1 \\ &= \left(\left[\frac{p+3}{2}\right]+\left[\frac{p+3}{2^2}\right]+\left[\frac{p+3}{2^3}\right]+\cdots\right)-1 \\ &\leq \left(\frac{p+3}{2}+\frac{p+3}{2^2}+\frac{p+3}{2^3}+\cdots\right)-1 \\ &= (p+3)\left(\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots\right)-1=p+2. \end{aligned}$$

Hence

$$Exp(|A_{p+3}|,2)<p+3.$$

Thus (iv) follows.

By the same reason as above, we can prove

$$Exp(|A_{p+3}|,r)<\frac{p-1}{2},$$

where $5\leq r\in\pi(A_{p+3})$.

Clearly, if $r>\left[\frac{p+3}{2}\right]$, then we have

$$Exp(|A_{p+3}|,r)=1.$$

Hence (v) follows. This completes the proof of Lemma 3.4. ∎

Lemma 3.5 ([20]) *Let a be an arbitrary integer and m be a positive integer. If $(a,m)=1$, then the equation $a^x\equiv 1\,(mod\ m)$ has solutions. Moreover, if the order of a modulo m is $h(a)$, then $h(a)|\varphi(m)$, where $\varphi(m)$ is Euler's function of m.*

Lemma 3.6 *Let A_{p+3} be an alternating group of degree $p+3$, where $p+2$ is a composite number and $p+4$ is a prime and $97<p\in\pi(1000!)$. Set $P\in Syl_p(A_{p+3})$ and $Q\in Syl_q(A_{p+3})$, where $5\leq q<p$. Then, the following assertions hold.*

(i) $q^{s(q)}\nmid|N_G(P)|$, *where* $s(q)=Exp(|A_{p+3}|,q)$.

(ii) *If* $p\in\{103,109,163,193,223,229,277,349,439,463,499,613,643,739,769,823,853,877,907,967\}$, *then* $p\nmid|N_G(Q)|$.

(iii) *If* $p\in\{127,307,313,379,397,457,487,673,757,859,883,937\}$, *then there exists at least a prime number, say r, such that the order of r modulo p is less than $p-1$, where $5\leq r<p$ and $r\in\pi(A_{p+3})$.*

Proof. It is easy to see that the equation $q^x \equiv 1(\,mod\,p)$ has solutions by Lemma 3.5. Suppose that the order of q modulo p is $h(q)$. If $h(q) = p-1$, then q is a primitive root of modulo p. By hypothesis, using Magma soft of computational group theory, we know that there are only 32 such groups satisfying the conditions that $p+2$ is a composite number and $p+4$ is a prime number and $97 < p \in \pi(1000!)$. Again, using Magma of mathematics soft, we can obtain $h(q)$. For convenience, we have computed the values of p and $h(q)$ listed in Table 1 of this article.

Table 1 (p **and** $h(q)$)

p	$h(q)$	Condition	p	$h(q)$	Condition	p	$h(q)$	Condition
103	$2{\cdot}3{\cdot}17$	none	109	$2^2{\cdot}3^3$	none	163	$2{\cdot}3^4$	none
127	$2{\cdot}3^2{\cdot}7$	$q \neq 19$	127	3	$q = 19$	193	$2^6{\cdot}3$	none
223	$2{\cdot}3{\cdot}37$	none	229	$2^2{\cdot}3{\cdot}19$	none	277	$2^2{\cdot}3{\cdot}23$	none
307	$2{\cdot}3^2{\cdot}17$	$q \neq 17$	307	3	$q = 17$	349	$2^2{\cdot}3{\cdot}29$	none
313	$2^3{\cdot}3{\cdot}13$	$q \neq 5$	313	8	$q = 5$	439	$2{\cdot}3{\cdot}73$	none
379	$2{\cdot}3^3{\cdot}7$	$q \neq 5$	379	21	$q = 5$	397	$2^2{\cdot}3^2{\cdot}11$	$q \neq 31$
397	11	$q = 31$	457	$2^3{\cdot}3{\cdot}19$	$q \neq 109$	457	4	$q = 109$
463	$2{\cdot}3{\cdot}7{\cdot}11$	none	487	$2{\cdot}3^5$	$q \neq 5, 41$	487	54	$q = 5$
487	9	$q = 41$	499	$2{\cdot}3{\cdot}83$	none	613	$2^2{\cdot}3^2{\cdot}17$	none
643	$2{\cdot}3{\cdot}107$	none	673	$2^5{\cdot}3{\cdot}7$	$q \neq 23$	673	14	$q = 23$
739	$2{\cdot}3^2{\cdot}41$	none	757	$2^2{\cdot}3^3{\cdot}7$	$q \neq 59$	757	7	$q = 59$
769	$2^8{\cdot}3$	none	823	$2{\cdot}3{\cdot}137$	none	853	$2^2{\cdot}3{\cdot}71$	none
859	$2{\cdot}3{\cdot}11{\cdot}13$	$q \neq 13$	859	11	$q = 13$	877	$2{\cdot}3^2{\cdot}73$	none
883	$2{\cdot}3^2{\cdot}7^2$	$q \neq 71$	883	7	$q = 71$	907	$2{\cdot}3{\cdot}151$	none
937	$2^3{\cdot}3^2{\cdot}13$	$q \neq 13, 23$	937	18	$q = 13$	937	24	$q = 23$
967	$2{\cdot}3{\cdot}7{\cdot}23$	none						

By N-C Theorem, the factor group $N_G(P)/C_G(P)$ is isomorphic to a subgroup of $Aut(P) \cong \mathbb{Z}_{p-1}$. Thus, $|N_G(P)/C_G(P)|\big|(p-1)$. By Table 1, if there exists a prime q, where $5 \leq q < p$ and $q \in \pi(A_{p+3})$, such that $q^{s(q)}\big||N_G(P)|$, then $q\big||C_G(P)|$. Hence $deg(p) \geq 2$, a contradiction, and so (i) is proved.

We next assume that

$$p \in \{103, 109, 163, 193, 223, 229, 277, 349, 439, 463, 499, 613, 643, 739, 769, 823, 853, 877, 907, 967\}.$$

If $p\big||N_G(Q)|$, by Table 1, and $Exp(|A_{p+3}|, q) < p$, then $p\big||C_G(Q)|$, which leads to a similar contradiction. Thus (ii) holds. The remaining parts of (iii) follows at once from Table 1. This completes the proof of Lemma 3.6.

Lemma 3.7 ([4], [21]) *Let M be a finite nonabelian simple group with order having prime divisors at most* 997*. Then M is isomorphic to one of the simple groups listed in Table* 1-3 *in* [21]*. Particularly, if* $|\, \pi(Out(M))\, | \neq 1$*, then* $\pi(Out(M)) \subseteq \{2, 3, 5, 7\}$.

Lemma 3.8 ([22]) *Let $S = P_1 \times P_2 \times \cdots \times P_r$, where P_is are isomorphic non-abelian simple groups. Then $Aut(S) = (Aut(P_1) \times Aut(P_2) \times \cdots \times Aut(P_r)) \rtimes \mathbb{S}_r$.*

4. OD-Characterization of Alternating Group A_{p+3}

We again recall that all the alternating groups A_p, A_{p+1} and A_{p+2}, where p is a prime number, are OD-characterizable (see Proposition 1.3 (1)). On the other hand, it has been shown that all the alternating groups A_{p+3} with $7 \neq p \in \pi(100!)$ are OD-characterizable (see Proposition 1.3 (2)). It is worth to mention that alternating group A_{10} is 2-fold OD-characterizable (see Proposition 1.4). Moreover, So far we have not found an alternating group which is not OD-characterizable. Hence, Professor Moghaddamfar, A. R. in [11] put forward the following conjecture 1 of this article:

Conjecture 1 ([11]) *All alternating groups A_{p+3} with $p \neq 7$ are OD-characterizable.*

In this section, we are going to give an affirmative answer to this conjecture for all the alternating groups A_{p+3}, where $p+2$ is a composite number and $p+4$ is a prime and $7 \neq p \in \pi(1000!)$. In other words, we will prove the following Main Theorem. We need to mention that this result not only generalizes the results in [1] but also gives an affirmative answer to the **Question** of this article for the alternating group A_{p+3}.

Main Theorem *All alternating groups A_{p+3}, where $p+2$ is a composite and $p+4$ is a prime and $7 \neq p \in \pi(1000!)$, are OD-characterizable.*

Proof. Let G be a finite group satisfying the conditions: (1) $|G| = |S_{p+3}|$ and (2) $D(G) = D(S_{p+3})$, where $p+2$ is a composite number and $p+4$ is a prime and $7 \neq p \in \pi(1000!)$. By [10], we only discuss the alternating groups A_{p+3}, where $p+2$ is a composite and $p+4$ is a prime and $97 < p \in \pi(1000!)$. By these hypotheses, we deduce that $\{r\} \cup \{rs | r+s \leq p+3\} \subseteq \pi_e(G)$ and $\{rs | r+s > p+3\} \cap \pi_e(G) = \varnothing$, where $r, s \in \pi(G)$. By Lemma 3.4, the prime graph of G is connected since $deg(3) = |\pi(G)| - 1$. Moreover, by the structure of $D(G)$, it is easy to check that $\Gamma(G) = \Gamma(A_{p+3})$. In the following, we write the proof in a number of separate cases. ∎

Case 1. Let K be the maximal normal solvable subgroup of G. Then K is a $\{2,3\}$-group. Particularly, G is nonsolvable.

Proof. We first assert that K is a p'-group. If not, let p divides the order of K. Set $P \in Syl_p(G)$. By Lemma 3.6 (i), one has that $q^{s(q)} \nmid |N_G(P)|$ for each prime $q \in \pi(G)$ and $5 \leq q < p$. If $q \big| |N_G(P)|$, then either $q \big| |C_G(P)|$ or $q \in \pi(K)$. For the former, by Lemma 3.4 (iii), this leads to an obvious contradiction since $q \sim p$. In the latter case, i.e., $q \in \pi(K)$. In this case, by Table 1, it is easy to check that there necessarily exists such a prime r such that $r \not\sim q$, where $5 \leq r < p$

and $r \in \pi(K)$. In fact, by Lemma 3.2 (1), it is sufficient to find such a prime r such that $r+q>p$, then $r \not\sim q$. Since K is solvable, it possesses a Hall $\{p,q,r\}$-subgroup T. It follows that T is solvable. Since there exists no edge between p, q and r in $\Gamma(G)$, all elements in T are of prime power order. Hence $|\pi(T)| \leq 2$ by Lemma 3.3, a contradiction. Thus K is a p'-group.

Next, we show that K is a q'-group for each $q \in \pi(G) \backslash \{2,3,p\}$. Set $Q \in Syl_q(K)$, where $q \in \pi(K)$. Suppose that the order of q modulo p is $h(q)$. By the Frattini argument, $G = KN_G(Q)$, hence p divides the order of $N_G(Q)$. By Lemma 3.6 (ii) and (iii), it is easy to see that p is one of the possible primes: 127, 307, 313, 379, 397, 457, 487, 673, 757, 859, 883 and 937. In this case, there necessarily exists at least a prime, say q, such that $h(q) < p-1$. We prove the lemma up to choice of p one by one. The proof is divided into 3 cases.

Case 1.1. $p = 127$.

By Table 1, If there exists a prime q such that $p \big| |N_G(Q)|$, where $Q \in Syl_q(G)$, then $q = 19$. Now, by N-C Theorem, $N_G(Q)/C_G(Q) \lesssim Aut(Q)$. By Lemma 3.4 (v), we have $Exp(|G|, 19) = 6$, thus $|N_G(Q)/C_G(Q)| \big| \prod_{i=1}^{6} 19^{15} \cdot (19^i - 1)$. It is to check that $113 \nmid \prod_{i=1}^{6} 19^{15} \cdot (19^i - 1)$. If $113 \big| |N_G(Q)|$, then $113 \in \pi(C_G(Q))$. Thus $19 \sim 113$, a contradiction. Therefore, $113 \in \pi(K)$. Since K is a solvable group, it possesses a Hall $\{19, 113\}$-subgroup H of order $19^6 \cdot 113$. Clearly, H is nilpotent, so $19 \sim 113$, which leads to a contradiction as above.

Case 1.2. To prove the lemma follows if $p = 307$.

It is easy to see that there exists a prime, say q, such that $p \big| |N_G(Q)|$, where $Q \in Syl_q(G)$. Then $q = 17$ by Table 1. Since $N_G(Q)/C_G(Q) \lesssim Aut(Q)$ by N-C Theorem and $Exp(|G|, 17) = 19$ by Lemma 3.4, we have $|N_G(Q)/C_G(Q)| \big| \prod_{i=1}^{19} 17^{171} \cdot (17^i - 1)$. Using Magma soft, one has that $31 \nmid \prod_{i=1}^{19} 17^{171} \cdot (17^i - 1)$. If $31 \big| |N_G(Q)|$, then $31 \in \pi(C_G(Q))$. Set $N = N_G(Q)$, $C = C_G(Q)$ and $K_{31} \in Syl_{31}(C_G(Q))$. By Lemma 3.4, we have $Exp(|G|, 31) = 9$. By the Frattini argument, we can obtain that $N = CN_N(K_{31})$, and hence $p \nmid |N_N(K_{31})|$. Thus $p \big| |C_G(Q)|$, and so $deg(p) \geq 3$, which is a contradiction. Therefore, $31 \nmid |N_G(Q)|$ and $31 \in \pi(K)$. Set $P_{31} \in Syl_{31}(K)$. Then $P_{31} \in Syl_{31}(G)$. Since $G = KN_G(P_{31})$, then $p \big| |N_G(P_{31})|$. By Table 1, we see that this is impossible.

Case 1.3. Till now we have proved that K is a q'-group while $p = 127$ or 307. Assume that p is equal to one of the remaining primes. Now, we have to discuss ten cases up choice of p one by one. If K is a q-group for each $q \in \pi(G) \backslash \{2,3,p\}$, we can prove that this is impossible by checking each choice of p. Since the methods used below is completely the same as in Case 2, hence, we omitted the detailed processes. Therefore K is a $\{2,3\}$-group. Since $K \neq G$, it follows at once that G is a nonsolvable group, which completes the proof of Case 1.3 and also completes the proof of Case 1.

Case 2. G/K is an almost simple group. In other words, there exists a nonabelian simple group S such that $S \lesssim G/K \lesssim Aut(S)$.

Proof. Let $\overline{G} := G/K$ and $S := Soc(\overline{G})$. Then $S = P_1 \times P_2 \times \cdots \times P_s$, where P_i $(i = 1, 2, ..., s)$ are nonabelian simple groups and $S \lesssim \overline{G} \lesssim Aut(S)$. We will show that $s = 1$, and hence $S = P_1$.

Suppose that $s \geq 2$. We assert that p does not divide the order of S. Otherwise, there exists a prime r such that $r \sim p$, where $5 \leq r < p$ and $r \in \pi(G)$, an obvious contradiction to Lemma 3.4 (iii). Hence, for every i we have $P_i \in \mathcal{F}_p$ (cf.[21]). By Lemma 3.7, we observe that $p \in \pi(\overline{G}) \subseteq \pi(Aut(S))$. Thus $p\big||Out(S)|$. But $Out(S) = Out(S_1) \times Out(S_2) \times \cdots \times Out(S_r)$, where the groups S_j are direct products of all isomorphic $P_i^{'}s$ such that $S = S_1 \times S_2 \times \cdots \times S_r$. Therefore, for some j, p divides the order of an outer automorphism group of a direct product S_j of t isomorphic simple groups P_i for some $1 \leq i \leq s$. Since $P_i \in \mathcal{F}_p$, it follows that $|Out(P_i)|$ is not divided by p by Lemma 3.7. Now, by Lemma 3.8, we obtain that $|Aut(S_j)| = |Aut(P_i)|^t \cdot t!$. Therefore, $t \geq p$ and so 2^{2p} must divide the order of G. However, $Exp(|S_{p+3}|, 2) \leq p+3 < 2p$ by Lemma 3.4 (iv), which is a contradiction. Thus $s = 1$ and $S = P_1$. This completes the proof of Case 2.

Case 3 $G \cong A_{p+3}$. In other words, A_{p+3} is OD-characterizable.

Proof. By Lemma 3.7 and Case 1, we may assume that $|S| = \dfrac{|G|}{2^{k_1} \cdot 3^{k_2}} \cdot 2^{\beta_1} \cdot 3^{\beta_2}$, where $2 \leq \beta_1 \leq Exp(|A_{p+3}|, 2) = k_1$, $1 \leq \beta_2 \leq Exp\,(|A_{p+3}|, 3) = k_2$. Let $p_1, p_2, p_3, ..., p_s$ be distinct consecutive primes and $2 = p_1 < p_2 < p_3 < \cdots < p = p_s$, then $|G|_{p_j} = Exp\,(|A_{p+3}|, p_j)$ for any $j \geq 3$. Using Table 1-3 in [21], we deduce that S can only be isomorphic to one of the simple groups: A_p, A_{p+1}, A_{p+2} and A_{p+3}.

If $S \cong A_p$, then K is a 2-group. In this case, it is easy to see that $3p \in \pi_e(\overline{G}) \backslash \pi_e(S_p)$, a contradiction.

By the similar reason as above, $S \ncong A_{p+1}$ or A_{p+2}. Therefore, $S \cong A_{p+3}$. According to the consequence of Case 2, one has that

$$A_{p+3} \lesssim G/K \lesssim Aut(A_{p+3}) \cong S_{p+3}.$$

If $G/K \cong S_{p+3}$, then $2^{Exp(|S_{p+3}|,2)}\big||G|$, a contradiction.

If $G/K \cong A_{p+3}$, then $|K| = 1$. Therefore $G \cong A_{p+3}$. This completes the proof of Case 3 and also the proof of Main Theorem. ∎

In 1989, in [2] Professor Shi, W. J. put forward the following conjecture.

Conjecture 1 [2] *Let G be a group and M a finite simple group. Then $G \cong M$ if and only if*

(1) $|G| = |M|$, *and*

(2) $\pi_e(G) = \pi_e(M)$.

The above Conjecture 1 was proved by joint works of many mathematicians, the last part of the proof was given by V.D. Mozurov et al. in [23]. That is, the following theorem holds.

Theorem 4.1 [23] *Let G be a group and M a finite simple group. Then $G \cong M$ if and only if*

(1) $|G| = |M|$, *and*

(2) $\pi_e(G) = \pi_e(M)$.

In fact, Theorem 4.1 is valid for alternating group A_{p+3} since $\pi_e(G) = \pi_e(A_{p+3})$ implies G and A_{p+3} have the same prime graph. Thus they have the same degree pattern. Therefore, we can get the following corollary.

Corollary 4.2 *If G is a finite group such that*

(1) $|G| = |A_{p+3}|$,*and*

(2) $\pi_e(G) = \pi_e(A_{p+3})$, *where* $7 \neq p \in \pi(1000!)$,

then $G \cong A_{p+3}$.

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PARTITIONED FRAMES IN DISCRETE BAK SNEPPEN MODELS

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Abstract. In this paper, we wish to present some simplified cases of discrete Bak-Sneppen models in which explicit computations via Markov chains are possible, hence reaching a better understanding of some rather hidden phenomena of the general case: in particular "avalanches" can be read in terms of mean waiting times and in terms of transitions between structures. The simple models allow us to introduce new frames that do not seem to have been considered in the previous literature, namely the case of partitioned Bak-Sneppen frames, that appear more realistic from the point of view of speed of evolution and do not present a unique criticality level, but a staircase tending towards a final equilibrium level, cadenced by an increasing sequence of footholds. The introduction summarizes Bak-Sneppen models, starting from the central model due to Bak and Sneppen, and recalls their use in applied sciences. The first section gives the general frame of models where locality and globality coexist, the second section shows the simplest case of a matching between locality and globality, that will become exemplar in the most complex frames of Bak-Sneppen processes. The main quantitative theorems are stated and proved in the third section and finally the fourth section presents examples that illustrate the more sophisticated points of our paper and the use (and limits) of experimental results, while the fifth section considers real world situations where Bak Sneppen partitioned schemes can be tailored to represent the core of their evolution.

Keywords: Self organizing criticalities, Markov Chains, Bak Sneppen processes, Econophysics, Socio-economic evolution staircase.

AMS Mathematics Subject Classification: 34C28, 37H99, 60J10, 60J22, 91B80.

Introduction

Bak-Sneppen ([1], [2]) original model (BS) can be defined as follows. There are n species arranged on a circle, each of which has been assigned a random fitness. The fitness values are independent and uniformly distributed on $(0;1)$. At each discrete time step the system is updated by locating the lowest fitness and replacing this fitness, and those of its two neighbors, by independent and uniform $(0;1)$ random variables. This model is the result of a powerful synthesis of non equilibrium systems displaying self organized criticality, a concept introduced by P. Bak, C. Tang and K. Wiesenfeld [3]. One of the most fundamental characteristics of a system in a self-organized critical state is to exhibit a stationary state with a long-range power law decay of both spatial and temporal correlations.

Usually, the self organized state is attained only after a very long period of transient, a minor change in the system can cause colossal instabilities called avalanches. Intermittent burst of activity separating long periods of quiescence is called punctuated equilibrium.

An f_0-avalanche is defined as the event when all fitness initially above a threshold f_0 are perturbed such that for a certain time there are some below f_0. The event ends as soon as all fitness are again above f_0. For a certain value of f_0, namely $f_0 = f_c$ one obtains scale free avalanches, i.e. their distribution in size and duration follows a power law. The exponent of this power law is not easy to measure and still debated ([12], [23]). The distribution above f_c seems to be uniform, and asymptotically one expects a step function for the distribution of fitness. The value of f_c is given in ([33], [23]) as $f_c = 0,66702$ (a simulation value).

BS models can be defined on a wide range of graphs using the same update rule as above. What the BS model illustrates is that even random processes can result in self-organization to a critical state, see [31] for a discussion. We cannot describe the many studies that have arisen in physics, probability, econophysic following the first paper; we just recall some developments that are connected with the present paper. There is anyhow a strong believing in the power of these methods for constructing economic models that should suggest what actually happens in many-agent phaenomena. Though the provisional validity is small, these models can be richer than many econometric sophisticated simulations.

The authors in particular think that a sound basis for applying BS model of contact with neighbors is given by Duesenberry demonstration effect. Its first presentation can be found in Duesenberry's [17], while many application in consumer's economy and sociology can be found for example in Cavalli [6]. We may also recall Cuniberti et al. [11] and Rotundo–Scozzari [38], Rotundo–Ausloos [37].

It has been debated if changing the microscopic dynamical rules in BS model does or does not change the self organized universality class of the system [34], [15]. It is therefore interesting to study the robustness of BS-type models when the interaction rules are changed. A number of variants of Bak and Sneppen original model have been introduced which evolve according to different criteria. One variant is the anisotropic Bak Sneppen model [26], [30], [22], in which, in addition to the least fit species, only its right-hand nearest neighbor is replaced.

This model also gives rise to a threshold value $f_c = 0.724$ [22]. Another variant on the BS model which eliminates topology is the mean-field version analysed in [21], [13], [29], in which one replaces the smallest fitness and a fixed number of randomly chosen other ones.

In [22] the authors perform a detailed investigation of the effect of symmetry on the scaling behavior of the BS model. In their generalized model, the site with the minimum fitness plus a neighbors on the left side and b neighbors on the right are replaced with independent random numbers. If $a = b = 1$ it is recovered the original model; if $a = 0$ and $b = 1$ the anisotropic BS model; if $a \neq b$ is obtained the modified BS models with asymmetric dynamics. They conjectured that all dynamics which preserve the reflection symmetry of the original BS model possess the same critical exponents as the original model, while asymmetric dynamics lead to the exponents of the anisotropic BS model, reinforcing the evidence for two symmetry-based universality classes [26]. We suppose that, apart segmentation, that obviously perturbs also the symmetric class, the asymmetric models do not belong to a unique class, but rather to a certain number of slightly differentiated classes as it is shown in the examples of 19 and 24 nodes (see Example 3).

Motivated by the difficulty of analyzing rigorously even the one-dimensional version of the BS model, J. Barbay and C. Kenyon [5] proposed a still simpler model with discrete fitness values.[1] In their model, each species has fitness 0 or 1, and each new fitness is drawn from the Bernoulli distribution with parameter p. Since there are typically several least fit species, the process then repeatedly chooses a species at random for mutation among the least fit species. They proved bounds on the average numbers of ones in the stationary distribution and presented experimental results. Parameter p can substitute up to a certain level a plurality of values, but it cannot explain the staircase phenomenon of Example 1 in Section 5. Hence binary structure, though simple and appealing, is not sufficient for a thorough description of what may happen.[2]

In the first section we give a frame for studying local and global evolution.

Section 2 discusses a basic model of teleological local-global evolution, that will be used to clarify the content of Sections 3 and 4 (the first example, where overtaking and footholds arise).

Partitioned BS frames are the object of Section 3, while Section 4 shows how this model clearly explains the overtaking of competitors with respect to species that seem to be well assessed and recalls some of our experimental data.

Finally, Section 5 considers real world situations where Bak Sneppen partitioned schemes can be tailored to represent the core of their evolution.

[1] Also this case is by no means trivial, as it was shown by Meester and Znamenski (see [32]).

[2] A further case is dealt by C. Bandt in [4], who shows that the discrete BS model behaves exactly like the contact process, on an arbitrary graph, thus all results which have been shown for Contact process will immediately extend to discrete BS model.

1. A frame for studying local and global evolution

The idea of BS model is to glue globality with locality. For this reason in this section it is pointed out the way how locality and globality coexist in the same system and how they interact. For sake of simplicity the present paper will deal with finite graphs and with discrete time evolution systems.

Let us consider a finite set of nodes

$$X = \{K_1, K_2, ..., K_n\}$$

To each node it is appended a unique cell that contains the node itself and in general some other nodes. This will be called locality cell of the node:

$$L_k = \{K_k, K_{(k1)}, ..., K_{(kh)}\}$$

The node that labels the locality cell is called the kernel. Remark that unlike in a partition the cells usually overlap, so that a node could belong to different locality cells, but in particular it belongs to its own cell, in which it is the kernel. The two trivial cases are the atomistic locality system, where $h = 0$ for all k, so that each cell is built only by its kernel and the cells form a partition of the set X, and the totally global system where all cells coincide with the whole set X. To each node K_k, a vector of features is associated. The dimension of the vector does not need to be the same at all nodes, and it can reduce to 1. In many cases, the features are expressed by values taken from some ordered set, or even from numeric sets, discrete or continuous. In order to apply the scheme to numerical calculus, it is useful to associate a duplicate of the vector that registers the next step of the evolution.

The meaning of the cell is that all evolutions in the cell depend only on the data contained in the cell, and affect only elements of the cell. Generally the evolution affects only the features of the kernel of the cell, but sometimes it can affect also other elements, as it happens in BS processes. Usually more than one type of evolution can affect a cell, or a parametric evolution, so when the cell is chosen for evolution it must receive also the information of the action that it must perform.

Globality is controlled by the Global Controller (GC). He has at least some counters and can access all data stored in the kernels and using these data he can perform global operations, if it is required. According to the counters and the result of operations he will define the next cell that will evolve, the choice of the type of evolution and its parameters, if any. For example the approximate solution of a differential equation (using for example fourth order Runge Kutta method) has at least one more parameter, namely the length of the step. More sophisticated controls arise when there is a guess of the error by dividing the step and comparing the solutions. The choice of the step may thus become adaptive.

A case in which globality performs a fundamental task is when the choice of the evolving cell is no longer sequential, but is connected to some global evaluation of all the cells. This happens for example in DNA analysis, where longest matching is chosen for improvement. The simplest case is the random maximization, that is performed on atomistic local system: a cell is chosen randomly and then to its kernel is assigned a random number. After a transitory phase in which some memory of the original values still survives, it is obtained a purely random distribution of values, that coincides in probability with a casual extraction with replacement. The process becomes much more interesting if some rule for the choice of the evolution cell is added. Till now, no local relation is at hand. The idea of Bak and Sneppen is to join each kernel with two adjacent nodes, getting three-node local cells. The cell that evolves is the one whose kernel is minimum, but all the three nodes, not only the kernel, will receive a new value. In this case, very interesting connection structures tend to become more likely than purely casual distributions, usually long chains of high values, alternated with shorter chains of lower values is obtained.

An interesting process can arise also without ordered structures of values. It is enough to choose a cell where the value is not modal, and replace it at random. The structure will converge anyhow to a mode involving all kernels but one; in particular if, at the starting time, more than half of the kernels have the same value, this will also be the final mode, otherwise we have some function of membership depending on the original distribution of features. A somewhat opposite process arises when it is chosen one of the modal cells; in this case the evolution tends to a division into groups that have all the same mode, or a mode differing by one. Both schemes arise in political analysis (band-wagon in one case, segmentation in the other case, see [36])

2. Binary Bak Sneppen linear models

Let us introduce a modified discrete BS process denoted by $(l_1, l_2)-$BS. Let $X = \{0, 1, ..., n-1\}$ be the set of nodes in the global system. The nodes will be arranged on a circle (the operation are *mod n*), each of which has been assigned a random fitness. The fitness values are independent and uniformly distributed on the set $\{0, 1, ..., s-1\}$. Sometimes it will be useful to think the set $\{0, 1, ..., s-1\}$ as a partition of the unitary interval $[0, 1]$ so that the element $i \in \{0, 1, ..., s-1\}$ will be identified with the central value $\frac{i}{s} + \frac{1}{2s}$. Let l_1 and l_2 be natural numbers $(l_1 < l_2)$.

> $(l_1, l_2)-$*BS is a process such that at each discrete time step, the node* $i \in X$ *with the lowest fitness* (*in the case of more then one element the choice will be made randomly between all the candidates*) *is chosen. Then the fitness of* i, $i + l_1$ *and* $i + l_2$, (*so the locality cell of* i *is* $L_i = \{i, i + l_1, i + l_2\}$) *will be replaced by independent and uniform* $\{0, 1, ..., s-1\}$ *random variables.*

Since the states may be described by a number of n digits in base s, the total number of states is s^n and the evolution of the system beginning from an initial state chosen randomly to the successive one, according to the law just described, can be modeled by an irreducible Markov chain with transition matrix, say M (to simplify the reading we will consider the transposed matrix of the usual one). Let us recall, for the convenience of the reader, some important definitions about Markov chains.

Definition 1 Two states x and y communicate each other, and we write $x \leftrightarrow y$, if they are equal or if it is possible (with positive probability) to get from either one to the other in a finite amount of time. This is an equivalence relation and the equivalence classes are called communication classes.

There are two types of communication classes: recurrent and transient.

Definition 2 A communication class Z is called transient if, starting from any $x \in Z$, it is possible to return to x only a finite number of times with probability one, otherwise it is called recurrent or persistent.

It is also useful to remaind the following result.

Theorem 1 *Given an irridducible Markov chain with transition matrix M, there is a unique probability distribution π on the state space such that*

$$M\pi = \pi \tag{1}$$

π is called the stationary distribution of the Markov chain.

It is easy to see that every stochastic matrix has the all-1 vector as a left eigenvector corresponding to the eigenvalue 1. The above theorem says that the corresponding right eigenvector is also non-negative, and that there is only one eigenvector corresponding to eigenvalue 1 if the matrix corresponds to an irreducible Markov chain.

The mean waiting time from state x to state y is the expected number of iterations for reaching state y for the first time starting from state x.

Definition 3 Let Z be a communication class, a state $e \in Z$ will be called an exchange state of Z if it is the unique state such that $Z \backslash \{e\}$ is not a communication class.

From a theoretical point of view, it is possible to construct the transition matrix M of the process for any finite dimension, but dimension grows as s^n. Hence the most critical factor is the number of sections, and this explains the importance of two section frame, where critical information are left to the probability of the digits as in [5]. The authors have developed a computer program that draws the transition matrix. From this matrix many information can be derived: the most important parameter associated to the process is its average:

$$A_s(n; l_1, l_2) = \sum_{i=0}^{s^n-1} \pi(i) w(i) \tag{2}$$

where $w(i)$ is the value of the state, in particular, in the standard case of sections on the unitary interval, denoting by $C_s(i,j)$ the number of j-digits present in the representation of i in base s, it amounts to

$$w(i) = \frac{1}{s}\sum_{j=0}^{s-1} C_s(i,j)j + \frac{1}{2s} \tag{3}$$

When the number of states is greater than 2 an important value is also the frequency of the sections (associated to the digit j ranging from 0 to $s-1$):

$$F_s(j;n;l_1,l_2) = \frac{1}{n}\sum_{i=0}^{s^n-1} \pi(i)C_s(i,j) \tag{4}$$

2.1. Example

As an example of non trivial discrete $(l_1,l_2)-$BS model we choose what seems to be the simplest meaningful case: 6 nodes and 2 values. In this case, when we identify the states apart the rotation, the 64 states can be reduced to the following 14:

0=000000	1=000001	3=000011	5=000101	7=000111	9=001001
11=001011	13=001101	15=001111	21=010101	23=010111	27=011011
31=011111	63=111111				

The transition matrix depends on the probability of each digit and on the system of locality cells. Since many minima may arise, we suppose that the locality cell is chosen at random among the candidates (different rules give rise to different transition matrix and may even be not consistent with a representation of states reduced by rotation, these evolutions will be discussed in another paper).

For sake of simplicity, we consider only the case of constant probability. For the couples (l_1,l_2) there are the following cases:

- $(1,5)$ (that is equal to the classical BS model)
- $(1,2)$ (that a posteriori coincides with the results of (1,3) and of (2,3))
- $(1,4)$ (that a posteriori coincides with the results of (2,5))
- $(2,4)$ (that as shown later allows also a more suitable state representation)

Next table will summarize the results, the first part shows the frequencies of each state, the second one shows the averages. We have added also a non BS case, namely the random one, where the locality cell coincides with the whole space for each kernel, and the two neighbors are chosen at random (compare [21], [13], [29]).

Configuration	Structure (1,5)	Structures (1,2),(1,3),(2,3)	Structures (1,4),(2,5)	Structure (2,4)	Structure Random
0	0,002643	0,0033320	0,004112	0,000000	0,003662
1	0,024136	0,0304099	0,035908	0,000000	0,032801
3	0,049081	0,0446415	0,050715	0,000000	0,050041
5	0,035378	0,0446415	0,050715	0,000000	0,050041
7	0,108473	0,0626871	0,069095	0,000000	0,079790
9	0,016208	0,0280814	0,028844	0,000000	0,025020
11	0,060323	0,1002903	0,076068	0,000000	0,079790
13	0,060323	0,0742085	0,098396	0,000000	0,079790
15	0,167864	0,1221499	0,120348	0,000000	0,128153
21	0,016527	0,0208957	0,023032	0,125000	0,026597
23	0,122677	0,1221499	0,120348	0,375000	0,128153
27	0,044115	0,0798766	0,074824	0,000000	0,064076
31	0,230218	0,2115085	0,197502	0,375000	0,201231
63	0,062034	0,0551272	0,050093	0,125000	0,050854

l_1	l_2	$A_2(6; l_1, l_2)$
1	5	0,568695
1	2	0,560366
1	4	0,553865
2	4	0,625
random	random	0,556145

Table 1. *Binary BS model on a frame of* 6 *nodes: frequencies and averages*

The most striking value is the case of (2,4) and will be discussed in detail below. Usually it is expected that in BS processes (self organizing criticality) the average is higher than in a random process, since BS lets arise chains of adjoining high values and chains are in some sense stable. Table 1 shows that, in a structured process, it is possible to get an average lower than in a random process, namely the strongly asymmetric case (1,4) in which the two operating cells are just opposite, hence tend to break long chains of maximums wherever they could be located.

In the case (2,4), the set of nodes is split in two subsets: even $E = \{0, 2, 4\}$ and odd $O = \{1, 3, 5\}$. For each even kernel its locality cell is given by E, while for odd kernels their locality cell is given by O. There is no overlapping between the two sets of locality cells. Hence the system operates as if it was built by two subsystems of three nodes each. Until both in E and in O there exist 0's, the change may happen in any of the two sets, and the probability is given according to the frequency of 0's (in a set of three elements BS is trivial since it coincides with random process). When the state 111 is reached in one of the subsets, only the other one may be changed, until the exchange state 111, 111 is reached. From this state the process will go on in one of the two subsets, but the other one will maintain the values of 111, so that 0's can appear at most in one of two subsets. This means that the original set of 14 states reduces to a persistent class formed only by states 21, 23, 31, 63, that is one set with three 0's, one set with two 0's, one set with one 0 and the exchange state of all 1's. This reduced system does not give a complete information since it identifies the two subsets O and E, that up to a rotation are coincident.

An equivalent (but more expressive) transition matrix could be derived from the 14-element matrix taking into account only the couples of numbers of 1's in the two sets. We get the following 10 states

$$\boxed{00\ 10\ 11\ 20\ 21\ 22\ 30\ 31\ 32\ 33}$$

where ij means i ones in one set and j ones in the other. The correspondences are obtained by direct check: 00={0}; 10={1}; 11={3, 9}; 20={5}; 21={7,11,13}; 22={15,27}; 30={21}; 31={23}; 32={31}; 33={63}.

Gluing together the rows and columns in the 14 nodes transition matrix actually we get the reduced transition matrix M, given by:

	00	10	11	20	21	22	30	31	32	33
00	1/8	2/40	0	1/32	0	0	0	0	0	0
10	3/8	9/40	1/8	3/32	1/24	0	0	0	0	0
11	0	9/40	3/8	0	3/24	0	0	0	0	0
20	3/8	6/40	0	6/32	2/24	1/8	0	0	0	0
21	0	9/40	3/8	9/32	9/24	3/8	0	0	0	0
22	0	0	0	9/32	6/24	3/8	0	0	0	0
30	1/8	2/40	0	1/32	0	0	1/8	1/8	1/8	1/8
31	0	3/40	1/8	0	1/24	0	3/8	3/8	3/8	3/8
32	0	0	0	3/32	2/24	1/8	3/8	3/8	3/8	3/8
33	0	0	0	0	0	0	1/8	1/8	1/8	1/8

Table 2. *Transition matrix of the reduced representation for the* $(2,4)$*-BS process of* 6 *nodes.*

Let us remark that from all the six non persistent states $(00, 10, 11, 20, 21, 22)$ the total probability of reaching a persistent state $(30, 31, 32, 33)$ is 1/8, but it is not possible to reach directly the exchange state 33, that can be attained only starting from a persistent state.

In the next section we shall review this example in the context of a general theory of partitioned structures. This section ends with some remarks about equivalence of different (l_1, l_2)-BS like processes. In fact a condition of equivalence is the possibility of rearrangement of rows and columns of the transition matrix. A trivial condition of equivalence is symmetry: in a non symmetric BS process such as (1,2), this is equivalent to $(n-1, n-2)$. A more sophisticate condition can be achieved when a Hamiltonian permutation of the nodes is available. In order to avoid complication due to partitioned frames, consider a prime number of nodes n. Then, for any given couple (l_1, l_2), all the couples ([(k l_1) mod n],[(k l_2) mod n]) are obtained by a Hamiltonian permutation that allows to rearrange the transition matrix so that it coincides with the original matrix associated to (l_1, l_2). Let us remark that this result does not require the actual construction of the transition matrix, and it holds for any dimension of nodes and sections. Consider, for example, the case of 5 nodes: the couples are (1,2); (1,3); (1,4) (equivalent to standard BS); (2,3); (2,4); (3,4). The couple (3,4) is the symmetric of (1,2) and (2,4) is the symmetric of (1,3). We get $(1,2) \equiv (2,4)$, hence $(1,2) \equiv (1,3)$. Finally, $(1,4) \equiv (2,3)$; the latter equivalence is usually read as $(1,-1) \equiv (2,-2)$ (that is a formal extension of symmetric BS, compare [22]).

A somewhat more complex model is obtained for a higher prime number of nodes. For example, in the case of 19 nodes there are at most 9 possible equivalent structures. We list them according to decreasing (statistical) average. There is statistical evidence that the two couples (1,2), (1,17) and (1,3), (1,16) are actually different even if it is not proved that the two components of the couple are different. On the contrary, there is no statistical evidence that the last four cases have really different averages.

For reader's information, next table gets the statistical average values for 19 nodes. Even if this section is dealing with binary processes, it is added also information about the division in 4 sections. The frequencies in the case of 2 sections are immediately recovered from the average subtracting 0.25 and multiplying by 2. The first five structures are kept distinct, while the remaining 4 are merged into the class "other".

Name	Av. 2 sect.	Av. 4 sect.	$F_s(0)$	$F_s(1)$	$F_s(2)$	$F_s(3)$
(1,18)	0.67826	0.70015	0.045065	0.090301	0.383608	0.481026
(1,2)	0.653533	0.66260	0.049383	0.156482	0.387682	0.406453
(1,17)	0.640368	0.661908	0.050324	0.157052	0.388094	0.40453
(1,3)	0.62702	0.64451	0.052896	0.195466	0.373361	0.378277
(1,16)	0.62455	0.64583	0.053179	0.191047	0.37481	0.380964
other	0.621289	0.64137	0.05363	0.20193	0.36977	0.37468

Table 3. *Statistical results for a frame of* 19 *nodes,* 2 (*respectively* 4) *sections.*

The reader can make many remarks. The most obvious is that passing from 2 sections to 4 sections the average is increased; this fact depends from the central value that is assigned to each section. Recall that the limit distribution in Bak and Sneppen infinite dimensional model is a step function of 0 value up to the critical value, and then it is constant. In general, it is a monotone function that becomes asymptotically constant. Giving the central value to each interval it is obtained a good estimate for the asymptotical section of the distribution, but we underestimate the lower sections, the more the less is the number of sections.

3. Partitioned frames

In this section we will analyze a much more interesting phenomenon, that may arise only in the case of a non prime number of nodes (hence the reason for choosing 6 nodes for the simplest case). In Table 1 the average of the last case (2,4) is much greater than all the remaining averages. In the transition matrix of the associated Markov chain there exists a persistent class that is smaller than the whole set of 14 structures. Namely it contains only 4 structures (apart rotation): 01 01 01, 01 01 11, 01 11 11, 11 11 11, while the remaining 10 form a transient class: this means that there exist sets of transformations that allow to pass from any of them to any other, but there is the possibility of falling outside into the persistent class without the possibility of returning back. In this simple case the analysis is straightforward: nodes 2 and 4, together with the minimum conventionally placed at 0, form a subset that has no interference with nodes 1-3-5, that, on the contrary, are activated when the minimum is attained in one of those nodes. Whenever an operation is performed, one 3-element subset is left unchanged, while the other one is totally changed at random. The trick is that in this case the two subsets do not change between different operations, what on the contrary happens in all the remaining cases. The process cannot anyhow be divided into two independent subprocesses, since the minimum must be looked for among the elements of both subsets (for example a subset containing 011 enters in this search, while a subset 111 enters in the search only if also the other one is a 111 subset, in which case there are 6 minima of value 1). During the transient phase a further interaction is given by the number of minima; for example in the case 000 011 the first subset has probability 3/4 of being chosen, while the other one only 1/4. When one set reaches the configuration 111 (probability 1/8) the transient phase is terminated and the process becomes stable, in the sense that it can be changed only if the global minimum is 1, that is the configuration 111 111. This is the exchange state. In this case at least one of the two subsets will save the configuration 111, that can no longer be destroyed. The four structures above in fact can be read, using the reduced state description of the last section, as 03= 000 111, 13 = 001 111, 23 = 011 111, 33 = 111 111. In the particular case of 6 nodes the transition matrix becomes trivial since there is no longer a BS structure.

In particular the mean waiting time for reaching the persistent class starting from any configuration of the transient class is 8, and does not depend on the initial structure. The mean waiting time for reaching the top (and exchange) configuration 111 111 starting from 000 000 (or any other transient) is 16. Remark that in the standard BS process this last mean waiting time is 23.07572. The increase in time is due to the impossibility of protecting the structure 111 from decay once it is achieved for the first time.

A richer, but similar situation, arises in the binary case of 8 nodes. The subsets are formed by four elements and the different structures are 0000, 0001, 0011, 0101, 0111, 1111; the persistent class is thus formed by a subset 1111 coupled with any of these six structures. The estimations become less trivial because the structure with four elements is already BS non trivial, inasmuch one element is not changed at random and preserves memory of the past. The average is therefore somewhat increased, up to 0.63785. Using the transition matrix, one can compute the mean waiting time. In particular, in this case it depends on the starting configuration, in particular from 0000 0000 it is 14.79873, while it attains a minimum from 0011 0011 or 0111 0111 where anyhow the 1 is saved. In this case the mean waiting time is 10.98887.

We wish now to state formally the theorems that describe the main factors of partitioned frames. In the first theorem there is no need to define explicitly what a partitioned frame is, since it is simply built by the distinct cosets of the cyclic group of n elements when n is not prime. In the general case of the second theorem some definition will be needed, and also more information about the substructures will be required.

Theorem 2 *Let n be a non prime number of nodes. Let $m = MCD(n, l_1, l_2)$ [3] be the greatest common divisor, and denote by $Max = (s - 1/2)/s$ the central value of the upmost section, then the average is*

$$A_s(n; l_1, l_2) = \frac{m-1}{m} Max + \frac{1}{m} A_s\left(\frac{n}{m}; \frac{l_1}{m}, \frac{l_2}{m}\right) \tag{5}$$

and the frequencies of the single sections are

$$F_s(j; n; l_1, l_2) = \frac{1}{m} F_s\left(j; \frac{n}{m}; \frac{l_1}{m}, \frac{l_2}{m}\right) \tag{6}$$

for $j < s - 1$, and

$$F_s(s-1; n; l_1, l_2) = \frac{m-1}{m} + \frac{1}{m} F_s\left(s-1; \frac{n}{m}; \frac{l_1}{m}, \frac{l_2}{m}\right) \tag{7}$$

Proof. The theorem is a particular case of Theorem 2. Symmetry considerations would allow a straightforward proof, but we prefer to use the proof of the much more powerful Theorem 2. As for the numerical aspect see the comment after the proof of theorem 2. ∎

We give now a formal definition of partitioned scheme.

Definition 4 Let X be the space of nodes on which a locality cell system L_i is given; X is said to be partitioned into a subsystem $(X_1, X_2, ..., X_k)$ if for any cell L_h it holds

$$\#\{i : X_i \cap L_h \neq \} = 1$$ [4]

[3] Of course, if $m = 1$, the formulas still hold, but are not meaningful.

[4] That is, each locality cell is contained exactly in one subset of the partition.

With n_i we will denote the cardinality of set X_i and so $n = \sum_{i=1}^{k} n_i$.[5]

In a partitioned scheme, the configurations may be rearranged in order to be consistent with the partition in subsets. To a set X_k we associate a set S_k formed by the configurations that involve the nodes of the component X_k. A set of configurations suitable for the construction of the transition matrix is thus represented by the Cartesian product

$$S = S_1 S_2 ... S_k \text{ }^{6}$$

The use of a reduced transition space is allowed by the fact that changes can happen only inside one subspace at each time. When the maximum is the same for the whole system (as it usually happens) the exchange configuration is $E = m_1 m_2 ... m_h$, where m_i denotes the state in which all nodes in the subset X_i attain the maximum value. If in the punctuated k-dimensional structure we associate the state m_i with the plane $x_i = 0$, then the persistent class is formed by the coordinate axes and the exchange configuration is the origin. In the partitioned BS scheme the transition matrix thus assumes the form given in the following table.

	E	C_1	C_2	...	C_h	Z
E	$*$	$*$	$*$	$*$	$*$	0
C_1	$*$	$*$	0	0	0	$*$
C_2	$*$	0	$*$	0	0	$*$
...	$*$	0	0	$*$	0	$*$
C_h	$*$	0	0	0	$*$	$*$
Z	0	0	0	0	0	T

Table 4. *Transition matrix in a partitioned BS scheme.*

Let us remark that C_i denotes the set of states derived from the exchange state E changing only the i-th component of the Cartesian product, and leaving all the remaining components at the value m_s.[7] For passing from one state to another it is compulsory to pass through the exchange state E. Here $*$ denotes the elements that can be non zero, while 0 indicates the components that are necessarily 0. T is the transition matrix of the transient class Z in itself, and also can be built up by a recursive matrix of partitioned frames. This happens in particular whenever the sections are more than 2, as it will be shown in the examples of the next section.

[5]Very often it may be convenient to look for the finest partition available, but our results do not require minimality conditions.

[6]An example was given for the case of 6 nodes when we passed from the 14 state full representation to the 10 state reduced representation.

[7]Hence it must not be confounded with S_i since it is a proper subset of the whole Cartesian product.

The restrictions of the process to each subset X_i can be described using a transition matrix M_i defined on the system of configurations S_i. Thus we get the usual information such as frequencies of each configuration y_i that we shall denote by $F_i\ (y_i)$, in particular we shall be particularly interested to $F_i\ (E_i)$, where E_i denotes the projection of E on the space X_i. Another important information is of course the average value A_i. When we pass from the partitioned matrixes M_i to the global matrix M, we require one more information, namely the probability p_i of going from the exchange state E to each state of the subsystems C_i. Let us remark that in BS processes this is given by the ratio between the number of maximum elements in each set (i.e n_i/n). In particular if all the sets X_i have the same number of nodes, p_i is constant. We describe the persistent matrix, dropping the transitions described by the last column and the last row. All data outside the first column of the transition matrix of Table 4 are the same as in the single transition matrixes M_h, and a formal change is the addition of the full description of the other subsets that remain unchanged. The only change happens in the first column, where the terms outside the first row are the corresponding terms of the first rows of matrixes M_i multiplied by p_i, and $M(E, E)$ is the complement to 1 of the column. Different schemes of course could be foreseen, provided the exchange rule is maintained.

The most interesting results allow to calculate global frequencies and averages from the frequencies and averages of the single subsystems, taking obviously into account the cardinality of the subsets, the value of the exchange state (usually the maximum) and the probability of different exits from the exchange state. For sake of generality we denote by V_i the value of each component of the exchange state, hence its value is given by the sum of these items.

Theorem 3 *Let M be the transition matrix of a partitioned frame generated by matrixes M_i. Let E be the exchange state, p_i the probability of different exits from the exchange state, F_i the family of frequencies for each subsystem. Then the average A is a weighted sum of the single averages according to the formula*

$$A = \frac{1}{N}\sum_{i=1}^{k}\frac{p_i}{F_i(E_i))}\left(\sum_{j\neq i}\frac{n_j}{n}V_j + \frac{n_i}{n}A_i\right) \tag{8}$$

where the normalization factor satisfies the relation

$$N = \sum_{i=1}^{k}\frac{p_i}{F_i(E_i)} \tag{9}$$

The frequency of the global configurations is 0 for the transient configurations. As for the persistent configurations, we denote by B_i the configuration

$$(E_1, ..., E_{i-1}, b_i, E_{i+1}, ..., E_k)$$

and we get

$$F(B_i) = \frac{1}{N}\frac{p_i}{F(E_i)}F_i(b_i) \tag{10}$$

for $B_i \neq E$, and

$$F(E) = \frac{1}{N}\cdot \tag{11}$$

Proof. For sake of simplicity, we can consider the case of two subsets, since the general case may be proved by induction. We represent the transition matrixes M_1 and M_2 by the following notations

M_a	E_a	a_1	...	a_r
E_a	e_{a0}	a_{01}	...	a_{0r}
a_1	e_{a1}	a_{11}	...	a_{1r}
...	...	...	...	...
a_r	e_{ar}	a_{r1}	...	a_{rr}

M_b	E_b	b_1	...	b_s
E_b	e_{b0}	b_{01}	...	b_{0s}
b_1	e_{b1}	b_{11}	...	b_{1s}
...	...	...	...	...
b_s	e_{bs}	b_{s1}	...	b_{ss}

Let $v_0 = F(E_a)$, $v_1 = F(a_1)$,..., $v_r = F(a_r)$, and $w_0 = F(E_b)$, $w_1 = F(b_1), ...,$ $w_s = F(b_s)$, and denote by A the submatrix of M_a without the first row and the first column, and similarly by B the submatrix of M_b without the first row and the first column, by e_a the column vector $(e_{a1}, ..., e_{ar})^*$, by e_b the column vector $(e_{b1}, ..., e_{bs})^*$, an eigenvector associated to eigenvalue 1 is given by $(x_0, x_1, ..., x_r)$ where $x_0 = 1$ and $x_1, ..., x_n$ satisfy the system

$$(A - I)\mathbf{x} = -e_a \tag{12}$$

Normalizing, the frequency vector is given by $\mathbf{v} \equiv (v_0, x_1 v_0, ..., x_r v_0)$ or $\mathbf{x} = \mathbf{v}/F(E_a)$. Respectively, let $y_1, y_2, ..., y_s$ satisfy the system

$$(B - I)\mathbf{y} = -e_b \tag{13}$$

Normalizing, we get $\mathbf{w} \equiv (w_0, y_1 w_0, ...y_s w_0)$, or $\mathbf{y} = \mathbf{w}/F(E_b)$

In the complete system, we are interested only to the part that corresponds to the persistent class. The matrix is thus

M	E	A_1	...	A_r	B_1	...	B_s
E	$p_a e_{a0} + p_b e_{b0}$	a_{01}	...	a_{0r}	b_{01}	...	b_{0s}
A_1	$p_a e_{a1}$	a_{11}	...	a_{1r}	0	...	0
...	...	...	...	...	...	...	
A_r	$p_a e_{ar}$	a_{r1}	...	a_{rr}	0	...	0
B_1	$p_b e_{b1}$	0	...	0	b_{11}	...	b_{1s}
...	...	...	...	...	...	...	
B_s	$p_b e_{bs}$	0	...	0	b_{s1}	...	b_{ss}

Table 5: *Joint transition matrix of a partitioned process*

In order to obtain the eigenvectors we solve thus the system

$$\begin{cases} (A-I)\mathbf{x}' + 0\mathbf{y}' = -p_a e_a \\ 0\mathbf{x}' + (B-I)\mathbf{y}' = -p_b e_b \end{cases}$$

obtaining

$$\mathbf{x}' = p_a\mathbf{x} = p_a/F(E_a)\mathbf{v},\ \mathbf{y}' = p_b\mathbf{y} = p_b/F(E_b)\mathbf{w}.$$

The normalization of the vector leads to the normalizing factor N given in equation (9) that appears in the statement of the theorem.

$$\begin{aligned} 1+\sum_{i=1}^{r} x_i' + \sum_{j=1}^{s} y_i' &= 1+\frac{p_a}{F_a(E_a)}\sum_{i=1}^{r} v_i + \frac{p_b}{F_b(E_b)}\sum_{i=1}^{s} w_i \\ &= 1+\frac{p_a}{F_a(E_a)}(1-F_a(E_a)) + \frac{p_b}{F_b(E_b)}(1-F_b(E_b)) \\ &= 1+\frac{p_a}{F_a(E_a)} + \frac{p_b}{F_b(E_b)} - p_a - p_b \\ &= \frac{p_a}{F_a(E_a)} + \frac{p_b}{F_b(E_b)} = N \end{aligned} \tag{14}$$

Once formulas (10) and (11) are proved, the main formula (8) is easily derived keeping in mind that the average value of the configuration is the weighted sum of the averages of the single cartesian components, and the weight is just given by the proportion of configurations that belong to each S_i. ■

We wish to remark that there is an essential difference between the two theorems: in fact in Theorem 1 in order to get the average it is not required to know the frequencies $F_i(E_i)$. From a numerical point of view these frequencies, very near to 0, are difficult to be experimentally determinated, while on the contrary the averages in non partitioned schemes can be easily experimentally estimated. Let us remark that this is not true for the case of partitioned schemes, where transitory phase may be very long. A computational trick is thus to start directly from the exchange configuration or anyhow from the interior of the persistent class, so that transitory period is skipped away.

4. Examples and staircase of critical configurations with overtaking

In this section, we give some examples for the theorems of the previous section, but before we highlight the main features of the partitions that do not allow to study these processes simply using binary representations joint to a change in the probability system.

4.1. Example 1

We come to a more general case of partition of the BS process. The simplest case that shows the main features requires 2 subsets of 3 nodes each, and we consider a ternary set of values, say 0,1,2 [8]. This corresponds to 6 nodes and displacements of 2-4. A simplified analysis is the following: we attach label 0 to any subset in which the minimum is 0 (not regarding their number), label 1 to any subset in which the minimum is 1 (not regarding their number), and label 2 to the set $\{2, 2, 2\}$.

Next table shows the transition matrix.

	00	01	10	11	02	20	12	21	22
00	38/54								
01	7/54	38/54		19/54					
10	7/54		38/54	19/54					
11		14/54	14/54	14/54					
02	1/54				38/54		38/54		19/54
20	1/54					38/54		38/54	19/54
12			2/54	1/54	14/54		14/54		7/54
21		2/54		1/54		14/54		14/54	7/54
22					2/54	2/54	2/54	2/54	2/54

Tab le 6. *Transitions in a* 6 *node bipartite Bak Sneppen set with* 3 *values.*

There is one transient class (01, 11, 10) with exchange state 11 and one persistent class (20, 02, 22, 21, 12) with exchange state 22. The asymptotic average, normalized to the scale $[0, 1]$ is thus 0.667, since one subset has the form $\{2, 2, 2\}$, that corresponds to 0.8333 and the second one is random on the three values 012 and corresponds to 0.5. A more complex frame with a ternary partition, but in reduced form, was introduced in [35] and [36]. From states 00, 01, 10, 11 the mean waiting time to the persistent class is 27. Let us remark that the topological structure of our representation is the same as what would be achieved in section 3 when dealing with random optimization for the case of three values on three nodes, and becomes exactly the same if we set $p(0) = 19/27$, $p(1) = 7/27$, $p(2) = 1/27$.

Unlike the ternary partition there exists a path that touches all states even if mean waiting time between non persistent states still remains infinity. This remark a peculiar "overtaking law", that namely concerns the transition from 10 to 12 and from 01 to 21. The subset that is changed into the optimal label 2 is not one labeled with 1, but one that is labeled with 0, since its minimum must be lower than that of the best subset. For example, in a situation $A = \{1, 2, 2\}$, $B = \{0, 1, 1\}$, it is impossible that A is transformed into $\{2, 2, 2\}$, while this is possible, even if unlikely, for B.

[8] The example was first presented during the AMASES meeting of 2011 [35], see also [36]

4.2. Example 2

We go back now to the examples that illustrate theorems 1 and 2. We give an example of a non symmetric partitioned frame in which BS processes happen. We suppose that the total number of nodes is $n = 8$, and that we have a subset X_a of $n_a = 5$ nodes and a subset X_b of $n_b = 3$ nodes. The number of sections is 2. On both subsets the locality cells are whose of the standard BS process, namely each cell contains the kernel and the right and left neighbor; obviously in the case of three nodes (random case) the locality cell is the same for all the kernels and coincides with the subset.

We write down separately the two evolution matrixes and we make the usual computations of frequency and of average. The representation is minimal, hence S_a contains 8 items instead of 32 and S_b contains 4 items instead of 8. In the case of S_a we get the transition matrix of Table 7, while the trivial transition matrix of S_b in presented in Table 8.

	31	0	1	3	5	7	11	15	F	$\sharp 1's$
31	0	0	0	0.0416667	0	0.125	0	0.125	0.071267	5
0	0	0.125	0.0625	0	0.0416667	0	0	0	0.0098982	0
1	0.125	0.375	0.25	0.0833333	0.2083333	0	0.125	0	0.0735294	1
3	0	0.25	0.1875	0.125	0.1666667	0.125	0.125	0.125	0.1348982	2
5	0.25	0.125	0.1875	0.1666667	0.2083333	0	0.25	0	0.0975679	2
7	0.125	0.125	0.125	0.1666667	0.125	0.25	0.125	0.25	0.196267	3
11	0.375	0	0.125	0.2083333	0.1666667	0.125	0.25	0.125	0.1589367	3
15	0.125	0	0.0625	0.2083333	0.0833333	0.375	0.125	0.375	0.2576357	4
								Average	0.549095	

Table 7. *Transition matrix for Bak Sneppen binary process on five nodes.*

	7	0	1	3	F	$\sharp 1's$
7	0.125	0.125	0.125	0.125	0.125	3
0	0.125	0.125	0.125	0.125	0.125	0
1	0.375	0.375	0.375	0.375	0.375	1
3	0.375	0.375	0.375	0.375	0.375	2
				Average	0.5	

Table 8. *Trivial transition matrix of binary BS process on three nodes.*

The exchange state is $(31, 7)$, that corresponds to section 1 in all the 8 nodes. The probability of changing state S_a (resp. S_b) is proportional to the number of nodes, so we have $p_a = 0.625$, $p_b = 0.325$, $n_a = 5$, $n_b = 3$. The remaining coefficients are already known from the elaboration of the transition matrixes, we have namely $F(E_a) = 0.071267$, $F(E_b) = 0,125$, $m_a = 0.5491$ and $m_b = 0.5$ (obviously, since it is a random process). We get thus the coefficients $p_a/F(E_a) = 8.76984$, $p_b/F(E_b) = 3$, $N = 11.7698$. We recall that, in view of our convention about the middle of the sections, $V_a = V_b = 0.75$. Finally we get from theorem

2, formula (8), the final value $A = 0.632544$. The case is simple enough that it can be handled directly by constructing the transition matrix. We obtain the following table:

	31-7	0-7	1-7	3-7	5-7	7-7	11-7	15-7	31-0	31-1	31-3	F	$1's$
31-7	0.125	0	0	0.041	0	0.125	0	0.125	0.125	0.125	0.125	0.084	8
0-7	0	0.125	0.062	0	0.041	0	0	0	0	0	0	0.007	3
1-7	0	0.375	0.25	0.083	0.208	0	0.125	0	0	0	0	0.055	4
3-7	0.078	0.25	0.187	0.125	0.166	0.125	0.125	0.125	0	0	0	0.100	5
5-7	0	0.125	0.187	0.166	0.208	0	0.25	0	0	0	0	0.072	5
7-7	0.156	0.125	0.125	0.166	0.125	0.25	0.125	0.25	0	0	0	0.146	6
11-7	0.078	0	0.125	0.208	0.166	0.125	0.25	0.125	0	0	0	0.118	6
15-7	0.234	0	0.062	0.208	0.083	0.375	0.125	0.375	0	0	0	0.191	7
31-0	0.046	0	0	0	0	0	0	0	0.125	0.125	0.125	0.031	5
31-1	0.140	0	0	0	0	0	0	0	0.125	0.125	0.125	0.095	6
31-3	0.140	0	0	0	0	0	0	0	0.375	0.375	0.375	0.095	7
										Average	0.632		

Table 9. *The transition matrix of an asymmetrically partitioned Bak Sneppen frame*

A further comparison of the frequencies is consistent with formulas (10) and (11).

We end this example comparing the result with the symmetrical partition of the 8 nodes system as given by formula (5) in Theorem 1. We need to know $A_2(4;1,3)$.

The transition matrix is given below and the average is 0.525735.

	15	0	1	3	5	7	F	$\sharp 1's$
15	0.125	0	0.041667	0.125	0	0.125	0.088235	4
0	0	0.125	0.083333	0	0.125	0	0.036765	0
1	0.125	0.375	0.291667	0.125	0.375	0.125	0.198529	1
3	0.25	0.25	0.25	0.25	0.25	0.25	0.25	2
5	0.125	0.125	0.125	0.125	0.125	0.125	0.125	2
7	0.375	0.125	0.208333	0.375	0.125	0.375	0.301471	3
						average	0.525735	

Table 10. *Transition matrix for BS binary process on four nodes*

By this computation we also know $F(15)$, but we do not require it [9]. We remark that, as usual, $V(E_a) = V(E_b) = 0.75$. Hence we get

$$A_4(8;2,6) = [0.75 + A_2(4;1,3)]/2 = 0.637868.$$

The symmetrical partition gives an average slightly greater than the unbalanced partition, according to the general conjecture of [26] and to some experimental results we recall now in the end of this section.

[9]The advantage of not requiring the knowledge of $F(E)$ will become evident in the next example.

4.3. Example 3

This example shows a partitioned frame in a case particularly rich, namely 24 nodes. Here also some of the partitions may still be decomposed. The results of Theorem 2 do not depend on the number of sections, but the numerical simulation can become heavily unstable, since it can take a very long transient period before the exchange state is reached for the first time. If the numerical simulations are needed for analyzing the persistent state, the best strategy is to start the simulation from the exchange state or anyhow from the inside of the persistent class. Since in Theorem 1 all the partitions are equal, there is no problem if it takes a great number of iterations to rejoin the exchange state in order to pass to another partition.

The problem would arise in the case of Theorem 3, where the single partitions must necessarily be simulated separately, and the further estimate of the frequency of exchange state for each partition would be needed.

We describe now the structure of the 24 node system. The experimental data (*) are referred to 4 sections, where in particular $M = 0.875$. In the next table we report only the reduced classical BS of each partition.
8 subsets of 3 elements, 6 subsets of 4 elements, 4 subsets of 6 elements, 3 subsets of 8 elements, 2 subsets of 12 elements:

$$\begin{array}{lll}
A_s(24,8,16) = 7/8M + 1/8A_s(3,1,2) & A_4(24,8,16)^* & = 0.828166 \\
A_s(24,6,18) = 5/6M + 1/6A_s(4,1,3) & A_4(24,6,18)^* & = 0.817772 \\
A_s(24,4,20) = 3/4M + 1/4A_s(6,1,5) & A_4(24,4,20)^* & = 0.802184 \\
A_s(24,3,21) = 2/3M + 1/3A_s(8,1,7) & A_4(24,3,21)^* & = 0.789492 \\
A_s(24,2,22) = 1/2M + 1/2A_s(12,1,11) & A_4(24,2,22)^* & = 0.768288 \\
A_4(24,1,23) & & = 0.715446
\end{array}$$

For reader's convenience we add also

$$A_4(24,1,2)^* = A_4(24,5,10)^* = A_4(24,7,14)^* = A_4(24,11,22)^* = 0.676339$$

The remaining non reducible structures have averages ranging between 0.649575 and 0.663925.

The 19 (prime number) case is non decomposable and its range should be compared only with non decomposable cases of 24; but the whole range reaches, as shown before, 0.8281, hence much greater than the case of 19 nodes. These results are to be compared with the case of a prime number of nodes, where the dependence on the structure has much smaller effects. In the case of 19 nodes the total range is between 0.640 and 0.700, while here the average ranges between 0.660 and 0.828. While the non decomposable frames show only small differences due to the increase in the number of nodes, the main differences arise as the dimension of the subsets diminishes down to the elementary case of 3 elements.

5. Examples from the socio-economic world

As mentioned in the introduction, the BS model is suitable to analyze multi-agent economic and social phenomena. Many variations are obtained by structuring the network of connections differently, giving different laws of transition, establishing different criteria for choosing the item that has to undergo the change. In the case of partitioned schemes the basic element is the segmentation of the system of nodes, which may possibly accompany the other structural changes. The situation presented in the theoretical sections and in the section of mathematical examples is clearly an extreme situation, but it summarizes trends that can be found in the external reality: the links that join subsets can become very weak at the level of connection between different sets, while can remain strong within the subsets. An analysis of the fundamental components then leads to an approximation of the model that ends up consisting of separate components as it happens in Hsu technique of generalized cell-mapping ([27]), used by the authors in territorial analysis ([9]). Example 3 of the previous section shows that there are large differences in mean values as the division into subsets becomes finer, while in the absence of splitting the average values are much lower. Actually the process of approximation is more articulated, because the transition takes place progressively changing the laws of proximity between nodes. In the examples there was a pure dichotomy, with the node connected or not connected without gradients, while it could be supposed that the transition to the partitioned cases occurs by a progressive change of the probability of proximity. So the model provides a useful partitioned limit-schema for understanding (but not for quantitative prediction).

An aspect of particular interest was highlighted in Example 1 where the phenomenon of overtaking is presented. It is typically characterized by a low probability, but becomes gradually higher as the set shrinks. The phenomenon of overtaking can occur along a scale of discrete values more complex than the ternary system (and as it is obvious it cannot be recognized in the binary system), and as the steps go up it becomes more and more unlikely, but instead acquires great stability. A first discussion of the phenomenon was given in [36].

The territorial systems are those which because of their nature are more subjected to segmentation, both for the influence of metric distance, and for the effect of border barriers, that may be physical but also regulatory, economic, social. In urbanism the various districts are subject to town plans that generally do not interface with their neighbors (even if it seems absurd), and thus to a BS evolution model that is subject to segmentation. This encourages overtaking, which often is found in the restoration of old urban centers or deteriorated areas of the city, as has often occurred in the case of the old port areas (Liverpool, Valencia, the East End of London.) The drag effects fall in the broader study of spontaneous synchronization phenomena that characterize complex systems. A regional system of central places (as proposed by Christaller and perfected by many geographers) can enter in the pattern of BS models provided the local districts are normalized to make them balanced in dimensions (combinations of secondary sites or subdivisions of the central place) or in connections (asymmetrical links). However, when the system is substantially changed, becoming a lattice, it is necessary to pass to a partitioned model, where the restricted set of nodes in the network is separate from the system of central places (which is rated lower). All the same, if in this system subsystems of small size are created, it is possible that in turn some of them can perform overtaking. That is the case for many small countries that were not originally nodes in the network system, such as Luxembourg, San Marino, Monaco, United Arab Emirates, where the ability to get out of law restrictions allows the creation of nuclei inserted in a strong economic system both because of the connections with the rich system and for the possibility of tax havens. Singapore has not been added to this list, as it, like other strategic ports, has always represented an exceptional situation, so it would be improper to speak of overtaking, while it is more correct to see a continuity in its position of privilege.

Partitioned models typically represent abnormal situations, as the evolution of the individual subsets may differ significantly. At the global level between the US and Europe there are sensitive differences in terms of the law even where there are international rules. An interesting case concerning different rules for the production of compost was discussed in [20], particularly about the rights on the green waste, where a standard obedience was expected. Even at the local level BS models of evolution bring to unexpected gradients as soon as region borders are overcome ([28]) and it seems that these gradients may not be statistically included in a simple random fluctuation.

Economic interdependence is a field that suggests various applications. There is a horizontal network of relationships between companies operating in the same industry and a network of vertical relations of sale and purchase between the different branches. This is assessed at national level through various types of input-output matrices based on Leontiev Model. The input-output matrices correspond to an oriented network connecting the various nodes (branches) of the economy and allow to study how internal relations influence economic evolution. There are several ways to reorder the matrix and then to analyze the interrelationships between the subsystems of the network. This leads to the possibility of comparisons between different countries, as was done in section 3 of [10]. You may recall that in theory the French model emphasizes the Hamiltonian circuit, so aims to an economy in which the structure should recall the basic circular model of BS. However, the partitioned models can lead to higher average returns even if the imbalances between single subsets may be higher.

The segmentation is due to different skills and different technologies but also to the entry barriers that individual subsystems seek to set up for their own defense. On the other side they evaluate the possibility of overcoming thus becoming aggressive and the effects of this unstable equilibrium can be evaluated by means of dedicated indicators (see for example [7]). At the horizontal level the slight initial segmentation due to natural aggregations disappears along the trend towards oligopoly, which creates a smaller number of subsets, moreover strongly segmented between them. The vertical expansion of these oligopolistic subsets is not very strong but it is an important phenomenon in terms of the BS model, as it enhances concentration and promotes the tendency to a network system of oligopolists that crosses the widespread system of small agents. The problem has been studied in particular in two fields where the concentrated system coexists with the distributed system, namely in the field of hotels and restaurants by [16] and in the agrifood chain by [8].

In a socio-cultural context, interesting cases occur in the evolution of science when overtaking happens in those areas where large investments of capital are not required. Mathematics is one of these areas. Countries with good basic skills but no former tradition of innovative research may suddenly succeed since there is the possibility to bypass the plethora of detailed information by directly accessing the nodal points of the evolution of the search. According to some scholars, in particular Guerraggio (see [25]), that is what happened in the golden age of Italian mathematics in the years 1880 to 1910, when the new state saw active almost simultaneously U. Dini, V. Volterra, G. Peano, F. Enriques, Severi F.G. Castelnuovo, just to remember some of the famous names who were honored by the greatest international appraisal. The importance of Italy was confirmed by the allocation of the third International Congress (Rome 1908) and by the astonishing development of Mathematical Circle of Palermo. In its acts (e.g. [24]) the formidable team of world-famous associates at the beginning of the century can still be read: among them Poincaré considered Palermo Circle as the most important mathematical organization in the world. It is worth to mention the sudden overtaking in regularity theorems performed by De Giorgi in the 50's of last century (De Giorgi-Nash theorem). The very De Giorgi in a private conversation said he had profited of his restricted knowledge of the relevant literature, so that he did not follow the paths already beaten. In fact he pointed strongly on the knowledge of Caccioppoli inequality and on the isoperimetric refinement of Sobolev inequality, adding the construction and the solution of an ingenious system of finite difference equations that allowed him to close the chain of inequalities.

In the world economic development, the examples of sudden overtaking are very common. Starting two centuries ago from Germany industry, when England compelled German production to advertise that it was "made in Germany", and not in England, passing through Japan and, again, Germany after the Second World War, and arriving at the emerging economic and industrial powers of China, India, Brazil, South Africa. Self defense of the leading countries creates the segmentation that can lead to the overtaking, that will be discovered only when it is actually too late. A counterpart can be found in protectionism, that encourages the first phases of development, defending from the risks of global, and mostly unfair, competition, but according to Example 1 bears the risk of stopping a long time on one of the lower steps of the development staircase, without the incentive to new (and perhaps risky) steps in connection with global evolution. Bak and Sneppen partitioned schemes are sensitive to these real world situations. In these models, as sometimes in the real life, the principle is *quieta non movere* (Let quiet things stay). The idea is that movement requires material or intellectual dissatisfaction, while further movements are caused by some form of nested neighborhood, with unsatisfied people that can in turn generate new dissatisfaction. To this purpose we can remind the *happiness paradox* of Easterlin ([18], [19]).

6. Conclusions

In this paper, the authors have shown that discrete models of local-global processes may give relevant information on systems where self organizing criticalities arise. They have shown that a relevant feature is the existence of sequences of footholds that allow also long time persistency, but are subjected to phenomena of overtaking (what has also a very important economical and behavioural meaning). From a technical point of view the authors have shown that among the many generalizations of BS processes, there exist a lot of cases where some form of decomposition is possible, allowing optimization at higher levels. Random optimization, that seems to represent the simplest form of local-global process, allows a comparison with BS processes on partitioned frames, even if it does not allow sharp estimates on the average time required to ascend all the scale of increasing footholds. The definition of partitioned frames is somewhat more general than the pure definition that arises from BS models. The main theorems give a sharp information about the average and the frequencies of the states of the global processes when the single processes that are glued together are known, thus allowing in particular a reduction of the numerical instabilities that conflict with a good knowledge of self organizing criticalities.

This scheme could be used in very general partitioned processes; in some cases two processes could coexist independently from each other, if the transition matrix of the system S_k does not depend from the remaining S_i's, but this case would have little interest. In general the evolution depends also on what is the actual global state of the system even if it perturbs only one subset at a time. The single transition matrix might even not depend explicitly on the rest of the system, but it is enough that some selection rules are given from the GC that allow or forbid some subsystems to evolve, according to a suitable law of choice, deterministic or probabilistic. Actually when dealing with BS processes, much harder bounds are put on partitioned schemes. In fact a subset can be changed only if it has at least one node that attains the minimum. When in a configuration all the nodes of a subset attain the maximum, they cannot be changed until some lower term exists in the system. This means that in this terminal (persistent) class the only admissible configurations are those in which all the subsets but one get the maximum value. In particular should it happen that some subset can attain values greater than the rest of the system, then when it attains this "exceptional" maximum it will never change any longer. The socioeconomic examples of Section 5 suggest the use of these generalizations.

In our numerical simulations we experimented mainly the case of four values since for lower number of values the standard BS distribution is anyhow too concentrated on the top value, hence the different steps are confounded with the casual fluctuation. As soon as the dimension of the subsets is increased it becomes more and more difficult to reach the stable steps; for example already subsets of 18 nodes very often require more than 100,000 iterations in order to reach the persistent class. Let us remark that the transition, when it happens, is very similar to an "avalanche" and average suddenly increase. In some cases the process is not complete, hence it can be reverted, but finally it happens that the threshold is reached. Increasing the number of values, multiple thresholds arise; the lower levels can be easily overcome, while the top levels can prove to be almost unreachable. This is for example the case of ten values and forty nodes, in which the top level has never been reached in ten simulations of 1,000,000 iterations.

A further remark is that in complex cases (more than two subsets, more than two levels) the evolution staircase can change between simulations, since each step is not reversible, but different steps have different compatibility, so that different development paths can arise starting from similar original situations.

The experienced reader can remark that the Global Controller is not a new concept, since such a figure was introduced in full detail already in the Middle Ages by Dante's poetry, and was called Fortuna (Fortune):

> *"He made earth's splendors by a like decree*
> *and posted as their minister this high Dame,*
> *The Lady of Permutations. All earth's gear*
> *she changes from nation to nation, from house to house,*
> *in changeless change through every turning year."*

Dante, Inferno, 7, 77-81 translation J. Ciardi

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EXISTENCE OF THREE SOLUTIONS FOR NONLOCAL ELLIPTIC SYSTEM OF $(p_1, ..., p_n)$-KIRCHHOFF TYPE

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Abstract. In this paper, we establish the existence of at least three solutions to a Dirichlet boundary problem involving the $(p_1, ..., p_n)$-Kirchhoff type systems. Our technical approach is mainly based on the general three critical points theorem obtained by Ricceri.

Keywords: $(p_1, ..., p_n)$-Kirchhoff type system; multiple solutions; three critical points theory.

1. Introduction and main results

In the present paper, we deal with the existence of at least three solutions for nonlinear elliptic equations of $(p_1, ..., p_n)$-Kirchhoff type under Dirichlet boundary conditions:

$$(1.1)\quad \begin{cases} -\left[M_1\left(\int_\Omega |\nabla u_1|^{p_1}\right)\right]^{p_1-1}\Delta_{p_1}u_1=\lambda F_{u_1}(x,u_1,...,u_n)+\mu G_{u_1}(x,u_1,...,u_n), & \text{in } \Omega,\\ -\left[M_2\left(\int_\Omega |\nabla u_2|^{p_2}\right)\right]^{p_2-1}\Delta_{p_2}u_2=\lambda F_{u_2}(x,u_1,...,u_n)+\mu G_{u_2}(x,u_1,...,u_n), & \text{in } \Omega,\\ \cdots \\ -\left[M_n\left(\int_\Omega |\nabla u_n|^{p_n}\right)\right]^{p_n-1}\Delta_{p_n}u_n=\lambda F_{u_n}(x,u_1,...,u_n)+\mu G_{u_n}(x,u_1,...,u_n), & \text{in } \Omega,\\ u_i=0 \quad \text{for } 1\le i\le n, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset R^N (N \geq 1)$ is a non-empty bounded open set with a sufficient smooth boundary $\partial\Omega$, $\lambda, \mu \in [0, +\infty)$, $p_i > N$, Δ_p is the p-Laplacian operator $\Delta_p u = div\left(|\nabla u|^{p-2}\nabla u\right)$. $F, G : \Omega\times R^n \mapsto \mathrm{R}$ are functions such that $F(\cdot, t_1, ..., t_n)$, $G(\cdot, t_1, ..., t_n)$ are measurable in Ω for all $(t_1, ..., t_n) \in R^n$ and $F(x, \cdot)$, $G(x, \cdot)$ are continuously differentiable in R^n for a.e. $x \in \Omega$. F_{u_i} is the partial derivative of F with respect to u_i, $1 \leq i \leq n$, so does G_{u_i}. $M_i : R^+ \to R$, $i = 1, 2, ..., n$ are continuous functions, which satisfy the bounded conditions as follows.
(M) There are two positive constants m_0, m_1 such that

$$(1.2)\qquad m_0 \leq M_i(t) \leq m_1, \quad \forall t \geq 0, \quad i = 1, 2, ..., n.$$

In what follows, $|\Omega|$ denotes the Lebesgue measure of Ω, X denotes the Cartesian product of Sobolev spaces $W_0^{1,p_1}(\Omega), \dots, W_0^{1,p_{n-1}}(\Omega)$ and $W_0^{1,p_n}(\Omega)$, i.e., $X = W_0^{1,p_1}(\Omega) \times \cdots \times W_0^{1,p_n}(\Omega)$. The space X is endowed with the norm

$$\|(u_1, \dots, u_n)\| = \sum_{i=1}^{n} \|u_i\|_{p_i}, \quad \|u_i\|_p = \left(\int_\Omega |\nabla u_i|^{p_i}\right)^{1/p_i}, \ 1 \leq i \leq n.$$

Let

$$(1.3)\qquad C = \max\left\{ \sup_{u_i\in W_0^{1,p_i}(\Omega)\setminus\{0\}} \frac{\max_{x\in\bar\Omega}\{|u_i(x)|^{p_i}\}}{\|u_i\|_{p_i}^{p_i}} \right\}.$$

Since $p_i > N$, $W_0^{1,p_i}(\Omega) \to C^0(\bar\Omega)$, $1 \leq i \leq n$, are compact, and one has $C < +\infty$. As usual, a weak solution of system (1.1) is any $(u_1, ..., u_n) \in X$ such that

$$(1.4)\qquad \begin{aligned} &\sum_{i=1}^{n}\left[M_i\left(\int_\Omega |\nabla u_i|^{p_i}\right)\right]^{p_i-1}\int_\Omega |\nabla u_i|^{p_i-2}\nabla u_i \nabla\xi_i \\ &\quad -\sum_{i=1}^{n}\lambda\int_\Omega F_{u_i}(x,u_1,...,u_n)\,\xi_i dx - \sum_{i=1}^{n}\lambda\int_\Omega G_{u_i}(x,u_1,...,u_n)\,\xi_i dx = 0 \end{aligned}$$

for all $(\xi_1, ..., \xi_n) \in X$.

The system (1.1) is related to the stationary version of a model, the so-called Kirchhoff equation which was introduced by [1]. More precisely, Kirchhoff proposed the following mathematical model.

$$\rho\frac{\partial^2 u}{\partial t^2}-\left(\frac{P_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2dx\right)\frac{\partial^2 u}{\partial x^2}=0, \tag{1.5}$$

which generalizes the D'Alembert's wave equation involving free vibrations of elastic strings, where ρ is the mass density, P_0 is the initial tension, h is the area of the cross-section, E is the Young modulus of the material, and L is the length of the string.

Later, (1.5) was developed to the following result

$$u_{tt}-M\left(\int_\Omega|\nabla u|^2\right)\Delta u=f(x,u)\text{ in }\Omega, \tag{1.6}$$

where $M : R^+ \to R$ is a given function. After that, some authors studied the following problem

$$-M\left(\int_\Omega|\nabla u|^2\right)\Delta u=f(x,u)\text{ in }\Omega,\ u=0\text{ on }\partial\Omega, \tag{1.7}$$

which is the stationary counterpart of (1.6). By using variational methods and other techniques, many results of (1.7) were obtained, please refer to [2]-[12] and the references therein. In particular, Alves et al. [2, Theorem 4] assumed that M satisfies bounded condition (M) and $f(x,t)$ satisfies the following condition.

$$0<\upsilon F(x,t)\le f(x,t)t,\ \text{ for all }\ |t|\ge R, x\in\Omega \text{ for some } \upsilon>2 \text{ and } R>0, \tag{AR}$$

where $F(x,t)=\int_0^t f(x,s)ds$. One positive solutions for (1.7) was obtained.

In [13], applying Ekeland's Variational Principle, the authors established the existence of a weak solution for boundary problem involving the nonlocal elliptic system of p-Kirchhoff type

$$\begin{cases} -\left[M_1\left(\int_\Omega|\nabla u|^p\right)\right]^{p-1}\Delta_p u=f(u,\upsilon)+\rho_1(x), & \text{in }\Omega,\\ -\left[M_2\left(\int_\Omega|\nabla \upsilon|^p\right)\right]^{p-1}\Delta_p \upsilon=g(,u,\upsilon)+\rho_2(x), & \text{in }\Omega,\\ \dfrac{\partial u}{\partial\eta}=\dfrac{\partial\upsilon}{\partial\eta}=0, & \text{on }\partial\Omega,\end{cases} \tag{1.8}$$

where η is the unit exterior vector on $\partial\Omega$, and $M_i, \rho_i(i=1,2)$, f, g satisfy suitable assumptions.

In [14], when $\mu=0, n=2$ in (1.1), Cheng et al. studied the existence of two solutions and three solutions of the following nonlocal elliptic system

$$\begin{cases} -\left[M_1\left(\int_\Omega|\nabla u|^p\right)\right]^{p-1}\Delta_p u=\lambda F_u(x,u,\upsilon), & \text{in }\Omega,\\ -\left[M_2\left(\int_\Omega|\nabla \upsilon|^q\right)\right]^{q-1}\Delta_q \upsilon=\lambda F_\upsilon(x,u,\upsilon), & \text{in }\Omega,\\ u=\upsilon=0, & \text{on }\partial\Omega.\end{cases} \tag{1.9}$$

In [15], when $n=2$ in (1.1), Chen et al. proved the existence of three solutions of the following problem

$$
(1.10)\qquad \begin{cases} -\left[M_1\left(\int_\Omega |\nabla u|^p\right)\right]^{p-1}\Delta_p u = \lambda F_u(x,u,\upsilon)+\mu G_u(x,u,\upsilon), & \text{in } \Omega,\\ -\left[M_2\left(\int_\Omega |\nabla \upsilon|^q\right)\right]^{q-1}\Delta_q \upsilon = \lambda F_\upsilon(x,u,\upsilon)+\mu G_\upsilon(x,u,\upsilon), & \text{in } \Omega,\\ u=\upsilon=0, & \text{on } \partial\Omega. \end{cases}
$$

In this paper, our objective is to prove the existence of three solutions of problem (1.1) by applying three critical points theorem introduced by Ricceri [16]. Our result, under some suitable conditions, ensures the existence of an open interval $\Lambda\subset[0,+\infty)$ and a positive real number ρ such that, for each $\lambda\in\Lambda$, problem (1.1) admits at least three weak solutions whose norms in X are less than ρ. The purpose of the present paper is to generalize the main result of [15] to the general case.

Now, for every $x_0\in\Omega$ and choosing R_1, R_2 with $R_2>R_1>0$, such that $B(x_0,R_2)\subseteq\Omega$, where $B(x,R)=\{y\in R^N: |y-x|<R\}$, let

$$
(1.11)\qquad \alpha_i=\alpha_i(N,p_i,R_1,R_2)=\frac{C^{1/p_i}(R_2^N-R_1^N)^{1/p_i}}{R_2-R_1}\left(\frac{\pi^{N/2}}{\Gamma(1+N/2)}\right)^{1/p_i}, \quad 1\le i\le n,
$$

where Γ is the Gamma function. Moreover, assume that a, c are positive constants, define

$$
\begin{aligned}
y(x) &= \frac{a}{R_2-R_1}\left(R_2-\left\{\sum_{i=1}^N (x^i-x_0^i)^2\right\}^{1/2}\right),\ \forall x\in B(x_0,R_2)\backslash B(x_0,R_1),\\
A(c) &= \left\{(t_1,\dots,t_n)\in R^n: \sum_{i=1}^n |t_i|^{p_i}\le c\right\},\\
M^+ &= \max\left\{\frac{m_1^{p_i-1}}{p_i},\ i=1,...,n\right\},\ M_-=\min\left\{\frac{m_0^{p_i-1}}{p_i},\ i=1,...,n\right\}.
\end{aligned}
$$

Our main result is the following theorem.

Theorem 1.1 *Let $R_2>R_1>0$, such that $B(x_0,R_2)\subseteq\Omega$. Assume that there exist $n+2$ positive constants a, b, γ_i for $1\le i\le n$, with $\gamma_i<p_i$, $\sum_{i=1}^n(a\alpha_i)^{p_i}>bM^+/M_-$, and a function $\alpha(x)\in L^\infty(\Omega)$ such that*

(j1) *$F(x,t_1,...,t_n)\ge 0$, for a.e. $x\in\Omega\backslash B(x_0,R_1)$ and all $(t_1,...,t_n)\in[0,a]\times\cdots\times[0,a]$;*

(j2) $\sum_{i=1}^n(a\alpha_i)^{p_i}|\Omega|\sup_{(x,t_1,...,t_n)\in\Omega\times A(bM^+/M_-)}F(x,t_1,...,t_n)<b\int_{B(x_0,R_1)}F(x,a,...,a)dx;$

(j3) $F(x, t_1, ..., t_n) \leq \alpha(x) \left(1 + \sum_{i=1}^{n} |t_i|^{\gamma_i}\right)$ *for a.e.* $x \in \Omega$ *and all* $(t_1, ..., t_n) \in R^n$;

(j4) $F(x, 0, ..., 0) = 0$, *for a.e.* $x \in \Omega$.

Then there exist an open interval $\Lambda \subseteq [0, \infty)$ *and a positive real number* ρ *with the following property:*

for each $\lambda \in \Lambda$ *and for Carathéodory functions* $G_{u_i} : \Omega \times R^n \mapsto R$ *satisfying*

(j5) $\sup_{\{|t_i| \leq \xi, 1 \leq i \leq n\}} \left(\sum_{i=1}^{n} |G_{u_i}(\cdot, s, t)|\right) \in L^1(\Omega)$ *for all* $\xi > 0$,

there exists $\delta > 0$ *such that, for each* $\mu \in [0, \delta]$, *problem* (1.1) *has at least three weak solutions* $w_i = (u_{i1}, ..., u_{in}) \in X$ $(i = 1, 2, 3)$ *whose norms* $\|w_i\|$ *are less than* ρ.

2. Proof of the main result

Our analysis is based on the following modified form of Ricceri's three critical points theorem (Theorem 1 in [16]) and Proposition 3.1 of [17], which is our mainly tool in proving our main result.

Theorem 2.1 ([16], Theorem 1) *Let* X *be a reflexive real Banach space and* $\Phi : X \mapsto R$ *be a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on* X^* *and* Φ *is bounded on each bounded subset of* X; $\Psi : X \mapsto R$ *is a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact;* $I \subseteq R$ *an interval. Suppose that*

$$\lim_{\|x\| \to +\infty} (\Phi(x) + \lambda \Psi(x)) = +\infty$$

for all $\lambda \in I$, *and that there exists* $h \in R$ *such that*

$$\sup_{\lambda \in I} \inf_{x \in X} (\Phi(x) + \lambda(\Psi(x) + h)) < \inf_{x \in X} \sup_{\lambda \in I} (\Phi(x) + \lambda(\Psi(x) + h)). \tag{2.1}$$

Then, there exists an open interval $\Lambda \subseteq I$ *and a positive real number* ρ *with the following property: for every* $\lambda \in \Lambda$ *and every* C^1 *functional* $J : X \mapsto R$ *with compact derivative, there exists* $\delta > 0$ *such that, for each* $\mu \in [0, \delta]$ *the equation*

$$\Phi'(x) + \lambda \Psi'(x) + \mu J'(x) = 0$$

has at least three solutions in X *whose norms are less than* ρ.

Proposition 2.1 ([17], Proposition 3.1) *Suppose that X is a non-empty set and Φ, Ψ are two real functions on X. Assume that there exist $r > 0$ and $x_0, x_1 \in X$ such that*

$$\Phi(x_0) = -\Psi(x_0) = 0, \quad \Phi(x_1) > 1, \quad \sup_{x\in\Phi^{-1}([-\infty,r])} -\Psi(x) < r\frac{-\Psi(x_1)}{\Phi(x_1)}.$$

Then, for each h satisfying

$$\sup_{x\in\Phi^{-1}([-\infty,r])} -\Psi(x) < h < r\frac{-\Psi(x_1)}{\Phi(x_1)}$$

one has

$$\sup_{\lambda\geq 0}\inf_{x\in X}(\Phi(x) + \lambda(\Psi(x) + h)) < \inf_{x\in X}\sup_{\lambda\geq 0}(\Phi(x) + \lambda(\Psi(x) + h)).$$

Before giving the proof of Theorem 1.1, we define a functional and give a lemma.

The functional $H : X \to R$ is defined by

$$\begin{aligned} H(u_1,...,u_n) &= \Phi(u_1,...,u_n) + \lambda J(u_1,...,u_n) + \mu\psi(u_1,...,u_n) \\ &= \sum_{i=1}^{n}\frac{1}{p_i}\widehat{M_i}\left(\int_\Omega |\nabla u_i|^{p_i}\right) - \lambda\int_\Omega F(x,u_1,...,u_n)dx - \mu\int_\Omega G(x,u_1,...,u_n)dx \end{aligned} \tag{2.2}$$

for all $(u_1,...,u_n) \in X$, and where

$$\widehat{M_i}(t) = \int_0^t [M_i(s)]^{p_i-1}\,ds, \quad 1\leq i\leq n, \quad \text{for all } t\geq 0. \tag{2.3}$$

By the conditions (M) and (j3), it is easy to see that $H \in C^1(X,R)$ and a critical point of H corresponds to a weak solution of the system (1.1).

Lemma 2.2 *Suppose that there exist two positive constants a, b with $\sum\limits_{i=1}^{n}(a\alpha_i)^p > bM^+/M_-$, such that*

(j1) $F(x,t_1,...,t_n)\geq 0$, *for a.e.* $x\in\Omega\backslash B(x_0,R_1)$ *and all* $(t_1,...,t_n)\in[0,a]\times\cdots\times[0,a]$;

(j2) $\sum\limits_{i=1}^{n}(a\alpha_i)^{p_i}|\Omega| \sup\limits_{(x,t_1,...,t_n)\in\Omega\times A(bM^+/M_-)} F(x,t_1,...,t_n) < b\int\limits_{B(x_0,R_1)} F(x,a,...,a)dx.$

Then there exist $r > 0$ and $u_{i0} \in W_0^{1,p_i}(\Omega)$, $1\leq i\leq n$, such that

$$\Phi(u_{10},...,u_{n0}) > r$$

and

$$|\Omega|\sup_{(x,t_1,...,t_n)\in\Omega\times A(bM^+/M_-)} F(x,t_1,\ldots,t_n) \leq \frac{bM^+}{C}\frac{\int_\Omega F(x,u_{10},...,u_{n0})dx}{\Phi(u_{10},...,u_{n0})}.$$

Proof. Let

$$w_0(x) = \begin{cases} 0, & x \in \bar{\Omega}\backslash B(x_0, R_2), \\ \dfrac{a}{R_2 - R_1}\left(R_2 - \left\{\displaystyle\sum_{i=1}^{N}\left(x^i - x_0^i\right)\right\}^{1/2}\right), & x \in B(x_0, R_2)\backslash B(x_0, R_1), \\ a, & x \in B(x_0, R_1). \end{cases}$$

and $u_{10}(x) = \cdots = u_{n0}(x) = w_0(x)$. It is obvious to verify $(u_{10}, ..., u_{n0}) \in X$, and in particular, we have

$$\|u_{i0}\|_{p_i}^{p_i} = (R_2^N - R_1^N)\frac{\pi^{N/2}}{\Gamma(1 + N/2)}\left(\frac{a}{R_2 - R_1}\right)^{p_i}, \quad 1 \leq i \leq n. \tag{2.4}$$

Hence, it follows from (1.11) and (2.4) that

$$\|u_{i0}\|_{p_i}^{p_i} = \|w_0\|_{p_i}^{p_i} = \frac{(a\alpha_i)^{p_i}}{C}, 1 \leq i \leq n. \tag{2.5}$$

Under condition (M), by a direct computation, one has

$$M_-\left(\sum_{i=1}^{n}\|u_i\|_{p_i}^{p_i}\right) \leq \Phi\left(u_1, \ldots, u_n\right) \leq M^+\left(\sum_{i=1}^{n}\|u_i\|_{p_i}^{p_i}\right). \tag{2.6}$$

Put $r = \dfrac{bM^+}{C}$, and using the assumption of Lemma 2.2

$$\sum_{i=1}^{n}(a\alpha_i)^{p_i} > bM^+/M_-,$$

it follows from (2.5)and (2.6) that

$$\Phi\left(u_{10}, \ldots, u_{n0}\right) \geq M_-\left(\sum_{i=1}^{n}\|u_{i0}\|_{p_i}^{p_i}\right) = \frac{M_-}{C}\sum_{i=1}^{n}(a\alpha_i)^{p_i} > \frac{M_-}{C}\frac{bM^+}{M_-} = r.$$

Since, $0 \leq u_{i0} \leq a$, $1 \leq i \leq n$, for each $x \in \Omega$, condition (j1) ensures that

$$\int_{\Omega\backslash B(x_0,R_2)} F(x, u_{10}, \ldots, u_{n0})dx + \int_{B(x_0,R_2)\backslash B(x_0,R_1)} F(x, u_{10}, \ldots, u_{n0})dx \geq 0.$$

Hence, from condition (j2), we get

$$
\begin{aligned}
|\Omega| \sup_{(x,t_1,\ldots,t_n)\in\Omega\times A(bM^+/M_-)} F(x,t_1,\ldots,t_n) &< \frac{b}{\sum\limits_{i=1}^{n}(a\alpha_1)^p}\int_{B(x_0,R_1)} F(x,a,...,a)dx \\
&= \frac{bM^+}{C}\frac{\int_{B(x_0,R_1)} F(x,a,\ldots,a)dx}{M^+\sum\limits_{i=1}^{n}(a\alpha_1)^p/C} \\
&\leq \frac{bM^+}{C}\frac{\int_{\Omega\setminus B(x_0,R_1)} F(x,u_{10},...,u_{n0})dx+\int_{B(x_0,R_1)} F(x,u_{10},...,u_{n0})dx}{M^+\left(\sum\limits_{i=1}^{n}\|u_{i0}\|_{p_i}^{p_i}\right)} \\
&\leq \frac{bM^+}{C}\frac{\int_{\Omega} F(x,u_{10},...,u_{n0})dx}{\Psi(u_{10},...,u_{n0})}.
\end{aligned}
$$

Next, we can give the proof of our main result.

Proof of Theorem 1.1. For each $(u_1,\ldots,u_n)\in X$, $1\leq i\leq n$, assume that

$$
\begin{aligned}
\Phi(u_1,...,u_n) &= \sum_{i=1}^{n}\frac{\widehat{M_i}(||u_i||_{p_i}^{p_i})}{p_i}, \\
\Psi(u_1,...,u_n) &= -\int_{\Omega} F(x,u_1,...,u_n)dx, \\
J(u,v) &= -\int_{\Omega} G(x,u_1,...,u_n)dx.
\end{aligned}
$$

based on conditions of Theorem 1.1, it is easy to know that Φ is a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional. Additionally from Lemma 2.2 the Gâteaux derivative of Φ has a continuous inverse on X^*. Ψ and J are continuously Gâteaux differential functionals whose Gâteaux derivatives are compact. Obviously, Φ is bounded on each bounded subset of X. In particular, for each $(u_1,\ldots,u_n),(\xi_1,\ldots,\xi_n)\in X$, we have

$$
\begin{aligned}
\left\langle \Phi'(u_1,...,u_n),(\xi_1,...,\xi_n)\right\rangle &= \sum_{i=1}^{n}\left[M_i\left(\int_{\Omega}|\nabla u_i|^{p_i}\right)\right]^{p_i-1}\int_{\Omega}|\nabla u_i|^{p_i-2}\nabla u_i\nabla\xi_i \\
\left\langle \Psi'(u_1,...,u_n),(\xi_1,...,\xi_n)\right\rangle &= -\sum_{i=1}^{n}\int_{\Omega} F_{u_i}(x,u_1,...,u_n)\xi_i dx, \\
\left\langle J'(u_1,...,u_n),(\xi_1,...,\xi_n)\right\rangle &= -\sum_{i=1}^{n}\int_{\Omega} G_{u_i}(x,u_1,...,u_n)\xi_i dx.
\end{aligned}
$$

Hence, it follows from (1.4) that the weak solutions of problem (1.1) are exactly the solutions of the following equation

$$
\Phi'(u_1,...,u_n)+\lambda\Psi'(u_1,...,u_n)+\mu J'(u_1,...,u_n)=0\ .
$$

Thanks to (j3), for each $\lambda > 0$, one has

$$\lim_{\|(u,v)\|\to+\infty} \left(\lambda\Phi(u_1,\ldots,u_n) + \mu\Psi(u_1,\ldots,u_n)\right) = +\infty, \tag{2.7}$$

and so the first condition of Theorem 2.1 holds.

By Lemma 2.2, there exists $(u_{10},...,u_{n0}) \in X$ such that

$$\begin{aligned}\Phi(u_{10},...,u_{n0}) &= \sum_{i=1}^{n}\frac{\widehat{M}_i(||u_{i0}||_{p_i}^{p_i})}{p_i}\\ &\geq M_-\left(\sum_{i=1}^{n}||u_{i0}||_{p_i}^{p_i}\right) = \frac{M_-}{C}\sum_{i=1}^{n}(a\alpha_i)^{p_i}\\ &> \frac{M_-}{C}\frac{bM^+}{M_-} = \frac{bM^+}{C} > 0 = \Phi\left(0,...,0\right)\end{aligned} \tag{2.8}$$

and

$$|\Omega| \sup_{(x,t_1,\ldots,t_n)\in\Omega\times A(bM^+/M_-)} F(x,t_1,\ldots,t_n) \leq \frac{bM^+}{C}\frac{\int_\Omega F(x,u_{10},\ldots,u_{n0})dx}{\Phi\left(u_{10},\ldots,u_{n0}\right)}. \tag{2.9}$$

From (1.3), we have

$$\max_{x\in\bar{\Omega}}\{|u_i(x)|^{p_i}\} \leq C\,\|u\|_{p_i}^{p_i}, \quad 1\leq i\leq n,$$

for each $(u_1,\ldots,u_n) \in X$. One has

$$\max_{x\in\bar{\Omega}}\left\{\sum_{i=1}^{n}\frac{|u_i\,(x)|^{p_i}}{p_i}, 1\leq i\leq n\right\} \leq C\left\{\sum_{i=1}^{n}\frac{\|u\|_{p_i}^{p_i}}{p_i}, 1\leq i\leq n\right\}, \tag{2.10}$$

for each $(u_1,\ldots,u_n) \in X$.

Suppose that $r = \dfrac{bM^+}{C}$, for each $(u_1,...,u_n) \in X$ such that

$$\Phi(u_1,...,u_n) = \sum_{i=1}^{n}\frac{\widehat{M}_i(||u_i||_{p_i}^{p_i})}{p_i} \leq r.$$

Thanks to (2.10), we get

$$\sum_{i=1}^{n}|u_i(x)|^{p_i} \leq C\sum_{i=1}^{n}\|u_i\|_{p_i}^{p_i} \leq \frac{Cr}{M_-} = \frac{C}{M_-}\frac{bM^+}{C} = \frac{bM^+}{M_-}. \tag{2.11}$$

Then, from (2.9) and (2.11), we obtain

$$\begin{aligned}
\sup_{(u_1,\ldots,u_n)\in\Phi^{-1}(-\infty,r)} (-\Psi(u_1,\ldots,u_n)) &= \sup_{\{(u_1,\ldots,u_n)|\Phi(u_1,\ldots,u_n)\le r\}} \int_\Omega F(x,u_1,\ldots,u_n)dx \\
&\le \sup_{\left\{(u_1,\ldots,u_n)|\sum_{i=1}^{n}|u_i(x)|^{p_i}\le bM^+/M_-\right\}} \int_\Omega F(x,u_1,\ldots,u_n)dx \\
&\le \int_\Omega \sup_{(t_1,\ldots,t_n)\in A(bM^+/M_-)} F(x,t_1,\ldots,t_n)dx \\
&\le |\Omega| \sup_{(x,t_1,\ldots,t_n)\in\Omega\times A(bM^+/M_-)} F(x,t_1,\ldots,t_n) \\
&\le \frac{bM^+}{C}\frac{\int_\Omega F(x,u_{10},\ldots,u_{n0})dx}{\Phi(u_{10},\ldots,u_{n0})} \\
&= r\frac{-\Psi(u_{10},\ldots,u_{n0})}{\Phi(u_{10},\ldots,u_{n0})}.
\end{aligned}$$

Consequently we have

$$\sup_{\{(u_1,\ldots,u_n)|\Phi(u_1,\ldots,u_n)\le r\}} (-\Psi(u_1,\ldots,u_n)) < r\frac{-\Psi(u_{10},\ldots,u_{n0})}{\Phi(u_{10},\ldots,u_{n0})}. \tag{2.12}$$

Fix h such that

$$\sup_{\{(u_1,\ldots,u_n)|\Phi(u_1,\ldots,u_n)\le r\}} (-\Psi(u_1,\ldots,u_n)) < h < r\frac{-\Psi\left(u_{10},\ldots,u_{n0}\right)}{\Phi\left(u_{10},\ldots,u_{n0}\right)},$$

by (2.8), (2.12) and Proposition 2.1, with $(u_{11},\ldots,v_{n1}) = (0,\ldots,0)$ and $(u_1^*,\ldots,u_n^*) = (u_{10},\ldots,u_{n0})$, we have

$$\sup_{\lambda\ge 0}\inf_{x\in X}\left(\Phi\left(x\right)+\lambda\left(h+\Psi\left(x\right)\right)\right) < \inf_{x\in X}\sup_{\lambda\ge 0}\left(\Phi\left(x\right)+\lambda\left(h+\Psi\left(x\right)\right)\right), \tag{2.13}$$

and so the condition (2.1) of Theorem 2.1 holds.

Now, all the conditions of Theorem 2.1 hold. Hence, applying Theorem 2.1, our conclusion is obtained.

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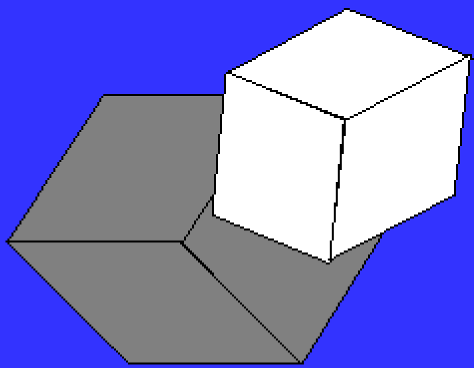

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